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Closed Convex Hulls and Polyhedral Approximation in C^* - Algebras and Banach Spaces with Nontrivial Twisted Sums

C^*الهياكل المحدبة المغلقة وتقريب متعدد السطوح في جبريات - وفضاءات باناخ مع مجموع الالتواء غير البديهي

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Dedication

To my parents, brothers and sisters

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Abstract

We study closed convex hulls of unitary orbits in various C^*-algebras. For unital C^*-algebras with real rank zero and a faithful tracial state determining equivalence of projections, a notion of majorization which describes the closed convex hulls of unitary orbits for self-adjoint operators are considered. Other notions of majorization are examined in these C^*-algebras. We show that norms on certain Banach spaces can be approximated uniformly, and with arbitrary precision, on bounded subsets by C^∞ smooth norms and polyhedral norms. We employ the pinching theorem, ensuring that some operators admit *any* sequence of contractions as an operator diagonal. Nontrivial twisted are shown.

الخلاصة

درسنا الهياكل المعيارية المغلقة إلى المسارات الواحدية في جبريات - C^* الأحادية مع صفر الرتبة الحقيقي والحالة C^* المتنوعة . أعتبرنا لأجل جبريات - الأثرية المعتقدة المحددة لتكافؤ المساقط والفكرة الرائدية لوصف الهياكل المحدبة المغلقة للمسارات الأحادية لأجل مؤثرات المرافق الذاتي تمت مدروسية أفكار أخرى هذه. أوضحنا أن النظام عن فضاءات باناخ المعينة يمكن C^* للرائدية في جيريات - أن تقرب بانتظام ومع دقة اختيارية وعلى الفئات الجزئية المحدودة بواسطة النظائم والنظائم متعددة السطوح. استخدمنا مبرهنة الانضغاط وضامنين أن C^∞ الملساء بعض المؤثرات تسمح لأي متتالية للانكماش كمؤثر قطري. أوضحنا مجموع الالتواء غير البديهي.

The Contents

Chapter 1

Unitary Orbit In C^*-Algebras of Real Rank Zero

Combining the ideas with the Dixmier property, we demonstrate unital, infinite dimensional C^*-algebras of real rank zero and strict comparison of projections with respect to a faithful tracial state must be simple and have a unique tracial state, closed convex hulls of unitary orbits of self-adjoint operators are fully described in unital, simple, purely infinite C^*-algebras.

Section (1.1): Scalars and Unitary Orbits of Convex Hulls

Unitary orbits of operators are important objects that provide significant information about operators. In the infinite dimensional setting, the norm closure of the unitary orbits must be taken as unitary groups are no longer compact. For all intents and purposes, two operators that are approximately unitarily equivalent (that is, have the same closed unitary orbits) may be treated as the same operator inside a C^* $-$algebra and the question of when two (normal) operators are approximately unitarily equivalent has been studied.

When two operators are not approximately unitarily equivalent, it is interesting to ask, "How far are the operators from being approximately unitarily equivalent?" This question is quantified by describing the distance between the operators' unitary orbits and has a long history. For self-adjoint matrices S and T with eigenvalues $\{\mu_k\}_{k=1}^n$ and $\{\lambda_k\}_{k=1}^n$ respectively, the distance between the unitary orbits of S and T was computed to be the optimal matching distance

$$\min_{\sigma\in S_n} \max\{|\lambda_k - \mu_{\sigma(k)}| \mid k \in \{1,\dots,n\}\}$$

where S_n is the permutation group on $\{1, \dots, n\}$. However, if S and T are normal matrices, the distance between the unitary orbits of S and

T need not equal the optimal matching distance. For bounded normal operators on Hilbertspace, results have been obtained analogous to the known matricial results. This question has been active in other C^*-algebras where the most recent work has made use of K-theoretic properties and ideas.

Another important concept is that of majorization for self-adjoint matrices. A notion of majorization for real-valued functions in $L_1[0, 1]$ was first developed by Hardy, Littlewood, and Pólya using non-increasing rearrangements and this notion has been widely studied. When applied to self-adjoint matrices through their eigenvalues, a fascinating concept is obtained. Majorization of self-adjoint matrices has been thoroughly analyzed and has relations to a wide range of problems in linear algebra, such as classical theorem of Schur and Horn characterizing the possible diagonal n-tuples of a self-adjoint matrix based on its eigenvalues and applications to generalized numerical ranges of matrices.

Majorization has an immediate analogue in II_1 factors by replacing eigenvalues with spectral distributions. By using the notion of majorization via eigenvalue functions (also known as spectral scales) of self-adjoint operators in II_1 factors, several analogues of matricial results have been obtained. For example, an analogue of the Schur-Horn Theorem for II_1 factors was first postulated and proved by Ravichandran and analogues of generalized numerical ranges were developed.

The notion of majorization of self-adjoint operators in both matrix algebras and II_1 factors as a deep connection with unitary orbits. Indeed, given two self-adjoint operators S and T, it was shown for matrix algebras and II_1 factors that T majorizes S if and only if S is in the (norm) closure of the convex hull (the convex hull may be defined as the intersection of all convex sets containing x or as the set of all convex combinations of point in x) [5] of the unitary orbit of T, denoted $\overline{\text{conv}}(\mathcal{U}(T))$. Consequently, the question of whether T majorizes S is a question of whether S can be obtained by 'averaging' copies of T.

Analysis of the closure of convex hulls of unitary orbits has yielded some interesting results. For example, the Dixmier property for a C*-algebra asks that the centre of the C*-algebra interests every such orbit. One need only consider self-adjoint operators to verify the Dixmier property show that a unital C*-algebra A has the Dixmier property if and only if $\mathfrak{A}$ is simple and has at most one faithful tracial state.

We describe the closure of convex hulls of unitary orbits of self-adjoint operators in various C*-algebras. Taking inspiration from von Neumann algebra theory, we will focus on C*-algebras that behave like type III and type II_1 factors. In particular, unital, simple, purely infinite C*-algebras are our analogues of type III factors and unital C*-algebras with real rank zero and a faithful tracial state determining equivalence of projections are our analogues of type II_1 factors.

Develops and extends the necessarily preliminary results on majorization of self-adjoint operators in matrix algebras and II_1 factors. In particular, the notion of eigenvalue functions is adapted from II_1 factors to C*-algebras with faithful tracial states by replacing spectral distributions with dimension functions. The properties of eigenvalue functions are immediately transferred to this setting.

There are scalars in convex hulls of unitary orbits in C*-

algebras with faithful tracial states. Notice if $\mathfrak{A}$ is a unital C*-algebra with a faithful tracial state τ and $T \in \mathfrak{A}$, then $\tau(S) = \tau(T)$ for all $S \in \overline{\text{conv}}(\mathcal{U}(T))$. Consequently $\overline{\text{conv}}(\mathcal{U}(T)) \cap \{\mathbb{C}I_{\mathfrak{A}}\}$ is either empty or $\{\tau(T)I_{\mathfrak{A}}\}$. Using an averaging process along with manipulations of projections, it is demonstrated that if $\mathfrak{A}$ is a unital, infinite dimensional C*-algebra with real rank zero and strict comparison of projections with respect to a faithful tracial state , then $\tau(T) \in \overline{\text{conv}}(\mathcal{U}(T))$ for all $T \in \mathfrak{A}$. Combined with the Dixmier property, this implies $\mathfrak{A}$ must be simple and τ must be the unique faithful tracial state on $\mathfrak{A}$. We investigated the ability of faithful tracial states to imply simplicity of C*-algebras.

We analyze $\overline{\text{conv}}(\mathcal{U}(T))$ for self-adjoint *T* in unital C*-algebras $\mathfrak{A}$ that have real rank zero and a faithful tracial state τ with the property that if $P, Q \in \mathfrak{A}$ are projections, then $\tau(P) \leq \tau(Q)$ if and only if P is Murray-von Neumann equivalent to a subprojection of Q. In particular, theorm (1.1.46) shows for such C*-algebras that $S \in \overline{\text{conv}}(\mathcal{U}(T))$if and only if *T* majorizes *S* with respect to τ. $\mathfrak{A}$ though the assumptions on $\mathfrak{A}$ are restrictive in the classification theory world, they do apply to several C*-algebras such as UHF C*-algebras, the Bunce-Deddens C*-algebras, irrational rotations algebras, and many others.

Trying to generalize Theorem (1.1.37) to other C*-algebras may be a difficult task. Indeed, it is the case that there are self-adjoint operators with the same eigenvalue functions that are not approximately unitarily equivalent when the assumption '$\tau(P) = \tau(Q)$ implies P and Q are equivalent' is removed. In addition, the question of characterizing $\overline{\text{conv}}(\mathcal{U}(T))$ appear very complicated if $\mathfrak{A}$

has more than one tracialstate as, by above discussions, $\overline{\text{conv}}(\mathcal{U}(T)) \cap \{\mathbb{C}I_{\mathfrak{A}}\} = \emptyset$.

We devote to investigating other closed orbits and notions of majorization of operators. We begin by using eigenvalue, which computes the distance between unitary orbits of self-adjoint operators via an analogue of the optimal matching distance. In addition, an analogue of singular value decomposition of matrices is obtained. Furthermore, descriptions of when one operator's eigenvalue (singular value) function dominants another operator's eigenvalue (respectively singular value) function and when one operator (absolutely) submajorizes another operator are described.

We describing $\overline{\text{conv}}(\mathcal{U}(T))$ for self-adjoint operators T in unital, simple, purely infinite C*-algebras. In particular, $\overline{\text{conv}}(\mathcal{U}(T))$ is precisely all self-adjoint operators S such that the spectrum of S is contained in the convex hull of the spectrum of T.

We will extend the notion and properties of eigenvalue functions to C*-algebras with faithful tracial states.

Definition (1.1.1)[1]: For a unital C*-algebra $\mathfrak{A}$ and an element $T \in \mathfrak{A}$, the *unitary orbit* of T is

$$\mathcal{U}(T) := \{U^*T\,U \mid U \text{ a unitary in } \mathfrak{A}\}.$$

The *closed unitary orbit* of $T \in \mathfrak{A}$ is $\mathcal{O}(T) := \mathcal{U}(T)$, the norm closure of (T).

The convex hull of $\mathcal{U}(T)$ will be denoted by conv$(\mathcal{U}(T))$ and its norm closure by $\overline{\text{conv}}(\mathcal{U}(T))$.

The main component is the generalization of the following notions from tracial von Neumann algebras to tracial C*-algebras.

Definition (1.1.2)[1]: Let $\mathfrak{M}$ be a von Neumann algebra a tracial state τ .

(i) For a self-adjoint operator $T \in \mathfrak{M}$, the *eigenvalue function of* T *associated with* τ , denoted λ_T^τ is defined for $s \in [0,1)$ by

$$\lambda_T^\tau(s) := \inf\{t \in \mathbb{R} | m_T((t,\infty)) \leq s\}$$

where m_T is the spectral distribution of T with respect to τ .

(i) For an arbitrary $T \in \mathfrak{M}$, the *singular value function of* T *associated with* τ, denoted μ_τ is defined for $s \in [0,1)$ by

$$,\mu_T^\tau(s) := \lambda_{|T|}^T(s).$$

Example (1.1.3)[1]: Let $T \in \mathcal{M}_n(\mathbb{C})$ be self-adjoint with eigenvalues $\{\lambda_k\}_{k=1}^n$ where $\lambda_k \geq \lambda_{k+1}$ for all k. If τ is the normalized trace on $\mathcal{M}_n(\mathbb{C})$, then $\lambda_T^\tau(s) = \lambda_k$ for all $s \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$.Similarly ,if $T \in \mathcal{M}_n(\mathbb{C})$has singular values $\{\sigma_k\}_{k=1}^n$ where $\sigma_k \geq \sigma_{k+1}$ for all k, then $\mu_T^\tau(s) = \mu_k$ for all $s \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$.

Example (1.1.4)[1]: Let $\mathfrak{M} = L_\infty[0,1]$ equipped with the tracial state τ defined by integrating against the Lebesgue measure m. If $f \in \mathfrak{M}$ is real-valued, then $\lambda_f^\tau(s) = f^*(s)$ where f^* is the non-increasing rearrangement of f, which may be defined by

$$f^*(s) := \inf\{t \in \mathbb{R} \mid m(\{x \in [0,1] \mid f(x) > t\}) \leq s\}.$$

It can be shown (see Theorem (1.1.9) that f^* is a non-increasing, right continuous function. Consequently, if f is non-increasing and right continuous, then $f = f^*$.

To generalize these notions to C*-algebras with faithful tracial states, we will use the following as a replacement for spectral distributions.

Definition (1.1.5)[1]: Let $\epsilon > 0$ and let f_ϵ denote the continuous function on $[0, \infty)$ such that $f_\epsilon(x) = 1$ if $x \in [\epsilon,\infty)$, $f\epsilon(x) = 0$ if $x \in \left[0,\frac{\epsilon}{2}\right]$,and $f_\epsilon(x)$ is linear on $\left(\frac{\epsilon}{2},\epsilon\right)$.

Let $\mathfrak{A}$ be a unital C*-algebra with faithful tracial state τ. The *dimension function associated with* τ, denoted d_τ , is defined for positive operators $A \in \mathfrak{A}$ by

$$d_\tau(A) := \lim_{\epsilon \to 0} \tau(f_\epsilon(A)) .$$

Definition (1.1.6)[1]. Let $\mathfrak{A}$ be a unital C*-algebra with a faithful tracial state .

(i) For a self-adjoint operator $T \in \mathfrak{A}$, the *eigenvalue function of* T *associated with* τ , denoted λ_T^τ, is defined for $s \in [0,1)$ by

$$\lambda_T^\tau(s) := \inf\{t \in \mathbb{R} | d\tau\ ((T - tI_{\mathfrak{A}})_+) \leq s\}$$

where $(T - tI_{\mathfrak{A}})_+$ denotes the positive part of $T - tI_{\mathfrak{A}}$.

(ii) For an arbitrary $T \in \mathfrak{A}$, the *singular value function of* T *associated with* τ , denoted μ_T^τ, is defined for $s \in [0,1)$ by

$$\mu_T^\tau(s) := \lambda_{|T|}^\tau(s).$$

Lemma (1.1.8) will demonstrate that Definitions (1.1.2) and (1.1.6) agree when $\mathfrak{A}$ is a von Neumann algebra.

Example (1.1.7)[1]: Let $\mathfrak{A}$ be a unital C*-algebra with a faithful tracial state τ . Let $\{\lambda_k\}_{k=1}^n \subseteq \mathbb{R}$ be such that $\lambda_k \geq \lambda_{k+1}$for all k and let $\{P_k\}_{k=1}^n \subseteq \mathfrak{A}$ be a collection of pairwise orthogonal projections such that $\sum_{k=1}^n P_k I_{\mathfrak{A}}$. For each $k \in \{0, 1, \ldots, n\}$, let $s_k = \sum_{j=1}^k \tau(P_j)$. If $T = \sum_{k=1}^n \lambda_k P_k$, then $\lambda_T^\tau(s) = \lambda_k$ for all $s \in [s_{k-1}, s_k)$.

Eigenvalue and singular value functions have several important properties. Although most (if not all) of these properties can be demonstrated using C*-algebraic techniques, we will appeal to von Neumann algebra theory to shorten the exposition.

For a unital C*-algebra $\mathfrak{A}$ with a faithful tracial state , let $\pi_\tau: \mathfrak{A} \to \mathcal{B}(L_2(\mathfrak{A}, \tau))$ be the GNS representation of $\mathfrak{A}$ with respect to τ. Note π_τ is faithful and τ is a vector state on $\mathcal{B}(L_2(\mathfrak{A}, \tau))$. If $\mathfrak{M}$ is the von Neumann algebra generated by $\pi_\tau(\mathfrak{A})$, specifically $\pi_\tau(\mathfrak{A})''$, then τ extends to a tracial state on $\mathfrak{M}$.

Lemma (1.1.8)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with faithful tracial state τ and let $\mathfrak{M}$ be the von Neumann algebra described above. If $T \in \mathfrak{A}$ is self-adjoint, then*

$$\lambda_T^\tau(s) = \lambda_\pi^\tau(T)(s)$$

for all $s \in [0,1)$, where $\lambda_{\pi_\tau(T)}^\tau$ is as defined in Definition (1.1.2).

Proof. If $m_{\pi_\tau}(T)$ denote s the spectral distribution of $\pi_\tau(T)$ with respect to τ, we obtain for all $t \in \mathbb{R}$ that

$$\begin{aligned} d_\tau((T - tI_{\mathfrak{A}})_+) &= \lim_{\epsilon\to 0} \tau\left(f_\epsilon((T - tI_{\mathfrak{A}})_+)\right) \\ &= \lim_{\epsilon\to 0} \tau\left(\pi_\tau((T - tI_{\mathfrak{A}})_+)\right) \\ &= \lim_{\epsilon\to 0} \tau\left(\pi_\tau((T - tI_{\mathfrak{A}})_+)\right) = m_{\pi_\tau}(T)((t,\infty)) \end{aligned}$$

as $f_\epsilon((T - tI_{\mathfrak{A}})_+)$ converges in the weak*-topology to the spectral projection of $\pi_\tau(T)$ onto (t,∞). The result then follows by definitions.

Using Lemma (1.1.8), the known properties of eigenvalue and singular value functions on von Neumann algebras automatically transfer to the tracial C*-algebra setting.

Theorem (1.1.9)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with faithful tracial state τ and let $T, S \in \mathfrak{A}$ be self-adjoint operators. Then:*

(i) *The map $s \to \lambda_T^\tau(s)$ is non-increasing and right continuous.*

(ii) *If $T \geq 0$, $\lim_{s\searrow 0} \lambda_T^\tau(s) = \|T\|$and $\lambda_T^\tau(s) \geq 0$ for all $s \in [0,1)$.*

(iii) *If $\sigma(T)$ denotes the spectrum of T, then $\lim_{s\nearrow 1} \lambda_T^\tau(s) = inf\{t | t \in \sigma(T)\}$and $\lim_{s\searrow 0} \lambda_T^\tau(s) = \sup\{t | t \in \sigma(T)\}$.*

(iv) *If $S \leq T$, then $\lambda_S^\tau(s) \leq \lambda_T^\tau(s)$for all $s \in [0,1)$.*

(v) *If $\alpha \geq 0$, then $\lambda\tau\ \lambda_{\alpha T}^\tau(s) = \alpha\lambda_T^\tau(s)$ and $\lambda_{T+\alpha I_{\mathfrak{A}}}^\tau = \lambda_T^\tau(s) + \alpha$ for all $s \in [0,1)$.*

(vi) *$\lambda_{S+T}^\tau(s + t) \leq \lambda_S^\tau(s) + \lambda_T^\tau(t)$ for all $s, t \in [0,1)$ with $s + t < 1$.*

(vii) *$|\lambda_S^\tau(s) - \lambda_T^\tau(s)| \|S - T\|$ for all $s \in [0,1)$.*

(viii) $\tau\,(f(T)) = \int_0^1 f(\lambda_T^\tau(s))ds$ *for all continuous functions* $f: \mathbb{R} \to \mathbb{R}$.

(ix) *If* $T \geq 0$*, then* $\lambda_{V^*TV}^\tau(s) \leq \|V\|^2\, \lambda_T^\tau(s)$ *for all* $s \in [0,1)$ *and* $V \in \mathfrak{A}$.

(x) *If* $U \in \mathfrak{A}$ *is a unitary, then* $\lambda_{U^*TU}^\tau s) = \lambda_T^\tau(s)$ *for all* $s \in [0,1)$.

(xi) *If* $T \geq 0, \lambda_{f(T)}^\tau(s) = f(\lambda_T^\tau(s))$ *for all* $s \in [0,1)$ *and all continuous increasing functions* $f : [0,\infty) \to \mathbb{R}$ *with* $f(0) \geq 0$.

(xii) *If* $S, T \geq 0$*, then* $\int_0^t f\left(\lambda_{S+T}^\tau(s)\right)ds \leq \int_0^t f(\lambda_S^\tau(s) + \lambda_T^\tau(s))ds$ *for all* $t \in [0,1]$ *and all continuous, increasing, convex functions* $f : \mathbb{R} \to \mathbb{R}$.

(xiii) *If* $S, T \geq 0$*, then* $\int_0^t f\left(\lambda_{S+T}^\tau(s)\right)ds \leq \int_0^t f\left(\lambda_S^\tau(s)\right) + f(\lambda_T^\tau(s))ds$ *for all* $t \in [0,1]$ *and all increasing concave functions* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(0) = 0$.

Theorem (1.1.10)[1]: *Let* $\mathfrak{A}$ *be a unital* C^**-algebra with faithful tracial state* τ *and let* $T, S, R \in \mathfrak{A}$*. Then:*

(i) $\mu_T^\tau(s) = \mu_{|T|}^\tau(s) = \mu_{T^*}^\tau(s)$ *for all* $s \in [0,1)$.

(ii) $\mu_{\alpha T}^\tau(s) = |\alpha|\mu_T^\tau(s)$ *for all* $s \in [0,1)$ *and* $\alpha \in \mathbb{C}$.

(iii) $\mu_{RTS}^\tau(s)\|R\|\|S\|\mu_T^\tau(s)$ for all $s \in [0,1)$.

(iv) $\mu_{ST}^\tau(s + t) \leq \mu_S^\tau(s)\mu_T^\tau(t)$ *for all* $s, t \in [0,1)$ *with* $s + t < 1$

(v) $\int_0^t f\left(\mu_{S+T}^\tau(s)\right)ds \leq \int_0^t f(\mu_S^\tau(s) + \mu_T^\tau(s))ds$ *for all* $t \in [0,1]$ *and all continuous increasing, convex functions* $f: \mathbb{R} \to \mathbb{R}$.

(vi) $\int_0^t f\left(\mu_{S+T}^\tau(s)\right)ds \leq \int_0^t f\left(\mu_S^\tau(s)\right) + f(\mu_T^\tau(s))ds$ *for all* $t \in [0,1]$ *and all increasing concave functions* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(0) = 0$.

To define a notion of majorization for self-adjoint operators, we recall the following.

Definition (1.1.11)[1]: For real-valued functions $f, g \in L_\infty[0,1]$, it is said that f *majorizes* g, denoted $g \prec f$, if

$$\int_0^t g^*(s)ds \le \int_0^t f^*(s)ds \text{ for all } t \in [0,1] \text{and } \int_0^1 g^*(s)ds = \int_0^1 f^*(s)ds$$

where f^* and g^* are the non-increasing rearrangements of f and g (see Example (1.1.4)).

The following example provides some intuition for majorization.

Example (1.1.12)[1]: Let $f \in L_\infty[0,1]$ be a real-valued function and fix $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$. For $k \in \{1,\ldots,n\}$, let

$$\alpha_k = \frac{1}{s_k - s_{k-1}} \int_{s_{k-1}}^{s_k} f^*(s)ds$$

and let $g = \sum_{k=1}^n 1_{[s_{k-1},s_k)}$, where 1_X denotes the characteristic function of X.

We claim that $\prec f$. Note g is non-increasing and right continuous so $g^* = g$.

Furthermore, note

$$\int_0^{s_k} f^*(s)ds = \int_0^{s_k} g(s)ds$$

for all $k \in \{0,1,\ldots,n\}$.

Suppose $t \in [s_{k-1}, s_k]$. *If* $g(t) \le f^*(t)$, *then* $g(s) \le f^*(s)$ *for all* $s \in [s_{k-1}, t]$ *as* g is constant on $s_{k-1}, s_k]$ and f^* is non-increasing. Thus

$$\int_0^t f^*(s) - g(s)ds = \int_{s_{k-1}}^t f^*(s) - g(s)ds \ge 0.$$

Otherwise $g(t) > f^*(t)$. Hence $g(s) > f^*(s)$ for all $s \in [t, s_k)$ as g is constant on $[s_{k-1}, s_k)$ and f^* is non-increasing. Thus

$$\int_0^t f^*(s) - g(s)ds = \int_{s_{k-1}}^t f^*(s) - g(s)ds \ge \int_{s_{k-1}}^{s_k} f^*(s) - g(s)ds = 0.$$

Hence $g \prec f$ as claimed.

Definition (1.1.13)[1]: Let $\mathfrak{A}$ be a unital C*-algebra with a faithful tracial state τ . For self-adjoint elements $T, S \in \mathfrak{A}$, it is said that T *majorizes* S *with respect to* τ, denoted $\prec_\tau T$, if $\lambda_S^\tau \prec \lambda_T^\tau$.

Example(1.1.14)[1]: Let $T, S \in \mathcal{M}_n(\mathbb{C})$ be self-adjoint with eigenvalues $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$ and $\{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$ respectively. If τ is the normalized trace on $M_n(C)$, then $S \prec_\tau T$ if and only if

$$\sum_{k=1}^{m} \mu_k \leq \sum_{k=1}^{m} \lambda_k \text{ for all } m \in \{1, \ldots, n-1\} \text{and} \sum_{k=1}^{n} \mu_k = \sum_{k=1}^{n} \lambda_k .$$

There are several equivalent formulations of majorization of self-adjoint operators in tracial von Neuman algebras as the following theorem demonstrates.

Theorem (1.1. 15)[1]: *Let $\mathfrak{M}$ be a von Neumann algebra with a faithful tracial state τ . Let $T , S \in \mathfrak{M}$ be positive operators. Then the following are equivalent:*

(i) $S \prec_\tau T$.

(ii) $\tau\left(\left(S - rI_{\mathfrak{M}}\right)_+\right) \leq \tau\left(\left(T - rI_{\mathfrak{M}}\right)_+\right)$ *for all* $r > 0$ *and* $\tau(T) = \tau(S)$.

(iii) $\tau(f(S)) \leq \tau(f(T))$ *for every continuous convex function* $f: \mathbb{R} \to \mathbb{R}$.

(iv) *If* $\mathfrak{M}$ *is a factor, then for all self-adjoint* $S, T \in \mathfrak{M}, S \prec_\tau T$ *is equivalent to:*

$S \in \overline{\text{conv}}(\mathcal{U}(T))$.

$S \in \overline{\text{conv}(\mathcal{U}(T))}^{w^*}$.

There exists a unital, trace-preserving, positive map $\Phi : \mathfrak{M} \to \mathfrak{M}$ *such that* $\Phi(T) = S$.

(v) There exists a unital, trace-preserving, completely positive map $\Phi : \mathfrak{M} \to \mathfrak{M}$ such that $\Phi(T) = S$.

We see to what extent Theorem (1.1.15) generalizes to tracial C^*-algebras. Note Lemma (1.1.8) immediately implies the following.

Corollary (1.1.16)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with a faithful tracial state τ. Let $T, S \in \mathfrak{A}$ be positive operators. Then the following are equivalent:*

(i) $S \prec_\tau T$.

(ii) $\tau\left(\left(S - rI_{\mathfrak{M}}\right)_+\right) \leq \tau\left(\left(T - rI_{\mathfrak{M}}\right)_+\right)$ for all $r > 0$ and $\tau(T) = \tau(S)$.

(iii) $\tau(f(S)) \leq \tau(f(T))$ for every continuous convex function $f : \mathbb{R} \to \mathbb{R}$.

For the remaining equivalences in Theorem (1.1.15), note part (v) does not make sense in an arbitrary C^*-algebra. We will mainly focus on part (iv) of Theorem (1.1.15) to which we have the following preliminary result.

Lemma (1.1.17)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with a faithful tracial state τ and let $T \in \mathfrak{A}$ be self-adjoint. Then*

(i) If $\lambda \in \mathbb{R}$, then $\lambda I_{\mathfrak{A}} \prec_\tau T$ if and only if $\lambda = \tau(T)$

(ii) If $S \in \overline{\text{conv}}(\mathcal{U}(T))$, then $S = S^*$ and $\prec_\tau T$.

Proof. The first claim follows from Example (1.1.12) and part (viii) of Theorem (1.1.9).

For the second claim, suppose $\{U_k\}_{k=1}^n \subseteq \mathfrak{A}$ are unitary operators, $\{t_k\}_{k=1}^n \subseteq [0,1]$ are such that $\sum_{k=1}^n t_k$, and $R = \sum_{k=1}^n t_k\, U_k^* T U_k$. Then R is self-adjoint and $\tau(R) = \tau(T)$. Moreover, by parts (v, x, xii) of Theorem (1.1.9),

$$\int_0^t \lambda_R^\tau(s)ds \leq \int_0^t \sum_{k=1}^n t_k\, \lambda_{U_k^* T U_k}^\tau(s)ds = \int_0^t \sum_{k=1}^n t_k\, \lambda_T^\tau(s)ds$$

$$= \int_0^t \lambda_T^\tau(s)ds$$

for all $t \in [0,1]$. Thus $R \prec_\tau T$ for all $R \in \text{conv}(\mathcal{U}(T))$.

If $S \in \overline{\text{conv}}(\mathcal{U}(T))$, then clearly $S = S^*$. The fact that $R \prec_\tau T$ then follows by part (vii), the above paragraph, the fact that τ is norm continuous, and the fact that

$$\left|\int_0^t f(s) - g(s)ds\right| \leq \|f - g\|_\infty \leq 0$$

for all $t \in [0,1]$ and all bounded functions f and g.

It is unlikely that parts (vi, viii) of Theorem (1.1.15) holds in arbitrary tracial C^*-algebras due to the lack of ability to take weak*-limits of convex combinations of inner automorphisms. However, we have the following .

