Chapter (1)

Exterior Calculus , Lie Algebra and , Lifted Action

Section (1.1): Classical Mechanics , Exterior Calculus and Fermat`s Theorem

We start with some review of Newton , Lagrange's equations and Hamilton. In Newton's law:

$m\ddot{q}=F(q,\dot{q})$ **,** inertial frames , uniform motion , etc

In The Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}=\frac{\partial L}{\partial q} \text{ for Lagrangian } L(q,\dot{q},t) .$$

Define on the tangent bundle TQ of the configuration space Q with coordinates $(q,\dot{q})\in TQ$, the solution is a curve (or trajectory) in Q parametrized by time t.

The tangent vector of the curve $q(t)$ through each point $q\in Q$ is the velocity $\dot{q}$ along the trajectory that passes though the point q at time t .

This vector is written $\dot{q}\in T_q Q$.

Lagrang's equations may be expressed compactly in terms of vector fields and one – forms (differentials). Namely , the Lagrangian vector field $X_L=\dot{q}\frac{\partial}{\partial q}+F(q,\dot{q})\frac{\partial}{\partial \dot{q}}$ acts on the one –form $\left(\frac{\partial L}{\partial \dot{q}}dq\right)$ just as a time- derivative does. To yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}dq\right)=\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)dq+\left(\frac{\partial L}{\partial \dot{q}}\right)d\dot{q}=dL\Rightarrow\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}=\frac{\partial L}{\partial q}$$

In The Hamiltonian $H(p.q)=p\dot{q}-L$ and Hamilton's canonical equations:

$$\dot{q}=\frac{\partial H}{\partial p} , \dot{p}=-\frac{\partial H}{\partial q}$$

The configuration space Q has coordinates $q\in Q$. Its phase space , or cotangent bundle T^*Q has coordinates $(q,p)\in T^*Q$.

Hamilton's canonical equations are associated to the canonical Poisson bracket for functions on phase space , by

$$\dot{p}=\{p,H\}\,,\dot{q}=\{q,H\}\Leftrightarrow \dot{F}(q,p)=\{F,H\}=\frac{\partial F}{\partial q}\frac{\partial H}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial H}{\partial q}$$

The canonical Poisson bracket has the following familiar properties , which may be readily verified:

1- It is bilinear,
2- skew symmetric , $\{F,H\}=-\{H,F\}$,
3- satisfies the Leibniz rule (chain rule) ,

$$\{FG,H\}=\{F,H\}G+F\{G,H\}$$

for the product of any two phase space functions F and G ,

4- and satisfies the Jacobi identity

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0$$

for any three phase space functions F,G and H

Its Leibniz property (chain rule) means the canonical Poisson bracket is a type of derivative. This derivation property of the Poisson bracket allows its use in defining the Hamiltonian vector field X_H ,by

$$X_H=\{.,H\}=\frac{\partial H}{\partial p}\frac{\partial}{\partial q}-\frac{\partial H}{\partial q}\frac{\partial}{\partial p}\quad ,$$

for any phase space function H .The action of X_H on phase space functions is given by

$$\dot{p}=X_{Hp}\quad ,\quad \dot{q}=X_{Hq}\quad \text{and}\quad X_H(FG)=(X_HF)G+F(X_HG)=\dot{F}G+F\dot{G}\quad .$$

Thus ,solutions of Hamilton's canonical equations are the characteristic paths of the first order linear partial differential operator X_H . That is , X_H corresponds to the time – derivative along these characteristic paths, given by

$$dt=\frac{dq}{\partial H/\partial p}=-\frac{dp}{\partial H/\partial q} \tag{1.1}$$

The union of these paths in phase space is called the flow of the Hamiltonian vector field X_H .

Proposition(1.1.1): (Poisson Bracket as Commutator of Hamiltonian Vector Field)

The Poisson bracket $\{F,H\}$ is associated to the commutator of the corresponding Hamiltonian vector fields X_F and X_H by

$$X_{\{F,H\}} = X_H X_F - X_F X_H =: -[X_F, X_H]$$

Corollary (1.1.2) :

Thus , the Jacobi identity for the canonical Poisson bracket $\{.,.\}$ is associated to the Jacobi identity for the commutator $[.,.]$ of the corresponding Hamiltonian vector fields ,

$$[X_F,[X_G,X_H]]+[X_G,[X_H,X_F]]+[X_H,[X_F,X_G]]=0.$$

The differential , or exterior derivative of a function F on phase space is written

$$dF = F_q dq + F_p dp ,$$

In which subscripts denote partial derivatives. For the Hamiltonian itself, the exterior derivative and the canonical equations yield

$$dH = H_q dq + H_p dp = -\dot{p}dq + \dot{q}dp$$

The action of a Hamiltonian vector field X_H on phase space function F commutes with its differential, or exterior derivative. Thus,

$$d(X_H F) = X_H (dF).$$

This means X_H may also acts as a time derivative on differential forms defined on phase space . For example, it acts on the time dependent one- form $pdq(t)$ along solutions of Hamilton's equations as

$$X_H(pdq) = \frac{d}{dt}(pdq) = \dot{p}dq + pd\dot{q}$$

$$= \dot{p}dq - pd\dot{q} + d(p\dot{q})$$

$$= -H_q dq - H_p dp + d\left(p\dot{q}\right)$$

$$= d\left(-H + p\dot{q}\right) = dL\left(q, p\right),$$

Upon substituting Hamilton's canonical equations .

The exterior derivative of the one – form pdq yields the canonical , or symplectic two – form

$$d\left(pdq\right) = dp \wedge dq$$

Here we have used the chain rule for the exterior derivative and its property that $d^2 = 0$. (The latter amounts to equality of cross derivatives for continuous functions .)

The result is written in terms of the wedge product $\wedge$, which combines two one – forms (the line elements dq and dp).

As a result , the two – form $\omega = dq + dp$ representing area in phase space is conserved along the Hamiltonian flows ,

$$X_H\left(dq \wedge dp\right) = \frac{d}{dt}\left(dq \wedge dp\right) = 0$$

This proves .

Theorem (1.1.3) : (Poincare's Theorem)

Hamiltonian flows preserve area in phase space .

Definition (1.1.4) : (Symplectic two – Form)

The phase space area $\omega = dq \wedge dp$ is called the symplectic two – form .

Definition (1.1.5) : (Symplectic Flows)

Flows that preserve area in phase space are said to be symplectic .

Remark (1.1.6) : (Poincare's Theorem)

Hamiltonian flows are symplectic .

Now we discuss the Handout on exterior calculus , Symplectic forms and Poincare's theorem in higher dimensions .

Exterior calculus on symplectic manifolds is the geometric language of Hamiltonian mechanics .

As an introduction and motivation for more detailed study , we begin with a preliminary discussion .

In differential geometry , the operation of contraction denoted as $\rfloor$ introduce a pairing between vector fields and differential forms . Contraction is also called substitution of a vector field into a differential form . For example , there are the dual relation ,

$$\partial q \rfloor dq = 1 = \partial p \rfloor dp \text{ , and } \partial q \rfloor dp = 0 = \partial p \rfloor dq$$

A Hamiltonian vector field

$$X_H = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} = H_P \partial q - H_q \partial p = \{.,H\}$$

Satisfies

$$X_H \rfloor dq = H_P \quad \text{and} \quad X_H \rfloor dp = H_q$$

The rule for contraction or substitution of a vector field into a differential form is to sum the substitutions of X_H over the permulations of the factors in the differential form that bring the corresponding dual basis element into its leftmost position . For example , substitution of the Hamiltonian vector field X_H into the symplectic form $\omega = dq \wedge dp$ yields

$$X_H \rfloor \omega = X_H \rfloor (dq \wedge dp) = (X_H \rfloor dq) dp - (X_H \rfloor dp) dq$$

In this example , $X_H \rfloor dp = -H_q$ and $X_H \rfloor dq = H_p$,so which follows because

$$\partial q \rfloor dp = 0 = \partial p \rfloor dq \quad \text{and} \quad \partial q \rfloor dq = 1 = \partial p \rfloor dp .$$

This calculation proves .

Theorem (1.1.7) : (Hamiltonian Vector Field)

The Hamiltonian vector field $X_H = \{.,H\}$ satisfies

$$X_H \lrcorner\, \omega = dH \quad \text{with} \quad \omega = dq \wedge dp \tag{1.2}$$

Relation (1,2) may be taken as the definition of a Hamiltonian vector field .

As a consequence of this formula , flow of X_H preserve the closed exact two form ω for any Hamiltonian H . This preservation may be verified by a formula calculation using

Along $$\left(\frac{dq}{dt}, \frac{dp}{dt}\right) = (\dot{q}, \dot{p}) = \left(H_p, -H_q\right) \ , \tag{1.3}$$

We have

$$\frac{d\omega}{dt} = d\dot{q} \wedge dp + dq \wedge d\dot{p} = dH_p \wedge dp - dq \wedge dH_q$$

$$= d\left(H_p dp + H_q dq\right)$$

$$= d\left(X_H \lrcorner\, \omega\right)$$

$$= d\left(dH\right) = 0$$

The first step uses the chain rule for differential forms and the third and last steps use the property of the exterior derivative d that $d^2 = 0$ for continuous forms . The latter is due to equality of cross derivatives $H_{pq} = H_{qp}$ and antisymmetry of the wedge product : $dq \wedge dp = -dp \wedge dq$.

Consequently , the relation $d\left(X_H \lrcorner\, \omega\right) = d^2 H = 0$ for Hamiltonian vector fields shows the following .

Theorem (1.1.8) : (Poincare's Theorem for One Degree of Freedom)

The flow of a Hamiltonian vector field is symplectic , which means it preserve the phase space area , or two – form , $\omega = dq \wedge dp$.

Definition (1.1.19) : (Cartan's Formula for The Lie Derivative)

The operation of Lie derivative of a differential form ω by a vector field X_H is defined by

$$\pounds_{X_H}\omega = d\left(X_H \rfloor \omega\right) + X_H \rfloor d\omega \tag{1.3}$$

Corollary (1.1.10) :

Because $d\omega = 0$, the symplectic property $\frac{d\omega}{dt} = d\left(X_H \rfloor \omega\right) = 0$ in Poincare's Theorem (1,1,8) may be rewritten using Lie derivative notation as

$$0 = \frac{d\omega}{dt} = \pounds_{X_H}\omega = d\left(X_H \rfloor \omega\right) + X_H \rfloor d\omega = \left(divX_H\right)\omega \tag{1.4}$$

The last equality defines the divergence of the vector field X_H in terms of the Lie derivative .

Remark (1.1.11) :

1- Relation (1,4) associates Hamiltonian dynamics with the symplectic flow in phase space of the Hamiltonian vector field X_H , which is divergenceless with respect to the symplectic form ω .

2- The Lie derivative operation defined in (1,4) is equivalent to the time derivative along the characteristic paths (flow) of the first order linear partial differential operator X_H , which are obtained from its characteristic equation in (1,1) . This is the dynamical meaning of the Lie derivative $\pounds_{X_H}$ in (1,3) for which invariance $\pounds_{X_H}\omega = 0$ gives the geometric definition of symplectic flows in phase space .

Theorem (1.1.12) : (Poincare's Theorem for N Degrees of Freedom)

For a system of N particles , or N degrees of freedom , the flow of a Hamiltonian vector field preserves each subvolume in the phase space T^*R^N . That is , let $\omega_n = dq_n \wedge dp_n$ be the symplectic form expressed in terms of the position and momentum of the nth particle . Then

$$\frac{d\omega_M}{dt} = 0 \text{ , for } \omega_M = \prod_{n=1}^{M}\omega_n \text{ ,} \quad \forall_M \le N \text{ .}$$

The proof of the preservation of these Poincare's invariants ω_M with $M = 1, 2, 3, ..., N$ follows the same pattern as the verification above for a single degree of freedom .

Basically , this is because each factor $\omega_n = dq_n \wedge dp_n$ in the wedge product of symplectic forms is preserved by its corresponding Hamiltonian flow in the sum

$$X_H = \sum_{n=1}^{M} \left(\dot{q}_n \frac{\partial}{\partial q_n} + \dot{p}_n \frac{\partial}{\partial p} \right) = \sum_{n=1}^{M} \left(H_{p_n} \partial q_n - H_{q_n} \partial p_n \right)$$
$$= \sum_{n=1}^{M} X_{H_n} = \{ ., H \}$$

That is , $£_{X_{H_n}} \omega_M$ vanishes for each term in the sum

$$£_{X_H} \omega_M = \sum_{n=1}^{M} £_{X_{H_n}} \omega_M$$

Since

$$\partial q_m \rfloor dq_n = \delta_{mn} = \partial p_m \rfloor dp_n \quad \text{and} \quad \partial q_m \rfloor dq_n = 0 = \partial p_m \rfloor dp_n \ .$$

Now we discuss the Fermat's theorem in geometrical ray optics .

In geometrical optics , the ray path is determined by Fermat's principle of least optical length ,

$$\delta \int n(x, y, z) ds = 0 .$$

Here $n(x, y, z)$ is index of refraction at the spatial point (x, y, z) and ds is the element of arc length along the ray path through that point . Choosing coordinates ,so that the z-axis coincides with the optical axis (the general direction of propagation) , gives

$$ds = \left[(dx)^2 + (dy)^2 + (dz)^2 \right]^{\frac{1}{2}} = \left[1 + \dot{x}^2 + \dot{y}^2 \right]^{\frac{1}{2}} dz \ ,$$

with $\dot{x} = \frac{dx}{dz}$ and $\dot{y} = \frac{dy}{dz}$ .Thus , Fermat's principle can be written in Lagrangian form , with z playing the rule of time.

$$\delta \int L(x, y, \dot{x}, \dot{y}, z) dz = 0 .$$

Here , the optical Lagrangian is ,

$$L\left(x\,,y\,,\dot{x}\,,\dot{y}\,,z\,\right)=n\left(x\,,y\,,z\,\right)\left[1+\dot{x}^{\,2}+\dot{y}^{\,2}\right]=n\big/\gamma \quad ,$$

or , equivalently , in tow – dimensional vector notation with $q=\left(x\,,y\,\right)$,

$$L\left(q,\dot{q},z\,\right)=n\left(q,z\,\right)\left[1+\left|\dot{q}\right|^2\right]^{\frac{1}{2}}=n\big/\gamma \quad \text{with} \quad \gamma=\left[1+\left|\dot{q}\right|^2\right]^{\frac{-1}{2}}\leq 1 \ .$$

Consequently , the vector Euler – Lagrange equation of the light rays is

$$\frac{d}{ds}\left(n\,\frac{dq}{ds}\right)=\gamma\,\frac{d}{dz}\left(n\gamma\,\frac{dq}{dz}\right)=\frac{\partial n}{\partial q} \ .$$

The momentum p canonically conjugate to the ray path position q in an "image plane" , or on an "image screen" , at a fixed value of z is given by

$$p=\frac{\partial L}{\partial \dot{q}}=n\gamma\dot{q}$$

Which satisfies

$$\left|p\right|^2=n^2\left(1-\gamma^2\right).$$

This implies the velocity $\dot{q}=p\Big/\left(n^2-\left|p\right|^2\right)^{1/2}$.

Hence , the momentum is real – valued and the Lagrangian is hyperregular , provided $n^2-\left|p\right|^2>0$ .When $n^2=\left|p\right|^2$, the ray trajectory is vertical and has grazing incidence with the image screen .

Defining $\sin\theta=dz/ds=\gamma$ leads to $\left|p\right|=n\cos\theta$, and gives the following geometrical picture of the ray path . Along the optical axis (the z - axis) each image plane normal to the axis is pierced at a point $q=\left(x\,,y\,\right)$ by a vector of magnitude $n\left(q,z\,\right)$ tangent to the ray path . This vector makes an angle θ to the plane .

