Sudan University of Science and Techno

College of Graduate Studies

On space - Time Estimates and Spectral Theory of Schrödinger Operators

حول تقديرات الزمن – الفضاء ونظرية الطيف لمؤثرات شرودنجر

A Thesis submitted in fulfillment for the

Philosophy Doctorate Degree in Mathematics

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Dedication

my special thanks to my family for their help throughout the entire doctorate program.

Full thanks to my many friends who have supported me throughout the process.

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Abstract

We give the global and Strichartz estimates for the Schrödinger maximal operators, end point maximal and the local smoothing estimates for Schrödinger equation. The singular continuous and pure point spectrum of self-adjoint extensions and Laplaceians of fractul graphs are shown with the spectral Localization in the hierarchical Anderson model. The radial positive definite function with bases of subspaces, property of x-positive definiteness, general Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities are investigated. The space time estimates and the negative spectrum of the three dimentional hierarchical Schrödinger operaters with pure point spectrum interactions are discussed.

الخلاصة

اعطينا التقديرات العالمية وستريشارتزلاجل المؤثرات الاعظمية لشرودنجر واعظمية النقطة الاخيرة ؛ وتقديرات الملسان الموضعي لمعادلة شرودنجر . اوضحنا الاستمرارية الشاذة وطيف النقطة البحت , لتمديدات المرافق ــ الذاتي واللابلسينات والبيانات الكسرية مع الموضوعية الطيفية في نموذج هيرارشيكال اندرسون. تمت مناقشة الدالة المحددة الموجبة الاحادية مع الاساس للفضاء الجزئي والخاصية المحددة x ـ الموجبة ومتباينات سويكل – ليب – روزنبلم و ليب – ثيرنج.درسنا تقديرات زمان المكان والطيف السالب لمؤثرات هيرارشيكال شرودنجر للابعاد الثلاثة مع تدخلات طيف النقطة البحت.

Introduction

In higher dimensions, we show that $\sup_{t}|e^{it\Delta}f|$and $\sup_{0<t<1}|e^{it\Delta}f|$ are bounded from $H^s(R^n)$ to $L^q(R^n)$ only if $s \geq \frac{1}{2} - \frac{1}{2(n+1)}$.We also show that the Schrödinger maximal operator $\sup_{0<t<1}|e^{it\Delta}f|$is bounded from$H^s(R^n)$ to$L^2_{loc}(R^n)$when$s > s_0$ if and only if it is bounded from$H^s(R^n)$ to $L^2(R^n)$ when $s > 2s_0$. A corollary isthat $\sup_{0<t<1}|e^{it\Delta}f|$is bounded from$H^s(R^2)$ to $L^2(R^2)$when s >3/4.

When n = 2, we unconditionally improve the rangefor which the mixed norm estimates hold.We shall show that a symmetric operator with infinite deficiency indices and some gap has self-adjoint extensions with non-empty singular continuous spectrum.

We establish the pure point spectrum of Laplacians o two point self-similar fractal graph.We show that a large class of hierarchical Anderson models withspectral dimension$d \geq 2$ has only pure point spectrum.

We strengthen the fixed time estimates due to Fefferman and Stein, and Miyachi. As anessential tool we establish sharpL^p space-time estimates (local in time) for the samerange of p.We show mixed norm space-time estimates for solutions of the Schrodingerequation, with initial data inL^p Sobolev (or Besov) spaces, and clarify the relation withadjoint restriction.

A number of results on radial positive definite functions onR^n related to Schoenberg's integral representationtheorem are obtained. They are applied to the study of spectral properties of self-adjoint realizationsof two- and three-dimensional Schrödinger operators with countably many point interactions.

These classical inequalities allow one to estimate the number of negative eigen-values and the sums $S_\gamma = \sum|\gamma_i|^\gamma$ for a wide class of Schrodinger operators. We provide a detailed proof of these inequalities for operators on functions in metric spaces using the classical Lieb approach based on the Kac-Feynman formula. The main goal is a new set of examples which include perturbations of the Anderson operator, operators on free, nilpotent and solvable groups, operators on quantum graphs, Markov processes with independent increments. Since the spectral dimension of the operator under consideration can be an arbitrary positive number, the model allows a continuous phase transitionfrom recurrent to transient underlying Markov process. This transition is also studied.

Chapter 1
Global and Local Smoothing Estimates

The Schrödinger equation, $i\partial_t u + \Delta u = 0$, in R^{n+1}, with initial datum f contained in a Sobolev space $H^s(R^n)$, has solution $e^{it\Delta}f$. We give sharp conditions under which $\sup_t |e^{it\Delta}f|$isbounded from $H^s(R)$to$L^q(R)$ for all q, and give sharp conditions under which $\sup_{0<t<1} |e^{it\Delta}f|$is bounded from $H^s(R)$ to $L^q(R)$ for all$q \neq 2$.We show that the Schrödinger operator$e^{it\Delta}$ is bounded from $W^{\alpha,q}(R^n)$to $L^q(R^n \times [0,1])$for all$\alpha > 2n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{q}andq \geq 2+\frac{4}{(n+1)}$. this is almost sharp with respect to the Sobolev index.

Section (1.1): Schrödinger Maximal Operator and Global Estimates:

The Schrödinger equation, $i\partial_t u + \Delta u = 0$, in R^{n+1}, with initial datum f contained in a Sobolev space $H^s(R^n)$, has solution $e^{it\Delta}f$which can be formally written as

$$e^{it\Delta}f(x) = \int \hat{f}(\xi)e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2)}d\xi. \tag{1}$$

We will consider the Schrödinger maximal operators $S^*$and S^{**}, defined by

$$S^*f = \sup_{0<t<1}|e^{it\Delta}f| \text{ and } S^{**}f = \sup_{t\in R}|e^{it\Delta}f|.$$

The minimal regularity of $f$under which $e^{it\Delta}f$converges almost everywhere to f, as $t$tends to zero, has been studied extensively. By standard arguments, the problem reduces to the minimal value of s for which

$$\|S^*f\|_{L^q(B^n)} \leq C_{n,q,s}\|f\|_{H^s(R^n)} \tag{2}$$

holds, where $B^n$is the unit ball in R^n.

In two dimensions, that is one spatial dimension, Carleson [4] (see also [10]) showed that (2) holds when $s \geq 1/4$. Dahlberg and Kenig [6] showed that this is sharp in the sense that it is not true when $s < 1/4$.

In three dimensions, significant contributions have been made by Bourgain [1, 2], Moyua, Vargas and Vega [12, 13], and Tao and Vargas [21, 22]. The best known result is due to Lee [11] who showed that (2) holds when $s > 3/8$.

In higher dimensions, Sjölin [15] and Vega [23, 24] independently showed that (2) holds when $s > 1/2$. It is conjectured that, in all dimensions, the minimal value of $s$for which (2) holds is $1/4$.

Replacing the unit ball B^nin (2) by the whole space R^n, we consider the global estimates

$$\|S^*f\|_{L^q(R^n)} \leq C_{n,q,s}\|f\|_{H^s(R^n)} \tag{3}$$

and

$$\|S^{**}f\|_{L^q(R^n)} \leq C_{n,q,s}\|f\|_{H^s(R^n)}. \tag{4}$$

In one spatial dimension, Kenig, Ponce and Vega [9] proved that (4) holds when $q = 4$ and $s = \frac{1}{4}$. This was extended by Gülkan [7] who proved that (4) holds when $q \in [4,\infty)$ if and only if $s \geq 1/2 - 1/q$, and it is well known that (4) holds when $q = \infty$if and only if $s > 1/2$ (see [19]). Sjölin [16] proved that if $q = 2$, then (4) does not hold for any$s$, and we will show that this is also the case when $q \in (2,4)$. Thus, we have the following theorem.

Theorem (1.1.1)[25]: Let $n = 1$. Then (4) holds if and only if $q \in [4,\infty)$ and $s \geq 1/2 - 1/q$, *or* $q = \infty$ and $s > 1/2$.
The following theorem extends a result of Vega [23, 8] (see also [17]) by the endpoint $s = 1/q$ in the range $q \in (2,4)$.
Theorem (1.1.2)[25]: *Let* $n = 1$ and $q \in (2,\infty)$*: Then (3) holds if and only if* $s \geq \max\{1/q, 1/2 - 1/q\}$.
Vega [23, 8] (see also [16]) proved that (3) holds when $q = 2$ and $s > 1/2$, and this is not true when $q = 2$ and $s < 1/2$, or for any value of s when $q < 2$. As in Theorem (1.1.1), when $q = \infty$, (3) holds if and only if $s > 1/2$ (see [19]). Thus, in order to have complete results in Theorem (1.1.2), the only case that remains undecided is $q = 2, s = 1/2$.
In higher dimensions, we show that (3) holds only if

$$s \geq \frac{n}{2(n+1)}.$$

We note that the minimal s is thus strictly greater than $1/4$ when $n \geq 2$. A plausible conjecture is that these are indeed the minimal values of s that can appear in (3).
Throughout, C will denote an absolute constant whose value may change from line to line.
First, we consider one spatial dimension, and extend the argument of Carleson as in [14]. We employ the Kolmogorov–Seliverstov–Plessner method and the following two lemmas. The first is proved by a very slight modification of a lemma due to Sjölin [20]; The second is proved by refining the ideas of Carleson.
Lemma (1.1.3)[25]: *Let* $x, t \in R$ *and* $\alpha \in [1/2,1)$. *Then there is a constant* C *such that*

$$\left|\int_R \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}d\xi\right| \leq \frac{C}{|x|^{1-\alpha}}.$$

Lemma (1.1.4)[25]: *Let* $x \in R, t \in [-1,1]$ *and* $\alpha \in [1/2,1]$. *Then there is a constant* C *such that*

$$\left|\int_R \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}d\xi\right| \leq \frac{C}{|x|^{\alpha}}.$$

Proof. Splitting the integral in two and taking the complex conjugate if necessary we can suppose that $x > 0$, and consider the integral over $(0,\infty)$. When $x \leq 4$ and $\alpha < 1$, we are done by Lemma (1.1.3), so we can suppose that $x \geq 4$ and $1/x \leq C/x^\alpha$.
When $t \leq 0$, there exist $c_1, c_2 \in (0,1)$ such that

$$\left|\int_0^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}d\xi\right| \leq \left|\int_0^{c_1} \cos(2\pi(x\xi - t\xi^2))d\xi\right| + \left|\int_0^{c_2} \sin(2\pi(x\xi - t\xi^2))d\xi\right|,$$

by the Bonnet form of the second mean value theorem for integrals. The derivative of the phase, $x - 2t\xi$, is monotone, and bounded below by x, so by van der Corput's lemma,

$$\left|\int_0^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}d\xi\right| \leq \frac{C}{x} \leq \frac{C}{x^\alpha}.$$

and we are done.
Now we suppose that $t > 0$, and make the change of variables $\xi \to \xi + 1$, so that

$$\left|\int_0^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{(1+|\xi|)^\alpha}\,d\xi\right| = \left|\int_1^\infty \frac{e^{2\pi i\left((x+2t)\xi - t\xi^2\right)}}{\xi^\alpha}\,d\xi\right|.$$

As $x + 2t > x$, it will suffice to show that

$$\left|\int_1^\infty \frac{e^{2\pi i(x\xi - t\xi^2)}}{\xi^\alpha}\,d\xi\right| \le \frac{C}{x^\alpha}.$$

Changing variables again, $\xi \to \sqrt{t}\xi$, and denoting $2A = x/\sqrt{t}$, we are required toshow that

$$\frac{1}{\sqrt{t}^{1-\alpha}}\left|\int_{\sqrt{t}}^\infty \frac{e^{2\pi i(2A - \xi^2)}}{\xi^\alpha}\,d\xi\right| \le \frac{C}{x^\alpha}.$$

Note that $A > 2$, as we have that $x \ge 4$.

Consider first the integral over $(\sqrt{t}, A/2)$. By the change of variables, $\xi \to A\xi$, we are required to show that

$$\frac{1}{x^{1-\alpha}}\left|\int_{x/2}^{A^2/2} \frac{e^{2\pi i(2\xi - \xi^2/A^2)}}{\xi^\alpha}\,d\xi\right| \le \frac{C}{x^\alpha}.$$

The derivative of the phase, $2 - 2\xi/A^2$, is bounded below by one on $(x/2, A^2/2)$, so that, by the mean value theorem and van der Corput's lemma,

$$\frac{1}{x^{1-\alpha}}\left|\int_{x/2}^{A^2/2} \frac{e^{2\pi i(2\xi - \xi^2/A^2)}}{\xi^\alpha}\,d\xi\right| \le \frac{C}{x} \le \frac{C}{x^\alpha},$$

and we are done.

Finally, we are required to show that

$$\frac{1}{\sqrt{t}^{1-\alpha}}\left|\int_{A/2}^\infty \frac{e^{2\pi i(2A\xi - \xi^2)}}{\xi^\alpha}\,d\xi\right| \le \frac{C}{x^\alpha}.$$

By the mean value theorem, and the fact that modulus of the second derivative of the phase is bounded below by one,

$$\frac{1}{\sqrt{t}^{1-\alpha}}\left|\int_{A/2}^\infty \frac{e^{2\pi i(2A\xi - \xi^2)}}{\xi^\alpha}\,d\xi\right| \le \frac{C\sqrt{t}^{2\alpha-1}}{x^\alpha}\left|\int_{A/2}^c e^{2\pi i(2A\xi - \xi^2)}\,d\xi\right| \le \frac{C}{x^\alpha},$$

and we are done.

The following theorem is an endpoint improvement of result of Vega [23, 8] (see also [17]) in the range (2*;* 4).

Theorem (1.1.5)[25]:*Let $n = 1$. If $q \in [4, \infty)$and $s \ge 1/2 - 1/q$, then (4) holds. If $q \in (2, \infty)$and $s \ge \max\{1/q, 1/2 - 1/q\}$, then (3) holds.*

Proof. By duality, it will suffice to show that

$$\left|\int_R e^{it(x)\Delta} f(x) w(x)\,dx\right|^2 \le C_q \|f\|_{H^s(R)}^2 \|w\|_{L^{q'}(R)}^2$$

for all positive $w \in L^{q'}(R)$, where the measurable function $t$maps into R when weare considering the bound (4) and into $(0,1)$ when we consider (3).

By Fubini's theorem and the Cauchy–Schwarz inequality, the left hand side of this inequality is bounded by

$$\int_R \left|\hat{f}(\xi)\right|^2 (1+|\xi|)^{2s} d\xi \int_R \left|\int_R e^{2\pi i(x\xi - t(x)\xi^2)} w(x)dx\right|^2 \frac{d\xi}{(1+|\xi|)^{2s}}.$$

Thus, by writing the squared integral as a double integral, it will suffice to show that

$$\int_R \int_R \int_R e^{2\pi i\left((x-y)\xi - (t(x)-t(y))\xi^2\right)} w(x)w(y)dxdy \frac{d\xi}{(1+|\xi|)^{2s}} \le C_p \|w\|^2_{L^{q'}(R)}. \qquad (5)$$

By Lemma (1.1.3), we have

$$\left|\int_R \frac{e^{2\pi i\left((x-y)\xi - (t(x)-t(y))\xi^2\right)}}{(1+|\xi|)^{2s}} d\xi\right| \le \frac{C}{|x-y|^{1-2s}}$$

when $t$takes values in R, and $2s \in [1/2,1)$, and by Lemmas (1.1.3) and (1.1.4), we have

$$\left|\int_R \frac{e^{2\pi i\left((x-y)\xi - (t(x)-t(y))\xi^2\right)}}{(1+|\xi|)^{2s}} d\xi\right| \le \frac{C}{|x-y|^{\max\{2s,1-2s\}}}$$

when $t$takes values in $(0,1)$. Thus, by Fubini's theorem, the left hand side of (5) is bounded by a constant multiple of

$$\int_R \int_R \frac{w(x)w(y)}{|x-y|^{1-2s}} dxdy$$

in the first case, and

$$\int_R \int_R \frac{w(x)w(y)}{|x-y|^{\max\{2s,1-2s\}}} dxdy$$

in the second. Finally, by Hölder's inequality and the Hardy–Littlewood–Sobolev inequality, these are bounded by

$$\|w\|_{L^{q'}(R)} \left\|\int_R \frac{w(x)}{|x-\cdot|^{1-2s}} dx\right\|_{L^q(R)} \le C_q \|w\|^2_{L^{q'}(R)},$$

where $s = 1/2 - 1/q$and $q \ge 4$ when we are considering the bound in (4), and

$$\|w\|_{L^{q'}(R)} \left\|\int_R \frac{w(x)}{|x-\cdot|^{\max\{2s,1-2s\}}} dx\right\|_{L^q(R)} \le C_q \|w\|^2_{L^{q'}(R)},$$

where $s = \max\{1/q, 1/2 - 1/q\}$and $q > 2$ when we consider (3).

In higher dimensions, we simply interpret the known results. By modifying very slightly the proof of Theorem 2.2 in [21] due to Tao and Vargas, the following result is proved using bilinear restriction estimates.

Theorem (1.1.6)[25]:_Let_ $q\in\left(2+\frac{4}{n+1},\infty\right]$, $p\in\left(\max\left\{q,\frac{2q}{nq-2(n+1)}\right\},\infty\right]$, _and_ $s>n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{p}$. _Then there exists a constant_ $C_{n,q,p,s}$ _such that_

$$\left\|e^{it\Delta}f\right\|_{L^q(R^n,L^p(R))}\le C_{n,q,p,s}\|f\|_{H^s(R^n)}.$$

As usual, we define ∂_t^α by $\widehat{\partial_t^\alpha g}(\tau)=(2\pi|\tau|)^\alpha\hat{g}(\tau)$, where $\alpha>0$. Observingthat $\partial_t^\alpha e^{it\Delta}f=e^{it\Delta}f_\alpha$, where b $\hat{f}_\alpha(\xi)=(4\pi^2|\xi|^2)^\alpha\hat{f}(\xi)$, and applying the Sobolevimbedding theorem with $\alpha>1/p$, we recover their theorem in the following corollary.

Corollary(1.1.7)[25]: If $q\in\left(2+\frac{4}{n+1},\infty\right]$ and $s>n(1/2-1/q)$, then (3) and (4) hold.

We will see below that these kind of global bounds do not hold when $q<2$. Thus, for completeness, we provide sufficient conditions, albeit not sharp, for the remaining values of $q$in (3).

Theorem (1.1.8) [25]: If$q\in\left[2,2+\frac{4}{n+1}\right]$and $s>3/q-1/2$, then (3) holds.

Proof. Carbery [3] and Cowling [5] independently proved that if $q=2$ and $s>1$, then (3) holds. Considering $H^s$to be a weighted L^2space, we can interpolate between this and the bound in Corollary 1 to get the result.

We consider one spatial dimension and complete the proof of Theorem (1.1.1). The novelty in the following is that if $n=1$ and $q\in(2,4)$, then (4) cannot hold for any value of s.

Theorem (1.1.9)[25]:_Let_ $n=1$. _If (4) holds, then_ $q\in[4,\infty)$_and_ $s\ge 1/2-1/q$, _or_ $q=\infty$ _and_ $s>1/2$.

The following theorem is due to Sjölin [17], but it will also follow easily from the following proof of Theorem (1.1.9).

Theorem (1.1.10)[25]:_Let_ $n=1$. _If (3) holds then_ $q\in[2,\infty)$_and_ $s\ge\max\{1/q,1/2-1/q\}$, _or_ $q=\infty$ _and_ $s>1/2$.

Proof. By a change of variables,

$$S^{**}f(x)=\sup_{t\in R}\left|\frac{1}{2\pi}\int\hat{f}\left(\frac{\xi}{2\pi}\right)e^{i(x\xi-t\xi^2)}d\xi\right|.$$

Define $A=[N,N+N^\lambda]$, where $N\gg 1$ and $\lambda\in(-\infty,1]$, and consider $f_A$defined by$\hat{f}_A(\xi/2\pi)=e^{-iN^{-\lambda}\xi}\chi_A(\xi)$. We will show that for a range of values of x, a time $t(x)$ can be chosen so that the phase,

$$\phi_x(\xi)=(x-N^{-\lambda})\xi-t(x)\xi^2,$$

is roughly constant on A. With the phase roughly constant, we have

$$S^{**}f_A(x)\ge C\left|\int_A e^{i\left((x-N^{-\lambda})\xi-t(x)\xi^2\right)}d\xi\right|\ge C|A|.$$

As $A$is an interval of length N^λ, in order to insure that the phase is roughly constant, we impose the condition $|\phi_x'(\xi)|\le N^{-\lambda}$on A. This insures that for all N and λ, there exists a $\theta_x$such that

$$\theta_x-1/2\le\phi_x(\xi)\le\theta_x+1/2.$$

As $\phi_x'(\xi)=x-N^{-\lambda}-2t(x)\xi$, the condition can be rewritten as

$$\frac{x-2N^{-\lambda}}{2\xi}\le t(x)\le\frac{x}{2\xi}$$

for all $\xi\in A$. Define $a$and $b$by

$$a(x) = \sup_{\xi \in A} \frac{x - 2N^{-\lambda}}{2\xi} \text{ and } b(x) = \inf_{\xi \in A} \frac{x}{2\xi}.$$

To be able to choose the time $t(x)$ we require that $a(x) \le b(x)$. This is clear when $x \in [0{,}2N^{-\lambda}]$, so we suppose that $x > 2N^{-\lambda}$. Now, when $x > 2N^{-\lambda}$,

$$a(x) = \frac{x - 2N^{-\lambda}}{2N} \text{ and } b(x) = \frac{x}{2(N + N^{\lambda})},$$

so that we can choose a $t(x)$ when

$$\frac{x - 2N^{-\lambda}}{2N} \le \frac{x}{2(N + N^{\lambda})}.$$

This condition can be rewritten as $x \le 2N^{-\lambda} + 2N^{1-2\lambda}$, so we will consider the set $E = [0, N^{1-2\lambda}]$.

As $S^{**}f_A \ge C|A|$ on E, we see that

$$\|S^{**}f_A\|_{L^q(R)} \ge C|A||E|^{1/q}.$$

On the other hand,

$$\|f_A\|_{H^s(R)} \le C\left(\int_A (1 + |\xi|)^{2s}\right)^{\frac{1}{2}} \le C|A|^{1/2}\left(1 + N + N^{\lambda}\right)^s,$$

so that, as $\|S^{**}f_A\|_{L^q(R)} \le C\|f_A\|_{H^s(R)}$, we have

$$|A||E|^{1/q} \le C|A|^{1/2}\left(1 + N + N^{\lambda}\right)^s.$$

Recalling that $|A| = N^{\lambda}$ and $|E| = N^{1-2\lambda}$, we see that

$$N^{\frac{\lambda}{2}} N^{\frac{1-2\lambda}{q}} \le CN^s,$$

so that, letting N tend to infinity,

$$s \ge \frac{1}{q} + \lambda\left(\frac{1}{2} - \frac{2}{q}\right)$$

for all $\lambda \in (-\infty, 1]$. When $q < 4$, we let λ tend to $-\infty$ to obtain a contradiction for all s. Letting $\lambda = 1$ we recover the fact that $s \ge 1/2 - 1/q$.

Finally, by a well–known counterexample (see [19]), $s > 1/2$ is necessary when $q = \infty$, and we are done.

In order to prove results for S^*, we have the added requirement that

$$[a(x), b(x)] \cap (0{,}1) \ne \emptyset$$

for all $x \in E$. We have that $a(x) < 1$ when

$$\frac{x - 2N^{-\lambda}}{2N} < 1,$$

which we rewrite as

$$x < 2N + 2N^{-\lambda}.$$

When $\lambda < 0$, this is an added restriction so we reanalyze in this case. Redefining a smaller $E = [0{,}2N + 2N^{-\lambda}]$, we see that

$$N^{\lambda/2}\left(N + N^{-\lambda}\right)^{1/q} \le CN^s$$

for all $\lambda \in (-\infty, 0]$, so that, letting N tend to infinity,

$$s \ge \frac{1}{q} + \frac{\lambda}{2} \tag{6}$$

and

$$s \geq \lambda\left(\frac{1}{2}-\frac{1}{q}\right). \quad (7)$$

When $q < 2$, we see by (7) that, letting λ tend to $-\infty$, we have a contradiction for all s. If we let $\lambda = 0$ in (6), we see that $s \geq 1/q$, and from before, when $\lambda = 1$, we have that $s \geq 1/2 - 1/q$.
Again, by the well-known counterexample (see [19]), $s > 1/2$ is necessary when $q = \infty$, and so we are done.

Remark (1.1.11)[25]: We note that taking $\lambda = 1/2$ in the above proof, $E = [0,1]$, the time $t(x)$ can be chosen to be a member of $(0,1)$ for all $x \in E$, and $s \geq 1/4$ for all q, so we recover the fact that $s \geq 1/4$ is necessary in (2). It is easy to generalize this to higher dimensions. Indeed, it can be shown that g defined by

$$\hat{g} = \sum_{j=2}^{\infty} 2^{-\alpha j}\chi_{[2^{2j},2^{2j}+2^{j-3}]\times[1,9/8]^{n-1}},$$

where $\alpha \in (2s + 1/2,1)$ and $s < 1/4$, is a member of $H^s(R^n)$ such that $e^{it\Delta}g$ diverges on the set $[8/9,1]^n$ as t tends to zero.

We now consider higher dimensions. A corollary of the following theorems is that the minimal value of s that can appear in (3) or (4) is greater than or equal to $\frac{1}{2}-\frac{1}{2(n+1)}$. Again, both theorems will follow from the same proof.

It can be seen by scaling that if $q < 2$ or $s < n(1/2 - 1/q)$, then (4) does not hold. Theorem (1.1.12) is that if $q \in (2,2 + 2/n)$, then (4) cannot hold for any value of s. That q cannot equal 2 is due to Sjölin [16].

Theorem (1.1.12) [25]: *If (4) holds, then* $q \in \left[2 + \frac{2}{n}, \infty\right)$ *and* $s \geq n(1/2 - 1/q)$, *or* $q = \infty$ *and* $s > n/2$.

Theorem (1.1.13) [25]: *If (3) holds, then* $q \in [2, \infty)$ *and* $s \geq \max\{1/q, n(1/2 - 1/q)\}$, *or* $q = \infty$ *and* $s > n/2$.

Proof. We consider S^{**} and argue as in the proof of Theorem (1.1.9). Define A by

$$A = \left[N, N + N^{\lambda}\right]^n,$$

where $N \gg 1$ and $\lambda \in (-\infty, 1]$, and consider f_A defined by $\hat{f}_A(\xi/2\pi) = e^{-i\widetilde{N}_\lambda\cdot\xi}\chi_A(\xi)$, where $\widetilde{N}_\lambda = \left(N^{-\lambda}, \dots, N^{-\lambda}\right)$.

In order to show that the phase in (1) is roughly constant on A, we will need that the partial derivatives of the phase are small. we require that

$$\left|x_j - N^{-\lambda} - 2t(x)\xi_j\right| \leq N^{-\lambda},$$

for all $j = 1, \dots, n$. Rewriting this condition, for each x we need to choose a $t(x)$ so that

$$\frac{x_j - 2N^{-\lambda}}{2\xi_j} \leq t(x) \leq \frac{x_j}{2\xi_j}$$

for all $\xi \in A$ and $j = 1, \dots, n$. Define a and b by

$$a(x) = \sup_{1\leq j\leq n}\sup_{\xi\in A}\frac{x_j - 2N^{-\lambda}}{2\xi_j} \quad \text{and} \quad b(x) = \inf_{1\leq j\leq n}\inf_{\xi\in A}\frac{x_j}{2\xi_j}.$$

To be able to choose the time $t(x)$ we need that $a(x) \leq b(x)$. As before, we require that $x_j \geq 0$ and

$$\frac{x_j - 2N^{-\lambda}}{2N} \leq \frac{x_k}{2(N + N^{\lambda})},$$

for all $j, k = 1, \dots, n$. We rewrite this as

$$0 \leq x_j \leq 2N^{-\lambda} + \frac{N}{2(N + N^{\lambda})} x_k$$

for all $j, k = 1, \dots, n$. Now, the set Edefined by these conditions, is the convex solid body with vertices $(0, \dots, 0), 2(N^{1-2\lambda} + N^{-\lambda})(1, \dots, 1)$, and $2N^{-\lambda} e_j$for all $j = 1, \dots, n$, where $e_j$are the standard basis vectors. Thus,

$$|E| \geq CN^{-\lambda(n-1)} N^{1-2\lambda}.$$

As $S^{***} f_A \geq C|A|$on E, we see that

$$\|S^{**} f_A\|_{L^q(R^n)} \geq C|A||E|^{1/q}.$$

As before,

$$\|f_A\|_{H^s(R^n)} \leq C \left(\int_A (1 + |\xi|)^{2s} \right)^{1/2} \leq C|A|^{1/2} (1 + N + N^{\lambda})^s,$$

so that, as $\|s^{**} f_A\|_{L^q(R^n)} \leq C\|f_A\|_{H^s(R^n)}$, we have

$$C|A||E|^{1/q} \leq C|A|^{1/2} (1 + N + N^{\lambda})^s.$$

Recalling that $|A| = N^{n\lambda}$and $|E| \geq CN^{1-(n+1)^{\lambda}}$, we see that

$$N^{\frac{n\lambda}{2}} N^{\frac{1-(n+1)\lambda}{q}} \leq CN^s$$

for all $\lambda \in (-\infty, 1]$, so that

$$s \geq \frac{1}{q} + \lambda \left(\frac{n}{2} - \frac{n+1}{q} \right).$$

When $q < 2 + 2/n$, we let λtend to $-\infty$to obtain a contradiction for all s, and letting $\lambda = 1$ we recover the fact that $s \geq n(1/2 - 1/q)$. We also note for later thatby letting $\lambda = 0$, we have $s \geq 1/q$.

By a well-known counterexample (see [19]), $s > n/2$ is necessary when $q = \infty$, so we have finished the proof of Theorem (1.1.12).

In order to prove results for S^*, we have the added requirement that

$$[a(x), b(x)] \cap (0,1) \neq \emptyset$$

for all $x \in E$. Now, we can ensure that $a(x) < 1$ when

$$\frac{x_j - 2N^{-\lambda}}{2N} < 1$$

for all $j = 1 \dots n$, which we rewrite as

$$x_j < 2N^{-\lambda} + 2N.$$

When $\lambda < 0$, this is an added restriction so we reanalyze the case when λtends to negative infinity. As before, we consider the set Edefined by

$$0 \leq x_j \leq 2N^{-\lambda} + \min\left\{ \frac{Nx_k}{N + N^{\lambda}}, 2N \right\}$$

for all $j, k = 1 \dots n$. It is clear from here that

$$|E| \geq CN^{-\lambda n},$$

so that, as before,

$$N^{n\lambda/2}N^{-n\lambda/q} \le CN^s.$$

Letting N tend to infinity, we have

$$s \ge n\lambda\left(\frac{1}{2}-\frac{1}{q}\right),$$

so that when $q < 2$, we can let λ tend to $-\infty$ to obtain a contradiction for all s.

From before we have that $s \ge n(1/2 - 1/q)$ and $s \ge 1/q$ are necessary conditions, and by the well-known counterexample (see [19]), $s > n/2$ is necessary when $q = \infty$, and so we are done.

Section (1.2): Schrödinger Equation and Local Smoothing Estimate:

The solution to the wave equation, $\partial_{tt}u = \Delta u$, with initial data $u(\cdot,0) = f$ and $u'(\cdot,0) = 0$, can be formally written as the real part of

$$e^{it\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i(x\cdot\xi - t|\xi|)}d\xi. \qquad (8)$$

Let $\|\cdot\|_{q,\alpha}$ denote the inhomogeneous Sobolev norm with α derivatives in $L^q(\mathbb{R}^n)$. J.C. Peral [39] proved that for any fixed time t and $q \in (1,\infty)$,

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^q(\mathbb{R}^n)} \le C_{t,q}\|f\|_{q,\alpha}$$

for all $\alpha \ge (n-1)\left|\frac{1}{2}-\frac{1}{q}\right|$, and this is sharp. Sogge [41] conjectured that

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^q(\mathbb{R}^n\times[1,2])} \le C_{q,\alpha}\|f\|_{q,\alpha}$$

for all $\alpha > (n-1)\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{q}$ and $q > 2+\frac{2}{n-1}$. This is known as the local smoothing conjecture due to the potential gain of $1/q$ derivatives.

In two spatial dimensions, Mockenhaupt, Seeger and Sogge [38] showed that the local smoothing estimate holds at the critical exponent $q = 4$ for all $\alpha > 1/8$, and this was improved by Bourgain [2], Tao and Vargas [22], and Wolff [45] to $\alpha > 5/44$.

Moving away from the critical exponent, but remaining in two spatial dimensions, Wolff [44] proved the (almost) sharp estimate in the range $q > 74$, and Łaba and Wolff [33] generalized this to higher dimensions. Garrigós and Seeger [32] have recently refined their arguments, so that, in higher dimensions for example, the (almost) sharp estimate holds in the range

$$q > 2 + \frac{8}{n-3}\left(1-\frac{1}{n+1}\right).$$

The Schrödinger equation, $i\partial_t u + \Delta u = 0$, with initial datum f has solution $e^{it\Delta}f$ which can be formally written as

$$e^{it\Delta}f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i\left(x\cdot\xi - 2\pi t|\xi|^2\right)}d\xi. \qquad (9)$$

Miyachi [37] (see also [31]) proved that for any fixed time t and $q \in (1,\infty)$,

$$\left\|e^{it\Delta}f\right\|_{L^q(\mathbb{R}^n)} \le C_{t,\alpha}\|f\|_{q,\alpha}$$

for all $\alpha \ge 2n\left|\frac{1}{2}-\frac{1}{q}\right|$, and this is sharp. When $n \ge 2$, square function estimates (see [27, 34, 36]) yield

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[1,2])} \le C_{q,\alpha}\|f\|_{q,\alpha}$$

for all $\alpha > 2n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{q}$and $q > 2+4/n$. We see that averaging locally in time yields a gain of $2/q$derivatives.

We extend the range of $q$by taking advantage of all $n+1$ dimensions of curvature. This also allows us to treat the $n=1$ case for which we obtain almost sharp estimates. In higher dimensions, it may be possible to extend the range to $q > 2+2/n$, and we shall see later that this would follow from the restriction conjecture for paraboloids.

Theorem (1.2.1) [46]: *Let* $q > 2+\frac{4}{n+1}$*and* $\alpha > 2n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{q}$. *Then there exists a constant* $C_{q,\alpha}$*suchthat*

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[0,1])} \le C_{q,\alpha}\|f\|_{q,\alpha}.$$

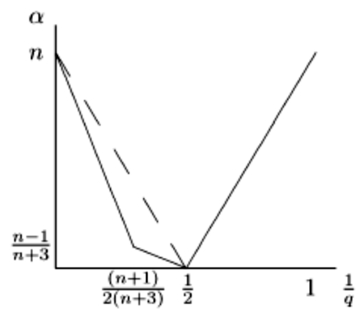

Fig. 1. Region of local smoothing in Corollary (1. 2. 2)

Although there is a formal similarity between this and the estimates of Wolff et al., the question for the Schrödinger equation is not as deep, and the arguments will bear no resemblance. An obvious difference is that the wave operator, for finite time, is a local operator, whereas the Schrödinger operator is not. We will see however, that one can decompose the initial data so that the Schrödinger operator, for finite time, may essentially be treated as a local operator.

Before proceeding further, we should mention that there are estimates for the Schrödinger equation, independently due to Sjölin [15], Vega [23, 24], and Constantin and Saut [29], which are more deserving of the description 'local smoothing.' They proved that

$$\|e^{it\Delta}f\|_{L^2(\mathbb{B}^n\times[0,1])} \le C_s\|f\|_{H^{-1/2}(\mathbb{R}^n)},$$

where $\mathbb{B}^n$is the unit ball in $\mathbb{R}^n$, and $\|\cdot\|_{H^s(\mathbb{R}^n)}$denotes $\|\cdot\|_{2,\alpha}$. Thus, the solution is locally half a derivative smoother than the initial datum. We will see later that this is equivalent up to endpoints with the global estimate

$$\|e^{it\Delta}f\|_{L^2(\mathbb{R}^n\times[0,1])} \le C_s\|f\|_{L^2(\mathbb{R}^n)},$$

which we will refer to as simply the conservation of charge.

Interpolating between this and the bound in Theorem (1. 2. 1)yields the following corollary. In one spatial dimension, it is almost sharp in the range $q\in[1,\infty]$, and in higher dimensions it is almost sharp in the ranges $q\in[1,2]$and $q\in\left[2+\frac{4}{n+1},\infty\right]$.

Corollary (1.2.2)[46]: *Let* $q\in[1,\infty]$*and* $\alpha > \max\left\{2n\left(\frac{1}{q}-\frac{1}{2}\right),(n-1)\left(\frac{1}{2}-\frac{1}{q}\right),2n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{q}\right\}$. *Thenthere exists a constant* $C_{T,\alpha}$*such that*

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[-T,T])} \le C_{T,\alpha}\|f\|_{q,\alpha}$$

(see fig 1)
We will consider the minimal value of s for which

$$\left\| \sup_{0<t<1} \left| e^{it\Delta} f \right| \right\|_{L^2(\mathbb{B}^n)} \le C_{n,s} \|f\|_{H^s(\mathbb{R}^n)} \quad (10)$$

holds. By standard arguments, the estimate implies the almost everywhere convergence of $e^{it\Delta}f$ to f, as t tends to zero. The minimal s for which the global bound

$$\left\| \sup_{0<t<1} \left| e^{it\Delta} f \right| \right\|_{L^2(\mathbb{R}^n)} \le C_{n,s} \|f\|_{H^s(\mathbb{R}^n)} \quad (11)$$

holds, has also been considered in connection with the well-posedness of certain initial value problems (see [8]).
In one spatial dimension, Carleson, Kenig and Ruiz [4, 10] showed that (10) holds when $s \ge 1/4$, and Dahlberg and Kenig [6] showed that this is sharp. Vega [8, 23] (see also [16]) showed that the global bound (4) holds when $s > 1/2$, and this is also sharp.
In higher dimensions, it was independently proven by Sjölin [15] and Vega [24] that (10) holds when $s > 1/2$, and the bound cannot hold when $s < 1/4$. Carbery [3] and Cowling [5] independently showed that (11) holds when $s > 1$, and in this case, the bound cannot hold when $s < 1/2$. It is conjectured that, the minimal value of s for which (10) holds is $1/4$, and the minimal value for which (11) holds is $1/2$.
We will put these results and conjectures in proving the following theorem.
Theorem (1.2.3) [46]: (10) holds for $s > s_0$ ⇔(11) holds for $s > 2s_0$.
In two spatial dimensions, more was known for the local bound than for the global bound. Bourgain [1] showed that there exists an s strictly less that $1/2$ for which (10) holds, and this was improved by Moyua, Vargas and Vega [13], and Tao and Vargas [21, 22]. The best known result is due to S. Lee [11], who showed that (10) holds when $s > 3/8$.
Therefore, as a consequence of the equivalence, we have the following corollary, which improves the result of Carbery and Cowling in two spatial dimensions.
Corollary (1.2.4)[46]: *For all $s > 3/4$, there exists a constant C_s such that*

$$\left\| \sup_{0<t<1} \left| e^{it\Delta} f \right| \right\|_{L^2(\mathbb{R}^2)} \le C_s \|f\|_{H^s(\mathbb{R}^2)}.$$

The result of Cowling also holds when the Laplacian is replaced by a more general class of operators that includes

$$\square = \partial^2_{x_1} - \partial^2_{x_2} \pm \partial^2_{x_3} \pm \cdots \pm \partial^2_{x_n}.$$

For physical applications of the nonelliptic Schrödinger equation, see for example [42]. We will also prove the equivalence in this case, so that, by a local result of Vargas, Vega and [14], the global result of Cowling is almost sharp. We state this as a corollary.
Corollary (1.2.5) [46]: *For all $s > 1$, there exists a constant C_s such that*

$$\left\| \sup_{0<t<1} \left| e^{it\square} f \right| \right\|_{L^2(\mathbb{R}^2)} \le C_s \|f\|_{H^s(\mathbb{R}^n)},$$

and this is not true when $s < 1$.
Throughout, c and C will denote positive constants that may depend on the dimensions and exponents of the Sobolev spaces. It will be made explicit when they depend on other factors like, for example, the Sobolev index. Their values may change from line to line. The following are

notations that will be used frequently:

$L_x^q\big(\mathbb{R}^n, L_t^r(I)\big)$: The Lebesgue space with norms$\left(\int_{\mathbb{R}^n}\left|\int_I \ |f(x,t)|^r dt\right|^{q/r} dx\right)^{1/q}$.

$W^{\alpha,q}(\mathbb{R}^n)$: The inhomogeneous Sobolev space with αderivatives in $L^q(\mathbb{R}^n)$.

$\|\cdot\|_{q,\alpha}$: The inhomogeneous Sobolev norm with αderivatives in $L^q(\mathbb{R}^n)$.

$H^s(\mathbb{R}^n) \coloneqq W^{s,2}(\mathbb{R}^n)$.

$\square = \partial_{x_1}^2 - \partial_{x_2}^2 \pm \partial_{x_3}^2 \pm \cdots \pm \partial_{x_n}^2$.

$\mathbb{B}^n \coloneqq \{x \in \mathbb{R}^n : |x| \leq 1\}$.

$\mathbb{A}^n \coloneqq \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 1\}$.

$B_R \coloneqq \{x \in \mathbb{R}^n : |x| \leq R\}$.

$A_R \coloneqq \{x \in \mathbb{R}^n : R/2 \leq |x| \leq R\}$.

χ_{B_R}: the indicator function of B_R.

$\varphi_{R^2}(x) \coloneqq R^{-2n}\left(1 + \frac{|x|}{R^2}\right)^{-2n}$.

$L_{R^2}f \coloneqq \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|$.

v_j: a member of the lattice $R^{-2}\mathbb{Z}^n$.

x_k: a member of the lattice $R^2\mathbb{Z}^n$.

$T_{jk} \coloneqq \left\{(x,t) \in \mathbb{R}^n \times [0,R^4] : \left|x - \left(x_k + 4\pi t v_j\right)\right| \leq R^2\right\}$.

$\{Q_l\}_{l\in\mathbb{N}}$: a partition of $\mathbb{R}^n$into cubes of side R^2, centred at $x_l \in R^2\mathbb{Z}^n$.

$\hat{\psi}$: a positive and smooth function, supported in $B_{\sqrt{n}}$.

$\hat{\eta}$: a positive and smooth function, supported in $\mathbb{B}^n$, and equal to 1 at the origin.

Let $\hat{\eta}$be a positive and smooth function supported in $\mathbb{B}^n$, and denote by$\hat{\eta}_{R^{-1}}$ the scaledversion $\hat{\eta}\left(\frac{\cdot}{R}\right)$. Correspondingly, we let $\eta_{R^{-1}}$ denote its inverse Fourier transform $R^n\eta(R\ \cdot)$. Weconsider initial data $f_R$defined by

$$\hat{f}_R(\xi) = e^{2\pi^2 i|\xi|^2}\frac{\hat{\eta}_{R^{-1}}(\xi)}{(1+|\xi|^2)^{\alpha/2}}.$$

We note that

$$\|f_R\|_{r,\alpha} = \left\|e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}\right\|_{L^r(\mathbb{R}^n)},$$

and by a change of variables,

$$e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}(x) = R^n\int_{\mathbb{R}^n}\hat{\eta}(\xi)e^{2\pi i\left(Rx\cdot\xi + R^2\pi|\xi|^2\right)}d\xi.$$

When $|x| > 2\pi R$, by repeated integration by parts, there exists constants $C_N$such that

$$\left|e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}(x)\right| \leq C_N\left(\frac{|x|}{2\pi R}\right)^{-N} \tag{12}$$

for all $N \in \mathbb{N}$. When $|x| \leq 2\pi R$, by the dispersive estimate,

$$\left|e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}(x)\right| \leq C\|\eta_{R^{-1}}\|_{L^1(\mathbb{R}^n)} \leq C. \tag{13}$$

Combining these two bounds, we see that

$$\|f_R\|_{r,\alpha} = \left\|e^{-i\frac{1}{2}\Delta}\eta_{R^{-1}}\right\|_{L^r(\mathbb{R}^n)} \leq CR^{\frac{n}{r}}. \tag{14}$$

On the other hand, by a change of variables,

$$\left|e^{it\Delta}f_R(x)\right| = \left|\int_{\mathbb{R}^n} \frac{\hat{\eta}\left(\frac{\xi}{R}\right)}{(1+|\xi|^2)^{\alpha/2}} e^{2\pi i\left(x\cdot\xi - 2\pi\left(t-\frac{1}{2}\right)|\xi|^2\right)} d\xi\right|$$

$$= \left|R^{n-\alpha}\int_{\mathbb{R}^n} \frac{\hat{\eta}(\xi)}{\left(\frac{1}{R^2}+|\xi|^2\right)^{\alpha/2}} e^{2\pi i\left(Rx\cdot\xi - 2\pi R^2\left(t-\frac{1}{2}\right)|\xi|^2\right)} d\xi\right|,$$

so when $|x| \le \frac{1}{10R}$and $\left|t-\frac{1}{2}\right| \le \frac{1}{20\pi R^2}$, we have $\left|e^{it\Delta}f_R(x)\right| \le CR^{n-\alpha}$. Thus,

$$\left\|e^{it\Delta}f_R\right\|_{L^q(\mathbb{R}^n\times[0,1])} \ge CR^{n-\alpha}R^{-\frac{n+2}{q}},$$

and combining this with (14), we see that for

$$\left\|e^{it\Delta}f\right\|_{L^q(\mathbb{R}^n\times[0,1])} \le C_\alpha\|f\|_{r,\alpha} \quad (15)$$

to hold, it is necessary that $\alpha \ge n\left(1-\frac{1}{q}-\frac{1}{r}\right)-\frac{2}{q}$.

By considering $f_R$defined by$\hat{f}_R = \hat{\eta}_{R^{-1}}$, we reverse the previous focusing example. Note thatthe rapid decay (12) and upper bound (13) remain true for all $t \in [1/2, 1]$. This forces $\left|e^{it\Delta}f_R\right| \ge c$on a set of measure cR^nas otherwise the conservation of charge would be violated. We see that

$$\left\|e^{it\Delta}f_R\right\|_{L^q(\mathbb{R}^n\times[0,1])} \ge CR^{\frac{n}{q}},$$

and as $\|f_R\|_{r,\alpha} \le CR^\alpha R^{n-\frac{n}{r}}$, for (15) to hold it is also necessary that $\alpha \ge n\left(\frac{1}{q}+\frac{1}{r}-1\right)$.

Finally, we consider initial data $f_R$defined by$\hat{f}_R(\xi) = \hat{\eta}\left(R^\lambda(\xi-(R,\ldots,R))\right)$, where $\lambda \ge 1$, so that

$$e^{it\Delta}f_R(x) = \int \hat{\eta}\left(R^\lambda(\xi-(R,\ldots,R))\right)e^{2\pi i(x\cdot\xi-2\pi t|\xi|^2)}d\,\xi.$$

One can calculate that $\left|2\pi\nabla_\xi(x\cdot\xi-2\pi t|\xi|^2)\right| \le \frac{R^\lambda}{10}$ in the region defined by

$$|x| \le \frac{R^\lambda}{100}, |t| \le \frac{1}{1000}, \text{and} |\xi-(R,\ldots,R)| \le \frac{1}{R^\lambda},$$

so that the phase is almost constant for each pair (x,t)in the region. Thus,

$$\left\|e^{it\Delta}f_R\right\|_{L^q(\mathbb{R}^n\times[0,1])} \ge CR^{-n\lambda}R^{\frac{n\lambda}{q}},$$

and combining this with

$$\|f_R\|_{r,\alpha} \le R^\alpha R^{-n\lambda+\frac{n\lambda}{r}},$$

we see that

$$\alpha \ge \lambda n\left(\frac{1}{q}-\frac{1}{r}\right).$$

Setting $\lambda = 1$ and letting $\lambda \to \infty$yield the necessary conditions $\alpha \ge n\left(\frac{1}{q}-\frac{1}{r}\right)$and $q \ge r$, respectively.

In particular, ignoring endpoint issues, one may hope that

$$\left\|e^{it\Delta}f\right\|_{L^q(\mathbb{R}^n\times[0,1])} \le C_\alpha\|f\|_{q,\alpha}$$

for all $\alpha > \max\left\{2n\left(\frac{1}{q}-\frac{1}{2}\right), 0, 2n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{q}\right\}$.

As in the arguments of Fefferman [30], Bourgain [2], Wolff [45], Tao [21], and others, we decompose the solution of the Schrödinger equation into wave packets at scale $R^2 \gg 1$.

Fix a positive and smooth function $\widehat{\psi}$, supported in $B_{\sqrt{n}}$, such that

$$\sum_j \widehat{\psi}(\xi - R^2 v_j) = 1,$$

where $v_j \in R^{-2}\mathbb{Z}^n$.We also fix a positive and smooth $\hat{\eta}$, supported in $\mathbb{B}^n$, that satisfies $\hat{\eta}(0) = 1$,so that, by the Poisson summation formula,

$$\sum_k \eta\left(x - \frac{x_k}{R^2}\right) = 1,$$

where $x_k \in R^{-2}\mathbb{Z}^n$. Now for any Schwartz function f, we define $f_j$and f_{jk}implicitly in the following decomposition:

$$\hat{f}(\xi) = \sum_j \hat{f}_j(\xi) = \sum_j \widehat{\psi}\left(R^2(\xi - v_j)\right)\hat{f}(\xi), \qquad (16)$$

$$f(x) = \sum_{j,k} f_{jk}(x) = \sum_{j,k} \eta\left(\frac{x - x_k}{R^2}\right) f_j(x). \qquad (17)$$

Note that $\hat{f}_{jk}$is supported in the ball of radius $(\sqrt{n} + 1)R^{-2}$ with centre v_j.

We also partition $\mathbb{R}^n$into cubes $Q_l$of side R^2, centred at $x_l \in R^2\mathbb{Z}^n$, and define the function φ_{R^2} by

$$\varphi_{R^2}(x) = R^{-2n}\left(1 + \frac{|x|}{R^2}\right)^{-2n},$$

and the operator L_{R^2} by

$$L_{R^2}f = \varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|.$$

We state a slightly refined version of a lemma which can be found in [21], or more explicitly in [35], where we replace the Hardy–Littlewood maximal operator by a convolution operator. It is clear from their proofs that this is permissible.

Lemma (1.2.6)[46]: *Let $t \in [0, R^4]$. Then for all $N \in \mathbb{N}$there exists a constant $C_N$such that*

$$\left|e^{it\Delta} f_{jk}(x)\right| \le C_N \varphi_{R^2} * \left|f_j(x_k)\right|\left(1 + \frac{\left|x - (x_k + 4\pi t v_j)\right|}{R^2}\right)^{-N}.$$

We note that when $t \in [0, R^4]$, the wave packets $e^{it\Delta} f_{jk}$are essentially supported in the tubes T_{jk}defined by

$$T_{jk} = \left\{(x,t) \in \mathbb{R}^n \times [0, R^4] : x - (x_k + 4\pi t v_j) \le R^2\right\}.$$

Lemma (1.2.7)[46]: *For all $f$frequency supported in $\mathbb{B}^n$and $\varepsilon > 0$, there exists functions $f_l, \tilde{f}_l$satisfying*

(i) $\|f_l\|_{L^p(\mathbb{R}^n)} \le CR^{2n\left(\frac{1}{p} - \frac{1}{q}\right) + \varepsilon}\left\|\tilde{f}_l\right\|_{L^q(\mathbb{R}^n)}$

for all $p \le q$,

(ii) $\sum_l \left\|\tilde{f}_l\right\|_{L^q(\mathbb{R}^n)}^q \le CR^{\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}^q$,

and for all $l, N \in \mathbb{N}$and $(x,t) \in Q_l \times [0, R^2]$,

(iii) $\left|e^{it\Delta} f(x)\right| \le \left|e^{it\Delta} f_l(x)\right| + C_N R^{-N} L_{R^2} f(x)$.

Proof. We decompose the solution into wave packets, $e^{it\Delta}f = \sum_{j,k} e^{it\Delta}f_{jk}$, at scale R^2. We recall that

$$f_{jk}(x) = \eta\left(\frac{x - x_k}{R^2}\right) f_j(x),$$

and we define $\tilde{f}_{jk}$by

$$\tilde{f}_{jk}(x) = |\eta|^{1/2}\left(\frac{x - x_k}{R^2}\right) f_j(x).$$

As ηdecays rapidly and $\sum_k \eta\left(x - \frac{x_k}{R^2}\right) = 1$, it is easy to see that

$$\sum_k |\eta|^{1/2}\left(x - \frac{x_k}{R^2}\right) \leq C,$$

so that

$$\sum_k \left\|\sum_j \tilde{f}_{jk}\right\|^q_{L^q(\mathbb{R}^n)} \leq C\left\|\sum_{j,k} \tilde{f}_{jk}\right\|^q_{L^q(\mathbb{R}^n)} \leq C\|f\|^q_{L^q(\mathbb{R}^n)}. \qquad (18)$$

As $\operatorname{supp}\hat{f} \subset \mathbb{B}^n$, we have that the v_j's are contained in a slight enlargement of $\mathbb{B}^n$. Thus, the tubes T_{jk}make angles with the spatial hyperplane which are uniformly bounded below. Letting $R^\varepsilon Q_l$denote the cube of side $R^{2+\varepsilon}$with centre x_l, we write

$$f_l = \sum_{k:Q_k \cap R^\varepsilon Q_l \neq \emptyset} \sum_j f_{jk},$$

so that $e^{it\Delta}f_l$consists of the wave packets that pass through or near to $Q_l \times [0, R^2]$. Similarly, we define $\tilde{f}_l$by

$$\tilde{f}_l = \sum_{k:Q_k \cap R^\varepsilon Q_l \neq \emptyset} \sum_j \tilde{f}_{jk}.$$

To prove property (i), we note that

$$|f_l(x)| = \left|\sum_{k:Q_k \cap R^\varepsilon Q_l \neq \emptyset} \eta\left(\frac{x - x_k}{R^2}\right) f(x)\right|$$

$$\leq C\left(1 + \frac{|x - x_l|}{R^{2+2\varepsilon}}\right)^{-M} \left|\sum_{k:Q_k \cap R^\varepsilon Q_l \neq \emptyset} |\eta|^{1/2}\left(\frac{x - x_k}{R^2}\right) f(x)\right|$$

$$= C\left(1 + \frac{|x - x_l|}{R^{2+2\varepsilon}}\right)^{-M} \left|\tilde{f}_l(x)\right|$$

for some large $M \in \mathbb{N}$, so that, by Hölder,

$$\|f_l\|_{L^p(\mathbb{R}^n)} \leq CR^{2(1+\varepsilon)n\left(\frac{1}{p}-\frac{1}{q}\right)}\|\tilde{f}_l\|_{L^q(\mathbb{R}^n)}.$$

To prove property (ii), we note that a cube $Q_k$can intersect $R^\varepsilon Q_l$for at most $2R^{n\varepsilon}$different cubes Q_l, so that

$$\sum_l \|\tilde{f}_l\|^q_{L^q(\mathbb{R}^n)} \leq C\sum_l \sum_{k:Q_k \cap R^\varepsilon Q_l \neq \emptyset} \left\|\sum_j \tilde{f}_{jk}\right\|^q_{L^q(\mathbb{R}^n)}$$

$$\leq CR^{n\varepsilon}\sum_k\left\|\sum_j \tilde{f}_{jk}\right\|^q_{L^q(\mathbb{R}^n)}.$$

Thus, by (18), we see that

$$\sum_l \|\tilde{f}_l\|^q_{L^q(\mathbb{R}^n)} \leq CR^{n\varepsilon}\|f\|^q_{L^q(\mathbb{R}^n)}.$$

To prove property (iii), we consider the pointwise bound

$$\left|e^{it\Delta}f\right| \leq \left|e^{it\Delta}f_l\right| + \left|\sum_{k:Q_k\cap R^\varepsilon Q_l=\emptyset}\sum_j e^{it\Delta}f_{jk}\right|. \quad (19)$$

By construction and Lemma (1.2.6),

$$\left|\sum_{k:Q_k\cap R^\varepsilon Q_l=\emptyset}\sum_j e^{it\Delta}f_{jk}(x)\right| \leq C_{N'}R^{2N'}\sum_{j=1}^{cR^{2n}}\sum_{k:\,|x_k-x_l|\geq\frac{1}{2}R^{2+\varepsilon}}\frac{\varphi_{R^2}*|f_j|(x_k)}{|x_k-x_l|^{N'}}$$

for all $(x,t)\in Q_l\times[0,R^2]$, and all $N'\in\mathbb{N}$. Choosing an $N' > (4n+N)/\varepsilon + 2n$, we have

$$\left|\sum_{k:Q_k\cap R^\varepsilon Q_l=\emptyset}\sum_j e^{it\Delta}f_{jk}(x)\right| \leq C_N R^{-N}\sum_{j=1}^{cR^{2n}}\sum_{k:\,|x_k-x_l|\geq\frac{1}{2}R^{2+\varepsilon}}\frac{\varphi_{R^2}*|f_j|(x_k)}{|x_k-x_l|^{2n}}$$

for all $N\in\mathbb{N}$. Now, by (16),

$$|f_j| \leq R^{-2n}\psi(R^{-2}\,\cdot)*|f| \leq C\varphi_{R^2}*|f|,$$

so that

$$\left|\sum_{k:Q_k\cap R^\varepsilon Q_l\neq\emptyset}\sum_j e^{it\Delta}f_{jk}(x)\right| \leq C_N R^{-N}\sum_{j=1}^{cR^{2n}}\sum_{k:\,|x_k-x_l|\geq\frac{1}{2}R^{2+\varepsilon}}\frac{\varphi_{R^2}*\varphi_{R^2}*|f|(x_k)}{|x_k-x_l|^{2n}}.$$

Now, it is easy to see that

$$\varphi_{R^2}*\varphi_{R^2}*|f|(x) \approx \varphi_{R^2}*\varphi_{R^2}*|f|(x')$$

when $|x-x'|\leq\sqrt{n}R^2$, so that

$$\sum_{k:\,|x_k-x_l|\geq\frac{1}{2}R^{2+\varepsilon}}\frac{\varphi_{R^2}*\varphi_{R^2}*|f|(x_k)}{|x_k-x_l|^{2n}} \leq C\varphi_{R^2}*\varphi_{R^2}*\varphi_{R^2}*|f|(x_l)$$

$$\leq C\varphi_{R^2}*\varphi_{R^2}*\varphi_{R^2}*|f|(x)$$

for all $x\in Q_l$. Substituting into (19), we have

$$\left|e^{it\Delta}f(x)\right| \leq \left|e^{it\Delta}f_l(x)\right| + C_N R^{-N}\varphi_{R^2}*\varphi_{R^2}*\varphi_{R^2}*|f|(x)$$

for all $(x,t)\in Q_l\times[0,R^2]$, and we are done.

Lemma (1.2.8) [46]: *Let $q\geq p_1\geq p_0$ and $I\subset[0,R^2]$. Suppose that*

$$\left\|e^{it\Delta}f\right\|_{L^q_x\left(B_{R^2},L^r_t(I)\right)} \leq CR^s\|f\|_{L^{p_0}(\mathbb{R}^n)}$$

whenever $R\gg 1$, and f is frequency supported in $\mathbb{B}^n$. Then for all $\varepsilon>0$,

$$\left\|e^{it\Delta}f\right\|_{L^q_x\left(B_{R^2},L^r_t(I)\right)} \leq C_\varepsilon R^{s+2n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)+\varepsilon}\|f\|_{L^{p_1}(\mathbb{R}^n)}.$$

Proof. By Lemma (1.2.7), for all $\varepsilon>0$, there exists functions f_l and $\tilde{f}_l$ such that

$$\|f_l\|_{L^{p_0}(\mathbb{R}^n)} \leq CR^{2n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)+\varepsilon}\|\tilde{f}_l\|_{L^{p_1}(\mathbb{R}^n)}, \tag{20}$$

$$\sum_l \|\tilde{f}_l\|_{L^{p_1}(\mathbb{R}^n)}^{p_1} \leq CR^{\varepsilon}\|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_1}, \tag{21}$$

and for all $N, l \in \mathbb{N}$ and $(x,t) \in Q_l \times [0, R^2]$,

$$|e^{it\Delta}f(x)| \leq |e^{it\Delta}f_l(x)| + C_N R^{-N} L_{R^2}f(x).$$

We use these pointwise bounds on cubes, to obtain an $L^q(\mathbb{R}^n, L_t^r(I))$bound. We have

$$\begin{aligned}\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n, L_t^r(I))}^q &= \sum_l \|e^{it\Delta}f\|_{L^q(Q_l, L_t^r(I))}^q \\ &\leq \sum_l \left\| |e^{it\Delta}f_l| + C_N R^{-N} L_{R^2}f \right\|_{L^q(Q_l, L_t^r(I))}^q,\end{aligned}$$

and using the fact that $\|g+h\|^q \leq 2^q(\|g\|^q + \|h\|^q)$, we see that

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n, L_t^r(I))}^q \leq C\sum_l \|e^{it\Delta}f_l\|_{L^q(Q_l, L_t^r(I))}^q + C_N R^{-N}\sum_l \|L_{R^2}f\|_{L^q(Q_l, L_t^r(I))}^q.$$

Now, byYoung's inequality,

$$\begin{aligned}\sum_l \|L_{R^2}f\|_{L^q(Q_l, L_t^r(I))}^q &\leq R^{2q}\|\varphi_{R^2} * \varphi_{R^2} * \varphi_{R^2} * |f|\|_{L^q(\mathbb{R}^n)}^q \\ &\leq CR^{2q}\|f\|_{L^q(\mathbb{R}^n)}^q,\end{aligned}$$

so that

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n, L_t^r(I))}^q \leq C\sum_l \|e^{it\Delta}f_l\|_{L^q(Q_l, L_t^r(I))}^q + C_N R^{-N}\|f\|_{L^q(\mathbb{R}^n)}^q. \tag{22}$$

By translation invariance and the hypothesis,

$$\|e^{it\Delta}f_l\|_{L^q(Q_l, L_t^r(I))} \leq CR^s\|f_l\|_{L^{p_0}(\mathbb{R}^n)}$$

for all $l \in \mathbb{N}$, and combining this with (20),

$$\|e^{it\Delta}f_l\|_{L^q(Q_l, L_t^r(I))} \leq CR^{s+2n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)+\varepsilon}\|\tilde{f}_l\|_{L^{p_1}(\mathbb{R}^n)}. \tag{23}$$

On the other hand, as $\operatorname{supp}\hat{f} \subset \mathbb{B}^n$and $p_1 \leq q$, by Bernstein's inequality,

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}. \tag{24}$$

Substituting (23) and (24) into (22), we see that

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n, L_t^r(I))}^q \leq CR^{q\left(s+2n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)+\varepsilon\right)}\sum_l \|\tilde{f}_l\|_{L^{p_1}(\mathbb{R}^n)}^q + C_N R^{-N}\|f\|_{L^{p_1}(\mathbb{R}^n)}^q.$$

Finally, as $q \geq p_1$, by convexity,

$$\sum_l \|\tilde{f}_l\|_{L^{p_1}(\mathbb{R}^n)}^q \leq \left(\sum_l \|\tilde{f}_l\|_{L^{p_1}(\mathbb{R}^n)}^{p_1}\right)^{q/p_1},$$

so that, by (21), we can sum to obtain the required bound.

We denote by$L^S(q \to q)$the estimate

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n \times [0,1])} \leq C_\alpha\|\hat{f}\|_{L^p(\mathbb{R}^n)}$$

for all $\alpha > 2n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{2}{q}$.

We denote by $R^*(p \to q)$ the (adjoint) restriction estimate

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^{n+1})} \le C\|\hat{f}\|_{L^p(\mathbb{R}^n)},$$

where $p' = \frac{nq}{n+2}$. It is conjectured that $R^*(p \to q)$ holds for all $> 2 + \frac{2}{n}$, and it has been provenin the affirmative by Tao [32] in the range $q > 2 + \frac{4}{n+1}$.

Theorem (1.2.9)[46]: $R^*(p \to q) \Rightarrow LS(q \to q)$.

Proof. Suppose first that supp $\hat{f} \subset \mathbb{B}^n$. Considering (9), we see that $e^{it\Delta}f$ can be viewed as the convolution of f with the Fourier transform of $e^{-4\pi^2 i|\xi|^2 t}$, so that we can also write

$$e^{it\Delta}f(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(y)e^{\frac{i|x-y|^2}{4t}}dy. \tag{25}$$

As in [28], we 'complete the square' in (9), and compare the representations, so that

$$|e^{it\Delta}f(x)| = \left|\frac{c^{n/2}}{t^{n/2}}e^{-ic^2\frac{1}{t}\Delta}\hat{f}\left(\frac{cx}{t}\right)\right|. \tag{26}$$

Making a 'pseudo-conformal' change of variables, we have

$$\|e^{it\Delta}f\|_{L^q(B_{R^2}\times[R^2/2,R^2])} \le CR^{-n}\left\|e^{i\frac{1}{t}\Delta}\hat{f}\left(\frac{\cdot}{t}\right)\right\|_{L^q(B_{R^2}\times[R^2/2,R^2])}$$

$$\le CR^{-n+\frac{2(n+2)}{q}}\|e^{it\Delta}\hat{f}\|_{L^q(\mathbb{B}^{n+1})}.$$

Now, by hypothesis,

$$\|e^{it\Delta}\hat{f}\|_{L^q(\mathbb{B}^{n+1})} \le C\|f\|_{L^p(\mathbb{R}^n)},$$

where $p' = \frac{nq}{n+2}$, so that

$$\|e^{it\Delta}f\|_{L^q(B_{R^2}\times[R^2/2,R^2])} \le CR^{-n+\frac{2(n+2)}{q}}\|f\|_{L^p(\mathbb{R}^n)}.$$

Thus, by Lemma (1. 2. 8)

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[R^2/2,R^2])} \le CR^{-n+\frac{2(n+2)}{q}+2n\left(\frac{1}{p}-\frac{1}{q}\right)+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}$$

$$= CR^{n\left(1-\frac{2}{q}\right)+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}.$$

Finally we scale, so that

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[2^{-k},2^{-k+1}])} \le C2^{-\frac{2k}{q}}R^{n\left(1-\frac{2}{q}\right)-\frac{2}{q}+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}$$

whenever supp $\hat{f} \subset B_{2^k R}$ with $k \ge 0$. Summing, we see that

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}^n\times[0,1])} \le CR^{n\left(1-\frac{2}{q}\right)-\frac{2}{q}+\varepsilon}\|f\|_{L^q(\mathbb{R}^n)}$$

whenever supp $\hat{f} \subset B_R$, and the proof is completed with the standard Littlewood–Paley arguments.

We consider the local bound,

$$\|e^{it\Delta}f\|_{L^q_x(\mathbb{B}^n,L^r_t[0,1])} \le C_s\|f\|_{H^s(\mathbb{R}^n)}, \tag{27}$$

and the global bound,

$$\|e^{it\Delta}f\|_{L^q_x(\mathbb{R}^n,L^r_t[0,1])} \le C_s\|f\|_{H^s(\mathbb{R}^n)}. \tag{28}$$

Theorem (1.2.10) [46]: *Let* $q, r \ge 2$. *Then* (27) *holds for all* $s > s_0$ *if and only if* (28) *holds for all*

$> 2s_0 - n\left(\frac{1}{2} - \frac{1}{q}\right) + \frac{2}{r}$.

Letting $q = 2$ and $r = \infty$, we obtain Theorem (1.2.3). Letting $q = r = 2$, we see the equivalence up to endpoints of the conservation of charge and the local smoothing theorem of Sjölin, Vega, and Constantin and Saut, mentioned.

We will need the following lemma due to Lee.

Lemma (1.2.11)[46]: *(See [31].) Let $q, r \geq 2$. Suppose that*

$$\|e^{it\Delta}f\|_{L_x^q(B_R, L_t^r[0,R])} \leq CR^s\|f\|_{L^2(\mathbb{R}^n)},$$

whenever $R \gg 1$, and f is frequency supported in $\mathbb{A}^n$. Then for all $\varepsilon > 0$,

$$\|e^{it\Delta}f\|_{L_x^q(B_R, L_t^r[0,R^2])} \leq C_\varepsilon R^{s+\varepsilon}\|f\|_{L^2(\mathbb{R}^n)}.$$

By the standard Littlewood–Paley arguments and scaling, to prove Theorem (1.2.10), it will suffice to prove the following theorem, where (ii) and (iii) correspond to (27) and (28), respectively.

Theorem (1.2.12) [46]: *Let $q, r \geq 2$, and consider functions f which are frequency supported in $\mathbb{A}^n$. Then the following bounds are equivalent*:

(i) $\|e^{it\Delta}f\|_{L_x^q(B_R, L_t^r[0,R])} \leq CR^s\|f\|_{L^2(\mathbb{R}^n)}$ *for all $R \gg 1$ and $s > s_0$,*

(ii) $\|e^{it\Delta}f\|_{L_x^q(B_R, L_t^r[0,R^2])} \leq CR^s\|f\|_{L^2(\mathbb{R}^n)}$ *for all $R \gg 1$ and $s > s_0$,*

(iii) $\|e^{it\Delta}f\|_{L_x^q(\mathbb{R}^n, L_t^r[0,R^2])} \leq CR^{2s}\|f\|_{L^2(\mathbb{R}^n)}$ *for all $R \gg 1$ and $s > s_0$.*

Proof. By changing variables $R \to R^{1/2}$ in (iii), we see that (ii) and (iii) trivially imply (i). Thus, it will suffice to show that (i) implies (ii) and (iii). Now, (i) implies (ii) is precisely the content of Lemma (1.2.11). Similarly, by changing variables and letting $p_0 = p_1 = 2$ and $I = [0, R^2]$ in Lemma (1.2.8), we see that (i) implies (iii).

By the local result of Lee [11], mentioned, and Theorem (1.2.10) with q and r taken to be 2 and ∞, respectively, we obtain the following corollary.

Corollary (1.2.13)[46]: *For all $s > 3/4$, there exists a constant C_s such that*

$$\left\|\sup_{0<t<1}\left|e^{it\Delta}f\right|\right\|_{L^2(\mathbb{R}^2)} \leq C_s\|f\|_{H^s(\mathbb{R}^2)}.$$

We note that as (28) cannot hold for any value of s when $q < 2$ (see for example [25]), there can be no such equivalence when $q < 2$. Letting $r = \infty$, we also see that the necessary conditions for (28) to hold given in [25], are equivalent to the necessary conditions for (27) to hold given in [40].

The generalised Schrödinger equation, $i\partial_t u + \phi(D)u = 0$, where $\widehat{\phi(D)u} = \phi(\xi)\hat{u}(\xi)$ and $\phi(\xi)$ is real, has solution $e^{it\phi(D)}f$ which can be formally written as

$$e^{it\phi(D)}f(x) = \int \hat{f}(\xi)e^{2\pi ix\cdot\xi + it\phi(\xi)}d\xi.$$

In the local case, Kenig, Ponce and Vega [9] showed that if there are at most $N \in \mathbb{N}$ solutions to

$$\phi(\xi_1, \ldots, \xi_k, x, \xi_{k+1}, \ldots, \xi_{n-1}) = r \qquad (29)$$

for all $\xi \in \mathbb{R}^{n-1}, r \in \mathbb{R}, k = 0, \ldots, n-1$, and

$$\frac{|\phi(\xi)|}{|\nabla\phi(\xi)|} \leq C(1 + |\xi|^2)^{s_0},$$

then for $s > s_0$,

$$\left\| \sup_{0<t<1} \left| e^{it\phi(D)} f \right| \right\|_{L^2(\mathbb{B}^n)} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}. \quad (30)$$

In the global case, Cowling [5] showed that if $|\phi(\xi)| \leq C(1 + |\xi|^2)^{s_0}$, then for $s > s_0$,

$$\left\| \sup_{0<t<1} \left| e^{it\phi(D)} f \right| \right\|_{L^2(\mathbb{R}^n)} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}. \quad (31)$$

In particular, both these results hold for smooth ϕthat are homogeneous of degree $m \geq 1$. The injectivity condition (29) is fulfilled and

$$\frac{|\phi(\xi)|}{|\nabla\phi(\xi)|} \leq C(1 + |\xi|^2)^{1/2},$$

so that (30) holds for all $s > 1/2$. On the other hand $|\phi(\xi)| \leq C(1 + |\xi|^2)^{m/2}$, so that (31) holds for all $s > m/2$.

For such ϕ, these results are again equivalent. Indeed, for anyϕsatisfying $|D^\alpha \phi(\xi)| \leq C_0 |\xi|^{m-|\alpha|}$, where $|\alpha| \leq 2$, and $|\nabla\phi(\xi)| \geq C_0^{-1} |\xi|^{m-1}$, there is an equivalence.

We consider the local bound,

$$\left\| e^{it\phi(D)} f \right\|_{L_x^q(\mathbb{B}^n, L_t^r[0,1])} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}, \quad (32)$$

and the global bound,

$$\left\| e^{it\phi(D)} f \right\|_{L_x^q(\mathbb{R}^n, L_t^r[0,1])} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}. \quad (33)$$

By scaling, it will suffice to consider $e^{it\phi_R(D)} f$defined by

$$e^{it\phi_R(D)} f = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi + tR^{-m}\phi(R\xi)} d\xi ,$$

where $\phi_R = R^{-m}\phi(R\,\cdot), \hat{f}$is supported in $\mathbb{A}^n$and $t \in [0, R^m]$. It is easy to see that $|D^\alpha \phi_R(\xi)| \leq C_0 |\xi|^{m-|\alpha|}$and $|\nabla\phi_R(\xi)| \geq C_0^{-1} |\xi|^{m-1}$ for all R, so that $\left|\nabla\phi_R(v_j)\right| \approx \left|v_j\right|^{m-1}$.

Now, Lemma (1.2.6) generalises to ϕsuch that $|D^\alpha \phi(\xi)| \leq C_0 |\xi|^{m-|\alpha|}$for $|\alpha| \leq 2$ (see [35]). The $2v_j$is replaced by$\nabla\phi(v_j)$, and the constants depend only on C_0.

To prove versions of Lemmas (1.2.7) and (1.2.8) with $e^{it\phi_R(D)} f$in place of $e^{it\Delta} f$, only the numerology changes. The important point is that the tubes make angles with the spatial plane which are uniformly bounded away from zero, which we have insured by requiring that $|\nabla\phi_R(\xi)| \leq C_0$ for all $\xi \in \mathbb{A}^n$.

Lemma (1.2.11) can be similarly generalised. The important point there is that the tubes make angles with the t-axis which are uniformly bounded away from zero, which we have insured by requiring that $|\nabla\phi_R(\xi)| \geq \frac{1}{2} C_0^{-1}$ for all $\xi \in \mathbb{A}^n$.

Thus, considering $f$frequency supported in $\mathbb{A}^n$, and $q, r \geq 2$, the following bounds are equivalent:

(i) $\left\| e^{it\phi_R(D)} f \right\|_{L_x^q(B_R, L_t^r[0,R])} \leq CR^s \|f\|_{L^2(\mathbb{R}^n)}$ *for all* $R \gg 1$ *and* $s > s_0$,

(ii) $\left\| e^{it\phi_R(D)} f \right\|_{L_x^q(B_R, L_t^r[0,R^m])} \leq CR^s \|f\|_{L^2(\mathbb{R}^n)}$ *for all* $R \gg 1$ *and* $s > s_0$,

(iii) $\left\| e^{it\phi_R(D)} f \right\|_{L_x^q(\mathbb{R}^n, L_t^r[0,R^m])} \leq CR^{ms} \|f\|_{L^2(\mathbb{R}^n)}$ *for all* $R \gg 1$ *and* $s > s_0$.

By scaling and the usual arguments of Littlewood and Paley, this yields the following theorem.

Theorem (1.2.14) [46]: *Let* $q, r \geq 2$. *Suppose that*$|D^\alpha \phi(\xi)| \leq C_0 |\xi|^{m-|\alpha|}$*and* $|\nabla\phi(\xi)| \geq C_0^{-1} |\xi|^{m-1}$*forall* $\xi \in \mathbb{R}^n \backslash \{0\}$, *where* $|\alpha| \leq 2$*and*$m > 1$. *Then* (32) *holds for all* $s > s_0$*if and only if*

(33) *holdsfor all* $s > ms_0 - (m-1)\left(n\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{m}{r}\right)$.

A corollary of this and the generalised result of Lee [19], is that Corollary (1.2.13) also holds for the generalised Schrödinger equation; where $|D^\alpha\phi(\xi)| \le C|\xi|^{2-|\alpha|}$and $|\nabla\phi(\xi)| \ge C^{-1}|\xi|$, and the Hessian of ϕhas two nonzero eigenvalues of the same sign.

For completeness, we note that when $m \le 1$, we no longer need Lemma (1.2.11), so that we have the following theorem.

Theorem (1.2.15) [46]: *Let* $q \ge 2$*and suppose that* $|D^\alpha\phi(\xi)| \le C_0|\xi|^{m-|\alpha|}$*for all* $\xi \in \mathbb{R}^n\backslash\{0\}$*, where* $|\alpha| \le 2$*and* $m \le 1$*. Then* (32) *holds for all* $s > s_0$*if and only if* (33) *holds for all* $s > s_0$*.*

In particular, we consider $\phi(\xi) = (2\pi|\xi|)^m$so that $\phi(D) = (-\Delta)^{m/2}$ with $m \in (0,1)$. The conditions of Theorem (1.2.15) are fulfilled, and we see that global bounds are equivalent to local bounds.

We consider the nonelliptic Schrödinger equation; where $\phi$is defined by $\phi(\xi) = -4\pi^2(\xi_1^2 - \xi_2^2 \pm \xi_3^2 \pm \cdots \pm \xi_n^2)$, and

$$\phi(D) == \partial_{x_1}^2 - \partial_{x_2}^2 \pm \partial_{x_3}^2 \pm \cdots \pm \partial_{x_n}^2.$$

Note that the conditions of Theorem (1.2.14) are fulfilled with $m = 2$. Vargas, Vega and the author [14] showed that, in this case, the bound of Kenig, Ponce and Vega is almost sharp, in the sense that

$$\left\|\sup_{0<t<1}\left|e^{it\square}f\right|\right\|_{L^2(\mathbb{B}^n)} \le C_s\|f\|_{H^s(\mathbb{R}^n)}$$

does not hold when $s < 1/2$.

Therefore, by Theorem (1.2.14), we see that the bound of Cowling is similarly sharp, and we state this as a corollary.

Corollary (1.2.16) [46]: *For all* $s > 1$*, there exists a constant* C_s*such that*

$$\left\|\sup_{0<t<1}\left|e^{it\square}f\right|\right\|_{L^2(\mathbb{R}^n)} \le C_s\|f\|_{H^s(\mathbb{R}^n)},$$

and this is not true when $s < 1$*.*

Theorem (1.2.9) also generalises to the nonelliptic case, so the well-known Stein–Tomas–Strichartz estimate yields an almost sharp local smoothing estimate in the range $q \ge 2 + 4/n$. In two spatial dimensions, by a restriction theorem independently due to Vargas [43] and Lee [35], we have the result in the range $q \ge 10/3$.

Corollary (1.2.17) [210]. Let $n = 1$. If $\epsilon \ge 0$ and $4\epsilon^2 + 15\epsilon \ge 0$, then (4) holds. If $\epsilon > 0$and $\frac{1}{2} + \epsilon \ge \max\{1/(2+\epsilon), 1/2 - 1/(2+\epsilon)\}$, then (3) holds.

Proof. By duality, it will suffice to show that

$$\left|\int_{\mathrm{R}} e^{it(x)\Delta}f(x)w(x)dx\right|^2 \le C_{(4+\epsilon)}\|f\|^2_{H^{\frac{1}{2}+\epsilon}(\mathrm{R})}\|w\|^2_{L^{(4+\epsilon)'}(\mathrm{R})}$$

for all positive $w \in L^{(4+\epsilon)'}(\mathrm{R})$, where the measurable function t maps into R when we are considering the bound (4) and into $(0,1)$ when we consider (3).

By Fubini's theorem and the Cauchy–Schwarz inequality, the left hand side of this inequality is bounded by

$$\int_R \left|\hat{f}(\xi)\right|^2 (1+|\xi|)^{2(\frac{1}{2}+\epsilon)} d\xi \int_R \left|\int_R e^{2\pi i(x\xi - t(x)\xi^2)} w(x) dx\right|^2 \frac{d\xi}{(1+|\xi|)^{2(\frac{1}{2}+\epsilon)}}.$$

Thus, by writing the squared integral as a double integral, it will suffice to show that

$$\int_R \int_R \int_R e^{2\pi i(\epsilon\xi - (t(x)-t(x-\epsilon))\xi^2)} w(x) w(x-\epsilon) dx d(x-\epsilon) \frac{d\xi}{(1+|\xi|)^{2(\frac{1}{2}+\epsilon)}} \leq C_p \|w\|^2_{L^{(2+\epsilon)'}(R)}. \quad (5)$$

By Lemma 1, we have

$$\left|\int_R \frac{e^{2\pi i(\epsilon\,\xi - (t(x)-t(x-\epsilon))\xi^2)}}{(1+|\xi|)^{2(\frac{1}{2}+\epsilon)}} d\xi\right| \leq \frac{C}{|\epsilon|^{-2\epsilon}}$$

when $(1-\epsilon)$ takes values in R, and $-\frac{1}{4} \leq \epsilon < 0$, and by Lemmas 1 and 2, we have

$$\left|\int_R \frac{e^{2\pi i(\epsilon\,\xi - (t(x)-t(x-\epsilon))\xi^2)}}{(1+|\xi|)^{2(\frac{1}{2}+\epsilon)}} d\xi\right| \leq \frac{C}{|\epsilon|^{\max\{2(\frac{1}{2}+\epsilon),1-2(\frac{1}{2}+\epsilon)\}}}$$

when (ϵ) takes values in $0 < \epsilon < 1$ Thus, by Fubini's theorem, the left hand side of (5) is bounded by a constant multiple of

$$\int_R \int_R \frac{w(x)w(x-\epsilon)}{|\epsilon|^{1-2(\frac{1}{2}+\epsilon)}} dx d(x-\epsilon)$$

in the first case, and

$$\int_R \int_R \frac{w(x)w(x-\epsilon)}{|\epsilon|^{\max\{(1+2\epsilon),-2\epsilon\}}} dx d(x-\epsilon)$$

In the second. Finally, by Hölder's inequality and the Hardy–Littlewood–Sobolev inequality, these are bounded by

$$\|w\|_{L^{(2+\epsilon)'}(R)} \left\|\int_R \frac{w(x)}{|x-\cdot|^{-2\epsilon}} dx\right\|_{L^{(2+\epsilon)}(R)} \leq C_{(2+\epsilon)} \|w\|^2_{L^{(2+\epsilon)'}(R)},$$

Where $\epsilon^2 + 2\epsilon + 1 = 0$ and $\epsilon \geq 2$ when we are considering the bound in (4), and

$$\|w\|_{L^{(2+\epsilon)'}(R)} \left\|\int_R \frac{w(x)}{|x-\cdot|^{\max\{2(\frac{1}{2}+\epsilon)1-2(\frac{1}{2}+\epsilon)\}}} dx\right\|_{L^{(2+\epsilon)}(R)} \leq C_{(2+\epsilon)} \|w\|^2_{L^{(2+\epsilon)'}(R)},$$

Where $\frac{1}{2} + \epsilon = \max\{1/(2+\epsilon), 1/2 - 1/(2+\epsilon)\}$ and $(2+\in) > 2$ when we consider (3).

in [21] due to Tao and Vargas, the following result is proved using bilinear restriction estimates.

Chapter 2
Strichartz Estimates and Singular Continuous Spectrum

We consider the Schrödinger operator$e^{it\Delta}$ acting on initial datafin H^s.We show that an affirmative answer to a question of Carleson, concerning the sharp range of s forwhich$\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ a.e. $x \in R^n$,would imply an affirmativeanswer to a question of Planchon, concerning the sharp range of q and r for which$e^{it\Delta}$ is bounded in $L_x^q(R^n, L_t^r(R)$. We have shown that every kind of absolutely continuous spectrum within a gap J of H can be generated by a self-adjoint extension$H^{\sim}$ of$H$, cf. [61.

Section (2.1): The Schrödinger Maximal Operator

The Schrödinger equation, $i\partial_t u + \Delta u = 0$, in $\mathbb{R}^{n+1}$, with initial datum $f$in the Sobolev space $\dot{H}^s(\mathbb{R}^n)$, has solution $e^{it\Delta}f$which can be formally written as

$$e^{it\Delta}f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2)}d\xi. \quad (1)$$

We define the dimensional or scaling relation $s(q,r)$by

$$s(q,r) = n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{2}{r}.$$

Stein [55], Tomas [58], Strichartz [56], Ginibre and Velo [47], and Keel and Tao [49] have all played a role in proving the following theorem.

Theorem (2.1.1)[59]: [49] *Let* $q \in [2,\infty), r \in [2,\infty]$*and* $\frac{n}{q} + \frac{2}{r} \le \frac{n}{2}$. *Then*

$$\|e^{it\Delta}f\|_{L_t^r(\mathbb{R},L_x^q(\mathbb{R}^n))} \le C\|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

The theorem is sharp in the sense that it is not true when $q < 2, r < 2$, or $\frac{n}{q} + \frac{2}{r} > \frac{n}{2}$.When $q = \infty$, the estimate holds only occasionally (see [51,19]).

Changing the order of the integrals, the problem is more difficult. We will ignore the subtle endpoint questions. In connection with his work on the cubic semilinear Schrödinger equation, Planchon [52] asked whether the following is true:

Conjecture(2.1.2) [59] Let $q \in \left(\frac{2(n+1)}{n}, \infty\right], r \in [2,\infty)$and $\frac{n+1}{q} + \frac{1}{r} < \frac{n}{2}$. Then

$$\|e^{it\Delta}f\|_{L_x^q(\mathbb{R}^n,L_t^r(\mathbb{R}))} \le C\|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

In one spatial dimension, this had already been proven in the affirmative, including the endpoints, by Kenig, Ponce and Vega [9, 23].

In higher dimensions, arguments originally due to Tao and Vargas [22] which were then refined by Planchon [52] (see also [25]), can be combined with Tao's bilinear restriction estimate [21] to yield the conjecture in the range $q > \frac{2(n+3)}{n+1}$. When $q > r$, the endpoints can be included, and the key bound follows from the original Stein–Tomas theorem (see [48,52,23]). Note that $s(q,r)$can be negative in this range.

We will prove that the conjecture would follow from a positive resolution of a question of Carleson concerning the sharp range of $s$for which

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \quad f \in H^s(\mathbb{R}^n).$$

By standard arguments, the convergence follows from the estimate

$$\left\| \sup_{0<t<1} \left| e^{it\Delta} f \right| \right\|_{L_x^2(\mathbb{B}^n)} \le C_s \|f\|_{H^s(\mathbb{R}^n)}, \tag{A}$$

where $\mathbb{B}^n$is the unit ball in $\mathbb{R}^n$. If we restrict time to a sequence, then the convergence and a nonendpoint version of the maximal estimate are equivalent (see [54]).

Conjecture (2.1.3) [59] (A) holds for all $s > 1/4$.

In one spatial dimension, the convergence was originally proven by Carleson [4] via an L^1-estimate, and Kenig and Ruiz [10] showed that (A) holds for all $s \ge 1/4$. Dahlberg and Kenig [6] showed that this is sharp in the sense that (A) cannot hold when$s < 1/4$.

In two spatial dimensions, significant contributions were made by Bourgain [1,2], Moyua et al. [12, 13], and Tao and Vargas [21 - 22]. The best known result is due to Lee [11] who showed that (A) holds when $s > 3/8$.

In higher dimensions, significant contributions were made by Carbery [3] and Cowling [5]. The best known result is independently due to Sjölin [15] and Vega [24] who showed that (A) holds when $s > 1/2$.

We rewrite estimate (A) as

$$\left\| \sup_{0<t<1} \left| e^{it\Delta} f \right| \right\|_{L_x^2(\mathbb{B}^n)} \le C \|f\|_{H^{1/4+\kappa}(\mathbb{R}^n)}, \tag{A_κ}$$

where $\kappa \ge 0$, and define the dual exponents $q_\kappa$and $q'_\kappa$by

$$q_\kappa = \frac{n+1+8\kappa}{n+4\kappa} \text{ and } q'_\kappa = \frac{n+1+8\kappa}{1+4\kappa}.$$

Theorem (2.1.4) [59] *Let* $q \in (2q_\kappa, \infty], r \in (2q'_\kappa, \infty)$ *and* $\frac{n}{2q'_\kappa} + \frac{n}{q} + \frac{1}{r} < \frac{n}{2}$. *If* (A_κ) *holds,then*

$$\left\| e^{it\Delta} f \right\|_{L_x^q(\mathbb{R}^n, L_t^r(\mathbb{R}))} \le C \|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

Note that $2q_\kappa$and $\frac{q'_\kappa}{n}$both tend to $\frac{2(n+1)}{n}$as κtends to zero. Comparing with Conjecture (2.1.2), we see that (q, r)can approach the endpoint $\left(\frac{2(n+1)}{n}, \infty\right)$;

Corollary (2.1.5)[59]: Conjecture(2.1.3)⇒ Conjecture (2.1.3).