Proposition (1.1.18)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with a faithful tracial state τ and let $\varphi : \mathfrak{A} \to \mathfrak{A}$ be a positive map. Then φ is unital and τ -preserving if and only if $\varphi(T) \prec_\tau T$ for all positive operators $T \in \mathfrak{A}$.*

Proof: Suppose φ is unital, positive, and τ -preserving. Let $T \in \mathfrak{A}$ be positive.

Then $\tau(\varphi(T)) = \tau(T)$. Furthermore, for all $r > 0$ notice

$$\varphi(T) - rI_{\mathfrak{A}} = \varphi(T - rI_{\mathfrak{A}}) \leq \varphi((T - rI_{\mathfrak{A}})_+)$$

so

$$\tau((\varphi(T) - rI_{\mathfrak{A}})_+) \leq \tau\left(\varphi((T - rI_{\mathfrak{A}})_+)\right) = \tau((T - rI_{\mathfrak{A}})_+).$$

Hence Corollary (1.1.16) implies that $(T) \prec_\tau T$.

Conversely, suppose $\varphi : \mathfrak{A} \to \mathfrak{A}$ is a positive map such that $\varphi(T) \prec_\tau T$ for all positive operators $T \in \mathfrak{A}$. By part (viii) of Theorem (1.1.9),

$$\tau\,(\varphi(T)) = \int_0^1 \lambda^{\tau}_{\varphi(T)}(s)ds = \int_0^1 \lambda^{\tau}_{T}(s)ds = \tau(T)$$

for all positive operators $T \in \mathfrak{A}$. Hence φ is τ -preserving. Since $\lambda^{\tau}_{I_{\mathfrak{A}}}(s) = 1$ for all $s \in [0,1)$, by parts (i, ii) of Theorem (1.1.9),

$$\|\varphi(I_{\mathfrak{A}})\| = \lim_{t\searrow 0} \int_0^1 \lambda^{\tau}_{\varphi(I_{\mathfrak{A}})}(s)ds \le \lim_{t\searrow 0} \int_0^1 \lambda^{\tau}_{I_{\mathfrak{A}}}(s)ds = 1.$$

Hence $0 \le \varphi(I_{\mathfrak{A}}) \le I_{\mathfrak{A}}$. If $\varphi(I_{\mathfrak{A}}) \neq I_{\mathfrak{A}}$, then

$$0 = \tau(I_{\mathfrak{A}}) - \tau(\varphi(I_{\mathfrak{A}})) = \tau(I_{\mathfrak{A}} - \varphi(I_{\mathfrak{A}})) > 0,$$

a clear contradiction. Hence $\varphi(I_{\mathfrak{A}}) = I_{\mathfrak{A}}$.

There are many other forms of majorization for elements of $L_{\infty}[0,1]$. We have the following.

Definition (1.1.19)[1]: Let $\mathfrak{A}$ be a unital C^*-algebra with a faithful tracial state τ. For $T, S \in \mathfrak{A}$, it is said that T *(absolutely) submajorizes* S *with respect to* τ, denoted $S \prec^{w}_{\tau} T$, if

$$\int_0^t \mu^{\tau}_{S}(s)ds \le \int_0^t \mu^{\tau}_{T}(s)ds \text{ for all } t \in [0,1].$$

In this part, we will demonstrate for certain unital C^*-algebras A with a faithful tracial state τ that $\tau\,(T)I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$ for all self-adjoint $T \in A$. Combined with the Dixmier property, this implies these C∗-algebras are simple; that is, have no closed ideals. We begin with definitions and examples of C^*-algebras for which these results apply.

Definition (1.1.20)[1]: A unital C^*-algebra $\mathfrak{A}$ is said to have real rank zero if the set of invertible self-adjoint operators of $\mathfrak{A}$ is dense in the set of self-adjoint operators. Equivalently, $\mathfrak{A}$ has real rank zero if and only if every self-adjoint element of $\mathfrak{A}$ can be approximated by self-adjoint elements with finite spectrum. Also $\mathfrak{A}$ is said to have stable rank one if the set of invertible elements is dense in $\mathfrak{A}$.

Definition (1.1.21): Let $\mathfrak{A}$ be a unital C^*-algebra and let $P, Q \in \mathfrak{A}$ be projections. It is said that P and Q are Murray-von Neumann equivalent (or simply equivalent), denoted $P \sim Q$, if there exists an element $V \in \mathfrak{A}$ such that $P = V^*V$ and $Q = VV^*$. It is said that P is equivalent to a subprojection of Q, denoted $P \precsim Q$, if there exists a projection $Q' \leq Q$ such that $P \sim Q'$.

Definition (1.1.22)[1]: Let $\mathfrak{A}$ be a unital C^*-algebra with a faithful tracial state . Then:

(i) $\mathfrak{A}$ is said to have strong comparison of projections with respect to τ if for all projections $P, Q \in \mathfrak{A}, \tau(P) \leq \tau(Q)$ implies $P \precsim Q$.

(ii) $\mathfrak{A}$ is said to have strict comparison of projections with respect to τ if for all projections $P, Q \in \mathfrak{A}, \tau(P) < \tau(Q)$ implies $P \precsim Q$.

There are several C^*-algebras that are known to have the above properties.

Example (1.1.23)[1]: Type II_1 factors are well known to be unital C^*-algebras that are simple, have real rank zero, and have strong comparison of projections with respect to a faithful tracial state, which happens to be unique.

Example (1.1.24)[1]: It is not difficult to verify that UHF C^*-algebras and the Bunce-Deddens algebras (specific direct limits of $\mathcal{M}_n(C(T))$) are unital, simple, real rank zero C^*-algebras that have strong comparison of projections with respect to a faithful tracial state, which happens to be unique. However, as mentioned, there exists unital, simple, AF C^*-algebras with unique tracial states that do not have strong comparison of projections.

Example (1.1.25)[1]: As mentioned, irrational rotation algebras and, more generally, simple non-commutative tori for which the map from K_0 to $\mathbb{R}$ induced by the tracial state is faithful are examples of unital, simple, real rank zero C^*-algebras that have strong comparison of projections with respect to a faithful tracial state, which happens to be unique.

Example (1.1.26): More generally, if $\mathfrak{A}$ is a unital, simple, C^*-algebra with real rank zero, stable rank one, and a tracial state τ such that the induced map $\tau_* : K_0(A) \to \mathbb{R}$ defined by $\tau^*([x]_0) = \tau(x)$ is injective, then $\mathfrak{A}$ will have strong comparison of projections with respect to τ by cancellation.

Example (1.1.27)[1]: It was demonstrated free minimal actions of $\mathbb{Z}^d$ on Cantor sets give rise to crossed product C^* −algebras that have real rank zero, stable rank one, and strict comparison of projections with respect to their tracial states.

Example (1.1.28)[1]: For certain tracial reduced free product C^*-algebras, implies simplicity, implies stable rank one, and implies real rank zero and strict comparison of projections.

Notice that all of the C^*-algebras presented above are simple. This turns out to be no coincidence. To see this, we prove the following result.

Theorem (1.1.29)[1]: *If $\mathfrak{A}$ and τ are as in the hypotheses of Theorem (1.1.37), then $\mathfrak{A}$ is simple and τ is the unique tracial state on $\mathfrak{A}$.*

Proof. The following argument can be found, but is repeated for convenience of the reader. Suppose $\mathcal{J}$ is a non-zero ideal in $\mathfrak{A}$. Let $T \in \mathcal{J} \setminus \{0\}$ be positive. Therefore $\tau(T)I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T)) \subseteq \mathcal{J}$ by Theorem (1.1.37). As τ is faithful, $\tau(T) \neq 0$ so $\mathcal{J} = \mathfrak{A}$. Hence $\mathfrak{A}$ is simple.

Suppose τ_0 is another tracial state on $\mathfrak{A}$. $\tau_0(S) = \tau_0(T)$ for all $S \in \overline{\text{conv}}(\mathcal{U}(T))$. Hence Theorem (1.1.37) implies.

$$\tau_0(T) = \tau_0(\tau(T)I_{\mathfrak{A}}) = \tau(T).$$

As this holds for all self-adjoint $T \in \mathfrak{A}$, we obtain that $\tau_0 = \tau$.

Example (1.1.30)[1]: To see why strict comparison of projections without arbitrarily small projections is not sufficient in Theorem (1.1.29), consider the C^*-algebra $\mathfrak{A} = \mathbb{C} \oplus \mathbb{C}$ with the faithful tracial state $\tau((a,b)) = \frac{1}{2}(a + b)$. It is clear that $\mathfrak{A}$ is a unital C^*-algebra with real rank zero and strict comparison of projections with respect to τ. However, $\mathfrak{A}$ is not simple.

Note the following easily verified lemma which will be used often without citation.

Lemma (1.1.31)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra and let $T, S, R \in \mathfrak{A}$. if $T \in \overline{\text{conv}}(\mathcal{U}(s))$ and $S \in \overline{\text{conv}}(\mathcal{U}(R))$, then $T \in \overline{\text{conv}}(\mathcal{U}(R))$.*

To prove Theorem (1.1.37), it will suffice to prove the theorem for self-adjoint operators with finite spectrum by the assumption that $\mathfrak{A}$ has real rank zero. Combined with the following remark, it will suffice to consider self-adjoint operators with two points in their spectra.

To prove Theorem (1.1.37) for self-adjoint operators with two points in their spectra, we will use equivalence of projections to construct matrix algebras and apply results on majorization for self-adjoint matrices, to average part of one spectral projection with the other. Using a back-and-forth-type argument, we eventually obtain an operator in conv($\mathcal{U}(T)$) that is almost $\tau(T)I_{\mathfrak{A}}$.

As $\tau(\mathfrak{A})$ may not equal [0, 1], we may only divide projections up based on the size of another projection. As such, the following division algorithm result will be of use to us and is easily verified.

Lemma (1.1.32)[1]: *Let $t \in \left(0, \frac{1}{2}\right)$ and write $1 = k_1 t + r_1$ where $k_1 \in \mathbb{N}$ and $0 \leq r_1 < t$. Then $k_1 \geq 2$ and $0 \leq r_1 < \frac{1}{k_1+1}$ Furthermore, if $r_1 \neq 0$ and $1 = k_2 r_1 + r_2$ for some $k_2 \in \mathbb{N}$. and $0 \leq$*

$r_2 \leq r_1$, *then* $k_2 \geq k_1$.

The following lemma will be our method of constructing matrix algebras. However, the embedding of each matrix algebra into $\mathfrak{A}$ need not be a unital embedding.

Lemma (1.1.33)[1]: *Let* $\mathfrak{A}$ *be a unital* C^**-algebra with a faithful tracial state* τ *and let* $P \in \mathfrak{A}$ *be a projection with* $(P) \in \left(\frac{0,1}{2}\right]$ *. Write* $1 = k\tau(P) + r$ *where* $k \in \mathbb{N}$ *and* $0 \leq r < \tau(P)$*. If*

(i) $\mathfrak{A}$ *has strong comparison of projections with respect to* τ *and* $\ell = k - 1$*, or*

(ii) $\mathfrak{A}$ *has strict comparison of projections with respect to* τ , $r \neq 0$*, and* $\ell = k - 1$*, or*

(iii) $\mathfrak{A}$ *has strict comparison of projections with respect to* τ *, and* $\ell = k - 2$*, then there exists pairwise orthogonal subprojections* $\{P_j\}_{j=1}^{\ell}$ *of* $I_{\mathfrak{A}} - P$ *such that* $\{P\} \cup \{P_j\}_{j=1}^{\ell}$ *are equivalent in* $\mathfrak{A}$*.*

Proof: Notice $\tau(I_{\mathfrak{A}} - P) = (k - 1)\tau(P) + r$. Since $k \geq 2, \tau(P) \leq \tau(I_{\mathfrak{A}} - P)$ with strict inequality when $r \neq 0$. Therefore, by assumptions, there exists a subprojection P_1 of $I_{\mathfrak{A}} - P$ such that $P_1 \sim P$. If $k \geq 3$ (and $\ell \geq 2$), there exists a subprojection P_2 of $I_{\mathfrak{A}} - P - P_1$ such that $P_2 \sim P$. By repeating this argument, we obtain pairwise orthogonal subprojections $\{P_j\}_{j=1}^{\ell}$ of $I_{\mathfrak{A}} - P$ such that $P_j \sim P$ for all j. As Murray-von Neumann equivalence is an equivalence relation, the result follows.

We now divide the proof of Theorem (1.1.37) for T with two point spectra into twoparts: Lemma (1.1.34) proves the result when $\mathfrak{A}$ has strong comparison of projections, and Lemma (1.1.33) will modify the argument to obtain the result in the other case. In that which follows, diag$(a_1, \ldots, a_n)$ denotes the diagonal $n \times n$ matrix with diagonal entries $a_1, \dots, a_n$.

Lemma (1.1.34)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero that has strong comparison of projections with respect to a faithful tracial state . If $P \in \mathfrak{A}$ is a projection, $a, b \in \mathbb{R}$, and $T = aP + b(I_{\mathfrak{A}} - P)$, then $\tau(T)I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$.*

Proof. By interchanging P and $I_{\mathfrak{A}} - P$, we may assume that $\tau(P) \leq \frac{1}{2}$. Let $r_0 = \tau(P)$ and write $1 = k_1 r_0 + r_1$ where $k_1 \in \mathbb{N}$, $k_1 \geq 2$ and $0 \leq r_1 \leq \min\left\{r_0, < \frac{1}{k_1+1}\right\} < \frac{1}{2}$. By Lemma (1.1.40) there are pairwise orthogonal subprojections $\{Q_j\}_{j=1}^{k_1-1}$ of $I_{\mathfrak{A}} - P$ such that $\{P\} \cup \{Q_j\}_{j=1}^{k_1-1}$ are equivalent in $\mathfrak{A}$. Let $P_1 = I_{\mathfrak{A}} - P \sum_{j=1}^{k_1-1} Q_j$. Using the equivalence of $\{P\} \cup \{Q_j\}_{j=1}^{k_1-1}$, a copy of $\mathcal{M}_{k_1}(\mathbb{C})$ may be constructed in $\mathfrak{A}$ with unit $I_{\mathfrak{A}} - P_1$. Using this matrix subalgebra, T can be viewed as the operator

$$T = \text{diag}(a, b, \dots, b) \oplus bP_1 \in \mathcal{M}_{k_1}(\mathbb{C}) \oplus P_1 \mathfrak{A} P_1 \subseteq \mathfrak{A}.$$

Since any self-adjoint matrix majorizes its normalized trace, we obtain by that

$$\frac{a + (k_1 - 1)b}{k_1} I_{k_1} \in \overline{\text{conv}}(\mathcal{U}(\text{diag}(a, b, \dots, b)))$$

where the unitary orbit is computed in $\mathcal{M}_{k_1}(\mathbb{C})$. Therefore, if $a_1 = \frac{a+(k_1-1)b}{(k_1}$, we obtain by using a direct sum argument that

$$T_1 := a_1(I_{\mathfrak{A}} - P_1) + bP_1 \in \overline{\text{conv}}(\mathcal{U}(T)).$$

Notice $(P_1) = r_1$. If $r_1 = 0$, the proof is complete (as $\tau(T_1) = \tau(T)$). Otherwise, by writing $1 = k_2 r_1 + r_2$ where $k_2 \in N, k_2 \geq k_1$, and $0 \leq r_2 \leq \min\left\{r_1, \frac{1}{k_2+1}\right\}$ and by repeating the above argument, there exists a projection $P_2 \in \mathfrak{A}$ such that $\tau(P_2) = r_2$ and

$$T_2 := a_1P_2 + \frac{b + (k_2 - 1)a_1}{k_2} (I_{\mathfrak{A}} - P_2) \in \overline{\text{conv}}(\mathcal{U}(T_1))$$
$$\subseteq \overline{\text{conv}}(\mathcal{U}(T)).$$

Notice if $r_2 = 0$, the proof is again complete.

Repeat the above process ad infinitum. Notice that the proof is complete if the process ever terminates via a zero remainder. As such, we may assume that we have found a non-decreasing sequence $(k_n)_{n\geq 1} \subseteq \mathbb{N}$ with $k_1 \geq 2$, a sequence $(r_n)_{n\geq 1} \subseteq \left(0, \frac{1}{2}\right]$ with $1 = k_{n+1}r_n + r_{n+1}$, projections $\{P_n\}_{n\geq 1} \subseteq \mathfrak{A}$ with $\tau(P_n) = r_n$, sequences $(a_n)_{n\geq 1}, (b_n)_{n\geq 1} \subseteq \mathbb{R}$ such that

$$a_{n+1} = \frac{(k_{2n+1} - 1)b_n}{k_{2n+1}} \text{ and } b_{n+1} = \frac{(k_{2n+1} - 1)a_{n+1}}{k_{2n+1}},$$

and operators

$$T_{2n} = a_nP_{2n} + b_n(I_{\mathfrak{A}} - P_{2n}) \text{ and } T_{2n+1} = b_nP_{2n+1} + a_{n+1}(I_{\mathfrak{A}} - P_{2n+1})$$

such that $T_n \in \overline{\text{conv}}(\mathcal{U}(T))$ for all n.

If $a \leq b$, it is elementary to verify that

$$a \leq a_1 \leq a_2 \leq \cdots \leq b_2 \leq b_1 \leq b$$

(as averages are used to construct each a_n and b_n). Similarly, if $b \leq a$, then

$$b \leq b_1 \leq b_2 \leq \cdots \leq a_2 \leq a_1 \leq a.$$

As a result, $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are bounded monotone sequence of R and thus converge. Let

$$a' = \lim_{n\to\infty} a_n \quad and\ b' = \lim_{n\to\infty} b_n .$$

If the non-decreasing sequence $(k_n)_{n\geq 1}$ is bounded (and thus eventually constant), using the fact that $k_1 \geq 2$ we obtain $a' = b'$ by taking the limit of one relations between an and bn. If $(k_n)_{n\geq 1}$ is unbounded, then by using the fact that

$$\lim_{m\to\infty}\left|c-\frac{c+md}{m+1}\right| = |c-d|$$

we again obtain $a' = b'$.

Let $\epsilon > 0$ and choose n such that $|a_n - a'| < \epsilon$ and $|b_n - a'| < \epsilon$. Then $\|T_{2n} - a'I_{\mathfrak{A}}\| < \epsilon$ so

$$\text{dist}\,(a'I_{\mathfrak{A}}, \overline{\text{conv}}(\mathcal{U}(T))) \leq \epsilon.$$

Hence $a'I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$. Since every element of $\overline{\text{conv}}(\mathcal{U}(T))$ has trace equal to $\tau(T)$, we obtain $a' = \tau(T)$ thereby completing the result.

Lemma (1.1.35)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero and property (b) of Theorem (1.1.37) with respect to a faithful tracial state . If $P \in \mathfrak{A}$ is a projection, $a, b \in \mathbb{R}$, and $T = aP + b(I_{\mathfrak{A}} - P)$, then $\tau(T)I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$.*

Proof. Notice, by case (ii) of Lemma (1.1.40), that the recursive algorithm in the proof of Lemma (1.1.41) works at the n^{th} stage in this setting provided $r_n \neq 0$. Therefore, if $r_n \neq 0$ for all $n \in \mathbb{N}$, the proof is complete. Otherwise, if n is the first number in the algorithm for which $r_n = 0$, notice $r_{n-1} = \frac{1}{k_n}$. Thus it suffices to prove the result in the case that $\tau(P) = \frac{1}{k}$ for some $k \in \mathbb{N}$ with $k \geq 2$.

If $k \geq 3$, we can apply the algorithm in Lemma (1.1.41) by viewing the remainders as being $\frac{1}{k}$ instead of zero. Indeed the proof of Lemma(1.1.34) may be adapted using case (iii) instead of case (iii) of Lemma (1.1.33) to construct $(k_n - 1) \times (k_n - 1)$ matrix algebras (instead $of\ k_n \times k_n$) and by using the new scalars

$$a_{n+1} = \frac{a_n + (k_{2n+1} - 2)b_n}{k_{2n+1}} \text{ and } b_{n+1} = \frac{b_n(k_{2n+2} - 2)a_{n+1}}{k_{2n+2} - 1},$$

The remainder of the proof then follows as in Lemma (1.1.41). Thus it remains to prove the result in the case $\tau(P) = \frac{1}{2}$.

Since $\mathfrak{A}$ has property (b), there exists a projection $P_0 \leq I_{\mathfrak{A}} - P$ with $\tau(P_0) < \frac{1}{2}$. Consider

$$T_0 = aP + bP_0 \in (P + P_0)A(P + P_0).$$

As $(P + P_0)\mathfrak{A}(P + P_0)$ satisfies the assumptions of this lemma and since

$$\tau_{(P + P_0)}(P) = \frac{1}{\tau(P + P_0)}\tau(P) \neq \frac{1}{2},$$

the above cases imply there exists $\alpha_0 \in \mathbb{R}$ such that $\alpha_0(P + P_0) \in \overline{\text{conv}}(\mathcal{U}(T_0))$ where $\overline{\text{conv}}(\mathcal{U}(T_0))$ is computed in $(P + P_0)\mathfrak{A}(P + P_0)$. Consequently

$$\alpha_0(P + P_0) + b(I_{\mathfrak{A}} - P - P_0) \in \overline{\text{conv}}(\mathcal{U}(T))$$

by a direct sum argument. As $\tau(P + P_0) = \frac{1}{2}$, the above cases imply there exists $\alpha \in \mathbb{R}$ such that $\alpha I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$. As every element of $\overline{\text{conv}}(\mathcal{U}(T))$ has trace $\tau(T)$, $\alpha = \tau(T)$ completing the result.

Lemma (1.1.36)[1]: *Let $\mathfrak{A}$ and τ be as in the hypotheses of Theorem (1.1.37). If $T \in \mathfrak{A}$ is a self-adjoint operator with finite spectrum, then $\tau(T)I_{\mathfrak{A}} \in \overline{conv}(\mathcal{U}(T))$.*

Proof. By assumption there exist pairwise orthogonal non-zero projections $\{P_k\}_{k=1}^{n}$ and scalars $\{\alpha_k\}_{k=1}^{n} \subseteq \mathbb{R}$ such that $= \sum_{k=1}^{n} \alpha_k P_k$. By applying Lemma(1.1.34) (1.1.35) to $\alpha_1 P_1 + \alpha_2 P_2$ in $(P_1 + P_2)\mathfrak{A}(P_1 + P_2)$ and by appealing to a direct sum argument, there exists a $\beta_0 \in \mathbb{R}$ such that

$$\beta_0(P_1 + P_2) + \sum_{k=3}^{n} \alpha_k P_k \in \overline{\text{conv}}(\mathcal{U}(T)).$$

By iterating this argument another $n - 2$ times, there exists a $\beta \in \mathbb{R}$ such that $\beta I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$. As every element of $\overline{\text{conv}}(\mathcal{U}\ (T))$ has trace $\tau(T), \beta\ =\ \tau(T)$ completing the result.

Theorem (1.1.37)[1]: *Let $\mathfrak{A}$ be a unital $C*$-algebra with real rank zero. Suppose τ is a faithful tracial state on $\mathfrak{A}$ such that either:*

(a) $\mathfrak{A}$ *has strong comparison of projections with respect to τ , or*

(b) $\mathfrak{A}$ *has strict comparison of projections with respect to τ and for every $n\ \in \mathbb{N}$ there exists a projection $P\ \in\ \mathfrak{A}$ such that* $0\ <\ \tau\ (P)\ < \frac{1}{n}$.

Then $\tau\ (T)I_{\mathfrak{A}}\ \in\ \overline{\text{conv}}(\mathcal{U}(T))$ for all self-adjoint $T\ \in \mathfrak{A}$.

Once Theorem (1.1.29) is established, we easily obtain the following .

Proof: Let $T\ \in\ \mathfrak{A}$ be self-adjoint. Let $\epsilon\ >\ 0$. Since $\mathfrak{A}$ has real rank zero, there exists a self-adjoint operator T0 ∈ $\mathfrak{A}$ with finite spectrum such that $\|T - T_0\|\ < \epsilon$. Notice this implies dist $(R,\ \overline{\text{conv}}(\mathcal{U}(T)))\ \leq$ ϵ for all $R\ \in\ \overline{\text{conv}}(\mathcal{U}\ (T_0))$.

By Lemma (1.1.44), $\tau(T_0)I_{\mathfrak{A}} \in\ \overline{\text{conv}}(\mathcal{U}T_0))$. Since $|\tau(T_0)\ -\ \tau(T)| < \epsilon$, we obtain

$$\text{dist}\ (\tau(T)I_{\mathfrak{A}}, \overline{\text{conv}}(\mathcal{U}(T)))\ <\ 2\epsilon.$$

As ϵ was arbitrary, the result follows.

Note the set of projections contained in $\mathcal{J}$ is closed under taking subprojections (as $\mathcal{J}$ is hereditary) and is closed under Murray-von Neumann equivalence (as $\mathcal{J}$is an ideal). Therefore, by part (iii) of Lemma (1.1.33), there exists a projection $P\ \in\ \mathcal{J}$ with $\tau(P)\ \geq \frac{1}{2}$.

If $\tau(P)\ = \frac{1}{2}$, choose a non-zero projection $P'\ \leq$P with $\tau(P')\ < \frac{1}{2}$ and a subprojection Q of $I_{\mathfrak{A}} -$ P with $\tau(Q)\ =\ \tau(P')$ such that $Q\ \sim\ P'$. Hence $\in \mathcal{J}I\ so\ P\ +\ Q\ \in \mathcal{J}$. As $\tau(P\ +\ Q)\ > \frac{1}{2}$, we have reduced to the case $\tau(P)\ > \frac{1}{2}$.

If $\tau(P)\ > \frac{1}{2}$, then $I_{\mathfrak{A}} - P$ is equivalent to a subprojection of P

and thus $I_{\mathfrak{A}} - P \in \mathcal{I}$. Since $P \in \mathcal{I}$, this implies $I_{\mathfrak{A}} \in \mathcal{I}$ so $\mathcal{I} = \mathfrak{A}$.

We will demonstrate the following theorem which characterizes $\overline{\text{conv}}(\mathcal{U}(T))$ for self-adjoint T in various C^*-algebras using the notion of majorization.

Lemma (1.1.38)[1]: *Let $\mathfrak{A}$ and τ be as in Theorem (1.1.42). Suppose $S, T \in \mathfrak{A}$ are self-adjoint operators with finite spectrum. Then there exists two collections of pairwise orthogonal non-zero projections $\{P_k\}_{k=1}^n$ and $\{Q_k\}_{k=1}^n$ with*

$$\sum_{k=1}^{n} P_k = \sum_{k=1}^{n} Q_k = I_{\mathfrak{A}} \ and \ \tau(P_k) = \tau(Q_k) \ for \ all \ k$$

and scalars $\{\alpha_k\}_{k=1}^n, \{\beta_k\}_{k=1}^n \subseteq \mathbb{R}$ with $\alpha_k \geq \alpha_{k+1}$ and $\beta_k \geq \beta_{k+1}$ such that

$$T = \sum_{k=1}^{m} \alpha_k P_k \ \ and \ S = \sum_{k=1}^{n} \beta_k Q_k \ .$$

Proof. Since T and S have finite spectrum, there exists two collections of pairwise orthogonal non-zero projections $\{P'_k\}_{k=1}^l$ and $\{Q'_k\}_{k=1}^l$ with $\sum_{k=1}^m P_k = \sum_{k=1}^l Q_k = I_{\mathfrak{A}}$ and scalars $\{\alpha'_k\}_{k=1}^m, \{\beta'_k\}_{k=1}^l \subseteq \mathbb{R}$ with $\alpha'_k > \alpha'_{k+1}\alpha \ and \ \beta'_k > \beta'_{k+1}$ such that

$$T = \sum_{k=1}^{m} \alpha'_k P'_k \ \ and \ S = \sum_{k=1}^{l} \beta'_k Q'_k \ .$$

Suppose $\tau(P'_1) \geq \tau(Q'_1)$. Since $\mathfrak{A}$ has strong comparison of projections, there exists a projection $P_1 \in \mathfrak{A}$ such that $\tau(P_1) = \tau(Q'_1)$ and $P_1 \leq P'_1$. Letting $Q_1 = Q'_1$, we have

$$T = \alpha'_1 P_1 + \alpha'_1(P_1 - P'_1) + \sum_{k=2}^{m} \alpha'_k P'_k \ \text{ and } S = \beta'_1 Q_1 + \sum_{k=2}^{l} \beta'_k Q'_k \ .$$

Similarly, if $\tau(P'_1) \leq \tau(Q'_1)$, there exists a projection $Q_1 \in \mathfrak{A}$ such that $\tau(Q_1) = \tau(P'_1)$ and $Q_1 \leq Q'_1$. Letting $P_1 = P'_1$, we have

$$T = \alpha_1' P_1 + \sum_{k=2}^{m} \alpha_k' P_k' \text{ and } S = \beta_1' Q_1 + + \beta_1'(Q_1' - Q_1) + \sum_{k=2}^{l} \beta_k' Q_k' .$$

By repeating this argument at most another $m + l - 1$ times (for the next iteration, using P_2' and Q_2' when $\tau(P_1') = \tau(Q_1')$ and otherwise using $P_1' - P_1$ and Q_2' in the first case and P_2' and $Q_1' - Q_1$ in the second case), the result follows.

The following result enables us to reduce Theorem (1.1.42) to the case of self-adjoint operators with finite spectrum.