The projection of this vector onto the image plane is the canonical momentum p . This picture of the ray paths captures all but the rays of grazing incidence to the image planes . Such grazing rays are ignored in what follows .

Passing now via the usual Legendre transformation from the Lagrangian to the Hamiltonian description gives

$$\dot{q} = p.\dot{q} - L = n\gamma|\dot{q}|^2 - n/\gamma = -\left[n(q,z)^2 - |p|^2 \right]^{1/2}$$

Thus , in the geometrical picture , the component of the tangent vector of the ray – path along the optical axis is (minus) the Hamiltonian , that is , $n(q,z)\sin\theta = -H$.

The phase space description of the ray path now follows from Hamilton's equations ,

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{-1}{H}p \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} = \frac{-1}{2H}\frac{\partial n^2}{\partial q} \quad .$$

Remark (1.1.13) : (Translation Invariant Media)

If $n = n(q)$, so that the medium is translation invariant along the optical axis , then $H = -n\sin\theta$ is conserved . (conservation of H at an interface is Snell's law.)

For translation – invariant media , the vector ray – path equation simplifies to

$$\ddot{q} = \frac{-1}{2H^2}\frac{\partial n^2}{\partial q} \quad ,$$

Newtonian dynamics for $q \in R^2$.

Thus , in this case geometrical ray tracing reduces to "Newtonian dynamics " in z , with potential $-n^2(q)$ and with " time " rescaled along each path by the value of $\sqrt{2}H$ determined from the initial conditions for each ray .

In axisymmetric , translation invariant media , the index of refraction is a function of the redius alone . Axisymmetry implies an additional constant of motion and , hence , reduction of the Hamiltonian system for the index of refraction satisfies

$$n(q,z) = n(r) \qquad , \qquad r = |q| \ .$$

Passing to polar coordinates (r,ϕ) with $q = (x,y) = r(\cos\phi,\sin\phi)$ leads in the usual way to

$$|p|^2 = p_r^2 + p_\phi^2 / r^2 \ .$$

Consequently , the 0ptical Hamiltonian ;

$$H = -\left[n(r)^2 - p_r^2 - p_\phi^2 / r^2 \right]^{1/2}$$

is independent of the a zimuthal angle ϕ ; so its canonically conjugate " angular momentum " p_ϕ is conserved .

Using the relation $q.p = rp_r$ leads to an interpretation of p_ϕ in terms of the image – screen phase space variables p and q . Namely ,

$$|p \times q|^2 = |p|^2 |q|^2 - (p.q)^2 = p_\phi^2$$

The conserved quantity $p_\phi = p \times q = yp_x - xp_y$ is called the skewness function , or the Petzval invariant for axisymmetric media . Vanishing of p_ϕ occurs for meridional rays , for which p and q are collinear in the image plane .On the other hand , p_ϕ takes its maximum value for sagittal rays , for which $p.q = 0$, so that p and q are orthogonal in the image plane .

Now we discuss the Petzval invariant and its Poisson bracket relations .

The skewness function

$$S = p_\phi = p \times q = yp_x - xp_y$$

Generates rotations of phase space , of q and p jointly , each in its plane , around the optical axis. Its square , S^2 (called the Petzval invariant) is conserved for ray optics in axisymmetric media. That is , $\{S^2, H\} = 0$ for optical Hamiltonians of the form ,

$$H = -\left[n\left(|q|^2\right)^2 - |p|^2 \right]^{1/2}$$

We define the axisymmetric invariant coordinates by the map $T^*R^2 \mapsto R^3$, $(q, p) \mapsto (X, Y, Z)$, $X = |q|^2 \geq 0$, $Y = |p|^2 \geq 0$, $Z = p.q$.

The following Poisson bracket relations hold

$$\{S^2, X\} = 0 \ , \ \{S^2, Y\} = 0 \ , \ \{S^2, Z\} = 0,$$

Since rotations preserve dot products. In terms of these invariant coordinates , the Petzval invariant and optical Hamiltonian satisfy

$$S^2 = XY - Z^2 \geq 0 \ , \quad \text{and} \quad H^2 = n^2(X) - Y \geq 0 \ .$$

The level sets of S^2 are hyperboloids of revolution around the $X = Y$ axis , extending up through the interior of the $S = 0$ cone , and lying between the X and Y - axis . The level sets of H^2 depend on the functional form of the index of refraction , but they are Z - independent .

Now we discuss the R^3 Poisson bracket for ray optics .

The Poisson brackets among the axisymmetric variables X, Y and Z close among themselves ,

$$\{X, Y\} = 4Z \quad , \quad \{Y, Z\} = -2Y \quad , \quad \{Z, X\} = -2X$$

The Poisson brackets derive from a single R^2 Poisson bracket for $X = (x, y, z)$ given by

$$\{F, H\} = -\nabla S^2 . \nabla F \times \nabla H$$

Consequently , we may re – express the equations of Hamiltonian ray optics in axisymmetric media with $H = H(X, Y)$ as

$$\dot{X} = \nabla S^2 \times \nabla H \ .$$

With Casimir S^2 , for which $\{S^2, H\} = 0$, for every H . Thus , the flow preserves volume $(div\dot{X} = 0)$ and the evolution takes place on intersections of level surfaces of the axisymmetric media invariants S^2 and $H(X, Y)$.

Now we discuss the Recognition of the Lie – Poisson bracket for ray optics .

The Casimir invariant $S^2 = XY - Z^2$ is quadratic . In such cases , one may write the R^3 Poisson bracket in the suggestive form

$$\{F, H\} = -C_{ij}^{k} X_k \frac{\partial F}{\partial X_i} \frac{\partial H}{\partial X_j} \ .$$

In this particular case , $C_{12}^{3}=4$, $C_{23}^{2}=2$ and $C_{31}^{1}=2$, and the rest either vanish , or are obtained from antisymmetry of C_{ij}^{k} under exchange of any pair of its indices.

These values are the structure constants of any of the Lie algebras $sp(2,R)$, $so(2,1)$, $su(1,1)$, or $sl(2,R)$. Thus , the reduced description of Hamiltonian ray optics in terms of axisymmetric R^3 variables is said to be " Lie – Poisson " on the dual space of any of these Lie algebras , say , $sp(2,R)^*$ for definiteness . We will have more to say about Lie – Poisson brackets later , when we reach the Euler – Poincare reduction theorem .

Remark (1.1.14) : (Coadjoint Orbits)

As one might expect , the coadjoint orbits of the group $SP(2,R)$ are the hyperboloids corresponding to the level sets of S^2 .

Remark (1.1.15) :

As we shall see later , the map $T^*R^2 \mapsto sp(2,R)^*$ taking $(q,p) \mapsto (X,Y,Z)$ is an example of a momentum map .

Now we discuss the Geometrical structure of classical mechanics .

Configuration space : coordinates $q \in M$, where M is a smooth manifold . The composition $\phi_\beta o \phi_\alpha^{-1}$ is a smooth change of variables .

For later , smooth coordinate transformations :

$$q \to Q \quad \text{with} \quad dQ = \frac{\partial Q}{\partial q} dq \ .$$

Definition (1.1.16) :

A smooth manifold M is a set of points together with a finite (or perhaps countable) set of subsets $U_\alpha \subset M$ and one – to – one mappings $\phi_\alpha : U_\alpha \to R^n$ such that

1- $\bigcup_\alpha U_\alpha = M$

2- For every nonempty intersection $U_\alpha \cap U_\beta$, the set $\phi_\alpha\left(U_\alpha \cap U_\beta\right)$ is an open subset of R^n and the one – to – one mapping $\phi_\beta o \phi_\alpha^{-1}$ is a smooth function on $\phi_\alpha\left(U_\alpha \cap U_\beta\right)$.

Remark (1.1.17) :

The sets U_α in the definition are called coordinate charts . The mappings ϕ_α are called coordinate functions or local coordinates . A collection of charts satisfying 1 and 2 is called an atlas . condition 3 allows the definition of manifold to be made independently of a choice of atlas .A set of charts satisfying 1 and 2 can always be extended to a maximal set ; so , in practice , conditions 1 and 2 define the manifold.

Example (1.1.18) :

Manifolds often arise as intersections of zero level sets
$M = \left\{x : f_i\left(x\right) = 0, i = 1, \ldots\ldots\ldots\ldots, k\right\}$

For a given set of functions $f_i : R^n \to R$, $i = 1, \ldots\ldots\ldots, k$.

If the gradients ∇f_i are linearly independent , or more generally if the rank of $\left\{\nabla f\ \left(x\right)\right\}$ is a constant r for all x , then M is a smooth manifold of dimension $n - r$.

The proof uses the Implicit Function Theorem to show that an $\left(n - r\right)$ - dimensional coordinate chart may be defined in a neighbourhood of each point on M . In this situation , the set M is called a submanifold of R^n .

Definition (1.1.19) :

If $r = k$, then the map $\{f_i\}$ is called a submersion .

Definition (1.1.20) : (Tangent Space to Level Sets)

Let $M = \left\{x : f_i\left(x\right) = 0, i = 1, \ldots\ldots\ldots, k\right\}$ be a manifold in R^n . The tangent space at each $x \in M$, is defined by

$$T_x M = \left\{v \in R^n : \frac{\partial f_i}{\partial x^\alpha}\left(x\right) v^\alpha = 0, i = 1, \ldots\ldots\ldots, k\right\}.$$

Note : we use the summation convention , that is , repeated indices are summed over their range .

Remark (1.1.21):

The tangent space is a linear vector space .

Example (1.1.22) : (Tangent Space to The Sphere in R^3)

The sphere S^2 is the set of points $(x,y,z)\in R^3$ solving $x^2+y^2+z^2=1$. The tangent space to the sphere at such a point (x,y,z) is the plane containing vectors (u,v,w) satisfying $xu+yv+zw=0$.

Definition (1.1.23) : (Tangent Bundle)

The tangent bundle of a manifold M , denoted by TM , is the smooth manifold whose underlying set is the disjoint of the tangent spaces to M at the points $x\in M$; that is ,

$$TM=\bigcup_{x\in M} T_x M$$

Thus , a point of TM is a vector v which is tangent to M at some point $x\in M$.

Example (1.1.24) : (Tangent Bundle TS^2 of S^2)

The tangent bundle TS^2 of $S^2\in R^3$ is the union of the tangent space of S^2 :

$$TS^2=\{(x,y,z;u,v,w)\in R^6 : x^2+y^2+z^2=1 \, and \; xu+yv+zw=0\}.$$

Remark (1.1.25) : (Dimension of Tangent Bundle TS^2)

Defining TS^2 requires two independent conditions in R^6 ; so $\dim TS^2=4$.

Example (1.1.26): (The Two Stereographic Projections of $S^2\to R^2$)

The unit sphere $S^2=\{(x,y,z):x^2+y^2+z^2=1\}$ is a smooth two – dimensional manifold realized as a submersion in R^3 .

Let $U_N=S^2\setminus\{0,0,1\}$, and $U_S=S^2\setminus\{0,0,-1\}$ be the subsets obtained by deleting the North and South poles of S^2 , respectively .

Let $\chi_N : U_N \to (\xi_N, \eta_N) \in R^2$ and $\chi_S : U_S \to (\xi_S, \eta_S) \in R^2$ be stereographic projections from the North and South poles onto the equatorial plane , $z = 0$. Thus , one may place two different coordinate patches in S^2 intersecting everywhere except at the points along the z – axis at $z = 1$ (North pole) and $z = -1$ (South pole).

In the equatorial plane $z = 0$, one may define two sets of (right – handed) coordinates .

$$\phi_\alpha : U_\alpha \to R^2 \setminus \{0\} \quad , \quad \alpha = N, S$$

obtained by the following two stereographic projections from the North and South poles:

1- (Valid everywhere except $z = 1$)

$$\phi_N(x, y, z) = (\xi_N, \eta_N) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$

2- (Valid everywhere except $z = -1$)

$$\phi_S(x, y, z) = (\xi_S, \eta_S) = \left(\frac{x}{1+z}, \frac{-y}{1+z}\right).$$

(The two complex planes are identified differently with the plane $z = 0$. An orientation – reversal is necessary to maintain consistent coordinates on the sphere .) One may check directly that on the overlap $U_N \cap U_S$ the map ,

$$\phi_N o \phi_S^{-1} : R^2 \setminus \{0\} \to R^2 \setminus \{0\}$$

is a smooth diffeomorphism , given by the inversion

$$\phi_N o \phi_S^{-1}(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right).$$

Answer (1.1.27):

$$(\xi S, \eta S) = \frac{1-z}{1+z}(\xi N, \eta N) = \frac{1}{\xi^2 N + \eta^2 N}(\xi N, \eta N).$$

The map $(\xi N, \eta N) \to (\xi S, \eta S)$ is smooth and invertible at $(\xi N, \eta N) = (0,0)$.

Example (1.1.28):

If we start with two identical circles in the $xz-$ plane , of redius r and centeral at $x = \pm 2r$, then rotate them round the z – axis in R^3 , we get a torus , written T^2 . It's a manifold .

Answer (1.1.29):

The origin has a neighbourhood diffeomorphic to a double cone . This is not diffeomorphic to R^2 .

Remark (1.1.30):

The sphere will appear in several examples as a reduced space in which motion takes place after applying a symmetry . Reduction by symmetry is associated with a classical topic in celestial mechanics known as normal form theory . Reduction may be " singular" in which case it leads to " pointed " spaces . For example different resonances of coupled spaces : 1:1 resonance – sphere ; 1:2 resonance – pinched sphere with one cone point ; 1:3 resonance – pinched sphere with one cusp point ; 2:3 resonance – pinched sphere with one cone point and one cusp point .

Now we discuss the tangent vectors and flows .

Envisioning our later considerations of dynamical systems , we shall consider motion along curves $c(t)$ parametrized by time t on a smooth manifold M .

Suppose these curves are trajectories of a flow ϕ_t of a vector field . We anticipate this means $\phi_t(c(0)) = c(t)$ and $\phi_t o \phi_s = \phi_{t+s}$ (flow property). The flow will be tangent to M along the curve . To deal with such flows , we will need the concept of tangent vectors .

Recall from Definition $(1,1,23)$ that the tangent bundle of M is

$$TM = \bigcup_{x \in M} T_x M \ .$$

We will now add a bit more to that definition . The tangent bundle is an example of a more general structure than a manifold .

Definition (1.1.31): (Bundle)

A bundle consists of a manifold B , another manifold M called the " base space " and a projection between them $\Pi : B \to M$. Locally , in small enough regions of x the inverse images of the projection Π exists . These are called the fibers of the bundle . Thus , subsets of the bundle B locally have the structure of a Cartesian product . An example is (B, M, Π) consisting of $\left(R^2, R^1, \Pi : R^2 \to R^1\right)$. In this case , $\Pi : (x, y) \in R^2 \to x \in R^1$. Likewise , the tangent bundle consists of M , TM and a map $\tau_M : TM \to M$.

Let $x = \left(x^1, \ldots\ldots, x^n\right)$ be local coordinates on M , and let $v = \left(v^1, \ldots\ldots\ldots, v^n\right)$ be components of a tangent vector .

$$T_x M = \left\{ v \in R^n : \frac{\partial f_i}{\partial x} . v = 0, i = 1, \ldots\ldots\ldots, m \right\}$$

for

$$M = \left\{ x \in R^n : f_i(x) = 0, i = 1, \ldots\ldots\ldots\ldots\ldots, m \right\}$$

These $2n$ numbers (x, v) give local coordinates on TM , where $\dim TM = 2 \dim M$. The tangent bundle projection is a map $\tau_M : TM \to M$ which takes a tangent vector v to a point $x \in M$ where the tangent vector v is attached (that is , $v \in T_x M$) . The inverse of this projection $\tau_M^{-1}(x)$ is called the fiber over x in the tangent bundle .