Combining the identity$D_t^s e^{it\Delta} f = e^{it\Delta} D_x^{2s} f$with Sobolev embedding, Theorem (2.1.1) also yields estimates for the maximal operator. Indeed, applying Hölder to obtain localL^2-bounds, we see that

$$(A_\kappa) \Rightarrow \quad (A_{\kappa'}), \qquad \kappa' > n\left(\frac{1}{2} - \frac{1}{2q_\kappa}\right) - \frac{1}{4}.$$

There is an improvement in regularity when$\kappa > (n-1)/8$. Taking $n = 2$ and iterating, we can suppress $\kappa$to be arbitrarily close to 1/8, which recovers Lee's result.

We see that a global version holds;

Corollary (2.1.6)[59]: *Let* $q > 16/5$. *Then for all* $s > 1 - 2/q$,

$$\left\| \sup_{t\in\mathbb{R}} \left| e^{it\Delta} f \right| \right\|_{L^q(\mathbb{R}^2)} \le C_s \|f\|_{H^s(\mathbb{R}^2)}.$$

Taking more care with the range of r, we will also improve Planchon's estimate.

Theorem (2.1.7)[59]: *Let* $n = 2$. *Then Conjecture 1 is true for* $q > 16/5$.

To illustrate, this is a nonendpoint version of

$$\|e^{it\Delta}f\|_{L_x^{16/5}(\mathbb{R}^2, L_t^{16}(\mathbb{R}))} \le C\|f\|_{\dot H^{1/4}(\mathbb{R}^2)}.$$

We follow the approach of Lee in that we adapt the proof of Tao's bilinear theorem [21], rather than applying the estimate directly.

Throughout, c and C will denote positive constants that may depend on the dimensions and exponents of the Lebesgue spaces. The constants C will sometimes depend on the small parameters ε, δ and β, but never on the functions f or g, and never on the large parameters R or N. It will occasionally be made explicit when they depend on other factors like the Sobolev index. Their values may change from line to line. The following are notations that will be used frequently:

$L_x^q(\mathbb{R}^n, L_t^r(I))$: the Lebesgue space with norms $\left(\int_{\mathbb{R}^n}\left|\int_I \ |f(x,t)|^r dt\right|^{q/r} dx\right)^{1/q}$

D^s: the derivative defined by $\widehat{D^s g}(\xi) = (2\pi|\xi|)^s \hat g(\xi)$

$\dot H^s(\mathbb{R}^n)$: the homogeneous Sobolev space with s derivatives in $L^2(\mathbb{R}^n)$

$H^s(\mathbb{R}^n)$: the inhomogeneous Sobolev space with s derivatives in $L^2(\mathbb{R}^n)$

$\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| \le 1\}$

$B_1(Ne_1) := \{\xi \in \mathbb{R}^n : |\xi - Ne_1| \le 1\}$

ξ_j: a member of the lattice $R^{-1/2}\mathbb{Z}^n$

x_k: a member of the lattice $R^{1/2}\mathbb{Z}^n$

$T_{jk} := \{(x,t) \in \mathbb{R}^n \times [0,R] : |x - (x_k + 4\pi t\xi_j)| \le R^{1/2}\}$.

$Q_R := [-R/4, R/4] \times \ldots \times [-R/4, R/4]$

$P_R(l) := \{(x,t) \in \mathbb{R}^n \times [R/2, R] : x - (lR/2 + 4\pi tN)e_1 \in Q_R\}$

$s(q,r) := n(1/2 - 1/q) - 2/r$

$q_\kappa := \dfrac{n + 1 + 8\kappa}{n + 4\kappa}$

$\hat\psi$: a positive and smooth function, supported in $B_{\sqrt{n}}$.

$\hat\eta$: a positive and smooth function, supported in $\mathbb{B}^n$, and equal to 1 at the origin.

The following lemma provides convenient estimates with which we will interpolate.

Lemma (2.1.8)[59]: *For all $N \gg 1, r \ge 2$, and f frequency supported in $B_1(Ne_1)$,*

$$\|e^{it\Delta}f\|_{L_x^\infty(\mathbb{R}^n, L_t^r(\mathbb{R}))} \le CN^{-1/r}\|f\|_{L^2(\mathbb{R}^n)}.$$

Proof. We suppose that $n \ge 2$; the 1-dimensional case was proven in [9]. By interpolation with the trivial L^∞-estimate, we may also take $r = 2$. By writing the square as a double integral,

$$\|e^{it\Delta}f(x)\|_{L_t^2(\mathbb{R})}^2 = \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat f(\xi)\hat f(y) e^{2\pi i\left(x\cdot(\xi-y) - 4\pi t\left(|\xi|^2 - |y|^2\right)\right)} d\xi dy dt,$$

so that, by an application of Fubini, and integrating in t,

$$\|e^{it\Delta}f(x)\|_{L_t^2(\mathbb{R})}^2 \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\hat f(\xi)\hat f(y)|}{\left||\xi|^2 - |y|^2\right|} d\xi dy.$$

Writing $|\xi|^2 - |y|^2 = (\xi + y)\cdot(\xi - y)$, and recalling that $y, \xi \in B_1(Ne_1)$, we see that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\hat{f}(\xi)\hat{f}(y)|}{||\xi|^2-|y|^2|}d\xi dy \le \frac{C}{N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\hat{f}(\xi)\hat{f}(y)|}{|\xi-y|}d\xi dy\,.$$

Thus, by the Hardy–Littlewood–Sobolev inequality,

$$\|e^{it\Delta}f(x)\|^2_{L^2_t(\mathbb{R})} \le CN^{-1}\|\hat{f}\|^2_{L^{\frac{2n}{2n-1}}(\mathbb{R}^n)},$$

and, as $\operatorname{supp}\hat{f} \subset B_1(Ne_1)$, by Hölder and Plancherel we complete the proof.

As in the arguments of Fefferman [30], Bourgain [26], Wolff [45], Tao [21], and Lee [11], we decompose into wave-packets at scale $R \gg 1$.

Fix a positive and smooth function $\hat{\psi}$, supported in $B_{\sqrt{n}}$, such that

$$\sum_j \hat{\psi}(\xi - R^{1/2}\xi_j) = 1,$$

where $\xi_j \in R^{-1/2}\mathbb{Z}^n$. We also fix a positive and smooth function $\hat{\eta}$, supported in $\mathbb{B}^n$and equal to one at the origin, so that by the Poisson summation formula,

$$\sum_k \eta\left(x - \frac{x_k}{R^{1/2}}\right) = 1,$$

where $x_k \in R^{1/2}\mathbb{Z}^n$. Now, for any Schwartz function $f$we have the decompositions

$$\hat{f}(\xi) = \sum_j \hat{f}_j(\xi) = \sum_j \hat{\psi}\left(R^{1/2}(\xi - \xi_j)\right)\hat{f}(\xi), \tag{2}$$

$$f(x) = \sum_{j,k} f_{jk}(x) = \sum_{j,k}\eta\left(\frac{x-x_k}{R^{1/2}}\right)f_j(x). \tag{3}$$

Note that $\hat{f}_{jk}$is supported in the ball of radius $(\sqrt{n}+1)R^{-1/2}$ with centre ξ_j.

We recall the Hardy–Littlewood maximal operator $M : L^1_{loc}(\mathbb{R}^n) \to L^1_{loc}(\mathbb{R}^n)$ defined by

$$Mf(x) = \sup_{r>0}\frac{1}{|B_r|}\int_{B_r}|f(y-x)|dy\,.$$

For a proof of the following lemma see [21] or [35].

Lemma (2.1.9)[59]: *Let $t \in [-R, R]$. Then for all $K \in \mathbb{N}$there exist constants C_K, such that*

$$|e^{it\Delta}f_{jk}(x)| \le C_K Mf_j(x_k)\left(1 + \frac{|x-(x_k+4\pi t\xi_j|}{R^{1/2}}\right)^{-K}.$$

We note that when $t \in [0, R]$, the wave-packets $e^{it\Delta}f_{jk}$are essentially supported in the tubes $T_{jk}$with direction $(4\pi\xi_j, 1)$defined by

$$T_{jk} = \{(x,t) \in \mathbb{R}^n \times [0,R] : |x-(x_k+4\pi t\xi_j)| \le R^{1/2}\}.$$

We see that a translation of the frequency support of the data corresponds to an affine translation of the essential supports of the wave-packets.

Similarly, for $l \in \mathbb{Z}$, we define parallelepipeds $P_R(l)$by

$$P_R(l) = \{(x,t) \in \mathbb{R}^n \times [R/2, R] : x - (lR/2 + 4\pi tN)e_1 \in Q_R\},$$

where $Q_R$is the n-dimensional cube of side $R/2$, centred at the origin. Note that when $\xi_j \in B_1(Ne_1)$, the tubes and parallelepipeds point approximately in the same direction.

Definition (2.1.10)[59]: We say that E_1 and E_2 are *1-separated* if they are measurable sets that satisfy

$$\inf\{|\xi_1-\xi_2| : \xi_1 \in E_1, \xi_2 \in E_2\} \geq 1/2.$$

The following lemma is a key ingredient. It allows us to deduce estimates on balls from estimates restricted to parallelepipeds. We will see later that parallelepipeds are the natural domain on which to attack the problem.

Lemma (2.1.11)[59]: *Let $r \geq q$ and $\alpha \geq \frac{1}{q}-\frac{1}{r}$. Suppose that*

$$\left\|e^{it\Delta}f e^{it\Delta}g\right\|_{L_x^q L_t^r(P_R(0))} \leq CR^{\varepsilon}N^{\alpha}\|f\|_2\|g\|_2$$

whenever $R, N \gg 1$, and $\hat{f}, \hat{g}$ are supported on 1-separated subsets of $B_1(Ne_1)$. Then

$$\left\|e^{it\Delta}f e^{it\Delta}g\right\|_{L_x^q(Q_R, L_t^r[R/2,R])} \leq CR^{\varepsilon}N^{\alpha}\|f\|_2\|g\|_2.$$

Proof. We decompose the solution into wave-packets at scale R,

$$e^{it\Delta}f = \sum_{j,k} e^{it\Delta}f_{jk}\,.$$

Letting P_l denote the short, fat tubes defined by

$$P_l = \{(x,t) \in \mathbb{R}^n \times [R/2, R] : |x-(lR/2+4\pi tN)e_1| \leq 50R\},$$

where $l \in \mathbb{Z}$, we write

$$f_l = \sum_{j,k:T_{jk}\cap P_l \neq \emptyset} f_{jk}\,,$$

so that $e^{it\Delta}f_l$ consists of the wave-packets that pass near to $P_R(l)$. As the tubes and the parallelepipeds point in essentially the same direction, a tube T_{jk} can intersect P_l for at most a constant number of l, so we note for later that

$$\begin{aligned}\sum_l \|f_l\|_{L^2(\mathbb{R}^n)}^2 &\leq C\sum_l \sum_{j,k:T_{jk}\cap P_l\neq\emptyset} \left\|f_{jk}\right\|_{L^2(\mathbb{R}^n)}^2\\ &\leq C\sum_{j,k}\left\|f_{jk}\right\|_{L^2(\mathbb{R}^n)}^2\\ &\leq C\|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

and we will refer to this as almost orthogonality.

We consider the pointwise bound

$$\left|e^{it\Delta}f\right| \leq \left|e^{it\Delta}f_l\right| + \left|\sum_{j,k:T_{jk}\cap P_l\neq\emptyset} e^{it\Delta}f_{jk}\right|, \tag{4}$$

and use the rapid decay to show that the last term is of negligible size on $P_R(l)$.

Writing $x = x - 4\pi tNe_1$, we have $\left|x-\left(x_k+4\pi t\xi_j\right)\right| \approx |x-x_k|$ whenever $(x,t) \in P_R(l)$ and $T_{jk} \cap P_l = \emptyset$, so by Lemma (2.1.9),

$$\left|\sum_{j,k:T_{jk}\cap P_l=\emptyset} e^{it\Delta}f_{jk}(x)\right| \leq C_{K'}R^{K'/2}\sum_{j=1}^{cR^{n/2}}\sum_{k:|\bar{x}-x_k|\geq R}\frac{Mf_j(x_k)}{|\bar{x}-x_k|^{K'}}$$

for all $K' \in \mathbb{N}$. Choosing K' sufficiently large, we see that for all $K \in \mathbb{N}$,

$$\left|\sum_{j,k:T_{jk}\cap P_l=\emptyset} e^{it\Delta}f_{jk}(x)\right| \leq C_K R^{-K} \sum_{j=1}^{cR^{n/2}} \sum_{k:|\bar{x}-x_k|\geq R} \frac{Mf_j(x_k)}{|\bar{x}-x_k|^{2n}}. \tag{5}$$

Writing $\psi_R = R^{-n/2}\psi(R^{-1/2}\,\cdot)$, by (2) we have

$$|f_j| = |\psi_R * f|, \tag{6}$$

so that $Mf_j(x') \approx Mf_j(x_k)$whenever $|x'-x_k| \leq \sqrt{n}R^{1/2}$. Now observe that

$$\begin{aligned}\sum_{k:|\bar{x}-x_k|\geq R} \frac{Mf_j(x_k)}{|\bar{x}-x_k|^{2n}} &\leq CR^{-n/2}\left(1+\frac{|\cdot|}{R^{1/2}}\right)^{-2n} * Mf_j(\bar{x}) \\ &\leq CMMf_j(\bar{x}),\end{aligned} \tag{7}$$

so the error term is not only going to be small, but also square integrable. Substituting (6) and (7) into (5),

$$\left|\sum_{j,k:T_{jk}\cap P_l=\emptyset} e^{it\Delta}f_{jk}(x)\right| \leq C_K R^{-K} MM[\psi_R * f](x),$$

and substituting this into (4), we see that for all $K \in \mathbb{N}$ there exist $C_K$such that

$$|e^{it\Delta}f(x)| \leq |e^{it\Delta}f_l(x)| + C_K R^{-K} MM[\psi_R * f](x - 4\pi t N e_1)$$

whenever $(x,t) \in P_R(l)$.

We use these pointwise bounds on parallelepipeds, to obtain an $L^q(Q_R, L_t^r[R/2,R])$bound. Fix a large $K$and define $Lf(x,t) := R^{-K}MM[\psi_R * f](x-4\pi tNe_1)$. We also write $\bar{P}_R(l) := Q_R \times [R/2,R] \cap P_R(l)$, so that by concavity

$$\begin{aligned}\|e^{it\Delta}fe^{it\Delta}g\|^q_{L^q(Q_R,L_t^r[R/2,R])} &\leq \sum_l \|e^{it\Delta}fe^{it\Delta}g\|^q_{L_x^qL_t^r(\bar{P}_R(l))} \\ &\leq C_K^{2q}\sum_l \|(|e^{it\Delta}f_l| + Lf)(|e^{it\Delta}g_l| + Lg)\|^q_{L_x^qL_t^r(\bar{P}_R(l))} \\ &\leq C_K^{2q}\sum_l \|e^{it\Delta}f_le^{it\Delta}g_l\|^q_{L_x^qL_t^r(\bar{P}_R(l))} + \|Lfe^{it\Delta}g_l\|^q_{L_x^qL_t^r(\bar{P}_R(l))} \\ &\quad + \|e^{it\Delta}f_lLg\|^q_{L_x^qL_t^r(\bar{P}_R(l))} + \|Lf\,Lg\|^q_{L_x^qL_t^r(\bar{P}_R(l))}.\end{aligned} \tag{8}$$

Now, by two applications of Hölder,

$$\begin{aligned}\sum_l \|Lf\|^{2q}_{L_x^{2q}L_t^{2r}(Q_R\times[R/2,R]\cap P_R(l))} &\leq CR^{nq\left(\frac{1}{q}-\frac{1}{r}\right)}\sum_{l=-N}^{N}\|Lf\|^{2q}_{L_x^{2r}L_t^{2r}(P_R(l))} \\ &\leq CR^{nq\left(\frac{1}{q}-\frac{1}{r}\right)}N^{q\left(\frac{1}{q}-\frac{1}{r}\right)}\left(\sum_l\|Lf\|^{2r}_{L_x^{2r}L_t^{2r}(P_R(l))}\right)^{\frac{q}{r}}\end{aligned}$$

By summing up, applying Fubini and making an affine change of variables,

$$\begin{aligned}\sum_l \|Lf\|^{2r}_{L_x^{2r}L_t^{2r}(P_R(l))} &\leq CR^{-2rK+1}\|MM[\psi_R * f]\|^{2r}_{L_x^{2r}(\mathbb{R}^n)} \\ &\leq CR^{-2rK+1}\|f\|^{2r}_{L^{2r}(\mathbb{R}^n)},\end{aligned}$$

where the second inequality is by the Hardy–Littlewood maximal theorem and Young's inequality.

As $\hat{f}$ is supported in $B_1(Ne_1)$, together with Bernstein's inequality, these estimates yield

$$\sum_l \|Lf\|^{2r}_{L_x^{2r}L_t^{2r}(P_R(l))} \leq CR^{-qK}N^{q\left(\frac{1}{q}-\frac{1}{r}\right)}\|f\|^{2q}_{L^2(\mathbb{R}^n)}.$$

We have the same inequality for g, so that, by two applications of Cauchy–Schwarz,

$$\|Lf\ Lg\|^q_{L_x^qL_t^r(\bar{P}_R(l))} \leq CR^{-qK}N^{q\left(\frac{1}{q}-\frac{1}{r}\right)}\|f\|_2^q\|g\|_2^q.$$

On the other hand, by Hölder and Lemma (2.1.8),

$$\begin{aligned}\left\|e^{it\Delta}f_l\right\|_{L_x^{2q}L_t^{2r}(\bar{P}_R(l))} &\leq CR^{\frac{n}{2q}}\left\|e^{it\Delta}f_l\right\|_{L_x^\infty L_t^{2r}(\mathbb{R}^{n+1})}\\ &\leq CR^{\frac{n}{2q}}N^{-\frac{1}{2r}}\|f_l\|_2.\end{aligned}$$

Thus, by two applications of Cauchy–Schwarz,

$$\begin{aligned}\sum_l \left\|e^{it\Delta}f_lLg\right\|^q_{L_x^qL_t^r(\bar{P}_R(l))} &\leq CR^{-\frac{qK}{2}}N^{\frac{q}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\left(\sum_l\left\|e^{it\Delta}f_l\right\|^{2q}_{L_x^{2q}L_t^{2r}(\bar{P}_R(l))}\right)^{1/2}_{\|g\|_2^q}\\ &\leq CR^{-\frac{qK}{4}}N^{q\left(\frac{1}{q}-\frac{1}{r}\right)}\left(\sum_l\|f_l\|_2^{2q}\right)^{1/2}\|g\|_2^q\\ &\leq CR^{-\frac{qK}{4}}N^{q\left(\frac{1}{q}-\frac{1}{r}\right)}\|f\|_2^q\|g\|_2^q,\end{aligned}$$

where in the third inequality we have used convexity and the almost orthogonality derived earlier. Similarly, we have

$$\sum_l\left\|Lfe^{it\Delta}g_l\right\|^q_{L_x^qL_t^r(\bar{P}_R(l))} \leq CR^{-\frac{qK}{4}}N^{q\left(\frac{1}{q}-\frac{1}{r}\right)}\|f\|_2^q\|g\|_2^q.$$

Finally, by spatial translation invariance and the hypothesis,

$$\left\|e^{it\Delta}f_le^{it\Delta}g_l\right\|_{L_x^qL_t^r(P_R(l))} \leq CR^\varepsilon N^\alpha\|f_l\|_2\|g_l\|_2,$$

so that, by Cauchy–Schwarz,

$$\begin{aligned}\sum_l\left\|e^{it\Delta}f_le^{it\Delta}g_l\right\|^q_{L_x^qL_t^r(P_R(l))} &\leq CR^{q\varepsilon}N^{q\alpha}\left(\sum_l\|f_l\|_2^{q2}\right)^{1/2}\left(\sum_l\|g_l\|_2^{q2}\right)^{1/2}\\ &\leq CR^{q\varepsilon}N^{q\alpha}\|f\|_2^q\|g\|_2^q,\end{aligned}$$

again using convexity and the almost orthogonality.

Comparing the terms in (8), we see that

$$\left\|e^{it\Delta}fe^{it\Delta}g\right\|_{L^q(Q_R,L_t^r[R/2,R])} \leq R^\varepsilon N^\alpha\|f\|_2\|g\|_2,$$

and we are done.

The following mixed norm 'epsilon removal' lemma is due to Lee and Vargas [50] (see also [2,57]). In their work, the spatial integral is evaluated before the temporal integral and as such the estimates are invariant under translation on the frequency side. A careful reading of the proof reveals that only small changes are required to reverse the order.

Lemma (2.1.12)[59]: *Suppose that for all* $\varepsilon > 0$ *and* $\alpha > \frac{1}{q_0}-\frac{1}{r_0}$,

$$\left\|e^{it\Delta}fe^{it\Delta}g\right\|_{L_x^{q_0}(Q_R,L_t^{r_0}[R/2,R])} \leq C_{\varepsilon,\alpha}R^\varepsilon N^\alpha\|f\|_2\|g\|_2$$

whenever $R, N \gg 1$, *and* $\hat{f}, \hat{g}$ *are supported on* 1*-separated subsets of* $B_1(Ne_1)$. *Then provided that*

$\frac{q}{r} > \frac{q_0}{r_0}, q\left(1-\frac{1}{r}\right) > q_0\left(1-\frac{1}{r_0}\right)$, *and* $\alpha > \frac{1}{q}-\frac{1}{r}$,

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^qL_t^r(\mathbb{R}^{n+1})} \leq C_{q,r,\alpha}N^{\alpha}\|f\|_2\|g\|_2.$$

Proof. The proof is the same as that of Lemma 4.4 and Remark 4.5 in [50], with the following changes:

The measures $d\sigma_i$ are replaced by the canonical pull-back measure on

$$\{(\xi,-2\pi|\xi|^2)\in\mathbb{R}^{n+1}:\xi\in B_1(Ne_1)\}$$

which we denote by $d\sigma_N$. By a well-known calculation,

$$\left|\widehat{d\sigma_N}(x,t)\right| = \left|e^{it\Delta}\left(\chi_{B_1}(Ne_1)\right)^{\vee}(x)\right| \leq C(1+|x-4\pi tNe_1|+|t|)^{-n/2}$$
$$\leq CN^{n/2}(1+|x|+|t|)^{-n/2}.$$

We replace the estimate

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^{q_0}L_t^{r_0}(Q)} \leq C_\varepsilon R^\varepsilon\|f\|_2\|g\|_2$$

for all $n+1$ dimensional cubes Q of side length $R/2$, by

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^{q_0}L_t^{r_0}(Q)} \leq C_{\varepsilon,\alpha}R^\varepsilon N^\alpha\|f\|_2\|g\|_2 \quad (9)$$

for all $\alpha > \frac{1}{q_0}-\frac{1}{r_0}$, which follows from the hypothesis and translation invariance. The estimate

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^\infty L_x^1(\mathbb{R}^{n+1})} \leq \|f\|_2\|g\|_2,$$

is replaced with

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^\infty L_x^1(\mathbb{R}^{n+1})} \leq CN^{-1}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$$
$$= CN^{\frac{1}{\infty}-\frac{1}{1}}\|f\|_2\|g\|_2, \quad (10)$$

Which follows by Cauchy-Schwarz from Lemma (2.1.8). The third interpolation point is unchanged

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^\infty L_x^\infty(\mathbb{R}^{n+1})} \leq C\|f\|_2\|g\|_2$$
$$= CN^{\frac{1}{\infty}-\frac{1}{\infty}}\|f\|_2\|g\|_2. \quad (11)$$

Interpolating between (9), (10), and (11), we note that

$$\begin{aligned}\alpha_\theta &:= \theta\alpha_0+(1-\theta)\alpha_1\\ &\geq \theta\left(\frac{1}{q_0}-\frac{1}{r_0}\right)+(1-\theta)\left(\frac{1}{q_1}-\frac{1}{r_1}\right)\\ &= \left(\frac{\theta}{q_0}+\frac{1-\theta}{q_1}\right)-\left(\frac{\theta}{r_0}+\frac{1-\theta}{r_1}\right)\\ &=: \frac{1}{q_\theta}-\frac{1}{r_\theta},\end{aligned}$$

so that the powers of N behave as desired.

We will require a version of the previous lemma for dealing with nonsharp powers of N. Note that the interpolation points with $q=\infty$ of the previous proof are α-improving so that the following lemma follows in the same way.

Lemma (2.1.13)[59]: *Suppose that for some* $\alpha_0 > 0$ *and for all* $\varepsilon > 0$,

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^{q_0}(Q_R,L_t^{r_0}[R/2,R])} \leq C_\varepsilon R^\varepsilon N^{\alpha_0}\|f\|_2\|g\|_2$$

whenever $R,N \gg 1$, and $\hat{f},\hat{g}$ are supported on 1-separated subsets of $B_2(Ne_1)$. Then provided that $\frac{q}{r} >$

$\frac{q_0}{r_0}, q\left(1-\frac{1}{r}\right) > q_0\left(1-\frac{1}{r_0}\right)$, *and* $\alpha > \alpha_0$,

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^qL_t^r(\mathbb{R}^{n+1})} \le C_{q,r,\alpha}N^\alpha\|f\|_2\|g\|_2.$$

By the globalizing lemmas, it will suffice to prove local estimates.

Definition (2.1.14)[59]: Let $R^*(2\times 2 \to q,r,\alpha,\beta)$ denote the estimate

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^qL_t^r(P_R)} \le CR^\beta N^\alpha\|f\|_2\|g\|_2$$

whenever $R, N \gg 1, \hat{f}, \hat{g}$ are supported on 1-separated subsets of $B_1(Ne_1)$, and P_R is a parallelepiped of side $R/2$ and direction $(4\pi Ne_1, 1)$.

Recall the notional estimate

$$\left\|\sup_{0<t<1}\left|e^{it\Delta}f\right|\right\|_{L_x^2(\mathbb{B}^n)} \le C\|f\|_{H^{1/4+\kappa}}, \qquad (A_\kappa)$$

and the dual exponents q_κ and q'_κ defined by

$$q_\kappa = \frac{n+1+8\kappa}{n+4\kappa} \text{ and } q'_\kappa = \frac{n+1+8\kappa}{1+4\kappa}.$$

Theorem (2.1.15)[59]: Suppose that (A_κ) holds. Then for all $q > q_\kappa, r > q'_\kappa$ and $\alpha > \frac{n}{q'_\kappa} - \frac{1}{r}$,

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^qL_t^r\left(\mathbb{R}^n, L_t^r(\mathbb{R})\right)} \le C_\alpha N^\alpha\|f\|_2\|g\|_2$$

whenever $N \gg 1$, and $\hat{f}, \hat{g}$ are supported on *1*-separated subsets of $B_1(Ne_1)$.

Proof. As f is frequency supported in $B_1(Ne_1)$, it is easy to calculate that the temporal Fourier transform of $e^{it\Delta}f$ is supported in an interval of length CN. Similarly this is true for $e^{it\Delta}fe^{it\Delta}g$, so that by Bernstein's inequality,

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^r(\mathbb{R})} \le CN^{\frac{1}{p}-\frac{1}{r}}\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^p(\mathbb{R})}.$$

Thus, by Lemmas (2.1.11) and (2.1.13), it will be enough to show that

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P_R)} \le C_\beta R^\beta N^{\frac{n}{q'_\kappa}-\frac{1}{q'_\kappa}}\|f\|_2\|g\|_2 \qquad (12)$$

whenever $R \gg 1, \beta > 0$, and P_R is of side $R/2$ and direction $(4\pi Ne_1, 1)$.

We proceed by induction on scales. As P_R is contained in a cuboid, with long side $4\pi RN$, and short side R, by Hölder,

$$\begin{aligned}\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P_R)} &\le C(R^nN)^{\frac{1}{q_\kappa}-\frac{1}{q'_\kappa}}\|e^{it\Delta}fe^{it\Delta}g\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(\mathbb{R}^{n+1})} \\ &\le C(R^nN)^{\frac{1}{q_\kappa}-\frac{1}{q'_\kappa}}\|f\|_2\|g\|_2,\end{aligned}$$

where the second inequality is by Cauchy–Schwarz, Fubini, and the linear Strichartz estimates of Theorem (2.1.11). Thus we have $R^*(2\times 2 \to q_\kappa, q'_\kappa, (n-1)/q'_\kappa, \beta)$ for some large β. In fact we have a better power of α here than the $(n-1)/q'_\kappa$ that we get in the induction step. From now on we denote $(n-1)/q'_\kappa$ by α_κ. It will suffice to prove

$$R^*(2\times 2 \to q_\kappa, q'_\kappa, \alpha_\kappa, \beta) \Rightarrow R^*(2\times 2 \to q_\kappa, q'_\kappa, \alpha_\kappa, \max\{(1-\delta)\beta, c\delta\} + \varepsilon)$$

for all δ and $\varepsilon > 0$, where c is independent of δ and ε, as (12) would follow by iteration.

First we consider the problem when the frequency supports are close to the origin. We define $\tilde{f}$ and $\tilde{g}$ by

$$\hat{\tilde{f}} = \hat{f}(\xi - Ne_1) \text{ and } \hat{\tilde{g}} = \hat{g}(\xi - Ne_1),$$

and we break up the solutions into wave-packets at scale R, so that

$$e^{it\Delta}\tilde{f} = \sum_{j,k} e^{it\Delta}\tilde{f}_{jk} \text{ and } e^{it\Delta}\tilde{g} = \sum_{j,k} e^{it\Delta}\tilde{g}_{jk}.$$

Recall that the wave-packets $e^{it\Delta}\tilde{f}_{jk}$ are essentially supported on tubes $\tilde{T}_{jk}$, and we denote the tubes associated to $e^{it\Delta}\tilde{g}_{jk}$ by $\tilde{T}'_{jk}$. We also cover the cube $Q_R \times [R/2, R]$ by cubes $\tilde{P} \in \tilde{\mathcal{P}}$ of side $R^{1-\delta}$. The following orthogonality lemma is the key ingredient of Tao's bilinear restriction theorem.

Lemma (2.1.16)[59]: [21] *There exists a relationship* $\sim$ *between tubes* $\tilde{T}_{jk}$ *and cubes* $\tilde{P}$ *such that, for all tubes* $\tilde{T}_{jk}$,

$$\# \quad \{\tilde{P} \in \tilde{\mathcal{P}} : \tilde{T}_{jk} \sim \tilde{P}\} \leq CR^{\varepsilon}, \tag{13}$$

and for a constant c independent of δ and ε,

$$\left\| \left(\sum_{\tilde{T}_{jk} \sim \tilde{P}} e^{it\Delta}\tilde{f}_{jk} \right) \left(\sum_{\tilde{T}'_{jk} \nsim \tilde{P}} e^{it\Delta}\tilde{g}_{jk} \right) \right\|_{L^2(\tilde{P})} \leq CR^{\varepsilon + c\delta - \frac{n-1}{4}} \|f\|_2 \|g\|_2,$$

and

$$\left\| \left(\sum_{\tilde{T}_{jk} \nsim \tilde{P}} e^{it\Delta}\tilde{f}_{jk} \right) \left(\sum_{\tilde{T}'_{jk} \nsim \tilde{P}} e^{it\Delta}\tilde{g}_{jk} \right) \right\|_{L^2(\tilde{P})} \leq CR^{\varepsilon + c\delta - \frac{n-1}{4}} \|f\|_2 \|g\|_2.$$

see [21] for the precise definition of the relation $\sim$. It can be thought of as saying that the wave-packets are concentrated on the cubes.

As a translation of the frequency supports corresponds to an affine translation of the spatial support, returning to the original problem, we can suppose that P_R is the affine translation of $Q_R \times [R/2, R]$ under the mapping $x_1 \to x_1 + 4\pi t N e_1$. We cover this by parallelepipeds $P \in \mathcal{P}$ that correspond to the cubes $\tilde{P}$ under the same affine translation. Similarly we break up the solutions into wave-packets with associated tubes T_{jk} and T'_{jk}, that correspond to $\tilde{T}_{jk}$ and $\tilde{T}'_{jk}$ under the affine translation. Thus, we have the induced relation $T_{jk} \sim P$ if $\tilde{T}_{jk} \sim \tilde{P}$.

As we have covered P_R by smaller parallelepipeds P, by the triangle inequality, it will suffice to show

$$\sum_{P \in \mathcal{P}} \left\| \left(\sum_{T_{jk}} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk}} e^{it\Delta} g_{jk} \right) \right\|_{L_x^{q_\kappa} L_t^{q'_\kappa}(P)} \leq C_\beta R^{\max\{(1-\delta)\beta, c\delta\} + \varepsilon} N^{\alpha_\kappa} \|f\|_2 \|g\|_2.$$

By the triangle inequality again, it will suffice to bound the 'local' part,

$$\sum_{P \in \mathcal{P}} \left\| \left(\sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk} \sim P} e^{it\Delta} g_{jk} \right) \right\|_{L_x^{q_\kappa} L_t^{q'_\kappa}(P)}$$

and the 'global' parts,

$$\sum_{P \in \mathcal{P}} \left\| \left(\sum_{T_{jk} \sim P} e^{it\Delta} f_{jk} \right) \left(\sum_{T'_{jk} \nsim P} e^{it\Delta} g_{jk} \right) \right\|_{L_x^{q_\kappa} L_t^{q'_\kappa}(P)},$$

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\nsim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\sim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)},$$

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\nsim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)}.$$

To bound the local part, we simply invoke the induction hypothesis;

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^{q_\kappa}L_t^{q'_\kappa}(P)}$$

$$\le\sum_{P\in\mathcal{P}}CR^{(1-\delta)\beta}N^{\alpha_\kappa}\left\|\sum_{T_{jk}\sim P}f_{jk}\right\|_2\left\|\sum_{T'_{jk}\nsim P}g_{jk}\right\|_2$$

$$\le CR^{(1-\delta)\beta}N^{\alpha_\kappa}\left(\sum_{P\in\mathcal{P}}\left\|\sum_{T_{jk}\sim P}f_{jk}\right\|_2^2\right)^{1/2}\left(\sum_{P\in\mathcal{P}}\left\|\sum_{T'_{jk}\nsim P}g_{jk}\right\|_2^2\right)^{1/2}$$

$$\le CR^{(1-\delta)\beta+\varepsilon}N^{\alpha_\kappa}\|f\|_2\|g\|_2,$$

where the second inequality is by Cauchy–Schwarz, and the third by (13) and almost orthogonality. This bound is acceptable.

Considering the first global part, by Fubini and the affine change of variables $x_1\to x_1+4\pi tNe_1$, followed by Lemma (2.1.16), we have

$$\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^2L_t^2(P)}\le CR^{\varepsilon+c\delta-\frac{n-1}{4}}\|f\|_2\|g\|_2. \qquad (14)$$

On the other hand, by scaling and the hypothesis,

$$\left\|\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right\|_{L_x^2L_t^\infty(B_{NR})}\le C(RN^2)^{1/4+\kappa}\left\|\sum_{T_{jk}\sim P}f_{jk}\right\|_2$$

$$\le C(RN^2)^{1/4+\kappa}\|f\|_2.$$

Similarly

$$\left\|\sum_{T_{jk}\nsim P}e^{it\Delta}g_{jk}\right\|_{L_x^2L_t^\infty(B_{NR})}\le C(RN^2)^{1/4+\kappa}\|g\|_2,$$

so that by Cauchy–Schwarz,

$$\left\|\left(\sum_{T_{jk}\sim P}e^{it\Delta}f_{jk}\right)\left(\sum_{T'_{jk}\nsim P}e^{it\Delta}g_{jk}\right)\right\|_{L_x^1L_t^\infty(P)}\le C(RN^2)^{1/2+2\kappa}\|f\|_2\|g\|_2. \qquad (15)$$

Interpolating between (14) and (15), using Hölder, gives

$$\left\|\left(\sum_{T_{jk}\sim P} e^{it\Delta} f_{jk}\right)\left(\sum_{T'_{jk}\nsim P} e^{it\Delta} g_{jk}\right)\right\|_{L_x^{q_\kappa} L_t^{q'_\kappa}(P)} \leq CR^{\varepsilon+c\delta} N^{\alpha_\kappa} \|f\|_2 \|g\|_2,$$

so that, by summing,

$$\sum_P \left\|\left(\sum_{T_{jk}\sim P} e^{it\Delta} f_{jk}\right)\left(\sum_{T'_{jk}\nsim P} e^{it\Delta} g_{jk}\right)\right\|_{L_x^{q_\kappa} L_t^{q'_\kappa}(P)} \leq CR^{(c+n+1)\delta+\varepsilon} N^{\alpha_\kappa} \|f\|_2 \|g\|_2,$$

which is acceptable. The other two global parts are bounded in the same way, which completes the proof.

We now pass to the unconditional result in which the powers of $N$are improved. we will see that this improvement allows us to obtain the almost optimal range of $r$in Theorem (2.1.15). A refinement of Lemma (2.1.12), which preserved the precise powers of N, would allow $\alpha$to equal $1/q-1/r$in the following.

Theorem (2.1.17)[59]: *Suppose that $q \in \left(\frac{8}{5}, \frac{5}{3}\right)$and $\frac{4}{q}+\frac{1}{r}<3$. Then for all $\alpha > \frac{1}{q}-\frac{1}{r}$,*

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_x^q\left(\mathbb{R}^2, L_t^r(\mathbb{R})\right)} \leq C_\alpha N^\alpha \|f\|_2 \|g\|_2$$

whenever$N \gg 1$, and$\hat{f}, \hat{g}$are supported on1-separated subsetsof$B_1(Ne_1)$.

Proof. Combining the bilinear theorem of Tao [28] with Bernstein's inequality as before, we see that

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_x^q\left(\mathbb{R}^2, L_t^r(\mathbb{R})\right)} \leq CN^{\frac{1}{q}-\frac{1}{r}} \|f\|_2 \|g\|_2 \qquad (16)$$

for all $r \geq q > 5/3$. Now, by interpolation combined with Lemmas (2.1.11) and (2.1.12), it will suffice to show that

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_x^{8/5} L_t^2(P_R)} \leq CR^\beta N^{1/8} \|f\|_2 \|g\|_2$$

whenever $R \gg 1, \beta > 0$, and $P_R$has side $R/2$ and direction $(4\pi N e_1, 1)$.

Again, we proceed by induction on scales. As $P_R$is contained in a cuboid, with long side $4\pi RN$, and short side R, by Hölder,

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_x^{8/5} L_t^2(P_R)} \leq C(R^2 N)^{1/8} \|e^{it\Delta} f e^{it\Delta} g\|_{L_x^{8/5} L_t^2(\mathbb{R}^{2+1})},$$

so that by (16), we have

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_x^{8/5} L_t^2(P_R)} \leq C(R^2 N)^{1/8} \|f\|_2 \|g\|_2.$$

We see that $R^*(2\times 2 \to 8/5, 2, 1/8, \beta)$holds for a large β. Therefore, by iterating,it will suffice to prove that

$$R^*(2\times 2 \to 8/5, 2, 1/8, \beta) \Rightarrow R^*(2\times 2 \to 8/5, 2, 1/8, \max\{(1-\delta)\beta, c\delta\}+\varepsilon)$$

for all $\delta$and $\varepsilon > 0$, where the constant $c$is independent of $\delta$and ε.

As before, we cover $P_R$by smaller parallelepipeds P, so that it will suffice to bound the local part,

$$\sum_{P\in\mathcal{P}} \left\|\left(\sum_{T_{jk}\sim P} e^{it\Delta} f_{jk}\right)\left(\sum_{T'_{jk}\sim P} e^{it\Delta} g_{jk}\right)\right\|_{L_x^{8/5} L_t^2(P)},$$

which is dealt with via the induction hypothesis, and the global parts of type

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P} e^{it\Delta} f_{jk}\right)\left(\sum_{T'_{jk}\nsim P} e^{it\Delta} g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)}.$$

By Hölder, followed by Fubini and the affine change of variables $x_1 \to x_1 + 4\pi t N e_1$,

$$\left\|\left(\sum_{T_{jk}\sim P} e^{it\Delta} f_{jk}\right)\left(\sum_{T'_{jk}\nsim P} e^{it\Delta} g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)}$$
$$\leq (R^2N)^{1/8}\left\|\left(\sum_{\tilde{T}_{jk}\sim P} e^{it\Delta} \tilde{f}_{jk}\right)\left(\sum_{\tilde{T}'_{jk}\nsim P} e^{it\Delta} \tilde{g}_{jk}\right)\right\|_{L_t^2(P)},$$

so that by Lemma (2.1.16),

$$\left\|\left(\sum_{T_{jk}\sim P} e^{it\Delta} f_{jk}\right)\left(\sum_{T'_{jk}\nsim P} e^{it\Delta} g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)} \leq CR^{\varepsilon+c\delta}N^{1/8}\|f\|_2\|g\|_2,$$

where the constant $c$is independent of $\delta$and ε. Summing, this yields

$$\sum_{P\in\mathcal{P}}\left\|\left(\sum_{T_{jk}\sim P} e^{it\Delta} f_{jk}\right)\left(\sum_{T'_{jk}\nsim P} e^{it\Delta} g_{jk}\right)\right\|_{L_x^{8/5}L_t^2(P)} \leq CR^{(c+3)\delta+\varepsilon}N^{1/8}\|f\|_2\|g\|_2,$$

which is acceptable. The other two global parts are bounded in the same way, which completes the proof.

The following lemma is a simple consequence of the Littlewood–Paley inequality (see [24]). Let $\vartheta \in C_0^\infty(\mathbb{R})$and $\phi = \vartheta(2\pi|\cdot|^2)$satisfy

$$\sum_{k=-\infty}^{\infty}\vartheta(4^{-k}|\cdot|) = 1 \quad \text{and} \sum_{k=-\infty}^{\infty}\phi(2^{-k}|\cdot|) = 1.$$

Defining $f_k$by$\hat{f}_k = \phi(2^{-k}|\cdot|)\hat{f}$, it can be calculated that

$$\left(\vartheta(4^{-k}|\tau|)\left(e^{it\Delta}f\right)^{\wedge_t}(\tau)\right)^{\vee_t}(t) = e^{it\Delta}f_k.$$

Lemma (2.1.18)[59]: *Let $q \in [2,\infty]$and $r \in [2,\infty)$. Then*

$$\left\|e^{it\Delta}f\right\|^2_{L_x^q\left(\mathbb{R}^n,L_t^r(\mathbb{R})\right)} \leq C\sum_{k=-\infty}^{\infty}\left\|e^{it\Delta}f_k\right\|^2_{L_x^q\left(\mathbb{R}^n,L_t^r(\mathbb{R})\right)}.$$

We are now in a position to prove the linear estimates. There are two types of restriction on r; those which come from the restriction on $r$in the bilinear theorem are generally less restrictive than those related to the power of N.

Theorem (2.1.19)[59]: *Let $q \in (2q_\kappa,\infty]$, $r \in (2q'_\kappa,\infty)$and $\frac{n}{2q'_\kappa}+\frac{n}{q}+\frac{1}{r}<\frac{n}{2}$. If (A_κ) holds,then*

$$\left\|e^{it\Delta}f\right\|_{L_x^q\left(\mathbb{R}^n,L_t^r(\mathbb{R})\right)} \leq C\|f\|_{\dot{H}^{s(q,r)}(\mathbb{R}^n)}.$$

Proof. By scaling and Lemma (2.1.18), it will suffice to prove that

$$\left\|e^{it\Delta}f\right\|_{L_x^qL_t^r(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^n)}$$

whenever $\hat{f}$is supported in $\{1/2 \leq |\xi| \leq 1\}$. In order to apply our bilinear theorem, we square the integral, so that

$$\|e^{it\Delta}f\|^2_{L^q_xL^r_t(\mathbb{R}^{n+1})} = \|e^{it\Delta}f\ e^{it\Delta}f\|_{L^{q/2}_xL^{r/2}_t(\mathbb{R}^{n+1})}.$$

Now, for each $j \in \mathbb{N}$ we can break up the support of $\hat{f}$into dyadic cubes $\tau^j_k$of side2^{-j}. We write $\tau^j_k \sim \tau^j_{k'}$if $\tau^j_k$and $\tau^j_{k'}$have adjacent parents, but are not adjacent.Writing $\hat{f} = \sum_k \hat{f}^j_k$, where $\hat{f}^j_k = \hat{f}_{\chi_{\tau^j_k}}$, we have

$$\begin{aligned} e^{it\Delta}f(x)e^{it\Delta}f(x) &= \int\int \hat{f}(\xi)\hat{f}(y)e^{2\pi i\left(x\cdot(\xi+y)-2\pi t(|\xi|^2+|y|^2)\right)}d\xi dy \\ &= \sum_{j,k,k':\tau^j_k\sim\tau^j_{k'}} \int\int \hat{f}^j_k(\xi)\hat{f}^j_{k'}(y)e^{2\pi i\left(x\cdot(\xi+y)-2\pi t(|\xi|^2+|y|^2)\right)}d\xi dy \\ &= \sum_{j,k,k':\tau^j_k\sim\tau^j_{k'}} e^{it\Delta}f^j_k(x)e^{it\Delta}f^j_{k'}(x). \end{aligned}$$

By the triangle inequality, we see that

$$\|e^{it\Delta}f\|^2_{L^q_xL^r_t(\mathbb{R}^{n+1})} \le \sum_{j,k,k':\tau^j_k\sim\tau^j_{k'}} \|e^{it\Delta}f^j_k(x)e^{it\Delta}f^j_{k'}(x)\|_{L^{q/2}_xL^{r/2}_t(\mathbb{R}^{n+1})}.$$

Now, scaling out, applying Theorem (2.1.15) taking into account the rotational symmetry, then scaling in again, we see that

$$\|e^{it\Delta}f\|^2_{L^q_xL^r_t(\mathbb{R}^{n+1})} \le C_\alpha \sum_{j,k,k':\tau^j_k\sim\tau^j_{k'}} 2^{-j\left(n-\frac{2n}{q}-\frac{4}{r}\right)}2^{j\alpha}\|f^j_k\|_{L^2(\mathbb{R}^n)}\|f^j_{k'}\|_{L^2(\mathbb{R}^n)}$$

for all $\alpha > \frac{n}{q'_\kappa} - \frac{2}{r}$, where $q > 2q_\kappa$and $r > 2q'_\kappa$.

Finally, as $\operatorname{supp}\hat{f}^j_k$, $\operatorname{supp}\hat{f}^j_{k'} \subset \operatorname{supp}\hat{f}^{j-2}_{k''}$ for some k'', we have

$$\sum_{k,k':\tau^j_k\sim\tau^j_{k'}} \|f^j_k\|_{L^2(\mathbb{R}^n)}\|f^j_{k'}\|_{L^2(\mathbb{R}^n)} \le C\|f\|^2_{L^2(\mathbb{R}^n)},$$

and the sum in $j$converges by hypothesis, which completes the proof.

Observe that if the power of Nin the bilinear estimate was improved to $\alpha > 1/q - 1/r$, then we would obtain the almost sharp restriction, $\frac{n+1}{q} + \frac{1}{r} < \frac{n}{2}$, in thelinear estimates. We state this formally.

Definition (2.1.20)[59]: Let $R^*(2\times 2 \to q, r)$denote the estimate

$$\|e^{it\Delta}f\ e^{it\Delta}g\|_{L^q_x\left(\mathbb{R}^n,L^r_t(\mathbb{R})\right)} \le C_\alpha N^\alpha\|f\|_2\|g\|_2$$

whenever $N \gg 1$, $\alpha > \frac{1}{q} - \frac{1}{r}$, and $\hat{f}, \hat{g}$are supported on 1-separated subsets of $B_1(Ne_1)$.

Definition (2.1.21)[59]: Let $R^*(2 \to q, r)$denote the estimate

$$\|e^{it\Delta}f\|_{L^q_x\left(\mathbb{R}^n,L^r_t(\mathbb{R})\right)} \le C\|f\|_{L^2(\mathbb{R}^n)}$$

whenever $\hat{f}$is supported in $\{1/2 \le |\xi| \le 1\}$.

Lemma (2.1.22)[59]: *Let* $\frac{n+1}{q} + \frac{1}{r} < \frac{n}{2}$. *Then* $R^*\left(2\times 2 \to \frac{q}{2}, \frac{r}{2}\right) \Rightarrow R^*(2 \to q, r)$.

It remains to prove Theorem (2.1.7). By scaling and Lemma (2.1.18), it suffices to consider functions with frequency support in the unit annulus. Combining Theorem (2.1.17) with Lemma (2.1.22), we note that the condition $8/q + 2/r < 3$ that comes from the former is less restrictive than $3/q + 1/r < 1$ which comes from the latter, and we are done

Section (2.2): Self-adjoint Extensions and Singular Continuous Spectrum:

In [66] and [68] by Friedrichs and Krein it has been shown that every closed symmetric operator H in a Hilbert $\mathscr{k}$ space with gap J has a self-adjoint extension $\widetilde{H}$ such that J is contained in the resolvent set of $\widehat{H}$; an open interval (a,b) is called a gap of H if

$$\left\|\left(H - \frac{a+b}{2}\right)f\right\| \geqq \frac{b-a}{2}\|f\|, \quad f \in D(H), \ if -\infty < a < b < \infty,$$

$$(Hf, f) \geqq b\|f\|^2, \quad f \in D(H), \quad if -\infty = a < b < \infty.$$

Moreover Krein has found that if in addition H has finite deficiency indices (n, n), then within the gap J the spectrum of every self-adjoint extension consists of a finite number of eigenvalues such that the sum of their multiplicities does not exceed n, cf. [68], Conversely, if $\{\lambda_j\}_{j=1}^{s}, 1 \leqq s < \infty$, is an arbitrary sequence of points of J and $\{p_j\}_{j=1}^{s}$ is an arbitrary sequence of positive integers obeying $\Sigma_{j=1}^{s} p_j \leqq n$, then there exists a self-adjoint extension $\widetilde{H}$ of H such that within the gap J the spectrum of $\widetilde{H}$ coincides with the points λ_j which are eigenvalues of multiplicity$p_j, 1 \leqq j \leqq s$[68], So the problem which spectrum can the self-adjoint extensions have within the gap is completely solved for finite deficiency indices.

In [62, 63, 64] and [69] an attempt was made to extend these results to the case of infinite deficiency indices. It turned out that Theorem 23 of [68] has a straightforward generalization. Let $\oint$ be a countable set within the gap J and let $p: \oint \to \mathrm{N} \cup (\mathcal{N}_0)$ be an arbitrary function. Then there exists a self-adjoint extension $\widetilde{H}$of H such that $\sigma_p(\widetilde{H}) \cap J = \oint$, the multiplicity of each eigenvalue $\lambda \in \oint$ equals $p(\lambda)$ and no point of the gap J belongs to the continuous spectrum of $\widetilde{H}$. In other words, any pure point spectrum can be generated within the gap J by choosing an appropriate extension. Here $\sigma_p(\cdot)$ denotes the set of eigenvalues of an operator.

However, provided the deficiency indices of H are infinite it seems naturally to believe that other kinds of spectra (singular and absolutely continuous spectra) can arise within the gap J. In fact, for a large class of operators H, including all symmetric operators with infinite deficiency indices and compact resolvent, we have shown that every kind of absolutely continuous spectrum within a gap J of H can be generated by a self-adjoint extension$H$ of H, cf. [61. we shall show that a symmetric operator with infinite deficiency indices and some gap has self-adjoint extensions with non-empty singular continuous spectrum.

Theorem (2.2.1) [70]: (A. Gordon [67]; R. del Rio, N. Makarov, B. Simon [65], Theorem 3) Let A be a self-adjoint operator and g a cyclic vector of A. Then the set$\{\alpha \in R: A + \alpha(g,\cdot)g\sigma(A)$ has no eigenvaue in $\sigma(A)\}$is a dense G_δ subset of R.

we shall give a proof of the existence of the auxiliary operator H_{aux} which is more simple and much shorter than our original proof in [62]. Moreover we shall need the mentioned result by A. Gordon and by R. del Rio, N. Makarov and B. Simon only in a very special case. Instead to show that this result can be used in our situation we shall give a short direct proof that the operator $\widetilde{H}_\alpha$ has the required spectral properties.

In our very special case we get absence of eigenvalues in $\overline{J_0} \cap J$ even for every$\alpha \in\neq 0$.

Finally we mention that Theorem (2.2.3) allow only to generate so-called "fat" singular continuous spectrum by extensions, i.e., singular continuous spectrum which coincides with the closure of its inner points. For spectrum which does not have this property (so-called "thin" spectrum) we cannot make any conclusions, we cannot generate singular continuous spectrum which is a Cantor set. The problem is that for thin sets the used proof technique does not allow to decide whether the generated spectrum is really singular continuous or results from the closure of the discrete spectrum which is outside the thin set.

Lemma (2.2.2) [70] Let H be a symmetric operator in some separable Hilbert space $\hbar$. Let b he a strictly positive real number and $J = (-b, b)$ or $J = (-\infty, b)$. Suppose that J is a gap of H. For every$\lambda \in J let P_\lambda$: $\ker(H^*) \to \ker(H^* - \lambda)$ be the mapping given by

$$P_\lambda f := P_{\ker(H^*-\lambda)} f, \quad f \in \ker(H^*), \tag{17}$$

where P_ℓ denote the orthogonal projection in $\hbar$ onto the subspace ℓ. Then for every$\lambda \in J$ the mapping P_λ is bijective and

$$\|P_\lambda^{-1} g\| \leqq \frac{b + |\lambda|}{b - |\lambda|} \|g\|, \qquad g \in \operatorname{ran}(P_\lambda), \tag{18}$$

when $J = (-b, b)$ and

$$\|P_\lambda^{-1} g\| \leqq \max\left\{\frac{b}{b-\lambda}, \frac{b-\lambda}{b}\right\}, g \in ran(P_\lambda), \tag{19}$$

When $J = (-\infty, b)$.

Proof. Since J is a gap of H the symmetric operator H has a self-adjoint extension $\widehat{H}$ such that $J \cap \sigma(\widehat{H}) = \emptyset$, e.g., the Friedericsh and the Krein extension of H in the case when $J = (-\infty, b)$ and $J = (-b, b)$, respectively. Note that

$$\int_J F(t) d(E(t)f, g) = 0$$

for all $f, g \in \hbar$ and every Borel-measurable function F where $\{E(t)\}_{t \in \mathbf{R}}$ denotes the spectral family of the self-adjoint operator $\widehat{H}$.

Let $\lambda \in J$. Let $f \in \ker(H^*) = rank(H)^\perp, f \neq 0$ and $g \in D(H)$. We have

$$\left(\widehat{H}(\widehat{H} - \lambda)^{-1} f, (H - \lambda) g\right) = \int \frac{t}{t - \lambda}(t - \lambda) d(E(t)f, g) = \int t d(E(t)f, g) = (f, Hg) = 0$$

Thus $\tilde{f}$: $\widehat{H}(\widehat{H} - \lambda)^{-1} f \in ran(H - \lambda)^\perp \ker(H^* - \lambda)$ and consequently we have

$$\|P_\lambda f\| \geq \left(\frac{\tilde{f}}{\|\tilde{f}\|}, f\right) = \frac{\int_{R\setminus} t/(t-\lambda) d\|E(t)f\|^2}{\left\{\int_{R\setminus J} \left(t/(t-\lambda)\right)^2 d\|E(t)f\|^2\right\}^{1/2}}. \tag{20}$$

Since

$$\frac{b}{b + |\lambda|} \leqq \frac{t}{t - \lambda} \leqq \frac{b}{b - |\lambda|}, \qquad t \in \mathrm{R} \setminus J,$$

when $J = (-b, b)$ and

$$\min\left\{1, \frac{b}{b - \lambda}\right\} \leqq \frac{t}{t - \lambda} \leqq \max\left\{1, \frac{b}{b - \lambda}\right\}, t \in \mathrm{R} \setminus J,$$

when $J = (-\infty, b)$ this implies that

$$\|P_\lambda f\| \geqq \frac{b-|\lambda|}{b+|\lambda|}\|f\| (21)$$

and

$$\|P_\lambda f\| \geqq \frac{\min\{1, b/(b-\lambda)\}}{\max\{1, b/(b-\lambda)\}}\|f\| (22)$$

when $J=(-b,b)$ and $J=(-\infty,b)$, respectively. Thus P_λ is invertible and (18) and (19) hold.
By (21) and (22) the operator P_λ has a trivial kernel and a range. Hence it remains to show that $f \in \ker(H^*-\lambda)$ and $(f,h)=0$ for each $h \in \ker(H^*)$yields $f=0$. Since

$$D(H^*)=D(\widehat{H})+\ker(H^*),$$

we obtain elements $g \in \ker(H^*)$ such that $f=g+k$. By$H^*f=\lambda f$ and $(f,h)=0, h \in \ker(H^*)$, we find $H^*f \in ran(H)$. Hence one gets $H^*f=H_g \in ran(H)$. However, this yields $g \in D(H)$. Using that we obtain.

$$(H-\lambda)g=\lambda k.$$

Since $k \in \ker(H^*)$ we have

$$(Hg,(H-\lambda)g)=\|Hg\|^2-\lambda(Hg,g)=0$$

which implies

$$\|Hg\| \leqq |\lambda|\|g\|.$$

Let $|\lambda|<b$. Since $b\|g\| \leqq \|H_g\|$ we immediately find.

$$b\|g\| \leqq \|Hg\| \leqq |\lambda|\|g\|$$

which proves $g=0$. If $\lambda \leqq -b$, then the result is obvious. Therefore $k=0$ and $f=0$.

Theorem (2.2.3) [70]: Let H be a symmetric operator in some Hilbert space $\mathfrak{f}$. Suppose that the operator H has some gap J and infinite deficiency indices. Let J_0 be any open subset of J. Then H has a seif-adjoint extension $\widetilde{H}$ with the following properties:

$\sigma_{sc}(\widetilde{H}) \cap J=\sigma_{ess}(\widetilde{H}) \cap \overline{J_0} \cap J$.

$\sigma_{ac}(\widetilde{H}) \cap J=\emptyset$.

$\widetilde{H}$ has no eigenvalue in $\overline{J_0} \cap J$.

Here $\sigma, \sigma_{ac}, \sigma_{sc}$ and σ_{ess} denote the spectrum, the absolutely continuous, the singular continuous and the essential spectrum, respectively. $\tilde{S}$ denotes the closure of the set S.

Without loss of generality we assume $0 \in J$. First one constructs an auxiliary invertible self-adjoint extension H_{aux} of H such that H_{aux} has pure point spectrum within the gap J of H, the eigenvalues of H_{aux} within J are simple and form a dense subset of J_0. Then one chooses a vector $g \in ran(H)^\perp$ such that $(g,e) \neq 0$ for every eigenvector e of H_{aux} corresponding to an eigenvalue in J and shows that the operator $H_{aux}^{-1}+\alpha(g,\cdot)g$ is invertible and its inverse $\widetilde{H}_\alpha$ is a seif-adjoint extension of H for every real number α. Finally one proves that for everyα in some dense G_δ-subset of R the operator $\widetilde{H}_\alpha$ has the required spectral properties. This easily follows from the following recent result by A. Gordon resp. by R. del Rio, N. Makarov and B. Simon.

Proof. Since H has a self-adjoint extension $\widehat{H}$ such that the gap J is contained in the resolvent set of $\widehat{H}$ the theorem is true (with $\widetilde{H}=\widehat{H}$) in the special case when $J_0=\emptyset$. Moreover we may assume that $J=(-b,b)$ or $J=(-\infty,b)$ for some strictly positive real number$b$.

It suffices to show that there exists a self-adjoint extension $\widetilde{H}$ of H such that $\sigma_{ess}(\widetilde{H}) \cap J=\overline{J_0} \cap J, \sigma_{ac}(\widetilde{H}) \cap J=\emptyset$ and $\widetilde{H}$ has no eigenvalue in $\overline{J_0} \cap J$. In fact, then on the one hand every$\lambda \in J_0$

belongs to the singular continuous spectrum of $\widetilde{H}$ and consequently we have $\overline{J_0} \cap \sigma_{sc}(\widetilde{H})$, on the other hand we have $\sigma_{sc}(\widetilde{H}) \subset \sigma_{ess}(\widetilde{H})$ and consequently$\sigma_{sc}(\widetilde{H}) \cap J \subset \overline{J_0}$.
We chose any square summable sequence $\{\alpha_n\}_{n\in\mathbf{N}}$ of numbers such that $\alpha_n \neq 0$ for every$n \in \mathrm{N}$ and any sequence $\{\eta_n\}_{n\in\mathbf{N}}$ in $J_0^{-1} := \{1/t : t \in J_0, t \neq 0\}$ such that $\eta_n \neq \eta_m$ for $n \neq m$ and for every$\eta \in J_0^{-1}$.

$$|\eta_n - \eta| < |\alpha_n| \quad (23)$$

for infinitely many$n \in N$.
Such sequences always exist. For instance we start with a partion Γ_1 of the real axis into intervals $[k, k+1), k \in Z$. Dividing the intervals $[k, k+1)$ into two intervals $[k, k+\frac{1}{2})$ and $[k+\frac{1}{2}, k+1)$ into two subintervals of half length we get a further partions Γ_3. Repeating this procedure again and we obtain a sequence of partions $\{\Gamma_l\}_{l\in\mathbf{N}}$. Choosing now from the intersection of J_0^{-1} with the intervals of the partion Γ_l, provided this intersection is not empty, points we get for each $l \in \mathrm{N}$ a sequence of points $\{\eta_{lm}\}_{l\in\mathbf{Z}}$. Obviously all those points η_{lm} can be chosen different from each other. Making a suitable renumeration of the sequence $\{\eta_{lm}\}_{l\in\mathbf{N}, m\in\mathbf{Z}}$ we find the desired sequence $\{\eta_n\}_{n\in\mathbf{N}}$ of J_0^{-1}.
For notational brevity we put $\lambda_n := 1/\eta_n$ and $p_n := P_{\lambda_n}$ for every $n \in N$ where for every$\lambda \in J$ the linear mapping $P_\lambda : \ker(H^*) \to \ker(H^* - \lambda)$ is given by (17).
We choose any$e_1 \in \ker(H^* - \lambda_1)$ such that$\|e_1\| = 1$. Let$n \in N$ and suppose that $e_j \in \ker(H^* - \lambda_j), 1 \leqq j \leqq n$, have been chosen. Then we choose any$e_{n+1} \in \ker(H^* - \lambda_{n+1})$ such that $\|e_{n+1}\| = 1$,

$$e_{n+1} \perp e_j, \qquad e_{n+1} \perp p_j^{-1} e_j,$$
$$p_{n+l}^{-1} e_{n+1} \perp p_j^{-1} e_j, \quad p_{n+1}^{-1} e_{n+1} \perp e_j$$

$1 \leqq j \leqq n$. Since, by Lemma (2.2.2), for every$\lambda \in J$ the linear mapping P_λ is bijective and consequently the space $\ker(H^* - \lambda)$ is infinite dimensional each of these choices is possible. we get, by induction, an orthonormal system $\{e_n\}_{n\in\mathbf{N}}$ with the following properties:

$$e_n \in \ker(H^* - \lambda_n), \quad n \in \mathrm{N}, \qquad (24)$$
$$(g_n, g_m) = 0 = (g_n, e_m) for n \neq m \quad (25)$$

where

$$g_n := p_n^{-1} e_n, \qquad n \in \mathrm{N}. \qquad (26)$$

Next we shall show that there exists an auxiliary self-adjoint extensions H_{aux} of H with the following properties:

(i) H_{aux} has a pure point spectrum within J.
(ii) λ_n is a simple eigenvalue of H_{aux} and e_n a corresponding eigenvctor for every$n \in N$.
(iii) $\sigma_p(H_{aux}) \cap J = \{\lambda_n : n \in \mathrm{N}\}$.

Since $\{\lambda_n : n \in \mathrm{N}\}$ is a dense subset of $\overline{J_0}$ and $\lambda_n \neq 0$ for every$n \in \mathrm{N}$ it follows from (i) and (iii) that such an operator also satisfies

(iv) $\sigma_{ess}(H_{aux}) \cap J = \overline{J_0} \cap J$.
(v) H_{aux} is invertible.

We denote by$\hbar_0$ the closure of the span of the span of $\{e_n : n \in \mathrm{N}\}$ and by$M$ the self-adjoint operator in the Hilbert space $\hbar_0$ given by

$$D(M) := \left\{ \sum_{n=1}^{\infty} \beta_n e_n : \sum_{n=1}^{\infty} (1 + \lambda_n^2)|\beta_n|^2 < \infty \right\},$$
$$M \sum_{n=1}^{\infty} \beta_n e_n := \sum_{n=1}^{\infty} \lambda_n \beta_n e_n, \quad \sum_{n=1}^{\infty} (1 + \lambda_n^2)|\beta_n|^2 < \infty.$$
Obviously the operator M has a pure point spectrum, λ_n is a simple eigenvlaue of M and e_n a corresponding eigenvector for every$n \in \mathrm{N}$.
$$\sigma_p(M) = \{\lambda_n : n \in \mathrm{N}\}$$
and
$$(Mf, f) \leqq b\|f\|^2, \quad f \in D(M) (27)$$
in the case when $J = (-\infty, b)$ and
$$\|Mf\| \leqq b\|f\|, \qquad f \in D(M),$$
in the case when $J = (-b, b)$.
M is a restriction of H^*since $e_n \in \ker(H^* - \lambda_n)$ for every$n \in \mathrm{N}$ and H^* is a closed operator. Thus we can define an extension H' of H by
$$D(H') := D(H) \dot{+} D(M), \quad H'g := H^*g, \quad g \in D(H').$$
A short computation shows that H' is a symmetric operator.
Let $f \in D(H')$. For every$n \in \mathrm{N}$ we have
$$(H'f, e_n) - (f, Me_n) = \lambda_n (f, e_n).$$
Thus
$$\sum_{n=1}^{\infty} \lambda_n^2 |(f, e_n)|^2 = \left\|P_{\hbar_0} H'f\right\|^2 < \infty.$$
Hence $P_{\hbar_0} f \in D(M)$. For every$n \in \mathrm{N}$ we have
$$\left(P_{\hbar_0} H'f, e_n\right) = (f, Me_n) = \left(MP_{\hbar_0} f, e_n\right)$$
Thus
$$P_{\hbar_0} H'f = MP_{\hbar_0} f, \qquad f \in D(H').$$
This implies that the operator H' can be written in the form
$$H' = M \oplus G_0,$$
where the symmetric operator G_0 in the Hilbert space $\hbar_0^{\perp}$ is given by
$$G_0 := H'_{|D(H') \cap \hbar_0^{\perp}}.$$
We shall show by contradiction that the gap J of H is also a gap of G_0. We shall give the proof for $J = (-\infty, b)$. The proof in the other case is virtually the same. Suppose that
$$(G_0 f, f) < b\|f\|^2 (28)$$
for some $f \in D(G_0)$. We choose $g \in D(H)$ and $h \in D(M)$ such that $f = g + h$. Then we have
$$(Hg, g) = (H'(f - h), f - h) = (G_0 f, f) + (Mh, h) < b\|f\|^2 + b\|h\|^2 = b\|f - h\|^2 = b\|g\|^2.$$
Here we have used that $H' = M \oplus G_0$, as well as our assumption (27) and (28). Thus the assumption (28) leads to a contradiction to the hypothesis that $(-\infty, b)$ is a gap of H. Thus J is also a gap of G_0.

Since J is a gap of symmetric operator G_0 in $\hbar_0^{\perp}$ there exists a self-adjoint operator G in $\hbar_0^{\perp}$ such that $G_0 \subset G$ and $\sigma(G) \cap J = \emptyset$. We put
$$H_{aux} := M \oplus G.$$
ObviouslyH_{aux} has the required properties.

We put

$$g := \sum_{n=1}^{\infty} \alpha_n \frac{g_n}{\|g_n\|},$$

where the $g_n, n \in \mathrm{N}$, are given by (26) and the $\alpha_n, n \in \mathrm{N}$, are any numbers different from zero such that the sequence $\{\alpha_n\}_{n\in\mathbf{N}}$ is square summabe (23) holds. Since, by (25), $\{g_n/\|g_n\|\}_{n\in\mathbf{N}}$ is an orthonormal system the series converges and g is well-defined. Since $g_n \in \ker(H^*)$ for every$n \in \mathrm{N}$ and $\ker(H^*)$ is closed we have that $g \in \ker(H^*)$.

Obviously$g \neq 0$.

we choose any$\alpha \in \mathrm{R}, \alpha \neq 0$. Since along with H_{aux} also the inverse H_{aux}^{-1} of H_{aux} is a self-adjoint operator and $\alpha \in \mathrm{R}, \alpha \neq$. Since along with H_{aux} also the inverse H_{aux}^{-1} of H_{aux} a self-adjoint operator and $\alpha(g,.)$ is a bounded self-adjoint operator the sum $H_{aux}^{-1} + \alpha(g,.)g$ is also self-adjoint. Let $h \in D(H_{aux}^{-1})$ be such that

$$H_{aux}^{-1}h + \alpha(g,h)g = 0.$$

Then $(g,h)g \in ran(H_{aux}^{-1}) = D(H_{aux})$. If g would be in $D(H_{aux})$ then we would have $H_{aux}g = H^*g = 0$ with is impossible since H_{aux} is invertible. Thus we have $(g,h) = 0$. It follows that $H_{aux}^{-1}h = 0$ which implies that $h = 0$. Thus we have shown that the operator $H_{aux}^{-1} + \alpha(g,\cdot)g$ is invertible. Along with this operator also it's inverse

$$\widetilde{H} := (H_{aux}^{-1} + \alpha(g,\cdot)g)^{-1}$$

is self-adjoint

Let $h \in D(H^{-1}) = ran(H)$. Since $H \subset H_{aux}$ and $g \in \ker(H^*) = ran(H)^{\perp}$ we have that $H^{-1}h = H_{aux}^{-1}h = \widetilde{H}^{-1}h$. Thus $\widetilde{H}$ is a self-adjoint extension of H. Since the resolvent difference $\widetilde{H}^{-1} - H_{aux}^{-1}$ of the self-ajoint operator $\widetilde{H}$ and H_{aux} is nuclear we have that $\sigma_{ac}(\widetilde{H}) = \sigma_{ac}(H_{aux})$ and $\sigma_{ess}(\widetilde{H}) = \sigma_{ess}(H_{aux})$. In particular, we have

$$\sigma_{ac}(\widetilde{H}) \cap J = \emptyset, \qquad \sigma_{ess}(\widetilde{H}) \cap J = \overline{J_0} \cap J.$$

Thus we have only to show that $\widetilde{H}$ has no eigenvalue in $\overline{J_0} \cap J$.

The point zero is not an eigenvalue of $\widetilde{H}$ since $\widetilde{H}$ is invertible. Let $\lambda \in \overline{J_0} \cap J$ and $\lambda \neq 0$. We have only to show that $\eta := 1/\lambda$ is not an eigenvalue of $\widetilde{H}^{-1}$. Let $h \in D(\widetilde{H}^{-}) = D(H_{aux}^{-1})$ and

$$\widetilde{H}^{-1}h = H_{aux}^{-1}h + \alpha(g,h)g = \eta h.$$

By taking the scalar product with e_n we get from the last realtion that

$$\eta_n(e_n,h) + \alpha(g,h)\frac{\alpha_n}{\|g_n\|} = \eta(e_n,h)$$

for every$n \in \mathrm{N}$. Thus we have

$$|\eta_n - ||e_n,h| = |\alpha(g,h)|\frac{|\alpha_n|}{\|g_n\|}, \quad n \in \mathrm{N}. \tag{29}$$

By (23), there exists a subsequence $\{\eta_{n_j}\}_{j\in\mathbf{N}}$ of $\{\eta_n\}_{n\in\mathbf{N}}$ such that

$$\left|\eta_{n_j} - \eta\right| < \alpha_{n_j}, \qquad j \in \mathrm{N}. \tag{30}$$

By (18) resp. (19) in the Lemma (2.2.2) and (26) there exists a finite constant c such that

$$\left\|g_{n_j}\right\| < c, \quad j \in \mathrm{N}. \tag{31}$$

Since $\sum_{n=1}^{\infty}|e_n,h|^2 = \left\|P_{\hbar_0}h\right\|^2 < \infty$ it follows from (29), (30) and (31) that

$$(g, h) = 0.$$

Thus we have

$$H_{aux}^{-1}h = \eta h.$$

Since the only eigenvlues of operator H_{aux}^{-1} in J^{-1} are the numbers $\eta_n, n \in \mathrm{N}$, and η_n is a simple eigenvalue H_{aux} with corresponding eigenvector e_n for every$n \in \mathrm{N}$ this implies that $h = ae_n$ for some constant a and some $n \in \mathrm{N}$. Since

$$0 = (g, h) = a\frac{a_n}{\|g_n\|}$$

It follows that $a = 0$ and $h = 0$. Thus η is not an eigenvalue of the operator $\widetilde{H}^{-1}$ and the theorem is proven.

Example (2.2.4) [70]: Let Ω be a bounded non-empty domain in $\mathrm{R}^d, d > 1$. Then the minimal Laplacian on Ω, i.e. the operator $-\Delta_{min}^{\Omega}$ in $L^2(\Omega)$ given by

$$D\left(-\Delta_{min}^{\Omega}\right) := C_0^{\infty}(\Omega),$$

$$-\Delta_{min}^{\Omega}f := -\Delta f, \quad f \in C_0^{\infty}(\Omega),$$

Is a symmetric operator with infinite deficiency indices. Here $C_0^{\infty}(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. Thus, by Theorem (2.2.3), there exist self-adjoint realizations of the Laplacian on Ω, i.e. self-adjoint extension of $-\Delta_{min}^{\Omega}$, with non-empty singular continuous spectrum. Thus (the proof of) Theorem (2.2.3) enables us to construct self-adoint realizations of the Laplacian on a bounded domain Ω in $d^d, d > 1$, with spectral properties very different from the properties of the self-adjoint realizations investigated before.

Chapter 3
Pure point Spectrum and Spectral Localization

All eigenvalues have infinite multiplicity and acountable system of orthonormal eigenfunctions with compact support is the corresponding Hilbert space.

Section (3.1): The Laplacians on Fractal Graphs:

Considerable attention has been paid by graph theorists to the study of spectra of the difference Laplacians on infinite graphs. We refer to Mohar and Woess [82], which is an excellent survey of this theory, Explicit computational results about the spectrum of the Laplacians are known only when the graph under consideration satisfies certain kind of regularity property that leads to the existence of the absolutely continuous spectrum (see [82, 71]).

If we study fractal or disordered materials and the difference Laplacians are some discrete approximations, we should expect the spectrum to be pure point.

The first result is [83] where the spectrum of the Laplacian on the Sierpinski lattice is considered, an application of the very interesting Renormalization Group method to this case was given by Bellissard in [73].

We study the spectrum of Lablacians on so-called two-point self-similar fractal graphs (TPSG) (we mean the Lablacians which correspond to the adjacency matrix and the simple random walk). A good example of such a kind of graphs is modified Koch graph which can beconsidered as the discrete approximation of the fractal set, namely the modified Koch curve [789].

We will show that if the TPSG has an infinite number of cycles and the length of these cycles approaches infinity, then the spectrum of the Laplacians is pure point.

The problem of the description of the spectrum as a set in IR is not trivial as shown by the example of the modified Koch graph. The spectrum for this graph is the union of two sets. The first set is the Julia set of the rational function

$$R(z) = 9z(z-1)\left(z-\frac{4}{3}\right)\left(z-\frac{5}{3}\right)(z-\frac{3}{2})^{-1}.$$

This is a Cantor set of Lebesgue measure zero which may be obtained as a closure of a countable set of eigenvalues of the Laplacian with infinite multiplicity. The second set is a discrete countable set of eigenvalues with infinite multiplicity which has the limit points in the first set.

We note the new property of the eigenfunction of the Laplacians on TPSG: a countable system of orthonormal eigenfunction with compact support is complete in the Hilbert space where this operator is defined.

We consider in Theorem (3.1.11) the Anderson localization for the Schrodinger operator with Bernoulli potential on TPSG. It was proven that any eigenvalue of the Laplacian is an eigenvalue of infinite multiplicity of the Schrodinger operator for any coupling constant. Unfortunately, we cannot prove that the spectrum of such operator is pure point. However, we note that Aizenman and Mo1chanov [72] proved the localization of the spectrum in the standard Anderson model for suffiently large disorders on general graphs.

The two-point self-similar fractal graphs can be considered as nested pre-fractals with two essential fixed points introduced by Lindstrom [78]. We also note that some questions about the integrated density of states of the Laplacian on fractal graphs were studied in [80, 75].

Some special examples of TPSG were considered in physical models (see [85, 74])

i. Let $G = (V, E)$ be a connected infinite locally finite graph, with vertex set V and edge set E. We suppose that the degree d, of all vertices x E V is finite.
Let $A = A(G)$be the adjacency matrix of the graph G and $P = P(G) = (PII._{,,.})$ U, vEV be the transition matrix, where

$$P_{u,r} = a_{u,r}/d_u$$

And $a_{u,r}$, is the number of edges between N and E.
Associated with each of the preceding two matrices are the difference Laplacians

$$\Delta_A = D(G) - A(G) \qquad (1)$$

And

$$\Delta_p = I(G) - P(G) \qquad (2)$$

Where $D(G)$ is the diagonal matrix of $d.,x \in V$, and $I(G)$ is the identity matrix over V
Let us introduce the space of functions on V

$$I_2(V) = \left\{f(x), x \in V; \sum_{x \in V} |f(x)|^2 < \infty\right\} \quad (3)$$

With inner product

$$(g, f) = \sum_{x \in V} |f(x)|^2 < \infty$$

And

$$I_2^{\#}(V) = \left\{f(x), x \in V; \sum_{x \in V} |f(x)|^2 < \infty\right\} \qquad (4)$$

With inner product

$$(g, f) = \sum_{x \in V} d_x |f(x)|^2 < \infty$$

If the function $\deg(x) = d.,x \in V$ is bonded, then the operators Δ_A and Δ_p are self-adjoint bounded operators in $I_2(V)$ and $I_2^{\#}(V)$, respectively.
ii. Let us introduce so-called two point self-similar graphs. Suppose $M = (V_M, E_M)$ and $G_0 = (V_0, E_0)$ are finite connected graphs and M is an ordered graph. We fix some $e_0 \epsilon E_M$, which is not a loop, and vertices $\alpha, \beta \epsilon V_M$ and $a_0, \beta_0 \epsilon V_0, a \neq \beta, a_0 \neq \beta_0$.
Informally speaking, the construction of a TPSG G is as follows: to get G_1 from M and G_0 we replace every edge $(a, b) \epsilon E_M, a, b \epsilon V_M$, by a copy of G_0 such that a_0 goes to a and β_0 to b. Then we take $a_0 = a, \beta_1 = \beta$ and proceed by induction. If a graph $G_n = V_n, E_n$. with fixed vertices a_n, β_n, V_n is defined then the graph $G_{n+1}$is obtained by replacement of every edge (a, b) ofM by the copy of G_n such that a_n goes to a and β_n goes to b. The vertices a_{n+1}, β_{n+1} are the vertices a, β after this replacement.
We can assume that $G_n \subseteq G_{n+1}$ is the copy corsponding to e_0 and define infinit graph $G = \cup_{n=1}^{\alpha} G_n$. Let us give a more formal.
Definition (3.1.1) [86]: A graph G is called TPSG with model graph M and infinite graph G_0 if the following holds:

(i) There are finite subgraphs $G_0, G_1, G_2, \ldots$ such that $G_n \subseteq G_{n+1}, n \geq 0$ and $G = \cup_{n \geq 0} G_n$.

(ii) For any $n \geq 0$ and $e \epsilon E_M$ there is graph homomorphism $\Psi_n^e : G_{n+1} \rightarrow G_{n+1}$ such that $G_{n+1} = \cup_{e \epsilon E_M} \Psi_n^e(G_n)$ and $\Psi_n^{e_0}$ is inclusion of G_n to G_{n+1}.

(iii) For all $n \geq 0$ there are two vertices $\alpha_n, \beta_n \epsilon V_n$ such that Ψ_n^e restricted to $G_n \backslash \{\alpha_n, \beta_n\}$ is a one-to-one mapping for every $e \epsilon E_M$. Moreover $\Psi_n^{e1}(V_n \backslash \{\alpha_n, \beta_n\}) \cap \Psi_n^{e2}(V_n \backslash \{\alpha_n, \beta_n\}) = \emptyset$ if $e_1 \neq e_2$.

(iv) For $n \geq 1$, there is an injection $K_n : V_M \to V_n$ such that $\alpha_n = K_n(\alpha), \beta_n = K_n(\beta)$ and for every edge $e = (\alpha, b)\epsilon E_M, \Psi_n^e(\alpha_{n-1}) = K_n(\alpha), \Psi_n^e(\beta_{n-1}) = K_n(b)$.

We say that the vertices α_n, β_n are the boundary vertices of G_n, i.e., $\partial G_n = \{\alpha_n, \beta_n\}$ and interior vertices of G_n.
Remark (3.1.2) [86]: One we can see the vertices α_n, β_n are the boundary vertices of G_n, i.e., $\partial G_n = \{\alpha_n, \beta_n\}$ and int $G_n = V_n \backslash \{\alpha_n, \beta_n\}$ are interior vertices of G_0 are given.
Suppose M dose not have loops and G_0 is just two vertices and one edge. Then two-point self-similar graphs are in one-to-one correspondence to so-called post-critically finite (p.c.f) self-similar sets with post-critically for such p.c.f. sets. However, G is not a p.c.f. set since the limiting procedures in these two cases are different. The definition of a p.c.f. set can be found in [76] or [77].
3. We need some auxiliary result on the structure of graph G.
Definition (3.1.3) [86]: Two different vertices x and y of graph Γ are equivalent if there is automorphisim φ of Γ such that$\varphi(x) = y, \varphi(y) = x$.
By induction it is easy to prove the following lemma.
Lemma (3.1.4) [86]:if the vetices $\alpha_n, \beta_n \epsilon V_M$ and $\alpha_0, \beta_0 \epsilon \beta_0$ are equivalent in M and G_0, respectively, then vertices α_n, β_n are equivalent in G_n for all n.
Remark (3.1.5) [86]: We will suppose in what follows that M and G_0 satisfy assumptions of Lemma 1.1 We call such graph G symmetric. In this case the graph$G$ does not depend in orientation of M.
Although our results are valid for nonsymmetrical graphs (with some additional assumption on the orientation of M) we do not consider such graphs for the sake of simplicity.
Let us introduce the graph $M = (V_{\widetilde{M}}, E_{\widetilde{M}})G_1$ which can be obtained in the same way as G_1 if we take the graph M instead of G_0 and the verticesα, β play the role of α_0, β_0.
We define the graph $\tilde{G}_{n+2}$ by replacement of every edge of $\widetilde{M}$ by the copy of G_n such that for every edge $(\alpha, b)\epsilon E_{\widetilde{M}}$, $\alpha, b \epsilon V_{\widetilde{M}}$ we say x_n goes to α and β_n to b.

iii. we neet some axillary result on structure of graph G.
Lemma (3.1.6) [86]: The graphs $\tilde{G}_{n+2}$ and G_{n+2} are isomorphic.
Proof. By definition $\tilde{G}_{n+2}$ can be written as

$$\tilde{G}_{n+2} = \bigcup_{e\epsilon E_{\widetilde{M}}} \widetilde{\Psi}_n^e(G_n) \tag{5}$$

Where the maps $\widetilde{\Psi}_n^e$ have the same properties as Ψ_n^e in definition (3.1.1) The proof follows by induction.
Let us introduce the space $l_2(X)$ by $l_2(X) = \{f \epsilon l_2(V) : f(x) = 0 \, for x \epsilon V \backslash X\}$, where $X \subset V$. $l_2^{\#}(X)$ is defined analogously. By $\Delta_A(X), \Delta_p(X)$ we denote the restriction of Δ_A, Δ_p to $l_2(X), l_2^{\#}(X)$. More precisely, $\Delta_{A.p}P$, where P is orthogonal projector to $l_2(X)$ or $l_2^{\#}(X)$. We will call this operators the Laplacians with zero boundary conditions on $V \backslash X$. For simplicity, we denote the Laplacians with zero boundary conditions on ∂G_n by $\Delta_A(n)$ and $\Delta_p(n)$.
By Lemma (3.1.4) there is isomorphism $\varphi_n : G_n \to G_n$ such that $\varphi_n(a_n) = \beta_n, \varphi_n(\beta_n) = a_n$. This isomorphism induces unitary maps $U_n : l_2(G_n) \to l_2(G_n)$ and $U_n^{\#} : l_2^{\#}(G_n) \to l_2^{\#}(G_n)$ by formula $U_n^{\#} f = f\varphi_n$.
Lemma (3.1.7) [86]:$U_n(U_n^{\#})$ commutes with $\Delta_A(G_n)$ and $\Delta_A(n) \left(\Delta_p(G_n) \text{and} \Delta_p(n)\right)$.
Proof of this lemma immediately follows from the definition of Δ_A and Δ_p. Let us consider the function $\deg(x) = d_x$. It can occur that the function $\deg(\cdot)$ is not bunded in general. Moreover, there can exist a point $x_0 \epsilon \Delta_A$ such that $\deg(x_0) = \infty$. The next Lemma should be more clear from

the following examples (see Figs. 2 and 3).

For an arbitrary graph G let us denote by $d, \tilde{G}$ the degree of the vertex x in $\tilde{G}$.

Lemma (3.1.8) [86]:

(i) $d_{2n}(G_n) = d_{2n}(G_0) \cdot (d_2(M))^n = d_{2n-1}(G_{n-1}) \cdot G_{n-1}(M)$.

(ii) $if x \epsilon$ int $G_{n.}$ then $(deg(x) = d_x(G_n) = d_x(G_{n-1}) foreveryn \geq 1$.

(iii) $The functiom\, deg(x)\, isbundedifandonlyif d_2(M) = 1$.

(iv) $ifx \epsilon V andx \neq x_0, \beta_0 then\, deg(x) < \infty$.

(v) $\deg(x_0) = \infty(deg(\beta_0) = \infty)$ if and only if e_0 is indicent to and $d_2(M) \geq 2$ (β is incident to e_0 and $d_\beta(M) \geq 2$).

Proof. The first statement can be proved by induction. The second follow from (ii) and (iii) of Definition (3.1.1) Statement (iii) follow from (i) and equality $max_{V \epsilon G_{n+1}} d_x(G_{n+1}) = max\{max_{V \epsilon M} d_x(G_n), max_{V \epsilon G_n} d_V(M)\}$.

(iv) There exists $n_0 \in \mathbb{N}$ such that $x \in V_n$ for every $n \geq n_0$. if $x \in int G_n$ the statement follows from (ii). Otherwise, $x \in \partial G_n$ for every $n \in n_0$ and consequently x is equal to α_0 or β_0.

(v) By (iV), it follows that$\alpha_0 \in \partial G_n$ for any $n \geq n_{0,} n_0 \in \mathbb{N}$. if a is not incident to e_0, thenα_0 is an interior point of G_{n_1} for some n_1. Let a be incident to e_0 and$(M) \geq 2$. Then statement (V) follows from (i).

Definition (3.1.9) [86]: We denote by$\partial G = \{x, deg(x) = \infty\}$

The boundary of the graph G. If $\partial G = \emptyset$, we say that G is a graph without boundary.

By Lemma (3.1.10) we obtain the following lemma:

Lemma (3.1.10) [86]:

(i) $e_0 = (\alpha, \beta)$ and $d_V(M) \geq 2$, if and only if $\partial G = \{\alpha, \beta\}$.

(ii) The boundary ∂G has only one point if and only if the points α vertex of e_0 and the degree of this vertex in M is not less than 2.

(iii) If conditions (i), (ii), are not satisfied for the graph G then $\partial G = \emptyset$.

If G has the boundary, we define the operator Δ_p with zero boundary condition, i.e.,

$$\Delta_p^0 : l_2^{\#}(V^0) \to l_2^{*}(V^0),$$

where

$$l_2^{\#}(V^0) = \{f \in l_2^{\#}(V), f(x) = 0, x \in \partial G\}.$$

The Δ_p^0 is a self-adjoint bounded operator, too.

Theorem (3.1.11) [86]: Let $m \in \mathbb{N}, \delta > 0$ and $c < \infty$ be fixed numbers and for every$n = 1,2, \dots,$ there exists a linear operator $\Phi_n : \mathscr{k}_n \to \mathscr{k}_{n+m}$ such that $\|\Phi_n\| \leq c, (f, \Phi_n(f)) \geq \delta \|f\|^2$ for any$f \in \mathscr{k}_n$ and $H\Phi_n(f) = \lambda_n^i \Phi_n(f)$ for any$f \in \tilde{F}_n^i, i = 1, \dots, K_{(n)}$.