Lemma (1.1.39)[1]: *Let $\mathfrak{A}$ and τ be as in Theorem (1.1.42). If $S, T \in \mathfrak{A}$ are self-adjoint operators, then for every $\epsilon > 0$ there exists self-adjoint operators $S', T' \in \mathfrak{A}$ with finite spectrum such that*

$$\|T - T'\| < \epsilon, \quad and\ \|S - S'\| < \epsilon.$$

Furthermore:

(i) $T' \prec_\tau T\ and\ S' \prec \tau\ S$.

(ii) *If S, T ≥ 0, then $S', T' \geq 0$.*

(iii) *If $\prec_\tau T$, then $S' \prec_\tau T'$.*

(iv) *If $S, T \geq 0$ and $S \prec_\tau^w T$, then $S' \prec w_\tau^w T'$.*

(v) If $\lambda_S^\tau(s) \leq \lambda_T^\tau(s)$ for all $s \in [0,1)$, then $\lambda_{S'}^\tau(s) \leq \lambda_{T'}^\tau(s)$ for all $s \in [0,1)$.

Proof. Let $\epsilon > 0$. Since $\mathfrak{A}$ has real rank zero, there exists self-adjoint operators $T_0, S_0 \in \mathfrak{A}$ with finite spectrum such that

$$\|T - T_0\| \leq \frac{\epsilon}{2}\ and\ \|S - S_0\| \leq \frac{\epsilon}{2} .$$

Let $\{P_k\}_{k=1}^n$, $\{Q_k\}_{k=1}^n$, $\{\alpha_k\}_{k=1}^n$, and $\{\beta_k\}_{k=1}^n$ be as in the conclusions of Lemma(1.1.22) so that

$$T_0 = \sum_{k=1}^{m} \alpha_k P_k \quad and\ S_0 = \sum_{k=1}^{n} \beta_k Q_k .$$

and, for each $k \in \{0, 1, \ldots, n\}$, let $s_k = \sum_{j=1}^{k} \tau(P_j)$. Notice $s_k < k + 1$

for all k, $s_0 = 0, s_k < s_n = 1$, and $\lambda_{s_0}^{\tau}(s) = \alpha_k$ and $\beta_{s_0}^{\tau}(s) = \alpha_k$ for all $s \in [s_{k-1}, s_k)$. For each $k \in \{0, 1, \ldots, n\}$, let

$$\alpha_k' = \frac{1}{s_k - s_{k-1}} \int_{s_{k-1}}^{s_k} \lambda_T^{\tau}(s)\, ds \text{ and } \beta_k' = \frac{1}{s_k - s_{k-1}} \int_{s_{k-1}}^{s_k} \lambda_T^{\tau}(s)\, ds\,,$$

and let

$$T' = \sum_{k=1}^{n} \alpha_k' P_k' \quad and \quad S' = \sum_{k=1}^{n} \beta_k' Q_k' \,.$$

We claim T' and S' are the desired self-adjoint operators, implies $T' \prec_\tau T$ and $S' \prec_\tau S$. Furthermore, if $S, T \geq 0$, then $\lambda_S^\tau(s)$ and $\lambda_T^\tau(s)$ are non-negative functions. Consequently $\alpha_k', \beta_k' \geq 0$ for all $k, so\, S', T' \geq 0$.

To see that $\|T - T'\| < \epsilon$, it suffices to show that $\|T_0 - T'\| \leq \frac{\epsilon}{2}$. For each k, notice

$$|\alpha_k - \alpha_k'| \leq \frac{1}{s_k - s_{k-1}} \int_{s_{k-1}}^{s_k} |\alpha_k - \lambda_T^\tau(s)|\, ds$$

$$= \frac{1}{s_k - s_{k-1}} \int_{s_{k-1}}^{s_k} \left|\lambda_{T_0}^\tau(s) - \lambda_T^\tau(s)\right| ds$$

$$= \frac{1}{s_k - s_{k-1}} \int_{s_{k-1}}^{s_k} \|T_0 - T\|\, ds = \|T_0 - T\| < \frac{\epsilon}{2}$$

As this holds for all k, we obtain $\|T_0 - T'\| \leq \frac{\epsilon}{2}$. The same arguments show $\|S - S'\| < \epsilon$.

Suppose $S \prec_\tau T$. Notice , that $\alpha_k' \geq \alpha_{k+1}'$\ and $\beta_k' \geq \beta_{k+1}'$ for all k. Consequently $\lambda_{T'}^\tau(s) = \alpha_k'$and $\lambda_{S'}^\tau(s) = \beta_k'$ for all s $\in [s_{k-1}, s_k)$. This along with the definition of α_k' and β_k' implies

$$\int_{s_{k-1}}^{s_k} \lambda_{T'}^\tau(s)\, ds = \int_{s_{k-1}}^{s_k} \lambda_T^\tau(s)\, ds \text{ and } \int_{s_{k-1}}^{s_k} \lambda_{S'}^\tau(s)\, ds = \int_{s_{k-1}}^{s_k} \lambda_S^\tau(s)\, ds$$

for all k. In particular, by adding integrals, we obtain

$$\int_0^1 \lambda_{T'}^\tau(s)\, ds = \int_0^{s_{k-1}} \lambda_T^\tau(s)\, ds = \int_0^1 \lambda_S^\tau(s)\, ds = \int_0^1 \lambda_{S'}^\tau(s)\, ds\,.$$

For an arbitrary $t \in [0,1]$, choose $k \in \{1,\dots,n\}$ such that $t \in [s_{k-1}, s_k]$ and notice

$$\int_0^t \lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s)\, ds$$

$$= \int_0^{s_{k-1}} \lambda_T^\tau(s) - \lambda_S^\tau(s) ds + \int_{s_{k-1}}^t \lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s)\, ds\,.$$

To see the left-hand-side is always non-negative, we note that $\lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s)$ is constant on $[s_{k-1,} sk)$. If $\lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s) \geq 0$ on $[s_{k-1,} sk)$, then

$$\int_0^t \lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s)\, ds \geq \int_0^{s_{k-1}} \lambda_T^\tau(s) - \lambda_S^\tau(s) ds \geq 0.$$

Otherwise $\lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s) < 0$ on $[s_{k-1,} sk)$ so

$$\int_0^t \lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s)\, ds$$

$$\geq \int_0^{s_{k-1}} \lambda_T^\tau(s) - \lambda_S^\tau(s) ds + \int_{s_{k-1}}^{s_k} \lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s)\, ds$$

$$= \int_0^{s_{k-1}} \lambda_T^\tau(s) - \lambda_S^\tau(s) ds + \int_{s_{k-1}}^{s_k} \lambda_T^\tau(s) - \lambda_S^\tau(s) ds \geq 0$$

Hence, $S' \prec_\tau T'$ when $\prec_\tau T$.

If $S, T \geq 0$ and $S \prec_\tau^w T$, then the proof that $S' \prec_\tau^w T'$ follows from the above proof (ignoring the part that shows $\int_0^1 \lambda_{S'}^\tau ds(s) = \int_0^1 \lambda_{T'}^\tau ds(s)$.

If $\lambda_S^\tau(s) \leq \lambda_T^\tau(s)$ for all $s \in [0,1)$, then $\beta_k' \leq \alpha_k'$ for all k and thus $\lambda_{S'}^\tau(s) \leq \lambda_{T'}^\tau(s)$ for all $s \in [0,1)$.

The following result for elements of $\mathcal{M}_2(\mathbb{C})$ is referred to as a pinching.

Lemma (1.1.40)[1]: Let $\mathfrak{A}$ and τ be as in Theorem (1.1.42). If $P \in \mathfrak{A}$ is a projection, $a, b \in \mathbb{R}$, and $T = aP + b(I_{\mathfrak{A}} - P)$, then for all $t \in [0,1]$,

$$\begin{aligned} tT + (1-t)\tau(T)I_{\mathfrak{A}} &= (at + \tau(T)(1-t))P + (bt + \tau(T)(1-t))(I_{\mathfrak{A}} - P) \\ &\in \overline{\mathrm{conv}}(\mathcal{U}(T)). \end{aligned}$$

Proof. Fix $t \in [0,1]$ and let

$$a' = at + \tau(T)(1-t) \text{ and } b' = bt + \tau(T)(1-t).$$

Since $\tau(T) = a\tau(P) + b\tau(I_{\mathfrak{A}} - P) \in \mathrm{conv}(\{a,b\})$, we obtain that $a', b' \in \mathrm{conv}(\{a,b\})$.

By interchanging P and $I_{\mathfrak{A}} - P$, we may assume that $\tau(P) \leq \frac{1}{2}$. Since $\mathfrak{A}$ has strong comparison of projections, there exists a projection $Q \in \mathfrak{A}$ such that $Q \sim P$ and $\leq I_{\mathfrak{A}} - P$. Consequently, using the partial isometry implementing the equivalence of P and Q, a copy of $\mathcal{M}_2(\mathbb{C})$ may be constructed in $(P+Q)\mathfrak{A}(P+Q)$ so that P and Q are the two diagonal rank one projections. Hence T can be viewed as the operator

$$\begin{aligned} T &= (aP + bQ) \oplus b(I_{\mathfrak{A}} - P - Q) \\ &\in \mathcal{M}_2(\mathbb{C}) \oplus (I_{\mathfrak{A}} - P - Q)\mathfrak{A}(I_{\mathfrak{A}} - P - Q) \subseteq A. \end{aligned}$$

Choose $b'' \in \mathbb{R}$ so that $b'' + a' = a + b$. Notice $b'' \in \mathrm{conv}(\{a,b\})$ as $a' \in \mathrm{conv}(\{a,b\})$. We see that

$$\mathrm{diag}(a', b'') \prec_{\frac{1}{2}\mathrm{Tr}} \mathrm{diag}(a,b)$$

where $\frac{1}{2}\mathrm{Tr}$ is the normalized trace on $\mathcal{M}_2(\mathbb{C})$ (which agrees with τ_{P+Q}). Thus

$$a'P + b''Q + b(I_{\mathfrak{A}} - P - Q) \in \overline{\mathrm{conv}}(\mathcal{U}(T)).$$

By applying Theorem (1.1.37) to $b''Q + b(I_{\mathfrak{A}} - P - Q)$ in $(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ and by applying a direct sum argument, we obtain that

$$a'P + b'''(I_{\mathfrak{A}} - P) \in \overline{\text{conv}}(\mathcal{U}(T))$$

for some $b''' \in \mathbb{R}$. As every element of $\overline{\text{conv}}(\mathcal{U}(T))$ has trace $\tau(T)$, one can verify that $b''' = b'$.

The following result contains the main technical details necessary for a recursive argument in the proof of Theorem (1.1.42). In particular, it will enable us to systematically apply pinchings.

Lemma (1.1.41)[1]: *Let $\mathfrak{A}$ and τ be as in Theorem* (1.1.42)*. Suppose $\{P_k\}_{k=1}^n$ is a collection of pairwise orthogonal projections with $\sum_{k=1}^n P_k = I_{\mathfrak{A}}, \{\alpha_k\}_{k=1}^n, \{\beta_k\}_{k=1}^n \subseteq \mathbb{R}$ with $\beta_k \geq \beta_{k+1}$ for all k, and*

$$T = \sum_{k=1}^{n} \alpha_k P_k \quad and \quad S = \sum_{k=1}^{n} \beta_k Q_k \ .$$

Suppose further that $S \prec_\tau T$ and there exists a j such that $\alpha_k \geq \beta_1$ for all $k < j, \alpha_j < \beta_1$, and $\alpha_k \geq \alpha_{k+1}$ for all $k \geq j$. Then there exists $\{\alpha'_k\}_{k=1}^n \subseteq \mathbb{R}$ such that $\alpha'_1 = \beta_1, \alpha_k = \alpha_k \geq \beta_1$ for all 1 $< k < j$*, $\alpha'_k \geq \alpha'_{k+1}$ for all $k \geq j$, and*

$$T' = \sum_{k=1}^{n} \alpha'_k P_k \in \overline{\text{conv}}(\mathcal{U}(T)). k = 1\textstyle\sum n$$

Furthermore, if $Q = \sum_{k=2}^n P_k$, then $QSQ \prec_\tau QT'Q$ in $Q\mathfrak{A}Q$.

Proof. Note $j \geq 2$ along with the fact that $S \prec_\tau T$. In addition, note $\alpha_1 > \alpha_j$.

Consider

$$T_0 = \alpha_1 P_1 + \alpha_j P_j \in (P_1 + P_j)\mathfrak{A}(P_1 + P_j).$$

If $\beta_1 \in [\tau_{P_1+P_j}(T_0), \alpha_1]$ write $\beta_1 = t\alpha_1 + (1-t)\tau_{P_1+P_j}(T_0)$ with $t \in [0,1]$ and let

$\alpha'_1 = \beta_1, \alpha_j = t\alpha_j + (1- t)\tau_{P_1+P_j}(T_0)$, and $\alpha'_k = \alpha_k$ for all $k \neq 1, j$.

Otherwise, if $\beta_1 \notin [\tau_{P_1+P_j}(T_0), \alpha_1]$, let

$$\alpha_1' = \alpha_j = \tau_{P_1+P_j}(T_0), \text{and } \alpha_k' = \alpha_k \text{ for all } k \neq 1, j.$$

Notice, in this later case, that $\alpha_1' = \alpha_j' > \beta_1$ so α_1 and T' will need to be modified again later to obtain the desired (see the last paragraph of the proof). Furthermore, in both cases,

$$\alpha_1'\tau(P-1) + \alpha_j'\tau(P_j) = \alpha_1\tau(P_1) + \alpha_j\tau(P_j).$$

If $T' = \sum_{k=1}^{n} \alpha_k' P_k$, then by applying Lemma(1.1.23) to $T_0 \in (P_1 + P_j)$ by appealing to a direct sum argument, we obtain T' $\overline{\text{conv}}(\mathcal{U}(T))$.

We claim that $S \prec_\tau T'$. For each $k \in \{0, 1, \dots, n\}$, let $s_k = \sum_{j=1}^{k} \tau(P_j)$. Notice $s_k < s_{k+1}$ for all $k, s_0 = 0, s_n = 1$, and $\lambda_T^\tau(s) = \alpha_k$ and $\lambda_S^\tau(s) = \beta_k$ for all s $\in [s_{k-1}, s_k)$. Notice, in both of the above cases, that $\alpha_k' \geq \beta_1$ for all $k < j$ and $k \geq \alpha_k' \geq \alpha_{k+1}'$ for all $k \geq j$ (as $\alpha_j' \geq \alpha_j$). Therefore, $\lambda_{T'}^\tau(s) = \alpha_k' = \lambda_T^\tau(s)$ for all $s \in [s_{k-1}, s_k)$ with $k > j$,

$$\int_0^{s_j} \lambda_{T'}^\tau(s)ds = \sum_{k=1}^{j} \alpha_k'\tau(P_k) = \sum_{k=1}^{j} \alpha_k\tau(P_k) = \int_0^{s_j} \lambda_T^\tau(s)ds ,$$

and $\lambda_{T'}^\tau(s) \geq \beta_1$ for all $s < s_{j-1}$. Consequently, if $t \in [0, s_{j-1}]$, we see that

$$\int_0^t \lambda_T^\tau(s) - \lambda_S^\tau(s)\, ds \geq \int_0^t \beta_1 - \beta_1 ds = 0.$$

For $t \in [s_{j-1}, s_j)$, we will need to divide the proof into two cases. First if $\alpha_j' \geq \beta_j$, then if $\alpha_k' \geq \beta_j$ for all $k < j$. Consequently $\lambda_{T'}^\tau(s) \geq \beta_j$ on $[o, s_j)$ so

$T'(s) \geq \beta j$ on

$$\int_0^t \lambda_{T'}^\tau(s) - \lambda_S^\tau(s)\, ds = \int_0^{s_{j-1}} \lambda_{T'}^\tau(s) - \lambda_S^\tau(s)ds + \int_{s_{j-1}}^t \lambda_{T'}^\tau(s) - \beta_1 ds$$

$$\geq 0 + 0.$$

Otherwise suppose $\alpha_j' < \beta_j$. Notice $\alpha_k' < \alpha_j' < \beta_j \leq \beta_1 \leq \alpha_j'$ for all $k \geq j$ and $l < j$. Thus

$$\int_0^{s_{j-1}} \lambda_{T'}^{\tau}(s)ds = \sum_{k=1}^{j-1} \alpha_k' \tau(P_k)$$

and $\lambda_{T'}^{\tau}(s) = \alpha_j'$ for all $s \in [s_{j-1}, s_j)$. Consequently,

$$\int_0^t \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s)\, ds \sum_{k=1}^{j-1} (\alpha_k' - \beta_k)\tau(P_k) + \int_{s_{j-1}}^t \alpha_j' - \beta_j ds$$

$$\geq \sum_{k=1}^{j-1} (\alpha_k' - \beta_k)\tau(P_k) + \int_{s_{j-1}}^{s_j} \alpha_j' - \beta_j ds$$

$$= \sum_{k=1}^{j} (\alpha_k' - \beta_k)\tau(P_k)$$

$$= \sum_{k=1}^{j} (\alpha_k - \beta_k)\tau(P_k) = \int_0^{s_j} \lambda_T^{\tau}(s) - \lambda_S^{\tau}(s)ds \geq 0.$$

Finally, if $\geq s_j$, then

$$\int_0^t \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s)ds = \int_0^{s_j} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s)ds + \int_{s_j}^t \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s)ds$$

$$= \int_0^{s_j} \lambda_T^{\tau}(s) - \lambda_S^{\tau}(s)ds + \int_{s_j}^t \lambda_T^{\tau}(s) - \lambda_S^{\tau}(s)ds \geq 0$$

with equality when $t = 1$. Thus, the proof that $S \prec_\tau T'$ is complete.

Postponing the discussion of the $\alpha_1' \neq \beta_1$ case, we demonstrate that if $\alpha_1' = \beta_1$ then $QSQ \prec_{\tau_Q} T'Q$ in $Q\mathfrak{A}Q$. For each $k \in \{1, \ldots, n\}$, let $s_k' \sum_{j=2}^{k} \tau_Q(P_j)$. Notice $s_k' < s_{k-1}'$ for all $k, s_n' = 1$ and $\lambda_{QSQ}^{\tau_Q}(s) = \beta_k$ for all $s \in [s_{k-1}, s_k)$. In the case $\alpha_1' = \beta_1$, we note that $\alpha_j' \leq \beta_1 \leq \alpha_l'$ for all $l < j$, and $\alpha_k' \geq \alpha_{k+1}$ for all $k \geq j$. Consequently, $\lambda_{QT'Q}^{\tau_Q}(s) \geq \beta_1$ for all $s < s_{j-1}', \lambda_{T'}^{\tau_Q}(s) = \alpha_k'$ for all $s \in [s_{k-1}', s_k')$with $k \geq j$, and

$$\int_0^{s'_{j-1}} \lambda_{T'}^{\tau_Q}(s)ds \ = \sum_{k=1}^{j-1} \alpha'_k \tau_Q(P_k).$$

Moreover, one can verify that

$$\lambda_{QT'Q}^{\tau_Q}\left(\frac{s \ - \ \tau(P_1)}{\tau(Q)} - 1\right)\lambda_{T'}^{\tau}(s) \text{ and } \lambda_{QSQ}^{\tau_Q}\left(\frac{s \ - \ \tau(P_1)}{\tau(Q)} - 1\right)\lambda_{S}^{\tau}(s)$$

for all $s \ \geq \ s_j$.

If $t \ < \ s'_{j-1}$, then

$$\int_0^t \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds \ \geq \int_0^t \beta_1 - \beta_2 ds \geq 0.$$

If $t \in \left[s'_{j-1}, s'_j\right]$ we see that

$$\int_0^t \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds$$

$$= \int_0^{s'_{j-1}} \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds \ + \int_{s'_{j-1}}^t \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds$$

$$= \frac{1}{\tau(Q)}\sum_{k=2}^{j-1}(\alpha'_k - \beta_k)\tau(P_k) + \int_{s'_{j-1}}^t \alpha'_j \ - \ \beta_j ds$$

$$= \frac{1}{\tau(Q)}\sum_{k=1}^{j-1}(\alpha'_k - \beta_k)\tau(P_k) + \int_{s'_{j-1}}^t \alpha'_j \ - \ \beta_j ds$$

$$= \frac{1}{\tau(Q)}\int_0^{s_{j-1}} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s)ds \ + \int_{s'_{j-1}}^t \alpha'_j \ - \ \beta_j ds$$

In particular, for $t \ = \ s'_j$, we see that

$$\int_0^{s'_j} \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds$$

$$= \frac{1}{\tau(Q)}\int_0^{s_{j-1}} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s)ds \ + \int_{s'_{j-1}}^{s'_j} \alpha'_j \ - \beta_j ds$$

$$= \frac{1}{\tau(Q)}\int_0^{s_{j-1}} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s)ds \ + \left(\alpha'_j \ - \beta_j\right)\frac{\tau(P_j)}{\tau(Q)}$$

$$= \frac{1}{\tau(Q)} \int_0^{s_j} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s) ds \,.$$

If $\alpha_j' \geq \beta_j$, then

$$\int_0^t \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds \geq \int_0^{s_j'} \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds$$

$$= \frac{1}{\tau(Q)} \int_0^{s_j} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s) ds \geq 0$$

Finally, if $t > s_j'$,

$$\int_0^t \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds$$

$$= \int_0^{s_j'} \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds + \int_{s_j'}^{t} \lambda_{QT'Q}^{\tau_Q}(s) - \lambda_{QSQ}^{\tau_Q}(s)ds$$

$$= \frac{1}{\tau(Q)} \int_0^{s_j} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s) ds$$

$$+ \frac{1}{\tau(Q)} \int_{s_j}^{\tau(Q)t+\tau(P_1)} \lambda_{QT'Q}^{\tau_Q}\left(\frac{s-\tau(P_1)}{\tau(Q)}\right) - \lambda_{QSQ}^{\tau_Q}\left(\frac{s-\tau(P_1)}{\tau(Q)}\right) ds$$

$$= \frac{1}{\tau(Q)} \int_0^{s_j} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s) ds$$

$$+ \frac{1}{\tau(Q)} \int_{s_j}^{\tau(Q)t+\tau(P_1)} \lambda_{T'}^{\tau}(s) - \lambda_S^{\tau}(s) ds \geq 0$$

with equality to zero when $t = 1$. Hence $QSQ \prec_{\tau_Q} QT'Q$ in $Q\mathfrak{A}Q$.

To complete the proof, we notice the proof is complete when $\beta_1 \in \left[\tau_{P_1+P_j}(T_0), \alpha_1\right]$(i.e. the $\alpha_1' = \beta_1$ case). Otherwise, repeat the above proof with j replaced with $j + 1$ and T replaced with T'. Note we end up obtaining that $\alpha_j' \geq \alpha_{j+1}'$ under this recursion as the first iteration yields $\alpha_1' = \alpha_j'$ and the second iteration would average α_1' with $\alpha_{j+1}' \leq \alpha_j < \alpha_j'$ to yield α_k'' with $\alpha_j'' = \alpha_j' > \alpha_{j+1}''$. This process must eventually obtain $\alpha_1' = \beta_1$ by reaching the case that $\beta_1 \in$

$\left[\tau_{P_1+P_j}(T_0), \alpha_1\right]$ for if we must apply the proof with $j = n$ and we produce a self-adjoint operator T' with $S \prec_\tau T'$, $\alpha'_1 > \beta_1$, and $\alpha'_k \geq \alpha'_1 \geq \beta_l$ for all k and l, we have a contradiction to the act that $S \prec_\tau T'$ (which guarantees $\tau(S) = \tau(T')$). Furthermore, note we obtain $QSQ \prec_{\tau_Q} QT'T$ at the last step of this iterative process.

Theorem (1.1.42)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero that has strong comparison of projections with respect to a faithful tracial state . If $T \in \mathfrak{A}$ is self-adjoint, then*

$$\overline{\text{conv}}(\mathcal{U}(T)) = \{S \in \mathfrak{A} \,|S* = S, S \prec_\tau T\}.$$

Before proceeding, we briefly outline the approach to the proof. First, we reduce to the case that T and S have finite spectrum. This is done by showing T and S can be approximated by self-adjoint operators T' and S' such that $S' \prec_\tau T'$. We then demonstrate a 'pinching' on self-adjoint operators T′ with exactly two points in their spectrum to show that all convex combinations of T' and $\tau(T')I_{\mathfrak{A}}$are in $\overline{\text{conv}}(\mathcal{U}(T'))$. Appealing to a specific decomposition result and by progressively applying pinchings, the result is obtained.

We begin with the decomposition result.

Proof : Let $T \in \mathfrak{A}$ be self-adjoint. Note the inclusion

$$\overline{\text{conv}}(\mathcal{U}(T)) \subseteq \{S \in \mathfrak{A} \,|S^* = S, S \prec_\tau T\}$$

To prove the other inclusion, let $S \in \mathfrak{A}$ be self-adjoint with $S \prec_\tau T$ By Lemma (1.1.26), we may assume without loss of generality that S and T have finite spectrum.

Let $\{P_k\}_{k=1}^n$, $\{Q_k\}_{k=1}^n$, $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$ be as in Lemma(1.1.25)so that

$$T = \sum_{k=1}^{n} \alpha_k Q_k \quad \text{and} \quad S = \sum_{k=1}^{n} \beta_k P_k .$$

Since $\mathfrak{A}$ has strong comparison of projections, there exists a unitary $U \in \mathfrak{A}$ such that $U^*Q_kU = P_k$ for all k. Hence $U^*TU = \sum_{k=1}^{n} \alpha_k Q_k$. Since $\lambda^{\tau}_{U^*TU}(s) = \lambda^{\tau}_{T}(s)$for all $s \in [0,1)$, $S \prec_{\tau} U^*TU$. Consequently, $\alpha_1 \geq \beta_1 \geq \beta_n \geq \alpha_n$.

If $\alpha_1 = \alpha_n$, then $T = S = \tau(T)I_{\mathfrak{A}}$and there is nothing to prove. Otherwise, we may apply Lemma (1.1.28) to obtain, for some $\{\alpha'_k\}_{k=2}^{n} \subseteq \mathbb{R}$, that

$$T' = \beta_1 P_1 + \sum_{k=2}^{n} \alpha'_k P_k \in \overline{\mathrm{conv}}\big(\mathcal{U}(U^*T\,U)\big) \text{and } QSQ \prec_{\tau_Q} QT'Q in\ QSQ$$

where $Q = \sum_{k=2}^{n} P_k$. In addition, note Lemma(1.1.28) produces $\{\alpha'_k\}_{k=2}^{n}$ so that QSQ and $QT'Q$ in QAQ are either equal or satisfy the hypotheses of Lemma (1.1.28); that is, $QSQ \prec_{\tau_Q} QT'Q$, $\alpha'_{k+1} \leq \alpha'_k$ for all $k \geq j$, $\alpha'_k = \alpha_k \geq \beta_1 \geq \beta_2$ for all $1 < k < j$, and, if $j = 2$, $\alpha'_2 \geq \beta_2 \geq \beta_n \geq \alpha_n$ .Therefore, by applying Lemma (1.1.28) at most another $n - 1$ times, we obtain that

$$S \in c\overline{\mathrm{conv}}(\mathcal{U}(U^*TU)) = \overline{\mathrm{conv}}(\mathcal{U}(T)).$$

Section(1.2): Classification of Sets and Purely Infinite C^*-Algebras

We will study additional sets based on eigenvalue and singular value functions in C^*-algebras. We begin by studying the distance between unitary orbits of self-adjoint operators. The following result is the main result.

Theorem (1.2.1)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero that has strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$ are self-adjoint, then*

$$\mathrm{dist}\big(\mathcal{U}(S), \mathcal{U}(T)\big) = \sup\{|\lambda^{\tau}_{S}(s) - \lambda^{\tau}_{T}(s)| \mid s \in [0,1)\}.$$

In particular, S and T are approximately unitarily equivalent if and *only if* $\lambda_S^\tau(s) = \lambda_T^\tau(s)$ *for all* $s \in [0,1)$ *if and only if* $S \prec_\tau T$ *and* $T \prec_\tau S$.

Proof. We have

$$|\lambda_S^\tau(s) - \lambda_T^\tau(s)| = |\lambda_{U^*SU}^\tau(s) - \lambda_{V^*TV}^\tau(s)| \le \|U^*SU - V^*TV\|$$

for all unitaries $U, V \in \mathfrak{A}$ and $s \in [0,1)$. Hence

$$\sup\{|\lambda_S^\tau(s) - \lambda_T^\tau(s)| \mid s \in [0,1)\} \le \operatorname{dist}(\mathcal{U}(S), \mathcal{U}(T)).$$

For the other inequality, fix $\epsilon > 0$. Since $\mathfrak{A}$ has real rank zero, there exists self-adjoint operators $S', T' \in \mathfrak{A}$ with finite spectrum such that

$$\|S - S'\| < \epsilon \text{ and } \|T - T'\| < \epsilon.$$

Note

$$|\lambda_T^\tau(s) - \lambda_{T'}^\tau(s)| < \epsilon \quad \text{and } |\lambda_S^\tau(s) - \lambda_{S'}^\tau(s)| < \epsilon$$

for all $s \in [0,1)$.