Now we discuss the vector fields , integral curves and flows .

Definition (1.1.32):

A vector field on a manifold M is a map $X : M \to TM$ that assigns a vector $X(x)$ at each point $x \in M$. This implies that $\tau_M o X = Id$.

Definition (1.1.33):

An integral curve of X with initial conditions x_0 at $t = 0$ is a differentiable map $c :]a, b[\to M$, where $]a, b[$ is an open interval containing 0, such that $c(0) = 0$ and $c'(t) = X(c(t))$ for all $t \in]a, b[$.

Remark (1.1.34):

A standard result from the theory of ordinary differential equations states that X being Lipschitz implies its integral curves are unique and c' . The integral curves $c(t)$ are differentiable for smooth X .

Definition (1.1.35):

The flow of X is the collection of maps $\phi_t : M \to M$, where $t \to \phi_t(x)$ is the integral curve of X with initial condition x .

Remark (1.1.36):

1- Existence and uniqueness results for solution of $c'(t) = X(c(t))$ guarantee that flow ϕ of X is smooth in (x,t) , for smooth X .

2- Uniqueness implies the flow property

$$\phi_{t+s} = \phi_t o \phi_s \qquad (1.5)$$

for initial condition $\phi_0 = Id$.

3- The flow property (FP)(1,5) generalized to the nonlinear case the familiar linear situation where M is a vector space , $X(x) = Ax$ is a linear vector field for a bounded linear operator A , and $\phi_t(x) = e^{At}x$.

We are now ready to define differentials of smooth functions and the cotangent bundle .

Let $f : M \to R$ be a smooth function . We differentiate f at $x \in M$ to obtain $T_x f : T_x M \to T_{f(x)} R$. As is standard , we identify $T_{f(x)}R$ with R itself , thereby obtaining a linear map $df(x) : T_x M \to R$. The result $df(x)$ is an element of the cotangent space $T_x^* M$, the dual space of the tangent space $T_x M$. The natural pairing between elements of the tangent space and the cotangent space is denoted as $\langle .,. \rangle : T_x^* M \times T_x M \mapsto R$.

In coordinates , the linear map $df(x) : T_x M \to R$ may be written as the directional derivative ,

$$\langle df(x), v \rangle = df(x).v = \frac{\partial f}{\partial x^i}.v^i ,$$

for all $v \in T_x M$.

(Reminder : the summation convention is intended over repeated indices.)

Hence, elements $df(x) \in T_x^* M$ are dual to vectors $v \in T_x M$ with respect to the pairing $\langle .,. \rangle$.

Definition (1.1.37):

df is the differential of the function f .

Definition (1.1.38):

The dual space of the tangent bundle TM is the cotangent bundle T^*M , that is ,

$$(T_x M)^* = T_x^* M \quad \text{and} \quad T^*M = \bigcup_x T_x^* M$$

Thus , replacing $v \in T_x M$ with $df \in T_x^* M$, for all $x \in M$ and for all smooth functions $f : M \to R$, yields the cotangent bundle T^*M . When the basis of vector fields is denoted as $\frac{\partial}{\partial x^i}$, for $i = 1, \ldots\ldots\ldots, n$, its dual basis is often denoted as dx^i . In this notation , the differential of a function at a point $x \in M$ is expressed as

$$df(x) = \frac{\partial f}{\partial x^i} dx^i$$

The corresponding pairing $\langle .,. \rangle$ of bases is written in this notation as

$$\left\langle dx^j, \frac{\partial}{\partial x^i} \right\rangle = \delta_i^j$$

Here δ_i^j is the Kronecker delta , which equals unity for $i = j$ and vanishes otherwise . That is , defining T^*M requires a pairing $\langle .,. \rangle : T^*M \times TM \to R$. (Different pairing exist for curvilinear coordinates , Riemannian manifolds . etc.).

Section (1.2):The Tangent Lift, Lie Algebra and Lifted Actions

We start by discussing of the tangent lift .

We next define derivatives of differentiable maps between manifolds (tangent lifts).

We expect that a smooth map $f : U \to V$ from a chart $U \subset M$ to a chart $V \subset N$, will lift to a map between the tangent bundles TM and TN so as to make sense from the viewpoint of ordinary calculus ,

$$U \times R^m \subset TM \to V \times R^n \subset TN$$

$$\left(q^1,........,q^m;X^1,.........,X^m\right) \mapsto \left(Q^1,.........,Q^n;Y^1,...........,Y^n\right)$$

Namely , the relations between the vector field components should be obtained from the differential of the map $f : U \to V$. Perhaps not unexpectedly , these vector field components will be related by

$$Y^i \frac{\partial}{\partial Q^i} = X^j \frac{\partial}{\partial q^j} \quad , \quad \text{so } Y^i = \frac{\partial Q^i}{\partial q^j} X^j$$

in which the quantity called the tangent lift

$$Tf = \frac{\partial Q}{\partial q}$$

of the function f arises from the chain rule is equal to the Jacobian for the transformation $Tf : TM \mapsto TN$. The dual of the tangent lift is the cotangent lift .

Roughly speaking , the cotangent lift of the function f ,

$$T^* f = \frac{\partial q}{\partial Q}$$

arises from

$$\beta_i dQ^i = \alpha_j dq^j \quad , \text{ so } \quad \beta_i = \alpha_j \frac{\partial q^j}{\partial Q^i}$$

and $T^* f : T^* N \mapsto T^* M$. Note the directions of these maps :

$$Tf : q, X \in TM \mapsto Q, Y \in TN$$

$$f : q \in M \mapsto Q \in N$$

$$T^* f : Q, \beta \in T^* N \mapsto q, \alpha \in T^* M \qquad \text{(map goes the other way)}$$

Definition (1.2.1): (Differentiable Map)

A map $f : M \to N$ from manifold M to manifold N is said to be differentiable (resp.C^k) if it is represented in local coordinates on M and N by differentiable (resp.C^k) functions .

Definition (1.2.2): (Derivative of a Differentiable Map)

The derivative of a differentiable map $f : M \to N$ at a point $x \in M$ is defined to be the linear map

$$T_x f : T_x M \to T_x N$$

constructed , as follows . For $v \in T_x M$, choose a curve $c(t)$ that maps an open interval $t \in (-\varepsilon, \varepsilon)$ around the point $t = 0$ to the manifold M .

$$c : (-\varepsilon, \varepsilon) \to M$$

with $c(0) = x$ and velocity vector $c'(o) := \frac{dc}{dt}|_{t=0} = v$ .Then $T_x f .v$ is the velocity vector at $t = 0$ of the curve $f\, o c : R \to N$ .That is ,

$$T_x f .v = \frac{d}{dt} f\left(c(t)\right)\Big|_{t=0} = \frac{\partial f}{\partial c} \frac{d}{dt} c(t)\Big|_{t=0}$$

Definition (1.2.3):

The union $Tf = \bigcup_x T_x f$ of the derivatives $T_x f : T_x M \to T_x N$ over points $x \in M$ is called the tangent lift of the map $f : M \to N$.

Remark (1.2.4):

The chain – rule definition of the derivative $T_x f$ of a differentiable map at a point x depends on the function f and the vector v . Other degrees of differentiability are possible . For example , if M and N are manifolds and $f : M \to N$ is of class C^{k+1} , then the tangent lift (Jacobian) $T_x f : T_x M \to T_x N$ is C^k .

Now we discuss the Lie groups and Lie algebras .

Definition (1.2.5):

A group is a set of elements with :

1- A binary product (multiplication) , $G \times G \to G$, such that
   - the product of g and h is written gh , and
   - the product is associative : $(gh)k = g(hk)$.
2- An identity element e such that $eg = g$ and $ge = g$, $\forall_g \in G$
3- An inverse operation $G \to G$, such that $gg^{-1} = g^{-1}g = e$

Definition (1.2.6):

A Lie group is a smooth manifold G which is a group and for which the group operations of multiplication , $(g,h) \to gh$, for $h \in G$, and inversion , $g \to g^{-1}$ with $gg^{-1} = g^{-1}g = e$, are smooth .

Definition (1.2.7):

A matrix Lie group is a set of invertible $n \times n$ matrices which is closed under matrix multiplication and which is submanifold of $R^{n \times n}$.

The conditions showing that a matrix Lie group is a Lie group are easily checked :

1- A matrix Lie group is a manifold , because it is a submanifold of $R^{n\times n}$
2- Its group operations are smooth , since they are algebraic operations on the matrix entries .

Example (1.2.8): (The General Linear Group $GL(n,R)$)

The matrix Lie group $GL(n,R)$ is the group of linear isomorphisms of R^n to itself . The dimension of the matrices in $GL(n,R)$ is n^2 .

Proposition (1.2.9):

Let $K \in GL(n,R)$ be a symmetric matrix , $K^T = K$. Then the subgroup S of $GL(n,R)$ defined by the mapping

$$S = \{U \in GL(n,R) : U^T K U = K\}$$

is a submanifold of $R^{n\times n}$ of dimension $n(n-1)/2$.

Remark (1.2.10):

The subgroup S leaves invariant a certain symmetric quadratic form under linear transformations , $S \times R^n \to R^n$ given by $x \to U_x$, since

$$x^T Kx = x^T U^T KU_x \ .$$

So the matrices $U \in S$ change the basis for this quadratic form , but they leave its value unchanged . Thus , S is the isotropy subgroup of the quadratic form associated with K .

Proof:

(i) Is S a subgroup ? We check the following three defining properties

1 – Identity : $I \in S$ because $I^T KI = K$.

2 – Inverse : $U \in S \Rightarrow U^{-1} \in S$,because

$$K = U^{-T}\left(U^T KU\right)U^{-1} = U^{-T}\left(K\right)U^{-1}$$

3 – Closed under multiplication : $U,V \in S \Rightarrow UV \in S$, because

$$\left(UV\right)^T KUV = V^T\left(U^T KU\right)V = V^T\left(K\right)V = K$$

(ii) Hence , S is a subgroup of $GL\left(n,R\right)$.

(iii) Now one come ask S is a submanifold of $R^{n\times n}$ of dimension $n\left(n-1\right)/2$.

Indeed , S is the zero locus of the mapping $UKU^T - K$ .This makes it a submanifold , because it turns out to be a submersion .
For a submersion , the dimension of the level set is the dimension of the domain minus the dimension of the range space . In this case , this dimension is

$$n^2 - n\left(n+1\right)/2 = n\left(n-1\right)/2 \ .$$

Example (1.2.11):

Explain why one can conclude that the zero locus map for S is a submersion . In particular , pay close attention to establishing the constant rank condition for the linearization of this map .

Solution:

Here is why is a submanifold of $R^{n\times n}$.

First , S is the zero locus of the mapping

$$U \to U^T KU - K \quad , \quad \text{(locus map)}$$

Let $U \in S$, and let δU be an arbitrary element of $R^{n\times n}$ .Then linearize to find

$$(U+\delta U)^T K(U+\delta U)-K = U^T KU - K + \delta U^T KU + U^T K\delta U + O(\delta U)^2 \ .$$

We may conclude that S is a submanifold of $R^{n\times n}$ if we can show that the linearization of the locus map , namely the linear mapping defined by

$$L \equiv \delta U \to \delta U^T KU + U^T K\delta U \quad , \quad R^{n\times n} \to R^{n\times n}$$

has constant rank for all $U \in S$.

Lemma (1.2.12):

The linearization map L is onto the space of $n\times n$ of symmetric matrices and hence the original map is a submersion .

Proof:

That L is onto:

1- Both the original locus map and the image of L lie in the subspace of $n\times n$ symmetric matrices .

2- Indeed , given U and any symmetric matrix S we can find δU such that

$$\delta U^T KU + U^T K\delta U = S \ .$$

Namely

$$\delta U = K^{-1}U^{-T}S/2 \ .$$

3- Thus , the linearization map L is onto the space of $n\times n$ of symmetric matrices and the original locus map $U \to UKU^T$ to the space of symmetric matrices is a submersion .

For a submersion , the dimension of the level set is the dimension of the domain minus the dimension of the range space . In this case , this dimension is $n^2 - n(n+1)/2 = n(n-1)/2$.

Corollary (1.2.13): (S is A Matrix Lie Group)

S is both a subgroup and a submanifold of the general linear group $GL(n,R)$. Thus , by Definition (1,2,3) , S is a matrix Lie group .

Proposition (1.2.14):

The linear space of matrices A satisfying

$$A^T K + KA = 0$$

defines $T_I S$, the tangent space at the identity of the matrix Lie group S defined in Proposition (1,2,9) .

Proof:

Near the identity the defining condition for S expands to

$$\left(I + \varepsilon A^T + O\left(\varepsilon^2\right)\right)K\left(I + \varepsilon A + O\left(\varepsilon^2\right)\right) = K \quad , \quad \text{for } \varepsilon << 1.$$

At linear order $O(\varepsilon)$ one finds ,

$$A^T K + KA = 0 \ .$$

This relation defines the linear space of matrices $A \in T_I S$.

If $A, B \in T_I S$, it follow that $[A,B] \in T_I S$. Using $[A,B]^T = \left[B^T, A^T\right]$, we check closure by a direct computation ,

$$\left[B^T, A^T\right]K + K[A,B] = B^T A^T K - A^T B^T K + KAB - KBA = B^T A^T K - A^T B^T K - A^T KB + B^T KA = 0$$

Hence , the tangent space of S at the identity $T_I S$ is closed under the matrix commutator $[.,.]$.

Remark (1.2.15):

In a moment , we will show that the matrix commutator for $T_I S$ also satisfies the Jacobi identity . This imply that the condition $A^T K + KA = 0$ defines a matrix Lie algebra .

We are ready to prove the following , in preparation for defining matrix Lie algebras .

Proposition (1.2.16):

Let S be a matrix Lie group , and let $A,B \in T_I S$ (the tangent space to S at the identity element). Then

$$AB - BA \in T_I S \text{ .}$$

Proof:

Let $R_A(s)$ be a curve in S such that $R_A(0) = I$ and $R_A'(0) = A$. Define $S(t) = R_A(t) B R_A(t)^{-1} \in T_I S$. Then $S(t) \in T_I S$ for every t . Hence , $S'(t) \in T_I S$, and in particular , $S'(0) = AB - BA \in T_I S$.

Lemma (1.2.17):

Let R be an arbitrary element of a matrix Lie group S , and let $B \in T_I S$. Then $RBR^{-1} \in T_I S$.

Proof:

Let $R_B(t)$ be a curve in S such that $R_B(0) = I$ and $R'(0) = B$. Define $S(t) = R R_B(t) R^{-1} \in T_I S$ for all t . Then $S(0) = I$ and $S'(0) = RBR^{-1}$. Hence , $S'(0) \in T_I S$, thereby proving the lemma .

Definition (1.2.18): (Matrix Commutator)

For any pair of $n \times n$ matrices A,B , the matrix commutator is defined as $[A,B] = AB - BA$.

Proposition (1.2.19): (Properties of The Matrix Commutator)

The matrix commutator has the following two properties :

(i) Any two $n \times n$ matrices A,B satisfy

$$[B,A] = -[A,B]$$

(This is the property of skew – symmetry .)

(ii) Any three $n \times n$ matrices A,B and C satisfy

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$$

(This is known as the Jacobi identity .)

Definition (1.2.20): (Matrix Lie Algebra)

A matrix Lie algebra ϑ is a set of $n \times n$ matrices which is a vector space with respect to the usual operations of matrix addition and multiplication by real numbers (scalars) and which is closed under the matrix commutator $[.,.]$.

Proposition (1.2.21):

For any matrix Lie group S , the tangent space at the identity $T_I S$ is a matrix Lie algebra.