Then the following statements hold:

(i) The operator H has only pure point spectrum. The set of eigenvalues is $\bigcup_{n \geq 1} \bigcup_{1 \leq i \leq K_{(n)}} \{\lambda_n^i\}$.

(ii) There is a countable set $S \subset \tilde{\mathscr{k}}$ of orthonormal eigenfunctions of the operator H which is complete in $\mathscr{k}$.

(iii) If $\Phi_n(f) \notin \mathcal{k}_n$ for any nonzero $f \in \mathcal{k}_n$ and every $n \geq 1$, then each eigenvalue of H has infinite multiplicity.

(iv) H is a self-adjoint operator in $\mathcal{k}$.

Proof. At first we note from the definition of H_n that $\mathcal{k}_n = \oplus_{i=1}^{K(n)} \tilde{F}_n^i$.

Let

$$S_n = \{f \in \mathcal{k}_n : Hf \in \mathcal{k}_n\}.$$

It is easy to see that $S_n \subset S_{n+1}$ for every $n \geq 1$.

We introduce the set S by the formula

$$S = \bigcup_{n\geq 1} \bigcup_{1\leq i\leq K_{(n)}} \left(F_n^i \cap S_n\right)$$

and we note that the set $S_n \cap F_n^i$ is not empty for $n \geq m+1$ because $\Phi_n(f) \in \mathcal{k}_{n+m}$ for every $f \in \mathcal{k}_n$ and

$$H_{n+m}\Phi_n(f) = P_{n+m}HP_{n+m}\Phi_n(f) = P_{n+m}\left(\lambda_n^i\Phi_n(f)\right) = \lambda_n^i\Phi_n(f), f \in F_n^i (6)$$

One can see from the condition of Theorem (3.1.11) and (6) that if $\lambda \in \sigma(H_n)$ then λ is an eigenvalue of H. That gives us the inclusion

$$\bigcup_{n\geq 1} \bigcup_{1\leq i\leq K_{(n)}} \{\lambda_n^i\} \subset \sigma(H). \qquad (7)$$

We will prove that the set S is complete in $\mathcal{k}$. Suppose that there exists $f \in \mathcal{k}$ such that $(f, g) = 0$ for any $g \in S$.

Let A be a subspace of $\mathcal{k}$ and P_A be the orthogonal projection to A.

Then

$$\|P_A f\| \geq \frac{1}{\|g\|}|(g, f)| \qquad (8)$$

for every $g \in A$, $g \neq 0$, and $f \in \mathcal{k}$. This follows from the expression

$$|\|g\|^{-1}(g, f) = \|g\|^{-1}|(P_A g, f)| = \|g\|^{-1}|(P_A^2 g, f) = \|g\|^{-1}(g, P_A f)| \leq \|g\|^{-1}\|g\|\|P_A f\| \leq \|P_A f\|$$

Let us introduce the subspace A_n of $\mathcal{k}_n$ by the formula

$$A_n = \bigoplus_{i=1}^{K(n)} \left(\tilde{F}_n^i \cap S_n\right)$$

and let Q_n be the orthogonal projector to A_n.

If $f_n = P_n f, n = 1, 2, \ldots$, by (8) and the conditions of Theorem (3.1.11) we have

$$\|Q_{n+m}f_n\| \geq |\Phi_n(f_n), f_n|\|\Phi_n(f_n)\|^{-1} \geq (c\|f_n\|)^{-1}|(\Phi_n(f), f_n)| \geq c^{-1}\delta\|f_n\|. \qquad (9)$$

Since $A_{n+m} \subset$ Span S we obtain $Q_{n+m}f = 0$. Hence.

$$0 = \|Q_{n+m}f\| \geq \|Q_{n+m}f_n\| - \|f - f_n\| \geq c^{-1}\delta\|f_n\| - \|f - f_n\|.$$

This implies $f = 0$ since $\|f - f_n\| \to 0$ as $n \to \infty$. Therefore S is complete in $\mathcal{k}$ and (i), (ii) is proved.

(iii) For arbitrary eigenvalue λ of H there exists a corresponding egenfunction $f \in S$ and consequently there are such n_0, i that $f \in F_{n_0}^i \cap S_{n_0}$. We denote $g_0 = \Phi_{n_0}(f)$ and $g_{k+1} = \Phi_{n_0+km}(g_k)$. Then $\{g_j\}_{k=0}^{\infty}$ is a linearly independependent sequence of eigenfunctions of the operator H because, by the definition of $\Phi_{n,g_{k+1}} \notin \mathcal{k}_{n0+km}$.

(iv) It is enough to prove that Ran$(H \pm i)$ are complete sets in $\mathcal{k}$ (see [84, Vol. 1. Theorem VIII.3)

that follows from (ii) of our theorem.
The theorem is proved.

Theorem (3.1.12) [86]: Suppose that the graph M has a cycle and the edge e_0 belongs to this cycle. Then the spectrum of the operator $\Delta_p(\Delta_p^0)$ is pure point. Moreover, a countable set of orthonomral eigenfunctions of $\Delta_p(\Delta_p^0)$ with compact support is complete in $l_2^{\#}(V)\left(l_2^{\#}(V^0)\right)$ and every eigenvalue has infinite multicity.

If e_0 does not belong to the cycle, we do not know the structure of the spectrum in general. However, there is the following theorem in a particular case.

Theorem (3.1.13) [86]: Suppose all conditions for the graph G in Theorem (3.1.13) hold. Then:

(i) The operator $\Delta_A(\Delta_A^0)$ is self-adjoint.

(ii) All statements of Theorem (3.1.13) are true.

Proof. By Theorem (3.1.11) it is enough to construct the operator $\Phi_n\colon \mathscr{k}_n \to \mathscr{k}_{n+m}, m \geq 1$ with required properties. We will prove Theorem (3.1.12) only for the operator Δ_p because the case of the Δ_A is the same.

Let $\mathscr{k}_n = l_2^{\#}(\text{int}\, G_n)$. We suppose that the cycle in M is defined by the set of vertices $\{v_k\}_{k=0}^{i}, v, \in V_M, v_0 = v_i$.

If $l = 2m, m \in \mathbb{N}$, we can introduce sets of edges.

$$E^+ = \{(v_{2k}, v_{2k+1})\}_{k=0}^{m} \subset E_M,$$
$$E^- = \{(v_{2k-1}, v_{2k})\}_{k=1}^{m} \subset E_M,$$

We note that for any$x \in \Psi_n^{\varepsilon}(V_n \setminus \partial G_n)$ there is a unique $y \in V_n \setminus \partial G_n$ such that $x = \Psi_n^{\varepsilon}(y), e \in E_M$. We may suppose that the maps $\Psi_n^{\varepsilon}, e \in E^+ \cup E^-$ can be chosen such that if different edges e_1 and e_2 have a common vertex, then at least one of the following equalities holds.

$$\Psi_n^{\varepsilon_1}(\alpha_n) = \Psi_n^{\varepsilon_2}(\alpha_n) or \Psi_n^{e_1}(\beta_n) = \Psi_n^{e_2}(\beta_n) (10)$$

Let us define operators $\Phi_n^e\colon \mathscr{k}_n \to \mathscr{k}_{n+1}$ for any$e \in E_M$ as follows:

$$\Phi_n^e(f)(x) = \begin{cases} 0 & if x \notin \Psi_n^{\varepsilon}(V_n \setminus \partial G_n) \\ f(y) & if x = \Psi_n^{e}(y), \in V_n \setminus \partial G_n \end{cases}$$

Then we define the operator

$$\Phi_n = \sum_{e \epsilon E^+} \Phi_n^e - \sum_{e \epsilon E^-} \Phi_n^e,$$

which maps into $\mathscr{k}_{n+1}$. We will verify that it satisfies the conditions of Theorem (3.1.11).

we note that if $e_1, e_2 \in E_M$, and $e_1 \neq e_2$ then $\Phi_n^{e_1}(f)$ and $\Phi_n^{e_2}(f)$ have disjoint supports. Thus $\Phi_n^{e_1}(f)$ is orthogonal to $\Phi_n^{e_2}(f)$ and the bound $\|\Phi_n\| \leq c = l$ is obtained. By condition (ii) of Definition (3.1.1) we have $\Phi_n^{e_0}(f) = f$ and

$$\left(f, \Phi_n(f)\right) = \|f\|^2$$

for every$f \in \mathscr{k}_n$. Now if $f \in \tilde{F}_n^i$ then the equality.

$$-\Delta_p \Phi_n(f) = \lambda_n^i \Phi_n(f)$$

follows from the definition of the operator Φ_n.

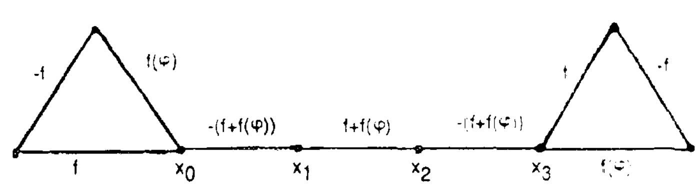

Diagram 1

Fig. 2

Since $\Phi_n(f)$ is an eigenfunction of the operator Δ_p with compact support by the definition of the set S in the proof of Theorem (3.1.11) we find that S is a set of eigenfunctions with compact supports

Let $l = 2m + 1, m \geq 1$. The construction of the operator Φ_n in this case is more delicate. In graph $\widetilde{M}$ (see Lemma (3.1.6)) we have at least two cycles of length l, joining by a path, and e_0 belongs to one of these cycles.

Say these cycles are $\{v_k\}_{k=0}^{l}, \{u_k\}_{k=0}^{l}, v_0 = n_l, u_0 = u_l$ and they are joined by a path $v_0 = x_0, x_l, \dots, x_r = u_0$.

Let $E_x^+, E_x^{\cdot}$ are defined similarly. Also, we define operators $\widetilde{\Phi}_n^e$ analgously to Φ_n^e, using Ψ_n^e instead of Ψ_n^e (see Lemma (3.1.6)).

Then

$$\Phi_n = \sum_{e \in E_x^+} \Phi_n^e - \sum_{e \in E_x^-} \Phi_n^e$$
$$- \sum_{e \in E_x^+} (\Phi_n^e + \Phi_n^e \circ U_n^{\#}) + \sum_{e \in E_x^-} (\Phi_n^e + \Phi_n^e \circ U_n^{\#}) + (-1)^{r+1} \left(\sum_{e \in E_n^+} \Phi_n^e - \sum_{e \in E_n^-} \Phi_n^e \right).$$

We suppose that condition (10) is satisfied in this case, too. This construction is sketched in Diagram 1 if r is odd and on Diagram 2 if r is even.

We note that $\Phi_n : G_{n+2}$ and this operator satisfies the condition of Theorem (3.1.11) that can be proved analogously to case 1 using Lemma (3.1.6) and (3.1.7) The theorem is proved.

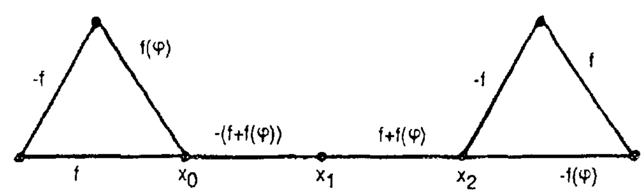

Fig. 3

Theorem (3.1.14) [86]: Suppose that the graph M has an odd cycle and there is an isomorphism $\varphi: M \to M$ such that $\varphi(\alpha) = \beta, \varphi(\beta) = \alpha$, and $(e_0)\# e_0$. If

(i) The edge e_0 belongs to a path joining α and β or

(ii) The edge e_0 belongs to a path joining α (or β) with the cycle then the conclusions of Theorem (3.1.13) hold for Δ_p and Δ_p^0.

Let us now consider the operator Δ_A. If the boundary of G is empty its action is well defined on all functions with compact support which form a dense subspace of $l^2(V)$. If $\partial G \neq \emptyset$ we define Δ_A^0 as an operator with zero boundary conditions (See above definition for Δ_A^0). This operator is symmetric and thus closable. We will denote its closure by the same symbol $\Delta_A(\Delta_A^0)$.

Theorem (3.1.15) [86]: if all conditions of Theorem (3.1.14) are satisfied for the graph G, then the operator $\Delta_A(\Delta_A^0)$ is self-adjoint and the statements of Theorem (3.1.14) hold for $\Delta_A(\Delta_A^0)$.

We note that the operator Δ_A is not self-adjoint in general. An example of a locally finite graph with no unique self-adjoint extension of Δ_A was given in [81].

The condition of the existence of a cycle in the graph M is not a necessary condition for the spectrum to be pure point. Moreover the graph G may be a tree in this case

Proof. We will consider only operator Δ_p because the case of Δ_A is the same. Also we assume that e_0 does not belong to a cycle, otherwise it is a special case of Theorem (3.1.12).

We define

$$\mathscr{k}_n = \{f \in l_2^{\#}(\mathrm{Int}\, G_n), \Delta_p f = \Delta_p(n) for U_n^{\#} f = f\}.$$

We have $\mathscr{k}_n \subset \mathscr{k}_{n+1}$. Let us show that $\mathscr{k} = \bigcup_{n\geq 1} \mathscr{k}_n$ is complete in $\mathscr{k} = l_2^{\#}(V)$. For any $f \in \mathscr{k}$ there is such n that $\|f - f_n\| \leq \frac{1}{4}\|f\|$, f_n is the restriction of f to V_n. Since $\varphi(e_0) \neq e_0$ we have $(U_{n+1}^{\#} f_n, f_n) = 0$ and so

$$\left|(f, f_n + U_{n+1}^{\#} f)\right| \geq \left|(f_n, f_n + U_{n+1}^{\#} f_n)\right| - \|f - f_n\| \cdot \left\|(f_n + U_{n+1}^{\#} f_n, f_n)\right\| \geq \|f_n\|^2 - \frac{\sqrt{2}}{4}\|f_n\|^2$$
$$\geq \frac{3}{16}\|f\|^2$$

because $\|f_n\| \geq \frac{3}{4}\|f\|$ and $\left\|f_n + U_{n+1}^{\#} f_n\right\| = \sqrt{2}\|f_n\|$. This implies that $\mathscr{k}$ is complete since f is arbitrary and $f_n + U_{n+1}^{\#} f_n \in \tilde{\mathscr{k}}$.

Therefore we need only construct operator Φ_n which satisfies the conditions of Theorem (3.1.11).

(i) One can see that the graph $\widetilde{M}$ has two odd cycles joining by a path such that e_0 belongs to this path. In this case, Φ_n can be defined exactly the same way as in the proof of Theorem (3.1.13) for an odd cycle.

(ii) If, for example, α is incident to e_0, then there is a path $x = x_0, x_1, \dots, x_i = u_0$ and an odd cycle $\{u_n\}_{k=0}^{n}, u_n$, where $e_0 = (x_0, x_1)$. Then Φ_n can be defined by

$$\Phi_n = \sum_{e\in E_x^+}(\Phi_n^e + \Phi_n^e \circ U_n^{\#}) - \sum_{e\in E_x^-}(\Phi_n^e + \Phi_n^e \circ U_n^{\#}) + (-1)'\left(\sum_{e\in E_m^-}\Phi_n^e - \sum_{e\in E_n^-}\Phi_n^e\right)$$

where $\Phi_n^e, E_x^+, E_x^-, E_u^+, E_u^-$ are defined the same way as in the proof of Theorem (3.1.12).

If α is not incident with e_0 the proof is analogously (i). The theorem is proved.

Theorem (3.1.16) [86]: Suppose there exist different vertices $y_0, y_1, y_2 \in V(M)$ such that there are edges $(y_0, y_1), (y_1, y_2) \in E(M), e_0 = (y_0, y_1), d_{y0}(M) = d_{y_2}(M) = 1$ and the set $\{y_0, y_2\}$ does not coincide with the set $\{x, \beta\}$.

Then all results of Theorem (3.1.11) and (3.1.13) hold.

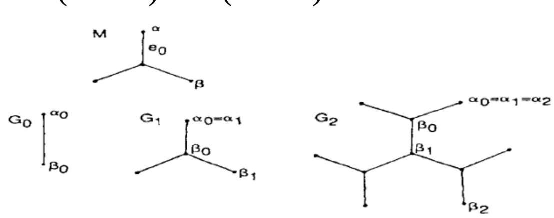

Figure 4

The simple example of a two-point self-similar graph such that the conditions of Theorem (3.1.13-

3.1.16) are not satisfied is the lattice $\mathbb{Z}$. It is well known that the spectrum of the Laplacian in this case is absolutely continuous.

Condition (iv) in Definition (3.1.1) defines the structure of eigenfucntions of the Laplacians. It is easy to see that conditions (i)-(iii) of Definition (3.1.1) are satisfied for Sierpinsky lattice but Theorem $1-2^0$ are not true in this case. By [75] it follows that there are such eigenvalues that if a function φ is an eigenfunction corresponding to one of them, then φ cannot have a compact support.

The problem of describing the spectrum as a set in $\mathbb{R}$ is hard enough as shown by the example of the operator Δ_p on the modified Koch graph in [79].

Let us introduce functions $W: V \to \mathbb{R}$ which do not change the nature of the spectrum of the Laplacian; i.e. the spectrum of the Schrödinger operator.

$$H = \Delta + W (11)$$

will be pure point, too. Here we denote Δ_A and Δ_p by the same symbol Δ.

We note that periodic functions are potentials of this sort for the Schrödinger operator in $l_2(\mathbb{Z}^n)$ but only in the case of absolutely continuous spectrum.

Suppose that $W_0: V_{n_0} \to \mathbb{R}$ is a function such that $W_0(\varphi(x)) = W_0(x)$, where $\varphi: G_n \to G_n$ is an automorphism of $G_n, \varphi(\alpha_n) = \beta_n, \varphi(\beta_n) = \alpha_n$. Let us define the potential $W: V \to \mathbb{R}$ by induction. We denote byW_{m+1} the restriction of W on V_{n_0+m+1} and we suppose $W_{m+1}(x) = W_m(y)$, where $x = \Psi^e_{n_0+m}(y) y \in V_{n_0+m}, e \in E_M$ for every$m \geq 0$.

Proof. At first we suppose that α, β are not from the set $\{y_0, y_2\}$. Without loss of generality we can assume that $d_{x_0}(G_n) < d_{\beta_0}(G_{n+1})$ and $\Psi_n^{(y_1,y_2)}(\beta_n) = \beta_n$.

Let us define

$$\mathscr{k}_n = \{f \in l_2^{\#}(G): f(x) = 0 \, if x \in (V \setminus \beta_n)\}.$$

The operator $\Phi_n: \mathscr{k}_n \to \mathscr{k}_{n+1}$ can be given by the formula

$$\Phi_n(f)(x) = \begin{cases} f(x) if x \in V_n \\ -f(x) if x \in \Psi_n^{(y_1,y_2)}(y), y \in G_n \\ 0 \quad otherwise \end{cases} (12)$$

If $\alpha = y_0$ the definition of the operator Φ_n is the same.

Let $\alpha = y_2$. Then we have to consider the graph $\widetilde{M}$ (Lemma (3.1.6)) instead of M which has the necessary properties to construct Φ_n by the formula (12). The theorem is proved.

Theorem (3.1.17) [86]: If the function W is defined as above, all results of Theorems(3.1.12), (3.1.15), (3.1.16) hold for the Schrödinger operator (6).

Let us consider the so-called Bernoulli potential $\{W(x), x \in V\}$ made of a sequence of i.i.d random variables taking only two values 0 and 1.

We set.

$$\mathbb{P}\{W(x) = 0\} = \mathbb{P}\{W(x) = 1\} = \frac{1}{2}, \ x \in V.$$

We are interested in the random Schrödinger operator.

$$H_\beta = \Delta + \beta W$$

with a coupling constant $\beta > 0$.

Proof. The proof is one-to-one to the proof of Theorem (3.1.12 – 3.1.15, 3.1.16).

Theorem (3.1.18) [86]: Let G satisfy conditions of one of the Theorem (3.1.12), (3.1.15), (3.1.16).

Then for any$\beta > 0$ with probability one, every eignavalue of Δ is an eigenvalue of H_β of infinite multiplicity.

Let $\mathscr{k}$ be a Hilbert space with the inner product (,) and $\mathscr{k}_n, = 1,2, \ldots,$ be a sequence of finite dimensional subspaces of $\mathscr{k}$ such that $\mathscr{k}_n \subset \mathscr{k}_{n+1}$ and $\mathscr{k} = \bigcup_{n=1}^{\infty} \mathscr{k}_n$ is dense in $\mathscr{k}$.

We suppose that H is a closed symmetric operator on $\mathscr{k}$ such that $\tilde{\mathscr{k}}$ belongs to the domain of definition of the operator H and $H_n = P_n H P_n$, where P_n is the orthogonal projector on $\mathscr{k}_n$.

Then $H_n: \mathscr{k}_n \to \mathscr{k}_n$ and H_n is symmetric, too.

Let $\lambda_n^1, \ldots, \lambda_n^{k_n}$ be all distinct eigenvalues of the operator H_n (restricted to $\mathscr{k}_n$).

Let $\tilde{F}_n^i$ be the eigenspace corresponding to λ_n^1 and let F_n^i be an orthonormal basis of $\tilde{F}_n^i$.

Proof. It is easy to see that if Ψ is an eigenfuction of the operator Δ with compact support and supp $\Psi \cap \operatorname{supp} W = \emptyset$ then the function Ψ is an eigenfucntion of the operator H_β.

Let Λ be a set of all eigenvlues of the Δ and let S be a countable set of orghonormal eigenfunctions of the Δ with compact support. For every$\lambda \in \Lambda$ there is an eigenfunction $f \in S$ and the integer n_0 such that supp $f \subset G_{n_0}$.

We note that graph G can be written as the union of copies of G_{n_0}. With the probability one there is an infinity set of disjoint copies of G_{n_0} where W is zero. Consequentlyλ is an eigenvalue of the operator H_β of infinite multiplicity. The theorem is proved.

Section (3.2): The Hierarchical Anderson Model

We devoted to study of the spectral properties of the hierarchical Anderson model and is motivated by the work of Molchanov [114]. we recall the definition of the model and its basic properties. For additional information about the hierarchical structures and the hierarchical Anderson model we refer to [111, 109, 108, 113, 114].

Let X be an infinite countable set. Throughout the sectionδ_x will denote the Kronecker delta function at $x \in X$. A partition $\mathcal{P}$ of X is a collection of its disjoint subsets whose union is equal to X. Let $\mathrm{n} = (n_r)_{r \geq 0}$ be a sequence of positive integers and $\mathrm{P} = (\mathcal{P}_r)_{r \geq 0}$ a sequence of partitions of X. The elements of $\mathcal{P}_r$ are called “cluster” of rank r. We say that $(X, \mathrm{P}, \mathrm{n})$ is a hierarchical if the following hold:

(i) $n_0 = 1$and every$Q \in \mathcal{P}_0$ has exactly one element.

(ii) For $r \geq 1$, every$Q \in \mathcal{P}_r$ is a disjoint union of n_r clusters in $\mathcal{P}_{r-1}$.

(iii) Given $x, \mathrm{y} \in X$, there is a cluster Q of some rank containing both x and y.

Let us state some immediate consequence of this definition. Every cluster of rank $r \geq 0$ has size $N_r: \prod_{s=0}^{r} n_s$. Given $x \in X$ and $r \geq 0$, there is a unique cluster of rank r containing x. We denote this cluster by$Q_r(x)$. The map.

$$d(x, \mathrm{y}) \coloneqq \min\{r: \mathrm{y} \in Q_r(x)\},$$

is a metric on X and $Q_r(x) = \{\mathrm{y}: d(x, \mathrm{y}) \leq r\}$. Note that $Q_r(x) = Q_r(\mathrm{y})$ whenever $d(x, \mathrm{y}) \leq r$. Given an integer $n \geq 2$, a hierarchical structure is called homogeneous of degree n if $n_r = n$ for all $r \geq 1$.

The free Laplacian on the hierarchical structure $(X, \mathrm{P}, \mathrm{n})$ is define as follows. For each $r \geq 0$, let $E_r: l^2(X) \to l^2(X)$ be the aver operator.

$$(E_r\psi)(x) := \frac{1}{N_r}\sum_{d(x,y)\le r}\psi(y).$$

Let $P = (p_r)_{r\ge 1}$ be a sequence of positive number such that $\sum_{r=1}^{\infty} p_r = 1$. In the sequel we set $p_0 = 0$ and

$$\curlywedge_r := \sum_{s=0}^{r} p_s, \quad r = 0,1,\dots,\infty.$$

The hierarchical Laplacian Δ on $l^2(X)$ is defined by

$$\Delta := \sum_{r=0}^{\infty} p_r E_r.$$

Clearly, Δ is a bounded self-adjont operator and $0 \le \Delta \le 1$.

A hierarchical model is a hierarchical structure (X, P, n) together with the hierarchical Laplacian Δ. The spectral properties of Δ only depend on $n$and P and are summarized in:

Theorem (3.2.1) [95]: (i) The spectrum of Δ is equal to $\{\curlywedge_r : r = 0,\dots,\infty\}$. Each $\curlywedge_r, r < \infty$, is an eigenvalue of Δ of infinite multiplicity. The point $\curlywedge_\infty = 1$ is not an eigenvalue.

(ii) $E_r - E_{r+1}$ is the orthogonal projection onto the eigenspace of $\curlywedge_r$ and

$$\Delta = \sum_{r=0}^{\infty} \curlywedge_r \,(E_r - E_{r+1}).$$

(iii) For every$x \in X$, the spectral measure for δ_x and Δ is given by

$$\mu = \sum_{r=0}^{\infty}\left(\frac{1}{N_r} - \frac{1}{N_{r+1}}\right)\delta(\curlywedge_r),$$

where $\delta(\curlywedge_r)$ stands for the Dirac unit mass at $\curlywedge_r$. Note that μ does not depend on x.

The spectra measure μ can be naturally interpreted as the integrated density of states of the operator Δ. Let $x_0 \in X$ be given and consider the increasing sequence of clusters $Q_r(x_0), r \ge 0$. Let P_r be the orthogonal projection onto the N_r-dimensional subspace.

$$l^2\big(Q_r(x_0)\big) := \{\psi \in l^2(X) : \psi(x) = 0 \; for x \notin Q_r(x_0)\}.$$

Let $e_1^{(r)} \le e_{N_r}^{(r)} \le \cdots \le e_{N_r}^{(r)}$, be the eigenvalues of the restricted Laplacian $P_r\Delta P_r$ acting on $l^2\big(Q_r(x_0)\big)$ and

$$v_r := \frac{1}{N_r} = \sum_{s=1}^{r}\delta\left(e_s^{(r)}\right),$$

the corresponding counting measure.

Proof. For $r \ge 0$, let $\mathcal{H}_r = Ran(E_r).\mathcal{H}_r$ is the closed subspace of $l^2(X)$ consisting of functions that are constant on each cluster of rank r. Note that

$$l^2(X) = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \mathcal{H}_3 \supset \cdots$$

and that $\bigcap \mathcal{H}_r = \{0\}$ since a nonzero function constant on every cluster would have infinite l^2 nom. These observations yield that

$$l^2(X) \quad \bigoplus_{r=0}^{\infty} \quad L_r, \qquad (13)$$

where L_r is the orthogonal complement of $\mathcal{H}_{r+1}$ in $\mathcal{H}_r$. Note that L_r is the infinite dimensional

subspace of function ψ s.t. $E_s\psi = \psi$ for $0 \le s \le r$ and $E_s\psi = 0$ for $s > r$. Hence for every$\psi \in L_r, \Delta\psi = \lambda_r\, \psi$, and this proves parts (1) (2).
The spectral measure $\mu_{x,\Delta}$ for δ_x and Δ is the unique Borel probability measure on $\mathbb{R}$ s.t.

$$\langle \delta_x | f(\Delta)\delta_x \rangle = \int_{\mathbb{R}} f(\xi)d\mu_{x,\Delta}(\xi),$$

for every bounded Borel function $f: \mathbb{R} \to \mathbb{C}$. To compute $\mu_{x,\Delta}$, we decompose δ_x according to (13):

$$\delta_x = \sum_{r=0}^{\infty}(E_r - E_{r+1})\delta_x = \sum_{r=0}^{\infty}\left(\frac{1}{N_r}1Q_{r(x)} - \frac{1}{N_{r+1}}1Q_{r+1(x)}\right),$$

where $1Q_{r+1(x)} \coloneqq \sum_{y\in Q_r(x)} \delta_x$. Hence

$$f(\Delta)\delta_x = \sum_{r=0}^{\infty} f(\lambda_r)\left(\frac{1}{N_r}1Q_{r(x)} - \frac{1}{N_{r+1}}1Q_{r+1(x)}\right),$$

and

$$\langle \delta_x | f(\Delta)\delta_x \rangle = \sum_{r=0}^{\infty} f(\lambda_r)\left\|\frac{1}{N_r}1Q_{r(x)} - \frac{1}{N_{r+1}}1Q_{r+1(x)}\right\|^2.$$

Since $\left\|\frac{1}{N_r}1Q_{r(x)} - \frac{1}{N_{r+1}}1Q_{r+1(x)}\right\|^2 = 1/N_r - 1/N_{r+1}$, (3) follows.
The analysis of the density of states of Δ us facilitated if one introduces the cut-off Laplacians

$$\Delta_r \coloneqq \sum_{s=0}^{r} p_s E_s, \qquad r \ge 0.$$

It is technically easier to work with Δ_r than with $P_r\Delta P_r$. Note that $l^2(Q_r(x_0))$ is an invariant subspace for Δ_r. One can exactly compute the eigenvalues and eigenvectors of restricted operator $P_r\Delta P_r$ acting on $l^2(Q_r(x_0))$. If $0 \le s \le r$, then every$\psi \in L_s \cap l^2(Q_r(x_0))$ is an eigenvector of $P_r\Delta P_r$ with eiegnvalue λ_r. The subspace $L_s \cap l^2(Q_r(x_0))$ has dimension $D_s^{(r)} \coloneqq N_r(1/N_s - 1/N_{s+1})$ for $0 \le s \le r-1$, and the subspace $L_r \cap l^2(Q_r(x_0))$ has dimension $D_r^{(r)} \coloneqq 1$. Since $\sum_{s=0}^{r} D_s^{(r)} = N_r$, the spectrum of $P_r\Delta_r$ is equal to $\{\lambda_s: s = 0, \dots, r\}$ and each eigenvalue λ_s has multiplicity$D_s^{(r)}$.

Proposition (3.2.2) [95]: The weak-* limit $\lim_{r\to\infty} v_r$ exists and is equal to μ. if

$$\lim_{t\downarrow 0}\frac{\log \mu([1-t,1])}{\log t} = d/2,$$

then the number d is called the spectral dimension of Δ. This definition is motivated by the analogy with the edge asymptotic of the density of states of the standard discrete Laplacian on $\mathbb{Z}^d$, for which the spectral and spatial dimensions coincide.
The relation $\sum_{y\in X}\langle \delta_x | \Delta\delta_y \rangle = 1$yields that Δ generates a random walk on X. We recall that the random walk on $\mathbb{Z}^d$ generated by the standard discrete Laplacian is recurrent if $d = 1, 2$ and transient if $d > 2$. The corresponding result for the hierarchical Laplacian is:

Proposition (3.2.3) [95]: Consider a homogeneous hierarchical structure of degree $n \ge 2$. Suppose that there exist constants $C_1 > 0, C_2 > 0$ and $\rho > 1$ such that

$$C_1\rho^{-r} \le p_r \le C_2\rho^{-r}.$$

for r bit enough. Then:

(i) The spectral dimension of the model is

$$\mathrm{d}(n,\rho) = 2\frac{\log n}{\log \rho}.$$

Hence $0 < d(n,\rho) \le 2$ if $n \le \rho$.

(ii) The random walk generated byΔ is recurrent if $0 < d(n,\rho) \le 2$ and transient if $\mathrm{d}(n,\rho) > 2$.

We now define the hierarchical Anderson model associated to $(X, \mathrm{P}, \mathrm{n})$ and the hierarchical Laplacian Δ. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega \coloneqq \mathbb{R}^X, \mathcal{F}$ is the usual Borel σ-algebra in Ω, and $\mathbb{P}$ is a given probability measure on $(\Omega, \mathcal{F})$. For $\omega \in \Omega$, we set

$$V_\omega \coloneqq \sum_{x\in X} \omega(x)\langle \delta_x|\cdot\rangle\, \delta_x.$$

V_ω is a self-adjoint (possibly unbounded) multiplication operator on $l^2(X)$. Let

$$H_\omega \coloneqq \Delta + V_\omega, \qquad \omega \in \Omega.$$

The family of self-adjoint operators $\{H_\omega\}_{\omega\in\Omega}$ indexed by the events of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called the hierarchical Anderson model.

Concerning the probability measure $\mathbb{P}$, we will need only one technical assumption having to do with the notion of conditional density. Throughout, m will denote the Lebesgue measure on $\mathbb{R}$. For any$x \in X, \Omega$ can be decomposed along the x'th coordinate as $\Omega = \mathbb{R} \times \widetilde{\Omega}, \widetilde{\Omega} = \mathbb{R}^{X\setminus\{x\}}$. Let $\widetilde{\mathbb{P}}_x$ be the corresponding marginal of $\mathbb{P}$ defined by$\widetilde{\mathbb{P}}_x(\tilde{B}) \coloneqq \mathbb{P}(\mathbb{R} \times \tilde{B})$, where $\tilde{B} \subset \widetilde{\Omega}$ is a Borel set. Then for $\widetilde{\mathbb{P}}_z$-a.e. $\widetilde{\omega} \in \widetilde{\Omega}$, there is a probability measure $\mathbb{P}_x^{\widetilde{\omega}}$ on $\mathbb{R}$ s.t. the conditional Fubini theorem holds: for all $f \in L^1(\Omega, P)$ we have.

$$\int_\Omega f(\omega)d\mathbb{P}(\omega) = \int_{\widetilde{\Omega}} \left(\int_{\mathbb{R}} f(\xi, \widetilde{\omega})\, d\mathbb{P}_x^{\widetilde{\omega}}(\xi)\right) d\widetilde{\mathbb{P}}_x(\widetilde{\omega}).$$

If for $\widetilde{\mathbb{P}}_x$-a.e. $\widetilde{\omega} \in \widetilde{\Omega}, \mathbb{P}_x^{\widetilde{\omega}}$ is absolutely continuous (a.c.) with respect to m, then we say that $\mathbb{P}$ has a conditional density along the x'th coordinate. An important special case of a conditionally a.c. probability measure is the product measure $\mathbb{P} = \otimes_{x\in X}\mathbb{P}_x$, where each $\mathbb{P}_x$, is a probability measure on $\mathbb{R}$ a.c. with respect to m.

We denote by$\sigma_{ac}(H_\omega)$ the absolutely continuous part of the spectrum of H_ω and by$\sigma_{\mathrm{cont}}(H_\omega)$ the continuous part.

Proof. Let v^* be a weak-* limit point of the sequence v_r. Let v_{r_k} be a subsequence converging to v^*. We claim that

$$v^*(\{\lambda_s\}) = \mu(\{\lambda_s\}), \qquad (14)$$

for all $s \ge 0$. Indeed, let $\delta \coloneqq \min_{j\neq s}|\lambda_s - \lambda_j|/2$ and $0 < \varepsilon < \delta/3$. Since $\|P_r\Delta P_r - P_r\Delta_r\| \le \sum_{j=r+1}^{\infty} p_j$, we have that $\|P_r\Delta P_r - P_r\Delta_r\| \le \varepsilon$ for all r big enough. For such r, the spectrum of $\Delta_r\Delta_r$ is contained in $\bigcup_{j=0}^{r}[\lambda_j - \varepsilon, \lambda_j + \varepsilon]$. Let R be the spectral projection of $P_r\Delta P_r$ on $[\lambda_s - \varepsilon, \lambda_s + \varepsilon]$ and T the spectral projection of $P_r\Delta P_r$ on the same interval. Let γ be the circle $\{z \in \mathbb{C}: |z - \lambda_s| = \delta\}$, oriented counterclockwise. Then

$$R - T = \frac{1}{2\pi i}\oint_\gamma (z - P_r\Delta P_r)^{-1}dz$$

$$- \frac{1}{2\pi i}\oint_\gamma (z - P_r\Delta_r)^{-1}dz = \frac{1}{\pi i}\oint_\gamma (z - P_r\Delta P_r)^{-1}(P_r\Delta P_r - P_r\Delta_r)(z - P_r\Delta_r)^{-1}dz,$$

and thus

$$\|R-T\| \leq \delta(2\delta/3)^{-1}\varepsilon(2\delta/3)^{-1} \leq 3/4 < 1.$$

It follows that $Ran(R)$ and $Ran(T)$ have the same dimension and that

$$\neq \left\{s: e_s^{(r)} \in [\lambda_s - \varepsilon, \lambda_s + \varepsilon]\right\} = D_s^{(r)}.$$

Then for all k big enough

$$v_{r_k}([\lambda_s - \varepsilon, \lambda_s + \varepsilon]) = D_s^{(r)}/N_r = 1/N_s - 1/N_{s+1}.$$

Letting $k \to \infty$, we get $v^*([\lambda_s - \varepsilon, \lambda_s + \varepsilon]) = 1/N_s - 1/N_{s+1}$, and (14) follows by taking $\varepsilon \downarrow 0$. Since $\sum_{s=0}^{\infty}(1/N_s - 1/N_{s+1}) = 1$ and v^* is a probability measure, we must have that $v^* = \mu$. Therefore μ is the unique weak-* limit point of the sequence v_r and $\lim_{r\to\infty} v_r = \mu$.

Note that $\mu([1-t,1])$ is a piecewise constant function of t with jump discontinuities at the points $1-\lambda_r$. Since

$$C_1(\rho-1)^{-1}\rho^{-r} \leq 1-\lambda_r = \sum_{s=r+1}^{\infty} p_s \leq C_2(\rho-1)^{-1}\rho^{-r},$$

and $\mu([1-\lambda_r, 1]) = 1/N_r = n^{-r}$, we have that

$$\lim_{t\downarrow 0} \frac{\log \mu([1-t,1])}{\log t} = \frac{\log n}{\log \rho},$$

which proves (i).

The random walk on X starting at x is transient if $R \coloneqq \sum_{k=0}^{\infty}\langle \delta_x | \Delta^k \delta_x \rangle < \infty$ and recurrent if $R = \infty$. Part (iii) of Theorem (3.2.1) allows to compute R explicitly:

$$R = \langle \delta_x | (1-\Delta)^{-1}\delta_x \rangle = \int \frac{d\mu(\xi)}{1-\xi} = \sum_{r=0}^{\infty} \frac{N_r^{-1} - N_{r+1}^{-1}}{1-\lambda_r}.$$

The bounds

$$C_2^{-1}(\rho-1)(1-1/n)\sum_{r=0}^{\infty}(\rho/n)^r \leq R \leq C_1^{-1}(\rho-1)(1-1/n)\sum_{r=0}^{\infty}(\rho/n)^r$$

show that $R < \infty$ for $\rho < n$ and $R = \infty$ for $\rho \geq n$, and part (2) follows.

We first derive a hierarchical approximation formula for the resolvent $(H_\omega - z)^{-1}$. Then we use the formula to obtain a bound on the resolvent matrix elements. This bound combined with the Simon-Wolff localization criterion yields the statement.

Set

$$H_{\omega,r} \coloneqq V_\omega + \sum_{s=0}^{r} p_s E_s, \quad r \geq 0.$$

Fix $\omega \in \Omega$. For any$Q_r \in \mathcal{P}_r$, the subspace $l^2(Q_r)$ is invariant for $H_{\omega,r}$. Let $\sigma(\omega, Q_r)$ be the set of the eigenvalues of the restricted operator $H_{\omega,r} \restriction l^2(Q_r)$ and $\sigma_\omega \colon \cup\, \sigma(\omega, Q_r)$ where the union is over all clusters of all ranks. Clearly, σ_ω is a countable subset of $\mathbb{R}$. For $z \in \mathbb{C} \setminus \sigma_\omega, r \geq 0$, and $x, \mathrm{y} \in X$, we set

$$C_{\omega,r}(x,\mathrm{y};z) \coloneqq \langle \delta_x \left| (H_{\omega,r} - z)^{-1} \delta_{\mathrm{y}} \right\rangle.$$

For $z \in \mathbb{C} \setminus \sigma_\omega, r \geq 0$ and $t \in X$, let $g_{\omega,r}(t;z)$ be the average of $C_{\omega,r}(\cdot, t; z)$ over the cluster $Q_r(t)$, i.e.

$$g_{\omega,r}(t;z) := \frac{1}{N_r}\sum_{d(t',t)\leq r} C_{\omega,r}(t',t;z).$$

Since the joint spectral measure for $\delta_t, \delta_{t'}$ and $H_{\omega,r}$ is real, $C_{\omega,r}(t',t;z) = C_{\omega,r}(t,t';z)$ and

$$g_{\omega,r}(t;z) = \frac{1}{N_r}\sum_{d(t',t)\leq r} C_{\omega,r}(t,t';z) = \frac{1}{N_r}\langle\delta_t|\left(H_{\omega,r}-z\right)^{-1}\mathbf{1}_{Qr(t)}\rangle. \quad (15)$$

Proposition (3.2.4) [95]: Let $\omega\in\Omega, x, \mathrm{y}\in\mathbb{C}\setminus\sigma_\omega$ and $r\geq 0$ be given. Then

$$C_{\omega,r}(x,\mathrm{y};z) = C_{\omega,0}(x,\mathrm{y};z) - \sum_{s=d(x,\mathrm{y})}^{r} p_s N_{s-1} g_{\omega,s}(\mathrm{y};z). \,(16)$$

Proof. The formula holds for $r = 0$ since $p_0 = 0$. For $s\geq 1$, the resolvent identityyields.

$$\left(H_{\omega,s}-z\right)^{-}\delta_{\mathrm{y}} - \left(H_{\omega,s-1}-z\right)^{-1}\delta_{\mathrm{y}} = -\left(H_{\omega,s-1}-z\right)^{-1} p_s E_s\left(H_{\omega,s}-z\right)^{-1}\delta_{\mathrm{y}}$$

Observe that $E_s\left(H_{\omega,s}-z\right)^{-1}\delta_{\mathrm{y}} = g_{\omega,s}(\mathrm{y};z)1_{Q_s(\mathrm{y})}$. Taking $\langle\delta_x|\cdot\rangle$ in the above equation yields

$$G_{\omega,s}(x,\mathrm{y};z) - G_{\omega,s-1}(x,\mathrm{y};z) = -p_s g_{\omega,s}(\mathrm{y};z)\,\langle\delta_x|\left(H_{\omega,s-1}-z\right)^{-1}1_{Q_s(\mathrm{y})}\rangle. \quad (17)$$

Note that by (15),

$$\langle\delta_x|\left(H_{\omega,s-1}-z\right)^{-1}1_{Q_s(\mathrm{y})}\rangle = \begin{cases} N_{s-1}g_{\omega,s-1}(x;z), & if d(x,\mathrm{y})\leq s,\\ 0, & if d(x,\mathrm{y})\leq s,\end{cases}$$

The formula (16) follows after adding (17) for $s = 1,2,\dots,r$

Theorem (3.2.5) [95]: Suppose that p_r and N_r satisfy (24). Let $\omega\in\Omega$ and $x\in X$ be fixed. Then for m-a.e. $e\in\mathbb{R}\setminus\sigma_\omega$,

$$\sup_{r\geq 0}\sum_{\mathrm{y}\in X}\left|G_{\omega,r}(x,\mathrm{y};e)\right|^2 < \infty. \,(18)$$

Proof. We shall use the following general results, proven in [M2]:

Let A be a hermitian $N\times N$ matrix and $v\in\mathbb{C}^N$. Then for all $M > 0$.

$$m(\{e: \|(A-e)^{-1}v\|_2^2 \geq M\}) \leq 4\sqrt{\frac{N}{M}}\|v\|_2, \quad (19)$$

where $\|\cdot\|_2$ stands for the l^2 norm on $\mathbb{C}^N$.

Since $l^2(Q_r(x))$ is an N_r-dimensional invariant subspace for $H_{\omega,x}$ and since $\left\|1_{Q_r(x)}\right\|_2 = \sqrt{N_r}$, we have from (19) that for $M_r > 0$,

$$m\left(\left\{e\in\mathbb{R}\setminus\sigma_\omega: \left\|\left(H_{\omega,r}-e\right)^{-1}1_{Q_r(x)}\right\|_2^2 \geq M_r\right\}\right) \leq \frac{4N_r}{\sqrt{M_r}}.$$

Let $M_r > 0$ be a sequence satisfying $\sum_{r=1}^{\infty} N_r M_r^{-1/2} < \infty$. By the Borel-Cantelli lemma, for $m, e.ee\in\mathbb{R}\setminus\sigma_\omega$, there exists a finite constant C_e such that

$$\left\|\left(H_{\omega,r}-e\right)^{-1}1_{Q_r(x)}\right\|_2^2 < C_e M_r, \quad (20)$$

for all $r\geq 0$. From now on, such an $e\in\mathbb{R}\setminus\sigma_\omega$ is fixed. Using the representation formula (16), we get the estimate.

$$\left(\sum_{\mathrm{y}\in X}\left|G_{\omega,r}(x,\mathrm{y};e)\right|^2\right)^{1/2} \leq \left|G_{\omega,0}(x,x;e)\right|\sum_{s=1}^{r} p_s N_{s-1}\left|g_{\omega,s-1}(x;e)\right|\left(\sum_{d(x,\mathrm{y})\leq s}\left|g_{\omega,s}(\mathrm{y};e)\right|\right) (21)$$

Observe that

$$\left(\sum_{d(x,y)\le s}\left|g_{\omega,s}(y;e)\right|^2\right)^{1/2} = \left(\sum_{d(x,y)\le s}\left|\frac{1}{N_s}\langle\delta_y|\left(H_{\omega,s}-e\right)^{-1}1_{Q_s(y)}\rangle\right|^2\right)^{1/2}$$

$$= \frac{1}{N_s}\left(\sum_{d(x,y)\le s}\left|\langle\delta_y|\left(H_{\omega,s}-e\right)^{-1}1_{Q_s(x)}\rangle\right|^2\right)^{1/2} = \frac{1}{N_s}\left\|\left(H_{\omega,s}-e\right)^{-1}1_{Q_s(x)}\right\|_2.$$

Inequality (20) gives the bound

$$\left(\sum_{d(x,y)\le s}\left|g_{\omega,s}(y;e)\right|^2\right)^{1/2} \le C_e^{1/2}\frac{\sqrt{M_s}}{N_s}. \quad (22)$$

Moreover,

$$N_{s-1}\left|g_{\omega,s-1}(x;e)\right| = \left|\langle\delta_x|\left(H_{\omega,s-1}-e\right)^{-1}1_{Q_{s-1}(x)}\rangle\right| \le C_e^{1/2}\sqrt{M_{s-1}}. \quad (23)$$

Combination of (21) with (23) and (22) yields the estimate

$$\left(\sum_{y\in X}\left|G_{\omega,r}(x,y;e)\right|^2\right)^{1/2} \le \left|G_{\omega,0}(x,x;e)\right| + C_e\sum_{s=1}^{r} p_s\frac{\sqrt{M_s}\sqrt{M_{s-1}}}{N_s}.$$

By hypothesis (24), the sequence $M_r = (u_r N_r)^2$ satisfies

$$\sum_{r=1}^{\infty} N_r M_r^{-1/2} = \sum_{r=1}^{\infty} u_r^{-1} < \infty.$$

Since

$$\sum_{r=1}^{\infty} p_r\frac{\sqrt{M_s}\sqrt{M_{s-1}}}{N_s} = \sum_{r=1}^{\infty} p_r N_{r-1}u_{r-1}u_r < \infty,$$

the result follows.

Let us recall the Simon-Wolff localization criterion. For $x \in X$ and $\omega \in \Omega$, denote byμ_x^ω the spectral measure for $\Delta + V_\omega$ and δ_x, by$\mu_{x,cont}^\omega$ the continuous part of μ_x^ω and by$\mu_{x,ac}^\omega$ the a.c. part. Define the function $G_{\omega,x}: \mathbb{R} \to [0, +\infty]$ by

$$G_{\omega,x}(e) \coloneqq \int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2} = \lim_{\epsilon\downarrow 0}\|(\Delta + V_\omega - e - i\epsilon)^{-1}\delta_x\|^2.$$

By the Theorem of de la Valle Poussin,

$$d\mu_{x,ac}^\omega(e) = \pi^{-1}\left(\lim_{\epsilon\downarrow 0}\|(\Delta + V_\omega - e - i\epsilon)^{-1}\delta_x\|^2\right) de.$$

Hence, if for a fixed $\omega \in \Omega$ we have that $G_{\omega,x}(e) < \infty$ for m-a.e. $e \in \mathbb{R}$, then $\mu_{x,ac}^\omega = 0$.

The Simon-Wolff localization criterion is summarized in:

Theorem(3.2.6) [95]: Assume that $\mathbb{P}$ has a conditional density along the x'th coordinate. Let $B \subset \mathbb{R}$ be Borel set such that $G_{\omega,x}(e) < \infty$ for $\mathbb{P} \otimes m$-a.e. $(\omega, e) \in \Omega \times B$. Then $\mu_{x,cont}^\omega(B) = 0\, for\ \mathbb{P}$-a.e. $\omega \in \Omega$.

Theorem (3.2.6) is a well known consequence of the rank-1 Simon-Wolff theorem [115] and the conditional Fubini theorem.

Theorem (3.2.7) [95]: Assume that there exists a sequence $u_r > 0$ such that $\sum_{r=1}^{\infty} u_r^{-1} < \infty$ and

$$\sum_{r=1}^{\infty} p_r N_{r-1} u_{r-1} u_r < \infty. \qquad (24)$$

Then:

(i) For all $\omega \in \Omega, \sigma_{ac}(H_\omega) = \emptyset$.

(ii) If $\mathbb{P}$ is conditionally $a.c.$ then $\sigma_{cont}(H_\omega) = \emptyset\ for$ $\mathbb{P}$-a.e. ω.

Proof. Fix $\omega \in \Omega$ and fix $e \in \mathbb{R} \setminus \sigma_\omega$ for which the bound (18) holds. By monotone convergence

$$\int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2} = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2 + \epsilon^2} = \sup_{\epsilon>0} \int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2 + \epsilon^2}.$$

Since for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{r\to\infty} \left\| \left(H_{\omega,r} - z\right)^{-1} - (H_\omega - z)^{-1} \right\| = 0.$$

we have that the weak-* limit $\lim_{r\to\infty} \mu_{x,r}^\omega$ equals μ_x^ω, where $\mu_{x,r}^\omega$ is the spectral

$$\int_{\mathbb{R}} \frac{d\mu_x^\omega(\lambda)}{(e-\lambda)^2} = \sup_{\epsilon>0} \lim_{r\to\infty} \int_{\mathbb{R}} \frac{d\mu_x^{\omega,r}(\lambda)}{(e-\lambda)^2+\epsilon^2} \le \sup_{\epsilon>0, r\ge 1} \int_{\mathbb{R}} \frac{d\mu_x^{\omega,r}(\lambda)}{(e-\lambda)^2+\epsilon^2} = \sup_{r\ge 1} \int_{\mathbb{R}} \frac{d\mu_x^{\omega,r}(\lambda)}{(e-\lambda)^2}$$

$$= \sup_{r\ge 1} \left\| \left(H_{\omega,r} - e\right)^{-1} \delta_x \right\|^2 = \sup_{r\ge 1} \sum_{y\in X} \left| C_{\omega,r}(x,y) \right|^2 < \infty.$$

In the final equality we used the fact that $\{\delta_y : y \in X\}$ is an orthonormal basis for $l^2(X)$. Since $m(\sigma_\omega) = 0$ and since the bound (18) holds for m-a.e. $e \in \mathbb{R} \setminus \sigma_\omega$, we have that for every fixed $\omega \in \Omega$, $G_{\omega,x}(e) < \infty$ for m-a.e. $e \in \mathbb{R}$. This proves part (i). Part (ii) follows from the fact that $G_{\omega,x}(e) < \infty$ for $\mathbb{P} \otimes m$-a.e. $(\omega, e) \in \Omega \times \mathbb{R}$ and the Simon-Wolff criterion.

Remark (3.2.8) [95]: Theorem (3.2.7) and Proposition (3.2.3) allow to construct hierarchical models with spectral dimension $d \le 2$ that exhibit Anderson localization at arbitrary disorder. If (X, P, n) is a homogeneous hierarchical structure of degree $n \ge 2$ and $p_r = C\rho^{-r}$ with $\rho > n$, then the hypothesis (24) is fulfilled for $u_r = r^{1+\varepsilon}$. Given $0 < d < 2$, one can adjust $\rho > n$ to make $d(n,\rho) = d$. If $p_r = Cr^{-3-\varepsilon} n^{-r}$, then the model has spectral dimension $d = 2$ and (24) is verified for $u_r = r^{1+\varepsilon/3}$. One can also construct trivial models with $d = 0$ by taking p_r to decrease faster than ρ^{-r} for anyρ. We emphasize that homogeneity of the hierarchical structure is not required for Theorem (3.2.7).

Chapter 4
Endpoint Maximal and Space-TimeEstimates

For $\alpha > 1$we consider the initial value problem for the dispersive equation$i\partial_t u + (-\Delta)^{\alpha/2}u = 0$.We show an endpoint$L^p$inequality for the maximal function $\sup_{t\in[0,1]} |u(\cdot,t)|$with initial values in L^p-Sobolev spaces, for $p \in (2 + 4/(d+1), \infty)$.

Section (4.1): Smoothing Estimates for Schrödinger Equation

For α 1 we consider L^p estimates for solutions to the initial value problem

$$\begin{cases} i\,\partial_t u + (-\Delta)^{\alpha/2}u = 0 \\ u(.,0) = f. \end{cases}$$

The case $\alpha = 2$ corresponds to the Schrodinger equation. We will not consider $\alpha = 1$ which corresponds to the wave equation and exhibits different mathematical features.

When fis a Schwartz function, the solution can be written as $u(x,t) = U_t^{\alpha} f(x)$, where

$$\widehat{U_t^{\alpha}} f(\xi) = e^{it|\varepsilon|^{\alpha}}\hat{f}(\xi) \qquad (1)$$

with $\hat{f}(\xi) = \int f(y)e^{-i\langle y,\xi\rangle}dy$ as the definition of the Fourier transform. The sharp end point L^p-Sobolev bounds for fixed t are due to Fefferman and Stein [31] and Miyachi [37]. Their result states that for any compact time interval I and any$p\epsilon(1,\infty)$, $\sup_{t\in I}\|U_t^{\alpha} f\|_{L^p(\mathbb{R}^d)}, \frac{\beta}{\alpha} = d\left|\frac{1}{2} - \frac{1}{p}\right|$;

This is sharp with respect to the regularity index β and can also be deduced from certain endpoint versions of the Hörmander multiplier theorem (96, 103).

We strengthen the fixed time estimates as follows.

Theorem (4.1.1) [108]: Let$p\epsilon\left(2 + \frac{4}{d+1}\right), \infty$ and $\alpha > 1$. Then, for any compact time interval I,

$$\| \sup_{t\in I} |U_t^{\alpha} f| \|_{Lp(\mathbb{R}^d)} \le C_{I,p,\alpha} \| f \|_{L^p_{\beta}(\mathbb{R}^d)}, \quad \frac{\beta}{\alpha} = d\left(\frac{1}{2} - \frac{1}{p}\right). \qquad (2)$$

This implies point wise convergence results; indeed we shall prove a little more, namely if $\chi\epsilon C_e^{\infty}(\mathbb{R})$ then the function $t \longmapsto (t)U_t^{\alpha} f(x)$ belongs to the Besov space $B^p_{1/p.1}(\mathbb{R})$, for almost every $x\epsilon\mathbb{R}^d$. These functions are continuous (for almost every$x$) and there for this implies almost everywhere convergence to the initial datum as $t \longrightarrow 0$.

The maximal function result is closely related to certain space-time estimates which improve the regularity index. The first such bounds are due to Constantin and Saut [29], Sjölin [15], and vega [24] who showed that better L^2 regularity properties that hold locally when $\alpha\epsilon(1,\infty)$; namely, if $f\epsilon L^2_{-(\alpha-1)/2}(\mathbb{R}^d)$ then $u\epsilon L^2_{loc}(\mathbb{R}^{d+1})$. However it is not possible to replace the L^2-norms over compact sets byL^2-norms which are global in space. This is known as the local smoothing phenomenon. For functions in L^2 -Sobolev spaces the various local and global problems for smoothing and for maximal operators have received a lot of attention, starting with [4]. We do not have a contribution to the L^p –Sobolev problems but rather consider corresponding questions with initial data in L^p –Sobolev spaces for $p > 2$, wit p not close to 2.

In [46] considered L^p regularity estimates which are global in space but involve an integration over a compact time interval 1,

$$\left(\int_{1} \|U_t^{\alpha}f\|_p^p dt\right)^{1/p} \le C_1 \|f\|_{L_\beta^p(\mathbb{R}^d)}. \qquad (3)$$

This question was motivated by the similar (although deeper) question for the wave equation (cf. [41]). In [46], it was proven that (3) holds for $\alpha = 2$ when $p > 2 + 4/(d+1)$ with $\beta/2 > d(1/2 - 1/p) - 1/p$. We remark that smoothing results of this type could also be deduced from square-function estimates related to Bochner-Riesz multipliers such as in [27], [98], [102] and [36] however these arguments do not apply when $d = 1$, an din dimensions $p \ge 2$ they are currently limited to the smaller range $p > 2 + 4/d$.

The L^p smoothing result in [46] was obtained from an $L^p \to L^p$ estimate for the adjoint Fourier restriction (or 'extension') operator associated to the paraboloid, and the range $p > 2 + \frac{4}{d+1}$ corresponds to the known range of $L^q \to L^p$ bounds for the extensions operator; see [99], [100] and [107] for the sharp bounds when $d = 2$. The reduction in [46] to the extension estimate used the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int e^{i|x-y|^{2/4t}} f(y)\,dy$$

Together with 'completing of the square' trick; see [28] for similar argument. Unfortunately this reasoning is not available when $\alpha \ne 2$.

We generalize to all $\alpha > 1$, and establish the endpoint regularity result.

Theorem (4.1.2) [108]: Let $p \in \left(2 + \frac{4}{d+1}, \infty\right)$ and $\alpha > 1$. Then for any compact time interval I.

$$\left(\int_{I} \|U_t^{\alpha}\|_p^p dt\right)^{1/p} \le C_{I,p,\alpha} \|f\|_{L_\beta^p(\mathbb{R}^d)}, \qquad \frac{\beta}{\alpha} = d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p}.$$

In Theorem (4.1.9) below we formulate a slightly improved version of this result which canalso be used to prove Theorem (4.1.1) We remark that for $d = 1$ our argument also give the analogous results for the range $0 < \alpha < 1$.

We mention an application in one spatial dimension where we obtain sharp estimates for the initial value problem for the Airy equation

$$u_t + u_{rrr} = 0. \qquad (4)$$

For $f := u(.,0)$ a Schwartz function, we can write $u(.,t) = U_{-t}^3 p + f + U_{-t}^3 p - f$, where p_+ and p_- are the projection operators with Fourier multipliers $\chi_{(0,\infty)}$ and $\chi_{(-\infty,0)}$, respectively.

Thus, for initial values in L_β^p the solution of (4) satisfies the sharp bound

$$\|u\|_{L^p(\mathbb{R}_X[-T,T])} \le C_T \|u(.,0)\|_{L_\beta^p(\mathbb{R})}, \beta = \frac{3(p-4)}{2p}, 4 < p < \infty.$$

And if $u(.,0) \epsilon L_c^p(\mathbb{R})$ for any $\varepsilon > 0$ with $2 < p \le 4$, then $u \epsilon L^p(\mathbb{R} \times [-T,T])$.

The proofs will be based on the bilinear adjoint restrictions theorem for elliptic surfaces due to Tao [21], having discussed the necessary conditions, we combine Tao's theorem with a variation of a localization technique employed in [30] to prove L^p estimates for same oseillatory integrals with elliptic phases; this yields the smoothing estimate for functions which are frequency supported in annulus. we extend to the general case by decomposing the Fefferman-Stein sharp function; here we use a variant of an argument in [103].

Throughout, c and C will denote positive constants that may depend on the dimension, exponents or indices of the Sobolev spaces, or the parameter α, but never on the functions. Such constants are called admissible and their values may change from line to line. We shall mostly use the notation $A \lesssim B$ if $A \le CB$ for an admissible constant C. We may sometimes indicates the dependence on a specific parameter c by using the notation $\lesssim$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Let θ be a nonnegative and smooth function supported in $\{2^{-1} < |\xi| < 2\}$ and equal to 1 in $\{2^{1/2} < |\xi| < 2^{1/2}\}$. For large λ, we consider initial data f_λ defined by $\widehat{f_\lambda}(\xi) = e^{-i|\xi|^{\alpha}}\theta(\lambda^{-1}\xi)$ and note that, by a change of variables,

$$f_{\lambda(x)} = \left(\frac{\lambda}{2\pi}\right)^d \int \theta(\xi) e^{i(\langle \lambda x.\xi\rangle - \lambda^{\alpha}|\xi|^{\alpha})} d\xi$$

Thus $|f_\lambda(x)| \lesssim \lambda^{d-\frac{d\alpha}{2}}$, by the method of the stationary phase (keeping in mind that $\alpha \neq 1$). On the other hand, when $|x|\, \lambda^{\alpha-1}$, by repeated integration by parts, there exists constants C_N such that $|f_\lambda(x)| < C_N(|x|\, \lambda^{1-\alpha})^{-N}$ for all $N \in \mathbb{N}$. Combining the low bounds, we see that

$$\|f_\lambda\|_{L^p_\beta(\mathbb{R}^d)} \approx \lambda^{\beta} \|f_\lambda\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{d-\frac{d\alpha}{2}+\frac{d(\alpha-1)}{p}+\beta.}$$

Next we consider

$$\left|U_i^\alpha f_\lambda(x)\right| = \left|\left(\frac{\lambda}{2\pi}\right)^d \int_{\mathbb{R}^d} \theta(\xi)\, e^{i(\langle \lambda x,\xi\rangle + \lambda^{\alpha}(t-1))|\xi|^{\alpha}} d\xi\right|,$$

So when $|x| \le (10\,\lambda)^{-1}$ and $|t-1| \le (10\,\lambda^{\alpha})^{-1}$, we have $\left|U_t^\alpha f_{\lambda(x)}\right| \ge c\,\lambda^d$ for some positive constant c. Thus,

$$\left(\int_{1-(10\lambda^{\alpha})^{-1}}^{1} \|U_t^\alpha f_\lambda\|_p^p dt\right)^{1/p} \ge C\,\lambda^{d-\frac{d+\alpha}{p}}.$$

Comparing this with upper bound for $\|f_\lambda\|_{L^p_\beta(\mathbb{R}^d)}$, and letting $\lambda \to \infty$, we see that $\beta/\alpha \ge d(1/2 - 1/p) - 1/p$ is necessary condition for (3) to hold when $\alpha \neq 1$.

Note that alternatively one can argue that by Sobolev embedding any improvement in the smoothing would give a better fixed time estimate than the sharp known bounds in [31], [37], which is impossible.

The range $p > 2 + 4/(d+1)$ for the smoothing estimate in Theorem (4.1.2) is sharp for $d = 1$, and for $d \ge 2$ it is conceivable that it holds for $p > 2 + 2/d$, see [46].

For Theorem (4.1.1) however our range may not be sharp even in one dimension. We can say that the maximal estimate (2) cannot hold when $p + 1/d$. this follow from the necessary condition $\beta/\alpha \ge 1/2p$ which we now show, modifying a calculation in [6].

Let χ be a nonnegative and smooth function supported in $(-\varepsilon, \varepsilon)$ where ε will be small depending only on α, Let $e_1 = (1,0,\dots,0)$ and define

$$g_\lambda(x) = \frac{1}{(2\pi)^d} \int \chi\left(\lambda^{\frac{\alpha-2}{2}} |\xi + e_1|\right) e^{i\langle x.\xi\rangle} d\xi.$$

Then immediately

$$\|g_\lambda\|_{L^p_\beta} \lesssim \lambda^{\beta + \frac{d(\alpha-2)}{2}\left(\frac{1}{p}-1\right)}.$$

Now

$$U_t^\alpha g_{\lambda(x)} = \frac{1}{(2\pi)^d} \int \chi(\lambda^{\frac{\alpha-2}{2}} |\xi + \lambda\, e_1|) e^{i(\langle x,\xi\rangle + t|\xi|^{\alpha})} d\xi$$

$$= \frac{1}{(2\pi)^d}\int \chi(\chi^{\frac{\alpha-2}{2}}|h|)e^{i\phi\lambda(x,t,h)}dh$$

Where $\phi\lambda(x,t,h) = t\lambda^{\alpha}|-e_1 + h/\lambda|^{\alpha} + \langle x, -\lambda e_1 + h\rangle$. A Taylor expansion gives for term in the phase is $\ll 1$ on the support of the cutoff function (provided that ε is sufficiently small).

Let $0 < c \ll \alpha$ and let R be the rectangle where $0 \le x_I \le c\lambda^{\alpha-1}$, and $|x_i| \le \lambda^{(\alpha-2)/2}$ for $i = 2\ldots,d$. We define $t(x) = \alpha^{-1}\lambda^{1-\alpha}x_1$ for $x \in R$ so that $t(x) \in [0,1]$ for $x \in R$, and for $x \notin R$ we may choose any (measurable) $t(x) \in [0,1]$. Then for $x \in R$, we have $|U^{\alpha}_{t x}g_{\lambda}(x)| \ge c_0\lambda^{-d(\alpha-2)/2}$ and thus

$$\left\|\sup_{0\le s\le 1}|U^{\alpha}_t g\lambda|\right\|_p \ge \|U^{\alpha}_t g\lambda\|_p \gtrsim \lambda^{\frac{\alpha-1}{p}+\frac{(\alpha-2)(d-1)}{2p}+\frac{(\alpha+1)d}{2}}.$$

Comparing with upper bond for $\|g\lambda\|_{L^p_\beta}$ leads to the condition $\beta/\alpha \ge 1/2p$.

We will rescale inequalities for U^{α}_t when acting on functions with compact frequency support. This process will give rise to the operator S define by

$$Sf(xt) \equiv S^{\emptyset}_{\chi}f(x,t) = \frac{1}{(2\pi)^d}\int \chi(\xi)e^{it\emptyset(\xi)}\hat{f}(\xi)e^{i(x,\xi)}d\xi \qquad (5)$$

Where $\chi \in C^{\infty}_0(\mathcal{U})$ and $\emptyset$is elliptic C^{∞} function ϕ on an open set $\mathcal{U}$ in $\mathbb{R}^d$ is called elliptic if for ever $\xi \in \iota$ the Hessian ϕ'' is positive define.

We ask for $L^p - (\mathbb{R}^d \times [0,\lambda])$ bounds for S. Note that for $|t| \le 1$ and $\chi \in C^{\infty}_0$ the function $\chi e^{it\phi}$ is Fourier multiplier of $L^p, 1 \le p \le \infty$, and consequently the question is only nontrivial for large λ.

The key ingredient will be Tao's bilinear estimate for the adjoint restriction operator [21] which applies to phase which are small perturbations 0f $|\xi|^2/2$. We need to formulate more specific assumptions on the phases allowed and follow [105]. Let$N \ge 10d$. We say$\phi : [-2,2]^d \to \mathbb{R}$ is a class $\Phi(N,A)$ if $\left|\partial^{\alpha_j}_{x_j}\phi(x)\right| \le A$ for all $x \in [-2,2]^d$ and all $|\alpha_j| \le N$, where $j = 1\ldots,d$. To add an ellipticity condition we say that ϕ is of class $\Phi_{ell}(\varepsilon,N,A)$ if $\phi(0) = \nabla\phi(0) = 0$, and if for all $x \in [-2,2]^d$ the eigenvalues of the Hessian $\phi''(x)$ lie in $[1-\mathcal{E}, 1+\mathcal{E}]$.

We define the adjoint restriction operator $\mathcal{E} \equiv \varepsilon^{\phi}$ by

$$\varepsilon h(x,t) = \int_{[-2,2]^d} e^{i\langle x,\xi\rangle + t\phi(\xi)}h(\xi)d\xi\, h$$

So that $Sf = (2\pi)^{-d}\varepsilon\hat{f}_i$,where $u = (-2,2)^d$. Now Tao's theorem can be stated as follows: Suppose $p > 2 + \frac{4}{d+1}$. Then there exists an N (depending on d andp) and for $A \ge 1$ there exists $\varepsilon = \varepsilon(A,N,d,p) > 0$ so that the following holds for $\phi \in \Phi(\varepsilon,N,A)$: For all pairs of L^2 functions h_1, h_2 so that dist(supp (h_1), supp(h_2))$\ge c > 0$ the inequality

$$\|\varepsilon h_1 h_2\|_{p/2 \lesssim_c}\|h_1\|_2\|h_2\|_2, p > 2 + \frac{4}{d+1}, \qquad (6)$$

Holds. In what follows we fix N, A and ε for which Tao's theorem applies. The constants may all depend on these parameters.

Lemma (4.1.3) [108]: Let$p > 2 + \frac{4}{d+1}, B_1, B_2 \subset [-1,1]^d$ be bals so that dist $(B_1, B_2)c$, and let $\phi\epsilon\Phi_{ell}(\varepsilon,N,A)$. Then for ff, g with supp $\hat{f} \subset B_1$ supp $\hat{f} \subset B_2$, $\|SfSg\|_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])} \lesssim_{c.p} \lambda^{d(1-2/p)}\|f\|_{L^p(\mathbb{R}^d)}g_{L^p(\mathbb{R}^p)}$.

Proof. Let$C_0 = 10(1 + \max_{\mathcal{E}\epsilon[-2,2]^d}|\nabla\phi(\xi)|)$, and let$\eta_1, \eta_2 \epsilon C^{\infty}_0$ be supported in $(-2,2)^d$ so that $\eta_1(\xi) = 1$ on B_1 and $\eta_2(\xi_2) = 1$ on B_2. Moreover assume that η_1 and η_2 are supported onlyslightly larger concentric balls $\widetilde{B}_1, \widetilde{B}_2$ with property that dist $(\widetilde{B}_1, \widetilde{B}_2) \geq c/2$. We also set

$$P_i f = \mathcal{F}^{-1}[\eta_i \hat{f}], \qquad i = 1,2.$$

Let $K_t^i = \mathcal{F}^{-1}[e^{it\phi}\eta_i\chi]$, for $i = 1,2$, so that

$$S_i f(x,t) := SP_i f(x,t) = K_t^i * f(x).$$

Then $SfSg = S_1 f S_2 g$. We first note that for all $t \in [-\lambda, \lambda]$

$$\left|K_t^i|x|\right| \lesssim |x|^{-N}, \qquad \text{if } |x| \geq C_0 \lambda \qquad (7)$$

This follows by a straightforward N-fold integration by parts, which uses the inequality $\left|\nabla_\xi(\langle x,\xi\rangle + t\phi(\xi))\right| \geq |x|/2$ if $|x| \geq C_0 \lambda$, $|t| \leq \lambda$.

Now let $\mathcal{Q}(\lambda)$ to be a tiling of $\mathbb{R}^d$ by cubes of sidelength λ, and for each $Q \in \mathcal{Q}(\lambda)$ letQ_* denote the enlarged cube with sidelength $2C_0 \lambda$, with same center as Q. For each cube we split each function into a part supported$Q_*$and a part supported in its complement.

Thus we can write

$$\|SfSg\|_{L^{p/2}(\mathbb{R}^d \times [0,\lambda])}^{p/2} = I + II + III + IV$$

Where

$$I = \sum_{Q \in \mathcal{Q}(\lambda)} \left\|S_1[f\chi_{Q_*}]S_2[g\chi_{Q_*}]\right\|_{L^{p/2}(Q\times[0,\lambda])}^{p/2},$$

$$II = \sum_{Q \in \mathcal{Q}(\lambda)} \left\|S_1[f\chi_{Q_*}]S_2[g\chi_{\mathbb{R}^d \setminus Q_*}]\right\|_{L^{p/2}(Q\times[0,\lambda])}^{p/2},$$

$$III = \sum_{Q \in \mathcal{Q}(\lambda)} \left\|S_1[f\chi_{\mathbb{R}^d \setminus Q_*}]S_2[g\chi_{Q_*}]\right\|_{L^{p/2}(Q\times[0,\lambda])}^{p/2},$$

$$IV = \sum_{Q \in \mathcal{Q}(\lambda)} \left\|S_1[f\chi_{\mathbb{R}^d \setminus Q_*}]S_2[g\chi_{\mathbb{R}^d \setminus Q_*}]\right\|_{L^{p/2}(Q\times[0,\lambda])}^{p/2},$$

The first term gives the main contribution and estimated using Tao's theorem, i.e. (6). One obtains,

$$|I| \leq \sum_{Q \in \mathcal{Q}(\lambda)} \left\|SP_1[f\chi_{Q_*}]SP_2[g\chi_{Q_*}]\right\|_{L^{p/2}(\mathbb{R}^d\times\mathbb{R})}^{p/2} \lesssim_c \sum \left\|P_1[g\chi_{Q_*}]\right\|_2^{p/2} \left\|P_2[g\chi_{Q_*}]\right\|_2^{p/2}$$

$$\lesssim \sum_Q \|f\chi_{Q_*}\|_2^{p/2} \|g\chi_{Q_*}\|_2^{p/2} \lesssim \left(\sum_Q \|f\chi_{Q_*}\|_2^p\right)^{1/2} \left(\sum_Q \|g\chi_{Q_*}\|_2^p\right)^{1/2}$$

By Hölder's inequality,

$$\left(\sum_Q \|f\chi_{Q_*}\|_2^p\right)^{1/p} \lesssim \left(\sum_Q |Q_*|^{p/2-1} \|f\chi_{Q_*}\|_p^p\right)^{1/p} \lesssim \lambda^{d(1/2-1/p)} \|f\|_p,$$

And we have the same estimate for g. Thus $I^{2/p} \lesssim_c \lambda^{d(1-2/p)} \|f\|_p \|g\|_p$ which is the desired bound for the main term.

The corresponding estimates for II, III, IV are straightforward as we use (7) for the terms supported in $\mathbb{R}^d \setminus Q_*$. We examines II and begin with

$$|III| \lesssim \sum_{Q\in\mathcal{Q}(\lambda)} \|S_1[f\chi_{Q_*}]\|^{p/2}_{L^p(Q\times[0,\lambda])} \|S_2[g\chi_{\mathbb{R}^d\setminus Q_*}]\|^{p/2}_{L^p(Q\times[0,\lambda])}$$

$$\leq \left(\sum_{Q\in\mathcal{Q}(\lambda)}\|S_1[f\chi_{Q_*}]\|^p_{L^p(Q\times[0,\lambda])}\right)^{1/2} \left(\sum_{Q\in\mathcal{Q}(\lambda)}\|S_2[f\chi_{\mathbb{R}^d}]\|^p_{L^p(Q\times[0,\lambda])}\right)^{1/2} \quad (8)$$

We use the trivial bound $\|S_1 f(.,t)\|_p \lesssim (1+|t|)^d\|f\|_p$ for f replaced with $f\chi_{Q_*}$, so that the first factor in (8) is bounded by$\left(C\lambda^{d+1}\|f\|_p\right)^{p/2}$. By (7) we get

$$\left(\sum\|S_2[g\chi_{\mathbb{R}^d\setminus Q_*}]\|^{p/2}_{L^p(Q\times[0,\lambda])}\right)^{1/p}$$

$$\lesssim \left(\int_{-\lambda}^{\lambda}\int_{x\in\mathbb{R}^d}\left[\int_{|z|\geq\lambda}|z|^{-N}|g(x-z)|dz\right]^p dx dt\right)^{1/p} \lesssim \lambda^{d+1-N}\|g\|_p.$$

Hence $II^{2/p} \lesssim_c \lambda^{2(d+1)-N}\|f\|_p\|g\|_p$. As $N \geq 10d$ this estimate is negligible. Because of symmetryIII is estimated by the same term. For the estimation of IV we proceed in the same way but use (7) for both terms, the result is the (again negligible) bound $IV^{2/p} \lesssim \lambda^{d+1-N}\|g\|_p$.

We now formulate an analogous result for functions with smaller frequency support and smaller separation.

Lemma (4.1.4) [108]: Let $p > 2 + \frac{4}{d+1}$ and $\lambda^{1/2} \geq 2^j \geq 1$. Let$Q_1, Q_2 \subset [-1,1]^d$ be cubes of side $2^j\lambda^{-1/2}$, so that dist $(Q_1,Q_2) \geq c2^j\lambda^{-1/2}$ and let $\phi\epsilon\Phi_{ell}(\varepsilon,N,A)$. Then for all f and g such that supp $\hat{f} \subset Q_{2,}\|Sf(Sg)\|_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])} \lesssim_c 2^{4j\left(\frac{d}{2}-\frac{d+1}{p}\right)}\lambda^{\frac{2}{p}}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}$.

Proof. By finite partitions and the triangle inequality, we may suppose that Q_1 and Q_2 are balls of radius $2^j\lambda^{-1/2}$. We reduce matters to the statement of Lemma (4.1.3) by scaling. Let ξ_0 be midpoint of the interval connecting the center of the balls. We change variables $\xi = \xi_0 + \delta\eta$ where $\delta = 2^j\lambda^{-1/2}$. Then a short computation shows that

$S^\Phi f(x,t) = e^{i(\langle x,\xi_0\rangle + t\phi(\xi_0))}S^\phi f_*\left(\delta\left(x + t\nabla\phi(\xi_0)\right),\delta^2 t\right)$ where $f_*(y) = f(\delta^{-1}y)e^{i\delta^{-1}\langle y,\xi_0\rangle}$ and the phase ψ is given by

$$\psi(\eta) = \frac{1}{2}\int_0^1\langle\phi''(\xi_0 + s\delta\eta)\eta,\eta\rangle ds.$$

The same consideration is applied to $S^\phi g$. Note that ψ is elliptic (with estimates uniform in ξ_0 and δ) and the frequency supports of f_* and g_* are now separated, independently of δ, $j$and λ. Thus we can apply Lemma (4.1.3) to obtain

$$\|S^\phi f S^\phi g\|_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])} = \delta^{-(d+2)/p/2}\|S^\psi f_* S^\psi g_*\|_{L^{p/2}(\mathbb{R}^d\times[0,\lambda\delta^2])}$$
$$\lesssim \delta^{-(2d+4)/p}(\lambda\delta^2)^{d(1-2/p)}\|f_*\|_p\|g_*\|_p$$
$$\lesssim \delta^{2d-4(d+1)/p}\lambda^{d(1-2/p)}\|f\|_p\|g\|_p.$$

As $\delta = 2^j\lambda^{-1/2}$ the assertion follows.

We will also require the following lemma for when we have no frequency separation.

Lemma (4.1.5) [108]: Let $P \geq 1$, let $Q \subset [-1,1]^d$ be a cube of side $\lambda^{-1/2}$, and let $\phi\,\epsilon\,\emptyset\,(N,A)$. Then for all f such that supp $\hat{f} \subset Q$, $\|Sf(\cdot,t)\|_{L^p}(\mathbb{R}^d)$, $|t| \leq \lambda$.

Proof. Letξ_B bethecenterofthecubeQ, and let $\chi\epsilon C_0^\infty$ so that $\chi(\xi) =$ for $|\xi| \leq \sqrt{d}$. It suffices to show that $\chi(\lambda^{1/2}(\xi-\xi_B))e^{it\phi(\xi)}$ is a Fourier multiplier of L^p for all $|t|\geq\lambda$, with bound uniform in t. By modulation, translation and dilation invariance of the multiplier norm it suffices to check that $h(.,t)$ defined by

$$h(\eta.t)=\chi(\eta)e^{it(\phi(\lambda^{-1/2}\eta+\xi_B)-\phi(\xi_B)-\langle\lambda^{-1/2}\eta.\nabla\phi(\xi_B)\rangle)}.$$

is a Fourier multiplier of L^p, uniformly in $|t|\geq\lambda$. However this follow since $\partial_\eta^\alpha h(\eta,t)=O(1)$ for $|t|\leq\lambda$ as one can easily check.

Propsition (4.1.6) [108]: Let $s>2+\frac{4}{d+1}$, $\chi\in C_0^\infty(\mathcal{U})$, and let ϕ be an ellipitic phase on $\mathcal{U}$. Then

$$\|Sf\|_{L^p(\mathbb{R}^d\times[-\lambda,\lambda])}\lesssim\lambda^{d(1/2-1/p)}\|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. By partition of unity and compactness argument it suffices to show that for every $\xi_0\in\mathcal{U}$ there is neighborhood $\mathcal{U}(\xi_0)$ so that the statement of the theorem holds with χ replaced by $\chi_0\in C_0^\infty$ supported in $\mathcal{U}(\xi_0)$. Now let $\mathcal{H}$ be the (symmetric) positive definite square root of $\phi''(\xi_0)$ and let

$$\psi(\eta)=\varepsilon_1^{-2}(\phi(\xi_0+\varepsilon_1\mathcal{H}^{-1}\eta)-\phi(\xi_0)-\varepsilon_1\langle\mathcal{H}^{-1}\eta,\nabla\phi(\xi_0)\rangle).$$

Then it suffices to show that S^ψ(defined with amplitude $\chi(\xi_0+\varepsilon_1\mathcal{H}^{-1}\eta)$) satisfies the asserted estimates, with a dependence on ε_1. If ε_1 is chosen sufficiently small then we have reduced matters to a phase function in $\Phi_{ell(\varepsilon,N,A)}$ with parameters for which Tao's Theorem and therefore Lemma (4.1.4) applies.

We now return to our original notation and work with ϕ a phase function but assume now that $\phi\in\Phi_{ell(\varepsilon,N,A)}$; we may also assume that the amplitude function χ is smooth and supported in $[-(2d)^{-10},2d^{-10}]^{-d}$. We make a decomposition of the product $SfSg$ in terms of bilinear operators, localizing the frequency variables in terms of nearness to the diagonal in (ξ,η)-space; this is similar to arguments in [34], [104] and [105].

Let χ_0 be a radial $C_0^\infty(\mathbb{R}^d)$ function so that $\chi_0(\omega)=1$ for $|\omega|\leq 8d^{1/2}$ and so that supp χ_0 is contained in $\{\omega:\ |\omega|<16d^{1/2}\}$. Fix $\lambda>1$ and set

$$\Theta_0(\xi,\eta)=\chi_0(\lambda^{1/2}(\xi-\eta))$$

$$\Theta_j(\xi,\eta)=\chi_0(\lambda^{1/2}2^{-j}(\xi-\eta))-\chi_0(2\lambda^{1/2}2^{-j}(\xi-\eta)),\quad j\geq 1,$$

So that Θ_0 is supported where $|\xi-\eta|\geq 16d^{1/2}\lambda^{-1/2}$ and, Θ_j is supported in the region

$$4d^{1/2}2^j\lambda^{-1/2}\leq|\xi-\eta|\geq 16d^{1/2}\lambda^{-1/2}.$$

We may then decompose

$$SfSg=\sum_{j\geq 0}B_j[f,g]$$

Where

$$B_j[f,g](x,t)=\frac{1}{(2\pi)^{2d}}\iint e^{i(x,\xi+\eta)}e^{it(\phi(\xi)+\phi(\eta))}\Theta_j(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta$$

Only values of $j\geq 0$ with $2^j\leq\lambda^{1/2}$ will be relevant, as otherwise B_j is identically zero. We will prove the estimate

$$\|B_j[f,g]\|_{p/2}\lesssim\begin{cases}2^{4j(\frac{d}{2}-\frac{d+1}{p})}\lambda^{\frac{2}{p}}\|f\|_p\|g\|_p, & \frac{2(d+3)}{d+1}<p\leq 4,\\ 2^{j(\frac{d}{2}-\frac{d+1}{p})}\lambda^{\frac{d}{2}-\frac{2(d-1)}{p}}\|f\|_p\|g\|_p, & 4<p<\infty\end{cases}\qquad(9)$$

And use this to bound

$$\|Sf\|_{L^p(\mathbb{R}^d\times[0,\lambda])} = \|(Sf)^2\|^{1/2}_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])} \leq \left(\sum_{0\leq j\geq \log_2(\lambda^{1/2})} \left\|B_j[f,f]\right\|_{p/2}\right)$$

And then sum a geometric series.

In order to prove (9), we decompose B_j into pieces on which we may apply Lemma (4.1.4) Let $\vartheta \in C_0^\infty(\mathbb{R}^d)$ a function supported in $[-3/5, 3/5]^d$, equal to 1 on $[-2/5, 2/5]^d$, and satisfying

$$\sum_{n\in\mathbb{Z}^d} \vartheta(\xi - n) = 1$$

For all $\mathbb{Z} \in \mathbb{R}^d$. For $j \geq 0, n \in \mathbb{Z}^d$, define

$$\beta_{j.n}(\xi) = \vartheta\left(\lambda^{1/2}\, 2^{-j}\xi - n\right)$$

And, for $(n, n') \in \mathbb{Z}^d \times \mathbb{Z}^d$,

$$\vartheta_{j,n,n'}(\xi,\eta) = \Theta_j(\xi, n)\beta_{j,n}(\xi)\beta_{j,n'}(\eta)$$

Observe that $\beta_{j,n}, \beta_{j,n'}$ are supported in cubes $Q_{j,n}, Q_{j,n'}$ which have sidelengths slightly larger than $\lambda^{-1/2}\, 2^j$, and that aare centered at the points $\xi_{j,n} = \lambda^{-1/2}\, 2^j n$ and $\xi_{j,n'} = \lambda^{-1/2}\, 2^j n'$, respectively.

Now let

$$\Delta_0 = \left\{(n, n') \in \mathbb{Z}^d \times \mathbb{Z}^d :\ |n - n'| \leq 18d^{1/2}\right\},$$
$$\Delta = \left\{(n, n') \in \mathbb{Z}^d \times \mathbb{Z}^d : 2d^{1/2} \leq |n - n'| \leq 18d^{1/2}\right\}.$$

Then if $\vartheta_{0,n,n'}$ is not identically zero then we necessarily have $(n, n') \in \Delta_0$ and if, for $j \geq 1$ the function $\vartheta_{0,n,n'}$ is not identically zero then we necessarily have $(n, n') \in \Delta_0$. These statements follow by the definitions of our cutoff functions. Moreover,

$$\mathrm{dist}\left(Q_{j,n}, Q_{j,n'}\right) \leq 18d^{1/2}2^j\, \lambda^{-1/2} \text{ if} (n, n') \in \Delta_0,$$

and

$$2^{-1}d^{1/2}2^j\, \lambda^{-1/2} \leq \mathrm{dist}\left(Q_{j,n}, Q_{j,n'}\right) \leq 18d^{1/2}2^j\, \lambda^{-1/2} \text{ if } j \geq 1 \text{ and } (n, n') \in \Delta_0$$

For the application of Lemma (4.1.4) it is convenient to eliminate the cutoff Θ_j but still keep the separation of the supports off $\beta_{j,n}$and $\beta_{j,n'}$. Set, for $j \geq 1$,

$$\widehat{B}_j[f, g](x, t) = \frac{1}{(2\pi)^{2d}} \iint e^{i\langle x,\xi\xi+\eta\rangle} e^{it(\phi(\xi)+\phi(\eta))} \sum_{n,n'\epsilon\Delta} \beta_{j,n}(\xi)\beta_{j,n'}(\eta)\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta$$

And define $\widehat{B}_j[f, g]$ similarly by letting (n, n') sum run over Δ_0. The reduction of the estimate for B_j to the estimate for $\widehat{B}_j$ is straightforward; by an averaging argument. Indeed, $\chi_1 = \chi_0 - \chi_0(2\,\cdot)$ and use the Fourier inversion formula

$$\Theta_j(\xi, \eta) = \frac{1}{2\pi^d} \int \hat{\chi}_1(y) e^{i\lambda^{1/2}2^{-j}\langle\xi-\eta,y\rangle} dy, \quad j \geq 1;$$

Then

$$\mathcal{B}_j[f, g] = \frac{1}{(2\pi)^d} \int \hat{\chi}_1(y)\widetilde{\mathcal{B}}_j\left[f_y, g_y\right]dy$$

Where $f_{-y}(x) = f\left(x + \lambda^{1/2}\, 2^{-j}y\right)$ and $g_y(x) = g\left(x - \lambda^{1/2}\, 2^{-j}y\right)$. A similar formula holds for $j = 0$, only then χ_1 is replaced with χ_0. Thus in order to finish the argument it is enough to show that $\left\|\widetilde{B}_j[f, g]\right\|_{p/2}$ is dominated by the right hand side of (9).