Let $\{P_k\}_{k=1}^n$, $\{Q_k\}_{k=1}^n$, $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$ so that

$$T' = \sum_{k=1}^n \alpha_k P_k \quad \text{and} \quad S' = \sum_{k=1}^n \beta_k Q_k .$$

If $s_k = \sum_{j=1}^k \tau(Q_j)$ for all $k \in \{0,1,\dots,n\}$, implies $\lambda_{T'}^\tau(s) = \alpha_k$ and $\lambda_{S'}^\tau(s) = \beta_k$ for all $s \in [s_{k-1}, s_k)$. Furthermore, since $\tau(P_k) = \tau(Q_k)$ for all k and since $\mathfrak{A}$ has strong comparison of projections, there exists a unitary $U \in \mathfrak{A}$ such that $U^*P_kU = Q_k$ for all k and, consequently, $U^*T'U = \sum_{k=1}^n \alpha_k Q_k$. Hence

$$\begin{aligned}
\|U^*TU - S\| &\le 2\epsilon + \|U^*T'U - S'\| \\
&= 2\epsilon + \sup\{|\alpha_k - \beta_k| \mid k \in \{1,\dots,n\}\} \\
&= 2\epsilon + \sup\{|\lambda_{T'}^\tau(s) - \lambda_{S'}^\tau(s)| \mid s \in [0,1)\} \\
&\le 4\epsilon + \sup\{|\lambda_T^\tau(s) - \lambda_S^\tau(s)| \mid s \in [0,1)\}.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, the proof is complete.

We have the following result .

Theorem (1.2.2)[1]: *Let* $\mathfrak{A}$ *be a unital* C^**-algebra with real rank zero* *that has strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$ are self-adjoint, then*

$$\operatorname{dist}\left(S, \overline{\operatorname{conv}}(\mathcal{U}(T))\right)$$

$$= \sup_{t\in(0,1)} \frac{1}{t} \max\left\{\int_0^t \lambda_S^\tau(s) - \lambda_T^\tau(s)\, ds \geq \frac{1}{t}\int_0^t \lambda_T^\tau(s) - \lambda_S^\tau(s)\, ds\right\}$$

Proof. Let α be the quantity on the right-hand side of the desired equation. Suppose $T' \in \overline{\operatorname{conv}}(\mathcal{U}(T))$. Then $T' \prec_\tau T$. Consequently,

$$\|T' - S\| \geq \frac{1}{t}\int_0^t \lambda_S^\tau(s) - \lambda_{T'}^\tau(s)\, ds \geq \frac{1}{t}\int_0^t \lambda_S^\tau(s) - \lambda_T^\tau(s)\, ds$$

and

$$\|T' - S\| \geq \frac{1}{t}\int_{1-t}^1 \lambda_{T'}^\tau(s) - \lambda_S^\tau(s)\, ds \geq \frac{1}{t}\int_{1-t}^1 \lambda_T^\tau(s) - \lambda_S^\tau(s)\, ds$$

Therefore $\operatorname{dist}(S, \operatorname{conv}(U(T))) \geq \alpha$.

For the other inequality, first suppose $\alpha \leq 0$. Then

$$\int_0^t \lambda_S^\tau(s) - \lambda_T^\tau(s)\, ds \text{ and } \int_{1-t}^1 \lambda_T^\tau(s) - \lambda_S^\tau(s)\, ds \leq 0$$

for all $t \in (0,1)$. The first inequality implies

$$\int_0^t \lambda_S^\tau(s)\, ds \leq \int_0^t \lambda_T^\tau(s)\, ds$$

for all $t \in [0,1]$, and by letting t tend to 1, the second inequality then implies

$$\int_0^1 \lambda_S^\tau(s)\, ds \leq \int_0^1 \lambda_T^\tau(s)\, ds$$

Consequently, $\alpha = 0$ and $\prec_\tau T$. Thus implies $S \in \overline{\operatorname{conv}}(\mathcal{U}(T))$; so equality is obtained in this case.

Otherwise, suppose $\alpha > 0$. Let $\epsilon > 0$. Since $\mathfrak{A}$ has real rank zero, there exist self-adjoint operators $S', T' \in \mathfrak{A}$ with finite spectrum such that

$$\|S - S'\| < \epsilon \quad \text{and} \quad \|T - T'\| < \epsilon.$$

In addition

$$\left|\lambda_S^\tau(s)-\lambda_{S'}^\tau(s)\right| < \epsilon \text{ and } \left|\lambda_T^\tau(s)-\lambda_{T'}^\tau(s)\right| < \epsilon$$

for all $s \in [0,1)$. By the definition of α, we obtain

$$\int_0^t \lambda_{S'}^\tau(s)-\alpha-2\epsilon\,ds \leq \int_0^t \lambda_S^\tau(s)-\alpha-\epsilon\,ds \leq \int_0^t \lambda_T^\tau(s)-\epsilon\,ds$$

$$\leq \int_0^t \lambda_{T'}^\tau(s)\,ds$$

$$\int_0^t \lambda_{S'}^\tau(s)+\alpha+2\epsilon\,ds \geq \int_0^t \lambda_S^\tau(s)+\alpha+\epsilon\,ds \geq \int_0^t \lambda_T^\tau(s)+\epsilon\,ds$$

$$\geq \int_0^t \lambda_{T'}^\tau(s)\,ds$$

for all $t \in (0,1)$. Consequently, using non-increasing rearrangements and applied to $f_1(s) = \lambda_{S'}^\tau(s)-\alpha-2\epsilon, f_2(s) = \lambda_{S'}^\tau(s)+\alpha+2\epsilon$, and $g(s) = \lambda_{T'}^\tau(s)$, there exists a real-valued, non-increasing function $h \in L_\infty[0,1]$ such that

$$f_1(s) \leq h(s) \leq f_2(s) \qquad (1)$$

for all $s \in [0,1)$ and $h \prec \lambda_{T'}^\tau$.

Let $\{P_k\}_{k=1}^n$, $\{Q_k\}_{k=1}^n$, $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$ be as in Lemma (1.1.21) so that

$$T' = \sum_{k=1}^m \alpha_k P_k \qquad \text{and} \qquad S' = \sum_{k=1}^n \beta_k Q_k\ .$$

Furthermore, for $k \in \{0,1,\ldots,n\}$, let $s_k = \sum_{j=1}^k \tau(Q_k)$, let

$$\alpha_k' = \frac{1}{s_k-s_{k-1}}\int_{s_{k-1}}^{s_k} h(s)\,ds\,,$$

and let $T_0 = \sum_{k=1}^n \alpha_k' P_k'$. Notice $\alpha_k' \geq \alpha_{k+1}'$ for all k as h is non-increasing. Since $h \prec \lambda_{T'}^\tau \prec \tau$, $T_0 \prec_\tau T$ and $T_0 \in \overline{\text{conv}}(\mathcal{U}(T'))$.

Since $\mathfrak{A}$ has strong comparison of projections, there exists a unitary $U \in \mathfrak{A}$ such that $U^*P_kU = Q_k$ for all k. Therefore $U^*T_0U =$

$\sum_{k=1}^{m} \alpha'_k Q_k$. However, due to the definition of α'_k, equation (1), we see that

$$\|U^* T_0 U - S'\| \leq \alpha + 2\epsilon.$$

Therefore, since $U^* T_0 U \in \overline{\mathrm{conv}}(\mathcal{U}(T'))$, $\|T - T'\| < \epsilon$, and $\|S - S'\| < \epsilon$, we obtain that

$$\mathrm{dist}(S, \mathrm{conv}(\mathcal{U}(T))) \leq \alpha + 4\epsilon$$

thereby completing the proof.

Since tracial states are norm continuous, Theorem (1.2.2) immediately implies the following.

Corollary (1.2.3)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero that has strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$ are self-adjoint, then*

$$\mathrm{dist}(\mathrm{conv}(\mathcal{U}(S)), \mathrm{conv}(\mathcal{U}(T))) = |\tau(S) - \tau(T)|.$$

We are also able to study arbitrary operators based on their singular value functions. The following object will play the role of the singular value decomposition of matrices for infinite dimensional C^*-algebras.

Definition (1.2.4)[1]: For a unital C^*-algebra $\mathfrak{A}$ and an element $T \in \mathfrak{A}$, the closed two-sided unitary orbit of T is

$$\mathcal{N}(T) = \overline{\{U\,T\,V \mid U, V \text{ unitaries in } \mathfrak{A}\}}.$$

We classify closed two-sided unitary orbits using singular values. We restrict to C^*-algebras with stable rank one as the following well-known lemma directly implies every operator almost has a polar decomposition.

Lemma (1.2.5)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra and let $M, \epsilon > 0$. There exists a $0 < \delta < \epsilon$ such that if $A, B \in \mathfrak{A}$, $\|A\| \leq M$, and $\|A - B\| < \delta$, then $\||A| - |B|\| < \epsilon$.*

Corollary (1.2.6)[1]: Let $\mathfrak{A}$ *be a unital C^*-algebra with stable rank* *one and let $T \in \mathfrak{A}$. Then for all $\epsilon > 0$ there exists a unitary $U \in \mathfrak{A}$ such that $\|T - U|T|\| < \epsilon$.*

Lemma (1.2.7)[1]: Let $\mathfrak{A}$ *be a unital C^*-algebra with a faithful tracial state . If $(T_n)_{n\geq 1} \subseteq A$ converges in norm to $T \in \mathfrak{A}$, then $\mu_T^\tau(s) = \lim_{n\to\infty} \mu_{T_n}^\tau(s)$ for all $s \in [0,1)$.*

Proof. Recall $\mu_S^\tau(s) = \lambda_{|S|}^\tau(s)$ for all $S \in A$. Since $T = \lim_{n\to\infty} T_n$, we obtain $|T| = \lim_{n\to\infty} T_n$ by Lemma (1.2.5).

Proposition (1.2.8)[1]: *Let $\mathfrak{A}$ be a unital C^* −algebra with real rank zero, stable rank one, and strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$, then $S \in \mathcal{N}(T)$ if and only if $\mu_S^\tau(s) = \mu_T^\tau(s)$ for all $s \in [0,1)$.*

Proof. If $U, V \in \mathfrak{A}$ are unitaries, then

$$\mu_{UTV}^\tau(s) = \lambda_{|UTV|}^\tau(s) = \lambda_{|T|V}^\tau V^*(s) = \lambda_{|T|}^\tau(s) = \mu_T^\tau(s)$$

for all $s \in [0,1)$. Consequently, if $S \in \mathcal{N}(T)$, then $\mu_S^\tau(s) = \mu_T^\tau(s)$ for all $s \in [0,1)$ by Lemma (1.2.7).

For the converse direction, suppose $\mu_S^\tau(s) = \mu_T^\tau(s)$ for all $s \in [0,1)$ and let $\epsilon > 0$. By Corollary (1.2.6), there exists unitaries $U, V \in \mathfrak{A}$ such that

$$\|T - U|T|\| < \epsilon \qquad \text{and} \qquad \|S - V|S|\| < \epsilon.$$

Furthermore, since

$$\lambda_{|T|}^\tau(s) = \mu_T^\tau(s) = \mu_S^\tau(s) = \lambda_{|S|}^\tau(s)$$

for all $s \in [0,1)$, Theorem (1.2.1) implies there exists a unitary $W \in \mathfrak{A}$ such that

$\|W^*|T|W - |S|\| < \epsilon$. Hence

$$\|V W^* U^* T W - S\| \leq 2\epsilon + \|V W^*|T|W - V|S'|\| < 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, the proof is complete.

Our next results provide descriptions of all operators whose eigenvalue function is dominated by another operator's eigenvalue function. In particular, these notions of majorization are related to Cuntz equivalence, but are significantly stronger (i.e. requiring bounded sequences for approximations). We have following result .

Proposition (1.2.9)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero that has strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$ are positive operators, then*

$$S \in \overline{\{A^*TA \mid A \in \mathfrak{A}, \|A\| \leq 1\}}$$

if and only if $\lambda_S^\tau(s) = \lambda_T^\tau(s)$ for all $s \in [0,1)$.

Proof. If $A \in \mathfrak{A}$ is such that $\|A\| \leq 1$, then

$$\lambda_{A^*TA}^\tau(s) \leq \|A\|^2 \lambda_T^\tau(s) \leq \lambda_T^\tau(s)$$

for all $s \in [0,1)$.

For the other direction, suppose $\lambda_S^\tau(s) = \lambda_T^\tau(s)$ for all $s \in [0,1)$. Let $\epsilon > 0$. There exists positive operators $S', T' \in \mathfrak{A}$ with finite spectra such that $\|T - T'\| < \epsilon, \|S - S'\| < \epsilon$, and $\lambda_{S'}^\tau(s) \leq \lambda_{T'}^\tau(s)$for all $s \in [0,1)$. Let $\{P_k\}_{k=1}^n$, $\{Q_k\}_{k=1}^n$, $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$ so that

$$T' = \sum_{k=1}^{n} \alpha_k P_k \qquad \text{and} \qquad S' = \sum_{k=1}^{n} \beta_k Q_k \ .$$

Since $T', S' \geq 0, \alpha_k, \beta_k \geq 0$ for all k. Furthermore, along with the fact that $\lambda_{S'}^\tau(s) \leq \lambda_{T'}^\tau(s)$ for $all\ s \in [0,1)$ implies $\beta_k \leq \alpha_k$ for all k.

Since $\mathfrak{A}$ has strong comparison of projections, there exists a unitary $U \in \mathfrak{A}$ such that $U^*P_kU = Q_k$ for all k so that $U^*T'U = \sum_{k=1}^n \alpha_k Q_k$. For each k, let

$$\gamma = \begin{cases} \sqrt{\dfrac{\beta_k}{\alpha_k}} & \text{if } \beta_k = 0 \\ 0 & \text{if } \beta_k \neq 0 \end{cases}.$$

Consequently, if $A = \sum_{k=1}^n \gamma_k Q_k \in \mathfrak{A}$, then $\|A\| \leq 1$ and $A^*U^*T'UA = S'$. Hence

$$\|A^*U^*TUA - S\| \le 2\epsilon + \|A^*U^*T'UA - S'\| = 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, the result follows.

Proposition (1.2.10)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero, stable rank one, and strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$, then*

$$S \in \overline{\{ATB \mid A, B \in \mathfrak{A}, \|A\|, \|B\| \le 1\}}$$

if and only if $\mu_S^\tau(s) \le \mu_T^\tau(s)$for all $s \in [0,1)$.

Proof. If A, B ∈ $\mathfrak{A}$ are such that $\|A\|$, $\|B\| \le 1$, then

$$M_{ATB}^\tau(s) \le \|A\| \|B\| \mu_T^\tau(s) \le \mu_T^\tau(s)$$

for all $s \in [0,1)$. Consequently, one direction follows from Lemma (1.2.7).

For the other direction, suppose$\mu_S^\tau(s) \le \mu_T^\tau(s)$for all $s \in [0,1)$. Consequently $\lambda_{|S|}^\tau(s) \le \lambda_{|T|}^\tau(s)$ for all $s \in [0,1)$. Thus Proposition (1.2.9) implies for all $\epsilon > 0$ there exists an $A \in \mathfrak{A}$ with $\|A\| \le 1$ such that $\||S| - A^*|T|A\| < \epsilon$. Furthermore, Corollary (1.2.6) implies there exists unitaries $U, V \in \mathfrak{A}$ such that $\|S - V|S|\| < \epsilon$ and $\|T - U|T|\| < \epsilon$. Thus

$$\|S - VA^*U^*TA\| \le \|S - VA^*|T|A\| + \epsilon \le \|S - V|S|\| + 2\epsilon \le 3\epsilon.$$

The result follows.

We desire to analyze the notion of (absolute) submajorization as defined. In particular. The following useful lemma shows if one positive operator submajorizes an operator, then conjugating by a specific contractive operator almost yields majorization.

Lemma (1.2.11)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero and strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$ are positive operators such that $S \prec_\tau^w T$, then for all $\epsilon > 0$ there exists positive operators $S', T' \in \mathfrak{A}$ and an $A \in \mathfrak{A}$ with $\|A\| \le 1$ such that*

$$\|S - S'\| \leq \epsilon, \|T - T'\| \leq \epsilon, \quad \text{and} \quad S' \prec_\tau A^*T'A.$$

Proof. Fix $\epsilon > 0$. There exists positive operators $S', T' \in \mathfrak{A}$ with finite spectra such that

$$\|S - S'\| \leq \epsilon, \|T - T'\| \leq \epsilon, \quad \text{and}\, S' \prec_\tau^w T'.$$

Let $\{P_k\}_{k=1}^n$, $\{Q_k\}_{k=1}^n$, $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$, so that

$$T' = \sum_{k=1}^{n} \alpha_k P_k \qquad \text{and} \qquad S' = \sum_{k=1}^{n} \beta_k Q_k .$$

For each $k \in \{0, 1, \ldots, n\}$, let $s_k = \sum_{k=1}^{n} \tau(P_k)$.

Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) = \int_0^t \lambda_{T'}^\tau(s)ds - \int_0^1 \lambda_{S'}^\tau(s)ds .$$

Since f is continuous, $f(0) \leq 0$, and $f(1) \geq 0$, there exists a $t_0 \in [0,1]$ such that $f(t_0) = 0$. Let $t' = \sup\{t \in [0,1] \mid f(t) = 0\}$ and choose $k' \in \{1, \ldots, n\}$ such that $t' \in [s_{k'-1}, s_{k'})$ (with $k' = n$ if $t' = 1$). Notice this implies

$$\int_0^{s_{k'-1}} \lambda_{T'}^\tau(s)ds \leq \int_0^1 \lambda_{S'}^\tau(s)ds \leq \int_0^{s_{k'}} \lambda_{T'}^\tau(s)ds .$$

Choose $q \in [0,1]$ such that

$$\int_0^1 \lambda_{T'}^\tau(s)ds = \int_0^{s_{k'-1}} \lambda_{T'}^\tau(s)ds + q \int_0^{s_{k'}} \lambda_{T'}^\tau(s)ds$$

and let $\mathfrak{A} = qP_{k'} + \sum_{k=1}^{k'-1} P_k$. Clearly $\|A\| \leq 1$ and

$$A^*T'A = q\alpha_{k'} P_{k'} + \sum_{k=1}^{k'-1} \alpha_k P_k .$$

Furthermore, one may verify using integral arguments that $S' \prec_\tau A^*T'A$.

Proposition (1.2.12)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero and strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$ are positive operators, then*

$$S \in \overline{\text{conv}}(\{A^*TA \mid A \in \mathfrak{A}, \|A\| \leq 1\})$$

if and only if $S \prec_\tau^w T$.

Proof. If $\{A_k\}_{k=1}^n \subseteq \mathfrak{A}$ are such that $\|A_k\| \leq 1$ for all k, $\{t_k\}_{k=1}^n \subseteq [0,1]$ are such that $\sum_{k=1}^n t_k$ and $S' = \sum_{k=1}^n t_k A_k^* T A_k$, then $S' \geq 0$ and

$$\int_0^t \lambda_{S'}^\tau(s)\, ds \leq \int_0^t \sum_{k=1}^n t_k \|A_k\|^2 \lambda_T^\tau(s)\, ds \leq \int_0^t \lambda_T^\tau(s)\, ds$$

Thus one inclusion follows.

For the other direction, suppose $S \prec_\tau^w T$. Let $\epsilon > 0$. By Lemma (1.2.11) there exists positive operators $S', T' \in \mathfrak{A}$ and an $A \in \mathfrak{A}$ with $\|A\| \leq 1$ such that

$$\|S - S'\| \leq \epsilon, \|T - T'\| \leq \epsilon, \text{ and } S' \prec_\tau A^*T'A.$$

As

$$S' \in \overline{\text{conv}}(\mathcal{U}(A^*T'A))$$

the result follows.

Proposition (1.2.13)[1]: *Let $\mathfrak{A}$ be a unital C^*-algebra with real rank zero, stable rank one, and strong comparison of projections with respect to a faithful tracial state . If $S, T \in \mathfrak{A}$, then*

$$S \in \overline{\text{conv}}(\{ATB \mid A, B \in \mathfrak{A}, \|A\|, \|B\| \leq 1\})$$

if and only if $S \prec_\tau^w T$.

Proof. If $\{A_k\}_{k=1}^n$, $\|B_k\|_{k=1}^n \subseteq \mathfrak{A}$ are such that $\|A_k\|, \|B_k\| \leq 1$ for all k, $\{t_k\}_{k=1}^n \subseteq [0,1]$ are such that $\sum_{k=1}^n t_k = 1$ and $S' = \sum_{k=1}^n t_k A_k T B_k$, then

$$\int_0^t \mu_{S'}^\tau(s)\, ds \leq \int_0^t \sum_{k=1}^n t_k \|A_k\|^2 \mu_T^\tau(s)\, ds \leq \int_0^t \mu_T^\tau(s)\, ds$$

Thus one inclusion follows from Lemma (1.2.7).

For the other direction, suppose $S \prec_\tau^w T$. Thus $|S| \prec_\tau^w |T|$ so Proposition (1.2.12) implies

$$|S| \in \overline{\text{conv}}(\{A^*|T|A \mid A \in \mathfrak{A}, \|A\| \leq 1\}).$$

The result then follows by approximation arguments along with Lemma (1.2.5).

We will show the following result describing the closed convex hulls of unitary orbits of self-adjoint operators T in unital, simple, purely infinite C^*-algebras (pure infiniteness C^*-algebras A are compared, and equivalence between them is obtained if the primitive ideal space of A has real rank zero, of if A is Approximately divisble) [6] based on the spectrum of T, denoted $\sigma(T)$.

Since unital, simple, purely infinite C^*-algebras have real rank zero, to verify the reverse inclusion it suffices to consider self-adjoint $S, T \in \mathfrak{A}$ with finite spectrum and $\sigma(S) \subseteq \text{conv}(\sigma(T))$ by the continuous functional calculus. Furthermore, note this problem is invariant under simultaneous multiplying the operators by non-zero real numbers and simultaneous translation of the operators by a real constant. As such, it suffices to prove the result for positive T with $\|T\| = 1$ and $0, 1 \in \sigma(T)$.

We will demonstrate it suffices to prove the result when T is a projection. This will be done by constructing (possibly non-unital) embeddings of arbitrarily larger matrix algebras into $\mathfrak{A}$. Subsequently, we will verify that the result holds for T a projection and $S \in \mathbb{C}I_{\mathfrak{A}}$. The result will follow for arbitrary S with finite spectrum by an application of K-Theory.

We begin with the following well-known result for purely infinite C^*-algebras.

Lemma (1.2.14)[1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $P, Q \in \mathfrak{A}$ be orthogonal non-zero projections. For any $n \in \mathbb{N}$ there exists a collection $\{P_k\}_{k=1}^{n}$ of pair-wise orthogonal subprojections of P such that each P_k is Murray-von Neumann* *equivalent to Q.*

By ‘a non-trivial projection’, we mean a non-zero projection P with $P \neq I_{\mathfrak{A}}$.

Lemma (1.2.15) [1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $P \in \mathfrak{A}$ be a non-trivial projection. If $\alpha, \beta \in \mathbb{R}$ and $T = \alpha P + \beta(I_{\mathfrak{A}} - P)$, then $\alpha I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$.*

Proof. Clearly the result holds if $\alpha = \beta$ so suppose $\alpha = \beta$. By scaling and translating, we may assume that $\alpha = 1$ and $\beta = 0$.

Let $n \in \mathbb{N}$ be arbitrary. By Lemma (1.2.14) there exists a collection $\{P_k\}_{k=1}^n$ of pairwise orthogonal subprojections of P such that $P_k \sim I_{\mathfrak{A}} - P$ for all k. Using the partial isometries implementing the equivalence of $\{I_{\mathfrak{A}} - P\} \cup \{P_k\}_{k=1}^n$, a copy of $\mathcal{M}_{n+1}(\mathbb{C})$ may be constructed in $\mathfrak{A}$ such that the unit of $\mathcal{M}_{n+1}(\mathbb{C})$ is $P_n' := I_{\mathfrak{A}} - P + \sum_{k=1}^n P_k$ and T may be viewed as the operator

$$T = \text{diag}(0,1,\ldots,1) \oplus (I_{\mathfrak{A}} - P_n')$$
$$\in \mathcal{M}_{n+1}(\mathbb{C}) \oplus (I_{\mathfrak{A}} - P_n')\mathfrak{A}(I_{\mathfrak{A}} - P_n') \subseteq \mathfrak{A}.$$

Since any self-adjoint matrix majorizes its trace, we obtain that

$$\frac{n}{n+1} I_{n+1} \in \overline{\text{conv}}\Big(\mathcal{U}\big(\text{diag}(0,1,\ldots,1)\big)\Big)$$

where the unitary orbit is computed in $\mathcal{M}_{n+1}(\mathbb{C})$. Thus, by a direct sum argument, we obtain

$$\frac{n}{n+1} P_n' + (I_{\mathfrak{A}} - P_n') \in \text{conv}(\mathcal{U}(T)).$$

By taking the limit as $n \to \infty$, we obtain $I_{\mathfrak{A}} \in \overline{\text{conv}}(\mathcal{U}(T))$.

Lemma (1.2.16)[1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $\{P_k\}_{k=1}^n$ be a collection of pairwise orthogonal, non-zero projections. If $T = \sum_{k=1}^n \lambda_k P_k$ for some real numbers $\{\lambda_k\}_{k=1}^n \in \mathbb{R}$, then*

$$\lambda_1\left(\sum_{k=1}^{n-1} P_k\right) + \lambda_n P_n \in \overline{\text{conv}}(\mathcal{U}(T)).$$

Lemma (1.2.17)[1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $P \in \mathfrak{A}$ be a non-trivial projection. For each $\gamma \in [0,1] \cap \mathbb{Q}$, there exist pairwise orthogonal, non-zero projections Q_1, Q_2, Q_3 such that $Q_1 + Q_2 + Q_3 = I_{\mathfrak{A}}$ and*

$$0Q_1 + \gamma Q_2 + 1Q_3 \in \overline{\text{conv}}(\mathcal{U}(P)).$$

Proof. Note the cases $\gamma = 0, 1$ are trivial. Otherwise, fix $n \in \mathbb{N}$ and choose $k \in \{1, \dots, n-1\}$ so that $\gamma = \frac{k}{n}$. Let $Q \in \mathfrak{A}$ be any non-trivial projection. By Lemma (1.2.15) there exists a collection $\{P_j\}_{j=1}^{k+1}$ of pairwise orthogonal subprojections of Psuch that $P_j \sim Q$ for all j. Similarly there exists a collection $\{P_j'\}_{j=1}^{n-k+1}$ of pairwise orthogonal subprojections of $I_{\mathfrak{A}} - P$ such that $P_j' \sim Q$ for all j.

Let

$$Q_1 = (I_{\mathfrak{A}} - P) - \sum_{j=1}^{n-1} P_j' , Q_2 = \sum_{j=1}^{k} P_j + \sum_{j=1}^{n-1} P_j' \text{ and } Q_3 = \sum_{j=1}^{k} P_j .$$

Since $P_{k+1} \leq Q_3$ and $P_{n-k+1} \leq Q_1$, it is clear that Q_1, Q_2, and Q_3 are pairwise orthogonal, non-zero projections such that $Q_1 + Q_2 + Q_3 + I_{\mathfrak{A}}$. Using the partial isometries implementing the equivalence of $\{P_j\}_{j=1}^{k} \cup \{P_j\} = 1$, a copy o$\mathcal{M}_n(\mathbb{C})$can be constructed in $\mathfrak{A}$ such that the unit of $\mathcal{M}_n(\mathbb{C})$ is Q_2 and

$$P = 0Q_1 \oplus D \oplus 1Q_3 \in Q_1\mathfrak{A}Q_1 \oplus \mathcal{M}_n(\mathbb{C}) \oplus Q_3\mathfrak{A}Q_3 \subseteq \mathfrak{A}$$

where D is a diagonal matrix with 1 appearing along the diagonal exactly k times and 0 appearing along the diagonal exactly $n - k$ times. Since any self-adjoint matrix majorizes its trace, we obtain and a direct sum argument that

$$0Q_1 + \gamma Q_2 + 1Q_3 \in \mathrm{conv}(\mathcal{U}(P)).$$

Lemma (1.2.18)[1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $P \in \mathfrak{A}$ be a non-trivial projection. For each $\gamma \in [0,1], \gamma I_{\mathfrak{A}} \in \overline{\mathrm{conv}}(\mathcal{U}(P))$.*

Proof. By applying approximations, it suffices to prove the result for $\gamma \in (0,1) \cap \mathbb{Q}$. By Lemma (1.2.1816 there exists pairwise orthogonal, non-zero projections Q_1, Q_2, Q_3 such that $Q_1 + Q_2 + Q_3 = I_{\mathfrak{A}}$ and

$$0Q_1 + \gamma Q_2 + 1Q_3 \in \overline{\mathrm{conv}}(\mathcal{U}(P)).$$

Choose two non-zero subprojections Q_1' and Q_3 of Q_2 such that $Q_1 + Q_3 = Q_2$. By applying Lemma (1.2.14) to $0Q_1 + \gamma Q_1' \in (Q_1 + Q_1')\mathfrak{A}(Q_1 + Q_1')$, we obtain that

$$\gamma(Q_1 + Q_1') \in \overline{\mathrm{conv}}(\mathcal{U}(0Q_1 + \gamma Q_1'))$$

(where the quantity on the right-hand side is computed in $((Q_1 + Q_1')\mathfrak{A}(Q_1 + Q_1'))$. Similarly

$$\gamma(Q_3 + Q_3') \in \overline{\mathrm{conv}}(\mathcal{U}(1Q_3 + \gamma Q_3'))$$

Hence, by the fact that $0Q_1 + \gamma Q_2 + 1Q_3$ is a direct sum of $0Q_1 + \gamma Q_1'$ and $1Q_3 + \gamma Q_3'$, we obtain that

$$\gamma I_{\mathfrak{A}} = \gamma(Q_1 + Q_1') + \gamma(Q_1 + Q_3') \in \overline{\mathrm{conv}}(\mathcal{U}(P)).$$

Theorem (1.2.19)[1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $T \in \mathfrak{A}$ be self-adjoint. Then*

$$\overline{\mathrm{conv}}(\mathcal{U}(T)) = \{S \in \mathfrak{A} \mid S^* = S, \sigma(S) \subseteq \mathrm{conv}(\sigma(T))\}.$$

Proof: We may assume $\sigma(S)$ and $\sigma(T)$ are finite so that there exists $\{\lambda_j\}_{j=1}^m, \{\alpha_k\}_{k=1}^n \subseteq \mathbb{R}$ with $\lambda_k < \lambda_{k+1}$ for all k and $\alpha_k \in \mathrm{conv}\left(\{\lambda_j\}_{j=1}^m\right)$ for all k, and two collections of pairwise orthogonal non-zero projections $\{P_j\}_{j=1}^m$ and $\{Q_k\}_{k=1}^n$ with $\sum_{j=1}^m P_j = I_{\mathfrak{A}} = \sum_{k=1}^n Q_k$ such that

$$T = \sum_{k=1}^{m} \lambda_k P_k \qquad \text{and} \qquad S = \sum_{k=1}^{n} \alpha_k Q_k \ .$$

The result is trivial if $m = 1$ so we assume $m \geq 2$. Furthermore, by translation and scaling, it suffices to prove the result when $\lambda_1 = 0$ and $\lambda_m = 1$. Furthermore, by Lemma (1.2.16) and the fact that $\lambda_1 \leq \lambda_k \leq \lambda_m$ for all k, we may assume that $m = 2$. For simplicity, let $P = P_m \, so \, P_1 = I_{\mathfrak{A}} - P$ and $= P$.