Proof:

This follows by Proposition(1,2,16) and because $T_I S$ is a vector space .

Example (1.2.22): (The Orthogonal Group $O(n)$)

The mapping condition $U^T KU = K$ in Proposition(1,2,9) specializes for $K = I$ to $U^T U = I$, which defines the orthogonal group . Thus , in this case , S specializes to $O(n)$, the group of $n \times n$ orthogonal matrices .The orthogonal group is of special interest in mechanics .

Corollary (1.2.23): ($O(n)$ is A Matrix Lie Group)

By Proposition (1,2,9) the orthogonal group $O(n)$ is both a subgroup and a submanifold of the general linear group $GL(n,R)$. Thus , by Definition (1,2,7) the orthogonal group $O(n)$ is a matrix Lie group .

Example (1.2.24): (The Special Linear Group $SL(n,R)$)

The subgroup of $GL(n,R)$ with $\det(U)=1$ is called $SL(n,R)$.

Example (1.2.25): (The Special Orthogonal Group $SO(n,R)$)

The special case of S with $\det(U)=1$ and $K = I$ is called $SO(n)$. In this case , the mapping condition $U^T KU = K$ specializes to $U^T U = I$ with the extra condition $\det(U)=1$.

Example (1.2.26): (The Tangent Space of $SO(n)$ at The Identity $T_I SO(n)$)

The special case with $K = I$ of $T_I SO(n)$ yields,

$$A^T + A = 0 \ .$$

These are antisymmetric matrices . Lying in the tangent space at the identity of a matrix Lie group , this linear vector space forms a matrix Lie algebra .

Example (1.2.27): (The Symplectic Group)

Suppose $n = 2l$ (that is , let n be even) and consider the nonsingular skew – symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the $l \times l$ identity matrix . One may verify that

$$Sp(l) = \{U \in GL(2l, R) : U^T JU = J\}$$

Is a group . This is called the symplectic group . Reasoning as before , the matrix algebra A satisfying $JA^T + AJ = 0$. This algebra is denoted as $sp(l)$.

Example (1.2.28): (The Special Euclidean Group)

Consider the Lie group of 4×4 matrices of the form

$$E(R, v) = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}$$

where $R \in SO(3)$ and $v \in R^3$. This is the special Euclidean group , denoted $SE(3)$. The special Euclidean group is of central interest in mechanics since it describes the set of rigid motions and coordinate transformations of three – dimensional space .

The action of a Lie group G on a manifold M is a group of transformations of M associated to elements of the group G , whose composition acting on M is corresponds to group multiplication in G .

Definition (1.2.29):

Let M be a manifold and let G be a Lie group . A left action of a Lie group G on M is a smooth mapping $\Phi : G \times M \to M$ such that

(i) $\Phi(e, x) = x$ for all $x \in M$,

(ii) $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ for all $g, h \in G$ and $x \in M$, and

(iii) $\Phi(g, .)$ is a diffeomorphism on M for each $g \in G$.

We often use the convenient notation gx for $\Phi(g, x)$ and think of the group element g acting on the point $x \in M$. The associativity condition (ii) above then simply reads $(gh)x = g(hx)$.

Similarly , one can define a right action , which is a map $\Psi : M \times G \to M$ satisfying $\Psi(x, e) = x$ and $\Psi(\Psi(x, g), h) = \Psi(x, gh)$. The convenient notation for right action is xg for $\Psi(x, g)$, the right action of a group element g on the point $x \in M$. Associativity $\Psi(\Psi(x, g), h) = \Psi(x, gh)$ is then be expressed conveniently as $(xg)h = x(gh)$.

Example (1.2.30): (Properties of Group Actions)

The action $\Phi : G \times M \to M$ of a group G on a manifold M is said to be

(1) transitive , if for every $x, y \in M$ there exists a $g \in G$, such that $gx = y$;

(2) free , if it has no fixed points , that is , $\Phi_g(x) = x$ implies $g = e$;

(3) proper , if whenever a convergent subsequence $\{x_n\}$ in M exists , and the mapping $g_n x_n$ converges in M , then $\{g_n\}$ has a convergent subsequence in G .

Orbits . Given a group action of G on M , for a given point $x \in M$, the subset

$$Orb\, x = \{gx : g \in G\} \subset M ,$$

is called the group orbit through x . In finite dimensions , it can be shown that group orbits are always smooth (possibly immersed) manifolds . Group orbits generalize the notion of orbits of a dynamical system .

Theorem (1.2.31):

Orbits of proper group actions are embedded submanifolds .

Example (1.2.32): (Orbits of $SO(3)$)

A simple example of a group orbit is the action of $SO(3)$ on R^3 given by matrix multiplication :

The action of $A \in SO(3)$ on a point $x \in R^3$ is simply the product Ax . In this case , the orbit of the origin is a single point (the origin itself) , while the orbit of any other point is the sphere through that point .

Example (1.2.33): (Orbits of A Lie Group Acting on itself)

The action of a group G on itself from either the left , or the right , also produces group orbits . This action sets the stage for discussing the tangent lifted action of a Lie group on its tangent bundle .

Left and right translations on the group are denoted L_g and Rg respectively . For example , $L_g : G \to G$ is the map given by $h \to gh$, while $R_g : G \to G$ is the map given by $h \to hg$, for $g, h \in G$.

(a) Left translation $L_g : G \to G \;\; ; h \to gh$ defines a transitive and free action of G on itself . Right multiplication $R_g : G \to G \;\; ; h \to hg$ defines a right action , while $h \to hg^{-1}$ defines a left action of G on itself .

(b) G acts on G by conjugation , $g \to I_g = R_{g^{-1}} o L_g$. The map $I_g : G \to G$ given by $h \to ghg^{-1}$ is the inner automorphism associated with g . Orbits of this action are called conjugacy classes .

(c) Differentiating conjugation at e gives the adjoint action of G on ϑ :

$$Ad_g := T_e I_g : T_e G = \vartheta \to T_e G = \vartheta \; .$$

Explicitly ,the adjoint action of G on ϑ is given by

$$Ad : G \times \vartheta \to \vartheta \quad , \quad Ad_g(\xi) = T_e\left(R_{g^{-1}} o L_g\right)\xi$$

We have already seen an example of adjoint action for matrix Lie groups acting on matrix Lie algebras , when we defined $S(t) = R_A(t) B R_A(t)^{-1} \in T_I S$ as a key step in the proof of Proposition(1,2,16) .

(d) The coadjoint action of G on ϑ^* , the dual of the Lie algebra ϑ of G , is defined as follows . Let $Ad_g^*:\vartheta^*\to\vartheta^*$ be the dual of Ad_g , defined by

$$\left\langle Ad_g^*\alpha,\xi\right\rangle=\left\langle\alpha,Ad_g\xi\right\rangle$$

for $\alpha\in\vartheta^*$, $\xi\in\vartheta$ and pairing $\langle .,.\rangle:\vartheta^*\times\vartheta\to R$. Then the map

$$\Phi^*:G\times\vartheta^*\to\vartheta^* \text{ given by } (g,\alpha)\mapsto Ad_{g^{-1}}^*\alpha$$

is the coadjoint action of G on ϑ^* .

The Lie algebra of $SO(n)$ is called $so(n)$. A basis (e_1,e_2,e_3) for $so(3)$ when $n=3$ is given by

$$\hat{x}=\begin{pmatrix}0 & -z & y\\ z & 0 & -x\\ -y & x & 0\end{pmatrix}=xe_1+ye_2+ze_3$$

Example (1.2.34): (The Isomorphism between $so(3)$ and R^3)

The previous equation may be written equivalently by defining the hat – operation $\hat{}$ as

$$\hat{x}_{ij}=\varepsilon_{ijk}x^k \quad , \quad \text{where } \left(x^1,x^2,x^3\right)=(x,y,z)$$

Here $\varepsilon_{123}=1$ and $\varepsilon_{213}=-1$, with cyclic permutations .The totally antisymmetric tensor $\varepsilon_{ijk}=-\varepsilon_{jik}=-\varepsilon_{ikj}$ also defines the cross product of vectors in R^3 .Consequently , we may write,

$$(x\times y)_i=\varepsilon_{ijk}x^jy^k=\hat{x}_{ij}y^j \text{ , that is , } x\times y=\hat{x}y$$

Examples (1.2.35): (The Rotation Group $SO(3)$)

(1) The Lie algebra $so(3)$ and its dual:

The special orthogonal group is defined by

$$SO(3):=\left\{A:A\ a\,3\times3\,orthogonal\ \ matrix,\det(A)=1\right\}.$$

Its Lie algebra $so(3)$ is formed by 3×3 skew symmetric matrices , and its dual is denoted $so(3)^*$.

(2) The isomorphism $\hat{}$: $\left(so(3),[.,.]\right)\to\left(R^3,\times\right)$

The Lie algebra $(so(3),[.,.])$, where $[.,.]$ is the commutator bracket of matrices , is isomorphic to the Lie algebra $(R^3,\times)$, where $\times$ denotes the vector product in R^3 , by the isomorphism

$$u := (u^1,u^2,u^3) \in R^3 \mapsto \hat{u} := \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix} \in so(3),$$

that is , $\hat{u}_{ij} := -\varepsilon_{ij} u^k$.

Equivalently , this isomorphism is given by

$$\hat{u}v = u \times v \quad \text{for all } u,v \in R^3.$$

The following formulas for $u,v,w \in R^3$ may be easily verified :

$$(u \times v)^\wedge = [\hat{u},\hat{v}]$$

$$[\hat{u},\hat{v}]w = (u \times v) \times w$$

$$u.v = -\frac{1}{2} trace(\hat{u}\hat{v}) .$$

(3) The Ad action of $SO(3)$ on $so(3)$:

The corresponding adjoint action of $SO(3)$ on $so(3)$ may be obtained as follows . For $SO(3)$ we have $I_A(B) = ABA^{-1}$. Differentiating $B(t)$ at $B(0) = Id$ gives

$$Ad_A\hat{v} = \left.\frac{d}{dt}\right|_{t=0} AB(t)A^{-1} = A\hat{v}A^{-1} \quad , \quad \text{with } \hat{v} = B'(0) .$$

One calculates the pairing with a vector $w \in R^3$ as

$$Ad_A\hat{v}(w) = A\hat{v}(A^{-1}w) = A(v \times A^{-1}w) = Av \times w = (AV)^\wedge w$$

where we have used a relation

$$A(u \times v) = Au \times Av$$

which holds for any $u,v \in R^3$ and $A \in SO(3)$.

Consequently ,

$$Ad_A\hat{v} = (Av)^\wedge$$

Identifying $so(3) \simeq R^3$ then gives

$$Ad_A v = Av .$$

So (speaking prose all our lives) the adjoint action of $SO(3)$ on $so(3)$ may be identified with multiplication of a matrix in $SO(3)$ times a vector in R^3 .

(4) The ad action of $so(2)$ on $so(3)$:

Differentiating again gives the ad - action of the Lie algebra $so(3)$ on itself:

$$[\hat{u},\hat{v}] = ad_{\hat{u}}\hat{v} = \frac{d}{dt}\bigg|_{t=0}\left(e^{t\hat{u}}v\right)^{\wedge} = (\hat{u}v)^{\wedge} = (u \times v)^{\wedge}.$$

So in this isomorphism the vector cross product is identified with the matrix commutator of skew symmetric matrices .

(5) Infinitesimal generator :

Likewise , the infinitesimal generator corresponding to $u \in R^3$ has the expression

$$u_{R^3}(x) := \frac{d}{dt}\bigg|_{t=0} e^{t\hat{u}}x = \hat{u}x = u \times x \ .$$

Now we discuss the dual Lie algebra isomorphism .

The dual $so(3)^*$ is identified with R^3 by the isomorphism

$$\varepsilon R^3 \mapsto \tilde{} \ \varepsilon so(3)^* : \tilde{} \ (\hat{u}) := u \quad \text{for any } u \in R^3 .$$

In terms of this isomorphism , the Co – Adjoint action of $SO(3)$ on $so(3)^*$ is given by

$$Ad^*_{A^{-1}} \ \tilde{} = (A\Pi)^{\tilde{}}$$

and the coadjoint action of $so(3)$ on $so(3)^*$ is given by

$$ad^*_{\hat{u}}\tilde{\Pi} = (\Pi \times u)^{\tilde{}} \ . \qquad (1,7)$$

Now we discuss the Lifted actions .

Definition (1.2.36):

Let $\Phi : G \times M \to M$ be a left action , and write $\Phi_g(x) = \Phi(g,x)$ for $x \in M$. The tangent lift action of G on the tangent bundle TM is defined by $gv = T_x\Phi_g(v)$ for every $v \in T_xM$.

Remark (1.2.37):

In standard calculus notation , the expression for tangent lift may be written as

$$T_x \Phi v = \frac{d}{dt}\Phi(c(t))\Big|_{t=0} = \frac{\partial \Phi}{\partial c} c'(t)\Big|_{t=0} =: D\Phi(x) v \quad ,$$

with $c(0)=x$ and $c'(0)=v$.

Definition (1.2.38):

If X is a vector field on M and ϕ is a differentiable map from M to itself , then the push – forward of X by ϕ is the vector field $\phi_* X$ defined by $(\phi_* X)(\phi(x)) = T_x \phi(X(x))$.

That is , the following diagram commutes :

$$\begin{tikzcd} TM \arrow[r, "T\phi"] & TM \\ M \arrow[u, "X"] \arrow[r, "\phi"] & M \arrow[u, "\phi_* X"'] \end{tikzcd}$$

If ϕ is a diffeomorphism then the pull – back $\phi^* X$ is also defined : $(\phi^* X)(x) = T_{\phi(x)} \phi^{-1}(X(\phi(x)))$.

Definition (1.2.39):

Let $\Phi : G \times M \to M$ be a left action , and write $\Phi_g(m) = \Phi(g, m)$. Then G has a left action on $X \in \mathfrak{X}(M)$ (the set of vector fields on M) by the push –forward:

$$gX = (\Phi_g)_* X \ .$$

Definition (1.2.40):

Let G act on M on the left . A vector field X on M is invariant with respect to this actions (we often say " G – invariant " if the action is understood) if $gX = X$

for all $g \in G$; equivalently (using all of the above definitions !) $g\left(X(x)\right) = X\left(g(x)\right)$ for all $g \in G$ and all $x \in X$.

Definition (1.2.41):

Consider the left action of G on itself by left multiplication , $\Phi_g(h) = L_g(h) = gh$. A vector field on G that is invariant with respect to this action is called left – invariant if and only if $g\left(X(h)\right) = X(gh)$, which in less compact notation means $T_h L_g X(h) = X(gh)$. The set all such vector field is written $\mathcal{X}^L(G)$.

Proposition (1.2.42):

Given a $\xi \in T_e G$, define $X_\xi^L(g) = g\xi$ (recall : $g\xi = T_e L_g \xi$).Then X_ξ^L is the unique left – invariant vector field such that $X_\xi^L(e) = \xi$.

Proof:

To show that X_ξ^L is left – invariant , we need to show that $g\left(X_\xi^L(h)\right) = X_\xi^L(gh)$ for every $g, h \in G$. This follows from the definition of X_ξ^L and the associativity property of group actions :

$$g\left(X_\xi^L(h)\right) = g(h\xi) = (gh)\xi = X_\xi^L(gh)$$

We repeat the last line in less compact notation :

$$T_h L_g\left(X_\xi^L(h)\right) = T_h L_g(h\xi) = T_e L_{gh}\xi = X_\xi^L(gh)$$

For uniqueness , suppose X is left – invariant and $X(e) = \xi$. Then for any $g \in G$, we have

$$X(g) = g\left(X(e)\right) = g\xi = X_\xi^L(g).$$

Remark (1.2.43):

Note that the map $\xi \mapsto X_\xi^L$ is an vector space isomorphism from $T_e G$ to $\mathcal{X}^L(G)$ .All of the above definitions have analogues for right actions . The definitions of right – invariant , $\mathcal{X}^R(G)$ and X_ξ^R use the right action of G on itself defined by $\Phi(g, h) = R_g(h) = hg$.