Define convolution operators $P_{j,n}$ by $\widehat{P_{j,n} f} = \beta_{j,n}\hat{f}$. Note that for fixed j, each ξ is contained in only a bounded number of the sets $Q_{j,n} + Q_{j,n'}$. this implies, interpolation of $\ell^2(L^2)$ with trivial $\ell^1(L^1)$or $\ell^\infty(L^\infty)$ bounds that, for $j \geq 1, p \geq 2$,

$$\left\|\widetilde{B}_j[f,g]\right\|_{L^{p/2}(\mathbb{R}\times[0,\lambda])} \qquad (10)$$

$$\lesssim \max\left\{1, \left(\lambda^{1/2}\, 2^{-j}\right)^{d(1-4/p)}\right\}\left(\sum_{n,n'\epsilon\Delta}\left\|SP_{j,n}SP_{j,n'}g\right\|^{p/2}_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])}\right)^{2/p}$$

The analogous formula for $j = 0$ holds if we replace Δ by Δ_0. Notice that for all j,

$$\left(\sum_n\left\|p_{j,n}f\right\|_p^p\right)^{1/p} \lesssim \|f\|_p. \qquad p \geq 2. \qquad (11)$$

Now if $j = 0$ we use Lemma (4.1.5) to estimate

$$\left\|SP_{0,n}f(.,t)SP_{0,n'}g(.,t)\right\|_{L^{p/2}(\mathbb{R}^d)} \lesssim \left\|SP_{0,n}f(.,t)\right\|_p\left\|SP_{0,n'}g(.,t)\right\|_p$$
$$\lesssim \left\|P_{0,n}f\right\|_p\left\|P_{0,n'}g\right\|_{p;}$$

Hence, after integrating in t,

$$\left\|\widehat{B}_0[f,g]\right\|_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])} \lesssim \max\left\{1,\lambda^{d(1/2-2/p)}\right\}\lambda^{p/2}\left(\sum_{n,n'\epsilon\Delta_0}\left\|P_{0,n}f\right\|_p^{p/2}\left\|P_{0,n'}g\right\|_p^{p/2}\right)^{2/p}$$

$$\lesssim \max\left\{1,\lambda^{d(1/2-2/p)}\right\}\lambda^{2/p}\left(\sum_n\left\|P_{0,n}f\right\|_p^p\right)^{1/p}\left(\sum_{n'}\left\|P_{0,n'}g\right\|_p^p\right)^{1/p}$$

The asserted bound for $j = 0$ follows from (11).

Next for $j > 0$ we use Lemma (4.1.4)and thus the assumption $p > 2 + \frac{4}{d+1}$, and estimate

$$\left\|SP_{j,n}fSP_{j,n'}g\right\|_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])} \lesssim 2^{4j\left(\frac{d}{2}-\frac{d+1}{p}\right)}\lambda^{2/p}\left\|P_{j,n}f\right\|_p\left\|P_{j,n'}g\right\|_{p.}$$

Therefore by (10)

$$\left\|\widetilde{B}_j[f,g]\right\|_{L^{p/2}(\mathbb{R}^d\times[0,\lambda])}$$

$$\lesssim \max\left\{1, \left(\lambda^{1/2}\, 2^{-j}\right)^{d(1-4/p)}\right\}2^{4j\left(\frac{d}{2}-\frac{d+1}{p}\right)}\lambda^{2/p}\left(\sum_n\left\|P_{j,n}f\right\|_p^p\right)^{1/p}\left(\sum_{n'}\left\|P_{j,n'}g\right\|_p^p\right)^{1/p}$$

and again asserted bound for $\left\|\widetilde{B}_j[f,g]\right\|_{p/2}$ follows from (11).

We now prove the endpoint estimates of Theorems (4.1.1) and (4.1.2) First we remark that by various scaling and symmetry arguments we assume that $I = [0,1]$.

Consider $\chi_0, \chi \in c_0^\infty(\mathbb{R})$ supported in $(-2,2)$and $(1/2,2)$, respectively, such that

$$\chi_0 + \sum_{k\geq 1}\chi\left(2^{-k}\right) = 1.$$

We define the operators $T_k^\alpha \equiv T_k$ by

$$\widehat{T_0 f(.,t)}(\xi) = \chi_0(\xi)e^{it|\xi|^\alpha}\hat{f}(\xi).$$
$$\widehat{T_0 f(.,t)}(\xi) = \chi\left(2^{-k}|\xi|\right)e^{it|\xi|^\alpha}\hat{f}(\xi), \qquad k \geq 1,$$

So that $U_t^\alpha = \sum_{k\geq 0} T_k(.,t)$.

Our main result is the following inequality for vector-valued functions $\{f_k\}_{k=0}^{\infty} \in \ell^p(L^p)$.

Theorem (4.1.7) [108]: Let $p \in \left(2+\frac{4}{d+1},\infty\right)$, $\alpha \neq 1$, $d = 1$ or $\alpha > 1$, $d \geq 2$ and $\beta = \alpha d\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{\alpha}{p}$. Then

$$\left\|\sum_{k\geq 0}\left(\int_0^1 \left|2^{-k\beta}T_k\right|^p dt\right)^{1/p}\right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_{k\geq 0}\|f_k\|_p^p\right)^{1/p} \qquad (12)$$

We now discuss the implication to Theorem (4.1.1) 1nd (4.1.2) in fact strengthened versions involving Triebel-Lizorkin spaces $F_{\alpha.q}^{p}$.

Here the norms in this spaces are given by the $L^p(\ell^q)$ and $\ell^q(L^p)$ norms (resp.) of the sequence $\left\{2^{k\alpha}L_k f\right\}_{k=0}^{\infty}$, with usual inhomogeneous dyadic frequency composition $I = \sum_{k\geq 0} L_k$. See [26]. The following corollary is an immediate consequence of Theorem (4.1.7) by Minkowski's inquality and Fubini's theorem.

Proof. The localization of the multiplier near the origin T_0 is easily handled as

$$\left\|\mathcal{F}^{-1}\left[\chi_0(|\cdot|)e^{it|\cdot|^{\alpha}}\right]\right\|_{L^1} \leq C$$

uniformly for $t \in [0,1]$. To see this, since $\mathcal{F}^{-1}[\chi_0(|\cdot|)] \in L^1$, it suffices to show that for ϕ supported in $(1/2,2)$, the L^1 norm of $\mathcal{F}^{-1}\left[\chi_0\left(2^{-k}.\right)\left(e^{it2^{-\alpha\kappa}}|\cdot|^{\alpha}-1\right)\phi(|\cdot|)\right]$ is $O(2^{-\alpha\kappa})$ which follows from the standard Bernstein criterion.

Now, by scaling and Proposition (4.1.6) with $\lambda \approx 2^{\alpha\kappa}$, $u = \{\xi : 1/2 < |\xi| < 2\}$ and $\phi(\xi) = |\xi|^{\alpha}$, we have already proven the estimates

$$\|T_k f\|_{L^p(\mathbb{R}^d\times[0,1])} \lesssim 2^{\kappa\beta}\|f\|_{L^p(\mathbb{R}^d)}, \quad \beta \geq \beta(p) := \alpha d\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{\alpha}{p} \qquad (14)$$

for $k > 0$ and $p > 2+\frac{4}{d+1}$.

It suffices thus to show that if (14) holds for all $k > 0$ and all $p > q$, then (4.1.7) holds for all $p \in (q,\infty)$. Due to our restriction on (14) we let $q = 2+\frac{4}{d+1}$ and fix $2+\frac{4}{d+1} < r < p$. We can make the additional assumption that the k sum on the left hand side is extended over a finite set (with the constant in the inequality independent of this assumption); the general case then follows by the monotone convergence theorem.

For later reference we state a Sobolev inequality which is proved linking frequency decompositions in ξ and $\top$ and Young's inequality (just as in the argument used to deduce Corollary(4.1.9) from Theorem (4.1.7) Namely

$$\left\|\|T_k f\|_{L_t^p[0.1]}\right\|_{L_x^r} \lesssim 2^{\alpha\kappa\left(\frac{1}{r}-\frac{1}{p}\right)}\left\|\|T_k f\|_{L_t^r[0.1]}\right\|_{L_x^r}. \qquad (15)$$

holds for $r \leq p \leq \infty$ (including the endpoint). Alternatively one can also apply the fundamental theorem of calculus to $|T_k f(x,.)|^r$ (see e.g. [55]) for $p = \infty$ and the general inequality follows by convexity.

The main ingredient in the proof of (4.1.7) will be the Fefferman-Stein sharp function [31] and their inequality

$$\|F\|_p \lesssim \|F^{\#}\|_p,$$

Where $p \in (1,\infty)$ and a apriori $F \in L^p$. We apply this to $\sum_{k>0} 2^{-k\beta(p)}\|T_k f_k(x,.)\|_{L_t^p[0.1]}$ and by (14) this function is apriori in L^p as the sum in k is assumed to be finite. Thus it will suffice to prove that

$$\left\| \sup_{x\in Q} \fint_Q \sum_{k>0} 2^{-k\beta(p)} \|T_k f_k(y,\cdot)\|_{L_t^p[0,1]} - \fint_Q \sum_{k>0} 2^{-k\beta(p)} \|T_k f_k(y,\cdot)\|_{L_t^p[0,1]} dz \right\|_{L_x^p}.$$

is dominated by $C\left(\sum_{k>0} 2^{-k\beta(p)} \|f_k\|_p^p\right)^{1/p}$. Here the supremum is taken over all cubes containing x, and the slashed integral denotes the average $|Q|^{-1}\int_Q$. By the triangle inequality the previous bound follows from

$$\left\| \sup_{x\in Q} \fint_Q \sum_{k>0} \fint_Q 2^{-k\beta(p)} \|T_k f_k(y,\cdot) - T_k f_k(z,\cdot)\|_{L_t^p[0,1]} dzdy \right\|_{L_x^p} \lesssim \left(\sum_k f_{k_p}^p\right)^{1/p}$$

Denoting the sidelength of Q by $\ell(Q)$, we observe that, by Minkowski's inequality, this would follow from the inequalities

$$\left\| \sup_{x\in Q} \fint_Q \sum_{2^k\ell(Q)\le 1} \fint_Q 2^{-k\beta(p)} \|T_k f_k(y,\cdot) - T_k f_k(z,\cdot)\|_{L_t^p[0,1]} dzdy \right\|_{L_x^p} \lesssim \left(\sum_k f_{k_p}^p\right)^{1/p} \quad (16)$$

$$\left\| \sup_{x\in Q} \fint_Q \sum_{2^k\ell(Q)>2^{\alpha k}} \fint_Q 2^{-k\beta(p)} \|T_k f_k(y,\cdot) - T_k f_k(z,\cdot)\|_{L_t^p[0,1]} dzdy \right\|_{L_x^p} \lesssim \left(\sum_k f_{k_p}^p\right)^{1/p} \quad (17)$$

and

$$\left\| \sup_{x\in Q} \fint_Q \sum_{2^{\alpha k}\ge 2^k\ell(Q)>1} \fint_Q 2^{-k\beta(p)} \|T_k f_k(y,\cdot) - T_k f_k(z,\cdot)\|_{L_t^p[0,1]} dzdy \right\|_{L_x^p} \lesssim \left(\sum_k f_{k_p}^p\right)^{1/p} \quad (18)$$

Proof of (16). It is enough to consider cubes Q of diameter $\approx 2^j$ with $x, y, z \in Q$ and $j + k \le 0$. Let $H_k = \mathcal{F}^{-1}[\tilde{\chi}](2^{-k}|\cdot|)$, where $\tilde{\chi}$ is smooth, equal to one on (1/2, 2), and supported in (1/3, 3). Then

$$|\nabla H_k(w)| \lesssim 2^k \frac{2^{kd}}{(w)^{2N}}$$

With large $N \ge 10d$. Thus

$$T_k f_k(y,t) - T_k f_k(z,t) = \int [H_k(y-w) - H_k(z-w)] T_k f_k(w,t) dw$$
$$= \iint_0^1 \langle (y-z), \nabla H_k(z + s(y-z) - w)\rangle T_k f_k(w,t) dw$$

Which is controlled by a constant multiple of

$$2^{j+kd} \int \frac{2^{kd}}{(1+2^k|x-w|)^N} |T_k f_k(w,t)| dw.$$

Thus, using the embedding $\ell^p \hookrightarrow \ell^\infty$, the right hand side of bounded by

$$\left\| \left(\sum_j \left| \sum_{0>k\ge -j} \left\| 2^{j+k} \int \frac{2^{kd}}{(1+2^k|\cdot - w|)^N} 2^{-k\beta(p)} |T_k f_k(w,\cdot)| dw \right\|_{L_t^p[0,1]} \right|^p \right)^{1/p} \right\|_{L_x^p}$$

$$\lesssim \sum_{n\geq 0} 2^{-n}\left(\sum_{j<-n}\left\|\int \frac{2^{-(n+j)(d-\beta(p))}}{\left(1+2^{-(n+j|\cdot-w|)}\right)^N}\left|T_{-(n+j)}f_{-(n+j)}(w,\cdot)\right|dw\right\|^p_{L^p(\mathbb{R}^d\times[0,1])}\right)^{1/p}$$

$$\lesssim \sum_{n\geq 0} 2^{-n}\left(\sum_{j<-n}\left\|2^{(n+j)\beta(p)}T_{-(n+j)}f_{-(n+j)}\right\|^p_{L^p(\mathbb{R}^d\times[0,1])}\right)^{1/p}$$

By the (14) the last expression is dominated by a constant times

$$\sum_{n\geq 0} 2^{-n}\left(\sum_{j<-n}\left\|f_{-(n+j)}\right\|^p_p\right)^{1/p} \lesssim \left(\sum_k \|f_k\|^p_p\right)^{1/p}$$

And (16) is proved.

Proofof(17). For fixed t, the operator T_k has convolution kernel K^t_k given by

$$K^t_k(x) = \frac{2^{kd}}{(2\pi)^d}\int_{\mathbb{R}^d}\chi(|\xi|)e^{i(x.\xi)+2^{\alpha\kappa}t|\xi|^\alpha}d\xi$$

$$\mathfrak{B}_k(\alpha) = \left\{x : |x| \leq 4C(\alpha)2^{k(\alpha-1)}\right\}.$$

Integration by parts yields favorable bounds in the complement of this ball. Observe that

$$\left|\nabla_\xi\left(2^k(x,\xi) + 2^{\alpha\kappa}t|\xi|\right)\right| \geq c_\alpha 2^k|x| \text{ if } x \notin \mathfrak{B}_k(\alpha), \quad t\in[0,1],$$

And we obtain

$$\left|K^t_k(x)\right| \leq C_N 2^{kd}\left(1+2^k|x|\right)^{-N} \text{if } x \notin \mathfrak{B}_k(\alpha), \quad t\in[0,1], \qquad (18)$$

Consequently the main contribution of $K^t_k(x)$ comes when $|x| \leq 4C(\alpha)2^{k(\alpha-1)}$.

We prove the estimate (17) by interpolation between

$$\left\|\sup_{x\in Q}\int_Q \sum_{2^k\ell(Q)>2^{\alpha k}} 2^{-k\beta(p)}\|T_kf_k(y,\cdot)\|_{L^p_t[0,1]}dy\right\|_\infty \lesssim \sup_k \|f_k\|_\infty$$

And

$$\left\|\sup_{x\in Q}\int_Q \sum_{2^k\ell(Q)>2^{\alpha k}} 2^{-k\beta(p)}\|T_kf_k(y,\cdot)\|_{L^p_t[0,1]}dy\right\|_r \lesssim \left(\sum_k \|f_k\|^r_r\right)^{1/r}$$

Where $2+\frac{4}{d+1} < r < p$.

Now, as $\beta(p) > \beta(r) + \alpha\left(\frac{1}{r}-\frac{1}{p}\right)$, the L^r bound is proven by applying Hölder in k, followed by the inequality

$$\left\|\sup_{x\in Q}\int_Q \left(\sum_k 2^{-k\left(\beta(r)+\left(\frac{1}{r}-\frac{1}{p}\right)\right)r}\|T_kf_k(y,\cdot)\|^r_{L^p_t[0,1]}\right)^{1/r} dy\right\|_r \left(\sum_k \|f_k\|^r_r\right)^{1/r}$$

This is a consequence of the L^r -boundedness of the Hardy-Littlewood maximal operator, the interchange of the spatial integral and the sum, an application of (15), followed by Fubini and the estimate (14) (for the admissible exponent $r > 2 + 4/(d+1)$).

To prove the L^∞ bound, we let Q^* be a cube with same center as Q satisfying $\ell(Q^*) = 10dC(\alpha)\ell(Q)$. By Minkowski's inequality itwill suffice to prove that

$$\int_Q \sum_{2^k\ell(Q)>2^{\alpha k}} 2^{-k\beta(p)} \|T_k[f_k\chi Q^*](y,\cdot)\|_{L^p_{t[[0,1]]}} dy \lesssim \sup_k \|f_k\|_\infty \qquad (19)$$

And

$$\int_Q \sum_{2^k\ell(Q)>2^{\alpha k}} 2^{-k\beta(p)} \|T_k[f_k\chi Q^*](y,\cdot)\|_{L^p_{t[[0,1]]}} dy \lesssim \sup_k \|f_k\|_\infty \qquad (20)$$

Uniformly in Q.

To prove (19), again we apply Hölder a number of times and (15);

$$\int_Q \sum_k \sum_{2^k\ell(Q)>2^{\alpha k}} 2^{-k\beta(p)} \|T_k[f_k\chi Q^*](y,\cdot)\|_{L^p_{t[[0,1]]}} dy$$

$$\lesssim |Q|^{-1/r} \sum_k 2^{-k\beta(p)-\alpha\left(\frac{1}{r}-\frac{1}{p}\right)} \left(\int \|T_k[f_k\chi Q^*](y,\cdot)\|^r_{L^p_{t[[0,1]]}} dy\right)^{1/r}$$

$$\lesssim \sup_k |Q|^{-1/r} 2^{-k\beta(r)} \left(\int \|T_k[f_k\chi Q^*](y,\cdot)\|^r_{L^p_{t[[0,1]]}} dy\right)^{1/r}$$

$$\lesssim \sup_k |Q|^{-1/r} \left(\int |f_k\chi Q^*|^r\, dx\right)^{1/r} \lesssim \sup_k \|f_k\|_\infty$$

Where the third inequality holds again by the L^r version of (14).

For (20), we note that as $\ell(Q) > 2^{k(\alpha-1)}$, and the function is supported in the complement of Q^* we can use the rapid decay in formula (18). We have that

$$\int_Q \sum_{2^k\ell(Q)>2^{\alpha k}} 2^{-k\beta(p)} \|T_k[f_k\chi Q^*](y,\cdot)\|_{L^p_{t[[0,1]]}} dy$$

$$\lesssim \sup_k \int_Q \left\| \int \frac{2^{kd}}{(1+2^k|y-z|)^{2d}} |f_k(z)| dz \right\|_{L^p_{t[[0,1]]}} dy$$

$$\lesssim \sup_k \int_Q \left\| \int \frac{2^{kd}}{(1+2^k|\cdot-z|)^{2d}} |f_k(z)| dz \right\|_\infty \lesssim \sup_k \|f_k\|_\infty$$

This concludes the proof of (17)

Proof of (18). We let $\varsigma_j(x) = d2^{j-d}$ if $|x| \le d2^j$ and $\varsigma_j(x) = 0$ if $|x| \le d2^j$. replacing cubes bydyadic balls we see that (18) follows from

$$\left\| \sup_j \varsigma_j * \sum_{\substack{k+j>0 \\ (\alpha-1)k\ge j}} 2^{-k\beta(p)} \|T_k f_k\|_{L^p_{t[[0,1]]}} \right\|_{L^p_x} \lesssim \left(\sum_k \|f_k\|_p^p\right)^{1/p}. \qquad (21)$$

Now, for fixed k we cover $\mathbb{R}^d$ by a grid $\mathcal{R}_k^{\alpha-1}$ consisting of cubes of sidelength $2^{k(\alpha-1)}$. For each

$R \in \mathcal{R}_k^{\alpha-1}$ let R^* be the cube with same center as R and sidelength $C(\alpha)2^{k(\alpha-1+10d)}$ where $C(\alpha)$ is as in the proof of (17)

For $R \in \mathcal{R}_k^{\alpha-1}$ we let $f_k^R = \chi_R f_k$. We may then split the left hand side of (21) as $I + II$ where

$$I = \left\| \sup_j \varsigma_j * \left[\sum_{\substack{k+j>0 \\ (\alpha-1)k\geq j}} 2^{-k\beta(p)} \left\| \sum_{R\in\mathcal{R}_k^{\alpha-1}} \chi R^* T_k f_k^R \right\|_{L^p_{t[[0,1]]}} \right] \right\|_{L^p_x}$$

And II is analogous expression where χR^* is replaced with $\chi_{\mathbb{R}^d/R^*}$.

By Hardy-Littlewood, Minkowski, Fubini, (18), and Young's inequality, we dominate

$$II \lesssim \sum_{k\geq 0} 2^{-k\beta(p)} \left\| \sum_{R\in\mathcal{R}_k^{\alpha-1}} \chi_{\mathbb{R}^d/R^*} T_k f_k^R \right\|_{L^p(\mathbb{R}^d\times[0.1])}$$

$$\lesssim \sum_{k\geq 0} 2^{-k\beta(p)} \left(\int_0^1 \int \left[\int \frac{2^{kd}}{(1+2^k|x-y|)^{2d}} \sum_{R\in\mathcal{R}_k^{\alpha-1}} |f_k^R(y)| d\, y \right]^p dxdt \right)^{1/p} \lesssim$$

$$\lesssim \sum_{k\geq 0} 2^{-k\beta(p)} \left\| \sum_{R\in\mathcal{R}_k^{\alpha-1}} f_k^R \right\|_p \lesssim \sup_k \|f_k\|_p \lesssim \left(\sum_k \|f_k\|_p^p \right)^{1/p}.$$

Concerning the main term I we use the embedding $\ell^p \hookrightarrow \ell^\infty$, interchange a sum an integral, and apply Minkowski's, so that

$$I \lesssim \left(\sum_j \left\| \varsigma_j * \left[\sum_{\substack{k+j>0 \\ (\alpha-1)k\geq j}} 2^{-k\beta(p)} \sum_{R\in\mathcal{R}_k^{\alpha-1}} \chi R^* \left\| T_k f_k^R \right\|_{L^p_{t[[0,1]]}} \right] \right\|_{L^p_x}^p \right)^{1/p}.$$

Now for $R \in \mathcal{R}_k^{\alpha-1}$ has sidelength greater than 2^j, so for fixed k. Setting $n = k + j > 0$ and applying Minkowski's inequality, we get

$$I \lesssim \sum_{n>k} I_n$$

Where

$$I_n = \left(\sum_{j>n} \sum_{R\in R\in\mathcal{R}_{n-j}^{\alpha-1}} 2^{-(n-j)\beta(p)p} \left\| \varsigma_j * \left\| T_{n-j} f_{n-j}^R \right\|_{L^p_{t[[0,1]]}} \right\|_{L^p_x}^p \right)^{1/p}$$

As before chose r so that $2 + \frac{d}{d+1} < r < p$. It will suffice to show that

$$I_n \lesssim 2^{-nd\left(\frac{1}{r}-\frac{1}{p}\right)} \left(\sum_k \|f_k\|_p^p \right)^{1/p}. \tag{22}$$

Observe that byYoung's convolution with ς_j maps $L^r(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with operator norm $O\left(2^{jd(1/r-1/p)}\right)$. Moreover by (15) we have

$$\left\|\left\|T_{n-j}f^R_{n-j}\right\|_{L^p_{t[[0,1]]}}\right\|_{L^p_x} \lesssim 2^{(n-j)\alpha\left(\frac{1}{r}-\frac{1}{p}\right)}\left\|\left\|T_{n-j}f^R_{n-j}\right\|_{L^r_t[0.1]}\right\|_{L^p_x}.$$

Thus we can bound

$$I_n \lesssim \left(\sum_j 2^{-jd\left(\frac{1}{r}-\frac{1}{p}\right)p}2^{(n-j)\alpha\left(\frac{1}{r}-\frac{1}{p}\right)p}2^{-(n-j)\beta(p)p}\sum_{R\in R\in\mathcal{R}^{\alpha-1}_{n-j}}\left\|T_{n-j}f^R_{n-j}\right\|^p_{L^r(\mathbb{R}^d\times[0.1])}\right)^{\frac{1}{p}}.$$

Which by (14), is

$$\lesssim \left(\sum_j 2^{-jd\left(\frac{1}{r}-\frac{1}{p}\right)p}2^{(n-j)\alpha\left(\frac{1}{r}-\frac{1}{p}\right)p}2^{-(n-j)\beta(p)p}\sum_{R\in R\in\mathcal{R}^{\alpha-1}_{n-j}}\left\|T_{n-j}f^R_{n-j}\right\|^p_{L^r(\mathbb{R}^d\times[0.1])}\right)^{\frac{1}{p}}.$$

Since $f^{\alpha-1}_{n-j}$ is supported on the cube R of size $2^{(n-j)(\alpha-1)d}$ we see by Hölder's inequality that the last displayed expression is dominated by a constant times

$$\left(\sum_j 2^{-jd\left(\frac{1}{r}-\frac{1}{p}\right)p}2^{(n-j)\alpha\left(\frac{1}{r}-\frac{1}{p}\right)p}2^{-(n-j)\beta(p)p}2^{-(n-j)\beta(r)p}2^{(n-j)d\left(\frac{1}{r}-\frac{1}{p}\right)p}\sum_{R\in R\in\mathcal{R}^{\alpha-1}_{n-j}}\left\|f^R_{n-j}\right\|^p_p\right)^{\frac{1}{p}}.$$

Now this simplifies after summation in R, to

$$I_n \lesssim 2^{-nd\left(\frac{1}{r}-\frac{1}{p}\right)}\left(\sum_j\|f_n-j\|^p_p\right)^{\frac{1}{p}} \le C2^{-nd\left(\frac{1}{r}-\frac{1}{p}\right)}\left(\sum_k\|f_k\|^p_p\right)^{1/p}.$$

This finishes the proof of (18) and concludes the proof of Theorem (4.1.7).

Corollary(4.1.8) [108]Let p, α, β be as in Theorem (4.1.7) then

$$\int_0^1\|U^\alpha_t f\|^p_{F^p_0(\mathbb{R}^d)}dt^{1/p} \lesssim \|f\|_{B^p_{\beta,p}(\mathbb{R}^d)}.$$

This implies Theorem (4.1.2) since for $p \ge 2$ the space $B^p_{\beta,p} \equiv F^p_{\beta,p}$ contain the Sobolev space $F^p_\beta \equiv F^p_{\beta,2}$, via the embedding $\ell^p \hookrightarrow \ell^p$ followed by the Littlewood-Paley inequality, and by the same reasoning $F^p_{0,1}$ is imbedded in $L^p \equiv F^p_{0,2}$. We remark that a similar sharp inequality for the wave equation is proved in [101], in sufficiently high dimensions.

Another consequence of Theorem (4.1.7) is

Corollary(4.1.9) [108]: Let p, α be as in Theorem (4.1.7) Let $t \rightarrowtail \vartheta(t)$ be smooth and completly supported. Then

$$\left\|\left\|\vartheta(\cdot)U^\alpha_{(\cdot)}g\right\|_{B^p_{1/p.1}(\mathbb{R})}\right\|_{L^p(\mathbb{R}^d)} \lesssim \|g\|_{B^p_{,p}(\mathbb{R}^d)}, \quad = \alpha d(1/2-1/p). \qquad (13)$$

Theorem (4.1.1) is an immediate consequence of Corollary(4.1.9) since the Besov space $B^p_{1/p,1}(\mathbb{R}^d)$ is continuously embedded in the space CO of continuous bounded functions which vanish at infinity.

To see how Corollary(4.1.9) follows from Theorem (4.1.7) we introduce dyadic frequency cutoffs in the t variable. We decompose the identity as $I = \sum_{j=0}\mathcal{L}_j$ where $\widehat{\mathcal{L}_j}f(\tau) = \tilde{\chi}_j(\tau) = \tilde{\chi}_j(\tau)\tilde{f}(\tau)$ where

$\tilde{\chi}_j = \tilde{\chi}(2^{-j}|\cdot|)$ for $j \geq 1$, with suitable $\tilde{\chi} \in C_0^\infty$ supported in $(1/2, 2)$ and $\tilde{\chi}_0$ is smooth and vanishes for $|\tau| \geq 2$. Now we apply$L_j$to $\vartheta T_{k.g.}$ If $2^{j-\alpha k} \notin (2^{-10}, 2^{10})$, then we apply an integration by parts in s to terms of the form

$$\iint \chi(2^{-j}|\tau|)\chi(2^{-k}|\xi|)\tilde{g}(\xi)e^{i(\langle x.\xi\rangle+t)}\int \vartheta(s)e^{is(|\xi|^\alpha-)}dsd\xi d.$$

One finds that for this range the contribution of $\mathcal{L}_j[\vartheta T_k g]$ is negligible; namely

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left|\mathcal{L}_j[\vartheta T_k g](x,s)\right|^p dxds\right)^{1/p} \lesssim C_N \min\{2^{-\alpha kN}, 2^{-jN}\}\|g\|_p \text{if} 2^{j-\alpha k} \notin (2^{-10}, 2^{10}).$$

Thus a localization in ~ where corresponds to a localization in T where ITI We combine this with Theorem (4.1.7) applied to and obtain

Section (4.2): Schrödinger Operator and Space-TimeEstimates

We consider the Schrodinger equation, $i\,\partial_t u + \Delta u = 0$, with initial data $u(.,o) = f$.

When f is a Schwartz function, the solution can be written as $u = Uf$, where

$$Uf(x.t) \equiv e^{it\Delta}f(x) = \frac{1}{2\pi^d}\int_{\mathbb{R}^d}\hat{f}(\xi)e^{-it|\xi|^2+i\langle x,\xi\rangle}\,d\xi. \qquad (23)$$

And $\widehat{\ }$ denotes the Fourier transform defined by$\hat{f}(\xi) = \int f(y)e^{-i(y.\xi)}dy$. We fix a compact time interval I and $L^q(\mathbb{R}^d; L^r(I))$ be the space equipped with mixed norm

$$\|u\|_{L^q(\mathbb{R}^d;L^r(I))} = \left(\int_{\mathbb{R}^d}\left(\int_I |u(x,t)|^r\,dt\right)^{q/r} dx\right)^{1/q}.$$

Our aim is to bound the solution in this space whet h initial data are given in the Sobolev spaces L^p_α, with norm $\|f\|_{L^p_\alpha} = \left\|(I-\Delta)^{\alpha/2}f\right\|_{L^p(\mathbb{R}^d)}$. We shall always assume that $q, r \geq 2$, and we will mostly assume $p \geq 2$ as well. The cases $r = 2, r = q$ and $r = \infty$ are of particular interest.

Theorem (4.2.1) [118]: Let $2 \leq p \leq \infty$.

Then $U : L^p(\mathbb{R}) \to L^p(\mathbb{R};\, L^r(I))$ is bounded if and only if $r \leq 2$.

The sufficiency of the condition follows from [16]. The necessary is a consequence of the following more precise bounds for frequency localized functions which also illustrated the sharp of the necessary conditions of [16] (at least in the cases $r \leq q$ and $d = 1$).

Corollary (4.2.2) [118]: Suppose that $2 \leq r \leq p \leq q, \frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p}$.

$$\text{Then} U : B^p_{\alpha.q}(\mathbb{R}) \to L^r(I) \text{is bounded with} \alpha = 1 - \frac{1}{p} - \frac{1}{q} - \frac{2}{r}.$$

When $p = q$ one could hope for the following estimates.

Conjecture(4.2.3) [118]: Let$p \in [2,\infty]r \in [2,\infty]$satisfy$\frac{d}{p} + \frac{1}{r} < \frac{d}{2}and\frac{2d+1}{p} + \frac{1}{r} < d$.

$$\text{Then} U : B^p_{\alpha.q}(\mathbb{R}^d) \to L^p\left(\mathbb{R}^d; L^r(1)\right) \text{is bounded with} \alpha = d\left(1 - \frac{2}{p}\right) - \frac{2}{r}.$$

To prove the conjecture it would suffice to prove the sharp estimates with $r = \infty, p$ and 2. The estimates with $r = \infty$ strengthen the sharp L^p-Sobolev bounds for fixed t and $\alpha = 2d|1/2 - 1/p|$ due to Fefferman-Stein [31] and Miyachi [37]. In [114], the conjecture was proven in the reduced range $p \in \left(\frac{2(d+2)}{d}, \infty\right)$, and for $d = 1$ it was proven in the range $p \in (4, \infty)$. In [108], the conjecture was proven for $p \in \left(\frac{2(d+3)}{d+1}, \infty\right)$, with $r \geq p$; moreover a related result was proven for the semigroup exp $it\big((-\Delta)^{\alpha/2}\big)$ for $\alpha = 1$. A nonendpoint result for $\alpha = 2, p = r$ has been previously obtained in [46].

In the case of the Schrodinger semigroups $(\alpha = 2)$ it is well known that the local something and maximal inequalities are closely related to estimates for the adjoint restriction operator for a compact portion of the paraboloid in $\mathbb{R}^{d+1}$ (see [15], [24], [110], [11], [46]). Here we improve the known $L^q(L^r)$ bounds for $q = r$ by establishing the actual equivalence of the space-time regularity estimates with estimates for the adjoint restriction operator (a related result establishing the the equivalence between the ajoint restriction and Bochner-Riesz for paraboloids was found by Garbery [28]).

Let $\mathcal{E}$ denote the adjoint restriction (or Fourier extension) operator given by

$$\varepsilon f(\xi, s) = \int_{|y|\geq 1} f(y)e^{is|y|^2 - i}dy \ (\xi, s)\epsilon\mathbb{R}^d x\mathbb{R}. \qquad (24)$$

Definition (4.2.4) [118]: We say that $R^*(p \to q)$ holds true if $\varepsilon :\ L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^{d+1})$ is bounded. In the critical case $q(p) = \frac{d+2}{d}p'$ it follow from the explicit formula

$$Uf(\chi, t) = \frac{1}{(4\pi it)^{d/2}}\int \exp\left(\frac{i|\chi - y|^2}{4t}\right) f(y)dy \qquad (25)$$

And scaling that $R^*(p \to q(p))$ implies the $L^p(\mathbb{R}^d) \to L^{q(p)}(\mathbb{R}^d \text{ x } I)$ boundedness of U.

Moreover it was also shown in [46] it implies the $L^p_\alpha \to L^q(\mathbb{R}^d \text{ x } I)$ bound for $\alpha > 2d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2}{p}$. we strengthen these results as follows.

Corollary(4.2.5) [118]: Let $2 < q_0 < \infty,\ 1 \ll p_0 \leq q_0$, and suppose that $R^*(p_0 \to q_0)$ holds.

$$\text{Let} q_0 < q < \infty, q \leq r \leq \infty \text{and suppose that } 0 \leq \frac{1}{p} - \frac{1}{q} \geq \frac{1}{p_0} - \frac{1}{q_0}.$$

$$\text{Then} U :\ B^p_\alpha(\mathbb{R}^d) \to L^q\left(\mathbb{R}^d; L^r(I)\right) \text{is bounded with} \alpha = d\left(1 - \frac{1}{p} - \frac{1}{q}\right) - \frac{2}{r}.$$

Using also the trivial $R^*(1 \to \infty)$ one can deduce the conclusion in the larger range $p_1(q) < p \leq q$, where $p_1(q) < p_0$ is defined by $\frac{1}{p_1(q)} = \frac{1}{p_0} + \left(1 - \frac{q_0}{q}\right)\left(1 - \frac{1}{p_0}\right)$.

Given Theorem (4.2.8) the recent progress on $R^*(p \to p)$ by Bourgain and Guth [110] can be used to verify Conjecture (4.2.3) for new parameters (see also [16] below for the case $p \neq q$). In two dimensions their implies that the conjecture holds in the case $p = q \leq r$ for $p > 33/10$; moreover, in higher dimensions, it holds for $p = PBG(d)$ with $PBG(d) = 2 + 3d^{-1} + O(d^{-2})$ (see [110] for their exact range of p).

In two dimensions a better range for p can be obtained for large r; this is closely related to previous results on maximal operators for L^2_α function and result on Planchon's conjecture in $\mathbb{R}^2$ (cf. [52], [11], [59], [115]).

Corollary (4.2.6) [118]: Let $2 \leq p \leq 16/5$.

Then $U :\ B^p_\alpha(\mathbb{R}^2) \to L^p(\mathbb{R}^2; L^\infty(I))$ is bounded with $\alpha > 3/4$.

Unlike the rest of the estimates in this article, there is no reason to suspect that this is sharp with respect to the regularity in the range $2 \leq p < 16/5$.

By m(D) we denote the convolution operator with Fourier multiplier m; that is to say $\widehat{m(D)f} = m\hat{f}$. For two nonnegative quantities A, B the notation $A \lesssim B$ and $B \lesssim A$.

We formulate a more technical version of Theorem (4.2.8) that applies to mixed norm inequalities. In what follows let

$$A(p) := \{\xi\epsilon\mathbb{R}^d : 3p \le |\xi| \le 12p\}. \qquad (26)$$

Theorem (4.2.7) [118]: Let $p, q, r\epsilon[2,\infty], p \le q, \beta > -d\left(\frac{1}{2}-\frac{1}{p}\right)$. Then the inequality

$$\sup_{\lambda > 1} \lambda^{-\beta} \sup_{\|f\|_p \le 1} \left(\int_{A(\lambda)} \left(\int_{\lambda}^{2\lambda} \left|\varepsilon f\left(\frac{s}{\lambda}\xi, s\right)\right|^r ds\right)^{q/r} d\xi\right)^{1/q} < \infty (27)$$

Holds if and only if for $? = d\left(1-\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}+2\beta$,

$$\sup_{\|f\|_{B^p_{?,1}} \le 1} \left\|\left(\int_{-1}^{1} |e^{it\Delta}f|^r dt\right)^{1/r}\right\|_q < \infty. \qquad (28)$$

Taking Theorem (4.2.7) for granted we can quickly give

Theorem (4.2.8) [118]: Suppose $2 \le p \le q < \infty$. The the following are equivalent:

(i) $R^*(p \to q)$ holds.

(ii) The operator $U : B^p_{\alpha,1}(\mathbb{R}^d) \to L^q(\mathbb{R}^d \times I)$ is bunded with $\alpha = d\left(1-\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{q}$.

We can also obtain result on larger spaces (including the Sobolev space L^p_α) if we give up endpoint in the q-range.

Proof. By Theorem (4.2.7) we just have to show that $R^*(p \to q)$ with equivalent to (6) for large λ, in the case $q = r$ and $\beta = 0$. Clearly the later is implied by bounded above and below in the region where $s \approx \lambda$.Vice versa, supposing that (28) holds in the case $q = r$ and $\beta = 0$, by the chang of variables, we have that $\varepsilon : L^p(\mathbb{R}^d) \to L^q(W_\lambda)$, where

$$W_\lambda = \{(\xi.s) : s\epsilon[\lambda .2\lambda], \quad x\epsilon A(s)\}.$$

For $\omega\epsilon\mathbb{R}^{d+1}$ define $f^\omega(y) = e^{i(\omega.y)-i\omega d+1|y|^2}f(y)$ and observe that $\varepsilon f^\omega = \varepsilon f(.-\omega)$. Thus using a finit number of translations we see that $\varepsilon : L^p(\mathbb{R}^d) \to L^q(B_\lambda)$, where B_λ of radius λ centered at the origin, and the operator norm is uniformly bounded in λ. Letting $\lambda \to \infty$ yieds $R^*(p \to q)$.

Lemma (4.2.9) [118]: Let $p, q, r\epsilon[2,\infty]$ with $p \le q$ and let $\lambda \ge 1$. suppose that

$$\left(\int_{A(\lambda^2)} \left(\int_{\lambda^2} \left|\varepsilon f\left(\frac{s}{\lambda^2}\xi.s\right)\right|^r ds\right)^{q/r} d\xi\right)^{1/q} \le A\|f\|_p \qquad (29)$$

holds. Then, for $\psi\epsilon C_c^\infty$ with support in $\{\xi : < |\xi| < 5\}$,

$$\left\|\left(\int_{1/2}^{1} \left|e^{it\Delta}\psi\left(\frac{D}{\lambda}\right)f\right|^r\right)^{1/r}\right\|_q \lesssim A\lambda^\alpha \|f\|_p, \quad \alpha = d\left(1-\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}. \qquad (30)$$

Proof. If f_λ is characteristic function of a ball of radius $(100\lambda)^{-2}$ then $\left|\varepsilon(f_\lambda)\left(\frac{s}{\lambda^2}\xi, s\right)\right| \ge \lambda^{-2d}$ for $(\lambda\xi, s)\epsilon A((\lambda^2))x[\lambda^2, 2\lambda^2]$.The resulting lower bound $A \ge c\lambda^{2d(-1+1/p+1/q)+2/r}$ (which is far from being sharp) will be used repeatedly to dominate certain error terms which decayfast in λ.

The convolution kernel for $e^{it\Delta}\psi\left(\frac{D}{\lambda}\right)$ can be written as

$$k_t^\lambda(x) = \left(\frac{\lambda}{2\pi}\right)^d \int \psi(\xi)e^{-it\lambda^2|\xi|^2+i\lambda(x.\xi)}d\xi.$$

By integration by parts it follows that

$$k_t^\lambda(x)| \le C_N|x|^{-N}, | \ge 11\lambda. \qquad (31)$$

Hence, by a standard argument,

$$\left(\int_{|x|\leq 11\lambda}\left(\int_{1/2}^{1}|k_t^{\lambda} * f|^{q/r}dt\right)\right)^{q/r} dx \lesssim A\lambda^{\alpha}\|f\|_p, \quad \alpha = d - \frac{d}{p} - \frac{d}{q} - \frac{2}{r} \qquad (32)$$

For f supported in the cube of the sidelength $\lambda\, 2d^{-1}$ centered at the origin. Indeed, suppose that (32) is verified, let $\mathfrak{Q}_{\lambda} = \{Q\}$be a grid of cubes with sidelength $\lambda\, 2d^{-1}$, and centres x_Q, and let B_Q be the ball of radius 11λ centred r_Q. Then we may estimates the $L^q(\mathbb{R}^d;\, L^r([2,1]))$ norm of $e^{it\Delta}\psi\left(\frac{D}{\lambda}\right)$ by

$$\left(\int\sum_{Q}\chi Q^{(x)}\left(\int_{1/2}^{1}|k_t^{\lambda} * [f\chi_Q](x)|^{r}dt\right)^{q/r} dx\right)^{1/q}\left(\int\sum_{Q}\chi Q^{(x)}\left(\left|k_t^{\lambda} * \left[f\chi_{\mathbb{R}^d}\backslash B_Q\right](x)\right|^{r}dt\right)^{q/r} dx\right)^{1/q} \quad (33)$$

By Minkowski's inequality in L^r. We use the finite overlap of the balls, the translation invariance of the operators and (32) to estimate the first term by

$$CA\lambda^{\alpha}\left(\sum_{Q}\|f\chi_Q\|_p^q\right)^{1/q} \lesssim CA\lambda^{\alpha}\|f\|_p$$

Where for the last inequality we have used the assumption $p \leq q$. For the second term in (33) we use (31) with $N > 2d$ and then Young's to bound it by

$$C\left(\int\left[\int_{|w|\geq 10\lambda}|w|^{-N}f(x-w)dw\right]^{q} dx\right)^{1/q} \lesssim \lambda^{-N+d\left(1-\frac{1}{p}+\frac{1}{q}\right)}\|f\|_p \lesssim A\lambda^{\alpha}\|f\|_p.$$

We used the trivial lower bound for A in the last step.

Our task is now to prove (32). We use a stationary phase calculation to see that $k_t^{\lambda} = H_t^{\lambda} + E_t^{\lambda}$, where

$$k_t^{\lambda}(x) = \frac{e^{-i|x|^2/4t}}{(4\pi i t)^{d/2}}\sum_{v=0}^{M}\psi_v\left(\frac{x}{2\lambda t}\right)^{\lambda^{-v}}$$

And $$|E_{\lambda}(x,t)| \leq C_L\lambda^{-L}$$

Where we chose $L \gg d$. For the leading term $\psi_0 = \psi$, and the functions ψ_v are obtained by letting certain differential operators act on ψ; thus $\psi_v(w) = 0$ for $|w| \leq 4$ and $|w| \geq 5$.

For the error we use a trivial bound

$$\left(\int_{|x|\leq 11\lambda}\left(\int_{1/2}^{1}\left[\int|E_{\lambda}(x-y,t)||f(y)|dy\right]^{r}dt\right)^{q/r} dx\right)^{1/q} \lesssim \lambda^{d-L}\|f\|_p \lesssim A\lambda^{\alpha}\|f\|_p.$$

For the oscillatory terms we have to prove the inequality

$$\left(\int_{|x|\leq 11\lambda}\left(\int_{1/2}^{1}\left|\int\psi_v\left(\frac{x-y}{2\lambda t}\right)\exp\left(i\frac{|x-y|^2}{4t}\right)f(y)dy\right|^{r}dt\right)^{q/r} dx\right)^{1/q} \lesssim A\lambda^{\alpha}\|f\|_p. \qquad (34)$$

Whenever f is supported in $\{|y| \leq \lambda/2\}$. By a change of variable $t \to u = 1/t$ (with $u \approx t \approx 1$) and the support properties for ψ_v this follows from

$$\left(\int_{\frac{7}{2}\lambda\leq|x|\leq\frac{21}{2}\lambda}\left(\int_{1}^{2}\left|\int_{|y|\leq\lambda/2}\psi_v\left(\frac{u(x-y)}{2\lambda}\right)\exp\left(i\frac{u}{4}(|y|^2-2\langle x-y\rangle)\right)\right.\right.\right. \qquad (35)$$

Whenever f is supported in $\{|y| \leq \lambda/2\}$. We now use a parabolic scaling in the (x,u) variables and set $x = \lambda^{-1} w,\ u = \lambda^{-2} s; \quad y = 2\lambda z$.

The previous inequality becomes

$$\left(\int_{\frac{7}{2}\lambda^2 \le |w| \le \frac{21}{2}\lambda^2} \left(\int_{\lambda^2}^{2\lambda^2} \left| \int_{|z|\le 1} \psi_v\left(\frac{sw - 2\lambda^2 sz}{2\lambda^4}\right) e^{e\left(i\left(s|z|^2 - \langle\frac{sw}{\lambda^2}, z\rangle\right)\right)} f(2\lambda z)(2\lambda)^d dz\right|^r \frac{ds}{\lambda^2}\right)^{q/r} \frac{dw}{\lambda^d}\right)^{1/q}$$

$$\lesssim A\lambda^{\alpha} \|f\|_p. \qquad (36)$$

We have the Fourier series expansion $\psi_v(x) = \sum_{e\epsilon\mathbb{Z}^d} c_{e.v} e^{i(e.x)}$ for $x\epsilon\left[-\frac{9}{10}\pi, \frac{9}{10}\pi\right]^d$ and for each v the Fourier coefficients are rapidly decaying, $|C_{\ell.v}| \le C_{N.v}(1 + |\ell|)^{-N}$. Thus

$$\psi_v\left(\frac{sw - 2\lambda^2 sz}{2\lambda^4}\right) = \sum_{\ell} C_{\ell.v} e^{i\lambda^{-4}\langle sw.\ell\rangle/2} e^{-i\lambda^{-2} s\langle z.\ell\rangle}.$$

Using Minkowki's inequality for the sum and the rapid decay of the Fourier coefficients the previous inequality (35) follows from

$$\left(\int_{\frac{7}{2}\lambda^2 \le |w| \le \frac{21}{2}\lambda^2} \left(\int_{\lambda^2}^{2\lambda^2} \left| \int_{|z|\le 1} \exp\left(i\left(s|s|^2 - \langle\frac{s(w+\ell)}{\lambda^2}, z\rangle\right)\right) f(2\lambda z) dz\right|^r ds\right)^{q/r} dw\right)^{1/q}$$

$$\lesssim (1 + |\ell|)^M A \lambda^{\alpha - d + \frac{2}{r} + \frac{d}{q}} \|f\|_p. \qquad (37)$$

The left hand side is trivially bounded by $C\lambda^{2/r + 2d/q}$ and therefore the displayed inequality holds for $|\ell| \ge \lambda^2/4$. if $|\ell| \le \lambda^2/4$, we change variable and see that for (37) we only need to show

$$\left(\int_{3\lambda^2 \le |w| \le 11\lambda^2} \left(\int_{\lambda^2}^{2\lambda^2} \left| \int_{|z|\le 1} \left(i\left(s|z|^2 - \langle\frac{sw}{\lambda^2}, z\rangle\right)\right) g(z) dz\right|^r ds\right)^{q/r} dw\right)^{1/q}$$

$$\lesssim A\lambda^{\alpha - d + \frac{2}{r} + \frac{d}{q}} \lambda^{d/p} \|g\|_p.$$

The right hand side is just $A\|g\|_p$, So that this would follow from (29).

Lemma (4.2.10) [118]: Let $p, q, r\epsilon[2, \infty]$ and $\lambda \gg 1$. Let $2 < \alpha_0 < \alpha_1$ and let a radial C_c^{∞} function which satisfies $\eta(\xi) = 1$ for $\frac{\alpha_0 - 2}{4} \le |\xi| \le 2(\alpha_{1+2})$. Suppose

$$\sup_{\|f\|_p \le 1} \left\| \left(\int_{1/2}^{1} \left|e^{it\Delta} \eta\left(\frac{D}{\lambda}\right) f\right|^r dt\right)^{1/r} \right\|_q \le B. \qquad (38)$$

Then

$$\left(\int_{\alpha_0\lambda^2 \le |\xi| \le \alpha_1\lambda^2} \left(\int_{\lambda^2}^{2\lambda^2} \mathcal{E}f\left(\frac{s}{\lambda^2}\xi, s\right)^r\right)^{q/r} d\xi\right)^{1/q} \lesssim B\lambda^{-d + \frac{d}{p} + \frac{d}{q}} \|f\|_p. \qquad (39)$$

Proof. In what follows let $\alpha = d\left(1 - \frac{1}{p} - \frac{1}{q}\right) - \frac{2}{r}$. We begin by observing the lower bound $B \ge c\lambda^{\alpha}$ which follows from the example in (ii).

By a change of variable $\xi = \lambda x, s = \lambda^2 p, y = 2\lambda z$ we see that (39) is equivalent with

$$\left(\int_{\alpha_0\lambda^2 \le |\xi| \le \alpha_1\lambda} \left(\int_1^2 \left| \int_{|y| \le 2\lambda} f\left(\frac{y}{2\lambda}\right) e^{i(p|y|^2/4 - p\langle x, y\rangle/2)} dy\right|^2 dp\right)^{q/r} dx\right)^{1/q}$$

$$\le CB\lambda^{-\alpha} (2\lambda)^d \lambda^{-d/q - 2/r} \|f\|_p.$$

By inverting $t = 1/p$ the previous inequality follows from

$$\left(\int_{\alpha_0\lambda^2\le|\xi|\le\alpha_1\lambda}\left(\int_{1/2}^{1}\left|\frac{1}{(4\pi it)^{d/2}}\int_{|y|\le\lambda} g(y)e^{\frac{i|x-y|^2}{4t}}dy\right|^r dt\right)^{q/r} dx\right)^{1/q}$$
$$\lesssim CB\lambda^{-\alpha}\lambda^{d-d/p-2/r}\lambda^{-d/p}\|f\|_p.$$

Which can be rewritten as

$$\left(\int_{\alpha_0\lambda^2\le|\xi|\le\alpha_1\lambda}\left(\int_{1/2}^{1}\left|e^{it\Delta}g(x)\right|^r dt\right)^{q/r} dx\right)^{1/q} \lesssim B\|g\|_p. \qquad (40)$$

For g supported in $\{y: |y| \le 2\lambda\}$. By assumption

$$\left(\int_{\alpha_0\lambda^2\le|\xi|\le\alpha_1\lambda}\left(\int_{1/2}^{1}\left|e^{it\Delta}\eta\left(\frac{D}{\lambda}\right)g(x)\right|^r dt\right)^{q/r} dx\right)^{1/q} \le B\|g\|_p.$$

And thus (39) follows from the straightforward estimate

$$\left(\int_{\alpha_0\lambda^2\le|\xi|\le\alpha_1\lambda}\left(\int_{1/2}^{1}\left|e^{it\Delta}\left(1-\eta\left(\frac{D}{\lambda}\right)\right)g(x)\right|^r dt\right)^{q/r} dx\right)^{1/q} \le C_M\lambda^{-M}\|g\|_p. \qquad (41)$$

Whenever g is supported in $\{y: |y| \le 2\lambda\}$.

To see (41) we decompose the multiplier. Let χo be smooth and supported in $\{|\xi| < 2\}$ And $\chi o(\xi) = 1$ for $|\xi| \le 1$, and let $\chi\kappa(\xi) = \chi o(2^{-\kappa}\xi) - \chi o(2^{1-\kappa}\xi)$, for $\kappa \ge 1$. Let

$$E_{\lambda.\kappa}(x,t) = \frac{1}{(2\pi)^d}\int \chi\kappa\left(\frac{\xi}{\lambda}\right)\left(1-\eta\left(\frac{\xi}{\lambda}\right)\right)e^{-it|\xi|^2+i(x,\xi)}d\xi$$

And we need to bound the expression

$$\left(1-\eta\left(\frac{D}{\lambda}\right)\right)e^{it\Delta}g(x,t) = \sum_{\kappa\ge0}\int_{|y|\le2\lambda}E_{\lambda.\kappa}(x-y)\,g(y)dy.$$

We now examine $\nabla_\xi(\langle x-y,\xi\rangle - t\xi^2) = x - y - 2t\xi$. since $\alpha_0 > 2$, foe the relevant choices $\alpha_0|\lambda| \le |x| \le \alpha_1\lambda, 1/2 \le t \le 1, |y| \le 2\lambda$ we have

$$|x-y-2t\xi| \ge \begin{cases} \frac{1}{2}(\alpha_0-2)\lambda & \text{if } |\xi| \le \frac{\alpha_0-2}{4}\lambda, \\ \max\left\{\frac{|\xi|}{2},(\alpha_1-2)\lambda\right\} & \text{if } |\xi| \ge (\alpha_1-2)\lambda. \end{cases}$$

Since $1-\eta(\lambda) = 0$ for $\frac{\alpha_0-2}{4} \le |\xi| \le 2(\alpha_0+2)$, after an N-fold integration by parts we find that $|E_{\lambda.\kappa}(x-y,t)| \le C_N(2^\kappa\lambda)^{d-N}$ for this choice of x, y, t, and the estimate (19) follows.

To complete the Theorem (4.2.7) we also need the following scaling lemma.

Lemma (4.2.11) [118]: Let $\gamma > d\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}$. Suppose that for $\lambda \gg 1$

$$\left\|\left(\int_{1/2}^{1}\left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f\right|^r dt\right)^{1/q}\right\|_q \lesssim \lambda^\gamma\|f\|_p. \qquad (42)$$

where $\chi \epsilon C_c^\infty$ is supported in $(1/2, 2)$ (with suitable bounds). Then, for $\lambda \gg 1$.

$$\left\|\left(\int_{1/2}\left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f\right|^r dt\right)^{1/r}\right\|_q \lesssim \lambda^\gamma\|f\|_p. \qquad (43)$$

Proof. It is easy to calculate that

$$sup_{0\le t\le(8\lambda)^2}\left|\mathcal{F}^{-1}\left[\chi\left(\frac{\cdot}{\lambda}\right)\exp(-it|\cdot|^2)\right](x)\right|\le C_N\,\lambda^d\,(1+\lambda\,|x|)^{-N}$$

And thus, byYoung's inequality,

$$\left\|\left(\int_0^{(8\lambda)^{-2}}\left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f\right|^r dt\right)^{1/r}\right\|_q \lesssim \left\|\lambda^{-2/r}\int\lambda^d\,(1+\lambda\,|y|)^{-N}|f(\cdot-y)|dy\right\|_q$$

$$\lesssim\lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}}\,\|f\|_p. \qquad (44)$$

Now letting $(8\,\lambda)^{-2}\le b\le 1$,

$$\left(\int_{b/2}^{b}\left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f(x)\right|^{1/r}dt\right)^{1/r} = b^{1/r}\left(\int_{1/2}^{1}\left|\chi\left(\frac{D}{b^{1/2}\lambda}\right)e^{is\Delta}\left[f\left(b^{-1/2}.\right)\right]\left(b^{-1/2}x\right)\right|^r ds\right)^{1/r}$$

Thus by change of variable (42) implies

$$\left\|\left(\int_{b/2}^{b}\left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f(x)\right|^{1/r}dt\right)^{1/r}\right\|_q \lesssim \left(\sqrt{b}\right)^{-d\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}}\left(\lambda\sqrt{b}\right)^{\gamma}\|f\|_p.$$

We chose $b=2^{-1}$. and since $\gamma>d\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}$ we may sum over I with $(8\,\lambda)^{-2}\le 2^{-1}\le 1$ and combine with (44). Hence we get

$$\left\|\left(\int_0^{1}\left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f\right|^r dt\right)^{1/r}\right\|_q \lesssim\lambda^{\gamma}\,\|f\|_p.$$

Now (43) with $I=[-1,1]$ follows using the formula $e^{it\Delta}f=\overline{e^{\imath t\Delta}\bar{f}}$,and the triangle inequality. Finally, by scaling, we can enlarge the time interval (so that the implicit constant is of course dependent on the interval), and we are done.

Proposition (4.2.12) [118]: Let $2\le p,q,r\le\infty$, andsuposethatthereconstantCsuchthat

$$\|Uf\|_{L^q(\mathbb{R}^d;L^r(I))}\le C\|f\|_{L^p_\alpha(\mathbb{R}^d)} \qquad (45)$$

wheneverf$\epsilon L^p_\alpha(\mathbb{R}^d)$. Then

(i)$p\le q$,

(ii)$\alpha\ge d\left(1-\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}$,

(iii)$\alpha\ge\frac{1}{q}-\frac{1}{r}$,

(iv)$\alpha\ge\frac{1}{q}-\frac{1}{p}$,

(v)$\alpha>\frac{1}{q}-\frac{1}{p}ifr>2$,

(vi)$\alpha>0$ if$r=2, p=q>2, d\ge 2$.

The proposition can be strengthened by replacing the Sobolev norm by the Besov norm $B^p_\alpha.v$, for any$v>o$, where $\|f\|_{B^p_{\alpha.v}}=\left(\sum_{k\ge0}2^{k\alpha v}\|P_kf\|_p^v\right)^{1/v}$. Here, for $k\ge1$, the operators p_k localize frequencies to annuli of width $\approx 2^k$ and $p_0=1-\sum_{k\ge1}P_k$. Recall that $B^p_\alpha.v$ is contained in L^p_α for $v\le$ min $\{2,p\}$.

The inequality (45) has been considered in many especial cases and some of the necessary conditions in Proposition (4.2.12) are related to similar conditions for other problems in harmonic analysis. In what follow we set $\alpha_{cr}(p;q,r) := d(1-\frac{1}{p}-\frac{1}{q})-\frac{2}{r}$.

(a) If $p = 2$, then the condition (ii)coincides with (iii)if $\frac{d+1}{q}+\frac{1}{r}=\frac{d}{2}$. This is the condition in the end point version of Planchon's conjecture (cf. [52], [115]).

(b) If $p = 2$ and $r = \infty$, then the condition (iii) follow from the necessary conditions for carleson's problem [4, 15], via an equivalence between local and global estimates [46].

(c) If $p = 2$ and $2 \le r \le q$. then the condition $\alpha \ge \alpha_{cr}(p;p,r)$ is more restrictive than (iv) if $d\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{r} > 0$. In particular, if $r = 2$, and $\alpha = \alpha_{cr}(p;p,2)$, the range $p > \frac{2d}{d-1}$ is necessary (in analogy to the Bochner-Riesz conjecture in $\mathbb{R}^d$), and for $r = p, \alpha = \alpha_{cr}(p;p,r)$ the range $p > \frac{2(d+1)}{d}$ is necessary (as to equivalent adjoint restriction theorem for the sphere in $\mathbb{R}^{d+1}$,

. (d) If $p < q = r$ then the condition $\alpha \ge \alpha_{cr}(p;p,2)$ is more restrictive than (iv) if $\frac{d+1}{q} \le \frac{d-1}{p'}$, the familiar range for the adjoint restriction theorem for the sphere in $\mathbb{R}^d$. Likewise if, $p < q = r$ then the condition $\alpha \ge \max \alpha_{cr}(p;q,q)$ implies $\frac{d+2}{q} \le \frac{d}{p'}$, the range for the adjoint restriction theorem for the paraboloid in $\mathbb{R}^{d+1}$.

(e) The necessity of the strict inequalities in(v), (vi) is proved by considerations which involve the Besicovich set. The necessity of the condition (vi) in dimensions $d \ge 2$ comes from the fact that a sharp square function estimate for the Schrodinger operator implies sharp bounds on Bochner-Riesz multipliers. The necessity for the open range (v) in one dimension was left open in [16].

Proof. First we discuss the easier necessary conditions (i)-(iv).

i) The condition$p \le q$. This follows from the translation invariance (see an argument in [112]). More precisely, the $L^p_\alpha(\mathbb{R}^d) \to L^q(\mathbb{R}^d; L^r(I))$ boundedness is equivalent with the $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d; L^r(I))$ boundedness of the operator $U[(1-\Delta)^{\alpha/2}f]$ which commutes with translation on $\mathbb{R}^d$. Let $A = \sup_{\|f\|_p} \le 1\|U[(1-\Delta)^{\alpha/2}f]\|_{L^q(L^r)}$. Then by the density argument, for $\epsilon > 0$ there is a $g \in C_c^\infty(\mathbb{R}^d)$ such that $A - \epsilon > \|U[(1-\Delta)^{\alpha/2}g]\|_{L^q(L^r)}$ and $\|g\|_p = 1$. One may test the inequality with $f = g + g(.+\alpha e_1)$. Letting $\alpha \to \infty$, we see that $(A-\epsilon)2^{1/q} \le A2^{1/p}$, which gives $A2^{1/q} \le A2^{1/p}$ by letting $\epsilon \to 0$, and thus $p \le q$.

ii) The condition $\alpha \ge d\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}$. This condition follows by a focusing example (see for example [46]). Let $\eta \in C_c^\infty$ be radial and supported in $\{\xi : 1 < |\xi| < 2\}$. Moreover $|Uf(x,t)| \gtrsim \lambda^d$ if, for suitable $c > 0, |x| \le c\lambda^{-1}$ and $\left|t-\frac{1}{2}\right| \le \lambda^{-2}$. For Large λ this leads to the restriction $\alpha \ge d\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}$.

iii) The condition $\alpha \ge \frac{1}{q}-\frac{1}{r}$. Let $g_\lambda$be defined by$\hat{g}_\lambda(\xi) = \chi(|\xi - \lambda e_1|), \chi$ supported in an $\epsilon-$ niighborhood of 0 (see [7], [24]), so that $g\lambda_{L^p_\alpha} \lesssim \lambda^\alpha$. Also

$$Ug\lambda(x,t) = \frac{1}{(2\pi)^d}\int \chi(|h|)e^{i\phi\lambda(x,t,h)}dh$$

Where $i\phi\lambda(x,t,h) = -t|h|^2 - t\lambda^2 + r_1\lambda + \langle r - 2t\lambda e_1\rangle$. Then $|Ug\lambda(x,t)| \ge c_0 > 0$ if $|t-(2\lambda)^{-1}x1| \le c\lambda^{-1}$ for $0 \le x1 \le \lambda, |x_i| \le c, i = 2\ldots,d$. It follows that $\|Uf\|_{L^q(L^r(I))} \ge \lambda^{1/q-1/r}$. Hence the condition $\alpha \ge 1/q - 1/r$ follows.

iv) The condition $\alpha \ge \frac{1}{q}-\frac{1}{r}$. Let $\lambda \gg 1$ and set $\widehat{h_\lambda}(\eta) = \phi(\eta')\lambda\phi(\lambda(\eta_1 - \lambda))$ with $\phi \in C_c^\infty(\mathbb{R})$.

Then $h_{\lambda}{}_{L^p_\alpha} \lesssim \lambda^{\alpha}\lambda^{1/p}$. Note that

$$Uh_{\lambda}(x,t) = \frac{1}{(2\pi)^d}\int e^{-it|\eta'|^2+i\langle x',\eta'\rangle}\phi(|\eta'|)d\eta' e^{i\lambda^2 t+i\lambda x_1}\int e^{i(-t\xi_1^2-2\lambda t\xi_1+x_1\xi_1)}\lambda\,\phi(\lambda\,\xi_1)d\xi_1,$$

So that $|Uh\lambda\,(x,t)| \ge c > 0$ if $|t|, |x'| \le c$ and $|x_1| \le c\lambda$ for small enough $c > 0$. This shows the necessity of $\alpha \ge 1/q - 1/p$.

To show the conditions (v) and (vi), we use sharp bounds in the construction of Besicovich sets [113] and adapt Fefferman's argument for the disc multiplier [111] (see also [109]).

v) The condition $\alpha \ge \frac{1}{q} - \frac{1}{r}$ if $r > 2$. This follows from

Proposition (4.2.13) [118]: Let $p, q, r \in (2,\infty)$. Let η be a radial C_c^∞ function satisfying $\eta(\xi) = 1$ for $1/4 \le |\xi| \le 12$. Define α_λ by

$$\alpha_{\lambda}(p,q,r) = \sup_{\|f\|_p \le 1}\left\|\left(\int_{1/2}^{1}\left|e^{it\Delta}\eta\left(\frac{D}{\lambda}\right)f\right|^r dt\right)^{1/r}\right\|_{L^q(\mathbb{R}^d)}. \quad (46)$$

Then for $\lambda \gg 1$.

$$\alpha_{\lambda}(p,q,r) \ge c\,\lambda^{1/q-1/p}\,(\log\lambda)^{1/q-1/r}. \quad (47)$$

Proof. In what follows we set

$$A_4(\lambda^2) = \{x : 3\lambda^2 \le |\xi| \le 4\lambda^2\}.$$

By Lemma (4.2.10) wit parameters $\alpha_0 = 3, \alpha_1 = 4$, for $\lambda \gg 1$

$$\sup_{\|f\|_{L^p} \le 1}\left(\int_{A_4(\lambda^2)}\left(\int_{\lambda^2}^{2\lambda^2}\left|\varepsilon f\left(\frac{s}{\lambda^2}\xi, s\right)\right|^r ds\right)^{\frac{q}{r}} d\xi\right)^{\frac{1}{q}} \lesssim \alpha_{\lambda}(p,q,r)\,\lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}.$$

Let

$$Tf(\xi,s) = \varepsilon f\left(\frac{s}{\lambda^2}\xi, s\right).$$

Using Khintchine's inequality we also get

$$\sup_{\|f_j\|_{L_p(\ell^2)} \le 1}\left(\int_{A_4(\lambda^2)}\left(\int_{\lambda^2}^{2\lambda^2}\left(\sum_j |Tf_j|^2\right)^{\frac{r}{2}} ds\right)^{\frac{q}{r}} d\xi\right)^{\frac{1}{q}} \lesssim \alpha_{\lambda}(p,q,r)\,\lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}. \quad (48)$$

For integers $|j| \le \lambda/10$, Let $z^j = (\lambda^{-1} j, 0, \dots, 0)$ in $\mathbb{R}^d$. Let $I_j = \{y : |y - z^j \le (100d\,\lambda)^{-1}|\}$. Let $R_j = \{(\xi,s)\in\mathbb{R}^{d+1} : |\xi - 2j\,\lambda^{-1} s| \le 10^{-1}\lambda, |\xi_i| \le 10^{-1}\lambda, i = 2,\dots,d, |s| \le 100^{-1}\lambda^2\}$.

For a pointwise lower bound we use the following lemma.

Lemma (4.2.14) [118]: Let $\alpha \in \mathbb{R}^d$, $b \in \mathbb{R}$, and $gj(y) = \chi I_j(y)e^{i\langle\alpha,y\rangle - ib|y|^2}$. Then there is a constant c > 0, independent of λ, j so that

$$\operatorname{Re}\left[e^{i\langle\xi-a,z^j\rangle - i(s-b)|z^j|^2}\varepsilon[g_j]\xi(\xi,s)\right] \ge c\,\lambda^{-d}, \text{ if } (\xi,s) \in R_j + (a,b).$$

Proof. Let $I_0 = \{y : |y| \le (100d\,\lambda)^{-1}\}$. We have

$$\varepsilon g_j(\xi,s) = \int e^{is|y|^2 - i\langle\xi,v\rangle}g_j(y)dy = \int e^{-i\langle\xi-a,z^j+h\rangle + i(s-b)|z^j+h|^2}\chi I_j(z^j+h)dh$$

$$= e^{-i\langle\xi-a,z^j\rangle}e^{i(s-b)|z^j|^2}\int e^{-}\chi_{I_0}(h)dh$$

The pointwise lower bound follows quickly.
Let N_λ to be the largest integer which is smaller than $\lambda/10$. By making use of the Besicovich set construction of Keich [113]. There are vectors $v_i \in \mathbb{R}^{d+1}$ such that $v_j = a_j e_1 + b_j e_{d+1}$ for some $a_j, b_j \in \mathbb{R}, v_j + R_j \subset \{(\xi, s) : \lambda^2 \leq s \geq 2\lambda^2\}$, and

$$\text{meas}\left(\bigcup_{j=1}^{N_\lambda}(v_j + R_j)\right) \lesssim \frac{\lambda^{d+3}}{\log \lambda}.$$

This is just obvious extension of the two dimensional construction which gives a collection of rectangles $\{R_j^{|2|}\}$ and vectors a_j, b_j such that $\text{meas}\left(\cup_{j=1}^{N_\lambda}(v_j + R_j)\right) \lesssim \frac{\lambda^4}{\log \lambda}$ and $a_j, b_j + R_j^{|2|}\{\xi_1, s : \lambda^2 \leq s \leq 2\lambda^2\}$.
Let $\Phi(\xi, s) = \left(\frac{s}{\lambda^2}\xi, s\right)$ which is $1-1$ 0n $A_4(\lambda^2) \times [\lambda^2, 2\lambda^2]$, and has Jacobian J_Φ with $|\det(J_\Phi(\xi, s))| \sim 1$. Let

$$v_j := \Phi^{-1}(v_j + R_j) \cap (A_4(\lambda^2) \times [\lambda^2, 2\lambda^2]), \quad E := \bigcup_{j=1,\dots,N_\lambda} v_j.$$

Then it follows that

$$\lambda^{d+2} \lesssim \text{meas}(v_j), \qquad \text{meas}(E) \lesssim \frac{\lambda^{d+3}}{\log \lambda}. \tag{49}$$

Let $f_j(y) = \chi_{I_j}(y) e^{i\langle a_j, y\rangle - ib_j|y|^2}$. Then by Lemma (4.2.14),

$$|Tf_j(\xi)| \lesssim \lambda^{-d}, \qquad \xi \in V_j, \tag{50}$$

And

$$\left\|\left(\sum |f_j|^2\right)^{1/2}\right\|_p \lesssim \lambda^{1-d/p}. \tag{51}$$

We now modify argument in [109]. By (49), we have

$$\lambda^{d+2} \lesssim N_\lambda \lambda^{d+2} \lesssim \sum_{j=1}^{N_\lambda} \text{meas}(v_j) \tag{52}$$

$$= \int_E \sum_{j=1}^{N_\lambda} \chi v_j(\xi, s) ds\, d\xi \lesssim \lambda^{2d} \int_E \sum_{j=1}^{N_\lambda} |Tf_j(\xi, s)|^2 ds\, d\xi,$$

And by application of Hölder's inequality,

$$\lambda^{2d} \int_E \sum_{j=1}^{N_\lambda} |Tf_j(\xi, s)|^2 \lesssim \lambda^{2d} A.B, \tag{53}$$

Where

$$A = \left(\int_{A_4(\lambda^2)} \left(\int_{\lambda^2}^{2\lambda^2} \left(\sum_j |Tf_j(\xi, s)|^2\right)^{\frac{r}{2}} ds\right)^{\frac{2}{r}} d\xi\right)^{\frac{2}{q}},$$

$$B = \left(\int_{A_4(\lambda^2)}\left(\int_{\lambda^2}^{2\lambda^2}\chi_E(\xi,s)ds\right)^{\frac{(q/2)'}{(r/2)'}}d\xi\right)^{1-\frac{2}{q}}.$$

From (48) and (51) we obtain,

$$A \lesssim \left(\lambda^{\frac{1-d}{p}}\mathfrak{v}_\lambda(p;q,r)\,\lambda^{-d+\frac{1}{p}+\frac{d}{q}+\frac{2}{r}}\right)^2. \qquad (54)$$

In order to estimate B we set

$$\mathfrak{v}(\xi) = \int_{\lambda^2}^{2\lambda^2}\chi_E(\xi,s)ds,$$

The measure of the vertical cross section of E at ξ. For $M > 0$. we break

$$B \lesssim \left(\int_{\{\xi\in A_4(\lambda^2)\,:\,\mathfrak{v}(\xi)\le M\}}\mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}}d\xi\right)^{1-\frac{2}{q}} + \left(\int_{\{\xi\in A_4(\lambda^2)\,:\,\mathfrak{v}(\xi)\le M\}}\mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}}d\xi\right)^{1-\frac{2}{q}}.$$

From the construction of E it is obvious that $\mathfrak{v}$ is supported in a tube where $|\xi_1| \le C\lambda^2$ and $|\xi_1| \le C\lambda, 2 \le i \le d$, so that

$$\left(\int_{\{\xi\in A_4(\lambda^2)\,:\,\mathfrak{v}(\xi)\le M\}}\mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}}d\xi\right)^{1-\frac{2}{q}} \lesssim M^{1-\frac{2}{r}}\lambda^{(d+1)\left(1-\frac{2}{q}\right)}.$$

Moreover since $r \le q$ and therefore $(1-\frac{(q/2)'}{(r/2)'}) \ge 0$, by (49)

$$\left(\int_{\{\xi\in A_4(\lambda^2)\,:\,\mathfrak{v}(\xi)\le M\}}\mathfrak{v}(\xi)^{\frac{(q/2)'}{(r/2)'}}d\xi\right)^{1-\frac{2}{q}} \lesssim \int\left(\mathfrak{v}(\xi)M^{\frac{(q/2)'}{(r/2)'}}d\xi\right)^{1-\frac{2}{q}}$$

$$\le M^{\frac{2}{q}-\frac{2}{r}}\mathrm{meas}(E)^{1-\frac{2}{q}} \lesssim M^{\frac{2}{q}-\frac{2}{r}}\left(\frac{\lambda^{d+3}}{\log\lambda}\right)^{1-\frac{2}{q}}.$$

Combining these two bounds, we have

$$B \lesssim M^{2/r}\lambda^{(d+3)\left(1-\frac{2}{q}\right)}\left[M\lambda^{-2\left(1-\frac{2}{q}\right)} + M^{\frac{2}{q}}(\log\lambda)^{\frac{2}{q}-1}\right],$$

And choosing $M = \lambda^2(\log\lambda)^{-1}$, with optimizes the above, we obtain

$$B \lesssim \lambda^{(d+3)(1-\frac{2}{q})}\lambda^{\frac{4}{q}-\frac{4}{r}}(\log\lambda)^{\frac{2}{r}-1}. \qquad (55)$$

Finally, we combine (55), (54), (53) and (52) to obtain

$$\lambda^{(d+3)} \lesssim \lambda^{2d}\lambda^{(d+3)(1-\frac{2}{q})}\lambda^{\frac{4}{q}-\frac{4}{r}}(\log\lambda)^{\frac{2}{r}}\left[\lambda^{\frac{1-d}{p}}\mathfrak{v}_\lambda(p;q,r)\,\lambda^{d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}\right]^2,$$

Which yields $\mathfrak{v}_\lambda(p;q,r) \ge c(\log\lambda)^{\frac{1}{2}-\frac{1}{r}}\lambda^{\frac{1}{p}-\frac{1}{q}}$.

vi) Relation with Bochner – Riesz and the condition $\alpha > 0$ *if* $r = q > 2, d \ge 2$.

The $L^p \to L^p(L^2(I))$ estimate implies sharp results for the Bochner-Riesz multiplier in the same way as the wave equation in [116].

For small $\delta > 0$, let us set $h_\delta(\xi) = \phi(\delta^{-1}(1-|\xi|^2))$ with $\phi \in C_c^\infty(-1,1)$. Let ψ be radial, supported in $\{1/2 < |\xi| < 2\}$ so that $\psi = 1$ on the support of h_δ. Then by the Fourier inversion formula and the support property of ψ it follows that

$$h_\delta(D)f = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta\hat{\phi}(\delta s)e^{is}e^{is\Delta}\,\psi(D)f\,ds.$$

By the Schwarz inequality we get

$$|h_\delta(D)f| \le |\delta\hat{\phi}(\delta s)|ds^{1/2}|e^{is\Delta}\psi(D)f|^2|\delta\hat{\phi}(\delta s)|ds^{1/2}.$$

Thus we see that

$$\|h_\delta\|_{M_p} \lesssim \sup_{\|f\|_p\le 1}\left\|\left(\int |e^{is\Delta}\psi(D)f|^2|\delta\hat{\phi}(\delta s)|ds\right)^{1/2}\right\|_p,$$

which after rescaling becomes

$$\|h_\delta\|_{M_p} \lesssim \sup_{\|f\|_p\le 1}\left\|\left(\int |e^{is\Delta}\psi(\sqrt{\delta}D)f|\hat{\phi}(t)|dt|^2\right)^{1/2}\right\|_p.$$

Hence, using the rapid decay of $\hat{\phi}$ and a further rescaling we see that the sharp bound $\|h_\delta\|_{M_p} \lesssim \delta^{1/2-d(1/2-1/p)}$, for $p > 2+\frac{2}{d-1}$. would follow from $U:\ B^p_{\alpha.v} \to L^p(L^2(I))$, with $\alpha = d\left(1-\frac{2}{p}\right)-1$, for any$v > 0$.

We see that the $L^p(L^2(I))$ inequality for some $p > 2$ would imply that h_δ is a multiplier of $\mathcal{F}L^p$ with bounds independent of δ. However a variant of Fefferman's argument for the ball multiplier [111]. Based on a Kakeya set argument, shows that

$$\|h_\delta\|_{M_p} \lesssim \log(1/\delta)^{1/2-1/p}. \tag{56}$$

This establishes the final necessary condition (vi) in Proposition (4.2.12) For completeness we include some details of the argument.

Proof of (56). By de Leeuw's theorem it suffices to prove the lower bound for $d = 2$. We may assume that $\delta < 10^{-10}$. By Khintchine's inequality, we have

$$\left\|\left(\sum_v |h_\delta(D)f_v|^2\right)^{1/2}\right\|_p \lesssim \|h_\delta\|_{M_p}\left\|\left(\sum_v |f_v|^2\right)^{1/2}\right\|_p. \tag{57}$$

For $v \in \mathbb{Z}\ \cap[-10^{-2}\delta^{-1/2}, 10^{-2}\delta^{-1/2}]$, let us set

$$h_{\delta.v}(\xi) = h_\delta(\xi)\phi(\delta^{-1/2}\xi_1 - v), \qquad \xi = (\xi_1,\xi_2)\in\mathbb{R}^2$$

Where χ_+ is the characteristic function of the upper half plane. Define T_v by$\widehat{T_v f} = h_{\delta.v}\hat{f}$. Let η_v be the inverse Fourier transform of a bump function which is supported on a half of radius $C\delta^{-1/2}$ so that $\eta_v(\xi) = 1$ for ξ in the support of $h_{\delta.v}$. Define ϕ_v by$\hat{\phi}_v(\xi) = \eta_v(\xi)\phi(\delta^{-1/2}\xi_1 - v)\chi_+(\xi)$. Then $|\Phi_v(x)| \lesssim \delta^{-d/2}(1+\delta^{-1/2}|x|)^{-(d+1)}$ for the v's under consideration, so that $\|\{\Phi_v * g_v\}\|_{L^p(\ell^2)} \lesssim \|\{g_v\}\|_{L^p(\ell^2)}$. Since $T_v g = h_\delta(D)[\Phi_v * g]$, inequality (57) applied to $f_v = \Phi_v * g_v$ implies that

$$\left\|\left(\sum_v |T_v g_v|^2\right)^{1/2}\right\|_p \lesssim \|h_\delta\|_{M_p}\left\|\left(\sum_v |g_v|^2\right)^{1/2}\right\|_p. \tag{58}$$

Let $\theta_v = (\delta^{1/2}v, \sqrt{1-\delta v^2},)$ let $\theta_v^{\perp}$ be a unit vector perpendicular to θ_v and

$$R_v = \left\{(x_1,x_2):\ |\langle x,\theta_v\rangle| \le 10^{-2}\delta^{-1}, |\langle x,\theta_v^{\perp}\rangle| \le 10^{-1}\delta^{-1/2}\right\}.$$

Letting $f_v(y) = \chi_{R_v}(y)e^{\langle\theta_v\cdot y\rangle}$, we have that

$$|e^{-i(x.\theta_v)}T_v g_v(x)| \ge c > 0 \text{ for } x \in R_v. \tag{59}$$

Here we use again sharp bounds in the construction of Besicovich sets [113]. There are vectors $a_v, |v| \le 10^{-2}\delta^{-1/2}$ so that with $E := U_v R_v$ the measure of E is $O(\delta^{-2}/\log\delta^{-1})$ but the corresponding translation $a_v + R_v$ have $O(1)$ overlap. Define $g_v(x) = f_v(x - a_v)$, which is supported in $a_v + R_v$. Then $|T_v g_v| \geq c$ on $a_v + R_v$. Thus we get

$$\delta^{-2} \lesssim \sum_v |R_v| \lesssim \sum_v \int \chi_{a_v+R_v}(x)dx \lesssim \sum_v |T_v g_v|^2 dx$$

And also by Hölder's inequality and (58) the last one in the above string of inequalities is bounded by

$$\text{meas}(E)^{1-2/p} \left\| \left(\sum |T_v g_v|^2 \right)^{1/2} \right\|_p^2 \lesssim \|h_\delta\|_{M_p}^2 \left(\frac{\delta^{-2}}{\log \delta^{-1}} \right)^{1-2/p} \left\| \left(\sum_v |g_v|^2 \right)^{1/2} \right\|_p^2.$$

Now by the bounded overlap of the translated rectangles $a_v + R_v$, we see

$$\left\| \left(\sum_v |g_v|^2 \right)^{1/2} \right\|_p^2 \lesssim \left(\int \sum_v \chi_{a_v+R_v} dx \right)^{2/p} \lesssim \left(\sum_v |R_v| \right)^{2/p} \lesssim \delta^{-4/p}.$$

Combining the three displayed inequalities we get $\delta^{-2} \lesssim \|h_\delta\|_{M_p}^2 (\delta^{-2}/\log \delta^{-1})^{1-2/p} \delta^{-4/p}$ and thus the desired (55).

Theorem (4.2.15) [118]: For large λ, let

$\mathfrak{A}_\lambda(p;q,r) = \sup \{\|Uf\|_{L^q(\mathbb{R};L^r(I))} : \|f\|_p \leq 1, \text{supp}\, \hat{f} \subset \{\xi : \lambda/5 \leq |\xi| \leq 15\lambda\}\}$.

Then for $\lambda \geq 1$, the following norm equivalences hold.

(i) For $2 \geq r \leq p \leq q \leq \infty$,

$$\mathfrak{A}_\lambda(p;q,r) \approx \begin{cases} \lambda^{1/q-1/p} [\log \lambda]^{1/r-1/2} \text{ if } \frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}, \\ \lambda^{1-1/p-1/q-2/r} \text{ if } \frac{1}{q} + \frac{1}{r} < \frac{1}{2}. \end{cases}$$

(ii) For $2 \geq p \leq r \leq q \leq \infty$,

$$\mathfrak{A}_\lambda(p;q,r) \approx \begin{cases} \lambda^{1/q-1/p} \text{ if } \frac{2}{q} + \frac{1}{r} \geq 1 - \frac{1}{p}, \\ \lambda^{1-1/p-1/q-2/r} \text{ if } \frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p}. \end{cases}$$

One can obtain sharp estimates for functions in Sobolev and Besov spaces. In order to compare such results recall that $B^p_{\alpha q_1} \subset B^p_{\alpha . q_2}$ for $q_1 < q_2$, that $B^p_{\alpha . 2} \subset B^p_\alpha \subset B^p_{\alpha . p}$ when $p \geq 2$, and that $B^p_{\alpha . p}$ is the same as the Sobolev-Slobodecki space $W^{\alpha . p}$ when $0 < \alpha < 1$.

Proof. The lower bounds for $\mathfrak{A}_\lambda(p;q,r)$ were established in the previous. And here we prove the upper bounds. Mainly by interpolation arguments. By Lemma (4.2.11), we can take $I = [1/2 . 1]$. We consider the cases $\frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}$ and $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$ separately.

The case $\frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}$. Note that the set

$$\left\{ \left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r} \right) : 2 \leq r \leq p \leq q \leq \infty, \frac{1}{q} + \frac{1}{r} \geq \frac{1}{2} \right\}$$

Is closed tetrahedron with vertices $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, and $\left(0, 0, \frac{1}{2}\right)$. Hence by interpolation it is enough to show the estimate

$$\mathfrak{A}_\lambda(p;q,r) \lesssim \lambda^{\frac{1}{q}-\frac{1}{p}} [\log \lambda]^{\frac{1}{2}-\frac{1}{r}} \quad (60)$$

For $(p;q,r) = (4,4,4), (2,2,2), (2,\infty,2)$ and $(\infty,\infty,2)$. The estimate for $(p;q,r) = (2,2,2)$ is immediate from Plancherel's theorem. More generally we recall from [114] the estimate

$\mathfrak{A}_\lambda(p;q,r) \lesssim 1$ with $2 \le p \le \infty$, which is related to a sqare-function estimate for equally spaced intervals. So we also get the estimates for $(p;q,r) = (\infty,\infty,2)$. For $(2,\infty,2)$ we choose a nonnegative $\chi_0 \in C_c^\infty(\mathbb{R})$, so that $\chi_0(t) = 1$ on $[1/2, 1]$. We need to estimate, for fixed x,

$$\int \chi_0(t)\left|U\eta\left(\frac{D}{\lambda}\right) f(x,t)\right|^2 dt = \frac{1}{(2\pi)^{2d}} \iint e^{ix(\xi-w)}\hat{f}(\xi)\overline{\hat{f}(w)}\eta\left(\frac{\xi}{\lambda}\right)\overline{\eta\left(\frac{w}{\lambda}\right)}\widehat{\chi_0}(|\xi|^2-|w|^2)d\xi dw$$

And since $|\xi| + |w| \ge \lambda$, the above is bounded by

$$C_N \iint \left(1+\lambda\,\big||\xi|-|w|\big|\right)^{-N}|\hat{f}(\xi)||\hat{f}(w)|dw\, d\xi \lesssim \lambda^{-1}\|f\|_2^2.$$

This is gives the desired estimate for $(p,q,r) = (2,\infty,2)$.For $(p,q,r) = (4,4,4)$ we use the bound

$$\left(\iint\left|\psi(\xi,s)\int_{|y|\le 1} f(y)e^{i\lambda(s|y|^2-\xi y)}f(y)\,dy\right|^4 d\xi\, ds\right)^{1/4} \lesssim \lambda^{-\frac{1}{2}}(\log\lambda)^{\frac{1}{4}}\|f\|_4.$$

Where $\psi \in C_c^\infty$. This is implicit in [100] (see also [117] for more discussion and related issues).
The by rescaling, Lemma (4.2.9) and Lemma (4.2.11) we get (60) for $(p,q,r) = (4,4,4)$.
The case $\frac{1}{q}+\frac{1}{r}<\frac{1}{2}$. We begin as before by observing that the set

$$\Delta_1 = \left\{\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right) : 2 \le r \le p \le q \le \infty, \frac{1}{q}+\frac{1}{r} \ge \frac{1}{2}\right\}$$

Is closed tetrahedron with vertices (0, 0, 0), $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{2},0,\frac{1}{2}\right)$ and $\left(0,0,\frac{1}{2}\right)$ and $\left(0,0,\frac{1}{2}\right)$, from which the triangle with vertices $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{2},0,\frac{1}{2}\right)$ and $\left(0,0,\frac{1}{2}\right)$ is removed. We use a bilinear analogue of our adjoint restriction operator, and rely on rather elementary estimates from [100]. Define χ_ℓ so that $\sum_{\ell\in\mathbb{Z}}\chi_\ell \equiv 1, \chi_\ell = \chi_1(2^\ell .)$ and χ_1 is supported in (33). Let

$$\mathfrak{B}_{\lambda.\ell}[f,g] = \iint_{[-1,1]^2} e^{is(|y|^2+|z|^2)-i\frac{s}{\lambda^2}\xi(y+z)}\chi_\ell(|y-z|)\,f(y)g(z)d\,ydz,$$

So that

$$(\mathcal{E}f\mathcal{E}f)\left(\frac{s}{\lambda^2}\xi,s\right) = \sum_{\ell\ge 0}\mathfrak{B}_{\lambda.\ell}(f,f)(\xi,s).$$

We shall verify that for $\ell \ge 0$

$$\|\mathfrak{B}_{\lambda.\ell}(f,g)\|_{L^{q/2}(A(\lambda^2):L^{r/2}[\lambda^2.2\lambda^2])} \lesssim 2^{-2\ell\left(\frac{1}{2}-\frac{1}{q}-\frac{1}{r}\right)}\|f\|_p\|g\|_p \qquad (61)$$

When $\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right)$ is contained in the closed tetrahedron with vertices (0, 0, 0), $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{2},0,\frac{1}{2}\right)$ and $\left(0,0,\frac{1}{2}\right)$. By summing a geometric series, this yields (61)
For $\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right) \in \Delta_1$ which by Lemmata (4.2.9) and (4.2.11)yields the desired

$$\mathfrak{A}_\lambda(p,q,r) \lesssim \lambda^{1-\frac{1}{p}-\frac{1}{q}-\frac{2}{r}}. \qquad (62)$$

We remark that conversely, if (62) holds, then we can use Lemma (4.2.10) and a Fourier expansion of $\chi_\ell(y-z)$ to bound the left hand side of (61) by$C\|f\|_p\|g\|_p$. with C independent of ℓ.
It remains to show (61). By interpolation it is enough to do this with $(p,q,r) = (\infty,\infty,\infty),(2,\infty,2)$ The last two estimates were already obtained; not that the bounds (60) and (62) coincide for the cases $(p,q,r) = (2,\infty,2)$ and $(\infty,\infty,2)$ and the bounds for (61) are independent of ℓ. Hence from the bounds (60) previously obtained and the discussion above we have the required bounds for $(p,q,r) = (2,\infty,2)$ and $(\infty,\infty,2)$. We note that the argument of the poof of the endpoint adjoint restriction theorem in [100] gives

$$\|B_{\lambda\ell}(f,g)\|_{L^2_{\xi,s}} \lesssim \|f\|_4\|g\|_4. \qquad (63)$$

Uniformly in $\ell \geq 0$, where $B_{\lambda\ell}(f,g)(\xi,s) = \mathfrak{B}(f,g)\left(\frac{\lambda^2}{s}\xi, s\right)$, and by a change of variables we obtain (61) holds with $(p,q,r) = (4,4,4)$. To get the inequality (61) for $(p,q,r) = (\infty,\infty,\infty)$ we need to integrate $\lambda\ \ell(|y-z|)$ over $[-1,1]^2$ which yields the gain of $2^{-\ell}$.