Since $\mathfrak{A}$ is a unital, simple, purely infinite C^*-algebra, there exists a collection $\{P'_k\}_{k=1}^{n-1}1$ of non-zero, pairwise orthogonal subprojections of Pand a collection$\{P''_k\}_{k=1}^{n-1}$ of non-zero, pairwise orthogonal subprojections of $I_{\mathfrak{A}} - P$ such that $P' \sim Q_k, P'_n = P - \sum_{k=1}^{n-1} P'_k$ is non-zero, and $P''_n = \sum_{k=1}^{n-1} P''_k$ is non-zero. For each $k \in \{1, \ldots, n\}$, let $Q'_k = P'_k + P''_k$. Therefore

$$\sum_{k=1}^{n} [Q_k]_0 = [I_{\mathfrak{A}}]_0 = \sum_{k=1}^{n} [Q'_k]_0 = [Q'_n]_0 + \sum_{k=1}^{n-1} [Q_k]_0 \, .$$

Hence $[Q_n]_0 = [Q'_n]_0$ so $Qn \sim Q'_n$.

Notice

$$T = \oplus_{k=1}^{n} \ (1P'_k + 0P''_k) \in \bigoplus_{k=1}^{n} Q'_k \mathfrak{A} Q'_k \ .$$

Since $P'_k$and P''_k are non-zero for each kand since $Q'_k \mathfrak{A} Q'_k$ is a unital, simple, purely infinite C^*-algebra, by applying Lemma (1.2.18) in each $Q'_k \mathfrak{A} Q'_k$and by taking a direct sum, we obtain

$$\sum_{k=1}^{n} \alpha_k Q'_k \in \overline{\text{conv}}(\mathcal{U}(T)).$$

Since $\sum_{k=1}^{n} \alpha'_k Q'_k$ is unitarily equivalent to S by the fact that $Q_k \sim Q'_k$ for all k, we obtain that $S \in \overline{\text{conv}}(\mathcal{U}(T))$.

We note the following .

Corollary (1.2.20)[1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra. If $S, T \in \mathfrak{A}$ are self-adjoint, then*

$$\operatorname{dist}\left(S, \operatorname{conv}(\mathcal{U}(T))\right) = \sup_{x\in\sigma(S)} \operatorname{dist}(x, \operatorname{conv}(\sigma(T))).$$

Proof. First, suppose $T' \in \operatorname{conv}(\mathcal{U}(T))$. $Let\, \pi : \mathfrak{A} \to \mathcal{B}(H)$ be a faithful representation of $\mathfrak{A}$ (whose existence is guaranteed by the GNS construction). For every self-adjoint operator $A \in \mathcal{B}(H)$,

$$\operatorname{conv}(\sigma(A)) = \overline{\{\langle A\eta, \eta\rangle \mid \eta \in \mathcal{H}, \|\eta\| = 1\}}.$$

Let $\eta \in \mathcal{H}$ be such that $\|\eta\| = 1$. Since

$$\|T' - S\| \geq |\langle\pi(T' - S)\eta, \eta\rangle| \geq \operatorname{dist}(\langle\pi(S)\eta, \eta\rangle, \operatorname{conv}(\sigma(T))),$$

we obtain that

$$\operatorname{dist}\left(S, \operatorname{conv}(\mathcal{U}(T))\right) \geq \sup_{x\in\sigma(S)} \operatorname{dist}(x, \operatorname{conv}(\sigma(T))).$$

For the reverse inclusion, defined a continuous function $f : \mathbb{R} \to \mathbb{R}$ so that $f(x) \in \operatorname{conv}(\sigma(T))$ for all x and

$$|x - f(x)| = \operatorname{dist}(x, \operatorname{conv}(\sigma(T)))$$

for all $x \in \mathbb{R}$. Let $T' = f(S)$. Therefore, by the continuous functional calculus, $\sigma(T') = f(\sigma(S)) \subseteq \operatorname{conv}(\sigma(T))$. Hence $T' \in \operatorname{conv}(\sigma(T))$ by Theorem (1.2.19). Since

$$\|S - T'\| = \sup_{x\in\sigma(S)} \|x - f(x)\| = \sup_{x\in\sigma(S)} \operatorname{dist}(x, \operatorname{conv}(\sigma(T))),$$

the reverse inclusion holds.

We note the proof of Theorem ((1.2.19) can be improved to normal operators provided $K_1(\mathfrak{A})$ is trivial or, more generally, for normal operators N such that $\lambda I_{\mathfrak{A}} - N$ is an element of the connected component containing $I_{\mathfrak{A}}$ in the set of invertible elements of $\mathfrak{A}$, denoted $\mathfrak{A}_0^{-1}$, for all $\lambda \notin \sigma(N)$. This is a generalization and we only sketch the modifications to the proof.

Theorem (1.2.21)[1]: *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-*

algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators with $\lambda I_{\mathfrak{A}} - N_k \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$and for all k. Then $N_2 \in \overline{\text{conv}}(\mathcal{U}(N_1))$ if and only if $\sigma(N_2) \subseteq \text{conv}(\sigma(N_1))$.

Proof. Suppose $N_2 \in \overline{\text{conv}}(\mathcal{U}(N_1))$. Let $(M_n)_{n\geq 1} \subseteq \text{conv}(\mathcal{U}(N_1))$ be such that $N_2 = \lim_{n\to\infty} M_n$ and let $\pi : \mathfrak{A} \to \mathcal{B}(H)$ be a faithful representation of $\mathfrak{A}$. For every normal operator $\mathfrak{A} \in \mathcal{B}(H)$,

$$\text{conv}(\sigma(A)) = \overline{\{\langle A\eta, \eta\rangle \mid \eta \in \mathcal{H}, \|\eta\| = 1\}}.$$

Since $M_n \in \text{conv}(\mathcal{U}(N_1))$, we obtain $\langle \pi(M_n)\eta, \eta\rangle \in \text{conv}(\sigma(N_1))$ for all $\eta \in \mathcal{H}$ with $\|\eta\| = 1$. Therefore, since $\langle \pi(N_2)\eta, \eta\rangle = \lim_{n\to\infty}\langle \pi(M_n)\eta, \eta\rangle$, we obtain $\sigma(N_2) \subseteq \text{conv}(\sigma(N_1))$.

For the converse direction, note that N_1 and N_2 can be approximated by normal operators with finite spectra. Thus, by an application of the continuous functional calculus, it suffices to prove that if $\sigma(N_2)$ and $\sigma(N_1)$ are finite and $\sigma(N_2) \subseteq \text{conv}(\sigma(N_1))$, then $N_2 \in \overline{\text{conv}}(\mathcal{U}(N_1))$. Furthermore, by using similar direct sum arguments as in the proof of Theorem ((1.2.19), it suffices to prove the result in the case that $N_2 \in \mathbb{C}I_{\mathfrak{A}}$.

Note that Lemma (1.2.15) holds when α and β are complex numbers by applying rotations and translations. Hence by applying the same ideas as in Lemma (1.2.16), we may reduce to the case that N has exactly three points in its spectrum.

Suppose $\sigma(N_1) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\gamma \in \text{conv}(\sigma(N_1))$. Then there exist a permutation σ on {1, 2, 3} and $t, r \in [0,1]$ such that if $\gamma' = t\alpha_{\pi(1)} + (1-t)\alpha_{\pi(2)}$ then $\gamma = r\gamma' + (1-r)\alpha_{\pi(3)}$. Consequently, by applying rotations, translations, compressions, and Lemma (1.2.18) first with the spectral projections corresponding to $\alpha_{\pi(1)}$ and $\alpha_{\pi(2)}$, and then again with the result and the spectral projection corresponding to $\alpha_{\pi(3)}$, the result is obtained.

Chapter 2

Smooth Banach Spaces

We show that this holds for any equivalent norm on $c_0(\Gamma)$, where Γ is an arbitrary set. We also give a necessary condition for the existence of a polyhedral norm on a weakly compactly generated Banach space, which extends a well-known result of Fonf

Section (2.1): Approximation of Norms a Necessary Condition for Polyhedrality in WCG Spaces

Given a Banach space $(X, \|\cdot\|)$ and $\varepsilon > 0$, we say that a new norm $|||\cdot|||$ is ϵ-equivalent to $\|\cdot\|$if

$$|||x||| \leq \|x\| \leq (1+\varepsilon)|||x|||,$$

for all $x \in X$. Suppose that P is some geometric property of norms, such as smoothness or strict convexity. We shall say that a norm $\|\cdot\|$ can be approximated by norms having P if, given any $\varepsilon > 0$, there exists a norm having P that is ε-equivalent to $\|\cdot\|$. That is $\|\cdot\|$ may be approximated uniformly, and with arbitrary precision, on bounded subsets of X by norms having P.

The question of whether all equivalent norms on a given Banach space can be approximated by norms having P is a recurring theme in renorming theory. It is known to be true if P is the property of being strictly convex, or locally uniformly rotund. (In fact, in these two cases, it is possible to show that if $\|\cdot\|$has P, then the set of equivalent norms on X having P is residual in the space of all equivalent norms on X, which is completely metrisable).

Definition (2.1.1)[2]: We say the norm $\|\cdot\|$ of a Banach space X is C^k

smooth if its kth Fréchet derivative exists and is continuous at every point of $X \setminus \{0\}$. The norm said to be C^∞ smooth if this holds for all $k \in \mathbb{N}$.

For separable spaces, we have the following recent and conclusive result.

Theorem (2.1.2)[2]: *Let X be a separable Banach space with a C^k smooth norm. Then any equivalent norm on X can be approximated by C^k smooth norms.*

There is an analogous result to Theorem (2.1.2) for polyhedral norms.

Definition (2.1.3)[2]: We say a norm $\|\cdot\|$ on a Banach space X is polyhedral if, given any finite-dimensional subspace Y of X, the restriction of the unit ball of $\|\cdot\|$ to Y is a polytope.

Theorem (2.1.4) [2]: *Let X be a separable Banach space with a polyhedral norm. Then any equivalent norm on X can be approximated by polyhedral norms.*

Definition (2.1.5)[2]: Let Γ be a set. The set $c_0(\Gamma)$ consists of all functions $x : \Gamma \to \mathbb{R}$, with the property that $\{\gamma \in \Gamma : |x(\gamma)| \geq \varepsilon\}$ is finite whenever $\varepsilon > 0$. We equip $c_0(\Gamma)$ with the norm $\|\cdot\|_\infty$, where $\|x\|_\infty = \max\{|x(\gamma)| : \gamma \in \Gamma\}$.

When Γ is uncountable, $c_0(\Gamma)$ is non-separable. The structure of $c_0(\Gamma)$ strongly promotes the existence of the sorts of norms under discussion. For example, it is well known that the canonical norm on $c_0(\Gamma)$ is polyhedral, and that it can be approximated by C^∞ smooth norms. In terms of finding positive non-separable analogues of Theorems (2.1.2) and (2.1.4), this class of spaces is a very plausible candidate.

The most general result concerning this class to date is given

below. We shall call a norm $\|\cdot\|$on $c_0(\Gamma)$a lattice norm if $\|x\| \leq \|y\|$ whenever $x, y \in c_0(\Gamma)$ satisfy $|x(\gamma)| \leq |y(\gamma)|$
for each $\gamma \in \Gamma$.

Theorem (2.1.5)[2]: *Every equivalent lattice norm on $c_0(\Gamma)$can be approximated by C^∞ smooth norms.*

The following result completely settles the approximation problem in the case of $c_0(\Gamma)$, from the point of view of C^∞ smooth norms and polyhedral norms. It solves a special case .

Definition (1.1.6)[2]: Let $(X, \|\cdot\|)$ be a Banach space. A subset B of the closed unit ball B_{X^*} is a called a boundary of $\|\cdot\|$ if, for each x in the unit sphere S_X, there exists $f \in B$ such that $f(x) = 1$.

This is also known as a James boundary of X. The dual unit sphere S_{X^*} and the set ext (B_{X^*}) of extreme points of the dual unit ball B_{X^*} are always boundaries of $\|\cdot\|$, by the Hahn-Banach Theorem and (the proof of the) Krein-Milman Theorem, respectively. It is worth noting that the property of being a boundary is not preserved by isomorphisms in general: a boundary of $\|\cdot\|$ may not be a boundary of $|||\cdot|||$, where $|||\cdot|||$ is an equivalent norm. Since we will be changing norms, it will be necessary to bear this in mind.

Boundaries play a key role in the theory of both smooth norms and polyhedral norms. If $(X, \|\cdot\|)$ has a boundary that is countable or otherwise well-behaved, then X enjoys good geometric properties as a consequence .

Recall that an element $f \in B_{X^*}$ is called a w^*-strongly exposed point of B_{X^*} if there exists $x \in B_X$ such that $f(x) = 1$ and, moreover, $|f - f_n| \to 0$ whenever $(f_n) \subseteq B_{X^*}$ is a sequence satisfying $f_n(x) \to 1$. It is a simple matter to check that the (possibly empty) set w^*-str exp(B_{X^*}) of w∗-strongly exposed points of B_{X^*} is contained in any boundary of $\|\cdot\|$. We recall the following important result of Fonf, concerning polyhedral norms.

Theorem (2.1.7)[2]: *Let $\|\cdot\|$ be a polyhedral norm on a Banach space X having density character κ. Then w^*-str exp(B_{X^*}) has cardinality κ and is a boundary of $\|\cdot\|$ (so is the minimal boundary, with respect to inclusion). Moreover, given $f \in w^*$-str exp(B_{X^*}), the set $A_f \cap B_X$ has non-empty interior, relative to the affine hyperplane $A_f := \{x \in X : f(x) = 1\}$.*

In particular, if X is separable and $\|\cdot\|$ is polyhedral, then w^*-str exp(B_{X^*}) is a countable boundary. Conversely, if $(X, \|\cdot\|)$ is a Banach space and $\|\cdot\|$ has a countable boundary B, then X admits equivalent polyhedral norms that approximate $\|\cdot\|$. Thus, in the separable case, the existence of equivalent polyhedral norms can be characterised purely in terms of the cardinality of the boundary.

In the non-separable case however, any analogous characterizations, if they exist, must generally rely on more than the cardinality of the boundary alone. There exist Banach spaces $(X, \|\cdot\|)$having no equivalent polyhedral norms, yet X has density the continuum c, and $\|\cdot\|$ has boundary B of cardinality c. Such Banach spaces can take the form $X = C(T)$, where T is the 1-point compactification of a suitably chosen locally compact scattered tree.

Recall that a Banach space X is *weakly compactly generated* (WCG) if $X = \overline{\text{span}}^{\|\cdot\|}(K)$, where $K \subseteq X$ is weakly compact. Separable spaces and reflexive spaces are WCG. Examples of WCG spaces that are neither include the $c_0(\Gamma)$ spaces above. The following is the main result. It provides a little more information about the structure of the set w^*-str exp(B_{X^*}), besides cardinality, given a WCG polyhedral Banach space.

Definition (2.1.8)[2]: We call an indexed set of pairs $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma} \subseteq X \times X^*$ a Markushevich basis (or M-basis) if

(i) $e_\alpha^*(e_\beta) = \delta_{\alpha\beta}$, (that is, $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma}$ is a biorthogonal system);

(ii) $\overline{\text{span}}^{\|\cdot\|}(e_\gamma)_{\gamma\in\Gamma} = X$, and

(iii) $(e_\gamma^*)_{\gamma\in\Gamma}$ separates the points of X.

Furthermore, an M-basis is called strong if $x \in \overline{\text{span}}^{\|\cdot\|}\{e_\gamma: e_\gamma^*(x) \neq 0\}$ for all $x \in X$, shrinking if $X^* = \overline{\text{span}}^{\|\cdot\|}(e_\gamma^*)_{\gamma\in\Gamma}$, and weakly compact if $\{e_\gamma: \gamma \in \Gamma\} \cup \{0\}$ is weakly compact.

The existence of an M-basis allows us to define supports of functionals in the dual space.

Definition (2.1.9)[2]: Let X be a Banach space with an M-basis $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma}$ and let $f \in X^*$. Define the support of f (with respect to the basis) to be the set

$$\text{supp}(f) = \{\gamma \in \Gamma: f(e_\gamma) \neq 0\}.$$

We say f has finite support if $\text{supp}(f)$ is finite.

The main result of this section, Theorem (2.1.15), states that if X has a strong M-basis then, given the right circumstances, the norm on X can be approximated by norms having boundaries that consist solely of elements having finite support. The following result illustrates the relevance of such boundaries to the current discussion. It amalgamates two theorems, both of which are stated with broader hypotheses in their original forms.

Theorem (2.1.10)[2]: *Let a Banach space X have a strong M-basis, and suppose that the norm $\|\cdot\|$ has a boundary consisting solely of elements having finite support. Then $\|\cdot\|$ can be approximated by both C^∞ norms and polyhedral norms.*

Now we will assume that the Banach space X has a strong M-basis $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma}$, such that $\|e_\gamma\| = 1$ for all $\gamma \in \Gamma$. Furthermore, we will suppose that there is some fixed $L \geq 0$ satisfying $\|e_\gamma^*\| \leq L$ for all $\gamma \in \Gamma$.

Given $\in X^*$, set $\|f\|_1 = \sum_{\gamma\in\Gamma}|f(e_\gamma)|$, whenever this quantity is finite, and set $\|f\|_1 = \infty$ otherwise. Observe that if $x = \sum_{\gamma\in F} e_\gamma^*(x)e_\gamma$, for some finite $F \subseteq \Gamma$, then

$$|f(x)| \leq \sum_{\gamma\in F} |e_\gamma^*(x)||f(e_\gamma)| \leq L\|x\| \sum_{\gamma\in F} |f(e_\gamma)| \leq L\|x\|\|f\|_1,$$

whence $\|f\| \leq L\,\|f\|_1$for all $f \in X^*$. It is also easy to see that $\|\cdot\|_1$ is a w^*-lower semicontinuous function on X^*, and that given $r > 0$, the norm-bounded set

$$W_r = \{f \in X^*: \|f\|_1 \leq \lambda\},$$

is symmetric, convexand w^*-compact.

Let us consider the set $B = \{f \in S_{X^*}: \|f\|_1 < \infty\}$. Evidently, B is the countable union of the sets $S_{X^*} \cap W_r$, $r \in \mathbb{N}$, which are w^*-closed in S_{X^*}. If $S_{X^*} \cap W_r$ contains a non-empty norm-open subset of S_{X^*}, for some $r \in \mathbb{N}$, then it is a straightforward matter to show that there exists $M \geq 0$ such that $\|f\|_1 \leq M\,\|f\|$ for all $f \in X^*$, whence $S_{X^*} \cap W_M = S_{X^*}$ and X is isomorphic to $c_0(\Gamma)$ via the map $x \longmapsto (e_\gamma^*(x))_{\gamma\in\Gamma}$. If there is no such r, then of course B is of first category in S_{X^*}. If X is not isomorphic to any space of the form $c_0(\Gamma)$, *then* $B \neq X^*$, but B may still be a boundary of $\|\cdot\|$.

We shall be interested in cases where B is a boundary of $\|\cdot\|$.

The following lemma will be used in Theorem (2.1.12).

Lemma (2.1.11)[2]: *Suppose that B as defined above is a boundary of*

$\|\cdot\|$ *Then* $X^* = \overline{\text{span}}^{\|\cdot\|}(e_\gamma^*)$,*i.e., the* M-*basis of* X *is shrinking.*

Proof: Let $F \subseteq \Gamma$ be finite, and define

$$X_F = \overline{\text{span}}^{\|\cdot\|}(e_\gamma)_{\gamma\in\Gamma\setminus F} \text{ and } W_F = \overline{\text{span}}^{\|\cdot\|}(e_\gamma^*)_{\gamma\in F}.$$

Then $W_F = X_F^\perp$ (the inclusion $X_F^\perp \subseteq W_F$ follows from the fact that the basis is strong), and thus X^* / W_F naturally identifies with X_F^*, and $\|f \restriction_{X_F}\| = d(f, W_F)$ for all $f \in X^*$,

where

$$d(f, W_F) = \inf\{\|f - g\| : g \in W_F\} .$$

Suppose, for a contradiction, that there exists $f \in X^*$ and $\varepsilon > 0$, such that $d(f, W_F) > \varepsilon$ for all finite $F \subseteq \Gamma$. Let F_0 be empty. Since $\|f\| = d(f, W_{F_0}) > \varepsilon$, take a unit vector $x_0 \in X$ having finite support, such that $f(x_0) > \varepsilon$. Set $F_1 = \text{supp } x_0$. Since $\left\|f \restriction_{X_{F_1}}\right\| = d(f, W_{F_1}) > \varepsilon$, there exists a unit vector $x_1 \in X$ having finite support in $\Gamma \setminus F_1$, such that $f(x_1) > \varepsilon$. Define $F_2 = F_1 \cup \text{supp } x_1$. Continuing like this, we get a sequence of unit vectors (x_n) having finite, pairwise disjoint supports, such that $f(x_n) > \varepsilon$ for all i. Clearly, (x_n) is not weakly null.

On the other hand, if $f \in B$ and $y = \sum_{\gamma\in F} e_\gamma^*(x) e_\gamma$ is a unit vector, where $F \subseteq \Gamma$ is finite, then

$$|f(y)| \sum_{\gamma\in\Gamma} |e_\gamma^*(y)| \, |f(e_\gamma)| \leq \sum_{\gamma\in F} |f(e_\gamma)|.$$

It follows that $f(x_n) \to 0$ as $n \to \infty$. This holds for every element of B, which is a boundary, so $x_n \to 0$ weakly, by Rainwater's. This is a contradiction.

We can now prove Theorem (2.1.12), although the approximation scheme used in that result fails in the case under consideration here, and substantial modifications must be made.

Theorem (2.1.12)[2]: *Let a Banach space X have an M-basis as above, and suppose that B as above is a boundary. Given $\varepsilon > 0$, there exists an ε-approximation $|||\cdot|||$ of $\|\cdot\|$, which has a boundary consisting solely of elements having finite support. Consequently, by Theorem (2.1.10), $\|\cdot\|$ can be approximated by C^{∞} smooth norms and polyhedral norms.*

Proof. Fix $\varepsilon \in (0,1)$. Suppose $f \in X^*$ satisfies $\|f\|_1 < \infty$. We define a sequence of positive numbers and a sequence of subsets of Γ inductively. To begin, set

$$p(f,1) = \max\{|f(e_\gamma)| : \gamma \in \Gamma\} \text{ and } G(f,1) = \{\gamma \in \Gamma : |f(e_\gamma)| = p(z,1)\}.$$

Given $n \geq 2$, we define

$$p(f,n) = \begin{cases} \max\{|f(e_\gamma)| : \gamma \in \Gamma \backslash G(f,n-1)\} & \text{if } \Gamma \backslash G(z,n-1) \neq \emptyset \\ 0 & otherwise, \end{cases}$$

$$\text{and } G(f,n) = \{\gamma \in \Gamma : |f(e\,\gamma)| \geq p(f,n)\}$$

Observe that the set $G(f,n)$ is finite if and only if $p(f,n) \neq 0$ and, in this case, $\|f\|_1 \geq p(f,n)|G(f,n)|$. By induction, $|G(f,n)| \geq n$ for all n, so $p(f,n) \leq \|f\|_1 n^{-1}$ and, in particular, $p(f,n) \to 0$. By construction, the sequence $(p(f,n))$ is decreasing, and strictly decreasing on the set of indices n at which it is non-zero. If $p(f,n) = 0$ for some $n \in \mathbb{N}$,then $f(e_\gamma) \neq 0$ for at most finitely many γ and hence f has finite support. Thus, when f has infinite support, we get a strictly decreasing sequence of positive numbers $p(f,n) \to 0$, and a strictly increasing sequence of finite sets $(G(f,n))$.

Provided $G(f,n)$ is finite, we define

$$w(f,n) = \sum_{\gamma \in G} (f,n) \operatorname{sgn}\left(f(e_\gamma)\right) e_\gamma^* \quad ,$$

and

$$h(f,n) = \sum_{i=1}^{n} (p(f,i) - p(f,i\ +\ 1))w(f,i).$$

Let $\gamma\ \in\ \Gamma$. If $\gamma\ \in\ \Gamma \setminus\ \bigcup_{n=1}^{\infty} G(f,n)$, then $h(f,m)(e_\gamma)\ =\ 0\ =\ f(e_\gamma)$ for all m. Otherwise, let n be minimal, subject to the condition $\gamma\ \in$ $G(f,n)$. By minimality, we have $p(f,n) = |f(e_\gamma)|$. If $m < n$, then $h(f,m)(e_\gamma) = 0$. If $m\ \geq\ n$, then we can see that

$$h(f,m)(e_\gamma) = \sum_{i=1}^{n} (p(f,i) - p(f,i\ +\ 1))\mathrm{sgn}(f(\gamma))$$

$$= [p(f,n)\ -\ p(f,n\ +\ 1)$$

$$+\ p(f,n\ +\ 1)\ -\ p(f,n\ +\ 2)$$

$$+ \ldots - \ldots$$

$$+\ p(f,m)\ -\ p(f,m\ +\ 1)]\mathrm{sgn}\left(f(e_\gamma)\right)$$

$$=\ |f(e\_\gamma)|\ \mathrm{sgn}(f(e\_\gamma))\ -\ p(f,m\ +\ 1)\mathrm{sgn}\left(f(e_\gamma)\right)$$

$$=\ f(e_\gamma) - p(f,m\ +\ 1)\mathrm{sgn}\left(f(e_\gamma)\right).$$

From the calculation above and the fact that $p(f,m\ +\ 1)\ <\ |f(e_\gamma)|$, we have

$$|h(f,m)(e_\gamma)|\ =\ |\mathrm{sgn}(f(e_\gamma)(|f(e_\gamma)|\ -\ p(f,m\ +\ 1))|$$

$$=\ |f(e\_\gamma)|\ -\ p(f,m\ +\ 1).$$

Since $p(f,m\ +\ 1)\ \geq\ 0$, we obtain $|h(f,m)(e_\gamma)|\ \leq\ |f(e_\gamma)|$.

Therefore, for all $\gamma\ \in\ \Gamma, |h(f,m)(e_\gamma)|\ \leq\ |f(e_\gamma)|$ and $h(f,m)(e_\gamma)\ \rightarrow\ f(e_\gamma)$ as $m\ \rightarrow\ \infty$. We apply Lebesgue's Dominated Convergence Theorem to conclude that $\|f\ -\ h(f,m)\|_1 \rightarrow\ 0$.
Since $\|\cdot\|g \leq\ L\ \|\cdot\|$, we also get $\|f\ -\ h(f,m)\|\ \rightarrow\ 0$. Since the signs of $w(f,i)(e_\gamma)$ and $w(f,i)(e_\gamma)$ agree whenever they are non-zero,

$$\|h(f,n)\|_1 = \sum_{i=1}^{n}\big(p(f,i) - p(f,i+1)\big)\ \|w(f,i)\|_1$$

$$= \sum_{i=1}^{n}(p(f,i) - p(f,i+1))|G(f,i)|.$$

Therefore, if f has infinite support, then $\|f\|_1 = \sum_{i=1}^{n}(p(f,i) - p(f,i+1))|G(f,i)|$.