Definition (1.2.44):

The Jacobi – Lie bracket on $\mathfrak{X}(M)$ is defined in local coordinates by

$$[X, Y]_{J-L} \equiv (DX).Y - (DY).X$$

which , in finite dimensions , is equivalent to

$$[X, Y]_{J-L} \equiv -(X.\nabla)Y + (Y.\nabla)X \equiv -[X, Y]$$

Theorem (1.2.45): (Properties of The Jacobi – Lie Bracket)

(1) The Jacobi – Lie bracket satisfies

$$[X, Y]_{J-L} \equiv \mathcal{L}_X Y \equiv \frac{d}{dt}\Big|_{t=0} \Phi_t^* Y \ ,$$

where Φ is the flow of X .(This is coordinate – free , and can be used as an alternative definition .)

(2) This bracket makes $\mathfrak{X}^L(M)$ a Lie algebra with $[X, Y]_{J-L} = -[X, Y]$, where $[X, Y]$ is the Lie algebra bracket on $\mathfrak{X}(M)$.

(3) $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$ for any differentiable $\phi : M \to M$.

Theorem (1.2.46):

$\mathfrak{X}^L(G)$ is a subalgebra of $\mathfrak{X}(G)$.

Proof :

Let $X, Y \in \mathfrak{X}^L(G)$. Using the last item of the previous theorem , and then the G invariance of X and Y , gives the push – forward relations

$$(L_g)_*[X, Y]_{J-L} = \left[(L_g)_* X, (L_g)_* Y\right]_{J-L}$$

for all $g \in G$ .Hence $[X, Y]_{J-L} \in \mathfrak{X}^L(G)$.This is the second property in Theorem(1,2,45) .

Theorem (1.2.47):

Set $\left[X_\xi^L, X_\eta^L\right]_{J-L}(e) = [\xi, \eta]$ for every $\xi, \eta \in \vartheta$, where the bracket on the right is the Jacobi – Lie bracket . (We say : the Lie bracket on ϑ is the pull – back of the Jacobi – Lie bracket by the map $\xi \mapsto X_\xi^L$.)

Proof :

The proof of this theorem for matrix Lie algebra is relatively easy : we have already seen that $ad_A B = AB - BA$. On the other hand , since $X_A^L(C) = CA$ for all C , and this linear in C , we have $DX_B^L(I).\mathrm{A} = \mathrm{AB}$,so

$$[A,B] = \left[X_A^L, X_B^L\right]_{J-L}(I) = DX_B^L(I).X_A^L(I) - DX_A^L(I).X_B^L(I)$$

$$= DX_B^L(I).A - DX_A^L(I).B = AB - BA$$

This is the third property of the Jacobi – Lie bracket listed in Theorem(1,2,45) .

Remark (1.2.48):

This theorem , together with Item (2) in Theorem (1,2,45) , proves that the Jacobi – Lie bracket makes ϑ into a Lie algebra .

Remark (1.2.49):

By Theorem (1,2,46) , the vector field $\left[X_\xi^L, X_\eta^L\right]$ is left – invariant . Since $\left[X_\xi^L, X_\eta^L\right]_{J-L}(e) = [\xi,\eta]$, it follows that

$$\left[X_\xi^L, X_\eta^L\right] = X_{[\xi,\eta]}^L \ .$$

Definition (1.2.50):

Let $\Phi : G \times M \to M$ be a left action , and let $\xi \in \vartheta$. Let $g(t)$ be a path in G such that $g(0) = e$ and $g'(0) = \xi$. Then the infinitesimal generator of the action in the ξ direction is the vector field ξ_M on M defined by $\xi_M(x) = \frac{d}{dt}\Big|_{t=0} \Phi_{g(t)}(x)$.

Theorem (1.2.51):

For any left action of G , the Jacobi – Lie bracket of infinitesimal generators is related to the Lie bracket on ϑ as follows :

$$[\xi_M, \eta_M] = -[\xi,\eta]_M \ .$$

Chapter (2)

The Lie Derivative , Jacobi – Lie Bracket and Euler's Equations for Incompressible Flow

Section (2.1): Euler's Equations and Lie Algebra as A Tangent Bundle

We start by discussing the Lie derivative and the Jacobi – Lie bracket .

Let X and Y be two vector fields on the same manifold M .

Definition(2.1.1):

The Lie derivative of Y with respect to X is $\mathcal{L}_X Y \equiv \frac{d}{dt} \Phi_t^* Y \Big|_{t=0}$ where Φ is the flow of X .

The Lie derivative $\mathcal{L}_X Y$ is " the derivative of Y in the direction given by X " . Its definition is coordinate – independent . By contrast , $DY .X$ (also written as $X[Y]$)is also " the derivative of Y in the X direction " , but the value of $DY .X$ depends on the coordinate system , and in particular does not usually equal $\mathcal{L}_X Y$ in the chosen coordinate system .

Theorem (2.1.2):

$\mathcal{L}_X Y = [X ,Y]$, where the bracket on the right is the Jacobi – Lie bracket .

Proof:

In the following calculation , we assume that M is finite – dimensional , and we work in local coordinates . Thus we may consider everything as matrices , which allows us to use the product rule and the identities $\left(M^{-1}\right)' = -M^{-1} M' M^{-1}$ and

$$\frac{d}{dt}\left(D\Phi_t(x)\right) = D\left(\frac{d}{dt}\Phi_t\right)(x) .$$

$$\mathcal{L}_X Y(x) = \frac{d}{dt}\Phi_t^* Y(x)\Big|_{t=0} = \frac{d}{dt}\left(D\Phi_t(x)\right)^{-1} Y\left(\Phi_t(x)\right)\Big|_{t=0}$$

$$= \left[\left(\frac{d}{dt}\left(D\Phi_t(x)\right)^{-1}\right) Y\left(\Phi_t(x)\right) + \left(D\Phi_t(x)\right)^{-1}\frac{d}{dt} Y\left(\Phi_t(x)\right)\right]_{t=0}$$

$$= \left[-\left(D\Phi_t(x)\right)^{-1}\left(\frac{d}{dt}D\Phi_t(x)\right)\left(D\Phi_t(x)\right)^{-1} Y\left(\Phi_t(x)\right) + \left(D\Phi_t(x)\right)^{-1}\frac{d}{dt} Y\left(\Phi_t(x)\right)\right]_{t=0}$$

$$= \left[-\left(\frac{d}{dt}D\Phi_t(x)\right) Y(x) + \frac{d}{dt} Y\left(\Phi_t(x)\right)\right]_{t=0}$$

$$= -D\left(\frac{d}{dt}\Phi_t(x)\Big|_{t=0}\right) Y(x) + DY(x)\left(\frac{d}{dt}\Phi_t(x)\Big|_{t=0}\right)$$

$$= -DX(x).Y(x) + DY(x).X(x)$$

$$= [X, Y]_{J-L}(x)$$

Therefore $\mathcal{L}_X Y = [X, Y]_{J-L}$.

The same formula applies in infinite dimensions , although the proof is more elaborate . For example , the equation for the vorticity dynamics of an Euler fluid with velocity u (with $div\, u = 0$) and vorticity $\omega = curl\, u$ may be written as

$$\partial_t \omega = -\mathrm{u}.\nabla\omega + \omega.\nabla\mathrm{u}$$

$$= -[u, \omega]$$

$$= -ad_u\omega$$

$$= -\mathcal{L}_u\omega$$

All of these equations express the invariance of the vorticity vector field ω under the flow of its corresponding divergenceless velocity vector field u .

This is also encapsulated in the language of fluid dynamics in characteristic form as

$$\frac{d}{dt}\left(\omega.\frac{\partial}{\partial\mathbf{x}}\right) = 0 \ \text{ , along } \ \frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) = curl^{-1}\omega \ .$$

Hence , the curl – inverse operator is defined by the Biot – Savart Law ,

$$\mathbf{u} = curl^{-1}\omega = curl\left(-\Delta\right)^{-1}\omega \ ,$$

which follows from the identity

$$curl\, curl\, \mathbf{u} = -\Delta\mathbf{u} + \nabla \mathrm{div}\mathbf{u} \ ,$$

and application of $div\boldsymbol{u} = 0$. Thus , in coordinates ,

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x},t) \Rightarrow \mathbf{x}(t,\mathbf{x}_0) = \Phi_t \mathbf{x}_0$$

with $\Phi_0 = Id$, that is , $\mathbf{x}(0,\mathbf{x}_0) = \mathbf{x}_0$ at $t = 0$, and

$$\omega^j\left(\mathbf{x}(t,\mathbf{x}_0),t\right)\frac{\partial}{\partial x^j\left(t,\mathbf{x}_0\right)} = \omega^A\left(\mathbf{x}_0\right)\frac{\partial}{\partial x_0^A}\circ\Phi_t^{-1}$$

Consequently ,

$$\Phi_{t^*}\omega^j\left(\mathbf{x}(t,\mathbf{x}_0),t\right) = \omega^A\left(\mathbf{x}_0\right)\frac{\partial x^j\left(t,\mathbf{x}_0\right)}{\partial x_0^A} =: D\Phi_t.\omega .$$

This is the Cauchy solution of Euler's equation for vorticity ,

$$\frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{t}} = \left[\boldsymbol{\omega}, \boldsymbol{curl}^{-1}\boldsymbol{\omega}\right].$$

The vorticity ω evolves by the ad – action of the right – invariant vector field $u = curl^{-1}\omega$. That is ,

$$\frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{t}} = -\boldsymbol{ad}_{curl^{-1}\omega}\boldsymbol{\omega} \ .$$

The Cauchy solution is the tangent lift of this flow ,namely,

$$\Phi_{t^*}\omega\left(\Phi_t\left(\mathbf{x}_0\right)\right) = T_{\mathbf{x}_0}\Phi_t\left(\omega\left(\mathbf{x}_0\right)\right).$$

Now we discuss the Euler's equation for incompressible flow.

The Euler's equation of incompressible fluid motion.

$$\underbrace{\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u}}_{d\mathbf{u}/dt\ along\ d\mathbf{x}/dt} + \nabla p = 0$$

where $\mathbf{u} : \Re^3 \times \Re \to \Re^3$ satisfies $\mathrm{div}\mathbf{u} = 0$.

The Geometric dynamics of vorticity

$$\omega = \mathrm{curl}\,\mathbf{u}$$

$$\omega_t = -\mathbf{u}.\nabla\omega + \omega.\nabla\mathbf{u}$$

$$= -[\mathbf{u}, \omega]$$

$$= -\mathrm{ad}_{\mathbf{u}}\omega$$

$$= -\mathcal{L}_{\mathbf{u}}\omega$$

In these equations , one denotes $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}$ and , hence , may write Euler vorticity dynamics equivalently in any of the following three forms

$$\frac{d\,\omega}{dt} = \omega.\nabla\mathbf{u} \;\; ,$$

as well as

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\left(\omega.\frac{\partial}{\partial \mathbf{x}}\right) = 0$$

or

$$\frac{d}{dt}(\omega.\nabla) = 0 \quad \text{along} \quad \frac{d\mathbf{x}}{dt} = \mathbf{u}$$

The last form is found using the chain rule as

$$\frac{d}{dt}(\omega.\nabla) = \frac{d\,\omega}{dt}.\nabla + \omega.\frac{d}{dt}\nabla = \left(\frac{d}{dt}\omega - \omega.\nabla\mathbf{u}\right).\nabla = 0 \;\; .$$

Theorem (2.1.3): (Ertel's)

The operations d/dt and $\omega.\nabla$ commute on solutions of Euler's fluid equations . That is ,

$$\left[\frac{d}{dt}, \omega.\nabla\right] = 0 \, ,$$

so that

$$\frac{d}{dt}(\omega.\nabla \boldsymbol{A}) = \omega.\nabla\frac{d}{dt}\boldsymbol{A}$$

for all differentiable $\mathbf{A}$ when $\omega = \mathrm{curl}\,\mathbf{u}$ and $\mathbf{u}$ is a solution of Euler's equations for incompressible fluid flow.Consequently ,one finds the following infinite set of conservation laws:

$$\text{If } \frac{d\boldsymbol{A}}{dt} = 0 \quad , \text{ then } \int \Phi\left(\omega.\nabla \boldsymbol{A}\right) d^3\mathbf{x} = \text{const} \quad ,$$

for all differentiable Φ.

Ohktani's Formula (2.1.4):

$$\frac{d^2\omega}{dt^2} = \frac{d}{dt}\left(\omega.\nabla\mathbf{u}\right) = \omega.\nabla\frac{d\mathbf{u}}{dt} = -\omega.\nabla\nabla p = -\mathbb{P}\omega$$

where

$$\mathbb{P}_{ij} = \frac{\partial^2 p}{\partial x^i \partial x^j} \qquad \text{(" Hessian" of pressure) .}$$

In addition , one has the relations

$$p = -\Delta^{-1} tr\left(\nabla\mathbf{u}^T.\nabla\mathbf{u}\right)$$

$$S = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \qquad \text{(strain rate tensor)}$$

so that , the following system of equations results ,

$$\frac{d\,\omega}{dt} = S\,\omega \quad , \quad \frac{d^2\omega}{dt^2} = -\mathbb{P}\omega$$

Theorem (2.1.5): (Kelvin Circulation)

$$\omega.\frac{\partial}{\partial\mathbf{x}} = \omega^j\frac{\partial}{\partial x^j} \quad , \quad \frac{d\mathbf{u}}{dt} + \nabla p = 0 \quad ,$$

where $\mathrm{div}\mathbf{u} = 0$, or equivalently $u^j_{,j} = 0$ in index notation .

The motion equation may be rewritten equivalently as a 1 – form relation ,

$$\frac{du_i}{dt}dx^i = -dp = \nabla_i p dx^i \quad \text{along} \quad \frac{d\mathbf{x}}{dt} = \mathbf{u}$$

$$\frac{d}{dt}\left(u_i dx^i\right) - \underbrace{u_i\frac{d}{dt}dx^i}_{=u_i du^i = d|u|^2/2} = -dp$$

Consequently

$$\frac{d}{dt}(\mathbf{u}.d\mathbf{x}) = -d\left(p - \frac{1}{2}|\mathbf{u}|^2\right)$$

which becomes

$$\frac{d}{dt}\oint_{C(\mathbf{u})} \mathbf{u}.d\mathbf{x} = -\oint_{C(\mathbf{u})} d\left(p - \frac{1}{2}\mathbf{u}^2\right) = 0$$

upon integrating around a closed loop $C(u)$ moving with velocity u. The 1 – form relation above may be rewritten as

$$(\partial_t + \mathcal{L}_{\mathbf{u}})(\mathbf{u}.d\mathbf{x}) = -d\left(p - \frac{1}{2}\mathbf{u}^2\right)$$

whose exterior derivative yields using $d^2 = 0$

$$(\partial_t + \mathcal{L}_{\mathbf{u}})(\omega.d\mathbf{S}) = 0$$

where $\omega.d\,\mathbf{S} = \mathrm{curl}\mathbf{u}.d\,\mathbf{S} = d\,(\mathbf{u}.d\,\mathbf{x})$ **.**

For these geometric quantities , one sees that the characteristic , or advective derivative is equivalent to a Lie derivative . Namely,

$$\underbrace{\left.\frac{d}{dt}\right|_{\text{advect}}}_{fluids} = \underbrace{\partial_t + \mathcal{L}_{\mathbf{u}}}_{geometry}$$

Theorem (2.1.6): (Stokes)

The classical theorem due to Stokes

$$\oint_{\partial S} \mathbf{u}.d\mathbf{x} = \iint_S \mathrm{curl}\mathbf{u}.d\mathbf{S}$$

shows that Kelvin's circulation theorem is equivalent to conservation of flux of vorticity

$$\frac{d}{dt}\iint_S \omega.d\mathbf{S} = 0$$

with $\partial \mathrm{S} = \mathrm{C}(\mathbf{u})$ through any surface commoving with flow.