We also consider the cases $1-\frac{1}{p} \leq \frac{2}{q} + \frac{1}{r}$. We note that the set

$$\Delta_2 = \left\{\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right) : 2 \leq p < r \leq q \leq \infty, \frac{2}{q} + \frac{1}{r} \geq 1 - \frac{1}{p}\right\}$$

Is the closed tetrahedron with vertices $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right), \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)\left(\frac{1}{2},\ \frac{1}{6},\ \frac{1}{6}\right)$ and $\left(\frac{1}{2},0,\frac{1}{2}\right)$, from which the face with vertices $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right), \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$ and $\left(\frac{1}{2},0,\frac{1}{2}\right)$ is removed. Note that from the previous bounds (60) and (62) we already have the required bounds

$$\mathfrak{A}_{\lambda}(p,q,r) \lesssim \lambda^{\frac{1}{q}-\frac{1}{r}} \qquad (64)$$

For $(p,q,r) = (2,2,2)$ and $(2,\infty,2)$. ObviouslyΔ_2 is contained in the convex hull of $\left(\frac{1}{2},0,\frac{1}{2}\right), \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$, and the half open line segment $\left[\left(\frac{1}{2},\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right)$. Hence by it is enough to show (64) for $\frac{1}{p},\frac{1}{q},\frac{1}{r}$ containe in the half closed line segment $\left(\frac{1}{2},\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\Big)$ but these follow from Lemmata (4.2.9) and (4.2.11) combined with restriction estimate for the parabola which gives (29) for$\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right) \in \left[\left(\frac{1}{2},\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right)$.

The case $1-\frac{1}{p}+\frac{1}{r}$. We note that the set

$$\left\{\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right) : \in 2 \leq p < r \leq q \leq \infty, \frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p}\right\}$$

Is contained in the equatrangular pyramid Q with vertices $(0,0,0), \left(\frac{1}{2},0,0\right), \left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\left(\frac{1}{2},\frac{1}{6},\frac{1}{6}\right)$, and $\left(\frac{1}{2},0,\frac{1}{2}\right)$. We need to show (62) for $\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right)$ contained in the above set. Repeating the above argument, the asserted estimates follows if we establish, for $\ell \geq 0$ and $\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right) \in Q$.

$$\|\mathfrak{B}_{\lambda.\ell}(f,g)\|_{L^{q/2}(A(\lambda^2):L^{r/2}[\lambda^2.2\lambda^2])} \lesssim 2^{-2\ell\left(\frac{1}{2}-\frac{1}{q}-\frac{1}{r}\right)}\|f\|_p\|g\|_p \qquad (65)$$

We only need to verify it for $(p,q,r) = (\infty,\infty,\infty), (4,4,4), (2,\infty,2), (2,6,6)$, and $(2,\infty,\infty)$.

The first three cases were already obtained when we showed (61), and the case $(p,q,r) = (2,6,6)$ follows from the linear adjoint restriction estimate for the parabola as before. Finally the case $(p,q,r) = (2,\infty,\infty)$ wit a gain of $2^{-\ell/2}$ follows from the Schwarz inequality, and so we are done.

One can use the uniform regularity results for the frequency localized pieces to prove sharper bounds such as Sobolev estimates by using argument based on the Fefferman-Stein #-function supported in $\{\xi : 1/4 < |\xi| < 4\}$, not identically 0. Let $I = [-1,1]$and

$$\Gamma(p,q,r) = \sup_{\lambda > 1} \lambda^{-d\left(-\frac{1}{p}-\frac{1}{q}\right)+\frac{2}{r}} \left\|U\varphi\left(\frac{D}{\lambda}\right)\right\|_{L^P \to L^q\left(\mathbb{R}^d : L^r(I)\right)} \qquad (66)$$

It is not hard to verify that the finiteness of $\Gamma(p,q,r)$ is independent of the particular choice of φ. The following statement is a special case of the result in [114].

Proposition (4.2.16) [118]: Let$p_0, q_0, r_0 \in [1,\infty], q \in (q_0,\infty), r_0 \leq r < \infty, p_0 \leq q_0$ and assume $1/p_0 - 1/q_0 = 1/p - 1/q$, suppose that $\Gamma(p_0; q_0, r_0) < \infty$. Then

$$\left\|\left(\int_1 |Uf(\cdot,t)|^r dt\right)^{1/r}\right\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{B^p_{a,q}(\mathbb{R}^d)}, a = d\left(1-\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}.$$

The Sobolev estimates follow from this since for$q \geq p \geq 2$ one has $L^p_a \subset B^p_{a,p} \subset B^p_{a,q}$.

We note that the result in [16] is slightly sharper. Namely the left hand side can be replaced by the $L^q(\mathbb{R}^d)$ norm of $\left(\sum_{k>0}(f_t|Uf(\cdot,t)|^r dt)^{v/r}\right)^{1/v}$, where $v > 0$.

Proposition (4.2.17) [118]: Suppose that $R^*(q_0 \to q_0)$ holds for some $q_0 \in \left(2, \frac{2(d+3)}{d+1}\right)$. Then

(i)$R^*(p \to q)$ holds $q = \frac{d+2}{d}p'$ provided that

$$q > q_{*:=2}\ \frac{2(d+3)}{d+1}(1-\Upsilon(d,q_0)), \text{where } \Upsilon(d,q_0) = \frac{\frac{1}{q_0}-\frac{d+1}{2(d+3)}}{\frac{d+1}{2d}-\frac{d+2}{dq_0}}.$$

(ii) Let $q_* < q < \infty, q \leq \infty$ *and suppose that* $0 \leq \frac{1}{p}-\frac{1}{q} < 1-\frac{2(d+3)}{dq_*}$.

Then $U: L^p_\alpha(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded with $\alpha = d\left(1-\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}$.

In two dimensions $R^*(p \to q)$ was proven in [3] for $q > 33/10$ and the sharp inequality$R^*(p \to q)$ for $q > 63/19$.

Proof. By Theorem (4.2.8) and Proposition (4.2.16) it suffices to prove the first part.

Let E_1 and E_2 be $1/2$-separated sets in the unit ball of $\mathbb{R}^d$ and define $\varepsilon_i f = \varepsilon[f\chi_{E_1}]$. By Theorem 2.2 in [105], suffices to prove the estimate

$$\|\mathcal{E}_1 f_1 \mathcal{E}_2 f_2\|_{q/2} \lesssim \|f_1\|_p \|f_2\|_p \quad (67)$$

For $q > q_*$ and p in a neighborhood of $\frac{dg}{dq-d-2}$ (i.e. the p which satisfies $q = \frac{d+2}{d}p'$).

By hypothesis and Hölder's inequality, (67) holds with $p \geq q = q_0$. with $p \geq 2$ and $q/2 > \frac{d+3}{d+1}$. The theorem then follows by interpolation of bilinear operators. Indeed, we determine $\theta \in (0,1)$ and $q_* \in \left(q_0, \frac{2(d+3)}{d+1}\right)$ by

$$\frac{1-\theta}{2}+\frac{\theta}{q_0} = 1-\frac{d+2}{dq_*}, \quad (1-\theta)\frac{d+1}{d+3}+\theta\frac{2}{q_0} = \frac{2}{q_*}.$$

We compute $\theta = \left(\frac{d+2}{dq_*}-\frac{1}{2}\right)/\left(\frac{1}{2}-\frac{1}{q_0}\right)$ and $\theta = \left(\frac{1}{q_*}-\frac{d+1}{2(d+3)}\right)/\left(\frac{1}{q_*}-\frac{d+1}{2(d+3)}\right)$, from which we obtain $1/q_* = \left(\frac{d+1}{2(d+3)}-\frac{b}{2}\right)/\left(1-\frac{d+2}{d}b\right)$ with $b = \left(\frac{1}{q_0}-\frac{d+1}{2(d+3)}\right)/\left(\frac{1}{2}-\frac{1}{q_0}\right)$. A further computation shows that q_* is equal to $\frac{2(d+3)}{d+1}(1-\Upsilon(d,q_0))$ as in the statement of the Lemma.

Definition (4.2.18) [118]: Fix $d \geq 1$, and let $p, q, r \in [2, \infty]$. for $N > 1$, let

$$A_{p,q,r}(N,p) \equiv A_{p,q,r}(N,p,d) = \sup\|Uf_1Uf_2\|_{L^{q/2}(\mathbb{R}^d, L^{r/2}[0.p])}$$

Where the supremum is taken over all pairs of function (f_1, f_2) whose Fourier transforms are supported in 1-separated subsets of $\{\xi : |\xi - N_{e_1}| \leq 2d\}$, and which satisfy$\|f\|_p, \|f_2\|_p \leq 1$.

We remark that the unit vector e_1 does not play a special role here. It could replace by any unit vector, by rotational invariance.

By considering two bump functions, it is easy to calculate that

$$A_{p,q,r}(N,p) \gtrsim N^{\frac{2}{q}-\frac{2}{r}}, \quad 1 \leq p,q,r \leq \infty, \quad (68)$$

$$\sup_{p>1} A_{p,q,r}(N,p) \lesssim N^{\frac{2}{q}-\frac{2}{r}}, \quad q > 16/5, \quad r \geq 4, \quad (69)$$

Which was proven in [115] (see also [11] and [46]). We will combine this with following two lemmata.

Corollary (4.2.19) [118]: Let $2 \le p \le q \le r \le \frac{2d}{q-2}$. Suppose that

$$\sup_{p>1} A_{p,q,r}(N,p) \lesssim N^{\gamma}, \qquad \text{for some } \gamma < 2d\left(1-\frac{1}{p}-\frac{1}{q}\right)-4. \tag{70}$$

Then if $d\left(1-\frac{1}{p}-\frac{1}{q}\right) \ge 0$, then for all $\lambda > 1$,

$$\left\|U\psi\left(\frac{D}{\lambda}\right)f\right\|_{L^q(\mathbb{R}^d:L^r[0,1])} \lesssim \lambda^{d\left(1-\frac{1}{p}-\frac{1}{q}\right)-\frac{2}{r}} \|f\|_p. \tag{71}$$

Supposing this for the moment we give the

Theorem (4.2.20) [118]: Let $\frac{16}{5} < p < \infty$ and $4 \le r \le \infty$.

Then $U: B^p_{\alpha,p}(\mathbb{R}^2) \to L^p(\mathbb{R}^2; L^r(I))$ is bounded with $\alpha = 2\left(1-\frac{2}{p}\right)-\frac{2}{r}$.

The r-range can be further improved for $16/5 < p < 4$, by interpolating with above mentioned $L^p(L^p(I))$ bounds for $p > 33/10$ (|3|) and the $L^p(L^2(I))$ bounds in [114] for $p > 4$. Moreover one can intermediate $L^p_\alpha \to L^q(L^r(I))$ bounds with critical α by interpolating with the $L^2 \to L^q(L^r)$ bounds in [115].

One can also interpolate with best known $L^2(\mathbb{R}^2)$ estimates for the maximal operator $f \mapsto \sup_{t\in I}|Uf(.,t)|$, which are equivalent to the best known local estimates (see [34, 59]).

Proof. By Proposition (4.2.16) it suffices to prove, in two spatial dimensions, the estimate (71) for $p = q > 16/5$ and $r \ge 4$. Using (69), we put $\gamma = 2/q - 2/r$ and verify that the condition (70) with $d = 2$ in the range $p = q > 16/5$ and $r \ge 4$. Thus (71) holds in this range, and we are done.

Lemma (4.2.21) [118]: Let $p_0 \le p \le q \le r$ and $\varepsilon_0 > 0$. *Then, for* $N, p > 1$,

$$A_{p,q,r}(N,p) \lesssim N^{\varepsilon_0} p^{2d} A_{p_0,q,r}(N,p). \tag{71}$$

Proof. Let η_1, η_2 be smooth in balls of diameter 1/2 which are contained in $\{\xi : |\xi - N_{e_1}| \le 2d\}$, and which are separated by 1/2. Define the operators s_1, s_2 by $\widehat{s_i f}(\xi,t) = \eta_i(\xi)\widehat{Uf}(\xi), i = 1,2$. it suffices to prove that $\|S_1f_1S_2f_2\|_{L^{q/2}(\mathbb{R}^d:L^{r/2}[0,p])}$ is dominated by $\|f_1\|_p\|f_2\|_p$ times a constant multiple of the expression on the right hand side of (71).

We partition $\mathbb{R}^d$ into cubes $\mathcal{Q}_v$ of side p with centre $pv \in p\mathbb{Z}^d$, and define

$$\{p_v = (x,t) \in \mathbb{R}^d \times [o,p] : x - 2tNe_1 \in \mathcal{Q}_v\}. \tag{72}$$

The parallelipipeds form a partition of $\mathbb{R}^d \times [o,p]$. For fixed x the intervals $I^x_v = \{t : (x,t) \in p_v\}$ are disjoint. Thus

$$\|F\|^{q/2}_{L^{q/2}(\mathbb{R}^d:\,L^{r/2}[0,p])} \le \int_{\mathbb{R}^d}\left(\sum_v |F(x,t)|^{r/2}dt\right)^{q/r} dx \le \sum_v \|\chi p_v F\|^{q/2}_{L^{q/2}(\mathbb{R}^d;[0,p])};$$

Here we used the triangle inequality for $\|\cdot\|^{q/r}_{\ell q/r}$ as $q/r \le 1$.

Taking $F = S_1f_1S_2f_2$, and denoting by $\mathcal{Q}^*_v$, the enlarged cube with side $50dpN^{\varepsilon}$, where $0 < \varepsilon < 4d\varepsilon_0$, we obtain

$$\|S_1f_1S_2f_2\|^{q/2}_{L^{q/2}(\mathbb{R}^d:\,L^{r/2}[0,p])} \le \sum_v \|\chi p_v S_1f_1S_2f_2\|^{q/2}_{L^{q/2}(\mathbb{R}^d:\,L^{r/2}[0,p])}$$
$$\lesssim \sum_v \left(I^{q/2}_v + II^{q/2}_v + III^{q/2}_v IV^{q/2}_v\right),$$

Where

$$I_v = \|\chi p_v S_1[f_1\chi_{\mathcal{Q}_v^*}]S_2[f_2\chi_{\mathcal{Q}_v^*}]\|_{L^{q/2}(\mathbb{R}^d:\, L^{r/2}[0,p])},$$
$$II_v = \left\|\chi p_v S_1\left[f_1\chi_{\mathbb{R}^d\setminus\mathcal{Q}_v^*}\right]S_2[f_2\chi_{\mathcal{Q}_v^*}]\right\|_{L^{q/2}(\mathbb{R}^d:\, L^{r/2}[0,p])},$$
$$III_v = \left\|\chi p_v S_1[f_1\chi_{\mathcal{Q}_v^*}]S_2\left[f_2\chi_{\mathbb{R}^d\setminus\mathcal{Q}_v^*}\right]\right\|_{L^{q/2}(\mathbb{R}^d:\, L^{r/2}[0,p])},$$
$$IV_v = \left\|\chi p_v S_1\left[f_1\chi_{\mathbb{R}^d\setminus\mathcal{Q}_v^*}\right]S_2\left[f_2\chi_{\mathbb{R}^d\setminus\mathcal{Q}_v^*}\right]\right\|_{L^{q/2}(\mathbb{R}^d:\, L^{r/2}[0,p])}, \qquad (73)$$

First we consider the main terms I_v. By Hölder's inequality,

$$I_v \le A_{p_0,q,r}(N,p)\prod_{i=1}^{2}\|f_i\chi_{\mathcal{Q}_v^*}\|_{p_0} \lesssim A_{p_0,q,r}(N,p)(pN^{\varepsilon})^{2d(\frac{1}{p_0}-\frac{1}{p})}\prod_{i=1}^{2}\|f_i\chi_{\mathcal{Q}_v^*}\|_p$$

We use the Schwarz inequality, the embedding $\ell^p \subset \ell^q, p \le q$, and the fact that every$x$ is contained in only$O(N^{ed})$ of the cubes $\mathcal{Q}_v^*$ to get

$$\sum_v\prod_{i=1}^{2}\|f_i\chi_{\mathcal{Q}_v^*}\|_p^{q/2} \le \prod_{i=1}^{2}\left(\|f_i\chi_{\mathcal{Q}_v^*}\|_p^q\right)^{1/2} \lesssim N^{ed}\prod_{i=1}^{2}\|f_i\|_p^q.$$

Combining the previous two estimates we bound

$$(\sum_v I_v^{q/2})^{2/q} \lesssim N^{2de\left(\frac{1}{p_0}-\frac{1}{p}+\frac{1}{q}\right)}p^{2d(\frac{1}{p_0}-\frac{1}{p})}(N,p)\prod_{i=1}^{2}\|f_i\|_p. \qquad (74)$$

We use very crude estimates to handle the remaining three terms which can to be dominate by$C_{M,\varepsilon}(N^e p)^{-M}\|f_1\|_p\|f_2\|_p$. which finishes the proof since

$A_{p_0,q,r}(N,p) \gtrsim N^{\frac{2}{q}-\frac{2}{r}}$ By: (68)

We only give the argument to bound $\sum_v II_v^{q/2}$ as the other terms are handled similarly by the Schwarz inequality we estimate $\sum_v II_v^{q/2}$ by

$$\left(\sum_v\left\|\chi p_v S_1\left[f_1\chi_{\mathbb{R}^d\setminus\mathcal{Q}_v^*}\right]\right\|_{L^{q/2}(\mathbb{R}^d:\, L^{r/2}[0,p])}^q\right)^{1/2}\left(\sum_v\|S_1[f_2\chi_{\mathcal{Q}_v^*}]\|_{L^{q/2}(\mathbb{R}^d\,;\,[0,p])}^q\right)^{1/2} \qquad (75)$$

For the second factor we use a wasteful bound, namely that the $L^p \to L^q(\mathbb{R}^d\,;\,[0,p])$ Operator norm of S_2 is $O(p^{1/r}N^d)$. consequently, the second factor in (75) can be bounded $C_{p^{q/2r}N^{d(\varepsilon+q/2)}}\|f_2\|_p^{q/2}$.

We consider the first factor in (75) and write $S_1f(x,t) = K_tf(x)$wherewith $\chi \in C_c^\infty$ equal to one in the ball of radius $2d$ centered at the origin. Integration by parts yields that for every$t \in [0,p]$

$$|K_t(y)| \le C_M|y-2tNe_1|^{-M} \text{ if } |y-2tNe_1| \ge 4d_p.$$

Let c_v be the center of $\mathcal{Q}_v^*$. If $x- y \in \mathbb{R}^d\setminus\mathcal{Q}_v^*$ and $(x,t) \in p_v$, then $|x- y-c_v| \ge 10d_pN^{\varepsilon}, |x-2tNe_1-c_v| \le 2d_pN^d$. and therefore also $|y-2tNe_1| \ge 8d_pN^{\varepsilon}$. thus for this choice of (x,t) and y we have

$$\left|S_1\left[f_1\chi_{\mathbb{R}^d\setminus\mathcal{Q}_v^*}\right]\right| \lesssim (pN^{\varepsilon})^{-M+d+1}\int_{|y-2tNe_1|\ge 8d_pN^{\varepsilon}}\frac{|f_1(x-y)|}{|y-2tNe_1|^{d+1}}dy$$

And the integral is bounded by$(pN)^{d+1}\int(1+|y|)^{-d-1}|f_1(x-y)|\,dy$. Here we use $p > 1$.

Now Let $\mathcal{Q}_v^{**}$Be the cube of sidelength $p(2+N)$ centered at c_v; $\mathcal{Q}_v^{**}\times[0,p]$contains p_v. Letting

$C_{p,N} := p^{1/r}(pN^{\varepsilon})^{-M+d+1}(pN)^{d+1}$, we have

$$\sum_{v} \left\| \chi\, p_v\, S_1\left[f_1 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_v^*}\right]\right\|_{L^{q/2}(\mathbb{R}^d :\, L^{r/2}[0,p])}^{q} \sum_{v} \int_{\mathcal{Q}_v^{**}} \left| \int \frac{|f_1(x-y)|}{(1+|y|)^{d+1}} dy \right|^{q} dx$$

Which is $\lesssim C_{p,N}^{q}(pN)^{d+1}\|f_1\|_p^q$; here one uses young's inequality and the fact that each $x \in \mathbb{R}^d$ is contained in at most $O\big((pN)^{d+1}\big)$ of the cubes $\mathcal{Q}_v^{**}$. collecting the estimates yields the crude bound

$$\sum_{v} II_v^{q/2} \leq C_M (pN^{\varepsilon})^{-M}(pN)^{q/2}\|f_1\|_p^{q/2}\|f_2\|_p^{q/2}$$

And we conclude by choosing M sufficiently large.

Lemma (4.2.22) [118]: Let $2 \leq p \leq q \leq r \leq \frac{2q}{q-2}$ and $\varepsilon > 0$. Let $\psi \in C_c^{\infty}$ be supported in the annuls $\{\xi \in \mathbb{R}^d :\ 1/2 \leq |\xi| \leq 2\}$. Then, for $\lambda > 1$,

$$\left\| U\psi\left(\frac{D}{\lambda}\right) f \right\|_{L^q(\mathbb{R}^d : L^r[0,1])}$$

$$\lesssim \left(\lambda^{\frac{4}{q}-2d\left(\frac{1}{p}-\frac{1}{q}\right)} + \sup_{1 < N < \lambda} N^{\frac{4}{r}-2d\left(\frac{1}{p}-\frac{1}{q}\right)+\varepsilon} A_{p,q,r}(N, C\lambda^2/N^2) \right)^{1/2} \lambda^{-\frac{2}{r}+d} \|f\|_p. \quad (76)$$

Lemma (4.2.21) realize on localization argument such as in [34] and Lemma (4.2.22) relies on a by now standard scaling argument in [105] which reduces estimates for bilinear operators with separation assumptions to estimates for linear operators.

We may combine (71), with $p_0 = 2$, and (76) to obtain

Proof. for $j \geq 0$, we write $A(j,\lambda) := 2^{2j\left(\frac{2}{r}-d\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \sup_{2^{j-1} \leq N \leq 2^{j+1}} A_{p_0,q,r}\big(N, C\lambda^2\, 2^{-2j+1}\big)$.

Define $T = U\psi(D)$, and thus $U\psi(\frac{D}{\lambda})f(x,t) = T[f(\lambda^{-1}\cdot)](\lambda x, \lambda^2 t)$. By scaling.

$$\left\| U\psi\left(\frac{D}{\lambda}\right)\right\|_{L^p \to L^q(\mathbb{R}^d :\, L^r[0,p])} = \lambda^{-2+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|T\|_{L^p \to L^q(\mathbb{R}^d :\, L^r[0,\lambda^2])}. \quad (77)$$

So that the statement of the lemma is an immediate consequence of

$$\|T\|_{L^p \to L^q(\mathbb{R}^d :\, L^r[0,\lambda^2])} \lesssim \left(\lambda^{\frac{4}{q}-2d\left(\frac{1}{p}-\frac{1}{q}\right)} + \sum_{1 \leq 2j \leq \lambda} A(j,\lambda) \right)^{1/2}. \quad (78)$$

Now by scaling we have that

$$\|Tf_1\, Tf_2\|_{L^{q/2}(\mathbb{R}^d :\, L^{r/2}[0,\lambda^2])} \lesssim A(j,\lambda), \prod_{i=1}^{2} \|f_1\|_{p,} \quad (79)$$

Whenever $\widehat{f_1}$ and $\widehat{f_2}$ are supported in a 2^{-j+1} ball, contained in $\{\xi :< |\xi| \leq 2\}$, and their supports are 2^{-j} separated. We will also require the following simpler estimates

$$\|Tf_1\, Tf_2\|_{L^{q/2}(\mathbb{R}^d :\, L^{r/2}[0,\lambda^2])} \lesssim \lambda^{\frac{4}{q}-2d\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{2} \|f_1\|_{p,} \quad (80)$$

Whenever $\widehat{f_1}$ and $\widehat{f_2}$ are supported in an ball of radius λ^{-1}, contained in $\{\xi :< |\xi| \leq 2\}$, by the Schwarz inequality, this follows from $\|Tf_1\|_{L^q(\mathbb{R}^d :\, L^{r/2}[0,\lambda^2])} \lesssim \lambda^{\frac{2}{q}-d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f_1\|_p$. Let $t \to \varpi(t)$ be a Schwartz function which is positive on $[0,\ 4d]$ and whose Fourier transform is supported in $[-1,\ 1]$. by scaling and rotation this would follow from

$$\| \varpi Tf\|_{L^q\left(\mathbb{R}^d;\, L^r(\mathbb{R})\right)} \lesssim \lambda^{\frac{2}{q}-\frac{2}{r}} \|f\|_p \qquad (81)$$

Whenever $\hat{f}$ is supported in $\{\xi : |\xi - \lambda e_1| \le 2d\}$. by a change of variables and trivial estimates it is easy to see (81) for $1 \le p \le q = r \le \infty$. the estimate for $r > q$ follows by applying Brenstein's inequality in t since the temporal Fourier transform of ϖTf is contained in $\{s : s \sim \lambda^2\}$.

We now argue similarly as in [105]. Write $\| Tf\|_{L^q\left(\mathbb{R}^d;\, L^r(\mathbb{R})\right)} = \|Tf_1\, Tf_2\|_{L^{q/2}(\mathbb{R}^d;\, L^{r/2}[0,\lambda^2])}$. For each j, $1 \le 2^j \le \lambda$, we Write $\ell \sim_j \bar{\ell}$ if s^j_ℓ and $s^j_{\bar{\ell}}$ have adjacent parent, but are not adjacent. When $\lambda < 2^j \le 2\lambda$, we mean by $\ell \sim_j \bar{\ell}$ that the distance between s^j_ℓ and $s^j_{\bar{\ell}}$ is $\lesssim \lambda^{-1}$. then, we can write for every $(\xi, \eta) \in \mathbb{R}^d$, with $\xi \neq \eta$.

$$\sum_{1 \le 2j \le 2\lambda} \sum_{\substack{(\ell,\bar{\ell}) \\ \ell \sim_j \bar{\ell}}} \chi_{s^j_\ell}(\xi)\chi_{s^j_{\bar{\ell}}}(\eta) = 1 \qquad (82)$$

Define p^j_ℓ by $\widehat{p^j_\ell f} = \chi_{s^j_\ell}\hat{f}$; then the operators p^j_ℓ are bounded on L^p, $1 < p < \infty$, with operator norms independent of ℓ and j. For any Schwartz function f we have by (82)

$$[\, Tf(x,t)]^2 = \sum_{1 \le 2j \le 2\lambda} \sum_{(\ell,\bar{\ell})\ell \sim_j \bar{\ell}} TP^j_\ell f(x,t) TP^j_{\bar{\ell}} f(x,t)$$

Let $\varphi \in C_c^\infty$ be supported in $[-1,1]^d$, satisfying $\sum_{j \in \mathbb{Z}^4} \varphi(\xi - \eth) = 1$ for all $\xi \in \mathbb{R}^d$. Define $\tilde{p}^j_\eth$ as acting on $L^\alpha(L^b)$ functions by $\widehat{\tilde{p}^j_\eth}\, G(\xi, t) = \varphi(\xi - \eth)$. We use the inequality

$$\left\| \sum_\eth \tilde{p}^j_\eth\, G_\eth \right\|_{L^\alpha(L^b)} \le C \|\{G_\eth\}\|_{\ell^\alpha\left(L^\alpha(L^b)\right)}, \;\; 1 \le \alpha \le 2, \;\; \alpha \le b \le \alpha', \qquad (83)$$

The constant C in (83) is independent of j. the inequality follows from Plancherel's theorem in the case $\alpha = b = 2$, and from an application of Minkowski's inequality in the case $\alpha = 1, 1 \le b \le 2$, The intermediate case follow by interpolation. Note that for any j and any $\eth \in \xi^d$ the number of pairs $(\ell, \bar{\ell})$ with $\ell \sim_j \bar{\ell}$ for which $\tilde{p}^j \left[Tp^j_\ell f\, Tp^j_{\bar{\ell}}\right] \neq 0$ is uniformly bounded (independent of $j, \eth, f$). Thus inequality (83) applied with $\alpha = q/2$ implies.

$$\|Tf\|^2_{L^q(\, L^r[0,\lambda^2])} \lesssim \sum_{1 \le 2j \le 2\lambda} \left(\sum_{\ell \sim_j \bar{\ell}} \left\|Tp^j_\ell f\, Tp^j_{\bar{\ell}}\right\|^{q/2}_{L^{q/2}(\, L^{r/2}[0,\lambda^2])} \right)^{2/q} ; \qquad (84)$$

Here we use that $1 \le q/2 \le r/2 \le (q/2)'$ i.e. $q \le r \le \frac{2q}{q-2}$ which implies that $q/2 \le 2$.

Now by (79) and (80) the right hand side of (84) is dominated by constant times

$$\sum_{1 \le 2j \le \lambda} A(j,\lambda) \left(\sum_{\ell \sim_j \bar{\ell}} \|p\|^{q/2}_p \|p\|^{q/2}_p \right)^{2/q} + \lambda^{\frac{4}{q} - 2d\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\sum_{\ell \sim_j \bar{\ell}} \left\|p^{j\eth}_\ell f\right\|^{q/2}_p \left\|p^{j\eth}_{\bar{\ell}} f\right\|^{q/2}_p \right)^{2/q}$$

$$\lesssim \lambda^{\frac{4}{q} - 2d\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\sum_\ell \left\|p^{j\eth}_\ell\right\|^q_p \right)^{2/q} + \sum_{1 \le 2j \le \lambda} A(j,\lambda) \left(\sum_\ell \left\|p^{j\eth}_\ell\right\|^q_p \right)^{2/q}.$$

Here $j\eth$ is the integer such that $\lambda > j\eth \le 2\lambda$, and we have used the Schwarz inequality and the fact that for each (j, ℓ) the number of $\bar{\ell}$ with $\ell \sim_j \bar{\ell}$ is uniformly bounded. Since $2 \le p \le q$, We also have

$$\left(\sum_{\ell}\|p_{\ell}^{j}f\|_{p}^{q}\right)^{1/q} \lesssim \|f\|_{p},$$

And thus we have shown (78).

Corollary (4.2.23). Let $\gamma > \frac{3\epsilon^2-2\epsilon-4}{(2+2\epsilon)(2+3\epsilon)}$.. Supposethat for $\lambda \gg 1$

$$\left\|\left(\int_{\frac{1}{2}} \left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f^2\right|^{(2+\epsilon)} dt\right)^{\frac{1}{(2+2\epsilon)}}\right\|_{2+3\epsilon} \lesssim \lambda^{\gamma}\, \|f^2\|_{(2+\epsilon)}. \tag{85}$$

where $\chi \epsilon C_c^{\infty}$ is supported in $\left(\frac{1}{2},2\right)$ (with suitable bounds). Then, for $\lambda \gg 1$.

$$\left\|\left(\int_{\frac{1}{2}} \left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f^2\right|^{(2+\epsilon)} dt\right)^{\frac{1}{(2+2\epsilon)}}\right\|_{(2+2\epsilon)} \lesssim \lambda^{\gamma}\, \|f^2\|_{(2+\epsilon)}. \tag{86}$$

Proof. It is easy to calculate that

$$\sup_{0\le t\le (8\lambda)^2} \left|\mathcal{F}^{-1}\left[\chi\left(\frac{\cdot}{\lambda}\right)exp(-it|\cdot|^2)\right](x)\right| \le C_N\, \lambda^{(2+\epsilon)}\,(1+\lambda\,|x|)^{-N}$$

And thus, by Young's inequality,

$$\left\|\left(\int_{1/2} \left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f^2\right|^{(2+\epsilon)} dt\right)^{\frac{1}{(2+2\epsilon)}}\right\|_{(2+2\epsilon)} \lesssim \left\|\lambda^{\frac{-2}{(2+\epsilon)}}\int \lambda^{(2+\epsilon)}\,(1+\lambda\,|y|)^{-N}dy\right\|_{(2+2\epsilon)}$$

$$\lesssim \lambda^{\frac{3\epsilon^2-2\epsilon-4}{(2+2\epsilon)(2+3\epsilon)}}\,\|f^2\|_{(2+\epsilon)}. \tag{87}$$

Now letting $(8\lambda)^{-2} \le 1-\epsilon$,

$$\left(\int_{(1-\epsilon)/2}^{(1-\epsilon)} \left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f^2(x)\right|^{\frac{1}{(2+\epsilon)}} dt\right)^{\frac{1}{(2+\epsilon)}}$$

$$= (-\epsilon)^{\frac{1}{(2+\epsilon)}}\left(\int_{1/2}^{1} \left|\chi\left(\frac{D}{(1-\epsilon)^{1/2}\lambda}\right)e^{is\Delta}\left[f^2(1-\epsilon^{-1/2}.)\right]\left((1-\epsilon)^{-1/2}x\right)\right|^{(2+\epsilon)} ds^2\right)^{\frac{1}{(2+\epsilon)}}$$

Thus by change of variable (2.17) implies

$$\left\|\left(\int_{b/2}^{b} \left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f^2\right|^{(2+\epsilon)} dt\right)^{\frac{1}{(2+\epsilon)}}\right\|_{(2+2\epsilon)}$$

$$\lesssim \left(\sqrt{(1-\epsilon)}\right)^{-(2+\epsilon)\left(\frac{1}{(2+\epsilon)}-\frac{1}{(2+2\epsilon)}\right)+\frac{2}{(2+3\epsilon)}}\left(\lambda\,\sqrt{(1-\epsilon)}\right)^{\gamma}\|f^2\|_{(2+\epsilon)}.$$

We choose $b = 2^{-1}$. and since $\gamma > (2+\epsilon)\left(\frac{1}{(2+\epsilon)}-\frac{1}{(2+\epsilon)}\right)-\frac{2}{(2+\epsilon)}$ we may sum over I with $(8\lambda)^{-2} \le 2^{-1} \le 1$ and combine with (2.19). Hence we get

$$\left\|\left(\int_{0}^{1} \left|e^{it\Delta}\chi\left(\frac{D}{\lambda}\right)f^2\right|^{\frac{1}{(2+\epsilon)}} dt\right)^{\frac{1}{(2+\epsilon)}}\right\|_{(2+2\epsilon)} \lesssim \lambda^{\gamma}\, \|f^2\|_{(2+\epsilon)}.$$

Now (86) with $I = [-1,1]$ follows using the formula $e^{it\Delta}f^2 = \overline{e^{\imath t\Delta}\bar{f}}$,and the triangle inequality. Finally, by scaling, we can enlarge the time interval (so that the implicit constant is of course dependent on the interval), and we are done

Chapter 5
Spectral Theory of Schrödinger Operators

We find conditions on the configuration of point interactions such that any self-adjoint realization haspurely absolutely continuous non-negative spectrum. We also apply some results on Schrödinger operatorsto obtain new results on completely monotone functions.

Section (5.1): Radial Positive Definite Function with Bases of Subspace and Property of x-positive Definiteness

An important topic in quantum mechanics is the spectral theory of Schrödinger Hamiltonians with point interactions. These are Schrödinger operators on the Hilbert space $L^2(\mathbb{R}^d), 1 \le d \le 3$, with potentials supported on a discrete (finite or countable) set of points of $\mathbb{R}^d$. There is an extensive literature on such operators, see e.g. [122, 124, 129, 140, 145, 147, 149, 162].

Let $X = \{x_j\}_1^m$ be the set of points in $\mathbb{R}^d$ and let $\alpha = \{\alpha_j\}_1^m$ be a sequence of real numbers, where $m \in \mathbb{N} \cup \{\infty\}$. The mathematical problem is to associate a self-adjoint operator (Hamiltonian) on $L^2(\mathbb{R}^d)$ with the differential expression

$$\mathcal{L}_d := \mathcal{L}_d(X, \alpha) := -\Delta + \sum_{j=1}^{m} \alpha_j \delta(\cdot - x_j), \quad \alpha_j \in \mathbb{R}, m \in \mathbb{N} \cup \{\infty\}, \qquad (1)$$

and to describe its spectral properties.

There are at least two natural ways to associate a self-adjoint Hamiltonian $H_{X,\alpha}$ with the differential expression (1). The first one is the form approach. That is, the Hamiltonian $H_{X,\alpha}$ is defined by the self-adjoint operator associated with the quadratic form

$$\tilde{t}_{X,\alpha}^d[f] = \int_{\mathbb{R}^d} |\nabla f|^2 dx + \sum_{j=1}^{m} \alpha_j |f(x_j)|^2. \quad \operatorname{dom}(\tilde{t}_{X,\alpha}^d) = W_{comp}^{2,2}(\mathbb{R}^d). \quad (2)$$

This is possible for $d = 1$ and finite $m \in \mathbb{N}$, since in this case the quadratic form $\tilde{t}_{X,\alpha}^d$ is semi-bounded below and closable (cf. [164]). Its closure $t_{X,\alpha}^{(1)}$ is defined by the same expression (2) on the domain $\operatorname{dom}\left(t_{X,\alpha}^{(1)}\right) = W^{1,2}(\mathbb{R})$. For $m = \infty$ the form (2) is also closable whenever it is semibounded (see [125, Corollary 3.3]).

Another way to introduce local interactions on $X := \{x_j\}_{j=1}^m \subset \mathbb{R}$ is to consider the minimal operator corresponding to the expression $\mathcal{L}_1$ and to impose boundary conditions at the points x_j.

in the case $d = 1$ and $m < \infty$ the domain of the corresponding Hamiltonian $H_{X,\alpha}$ is given by

$$\operatorname{dom}(H_{X,\alpha}) = \{f \in W^{2,2}(\mathbb{R} \setminus X) \cap W^{1,2}(\mathbb{R}): f'(x_j +) - f'(x_j -) = \alpha_j f(x_j)\}.$$

In contrast to the one-dimensional case, the quadratic form (2) is not closable in $L^2(\mathbb{R}^d)$ for $d \ge 2$, so it does not define a self-adjoint operator. The latter happens because the point evaluations $f \to f(x)$ are no longer continuous on the Sobolev space $W^{1,2}(\mathbb{R}^d)$ in the case $d \ge 2$.

However, it is still possible to apply the extension theory of symmetric operators. F.A. Berezin and L.D. Faddeev proposed in [129] to consider the expression (1) (with $m = 1$and $d = 3$).

They defined the minimal symmetric operator H as a restriction of $-\Delta$ to the domain $\operatorname{dom} H = \{f \in W^{2,2}(\mathbb{R}^d): f(x_1) = 0\}$ and studied the spectral properties of all its self-adjoint extensions. Self-adjoint extensions (or realizations) of H for finitely many point interactions have been investigated since then in numerous sections (see [122]). In the case of infinitely many point interactions $X = \{x_j\}_1^\infty$ the minimal operator H_{min} is defined by

$$H_d := H_{d,min} := -\Delta \upharpoonright \operatorname{dom} H, \qquad \operatorname{dom}(H_d) = \{f \in W^{2,2}(\mathbb{R}^d): f(x_j) = 0, j \in \mathbb{N}\}. \quad (3)$$

we investigate the "operator" (1) (with $d = 3$ and $m = \infty$) in the framework of boundary triplets. This is a new approach to the extension theory of symmetric operators that has been developed during the last three decades (see [139, 64, 134, 166]). A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for the adjoint of a densely defined symmetric operator A consists of an auxiliary Hilbert space $\mathcal{H}$ and two linear mapping $\Gamma_0, \Gamma_1: \operatorname{dom}(A^*) \to \mathcal{H}$ such that the mapping $\Gamma := (\Gamma_0, \Gamma_1): \operatorname{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective. The main requirement is the abstract Green identity.

$$(A^* f, g)_{\mathfrak{H}} - (f, A^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_{0g})_{\mathcal{H}} - (\Gamma_0 f, \Gamma_{1g})_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(A^*) (4)$$

A boundary triplet for A^* exists whenever A has equal deficiency indices, but it is not unique. It plays the role of a "coordinate system" for the quotient space $\operatorname{dom}(A^*)/\operatorname{dom}(\bar{A})$ and leads to a natural parametrization of the self-adjoint extension of A by means of self-adjoint linear relation (multi-valued operators) in $\mathcal{H}$, see [139] and [166].

The main analytical tool is the abstract Weyl function $M(\cdot)$ which was introduced and studied in [64]. This Weyl function plays a similar role in the theory of boundary triplets as the classical Weyl-Titchmarsh function does in the theory of Strum-Liouville operators, its allows one to investigate spectral properties of extensions (see [133, 64, 155, 158]).

When studying boundary value problems for differential operators, one is searching for an appropriate boundary triplet such that:

The properties of the mapping $\Gamma = \{\Gamma_0, \Gamma_j\}$ should correlate with trace properties of functions from the maximal domain $\operatorname{dom}(A^*)$.

The Weyl function and the boundary operator should have "good" explicit forms.

Such a boundary triplet was constructed and applied to differential operators with infinite deficiency indices in the following cases:

(i) Smooth elliptic operators in bounded or unbounded domains ([141, 172], see also [142]),

(ii) The maximal Strum-Liouville operator $-d^2 dx^2 + T$ in $L^2([0,1]; \mathcal{H})$ with an unbounded operator potential $T = T^* \geq aI, T \in (\mathcal{H})$ ([139], see also [64] for the case of $L^2(\mathbb{R}_+; \mathcal{H})$),

(iii) The ID Schrödinger operator $\mathcal{L}_{1,X}$ in the cases $d_*(X) > 0[150, 160]$ and $d_*(X) = 0[151]$, where $d_*(X)$ is defined by (5) below.

Constructing such a "good" boundary triplet involves always non-trivial analytic results. For instance, Grubb's construction [141] for (i) (see also the adaptation to the case of Definition 4 in [156]) is based on trace theory for elliptic operators developed by Lions and Magenes [153] (see also [142]). The approach in (iii) is based on a general construction of a (regularized) boundary triplet for direct sums of symmetric operators (see [158, Theorem 5.3] and [151, Corollary(5.1.36)].

We study all (that is, not necessarily local) self-adjoint extensions of the operator $H = H_3$ (realizations of $\mathcal{L}_3$) in the framework of boundary triplets approach. As in [122] our crucial assumption is

$$d_*(X) := \inf_{j\neq k}|x_k - x_j| > 0. \qquad (5)$$

Our construction of a boundary triplet Π for H^* is based on the following result: The sequence

$$\left\{\frac{e^{-|x-x_j|}}{|x-x_j|}\right\}_{j=1}^{\infty} \quad (6)$$

forms a Riesz basis of the defect subspace $\mathfrak{N}_{-1}(H) = \ker(H^* + I)$ of H^* (cf. Theorem (5.1.43)). Using this boundary triplet Π we parameterize the set of self-adjoint extensions of H, compute the corresponding Weyl function $M(\cdot)$ and investigate various spectral properties of self-adjoint extensions (semiboundedness, non-negativity, negative spectrum, resolvent comparability, etc.).

The main result on spectral properties of Hamiltonians with point interactions concerns the absolutely continuous spectrum (ac-spectrum). For instance, if

$$C := \sum_{|j-k|>0} \frac{1}{|x_j - x_k|^2} < \infty, \qquad (7)$$

We prove that the part $\widetilde{H}E_{\widetilde{H}}(C,\infty)$ of every self-adjoint extension $\tilde{H}$ of H is absolutely continuous (cf. Theorems (5.2.25) and (5.2.26)). Moreover, under additional assumptions on X, we show that the singular part of $\widetilde{H}_+ := \widetilde{H}E_{\widetilde{H}}(0,\infty)$ is trivial, i.e. $\widetilde{H}_+ = \widetilde{H}_+^{ac}$.

The absolute continuity of self-adjoint realizations $\widetilde{H}$ of H has been studied only in very few cases. Assuming that $X = Y + \Lambda$, where $Y = \{y_j\}_1^N \in \mathbb{R}^3$ is a finite set and $\Lambda = \{\sum_1^3 n_j a_j \in \mathbb{R}^3 : (n_1, n_2, n_3) \in \mathbb{Z}^3\}$ is a Bravais lattice, it was proved in [121, 123, 135, 140, 145-147, 124] (see also [122] and the references in [122] and [124]) that the spectrum of some periodic realizationsis absolutely continuous and has a band structure with a finite number of gaps.

An important feature of the investigations is an apparently new connection between the spectral theory of operators (1) for $d = 3$ and the class Φ_3 of radial positive definite functions on $\mathbb{R}^3$. We exploit this connection in both directions. We combine the extension theory of the operator H with Theorem (5.1.34) to obtain results on positive definite functions and the corresponding Gram matrices (8), while positive definite functions are applied to the spectral theory of self-adjoint realizations of operators (1) with infinitely many point interactions.

We deal with radial positive definite functions on $\mathbb{R}^d$ and has been inspired by possible applications to the spectral theory of operators (1). If f is such a function and $X = \{x_n\}_1^\infty$ is a sequence of points of $\mathbb{R}^d$, we say that f is strongly$X$-positive definite if there exists a constant $c > 0$ such that for all $\xi_1, \dots, \xi_m \in \mathbb{C}$,

$$\sum_{j,k=1}^{m} \xi_k \bar{\xi}_j f(x_k - x_j) \geq c \sum_{k=1}^{m} |\xi_k|^2, \quad m \in \mathbb{N}.$$

Using Schoenberg's theorem we derive a number of results showing under certain assumptions on X that f is strongly$X$-positive definite and that the Gram matrix

$$Gr_x(f) := (f(|x_k - x_j|))_{k,j\in\mathbb{N}} \; (8)$$

defines a bounded operator on $l^2(\mathbb{N})$. The latter results correlate with the properties of the sequence $\{e^{i(\cdot,xk)}\}_{k\in\mathbb{N}}$ of exponential functions to form a Riesz–Fischer sequence or a Bessel sequence, respectively, in $L^2(S_r^n; \sigma_n)$ for some $r > 0$.

We prove that the sequence (6) forms a Riesz basis in the closure of its linear span if and only if X

satisfies (5). This result is applied to prove that for such X and any non-constant absolute monotone function f on $\mathbb{R}_+$ the function $f(|\cdot|_3)$ is strongly$X$-positive definite. Under an additional assumption it is shown that the matrix (8) defines a boundedly invertible bounded operator on $l^2(\mathbb{N})$.
We collect some basic definitions and facts on boundary triplets, the corresponding Weyl functions and spectral properties of self-adjoint extensions.
Also we construct a boundary triplet for the adjoint operator H^* for $d = 3$ and compute the corresponding Weyl function $M(\cdot)$. The explicit form of the Weyl function given by (101) plays crucial role in the sequel. For the proof of the surjectivity of the mapping $\Gamma = (\Gamma_0, \Gamma_1)$ the strong X-positive definiteness of the function $e^{-|\cdot|}$ on $\mathbb{R}^3$ is essentially used. The latter follows from the absolute monotonicity of the function e^{-t} on $\mathbb{R}_+$.
We describe the quadratic form generated by the semibounded operator $M(0)$ on $l^2(\mathbb{N})$ as strong resolvent limit of the corresponding Weyl function $M(-x)$ as $x \to +0$. For this we use the strong X-positive definiteness of the function $\frac{1-e^{-|\cdot|}}{|\cdot|}$ on $\mathbb{R}^3$ which follows from the absolute monotonicity of the function $\frac{1-e^{-t}}{t}$ on $\mathbb{R}_+$. The operator $M(0)$ enters into the description of the Krein extension of H for $d = 3$ and allows us to characterize all non-negative self-adjoint extensions as well as all self-adjoint extensions with κ $(\leq \infty)$ negative eigenvalues. Using the behavior of the Weyl function at $-\infty$ we show that any self-adjoint extension H_B of H is semibounded from below if and only if the corresponding boundary operator B is. A similar result for elliptic operators on exterior domains has recently been obtained by$G$. Grubb [143].
We apply a technique elaborated in [133,158] as well as a new general result to investigate the ac-spectrum of self-adjoint realizations, we prove that the part $\widetilde{H}E_{\widetilde{H}}(C,\infty)$ of any self-adjoint realization $\widetilde{H}$ of $\mathcal{L}_3$ is absolutely continuous provided that condition (7) holds. Moreover, under some additional assumptions on X we show that the singular non-negative part $\widetilde{H}^s E_{\widetilde{H}}(0,\infty)$ of any realization $\widetilde{H}$ is trivial. Among others, provide explicit examples which show that an analog of the Weyl-von Neumann theorem does not hold for non-additive (singular) compact (and even noncompact) perturbations. The proof of these results is based on the fact that the function $\frac{\sin st}{t}$ belongs to Φ_3 for each $s > 0$. Then, by Propositions (5.1.17) and (5.1.19), $\frac{\sin s|\cdot|}{|\cdot|}$ is strongly$X$-positive definite for certain subsets X of $\mathbb{R}^3$ and any$s > 0$. The latter is equivalent to the invertibility of the matrices

$$\mathcal{M}(t) := \left(\delta_{k_j} + \frac{\sin(\sqrt{t}|x_k - x_g|)}{\sqrt{t}|x_k - x_j| + \delta_{k_j}}\right)_{j.k=1}^{\infty} \quad \text{for } t \in \mathbb{R}_+$$

Throughout and $\mathcal{H}$ are separable complex Hilbert spaces. We denote by$B(\mathcal{H},\mathfrak{H})$ the bounded linear operators from $\mathcal{H}$ into $\mathfrak{H}$, by $B(\mathcal{H})$ the set $B(\mathcal{H},\mathcal{H})$, by$\mathcal{C}(H)$ the closed linear operators on $\mathcal{H}$ and by$\mathfrak{S}p(\mathcal{H})$ the Neumann–Schatten ideal on $\mathcal{H}$. In particular, $\mathfrak{S}_\infty(\mathcal{H})$ and $\mathfrak{S}_1(\mathcal{H})$ are the ideals of compact operators and trace class operators on $\mathcal{H}$, respectively.
For closed linear operator T on $\mathfrak{H}$, we write $\mathrm{dom}(T), \ker(T), \mathrm{ran}(T), \mathrm{gr}(T)$ for the domain, kernel, range, and graph of T, respectively, and $\sigma(T)$ and $\rho(T)$ for the spectrum and the resolvent set of T. The symbols $\sigma_c(T), \sigma_{ac}(T), \sigma_s(T), \sigma_{sc}(T), \sigma_p(T)$ denote the continuous, absolutely continuous, singular, singularly continuous and point spectrum, respectively, of a self-adjoint operator T. Note

that $\sigma_s(T) = \sigma_{sc}(T) \cup \sigma_p(T)$ and $\sigma(T) = \sigma_{ac}(T) \cup \sigma_s(T)$. The defect subspaces of a symmetric operator T are denoted by$\mathfrak{N}_z$. [164-166, 148].

By$C[0,\infty)$ we mean the Banach space of continuous bounded functions on $[0,\infty)$ and byS_r^n the sphere in $\mathbb{R}^n$ of radius r centered at the origin and $S^n := S_1^n$. Further, $\Sigma'_{k\in\mathbb{N}}$ denotes the sum over all $k \neq j$ and $\Sigma_{|j-j|>0}$ is the sum over all $k, j \in \mathbb{N}$ with $k \neq j$.

Let $(u, v) = u_1v_1 + \cdots + u_nv_n$ be the scalar product of two vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ from $\mathbb{R}^n, n \in \mathbb{N}$, and let $|u| = |u|_n = \sqrt{(u, u)}$ be the Euclidean norm of u. First we recall some basic facts and notions about positive definite functions [1].

Definition (5.1.1) [176]: (See [119]). A function $g: \mathbb{R}^n \to \mathbb{C}$ is called positive definite if g is continuous at 0 and for arbitrary finite sets $\{x_1, \dots, x_m\}$ and $\{\xi_1, \dots, \xi_m\}$, where $x_k \in \mathbb{C}$, we have

$$\sum_{k,j=1}^{m} \xi_k \bar{\xi}_j g(x_k - x_j) \geq 0. \qquad (9)$$

The set of positive definite function on $\mathbb{R}^n$ is denoted by$\Phi(\mathbb{R}^n)$.

Clearly, a function g on $\mathbb{R}^n$ positive definite if and only if it is continuous at 0 and the matrix $G(X) = \left(g_{kj} := g(x_k - x_j)\right)_{k,j=1}^{m}$ is positive semi-definite for any finite subset $X = \{x_j\}_1^m$ of $\mathbb{R}^n$.

The following classical Bochner theorem gives a description of the class $\Phi(\mathbb{R}^n)$.

Theorem (5.1.2) [176]: (See [132]). A function $g(\cdot)$ is positive definite on $\mathbb{R}^n$ if and only if there is a finite non-negative Bore measure μ on $\mathbb{R}^n$ if and only if there is a finite non-negative Borel measure such that

$$g(x) = \int_{\mathbb{R}^n} e^{i(u,x)} d\mu(u) \quad \text{for all } x \in \mathbb{R}^n. \qquad (10)$$

Let us continue with a number of further basic definitions.

Definition (5.1.3) [176]: Let g be a positive define function on $\mathbb{R}^n$ and let X be a subset of $\mathbb{R}^n$.

(i) We say that g is strongly X-positive definite if there exists a constant $c > 0$ such that.

$$\sum_{k,j=1}^{m} \xi_k \bar{\xi}_j g(x_k - x_j) > c \sum_{k=1}^{m} |\xi_k|^2, \qquad \xi = \{\xi_1, \dots, \xi_m\} \in \mathbb{C}^m \setminus \{0\} (11)$$

for any finite set $\{x_j\}_{j=1}^m$ of distinct points $x_j \in X$.

(ii) It is said g is strictly X-positive definite if (3) is satisfied with $c = 0$.

Any strongly$X$-positive definite g is also strictly$X$-positive definite. For finite sets $X = \{x_j\}_1^m$ both notions are equivalent by the compactness of the sphere in $\mathbb{C}^m$.

Definition (5.1.4) [176]: (See [173]). Let $F = \{f_k\}_{k=1}^{\infty}$ be a sequence of vectors of a Hilbert space$\mathcal{H}$.

(i) The sequence is called a Riesz-Fischer sequence if there exists a constant $c > 0$ such that

$$\left\|\sum_{k=1}^{m} \xi_k f_k\right\|_{\mathcal{H}}^2 \geq c \sum_{k=1}^{m} |\xi_k|^2 \text{ for all } (\xi_1, \dots, \xi_m) \in \mathbb{C}^m \text{and } m \in \mathbb{N}. \quad (12)$$

(ii) The sequence F is said to be Besel sequence if there is a constant $C > 0$ such that.

$$\left\|\sum_{k=1}^{m}\xi_k f_k\right\|_{\mathcal{H}}^2 \le C\sum_{k=1}^{m}|\xi_k|^2 \quad \text{for all } (\xi_1,\dots,\xi_m)\in\mathbb{C}^m \text{ and } m\in\mathbb{N} \quad (13)$$

(iii) The sequence F is called Riesx basis of the Hilbert space $\mathcal{H}$ if its linear span is dense in $\mathcal{H}$ and F is both a Riesz-Fischer sequence and a Bessel sequence.

Note that the definitions of Riesz-Fischer and Bessel sequences given in [173] are different, but they are equivalent to the preceding definition according to [173]

The following proposition contains some slight reformulations of these notions.

If $\mathcal{A} = (a_{kj})_{k,j\in\mathbb{N}}$ is an infinite matrix of complex entries a_{kj} we shall say that $\mathcal{A}$ defines a bounded operator A on the Hilbert space $l^2(\mathbb{N})$ if

$$\langle Ax, y\rangle = \sum_{k,j=1}^{\infty} a_{kj}x_k\bar{y}_j \quad \text{for } x = \{x_k\}_{k\in\mathbb{N}}, y = \{y_k\}_{k\in\mathbb{N}} \in l^2(\mathbb{N}). \quad (14)$$

Clearly, if $\mathcal{A}$ defines a bounded operator A, then A is uniquely determined by Eq. (14).

Proposition(5.1.5) [176]: Suppose that $X = \{x_k\}_1^\infty$ is a sequence of pairwise distinct points of $\mathbb{R}^n$ and g is a positive definite function given by (10) with measure μ. Let $F = \left\{f_k := e^{i(\cdot,x_k)}\right\}_{k=1}^{\infty}$ denote the sequence of exponential function in the Hilbert space $L^2(\mathbb{R}^n;\mu)$. Then:

(i) F is a Riesz-Fischer sequence in $L^2(\mathbb{R}^n;\mu)$ if and only if g is strongly X-positive definite.

(ii) F is a Bessel sequence if and only if the Gram matrix.

$$G_{r_F} = (\langle f_k, f_j\rangle_{L^2(\mathbb{R}^2;\mu)})_{k,j\in\mathbb{N}} =: Gr_x(g) \quad (15)$$

defines a bounded operator on $l^2(\mathbb{N})$.

Proof. Using Eq. (10) we easily derive

$$\sum_{k,j=1}^{m}\xi_k\bar{\xi}_j g(x_k - x_j) = \int_{\mathbb{R}^n}\left|\sum_{k=1}^{m}\xi_k e^{i(u,x_k)}\right|^2 d\mu(u) = \int_{\mathbb{R}^n}\left|\sum_{k=1}^{m}\xi_k f_k(u)\right|^2 d\mu(u) = \left\|\sum_{k=1}^{m}\xi_k f_k\right\|_{L^2(\mathbb{R}^n;\mu)} \quad (16)$$

for arbitrary$m\in\mathbb{N}$ and $\xi = \{\xi_1,\dots,\xi_m\}\in\mathbb{C}^m$. Both statements are immediate from (16).

Taking in mind further applications to the spectral theory of self-adjoint realizations of $\mathcal{L}_3$ we will be concerned with radial positive definite functions. Let us recall the corresponding concepts.

Definition (5.1.6) [176]: Let $n\in\mathbb{N}$. A function $f\in C([0,+\infty))$ is called a radial positive definite function of the class Φ_n if $f(|\cdot|_n)$ is a positive definite function on $\mathbb{R}^n$, i.e., if $f(|\cdot|_n)\in\Phi(\mathbb{R}^n)$.

It is known that $\Phi_{n+1}\subset\Phi_n$ and $\Phi_n\neq\Phi_{n+1}$ for any$n\in\mathbb{N}$ (see, for instance, [171, 175]).

A characterization of the class Φ_n is given by the following Schoenberg theorem [167, 168], see, e.g., [119] or [130, 170]. Let σ_n denote the normalized surface measure on the unit sphere S^n.

Theorem(5.1.7) [176]:A function f on $[0,+\infty)$ belong to the class Φ_n if and only if there exists a positive finite Borel measure v on $[0,+\infty)$ such that

$$f(t) = \int_0^{+\infty}\Omega_n(rt)\,dv(r)dv(r), \qquad t\in[0,+\infty). \quad (17)$$

where

$$\Omega_n(|x|) = \int_{S^n} e^{i(u,x)}d\sigma_n(u), \quad x\in\mathbb{R}^n. \quad (18)$$

Moreover, we have

$$\Omega_n(t) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{1}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(t) = \sum_{p=0}^{\infty}\left(-\frac{t^2}{4}\right)^p \frac{\Gamma\left(\frac{n}{2}\right)}{p!\,\Gamma\left(\frac{n}{2}+p\right)}.\ t \in [0,+\infty). \qquad (19)$$

The first three function $\Omega_n, n = 1,2,3$, can be computed as

$$\Omega_1(t) = \cos t, \qquad \Omega_2(t) = J_0(t), \qquad \Omega_3(t) = \frac{\sin t}{t}. \qquad (20)$$

where J_0 is the Bessel function of first kind and order zero (see e.g., [163]).

It was proved in [138] using Schoenberg's theorem that for each non-constant function $f \in \Phi_n, n \geq 2$, the function $f(|\cdot|)$ is strictlyX-positive definite for any finite subset X of $\mathbb{R}^n$.

Definition (5.1.8) [176]: A function $f \in C[0,\infty) \cap C^{\infty}(0,+\infty)$ is called completely monotone on $[0,\infty)$ if $(-1)^k f^k(t) \geq 0$ for all $k \in \mathbb{N} \cup \{0\}$ and $t > 0$. The set of such functions is denoted by$M[0,\infty)$.

By Bernstein's theorem [1], a function f on $[0,\infty)$ belongs to the class $M[0,\infty)$ if and only if there exists a finite positive Borel measure τ on $[0,\infty)$ such that

$$f(t) = \int_0^{\infty} e^{-ts} d\tau(s), \quad t \in [0,\infty). \qquad (21)$$

The measure τ is then uniquely determined by the function f.

Schoenberg noted in [167, 168] that a function f on $[0,\infty)$ belongs to $\bigcap_{n\in\mathbb{N}} \Phi_n$ if and only if $f(\sqrt{\cdot}) \in M[0,\infty)$. The following statement is an immediate consequence of Schoenberg's result.

Proposition(5.1.9) [176]: If $f \in M[0,\infty)$, then $f \in \bigcap_{n\in\mathbb{N}} \Phi_n$.

Proof. For $s \geq 0$ the function $g_s(t) := e^{-s\sqrt{1}}$ is completely monotone for $t > 0$. Schoenberg's result applies to $g_s(t^2)$ and shows that $g_s(t^2) = e^{-st} \in \bigcap_{n\in\mathbb{N}} \Phi_n$. Therefore the integral representation (21) implies that $f(\cdot) \in \bigcap_{n\in\mathbb{N}} \Phi_n$.

For any sequence $X = \{x_k\}_1^{\infty}$ of points of $\mathbb{R}^n$ we set

$$d_*(X) := \inf_{k\neq j} |x_k - x_j|.$$

The following proposition describes a large class of radial positive definite functions that are stronglyX-positive definite for any sequence X of points of $\mathbb{R}^3$ such that $d_*(X) > 0$.

Corollary(5.1.10) [176]: Suppose $X = \{x_j\}_{j=1}^{\infty}$ is a sequence of points of $\mathbb{R}^3$ and τ is a finite positive Borel measure on $[0,+\infty)$. Then:

If $d_*(X) > 0$ and $\tau((0,+\infty)) > 0$, then $\widetilde{\Phi}$ forms a Riesz-Fischer sequence in $L^2(\mathbb{R}^3)$.

If $d_*(X) > 0$ and (67) holds, then $\widetilde{\Phi}$ is a Bessel sequence in $L^2(\mathbb{R}^3)$.

If $d_*(X) > 0$ and (67) is satisfied, then $\widetilde{\Phi}$ forms a Riesz basis in its closed linear span.

If the sequence $\widetilde{\Phi}$ is both a Riesz-Fischer and a Bessel sequence in $L^2(\mathbb{R}^3)$, then $d_*(X) > 0$.

An immediate consequence of the preceding corollary is

Corollary(5.1.11) [176]: Let f, X and τ be as in Theorem (5.1.37) and assume that condition (67) holds. Then the sequence $\widetilde{\Phi} = \{\widetilde{\varphi}_j\}_1^{\infty}$ forms a Riesz basis in its closed linear span if and only if $d_*(X) > 0$.

Remark(5.1.12) [176]: Let f be an absolutely monotone function with integral representation (21). Then.

$$\mathrm{Gr}_X(f) = (f(|x_j - x_k|))_{j,k\in\mathbb{N}} = (\langle\widetilde{\varphi}_j, \widetilde{\varphi}_k\rangle_{L^2(\mathbb{R}^3)})_{j,k\in\mathbb{N}} = \mathrm{Gr}_\Phi. \quad (22)$$

Proposition(5.1.13) [176]: Suppose that $f \in \Phi_n$ and let ν be the corresponding representing measure form (17). Let $X = \{x_j\}_1^\infty$ be an arbitrary sequence from $\mathbb{R}^n$. Then f is stronglyX-positive definite if and only if there exists a Borel subset $\mathcal{K} \subset (0, +\infty)$ such that $\nu(\kappa) > 0$ and the system $\{e^{i(\cdot,x_k)}\}_{k=1}^\infty$ forms a Riesz-Fischer sequence in $L^2(S_r^n; \sigma_n)$ for every$r \in \kappa$.

Proof. From (17) and (18) it follows that for $(\xi_1, \dots, \xi_m) \in \mathbb{C}^m$ and $m \in \mathbb{N}$.

$$\sum_{j,k=1}^m \xi_j \bar{\xi}_k f(|x_j - x_j|) = \int_0^{+\infty} \left(\int_{S^n} \left| \sum_{k=1}^m \xi_k e^{i(u,rx_k)} \right|^2 d\sigma_n(u) \right) d\nu(r). \quad (23)$$

Suppose that there exists a set κ as stated above. Then for every$\tau \in \mathcal{K}$ there is a constant $c(r) > 0$ such that

$$\left\| \sum_{k=1}^m \xi_k e^{i(u,rx_k)} \right\|_{L^2(S^n)}^2 \geq c(r) \sum_{k=1}^m |\xi_k|^2. \quad (24)$$

Choosing $c(r)$ measuring and combining this inequality with (23) we obtain

$$\sum_{j,k=1}^m \xi_j \bar{\xi}_k f(|x_j - x_k|) = \int_\kappa \left(\left\| \sum_{k=1}^m \xi_k e^{i(u,rx_k)} \right\|_{L^2(S^n)}^2 \right) d\nu(r) \geq c \sum_{k=1}^m |\xi_k|^2, \quad (25)$$

where $c := \int_\kappa c(r) d\nu(r)$. Since $\nu(\kappa) > 0$ and $c(r) > 0$, we have $c > 0$. That is, f is stronglyX-positive definite.

The converse follows easily from E.q. (23).

Remark(5.1.14) [176]: Or course, the set κ in Proposition (5.1.13) is not unique in general. If the measure ν has an atom $r_0 \in (0, +\infty)$, i.e., $\nu(\{r_0\}) > 0$, then one can choose $k = \{r_0\}$. For instance, for the function $f(\cdot) = \Omega_n(r_0)$ the representative measure from formula (17) is the delta measure δ_{r_0} at r_0. Therefore, $f(\cdot) = \Omega_n(r_0)$ is stronglyX-positive definite if and only if the system $\{e^{i(\cdot,x_k)}\}_{k=1}^\infty$ forms a Riesz-Fischer sequence in $L^2(S_{r_0}^n : \sigma_n)$.

Let $\Lambda = \{\lambda_k\}_1^\infty$ be a sequence of reals. For $r > 0$ let $n(r)$ denote the largest number of points λ_k that are contained in an interval of length r. Then the upper density of Λ is defined by.

$$D^*(\Lambda) = \lim_{r\to+\infty} n(r) r^{-1}.$$

Since $n(r)$ is subadditive, it follows that this limit always exists (see e.g. [131]).

In what follows we need the classical result on Riesz-Fischer sequences of exponents in $L^2(-a, a)$.

Proposition(5.1.15) [176]: Let $\Lambda = \{\lambda_k\}_1^\infty$ be a real sequence and $a > 0$. Set $E(\Lambda) := \{e^{i\lambda_k x}\}_1^\infty$.

(i) If $d_*(\Lambda) > 0$ and $D^*(\Lambda) < a/\pi$, then $E(\Lambda)$ is a Riesz-Fischer sequence in $L^2(-a, a)$.

(ii) If $E(\Lambda)$ is a Riesz-Fischer sequence in $L^2(-a, a)$, then $d_*(\Lambda) > 0$ and $D^*(\Lambda) \leq a/\pi$.

Assertion (i) of Proposition (5.1.15) is a theorem of A. Beurling [131], while assertion (ii) is a result of H.J. Landau [152], see e.g. [174]. Proposition (5.1.15)yields following statement.

Corollary(5.1.16) [176]: If $d_*(\Lambda) > 0$ and $D^*(\Lambda) = 0$, then $E(\Lambda)$ is a Riesz-Fischer sequence in $L^2(-a, a)$ for all $a > 0$.

From this corollary it follows that $E(\Lambda)$ is a Riesz-Fischer sequence in $L^2(-a, a)$ for all $a > 0$ if

$\lim_{k\to\infty}(\lambda_{k+1}-\lambda_k) = +\infty.$

Now we are ready to state the main result of this subsection.

Proposition(5.1.18) [176]: Let $f \in \Phi_n, f \neq \text{const}$, and let $X = \{x_k\}_1^\infty$ be a sequence of points $x_k \in \mathbb{R}^n, n \geq 2$, of the form $x_k = (0, x_{k2}, \dots, x_{kn})$. If the sequence $X_n \coloneqq \{x_{kn}\}_{k=1}^\infty$ of n-th coordinates satisfies the conditions $d_*(X_n) > 0$ and $D^*(X_n) = 0$, then f is stronglyX-positive definite.

Proof. By Schoenberg's Theorem (5.1.7), f admits a representation (17). Let $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{C}^m, m \in \mathbb{N}$. It follows from (17) and (18) that

$$\sum_{k,j=1}^m \xi_k\bar{\xi}_j f(|x_k - x_j|) = \int_0^{+\infty}\left(\int_{S^n}\left|\sum_{k=1}^m \xi_k e^{i(u,rx_k)}\right|^2 d\sigma_n(u)\right)dv(r). \quad (26)$$

Next, we transform the integral over S^n in (26). Recall that in terms of spherical coordinates

$$u_1 = \cos\varphi_1, \quad u_{n-1} = \sin\varphi_1 \dots \sin\varphi_{m-2}\cos\varphi_{n-1},$$
$$u_n = \sin\varphi_1 \dots \sin\varphi_{n-2}\sin\varphi_{n-1}, \varphi_1, \dots, \varphi_{n-2} \in [0,\pi] \text{ and } \varphi_{n-1} \in [0,2\pi]$$

the surface measure σ_n on the unit sphere S^n is given by

$$d\sigma_n(u) \equiv d\sigma_n(u_n, \dots, u_n) = \sin^{n-1}\varphi_1 \sin^{n-3}\varphi_2 \dots \sin\varphi_{n-2} \dots d\varphi_{n-1}$$

Set $v = (u_2, \dots, u_n)$ and $B_{n-1} \coloneqq \{v \in \mathbb{R}^{n-1}: |\leq 1|\}$. Writing $u \in S^n$ as $u = (u_1, v)$, we derive from the previous formula.

$$d\sigma_n(u) = \frac{1}{\sqrt{1-|v|^2}}dv, \text{ where } u_1^2 + |v|^2 = 1, v \in B_{n-1}. \quad (27)$$

Further, we write $v = (\mathcal{W}, t)$, where $w \in \mathbb{R}^{n-2}$ and $t \in \mathbb{R}$, and $x_k = (0, x_{2k}, \dots, x_{nk}) = (0, y_k, x_{kn})$, where $y_k \in \mathbb{R}^{n-2}$. Then we have $(u, rx_k) = r(w, y_k) + rtx_{kn}$. Let B_{n-2} denote the unit ball $B_{n-2} \coloneqq \{w \in \mathbb{R}^{n-2}: |w| \leq 1\}$ in $\mathbb{R}^{n-2}$. Using the equality (27) we then compute

$$\int_{S^n}\left|\sum_{k=1}^m \xi_k e^{i(u,rx_k)}\right|^2 d\sigma_n(u) = \int_{B_{n-1}}\left|\sum_{k=1}^m \xi_k e^{ir(w,y_k)_e irx_{nk}}\right|^2 \frac{1}{\sqrt{1-|v|^2}}dv \quad (28)$$

$$\int_{B_{n-1}}\left|\sum_{k=1}^m \xi_k e^{ir(w,y_k)}e^{irtx_{nk}}\right|^2 dv = \int_{B_{n-1}}\left(\int_{\sqrt{1-|w|^2}}^{\sqrt{1-|w|^2}}\left|\sum_{k=1}^m \xi_k e^{ir(w,y_k)}e^{irx_{nk}}\right|^2 dt\right)dw$$

$$= \int_{B_{n-2}} r^{-1}\left(\int_{-r\sqrt{1-|w|^2}}^{r\sqrt{1-|w|^2}}\left|\sum_{k=1}^m \xi_k e^{ir(w,y_k)}e^{isx_{nk}}\right|^2 ds\right)dw. \quad (29)$$

Since $d_*(X_n) > 0$ and $D^*(X_n) = 0$ by assumption, Corollary(5.1.16) implies that for any$a > 0$ the sequence $\{e^{isx_{kn}}\}_{k=1}^\infty$ is a Riesz-Fischer sequence in $L^2(-a,a)$. That is, there exists a constant $c(a) > 0$ such that

$$\int_{-a}^{a}\left|\sum_{k=1}^m (\xi_k e^{ir(w,y_k)})e^{isx_{nk}}\right|^2 ds \geq c(a)\sum_{k=1}^m \left|\xi_k e^{ir(w,y_k)}\right|^2 = c(a)\sum_{k=1}^m |\xi_k|^2.$$

Inserting this inequality, applied with $a = \sqrt{1-|w|^2} > 0$, into (29) and then (29) into (26) we obtain.

$$\sum_{k,j=1}^{m}\xi_k\bar{\xi}_j f(|x_k-x_j|)\geq\int_0^{+\infty}\left(\int_{B_{n-2}} r^{-1}\left(\int_{-r\sqrt{1-|w|^2}}^{r\sqrt{1-|w|^2}}\left|\sum_{k=1}^{m}(\xi_k e^{ir(w,y_k)})e^{isx_{nk}}\right|^2 ds\right)dw\right)d\bar{\nu}(r)$$

$$\geq\int_0^{-\infty}\left(\int_0^{+\infty}\int_{B_{n-2}} r^{-1}c\left(r\sqrt{1-|w|^2}\right)\left(\sum_{k=1}^{m}|\xi_k|^2\right)dw\right)d\bar{\nu}(r)$$

$$\geq\left(\int_0^{+\infty}\int_{B_{n-2}} r^{-1}c\left(r\sqrt{1-|w|^2}\right)dwd\bar{\nu}(r)\right)\sum_{k=1}^{m}|\xi_k|^2.$$

The double integral in front of the last sum is a finite constant, sayγ, by Since f is not constant by assumption, $\bar{\nu}((0,+\infty))>0$. Therefore, since $r^{-1}c\left(r\sqrt{1-|w|^2}\right)>0$ for all $r>0$ and $|w|<1$, we conclude that $\gamma>0$. This shows that f is stronglyX-positive definite.

Assuming $f\in\Phi_{n+1}$ rather that $f\in\Phi_n$ we obtain the following corollary.

Corollary(5.1.18) [176]: Assume that $f\in\Phi_{n+1}$ and f is not constant. Let $X=\{x_k\}_1^\infty$ be sequence of points $x_k=(x_{k1},x_{k2},\ldots,x_{kn})\in\mathbb{R}^n$. If the sequence $X_n:=\{x_{kn}\}_{k=1}^\infty$ of n-th coordinate satisfies the conditions $d_*(X_n)>0$ and $D^*(X_n)=0$, then f is stronglyX-positive definite.

Proof. We identify$\mathbb{R}^n$ with the subspace $0\oplus\mathbb{R}^{n+1}$. Then X is identified with the sequence $X=\{(0,x_k)\}_{k=1}^\infty$. Since $f\in\Phi_{n+1}$, Proposition (5.1.17) applies to the sequence $\widehat{X}$, so f is strongly$\widehat{X}$-positive definite. Hence it is stronglyX-positive definite.

The next proposition gives a more precise result for a sequence $X=\{x_k\}_{k=1}^\infty$ of $\mathbb{R}^3$ which are located on a line.

Proposition(5.1.19) [176]: Suppose that $\Lambda=\{\lambda_k\}_1^\infty$ is a real sequence and $r>0$. Let X be the sequence $X=\{x_k:=(0,0,\lambda_k)\}_{k=1}^\infty$ in $\mathbb{R}^3$ and let $f_r(x):=\Omega_3(r|x|), x\in\mathbb{R}^3$.

If $d_*(\Lambda)>0$ and $D^*(\Lambda)<r/\pi$, then the function f_r is stronglyX-positive definite.

If f_r is stronglyX-positive definite, then $d_*(\Lambda)>0$ and $D^*(\Lambda)\leq r/\pi$.

Proof. Suppose that $\xi=(\xi_1,\ldots,\xi_m)\in\mathbb{C}^m, m\in\mathbb{N}$. We introduce spherical coordinates on the unit sphere S^2 in $\mathbb{R}^3$ by.

$$u_1=\sin\theta\cos\varphi,\qquad u_2=\sin\theta\sin\varphi, u_3=\cos\theta,\qquad \text{where } \theta\in[0,\pi].$$

Then the surface measure σ_2 on the sphere S^2 is given by$d\sigma_2(u)=\sin\theta d\varphi d\theta$ and $(u,rx_k)=r\lambda_k\cos\theta$. Using these facts and Eq. (18) we compute.

$$\sum_{k,j=1}^{m}\xi_k\bar{\xi}_j f_r(|x_k-x_j|)=\sum_{k,j=1}^{m}\xi_k\bar{\xi}_j\Omega_3(r|x_k-x_j|)=\int_{S^2}\left|\sum_{k=1}^{m}\xi_k e^{i(u,rx_k)}\right|^2 d\sigma_2(u)$$

$$=\int_0^{2\pi}\int_0^{\pi}\left|\sum_{k}^{m}\xi_k e^{ir\xi_k\cos\theta}\right|^2\sin\theta d\varphi d\theta=2\pi\int_0^{\pi}\left|\sum_{k=1}^{m}\xi_k e^{ir\lambda_k\cos\theta}\right|^2\sin\theta d\theta.$$

Transforming the latter integral by setting $t=r\cos\theta$ obtain

$$\sum_{k,j=1}^{m}\xi_k\bar{\xi}_j f(|x_k-x_j|)=\frac{2\pi}{r}\int_{-r}^{r}\left|\sum_{k=1}^{m}\xi_k e^{i\lambda_k t}\right|^2 dt. \qquad (30)$$

Equality (30) is the crucial step for the proof of Proposition (5.1.19).

Since $d_*(\Lambda) > 0$ and $D^*(\Lambda) < r/\pi$, $(\Lambda) = \{e^{i\lambda_k t}\}_{k=1}^{\infty}$ is Riesz-Fischer sequence in $L^2(-r, r)$ by Proposition (5.1.15) (i). This means that there exists a constant $c > 0$ such that

$$\int_{-r}^{r}\left|\sum_{k=1}^{m}\xi_k e^{i\lambda_k t}\right|^2 dt \geq c\sum_{k=1}^{m}|\xi_k|^2.$$

Combined with (30) it follows that f is stronglyX-positive definite.

Since f is stronglyX-positive definite, there is a constant $c > 0$ such that

$$\sum_{k,j=1}^{m}\xi_k\bar{\xi}_j f(|x_k - x_j|) \geq c\sum_{k=1}^{m}|\xi_k|^2.$$

Because of (30) this implies that $E(\Lambda)$ is stronglyX-positive definite. Therefore, $d_*(\Lambda) > 0$ and $D^*(\Lambda) \leq r/\pi$ by Proposition (5.1.15) (ii).

Corollary(5.1.20) [176]: Assume the conditions of Proposition (5.1.19) and $r_0 > 0$. Then the functions f_r are stronglyX-positive definite for any$r \in (0, r_0)$ if and only if $d_*(\Lambda) > 0$ and $D^*(\Lambda) = 0$.

Here we discuss the question of when the Gram matrix (15) defines a bounded operator on $l^2(\mathbb{N})$. A standard criterion for showing that a matrix defines a bounded operator is Schur's test. It can be stated as follows:

Lemma(5.1.21) [176]: Let $A = \left(a_{k_j}\right)_{k,j\in\mathbb{N}}$ be an infinite Hermitian matrix satisfying.

$$C := \sup_{j\in\mathbb{N}}\sum_{k=1}^{\infty}|a_{kj}| < \infty. \qquad (31)$$

Then the matrix A defines a bounded self-adjoint operator A on $t^2(\mathbb{N})$ and we have $\|A\| \leq C$.

A proof of Lemma (5.1.21) can be found, e.g., in [173, p. 159].

Lemma(5.1.22) [176]: Let $A = (a_{kj})_{k,j\in\mathbb{N}}$be on infinite Hermitian matrix. Suppose that $(akj)_{k=1}^{\infty} \in t^2$ for all $j \in \mathbb{N}$ and

$$\lim_{m\to\infty}\left(\begin{matrix} sup \\ j \geq m\end{matrix}\sum_{k\geq m}|akj|\right) = 0. \qquad (32)$$

Then the Hermitian matrix $A = (akj)k, j \in \mathbb{N}$ defines a compact self-adjoint operator on $t^2(\mathbb{N})$.

Proof. For $m \in \mathbb{N}$ let A_m denote the matrix $(a_{k_j}^{(m)})_{k,j\in\mathbb{N}}$, where $a_{k_j}^{(m)} := 0$ if either $k \geq m$ or $j \geq m$and $a_{k_j}^{(m)} = a_{kj}$ otherwise. Clearly, A_m defines a bounded operator A_m on $l^2(\mathbb{N})$. From (32) it follows that the matrix $A - A_m$ satisfies condition (31) for large m, so $A - A_m$ defines a bounded operator B_m. Therefore A defines the bounded self-adjoint operator $A := A_m + B_m$.

Let $\varepsilon > 0$ be given. By (32), there exists m_0 such that $\sum_{k\geq m}|a_{jk}| < \varepsilon$ for $m > m_0$ and $j > m_0$. Using the latter, the Cauchy-Chwarz inequality and the relation $a_{kj} = a_{jk}$ we derive

$$\|B_m x\|^2 = \sum_{j>m}\left|\sum_{k>m} a_j x_k\right|^2 \le \sum_{j>m}\left(\sum_{k>m}|a_j k|\right)\left(\sum_{k>m}|a_j k|\,|x_k|^2\right) =\le \varepsilon \sum_{k>m}\sum_{j>m}|a_{kj}|\,|x_k|^2$$
$$\le \varepsilon^2 \sum_{k>m}|x_k|^2 \le \varepsilon^2\|x\|^2$$

for $x = \{x_j\}_1^\infty \in l^2(\mathbb{N})$ and $m > m_0$. This proves that $\lim_m \|B_m\| = \lim_m \|A - A_m\| = 0$. Obviously, A_m is compact, because it has finite rank. Therefore, A is compact.

An immediate consequence of Lemma (5.1.22) is the following matrix satisfying

Corollary (5.1.23) [176]: If $A = (A_{kj})_{k,\in\mathbb{N}}$ is an infinite Hermitian matrix satisfying.

$$\lim_{m\to\infty}\left(\sup_{j\in\mathbb{N}}\sum_{k\ge m}|a_{jk}|\right) - 0, \qquad (33)$$

then the matrix A defines a compact self-adjoint operator on $l^2(\mathbb{N})$.

Proposition(5.1.24) [176]: Let $f \in \Phi_n, n \ge 2$, and let v be the representing measure in Eq. (17). Let $X = \{x_k\}_1^\infty$ be a sequence of pairwise different points $x_k \in \mathbb{R}^n$. Suppose that for each $j, k \in \mathbb{N}, j \ne k$, there are positive numbers α_{kj} such that

$$K{:}\sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}{}' \frac{1}{(a_{kj}|x_k - x_j|)^{\frac{n-1}{2}}} < \infty. \qquad (34)$$

$$L{:}\sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}{}' v([0, a_{kj}]) < \infty. \qquad (35)$$

Then the matrix $G_{rX}(f) := (f(|x_k - x_j|))_{k,j\in\mathbb{N}}$ defines a bounded self-adjoint operator on $l^2(\mathbb{N})$.

Proof. By (19) the function $\Omega_n(t)$ has an alternating power series expansion and $\Omega_n(0) = 1$. Therefore we have $\Omega_n(t) \le 1$ for $t \in [0,\infty)$. It is well known (see, e.g., [163. P. 266]) that the Bessel function $J_{\frac{n-2}{2}}(t)$ behaves asymptotically as $\sqrt{\frac{2}{\pi t}}$ as $t \to \infty$. Therefore. It follows from (19) that there exists a constant C_n such that

$$|\Omega_n(t)| \le C_n t^{\frac{1-n}{2}} \text{ for } t \in (0,\infty). \qquad (36)$$

Using these facts and the assumptions (34) and (35) we obtain.

$$\sum_{k\in\mathbb{N}}{}' f(|x_k - x_j|) = \sum_{k\in\mathbb{N}}{}' \int_0^\infty \Omega_n(r|x_k - x_j|)dv(r) \le \sum_{k\in\mathbb{N}}{}'\left(\int_0^{\alpha k_j} 1dv(r) + C_n \int_{\alpha_{kj}}^\infty (r|x_k - x_j|)^{\frac{1-n}{2}}dv(r)\right)$$

$$\le \sum_{k\in\mathbb{N}}{}' v([0,\alpha_{kj}]) + \sum_{k\in\mathbb{N}}{}' C_n \int_{\alpha_{kj}}^\infty (\alpha_{kj}|x_k - x_j|)^{\frac{1-m}{2}} dv(r)$$

$$= L + C_n\left(\sum_{k\in\mathbb{N}}{}'(\alpha_{kj}|x_k - x_j|)^{\frac{1-n}{2}}\right)v(\mathbb{R}) \le L + C_n K v(\mathbb{R}).$$

so that

$$\sup_{j\in\mathbb{N}}\sum_{k=1}^\infty f(|x_k - x_j|) \le f(0) + L + C_n K v(\mathbb{R}) < \infty. \qquad (37)$$

This shows that the assumption (5.1.24) of the Schur test is fulfilled, so the matrix $G_{rX}(f)$ defines a bounded operator by Lemma (5.1.21).

The assumptions (35) and (34) are a growth condition of the measure ν at zero combined with a density condition for the set of points x_k. Let us assume that $\nu([0,\varepsilon]) = 0$ for some $\varepsilon > 0$. Setting $a_{kj} = \varepsilon$ in Proposition (5.1.24), (35) is trivially satisfied and 34) holds whenever.

$$\sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}}{}' \frac{1}{|x_k - x_j|^{\frac{n-1}{2}}} < \infty. \qquad (38)$$

Because of its importance we restate this result in the special case when $\nu = \delta_r$ is a delta measure at $r \in (0,\infty)$ separately as

Corollary(5.1.25) [176]: If $X = \{x_k\}_1^\infty$ is a sequence of pairwise distinct points $x_k \in \mathbb{R}^n$ satisfying (38), then for any$r > 0$ the infinite matrix $(\Omega_n(r|x_k - x_j|))_{k,j\mathbb{N}}$ define bounded operator on $l^2(\mathbb{N})$. An example is the next proposition.

Proposition(5.1.26) [176]: Suppose $X = \{x_k\}_1^\infty$ is a sequence of distinct points $x_k \in \mathbb{R}^\varepsilon$ such that

$$K := \sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}}{}' \frac{1}{|x_k - x_j|} < \infty. \qquad (39)$$

Let $r \in (0,\infty)$ and let A be the infinite matrix given by

$$\Omega_3(t,X) := (\Omega_3(t|x_k - x_j|))_{k,j\in\mathbb{N}} = \left(\frac{\sin(t|x_k - x_j|)}{t|x_k - x_j|}\right)_{k,j\in\mathbb{N}} \qquad (40)$$

where we set $\frac{\sin 0}{0} := 1$. If $r^{-1}K < 1$, then A defines a bounded self-addjoint operator A on $l^2(\mathbb{N})$ with bounded inverse; moreover, $\|A\| \le 1 + r^{-1}K$ and $\|A^{-1}\| \le (1 - r^{-1}K)^{-1}$.