Given $m > n$, define

$$g(f,n,m) = \begin{cases} \dfrac{\|f - h(f,n)\|_1}{|G(f,m)|} w(f,m) & \text{if } |G(f,m)| < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

and $j(f,n,m) = h(f,n) + g(f,n,m), m > n$. Observe that supp$(j(f,n,m))$

$\subseteq G(f,m)$. Let $B_r = B_{X^*} \cap W_r = \{f \in B_{X^*} : \|f\|_1 \leq r\}$. Of course, $B \subseteq \bigcup_{r=1}^{\infty} B_r$. We let

$V_r = j(f,n,m): f \in B_r , m > n$ and $\|f - j(f,n,m)\| < 2^{-(r+2)\,\varepsilon}$,

and set

$$V = \bigcup_{r=1}^{\infty}(1 + 2^{-r}\,\varepsilon)V_r .$$

Define $|||x||| = \sup\{f(x) : f \in V\}$. This is the norm that we claim ε –approximates $\|\cdot\|$ and has a boundary consisting solely of elements having finite support.

First of all, we prove that $\|x\| < |||x||| \leq (1 + \varepsilon)|||x|||$ whenever $x \neq 0$. Take $x \in X$ with $\|x\| = 1$ and let $f \in B$ such that $f(x) = 1$ (which is possible as B is a boundary of $\|\cdot\|$). Let r be minimal, such that $\in B_r$. Since $\|f\| \leq L\,\|f\|_1$ for all $\in X^*$, and $\|f - j(f,n,m)\|_1 \leq 2\,\|f - h(f,n)\|_1$, it follows that there exists n such that $\|f - j(f,n,m)\| < 2^{-(r+2)\,\varepsilon}$ whenever m > n. In particular,

$$|||x||| \geq (1 + 2^{-r}\varepsilon) j(f, n, n + 1)(x) \geq (1 + 2^{-r}\varepsilon)(1 - 2^{-(r+2)}\varepsilon)$$
$$\geq 1 + 2^{-(r+1)}\varepsilon.$$

To secure the other inequality, simply observe that if $f \in B_r$, $m > n$ and $\|f - j(f, n, m)\| < 2^{-(r+2)}\varepsilon$, then

$$(1 + 2^{-r}\varepsilon) j(f, n, m)(x) \leq (1 + 2^{-r}\varepsilon)\left(1 + 2^{-(r+2)}\varepsilon\right)$$
$$\leq 1 + \left(2^{-r} + 2^{-(r+2)} + 2^{-(2r+2)}\right)\varepsilon \leq 1 + \varepsilon.$$

This means that $|||x||| \leq 1 + \varepsilon$. By homogeneity, $\|x\| < |||x||| \leq (1 + \varepsilon)\|x\|$ whenever $x \neq 0$.

Now we show that $|||\cdot|||$ has a boundary consisting solely of elements having finite support. By Milman's Theorem we know that $\mathrm{ext}(B_{(X,|||\cdot|||)^*}) \subseteq \bar{V}^{w^*}$. Define

$$D = \bigcap_{r=1}^{\infty}\left(\overline{\bigcup_{s=r}^{\infty}(1 + 2^{-s}\varepsilon) V_s}^{w^*}\right) ,$$

and let $d \in D$. For each $r \in \mathbb{N}$, $\|d\| \leq (1 + 2^{-r}\varepsilon)(1 + 2^{-(r+2)}\varepsilon)$, and hence $\|d\| \leq 1$. Therefore, if $|||x||| = 1$, then

$$d(x) \leq \|d\|\|x\| \leq \|x\| < 1.$$

It follows that, with respect to $|||\cdot|||$, none of the elements of D are norm-attaining. Consequently, $\tilde{B} = \mathrm{ext}(B_{(X,|||\cdot|||)^*})\backslash D$ is a boundary of $|||\cdot|||$. We claim that every element of e B has finite support.

Given $f \in \tilde{B}$ we have $f \in (1 + 2 - r\,\varepsilon)\bar{V}_r^{w^*}$ for some $\in \mathbb{N}$. For a contradiction, we will assume that f has infinite support. According to Lemma (2.1.11), our M-basis is shrinking. It follows that supp g is countable for all $\in X^*$. Thus, $\bar{V}_r^{w^*}$ is Corson compact in the w^*-topology which implies that it is a Fr´echet-Urysohn space . In particular, there exist sequences $(f_k) \subseteq B_r$, and $(n_k), (m_k) \subseteq \mathbb{N}$, with $n_k < m_k$ for all $k \in \mathbb{N}$, such that $\left(j(f_k, n_k, m_k)\right) \subseteq V_r$ and

$j(f_k, n_k, m_k) \xrightarrow{w^*} l$, where $l = (1 + 2^{-r}\varepsilon)^{-1} f$.

We claim that, in fact, $f_k \xrightarrow{w^*} l$. First, we show that $h(f_k, n_k) \xrightarrow{w^*} l$. To this end, suppose that $|G(f_k, m_k)| \nrightarrow \infty$. Then by taking a subsequence if necessary, there exists $N \in \mathbb{N}$ such that $|\operatorname{supp}(j(f_k, n_k, m_k))| \leq |G(f_k, m_k)| \leq N$ for all k. But as $j(f_k, n_k, m_k) \xrightarrow{w^*} l$, this would force $|\operatorname{supp}(l)| \leq N < \infty$, which is not the case. Thus we must have $|G(f_k, m_k)| \to \infty$. Therefore, for all $\gamma \in \Gamma$, $g(f_k, n_k, m_k)(e_\gamma) \to 0$ as $k \to \infty$. Since $\|\cdot\| \leq L\|\cdot\|_1$, the sequence $(g(f_k, n_k, m_k))$ is bounded. Therefore, $g(f_k, n_k, m_k) \xrightarrow{w^*} 0$ and hence $h(f_k, n_k) \xrightarrow{w^*} 1$.

We will now show that $f_k - h(f_k, n_k) \xrightarrow{w^*} 0$. For each $\gamma \in \Gamma$, $|f_k(\gamma) - h(f_k, n_k)(e_\gamma)| \leq |f_k(e_\gamma)|$, so $\|f_k - h(f_k, n_k)\|_1 \leq \|f_k\|_1$. Therefore, $(f_k - h(f_k, n_k))$ is a bounded sequence. Given $\gamma \in \Gamma$,

$$|(f_k - h(f_k, n_k))(e\_\gamma)| \leq p(f_k, n_k + 1) \leq \frac{\|f_k\|_1}{|G(f_k, n_k + 1)|}$$
$$\leq \frac{r}{|G(f_k, n_k + 1)|}.$$

Since $(f_k, n_k) \xrightarrow{w^*} l$, as above, the infinite support of l ensures that $|G(f_k, n_k)| \to \infty$. Therefore, $(f_k - h(f_k, n_k)(e_\gamma) \to 0$ and hence $f_k - h(f_k, n_k) \xrightarrow{w^*} 0$ as $k \to \infty$. It follows that $f_k \xrightarrow{w^*} l$ as claimed, and hence $\in B_r$.

Fix $n \in \mathbb{N}$ such that $\|l - h(l, n)\|_1 < L^{-1} 2^{-(r+3)}\varepsilon$. Then for all $m > n$,

$$\|l - j(l, n, m)\| \leq L\|l - j(l, n, m)\|_1 \leq 2L\|l - h(l, n)\|_1$$
$$< 2^{-(r+2)}\varepsilon.$$

So $j(l,n,m) \in V_r$ for all $m > n$. Let

$$\lambda_m = \frac{\big(p(l,m) - p(l,m+1)\big)|G(l,m)|}{\|l - h(l,n)\|_1}.$$

Note that $\lambda_m > 0$ whenever $m > n$. Since $\|l - h(l,n)\|_1 = \sum_{i=n+1}^{\infty}(p(l,i) - p(l,i+1))|G(l,i)|$, we get $\sum_{m=n+1}^{\infty}\lambda_m = 1$.

$$\sum_{m=n+1}^{\infty}\lambda_m\, j(l,n,m) = \sum_{m=n+1}^{\infty}\lambda_m\, h(l,n) + \sum_{m=n+1}^{\infty}\lambda_m\, g(l,n,m)$$

$$= h(l,n) + \sum_{m=n+1}^{\infty}\big(p(l,i) - p(l,i+1)\big)w(l,i) = l.$$

Therefore, f is a nontrivial convex combination of elements of $(1 + 2^{-r}\varepsilon)V\,r \subseteq B_{(X,|||\cdot|||)^*}$, so $f \notin \operatorname{ext}(B_{(X,|||\cdot|||)^*})$, and hence $f \notin \tilde{B}$. This gives us our desired contradiction. In conclusion, e B is a boundary of $|||\cdot|||$ consisting solely of functionals having finite support.

Theorem (2.1.13) becomes a trivial consequence of Theorem (2.1.12).

Theorem (2.1.13)[2]: 14 *Let Γ be an arbitrary set, and let $\|\cdot\|$ be an arbitrary equivalent norm on $c_0(\Gamma)$. Then $\|\cdot\|$ can be approximated by both C^∞ norms and polyhedral norms.*

Theorem (2.1.13) is a consequence of a more general result, Theorem (1.1.12), which involves spaces having Markushevich bases. The proofs of both results are given.

Proof: In this case $B = S_{(c_0(\Gamma),\|\cdot\|)^*}$, so it is a boundary of $\|\cdot\|$.

It is worth remarking that the implication (d) ⇒ (c) is essentially Theorem (2.1.12), but with the additional assumption that the M-basis is countable. The method of proof in that case is completely different from the one presented here.

We begin with a lemma. It is based on straightforward geometry and is probably folklore, but is included for completeness .

Lemma (2.1.14) [2]: *Suppose that $D \subseteq B_{X^*}$ has the property that for all $f \in D$, there exists $x_f \in X$ and $r_f > 0$ such that $\|x_f + z\| = f(x_f + z)$ whenever $\|z\| < r_f$. Then*

(i) $r_f \leq \|x_f\|$, and

(ii)$\|z\| < r_f$ and $g \in D \setminus \{f\}$ implies $g(x_f + z) < \|x_f + z\|$.

In particular, if $f, g \in D$ are distinct then $\|x_g - x_f\| \geq r f$.

Proof.

(i) Suppose that $\|x_f\| < r_f$. Let $y \in X$ satisfy $\|y\| < r_f - \|x_f\|$. Tnen $\|\pm y - x_f\| < r f$ and so

$$f(y) = \|y\| = \|-y\| = f(-y) = -f(y),$$

meaning that $y \in \ker f$. It follows that $f = 0$, which is impossible.

(ii) Suppose $\|z\| < r_f$, $g \in D\{f\}$ and $g\big(x_f + z\big) = \|x_f + z\|$. Since $g \neq f$ we can find $y \in \ker f$ such that $g(y) > 0$ and $\|y\| < r_f - \|z\|$. Otherwise we would have $\ker f \subseteq \ker g$, so $g = \alpha f$ for some α, and since $f(x_f + z) = \|x_f + z\| = g(x_f + z) = \alpha f(x_f + z)$, and$\|x_f + z\| > 0$ by (1), we conclude that $g = f$, which is not the case. Thus ky + zk < r f and so

$$\|x_f + y + z\| = f(x_f + y + z) = f(x_f + z).$$

On the other hand,

$$\|x_f + y + z\| \geq g(x_f + y + z) > g(x_f + z) = \|x_f + z\|$$
$$= f(x_f + z).$$

Finally, if $f, g \in D$ are distinct and $\|x_g - x_f\| < r_f$, then by (2) we would have

$$\|x_g\| = g(x_g) = g(x_f + (x_g - x_f)) < \|x_f + (x_g - x_f)\| = \|x_g\|.$$

Armed with this lemma, we can give the proof of Theorem (1.16).

Theorem (2.1.15)[2]: Let *X be* WCG*, and let the norm $\|\cdot\|$ on X be* *polyhedral. Then the boundary* w^*-str exp(B_{X^*}) *of* ‖·‖ *may be written as*

$$w^*\text{-str}\exp(B_{X^*}) = \bigcup_{n=1}^{\infty} D_n ,$$

where each D_n is relatively discrete in the w^*-topology.

The theorem above should be compared to the following sufficient condition: if the norm ‖·‖ on X admits a boundary B such that $B = \bigcup_{n=1}^{\infty} D_n$ and $= \bigcup_{m=1}^{\infty} K_m$, where each Dn is relatively discrete in the w^*-topology, and each K_m is w^*-compact, then ‖·‖ can be approximated by polyhedral norms. Thus Theorem (2.1.15) can be considered as a step towards a characterisation of the existence of polyhedral norms, in the WCG case.

The main results concern a class of spaces which include all spaces of the form $c_0(\Gamma)$, namely those that admit the following type of basis.

Proof: Since X is WCG, we can find a weakly compact M-basis $(e_\gamma , e_\gamma^*)_{\gamma\in\Gamma}$ of X . Let E_n be the set of $x \in X$ that can be written as a linear combination of at most n elements of $(e_\gamma)_{\gamma\in\Gamma}$. Let us define $B :$ $= w^*$ –str exp(B_{X^*}). for each $f \in B$, we can find a point $x \in$ span$(e_\gamma)_{\gamma\in\Gamma}$ that lies in the interior of $A_f \cap B_X$, where A_f is the supporting hyperplane as defined in that theorem. By a straightforward argument, it follows that there exists $r > 0$ such that $\|x + z\| = f(x + z)$ whenever $\|z\| < r$. Any such x belongs to some E_n . Therefore, given $f \in B$, we can define n_f to be the minimal $n \in \mathbb{N}$ for which we can find an x and r as above, with $x \in E_n$.

Define $D_{n,m}$ to be the set of all $f \in B$ such that $n_f = n$, and there exist x and r, as described above, which in addition satisfy $r \geq 2^{-m}$ and

$$x = \sum_{\gamma \in F} a_\gamma e_\gamma,$$

where $F \subseteq \Gamma$ has cardinality n and $|a_\gamma| \leq m$ for all $\gamma \in F$. Any such pair (x, r) will becalled a witness for $f \in D_{n,m}$.

Evidently, $= \cup_{n,m=1}^{\infty} D_{n,m}$. We claim that each $D_{n,m}$ is relatively discrete in the norm topology. For a contradiction, suppose otherwise and let $f, f_k \in D_{n,m}$ such that $\|f - f_k\| \to 0$. For each $\in \mathbb{N}$, select a witness (x_k , r_k) for f_k . The set

$$L = \left\{\sum_{\gamma \in F} a_\gamma e_\gamma : F \subseteq \Gamma \text{ has cardinality } n \text{ and } |a_\gamma| \leq m \text{ for all } \gamma \in F\right\},$$

is weakly compact, being a natural continuous image of $[-m, m]^n \times \left(\{e_\gamma : \gamma \in \Gamma\} \cup \{0\}\right)^n$.Thus, by the Eberlein- Smulyan Theorem, and by taking a subsequence of (x_k) if necessary, we can assume that the x_k tend weakly to some $y \in L$. We claim that $y \in E_j$ for some $j < n$. Indeed, if

$$y = \sum_{\gamma \in F} a_\gamma e_\gamma,$$

where $F \subseteq \Gamma$ has cardinality n and a $\gamma \neq 0$ for all $\gamma \in F$, then there exists a K for which $e_\gamma^* (x_k) \neq 0$ for all $\gamma \in F$ and all $k \geq K$. Because each x_k can be expressed as a linear combination of n elements of $(e_\gamma)_{\gamma \in \Gamma}$, it follows that $x_k \in \text{span}(e_\gamma)_{\gamma \in F}$ whenever $k \geq K$.

Indeed, if

$$w = \sum_{\gamma \in G} b_\gamma e_\gamma ,$$

where $G \subseteq \Gamma$ has cardinality n, and if $e_\gamma^*(w) \neq 0$ for all $\gamma \in F$, then necessarily $F \subseteq G$, and equality of these sets follows since their cardinalities agree. Because the $x_k, k \geq K$, belong to a finite-dimensional space, it follows that $\|y - x_k\| \to 0$. However, by Lemma (2.1.14), we know that the x_k are uniformly separated in norm by 2^{-m} ($\leq r_k$), so they cannot converge in norm to anything.

Thus $y \in E_j$ for some $j < n$, as claimed. Now fix $z \in X$ such that $\|z\| < 2^{-m}$. We have $\|x_k + z\| = f_k(x_k + z)$ for all k, because $2^{-m} \leq r_k$. As $\|f - f_k\| \to 0$ and $x_k + z \to y + z$ weakly, we get $\|x_k + z\| \to f(y + z) \leq \|y + z\|$. On the other hand, by w-lower semicontinuity of the norm, $\|y + z\| \leq f(y + z)$. So the equality $\|y + z\| = f(y + z)$ holds whenever $\|z\| < 2^{-m}$. In particular, $1 = \|x\| \to \|y\|$. However $y \in E_j$ and $j < n$, and this contradicts the minimal choice of $n_f = n$.

Thus each $D_{n,m}$ is relatively discrete in the norm topology. Since $D_{n,m} \subseteq B$ and since the norm and w^* −topologies agree on B, it follows that Dn,m is relatively discrete in the w^* −topology as well.

Finally, we recall that a Banach space X is called weakly Lindel¨of determined (WLD) if B_{X^*} is Corson compact in the w^* −topology. The class of WLD spaces includes all WCG spaces. Any polyhedral Banach space is an Asplund space, and any WLD Asplund space is WCG. Therefore Theorem (2.1.13) extends to all WLD polyhedral spaces.

Chapter 3

Positive Linear Maps and Pinchings

We deduce two recent theorems of Kennedy-Skoufranis and Loreaux-Weiss for conditional expectations onto a masa in the algebra of operators on a Hilbert space. We also get a few results for sums in a unitary orbit

Section (3.1): Pinching Theorem and Sums in A Unitary Orbit

We recall two theorems which are fundamental to obtain several results about positive linear maps, in particular conditional expectations, and unitary orbits. These theorems were established we also refer to this article for various definitions and properties of the essential numerical range $W_\epsilon(A)$ of an operator A in the algebra $L(\mathcal{H})$ of all (bounded linear) operators on an infinite dimensional, separable (real or complex) Hilbert space $\mathcal{H}$.

We denote by $\mathcal{D}$ the unit disc of $\mathbb{C}$. We write $A \simeq B$ to mean that the operators A and B are unitarily equivalent. This relation is extended to operators possibly acting on different Hilbert spaces, typically, A acts on $\mathcal{H}$ and B acts on an infinite dimensional subspace S of $\mathcal{H}$, or on the spaces $H \oplus H$ or $\oplus^\infty \mathcal{H}$.

Theorem (3.1.1)[3]: *Let $A \in L(\mathcal{H})$ with $W_e(A) \supset \mathcal{D}$and $\{X_i\}_{i=1}^\infty$ a sequence in $L(H)$ such that $\sup_i \|X_i\| < 1$. Then, a decomposition $\mathcal{H} = \oplus_{i=1}^\infty \mathcal{H}_i$ holds with $A_{\mathcal{H}_i} \simeq X_i$ for all i.*

The direct sum refers to an orthogonal decomposition, and AHi stands for the compression of A onto the subspace $\mathcal{H}_i$.

Theorem (3.1.1) tells us that we have a unitary congruence between an operator in $L(\oplus^\infty H)$ and a "pinching" of A,

$$\bigoplus_{i=1}^{\infty} X_i \simeq \sum_{i=1}^{\infty} E_i A E_i$$

for some sequence of mutually orthogonal infinite dimensional projections $\{E_i\}_{i=1}^{\infty}$ in $L(\mathcal{H})$ summing up to the identity I. Thus $\{H_i\}_{i=1}^{\infty}$ can be regarded as an operator diagonal of A. In particular, if X is an operator on $\mathcal{H}$ with $\|X\| < 1$, then, A is unitarily congruent to an operator on $\mathcal{H} \oplus \mathcal{H}$ of the form,

$$A \simeq \begin{pmatrix} X & * \\ * & * \end{pmatrix}. \qquad (1)$$

For a sequence of normal operators, Theorem (3.1.1) admits a variation. Given $\mathcal{A}, \mathcal{B} \subset \mathbb{C}$, the notation $A \subset_{st} B$ means that $A + r\mathcal{D} \subset \mathcal{B}$ for some $r > 0$.

Theorem (3.1.2)[3]: *Let $A \in L(\mathcal{H})$ with $W_\epsilon(A) \supset \mathcal{D}$ and $\{X_i\}_{i=1}^{\infty}$a sequence of normal operators in $L(\mathcal{H})$ such that $\bigcup_{i=1}^{\infty} W(X_i) \subset_{st} W_\epsilon(A)$. Then, a decomposition $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ holds with $A_{\mathcal{H}_i} \simeq X_i$ for all* i.

Our concern is the study of generalized diagonals, i.e., conditional expectations onto a masa in $L(\mathcal{H})$, of the unitary orbit of an operator. The pinching theorems are the good tools for this study; we easily obtain and considerably improve two recent theorems, of Kennedy and Skoufranis for normal operators, and Loreaux and Weiss for application to the class of unital, positive linear maps which are trace preserving.

The next gives applications which only require (1). These results mainly focus on sums of two operators in a unitary orbit.

We recall a straightforward consequence of (1) for the weak convergence.

Corollary (3.1.3)[3]: *Let $A, X \in L(\mathcal{H})$ with $W_\epsilon(A) \supset \mathcal{D}$ and $\|X\| \leq 1$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^{\infty}$ in $L(\mathcal{H})$ such that*

$$\operatorname{wot}\lim_{n\to+\infty} U_n A U_n^* = X.$$

We cannot replace the weak convergence by the strong convergence; for instance if A is invertible and $\|X_h\| < \|A^{-1}\|^{-1}$ for some unit vector h, then X cannot be a strong limit from the unitary orbit of A. However, the next best thing does happen.

Moreover, this is even true for the $*$-strong operator topology.

Corollary (3.1.4)[3]: *Let $A, X \in L(\mathcal{H})$ with $W_\epsilon(A) \supset \mathcal{D}$ and $\|X\| \leq 1$. Then there exist two sequences of unitaries $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ in $L(\mathcal{H})$ such that* $*$

$$* \operatorname{sot} \lim_{n\to+\infty} \frac{U_n A U_n^* + V_n A V_n^*}{2} = X.$$

Proof. From (1) we also have

$$A \simeq \begin{pmatrix} X & -R \\ -S & T \end{pmatrix}.$$

Hence there exist two unitaries $U, V : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ such that

$$\frac{UAU^* + V\,AV^*}{2} = \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix}. \qquad (2)$$

Now let $\{\{e_n\}_{n=1}^{\infty}$ be a basis of $\mathcal{H}$ and choose any unitary $W_n : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ such that $W_n(e_j \oplus 0) = e_j$ for all $j \leq n$. Then

$$X_n := W_n \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} W_n^*$$

strongly converges to X. Indeed, $\{X_n\}$ is bounded in norm and, for all $j, X_n e_j \to X_{e_j}$.

Taking adjoints,

$$X_n^* := W_n \begin{pmatrix} X^* & 0 \\ 0 & T^* \end{pmatrix} W_n^*,$$

we also have $X_n^* \to X$ strongly. Setting $U_n = W_n U$ and $V_n = W_n V$ and using (2) completes the proof.

Remark (3.1.5)[3]: Corollary (3.1.4) does not hold for the convergence in norm. We give an example. Consider the permutation matrix

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and set $A = 2 \bigoplus^{\infty} T$ regarded as an operator in $L(\mathcal{H})$. Then $W_{\epsilon}(A) \supset \mathcal{D}$, however $X = (1/2)I$ is not a norm limit from the unitary orbit of A. Equivalently, $(1/2)I$ is not a norm limit from the unitary orbit of $(A + A^*)/2$. Indeed, $(A + A^*)/2 = I - (3/2)P$ for some projection P.

We reserve the word "projection" for selfadjoint idempotent. A strong limit of idempotent operators is still idempotent; thus, the next corollary is rather surprising.

Corollary (3.1.6)[3]: *Fix $\alpha > 0$. There exists an idempotent $Q \in L(\mathcal{H})$ such that for every $X \in L(\mathcal{H})$ with $\|X\| \leq \alpha$ we have two sequences of unitaries $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ in $L(\mathcal{H})$ for which*

$$* \operatorname{sot} \lim_{n \to +\infty} U_n Q U_n^* + V_n Q V_n^* = X.$$

Proof. Let $\alpha > 0$, define a two-by-two idempotent matrix

$$M_a = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} \tag{3}$$

and set $Q = \bigoplus^{\infty} M_a$ regarded as an operator in $L(\mathcal{H})$. Since the numerical range $W(\cdot)$ of

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

is $\mathcal{D}$, we infer that $W(2\alpha^{-1}M_a) = W_{\epsilon}((2\alpha^{-1}Q) \supset \mathcal{D}$ for a large enough a. The result then follows from Corollary (3.1.4) with $A = 2\alpha^{-1}Q$ and the contraction $\alpha^{-1}X$.

Corollary (3.1.7) does not hold for the convergence in norm.

Proposition (3.1.7) [3]: *Let $X \in L(\mathcal{H})$ be of the form $\lambda I + K$ for a compact operator K and a scalar $\lambda \notin \{0, 1, 2\}$. Then X is not norm limit of $U_n Q U_n^* + V_n Q V_n^*$ for any sequences of unitaries $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ and any idempotent Q in $L(\mathcal{H})$.*

Proof. First observe that if $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are two bounded sequences in $L(\mathcal{H})$ such that $A_n - B_n \rightarrow 0$ in norm, then we also have $A_n^2 - B_n^2 \rightarrow 0$ in norm; indeed

$$A_n^2 - B_n^2 = A_n(A_n - B_n) + (A_n - B_n)B_n.$$

Now, suppose that $\lambda \neq 1$ and that we have the (norm) convergence,

$$U_n Q U_n^* + V_n Q V_n^* \rightarrow \lambda I + K.$$

Then we also have

$$W_n Q W_n^* - (-Q + \lambda I + U_n^* K U_n) \rightarrow 0 \qquad (4)$$

where $W_n := U_*^n V_n$. Hence, by the previous observation,

$$(W_n Q W_n^*)^2 - (-Q + \lambda I + U_n^* K U_n)^2 \rightarrow 0,$$

that is

$$W_n Q W_n^* - (-Q + \lambda I + U_n^* K U_n)^2 \rightarrow 0 \qquad (5)$$

Combining (4) and (5) we get

$$(-Q + \lambda I + U_n^* K U_n) - (-Q + \lambda I + U_n^* K U_n)^2$$

hence

$$(-2 + 2\lambda)Q + (\lambda - \lambda^2)I + K_n \rightarrow 0$$

for some bounded sequence of compact operators K_n. Since $\lambda \neq 1$, we have

$$Q = \frac{\lambda}{2} I + L$$

for some compact operator L. Since Q is idempotent, either $\lambda = 2$ or $\lambda = 0$.

The operator X in Proposition (3.1.7) has the special property that $W_\epsilon(X)$ is reduced to a single point. However Proposition(3.1.7) may also hold when $W_\epsilon(X)$ has positive measure.

Corollary (3.1.8)[3]: *Let Q be an idempotent in $L(\mathcal{H})$ and $z \in \mathbb{C} \setminus \{0, 1, 2\}$. Then, there exists $\alpha > 0$ such that the following property holds:*

If $X \in L(\mathcal{H})$ satisfies $\|X - zI\| \le \alpha$, then X is not norm limit of $U_n Q U_n^* + V_n Q V_n^*$

for any sequences of unitaries $\{\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ in $L(\mathcal{H})$.

More operators with large numerical and essential numerical ranges are given in the next proposition. An operator X is stable when its real part $(X + X^*)/2$ is negative definite (invertible).

Proposition (3.1.9)[3]: *If $X \in L(\mathcal{H})$ is stable, then X is not norm limit of $U_n Q U_n^* + V_n Q V_n^*$ for any sequences of unitaries $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ and any idempotent Q in $L(\mathcal{H})$.*

Proof. We have a decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_{ns}$ in two invariant subspaces of Q such that Q acts on Hs as a selfadjoint projection P, and Q acts on Hns as a purely nonselfadjoint idempotent, that is $A_{\mathcal{H}_{ns}}$ is unitarily equivalent to an operator on $\mathcal{F} \oplus \mathcal{F}$ of the form

$$Q_{\mathcal{H}_{ns}} \simeq \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix} \tag{6}$$

where R is a nonsingular positive operator on a Hilbert space $\mathcal{F}$, so

$$Q \simeq P \oplus \simeq \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix}. \tag{7}$$

Let Y be a norm limit of the sum of two sequences in the unitary orbit of Q. If the purely non-selfadjoint part $\mathcal{H}_{ns}$ is vacuous, then Y is positive, hence $Y \neq X$. If $\mathcal{H}_{ns}$ is not vacuous, (7) shows that

$$\begin{aligned} Q + Q^* &\simeq 2P \oplus \begin{pmatrix} 2I & R \\ R & 0 \end{pmatrix} \\ &\simeq 2P \oplus \left\{ \begin{pmatrix} I & I \\ I & I \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \right\}. \end{aligned}$$

This implies that $\|(Q + Q^*)_+\| \ge \|(Q + Q^*)_-\|$, therefore $Y + Y^*$ cannot be negative definite, hence $X \neq Y$.