Recall the definition ,

$$\omega.\frac{\partial}{\partial \mathbf{x}} \lrcorner d^3x = \omega.d\mathbf{S}$$

One may check this formula directly , by computing

$$\left(\omega^1 \frac{\partial}{\partial x^1} + \omega^2 \frac{\partial}{\partial x^2} + \omega^3 \frac{\partial}{\partial x^3}\right) \lrcorner \left(dx^1 \wedge dx^2 \wedge dx^3\right)$$

$$= \omega^1 dx^2 \wedge dx^3 + \omega^2 dx^3 \wedge dx^1 \omega dx^1 \wedge dx^2$$

$$= \omega.d\mathbf{S}$$

One may then use the vorticity equation in vector – field form ,

$$\left(\partial_t + \mathcal{L}_{\mathbf{u}}\right) \omega.\frac{\partial}{\partial \mathbf{x}} = 0$$

to prove that the flux of vorticity through any commoving surface is conserved , as follows .

$$\left(\partial_t + \mathcal{L}_{\mathbf{u}}\right)\left(\omega.\frac{\partial}{\partial \mathbf{x}} d^3x\right)$$

$$= \underbrace{\left(\left(\partial_t + \mathcal{L}_{\mathbf{u}}\right) \omega.\frac{\partial}{\partial \mathbf{x}}\right)}_{=0} \lrcorner d^3x + \omega.\frac{\partial}{\partial \mathbf{x}} \lrcorner \underbrace{\left(\partial_t + \mathcal{L}_{\mathbf{u}}\right) d^3x}_{= div\, \mathbf{u} d^3x = 0} = 0$$

That is , as computed above using the exterior derivative

$$\left(\partial_t + \mathcal{L}_{\mathbf{u}}\right) \omega.d\,\mathbf{S} = 0$$

From Euler's fluid equation $du_i / dt + \nabla_i p = 0$ with $u^j_{,j} = 0$ one finds,

$$\int \left(\partial_t u_i + u^j \partial_j u_i + \partial_i p\right) d^3x = 0$$

$$= \frac{d}{dt} \int u_i d^3x + \int \partial_j \left(u_i u^j + p\delta_i^j\right) d^3x$$

$$= \underbrace{\frac{d}{dt} M_i}_{=0} + \oint \underbrace{\hat{n}_j \left(u_i u^j + p\delta_i^j\right) dS}_{=0, if\ \hat{n}.u = 0}$$

Local conservation of fluid momentum is expressed using differentiation by parts as

$$\partial_t u_i = -\partial_j T_i^{\ j}$$

where $T_i^{\ j} := u_i u^j + p\delta_i^j$ is the fluid stress tensor.

Moreover ,each component of the total momentum $M_i = \int u_i d^3x$ for $i = 1,2,3$ is conserved for an incompressible Euler flow, provided the flow is tangential to any fixed boundaries, that is, $\hat{n}.u = 0$.

For mass density $D(x,t)$ with total mass $\int D(x,t)d^3x$,along $dx/dt = u(x,t)$ one finds,

$$\frac{d}{dt}Dd^3x = \left(\partial_t + \mathcal{L}_{\mathbf{u}}\right)\left(Dd^3x\right) = \underbrace{\left(\partial_t D + divD\,\mathbf{u}\right)}_{continuity\ equation} d^3x = 0$$

The solution of this equation is written in Lagrangian form as

$$\left(Dd^3x\right).g^{-1}(t) = D(\mathbf{x}_0)d^3x$$

For incompressible flow, this becomes

$$\frac{1}{D} = \det\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \frac{d^3x}{d^3x_0} = 1$$

Likewise, in the Eulerian representation one finds the equivalent relations,

$$\left.\begin{matrix} D=1 \\ \partial_t D + div\,(D\,\mathbf{u}) = 0 \end{matrix}\right\} \Rightarrow div\ \mathbf{u} = 0$$

Euler's fluid equation for incompressible flow $div\ \mathbf{u} = 0$

$$\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = 0$$

conserves the total kinetic energy, defined by

$$KE = \int \tfrac{1}{2}|\mathbf{u}|^2 d^3x$$

The vector calculus identity

$$\mathbf{u}.\nabla\mathbf{u} = -\mathbf{u}\times curl\ \mathbf{u} + \tfrac{1}{2}\nabla|\mathbf{u}|^2$$

recasts Euler's equation as

$$\partial_t \mathbf{u} - \mathbf{u} \times curl\ \mathbf{u} + \nabla \left(p + \tfrac{1}{2} \mathbf{u}^2 \right) = 0$$

So that

$$\frac{\partial}{\partial t} \tfrac{1}{2} |\mathbf{u}|^2 + div \left(p + \tfrac{1}{2} |\mathbf{u}|^2 \right) \mathbf{u} = 0$$

Consequently ,

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\mathbf{u}|^2 d^3 x = -\oint_{\partial\Omega} \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) .d\boldsymbol{S} \boldsymbol{= 0}$$

since $u.dS = u.\hat{n}\, dS = 0$ on any fixed boundary and one finds

$$KE = \int \tfrac{1}{2} |\mathbf{u}|^2 d^3 x = const$$

for Euler fluid motion.

Now we discuss the Lie group action on its tangent bundle.

Definition (2.1.7):

A Lie group G acts on its tangent bundle TG by tangent lifts. Given $X \in T_h G$ we can consider the action of G on X by either left or right translations, denoted as $T_h L_g X$ or $T_h R_g X$, respectively. These repression may be abbreviated as

$$T_h L_g X = L_g^* X = gX \quad \text{and} \quad T_h R_g X = R_g^* X = Xg \text{ .}$$

Left action of a Lie group G on its tangent bundle TG is illustrated in the figure below.

$$\begin{tikzcd}
TG \arrow[r, "TL_g"] & TG \\
G \arrow[u, "X"] \arrow[r, "L_g"] & G \arrow[u, "gX"']
\end{tikzcd}$$

For matrix Lie groups , this action is just multiplication on the left or right, respectively.

A vector field X on G is called left – invariant , if for every $g \in G$ one has $L_g^* X = X$, that is, if

$$\left(T_h L_g\right) X\,(h) = X\,(gh)$$

for every $h \in G$. The commutative diagram for a left – invariant vector field is illustrated in the figure below.

$$\begin{tikzcd} TG \arrow[r, "TL_g"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_g"'] & G \arrow[u, "X"'] \end{tikzcd}$$

Proposition (2.1.8):

The set $\mathfrak{X}_L(G)$ of left invariant vector fields on the Lie group G is a subalgebra of $\mathfrak{X}(G)$ the set of all vector fields on G .

Proof:

If $X, Y \in \mathfrak{X}_L(G)$ and $g \in G$,then

$$L_g^*[X, Y] = \left[L_g^* X, L_g^* Y\right] = [X, Y]$$

Consequently , the Lie bracket $[X, Y] \in \mathfrak{X}_L(G)$. Therefore, $\mathfrak{X}_L(G)$ is a subalgebra of $\mathfrak{X}(G)$, the set of all vector fields on G .

Proposition (2.1.9):

The linear maps $\mathfrak{X}_L(G)$ and $T_e G$ are isomorphic as vector spaces.

Demonstration of proposition . For each $\xi \in T_e G$, define a vector field X_ξ on G by letting $X_\xi(g) = T_e L_g(\xi)$. Then

$$X_\xi(gh) = T_e L_g h(\xi) = T_e(L_g o L_h)(\xi)$$

$$= T_h L_g(T_e L_h(\xi)) = T_h L_g(X_\xi(h))$$

which shows that X_ξ is left invariant .

Definition (2.1.10): (Jacobi – Lie Bracket of Vector Fields)

Let $g(t)$ and $h(s)$ be curves in G with $g(0) = e$, $h(0) = e$ and define vector fields at the identity of G by the tangent vectors $g'(0) = \xi$, $h'(0) = \eta$ .Compute the linearization of the Adjoint action of G on $T_e G$ as

$$[\xi, \eta] := \frac{d}{dt}\frac{d}{ds} g(t) h(s) g(t)^{-1} \Big|_{s=0, t=0} = \frac{d}{dt} g(t) \eta g(t)^{-1} \Big|_{t=0} = \xi\eta - \eta\xi .$$

This is the Jacobi – Lie bracket of the vector fields ξ and η.

Definition (2.1.11):

The Lie bracket in $T_e G$ is defined by

$$[\xi, \eta] := [X_\xi, X_\eta](e),$$

for $\xi, \eta \in T_e G$ and for $[X_\xi, X_\eta]$ the Jacobi – Lie bracket of vector fields. This makes $T_e G$ into a Lie algebra . Note that

$$[X_\xi, X_\eta] = X_{[\xi, \eta]} ,$$

for all $\xi, \eta \in T_e G$.

Definition (2.1.12):

The vector space $T_e G$ with this Lie algebra structure is called the Lie algebra of G and is denoted by ϑ.

If we let $\xi_L(g) = T_e L_g \xi$, then the Jacobi – Lie bracket of two such left – invariant vector fields in fact gives the Lie algebra bracket:

$$[\xi_L, \eta_L](g) = [\xi, \eta]_L (g)$$

For the right – invariant case, the right hand side obtains a minus sign ,

$$[\xi_R, \eta_R](g) = -[\xi, \eta]_R (g).$$

The relative minus sign arises becomes of the difference in action $(xh^{-1})g^{-1} = x(gh)^{-1}$ on the right versus $(gh)x = g(hx)$ on the left .

Infinitesimal generators in mechanics , group actions often appear as symmetry transformations , which arise through their infinitesimal generators , defined as follows.

Definition (2.1.13):

Suppose $\Phi : G \times M \to M$ is an action . For $\xi \in \vartheta$, $\Phi^{\xi}(t, x) : R \times M \to M$ defined by $\Phi^{\xi}(x) = \Phi(\exp t\xi, x) = \Phi_{\exp t\xi}(x)$ is an R – action on M . In other words $\Phi_{\exp t\xi} \to M$ is a flow on M .The vector field on M defined by

$$\xi_M(x) = \frac{d}{dt}\bigg|_{t=0} \Phi_{\exp t\xi}(x)$$

is called the infinitesimal generator of the action corresponding to ξ.

The Jacobi – Lie bracket of infinitesimal generators is related to the Lie algebra bracket as follows:

$$[\xi_M, \eta_M] = -[\xi, \eta]_M$$

Now we discuss the Lie algebras as vector fields.

Definition (2.1.14): (The ad – operation)

For $A \in \vartheta$ we define the operator ad_A to be the operator $ad : \vartheta \times \vartheta \to \vartheta$ that maps $B \in \vartheta$ to $[A, B]$. We write $ad_A B = [A, B]$.

Definition (2.1.15):

A representation of a Lie algebra ϑ on a vector space V is a mapping ρ from ϑ to the linear transformations of V such that for $A, B \in \vartheta$ and any constant scalar c ,

(i) $\rho(A+cB) = \rho(A)+c\rho(B)$,

(ii) $\rho([A,B]) = \rho(A)\rho(B)-\rho(B)\rho(A)$.

If the map ρ is one – to – one , the representation is said to faithful.

Example (2.1.16): (Vector Field Representations of Lie Algebras)

The Jacobi – Lie bracket of the vector field ξ and η in Theorem(2.1.2) may be represented in coordinate charts as

$$\eta = \frac{dx}{ds}\bigg|_{s=0} = v(x) \text{ , and } \xi = \frac{dx}{dt}\bigg|_{t=0} = u(x).$$

The Jacobi – Lie bracket of these two vector fields yields a third vector field ,

$$\xi\eta - \eta\xi = \frac{d\eta}{dt}\bigg|_{t=0} - \frac{d\xi}{ds}\bigg|_{s=0} = \frac{dv}{dx}\frac{dx}{dt}\bigg|_{t=0} - \frac{du}{dx}\frac{dx}{ds}\bigg|_{s=0}$$

$$= \frac{dv}{dx}.\mathrm{u} - \frac{du}{dx}.v$$

$$= u.\nabla v - v.\nabla u$$

Thus , the Jacobi – Lie bracket of vector fields at the tangent space of the identity T_eG is closed and may be represented in coordinate charts by the Lie bracket (commutator of vector fields)

$$[\xi,\eta] := \xi\eta - \eta\xi = u.\nabla v - v.\nabla u =: [u,v].$$

Proposition (2.1.17):

Let $\mathfrak{X}(\Re^n)$ be the set of vector fields defined on $\Re^n$ .A Lie algebra ϑ may be represented on coordinate charts by vector fields $X_\xi = X_\xi^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\Re^n)$ for each element $\xi \in \vartheta$. This vector field representation satisfies

$$X_{[\xi,\eta]} = [X_\xi, X_\eta]$$

where $[\xi,\eta] \in \vartheta$ is the Lie algebra product and $\left[X_\xi, X_\eta\right]$ is the vector field commutator .

Now we discuss the Lagrangian and Hamiltonian formulations .

Newton's equations

$$m_i \ddot{\mathbf{q}}_i = \mathbf{F}_i \;\; , \quad i = 1,.......,N \quad \text{(no sum on } i \text{)} \tag{2.1}$$

describe the accelerations $\ddot{\mathbf{q}}_i$ of N particles with

Masses $\quad m_i \; , i = 1,.........,N$,

Euclidean position $\quad \mathbf{q} := \left(\mathbf{q}_1,..........,\mathbf{q}_N\right) \in \Re^{3N}$,

in response to prescribed forces ,

$$\mathbf{F} = \left(\mathbf{F}_1,...........,\mathbf{F}_N\right),$$

acting on these particles . Suppose the forces arise from a potential . That is , let

$$F_i\left(\mathbf{q}\right) = -\frac{\partial V\left(\{\mathbf{q}\}\right)}{\partial \mathbf{q}_i} \; , V : \Re^{3N} \to \Re \, , \tag{2.2}$$

where $\partial V / \partial \mathbf{q}_i$ denotes the gradient of the potential with respect to the variable q_i . Then Newton's equations (2.1) become

$$m_i \ddot{\mathbf{q}}_i = -\frac{\partial V}{\partial \mathbf{q}_i} \quad , \quad i = 1,..........,N \tag{2.3}$$

Remark (2.1.18):

Newton introduced the gravitational potential for celestial mechanics , now called the Newtonian potential

$$V\left(\{\mathbf{q}\}\right) = \sum_{i,j=1}^{N} \frac{-G m_i m_j}{\left|\mathbf{q}_i - \mathbf{q}_j\right|} \tag{2.4}$$

Theorem (2.1.19): (Lagrangian and Hamiltonian Formulations)

Newton's equations in potential form ,

$$m_i \ddot{\mathbf{q}}_i = -\frac{\partial V}{\partial \mathbf{q}_i} \quad , \quad i = 1, \ldots\ldots\ldots, N \tag{2.5}$$

for particle motion in Euclidean space $\Re^{3N}$ are equivalent to the following four statements:

(i) The Euler – Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L}{\partial \mathbf{q}_i} = 0 \quad , \quad i = 1, \ldots\ldots\ldots, N \tag{2.6}$$

hold for the Lagrangian $L : \Re^{6N} = \{(\mathbf{q}, \dot{\mathbf{q}}) : \mathbf{p}, \dot{\mathbf{q}} \in \Re^{3N}\} \to \Re$, defined by

$$L(\mathbf{q}, \dot{\mathbf{q}}) := \sum_{i=1}^{N} \frac{m_i}{2} \|\dot{\mathbf{q}}_i\|^2 - V(\mathbf{q}) \ , \tag{2.7}$$

with $\|\dot{\mathbf{q}}_i\|^2 = \dot{\mathbf{q}}_i . \dot{\mathbf{q}}_i = \dot{q}_i^j \dot{q}_i^k \delta_{jk}$ (no sum on i).