Proof. Set $S \equiv (a_{kj})_{k,j\in\mathbb{N}} := A - I$, where I is the identity matrix. Since $a_{kj} = 0$, one has

$$\sup_{j\in\mathbb{N}} \sum_k |a_{kj}| = \sup_{j\in\mathbb{N}} \sum_k{}' \left|\frac{\sin(r|x_k - x_j|)}{r|x_k - x_j|}\right| \le r^{-1} \sup_{j\in\mathbb{N}} \sum_k{}' \frac{1}{|x_k - x_j|} = r^{-1}k$$

This shows that Hermitian matrix S satisfies the assumption ((5.1.24)) of Lemma (5.1.21) with $C \le r^{-1}K$. Thus S is the matrix of a bounded self-adjoint operator S such that $\|S\| \le r^{-1}K$. We have $S := A - I$. This implies that A is the matrix of a bounded self-adjoint operator $A = I + S$ and $\|A\| \le 1 + r^{-1}K < 1$, A has a bounded inverse and $\|A^{-1}\| \le (1 - r^{-1}K)^{-1}$.

Let Δ denote the Laplacian on $\mathbb{R}^3$ with domain $\operatorname{dom}(-\Delta) = W^{2.2}(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3)$. It is well known that $-\Delta$ is self-adjoint. We fix a sequence $X = \{x_k\}_1^\infty$ of pairwise distinct points $x_j \in \mathbb{R}^3$ and denote by$H$ the restriction

$$H := -\Delta \restriction \operatorname{dom}H \quad \operatorname{dom}H = \{f \in W^{2,2}(\mathbb{R}^3) : f(x_j) = 0 \text{ for all } j \in \mathbb{N}\}. \qquad (41)$$

We abbreviate $r_j := |x - x_j|$ for $x = (x^1, x^2, x^3) \in \mathbb{R}^3$. For $z \in \mathbb{C} \setminus [0,\infty)$ we denote by$\sqrt{z}$ the branch of the square root of z with positive imaginary part.

Further, let us recall the formula for the resolvent $(-\Delta - zI)^{-1}$ on $L^2(\mathbb{R}^3)$ (see [159]):

$$((-\Delta - zI)^{-1}f)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{x}|x-t|}}{|x - t|} f(t)dt, \quad f \in L^2(\mathbb{R}^3). \qquad (42)$$

Lemma (5.1.27) [176]: The sequence $E := \{\frac{1}{\sqrt{2\pi}}\varphi_j\}_{j=1}^\infty = \{\frac{1}{\sqrt{2\pi}} \frac{e^{-|x-x_j|}}{|x-x_j|}\}_{j=1}^\infty$ is normed and complete in the deficit subspace $\mathfrak{N}_{-1} (\subset L^2(\mathbb{R}^3))$ of the operator H.

Proof. Suppose that $f \in \mathfrak{N}_{-1}$ and $f \perp E$. Then $u := (1-\Delta)^{-1} f \in W^{2,2}(\mathbb{R}^3)$. By (42), we have

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-|x-r|}}{|x-t} f(t)dt. \qquad (43)$$

Therefore, the orthogonality condition $f \perp E$ means that

$$0 = \langle f, \varphi_j \rangle = \frac{1}{4\pi} \int_{\mathbb{R}^3} f(t) \frac{e^{-|x-r|}}{|x-t} dt = u(x_j), \qquad j \in \mathbb{N}. \qquad (44)$$

By (44) and (41), $u \in \mathrm{dom}(H)$ and $f = (I-\Delta)u = (I+H)u \in \mathrm{ran}(I+H)$. Thus

$$f \in \mathfrak{N}_{-1} \cap \mathrm{ran}(I+H) = \{0\},$$

i.e., $f = 0$ and the system E is complete.

The function $e^{|\cdot|} (\in W^{2,2}(\mathbb{R}^3))$ is a (generalized) solution of the equation $(I-\Delta)e^{-|x|} = 2\frac{\exp(-|x|)}{|x|}$.

Therefore it follows from (43) with $f = f_y(x) := \frac{e^{-|x-y|}}{|x-y|}$ that

$$\frac{e^{-|x-y|}}{2} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-|x-t|}}{|x-t|} \cdot \frac{e^{-|x-y|}}{|t-y|} dt. \qquad (45)$$

Setting here $x = y = x_j$ we get $\|\varphi_j\|^2 = 2\pi$, i.e., the system E is normed.

In order to state the next result we need the following definition.

Definition(5.1.28) [176]:A sequence $\{f_j\}_1^\infty$ of vector of a Hilbert space is called W-linearly independent if for any complex sequence $\{c_j\}_1^\infty$ the relations.

$$\sum_{j=1}^{\infty} c_j f_j = 0 \ \text{ and } \ \sum_{j=1}^{\infty} |c_j|^2 \|f_j\|^2 < \infty \qquad (46)$$

imply that $c_j = 0$ for all $j \in \mathbb{N}$.

Lemma(5.1.29) [176]: Assume that $X = \{x_j\}_1^\infty$ has no finite accumulation points. Then the sequence $E\{\frac{1}{\sqrt{2\pi}} \varphi_j\}_{j=1}^\infty = \{\frac{1}{\sqrt{2\pi}} \frac{e^{-|x-x_j|}}{|x-x_j|}\}_{j=1}^\infty$ is W-linearly independent in $\mathfrak{N} = L^2(\mathbb{R}^3)$.

Proof. Assume that for some complex sequence $\{c_j\}_1^\infty$ conditions (46) are satisfied with φ_j in place of f_j. By Lemma (5.1.27), $\|\varphi_j\| = \sqrt{2\pi}$. Hence the second of condition (46) is equivalent to $\{c_j\} \in l^2$. Furthermore, since each function $\varphi_j(x)$ is harmonic in $\mathbb{R}^3 \setminus \{x_j\}$, this implies that the series $\sum_{j=1}^\infty c_j \varphi_j$ converges uniformly on each compact subset of $\mathbb{R}^3 \setminus X$.

Fix $k \in \mathbb{N}$. Since the points x_j are pairwise distinct and the set X has no finite accumulation points, there exists a compact neighborhood U_k of x_k and such that $x_j \notin U_k$ for all $j \neq k$. Then, by the preceding considerations, the series $\sum_{j \neq k} c_j \varphi_j$ converges uniformly on U_k.

From the first equality of (46) it follows that

$$-c_k = \sum_{j \in \mathbb{N}} c_j e^{-|x-x_j|} |x-x_j|^{-1} |x-x_k|$$

for all $x \in U_k, x \neq x_k$. Therefore, passing to the limit as $x \to x_k$ we obtain $c_k = 0$.

Definition(5.1.30) [176]:

(i) A sequence $\{f_j\}_1^\infty$ in the Hilbert space $\mathfrak{H}$ is called minimal if for any k

$$\mathrm{dist}\{f_k, \mathfrak{H}^{(k)}\} = \varepsilon_k > 0, \quad \mathfrak{H}^{(k)} := \mathrm{span}\{f_j : j \in \mathbb{N} \setminus \{k\}\}, \qquad k \in \mathbb{N} \tag{47}$$

(ii) A sequence $\{f_j\}_1^\infty$ is said to be uniformly minimal if $\inf_{k\in\mathbb{N}} \varepsilon_k > 0$.

(iii) A sequence $\{g_j\}_1^\infty \subset \mathfrak{H}$ is called birothogonal to $\{f_j\}_1^\infty$ if $\langle f_j, g_k\rangle = \delta_{jk}$ for all $j, k \in \mathbb{N}$.

Let us recall two well-known facts (see. e.g. [137]): A birothogonal sequence to $\{f_j\}_1^\infty$ exist if and only if the sequence $\{f_j\}_1^\infty$ is minimal. If this is true, then the biorthogonal sequence is uniquely determined if and only if the set $\{f_j\}_1^\infty$ is complete in $\mathfrak{H}$.

Recall that the sequence $\{\varphi_j\}$ is complete in $\mathfrak{N}_{-1}$ according to Lemma (5.1.27).

Lemma(5.1.31) [176]: Assume that $X = \{x_k\}_1^\infty$ has no finite accumulation points.

(i) The sequence $E := \{\varphi_j\}_1^\infty$ is minimal in $\mathfrak{N}_{-1}$.

(ii) The corresponding biorthogonal sequence $\{\psi_j\}_1^\infty$ is also complete in $\mathfrak{N}_{-1}$.

Proof. (i) To prove minimality it suffices to construct a biorthogonal system. Since X has no finite accumulation point, for any $j \in \mathbb{N}$ there exists a function $\tilde{u}_j \in C_0^\infty(\mathbb{C}^3)$ such that

$$\tilde{u}_j(x_j) = 1 \quad \text{and} \quad \tilde{u}_j(x_k) = 0 \quad \text{for } k \neq j \tag{48}$$

Moreover, $\tilde{u}_j(\cdot)$ can be chosen compactly supported in a small neighborhood of x_j.

Let $\widetilde{\psi}_j := (I - \Delta)\tilde{u}_j, j \in \mathbb{N}$. In general, $\widetilde{\psi}_j \notin \mathfrak{N}_{-1}$. To avoid this drawback we put

$$\psi_j := P_{-1}\widetilde{\psi}_j \in \mathfrak{N}_{-1} \quad \text{and} \quad g_j := \widetilde{\psi}_j - \psi_j, \qquad j \in \mathbb{N}. \tag{49}$$

where P_{-1} is the orthogonal projection in $\mathfrak{N}$ onto $\mathfrak{N}_{-1}$. Then $g_j \in \mathrm{ran}(I + H) = \mathfrak{H} \ominus \mathfrak{N}_{-1}, j \in \mathbb{N}$. Setting $v_j = (I - \Delta)^{-1}g_j$, we get $v_j \in \mathrm{dom}(\Delta)$. Therefore, by the Sobolev embedding theorem, $v_j \in C(\mathbb{R}^3)$. Together with the sequence $\{\tilde{u}_j\}_1^\infty$ we consider the sequence of functions.

$$u_j := \tilde{u}_j - v_j \in W^{2,2}(\mathbb{R}^3), \qquad j \in \mathbb{N}. \tag{50}$$

Since $v_j \in \mathrm{dom}(H)$, the functions u_j satisfy relations (48) as well. Thus,

$$-\Delta u_j + u_j = \psi_j \in \mathfrak{N}_{-1} \quad \text{and} \quad u_j(x_k) = \delta_{kj} \quad \text{for } j, k \in \mathbb{N}. \tag{51}$$

Combining these relations with resolvent formula (42) we get

$$\langle \varphi_j, \varphi_k\rangle = \frac{1}{4\pi}\int_{\mathbb{R}^3} \psi_j(x)\frac{e^{-|x-x_j|}}{|x - x_j|}dx = (I - \Delta)^{-1}\psi_j = u_j(x_k) = \delta_{kj}, j \in \mathbb{N} \tag{52}$$

These relations means that the sequence $\{\psi_j\}_1^\infty$ is biorthogonal to the sequence $\{\psi_j\}_1^\infty$. Hence the latter is minimal.

(ii) Let $\mathfrak{H}_1$ denote the closed linear span of the set $\{u_j; j \in \mathbb{N}\}$ in $W^{2,2}(\mathbb{R}^3)$.

We prove that $W^{2,2}(\mathbb{R}^3)$ is the closed linear span of its subspaces $\mathfrak{H}_1$ and $\mathrm{dom}(H)$. Indeed, assume that $g \in W^{2.2}(\mathbb{R}^3)$ and has a compact support $K = \mathrm{supp}\, g$. Then the intersection $X \cap K$ is finite since X has no accumulation points. Therefore the function.

$$g_1 = \sum_{x_j \in K} g(x_j) u_j \qquad (53)$$

is well defined and $g_1 \in \mathfrak{H}_1$. It follows from (51) that $g_0 := g_1 \in \mathrm{dom}(H)$ and $g = g_1 + g_0$. It remains to note that $C_0^\infty(\mathbb{R}^3)$ is dense in $W^{2,2}(\mathbb{R}^3)$.

Suppose that $f \in \mathfrak{N}_{-1}$ and $\langle f, \psi_j \rangle = 0, j \in \mathbb{N}$. Then, by (51).

$$0 = \langle f, \psi_j \rangle = \langle f, (-\Delta + I) u_j \rangle. \quad j \in \mathbb{N}. \qquad (54)$$

The inclusion $f \in \mathfrak{N}_{-1}$ means that $f \perp (I - \Delta)\mathrm{dom}(H)$. Combining this with (54) and using that $W^{2,2}(\mathbb{R}^3)$ is the closure of $\mathfrak{H}_1 + \mathrm{dom}(H)$ as shown above, it follows that $f \perp \mathrm{ran}(I - \Delta) = L^2(\mathbb{R}^3)$. Thus $f = 0$ and the sequence $\{\psi_j\}_1^\infty$ is complete.

Lemma(5.1.32) [176]: If $E = \{\varphi_j\}_1^\infty$ is uniformly minimal, then X has no finite accumulation points.

Proof. Since $\{\varphi_j\}_1^\infty$ is minimal in $\mathfrak{N}_{-1}$, there exists the biorthogonal sequence $\{\psi_j\}_1^\infty$ in $\mathfrak{N}_{-1}$. It was already mentioned that the uniform minimality of $E = \{\varphi_j\}_1^\infty$ is equivalent to $\sup_{j \in \mathbb{N}} \|\varphi_j\|. \|\psi_j\| < \infty$. Therefore, since $\|\varphi_j\| = 2\sqrt{\pi}$, by Lemma (5.1.27), the sequence $(\psi_j; j \in \mathbb{N})$ is unitofmly bounded i.e., $\sup_j \|\psi_j\| =: C_0 < \infty$. Setting $u_j = (I - \Delta)^{-1} \psi_j \in W_2^2(\mathbb{R}^3)$ we conclude that the sequence $\{u_j\}_1^\infty$ is uniformly bounded in $W^{2,2}(\mathbb{R}^3)$, that is, $\sup_{j \in \mathbb{N}} \|u_j\|_{W^{2,2}} = C_1 < \infty$.

Now assume to the contrary that there is a finite accumulation point y_0 of X. Thus, there exists a subsequence $\{x_{j_m}\}_{m=1}^\infty$ such that $y_0 = \lim_{m \to \infty} x_{jm}$. By the Sobolve embedding theorem, the set $\{u_j; j \in \mathbb{N}\}$ is compact in $C(\mathbb{R}^3)$. Thus there exists a subsequence of $\{u_{j_m}\}$ which converges uniformly to $u_0 \in C(\mathbb{R}^3)$. Without loss of generality we assume that the sequence $\{u_{j_m}\}$ itself converges to u_0, i.e. $\lim_{m \to \infty} \|u_{j_m} - u_0\|_{C(\mathbb{R}^3)} = 0$. Hence

$$1 = u_{j_m}(x_{j_m}) \xrightarrow[m \to \infty]{} u_0(y_0) = 1, \qquad 0 = u_{j_m}(u_{j_{m-1}}) \xrightarrow[m \to \infty]{} u_0(y_0) = 0$$

which is the desired contradiction.

Lemma(5.1.33) [176]: Suppose that $d_*(X) = 0$. If the matrix $\mathcal{T}_1 := (\frac{1}{2} e^{-|x_j - x_k|})_{j,k \in \mathbb{N}}$ defines a bounded self-adjoint operator T_1 on $l^2(\mathbb{N})$, then $0 \in \sigma_c(T_1)$, hence T_1 has not bounded inverse.

Proof. Let $\varepsilon > 0$. Since $d_*(X) = 0$, there exist number $n_j \in \mathbb{N}$ such that $r_{jk} := |x_{n_j} - x_{n_k}| < \varepsilon$. Let e_n denote the vector $e_n := \{\delta_{p,m}\}_{p=1}^\infty$ of $l^2(\mathbb{N})$. Then $2\,T_1(e_j - e_k) = \{e^{-r_{pj}} - e^{-r_{pk}}\}_{p=1}^\infty \in l^2(\mathbb{N})$.

Since $|r_{pj} - r_{pk}| < r_{jk} < \varepsilon$ by the triangle inequality, $e^{-\varepsilon} \le \exp(r_{pj} - r_{pk}) \le e^\varepsilon$ and hence

$$|e^{-r_{pj}} - e^{-r_{pk}}| = e^{-r_{pj}} |1 - e^{r_{pj} - r_{pk}}| \le \varepsilon C e^{-r_{pj}}, \qquad j, k, p \in \mathbb{N}.$$

where $C > 0$ is a constant. Using the assumption that T_1 is bounded we get

$$4\|T_1(e_j - e_k)\|^2 \le \varepsilon^2 C^2 \sum_p e^{-2r_{p_j}} = 4\varepsilon^2 C^2 \|T_1 e_j\|^2 \le 4\varepsilon^2 C^2 \|T_1\|^2. \qquad (55)$$

Since $\varepsilon > 0$ is arbitrary and $\|e_j - e_k\| = \sqrt{2}$ for $j \ne k$, it follows that $0 \in \sigma_c(T_1)$.

Theorem(5.1.34) [176]: The sequence $E = \{\varphi_k\}_1^\infty$ forms a Riesz basis of the Hilbert space $\mathfrak{N}_{-1}$ and only if $d_*(X) > 0$.

Proof. Sufficiency. Suppose that $d_*(X) > 0$. By Lemma (5.1.27) and (5.1.31), both sequences

$\{\varphi_j\}_1^\infty$ and $\{\psi_j\}_1^\infty$ are complete in $\mathfrak{N}_{-1}$. Therefore, by [137, Theorem 6.2.1], the sequence $\{\varphi_j\}$ forms a Riesz basis in $\mathfrak{N}_{-1}$ if and only if.

$$\sum_{j=1}^{\infty}\left|\langle f,\varphi_j\rangle\right|^2 < \infty \quad and \quad \sum_{j=1}^{\infty}\left|\langle f,\varphi_j\rangle\right|^2 < \infty \quad for\, all\, f \in \mathfrak{N}_{-1}. \tag{56}$$

Let B_j denote the ball in $\mathbb{R}^3$ centered at x_j with the radius $r = d_*(X)/3, j \in \mathbb{N}$. Clearly$B_j \cap B_k = \emptyset$ for $j \neq k$. By the Sobolve embedding theorem, there is a constant $C > 0$ such that

$$\left|v(x_j)\right| \le C\|v\|_{W^{2,2}(B_j)}, \quad v \in W^{2,2}(B_j), \qquad j \in \mathbb{N}, \tag{57}$$

where C is independent of j and $v \in W^{2,2}(B_j)$.

Let $f \in \mathfrak{N}_{-1}$ and set $u = (I - \Delta)^1 fu \in W^{2,2}(\mathbb{R}^3)$. Combining (57) with the representation (5.1.28) for u we get

$$\sum_{j=1}^{\infty}\left|(f,\varphi_j)\right|^2 = \sum_{j=1}^{\infty}\left|u(x_j)\right|^2 \le C\sum_{j=1}^{\infty}\|u\|^2_{W^{2,2}(B_j)} \le C\|u\|^2_{W^{2,2}(\mathbb{R}^3)}, f \in \mathfrak{N}_{-1} \tag{58}$$

This proves the first inequality of (56).

We now derive the second inequality. Let B_0 be the ball centered at zero with the radius $r = d_*(X)/3$. We choose a function $\tilde{u}_0 \in C_0^\infty(\mathbb{R}^3)$ supported in B_0 and satisfying $\tilde{u}_0(0) = 1$. Put

$$\tilde{u}_j(x) := \tilde{u}_0(x - x_j), \quad \in \mathbb{N}. \tag{59}$$

Clearly, the sequence $\{\tilde{u}_j\}_1^\infty$ satisfies conditions (33). Then repeating the reasonings of the proof of Lemma (5.1.31) (i) we find a sequence $\{v_j\}_1^\infty$ of vectors from $\mathrm{dom}(H)$ such that the new sequence $\{u_j := \tilde{u}_j - v_j\}_1^\infty$ satisfies relations (51). Hence for any$f \in \mathfrak{N}_{-1}$.

$$\langle f,\psi_j\rangle = \langle f,(-\Delta + I)u_j\rangle = \langle f,(-\Delta + I)(\tilde{u}_j - v_k)\rangle = \langle f,(-\Delta + I)\tilde{u}_j\rangle, \qquad j \in \mathbb{N}. \tag{60}$$

Since $\tilde{u}_j(\cdot)$ is supported in the ball B_j, it follows form (59) and relation (60) that

$$\begin{aligned}\sum_{j=1}^{\infty}|\langle f,\psi_k\rangle|^2 &= \sum_{j=1}^{\infty}\left|\langle f,(-\Delta + I)\tilde{u}_j\rangle\right|^2 \\ &\le C\sum_{j=1}^{\infty}\|f\|^2_{L^2(B_j)}\left\|\tilde{u}_j\right\|^2_{W^{2,2}(B_j)} \\ &= C\sum_{j=1}^{\infty}\|f\|^2_{L^2(B_j)}\|\tilde{u}_0\|^2_{W^{2,2}(B_0)} \\ &= C\|u_0\|^2_{W^{2,2}(B_0)}\sum_{j=1}^{\infty}\|f\|^2_{L^2(B_j)} \le C\|\tilde{u}_0\|^2_{W^{2,2}(B_0)}\|f\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Thus, the second inequality (56) is also proved, hence $\{\varphi_j\}$ forms a Riesz basis.

Necessity. Suppose the $d_*(X) = 0$. By [137, Theorem 6.2.1], a sequence $\psi = \{\psi_j\}_1^\infty$ of vectors is a Riesz basis of a Hilbert space $\mathfrak{H}$ if and only it is complete in $\mathfrak{H}$ and its Gram matrix $G_{r_\psi} := (\langle\psi_j,\psi_k\rangle)_{j,k\in\mathbb{N}}$ defines a bounded operator on $l^2(\mathbb{N})$ with bounded inverse.

By (45), $E = \{\varphi_j\}_1^\infty$ has the Gram matrix $G_{r_E} = (\langle\psi_j,\psi_k\rangle)_{j,k\in\mathbb{N}} = \left(\pi e^{-|x_j - x_k|}\right)_{j,k\in\mathbb{N}} = 2\pi\mathcal{T}_1$.

Therefore, by Lemma (5.1.30), if G_{r_E} defies a bounded operator, this operator is not boundedly invertible. Hence $E = \{\varphi_j\}_1^\infty$ is not a Riesz basis by the preceding theorem.

Remark(5.1.35) [176]: Note that the proof of uniform minimality of the system E is much simpler. Combining (59) with (60) we obtain.

$$|\langle f, \psi_j\rangle| \le \|f\|_{L^2} \cdot \|(I-\Delta)\tilde{u}_j\|_{L^2} \le \|f\|_{L^2}\|\tilde{u}_j\|_{W^{2,2}(\mathbb{R}^3)} = \|f\|_{L^2}\|\tilde{u}_0\|_{W^{2,2}(\mathbb{R}^3)}, \qquad j \in \mathbb{N}. \qquad (61)$$

Since $f \in \mathfrak{N}_{-1}$ is arbitrary, one has $\sup_{j\in\mathbb{N}}\|\varphi_j\|_{L^2(\mathbb{R}^3)} \le \|\tilde{u}_0\|_{W^{2,2}(\mathbb{R}^3)}$, so $\{\psi_j\}_{j\in\mathbb{N}}$ is uniformly minimal

Next we set

$$\varphi_{j,z}(x) := \frac{e^{i\sqrt{x}|x-x_j|}}{|x-x_j|} \quad \text{and} \quad e_{j,z}(x) : e^{i\sqrt{z}|x-x_j|}, \qquad j \in \mathbb{N}. \qquad (62)$$

Clearly, $\varphi_{j,-1} = \varphi_j, j \in \mathbb{N}$.

Corollary(5.1.36) [176]: Suppose that $d_*(X) > 0$. Then for any $z \in \mathbb{C} \setminus [0,\infty)$, the sequence $E_z := \{\frac{1}{2\pi}\varphi_{j.z}\}_{j=1}^\infty$ forms a Riesz basis in the deficiency subspace $\mathfrak{N}_z$ of the operator H. Moreover, for $z = -a^2 < 0 (a > 0)$ the system $\sqrt{a}E_{-a^2} = \{\frac{\sqrt{a}}{\sqrt{2\pi}}\varphi_{j,-a^2}\}_{j=1}^\infty$. Is normed.

Proof. It is easily seen that

$$\int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} \cdot \frac{e^{i\sqrt{z}|u-x_j|}}{|y-x_j|} dy \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}|x-y|}}{|x-y|} \cdot \frac{e^{-|y-x_j|}}{|x-x_j|} dy, \quad \in \mathbb{N}. \qquad (63)$$

Using (42) we can rewrite this equality as

$$(I-\Delta)^{-1}\varphi_{j,z} = (-\Delta - z)^{-1}\varphi_j, \qquad j \in \mathbb{N}, z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+. \qquad (64)$$

Therefore, we have

$$\varphi_{j,z} = U_{z\varphi_j}, \qquad \text{where} \quad U_z := (I-\Delta)(-\Delta - z)^{-1} = I - (1+z)(\Delta + z)^{-1}. \qquad (65)$$

Obviously, U_z is a continuous bijection of $\mathfrak{N}_{-1}$ onto $\mathfrak{N}_z$. therefore, since $E = E_{-1} = \{\varphi_j\}_{j\in\mathbb{N}}$ is Riesz basis of $\mathfrak{N}_{-1}$ by Theorem (5.1.31), $E_z = \{\varphi_{j,z}\}_{j=1}^\infty$ is a Riesz basis of $\mathfrak{N}_z$.

To prove the second statement we note that for any $a > 0$ the function $e^{-a|\cdot|}(\in W^{2,2}(\mathbb{R}^3))$ is a (generalized) solution of the equation $(a^2 I - \Delta)e^{-a|x|} = 2a\frac{\exp(-a|x|)}{|x|}$. Taking this equality into account we obtain from (42) with $z = -a^2$ and $f = f_y(x) := \frac{e^{-a|x-y|}}{|t-y|}$ that

$$\frac{e^{-a|x-y|}}{2a} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-a|x-t|}}{|x-t|} \cdot \frac{e^{-a|t-y|}}{|t-y|} dt, \quad a > 0. \qquad (66)$$

Setting here $x = y = x_j$ we get $\|\varphi_{j,-a^2}\|^2 = 2\pi/a$, i.e., the system $\sqrt{a}E_{-a^2}$ is normed.

Theorem(5.1.37) [176]: Let f be a non-constant function of $M[0,\infty)$ and let τ be the representing measure in Eq. (21). Suppose that $X = \{x_k\}_1^\infty$ is a sequence of points $x_k \mathbb{R}^3$. Then:

(i) If $d_*(X) > 0$, then the function $f(|\cdot|)$ is strongly X-positive definite.

(ii) Suppose that $d_*(X) > 0$ and

$$\int_0^\infty (s + s^{-3}) d\tau(s) < \infty. \qquad (67)$$

Then the Gram matrix $Gr_X(f) = \left(f(|x_k - x_j|)\right)_{k,j\in\mathbb{N}}$ defines a bounded operator with bounded inverse on $l^2(\mathbb{N})$.

(iii) If the Gram matrix $Gr_X(f)$ defines a bounded operator with bounded inverse on $l^2(\mathbb{N})$, then $d_*(X) > 0$.

Proof.(i) Suppose that $s \in (0, +\infty)$ and set

$$g_s(x) \coloneqq s^{-1}e^{-s|x|},\ \ \varphi_{j,s}(x) \coloneqq \frac{1}{\sqrt{2\pi}}\varphi_{j,-s^2}(x)\frac{1}{\sqrt{2\pi}}\frac{e^{-s|x-x_j|}}{|x-x_j|}, \qquad j \in \mathbb{N}.$$

Eq. (45) shows that $Gr_X(g_s) = (g_s(x_k - x_j))_{k,j\in\mathbb{N}}$ is the Gram matrix of the sequence $E_{-s^2} \coloneqq \{\overline{\varphi}_{j,s}\}_{j=1}^{\infty}$. Since $d_*(X) > 0$ by assumption, E_{-s^2} forms a Riesz by Corollary(5.1.33). Therefore it follows from [137, Theorem 6.2.1] that for any$s > 0$ the Gram matrix $\left(\langle\widetilde{\varphi}_{j,s}, \widetilde{\varphi}_{k,s}\rangle_{L^2(\mathbb{R}^3)}\right)_{j,k\in\mathbb{N}} = G_{r_X}(g_s)$ defines a bounded operator on $l^2(\mathbb{N})$ with bounded invese. Hence for any$s > 0$ and $c(s) > 0$ such that

$$C(s)\sum_{j=1}^{m}\left|\xi_j\right|^2 \geq \sum_{j,k=1}^{m}\langle\widetilde{\varphi}_{j,s}, \widetilde{\varphi}_{k,s}\rangle_{L^2(\mathbb{R}^3)}\,\xi_j\bar{\xi}_k = \sum_{j,k=1}^{m} s^{-1}e^{-s|x_j-x_k|}\xi_j\bar{\xi}_k \geq c(s)\sum_{j=1}^{m}\left|\xi_j\right|^2 \quad (68)$$

for all $(\xi_1, \dots, \xi_m) \in \mathbb{C}^m$ and $m \in \mathbb{N}$. Clearly, the function $c(s)$ on $(0, +\infty)$ can be chosen to be measurable. Since $c(s) > 0$ on $\mathbb{R}_+$ and $\tau(\mathbb{R}_+) > 0$, we have $c \coloneqq \int_{(0,+\infty)} sc(s)d\tau(s) > 0$. Combining (21) with (68) we arrive at the inequality.

$$\begin{aligned}\sum_{j,k=1}^{m} f(|x_j - x_k|)\xi_j\bar{\xi}_k &= \int_0^{\infty}\left(\sum_{j,k=1}^{m} e^{-s|x_j-x_k|}\,\xi_j\bar{\xi}_k\right)d\tau(s)\\ &\geq \int_0^{\infty} s\left(c(s)\sum_{j=1}^{m}\left|\xi_j\right|^2\right)d\tau(s) = c\sum_{j=1}^{m}\left|\xi_j\right|^2, \qquad (69)\end{aligned}$$

This means that the function $f(|\cdot|)$ is strongly$X$-positive definite.

(ii) By (65), $U_{-s^2} = (I - \Delta + s^2)^{-1}$, hence $\|U_{-s^2}\| = \max(1.s^2)$. Moreover, by (65). $\widetilde{\varphi}_{j,s} = U_{-s^2}\widetilde{\varphi}_{j,1}$. Using the preceding facts we derive

$$\sum_{j,k=1}^{m} f(|x_j - x_k|)\xi_j\bar{\xi}_k = \int_0^{\infty}\left(\sum_{j,k=1}^{m} e^{-s|x_j-x_k|}\,\xi_j\bar{\xi}_k\right)d\tau(s)(70)$$

$$= \sum_{j,k=1}^{m}\int_0^{+\infty} s\langle\widetilde{\varphi}_{j,s},\widetilde{\varphi}_{k,s}\rangle\xi_j\bar{\xi}_k d\tau(s)$$

$$= \int_0^{+\infty} s\left\|\sum_{j=1}^{m}\xi_j\widetilde{\varphi}_{j,s}\right\|^2 d\tau(s)$$

$$= \int_0^{+\infty} s\left\|U_{-s^2}\left(\sum_{j=1}^{m}\xi_j\widetilde{\varphi}_{j,1}\right)\right\|^2 d\tau(s) \le \int_0^{+\infty} s\|U_{-s^2}\|^2\left\|\sum_{j=1}^{m}\xi_j\widetilde{\varphi}_{j,1}\right\|^2 d\tau(s)$$

$$= 2\int_0^{+\infty} s\|U_{-s^2}\|^2\sum_{j,k=1}^{m}\langle\widetilde{\varphi}_{j,1},\widetilde{\varphi}_{k,1}\rangle\,\xi_j\bar{\xi}_k d\tau(s) \le \int_0^{+\infty} s(1+s^{-4})C(1)\left(\sum_{j=1}^{m}|\xi_j|^2\right)d\tau(s)$$

$$= C\sum_{j=1}^{m}|\xi_j|^2, \qquad (71)$$

where $C := C(1)\int_0^{+\infty}(s+s^{-3})d\tau(s) < \infty$ by assumption (67).

It follows from (69) and (70) that the matrix $G_{rx}(f)$ defines a bounded operator with bounded inverse.

(iii) Suppose that $d_*(X) = 0$. Assume to the contrary that the Gram matrix $G_{rx}(f)$ defines a bounded operator, say$T$, with bounded inverse on $l^2(\mathbb{N})$.

Fix $\varepsilon \in (0, \tau([0,\infty)))$. Since the measure τ is finite, there exists $s_0 > 0$ such that

$$\int_{[s_0,\infty)} d\tau(s) < \varepsilon < \tau([0,\infty)). \qquad (72)$$

By the assumption $d_*(X) = 0$ we can find points $x_j, x_j \in X, k, l \in \mathbb{N}$, such that $r_{jk} = |x_j - x_k| \le s_0^{-1}\ln(1+\varepsilon(0,s_0])))^{-1})$. Fix a number $l \in \mathbb{N}$. First suppose $r_{jl} \le r_{kl}$. Then

$$0 \le \left(1 - e^{s(r_{kl}-r_{jl})}\right)^2 \le 1 - e^{-sr_{kj}} \le \frac{\varepsilon\left(\tau([0,s_0])\right)^{-1}}{1+\varepsilon\left(\tau([0,s_0))\right)^{-1}} \le \varepsilon\left(\tau([0,s_0])\right)^{-1}, \qquad s \in [0,s_0]. \quad (73)$$

Using (72) and (73) we derive

$$\left(\int_0^{\infty}(e^{sr_{jl}} - e^{-sr_{kl}})d\tau(s)\right)^2 = \left(\int_0^{\infty} 1 - e^{s(r_{kl}-r_{jl})}e^{-sr_{jl}}d\tau(s)\right)^2$$

$$= \left(\int_0^{\infty}\left(1 - e^{-s(r_{kl}-r_{jl})}\right)^2 d\tau(s) + \int_0^{s_0}\left(1 - e^{-s(r_{kl}-r_{jl})}\right)^2 d\tau(s)\right)\left(\int_0^{\infty} e^{-2sr_{jl}}d\tau(s)\right)$$

$$\le 2\varepsilon\int_0^{\infty} e^{-2sr_{jl}}d\tau(s). \qquad (74)$$

If $r_{jl} > r_{kl}$ then the same reasoning yields.

$$\left(\int_0^\infty (e^{-sr_{jl}} - e^{-s})d\tau(s)\right)^2 \le 2_\varepsilon \int_0^\infty e^{2sr_{kl}}d\tau(s). \qquad (75)$$

Summing over l in (74) respectively (75) we obtain.

$$\|T(e_j - e_k)\|^2_{l^2(\mathbb{N})}$$
$$= \sum_l |\langle T(e_j - e_k), e_l\rangle|^2 = \sum_l \left(\int_0^\infty (e^{-sr_{jl}} - e^{-sr_{kl}})d\tau(s)\right)^2$$
$$\le 2\varepsilon \sum_l \left(\int_0^\infty e^{-2sr_{jl}}d\tau(s) + \int_0^\infty e^{-2sr_{kl}}d\tau(s)\right) = 2\varepsilon\left(\|T_{e_j}\|^2 + \|T_{e_k}\|^2\right)$$
$$\le 4\varepsilon\|T\|^2. \qquad (76)$$

and hence

$$4 = \|e_j - e_k\|^2 \le \|T^{-1}\|^2\|T(e_j - e_k)\|^2 \le 4\varepsilon\|T^{-1}\|^2\|T\|^2 (77)$$

for $j \ne k$. Since $\varepsilon > 0$ is arbitrary, this is a contraction.

Now we return to be considerations related to Theorem (5.1.34) and recall the following.

Definition(5.1.38) [176]:

A basis $\{f_j\}_1^\infty$ of a Hilbert space $\mathfrak{H}$ is called a Bari basis if there exists an orthonormal basis $\{g_j\}_1^\infty$ of $\mathfrak{H}$ such that

$$\sum_{j\in\mathbb{N}} \|f_j - g_j\|^2 < \infty. \qquad (78)$$

It is known that each Bari basis is a Riesz basis. The converse statement is not true.

Proposition(5.1.39) [176]: Assume that X has no finite accumulation points. Then the sequence $E\{\frac{1}{\sqrt{2\pi}}\varphi_j\}_{j=1}^\infty \coloneqq \{\frac{1}{\sqrt{2\pi}}\frac{e^{-|x-x_j|}}{|x-x_j|}\}_{j=1}^\infty$ forms a Bari basis of $\mathfrak{R}_{-1}$ if and only if

$$\sum_{j,k\in\mathbb{N}, j\ne k} e^{-2|x_j - x_k|} < \infty. \qquad (79)$$

Moreover, this condition is equivalent to

$$D_\infty \coloneqq \lim_{n\to\infty} D(\varphi_1, \dots, \varphi_n) > 0, \qquad (80)$$

where $D(\varphi_1, \dots, \varphi_n)$ denotes the determinant of the matrix $(\langle\varphi_j, \varphi_k\rangle)_{j,k=1}^n$.

Proof. By (45), we have $\langle\varphi_j, \varphi_k\rangle = 2\pi\exp(-|x_j - x_k|)$ for $j, k \in \mathbb{N}$. By Lemma (5.1.29), the system E is ω-linearly independent. Therefore, by [137, Theorem 6.3.3], E is a Bari basis if and only if.

$$(\langle\varphi_j, \varphi_k\rangle - 2\pi\delta_{jk})_{j,k=1}^\infty = 2\pi(\exp(-|x_j - x_k|) - \delta_{jk})_{j,k=1}^\infty \in \mathfrak{S}_2(l^2),$$

i.e. condition (79) is satisfied. The second statement follows from [137, Theorem 6.3.1].

Section (5.2): Three Dimensional Schrödinger Operator with Point Interactions

Here we briefly recall basis notions and facts on boundary triplets (see [64, 139, 166] for details). In what follows A denotes a densely defined closed symmetric operator on a Hilbert space $\mathfrak{H}, \mathfrak{N}_z := \mathfrak{N}_z(\mathrm{A}) = \ker(\mathrm{A}^* - z), z \in \mathbb{C}_\pm$, is the defect subspace. We also assume that A has equal deficiency indices $n_+(\mathrm{A}) := \dim(\mathfrak{N}_i) = \dim(\mathfrak{N}_{-i}) =: n-(\mathrm{A})$.

Definition (5.2.1) [176]: (See [139]). A boundary triplet for the a joint operator A^* is a triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of an auxiliary Hilbert space $\mathcal{H}$ and of linear mapping $\Gamma_0, \Gamma_1: \mathrm{dom}(\mathrm{A}^*) \to \mathcal{H}$ such that

(i) The following abstract Green identity holds:

$$(\mathrm{A}^*f, g)_{\mathfrak{H}} - (f, \mathrm{A}^*g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0, f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \mathrm{dom}(\mathrm{A}^*); \ (81)$$

The mapping $(\Gamma_0, \Gamma_1): \mathrm{dom}(\mathrm{A}^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

With a boundary triplet Π one associates two self-extensions of A defined by

$$\mathrm{A}_0 := \mathrm{A}^* \upharpoonright \ker(\Gamma_0) \quad \text{and}\ \mathrm{A}_1 := \mathrm{A}^* \upharpoonright \ker(\Gamma_1). \qquad (82)$$

Definition (5.2.2) [176]:

(i) A closed extension $\widetilde{\mathrm{A}}$ of A is called proper if $\mathrm{A} \subset \widetilde{\mathrm{A}} \subset \mathrm{A}^*$. The set of all extensions of A is denoted by Ext_A.

(ii) Two proper extensions $\widetilde{\mathrm{A}}_1$ and $\widetilde{\mathrm{A}}_2$ of A are called disjoint if $\mathrm{dom}(\widetilde{\mathrm{A}}_1) \cap +\mathrm{dom}(\widetilde{\mathrm{A}}_2) = \mathrm{dom}(\mathrm{A}^*)$.

Remark(5.2.3) [176]:

(i) If the symmetric operator A has equal deficiency indices $n_+(\mathrm{A}) = n_-(\mathrm{A})$, then a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* always exists and we have $\dim \mathcal{H} = n_\pm(\mathrm{A})$. [139]

(ii) For each self-adjoint extension $\widetilde{\mathrm{A}}$ of A there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $\widetilde{\mathrm{A}} = \mathrm{A}^* \upharpoonright \ker(\Gamma_0) = \mathrm{A}_0$.

(iii) It $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* and $\mathrm{B} = \mathrm{B}^* \in \mathrm{B}(\mathcal{H})$, then the triplet $\Pi_\mathrm{B} = \{\mathcal{H}, \Gamma_0^\mathrm{B}, \Gamma_1^\mathrm{B}\}$ with $\Gamma_1^\mathrm{B} := \Gamma_0$ and $\Gamma_0^\mathrm{B} := \mathrm{B}\Gamma_0, \Gamma_1$ is also a boundary triplet for A^*.

Boundary triplet for A^* allow one to parameterize the set Ext_A in terms of closed linear relations. For this we recall the following definitions.

Definition (5.2.4) [176]:

(i) A linear relation Θ in $\mathcal{H}$ is a linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. It is called if the corresponding subspaces is closed in $\mathcal{H} \oplus \mathcal{H}$.

(ii) A linear relation Θ is called symmetric if $(g_1, f_2) - (f_1, g_2) = 0$ for all $\{f_1, g_1\}, \{f_2, g_2\} \in \Theta$.

(iii) The adjoint relation Θ^* of a linear relation Θ in $\mathcal{H}$ is defined by

$$\Theta^* = \{\{k, k^1\}: (h', k) = (h, k') \text{ for all } \{h, h'\} \in \Theta\}.$$

(iv) A closed linear relation Θ is called self-adjoint if $\Theta = \Theta^*$.

(v) The inverse of a relation Θ is the relation Θ^{-1} defined by $\Theta^{-1} = \{\{h', h\}: \{h, h'\} \in \Theta\}$.

Definition (5.2.5) [176]: Let Θ be a closed relation in $\mathcal{H}$. The resolvent set $\rho(\Theta)$ is the set of complex numbers λ such that the relation $(\Theta - \lambda I)^{-1} := \{\{h' - \lambda h, h\}: \{h, h'\} \in \Theta\}$ is the graph of a bounded operator of $\mathrm{B}(\mathcal{H})$. the complement set $\sigma(\Theta) := \mathbb{C} \setminus \rho(\Theta)$ is called the spectrum of Θ.

For a relation Θ in $\mathcal{H}$ we define the domain $\mathrm{dom}(\Theta)$ and the multi-valued part $\mathrm{mul}(\Theta)$ by

$$\mathrm{dom}(\Theta) = \{h \in \mathcal{H}: \{h, h'\} \in \Theta \ \text{ for some } h' \in \mathcal{H}\}. \ \mathrm{mul}(\Theta) = \{h' \in \mathcal{H}: \{0, h'\} \in \Theta\}.$$

Each closed relation Θ is the orthogonal sum of $\Theta_\infty := \{\{0, f'\} \in \Theta\}$ and $\Theta_{op} := \Theta \ominus \Theta_\infty$. Then Θ_{op} is the graph of a closed operator, called the operator part of Θ and denoted also byΘ_{op}, and Θ_∞ is a "pure" relation, that is $\mathrm{mul}(\Theta_\infty) = \mathrm{mul}(\Theta)$.

Suppose that Θ is a self-adjoint relation in $\mathcal{H}$. Then $\mathrm{mul}(\Theta)$ is the orthogonal complement of

$\mathrm{dom}(\Theta)$ in $\mathcal{H}$ and Θ_{op} is a self-adjoint operator in the Hilbert space $\mathcal{H}_{op} \coloneqq \overline{\mathrm{dom}(\Theta)}$. That is, Θ is the orthogonal sum of an "ordinary" self-adjoint operator Θ_{op} in $\mathcal{H}_{op}$ and a "pure" relation Θ_∞ in $\mathcal{H}_\infty \coloneqq \mathrm{mul}(\Theta)$.

Proposition(5.2.6) [176]: 4.6. (See [64, 139, 166] Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping.

$$\mathrm{Ext}_A \ni \widetilde{A} \coloneqq A_\Theta \to \Theta \coloneqq \Gamma\left(\mathrm{dom}\left(\widetilde{A}\right)\right) = \left\{\{\Gamma_0 f, \Gamma_1 f\}: f \in \mathrm{dom}\left(\widetilde{A}\right)\right\} \quad (83)$$

Is a bijection of the set Ext_A of all proper extensions of A and the set of all closed linear relations $\tilde{\mathcal{C}}(\mathcal{H})$ in $\mathcal{H}$. Moreover, the following equivalences hold:

(i) $(A_\Theta)^* = A_{\Theta^*}$ for any linear relation Θ in $\mathcal{H}$.

(ii) A_Θ is symmetric if and only if Θ is symmetric. Moreover, $n_\pm(A_\Theta) = n_\pm(\Theta)$. In particular, A_Θ is self-adjoint if and only if Θ is self-adjoint.

(iii) The closed extensions A_Θ and A_0 are disjoint if and only if $\Theta = B$ is a closed operator. In this case.

$$A_\Theta = A_B = A^* \upharpoonright \mathrm{dom}(A_B), \quad \mathrm{dom}(A_B) = \ker(\Gamma_0 - B\Gamma_0). \quad (84)$$

The notion of the Weyl function and theyγ-filed of a boundary triplet was introduced in [64].

Definition (5.2.7) [176]: (See [64, 166]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator-valued functions $\gamma(\cdot): \rho(A_0) \to B(\mathcal{H}, \mathfrak{H})$ and $M(\cdot): \rho(A_0) \to B(\mathcal{H})$ defined by

$$\gamma(z) \coloneqq (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} \text{ and } M(z) \coloneqq \Gamma_1 \gamma(z), \qquad z \in \rho(A_0), \quad (85)$$

are called the γ-field and the Weyl function, respectively, of $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

Note that the γ-field $\gamma(\cdot)$ and Weyl function $M(\cdot)$ are holomorphic on $\rho(A_0)$.

Recall that a symmetric operator A in $\mathfrak{H}$ is said to be simple if there is no non-trivial subspace which reduces it to a self-adjoint operator. In other words, A is simple if it does not admit an (orthogonal) decomposition $A = A' \oplus S$ where A' is a symmetric operator and S is a self-adjoint operator acting on a non-trivial Hilbert space.

It is easily seen (and well known) that A is simple if and only if span $\{\mathfrak{N}_z(A): z \in \mathbb{C} \setminus \mathbb{R}\} = \mathfrak{H}$.

If A is simple, then the Weyl function $M(\cdot)$ determines the boundary triplet Π uniquely up to the unitary equivalence (see [64]). In particular, $M(\cdot)$ contains the full information about the spectral properties of A_0. Moreover, the spectrum of a proper (not necessarily self-adjoint) extension $A_\Theta \in \mathrm{Ext}_A$ can be described by means of $M(\cdot)$ and the boundary relation Θ.

Proposition(5.2.8) [176]: (See [64, 166]). Let A be a simple densely defined symmetric operator in $\mathfrak{H}, \Theta \in \tilde{\mathcal{C}}(\mathcal{H})$ and $z \in \rho(A_0)$. Then:

(i) $z \in \rho$ if and only if $0 \in \rho(\Theta - M(z))$;

(ii) $z \in \sigma_\tau(A_\Theta)$ if and only if $0 \in \sigma_\tau(\Theta - M(z)), \tau \in \{p, c\}$

(iii) $f \in \ker(A_\Theta - z)$ if and only if $\Gamma_0 f \in \ker(\Theta - M(z))$ and

$$\dim\ker(A_\Theta - z) = \dim\ker(\Theta - M(z)).$$

For any boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* and any proper extension $A_\Theta \in \mathrm{Ext}_A$ with non-empty resolvent set the following Krein-type resolvent formula holds (cf. [64, 166])/

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma(z)\big(\Theta - M(z)\big)^{-1}\gamma(\bar{z})^*, \qquad z \in \rho(A_\Theta) \cap \rho(A_0). \quad (86)$$

It should be emphasized that formulas (82), (83), and (85) express all data occurring in (86) in terms of the boundary triplet. These expressions allow one to apply formula (86) to boundary value problems.

The following result is deduced from (86).

Proposition(5.2.9) [176]: (See [64, Theorem 2]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $\Theta', \Theta \in \tilde{\mathcal{C}}(\mathcal{H})$. Suppose that $\rho(A_{\Theta'}) \cap \rho(A_\Theta) \neq \emptyset$ and $\rho(\Theta') \cap \rho(\Theta) \neq \emptyset$.

(i) For $z \in \rho(A_{\Theta'}) \cap \rho(A_\Theta), \zeta \in \rho(\Theta') \cap \rho(\Theta)$, and $\rho \in [0, \infty]$ the following equivalence is valid:

$$(A_{\Theta'} - z)^{-1} - (A_\Theta - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Leftrightarrow (\Theta' - \zeta)^{-1} - (\Theta - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H}) \qquad (87)$$

In particular, $(A_\Theta - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$ of and only if $(\Theta - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H})$ for $\zeta \in \rho(\Theta)$.

(ii) If $\mathrm{dom}(\Theta') = \mathrm{dom}(\Theta)$, then the following implication holds:

$$\overline{\Theta' - \Theta} \in \mathfrak{S}_p(\mathcal{H}) \Longrightarrow (A_{\Theta'} - z)^{-1} - (A_\Theta - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), z \in \rho(A_\Theta) \cap \rho(A_\Theta). \qquad (88)$$

In particular, if $\Theta', \Theta \in (\mathcal{H})$, then (87) is equivalent to $\Theta' - \Theta \in \mathfrak{S}_p(\mathcal{H})$.

In this subsection we assume that the symmetric operator A on $\mathfrak{H}$ is non-negative. Then the set $\mathrm{Ext}_A(0, \infty)$ of all non-negative self-adjoint extensions of A on $\mathfrak{H}$ is not empty. Moreover, there exists a maximal non-negative extension A_F, called the Friedrichs extension, and a minimal non0negative extension A_K, called Krein extension, in the set $\mathrm{Ext}_A(0, \infty)$ and

$$(A_F + x)^{-1} \leq \left(\tilde{A} + x\right)^{-1} \leq (A_K + x)^{-1}, \qquad x \in (0, \infty), \tilde{A} \in \mathrm{Ext}_A(0, \infty).$$

Proposition(5.2.10) [176]: (See [117]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 \geq 0$ and let $M(\cdot)$ be the corresponding Weyl function.

(i) There exists a lower semibounded self-adjoint linear relation $M(0)$ in $\mathcal{H}$ which is the strong resolvent limit of $M(x)$ as $x \uparrow 0$. Moreover, $M(0)$ is associated with the closed quadratic form.

$$t_0[h] \coloneqq \lim_{x \uparrow 0} (M(x)h, h), \ \ \mathrm{dom}(t_0) = \left\{h: \lim_{x \uparrow 0} (M(x)h, h) < \infty\right\} = \mathrm{dom}\left(\left(M(0) - M(-a)\right)^{1/2}\right).$$

(ii) The Krein extension A_K is given by

$$A_K = A^* \upharpoonright \mathrm{dom}(A_K), \ \ \mathrm{dom}(A_K) = \{f \in \mathrm{dom}(A^*): \{\mathcal{H}, \Gamma_0, \Gamma_1\} \in M(0)\}. \ (89)$$

The extensions A_K and A_0 are disjoint if and only if $M(0) \in \mathcal{C}(\mathcal{H})$. In this case $\mathrm{dom}(A_K) = \ker(\Gamma_1 - M(0)\Gamma_0)$.

(iii) $A_0 = A_F$ if and only if $\lim_{x \uparrow -\infty} (M(x)f, f) = -\infty$ for $f \in \mathcal{H} \setminus \{0\}$.

(iv) $A_0 = A_K$ if and only if $\lim_{x \uparrow -\infty} (M(x)f, f) = +\infty$ for $f \in \mathcal{H} \setminus \{0\}$.

If A_Θ is lower semibounded, then Θ is lower semibounded too. The converse is not true in general. In order to state corresponding result we introduce the following definition.

We shall say that $M(\cdot)$ tends uniformly to $-\infty$ as $x \to -\infty$ if for any $a > 0$ there exists $x_a < 0$ such that $M(x_a) < -a. I_{\mathcal{H}}$. In this case we write $M(x) \rightrightarrows -\infty$ as $x \to -\infty$.

Proposition(5.2.11) [176]: (See [64]). Suppose that A is a non-negative symmetric operator on $\mathfrak{H}$ and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* such that $A_0 = A_F$. Let M be the corresponding Weyl function. Then the two assertions:

(i) a linear relation $\Theta \in \tilde{\mathcal{C}}_{\mathrm{self}}(\mathcal{H})$ is semibounded below.

(ii) a self-adjoint extension A_Θ is semibounded below.

are equivalent if and only if $M(x) \rightrightarrows -\infty$ for $x \to -\infty$.

Recall that the order relation for lower semibounded self-adjoint operators T_1, T_2 is defined by

$$T_1 \leq T_2 \text{ if } \mathrm{dom}\left(t_{T_1}\right) \subset \mathrm{dom}\left(t_{T_2}\right) \ \text{ and } \ t_{T_1}[u] \geq t_{T_2}[u], \ \ u \in \mathrm{dom}\left(t_{T_1}\right), \qquad (90)$$

where t_{T_1} is the quadratic form associated with T_j.

If T is a self-adjoint operator with spectral measure $E_T put k_-(T) := \dim \operatorname{ran}(E_T(-\infty,0))$. For a self-adjoint relation Θ we set $k_-(\Theta) :=_{k_-} (\Theta_{op})$, where Θ_{op} is the operator part of Θ. For a quadratic form t we denote by $k_-(t)$ the number of negative squares of t (cf. [155]).

Proposition(5.2.12) [176]:(See [64]). Suppose A is a densely defined non-negative symmetric operator on $\mathfrak{H}$ and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* such that $A_0 = A_F$. Let M be the Weyl function of this boundary triplet and let Θ be a self-adjoint relation on $\mathcal{H}$. Then:

(i) The self-adjoint extension A_Θ is non-negative if and only if $\Theta \geq M(0)$,

(ii) If A_Θ is lower semibounded and $\operatorname{dom}(t_\Theta) \subset \operatorname{dom}(t_{M(0)})$, then $k_-(A_\Theta) = k_-(t_\Theta - t_{M(0)})$. If, in addition, $M(0) \in (\mathcal{H})$, then $k_-(A_\Theta) = k_-(\Theta - M(0))$.

In what follows we will denote.

$$M_h(z) := (M(z)h, h), \ \ z \in \mathbb{C}_+ \ \text{ and } \ M_h(x+i0) := \lim_{y\downarrow 0} M_h(x+iy), h \in \mathcal{H}.$$

Since $\operatorname{Im}(M_h(z)) > 0, z \in \mathbb{C}_+$, the limit $M_h(x+i0)$ exists and is finite for a.e. $x \in \mathbb{R}$. We put

$$\Omega_{ac}(M_h) := \{x \in \mathbb{R} : 0 < lm M_h(x) < +\infty\}.$$

We also set $d_M(x) := \operatorname{rank}\left(\operatorname{Im}(M(x+i0))\right) \leq \infty$ provided that the weak limit $M(x+i0) := \omega - \lim_{y\downarrow 0} M(x+iy)$ exists.

Proposition(5.2.13) [176]: (See [133]). Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with Weyl function M. Assume that $\{h_k\}_{k=1}^N, 1 \leq N \leq \infty$, is a total set in $\mathcal{H}$. Recall that A_0 is the self-adjoint operator defined by $A_0 = A^* \upharpoonright \ker(\Gamma_0)$.

(i) A_0 has no point spectrum in the interval (a,b) if and only if $\lim_{y\downarrow 0} y M_{h_k}(x+iy) = 0$ for all $x \in (a,b)$ and $k \in \{1,2 \dots, N\}$.

(ii) A_0 has no singular continuous spectrum in the interval (a,b) if the set $(a,b) \setminus \Omega_{ac}(M_{h_k})$ is countable for each $k \in \{1,2,\dots,N\}$.

To state the next proposition we need the concept of the ac-closure $cl_{ac}(\delta)$ of a Borel subset $\delta \subset \mathbb{R}$ introduced independently in [133] and [136]. We refer to [136, 158] for the definition of this notion as well as for its basic properties.

Proposition(5.2.14) [176]: (See [157, 158]). Retain the assumptions of Proposition (5.2.13) Let B be a self adjoint operator on $\mathcal{H}, A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$, and $M_B(z) := (B - M(z))^{-1}$.

(i) If the limit $M(x+i0) := \omega - \log_{y\downarrow 0} M(x+iy)$ exists a.e. on $\mathbb{R}$, then $\sigma_{ac}(A_0) = cl_{ac}\left(\operatorname{supp}(d_M(x))\right)$.

(ii) For any Borel subset $\mathcal{D} \subset \mathbb{R}$ the ac-parts $A_0 E_{A_0}^{ac}(\mathcal{D})$ and $A_B E_{A_B}^{ac}(\mathcal{D})$ of the operators $A_0 E_{A_0}(\mathcal{D})$ and $A_B E_{A_B}(\mathcal{D})$ are unitarily equivalent if and only if $d_M(x) = d_{M_B}(x)$ a.e. on $\mathcal{D}$.

Throughout we fix a sequence $X = \{x_k\}_1^\infty$ of points $x_k \in \mathbb{R}^3$ satisfying.

$$d_*(X) = \inf_{k,j \in \mathbb{N}, k \neq j} |x_k - x_j| > 0.$$

denote by H the restriction of $-\Delta$ given by (41), and set.

$$\varphi_{j,z}(x) = \frac{e^{i\sqrt{z}|x - x|}}{|x - x_j|} \ \text{ and } \ e_{j,z}(x) = e^{i\sqrt{z}|x-x_j|}, \qquad z \in \mathbb{C} \setminus [0,+\infty), j \in \mathbb{C}. \qquad (91)$$

Clearly, $\varphi_j = \varphi_{j,-1}$ and $e_j = e_{j,-1}$. Recall from Lemma (5.1.33) that T_1 is the bounded operator on

$l^2(\mathbb{N})$ defined by the matrix $\mathcal{T}_1 := \left(2^{-}e^{-|x_j-x_k|}\right)_{j,k\in}$.

The following lemma is a special case of Example 14.3 in [166]

Lemma(5.2.15) [176]: Let A be densely defined closed symmetric operator on $\mathfrak{H}$. Suppose that $\widetilde{A}$ is a self-adjoint extension of A on $\mathfrak{H}$ and $-\in \rho(\widetilde{A})$. Then:

(i) $\operatorname{dom}(A^*) = \operatorname{dom} A$

$$+ \ker(A^* + I) + \left(\widetilde{A} + I\right)^{-1}\mathfrak{N}_-, A^*\left(f_A + f_0 + \left(\widetilde{A} + I\right)^{-1} f_1\right)$$

$$= Af_A - f_0 + \widetilde{A}\left(\widetilde{A} + I\right)^{-1} f_1.$$

where $f_A \in \operatorname{dom}(A)$ and $f_0, f_1 \in \mathfrak{N}_{-1} := \ker(A^* + I)$.

(ii) Definition $\mathcal{H}' = \mathfrak{N}_{-1}$ and $\Gamma_j'\left(f_A + f_0 + \left(\widetilde{A} + I\right)^{-1} f_1\right) = f_j$ for $j = 0{,}1$. Then $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ form a boundary triplet for A^*.

Proof. Assertion (i) is well known in extension theory (see e.g. [166], formula (14.17), so we prove only assertion (ii). Let $f = f_A + f_0 + \left(I + \widetilde{A}\right)^{-1} f_1$ and $g = g_A + g_0 + \left(I + \widetilde{A}\right)^{-1} g_1$, where $f_0, f_1, g_0, g_1 \in \mathfrak{N}_{-1}$. Then

$$\begin{aligned}
\langle A^* f, g\rangle - \langle f, A^* g\rangle \\
&= \langle \widetilde{A}\left(I + \widetilde{A}\right)^{-1} f_1, g_0\rangle - \langle f_0, (I + A^{-1}) g_1\rangle + \langle \widetilde{A}\left(I + \widetilde{A}\right)^{-1} f_1, \left(1 + \widetilde{A}\right)^{-1} g_1\rangle \\
&- \langle f_0, \widetilde{A}\left(I + \widetilde{A}\right)^{-1} g_1\rangle + \langle \left(I + \widetilde{A}\right)^{-1} f_1, g_0\rangle - \langle \left(I + \widetilde{A}\right)^{-1} f_1, \widetilde{A}\left(I + \widetilde{A}\right)^{-1} g_1\rangle \\
&= -\langle f_0\left(I + \widetilde{A}\right)\left(I + \widetilde{A}\right)^{-1} g_1\rangle + \langle \left(I + \widetilde{A}\right)\left(I + \widetilde{A}\right)^{-1} f_1, g_0\rangle = -\langle f_0, g_1\rangle_{\mathcal{H}'} + \langle f_1, g_0\rangle_{\mathcal{H}'} \\
&= \langle \Gamma_1' f, \Gamma_0' g\rangle_{\mathcal{H}'} - \langle \Gamma_0' f, \Gamma_1' g\rangle_{\mathcal{H}'} (92)
\end{aligned}$$

The surjectivity of the mapping $(\Gamma_0' f, \Gamma_1')$ is obvious.

Next we apply Lemma (5.2.15) to the minimal Schrödinger operator $A = H$.

Proposition(5.2.16) [176]: Suppose H is the minimal Schrödinger operator defined by (41) and $d_*(X) > 0$. Let T_1 be the bounded operator on $l^2(\mathbb{N})$ defined by the matrix $\mathcal{T}_1 := \left(2^{-1}e^{-|x_j-x_k|}\right)_{j,k\in\mathbb{N}}$. Then

(i) H is a closed symmetric operator with deficiency indices (∞, ∞). The defect subspace $\mathfrak{N}_{-1} = \ker(H^* + I)$ is given by

$$\mathfrak{N}_{-1} = \left\{\sum_{j=1}^{\infty} c_j \varphi_j : \{c_j\}_1^{\infty} \in l^2(\mathbb{N})\right\}. \qquad (93)$$

(ii) $\operatorname{dom}(H^*)$ is the direct sum of vector spaces $\operatorname{dom} H, \mathfrak{N}_{-1}$ and $(-\Delta + I)^{-1}\mathfrak{N}_{-1}$, that is,

$$\begin{aligned}
\operatorname{dom}(H^*) &= \{f = f_H + f_0 + (-\Delta + I)^{-1} f_1 : f_H \in \operatorname{dom} H, f_0, f_1 \in \mathfrak{N}_{-1}\} \\
&= \left\{f = f_H + \sum_{j=1}^{\infty}\left(\xi_{0j}\varphi_j + \xi_{1j} e_j\right) : f_H \in \operatorname{dom} H, \xi_0 := |\xi_{0j}|, \xi_1 = \{\xi_{1j}\} \in l^2(\mathbb{N})\right\}, (94)
\end{aligned}$$

$$H^* f = -\Delta f_H - f_0 + (-\Delta)(-\Delta + I)^{-1} f_1 = -f_H + \sum_{j=1}^{\infty}\left(-\xi_{0j}\varphi_j + \xi_{1j}\left(\varphi_j - e_j/2\right)\right). \qquad (95)$$

The triplet $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where

$$\mathcal{H} = l^2(\mathbb{N}), \quad \Gamma_0 f = \xi_0, \quad \Gamma_1 f = T_1\xi_1, \quad f \in \operatorname{dom}(H^*). \qquad (96)$$

is a boundary triplet for H^*.

Proof. (i) By the Sobolev embedding theorem, $f \to f(x_j)$ is a continuous linear functional on $W^{2,2}(\mathbb{R}^3)$ (see [159]). Therefore, $\mathrm{dom}(H) = W^{2,2}(\mathbb{R}^3) \upharpoonright \bigcap_{j=1}^{\infty} \ker\left(\delta_{x_j}\right)$ is closed in the graph norm of $-\Delta$, so the operator H is closed. Since $-\Delta$ is self-adjoint, H is symmetric.
Since $d_*(X) > 0$ by assumption. Theorem (3.1.34) applies and shows that $\{\varphi_j\}_1^{\infty}$ is a Riesz basis of the Hilbert space $\mathfrak{N}_{-1}$. In particular, $n_\pm(H) = \infty$.
(ii) All assertions of (ii) follow from (i) and Lemma (5.2.15) (i), applied to the self-adjoint operator $A = -\Delta$ on $L^2(\mathbb{R}^3)$. For the formula of H^*f we recall that $e_j/2 = (-\Delta + I)^{-1}\varphi_j$ and therefore, $H^*e_j = -\Delta(-\Delta + I)^{-1}\varphi_j = \varphi_j - e_j/2$.
(iii) From (45) it follows that $\langle\varphi_j, \varphi_k\rangle = 2^{-1}e^{-|x_j - x_k|}$, i.e., the Gram matrix of $E = \{\varphi_j\}_{j\in\mathbb{N}}$ is $\mathcal{T}_1$. $\mathcal{T}_1$ defines the bounded operator $\mathcal{T}_1$ on $l^2(\mathbb{N})$ with bounded inverse. Hence $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ are well defined and the map $(\tilde{\Gamma}_0, \tilde{\Gamma}_1)$ are well defined and the map $(\tilde{\Gamma}_0, \tilde{\Gamma}_1)\colon \mathrm{dom}(A^*) \to \mathcal{H}\oplus\mathcal{H}$ is surejctive.
Next we verify the Green formula. Let $f, g \in \mathrm{dom}(H^*)$. By (93), these vectors are of the form

$$f = f_H + f_0 + (-\Delta + I)^{-1}f_1, \qquad g = g_H + g_0 + (-\Delta + I)^{-1}g_1$$

with $f_H, g_H \in \mathrm{dom}\, H$ and $f_0, f_1 \in \mathfrak{N}_-$, f_0, f_1, g_0, g_1 can be written as

$$f_0 = \sum_{j=1}^{\infty} \xi_{0j}\varphi_j, \quad f_1 = \sum_{j=1}^{\infty} \xi_{1j}\varphi_j, \quad g_0 = \sum_{j=1}^{\infty} \eta_{0j}\varphi_j, \; g_1 = \sum_{j=1}^{\infty} \eta_{0j}\varphi_j.$$

where $\{\xi_{0j}\}_{j\in\mathbb{N}}, \{\xi_{1j}\}_{j\in\mathbb{N}}\{\eta_{0j}\}_{j\in\mathbb{N}}, \{\eta_{1j}\}_{j\in\mathbb{N}} \in l^2(\mathbb{N})$. Using the Green identity for the boundary triplet $\Pi' = (\mathcal{H}', \Gamma_0', \Gamma_1')$ in Lemma (5.2.15), applied to $A = H$ and $\widetilde{A} = -\Delta$, we derive the identity.

$$\begin{aligned}\langle H^*f, g\rangle - \langle f, H^*g\rangle &= \langle\Gamma_1'f, \Gamma_0'g\rangle - \langle\Gamma_0'f, \Gamma_1'g\rangle = \langle f_1, g_0\rangle_{\mathfrak{N}_{-1}} - \langle f_0, g_1\rangle_{\mathfrak{N}_{-1}} \\ &= \sum_{j,k=1}^{\infty}\left(\xi_{1j}\overline{\eta_{0k}} - \xi_{0j}\overline{\eta_{1k}}\right)\langle\varphi_j, \varphi_k\rangle \\ &= \sum_{k=1}^{\infty}\left((T_1\xi_1)_k\eta_{0k} - \xi_{0k}\left(T_{1\eta_1}\right)_k\right) = \langle T_1\xi_1, \eta_0\rangle - \langle\xi_1, T_1\eta_0\rangle = \langle\tilde{\Gamma}_1 f, \tilde{\Gamma}_0 g\rangle_{\mathcal{H}} \\ &\quad - \langle\Gamma_0 f \Gamma_1 g\rangle_{\mathcal{H}},\end{aligned}$$

which complete the proof.
However, we prefer to work with another boundary triple. For this purpose we define

$$(T_0(\xi_j))_k = -\xi_k + \sum_{j\in\mathbb{N}, j\neq k} \xi_j \frac{e^{-|x_k - x_j|}}{|x_k - x_j|}, \qquad \{\xi_j\}_{j\in\mathbb{N}} \in l^2(\mathbb{N}). \qquad (97)$$

It follows from the assumption $d_*(X) > 0$ and the fact that the matrix $\left(2^{-1}e^{|x_j - x_j|}\right)_{j,k\in\mathbb{N}}$ defines a bounded operator T_1 on $l^2(\mathbb{N})$ be Lemma (3.1.33), that T_0 is a bounded self-adjoint operator on $l^2(\mathbb{N})$.
Next we slightly modify the boundary triplet $\widetilde{\Pi}\{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ and express the trace mappings $\tilde{\Gamma}_j$ in terms of the “boundary values”. We abbreviate

$$G_{\sqrt{x}}(x) = \begin{cases} \dfrac{e^{i\sqrt{z}|x|}}{|x|}, & x \neq 0; \\ 0, & x = 0, \end{cases} \qquad (98)$$

Proposition(5.2.17) [176]: Let H be the Schrödinger operator defined by (41). Suppose that

$d_*(X) > 0$.

(i) The triplet $\Pi\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H} = l^2(\mathbb{N})$,

$$\Gamma_0 f\left\{\lim_{x\to x_k} f(x)|x-x_k|\right\}_1^{\infty} =: \{\xi_{0k}\}_1^{\infty},$$

$$\Gamma_1 f\left\{\lim_{x\to x_k} f(x) - \xi_{0k}|x-x_k|^{-1}\right\}_1^{\infty}, \qquad (99)$$

is a boundary triplet for H^*.

(ii) The deficiency subspace $\mathfrak{N}_z = \mathfrak{N}_z(H)$ is $\mathfrak{N}_z = \left\{\sum_{j=1}^{\infty} c_j\varphi_{j,z}: \{c_j\}_1^{\infty} \in l^2(\mathbb{N})\right\}, z \in \mathbb{C}\setminus\mathbb{R}$.

(iii) The gamma field $\gamma(\cdot)$ of the triplet $\Pi\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is given by

$$\gamma(z)(\{c_j\}) = \sum_{j=1}^{\infty} c_j\varphi_{j,z}, \qquad \{c_j\}_1^{\infty} \in l^2(\mathbb{N}), z \in \mathbb{C}\setminus[0,+\infty). \qquad (100)$$

(iv) The corresponding Weyl function acts by

$$(M(z)\{c_j\})_k = c_k i\sqrt{z} + \sum_{j\in\mathbb{N}}{}' c_j \frac{e^{i\sqrt{z}|x_k-x_j|}}{|x_k-x_j|}, \quad \{c_j\}_{j\in\mathbb{N}} \in l^2(\mathbb{N}), z \in \mathbb{N}\setminus[0,+\infty), \qquad (101)$$

that is, the operator $M(z)$ is given by the matrix.

$$\mathcal{M}(z) = \left(i\sqrt{z}\delta_{jk} + \overline{G}_{\sqrt{z}}(x_j - x_k)\right)_{j,k=1}^{\infty}. \qquad (102)$$

Proof. (i) Since $T_0 = T_0^* \in [\mathcal{H}]$ and $\widetilde{\Pi}$ is boundary triplet for H^* by Proposition (5.2.16) (iii), so is the triplet $\Pi' = \{\mathcal{H}, \Gamma_0', \Gamma_1'\}$, where

$$\mathcal{H} = l^2(\mathbb{N}), \qquad \Gamma_0' := \Gamma_0, \qquad \text{and} \quad \Gamma_1' = \tilde{\Gamma}_1 + T_0\tilde{\Gamma}_0. \qquad (103)$$

It therefore suffices to show that $\Gamma_j = \Gamma_j^1, j = 0.1$.

Let $f \in \mathrm{dom}H^*$. By Proposition (5.2.16) (ii), f is of the form $f = f_H + f_0 + (-\Delta + I)^{-1}f_1$, where $f_H \in \mathrm{dom}(H), f_0 = \sum_{j\in\mathbb{N}} \xi_{1j}\varphi_j$. Then $(-\Delta + I)^{-1}f_1 = 2^{-1}\sum_j \xi_{1j}e_j$.

Fix $k \in \mathbb{N}$. Since the series $f_0 = \sum_{j\in\mathbb{N}} \xi_{0j}'\varphi_j$ converges uniformly on compact subsets of $\mathbb{R}^3\setminus X$ and $f_H \in W^{2,2}(\mathbb{R}^3)$ is continuous and $f_H(x_j) = 0$ by (41), we get

$$\xi_{0k} = \lim_{x\to x_k} f(x)|x-x_k| = \xi_{0k}' = \left(\tilde{\Gamma}_0 f\right)_k = (\Gamma_0' f)_k.$$

This proves the first formula of (99). the second formula is derived by

$$\lim_{x\to x_k}(f(x) - \xi_{0k}|x-x_k|^{-1})$$

$$= \lim_{x\to x_k}\left(\xi_{0k}\frac{e^{-|x-x_j|}-1}{|x-x_k|} + \sum_{j\neq k}^{\infty}\xi_{0j}\frac{e^{-|x-x_j|}}{|x-x_j|} + 2^{-1}\sum_{j=1}^{\infty}\xi_{1j}e^{-|x-x_j|}\right) = -\xi_{0k}$$

$$+ \sum_{j\neq k}^{\infty}\xi_{0_j}\frac{e^{-|x_k-x_j|}}{|x_k-x_j|} + 2^{-}\sum_{j=1}^{\infty}\xi_{1j}e^{-|x_k-x_j|} = (T_0(\xi_{kj}))_k + (T_1(\xi_{1j}))_k = (\Gamma_1' f)_k.$$

where T_0 is defined by (97), and T_1 is introduced in Proposition (5.2.16).

(ii) follows at once from Corollary(5.1.36).

(iii) Clearly, $\lim_{x\to x_k}(\varphi_{k,z}(x) - \varphi_k(x))|x-x_k| = 0$. Therefore, by (99), $\Gamma_0(\varphi_{k,z} - \varphi_k) = 0$ and so $\Gamma_{0\varphi k,z} = \Gamma_{0\varphi k} = e_k = \{\delta_{jk}\}_{j=1}^{\infty}$ is the standard orthonormal basis of $l^2(\mathbb{N})$. Hence, by (85) combined with (ii), the gamma field is of the form given in (100).

(iv) Next we prove the formula for the Weyl function. Since M is linear and bounded, it suffices to prove this formula for the vectors $e_l, l \in \mathbb{N}$. The function $\varphi_{l,z} \in \mathrm{dom}(H^*) f_{1,z} \sum_{j\in\mathbb{N}} \xi_{1j}(z)\, \varphi_j$. Then, by (99) and (91),

$$\xi_{0j}(z) = \lim_{x\to x_j} \varphi_{l,z}(x)\,|x - x_j| = \delta_{jl}, \ \ j \in \mathbb{N}, \ \ \text{i.e.,} \qquad f_{0,z}(x) = |x - x_l|^{-1} e^{-|x-x_l|}, \qquad (104)$$

so $f_{0,z}$ does not depend on z. Since $\xi_{0k}(z) = 0$ for $k \neq l$, (99) and (91) yield.

$$(\Gamma_1 \varphi_{1,z})_k = \lim_{x\to x_k} \left(\varphi_{l,z} - \xi_{0k}|x - x_k|^{-1}\right) = \lim_{x\to x_k} \varphi_{l.z}(x) = \frac{e^{i\sqrt{z}|x_l - x_k|}}{|x_l - x_k|}, \qquad k \neq l, k, l \in \mathbb{N}$$

Similarly, using that $\xi_{0l}(z) = 1$ if follows from (99) and (91) that $(\Gamma_1\varphi_l, z)_l = i\sqrt{z}$. Inserting these expressions into (85) with account of (100) we arrive at the formula (101) for the Weyl function.

Proposition (5.2.18) [176]: Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplets for H^* defined in Proposition (5.2.17) (see (99)). Let T_0 be defined by (97) and $T_1 = 2^{-1}\left(e^{-|x_j - x_k|}\right)_{j,k\in\mathbb{N}}$. Then:

(i) The set of self-adjoint realization $\widetilde{H} \in \mathrm{Ext}_H$ is parameterized by the set of linear relations $\Theta = \Theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$ as follows: $H_\Theta = H^* \restriction \mathrm{dom}(H_\Theta)$, where

$$\mathrm{dom}(H_\Theta) = \left\{ f = f_H + \sum_{j=1}^{\infty} \left(\xi_{0j} \frac{e^{-|x-x_j|}}{|x - x_j|} + \xi_{1j} e^{-|x-x_j|} \right) : f_H \in \mathrm{dom}(H).\, (\xi_0, T_0\xi_0 + T_1\xi_1) \in \Theta \right\}. \quad (105)$$

Moreover, we have $\Theta = \Theta_{op} \oplus \Theta_\infty$ where Θ_{op} is the graph of an operator $B = B^*$ in $\mathcal{H}_0 \coloneqq \mathrm{dom}(\Theta)$ and Θ_∞ is the multi-valued part of Θ, and $\mathcal{H} = \mathcal{H}_{\oplus}\mathcal{H}_\infty$, where $\mathcal{H}_\infty \coloneqq \mathrm{mul}(\Theta)$ and

$$\Theta_\infty \coloneqq \{0, \mathcal{H}_\infty\} \coloneqq \left\{\{0, T_1\xi_1''\}: \xi_1'' \perp T_1\xi_0, \xi_0 \in \mathcal{H}_0\right\}. \qquad (106)$$

$$\Theta_{op} = \left\{\{\xi_0, T_0\xi_0 + T_1\xi_1'\}: \xi_0 \in \mathcal{H}_0, \xi_1' = T_1^{-1}(B\xi_0 - T_0\xi_0)\right\}. \qquad (107)$$

In particular, $\widetilde{H} = H_\Theta$ is disjoint with H_0 if and only if $\overline{\mathrm{dom}(\Theta)} = \mathcal{H} l^2(\mathbb{N})$. In this case $\Theta = \Theta_{op}$ is the graph of B, so that $H_\Theta = H^* \restriction (\ker(\Gamma_1 - B\Gamma_0))$.

(ii) Let $z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$. Then $z \in \sigma_p(H_\Theta)$ if and only if $0 \in \sigma_p\left(\Theta - \left(i\sqrt{z}\delta_{jk} + G_{\sqrt{z}}(x_l)\right)_{j,k=1}^{\infty}\right)$.

The corresponding eienfunctions ψ_z have the form

$$\psi_z = \sum_{j=1}^{\infty} \xi_j |x - x_j|^{-1} e^{i\sqrt{z}|x-x_j|}, \ \ \text{where } (\xi_j) \in \ker(\Theta M(z)) \subset l^2(\mathbb{N}). \qquad (108)$$

(iii) The resolvent of the extension $\Delta_{\Theta,X} \coloneqq H_\Theta$ admits the integral representation.

$$\left((-\Delta_{\Theta,X} - z\right)^{-1} f(x) = (x) \int_{\mathbb{R}^1} T_{\Theta,X}(x, y; z)\, f(y) dy, \qquad z \in \rho(-\Delta_{\Theta,X}), \qquad (109)$$

with kernel $T_{\Theta,X}(\cdot,,;z)$ defined by

$$T_{\Theta,X}(x, y; z) = \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x - y} + \sum_{j,k} \Theta_{jk}(z) \frac{e^{i\sqrt{z}|y-x_j|}}{|y - x_j|} . \frac{e^{i\sqrt{z}|x-x_k|}}{|x - |}. \qquad (110)$$

where $(\Theta_{jk}(z))_{j,\in\mathbb{N}}$ is the matrix representation of the operator $(\Theta - M(z))^{-1}$ on $l^2(\mathbb{N})$.

Proof. (i) Formula (105) is immediate from Proposition (5.2.6), formula (83).

Both formulas (106) and (107) are proved by direct computations. We show that (106) and (107) imply the self-adjointness of Θ; the proof of the converse implication is similar. Indeed, it follows,

(106) and (107) that $(T_1\xi_1'', \xi_0) = 0 = (\xi_0, T_1'')$ and

$$(T_1\xi_1', \xi_0) = (B\xi_0 - T_0\xi_0, \xi_0 - T_0\xi_0) = (\xi_0, T_1\xi_1'). \qquad (111)$$

Hence we have $(T_1\xi_1, \xi_0) = (\xi_0, T_1\xi_1)$ for all $(\xi_0, \xi_1) \in \Theta$. It is easily checked that the latter condition is equivalent to the self-adonintness of the relation Θ.

(ii) The symmetric operator H is in general not simple. It admits a direct sum decomposition $H = \widehat{H} \oplus H'$ where $\widehat{H}$ is a simple symmetric operator and H' is self-adjoint. Define $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, where $\widehat{\Gamma}_j: \widehat{\Gamma}_j \upharpoonright \operatorname{dom}(\widehat{H}^*), j \in \{0,1\}$. Clearly, $\widehat{\Pi}$ is a boundary triplet for $\widehat{H}^*$ and the corresponding Wely function $\widehat{M}(\cdot)$ coincides with the Weyl function $M(\cdot)$ of Π. Further, any proper extension $\widetilde{H} = H_\Theta$ of H admits a decomposition $H_\Theta = \widehat{H}_\Theta \oplus H'$. Being a part of H_0, the operator H' is non-negative. Therefore, for $z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$, we have $z \in \sigma_p(H_\Theta)$ is and only if $z \in \sigma_p(\widehat{H}_\Theta)$. Thus, it suffices to prove the assertion for extension $\widehat{H}_\Theta$ of the simple symmetric operator $\widehat{H}$. But then the statement follows from Proposition (5.2.8) and 93 (ii) and formula (100).

(iii) Noting that $i\sqrt{\bar{z}} = \overline{\imath\sqrt{z}}$ it follows from (91) that $\varphi_{j,\bar{z}} = \overline{\varphi_{j,z}}$. Therefore, (100) implies that

$$\gamma^*(\bar{z})f = \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^3} f(x)\overline{\varphi_{k,z}(x)}\,dx\right)e_k = \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^3} f(x)\frac{e^{i\sqrt{z}|x-x_k|}}{|x-x_k|}\right)e_k, \qquad (112)$$

where $e_k = \{\delta_{jk}\}_{j=1}^{\infty}$ is the standard basis of $l^2(\mathbb{N})$.

Inserting (112) and (100) into the Krein-type formula (86) and applying the formula (43) for the resolvent of the free Hamiltonian $-\Delta$, we obtain

$$((-\Delta_{\Theta,X} - z)^{-1}f)(x) = \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} f(y)dy + \sum_{j,k}^{\infty}\left[(\Theta - M(z))^{-1}\right]_{j,k}(f, \varphi_{k,z})\varphi_{j,z}(x).$$

Clearly, the latter is equivalent to the representations (109) – (110).

Next we turn to non-negative or lower semibounded self-adjoint extensions of H. For this we need the following technical result.

Lemma(5.2.19) [176]: Retain the assumptions of Proposition (5.2.17) and let $\Pi\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for H^* defined therein. Then;

(i) There exists a lower semibounded self-adjoint operator $M(0)$ on $\mathcal{H} = l^2(\mathbb{N})$ which is the limit of $M(-x)$ in the strong resolvent convergence as $x \to +0$.

(ii) The quadratic from $t_{M(0)}$ of $M(0)$ is given by

$$t_{M(0)}[\xi] = \sum_{|j-k|>0} \frac{1}{|x_j - x_k|}\xi_j\bar{\xi}_k < \infty, dom(t_{M(0)}) = \left\{\xi = \{\xi_j\} \in l^2(\mathbb{N}): \sum_{|j-k|>0} \frac{1}{|x_j - x_k|}\xi_j\bar{\xi}_k < \infty\right\}. \quad (113)$$

(iii) The operator $M(0) = M(0)^*$ associated with the form $t_{M(0)}$ is uniquely determined by the following conditions: $\operatorname{dom}(M(0)) \subset \operatorname{dom}(t_{M(0)})$ and

$$(M(0)\xi, \eta) = \sum_{|j-k|>0} \frac{1}{|x_j - x_k|}\xi_j\bar{\eta}_k, \xi = \{\xi_j\} \in \operatorname{dom}(M(0)), \qquad \eta = \{\eta_j\} \in (t_{M(0)}). \quad (114)$$

If, in addition, $\sum_{j\in\mathbb{N}}' |x_j - x_k|^{-2} < \infty$ for every$k \in \mathbb{N}$, then $e_k \in \operatorname{dom}(M(0)), k \in \mathbb{N}$, where $e_k = \{\delta_{jk}\}_{j=1}^{\infty}$ is the standard orthonormal basis of $l^2(\mathbb{N})$, and the matrix.

$$\mathcal{M}'(0) := \left(\frac{1-\delta_{kj}}{|x_k - x_j| + \delta_{kj}}\right)_{j,k=1}^{\infty}, \tag{115}$$

define a (minimal) closed symmetric operator $M'(0)$ on $l^2(\mathbb{N})$. Moreover,

$$\mathrm{dom}(M'(0)^*) = \left\{\{\xi_j\} \in l^2(\mathbb{N}): \sum_{j\in\mathbb{N}} \left|\sum_{k\in\mathbb{N}}{}' |x_j - x_k|^{-1}\xi_k\right|^2 < \infty\right\}. \tag{116}$$

The operator $M'(0)$ is semibounded from below and its Friedrichs extension $M'(0)_F$ coincides with $M(0)$, that is, $M'(0)_F = M(0)$.

Proof. (i) The assertion follows by combining Proposition (5.2.10) (i) and (5.2.17) (iv) (cf. formulas (102) and (98)).

(ii) By Proposition (5.2.10) (i).

$$t_{M(0)}[\xi] := \lim_{t\downarrow 0}(M(-t)\xi,\xi). \qquad \xi \in \mathrm{dom}(t_{M(0)}) := \left\{\eta: \lim_{t\downarrow 0}(M(-t)\eta,\eta) < \infty\right\}. \tag{117}$$

Let us denote for the moment the form defined in (113) by$t_0 = t_{M(0)}$.

Note that the function $f(t) = (1-e^t)/t = \int_0^1 e^{-st}ds$ is absolutely monotone $f \in M[0,\infty)$. Hence $f \in \Phi_3$. This fact together with (102) and (113) yields

$$t_0[\xi] - (M(-t)\xi,\xi) = \sum_{|k-j|>0} \frac{1-e^{-t|x_j-x_k|}}{|x_j - x_k|}\xi_j\bar{\xi}_k > 0, t > 0, \xi = \{\xi_j\}_1^{\infty} \in \mathrm{dom}(t_0). \tag{118}$$

Thus, for any$\xi \in \mathrm{dom}(t_0)$ the $\lim_{t\downarrow 0}(M(-t)\xi,\xi)$ is finite and by (117), $\mathrm{dom}(t_0) \subset \mathrm{dom}(t_{M(0)})$.

Now we prove that $t_{M(0)}[\xi] = t_0[\xi]$ for all $\xi \in \mathrm{dom}(t_0)$. For finite vectors this follows at once from (118) and (117). fix $\xi \in \mathrm{dom}(t_0)$. Given $\varepsilon > 0$ if follows from (113) and (117) that there exists $N \in \mathbb{N}$ such that the finite vector $\xi^{(N)} := \{\xi_j\}_1^N$ satisfies.

$$\left|t_0[\xi] - t_0[\xi^{(N)}]\right| < \varepsilon \text{ and } \left|t_{M(0)}[\xi] - t_{M(0)}[\xi^{(N)}]\right| < \varepsilon.$$

Then $\left|t_0[\xi] - t_{M(0)}[\xi]\right| < 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $t_{M(0)}[\xi] = t_0[\xi]$.

The equality$\mathrm{dom}t_0 = \mathrm{dom}(t_{M(0)})$ is obvious.

(iii) follows from (ii) and the first form representation theorem (cf. [121]. Theorem 6.2.1]).

(iv) By the assumption $\Sigma'_{j\in\mathbb{N}}|x_j - x_k|^{-2} < \infty$m we have $e_k \in \mathrm{dom}(M(0))$. Now [120, Theorem 56.4] gives the first assertion, while the second follows from [120, Theorem 56, 2].

(v) Define a quadratic from t'_0 by$t'_0[\xi] := (M'(0)\xi, \xi \in \mathrm{dom}(t'_0)) = \mathrm{dom}(M'(0))$. Clearly, the finite vectors are dense in $\mathrm{dom}(t_{M(0)})$ with respect to the norm $[\xi]_+^2 := t_{M(0)}[\xi] + C\|\xi\|^2$ for sufficiently large $C > 0$. Since $t'_0[\eta] = t_{M(0)}[\eta]$, the closure of the form t'_0 is $t_{M(0)}$. Since $M(0) = M(0)^*$ and $\mathrm{dom}(M(0)) \subset \mathrm{dom}t_{M(0)}$, this complete the proof.

Theorem(5.2.20) [176]: Let $\Pi\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be the boundary triplet for H^* defined in Proposition (5.2.17), M the corresponding Weyl function and let Θ be a self-adjoint relation on $\mathcal{H}$. Then:

(i) The operator $H_0 := H^* \upharpoonright \ker\Gamma_0$ is the free Lapacian $H_0 = -\Delta, \mathrm{dom}(H_0) = \mathrm{dom}(\Delta) = W^{2,2}(\mathbb{R}^3)$. Moreover, H_0 is the Friedrichs extension H_F of H and $\mathrm{dom}(t_{H_0}) = W^{1,2}(\mathbb{R}^3)$.