Section (3.2): Unital, Trace Preserving Positive Linear Maps

With Pinchings in Factors

Kennedy and Skoufranis have studied the following problem: Let $\mathfrak{X}$ be a maximal abelian $*$-subalgebra (masa) of a von Neumann algebra $\mathfrak{M}$, with corresponding expectation $\mathbb{E}_{\mathfrak{X}} : \mathfrak{M} \to \mathfrak{X}$ (i.e., a unital positive linear map such that $\mathbb{E}_{\mathfrak{X}}(XM) = X\,\mathbb{E}_{\mathfrak{X}}(M)$ for all $X \in \mathfrak{X}$ and $M \in \mathfrak{M}$) . Given a normal operator $\in \mathfrak{M}$, determine the image by $\mathbb{E}_{\mathfrak{X}}$ of the unitary orbit of A,

$$\Delta_{\mathfrak{X}}(A) = \{\, \mathbb{E}_{\mathfrak{X}}(UAU*) : U \text{ a unitary in } \mathfrak{M} \,\}.$$

In several cases, they determined the norm closure of $\Delta_{\mathfrak{X}}(A)$. We have the following two propositions.

Proposition (3.2.1)[3]: *Let $\mathfrak{X}$ be a masa in $(\mathcal{H})$, $X \in \mathfrak{X}$, and A a normal operator in $L(\mathcal{H})$.*

If $\sigma(X) \subset \operatorname{conv}\sigma_e(A)$, then X *lies in the norm closure of $\Delta_{\mathfrak{X}}(A)$.*

Proposition (3.2.2) [3]: *Let $\mathfrak{X}$ be a continuous masa in $L(\mathcal{H})$, $X \in \mathfrak{X}$, and A a normal operator in $L(\mathcal{H})$. If* X *lies in the norm closure of $\Delta_{\mathfrak{X}}(A)$, then $\sigma(X) \subset \operatorname{conv}\sigma_e(A)$.*

Since we deal with normal operators, $\sigma(X) \subset \operatorname{conv}\sigma_e(A)$ means $W(X) \subset W_\epsilon(A)$. Proposition (3.2.2) needs the continuous assumption. It is a rather simple fact; we generalize it in Lemma (3.2.4): *Conditional expectations reduce essential numerical ranges,* $W(\mathbb{E}_{\mathfrak{X}}(T)) \subset W_\epsilon(T)$ *for all* $T \in L(\mathcal{H})$. Thus, the main point which says that if $W(X) \subset W_\epsilon(A)$ then X can be approximated by operators of the form $\mathbb{E}_{\mathfrak{X}}(UAU^*)$ with unitaries U. With the slightly stronger assumption $W(X) \subset_{st} W_\epsilon(A)$, via the following corollary, that X is exactly of this form, furthermore the normality assumption on XA is not necessary.

Corollary (3.2.3)[3]: *Let $\mathfrak{X}$ be a masa in $L(\mathcal{H})$, $X \in \mathfrak{X}$ and $A \in L(\mathcal{H})$. If $W(X) \subset_{st} W_\epsilon(A)$, then $X = \mathbb{E}_{\mathfrak{X}}(UAU^*)$ for some unitary* *operator* $U \in L(\mathcal{H})$.

Proof. First, we note a simple fact: Let $\{P_i\}_{i=1}^{\infty}$ be a sequence of orthogonal projections in $\mathfrak{X}$ such that $\sum_{i=1}^{\infty} P_i = I$, and let $Z \in L(\mathcal{H})$ such that $P_i Z P_i \in \mathfrak{X}$ for all i. Then, we have a strong sum

$$\mathbb{E}_{\mathfrak{X}}(Z) = \sum_{i=1}^{\infty} P_i Z P_i .$$

Now, denote by $\mathcal{H}_i$ the range of P_i and assume dim Hi = ∞ for all i. We have $W_\epsilon(X_{\mathcal{H}_i}) \subset W_\epsilon(X)$, hence

$$\bigcup_{i=1}^{\infty} W(X_{\mathcal{H}_i}) \subset_{st} W_\epsilon(A).$$

We get a unitary U on $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ such that

$$A \simeq UAU^* = \begin{pmatrix} X_{\mathcal{H}_1} & * & \cdots & \cdots \\ * & X_{\mathcal{H}_2} & * & \cdots \\ \vdots & * & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} ????.?$$

Since $0 \oplus \cdots \oplus X_{\mathcal{H}_i} \oplus 0 \cdots \in \mathfrak{X}$ for all i, the previous simple fact shows that

$$\mathbb{E}_{\mathfrak{X}}(UAU^*) = \bigoplus_{i=1}^{\infty} X_{\mathcal{H}_i} = X.$$

The following result extends Proposition (3.2.2), the ”easy” part of Kennedy-Skoufranis’ theorem .

Lemma (3.2.4)[3]: *If* X *is a masa in* $L(\mathcal{H})$ *and* $Z \in L(\mathcal{H})$, *then* $W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset \overline{W}(Z)$ *and* $W_\epsilon(\mathbb{E}_{\mathfrak{X}}(Z)) \subset W_\epsilon(Z)$.

Proof. (i) Assume Z is normal. We may identify the unital C^*-algebra $\mathfrak{A}$spanned by Z with $C^0(\sigma(Z))$ via a $*$-isomorphism $\varphi: C^0(\sigma(Z)) \to \mathfrak{A}$ with $\varphi(z \mapsto z) = Z$. Let $h \in \mathcal{H}$ be a unit vector. For $f \in C^0(\sigma(Z))$, set

$$\psi(f) = \langle h, \mathbb{E}_{\mathfrak{X}}(\varphi(f))h \rangle.$$

Then ψ is a positive linear functional on $C^0(\sigma(Z))$ and $\psi(1) = 1$. Thus ψ is a Radon measure induced by a probabilty measure μ,

$$\psi(f) = \int_{\sigma(Z)} f(z) d\mu(z)$$

We then have $\langle h, \mathbb{E}_{\mathfrak{X}}(Z)h\rangle = \psi(z) \in \text{conv}(\sigma(Z))$. Since $\text{conv}(\sigma(Z)) = \overline{W}(Z)$, we obtain $W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset \overline{W}(Z)$.

(ii) Let Z be a general operator in $L(\mathcal{H})$ and define a conditional expectation

$$\mathbb{E}_2 : L(\mathcal{H} \oplus \mathcal{H}) \to \mathfrak{X} \oplus \mathfrak{X}$$

by

$$\mathbb{E}_2\left(\begin{pmatrix} A & B \\ D & C \end{pmatrix}\right) = \begin{pmatrix} A\mathbb{E}_{\mathfrak{X}}(A) & 0 \\ 0 & \mathbb{E}_{\mathfrak{X}}(B) \end{pmatrix}.$$

From the first part of the proof, we infer

$$W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset W\left(\begin{pmatrix} A\mathbb{E}_{\mathfrak{X}}(Z) & 0 \\ 0 & \mathbb{E}_{\mathfrak{X}}(B) \end{pmatrix}\right) \subset \overline{W}\left(\begin{pmatrix} Z & C \\ D & B \end{pmatrix}\right)$$

whener $\begin{pmatrix} Z & C \\ D & B \end{pmatrix}$ is normal. Since we have, by a simple classical fact

$$\overline{W}(Z) = \bigcap \overline{W}\left(\begin{pmatrix} Z & C \\ D & B \end{pmatrix}\right), \tag{7}$$

where the intersection runs over all B, C, D such that $\begin{pmatrix} Z & C \\ D & B \end{pmatrix}$ is normal, we obtain $W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset \overline{W}(Z)$.

(iii) We deal with the essential numerical range inclusion. We can split $\mathfrak{X}$ into its discrete part $\mathfrak{D}$ and continuous part $\mathfrak{C}$ with the corresponding decomposition of the Hilbert space,

$$\mathfrak{X} = \mathfrak{D} \oplus \mathfrak{C}, \qquad hh = \mathcal{H}_d \oplus \mathcal{H}_c.$$

We then have

$$W_\epsilon(\mathbb{E}_{\mathfrak{X}}(Z)) = \text{conv}\left\{W_\epsilon\left(\mathbb{E}_{\mathfrak{D}}(Z_{\mathcal{H}_d})\right); W_\epsilon\left(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c})\right)\right\}. \tag{8}$$

We have an obvious inclusion

$$W_\epsilon\left(\mathbb{E}_{\mathfrak{D}}(Z_{\mathcal{H}_d})\right) \subset W_\epsilon(Z_{\mathcal{H}_d}). \tag{9}$$

On the other hand, for all compact operators $K \in L(\mathcal{H})$,

$$W_\epsilon\left(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c})\right) = W_\epsilon\left(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c}) + K_{\mathcal{H}_c}\right) = W_\epsilon\left(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c} + K_{\mathcal{H}_c})\right)$$
$$\subset W(Z_{\mathcal{H}_c} + K_{\mathcal{H}_c})$$

by the simple folklore fact that a conditional expectation onto a continous masa vanishes on compact operators and part (ii) of the proof. Thus, when K runs over all compact operators, we obtain

$$W_\epsilon\left(\mathbb{E}_{\mathfrak{D}}(Z_{\mathcal{H}_c})\right) \subset W_\epsilon(Z_{\mathcal{H}_c}). \quad (10)$$

Combining (8) (9) and (10) completes the proof.

For discrete masas, unlike continuous masas , there is a unique conditional expectation, which merely consists in extracting the diagonal with respect to an orthonormal basis. In a recent article, Loreaux and Weiss give a detailed study of diagonals of idempotents in $L(\mathcal{H})$. They established that a nonzero idempotent Q has a zero diagonal with respect to some orthonormal basis if and only if Q is not a Hilbert-Schmidt perturbation of a projection (i.e., a self-adjoint idempotent). They also showed that any sequence $\{a_n\} \in l^\infty$ such that $|a_n| \leq \alpha$ for all n and, for some $a_{n_0}, a_k = a_{n_0}$ for infinitely many k, one has a idempotent Q such that $\|Q\| \leq 18\alpha + 4$ and Q admits $\{a_n\}$ as a diagonal with respect to some orthonormal basis. Using this, they proved that any sequence in l^∞ is the diagonal of some idempotent operator answering a question of Jasper. Further, it is not necessary to confine to diagonals, i.e., discrete masas, and the constant $18\alpha + 4$ can be improved; in the n$\mathbb{E}\mathfrak{X}$t corollary we explicit the best constant when $\alpha = 1$.

Corollary (3.2.5)[3]: *Let $\mathfrak{X}$ be a masa in $L(\mathcal{H})$ and $\alpha > 0$. There exists an idempotent $Q \in L(\mathcal{H})$, such that for all $X \in \mathfrak{X}$ with $\|X\| < \alpha$, we have $X = \mathbb{E}_{\mathfrak{X}}(UQU^*)$ for some unitary operator $U \in L(\mathcal{H})$. If $\alpha = 1$, $\|Q\| = \sqrt{5 + 2\sqrt{5}}$ is the smallest possible norm.*

Proof. We have an idempotent Q such that $W_\epsilon(Q) \supset \alpha\mathcal{D}$, hence the first and main part of Corollary (3.2.5) follows from Corollary (3.2.3). The remaining parts require a few computations.

To obtain the bound $\sqrt{5+2\sqrt{5}}$ when $\alpha = 1$ we get a closer look at $\bigoplus^\infty M_a$ with M_a where a is a positive scalar. We have

$$W(M_a) = \{\langle h, M_a h\rangle : \mathcal{H} \in \mathbb{C}^2, \|h\| = 1\}$$
$$= \{|h_1|^2 + a\overline{h_2}h_1 : \ |h_1|^2 + |h_2|^2 = 1\},$$

hence, with $h_1 = re^{i\theta}, h_2 = \sqrt{1-r^2e^{i\alpha}}$,

$$W(M_a) = \bigcup_{0\le r\le 1}\left\{r^2 + ar\sqrt{1-r^2e^{i(\theta-\alpha)}} : \ \theta, \alpha \in [0, 2\pi]\right\}.$$

Therefore $W(M_a)$ is a union of circles Γ_r with centers r^2 and radii $ar\sqrt{1-r^2}$. To have $\mathcal{D} \subset W(M_a)$ it is necessary and sufficient that $-1 \in \Gamma_r$ for some $r \in [0,1]$, hence

$$a = \frac{1+r^2}{\sqrt{1-r^2}}. \tag{11}$$

Now we minimize $a = a(r)$ given by (11) when $r \in (0,1)$ and thus obtain the matrix M_{a_*} with smallest norm such that $W(M_{a_*}) \supset \mathcal{D}$. Observe that $a(r) \to +\infty$ as $r \to 0$ and as $r \to 1$, and

$$r^2(1-r^2)^{3/2}a'(r) = r^4 + 4r^2 - 1.$$

Thus $a(r)$ takes its minimal value a_* when $r^2 = \sqrt{5} - 2$. We have $a_*^2 = 4 + 2\sqrt{5}$, hence

$$\|M_{a_*}\| = \sqrt{5+2\sqrt{5}}.$$

Now, letting $= \bigoplus^\infty M_{a_*}$, we have $W_\epsilon(Q) = W(M_{a_*})$, so that Q is an idempotent in $L(\mathcal{H})$ such that $W_\epsilon(Q) \supset \mathcal{D}$, and thus by Corollary (3.2.3) any operator X such that $\|X\| < 1$ satisfies $\mathbb{E}_{\mathfrak{X}}(UQU^*) = X$ for some unitary U.

It remains to check that if Q is an idempotent such that Corollary (3.2.5) holds for any operator X such that $\|X\| < 1$, then $\|Q\| \geq \sqrt{5+2\sqrt{5}}$. To this end, we consider the purely nonselfadjoint part $Q_{\mathcal{H}_{ns}}$ of ,

$$Q_{\mathcal{H}_{ns}} \simeq \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix}.$$

We have $W_\epsilon(Q) \supset \mathcal{D}$ if and only if $W_\epsilon(Q_{\mathcal{H}_{ns}}) \supset \mathcal{D}$. By Lemma (3.2.5) this is necessary. We may approximate $W_\epsilon(Q_{\mathcal{H}_{ns}})$ with sligthly larger essential numerical ranges, by using apositive diagonalizable operator R_ε such that $R_\varepsilon \geq R \geq R_\varepsilon - \varepsilon I$, for writing which

$$W_\epsilon\left(\begin{pmatrix} I & 0 \\ R_\varepsilon & 0 \end{pmatrix}\right) = W_\epsilon\left(\bigoplus_{i=1}^{\infty} \begin{pmatrix} I & 0 \\ a_n & 0 \end{pmatrix}\right)$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive scalars, the eigenvalues of R_ε. By the previous step of the proof, this essential numerical range contains $\mathcal{D}$if and only if $\overline{\lim}\, a_n \geq a_*$. If this holds for all $\varepsilon > 0$, then $\|Q\| \geq \sqrt{5+2\sqrt{5}}$.

Unital positive linear maps $\Phi : \mathbb{M}_n \to \mathbb{M}_n$, the matrix algebra, which preserve the trace play an important role in matrix analysis and its applications. These maps are sometimes called doubly stochastic.

We say that $\Phi: L(\mathcal{H}) \mapsto L(\mathcal{H})$ is trace preserving if it preserves the trace ideal $\mathcal{T}$and $\mathrm{Tr}\, \Phi(Z) = \mathrm{Tr}\, Z$ for all $\in \mathcal{T}$.

Corollary (3.2.6)[3]: *Let $A \in L(\mathcal{H})$. The following two conditions are equivalent:*

(i)$W_\epsilon(A) \supset \mathcal{D}$.

(ii) For all $X \in L(\mathcal{H})$ with $\|X\| < 1$, there exists a unital, trace preserving, positive linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$ *such that $\Phi(A) = X$.*

We may further require in (ii) that Φ *is completely positive and* sot-*and* wot-*sequentially continuous.*

Proof. Assume (i). We have a unitary $U : \mathcal{H} \to \bigoplus^{\infty} \mathcal{H}$ such that

$$A \simeq UAU^* = \begin{pmatrix} X & * & \cdots & \cdots \\ * & X & * & \cdots \\ \vdots & * & X & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Now consider the map $\Psi : L(\bigoplus^{\infty} \mathcal{H}) \to L(\mathcal{H})$,

$$\begin{pmatrix} Z_{1,1} & Z_{1,2} & \cdots \\ Z_{2,1} & Z_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

and define $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$ as $\Phi(T) = \Psi(UTU^*)$. Since both Ψ and the unitary congruence with U are sot- and wot-sequentially continuous, and trace preseverving, completely *positive* and unital, so is Φ. Further $\Phi(A) = X$.

Assume (ii) and suppose that $z \notin W_{\epsilon}(A)$ and $|z| < 1$ in order to reach a contradiction. If $z = |z|e^{i\theta}$, replacing A by $e^{-i\theta}A$, we may assume $1 > z \geq 0$. Hence,

$$W_{\epsilon}((A + A^*)/2) \subset (-\infty, z]$$

and there exists a selfadjoint compact operator L such that

$$\frac{A + A^*}{2} \leq zI + L.$$

This implies that $X := \frac{1+z}{2}I$ cannot be in the range of Φ for any unital, trace preserving positive linear map. Indeed, we would have

$$\frac{1 + z}{2}I = \frac{X + X^*}{2} = \Phi\left(\frac{A + A^*}{2}\right) \leq zI + \Phi(L)$$

which is not possible as $\Phi(L)$ is compact.

In the finite dimensional setting, two Hermitian matrices A and X satisfy the relation $X = \Phi(A)$ for some positive, unital, trace preserving linear map if and only if X is in the convex hull of the unitary orbit of A. In the infinite dimensional setting, if the norm closure of the unitary orbit of A. This is easily checked by approximating the operators with diagonal operators. Such an equivalence might not be brought out to the setting of Corollary (3.2.5).

Here we mention a result of Wu: *If* A ∈ $L(\mathcal{H})$ *is not of the form scalar plus compact, then every* X ∈ L($\mathcal{H}$) *is a linear combination of operators in the unitary orbit of* A*.*

If one deletes the positivity assumption, the most regular class of linear maps on $L(\mathcal{H})$ might be given in the following definition.

Definition (3.2.7) [3]: A linear map $\Psi : L(\mathcal{H}) \to L(\mathcal{H})$ is said ultra-regular if it fulfills two conditions:

(u1) $\Psi(I) = I$ and Ψ is trace preserving.

(u2) Whenever a sequence $A_n \to A$ for either the norm-, strong-, or weak-topology, then we also have $\Psi(A_n) \to \Psi(A)$ for the same type of convergence.

Any ultra-regular linear map preserves the set of essentially scalar operators (of the form $\lambda I + K$ with $\lambda \in \mathbb{C}$ and a compact operator K). For its complement, we state our last corollary.

Corollary (3.2.8)[3]: *Let $A \in L(\mathcal{H})$ be essentially nonscalar. Then, for all $X \in L(\mathcal{H})$ there exists a ultra-regular linear map $\Psi : L(\mathcal{H}) \to L(\mathcal{H})$ such that $\Psi(A) = X$.*

Proof. An operator is essentially nonscalar precisely when its essential numerical range is not reduced to a single point. So, let $a, b \in W_\epsilon(A), a \neq b$. By a lemma of Anderson and Stampfli, A is unitarily equivalent to an operator on $\mathcal{H} \oplus \mathcal{H}$ of the form

$$B = \begin{pmatrix} D & * \\ * & * \end{pmatrix}$$

Where $D = \bigoplus_{n=1}^{\infty} D_n$, with two by two matrices D_n,

$$D_n = \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix}$$

such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. We may assume that, for some $\alpha, \beta > 0$, we have $\alpha > |a_n| + |b_n|$ and $|a_n - b_n| > \beta$. Hence there exist $\gamma > 0$ and two by two intertible matrices T_n such that, for all $n, W(T_n D_n T_n^{-1}) \supset \mathcal{D}$and $\|T_n\| + \|T_n^{-1}\| \leq \gamma$. So, letting $T = (\bigoplus_{n=1}^{\infty} T_n) \oplus I$, we obtain an invertible operator T on $\mathcal{H} \oplus \mathcal{H}$ such that $W_\epsilon(TBT^{-1}) \supset \mathcal{D}$.

Hence we have an invertible operator S on $\mathcal{H}$ such that $W_\epsilon(SAS^{-1}) \supset \mathcal{D}$. Therefore we may apply Corollary (3.2.7)and obtain a wot- and sot-sequentially continuous, unital, trace preserving map Φ such that $\Phi(SAS^{-1}) = X$. Letting $\Psi(\cdot) = \Phi(S \cdot S^{-1})$ completes the proof.

We cannot find an alternative proof, not based on the pinching theorem, for Corollaries (3.2.6) and (3.2.8).

If we trust in Zorn, there exists a linear map $\Psi : L(\mathcal{H}) \to L(\mathcal{H})$ which satifies the condition (u1) but not the condition (u2). Indeed, let $\{a_p\}_{p\in\Omega}$ be a basis in the Calkin algebra $\mathfrak{C} = L(\mathcal{H})/K(\mathcal{H})$, indexed on an ordered set Ω, whose first element a_{p_0} is the image of I by the canonical projection $\pi : L(\mathcal{H}) \to \mathfrak{C}$. Thus, for each operator X, we have a unique decomposition $\pi(X) = \sum_{p\in\Omega}(\pi(X))_p a_p$ with only finitely many nonzero terms. Further $(\pi(X))_{p_0} = 0$ if X is compact, and $(\pi(I))_{p_0} = 1$. We then define a map $\psi : L(\mathcal{H}) \to L(\mathcal{H} \oplus \mathcal{H})$ by

$$\psi(X) = \begin{pmatrix} X & 0 \\ 0 & (\pi(I))_{p_0} I \end{pmatrix}.$$

Letting $\Psi(X) = V\psi(X)V^*$ where $V : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ is unitary, we obtain a linear map $\Psi : L(\mathcal{H}) \to L(\mathcal{H})$ which satifies (u1) but not (u2): it is not norm continuous.

Let ω be a Banach limit on l^∞ and define a map $\varphi : l^\infty \to l^\infty, \{a_n\} \mapsto \{b_n\}$, where $b_1 = \omega(\{a_n\})$ and $b_n = a_{n-1}, n \geq 2$. Letting $\Psi(X) = \varphi(diag(X))$, where $diag(X)$ is the diagonal of $X \in \mathcal{H}$ in an orthonormal basis, we obtain a linear map Ψ which is norm continuous, satisfies (u1) but not (u2): it is not strongly sequentially continuous.

However, it seems not possible to define explicitly a linear map $\Psi : L(\mathcal{H}) \to L(\mathcal{H})$ satisfying (u1) but not (u2).

We discuss possible extensions to our results to a von Neumann algebra $\mathfrak{R}$ acting on a separable Hilbert space $\mathcal{H}$. First, we need to define an essential numerical range $W_\epsilon^{\mathfrak{R}}$ for $\mathfrak{R}$. Let $\in \mathfrak{R}$. If $\mathfrak{R}$ is type-III, then $W_\epsilon^{\mathfrak{R}}(A) := W_\epsilon(A)$. If $\mathfrak{R}$ is type-II$^\infty$, then

$$W_\epsilon^{\mathfrak{R}}(A) := \bigcap_{K \in \mathcal{T}} \overline{W}(A + K)$$

where $\mathcal{T}$ is the trace ideal in $\mathfrak{R}$ (we may also use its norm closure K, the "compact" operators in $\mathfrak{R}$, or any dense sequence in K)

Recently, Dragan and Kaftal obtained some decompositions for positive operators in von Neumann factors, which, in the case of $L(\mathcal{H})$ were first investigated. This suggests that our questions dealing with a possible extension to type- II$^\infty$ and -III factors also have an affirmative answer.

Let $\mathfrak{R}$ be a type- II$^\infty$ or -III factor.

Definition (3.2.9): A sequence $\{V_i\}_{i=1}^\infty$ of isometries in $\mathfrak{R}$ such that $\sum_{i=1}^\infty V_i V_{i=I}^*$ is called an isometric decomposition of $\mathfrak{R}$.

Conjecture (3.2.10): *Let $A \in \mathfrak{R}$ with $W_\epsilon^{\mathfrak{R}}(A) \supset \mathcal{D}$ and $\{X_i\}_{i=1}^\infty$ a sequence in $\mathfrak{R}$ such that $\sup_i \|X_i\| < 1$. Then, there exists an isometric decomposition $\{V_i\}_{i=1}^\infty$ of $\mathfrak{R}$ such that $V_i^* A X_i = X_i$ for all i.*

Chapter 4

Nontrivial Twisted Sums of C_0 and C(K)

We obtain a new class of compact Hausdorff spaces K for which c_0 can be nontrivially twisted with C(K).

Section (4.1): Nontrivial Twisted Sums for Corson Compacta Toward the General Valdivia Case

We present a broad new class of compact Hausdorff spaces K such that there exists a nontrivial twisted sum of c_0 and $C(K)$, where $C(K)$ denotes the Banach space of continuous real-valued functions on K endowed with the supremum norm. By a twisted sum of the Banach spaces Y and X we mean a short exact sequence $0 \to Y \to Z \to X \to 0$, where Z is a Banach space and the maps are bounded linear operators. This twisted sum is called trivial if the exact sequence splits, i.e., if the map $Y \to Z$ admits a bounded linear left inverse (equivalently, if the map $Z \to X$ admits a bounded linear right inverse). In other words, the twisted sum is trivial if the range of the map $Y \to Z$ is complemented in Z; in this case, $\cong X \oplus Y$. We denote by Ext(X, Y) the set of equivalence classes of twisted sums of Y and X and we write Ext(X, Y) = 0 if every such twisted sum is trivial.

Many problems in Banach space theory are related to the quest for conditions under which Ext(X, Y) = 0. For instance, an equivalent statement for the classical Theorem of Sobczyk is that if X is a separable Banach space, then $\text{Ext}(X, c_0) = 0$. The converse of the latter statement clearly does not hold in general: for example, $\text{Ext}(\ell_1(I), c_0) \neq 0$, since $\ell_1(I)$ is a projective Banach space. However, the following question remains open: is it true that $\text{Ext}(C(K), c_0) \neq 0$ for any nonseparable $C(K)$ space? This problem was stated and further studied in which proves that, under the continuum hypothesis (CH), the space Ext$(C(K), c_0)$ is nonzero for a nonmetrizable compact Hausdorff space K of finite height. In addition to this result, everything else that is known about the problem is summarized namely that Ext$(C(K), c_0)$ is nonzero for a $C(K)$ space under any one of the following assumptions:

(i) K is a nonmetrizable Eberlein compact space.

(ii) K is a Valdivia compact space which does not satisfy the countable

(iii) chain condition (ccc);

(iv) the weight of K is equal to ω1 and the dual space of C(K) is not weak*-separable.

(v) K has the extension property and it does not have ccc;

(vi) C(K) contains an isomorphic copy of ℓ_∞

Note also that if Ext$(Y, c_0) \neq 0$ and X contains a complemented isomorphic copy of , then Ext$(X, c_0) \neq 0$= 0.

Here is an overview of the main results of this article. Theorem (4.1.3) gives a condition involving biorthogonal systems in a Banach space X which implies that Ext$(X, c_0) \neq 0$. We discuss some of its implications when X is of the form C(K). It is proven that if K contains a homeomorphic copy of $[0, \omega] \times [0, c]$ or of 2c, then Ext$(C(K), c_0)$ is nonzero, where c denotes the cardinality of the continuum. We investigate the consequences of the results Valdivia and Corson compacta. Recall that Valdivia compact spaces constitute a large superclass of Corson compact spaces (This, let x is Corson compact spaces with a Radon probability measure μ- by removing all open subset of x) [7] closed under arbitrary products; moreover, every Eberlein compact is a Corson compact devoted to the proof that, under CH, it holds that Ext$(C(K), c_0) \neq 0$ for every nonmetrizable Corson compact space K. The question of whether Ext$(C(K), c_0) \neq 0$ for an arbitrary nonmetrizable Valdivia compact space K remains open (even under CH), but we solve some particular cases of this problem.

The weight and the density character of a topological space X are denoted, respectively, by $w(\chi)$ and dens(χ) . Moreover, we always denote by χ_A the characteristic function of a set A and by $|A|$ the cardinality of A. We start with a technical lemma which is the heart of the proof of Theorem (4.1.3). A family of sets (Ai)i∈I is said to be almost disjoint if each Ai is infinite and $A_i \cap A_j$ is finite, for $all\ i, j \in I\ with\ i = j.$

Lemma (4.1.1)[4]: *There exists an almost disjoint family* $(A_{n,\alpha})_{n\in\omega,\alpha\in c}$ *of subsets of* ω *satisfying the following property: for every family* $(A'_{n,\alpha})_{n\in\omega,\alpha\in c}$ *with each* $A'_{n,\alpha} \subset A_{n,\alpha}$ *cofinite in An,α, it holds that* $\sup_{p\in\omega} |M_p| = +\infty$*, where:*

$$M_p = \left\{n \in \omega : p \in \bigcup\nolimits_{\alpha\in c} A'_{n,\alpha}\right\}.$$

Proof. We will obtain an almost disjoint family $(A_{n,\alpha})_{n\in\omega,\alpha\in c}$ of subsets of $2^{<\omega}$ with the desired property, where $2^{<\omega} = \bigcup_{k\in\omega} 2^k$ denotes the set of finite sequences in $2 = \{0,1\}$. For each $\epsilon \in 2^\omega$, we set:

$$\mathcal{A}_\epsilon = \{\epsilon|_k : k \in \omega\},$$

so that $(\mathcal{A}_\epsilon)_{\epsilon\in 2^\omega}$ is an almost disjoint family of subsets of $2^{<\omega}$. Let $(\mathcal{B}_\alpha)_{\alpha\in c}$ be an enumeration of the uncountable Borel subsets of 2^ω. Recalling that

$|\mathcal{B}_\alpha| = c$ for all $\alpha \in c$, one easily obtains by transfinite recursion a family $(\epsilon_{n,\alpha})_{n\in\omega,\alpha\in c}$ of pairwise distinct elements of s2^{ω}uch that $\epsilon_{n,\alpha} \in \mathcal{B}_{\alpha}$, for all $n \in \omega, \alpha \in c$. Set $A_{n,\alpha} = A_{\epsilon_{n,\alpha}}$ and let $(A'_{n,\alpha})_{n\in\omega,\alpha\in c}$ be as in the statement of the lemma. For $n \in \omega$, denote by D_n the set of those $\epsilon \in 2^{\omega}$ such that $n \in M_p$ for all but finitely many $p \in A_{\epsilon}$. Note that:

$$D_n = \bigcup_{k_0\in\omega} \bigcap_{k\geq k_0} \bigcup \{C_p : p \in 2^k \text{ with } n \in M_p\},$$

where C_p denotes the clopen subset of 2^{ω} consisting of the extensions of p.