(ii) Hamilton's principle of stationary action , $\delta \mathcal{S}$=0, holds for the action functional (dropping i s)

$$\mathcal{S}[\mathbf{q}(.)] := \int_a^b L(\mathbf{q}(t), \dot{\mathbf{q}}(t))dt \ . \tag{2.8}$$

(iii) Hamilton's equations of motion ,

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \quad , \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \ , \tag{2.9}$$

hold for the Hamiltonian resulting from the Lagendre transform ,

$$H(\mathbf{q}, \mathbf{p}) := \mathbf{p}.\dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}) - L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})) \ , \tag{2.10}$$

where $\dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})$ solves for $\dot{q}$ from the definition $\mathbf{p} := \partial L(\mathbf{q}, \dot{\mathbf{q}}) / \partial \dot{\mathbf{q}}$.

In the case of Newton's equations in potential form (2.5) , the Lagrangian in equation (2.7) yields $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$ and the resulting Hamiltonian is (restoring i s)

$$H = \underbrace{\sum_{i=1}^{N} \frac{1}{2m_i} \|\mathbf{p}_i\|^2}_{Kinetic\ energy} + \underbrace{V(\mathbf{q})}_{Potential}$$

(iv) Hamilton's equations in their Poisson bracket formulation ,

$$\dot{F} = \{F, H\} \quad \text{for all } F \in \mathcal{F}(\mathrm{P}) \ , \tag{2.11}$$

hold with Poisson bracket defined by

$$\{F, G\} := \sum_{i=1}^{N} \left(\frac{\partial F}{\partial \mathbf{q}_i} . \frac{\partial G}{\partial \mathbf{p}_i} - \frac{\partial F}{\partial \mathbf{p}_i} . \frac{\partial G}{\partial \mathbf{q}_i} \right) \quad \text{for all } F, G \in \mathcal{F}(\mathrm{p}) \ . \tag{2.12}$$

We will prove this theorem by proving a chain of linked equivalence relations : $(2.5) \Leftrightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ as propositions .

Step(I): Proof that Newton's equations $(2.5) \Leftrightarrow (i)$

Step (II): Proof that $(i) \Leftrightarrow (ii)$

The Euler – Lagrange equations (2.6) are equivalent to Hamilton's principle of stationary action .

To simplify notation , we momentarily suppress the particle index i .

We need to prove the solutions of (2.6) are critical points $\delta\mathcal{S}=0$ of the action functional

$$\mathcal{S}\left[\mathbf{q}(.)\right] := \int_a^b L\left(\mathbf{q}(t),\dot{\mathbf{q}}(t)\right)dt\,, \tag{2.13}$$

(where $\dot{\mathbf{q}} = \partial\mathbf{q}(t)/\partial t$) with respect to variations on $C^\infty\left(\left[a,b\right],\Re^{3N}\right)$, the space of smooth trajectories $\mathbf{q}:\left[a,b\right]\to\Re^{3N}$ with fixed endpoints $\mathbf{q}_a,\mathbf{q}_b$.

In $C^\infty\left(\left[a,b\right],\Re^{3N}\right)$ consider a deformation $\mathbf{q}(t,s)$, $s\in(-\varepsilon,\varepsilon)$, $\varepsilon>0$,with fixed endpoints $\mathbf{q}_a,\mathbf{q}_b$, of a curve $\mathbf{q}_0(t)$, that is , $\mathbf{q}(t,0)=\mathbf{q}_0(t)$ for all $t\in\left[a,b\right]$ and $\mathbf{q}(a,s)=\mathbf{q}_0(a)=\mathbf{q}_a$, $\mathbf{q}(b,s)=\mathbf{q}_0(b)=\mathbf{q}_b$ for all $s\in(-\varepsilon,\varepsilon)$.

Define a variation of the curve $\mathbf{q}_0(.)$ in $C^\infty\left(\left[a,b\right],\Re^{3N}\right)$ by

$$\delta\mathbf{q}(.) := \frac{d}{ds}\bigg|_{s=0}\mathbf{q}(.,s)\in T_{\mathbf{q}_0(.)}C^\infty\left(\left[a,b\right],\Re^{3N}\right),$$

and define the first variation of $\mathcal{S}$ at $\mathbf{q}_0(t)$ to be the derivative

$$\delta\mathcal{S} := \mathrm{D}\mathcal{S}[\mathbf{q}_0(.)]\left(\delta\mathbf{q}(.)\right) := \frac{d}{ds}\bigg|_{s=0}\mathcal{S}\left[\mathbf{q}(.,s)\right]. \tag{2.14}$$

Note that $\delta\mathbf{q}(a)=\delta\mathbf{q}(b)=0$. With these notations , Hamilton's principle of stationary action states that the curve $\mathbf{q}_0(t)$ satisfies the Euler – Lagrange equations (2.6) if and only if $\mathbf{q}_0(.)$ is a critical point of the action functional , that is $\mathrm{D}\mathcal{S}\left[\mathbf{q}_0(.)\right]=0$. Indeed, using the equality of mixed partials , integrating by parts , and taking into account that $\delta\mathbf{q}(a)=\delta\mathbf{q}(b)=0$, one finds

$$\delta \mathcal{S} := \mathrm{D}\mathcal{S}[\mathbf{q}_0(.)](\delta \mathbf{q}(.)) = \frac{d}{ds}\Big|_{s=0} \mathcal{S}[\mathbf{q}(.,s)]$$

$$= \frac{d}{ds}\Big|_{s=0} \int_a^b L(\mathbf{q}(t,s), \dot{\mathbf{q}}(t,s))dt$$

$$= \sum_{i=1}^{N} \int_a^b \left[\frac{\partial L}{\partial \mathbf{q}_i} . \delta \mathbf{q}_i (t,s) + \frac{\partial L}{\partial \dot{\mathbf{q}}_i} . \delta \dot{\mathbf{q}}_i \right] dt$$

$$= -\sum_{i=1}^{N} \int_a^b \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i} \right) - \frac{\partial L}{\partial \mathbf{q}_i} \right] . \delta \mathbf{q}_i \, dt = 0$$

for all smooth $\delta \mathbf{q}_i (t)$ satisfying $\delta \mathbf{q}_i (a) = \delta \mathbf{q}_i (b) = 0$ .These proves the equivalence of (i) and (ii) , upon restoring particle index i in the last two lines .

Definition (2.1.20):

The conjugate momenta for the Lagrangian in (2.7) are defined as

$$\mathbf{p}_i := \frac{\partial L}{\partial \dot{\mathbf{q}}_i} = m_i \dot{\mathbf{q}}_i \in \Re^3, \quad i = 1, \ldots\ldots, N \qquad \text{(no sum on } i \text{)} \tag{2.15}$$

Definition (2.1.21):

The Hamiltonian is defined via the change of variables $(\mathbf{q}, \dot{\mathbf{q}}) \mapsto (\mathbf{q}, \mathbf{p})$ called the Legendre transform ,

$$H(\mathbf{q}, \mathbf{p}) := \mathbf{p} . \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}) - L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}))$$

$$= \sum_{i=1}^{N} \frac{m_i}{2} \|\dot{\mathbf{q}}_i\|^2 + V(\mathbf{q})$$

$$= \underbrace{\sum_{i=0}^{N} \frac{1}{2m_i} \|\mathbf{p}_i\|^2}_{Kinetic\ energy} + \underbrace{V(\mathbf{q})}_{Potential} \tag{2.16}$$

Remark (2.1.22):

The value of the Hamiltonian coincides with the total energy of the system . This value will be shown momentarily to remain constant under the evolution of Euler – Lagrange equations (2.6) .

Remark (2.1.23):

The Hamiltonian H may be obtained from the Legendre transformation as a function of the variables $(\mathbf{q},\mathbf{p})$, provided one may solve for $\dot{\mathbf{q}}(\mathbf{q},\mathbf{p})$, which requires the Lagrangian to be regular , for example ,

$$\det \frac{\partial^2 L}{\partial \dot{\mathbf{q}}_i \partial \dot{\mathbf{q}}_i} \neq 0 \qquad \text{(no sum on } i \text{) .}$$

Step (III): Proof that $(ii) \Leftrightarrow (iii)$

(Hamilton's principle of stationary action is equivalent to Hamilton's canonical equations .) Lagrangian (2.1.7) is regular and the derivatives of the Hamiltonian may be shown to satisfy ,

$$\frac{\partial H}{\partial \mathbf{p}_i} = \frac{1}{m_i}\mathbf{p}_i = \dot{\mathbf{q}}_i = \frac{d\mathbf{q}_i}{dt} \quad \text{and} \quad \frac{\partial H}{\partial \mathbf{q}_i} = \frac{\partial V}{\partial \mathbf{q}_i} = -\frac{\partial L}{\partial \mathbf{q}_i} .$$

Consequently , the Euler – Lagrange equations (2.6) imply

$$\dot{\mathbf{p}}_i = \frac{d\,\mathbf{p}_i}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i}\right) = \frac{\partial L}{\partial \mathbf{q}_i} = -\frac{\partial H}{\partial \mathbf{q}_i} .$$

These calculations show that the Euler – Lagrange (2.6) are equivalent to Hamilton's canonical equations :

$$\dot{\mathbf{q}}_i = \frac{\partial H}{\partial \mathbf{p}_i} \quad , \quad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{q}_i} \quad , \qquad (2.17)$$

where $\partial H/\partial \mathbf{q}_i , \partial H/\partial \mathbf{p}_i \in \Re^3$ are the gradient of H with respect to $\mathbf{q}_i , \mathbf{p}_i \in \Re^3$, respectively . This proves the equivalence of (ii) and (iii) .

Remark (2.1.24):

The Euler – Lagrange equations are second order and they determine curves in configuration space $\mathbf{q}_i \in C^\infty([a,b],\Re^{3N})$. In contrast , Hamilton's equations are first order and they determine curves in phase space $(\mathbf{q}_i , \mathbf{p}_i) \in C^\infty([a,b],\Re^{6N})$, a space whose dimension is twice the dimension of the configuration space .

Step (IV): Proof that $(iii) \Leftrightarrow (iv)$

(Hamilton's canonical equations may be written using a Poisson bracket .)

By the chain rule and (2.17) any $F \in \mathcal{F}(P)$ satisfies

$$\frac{dF}{dt} = \sum_{i=0}^{N} \left(\frac{\partial F}{\partial \mathbf{q}_i} . \dot{\mathbf{q}}_i + \frac{\partial F}{\partial \mathbf{p}_i} . \dot{\mathbf{p}}_i \right)$$

$$= \sum_{i=1}^{N} \left(\frac{\partial F}{\partial \mathbf{q}_i} . \frac{\partial H}{\partial \mathbf{p}_i} - \frac{\partial F}{\partial \mathbf{p}_i} . \frac{\partial H}{\partial \mathbf{q}_i} \right) = \{F, H\}$$

This finishes the proof of the theorem , by proving the equivalence of (iii) and (iv) .

Remark (2.1.25): (Energy Conservation)

Since the Poisson bracket is skew symmetric , $\{H, F\} = -\{F, H\}$, one finds that $\dot{H} = \{H, H\} = 0$. Consequently , the value of the Hamiltonian is preserved by the evolution . Thus , the Hamiltonian is said to be a constant of the motion .

Section (2.2): Lie Derivative and Euler – Lagrange Equations of Manifolds

We start with some Hamilton's principle on manifolds .

Theorem (2.2.1): (Hamilton's Principle of Stationary Action)

Let the smooth function $L : TQ \to \Re$ be a Lagrangian on $TQ.AC^2$ curve $c : [a,b] \to Q$ joining $q_a = c(a)$ to $q_b = c(b)$ satisfies the Euler – Lagrange equations if and only if

$$\delta \int_a^b L\left(c(t), \dot{c}(t)\right) dt \ .$$

Proof :

The meaning of the variational derivative in the statement is the following . Consider a family of C^2 curves $c(t,s)$ for $|s| < \varepsilon$ satisfying $c_0(t) = c(t)$, $c(a,s) = q_a$, and $c(b,s) = q_b$ for all $s \in (-\varepsilon, \varepsilon)$. Then

$$\delta \int_a^b L\left(c(t), \dot{c}(t)\right) dt := \frac{d}{ds}\bigg|_{s=0} \int_a^b L\left(c(t,s), \dot{c}(t,s)\right) dt \ .$$

Differentiating under the integral sign , working in local coordinates (covering the curve $c(t)$ by a finite number of coordinate charts) , integrating by parts , denoting

$$v(t) := \frac{d}{ds}\bigg|_{s=0} c(t,s),$$

and taking into account that $v(a) = v(b) = 0$, yields

$$\int_a^b \left(\frac{\partial L}{\partial q^i} v^i + \frac{\partial L}{\partial \dot{q}^i} v^i \right) dt = \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) v^i dt \ .$$

This vanishes for any C^1 function $v(t)$ if and only if the Euler – Lagrange equations hold .

Remark (2.2.2):

The integral appearing in this theorem

$$\mathcal{S}\left(c\left(.\right)\right) := \int_a^b L\left(c\left(t\right), \dot{c}\left(t\right)\right) dt$$

is called the action integral . It is defined on C^2 curves $c : [a,b] \to Q$ with fixed endpoints , $c(a) = q_a$ and $c(b) = q_b$.

Remark (2.2.3): (Variational Derivatives of Functionals vs Lie Derivatives of Functions)

The variational of a functional $S[u]$ is defined as the linearization

$$\lim_{\varepsilon \to 0} \frac{S[u + \varepsilon v] - S[u]}{\varepsilon} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S[u + \varepsilon v] = \left\langle \frac{\delta S}{\delta v}, v \right\rangle .$$

Compare this to the expression for the Lie derivative of a function . If f is a real valued function on a manifold M and X is a vector field on M , the Lie derivative of f along X is defined as the directional derivative

$$\mathcal{L}_X f = X(f) := df .X .$$

If M is finite – dimensional , this is

$$\mathcal{L}_X f = X[f] := \mathbf{d}f .X = \frac{\partial f}{\partial x^i} X^i = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon X) - f(x)}{\varepsilon} .$$

The similarity is suggestive : Namely , the Lie derivative of a function and the variational derivative of a functional are both defined as linearizations of smooth maps in certain directions .

The next theorem emphasizes the role of Lagrangian one – forms and two –forms in the variational principle . The following is a direct corollary of the previous theorem .

Theorem (2.2.4):

Given a C^k Lagrangian $L : TQ \to \mathfrak{R}$ for $k \geq 2$, there exists a unique C^{k-2} map $\Upsilon L(L) : \ddot{Q} \to T^*Q$, where

$$\ddot{Q} := \left\{ \frac{d^2 q}{dt^2} \bigg|_{t=0} \in T(TQ) : q(t) \, is \; a \, C^2 \; curve \; in \; Q \right\}$$

is a submanifold of $T(TQ)$, and a unique C^{k-1} one – form $\Theta_L \in \wedge^1(TQ)$, such that for all C^2 variations $q(t,s)$ (defined on a fixed t – interval) of $q(t,0) = q_0(t) := q(t)$, we have

$$\delta \mathcal{S} := \frac{d}{ds} \bigg|_{s=0} \mathcal{S}[c(.,s)] = \mathrm{D}\mathcal{S}[q(.)].\delta \mathrm{q}(.)$$

$$= \int_a^b \Upsilon L(L)(q,\dot{q},\ddot{q}).\delta q \, dt + \underbrace{\Theta_L(q,\dot{q}).\delta q \, \Big|_a^b}_{cf\ .Noether's Theorem} \qquad (2.18)$$

where

$$\delta q(t) = \frac{d}{ds} \bigg|_{s=0} q(t,s)$$

Now we discuss the vector fields and 1 – forms .