(ii) The operator $H_{M(0)}$ is the Krein extension H_k of H and given by $H_K = H^* \upharpoonright \mathrm{dom}(H_K)$, where the domain $\mathrm{dom}(H_K)$ is the direct sum of $\mathrm{dom}(H)$ and the vector space

$$\left\{\sum_{j=1}^{\infty}(\xi_{0j}\varphi_j + \xi_{1j}e_j): \{\xi_{1j}\} = T_1^{-1}(M(0) - T_0)\xi_0\{\xi_{0j}\} \in \mathrm{dom}(M(0))\right\}.$$

The extensions $H_0 = H_F$ and H_K are disjoint. They are transversal if and only if the operator $M(0)$ is bounded on $l^2(\mathbb{N})$. For instance, this is true whenever condition (40) is satisfied.

(iii) $H_\Theta \geq 0$ if and only if Θ is semibounded below, $\mathrm{dom}(t_\Theta) \subset \mathrm{dom}(t_{M(0)})$ and $t_\Theta \geq t_{M(\Theta)}$. In particular, $H_\Theta \geq 0$ when $\mathrm{dom}(\Theta) \subset \mathrm{dom}(M(0))$ and $\Theta - M(0) \geq 0$.

(iv) H_Θ is lower semibounded if and only if Θ is. In this case the quadratic from $t_{H_\Theta}$is

$$\mathrm{dom}(t_{H_\Theta})W^{1,2}(\mathbb{R}^3) + \left\{\sum_{j=1}^{\infty}\xi_j\varphi_j: \xi = \{\xi_j\}_{j\in\mathbb{N}} \in (t_\Theta) \subset l^2(\mathbb{N})\right\}, \tag{119}$$

$$t_{H_\Theta}[f] + \|f\|_{L^2}^2 = \int_{\mathbb{R}^3}(|\nabla g(x)|^2 + |g(x)|^2)dx + t_\Theta[\xi] - \sum_{|k-j|>0}\frac{e^{-|x_j-x_k|}}{|x_j - x_k|}\xi_j\bar{\xi}_k, \tag{120}$$

where $f = g + \sum_{j\in\mathbb{N}}\xi_j\varphi_j \in \mathrm{dom}(t_{H_\Theta})$ with $g \in W^{1,2}(\mathbb{R}^3)$ and $\xi = \{\xi_j\}_{j\in\mathbb{N}} \in \mathrm{dom}(t_\Theta)$.

(v) In particular, for the quadratic form $t_{H_\Theta} = t_{H_{M(0)}}$we have

$$\mathrm{dom}(t_{H_K}) = W^{1,2}(\mathbb{R}^3) + \left\{\sum_{j=1}^{\infty}\xi_j\varphi_j: \{\xi_j\}_1^\infty \in l^2(\mathbb{N}), \sum_{|k-j|>0}|x_j - x_k|^{-1}\xi_j\bar{\xi}_k < \infty\right\}, \tag{121}$$

$$t_{H_\Theta}[f] + \|f\|_{L^2}^2 = \int_{\mathbb{R}^3}|\nabla g(x)|^2dx + \|g\|_{L^2}^2 + \sum_{|k-j|>0}\frac{1 - e^{-|x_j-x_k|}}{|x_j - x_k|}\xi_j\xi_k, \tag{122}$$

where $f = g + \sum_{j\in\mathbb{N}}\xi_j\varphi_j \in \mathrm{dom}\left(t_{H_{M(0)}}\right)$ with $g \in W^{1,2}(\mathbb{R}^3)$ and $\{\xi_j\}_{j\in\mathbb{N}} \in \mathrm{dom}(t_{M(0)})$.

(vi) If Θ is lower semiboudned and $\mathrm{dom}(t_\Theta) \subset \mathrm{dom}(t_{M(0)})$, then $k_-(H_\Theta) = k_-(t_{\Theta - M(0)})$. If, in addition, $\mathrm{dom}\,(\Theta) \subset \mathrm{dom}(M(0))$, then $k_-(\Theta - M(0))$.

(vii) If $M(0)$ is bounded, i.e., H_k and H_F are transversal, we have the implication.

$$(\Theta - M(0))E_{\Theta - M(0)}(-\infty, 0) \in \mathfrak{S}_p(\mathcal{H}) \Longrightarrow H_\Theta E_{H_\Theta}(-\infty, 0) \in \mathfrak{S}_p(\mathfrak{H}). \tag{123}$$

For instance, implication (123) holds whenever condition (123) is satisfied

Proof. (i) The first statement is immediate from (94) and definition (99) of Γ_0.

Further, integrating by part one gets

$$t'_H[f] + \|f\|_{L^2}^2 \coloneqq (Hf, f) + \|f\|_{L^2}^2 = \int_{\mathbb{R}^3}|\nabla f(x)|^2dx + \|f\|_{L^2}^2 =: \|f\|_{W^{1,2}}^2.\, f \in \mathrm{dom}(H). \tag{124}$$

Since $\mathrm{dom}(H)$ is dense in $W^{1,2}(\mathbb{R}^3)$, the closure t_H of t'_H is defined by (124) on the domain $\mathrm{dom}(t_H) = W^{1,2}(\mathbb{R}^3)$. Noting that $\mathrm{dom}(t_{H_0}) = W^{1,2}(\mathbb{R}^3) = \mathrm{dom}(t_H)$ we get the result.

We present another proof that is based on the Weyl function. it follows from (102) and (98) that $\lim_{x\downarrow-\infty}(M(x)h, h) = -\infty$ for $h \in \mathcal{H} \setminus \{0\}$. It follows from (102) and (98)

(ii) By Proposition (5.2.10), $\mathrm{dom}(H_K) = \ker(\Gamma_1 - M(0)\Gamma_0)$ since H_K and $H_0 = H_F$ are disjoint. Inserting the expressions from (99) and (103) for Γ_1 and Γ_0 we get the result.

(iii) follows immediately from Proposition (5.2.12) (i).

(iv) Let $\xi = \{\xi_j\}_1^\infty \in l^2(\mathbb{N})$. Set $|\xi| := \{|\xi_j|\}_{j\in\mathbb{N}}$. Then we derive from (102)

$$\left|\langle M(-t^2)\xi,\xi\rangle + \frac{t}{4\pi}\|\xi\|_{t^2}^2\right| \le \left|\sum_{|k-j|>0} \frac{e^{-t|x_j-x_k|}}{|x_j - x_k|}\xi_j\bar{\xi}_k\right|$$

$$\le \frac{1}{d_*(X)}\sum_{j,k\in\mathbb{N}} e^{-t|x_j-x_k|}|\xi_j\bar{\xi}_k| \le d_*(X)^{-1}e^{-(t-1)d_*(X)}\sum_{j,k\in\mathbb{N}} e^{-|x_j-x_k|}|\xi_j\bar{\xi}_k|$$

$$= d_*(X)^{-1}e^{-(t-1)d_*(X)}2.\,|\langle T_1|\xi|,|\xi|\rangle_{l^2(\mathbb{N})}| \le d_*(X)^{-1}e^{(1-t)d_*(X)}2.\,\|T_1\|\cdot\|\xi\|_{l^2(\mathbb{N})}^2 \quad (125)$$

For any $\varepsilon > 0, \varepsilon < \|T_1\|d_*(X)^{-1}$, we define $t_0 = t_0(\varepsilon)$ by

$$t_0 = t_0(\varepsilon) = 1 - \ln(\varepsilon d_*(X)\|T_1\|^{-1}). \quad (126)$$

Then it follows from (125) that

$$(M(-l^2)\xi,\xi) \ge -\left(\frac{1}{4\pi} + \varepsilon\right)\|\xi\|_{l^2}^2, \qquad t \ge t_0, \quad (127)$$

and hence $M(-l^2) \rightrightarrows -\infty$. Now Proposition (5.2.11)yield the first assertion.

Next we prove the second statement. By [155, Theorem 1], the domain $dom(t_{H_\Theta})$ is a direct sum

$$dom(t_{H_\Theta}) = dom(t_H) + \gamma(-\varepsilon^2)dom(t_\Theta), \quad \varepsilon > 0, \quad (128)$$

Hence any $f \in dom(t_{H_\Theta})$ can be written as $f = g + \gamma(-\varepsilon^2)h$, where $g \in dom(t_H)$ and $h \in dom(t_\Theta)$. Noting that $dom(t_H) = W^{1,2}(\mathbb{R}^3)$, and combining (128) with (100) yields (119).

Further, by [155, Theorem 1] we have the equality

$$t_{H_\Theta}[f] + \|f\|^2 = t_H[g] + \|g\|^2 + t_\Theta[h] - (M(-1)h,h), \qquad f := g + \gamma(-1)h. \quad (129)$$

Using Proposition (5.2.17) (iv) and the equality $t_H[g] = \int_{\mathbb{R}^3}|\nabla g(x)|^2\,dx$ we obtain (120).

(v) follows from (iv) with $\Theta = M(0)$.

(vi) By (i), $H_0 = H_F$. Hence the assertion is immediate from Proposition (5.2.12) (ii).

(vii) Since H0 is the Friedrichs extension of H, [155, Theorem 3] implies the assertion.

Remark(5.2.21) [176]: It follows from (5.2.21) and (9) that the inclusion

$$dom(t_{H_k}) = W^{2,2}(\mathbb{R}^3) + \gamma(-1)dom t_{M(0)} \supset W^{2,2}(\mathbb{R}^3) + \mathfrak{N}_{-1}dom H^* \quad (130)$$

holds if and only if the operator $M(0)$ is bounded. This fact illustrates the following general result: for any non-negative operator A the inclusion $dom(t_{A_K}) \supset dom(A^*)$ holds if and only if A_K and A_F are transversal (see [155, Remark 3]).

Remark(5.2.22) [176]: (i) The Krein-type formulas (109)–(110) were established in [122, Theorem 3.1.1.1] for a special family $H_{X,\alpha}^{(3)}$ of self-adjoint extensions by approximation method. In our notation this family is parameterized by the set of self-adjoint diagonal matrices $B_\alpha = diag(\alpha_1,\ldots,\alpha_m,\ldots)$. In this case

$$H_{X,\alpha}^{(3)} = H^* \upharpoonright \left\{f = f_H + \sum_{j=1}^{\infty}\xi_{0j}\frac{e^{|x-x_j|}}{|x-x_j|} + \sum_{k,j=1}^{\infty} b_{jk}(\alpha)\xi_{0k}e^{-|x-x_j|}\right\}, \quad (131)$$

where $\widetilde{B}_\alpha = (b_{jk}(\alpha))_{j,k=1}^\infty T_1^{-1}(B_\alpha - T_0)$. It is proved in [122] that $H_{X,\alpha}^{(3)}$ is self-adjoint. Other parameterizations of the set of self-adjoint realizations are also contained in [149] and [161]. Another version of formulas (109)–(110) as well as an abstract Krein-like formula for resolvents can also be found in [161].

(ii)the case of finitely many point interactions $(m < \infty)$ different descriptions of nonnegative realizations has been obtained in [127,144,138].

(iii) In connection with Theorem (5.2.20) (iv) we mention the sections [151] and [143] where similar statements have been obtained for realizations of 1D Schrödinger operators (1) with $d_*(X) \geq 0$ and elliptic operators in exterior domains, respectively.

Theorem (5.2.23) [176]: Let $d_*(X) > 0$ and let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be the boundary triplet for H^* defined in Proposition (5.2.17). Suppose that Θ is a self-adjoint relation on $\mathcal{H}$. Then:

(i) For any $p \in (0, \infty]$ we have the following equivalence:

$$(H_\Theta - i)^{-1}(H_0 - i)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \Leftrightarrow (\Theta - i)^{-1} \in \mathfrak{S}_p(\mathcal{H}). \quad (132)$$

(ii) If $(\Theta - i)^{-1} \in \mathfrak{S}_1(\mathcal{H})$, then the non-negative ac-part $H_\Theta^{ac} E_{H_\Theta}(\overline{\mathbb{R}}_+)$ of the operator $H_\Theta = H_\Theta^*$ is unitarily equivalent to the Laplacian $-\Delta$.

(iii) Suppose that $(\Theta - i)^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$ and condition (40) is satisfied, i.e.,

$$C_1 := \sup_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}}{}' \frac{1}{|x_k - x_j|} < \infty. \quad (133)$$

Then the ac-part $H_\Theta^{ac} = H_\Theta^{ac} E_{H_\Theta}(\overline{\mathbb{R}}_+)$ of H_Θ is unitarily equivalent to the Laplacian $-\Delta$.

Proof. (i) This assertion follows at once from Proposition (5.2.9).

(ii)By Proposition (5.2.20) (i) $H_0 = -\Delta$. Therefore, by (132) with $p = 1$, $\left[(H_\Theta - i)^{-1} — \Delta - i)^{-1}\right] \in \mathfrak{S}_1(\mathfrak{H})$. It remains to apply the Kato-Rosenblum theorem (see [148]).

(iv) (iii) Let $z = t + i\gamma \in \mathbb{C}_+, t > 0$ and $\sqrt{z} = \alpha + i\beta$. Clearly, $\alpha > 0, \beta > 0$ and $i\sqrt{z} = i\alpha - \beta$. It follows from (98) that

$$\tilde{G}_{\sqrt{z}}(|x_j - x_k|) = \frac{\left|e^{(-\beta + i\alpha)|x_j - x_k|}\right|}{|x_j - x_k|} = \frac{e^{-\beta|x_j - x_k|}}{|x_j - x_k|}, j \neq k, \quad (134)$$

It follows from (102) combined with (133) and (134) that

$$\|M(t + iy)\| \leq \sqrt{\alpha^2 + \beta^2} + e^{-\beta} \sup_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}}{}' \frac{1}{|x_k - x_j|} = \sqrt{\alpha^2 + \beta^2} + C_1 e^{-\beta}$$
$$\leq \sqrt{t} + 1 + 1 + C_1, \qquad y \in [0,1]$$

Thus, for any fixed $t > 0$ the family$M(t + iy)$ is uniformly bounded for $y \in (0, 1]$, hence the weak limit $M(t + iy) := \omega - \lim_{y \downarrow 0} M(t + iy)$ exist and

$$\omega - \lim_{y \downarrow 0} M(t + iy) =: M(t + i0) =: M(t) = i\sqrt{t}I + (\tilde{G}_{\sqrt{t}}(|x_j - x_k|))_{j,k=1}^{\infty} (135)$$

From (132), applied with $p = \infty$, we conclude that $[(H_\Theta - z)^{-1} - (H_0 - z)^{-1}] \in \mathfrak{S}_\infty(\mathfrak{N})$ since $(\Theta - i)^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$. To complete the proof it suffices to apply [122], Theorem 4.3] to H_Θ and $H_0 = -\Delta$.

We need the following auxiliary lemma which is of interest in itself.

Lemma (5.2.24) [1767]: Suppose that A is a simple symmetric operator in $\mathfrak{H}$ and $\{H, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* with Weyl function M. Assume that for any$t \in (\alpha, \beta)$ the uniform limit

$$M(t) := M(t + i0) := u - \lim_{y \downarrow 0} M(t + iy) (136)$$

exists and $0 \in \rho(M_I(t))$ for $t \in (\alpha, \beta)$. Then the spectrum of any self-adjoint extension $\tilde{A}$ of A on $\mathfrak{H}$ in the interval (α, β) is purely absolutely continuous, i.e.,

$$\delta_s(\tilde{A}) \cap (\alpha, \beta) = \emptyset. \quad (137)$$

The operator $\widetilde{A}E_{\widetilde{A}}(\alpha,\beta) = \widetilde{A}^{ac}E_{\widetilde{A}}(\alpha,B)$ is unitarily equivalent to $A_0E_{A_0}(\alpha,\beta)$, where $A_0 = A^*\lceil \ker\Gamma_0$.

Proof. Without loss of generality we can assume that the extensions $\widetilde{A}$ and A_0 are disjoint. Then, by Proposition (5.2.6) (iii), there is a self-adjoint operator B on $\mathcal{H}$ such that $\widetilde{A} = A_B$, where $A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$.

We set $M_B(t+iy) := \left(B - M(t+iy)\right)^{-1}$ and note that

$$\operatorname{Im}(M_B(t+iy)) = (B - M(t+iy))^{-1}\operatorname{Im}(<(t+iy))\left(B - M^*(t+iy)\right)^{-1}, y\in\mathbb{R}_0. \quad (138)$$

Fix $t\in(\alpha,\beta)$. By assumption we have $0\in\rho(M_1(t))$, i.e., there exists $\varepsilon = \varepsilon(t)$ such that

$$\langle M_1(t+iy)h,h\rangle \geq \varepsilon\|h\|^2, \qquad h\in\mathcal{H}, \quad (139)$$

It follows from (136) that there exists $y_0\in\mathbb{R}_+$ such that

$$\|M_I(t+iy) - M_I(t)\| \leq \varepsilon/2 \text{ for } \in[0,y_0). \quad (140)$$

Combining (139) with (140) we get

$$\langle M_I(t+iy)h,h\rangle = \langle M_I(t)h,h\rangle + \langle\left(M_I(t+iy) - M_I(t)\right)h,h\rangle \geq 2^{-1}\varepsilon\|h\|^2, y\in[0,y_0).$$

Hence, for any$h\in\operatorname{dom}(B)$,

$$\|(M(t+iy)-B)h\|\cdot\|h\| \geq |\langle(M(t+iy)-B)h,h\rangle| \geq \operatorname{Im}\langle(M(t+iy)-B)h,h\rangle = \langle M_I(t+iy)h,h\rangle$$
$$\geq 2^{-}\varepsilon\|h\|^2, y\in\lceil 0,y_0)$$

Since $0\in\rho(M(t+iy)-B)$, the latter inequality is equivalent to

$$\|(M(t+iy)-B)^{-1}\| \leq 2\varepsilon^{-1}, \ y\in[0,y_0). \quad (141)$$

It follows that

$$\|(B-M(t+iy))^{-1} - (B-M(t))^{-1}\|$$
$$= \left\|\left(B-M(t+iy)\right)^{-1}[M(t+iy) - M(t+iy) - M(t)]\left(B-M(t)\right)^{-1}\right\|$$
$$\leq 4\varepsilon^{-2}\|M(t+iy)-M(t)\|, \qquad y\in[0,y_0)$$

Hence

$$u - \lim_{y\downarrow 0}\left(B - M(t+iy)\right)^{-1} = (B-M(t))^{-1}. \quad (142)$$

Next, it is easily seen that $\prod_B = \{\mathcal{H}, \ \Gamma_0^B, \Gamma_1^B \ \}$, where $\Gamma_0^B = B\Gamma_0 - \Gamma_1$, $\Gamma_1^B = 0$, is a generalized boundary triplet for $A_*\subset A^*, \operatorname{dom}(A_*) = \operatorname{dom}(A0) + \operatorname{dom}(A_B)$ (see [64] for the definitions). The corresponding Weyl function is $M_B(\cdot) = (B - M(\cdot))^{-1}$. Therefore, combining (142) with [131,Theorem 4.3], we get $\tau_s(A_B) \cap (\alpha,\beta) = \emptyset$, i.e., $\widetilde{A}E_{\widetilde{A}}(\alpha,\beta) = \widetilde{A}^{ac}E_{\widetilde{A}}(\alpha,\beta)$.

Moreover, passing to the limit in (138) as $y\downarrow 0$, and using (136) and (142), we obtain

$$\operatorname{Im}(M_B(t+i0)) = (B - M(t+i0))^{-1}M_I(t+i0)(B - M^*(t+i0))^{-1}, t\,(\alpha,\beta). \quad (143)$$

Since $\ker \ B - M^*(t+i0))^{-1} = \{0\}$, we have

$$\operatorname{rank}\ \operatorname{Im}(M_B(t+i0))) = \operatorname{rank}\ \operatorname{Im}(M_I(t+i0))), t\in(\alpha,\beta). \quad (144)$$

By Proposition (5.2.14) the operators $A_BE_{A_B}(\alpha,\beta)$ and $A_0E_{A_0}(\alpha,\beta)$are unitarily equivalent.

Now we are ready to prove the main result of this section.

Theorem (5.2.25) [176]: Let $\widetilde{H}$ be a self-adjoint extension of H. Suppose that

$$C_2 := \sum_{|k-j|>0}\frac{1}{|x_j - x_k|^2} < \infty. \quad (145)$$

(i) Then the part $\widetilde{H}E_{\widetilde{H}}(C_2,\infty)$ of $\widetilde{H}$ is absolutely continuous, i.e.,

$$\sigma_s(\widetilde{H}) \cap (C_2,\infty) = \emptyset. \quad (146)$$

Moreover, $\widetilde{H}E_{\widetilde{H}}(C_2,\infty)$ is unitarily equivalent to the part $-\Delta E_{-\Delta}(C_2,\infty)$ of $-\Delta$.

(ii) Assume, in addition, that the conditions in Proposition (5.1.17) are satisfied, i.e., $d_*(X_n) > 0$ and $D^*(X_n) = 0$. Then $\widetilde{H}_+ := \widetilde{H}E_{\widetilde{H}}(\mathbb{R}_+)$ is unitarily equivalent to $H_0 = -\Delta$. In particular, $\widetilde{H}_+$ is purely absolutely continuous, $\widetilde{H}_+ = \widetilde{H}_+^{ac}$.

Proof. As in the proof of Proposition (5.2.18) (ii) we decompose the symmetric operator H in a direct sum $H = \widehat{H} \oplus H'$ of a simple symmetric operator $\widehat{H}$ and a self-adjoint operator H'. Next we definea boundary triplet $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ for $\widehat{H}^*$ by setting $\widehat{\Gamma}_j := j \restriction \mathrm{dom}(\widehat{H}^*), j \in \{0,1\}$, and note that the corresponding Weyl function $\widehat{M}(\cdot)$ coincides with the Weyl function $M(\cdot)$ of Π. Further, any proper extension $\widetilde{H} = H_\Theta$ of H admits a decomposition $H_\Theta = \widehat{H}_0 \oplus H'$. In particular, the operator $\widehat{H}_0 = -\Delta$ is decomposed as $H_0 = \widehat{H}_0 \oplus H'$, where $\widehat{H}_0 = \widehat{H}^* \restriction \ker(\widehat{\Gamma}_0) = \widehat{H}_0^*$. Being a part of H_0, the operator $H' = (H')^*$ is absolutely continuous and $\sigma(H') = \sigma_{ac}(H') \subset \mathbb{R}_+$, because $\sigma(H_0) = \sigma_{ac}(H_0) = \mathbb{R}_+$. Therefore, it suffices to prove all assertions for self-adjoint extensions $\widehat{H}_\Theta$ of the simple symmetric operator $\widehat{H}$.

(i) To prove (146) for any extension of $\widehat{H}$ it suffices to verify the conditions of Lemma (5.2.24) noting that $\widehat{M}(\cdot) = M(\cdot)$. First we prove that for any $t \in \mathbb{R}_+$ the uniform limit

$$M(t+i0) := u - \lim_{y\downarrow 0} M(t+iy) \cong = \left(i\sqrt{t}\,\delta k_j + \frac{e^{i\sqrt{t}|x_k - x_j|} - \delta k_j}{|x_k - x_j| + \delta k_j}\right)_{j,k=1}^{\infty}, t \in \mathbb{R}, \tag{147}$$

exists, where the symbol $T \cong T$ means that the operator $\mathcal{T}$ has the matrix $\mathcal{T}$ with respect to the standard basis of $l^2(\mathbb{N})$.

Indeed, it follows from (102) that for any $\xi,\eta \in l^2(\mathbb{N})$,

$$\langle (M(t+iy) - M(t)\xi),\eta\rangle = (\sqrt{t+iy} - \sqrt{t})\langle \xi,\eta\rangle + \sum\nolimits'_{j,k\in\mathbb{N}} (e^{-\beta}|x_j - x_k| - 1) \frac{e^{i\alpha|x_j - x_k|}}{|x_j - x_k|} \xi_j \bar{\eta}_k. \tag{148}$$

Fix $\varepsilon > 0$. By to the assumption (145) there exists $N = N(\varepsilon) \in N$ such that

$$\sum\nolimits_{j\ge\mathbb{N}} \sum\nolimits'_{k\in\mathbb{N}} \frac{1}{|x_j - x_k|^2} + \sum\nolimits_{k\ge\mathbb{N}} \sum\nolimits'_{j\in\mathbb{N}} \frac{1}{|x_j - x_k|^2} < (\varepsilon/2)^2. \tag{149}$$

Then

$$\begin{aligned} &\sum\nolimits_{j\ge\mathbb{N}} \sum\nolimits'_{k\in\mathbb{N}} \frac{1}{|x_j - x_k|} |\xi_j \bar{\eta}_k| + \sum\nolimits_{k\ge\mathbb{N}} \sum\nolimits'_{j\in\mathbb{N}} \frac{1}{|x_j - x_k|} |\xi_j \bar{\eta}_k| \\ &\le \left(\sum_{j\ge N} |\xi_j|^2\right)^{1/2} \left(\sum_{j\ge N}^{\infty} |\eta_k|^2\right)^{1/2} \left(\sum\nolimits_{j\ge\mathbb{N}} \sum\nolimits'_{k\in\mathbb{N}} \frac{1}{|x_j - x_k|^2}\right)^{1/2} \\ &+ \left(\sum_{j\ge N} |\eta_k|^2\right)^{1/2} \left(\sum_{j=1}^{\infty} |\xi_j|^2\right)^{1/2} \left(\sum\nolimits_{k\ge\mathbb{N}} \sum\nolimits'_{j\in\mathbb{N}} \frac{1}{|x_j - x_k|^2}\right)^{1/2} \\ &\le 2^{-1}\varepsilon \|\xi\|_{l^2} \cdot \|\eta\|_{l^2}. \end{aligned} \tag{150}$$

On the other hand, since $d_*(X) > 0$, we can find $\beta_0 = \beta_0(N)$ such that

$$\sum_{j,k=1}^{N} \frac{(1 - e^{-\beta|x_j - x_k|})}{|x_j - x_k|} \le \varepsilon d_*(X)^{-1} \text{ for } \beta \in (0,\beta_0). \tag{151}$$

Combining (148) with (160) and (161) we get

$$|\langle (M(t+iy)-M(t))\xi,\eta\rangle| \le \varepsilon\ (1+d_*(X)^{-1})\|\xi\|_{l^2}\cdot\|\eta\|_{l^2}, y\in(0,y_0), \quad (152)$$

that is,

$$\|M(t+iy)-M(t)\| \le \varepsilon\ (1+d_*(X)^{-1}) \text{for } y\in(0,y_0). \quad (153)$$

Thus, the uniform limit (147) exists for any$t\in\mathbb{R}_+$.
Further, it follows from (147) that

$$M_I(t) := M_I(t+i0) \cong \sqrt{t}\left(\delta k_j + \frac{\sin(\sqrt{t}|x_k - x_j\,|)}{\sqrt{t}(|x_k-x_j| + \delta k_j)}\right)_{j,k=1}^{\infty}, t\in\mathbb{R}_+. \quad (154)$$

This relation combined with assumption (145) yields $0\in\rho\ (M_I(t))$for $t > C_2$. The assertion 0follows now by applying Lemma (5.2.24) to the operator bH and the interval (C_2,∞).

(ii) By (20) the function $\Omega_3(t) = \frac{\sin t}{t}$is in Φ_3. Hence, by Proposition (5.1.17), the matrix function $\Omega_3(t\|\cdot\|)$ is stronglyX-positively definite for any$t>0$, i.e., the matrix $\Omega_3(t\|x_j-x_k\|)_{j,k\in\mathbb{N}}$ is positively definite for any t > 0. By (154) we have

$$M_I(t) := M_I(t+i0) \cong \sqrt{t}\Omega_3(\sqrt{t}\|x_j-x_k\|)_{j,k\in\mathbb{N}} t\in\mathbb{R}_+.$$

Hence MI(t) is positively definite for $t\in\mathbb{R}_+$. It remains to apply Lemma (5.2.24) to the boundary triplet $\widehat{\Pi}$ and the interval $\mathbb{R}_+$.

Next we present another result on the ac-spectrum of self-adjoint extensions that is based on Corollary(5.1.23).

Theorem (5.2.26) [176]: Let $\widetilde{H}$ be an arbitrary self-adjoint extension of H. Assume that

$$\lim_{p\to\infty}\left(\sup_{j\in N}\sum\nolimits'_{k\in\mathbb{N}}\frac{1}{|x_k-x_j\,|}\right) = 0 \quad (155)$$

and let $C1$ be defined by (133). Then:

(i) The part $\widetilde{H}E_{\widetilde{H}}\ (C_1^2,\infty)$ of $\widetilde{H}$ is absolutely continuous, i.e.

$$\sigma_s(\widetilde{H}) \cap (C_1^2,\infty) = \emptyset. \quad (156)$$

Moreover, $\widetilde{H}E_{\widetilde{H}}\ (C_1^2,\infty)$ is unitarily equivalent to the part $-\Delta E_{-\Delta}(C_1^2,\infty$ of$-\Delta$.

(ii) Assume, in addition, that the conditions of Proposition (5.1.17) are fulfilled, i.e. $d_*(X_n) > 0$ and $D^*(X_n) = 0$. Then $\widetilde{H}E_{\widetilde{H}}\ (\mathbb{R}_+)$ is unitarily equivalent to $H_0 = -\Delta$. In particular, $eH+$ is purely absolutely continuous, i.e. $\widetilde{H}_+ = \widetilde{H}_+^{ac}$.

Proof. (i) The proof is similar to that of Theorem (5.2.25) (i). Indeed, by assumption (155), for any$\varepsilon > 0$ one can find $N = N(\varepsilon)\in N\mathbb{N}$ such that

$$\sup_{j\ge N}\sum\nolimits'_{k\in\mathbb{N}}\frac{1}{|x_j-x_k|} + \sup_{k\ge N}\sum\nolimits'_{j\in\mathbb{N}}\frac{1}{|x_j-x_k|} < \varepsilon/2. \quad (157)$$

Starting with (157) instead of (149), we derive

$$\sum\nolimits_{j\ge\mathbb{N}}\sum\nolimits'_{k\in\mathbb{N}}\frac{1}{|x_j-x_k|}|\xi_j\bar{\eta}_k| + \sum\nolimits_{k\ge\mathbb{N}}\sum\nolimits'_{j\in\mathbb{N}}\frac{1}{|x_j-x_k|}|\xi_j\bar{\eta}_k| \le 2^{-1}\varepsilon\|\xi\|_{l^2}\cdot\|\eta\|_{l^2} \quad (158)$$

which implies (153). That the operator MI$(\cdot)$ has a bounded inverse if $t > C_1^2$ follows from (154) and Proposition (5.1.26). It remains to apply Lemma (5.2.24) to the operator $\widetilde{H}$ and the interval (C_1^2,∞).

(ii) follows by arguing in a similar manner as in the proof of Theorem (5.2.25) (ii).

Chapter 6
General Inequalities and Negative Spectrum

In some cases the kernel decays exponentially as $t \to \infty$This allows us to consider very slow decaying potentials and obtain some results that are precise in the logarithmical scale. We devoted to the spectral theory of the Schrödinger operator on the simplest fractal: Dyson'shierarchical lattice. An explicit description of the spectrum, eigenfunctions, resolvent and parabolic kernelare provided for the unperturbed operator, i.e., for the Dyson hierarchical Laplacian. Positive spectrum is studied for the perturbations of the hierarchical Laplacian.

Section (6.1): Cwikel-Lieb-Rozenblum and Lieb-Thirring Inequalities

Lets us recall the classical estimate concerning the negative eigenvalues of the operator $H = -\Delta + V(x)$ on $L^2(R^d)$, $d \geq 3$. Let $N_E(V)$ be the number of eigenvalues, E_i of the operator H that are below or equal to $E \leq 0$. In particular, $N_0(V)$ is the number of non-positive eigenvalues. Let

$$N(V) = \#\{E_i < 0\}$$

be the number of strictly negative eigenvalues of the operator H. Then the Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities have the following form, respectively, (see [180], [191]-[194],[198], [197]).

$$N(V) \leq C_d \int_{R^d} W^{\frac{d}{2}}(x)dx, \tag{1}$$

$$\sum_{i:E_i<0} |E_i|^\gamma \leq C_{d,\gamma} \int_{R^d} W^{\frac{d}{2}+\gamma}(x)dx. \tag{2}$$

Here $W = |V_-|$, $V_-(x) = \min(V(x), 0)$, $d \geq 3$, $g \geq 0$. The inequality (1) can be considered as a particular case of (2) with $\gamma = 0$. Conversely, the inequality (2) can be easily derived from (1) (see [197]). So, below we will mostly discuss the Cwikel-Lieb-Rozenblum inequality and its extensions, although some new results concerning the Lieb-Thirring inequality will also be stated.

A review of different approaches to the proof of (1) can be found in [200]. We will remind only several results. E. Lieb [191], [192] and I. Daubechies [181] offered the following general form of (1) and (2). Let $H = H_0 + V(x)$, and $V(x) = V_+(x) - V_-(x), V_\pm \geq 0$. Then

$$N(V) \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} dt \int_X G(tW(x))\mu(dx). \qquad (3)$$

$$\sum_{i:E_i<0} |E_i|^\gamma \leq \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} dt \int_X G(tW(x))W^\gamma \mu(dx). \qquad (4)$$

Here $W = V_- = \max(0, -V(x))$, G is a continuous, convex, non-negative function which grows at infinity not faster than a polynomial, and is such that $z^{-1}G(z)$ is integrable at zero (hence, $G(0) = 0$), and the integral (3) is finite. The function $g(\lambda), \lambda \geq 0$, is defined by

$$g(\lambda) = \int_0^\infty z^{-1}G(z)e^{-z\lambda}dz, \quad \text{i.e.}, g(1) = \int_0^\infty z^{-1}G(z)e^{-z}dz. \quad (5)$$

Note that $\pi(t) = (2\pi t)^{-\frac{d}{2}}$ in the classical case of $H_0 = -\Delta$ on $L^2(R^d)$, and (1) follows from (3) in this case by substitution $t \to \mathcal{T} = tW(x)$ if G is such that $\int_0^\infty z^{-1-\frac{d}{2}}G(z)dz < \infty$.

The inequalities above are meaningful only for those W for which integrals converge. They become particularly transparent (see [192]) if $G(z) = 0$ for $z \leq \sigma, G(z) = z - \sigma$ for $z > \sigma, \sigma \geq 0$. Then (3), (4) take the form

$$N(V) \leq \frac{1}{c(\sigma)} \int_X W(x) \int_{\frac{\sigma}{W(x)}}^\infty \pi(t)dt\mu(dx), \qquad (6)$$

$$\sum_{i:E_i<0} |E_i|^\gamma \leq \frac{1}{c(\sigma)} W^{\gamma+1}(x) \int_{\frac{\sigma}{W(x)}}^\infty \pi(t)dt\mu(dx), \qquad (7)$$

where $c(\sigma) = e^{-\sigma} \int_0^\infty \frac{ze^{-z}dz}{z+\sigma}$.

1. Daubichies [181] used Lieb method to justify the estimates above for some pseudo-differential operators in R^d. She also mentioned there that the Lieb method works in a wider setting. A slightly different approach based on the Trotter formula was used by G. Rozenblum and M. Solomyak [199], [200]. They proved (3) for a wide class of operators in $L^2(X, \mu)$ where X is a measure space with a σ-finite measure $\mu = \mu(dx)$. They also suggested the following form of (3). Assume that the function $\pi(t)$ has different power asymptotics as $t \to 0$ and $t \to \infty$. Let

$$p_0(t, x, x) \leq c/t^{\alpha/2}, \quad t \leq h, \qquad p_0(t, x, x) \leq c/t^{\alpha/2}, \quad t > h \qquad (8)$$

where $h > 0$ is arbitrary. The parameters α and β characterize the "local dimension" and the "global dimension" of X, respectively. For example $\alpha = \beta = d$ in the classical case of the Laplacian $H_0 = -\Delta$ in the Euclidean space $X = R^d$. If $H_0 = -\Delta$ is the difference Laplacian on the lattice $X = Z^d$, then $\alpha = 0, \beta = d$. If $X = S^n \times R^d$ is the product of n-dimensional sphere and R^d, then $\alpha = n + d, \beta = d$.

If $\alpha, \beta > 2$, inequality (3) implies (see [200]) that

$$N(V) \leq C(h)[\int_{\{W(x)\leq h^{-1}\}} W^{\frac{\beta}{2}}(x)\mu(dx) + \int_{\{W(x)>h^{-1}\}} W^{\frac{\alpha}{2}}(x)\mu(dx)], \qquad (9)$$

Note that the restriction $\beta > 2$ is essential here in the same way as the condition $d > 2$ in (1). We will show that the assumption on α can be omitted, but the form of the estimate in (9) changes in this case.

We will consider operators which may have different power asymptotics of $\pi(t)$ as $t \to 0$ or $t \to \infty$ or exponential asymptotics as $t \to \infty$. The latter case will allow us to consider the potentials which decay very slowly at infinity. This is particularly important in some applications, such as Anderson model, where the borderline between operators with a finite and infinite number of eigenvalues is defined by the decay of the perturbation in the logarithmic scale.

We will assume that X is a complete σ-compact metric space with Borel σ-algebra $B(X)$ and a σ-finite measure $\mu(dx)$. Let H_0 be a self-adjoint non-negative operator on $L^2(X, B, \mu)$ with the following two properties:

(a) Operator $-H_0$ is the generator of a semigroup P_t acting on $C(X)$. The kernel $p_0(t, x, y)$ of P_t is continuous with respect to all the variables when $t > 0$ and satisfies the relations

$$\frac{\partial p_0}{\partial t} = -H_0 p_0, t > 0, p_0(0, x, y) = \delta_y(x), \qquad \int_X p_0(t, x, y)\mu(dy) = 1, \quad (10)$$

i.e. p_0 is a fundamental solution of the corresponding parabolic problem. We assume that $p_0(t, x, y)$ is symmetric, non-negative, and it defines a Markov process $x_s, s \geq 0$, on X with the transition density$p_0(t, x, y)$ with respect to the measure μ.

Note that this assumption implies that $p_0(t, x, x)$ is strictly positive for all $x \in X, t > 0$, since

$$p_0(t, x, x) = \int_X p_0^2(\frac{t}{2}, x, y)\mu(dy) > 0. \qquad (11)$$

(b) There exists a function $\pi(t)$ such that $p_0(t, x, x) \leq \pi(t)$ for $t \geq 0$ and all $x \in X$. We also assume that $\pi(t)$ has at most power singularity at $t \to 0$ and is integrable at infinity, i.e. there exists m such that

$$\int_0^\infty \frac{t^m}{1 + t^m}\pi(t)dt < \infty. \qquad (12)$$

Note that condition (b) implies that

$$p_0(t, x, y) \leq \pi(t), \qquad x, y \in X. \qquad (13)$$

In fact,

$$p_0(t, x, y) = \int_X p_0(\frac{t}{2}, x, z)p_0(\frac{t}{2}, z, y)\mu(dz) \leq (\int_X p_0^2(\frac{t}{2}, x, z)\mu(dz))^{\frac{1}{2}}(\int_X p_0^2(\frac{t}{2}, z, y)\mu(dz))^{\frac{1}{2}},$$

which implies (13) due to (11). Let us note that (12), (13) imply that the process x_s is transient.

We decided to put an extra requirement on X to be a metric space in order to be able to assume that p_0 is continuous and use a standard version of the Kac-Feynman formula. This makes all the arguments more transparent. In fact, X is a metric space in all examples below. However, all the arguments can be modified to be applicable to the case when X is a measure space by using L^2-theory of Markov processes based on the Dirichlet forms.

Many examples of operators which satisfy conditions (a) and (b) will be given later. At this point we would like to mention only a couple of examples. First, note that self-adjoint uniformly elliptic operators of second order satisfy conditions (a) and (b). Condition (b) holds with $\pi(t) = Ct^{-d/2}$ due to Aronson inequality.

Another wide class of operators with conditions (a) and (b) consists of operators which satisfy condition (a) and are invariant with respect to transformations from a rich enough subgroup Γ of the group of isometries of X. The subgroup Γ has to be transitive, i.e., for some reference point $x_0 \in X$

and each $x \in X$ there exists an element $g_x \in \Gamma$ for which $g_x(x_0) = x$. Then $p_0(t, x, x) = p_0(t, x_0, x_0) = \pi(t)$. The simplest example of such an operator is given by$H_0 = -\Delta$ on $L^2(R^d, B(R^d), dx)$. The group Γ in this case is the group of translations or the group of all Euclidean transformations (translations and rotations). Another example is given by$X = Z^d$ being a lattice and $-H_0$ a difference Laplacian. Other examples will be given later.

(c) Our next assumption mostly concerns the potential. We need to know that the perturbed operator $H = H_0 + V(x)$ is well defined and has pure discrete spectrum on the negative semiaxis. For this purpose it is enough to assume that the operator $V(x)(H_0 - E)^{-1}$ is compact for some $E > 0$. This assumption can be weakened. If the domain of H_0 contains a dense in $L^2(X, B, \mu)$ set of bounded compactly supported functions, then it is enough to assume that $V_-(x)(H_0 - E)^{-1}$ is compact for some $E > 0$ and the positive part of the potential is locally integrable (see [177]).

Typically (in particular, in all the examples below) H_0 is an elliptic operator, the kernel of the resolvent $(H_0 - E)^{-1}$ has singularity only at $x = y$, this singularity is weak, and the assumptions (c) holds if the potential has an appropriate behavior at infinity. Therefore we do not need to discuss the validity of this assumption in the examples below.

Remark (6.1.1) [202]: Note that (16) differs from (3) only by inclusion of the dimension of the null space of the operator H into the left-hand side of (16). This difference is not very essential, and the first goal of this part of the section is to give an alternative proof of (3) suitable for readers with a background in probability theory.

Remark (6.1.2) [202]: If $G(z) = 0$ for $z \le \sigma, G(z) = z - \sigma$ for $z > \sigma, \sigma \ge 0$, then (16), (17) take the form

$$N_0(V) \le \frac{1}{c(\sigma)} \int_X W(x) \int_{\frac{\sigma}{W(x)}}^{\infty} \pi(t) dt \mu(dx), \qquad (14)$$

$$\sum_{i:E_i 0} |E_i|^\gamma \le \frac{1}{c(\sigma)} \int_X W^{\gamma+1}(x) \int_{\frac{\sigma}{W(x)}}^{\infty} \pi(t) dt \mu(dx), \qquad (15)$$

where $c(\sigma) = e^{-\sigma} \int_0^\infty \frac{ze^{-z}}{z+\sigma} dz$. Some applications of these inequalities will be given below.

Remark (6.1.3) [202]: Inequalities (16), (17) are valid with $\pi(t)$ moved under sign of the interior integrals and replaced by$p_0(t, x, x)$. For example, (16) holds in the following form

$$N_0(V) \le \frac{1}{g(1)} \int_0^\infty \frac{1}{t} \int_X p_0(t, x, x) G(tW(x)) \mu(dx) dt.$$

The same change can be made in (14), (15). A very minor change in the proof of the theorem is needed in order to justify this remark. Namely, one needs only to omit the last line in (32).

Theorem (6.1.4) [202]: Let (X, B, μ) be a complete σ-compact metric space with the Borel σ-algebra B and a σ-finite measure μ on B.

Let $H = H_0 + V(x)$, where H_0 is a self-adjoint, non-negative operator on $L^2(X, B, \mu)$, the potential $V = V(x) = V_+ - V_-, V_\pm \ge 0$, is real valued, and the assumptions (a)-(c) hold.

Then

$$N_0(V) \le \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X G(tW(x)) \mu(dx) dt, \qquad (16)$$

and

$$\sum_{i:E_i<0} |E_i|^\gamma \le \frac{1}{g(1)} \int_0^\infty \frac{\pi(t)}{t} \int_X G(tW(x))W(x)^\gamma \mu(dx)dt, \qquad (17)$$

where $W(x) = V_-(x)$, and functions G and g are introduced above in (3) and (5).

Proof. Step 1. Since the eigenvalues E_i depend monotonically on the potential $V(x)$, without loss of generality one can assume that $V(x) = -W(x) \le 0$.

First (steps 1-6), we'll prove inequality (16) for $N(V)$ instead of $N_0(V)$. Here we can assume that $V(x) \in C_{com}(X)$. Indeed, when $N(V)$ is considered, inequality (16) with $V(x) \in C_{com}(X)$ implies the same inequality with any$V$ such that the integral in (16) converges (see [197]). Then (step 7), we'll show that inequality (16) for $N(V)$ leads to the same inequality for $N_0(V)$. Finally (step 8), we will remind the reader of standard arguments which allow us to derive (17) from (16).

Step 2. We denote by$B$ and B_n the operators

$$B = W^{1/2}(H_0 + \varkappa^2)^{-1}W^{1/2}, \qquad B_n = W^{1/2}(H_0 + \kappa^2 + nW)^{-1}W^{1/2}, W = W(x).$$

If $N_{-\varkappa^2}(V) = \#\{E_i \le -\varkappa^2 < 0\}$,$\lambda_k$ are eigenvalues of the operator B and $n(\lambda, B) = \#\{k: \lambda_k \ge \lambda\}$, then the Birman-Schwinger principle implies

$$N_{-\varkappa^2}(V) = n(1, B). \qquad (18)$$

Thus, if $F = F(\lambda), \lambda \ge 0$, is a non-negative strictly monotonically growing function, and $\{\mu_k\}$ is the set of eigenvalues of the operator $F(B)$, then

$$N_{-\varkappa^2}(V) \le \sum_{k:\mu_k \ge F(1)} 1 \le \frac{1}{F(1)} \sum_{k:\mu_k \ge F(1)} \mu_k \le \frac{1}{F(1)} \mathrm{Tr} F(B). \qquad (19)$$

This inequality will be used with the function F of the form

$$F(\lambda) = \int_0^\infty P(e^{-z}) e^{\frac{-z}{\lambda}} dz, \quad P(t) = \sum_0^N c_n l^n, \qquad (20)$$

The exponential polynomial $P(e^{-z}), z > 0$, will be chosen later, but it will be a non-negative function with zero of order m at $z = 0$, i.e.

$$P(e^{-z}) \le C \frac{z^m}{1 + z^m}, \qquad z \ge 0, \qquad (21)$$

where m is defined in the condition (b). Since $P(e^{-z}) \ge 0$, (20) implies that F is nonnegative and monotonic, and therefore (19) holds.

From (20) it follows that

$$F(\lambda) = \sum_{n=0}^N c_n \frac{\lambda}{1 + n\lambda},$$

and the obvious relation $B_n = B(1 + nB)^{-1}$ implies that

$$F(B) = \sum_{n=0}^N c_n B_n = W^{\frac{1}{2}} \sum_{n=0}^N c_n (H_0 + \kappa^2 + nW)^{-1} W^{\frac{1}{2}}.$$

For an arbitrary operator K, we denote its kernel by$K(x, y)$. The kernel of the operator $F(B)$ can be expressed trough the fundamental solutions $p = p_n(t, x, y)$ of the parabolic problem

$$p_1 = (H_0 + nW(x))p, t > 0, p(0, x, y) = \delta_y(x).$$

Namely,

$$F(B)(x,y) = W^{\frac{1}{2}}(x)\int_0^\infty e^{-\kappa^2 t}\sum_{n=0}^N c_n p_n(t,x,y)dtW^{\frac{1}{2}}(y). \quad (22)$$

It will be shown below that the integral above converges uniformly in x and y when $\kappa = 0$. Hence, the kernel $F(B)(x,y)$ is continuous. Since the operator $F(B)$ is non-negative, from the last relation and (19), after passing to the limit as $\kappa \to 0$, it follows that

$$N(V) \leq \frac{1}{F(1)}\int_0^\infty \int_X W(x)\sum_{n=0}^N c_n p_n(t,x,x)dt\mu(dx). \qquad (23)$$

Step 3. The Kac-Feynman formula allows us to write an"explicit" representation for the Schrodinger semigroup $e^{t(-H_0 - nW(x))}$ using the Markov process x_s associated to the unperturbed operator H_0. Namely, the solution of the parabolic problem

$$\frac{\partial u}{\partial t} = -H_0 u - nW(x)u, \qquad t > 0, \qquad u(0,x) = \varphi(x) \in C(X), \qquad (24)$$

can be written in the form

$$u(t,x) = E_x e^{-n\int_0^t W(x_s)ds}\varphi(x_t).$$

Note that the finite-dimensional distributions of x_s (for $0 < t_1 < \cdots < t_n, \Gamma_1, \ldots \Gamma_n \in B(X)$) are given by the formula

$$P_x\left(x_{t_1} \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n\right)$$

$$= \int_{\Gamma_1} \cdots \int_{\Gamma_n} p_0(t_1,x,x_1)p_0(t_2 - t_1,x_1,x_2) \ldots p_0(t_n - t_{n-1},x_{n-1}x_n)\mu(dx_1) \ldots \mu(dx_n).$$

If $p_0(t,x,y) > 0$, then one can define the conditional process (bridge) $\hat{b}_s = \hat{b}_s^{x\to y,t}, \in [0,t]$, which starts at x and ends at y. Its finite-dimensional distributions are

$$P_{x\to y}\left(\hat{b}_{t_1} \in \Gamma_1, \ldots, \hat{b}_{t_n} \in \Gamma_n\right)$$

$$= \frac{\int_{\Gamma_1} \ldots \int_{\Gamma_n} p_0(t_1,x,x_1) \ldots p_0(t_n - t_{n-1},x_{n-1},x_n)p_0(t - t_n,x_n,y)\mu(dx_1) \ldots \mu(dx_n)}{p_0(t,x,y)}$$

In particular, the bridge $\hat{b}_s^{x\to x,t}, s \in [0,t]$, is defined, since $p_0(t,x,x) > 0$ (see condition (a)).
Let $p = p_n(t,x,y)$ be the fundamental solution of the problem (24). Then $p_n(t,x,y)$ can be expressed in terms of the bridge $\hat{b}_s = \hat{b}_s^{x\to y,t}, s \in [0,t]$:

$$p_n(t,x,y) = p_0(t,x,y)E_{x\to y}e^{-n\int_0^t W(\hat{b}_s)ds}. \qquad (25)$$

One of the consequence of (25) is that

$$p_n(t,x,y) \leq p_0(t,x,y). \qquad (26)$$

Another consequence of (25) is the uniform convergence of the integral in (22) (and in (23)). In fact, (21) implies that

$$\sum_{n=0}^N c_n e^{n\int_0^t W(\hat{b}_s)ds} \leq C\frac{t^m}{1+t^m}.$$

Hence from (25) and (13) it follows that the integrand in (22) can be estimated from above by$C\pi(t)\frac{t^m}{1+t^m}$. Then the uniform convergence of the integral in (22) follows from (12).
Now (23) and (25) imply

$$N(V) \le \frac{1}{F(1)}\int_0^\infty \int_X W(x)p_0(t,x,x)E_{x\to x}[\sum_{n=0}^N c_n e^{-n\int_0^t W(\hat{b}_s)ds}]\mu(dx)dt, \hat{b}_s = \hat{b}_s^{x\to x,t}.$$

Step 4. We would like to rewrite the last inequality in the form

$$N(V) \le \frac{1}{F(1)}\int_0^\infty \int_X p_0(t,x,x)E_{x\to x}[W(\hat{b}_{\mathcal{T}})\sum_{n=0}^N c_n e^{-n\int_0^t W(\hat{b}_s)ds}]\mu(dx)dt \quad (27)$$

with an arbitrary $\mathcal{T} \in [0,t]$. For that purpose, it is enough to show that

$$\int_X p_0(t,x,x)E_{x\to x}[W(\hat{b}_{\mathcal{T}})e^{-\int_0^t mW(\hat{b}_s)ds}]\mu(dx)$$

$$= \int_X p_0(t,x,x)W(x)E_{x\to x}\left[e^{-\int_0^t mW(\hat{b}_s)ds}\right]\mu(dx). \quad (28)$$

The validity of (28) can be justified using the Markov property of $\hat{b}_s$ and its symmetry (reversibility in time). We fix $\mathcal{T} \in (0,t)$. Let $y = \hat{b}_{\mathcal{T}}$. We spilt $\hat{b}_s$ into two bridges $\hat{b}_u^{x\to y,\mathcal{T}}, u \in [0,\mathcal{T}]$, and $\hat{b}_v^{y\to x,t}, v \in [\mathcal{T},t]$. The first bridge starts at x and ends at y, the second one starts at $y$and goes back to x. Using these bridges, one can represent the left hand side above as

$$\int_X\int_X W(y)[p_0(\mathcal{T},x,y)p_0(t-\mathcal{T},y,x) - p_m(\mathcal{T},x,y)p_m(t-\mathcal{T},y,x)]\mu(dx)\mu(dy)$$

$$= \int_X W(y)[p_0(t,y,y) - p_m(t,y,y)]\mu(dy),$$

which coincides with the right hand side of (28). This proves (27).

Step 5. We take the average of both sides of (27) with respect to $\mathcal{T} \in [0,t]$ and rewrite it in the form

$$N(V) \le \frac{1}{F(1)}\int_0^\infty \int_X \frac{p_0(t,x,x)}{t}E_{x\to x}\sum_0^N (c_m\int_0^t W(\hat{b}_s)dse^{-\int_0^t mW(\hat{b}_s)ds})\mu(dx)dt$$

$$= \frac{1}{F(1)}\int_0^\infty \int_X \frac{p_0(t,x,x)}{t}E_{x\to x}(u(P(e^{-u}))\mu(dx)dt, \qquad u = \int_0^t W(\hat{b}_s)ds, \quad (29)$$

where P is the polynomial defined in (20) and (23).

Let now P be such that

$$uP(e^{-u}) \le G(u), \quad (30)$$

where G is defined in the statement of Theorem (6.1.4) Then one can replace $uP(e^{-u})$ in (29) by$G(u)$. Then the Jensen inequality implies that

$$G(\int_0^t W(\hat{b}_s))ds = G(\frac{1}{t}\int_0^t tW(\hat{b}_s))ds \le \frac{1}{t}G(tW(\hat{b}_s))ds.$$

This allows us to rewrite (29) in the form

$$N(V) \le \frac{1}{F(1)}\int_0^\infty \int_X \frac{p_0(t,x,x)}{t}\frac{1}{t}\int_0^t E_{x\to x}G(tW(\hat{b}_s))ds\mu(dx)dt. \quad (31)$$

It is essential that one can use the exact formula for the distribution above:

$$E_{x\to x}G(tW(\hat{b}_s)) = \int_X G(tW(z))\frac{p_0(s,x,z)p_0(t-s,z,x)}{p_0(t,x,x)}\mu(dz).$$

Form here and (31) it follows that

$$N(V) \le \frac{1}{F(1)}\int_0^\infty \frac{1}{t^2}\int_0^t ds\int_X \int_X G(tW(z))p_0(s,x,z)p_0(t-s,z,x)\mu(dx)\mu(dz)dt$$
$$= \frac{1}{F(1)}\int_0^\infty \frac{1}{t^2}\int_0^t ds\int_X \mu(dz)G(tW(z))p_0(t,z,z)d$$
$$= \frac{1}{F(1)}\int_0^\infty \frac{1}{t}\int_X G(tW(z))p_0(t,z,z)\mu(dz)dt$$
$$\le \frac{1}{F(1)}\int_0^\infty \frac{\pi(t)}{t}\int_X G(tW(z))\mu(dz)dt, \qquad (32)$$

where $F(1)$ is defined in (20).

Step 6. Now we are going to specify the choice of the polynomial P which was used in the previous steps. It must be non-negative and satisfy (12) and (30). Polynomial P will be determined by the choice of the function G. Note that it is enough to prove (16) for functions G which are linear at infinity. In fact, for arbitrary$G$, let $G_N \le G$ be a continuous function which coincides with G when $z \le N$ and is linear when $z \ge N$. For example, if G is smooth, G_N can be obtained if the graph of G for $z \ge N$ is replaced by the tangent line through the point $(N, G(N)$. Since $G_N \le G$, the validity of (16) for G_N implies (16) with the function G in the integrand and $g(1)$ being replaced by$g_N(1)$. Passing to the limit as $N \to \infty$ in this inequality, one gets (16), since $g_N(1) \to g(1)$ as $N \to \infty$. Similar arguments allow us to assume that $G = 0$ in a neighborhood of the origin (The validity of (16) for $G_\varepsilon(z) = G(z-\varepsilon) \le G(z)$ implies (16)). Now consider $G^\varepsilon(z) = \max(G(z), y(\varepsilon,z))$ where $y(\varepsilon,z)) = z^{m+1}, z \le \varepsilon, y(\varepsilon,z) = (m+1)(z-\varepsilon) + \varepsilon^{m+1}, z > \varepsilon$, with m defined in condition (b). We will show later that the right-hand side of (16) is finite for $G = G^\varepsilon$. Thus if (16) is proved for $G = G^\varepsilon$, then passing to the limit as $\varepsilon \to 0$ one gets (16) for G. Hence we can assume that $G = az$ at infinity and $G = z^{m+1}$ in a neighborhood of the origin. Note that $a \ne 0$, since G is convex.

A special approximation of the function G by exponential polynomials will be used. Consider function $H(z) = \frac{G(z)}{z(1-e^{-z})^m}, z > 0$. It is continuous, nonnegative and has positive limits as $z \to 0$ and $z \to \infty$. Hence there is an exponential polynomial $P_\varepsilon(e^{-z})$ which approximates $H(z)$ from below, i.e.

$$|H(z) - p_\varepsilon(e^{-z})| < \varepsilon, 0 < p_\varepsilon(e^{-z}) \le 2p_\varepsilon(e^{-z}), \qquad z > 0.$$

In order to find p_ε, one can change the variable $t = e^{-z}$ and reduce the problem to the standard Weierstrass theorem on the interval $(0,1)$. If $P_\varepsilon(e^{-z}) = (1-e^{-z})^m P_\varepsilon(e^{-z})$ then

$$|z^{-1}G(z) - P_\varepsilon(e^{-z})| < \varepsilon, 0 < P_\varepsilon(e^{-z}) \le z^{-1}G(z), z > 0; \; P_\varepsilon(e^{-z}) < Cz^m, z \to 0. \qquad (33)$$

We will choose polynomial P in (20) and (23) to be equal to P_ε. The last two of relations (33) show that $P = P_\varepsilon$ satisfies all the properties used to obtain (32). Function F in (32) is defined by (20) with $P = P_\varepsilon$, and therefore $F(1) = F_\varepsilon(1)$ depends on ε. From the first relation of (33) it follows that $F_\varepsilon(1) \to g(1)$ as $\varepsilon \to 0$. Thus passing to the limit in (32) as $\varepsilon \to 0$ we complete the proof of inequality (16) for $N(V)$.

Step 7. Now we are going to show that inequality (16) for $N(V)$ implies the validity of this inequality for $N_0(V)$ under the assumption that integral (16) converges. We can assume that G is linear at infinity and $G(z) = z^{m+1}$ in a neighborhood of the origin (see step 6). Then $G(2tW(x)) \le CG(tW(x))$, and therefore the convergence of the integral (16) implies the convergence of the same integral with W replaced by$2W$.

Let n be the dimension of the null space of the operator H. We need to show that n is finite and $N(V)+n$ does not exceed the right-hand side of (16).
Consider the operator

$$H_\varepsilon = H + \varepsilon V(x) = H_0 + (1+\varepsilon)V(x), \varepsilon > 0.$$

The Dirichlet form of this operator

$$(H_\varepsilon\phi,\phi) = (H\phi,\phi) + \varepsilon\int_X V(x)|\phi(x)|^2\mu(dx)$$

is strictly negative on the space $T\backslash\{0\}$, where the $(N(V)+n)$-dimensional space T is spanned by the eigenfunctions of H with negative or zero eigenvalues. Indeed, both terms on the right in the formula above are non positive on T. If $\phi\in T$ does not belong to the null space N of H, then the first term is strictly negative. If $\phi\in N\backslash\{0\}$, then the second term is strictly negative since otherwise there exists $\phi=\phi_0\in N\setminus\{0\}$ such that $V\phi_0=0$. Then ϕ_0 belongs to the null space of the unperturbed operator H_0. This contradicts the assumption (b) on the decay (integrability) of the heat kernel $p_0(t,x,x)$ as $t\to\infty$ (since $p_0\ge|\phi_0(x)|^2$).
The negativity of the Dirichlet form on T\{0} implies that operator H has at least $N(V)+n$ strictly negative eigenvalues. Hence from inequality (16) for strictly negative eigenvalues of the operator H_ε it follows that

$$N(V)+n \le \frac{1}{g(1)}\int_0^\infty \frac{\pi(t)}{t}\int_X G(t(1+\varepsilon)W(x))\mu(dx)dt. \qquad (34)$$

One may assume that the double integral in (16) converges. It was shown above that this assumption leads to the convergence of the integral in (34) when $\varepsilon=1$. Then one can pass to the limit as $\varepsilon\to 0$ in (34) and get

$$N(V)+n \le \frac{1}{g(1)}\int_0^\infty \frac{\pi(t)}{t}\int_X G(tW(x))\mu(dx)dt.$$

Hence (16) is proved
Step 8. In order to prove (17), we note that

$$\begin{aligned}\sum_{i:E_i<0}|E_i|^\gamma &= \gamma\int_0^\infty E^{\gamma-1}N_E(V)dE\\ &\le \gamma\int_0^\infty E^{\gamma-1}N_0(-(W-E)_+)dE\\ &\le \frac{\gamma}{g(1)}\int_0^\infty E^{\gamma-1}\int_0^\infty\frac{\pi(t)}{t}\int_X G(t(W(x)-E)_+)\mu(dx)dtdE\\ &= \frac{\gamma}{g(1)}\int_0^\infty\frac{\pi(t)}{t}\int_X\int_0^W E^{\gamma-1}G\big(t(W(x)-E)\big)dE\mu(dx)dt\\ &= \frac{\gamma}{g(1)}\int_0^\infty\frac{\pi(t)}{t}\int_X\int_0^1 u^{\gamma-1}W^\gamma(x)G\big(tW(x)(1-u)\big)du\mu(dx)dt.\end{aligned}$$

One can replace $G(tW(x)(1-u))$ here by $G(tW(x))$, since G is monotonically increasing. This immediately implies (17).

Theorem (6.1.5) [202]: Let $H = H_0 + V(x)$, where H_0 is a self-adjoint, non-negative operator on $L^2(X, B, \mu)$, the potential $V = V(x)$ is real valued, and the assumptions (a)-(c) hold.
If

$$\pi(t) \le c/t^{\frac{\beta}{2}}, \quad t \to \infty; \quad \pi(t) \le ct^{\frac{\alpha}{2}}, \qquad t \to 0 \tag{35}$$

For some $\beta > 2$ and $\alpha \ge 0$, then

$$N_0(V) \le C(h)[\int_{X_h} W(x)^{\beta/2}\mu(dx) + \int_{X_h^+} bW(x)^{\max(\alpha,2/1)}\mu(dx)], \tag{36}$$

where $X_h^- = \{x: W(x) \le h^{-1}\}, X_h^+ = \{x: W(x) > h^{-1}\}, b = 1$ if $\alpha \neq 2, b = \ln(1 + W(x))$ if $\alpha = 2$, in some cases $(\alpha/2,1)$ can be replaced by $\alpha/2$, as will be discussed in Section 3.
Proof. We write (14) in the form $N_0(V) \le I_- + I_+$, where $I_\mp$ correspond to integration in (14) over $X_h^\mp$ respectively.
Let $x \in X_h^-$, i.e., $W < h^{-1}$. Then the interior integral in (14) does not exceed

$$C(h)\int_{\frac{\sigma}{W}}^{\infty} t^{-\beta/2}dt = C(h)W^{(\beta/2)-1}. \tag{37}$$

Thus I_- can be estimated by the first term in the right-hand side of (36). Similarly

$$I_+ \le C(h)\int_{X_h^+} W(\int_{\frac{\sigma}{W}}^{h} + \int_h^\infty)\pi(t)dt \le C(h)\int_{X_h^+} W(\int_{\frac{\sigma}{W}}^{h} t^{-\alpha/2}dt + \int_h^\infty t^{-\beta/2}dt)dx,$$

which does not exceed the second term in the right-hand side of (36).
Theorem (6.1.6) [202]: Let $H = H_0 + V(x)$, where H_0 is a self-adjoint, non-negative operator on $L^2(X, B, \mu)$, the potential $V = V(x)$ is real valued, and the assumptions (a)-(c) hold
If

$$\pi(t) \le ce^{-at^\gamma}, t \to \infty; \ \ \pi(t) \le c/t^{\frac{\alpha}{2}}, \ \ t \to 0 \tag{38}$$

for some $\gamma > 0$ and $\alpha \ge 0$, then for each $A > 0$,

$$N_0(V) \le C(h, A)[\int_{X_h^-} e^{-AW(x)^{-\gamma}}\mu(dx) + \int_{X_h^+} bW(x)^{\max(\alpha/2,1)}\mu(dx)], \tag{39}$$

where X_h^-, X_h^+, b are the same as in the theorem above,
Proof. The proof is the same as that of the theorem above. One only needs to replace (37) by the following estimate

$$\begin{aligned} c(h)\int_{\frac{\sigma}{W}}^{\infty} e^{-at^\gamma}dt \\ &= C(h)W^{-1}\int_\sigma^\infty e^{-\frac{a}{2}(\frac{\mathcal{T}}{W})^\gamma}d\mathcal{T} \\ &\le C(h)W^{-1}e^{e^{-\frac{a}{2}(\frac{\sigma}{W})^\gamma}}\int_\sigma^\infty e^{-\frac{a}{2}(\frac{\mathcal{T}}{W})^\gamma}d\mathcal{T} \\ &\le C(h)W^{-1}\int_0^\infty e^{-\frac{a}{2}(h\mathcal{T})^\gamma}d\mathcal{T}]e^{-\frac{a}{2}(\frac{\sigma}{W})^\gamma}, \end{aligned}$$

and note that σ can be chosen as large as we please.
1. Operators on lattices and groups. It is easy to see that Theorems 6.1.6 and 6.1.5 are not exact if $\alpha \le 2$. We are going to illustrate this fact now and provide a better result for the case $\alpha = 0$ which

occurs, for example, when operators on lattices and discrete groups are considered. An important example with $\alpha = 1$ will be discussed in next subsection (operators on quantum graphs).
Let $X = \{x\}$ be a countable set and H_0 be a difference operator on $L^2(X)$ which is defined by

$$(H_0\psi)(x) = \sum_{y\in X} a(x,y)\psi(y), \qquad (40)$$

where

$$a(x,x) > 0, \qquad a(x,y) = a(y,x) \le 0, \qquad \sum_{y\in X} a(x,y) = 0.$$

A typical example of H_0 is the negative difference Laplacian on the lattice $X = Z^d$, i.e.,

$$(H_0\psi)(x) = -\Delta\psi = \sum_{y\in Z^d:|y-x|=1} [\psi(x) - \psi(y)], \quad x \in Z^d, \quad (41)$$

We will assume that $0 < a(x,x) \le c_0 < \infty$. Then $\mathrm{Sp}H_0 \subset [0, 2c_0]$. The operator $-H_0$ defines the Markov chain $x(s)$ on X with continuous time $s \ge 0$ which spends exponential time with parameter $a(x,x)$ at each point $x \in X$ and then jumps to a point $y \in X$ with probability$r(x,y) = \frac{a(x,y)}{a(x,x)}, \sum_{y:y\neq x} r(x,y) = 1$. The transition matrix $p(t,x,y) = P_x(x_t = y)$ is the fundamental solution of the parabolic problem

$$\frac{\partial p}{\partial t} + H_0 p = 0, \quad p(0,x,y) = \delta_y(x).$$

Obviously, $p(t,x,x) \le \pi(t) \le 1$, and $\pi(t) \to 1$ uniformly in x as $t \to 0$. The asymptotic behavior of $\pi(t)$ as $t \to \infty$ depends on operator and can be more or less arbitrary.
Consider now the operator $H = H_0 - m\delta_y(x)$ with the potential supported on one point. The negative spectrum of H contains at most one eigenvalue (due to rank one perturbation arguments), and such an eigenvalue exists if $m \ge c_0$. The latter follows from the variational principle, since

$$< H_0\delta_y, \delta_y > -m < \delta_y, \delta_y > \quad \le c_0 - m < 0.$$

However, Theorems 6.1.5 and 6.1.6 estimate the number of negative eigenvalues $N(V)$ of the operator H byCm. Similarly, if

$$V = -\sum_{1\le i\le n} m_i\delta(x - x_i)$$

and $m_i \ge c_0$, then $N(V) = n$, but Theorems 6.1.5 and 6.1.6 give only that $N(V) \le C\sum m_i$. The following statement provides a better result for the case under consideration than the theorems above. The meaning of the statement below is that we replace $\max(\alpha/2, 1) = 1$ in (36), (39) by$\alpha/2 = 0$. Let us also mention that these theorems can not be strengthened in a similar way if $0 < \alpha \le 2$ (see Example 3).
Theorem (6.1.7) [202]: Let $H = H_0 + V(x)$, where H_0 is defined in (40), and let assumptions of Theorem 6.1.4 hold. Then for each $h > 0$,

$$N_0(V) \le C(h)[n(h) + \int_0^\infty \frac{\pi(t)}{t} \sum_{x\in X_h^-} G(tW(x))dt], \quad n(h) = \#\{x \in X_h^+\}.$$

If, additionally, either (35) or (38) is valid for $\pi(t)$ as $t \to \infty$, then for each $A > 0$,

$$N_0(V) \le C(h)[\sum_{x\in X_h^-} W(x)^{\frac{\beta}{2}} + n(h)], \qquad n(h) = \#\{x \in X_h^+\}, \quad (42)$$

$$N_0(V) \le C(h,A)[\sum_{x\in X_h^-} e^{-AW(x)^{-\gamma}} + n(h)], \quad n(h) = \#\{x \in X_h^+\},$$

respectively,

Remark (6.1.8) [202]: Estimate (42) for $N(V)$ in the case $X = Z^d$ can be found in [200].

Proof. In order to prove the first inequality, we split the potential $V(x) = V_1(x) + V_2(x)$, where $V_2(x) = V(x)$ for $x \in X_h^+, V_2(x) = 0$ for $x \in X_h^-$. Now for each $\varepsilon \in (0,1)$,

$$N_0(V) \le N_0(\varepsilon^{-1}V_1) + N_0((1-\varepsilon)^{-1}V_2) = N_0(\varepsilon^{-1}V_1) + n(h). \qquad (43)$$

It remains to apply Theorem 6.1.4 to the operator $-\Delta + \varepsilon^{-1}V_1$ and pass to the limit as $\varepsilon \to 1$. The next two inequalities follow from Theorems 6.1.5 and 6.1.6.

2. Operators on quantum graphs. We will consider a specific quantum graph Γ^d, the so called Avron-Exner-Last graph. Its vertices are the points of the lattice Z^d, and the edges are all segments of length one connecting neighboring vertices. Let $s \in [0,1]$ be the natural parameter on the edges (distance from one of the end points of the edge). Consider the space D of smooth functions φ on edges of Γ^d with the following (Kirchoff's) boundary conditions at vertices: at each vertex φ is continuous and

$$\sum_{i=1}^{d} \varphi_i' = 0, \qquad (44)$$

where φ_i' are the derivatives along the adjoint edges in the direction out of the vertex. The operator H_0 acts on functions $\varphi \in D$ as $-\frac{d^2}{ds^2}$. The closure of this operator in $L^2(\Gamma^d)$ is a self-adjoint operator with the spectrum $[0,\infty)$ (see [179])

Theorem 6.1.9 Let $d \ge 3$ and V be constant on each edge e_i of the graph: $V(x) = -v_i < 0, x \in e_i$. Then

$$N_0(V) \le c(h)(\sum_{i:v_i\le h^{-1}} v_i^{d/2} + \sum_{i:v_i>h^{-1}} \sqrt{v_i}).$$

Proof. Put $V(x) = V_1(x) + V_2(x)$, where $V_1(x) = V(x)$ if $|V(x)| > h^{-1}, V_1(x) = 0$ if $|V(x)| \le h^{-1}$. Then (see 43))

$$N_0(V) \le N_0(2V_1) + N_0(2V_2).$$

One can estimate $N(V_1)$ from above (below) by imposing the Neumann (Dirichlet) boundary conditions at all vertices of Γ. This leads to the estimates

$$\sum_{i:v_i>h^{-1}} \frac{\sqrt{2v_i}}{\pi} \le N_0(V) \le \sum_{i:v_i>h^{-1}} (\frac{\sqrt{2v_i}}{\pi} + 1) \le c(h) \sum_{i:v_i>h^{-1}} \sqrt{v_i},$$

which, together with Theorem 2.5 applied to $N_0(2V_2)$, justifies the statement of the theorem

The same arguments allow one to get a more general result.

Theorem (6.1.10) [202]: Let $d \ge 3$. Let Γ_-^d be the set of edges, e_i of the graph Γ^d where $W \le h^{-1}, \Gamma_+^d$ be the complementary set of edges, and

$$\frac{\sup_{x\in e_i} W(x)}{\min_{x\in e_i} W(x)} \le k_0 = k_0(h), x \in \Gamma_+^d,$$

where $W = V_-$. Then

$$N_0(V) \le c(h, k_0)(\int_{\Gamma^d_-} W(x)^{d/2}dx + \int_{\Gamma^d_+} \sqrt{W(x)}\,dx).$$

Example. The next example shows that there are singular potentials on Γ^d for which $\max(\alpha/2, 1)$ in (36) can not be replaced by any value less than one. Consider the potential $V(x) = -A\sum_{i=1}^{m} \delta(x - x_i)$, where x_i are middle points of some edges, and $A > 4$. One can easily modify the example by considering δ-sequences instead of δ-functions (in order to get a smooth potential.) Then

$$\int_{\Gamma^d} W^{\sigma}(x)dx = 0$$

for any $\sigma < 1$, while $N(V) \ge m$. In fact, consider the Sturm-Liouville problem on the interval $[1 - 2/2, 1/2]$:

$$-y'' - A\delta(x)y = \lambda y, y(-1/2) = y(1/2) = 0, \quad A > 4.$$

It has (a unique) negative eigenvalue which is the root of the equation $\tanh(\sqrt{-\lambda}/2) = 2\sqrt{-\lambda}/A$. The corresponding eigenfunction is $y = \sinh[\sqrt{-\lambda}\,(|x| + 1/2)]$. The estimate $N(V) \ge m$ follows by imposing the Dirichlet boundary conditions on the vertices of Γ^d.

I. Discrete case. Consider the classical Anderson Hamiltonian $H_0 = -\Delta + V(x, \omega)$ on $L^2(Z^d)$ with random potential $V(x, \omega)$. Here

$$\Delta\psi(x) = \sum_{x':|x'-x|=1} \psi(x') - 2d\psi(x).$$

We assume that random variables $V(x, \omega)$. on the probability space (Ω, F, P) have the Bernoulli structure, i.e., they are i.i.d. and $P\{V(\cdot) = 0\} = p > 0, P\{V(\cdot) = 1\} = q = 1 - p > 0$. The spectrum of H_0 is equal to (see [178])

$$Sp(H_0) = Sp(-\Delta)\oplus 1 = [0, 4d + 1].$$

Let us stress that $0 \in Sp(H_0)$ due to the existence P-a.s. of arbitrarily large clearings in realizations of V, i.e., there are balls $B_n = \{x : |x - x_n| < r_n\}$ such that $V(x) = 0, x \in B_n$, and $r_n \to \infty$ as $n \to \infty$ (see the proof of the theorem below for details).

Let

$$H = H_0 - W(x), W(x) \ge 0.$$

The operator H has discrete random spectrum on $(-\infty, 0]$ with possible accumulation point at $\lambda = 0$. Put $N_0(-W) = \#\{\lambda_i \le 0\}$. Obviously, $N_0(-W)$ is random. Denote by E the expectation of a r.v., i.e.

$$EN_0 = \int_{\Omega} N_0 P(d\omega).$$

Theorem (6.1.11) [202]: (a) For each $h > 0$ and $\gamma < \frac{d}{d+2}$,

$$EN_0(-W) \le c_1(h)\left[\#\{x \in Z^d: W(x) \ge h^{-1}\}\right] + c_2(h, \gamma) \sum_{x:W(x)<h^{-1}} e^{-\frac{1}{W^{\gamma}(x)}}$$

In particular, if $W(x) < \frac{C}{\log^{\sigma}|x|}, |x| \to \infty$, with some $\sigma > \frac{d+2}{d}$, then $EN_0(-W) < \infty, i.e, N_0(-W) < \infty$ almost surely.

(b) If

$$W(x) > \frac{C}{\log^{\sigma}|x|}, |x| \to \infty, \qquad \text{and } \sigma < \frac{2}{d}, \qquad (45)$$

then $N_0(-W) = \infty$ a.s. (in particular, $EN_0(-W) = \infty$).

Proof. Since $V \geq 0$, the kernel $p_0(t,x,y)$ of the semigroup $\exp(-tH_0) = \exp(t(\Delta - V))$ can be estimated by the kernel of $\exp(t\Delta)$, i. e., by the transition probability of the random walk with continuous time on Z^d. The diagonal part of this kernel $p_0(t,x,x,\omega)$ is a stationary field on Z^d. Due to the Donsker-Varadhan estimate (see [182],[183]),

$$Ep_0(t,x,x,\omega) = Ep_0(t,x,x,\omega) \overset{\log}{\sim} \exp(-c_d t^{\frac{d}{d+2}}), \qquad t \to \infty,$$

i.e.,

$$\log Ep_0 \sim -c_d t^{\frac{d}{d+2}}, \qquad t \to \infty.$$

On the rigorous level, the relations above must be understood as estimates from above and below, and the upper estimate has the following form: for each $\delta > 0$,

$$Ep_0 \leq C(\delta)\exp(-c_d t^{\frac{d}{d+2}-\delta}), \quad t \to \infty. \qquad (46)$$

Now the first part of the theorem is a consequence of Theorems 6.1.4 and 6.1.6 In fact, from Remarks 2.3 and 2.4 and (46) it follows that

$$EN_0(V) \leq \frac{1}{c(\sigma)} \int_X W(x) \int_{\frac{\sigma}{W(x)}}^{\infty} Ep_0(t,x,x,\omega)dt\mu(dx) \leq \frac{C(\delta)}{c(\sigma)} \int_X W(x) \int_{\frac{\sigma}{W(x)}}^{\infty} e^{-c_d t^{\frac{d}{d+2}-\delta}} dt\mu(dx).$$

Then it only remains to repeat the arguments used to prove Theorem 6.1.6.

The proof of the second part is based on the following lemma which indicates the existence of large clearings at the distances which are not too large. We denote by$C(r)$ the cube in the lattice,

$$C(r) = \{x \in Z^d : |x_i| < r, \leq i \leq d\}.$$

Let's divide Z^d into cubic layers $L_n = C(a^{n+1}) \backslash C(a^n)$ with some constant $a \geq 1$ which will be selected later. One can choose a set $\Gamma^{(n)} = \{z_i^{(n)} \in L_n\}$ in each layer L_n such that

$$\left|z_i^{(n)} - z_j^{(n)}\right| \geq 2n^{\frac{1}{d}} + 1, \qquad d\left(z_i^{(n)}, \partial L_n\right) > n^{\frac{1}{d}},$$

and

$$\left|\Gamma^{(n)}\right| \geq c\frac{(2a)^{n(d-1)}a^{n+1}}{(2n^{\frac{1}{d}})^d} \geq ca^{nd}, \qquad n \to \infty.$$

Let $C(n^{1/d}, i)$ be the cube $C(n^{1/d})$ with the center shifted to the point $z_i^{(n)}$. Obviously, cubes $C_n 1/d_{,i}$ do not intersect each other, $C(n^{1/d}, i) \subset L_n$ and $\left|C(n^{1/d}, i)\right| \leq c'n$.

Consider the following event $A_n = \{\text{each cube } C(n^{1/d}, i) \subset L_n$ contains at least one point where $V(x) = 1\}$. Obviously,

$$P(A_n) = (1 - p^{|c(n^{1/d},i|})^{\Gamma(n)}) \leq e^{-|\Gamma^{(n)}|p^{|c(n^{1/d},i)|}} \leq e^{-ca^{nd}c'p^n = e^{-c(a^d p^{c'})^n}}.$$

We will choose a big enough, so that $a^d p^{c'} > 1$. Then $\sum P(A_n) < \infty$, and the Borel-Cantelli lemma implies that P-a.s. there exists $n_0(\omega)$ such that each layer $L_n, n \geq n_0(\omega)$, contains at least one empty cube $C(n^{1/d}, i), i = i(n)$. Then from (45) it follows that

$$W(x) \geq \frac{C}{n^{\frac{2}{d}-\delta}} = \varepsilon_n, \quad x \in C\left(n^{1/d}, i\right), \ i = i(n).$$

One can easily show that the operator $H = -\Delta - \varepsilon$ in a cube $C \subset Z^d$ with the Dirichlet boundary condition at∂C has at least one negative eigenvalue if $|C|\varepsilon^{d/2}$ is big enough. Thus the operator H in $C(n^{1/d}, i(n))$ with the Dirichlet boundary condition has at least one eigenvalue if n is big enough, and therefore $N(-W) = \infty$.

II. Continuous case. Theorem 6.1.11 is also valid for Anderson operators in R^d. Let $H_0 = -\Delta + V(x, \omega)$ on $L^2(R^d)$ with the random potential

$$V(x, \omega) = \sum_{n \in Z^d} \varepsilon_n I_{Q_n}(x), x \in R^d, n = (n_1, \dots, n_d),$$

where $Q_n = \{x \in R^d : n_i \le x_i \le x_i < n_i + 1, i = 1,2, \dots d\}$ and ε_n are independent Bernoulli r.v. with $P\{\varepsilon_n = 0\} = p, P\{\varepsilon_n = 1\} = q = 1 - p$. Put $H = H_0 - W(x) = -\Delta + V(x, \omega) - W(x)$.

Theorem (6.1.12) [202]: (a) If $d \ge 3$, then for each $h > 0$ and $\gamma < \frac{d}{d+2}$,

$$EN_0(-W) \le c_1(h) \int_{W(x) \ge h^{-1}} W(x)^{d/2} dx + c_2(h, \gamma) \int_{W(x) < h^{-1}} e^{-\frac{1}{W^{\gamma}(x)}} dx.$$

In particular, if $W(x) < \frac{C}{\log^{\sigma}|x|}, |x| \to \infty$, with some $\sigma < \frac{d}{d+2}$ then $EN_0(-W) < \infty$, i.e., $N_0(-W) < \infty$ almost surely.

(b) if $W(x) > \frac{C}{\log^{\sigma}|x|}, |x| \to \infty$, and $\sigma < \frac{2}{d}$, then $N_0(-W) = \infty$ a.s. (in particular, $EN_0(-W) = \infty$).

The proof of this theorem is identical to the proof of Theorem 6.1.11 with the only difference that now $p_0(t, 0, 0)$ is not bounded as $t \to 0$, but $p_0(t, 0, 0) \le c/t^{d/2}, t \to 0$.

1. Lobachevsky plane (see [184], [196]). We will use the Poincare upper half plane model, where $X = \{z = x + iy : y > 0\}$ and the (Riemannian) metric on X has the form

$$ds^2 = y^{-2}(dx^2 + dy^2). \qquad (47)$$

The geodesic lines of this metric are circular arcs perpendicular to the real axis (halfcircles whose origin is on the real axis) and straight vertical lines ending on the real axis. The group of transformations preserving ds^2 is $SL(2, R)$, i.e. the group of real valued 2×2 matrices with the determinant equal to one. For each $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, R)$, the action $A(z)$ is defined by

$$A(z) = \frac{az + b}{cz + d}.$$

For each $z_0 \in X$, there is a one-parameter stationary subgroup which consists of A such that $Az_0 = z_0$. The Laplace-Beltrami operator Δ' (invariant with respect to $SL(2, R)$) is defined uniquely up to a constant factor, and is equal to

$$\Delta' = y^2 \Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad (48)$$

The operator $-\Delta'$ is self-adjoint with respect to the Riemannian measure

$$\mu(dz) = y^{-2} dx dy, \qquad (49)$$

and has absolutely continuous spectrum on $[1/4, \infty)$. In order to find the number $N'(V)$ of eigenvalues of the operator $-\Delta' + V(x)$ below $1/4$, one can apply Theorem 6.1.4 to the operator $H_0 = -\Delta' - \frac{1}{4} I$.

One needs to know constants α, β in order to apply Theorem 6.1.5. It is shown in [188] that the fundamental solution for the parabolic equation $u_t = -\Delta' u$ has the following asymptotic behavior

$$p(t, 0,0) \sim c_1/t, \quad t \to 0; \;\; p(t, 0,0) \sim c_2 e^{-t/4}/t^{3/2}, \;\; t \to \infty.$$

Thus $\alpha = 2, \beta = 3$ for the operator $H_0 = -\Delta' - \frac{1}{4} I$. A similar result for the Laplacian in the Hyperbolic space of the dimension $d \ge 3$ can be found in [200].

2. Markov processes with independent increments (homogeneous pseudo differential operators). We will estimate $N_0(V)$ for shift invariant pseudo differential operators H_0 associated with Markov processes with independent increments. Similar estimates were obtained in [181] for pseudo differential operators under assumptions that the symbol $f(p)$ of the operator is monotone and non-negative, and the parabolic semigroup e^{-tH_0} is positivity preserving. This class includes important cases of $f(p) = |p|^\alpha, \alpha < 2$ and $f(p) = \sqrt{p^2 + m^2} - m$. Note that necessary and sufficient conditions of the positivity of $p_0(t,x,x)$ are given by Levy-Khinchin formula. We will omit monotonicity condition. What is more important, the results will be expressed in terms of the Levy measure responsible for the positivity of $p_0(t,x,x)$. This will allow us to consider variety estimates with power and logarithmical decaying potentials.

Let H_0 be a pseudo-differential operator in $X = R^d$ of the form

$$H_0u = F^{-1}\Phi(\kappa)Fu, \quad (F\mu)(k) = \int_{R^d} u(x)e^{-i(x,k)}\,dx, \quad u \in S(R^d),$$

where the symbol $\Phi(k)$ of the operator H_0 has the following form

$$\Phi(k) = \int_{R^d} (1 - \cos(x,k))v(x)d. \qquad (50)$$

Here $\mu(dx) = v(x)dx$ is an arbitrary measure (for simplicity we assumed that it has a density) such that

$$\int_{|x|>1} v(x)dx + \int_{|x|<1} |x|^2 v(x)dx < \infty. \qquad (51)$$

Assumption (50) is needed (and is sufficient) to construct a Markov process with the generator $L = -H_0$ (see below). However, we will impose an additional restriction on the measure $\mu(dx)$ assuming that the density$v(x)$ has the following power asymptotics at zero and at infinity

$$v(x) \sim |x|^{-d-2+\rho}, \qquad x \to 0, v(x) \sim |x|^{-d-\delta}, x \to \infty,$$

with some $\rho, \delta \in (0,2)$. Note that assumption (51) holds in this case. To be more rigorous, we assume that

$$v(x) = a\left(\frac{x}{|x|}\right)|x|^{-d-\rho}(1 + O|x|^\varepsilon)), \qquad x \to 0, \qquad (52)$$

$$v(x) = b\left(\frac{x}{|x|}\right)|x|^{-d-\delta}(1 + O|x|^{-\varepsilon})), \qquad x \to \infty, \qquad (53)$$

where $a, b, \varepsilon > 0$. we also will consider another special case when the asymptotic behavior of $V(x)$ at infinity is at logarithmical borderline for the convergence of the integral (51).

Namely, we will assume that (52) holds and

$$V(x) > C|x|^{-d}\log^{-\sigma}|x|, x \to \infty, \sigma > 1.$$

The solution of problem (10) is given by

$$p_0(t, x - y) = \frac{1}{2\pi}\int_{R^d} e^{-t\Phi(k) + i(x-y,K)}dK.$$

A special form of the pseudo differential operator H_0 is chosen in order to guarantee that $p_0 \geq 0$. In fact, let $x_s, s > 0$, be a Markov process in R^d with symmetric independent increments. It means that for arbitrary$0 < s_1 <_2 < \cdots$, the random variables $x_{s_1} - x_0, x_{s_2} - x_{s_1}, \ldots$ are independent and the distribution of $x_{t+s} - x_s$ is independent of s. The symmetry condition means that $\mathrm{Law}(x_s - x_0) =$

$\mathrm{Law}(x_0 - x_s)$, or $p(s, x, y) = p(s, y, x)$, where p is the transition density of the process. According to the Levy-Khinchin theorem (see [186]), the Fourier transform (characteristic function) of this distribution has the form

$$Ee^{i(kx_{t+s} - x_s)} = e^{-t\Phi(k)},$$

with $\Phi(k)$ given by (50). Moreover, each measure (51) corresponds to some process. One can consider the family of processes $x_s^{(x0)} = x_0 + x_s, s > 0$, with an arbitrary initial point x_0. The generator L of this family can be evaluated in the Fourier space. If $\varphi(x) \in S(R^d)$ and $\hat{\varphi}(k) = F\varphi$, then

$$L\varphi(x) = \lim_{t\to 0} \frac{E\varphi(x + x_t^{(0)}) - \varphi(x)}{t} = \lim_{t\to 0} \frac{1}{(2\pi)^d} \int_{R^d} \frac{Ee^{i(x + x_t^{(0)}, k)} - e^{i(x,k)}}{t} \hat{\varphi}(k) dk$$

$$= \frac{-1}{(2\pi)^d} \int_{R^d} e^{i(x,k)}\, \Phi(k)\hat{\varphi}(k)\, dk = -H_0\varphi.$$

Thus, function (55) is the transition density of some process, and therefore $p_0(t, x) \geq 0$, i.e., assumption (a) of Theorem 6.1.4 holds. Since operator H_0 is translation invariant, assumption (b) also holds with $\pi(t) = p_0(t, 0)$. Hence, Theorem 6.1.4 can be applied to study negative eigenvalues of the operator $H_0 + V(x)$ when (Levy) measure vdx satisfies (51). If (52), (53) or (52), (54) hold, then Theorems 6.1.5, 6.1.6 can be used. Namely, the following statement is valid.