The above equality implies that D_n is an F_{σ} (and, in particular, a Borel subset of 2^{ω}. We claim that the complement of D_n in $2^{\omega}$is countable. Namely, if it were uncountable, there would exist $\alpha \in c$ with $\mathcal{B}_{\alpha} = 2^{\omega} \setminus D_n$. But, since $n \in M_p$ for all $p \in A'_{n,\alpha}$, we have that $\epsilon_{n,\alpha} \in D_n$, contradicting the fact that $\epsilon_{n,\alpha} \in \mathcal{B}_{\alpha}$ and proving the claim. To conclude the proof of the lemma, note that for each $n \geq 1$ the intersection $\bigcup_{i<n} D_i$ is nonempty; for $\epsilon \in \bigcup_{i<n} D_i$, we have that $\{i : i < n\} \subset M_p$, for all but finitely many $p \in A_{\epsilon}$.

Let X be a Banach space. Recall that a biorthogonal system in X is a family $(x_i, \gamma_i)_{i\in I}$ with $x_i \in X, \gamma_i \in X^*, \gamma_i(x_i) = 1$ and $\gamma_i(x_j) = 0$ for $i \neq j$. The cardinality of the biorthogonal system $(x_i, \gamma_i)_{i\in I}$ is defined as the cardinality of I.

Definition (4.1.2)[4]: Let $(x_i, \gamma_i)_{i\in I}$be a biorthogonal system in a Banach space X. We call $(x_i, \gamma_i)_{i\in I}$ bounded if $\sup_{i\in I}\|x_i\| < +\infty$ and $\sup_{i\in I}\|y_i\| < +\infty$ weak*-null if $(\gamma_i)_{i\in I}$ is a weak*-null family, i.e., if $(\gamma_i(x))_{i\in I}$ is in $c_0(I)$, for all $x \in X$.

Theorem (4.1.3) [4]: *Let X be a Banach space. Assume that there* *exist a weak*- null biorthogonal system* $(x_{n,\alpha}, \gamma_{n,\alpha})_{n\in\omega,\alpha\in c}$ *in* X *and a constant* $C \geq 0$ *such that:*

$$\left\|\sum_{i=1}^{k} x_{n_i,\alpha_i}\right\| \leq C,$$

for all $n_1, \ldots, n_k \in \omega$ pairwise distinct, all $\alpha_1, \ldots, \alpha_k \in c$, and all $k \geq 1$.

Then $\operatorname{Ext}(X, c_0) \neq 0$.

Proof. We have that $\operatorname{Ext}(X, c_0) = 0$ if and only if every bounded operator $T: X \to \ell_\infty / c_0$ admits a *lifting 1, i.e.,* a bounded operator $\hat{T}: X \to \ell_\infty$ with $T(x) = \hat{T}(x) + c_0$, for all $x \in X$. Let us then show that there exists an operator $T : X \to \ell_\infty / c_0$ that does not admit a lifting. To this aim, let $(A_{n,\alpha})_{n\in\omega,\alpha\in c}$ be an almost disjoint family as in Lemma (4.1.1) and consider the unique isometric embedding $S : c_0(\omega \times c) \to \ell_\infty / c_0$ such that $S(e_{n,\alpha}) = \chi_{A_{n,\alpha}} + c_0$, where $(e_{n,\alpha})_{n\in\omega,\alpha\in c}$ denotes the canonical basis of $c_0(\omega \times c)$. Denote by $\Gamma : X \to c_0(\omega \times c)$ the bounded operator with coordinate functionals $(\gamma_{n,\alpha})_{n\in\omega,\alpha\in c}$ and set $T = S \circ \Gamma : X \to \ell_\infty / c_0$. Assuming by contradiction that there exists a lifting $\hat{T}$ of T and denoting by $(\mu_p)_{p\in\omega}$ the sequence of coordinate functionals of $\hat{T}$ we have that the set:

$$A'_{n,\alpha} = \{p \in A_{n,\alpha} : \mu_p(x_{n,\alpha}) \geq \frac{1}{2}\}$$

is cofinite in $A_{n,\alpha}$. It follows that for each $k \geq 1$, there exist $p \in \omega$, $n_1, \ldots, n_k \in \omega$ pairwise distinct, and $\alpha_1, \ldots, \alpha_k \in c$ such that $p \in A'_{n_i,\alpha_i}$, for $i = 1, \ldots, k$. Hence:

$$\frac{k}{2} \leq \mu_p\left(\sum_{i=1}^{k} x_{n_i,\alpha_i}\right) \leq C\|\hat{T}\|,$$

which yields a contradiction.

Corollary (4.1.4)[4]: *Let K be a compact Hausdorff space. Assume that there exists a bounded weak*-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n\in\omega,\alpha\in c}$ in $C(K)$ such that $f_{n,\alpha}f_{m,\beta} = 0$, for all $n, m \in \omega$ with $n \neq m$ and all $\alpha, \beta \in c$. Then* $\mathrm{Ext}(C(K), c_0) \neq 0$.

Definition (4.1.5)[4]: We say that a compact Hausdorff space K satisfies property $(*)$if there exist a sequence $(F_n)_{n\in\omega}$ of closed subsets of K and a bounded weak*-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n\in\omega,\alpha\in c}$ in $C(K)$ such that:

$$F_n \cap \overline{\bigcup_{m\neq n} F_m} = \emptyset \qquad (1)$$

and supp $f_{n,\alpha} \subset F_n$, for all $n \in \omega$ and all $\alpha \in c$, where supp$f_{n,\alpha}$ denotes the support of $f_{n,\alpha}$.

In what follows, we denote by $M(K)$ the space of finite countably-additive signed regular Borel measures on K, endowed with the total variation norm. We identify as usual the dual space of $C(K)$ with $M(K)$.

Lemma (4.1.6)[4]: *Let K be a compact Hausdorff space and L be a closed subspace of K. If L satisfies property $(*)$, then so does K.*

Proof: Consider, as in Definition (4.1.5), a sequence $(F_n)_{n\in\omega}$ of closed subsets of L and a bounded weak*-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n\in\omega,\alpha\in c}$ in $C(L)$. By recursion on n, one easily obtains a sequence $(U_n)_{n\in\omega}$of pairwise disjoint open subsets of K with each Un containing F_n. Let V_n be an open subset of K with $F_n \subset V_n \subset \overline{V_n} \subset U_n$. Using Tietze's Extension Theorem and Urysohn's Lemma, we get a continuous extension $\tilde{f}_{n,\alpha}$ of $f_{n,\alpha}$ to K with support contained in $\overline{V_n}$ and having the same norm as $f_{n,\alpha}$. To conclude the proof, let $\tilde{\gamma}_{n,\alpha} \in M(K)$ be the extension of $\gamma_{n,\alpha} \in M(L)$ that vanishes identically outside of L and observe that $\left(\tilde{f}_{n,\alpha}, \tilde{\gamma}_{n,\alpha}\right)_{n\in\omega,\alpha\in c}$ is a bounded weak*-null biorthogonal system in $C(K)$.

As an immediate consequence of Lemma (4.1.6) and Corollary (4.1.4), we obtain the following result.

Theorem (4.1.7)[4]: *If a compact Hausdorff space L satisfies property* $(*)$*, then every compact Hausdorff space K containing a homeomorphic copy of L satisfies* $\text{Ext}(C(K), c_0) \neq 0$.

We now establish a few results which give sufficient conditions for a space K to satisfy property $(*)$. Recall that, given a closed subset F of a compact Hausdorff space K, an extension operator for F in K is a bounded operator $E : C(F) \to C(K)$ which is a right inverse for the restriction operator $C(K) \ni f \mapsto f|F \in C(F)$. Note that F admits an extension operator in K if and only if the kernel $C(K|F) = \{f \in C(K) : f|_F = 0\}$ of the restriction operator is complemented in $C(K)$. A point x of a topological space X is called a cluster point of a sequence $(S_n)_{n\in\omega}$ of subsets of X if every neighborhood of x intersects Sn for infinitely many $n \in \omega$.

Lemma (4.1.8)[4]: *Let K be a compact Hausdorff space. Assume that there exist a sequence* $(F_n)_{n\in\omega}$ *of pairwise disjoint closed subsets of* K *and a closed subset* F *of* K *satisfying the following conditions:*

(a) *F admits an extension operator in K;*

(b) *every cluster point of* $(F_n)_{n\in\omega}$*is in F and* $F_n \cap F = \emptyset$*, for all* $n \in \omega$;

(c) there exists a family $\left(f_{n,\alpha}, \gamma_{n,\alpha}\right)_{n\in\omega,\alpha\in c}$, where $\left(f_{n,\alpha}, \gamma_{n,\alpha}\right)_{\alpha\in c}$ is a weak*-null biorthogonal system in $C(F_n)$ for each $n \in \omega$ and

$$\sup_{n\in\omega,\alpha\in c}\|f_{n,\alpha}\| < +\infty, \sup_{n\in\omega,\alpha\in c}\|\gamma_{n,\alpha}\| < +\infty.$$

Then K satisfies property(*) .

Proof. From (b) and the fact that the F_n are pairwise disjoint it follows that (1) holds. Let $(U_n)_{n\in\omega}$, $(V_n)_{n\in\omega}$, and $(\tilde{f}_{n,\alpha})_{n\in\omega,\alpha\in c}$ be as in the proof of Lemma (4.1.6); we assume also that $\overline{V_n}\cap F = \emptyset$, for all $n \in \omega$. Let $\tilde{\gamma}_{n,\alpha} \in M(K)$ be the extension of $\gamma_{n,\alpha}\in M(F_n)$ that vanishes identically outside of F_n.

We have that $(\tilde{f}_{n,\alpha},\tilde{\gamma}_{n,\alpha})_{n\in\omega,\alpha\in c}$ is a bounded biorthogonal system in $C(K)$ and that$(\tilde{\gamma}_{n,\alpha})_{\alpha\in c}$ is weak*-null for each n, though it is not true in general that the entire family $(\tilde{\gamma}_{n,\alpha})_{n\in\omega,\alpha\in c}$is weak*-null. In order to take care of this problem, let $P : C(K) \to C(K|F)$ be a bounded projection and set $\tilde{\gamma}_{n,\alpha} = \tilde{\gamma}_{n,\alpha}\circ P$. Since all ˜fn,α are in $C(K|F)$, we have that $(\tilde{f}_{n,\alpha},\tilde{\gamma}_{n,\alpha})_{n\in\omega,\alpha\in c}$ is biorthogonal. To prove that $(\tilde{\gamma}_{n,\alpha})_{n\in\omega,\alpha\in c}$ is weak*-null, note that (b) implies that $\lim_{n\to+\infty}\|f|_{F_n}\| = 0$, for all $f \in C(K|F)$.

Corollary (4.1.9)[4]: *Let K be a compact Hausdorff space. If $C(K)$ admits a bounded weak*-null biorthogonal system of cardinality c, then the space* $[0,\omega] \times K$ *satisfies property* (*)*. In particular,* $L \times K$ *satisfies property* (*) *for every compact Hausdorff space L containing a nontrivial convergent sequence.*

Corollary (4.1.10)[4]: *The spaces* $[0,\omega] \times [0,c]$ *and* 2^c *satisfy property* (*)*. In particular, a product of at least c compact Hausdorff spaces with more than one point satisfies property* (*).

Proof. The family $\left(\chi_{[0,\alpha]},\delta_\alpha-\delta_{\alpha+1}\right)_{\alpha\in c}$ is a bounded weak*-null biorthogonal system in $C([0,c])$, where $\delta_\alpha\in M([0,c])$ denotes the probability measure with support $\{\alpha\}$. It follows from Corollary

(4.1.9) that $[0,\omega] \times [0,c]$ satisfies property $(*)$. To see that 2^c also does, note that the map $[0,c] \ni \alpha \mapsto \chi_\alpha \in 2^c$ embeds [0,c] into 2^c, so that $2^c \cong 2^\omega \times 2^c$ contains a homeomorphic copy of $[0,\omega] \times [0,c]$.

Recall that a projectional resolution of the identity (PRI) of a Banach space X is a family $(P_\alpha)_{\omega\leq\alpha\leq\text{dens}(X)}$ of projection operators $P_\alpha: X \to X$ satisfying the following conditions:

(i) $P\alpha = 1$, for $\omega \leq \alpha \leq \text{dens}(X)$;

(ii) $P_{\text{dens}(X)}$ is the identity of X;

(iii) $P_\alpha[X] \subset P_\beta[X]$and $\text{Ker}(P_\beta) \subset \text{Ker}(P_\alpha)$, for $\omega \leq \alpha \leq \beta \leq \text{dens}(X)$;

(iv) $P_\alpha(x) = \lim_{\beta<\alpha} P_\beta(x)$, for all $x \in X$, if $\omega < \alpha \leq \text{dens}(X)$ is a limit ordinal;

(v) $\text{dens}(P_\alpha[X]) \leq |\alpha|$, for $\omega \leq \alpha \leq \text{dens}(X)$.

We call the PRI strictly increasing if $P_\alpha[X]$ is a proper subspace of $P_\beta[X]$, for $\omega \leq \alpha < \beta \leq \text{dens}(X)$.

Corollary (4.1.11)[4]: *Let K and L be compact Hausdorff spaces such that L contains a nontrivial convergent sequence and $w(K) \geq c$. If $C(K)$ admits a strictly increasing* PRI, *then the space $L \times K$ satisfies property* (*).

Proof. This follows from Corollary (4.1.9) by observing that if a Banach space X admits a strictly increasing PRI, then X admits a weak*-null biorthogonal system $(x_\alpha, \gamma_\alpha)_{\omega\leq\alpha<\text{dens}(X)}$ with $\|x_\alpha\| = 1$ and $\|\gamma_\alpha\| \leq 2$, for all α. Namely, pick a unit vector x_α in $P_{\alpha+1}[X] \cap \text{Ker}(P_\alpha)$ and set $\gamma_\alpha = \phi_\alpha \circ (P_{\alpha+1} - P_\alpha)$, where $\phi_\alpha \in X^*$ is a norm-one functional satisfying $\phi_\alpha(x_\alpha) = 1$.

Let us recall some standard definitions. Given an index set I, we write $\Sigma(I) = \{x \in \mathbb{R}^I : \text{supp}x \text{ is countable}\}$, where the support supp$x$ of x is defined by supp$x = \{i \in I : x_i \neq 0\}$. Given a compact Hausdorff space K, we call A a Σ *-subset* of K if there exist an index set I and a continuous injection $\varphi : K \to \mathbb{R}^I$ such that $A = \varphi^{-1}[\Sigma(I)]$. The space K is called a Valdivia compactum if it admits a dense Σ-subset and it is called a Corson compactum if K is a Σ-subset of itself. We dedicated to the proof of the following result.

Lemma (4.1.12)[4]: *Let K be a compact Hausdorff space and F be a closed non- open $G\delta$ subset of K. Then there exists a sequence $(F_n)_{n\in\omega}$ of nonempty pairwise disjoint regular closed subsets of K such that condition (b) in the statement of Lemma (4.1.12) holds.*

Proof. We can write $F = \bigcap_{n\in\omega} V_n$, with each V_n open in K and $\overline{V_{n+1}}$ properly contained in V_n. Set $U_n = V_n \setminus \overline{V_{n+1}}$, so that all cluster points of $(U_n)_{n\in\omega}$ are in F. To conclude the proof, let F_n be a nonempty regular closed set contained inU_n.

Once we get the closed sets $(F_n)_{n\in\omega}$ from Lemma (4.1.12). First, we need an assumption ensuring that $w(F_n) \geq c$, for all n. To this aim, given a point x of a topological space χ, we define the weight of x in χ by:

$$w(x, X) = \min\{w(V) : V \text{ neighborhood of } x \text{ in } \chi\}.$$

Recall that if K is a Valdivia compact space, then $C(K)$ admits a PRI . Moreover, a trivial adaptation of the proof shows in fact that $C(K)$ admits a strictly increasing PRI. Thus, by the argument in the proof of Corollary (4.1.11), $C(K)$ admits a weak*-null biorthogonal system $(f_\alpha, \gamma_\alpha)\omega \leq \alpha < w(K)$ such that $f_\alpha \leq 1$ and $\gamma_\alpha \leq 2$, for all α. The following result is now immediately obtained.

Corollary (4.1.13)[4]: *Let K be a Valdivia compact space such that $w(x, K) \geq c$, for all $x \in K$. Assume that there exists a closed non-*

open G_δ subset F admitting an extension operator in K. Then K satisfies property $(*)$.

Assuming that K has ccc, the next lemma allows us to reduce the proof of Theorem (4.1.15) to the case when $w(x, K) \geq c$, for all $x \in K$.

Lemma (4.1.14)[4]: *Let K be a ccc Valdivia compact space and set*:

$$H = \{x \in K : w(x, K) \geq c\}.$$

Then:

(a) $H = \emptyset$, if $w(K) \geq c$;

(b) $w(K \setminus \text{int}(H)) < c$, where $\text{int}(H)$denotes the interior of H;

(c) H is a regular closed subset of K;

(d) $w(x, H) \geq c$, for all $x \in H$.

Proof. If $H = \emptyset$, then K can be covered by a finite number of open sets with weight less than c, so that $w(K) < c$. This proves (a). To prove (b), let $(U_i)_{i \in I}$ be maximal among antichains of open subsets of *K* with weight less than c. Since *I* is countable and *c* has uncountable cofinality, we have that $U = \bigcup_{i \in I} U_i$ has weight less than c. From the maximality of$(U_i)_{i \in I}$, it follows that $K \setminus H \subset \overline{U}$; then K \ $\text{int}(H) = \overline{K \setminus H} \subset \overline{U}$. To conclude the proof of (b), let us show that $w(\overline{U}) < c$. Let A be a dense Σ-subset of *K* and let D be a dense subset of $A \cap U$ with $|D| \leq w(U)$. Then $\overline{D}$is homeomorphic to a subspace of $\mathbb{R}^{w}(U)$, so that $w(\overline{U}) = w(\overline{D}) \leq w(U) < c$. To prove (c), note that H is clearly closed; moreover, by (b), the open set $K \setminus \overline{\text{int}(H)}$ has weight less than *c* and hence it is contained in $K \setminus H$. Finally, to prove (d), let V be a closed neighborhood in K of some x ∈ H. By (b), we have $w(V \setminus H) < c$. Recall that if a compact Hausdorff space is the union of not more than κ subsets of weight not greater than κ, then the weight of the space is not greater than κ. Since $w(V) \geq c$, it follows from such result that $w(V \cap H) \geq c$.

Theorem (4.1.15) [4]: If *K is a Corson compact space with* $w(K) \geq c$, *then* $\text{Ext}(C(K), c_0) = 0$. *In particular, under CH, we have* $\text{Ext}(C(K), c_0) = 0$ *for any nonmetrizable Corson compact space K.*

The fact that $\text{Ext}(C(K), c_0) \neq 0$ for a Valdivia compact space K which does not have ccc is already known . Our strategy for the proof of Theorem (4.1.15) is to use Lemma (4.1.14) to show that if K is a Corson compact space with $w(K) \geq c$ having ccc, then K satisfies property $(*)$. We start with a lemma that will be used as a tool for verifying the assumptions. Recall that a closed subset of a topological space is called regular if it is the closure of an open set (equivalently, if it is the closure of its own interior). Obviously, a closed subset of a Corson compact space is again Corson and a regular closed subset of a Valdivia compact space is again Valdivia.

Proof: By Lemma (4.1.14), it suffices to prove that if *K* is a nonempty Corson compact space such that $w(x, K) \geq$ c for all $x \in K$, then *K* satisfies property $(*)$. Since a nonempty Corson compact space *K* admits a G_δ point , this fact follows from Corollary (4.1.14) with $F = \{x\}$.

In this section we prove that $\text{Ext}(C(K), c_0) \neq 0$ for certain classes of nonmetrizable Valdivia compact spaces K and we propose a strategy for dealing with the general problem. First, let us state some results which are immediate consequences of what we have done so far.

Proposition (4.1.16)[4]: *If K is a Valdivia compact space with* $w(K) \geq c$ *and L is a compact Hausdorff space containing a nontrivial convergent sequence, then* $L \times K$ *satisfies property* $(*)$.

Proof. As we have observed, if K is a Valdivia compact space, then $C(K)$ admits a strictly increasing PRI. The conclusion follows from Corollary (4.1.13).

Proposition (4.1.17)[4]: Let *K be a Valdivia compact space admitting a* G_δ *point x with* $w(x, K) \geq c$*. Then* $\mathrm{Ext}(C(K), c_0) \neq 0$ *and, if K has ccc, then K satisfies property* $(*)$.

Proof. As mentioned before, the non-ccc case is already known. Assuming that *K* has ccc, define H as in Lemma (4.1.14) and conclude that H satisfies property $(*)$ using Corollary (4.1.18) with $F = \{x\}$.

Corollary (4.1.18)[4]: *Let K be a Valdivia compact space with* $w(K) \geq c$ *admitting a dense* Σ*-subset A such that* $K \setminus A$ *is of first category. Then* $\mathrm{Ext}(C(K), c_0) \neq 0$ *and, if* K *has ccc, then* K *satisfies property* $(*)$*.*

Proof. K has a dense subset of G_δ points. Assuming that K has ccc and defining H as in Lemma (4.1.14), we obtain that H contains a G_δ point of *K*, which implies that *K* satisfies the assumptions of Proposition (4.1.17).

Now we investigate conditions under which a Valdivia compact space K contains a homeomorphic copy of $[0, \omega] \times [0, c]$. Given an index set I and a subset J of I, we denote by $r_J : \mathbb{R}^I \to \mathbb{R}^I$ the map defined by setting $r_J(x)|_J = x|_J$ and $r_J(x)|_{I \setminus J} \equiv 0$, for all x ∈ RI. Given a subset K of RI, we say that $J \subset I$ is K-good if $r_J[K] \subset K$. It is proven that if K is a compact subset of RI and $\Sigma(I) \cap K$ is dense in K, then every infinite subset J of I is contained in a K-good set J' with $|J| = |J'|$.

Proposition (4.1.19) [4]: *Let K be a Valdivia compact space admitting a dense* Σ*-subset A such that some point of* $K \setminus A$ *is the limit of a nontrivial sequence in* K*. Then* K *contains a homeomorphic copy of*

$[0, \omega] \times [0, \omega 1]$. *In particular, assuming CH, we have that K satisfies property* $(*)$.

Proof. We can obviously assume that K is a compact subset of some $\mathbb{R}^I$and that $A = \Sigma(I) \cap K$. Since A is sequentially closed, our hypothesis implies that there exists a continuous injective map $[0, \omega] \ni n \mapsto x_n \in K \setminus A$. Let J be a countable subset of I such that $x_n|_J = x_m|_J$, for all $n, m \in [0, \omega]$ with n = m. Using and transfinite recursion, one easily obtains a family $(J_\alpha)_{\alpha \le \omega_1}$ of K-good subsets of I satisfying the following conditions:

(i) Jα is countable, for α<ω1;

(ii) $J \subset J_0$;

(iii) $J_\alpha \subset J_\beta$, for $0 \le \alpha \le \beta \le \omega_1$;

(iv) $J_\alpha = \cup_{\beta<\alpha} J_\beta$, for limit $\alpha \in [0, \omega_1]$;

(v) for all $n \in [0, \omega]$, the map $[0, \omega_1] \ni \alpha \mapsto J_\alpha \cap \operatorname{supp} x_n$ is injective.

Given these conditions, it is readily checked that the map

$$[0, \omega] \times [0, \omega_1] \ni (n, \alpha) \mapsto r_{J_\alpha}(x_n) \in K$$

is continuous and injective.

We observe that the validity of the following conjecture would imply, under CH, that $\operatorname{Ext}(C(K), c_0) \neq 0$ for any nonmetrizable Valdivia compact space K.

Conjecture. *If K is a nonempty Valdivia compact space having ccc, then either K has a G_δ point or K admits a nontrivial convergent sequence in the complement of a dense Σ-subset.*

To see that the conjecture implies the desired result, use Lemma (4.1.14) and Propositions (4.1.17) and (4.1.19), keeping in mind that a regular closed subset of a ccc space has ccc as well. The conjecture remains open, but in what follows we present an example showing that it is false if the assumption that K has ccc is removed.

Recall that a tree is a partially ordered set $(T, \leq)$ such that, for all $t \in T$, the set $(\cdot, t) = \{s \in T : s < t\}$ is well-ordered. We define a compact Hausdorff space from a tree T by considering the subspace $P(T)$ of 2^T consisting of all characteristic functions of paths of T; by a path of T we mean a totally ordered subset A of T such that $(\cdot, t) \subset A$, for all $t \in A$. It is easy to see that $P(T)$ is closed in 2^T ; we call it the path space of T.

Denote by $S(\omega_1)$ the set of countable successor ordinals and consider the tree $= \bigcup_{\alpha \in S(\omega_1)} \omega_1^\alpha$, partially ordered by inclusion. The path space $P(T)$ is the image of the injective map $\Lambda \ni \lambda \mapsto \chi_{A(\lambda)} \in 2^T$, where $\Lambda = \bigcup_{\alpha \leq \omega_1} \omega_1^\alpha$ and $A(\lambda) = \{t \in T : t \subset \lambda\}$.

Proposition (4.1.20)[4]: *If the tree T is defined as above, then its path space $P(T)$ is a compact subspace of $\mathbb{R}^T$ satisfying the following conditions:*

(i) $P(T) \cap \Sigma(T)$ is dense in $P(T)$, so that $P(T)$ is Valdivia;

(ii) P(T) has no G_δ points;

(iii) no point of $P(T) \setminus \Sigma(T)$ is the limit of a nontrivial sequence in $P(T)$.

Proof. To prove (i), note that $\chi_{A(\lambda)} = \lim\limits_{\alpha < \omega_1} \chi_{A(\lambda|_\alpha)}$ for all $\lambda \in \omega_1^{\omega_1}$. Let us prove (ii). Since $P(T)$ is Valdivia, every G_δ point of $P(T)$ must be in $\Sigma(T)$, i.e., it must be of the form $\chi_{A(\lambda)}$, with $\lambda \in \omega_1^\alpha$, $\alpha < \omega_1$. To see that $\chi_{A(\lambda)}$ cannot be a G_δ point of $P(T)$, it suffices to check that for any countable subset E of T, there exists $\mu \in \Lambda, \mu = \lambda$, such that $\chi_{A(\lambda)}$ and $\chi_{A(\mu)}$ are identical on E. To this aim, simply take $\mu = \lambda \cup \{(\alpha, \beta)\}$, with $\beta \in \omega_1 \setminus \{t(\alpha) : t \in E$ and $\alpha \in \mathrm{dom}(t)\}$. Finally, to prove (iii), let $\left(\chi_{(\lambda_n)} A\right)_{n \geq 1}$ be a sequence of pairwise distinct elements of $P(T)$ converging to some $\epsilon \in P(T)$ and note that the support of ϵ must be contained in the countable set $\bigcup_{n \neq m}(A(\lambda_n) \cap A(\lambda_m))$.

It is easy to see that, for T defined as above, the space $P(T)$ does not have ccc. Namely, setting $U_t = \{\epsilon \in P(T) : \epsilon(t) = 1\}$ for $t \in T$, we have that U_t is a nonempty open subset of $P(T)$ and that $U_t \cap U_s = \emptyset$, when $t, s \in T$ are incomparable.

List of Symbols

Symbol	page
min : minimum	1
max : maximum	1
conv: convex	3
inf : infimum	6
L_∞ : Lebesque space	6
L_2 : Hilbert space	7
$\bigoplus$: Orthogonal sum	17
diag: diagonal	19
dist : distant	23
sup : supermum	35
ext: extreme	54
str: strongly	54
WCG: weakly compactly generated	55
Supp: support	56
sgn: signal	59
WLD: weakly Lindelof determined	68
WOT: Weak operator topology	71
*SOT: Strong operator topology	71
ℓ^∞ : Hibber space	84
ccc: chain condition	86
CH: confinuum hypothesis	86
debs: dense	93
Ker: kernel	93
PRT: projectional resolution of the identity	93
int: interior	95

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