Theorem (6.1.13) [202]: If measure vdx satisfies (52) and (53), then (35) is valid with $\beta = 2d/\delta, \alpha 2d/\rho$.

If measure vdx satisfies (52) and (54), then (38) is valid with $\gamma = 1/\sigma, \alpha = 2d/\rho$.

Proof. Consider first the case when (52) and (53) hold. Let us prove that these relations imply the Following behavior of $\Phi(k)$ at zero and at infinity

$$\Phi(k) = f(\frac{k}{|k|})|k|^\delta(1 + O(|k|^{\varepsilon_1})), k \to 0;$$

$$\Phi(k) = g(\frac{k}{|k|})|k|^\rho(1 + O(|k|^{-\varepsilon_1})), k \to \infty, \tag{56}$$

with some $f, g, \varepsilon_1 > 0$. We write (50) in the form

$$\Phi(k) = \int_{|x|<1} 2\sin^2(x, k))v(x)dx + \int_{|x|>1} 2\sin(x, k))v(x)dx = \Phi_1(k) + \Phi_2(k). \tag{57}$$

The term $\Phi_1(k)$ is analytic in k and is of order $O(|k|^2)$ as $k \to 0$. We represent the second term as

$$\int_{R^d} 2\sin^2(x, k)b(x)|x|^{-d-\delta}dx - \int_{|x|<1} 2\sin^2(x, k))b\,(x)|x|^{-d-\delta}dx + \int_{|x|>1} 2\sin^2(x, k))h(x)dx,$$

where $x = x/|x|$ and

$$h(x) = v(x) - b(x)|x|^{-d-\delta}, |h| \leq C|x|^{-d-\delta-\varepsilon}.$$

The middle term above is of order $O(|x|^2)$ as $k \to 0$. The first term above can be evaluated by substitution $x \to x/|k|$. It coincides with $f(\frac{k}{|k|})|k|^\delta$. One can reduce ε to guarantee that $\delta + \varepsilon < 2$. Then the last term can be estimated using the same substitution. This leads to the asymptiotics (56) as $k \to 0$.

Now let $|k| \to \infty$. Since $\Phi_2(k)$ is bounded uniformly in k, it remains to show that $\Phi_1(k)$ has the appropriate asymptotics as $|k| \to \infty$. We write $v(x)$ in the integrand of $\Phi_1(k)$ as follows

$$v(x) = a(x)|x|^{-d-\rho} + g(x), \quad |g(x)| \le C|x|^{-d-\rho+\varepsilon}.$$

Then

$$\Phi_1(k) = \int_{R^d} 2\sin^2(x,k)a(x)|x|^{-d-\rho}dx$$
$$- \int_{|x|>1} 2\sin^2(x,k)a(x)|x|^{-d-\rho}dx + \int_{|x|<1} 2\sin(x,k))g(x)dx.$$

The middle term in the right hand side above is bounded uniformly in k. The substitution $x \to x/|k|$ justifies that the first term coincides with $g(\frac{k}{|k|})|k|^{\rho}$. The same substitution shows that the order of the last term is smaller if $\varepsilon < \rho$. This gives the second relation of (56), and therefore, (56) is proved. Let us estimate $\pi(t)$ when (56) holds. From (55) it follows that

$$\pi(t) = \frac{1}{(2\pi)^d}\int_{|k|<1} e^{-t\Phi(k)}dk + O(e^{-\eta t}) \text{as } t \to \infty, \qquad \eta > 0. \quad (58)$$

Now the substitution $k \to t^{-1/\delta}k$ leads to

$$\pi(t) \sim ct^{d/\delta}, \quad t \to \infty, \quad c = \frac{1}{(2\pi)^d}\int_{R^d} e^{-g(\frac{k}{|k|})|k|^{\delta}}dk.$$

Hence, the first of relations (35) holds with $\beta = 2d/\delta$. In order to estimate $\pi(t)$ as $t \to 0$, we put

$$\pi(t) = \frac{1}{(2\pi)^d}\int_{|k|<1} e^{-t\Phi(k)}dk + O(1) \quad \text{as } t \to 0,$$

and make the substitution $k \to t^{-1/p}k$. This leads to

$$\pi(t) \sim ct^{-d/\rho}, \qquad t \to 0, \qquad c = \frac{1}{(2\pi)^d}\int_{R^d} e^{-f(\frac{k}{|k|})|k|^{\rho}}dk.$$

Hence the second of relations (35) holds with $\alpha = 2d/\rho$. The first statement of the theorem is proved.

Let us prove the second statement. If (52) and (54) hold, then

$$\Phi(k) \ge c(\log\frac{1}{|k|})^{1-\sigma}, k \to 0; \quad \Phi(k) = g\left(\frac{k}{|k|}\right)|k|^{\rho}\left(1 + O(|k|^{-\varepsilon_1})\right), k \to \infty. \qquad (59)$$

In fact, only integrability of $v(x)$ at infinity, but not (53), was used in the proof of the second relation of (56). Thus the second relation of (59) is valid. Let us prove the first estimate. Let $\Omega_k = \{x: |k|^{-2} > |x| > |k|^{-1}\}, |k| < 1$. We have

$$\Phi(k) \ge \int_{\Omega_k} 2\sin^2(x,k))v(x)dx \ge C\int_{\Omega_k} \sin^2(x,k))|x|^{-d}\log^{-\sigma}|x|dx$$
$$\ge C(2\log\frac{1}{|k|})^{-\sigma}\int_{\Omega_k} \sin^2(x,k))|x|^{-d}dx, \qquad |k| \to 0.$$

It remains to show that

$$\int_{\Omega_k} \sin^2(x,k))|x|^{-d}dx \sim \log\frac{1}{|k|}, \quad |k| \to 0. \qquad (60)$$

After the substitution $x = y/|k|$, the last integral can be written in the form

$$\frac{1}{2}\int_{|k|^{-1}>|Y|>1}|y|^{-d}dy-\frac{1}{2}\int_{|k|^{-1}>|y|>1}\cos(y,k))|y|^{-d}dy.$$

This justifies (60), since the second term above converges as $|k|\to 0$. Hence (59) is proved.

Finally, we need to obtain (38). The estimation of $\pi(t)$ as $t\to 0$ remains the same as in the proof of the first statement of the theorem. To get the estimate as $t\to\infty$, we use(58) (with a smaller domain of integration) and (59). Then we obtain

$$\pi(t)\le\frac{1}{(2\pi)^d}\int_{|k|<1/2}e^{-ct\left(\log\frac{1}{|k|}\right)^{1-\sigma}}dk+O(e^{-\eta t})\text{as } t\to\infty,\qquad \eta>0.$$

After integrating with respect to angle variables substitution $\log\frac{1}{|k|}=z$, we get

$$\pi(t)\le\frac{1}{(2\pi)^d}\int_{\log 2}^{\infty}z^{d-1}e^{-z-ctz^{1-\sigma}}dz+O(e^{-\eta t})\ \text{ as }\ t\to\infty,\eta>0.$$

The asymptotic behavior of the last integral can be easily found using standard Laplace method, and the integral behaves as $C_1t^{\frac{2d-1}{2d\sigma}}e^{-c_1t^{\frac{1}{\sigma}}}$ when $t\to\infty$. This completes the proof of (38).

1. Free groups. Let X be a group Γ with generators $a_1,a_2,\ldots a_d$, inverse elements $a_{-1},a_{-2},\ldots a_{-d}$, the unit element e, and with no relations between generators except $a_ia_{-i}=a_{-i}a_i=e$. The elements $g\in\Gamma$ are the shortest versions of the words $g=a_{i_1}\cdot\ldots\cdot a_{i_n}$ (with all factors e and a_ja_{-j} being omitted). The metric on Γ is given by

$$d(g_1,g_2)=d(e,g_1^{-1}g_2)=m(g_1^{-1}g_2),$$

where $m(g)$ is the number of letters $a_{\pm i}$ in g. The measure μ on Γ is defined by$\mu(\{g\})=1$ for each $g\in\Gamma$. It is easy to see that$|\{g:d(e,g)=R\}|=2d(2d-1)^{R-1}$, i.e., the group Γ has an exponential growth rate.

Define the operator Δ_Γ on $X=\Gamma$ by the formula

$$\Delta_\Gamma\psi(g)=\sum_{-d\le i\le d,i\ne 0}[\psi(ga_i)-\psi(g)].\tag{61}$$

Obviously, the operator $-\Delta_\Gamma$ is bounded and non-negative in $L^2(\Gamma, \mu)$. In fact, $\|\Delta_\Gamma\|4d$. As it is easy to see, the operator Δ_Γ is left-invariant:

$$(\Delta_\Gamma\psi)(gx)=\Delta_\Gamma(\psi(gx)),\ \ x\in\Gamma,$$

for each fixed $g\in\Gamma$. Thus, conditions (a), (b) hold for operator $-\Delta_\Gamma$. In order to apply Theorem 2.5, one also needs to find the parameters α and β.

Remark 6.1.14 Since the absolutely continuous spectrum of the operator Δ_Γ is shifted (it starts from γ, not from zero), the natural question about the eigenvalues of the operator $-\Delta_\Gamma+V(g)$ is to estimate the number $N_\Gamma(V)$ of eigenvalues below the threshold γ. Obviously, $N_\Gamma(V)$ coincides with the number $N(V)$ of the negative eigenvalues of the operator $H_0+V(g)$, where $H_0=-\Delta_\Gamma-\gamma I$. Hence one can apply Theorems 2.1, 3.1 to this operator. From (62) it follows that constants α,β for the operator $H_0=-\Delta_\Gamma-\gamma I$ are equal to 0 and 3, respectively, and

$$N_\Gamma(V)\le c(h)[n(h)+\sum_{g\in\Gamma:W(g)\le h^{-1}}W(x)^{3/2}],\ \ n(h)=\#\{g\in\Gamma:W(g)>h^{-1}\}.$$

Theorem (6.1.15) [202]: a) The spectrum of the operator $-\Delta_\Gamma$ is absolutely continuous and coincides with the interval $I_d=[\gamma,\gamma+4\sqrt{2d-1}],\gamma=2d-2\sqrt{2d-1}\ge 0$.

b) The kernel of the parabolic semigroup $\pi_\Gamma(t)=(e^{t\Delta_\Gamma})(t,e,e)$ on the diagonal has the following asymptotic behavior at zero and infinity

$$\pi_\Gamma(t) \to c_1 \text{ as } t \to 0, \quad \pi_\Gamma(t) \sim c_2 \frac{e^{-\gamma t}}{t^{3/2}} \text{ as } t \to \infty. \qquad (62)$$

Let us find the kernel $R_\lambda(g_1, g_2)$ of the resolvent $(\Delta_\Gamma - \lambda)^{-1}$. From the Γ-invariance it follows that $R_\lambda(g_1, g_2) = R_\lambda(e, g_1^{-1} g_2)$. Hence it is enough to determine $u_\lambda = R_\lambda(e, g)$. This function satisfies the equation

$$\sum_{i \neq 0} u_\lambda(g a_i) - (2d + \lambda) u_\lambda(g) = -\delta_e(g), \qquad (63)$$

where $\delta_e(g) = 1$ if $g = e, \delta_e(g) = 0$ if $g \neq e$. Since the equation above is preserved under permutations of the generators, the solution $u_\lambda(g)$ depends only on $m(g)$. Let $\psi_\lambda(m) = u_\lambda(g), m = m(g)$. Obviously, if $g \neq e$, then $m(g a_i) = m(g) - 1$ for one of the elements $a_i, i \neq 0$, and $m(g a_i) = m(g) + 1$ for all other elements $a_i, i \neq 0$. Hence (63) implies

$$2d\psi_\lambda(1) - (2d + \lambda)\psi_\lambda(0) = -1, \qquad (64)$$
$$\psi_\lambda(m-1) + (2d-1)\psi_\lambda(m+1) - (2d+\lambda)\psi_\lambda(m) = 0, \quad m > 0.$$

Two linearly independent solutions of these equations have the form $\psi_\lambda(m) = v_\pm^m$, where $v_\pm$ are the roots of the equation

$$v^{-1} + (2d-1)v - (2d + \lambda) = 0$$

Thus,

$$v_\pm = \frac{2d + \lambda \pm \sqrt{(2d+\lambda)^2 - 4(2d-1)}}{2(2d-1)}.$$

The interval I_d was singled out as the set of real λ such that the discriminant above is not positive. Since $v_+ v_- = 1/(2d-1)$, we have

$$|v_\pm| = \frac{1}{\sqrt{2d-1}} \text{ for } \lambda \in I_d; \ |v_+| > \frac{1}{\sqrt{2d-1}}, \qquad |v_-| < \frac{1}{\sqrt{2d-1}} \text{ for real } \lambda \notin I_d.$$

Now, if we take into account the set $A_{m_0} = \{g \in \Gamma, m(g) = m_0\}$ has exactly$2d(2d-1)^{m_0 - 1}$ points, i.e., $\mu(A_{m_0}) = 2d(2d-1)^{m_0-1}$, we get that

$$v_-^{m(g)} \in L^2(\Gamma, \mu), v_+^{m(g)} \notin L^2(\Gamma, \mu) \text{ for real } \lambda \notin I_d, \qquad (65)$$

and

$$\int_{\Gamma \cap \{g: m(g) \leq m_0\}} |v_\pm|^{2m(g)} \mu(dg) \sim m_0 \text{ as } m_0 \to \infty \text{ for } \lambda \notin I_d (66)$$

Relations (65) imply that $R \backslash I_d$ belongs to the resolvent set of the operator Δ_Γ and that $R_\lambda(e, g) = c v_-^{m(g)}$. Relation (66) implies that I_d belongs to the absolutely continuous spectrum of the operator Δ_Γ with functions $(v_+^{m(g)} - v_-^{m(g)})$ being the eigenfunctions of the continuous spectrum. Hence statement a) is justified.

Note that the constant c in the formula for $R_\lambda(e, g)$ can be found from (64). This gives

$$R_\lambda(e, g) = \frac{1}{(2d + \lambda) - 2d v_-} v_-^{m(g)}.$$

Thus

$$R_\lambda(e, e) = \frac{1}{(2d + \lambda) - 2d v_-}.$$

Hence, for each $a > 0$,

$$\pi_\Gamma(t) = \frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty} e^{\lambda t} R_\lambda(e,e) d\lambda = \frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}\frac{d\lambda}{(2d+\lambda) - 2dv_-}.$$

The integrand here is analytic with branching points at the ends of the segment I_d, and the contour of integration can be bent into the left half plane $\mathrm{Re}\,\lambda < 0$ and replaced by an arbitrary closed contour around I_d. This immediately implies the first relation of (62). The asymptotic behavior of the integral as $t \to \infty$ is defined by the singularity of the integrand at the point $-\gamma$ (the right end of ld). Since the integrand there has the form $e^{\lambda t}[a + b\sqrt{\lambda+\gamma} + O(\lambda+\gamma)], \lambda+\gamma \to 0$, this leads to the second relation of (62).

The examples below concern differential operators on the continuous and discrete non-commutative groups Γ (processes with independent increments considered in the previous section are examples of operators on the abelian groups R^d).

First we will consider the Heisenberg (nilpotent) group $\Gamma = H^3$ of the upper triangular matrices

$$g = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, (x,y,z) \in R^3, \qquad (67)$$

with units on the diagonal, and its discrete subgroup ZH^3, where $(x,y,z) \in Z3$.

Then we study (solvable) group of the affine transformations of the real line: $x \to ax + b, a > 0$, which has the matrix representation:

$$\mathrm{Aff}(R^1) = \left\{g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, a > 0, (0,b) \in R^2\right\},$$

And its subgroup generated by $\alpha_1 = \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} e & -e \\ 0 & 1 \end{bmatrix}$ and their inverses $\alpha_{-1} = \begin{bmatrix} e^{-1} & -1 \\ 0 & 1 \end{bmatrix}$ and $\alpha_{-2} = \begin{bmatrix} e^{-1} & 1 \\ 0 & 1 \end{bmatrix}$.

There are two standard ways to construct the Laplacian on a Lie group. A usual differential-geometric approach starts with the Lie algebra $\mathfrak{A}\Gamma$ on Γ, which can be considered either as the algebra of the first order differential operators generated by the differentiations along the appropriate one-parameter subgroups of Γ, or simply as a tangent vector space $T\Gamma$ to Γ at the unit element I. The exponential mapping $\mathfrak{A}\Gamma \to \Gamma$ allows one to construct (at least locally) the general left invariant Laplacian Δ_Γ on Γ as the image of the differential operator $\sum_{ij} a_{ij} D_i D_j + \sum_i b_i D_i$ with constant coefficients on $\mathfrak{A}\Gamma$. The Riemannian metric ds^2 on Γ and the volume element dv can be defined now using the inverse matrix of the coefficients of the Laplacian Δ_Γ. It is important to note that additional symmetry conditions are needed to determine Δ_Γ uniquely.

The central object in the probabilistic construction of the Laplacian (see, for instance, McKean [14]) is the Brownian motion g_t on Γ. We impose the symmetry condition $g_t \overset{\text{law}}{=} g_t^{-1}$. Since $\mathfrak{A}\Gamma$ is a linear space, one can define the usual Brownian motion b_t on $\mathfrak{A}\Gamma$ with the generator $\sum_{ij} a_{ij} D_i D_j + \sum_i b_i D_i$. The symmetry condition holds if $(I + db_t) \overset{\text{law}}{=} (I + db_t)^{-1}$. The process g_t (diffusion on Γ) is given (formally) by the stochastic multiplicative integral

$$g_t = \prod_{s=0}^{t} (I + db_s),$$

or (more rigorously) by the Ito's stochastic differential equation

$$dg_t = g_t db_t. \qquad (68)$$

The Laplacian Δ_Γ is defined now as the generator of the diffusion

$$\Delta_\Gamma f(g) = \lim_{\Delta t \to 0} \frac{Ef(g(1 + b_{\Delta t})) - f(g)}{\Delta t}, \qquad f \in C^2(\Gamma). \qquad (69)$$

The Riemannian metric form is defined as above (by the inverse matrix of the coefficients of the Laplacian).

We will use the probabilistic approach to construct the Laplacian in the examples below, since it allows us to easily incorporate the symmetry condition.

3. Heisenberg group $\Gamma = H^3$ of the upper triangular matrices (67) with units on the diagonal. We have

$$\mathfrak{A}\Gamma = \left\{A = \begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}, (\alpha, \beta, \gamma) \in R^3 \right\}, \quad e^A = \begin{bmatrix} 1 & \alpha & \gamma + \frac{\alpha\beta}{2} \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $A \to \exp(A)$ is a one-to-one mapping of $\mathfrak{A}\Gamma$ onto Γ. Consider the following Brownian motion on $\mathfrak{A}\Gamma$:

$$b_t = \begin{bmatrix} 0 & u_t & \sigma w_t \\ 0 & 0 & v_t \\ 0 & 0 & 0 \end{bmatrix},$$

where σ is a constant and u_t, v_t, w_t are (standard) independent Wiener processes. Then equation (68) has the form

$$dg_t = \begin{bmatrix} 0 & dx_t & dz_t \\ 0 & 0 & dy_t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x_t & z_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & du_t & \sigma dw_t \\ 0 & 0 & dv_t \\ 0 & 0 & 0 \end{bmatrix},$$

which implies that

$$dx_t = du_t, \qquad dy_t = dv_t, \qquad dz_t = \sigma dw_t + x_t dv_t.$$

Under condition $g(0) = I$, we get

$$g_t \begin{bmatrix} 1 & u_t & \sigma w_t + \int_0^t u_s dv_s \\ 0 & 1 & v_t \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us note that the matrix

$$(g_t)^{-1} = \begin{bmatrix} 1 & -u_t & u_t v_t - \sigma w_t - \int_0^t u_s dv_s \\ 0 & 1 & -v_t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -u_t & -\sigma w_t + \int_0^t v_s du_s \\ 0 & 1 & -v_t \\ 0 & 0 & 1 \end{bmatrix}$$

Has the same law as g_t. Now from (69) it follows that

$$(\Delta_\Gamma f)(x, y, z) = \frac{1}{2}\left[f_{xx} + f_{yy} + (\sigma^2 + x^2) f_{zz} + 2\sigma x f_{yz}\right].$$

The matrix of the left invariant Riemannian metric has the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \sigma x \\ 0 & \sigma x & \sigma^2 + x^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma^2 + x^2 & -\sigma x \\ 0 & -\sigma x & 1 \end{bmatrix},$$

i.e.,

$$ds^2 = dx^2 + (\sigma^2 + x^2) dy^2 + dz^2 - 2\sigma x dy dz, \quad dV = dx dy dz.$$

Denote by $p_\sigma(t, x, y, z)$ the transition density for the process g_t (fundamental solution of the parabolic equation $u_t = \Delta_\Gamma$). Let $\pi_\sigma(t) = p_\sigma(t,0,0,0)$.

Theorem (6.1.16) [202]: Function $\pi_\sigma(t)$has the following asymptotic behavior at zero and infinity:

$$\pi_\sigma(t) \sim \frac{c_0}{t^{\frac{3}{2}}}, \qquad t \to 0; \;\; \pi_\sigma(t) \sim \frac{c}{t^2}, t \to \infty, \qquad c = p_0(1,0,0), \qquad (70)$$

i.e., Theorem 6.1.5 holds for operator $H = \Delta_\Gamma + V(x,y,z)$ with $\alpha = 3, \beta = 4$.

Proof. Since H^3 is a three dimensional manifold, the asymptotics at zero is obvious. Let us prove the second relation of (70). We start with the simple case of $\sigma = 0$. The operator Δ_Γ in this case is degenerate. However, the density$p_0(t,x,y,z)$ exists and can be found using H¨ormander hypoellipticity theory or by direct calculations. In fact, the joint distribution of (x_t, y_t, z_t) is self-similar

$$\left(\frac{u_t}{\sqrt{t}}, \frac{v_t}{\sqrt{t}}, \frac{\int_0^t u_s dv_s}{t}\right) = \left(u_1, v_1, \int_0^t u_s dv_s\right),$$

i.e.,

$$p_0(t,x,y,z) = \frac{1}{t^2} p_0\left(1, \frac{x}{\sqrt{t}}, \frac{x}{\sqrt{t}}, \frac{z}{t}\right),$$

and therefore,

$$p_0(t,0,0,0) = \frac{c}{t^2}, \qquad c = p_0(1,0,0,0).$$

Let $\sigma^2 > 0$. Then

$$p_\sigma(t,x,y,z) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{R^1} p_0(t,x,y,z_1) e^{-\frac{(z-zz_1)^2}{2\sigma^2 t}} dz_1.$$

After rescaling $\frac{x}{\sqrt{t}} \to x, \frac{y}{\sqrt{t}} \to y, \frac{z}{t} \to z$, we get

$$p_\sigma(t,x,y,z) = \frac{\sqrt{t}}{t^2\sqrt{2\pi\sigma^2}} \int_{R^1} p_0(1,x,y,z_1) e^{-\frac{t(z-zz_1)^2}{2\sigma^2}} dz_1.$$

From here it follows that $p_\sigma(t,0,0,0) \sim c/t^2, t \to \infty$, with $c = p_0(1,0,0,0)$.

Theorem 6.1.16 can be proved for the group H^n of $n \times n$ upper triangular matrices with units on the diagonal. In this case,

$$\alpha = \dim H^n = \frac{n(n-1)}{2}, \beta = (n-1) + 2(n-2) + 3(n-3) + \cdots = \frac{n(n^2-1)}{2}.$$

$\Gamma = ZH^3$ of integer valued matrices of the form

$$g = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in Z^1.$$

Consider the Markov process g_t on ZH^3 defined by the equation

$$g_{t+dt} = g_t \begin{pmatrix} 1 & d\xi_t & d\zeta_t \\ 0 & 1 & d\eta_t \\ 0 & 0 & 1 \end{pmatrix}, \qquad (71)$$

where ξ_t, η_t, ζ_t are there independent Markov process on Z^1 with generators

$$\Delta_1\psi(n) = \psi(n+1) + \psi(n-1) - 2\psi(n), \quad n \in Z^1.$$

Equation (71) can be solved using discretization of time. This gives

$$g_t = \begin{pmatrix} 1 & x_t & y_t \\ 0 & 1 & z_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi_t & \zeta_t + \int_0^t \xi_s d\eta_s \\ 0 & 1 & \eta_t \\ 0 & 0 & 1 \end{pmatrix}$$

The generator L of this process has the form (61) with

$$a_{\pm 1} = \begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_{\pm 2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix}, a_{\pm 3} \begin{pmatrix} 1 & 0 & \pm 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

i.e.,

$$L = \Delta_\Gamma \psi(g) = \sum_{i=\pm 1, \pm 2, \pm 3} [\psi(ga_i) - \psi(g)]. \qquad (72)$$

If $\psi = \psi(g)$ is considered as a function of $(x, y, z) \in Z^3$, then

$$L\psi(x, y, z) = \psi(x + 1, y, z) + \psi(x - 1, y, z) + \psi(x, y + 1, z + x) + \psi(x, y - 1, z - x) + \psi(x, y, z + 1) + \psi(x, y, z - 1) - 6\psi(x, y, z) (73)$$

The analysis of the transition probability in this case is similar to the continuous case, and it leads to the following result

Theorem (6.1.17) [202]: If g_t is the process on ZH^3 with the generator (73), then

$$P\{g_t = I\} = P\{x_t = y_t = z_t = 0\} \sim \frac{c}{t^2}, t \to \infty,$$

with c defined in (70). can be applied to operator $H_0 = L$ with $\beta = 4$.

This result is valid in a more general setting (see [13]). Consider three independent processes $\xi_t, \eta_t, \zeta_t, t \geq 0$, on Z^1 with independent increments and such that

$$Ee^{ik\xi_t} = e^{-t(1-\sum_{i=1}^{\infty} p_i \cos ki)}, \sum_{i=1}^{\infty} p_i = 1,$$

$$Ee^{ik\eta_t} = e^{-t(1-\sum_{i=1}^{\infty} q_i \cos ki)}, \sum_{i=1}^{\infty} q_i = 1,$$

$$Ee^{ik\zeta_t} = e^{-t(1-\sum_{i=1}^{\infty} r_i \cos ki)}, \sum_{i=1}^{\infty} r_i = 1,$$

Assume also that there exist $\alpha_1, \alpha_2, \alpha_3$ on the interval $(0,2)$ such that

$$p_i \sim \frac{c_1}{i^{1+\alpha_1}}, q_i \sim \frac{c_2}{i^{1+\alpha_2}}, r_i \sim \frac{c_3}{i^{1+\alpha_3}}$$

as $i \to \infty$, i.e., distributions with characteristic functions $\sum_{i=1}^{\infty} p_i \cos ki, \sum_{i=1}^{\infty} q_i \cos ki, \sum_{i=1}^{\infty} r_i \cos ki$ belong to the domain of attraction of the symmetric stable law with parameters $\alpha_1, \alpha_2, \alpha_3$. Let g_t be the process on ZH^3 defined by (71). Then

$$P\{g_t = I\} \sim \frac{c}{t^\gamma}, t \to \infty, \gamma = \max(\frac{2}{\alpha_1} + \frac{2}{\alpha_1}, \frac{1}{\alpha_3}).$$

This group of transformations $x \to ax + b, a > 0$, has a matrix representation:

$$\Gamma = \mathrm{Aff}(R^1) = \{g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, a > 0, (a, b) \in R^2\}.$$

We start with the Lie algebra for $\mathrm{Aff}(R^1)$:

$$\mathfrak{A}\Gamma = \left\{\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}, (\alpha, \beta) \in R^2\right\}.$$

Obviously, for arbitrary$A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$, one has

$$\exp(A) = \begin{bmatrix} e^{\alpha} & \beta \frac{e^{\alpha} - 1}{\alpha} \\ 0 & 1 \end{bmatrix},$$

i.e., the exponential mapping of $\mathfrak{A}\Gamma$ coincides with the group Γ. Consider the diffusion

$$b_t = \begin{bmatrix} w_t + \alpha t & v_t \\ 0 & 0 \end{bmatrix}$$

on $\mathfrak{A}\Gamma$, where (w_t, v_t) are independent Wiener processes. Consider the matrix valued process $g_t = \begin{bmatrix} x_t & y_t \\ 0 & 1 \end{bmatrix}, g_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, on Γ satisfying the equation

$$dg_t = g_t db_t = \begin{bmatrix} x_t & y_t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dw_t + \alpha dt & dv_t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_t(dw_t + \alpha dt) & x_t dv_t \\ 0 & 0 \end{bmatrix}.$$

This implies

$$dx_t = x_t(dw_t + \alpha dt),$$
$$dy_t = x_t dv_t,$$

i.e. (due to Ito's formula),

$$x_t = e^{w_i + (\alpha - \frac{1}{2})t}, y_t = \int_0^t x_s dv_s.$$

We impose the following symmetry conditions:

$$(g_t)^{-1} \stackrel{law}{=} g_t, \tag{74}$$

It holds if $\alpha = \frac{1}{2}$. In fact,

$$g_t = \begin{bmatrix} e^{w_t} & \int_0^t e^{w_s} dv_s \\ 0 & 1 \end{bmatrix}, g_t^{-1} = \begin{bmatrix} e^{-w_t} & -\int_0^t e^{w_s - w_t} dv_s \\ 0 & 1 \end{bmatrix}, \tag{75}$$

and (74) follows after the change of variables $s = t - \tau$ in the matrix g_t^{-1}. Then the generator of the process g_t has the form

$$\Delta_\Gamma f = \frac{x^2}{2} \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right] + \frac{x}{2} \frac{\partial f}{\partial x}.$$

Remark (6.1.18) [202]: Let $H = \Delta_\Gamma + V$, where the negative part $W = V_-$ of the potential is bounded: $W \le h^{-1}$. From (76) and Theorem 2.5 it follows that

$$N_0(V) \le C(h) \int_0^\infty \frac{W^{3/2}(x, y)}{x} dx dy.$$

Remark (6.1.19) [202]: The left-invariant Riemannian metric on $Aff(R^1)$ is given by the inverse diffusion matrix of Δ_Γ, i.e.,

$$d\xi^2 = x^{-2}(dx^2 + dy^2) \left(g = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, x > 0 \right)$$

After the change $(x, y) \to (y, x)$, this formula coincides with the metric on the Lobachevsky plane (see the previous section). However, one can not identity the Laplacian on $Aff(R^1)$ and on the Lobachevsky plane L^2, since they are defined by different symmetry conditions. The plane L^2 has a three dimensional group of transformations, and each point $z \in L^2$ has a one-parameter stationary subgroup. The Laplacian on the Lobachevsky plane was defined by the invariance with respect to this three dimensional group of transformations. In the case of $\mathfrak{A}\Gamma = Aff(R^1)$, the group of transformations is two dimensional. It acts as a left shift $g \to g_1 g, g_1, g \in \Gamma$, and the Laplacian is specified by the left invariance with respect to this two dimensional group and the symmetry condition (74).

Theorem (6.1.20) [202]: Operator Δ_Γ is self-adjoint with respect to the measure $x^{-d}dxdy$. The function $\pi(t) = p(t,0,0)$ has the following behavior at zero and infinity:

$$\pi(t) \sim \frac{c_0}{t}, t \to 0; \qquad \pi(t) \sim \frac{C}{t^{3/2}}, \qquad t \to \infty. \tag{76}$$

Proof. Since Γ is a two dimensional manifold, the asymptotics of $\pi(t)$ at zero is obvious. One needs only to justify the asymptotics of $\pi(t)$ at infinity.

Let's find the density of $(x_t, y_t) = (e^{w_t}, \int_0^t e^{w_s} dv_s)$. The second term, for a fixed realization of $w.$, has the Gaussian law with (conditional) variance $\sigma^2 = \int_0^t e^{2w_s} ds$, and

$$P\{x_t \in 1 + dx, y_t \in 0 + dy\} = p(t,0,0)dxdy = \frac{1}{\sqrt{2\pi t}} E \frac{1}{\sqrt{2\pi \int_0^t e^{2\widehat{w}_s} ds}} \tag{77}$$

Here $\widehat{w}_s, s \in [0,t]$, is the Brownian bridge on [0, t] . The distribution of the exponential functional $A(t) = \int_0^t e^{2\widehat{w}_s} ds$ and the joint distribution of $(A(t), w(t))$ were calculated in [201]. Together with (77), these easily imply the statement of the theorem.

Let Γ be a discrete group generated by elements $a_1, \dots, a_d, a_{-1} = a_1^{-1}, \dots, a_{-d} = a_d^{-1}$, with some identities. Define the Laplacian on Γ by the formula

$$\Delta\psi(g) = \sum_{i=-d}^{d} \psi(ga_i) - 2d\psi(g), \qquad g \in \Gamma.$$

Consider the Markov process g_t on Γ with continuous time and the generator Δ. Let $\tilde{g}_k, k = 0,1,2, \dots$, be the Markov chain on Γ with discrete time (symmetric random walk) such that

$$P\{\tilde{g}_0 = e\} = 1, \qquad P\{\tilde{g}_{n+1} = ga_i | \tilde{g}_n = g\} = \frac{1}{2d}, i = \pm 1, \pm 2, \dots \pm d.$$

Then there is a relation between transition probability $p(t,e,g)$ of the Markov process g_t and the transition probability $P\{\tilde{g}_k = g\}$ of the random walk. In particular, one can estimate $\pi(t) = p(t,e,e)$ for large t through $\tilde{\pi}(2k) = P\{\tilde{g}_{2k} = e\}$ under minimal assumptions on $\tilde{\pi}(2k)$. For example, it is enough to assume that $\tilde{\pi}(2k) = k^\gamma L(k), \gamma \geq 0$, where $L(k)$ for large k can be extended as slowly varying monotonic function of continuous argument k. We are not going to provide a general statement of this type, but we restrict ourself to a specific situation needed in the next section. Note that we consider here only even arguments of $\tilde{\pi}$, since $\tilde{\pi}(2k+1) = 0$.

Theorem (6.1.21) [202]: Let $\tilde{\pi}(2n) \leq e^{-c_0(2n)^\alpha}, n \to \infty, c_0 > 0, 0 < \alpha < 1$.

Then

$$\pi(t) \leq e^{-c_0(2dt)^\alpha}, t \geq t_0.$$

Proof. The number v_t of jumps of the process g_t on the interval $(0,t)$ has Poisson distribution. At the moments of jumps, the process performs the symmetric random walk with discrete time and transition probabilities $P\{g \to ga_i\} = 1/2d, i = \pm 1, \pm 2, \dots \pm d$. Thus (taking into account that $\tilde{\pi}(2k+1) = 0$),

$$\pi(t) = p(t,e,e) = \sum_{n=0}^{\infty} \tilde{\pi}(2n) P\{v_t = 2n\}.$$

Due to the exponential Chebyshev inequality

$$P\{|v_t - 2dt| \geq \varepsilon t\} \leq e^{-c\varepsilon^2 t}, t \to \infty.$$

Secondly,

$$P\{v_t \text{ is even}\} = \frac{1}{2} + O(e^{-4dt}), t \to \infty.$$

These relations imply that, for $t \to \infty$ and $\delta > 0$,

$$\pi(t) = \sum_{n:|2n-2dt|<\varepsilon t} \tilde{\pi}(2n) P\{v_t = 2n\} + O(e^{-c_0(2dt)^\alpha})$$

$$\leq \sum_{n:|2n-2dt|<\varepsilon t} e^{-c_0(2n)^\alpha} P\{v_t = 2n\} + O(e^{-c_0(2dt)^\alpha})$$

$$\leq (1+\delta) e^{-c_0(2dt)^\alpha} \sum_{n:|2n-2dt|<\varepsilon t} P\{v_t = 2n) + O(e^{-c_0(2dt)^\alpha}) \leq \frac{1+\delta}{2} e^{-c_0(2dt)^\alpha}$$

$$+ O(e^{-c_0(2dt)^\alpha}).$$

7. Random walk on the discrete subgroup of $Aff(R^1)$. Let us consider the following two matrices $\alpha_1 = \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} e & -e \\ 0 & 1 \end{bmatrix}$ in $Aff(R^1)$ and their inverses $\alpha_{-1} = \begin{bmatrix} e^{-1} & -1 \\ 0 & 1 \end{bmatrix}$ and $\alpha_{-2} = \begin{bmatrix} e^{-1} & 1 \\ 0 & 1 \end{bmatrix}$. Let G be a subgroup of $Aff(R^1)$ generated by$\alpha_{\pm 1}$ and $\alpha_{\pm 2}$. Consider the random walk on G of the form

$$g_n = h_1 h_2 \dots h_n,$$

where one step random matrices h_i coincide with one of the matrices $\alpha_{\pm 1}, \alpha_{\pm 2}$ with probability 1/4, i.e.,

$$h_i = \begin{bmatrix} e^{\varepsilon_i} & \delta_i \\ 0 & 1 \end{bmatrix},$$

where

$$P\{\varepsilon_i = 1, \delta_i = e\} = P\{\varepsilon_i = 1, \delta_i = -e\} = P\{\varepsilon_i = -1, \delta_i = -1\} = P\{\varepsilon_i = -1, \delta_i = 1\} = 1/4. \quad (78)$$

Let Δ_G be the Laplacian on G which corresponds to the generators $a_{\pm 1}, a_{\pm 2}$, i.e., (compare with (61) (72))

$$L = \Delta_\Gamma \psi(g) = \sum_{i=\pm 1, \pm 2} [\psi(g a_i) - \psi(g)].$$

Theorem (6.1.22) [202]: (a) The following estimate is valid for $\tilde{\pi}(2n)$:

$$\tilde{\pi}(2n) \leq e^{-c_0(2n)^{1/3}}, n \to \infty, c_0 > 0.$$

(b) Theorem 6.1.7 can be applied to operator $H = \Delta_G + V(g)$ with $\gamma = 1/3$, i.e.,

$$N_0(V) \leq C(h, A)[\sum_{g:V(g) \leq h^{-1}} e^{-AW(g)^{-1/3}} + n(h)], \qquad n(h) = \#\{g: W(g) > h^{-1}\}$$

Proof. The random variables $(\varepsilon_i, \delta_i)$ are dependent, but (78) implies that $(\varepsilon_i, \tilde{\delta}_i)$, where $\tilde{\delta}_i = \operatorname{sgn} \delta_i$, are independent symmetric Bernoulli r.v. It is easy to see that

$$g_n = \begin{bmatrix} e^{S_n} & \sum_{k=1}^{n} \delta_k e^{S_{k-1}} \\ 0 & 1 \end{bmatrix},$$

where $S_0 = 1, S_k = \varepsilon_1 + \dots + \varepsilon_k, k > 0$, is a symmetric random walk on Z^1. This formula is an obvious discrete analogue of (75). Our goal is to calculate the probability

$$\tilde{\pi}(2n) = P\{g_{2n} = I\} = P\{S_{2n} = 0, \sum_{k=1}^{2n} \delta_k e^{S_{k-1}} = 0\} = \binom{2n}{n}\frac{1}{2^{2n}} P\{\sum_{k=2}^{2n} \delta_k e^{\hat{S}_{k-1}} = 0\} \sim \frac{1}{\sqrt{\pi n}} P\{\sum_{k=1}^{2n-1} \delta_{k+1} e^{\hat{S}_k} = 0\}, n \to \infty.$$

Here $\hat{S}_k, k = 0,1,\dots,2n$, is the discrete bridge, i.e., the random walk S_k under conditions $S_0 = S_{2n} = 0$.

Put $M_{2n} = \max_{k\le 2n} \hat{S}_k, m_{2n} = \min_{k\le 2n} \hat{S}_k$. Let $\Gamma_{s-1}^+, \Gamma_s^-$ be the sets of moments of time k when the bridge $\hat{S}_k$ changes value from $s-1$ to s or from to $s-1$, respectively. Introduce local times $\tau_{s-1}^+ = \text{Card}\,\Gamma_{s-1}^+$ and $\tau_s^- = \text{Card}\,\Gamma_s^-$, i.e., $\tau_{s-1}^+ = \#$ (jumps of $\hat{S}_k$ from $s-1$ to s) and $\tau_s^- = \#$ (jumps of $\hat{S}_k$ from s to $s-1$). Note that $\delta_{k+1} e^{\hat{S}_k} = \tilde{\delta}_{k+1} e^s$ when $k \in \Gamma_{s-1}^+ \cup \Gamma_s^-$, and therefore

$$\sum_{k=1}^{2n-1} \delta_{k+1} e^{\hat{S}_k} = \sum_{s=m_{2n}+1}^{M_{2n}} e^s \sum_{j\in\Gamma_{s-1}^+\cup\Gamma_s^-} \tilde{\delta}_j.$$

Since r.v. $\{\tilde{\delta}_j\}$ are independent of the trajectoryS_k and numbers $e^s, s = 0, \pm 1, \pm 2, \dots$, are rationally independent, we have

$$P\{g_{2n} = I\} \sim \frac{1}{\sqrt{\pi n}} E \prod_{s=m_{2n}+1}^{M_{2n}} \binom{2\tau_s^-}{\tau_s^-} (\frac{1}{2})^{2\tau_s^-} \le \frac{1}{\sqrt{\pi n}} (\frac{1}{2})^{M_{2n}-m_{2n}}$$

$$= \frac{1}{\sqrt{\pi n}} (\frac{1}{2})^{M_{2n}-m_{2n}} [I_{M_{2n}-m_{2n} > \sqrt{2n}} + I_{M_{2n}-m_{2n} < \sqrt{2n}}]$$

$$\le \frac{1}{\pi n} (\frac{1}{2})^{\sqrt{2n}}$$

$$+ \sum_{r=1}^{\sqrt{2n}} (\frac{1}{2})^r P\{|S_k| \le r, k = 1,2,\dots 2n, S_{2n} = 0\}$$

$$\le e^{c_1\sqrt{2n}} + \sum_{r=1}^{\sqrt{2n}} (\frac{1}{2})^r P\{|S_k| \le r, k = 1,2,\dots 2n, S_{2n} = 0\}.$$

Lemma (6.1.23) [202]:$P\{|S_k| \le r, I = 1,2,\dots 2n, S_{2n} = 0\} \le (\cos\frac{\pi}{2(r+1)})^{2n}$.

Proof. Let us introduce the operator $H_0\psi(x) = \frac{\psi(x+1)+\psi(x-1)}{2}$ on the set $[-r, r] \in Z^1$ with the Dirichlet boundary conditions $\psi(r+1) = \psi(-r-1) = 0$. Then $\varphi(x) = \cos\frac{\pi x}{2(r+1)}$ is an eigenfunction of H_0 with the eigenvalue $\lambda_{0,r+1} = \cos\frac{\pi}{2(r+1)}$. Hence

$$H_0^{2n}\varphi(x) = \lambda_{0,r+1}^{2n}\,\varphi(x).$$

Let $p_r(k, x, z)$ be the transition probability of the random walk on $[-r, r] \in Z^1$ with the absorption at $\pm(r+1)$. Then

$$\sum_{|z|\le r} p_r(2n, x, z)\varphi(z) = \lambda_{0,r+1}^{2n}\,\varphi(x).$$

Since $\varphi(z) \le 1, \varphi(0) = 1$, the latter relation implies

$$\sum_{|z|\leq r} p_r(2n,x,z) \leq \lambda_{0,r+1}^{2n}.$$

Since $S_k, k = 0,1, \ldots 2n$, is the symmetric random walk on Z^1, we have

$$P\{|S_k| \leq r, k = 1,2, \ldots 2n, S_{2n} = 0\} = p_r(2n,0,0) \leq \lambda_{0,r+1}^{2n}.$$

Direct calculation shows that

$$\max_{r\leq\sqrt{2n}} (\frac{1}{2})^r (\cos\frac{\pi}{2(r+1)})^{2n} \leq e^{-c(2n)^{1/3}},$$

with the maximum achieved at $r = r_0 \sim c_1(2n)^{1/3}$. Thus

$$P\{g_{2n} = I\} \leq (\frac{1}{2})^{\sqrt{2n}} + \sqrt{2n}e^{c_{0(2n)^{1/3}}} \leq e^{-\tilde{c}_0(2n)^{1/3}}$$

for arbitrary $\tilde{c}_0 < c_0$ and sufficiently large n. This proves the first statement of the theorem. Now the second statement follows from Theorem 6.1.20.

Theorem (6.1.24) [202]: The assumptions of Theorems 6.1.4, 6.1.5 hold for operator $-H_0$ introduced in this section with the constants α, β in Theorem 6.1.5 equal to 1 and d, respectively.

One can easily see that there is a Markov process with the generator $-H_0$, and condition (a) of Theorem 6.1.5 holds, we'll estimate the function p_0 in order to show that condition (b) holds and find constants α, β defined in Theorem 6.1.5 In fact, the same arguments can be used to verify condition (a) analytically.

As we discussed above, Theorem 6.1.5 is not exact if $\alpha \leq 2$. Theorem 6.1.7 provides a better result in the case $\alpha = 0$. The situation is more complicated if $\alpha = 1$. We will illustrate it using the operator H_0 on quantum graph Γ^d defined above. We will consider two specific classes of potentials. In one case, inequality (36) is valid with $\max(\alpha/2,1) = 1$ replaced by $\alpha/2 = 1/2$. However, inequality (36) can not be improved for potentials of the second type. The first class (regular potentials) consists of piece-wise constant functions.

Proof. As it was mentioned after the statement of the theorem, it is enough to show the validity of condition (b) and evaluate α, β. Let

$$u_t = -H_0 u, t > 0, \qquad u|_{t=0} = f,$$

with a compactly supported f and

$$\varphi = \varphi(x,\lambda) = \int_0^\infty u e^{\lambda t} dt, \text{Re}\,\lambda \leq -a < 0, x \in \Gamma^d.$$

Note that we replaced $-\lambda$ by λ in the Laplace transform above. it is convenient for future notations. Then φ satisfies the equation

$$(H_0 - \lambda)\varphi = f, \qquad (79)$$

and u can be found using the inverse Laplace transform

$$u = \frac{1}{(2\pi)^d} \int_{-a-i\infty}^{-a+i\infty} \varphi e^{-\lambda t} d\lambda. \qquad (80)$$

The spectrum of H_0 is $[0,\infty)$, and φ is analytic in λ when $\lambda \in C \setminus [0,\infty)$. We are going to study the properties of φ when $\lambda \to 0$ and $\lambda \to \infty$. Let $\psi(z) = \psi(z,\lambda), z \in Z^d$, be the restriction of the function $\varphi(x,\lambda), x \in \Gamma^d$, on the lattice Z^d. Let e be an arbitrary edge of Γ^d with end points $z_1, z_2 \in Z^d$ and parametrization from z_1 to z_2. By solving the boundary value problem on e, we can represent φ on e in the form

$$\varphi = \frac{\psi(z_1 \sin k(1-s) + \psi(z_2) \sin ks}{\sin k} + \varphi_{par}, \varphi_{par} = \int_0^1 G(s,t) f(t) dt, \qquad (81)$$

where $k = \sqrt{\lambda}, \operatorname{Im} k > 0$, and

$$G = \frac{1}{k \sin k} \begin{cases} \sin ks \sin k(1-t), & s < t \\ \sin kt \sin k(1-s), & s \geq t \end{cases}.$$

Due to the invariance of H_0 with respect to translations and rotations in Z^d, it is enough to estimate $p_0(t,x,x)$ when x belongs to the edge e_0 with z_1 being the origin in Z^d and $z_2 = (1,0,\ldots,0)$. Let f be supported on one edge e_0. Then (81) is still valid, but $\varphi_{par} = 0$ on all the edges except e_0. We substitute (81) into (44) and get the following equation for ψ:

$$(\Delta - 2d \cos k)\psi(z) = \frac{1}{k}\int_0^1 \sin k(1-t) f(t) dt \delta_1 + \frac{1}{k}\int_0^1 \sin ktf(t) dt \delta_0, z \in Z^d.$$

Here Δ is the lattice Laplacian defined in (41) and δ_0, δ_1 are functions on Z^d equal to one at z, y, respectively, and equal to zero elsewhere. In particular, if f is the delta function at a point s of the edge e_0, then

$$(\Delta - 2d \cos k)\psi = \frac{1}{k}\sin k(1-s)\delta_1 + \frac{1}{k}\sin ks \delta_0. \qquad (82)$$

Let $R_\mu(z - z_0)$ be the kernel of the resolvent $(\Delta - \mu)^{-1}$ of the lattice Laplacian. Then (82) implies that

$$\psi(z) = \frac{1}{\lambda}\sin\sqrt{\lambda}\, s R_\mu(z) + \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}\,(1-s) R_\mu(z - z_2), \mu = 2d\cos\sqrt{\lambda} \ (83)$$

Function $R_\mu(z)$ has the form

$$R_\mu(z) = \int_T \frac{e^{i(\sigma,z)} d\sigma}{(\sum_{1\leq j\leq d} 2\cos\sigma_j) - \mu}, T = [-\pi,\pi]^d.$$

Hence, function $\sin(\sqrt{\lambda}\, s) R_\mu(z),\ s \in (0,1), \mu = 2d\cos\sqrt{\lambda}$, decays exponentially as $|\operatorname{Im}\sqrt{\lambda}| \to \infty$. This allows one to change the contour of integration in (80), when $z \in Z^d$, and rewrite (80) in the form

$$u(z,t) = \frac{1}{(2\pi)^d}\int_l \psi_\lambda(z) e^{\lambda t} d\lambda, z \in Z^d, \qquad (84)$$

where contour l consists of the ray $\lambda = \rho e^{-i\pi/4}, \rho \in (\infty, 1)$, a smooth arc starting at $\lambda = e^{-\pi/4}$, ending at $\lambda = e^{\pi/4}$, and crossing the real axis at $\lambda = -a$, and the ray $\lambda = \rho e^{i\pi/4}, \rho \in (1,\infty)$. It is easy to see that $|\psi(z,\lambda)| \leq C/|\sqrt{\lambda}|$ as $\lambda \in l$ uniformly in s and $z \in Z^d$. This immediately implies that $|u(z,t)| \leq C/\sqrt{t}$. Now from (81) it follows that the same estimate is valid for $p_0(t,x,x), x \in e_0$, i.e., condition (b) holds, and $\alpha = 1$.

From (84)it also follows that the asymptotic behavior of u as $t \to \infty$ is determined by the asymptotic expansion of $\psi(z,\lambda)$ as $\lambda \to 0, \lambda \notin [0,\infty)$. Note that the spectrum of the difference Laplacian is $[-2d, 2d]$, and $\mu = 2d - d\lambda + O(\lambda^2)$ as $\lambda \to 0$. From here and the well known expansions of the resolvent of the difference Laplacian near the edge of the spectrum it follows that the first singular term in the asymptotic expansion of $R_\mu(z)$ as $\lambda \to 0, \lambda \notin [0,\infty)$, has the form

$$\begin{cases} c_d \lambda^{d/2-1}(1 + O(\lambda)), & d \text{ is odd}, \\ c_d \lambda^{d/2-} \ln\lambda\,(1 + O(\lambda)), & d \text{ is even}. \end{cases}$$

Then (83) implies that a similar expansion is valid for $\psi(z,\lambda)$ with the main term independent of s

and the remainder estimated uniformly in s. This allows one to replace l in (84) by the contour which consists of the rays arg $\lambda = \pm\pi/4$. From here it follows that for each $z \in Z^d$ and uniformly in$s$,

$$u(z,t) \sim t^{-d/2}, t \to \infty.$$

This and (81) imply the same behavior for $p_0(t,x,x), x \in e_0$, i.e., $\beta = d$.

Section (6.2): The Hierarchical Schrödinger Operator

The spectral theory of the fractals, which are similar to the infinite Sierpinski gasket (i.e. the spectral theory of the corresponding Laplacians) is well understood (see [206, 86, 207]). It has several important features: the existence of a large number of eigenvalues of infinite multiplicity, pure point structure of the integrated density of states, compactly supported eigenfunctions. These features manifest themselves in the unusual asymptotes of the heat kernel, the specific structure of the corresponding ζ-function, etc., see [203].

Fig. 7

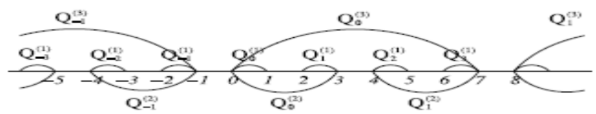

Fig. 7. An example of a hierarchical lattice with $X = \mathbb{Z}$ and $v = 2$.

The next natural step in the spectral theory is to study Schrödinger type operators, i.e., fractal Laplacian perturbed by a potential. There are two possible directions for such a development: analysis of the random Anderson Hamiltonians (the potential is stationary in space) or the study of the classical problem on the negative spectrum when the potential vanishes at infinity. For the first direction, see [88, 93, 95]. We will concentrate on the second problem in a particular case of the simplest fractal object: Dyson's hierarchical Laplacian perturbed by a decaying potential. Our goal is to prove the Cwikel-Lieb-Rozenblum (CLR) estimates for the number of negative eigenvalues and estimates for Lieb-Thirring (LT) sums. These estimates depend on the spectral dimension s_h of the fractal (which can take an arbitrary positive value).

The concept of the hierarchical structure was proposed by F. Dyson [205] in his theory of 1-D ferromagnetic phase transitions. There are several modifications of the hierarchical Laplacian (see [93]). We will study the simplest one, which is characterized by an integer-valued parameter $v \geq 2$ and a probabilistic parameter $p \in (0,1)$. More recent results in this area can be found in [204].

Consider a countable set X and a family of partitions $\Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \cdots$ of X (we write $\Pi_r \subset \Pi_{r+1}$ to mean that every element of Π_r is a subset of some element of Π_{r+1}). The elements of Π_0 are the singleton subsets of X. They are denoted by$Q_i^{(0)}$ and called cubes of rank zero. Each element $Q_i^{(1)}$ of Π_1 (cube of rank one) is a union of v different cubes of rank zero, i.e., $X = \bigcup Q_i^{(1)}, |Q_i^{(1)}| = v$ (see Fig. 7). Each element $Q_i^{(2)}$ of Π_2 (cube of rank two) is a union of v different cubes of rank one, i.e., $X = \bigcup Q_i^{(2)}, |Q_i^{(2)}| = v^2$, and so on. The parameter $v \geq 2$ is one of the two basic parameters of the model.

Each point x belongs to an increasing sequence of cubes of each rank $r \geq 0$ which we denote by$Q^{(r)}(x)$, i.e., $x = Q^{(0)}(x) \subset Q^{(1)}(x) \subset Q^{(2)}(x) \subset \cdots$.

The hierarchical distance $d_h(x,y)$ on X is defined as follows:

$$d_h(x,y) = \min\{r: \exists Q_i^{(r)} \ni x, y\}. \quad (85)$$

We assume the following connectivity condition holds: for each $x, y \in X$, the cubes $Q^{(n)}(x)$ contain y when n is large enough, i.e., $d_h(x, y) < \infty$.
Note that for arbitrary $z \in X, d_h(x, y) \leq \max\{d_h(x, z), d_h(y, z)\}$, i.e., $d_h(\cdot,\cdot)$ is a super-metric which implies that

$$\rho(x, y) = \rho_\beta(x, y) = e^{\beta d_h(x,y)} - 1, \qquad \beta > 0.$$

is also a metric. We will use it in the form

$$\rho(x, y) = \left(\frac{1}{\sqrt{p}}\right)^{d_h(x,y)} - 1, \tag{86}$$

i.e., $\beta = \ln\frac{1}{\sqrt{p}}$. Here $p \in (0,1)$ is the second parameter of the "Laplacian" Δ_h (see formula (3) below).
Now we denote by $l^2(X)$ the standard Hilbert space of square summable functions on the set X and define a self-adjoint bounded operator (the hierarchical Laplacian) depending on the parameter $p \in (0, 1)$:

$$\Delta_h\psi(x) = \sum_{r=1}^{\infty} a_r \left[\frac{\sum_{x'\in Q^{(r)}(x)} \psi(x')}{v^r} - \psi(x)\right], \text{where } a_r = (1-p)p^{r-1}, \sum_{r=1}^{\infty} \partial_r = 1. \tag{87}$$

The random walk on (X, d_h) related to the hierarchical Laplacian has a simple structure. It spends an exponentially distributed time τ (with parameter one) at each site x. At the moment $\tau + 0$ it randomly selects the rank k of a cube $Q^{(k)}(x), k \geq 1$, with $P\{k = r\} = a_r$ and jumps inside of $Q^{(k)}(x)$ with the new position $x' \in Q^{(k)}(x)$ being uniformly distributed.
It is clear that $\Delta_h = \Delta_h^*, \Delta_h \leq 0, Sp(\Delta_h) \in [-1,0]$. The following decomposition will play an essential role. Denote by $I_K(x)$ the indicator function of a set $K \in X$, i.e., $I_K = 1$ on K, $I_K = 0$ outside of K. Then, for each $y \in X$,

$$\delta_y(x) = \sum_{k=1}^{\infty} \left(\frac{I_{Q^{(k-1)}(y)}(x)}{v^{k-1}} - \frac{I_{Q^{(k)}(y)}(x)}{v^k}\right). \tag{88}$$

The validity of (4) is obvious. It is important that each term on the right is an eigenfunction of Δh and the kth term belongs to the eigenspace L_k defined in the following proposition.
Proposition (6.2.1) [209]:(a) The spectrum of Δ_h consists of isolated eigenvalues $\lambda_k = -p^{k-1}, k = 1,2, ..$, each of infinite multiplicity, and their limiting point $\lambda = 0$.
(b) The corresponding eigenspaces $L_k \subset l^2(X)$ have the following structure: For $k = 1$,

$$L_1 = \left\{\psi \in l^2(X): \sum_{x\in Q_i^{(1)}} \psi(x) = 0 \text{ for each } Q_i^{(1)} \in \Pi_1\right\}.$$

For $k > 1$, the space L_k consists of all $\psi \in l^2(x)$ which are constant on each cube $Q_i^{(k-1)}$, and have the property that $\sum_{x\in Q_i^{(1)}} \psi(x) = 0$ for each $Q_i^{(k)} \in \Pi_k$.
(c) The following decomposition holds: $l^2(X) = \bigoplus_{r=1}^{\infty} L_r$.
Indeed, one can easily check that the space L_k, defined above, consists of eigenfunctions with the eigenvalue $\lambda_k = -p^{k-1}$, and for each $y \in X$, the kth term in (4) belongs to L_k. Thus (4) immediately implies (c) which justifies (a).

Let us note that each eigenspace L_k has an orthogonal basis of compactly supported eigenfunctions. Such a basis in L_1 consists of functions which are zero outside of a fixed cube $Q_i^{(1)}$ and such that $\sum_{x \in Q_i^{(1)}} \psi(x) = 0$. There are $\nu - 1$ orthogonal functions with the latter property for each cube $Q_i^{(1)}$. The orthogonal complement of L_1 consists of the functions $\psi \in l^2(X)$ which are constant on each cube of rank one. The basis in L_2 is formed by functions supported by individual cubes of rank two such that $\psi(x) = c_i$ on sub-cubes $Q_i^{(1)}$ of rank one, and $\sum c_i = 0$. One needs to specify c_i to guarantee the orthogonality of the elements of the basis. The basis in L_k, $k > 1$, is formed by functions which are supported by individual cubes of rank k and which are constant on sub-cubes of rank $k - 1$ with the sum of those constants being zero.

Let's find the density of states for Δ_h and the spectral dimension s_h. We fix $x_0 \in X$ (the origin) and a positive integer N. Consider the spectral problem

$$-\Delta_h \psi = \lambda\psi; \qquad \psi \equiv 0 \text{ on } X \setminus Q^{(N)}(x_0).$$

(Now it is more convenient to work with $-\Delta_h$ instead of Δ_h.) It is easy to see (compare to Proposition 6.2.1) that the problem has the following eigenvalues:

$$\lambda_{0,N} = 1 \text{ with multiplicity } v^{N-1}(v-1),$$
$$\lambda_{1,N} = 1 \text{ with multiplicity } v^{N-2}(v-1),$$
$$\vdots$$
$$\lambda_{N-1,N} = p^{N-1} \text{ with multiplicity } (v-1)$$
$$\lambda_{N,N} = p^{N} \text{ with multiplicity } 1.$$

This implies the following relation for

$$\mathcal{N}_N(\lambda) = \frac{1}{v^N} \neq \{\lambda_{i,j} < \lambda\}.$$

Proposition (6.2.2) [209]: As $N \to \infty$,

$$\mathcal{N}_N(\lambda) \to N(\lambda) = \sum_{k \geq 0 : p^k < \lambda} \frac{1}{v^k}\left(1 - \frac{1}{v}\right) = \frac{1}{v^{k_0(\lambda)}},$$

where $k_0(\lambda) = \min\{k \geq 0 : p^k < \lambda\}$. Furthermore,

$$n(\lambda) = \frac{dN(\lambda)}{d\lambda} = \left(1 - \frac{1}{v}\right)\left[\delta_1(\lambda) + \frac{\delta_p(\lambda)}{v} + \frac{\delta_{p^2}(\lambda)}{v^2} + \cdots\right]$$

Proposition (6.2.3) [209]: As $\lambda \downarrow 0$,

$$N(\lambda) \asymp \lambda^{sh/2}, \qquad s_h = \frac{2\ln v}{\ln(1/p)},$$

or, more precisely

$$N(\lambda) \sim \lambda^{sh/2} h\left(\frac{\ln\lambda}{\ln p}\right)$$

for a positive, periodic function $h(z) = v^{-1-\{z\}} \equiv h(z+1)$. Here, $\{z\}$ is the fractional part of a number $z \in \mathbb{R}$. The latter proposition is a consequence of the following simple calculation. If $[z]$ is the integer part of $z \in R$, then

$$N(\lambda) = e^{-k_0(\lambda)\ln v} = e^{-[\frac{\ln\lambda}{\ln p}+1]\ln v} = e^{-\frac{\ln\lambda}{\ln p}\ln v} e^{(-\{\frac{\ln\lambda}{\ln p}\}-1)\ln v} = \lambda^{s_h/2} h\left(\frac{\ln\lambda}{\ln p}\right).$$

We will call the constant $s_h = \frac{2\ln v}{\ln 1/p}$ the spectral dimension of the triple $(X, d_n(.,.), \Delta_h)$.

Let $p(t,x,y) = P_x\{x(t) = y\}$ be the transition function of the hierarchical random walk $x(t)$, i.e.,

$$\frac{\partial p}{\partial t} = \Delta p, \quad p(0,x,y) = \delta_y(x),$$

and let

$$R_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)dt, \qquad \lambda > 0.$$

The functions p and R_λ define the bounded integral operators

$$(P_t f)(x) = \sum_{y\in X} p(t,x,y)f(y),$$

$$(R_\lambda f)(x) = \sum_{y\in X} R_\lambda(t,x,y)f(y),$$

acting in $l^\infty(X)$ and $l^2(X)$, respectively.

Formula (4) (where each term on the rights is an eigenfunction of Δ_h) and the Fourier method lead to the following statement:

Proposition (6.2.4) [209]: The transition kernel $p(t,x,y)$ has the form:

$$p(t,x,x) = \left(1-\frac{1}{v}\right)\left[e^{-t} + \frac{e^{-pt}}{v} + \cdots + \frac{e^{-p^k t}}{v^k} + \cdots\right] \text{for each } x \in X,$$

$$p(t,x,y) = -\frac{e^{p^{r-1}t}}{v^r} + \left(1-\frac{1}{v}\right)\left(\frac{e^{-p^{r-1}t}}{v^r} + \frac{e^{-p^{r+1}t}}{v^{r+1}} + \cdots\right), x \neq y. \quad (89)$$

Here, $r = d_h(x,y)$ is the minimal rank of the cube $Q^{(\cdot)}(x)$, containing the point y (see (1)).

Similar formulas for $R_\lambda(x,y)$ can be obtained from (88) or (easier) from the proposition above (by integration in t):

Proposition (6.2.5) [209]: For any$s_h > 0, \lambda > 0$,

$$R_\lambda(x_0,x) = -\frac{1}{(\lambda + p^{r-1})v^r} + \left(1-\frac{1}{v}\right)\left(\frac{1}{(\lambda + p^r)v^r} + \frac{1}{(\lambda + p^{r+1})v^{r+1}} + \cdots\right),$$

when $r = d_h(x_0,x) > 0$. If $x_0 = x$, then (independent of $x \in X$),

$$R_\lambda(x,x) = \left(1-\frac{1}{v}\right)\left[\frac{1}{\lambda+1} + \frac{1}{(\lambda+p)v} + \cdots + \frac{1}{(\lambda+p^s)v^s} + \cdots\right]. (90)$$

Corollary (6.2.6) [209]: (a) If $pv > 1 (s_h = \frac{2\ln v}{\ln 1/p} > 2)$, then for each $x \in X$.

$$R_0(x,x) = \int_0^\infty p(t,x,x)dt = \left(1-\frac{1}{v}\right)\left(1 + \frac{1}{pv} + \frac{1}{(pv)^2} + \cdots\right) = \frac{p(v-1)}{pv-1} < \infty.$$

If $pv \leq 1$ (i.e., $s_h = \frac{2\ln v}{\ln(1/p)} \leq 2$), then $\lim_{\lambda\to+0} R_\lambda(x,x) = \infty$. Thus the random walk $x(t)$ with the generator Δ_h is transient for $s_h > 2$ and recurrent for $s_h \leq 2$.

(b) If $s_h > 2$ and $\rho(x_0,x) \to \infty$ (see (2)), then

$$R_0(x_0,x) = \left(\frac{1}{p^r v^r} - \frac{1}{p^{r-1}v^r}\right) + \left(\frac{1}{p^{r+1}v^{r+1}} - \frac{1}{p^r v^{r+1}}\right) + \cdots = \frac{1-p}{(pr)^{r-1}(pv-1)} \sim \frac{c}{\rho^{s_h-2}(x_0,x)},$$

$$c = \frac{pv(1-p)}{pv-1}.$$

This is one more indication of a similarity between Δ_h and the lattice $\mathbb{Z}^d$ Laplacian.

Now let's find the asymptotic of $p(t,x,x)$ as $t \to \infty$. The asymptotics will play an essential role in the spectral theory of the Schrödinger operator $H = -\Delta_h + V(x)$.

Proposition (6.2.7) [209]: For arbitrary spectral dimension s_h.

$$p(t,x,x) \asymp \frac{1}{t^{s_h/2}}, \qquad t \to \infty,$$

and there exists a positive periodic function $h_1(z) \equiv h_1(z+1)$ such that

$$p(t,x,x) = \frac{h_1\left(\frac{\ln t}{\ln\left(\frac{1}{p}\right)}\right)}{t^{\frac{s_h}{2}}}\big(1 + o(1)\big) \text{ as } t \to \infty. \qquad (91)$$

Proof. The index of the maximal terms in the series $p(t,x,x) = (1 - \frac{1}{v})\sum_{s=0}^{\infty}\frac{e^{-p^s t}}{v^s}$ has order $s = O(\frac{\ln t}{\ln 1/p})$ when $t \to \infty$. We put $k = [\frac{\ln t}{\ln(1/p)}]$ and change the order of terms in the series representation of p, first taking the sum over $s \geq k$ and then taking the sum over $s < k$:

$$p(t,x,x) = \left(1 - \frac{1}{v}\right)\left(\frac{e^{-p^k t}}{v^k} + \frac{e^{-p^{(k+1)}t}}{v^{k+1}} + \cdots + \frac{e^{-p^{(k-1)}t}}{v^{k-1}} + \cdots\right)$$

$$= \left(1 - \frac{1}{v}\right)\frac{e^{-p^k t}}{v^k}\left[1 + \frac{e^{p^k t(1-p)}}{v} + \frac{e^{p^k t(1-p^2)}}{v^2} + \cdots + \frac{e^{p^k t\left(1-\frac{1}{p}\right)}}{v^{-1}} + \frac{e^{p^k t(1-\frac{1}{p^2})}}{v^{-2}} + \cdots\right]. \qquad (92)$$

The relation $\frac{\ln t}{\ln(1/p)} = k + \{\frac{\ln t}{\ln(1/p)}\}$ implies that

$$p^k t = p^{-\{\frac{\ln t}{\ln(1/p)}\}} \text{ and } \frac{1}{v^k} = e^{-\frac{\ln t}{\ln(1/p)}\ln v} v^{-\{\frac{\ln t}{\ln(1/p)}\}} = \frac{1}{t^{s_h}/2} v^{-\{\frac{\ln t}{\ln(1/p)}\}}.$$

We substitute the latter relations into (8) and note that $\{x\}$ is a periodic function of x with period one.

This and (8) would lead to (7) with zero reminder term if both series in square brackets in (8) had infinitely many terms. Since the second part in the square brackets has only k terms we obtain (7) with an exponentially small reminder.

The next statement provides the asymptotic expansion of $R_\lambda(x,x)$ as $\lambda \to +0$. We restrict ourselves to the more difficult and important case where $s_h < 2$. As in the previous proposition, the main term of the expansion contains a periodic function. We will use an alternative approach to show that:

Proposition (6.2.8) [209]: If $s_h < 2$, then

$$R_\lambda(x,x) = \lambda^{-\alpha} u\left(\frac{\ln\lambda}{\ln p}\right) + c_0 + O(\lambda), \qquad \lambda \to +0, \alpha = 1 - \frac{\ln v}{\ln 1/p} = -\frac{s_h}{2},$$

where $c_0 = \frac{p(v-1)}{pv-1}$ is a constant and $u(z+1) = u(z)$ is a positive periodic function with period one.

Proof. From series representation (6) it follows that

$$R_{p^\lambda} - \frac{1}{pv}R_\lambda = \frac{v-1}{v(p\lambda+1)}.$$

We put $R_\lambda = c_0 + f(\lambda)$. Then

$$f(p\lambda) - \frac{1}{pv} f(\lambda) = \frac{p(1-v)}{v(p\lambda + 1)} \lambda.$$

After the substitution $f(\lambda) = \lambda^{-\alpha} g(\lambda)$ we arrive at

$$g(p\lambda) - g(\lambda) = \zeta(\lambda) = \frac{p^2(1-v)}{p\lambda + 1} \lambda^{1+\alpha}. \quad (93)$$

The estimate $|\zeta(\lambda)| < C|\lambda^{1+\alpha}|, \lambda > 0$, is valid for the function ζ (this estimate was the goal of the subtraction of the constant c_0 from R_λ made above). Hence the series $g_{par} = \sum_0^\infty \zeta(p\lambda), \lambda > 0$, converges, has order $O(\lambda^{1+\alpha})$ as $\lambda \to +0$ and is a partial solution of Eq. (9). Any solution of the homogeneous equation (9) is a periodic function of $\ln_p \lambda = \frac{\ln \lambda}{\ln p}$ with period one. This completes the proof.

Rmark (6.2.9) [209]: The statement of the proposition and its proof remain valid if $\lambda \to 0$ in the complex plane, and $|\arg \lambda| \leq 3\pi/4$.

We conclude this section by defining two functions, $\theta(t)$ and $\varsigma(z)$, which are the analogues of the corresponding classical 1-D functions:

$$\theta(t) = \int_0^\infty e^{-\lambda t} dN(\lambda) = \left(1 - \frac{1}{v}\right)\left[e^{-t} + \frac{e^{-pt}}{v} + \frac{e^{-p^2 t}}{v^2} + \cdots\right],$$

$$\varsigma(z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \theta(t) dt = \left(1 - \frac{1}{v}\right) \sum_{r=0}^\infty \frac{1}{p^{rz} v^r} = \left(1 - \frac{1}{v}\right) \frac{p^z v}{p^z v - 1}.$$

The formula for $\varsigma(z)$ is obtained for $\operatorname{Re} z \in (0, \delta)$ with a small enough $\delta > 0$ ($p^{\operatorname{Re} z} \nu > 1$) and understood in the sense of the analytic continuation for other z. The function ς has no complex zeros, but (compare to [203]) has infinitely many poles at $z = z_n = \frac{s_h}{2} + \frac{i\pi n}{\ln 1/p}$.

The functions $p(t, x, y)$ and $R_\lambda(x, y)$ play a central role in the analysis of the positive spectrum of the hierarchical Schrödinger operator

$$H = \Delta_h + V(x), \quad V \geq 0. \quad (94)$$

With only weak assumptions on V, the positive spectrum $\lambda_n = \lambda_n(H) \geq 0$ of H is discrete (possibly, with accumulation at $\lambda = 0$). Our goals are to find upper bounds on $N_0(V) = \#\{\lambda_n \geq 0\}$ and on the Lieb–Thirring sums $S_\gamma(V) = \sum_n (\lambda_n)^\gamma, \gamma > 0$. Below, we will provide several estimates on N_0 and S_γ which are valid [202, 208] for general discrete operators and for the operator (10) in particular (the case of operators on the Euclidian lattice Z^d can be found in [200]).

Let X be an arbitrary countable set and let H_0 be a bounded self-adjoint operator on $l^2(X)$ given by

$$H_0 \psi(x) = \sum_{y: y \neq x} h(x, y)\big(\psi(y) - \psi(x)\big),$$

$$h(x, y) = h(y, x) \geq 0 \text{ for } x \neq y, \quad \sum_{y: y \neq x} h(x, y) \leq C_0 < \infty.$$

It is clear that $H_0 = H_0^*, H_0 \leq 0, \|H_0\| \leq 2C_0$.

Let $p(t, x, y) = P_x(x(t) = y)$ be the transition kernel of the continuous time Markov chain $x(t)$ generated by H_0. Of course,

$$\frac{\partial p}{\partial t} = H_0 p, \quad p(0, x, y) = \delta_y(x).$$

We assume that $x(t)$ is connected which means, since its time is continuous, that $p(t,x,y) > 0$ for arbitrary $x, y \in X$ and $t > 0$.

The bounds for the eigenvalues of H_0 depend essentially on whether the process $x(t)$ is transient or recurrent. If $\int_0^\infty p(t,x,x)dt < \infty$ for every $x \in X$, then $x(t)$ is transient, i.e., P-a.s., $x(t) \to \infty$ as $t \to \infty$. If $\int_0^\infty p(t,x,x)dt = \infty$ for every $x \in X$, then $x(t)$ visits each state $x \in X$ infinitely many times P-a.s. and the process is called recurrent. It is a well-known fact that, if the chain is connected, the convergence or divergence of $\int_0^\infty p(t,x,x)dt$ is independent of x, y.

Theorem (6.2.10) [209]: (General CLR estimate for discrete operators). If $\int_0^\infty p(t,x,x)dt < \infty$, then for any $a, \sigma > 0$ and some $c_1(\sigma)$,

$$N_0(V) \le \#\{x \in X: V(x) > a\} + c_1(\sigma) \sum_{x:V(x)\le a} V(x) \int_{\frac{\sigma}{V(x)}}^{\infty} p(t,x,x)dt.$$

Theorem (6.2.11) [209]: (LT estimate). If $\int_0^\infty p(t,x,x)dt < \infty$ then

$$S_\gamma(V) \le \frac{1}{c(\sigma)} \sum_{x\in X} V^{1+\gamma}(x) \int_{\frac{\sigma}{V(x)}}^{\infty} p(t,x,x)dt.$$

Theorem (6.2.12) [209]: If $\int_1^\infty t^{-\gamma} p(t,x,x)dt < \infty$ for some $\gamma > 0$, then

$$S_\gamma(V) \le \frac{2\gamma\Gamma(\gamma)}{c(\sigma)} \sum_{x\in X} V(x) \int_{\frac{\sigma}{V(x)}}^{\infty} t^{-\gamma} p(t,x,x)dt.$$

(Note that here, the process $x(t)$ may not be transient.)

The following two results are valid in both transient and recurrent cases. These results are based on the method of partial annihilation, proposed in [202, 208]. In the discrete situation it is equivalent to the rank-one perturbation technique.

Consider, for a fixed $x_0 \in X$, the process $x(t)$ with the condition of annihilation at x_0. The corresponding transition probability $p_1(t,x,y)$ is given by

$$\frac{\partial p_1}{\partial t} = H_0 p_1, \qquad x, y \ne x_0 p_1(t,x_0,y) \equiv 0; \quad p_1(0,x,y) = \delta_y(x). \; (95)$$

As easy to see, $\int_0^\infty p_1(t,x,x)dt < \infty$.

Theorem (6.2.13) [209]: (CLR estimate, the general case). For any $a, \sigma > 0$ and some $c_1(\sigma)$,

$$N_0(V) \le 1 + \#\{x: V(x) > a\} + c_1(\sigma) \sum_{x:V(x)\le a} V(x) \int_{\frac{\sigma}{V(x)}}^{\infty} p_1(t,x,x)dt.$$

Theorem (6.2.14) [209]: (LT estimates, the general case). The following two estimates hold for each $\sigma \ge 0$ and some $c(\sigma) > 0$:

$$S_\gamma(V) \le \Lambda^\gamma + \frac{1}{c(\sigma)} \sum_{X} V^{1+\gamma}(x) \int_{\frac{\sigma}{V(x)}}^{\infty} p_1(t,x,x)dt, \quad (96)$$

$$S_\gamma(V) \le \Lambda^\gamma + \frac{2\gamma\Gamma(\gamma)}{c(\sigma)}\sum_X V(x) \int_{\frac{\sigma}{V(x)}}^{\infty} t^{-\gamma} p_1(t,x,x)dt, \quad (97)$$

Here $\Lambda$is the largest eigenvalue of H.

Remark (6.2.15) [209]:(6.2.13) and (6.2.14) are valid without any assumptions on p_0, i.e., in both transient and recurrent cases.

Note that Theorem (6.2.13) not only covers the recurrent case, but also provides a better results than Theorems (6.2.10), (6.2.11) in the transient case when the operator $H = H^\alpha$ depends on a parameter α which approaches a threshold $\alpha = \alpha_0$, where the process becomes recurrent. In Theorem (6.2.10), (6.2.11) the integrals in t blow up when α approaches α_0 whereas they remain bounded in theorem (6.2.13). A similar remark is valid for Theorem (6.2.14)where the threshold depends on the values of α and γ.

In the case where $\sigma = 0$, [11] contains a more detailed description of the results obtained in Theorem (6.2.10), (6.2.14)

Theorems (6.2.10), (6.2.12) and Proposition (6.2.8), when applied to the operator (10), lead to the same bound on $N_0(V)$ and $S_\gamma(V)$ as in the case of the standard Schrödinger operator in $\mathbb{R}^d$ with the dimension d replaced by the spectral dimension S_h. and essential difference is that, while d must be an integer, the spectral dimension S_h can be an arbitrary positive number. The corresponding bound hold if $s > 2$, where $s = S_h$ in the estimate on $N_0(V)$ and $s = \gamma + \frac{S_h}{2}$ in the estimates on $S_\gamma(V)$. The right-hand sides in these estimates blow up when $s \downarrow 2$ (the integrals in t diverge when $s = 2$). For example, Theorem (6.3.10) with $\sigma = 0$ and Proposition (6.2.8) imply a usual estimate:

$$N_0(V) \le \#\{x \in X: V(x) > a\} + \frac{C(A)}{S_h - 2} \sum_{x:V(x)\le a} V^{S_h/2}(x), \quad 2 < S_h < A.$$

The case $s \le 2$ is covered by Theorems (6.2.13), (6.2.14). In fact, these theorem are valid for any$s > 0$ and the estimate proven there are (locally) uniform in s. Hence they provide a better result in the transient case $s > 2$ than do Theorems (6.2.10), (6.2.12) when $s \downarrow 2$, see [208].

In order to apply Theorems (6.2.13), (6.2.14), one needs to know an estimate on p_1 as $t \to \infty$ and both the annihilation point x_0 and x are arbitrary. If $\sigma = 0$, then only the integral $\int_0^\infty p_1\, dt$ is needed, not p_1 itself. The corresponding results can be found in [208] (we concentrated on $N_0(V)$ in [208], but $S_\gamma(V)$ can be studied similarly). Theorem (6.2.13) with $\sigma = 0$ implies [208] the following Bargmann type result:

$$N_0(V) \le 1 + \#\{x: V(x) \ge 1\} + C_1(S_h) \sum_{x:V(x)<1} V(x)\rho(x_0,x)^{2-S_h} S_h < 2, \quad (98)$$

with $C_1(S_h) \to \infty$ as $S_h \to 2$. A more accurate estimate of $\int_0^\infty p_1 dt$ leads [208] to estimates on $N_0(V)$ for all S_h and with a uniformly bounded constant:

Theorem (6.2.16) [209]: If $\varepsilon < S_h < \varepsilon^{-1}, S_h \ne 2$, then

$$N_0(V) \le 1 + \#\{x: V(x) \ge 1\} + C_2(\varepsilon) \sum_{x:V(x)<1} V(x) \frac{[1 + \rho(x_0,x)]^{2-S_h} - 1}{(\frac{1}{\sqrt{p}})^{2-S_h} - 1}. \quad (99)$$

If $S_h = 2$, then

$$N_0(V) \leq 1 + \#\{x: V(x) \geq 1\} + C_2 \sum_{x:V(x)<1} V(x) \frac{\ln[1 + \rho(x_0, x)]}{\ln \frac{1}{\sqrt{p}}}.$$

We will obtain an estimate for p_1 as $t \to \infty$, which allows one to use Theorems (6.2.13), (6.2.14) with arbitrary$\sigma > 0$. We will restrict ourselves to the case where $S_h < 2$ and provide an estimate only on $N_0(V)$. The following refined Bargmann type estimate is an immediate consequence of Theorem (6.2.13) and Proposition (6.2.19) which will be proven below.

Theorem (6.2.17) [209]: If $S_h < 2$, then

$$N_0(V) \leq 1 + \#\{x: V(x) \geq 1\} + C_1(S_h) \sum_{x:V(x)<1} V^{2-\frac{S_h}{2}}(x)[1 + \rho^2(x_0, x)]^{2-S_h}.$$

We will conclude with a proof of the estimate on p_1 as $t \to \infty$. This estimate is needed to justify the refined Bargmann estimate stated above and to prove similar estimates for S_γ.

Remark (6.3.18) [209]: We expect that, in the case of fractal lattices similar to the Sierpincki lattice, the same estimate will be valid for a random walk with annihilation at a point.

Proposition (6.2.19) [209]: The following estimate is valid.

$$p_1(t, x, x) \leq C \frac{(\rho^2 + 1)^{2\alpha}}{t^{1+\alpha}}, \qquad t \geq 1, \rho = \rho(x_0, x), \alpha = 1 - \frac{S_h}{2}.$$

Proof. Consider the function

$$R_\lambda^{(1)}(x, y) = \int_0^\infty e^{-\lambda t} p_1(t, x, y) dt. \tag{100}$$

It is well defined when $\mathrm{Re}\lambda > 0$ and understood in the sense of analytic continuation for complex $\lambda \in C_+ = \{\lambda \in \mathbb{C}: |\arg\lambda| < 3\pi/4\}$. From (95) it follows that $R_\lambda^{(1)}$ satisfies

$$(\Delta_h - \lambda) R_\lambda^{(1)}(x, y) = -\delta_y(x), \qquad x, y \neq x_0, \qquad R_\lambda^{(1)}(x_0, y) = 0.$$

Hence $R_\lambda^{(1)}(x, y) = R_\lambda(x, y) + cR_\lambda(x, x_0)$, which together with the second relation in the formula above implies that

$$R_\lambda^{(1)}(x, y) = R_\lambda(x, y) - \frac{R_\lambda(x_0, y)}{R_\lambda(x_0, x_0)} R_\lambda(x, x_0).$$

We put here $y = x$ and $R_\lambda(x_0, x) = R_\lambda(x_0, x_0) + \tilde{R}_\lambda(x_0, x)$ where (see Proposition (6.2.5))

$$\tilde{R}_\lambda(x_0, x) = -\frac{1}{(\lambda + p^{r-1})v^r} - \left(1 - \frac{1}{v}\right) \sum_{s=0}^{r-1} \frac{1}{(\lambda + p^s)}, r = d_h(x_0, x) \tag{101}$$

Taking also into account that $R_\lambda(x, x_0) = R_\lambda(x_0, x)$ and $R_\lambda(x, x)$ does not depend on x, we obtain that

$$R_\lambda^{(1)}(x, x) = -2\tilde{R}_\lambda(x_0, x) - \frac{\tilde{R}_\lambda^2(x_0, x)}{R_\lambda(x_0, x_0)}. \tag{102}$$

We not that (101) immediately implies the following two estimates:

$$\left|\tilde{R}_\lambda(x_0, x)\right| \leq \frac{c}{(pv)^r}, \quad \left|\tilde{R}_\lambda(x_0, x) - \tilde{R}_0(x_0, x)\right| \leq \frac{c|\lambda|}{(pv)^r} \text{ for all } \lambda \in C_+, r \geq 0,$$

which together with (18) and the Remark after Proposition (6.2.8) lead to

$$R_\lambda^{(1)}(x, x) = a(r) + g(\lambda, r), a(r) = -\tilde{R}_\lambda(x_0, x), |g| \leq \frac{2c|\lambda|}{(pv)^r} + \frac{c_1|\lambda|^\alpha}{(pv)^{2r}}. \tag{103}$$

The last estimate is valid for all $\lambda \in C_+$ with $|\lambda| < 1$ and all $r \geq 0$.
Applying the inverse Laplace transform to (100) we obtain

$$p_1(t, x, x) = \frac{1}{2\pi} \int_{b-i\infty}^{b+i\infty} e^{\lambda t} R_\lambda^{(1)}(x, x) d\lambda, \qquad b \gg 1.$$

Since $R_\lambda^{(1)}$ is analytic in $\lambda \in C_+$, and $\left|R_\lambda^{(1)}\right| \leq \frac{1}{|\mathrm{Im}\, \lambda|}$ (the resovlent does not exceed the inverse distance from the spectrum), the last integral can be rewritten as

$$p_1(t, x, x) = \frac{1}{2\pi} \int_\Gamma e^{\lambda t} R_\lambda^{(1)}(x, x) d\lambda,$$

where $\Gamma = \partial C_+$ with the direction on Γ such that $\mathrm{Im}\lambda$ increase along Γ. We now use (103), the decay of $R_\lambda^{(1)}$ on Γ at infinity, and the fact that $\int_\Gamma e^{\lambda t} d\lambda = 0, t > 0$. This leads to

$$p_1(t, x, x) \leq \frac{1}{2\pi} |e^{\lambda t}| \left(\frac{2c|\lambda|}{(pv)^r} + \frac{c_1 |\lambda|^\alpha}{(pv)^{2r}} \right) |d\lambda| = \frac{a_1}{t^2 (pv)^r} + \frac{a_2}{t^{1+\alpha} (pv)^{2r}}.$$

It remains to recall that $\alpha = 1 - \frac{\ln v}{\ln 1/p}$ (see Proposition (6.2.8). Thus $pv = p^\alpha$, and $\frac{1}{(pv)^r} = \frac{1}{p^{\alpha r}} = (\rho^2 + 1)^\alpha$.

List of Symbols

Symbol		Page
Sup	Supremum	1
H^S	Sobolev space	1
L^q	Dual Lebasgue space	1
$W^{\alpha,q}$	Sobolev space	1
max	Maximum	2
L^2	Helbert space	5
inf	Infimum	5
L^p	Lebasgue space	8
min	Minimum	14
Sup	Supremum	15
a.e.	Almost every where	22
L^∞	Essential Lebasgue space	24
Loc	Local	25
ker	Kernel	37
van	Vange	37
ess	Essential	38
ac	Absolutely	38
Sc	Singular continuous	38
Au	Auxiliary	40
$\oplus$	Orthogonal sum	43
TPSG	Two- point self- similar fractal	44
deg	Degree	46
int	Interior	50
l^2	Helbert space	56
L^1	Lebasgue space on real line	59
$\otimes$	Tensor produil	59
Cont	Conditionally	64
dist	Distance	69
l^∞	Lebasgue space	70
$F^p_{\alpha,q}$	Triebal- lizorkin-spaces	85

Re	Real	85
meas	Measene	85
det	Determinant	97
dom	Domain	97
comp	complete	99
Gr	Gram	100
gr	Graph	100
σ_p	Point spectrum	100
σ_s	Single Spectrum	105
Const	Constant	111
θ	Direct difference	117
ext	Extension	118
mul	Multi	118
op	Operator	120
Im	Imaginary	137
tr	Trace	145
p.a.s	Probably almost sure	145
r.v	Random variable	153
aff	Affine	162
Par	Parametrize	164
CLR	Cwikel – lieb rozenblum	164
LT	Lieb- Thirring	164

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