Let M be a manifold . In what follows , all maps may be assumed to be C^∞, although that's not necessary .

A vector field on M is a map $X : M \to TM$ such that $X(x) \in T_x M$ for every $x \in M$. The set of all smooth vector fields on M is written $\mathfrak{X}(M)$. (" Smooth " means differentiable or C^r for some $r \le \infty$, depending on context .)

A (differential) 1 – form on M is map $\theta : M \to T^*M$ such that $\theta(x) \in T_x^*M$ for every $x \in M$.

More generally , if $\pi : E \to M$ is a bundle , then a section of the bundle is a map $\varphi : M \to E$ such that $\pi \, o \, \varphi(x) = x$ for all $x \in M$. So a vector field is a section of the tangent bundle , while a 1 – form is section of the cotangent bundle .

Vector fields can added and also multiplied by scalar functions $k : M \to \Re$, as follows :

$$(X_1 + X_2)(x) = X_1(x) + X_2(x) \quad , \quad (kX)(x) = k(x) X(x) .$$

Differential forms can added and also multiplied by scalar functions $k : M \to R$, as follows :

$$(\alpha+\beta)(x)=\alpha(x)+\beta(x),(k\theta)(x)=k(x)\theta(x).$$

We have already defined the push – forward and pull – back of a vector field . The pull – back of a 1 – form θ on N by a map $\varphi: M \to N$ is the 1 – form $\varphi^*\theta$ on M defined by

$$(\varphi^*\theta)(x).v=\theta(\varphi(x)).T\varphi(v)$$

The push – forward of a 1 – form α on M by a diffeomorphism $\psi: M \to N$ is the pull – back of α by ψ^{-1}.

A vector field can be contracted with a differential form , using the pairing between tangent and cotangent vectors : $(X \lrcorner \theta)(x)=\theta(x).X(x)$. Note that $X \lrcorner \theta$ is a map from M to $\Re$. Many books write ${}^iX^{\theta}$ in place of $X \lrcorner \theta$, and the contraction operation is often called interior product .

The differential of $f: M \to \Re$ is a 1 – form df on M defined by

$$df(x).v=\frac{d}{dt}f(c(t))\Big|_{t=0}$$

for any $x \in M$, any $v \in T_xM$ and any path $c(t)$ in M such that $c(0)=0$ and $c'(0)=v$. The left hand side , $df(x).v$, means the pairing between cotangent and tangent vectors , which could also be written $df(x)(v)$ or $\langle df(x),v\rangle$.

Note :

$$X \lrcorner df = £_X f = X[f]$$

Remark (2.2.5):

df is very similar to Tf , but Tf is defined for all differentiable $f: M \to N$, whereas df is only defined when $N=\Re$. In this case , Tf is a map from TM to $T\Re$, and $Tf(v)=df(x).v \in T_{f(x)}\Re$ for every $v \in T_xM$ (we have identified $T_{f(x)}\Re$ with $\Re$.)

In coordinates . Let M be n – dimensional , and let $x^1,........,x^n$ be differentiable local coordinates for M . This means that there's an open subset U of M and an open subset V of $\Re^n$ such that the map $\varphi: U \to V$ defined by $\varphi(x)=(x^1(x),...,x^n(x))$ is a diffeomorphism . In particular , each x^i is a map from M to $\Re$, so the differential dx^i is defined . There is also a vector field $\frac{\partial}{\partial x^i}$ for every i , which is defined by $\frac{\partial}{\partial x^i}(x)=\frac{d}{dt}\varphi^{-1}(\varphi(x)+te_i)\Big|_{t=0}$, where e_i is the i^{th} standard basis vector .

Remark (2.2.6):

Of course , given a coordinate system $\varphi=(x^1,........,x^n)$, it is usual to write $x=(x^1,.......,x^n)$, which means x identified with $(x^1(x),.......,x^n(x))=\varphi(x)$.

For every $x\in M$, the vectors $\frac{\partial}{\partial x^i}(x)$ form a basis for T_xM , so every $v\in T^xM$ can be uniquely expressed as $v=v^i\frac{\partial}{\partial x^i}(x)$. This expression defines the tangent – lifted coordinates $x^1,......,x^n,v^1,........,v^n$ on TM (they are local coordinates , defined on $TU\subset TM$).

For every $x\in M$, the convectors $dx^i(x)$ form a basis for T_x^*M , so every $\alpha\in T^xM$ can be uniquely expressed as $\alpha=\alpha_i dx^i(x)$. This expression defines the cotangent – lifted coordinates $x^1,.......,x^n,\alpha_1,........,\alpha_n$ on T^*M (they are local coordinates , defined on $T^*U\subset T^*M$).

Note that the basis $\left(\frac{\partial}{\partial x^i}\right)$ is dual to basis $(dx^1,........,dx^n)$. It follows that

$$(\alpha_i dx^i).\left(v^i\frac{\partial}{\partial x^i}\right)=\alpha_i v^i$$

(we have used the summation convention) .

In mechanics , the configuration space is often called Q , and the lifted coordinates are written : $q^1,........,q^n,\dot{q}^1,........,\dot{q}^n$ (on TQ) and $q^1,......,q^n,p_1,........,p_n$ (on T^*Q) .

Since TQ is manifold , we can consider vector fields on it , which are sections of $T(TQ)$. In coordinates , every vector field on TTQ has the form $X=a^i\frac{\partial}{\partial q^i}+b^i\frac{\partial}{\partial\dot{q}^i}$, where the a^i and b^i are functions of q and $\dot{q}$. Note that the same symbol q^i has two interpretation : as a coordinate on TQ and as a coordinate on Q , so $\frac{\partial}{\partial q^i}$ can mean a vector field TQ (as above) or on Q .

The tangent lift of the bundle projection $\tau : TQ \to Q$ is a map $T\tau : TTQ \to TQ$. If X is written in coordinates as above , the $T\tau o X = a^i \frac{\partial}{\partial q^i}$. A vector field X on TTQ is second order if $T\tau o X(v) = v$; in coordinates , $a^i = \dot{q}^i$. The name comes from the process of reducing of second order equations to first order ones by introducing new variables $\dot{q}^i = \frac{dq^i}{dt}$.

One may also consider T^*TQ ,TT^*Q and T^*T^*Q . However , the subscript /superscript distinction is problematic here .

The 1 – forms on T^*Q are sections of T^*T^*Q . Given cotangent – lifted local coordinates

$$\left(q^1, \ldots\ldots\ldots, q^n, p_1, \ldots\ldots\ldots, p_n\right)$$

on T^*Q , the general 1 – form on T^*Q has the form $a_i dq^i + b_i dp_i$, where a_i and b_i are functions of (q, p). The canonical 1 – form on T^*Q is

$$\theta = p_i dq^i$$

also written in the short form pdq . Pairing $\theta(q, p)$ with an arbitrary tangent vector $v = a^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial p^i} \in T_{(q,p)} T^*Q$ gives

$$\left\langle \theta(q,p), v \right\rangle = \left\langle p_i dq^i, a^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial p^i} \right\rangle = p_i a^i = \left\langle p_i dq^i, a^i \frac{\partial}{\partial q^i} \right\rangle = \left\langle p, T\tau^*(v) \right\rangle ,$$

where $\tau^* : T^*Q \to Q$ is projection . In the last line we have interpreted q^i as a coordinate on Q , which implies that $p_i dq^i = p$, by definition of the coordinates p_i . Note that the last line is coordinate – free .

Recall that a 1 – form on M , evaluated at a point $x \in M$, is a linear map from $T_x M$ to $\Re$.

A 2 – form on M , evaluated at a point $x \in M$, is a skew – symmetric bilinear form on $T_x M$; and the bilinear form has to vary smoothly as x changes . (Confusingly bilinear forms can be skew – symmetric , symmetric or neither ; differential forms are assumed to be skew – symmetric .)

The pull – back of a 2 – form ω on N by a map $\varphi : M \to N$ is the 2 – form $\varphi^* \omega$ on M defined by

$$\left(\varphi^* \omega\right)(x)(v,w) = \theta\left(\varphi(x)\right)\left(T\varphi(v), T\varphi(w)\right)$$

The push – forward of a 2 – form ω on M by a diffeomorphism $\psi : M \to N$ is the pull – back of ω by ψ^{-1} .

A vector field X can be contracted with a 2 – form ω to get a 1 – form $X \lrcorner \omega$ defined by

$$\left(X \lrcorner \omega\right)(x)(v) = \omega(x)\left(X(x), v\right)$$

for any $v \in T_x M$. A shorthand for this $\left(X \lrcorner \omega\right)(v) = \omega(X, v)$, or just $X \lrcorner \omega = \omega(X, .)$.

The tensor product of two 1 – forms α and β is the 2 – form $\alpha \otimes \beta$ defined by

$$(\alpha \otimes \beta)(v,w) = \alpha(v)\beta(w)$$

for all $v, w \in T_x^* M$.

The wedge product of two 1 – forms α and β is the skew – symmetric 2 – form $\alpha \wedge \beta$ defined by

$$(\alpha \wedge \beta)(v,w) = \alpha(v)\beta(w) - \alpha(w)\beta(v) .$$

The differential df of a real – valued function is also called the exterior derivative of f . In this context , real – valued functions can be called 0 – forms . The exterior derivative is a linear operation form 0 – forms to 1 – forms that satisfies the Leibniz identity , a.k.a the product rule ,

$$d(fg) = fdg + gdf$$

The exterior derivative of a 1 – form is an alternating 2 – form , defined as follows:

$$d\left(a_i dx^i\right)=\frac{\partial a_i}{\partial x^j}dx^j\wedge dx^i\ .$$

Exterior derivative is a linear operation from 1 – forms to 2 – forms. The following identity is easily checked:

$$d\left(df\right)=0$$

for all scalar functions f .

Unless otherwise specified , n – forms are assumed to be alternating . Wedge products and contractions generalise .

It is a fact that all n – forms are linear combinations of wedge products of 1 – forms . Thus we can define exterior derivative recursively by the properties

$$d\left(\alpha\wedge\beta\right)=d\alpha\wedge\beta+\left(-1\right)^k\alpha\wedge d\beta\ ,$$

for all k – forms α and all forms β, and

$$d\,o\,d=0$$

In local coordinates , if $\alpha=\alpha_{i_1\dots i_k}dx^{i_1}\wedge\dots\dots\wedge dx^{i_k}$ (sum over all $i_1<\dots\dots<i_k$) , then

$$d\alpha=\frac{\partial\alpha_{i_1\dots i_k}}{\partial x^j}dx^j\wedge dx^{i_1}\wedge\dots\dots\wedge dx^{i_k}$$

The Lie derivative of an n – form θ in the direction of the vector field X is defined as

$$£_X\,\theta=\frac{d}{dt}\varphi_t^*\theta\,\bigg|_{t=0}\ ,$$

where φ is the flow of X .

Pull – back commutes with the operations $d\ ,\lrcorner,\wedge$ and Lie derivative .

In Cartan's magic formula:

$$£_X\,\alpha=d\left(X\lrcorner\alpha\right)+X\lrcorner d\alpha$$

This looks even more magic when written using the notation ${}^iX^{\alpha}=X\lrcorner\alpha$:

$$£_X=di_X+i_Xd$$

An n – form α is closed if $d\alpha = 0$, and exact if $\alpha = d\beta$ for some β . All exact forms are closed (since $d\circ d = 0$) , but the converse is false . It is true that all closed forms are locally exact ; this is the Poincare' Lemma .

Remark (2.2.7):

For a survey of the basis definitions , properties , and operations on differential forms , as well as useful of tables of relations between differential calculus and vector calculus .

Now we discuss the Euler – Lagrange equations of manifolds .

In Theorem (2.2.4) ,

$$\delta\mathcal{S} := \frac{d}{ds}\Big|_{s=0} \mathcal{S}\left[c(.,s)\right] = \mathrm{D}\mathcal{S}\left[q(.)\right].\delta \mathrm{q}(.)$$

$$= \int_a^b \Upsilon L(L)\left(q,\dot{q},\ddot{q}\right).\delta q\, dt + \underbrace{\Theta_L(q,\dot{q}).\delta q\Big|_a^b}_{cf\ .Noether's Theorem} \qquad (2.19)$$

where

$$\delta q(t) = \frac{d}{ds}\Big|_{s=0} q(t,s),$$

the map $\Upsilon\mathcal{L}: \ddot{Q} \to T^*Q$ is called the Euler – Lagrange operator and its expression in local coordinates is

$$\Upsilon\mathcal{L}(q,\dot{q},\ddot{\mathrm{q}})_i = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} .$$

One understands that the formal time derivative is taken in the second summand and everything is expressed as a function of $(q,\dot{q},\ddot{q})$.

Theorem (2.2.8): (Noether Symmetries and Conservation Laws)

If the action variation in (2.19) vanishes $\delta\mathcal{S}=0$ because of a symmetry transformation which does not preserve the end points and the Euler – Lagrange equations hold , then the term marked cf.Noether's Theorem must also vanish . However , vanishing of this term now is interpreted as a constant of motion . Namely , the term ,

$$A(v,w) := \langle \mathbb{F}L(v), w\rangle, \text{ or , in coordinates } A(q,\dot{q},\delta q) = \frac{\partial L}{\partial \dot{q}^i}\delta q^i ,$$

is constant for solutions of the Euler – Lagrange equations . In particular , in the PDE (Partial Differential Equation) setting one must also include the transformation of the volume element in the action principle .

Remark (2.2.9):

Conservation of energy results from Noether's Theorem if , in Hamilton's principle , the variations are chosen as

$$\delta q(t) = \frac{d}{ds}\bigg|_{s=0} q(t,s) \ ,$$

Corresponding to symmetry of the Lagrangian under reparametrizations of time along the given curve $q(t) \to q(\tau(t,s))$.

The canonical Lagrangian one – form and two – form . The one – form Θ_L , whose existence and uniqueness is guaranteed by Theorem (2.2.4) , appears as the boundary term of the derivative of the action integral , when the endpoints of the curves on the configuration manifold are free . In finite dimensions , its local expression is

$$\Theta_L(q,\dot{q}) := \frac{\partial L}{\partial \dot{q}^i}\mathbf{d}q^i \qquad \left(= p_i(q,\dot{q})\,\mathbf{d}q^i\right) .$$

The corresponding closed two – form $\Omega_L = \mathbf{d}\Theta_L$ obtained by taking its exterior derivative may be expressed as

$$\Omega_L := -\mathbf{d}\Theta_L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\mathbf{d}q^i \wedge \mathbf{d}q^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\mathbf{d}q^i \wedge \mathbf{d}\dot{q}^j \qquad \left(= \mathbf{d}p_i(q,\dot{q}) \wedge \mathbf{d}q^i\right) .$$

These coefficients may be written as the $2n \times 2n$ skew – symmetric matrix

$$\Omega_L = \begin{pmatrix} \mathcal{A} & \dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \\ -\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} & 0 \end{pmatrix} \tag{2.20}$$

where $\mathcal{A}$ is the skew – symmetric $n \times n$ matrix $\left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\right) - \left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\right)^T$.

Non – degeneracy of Ω_L is equivalent to the invertiblility of the matrix $\left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$.

Definition (2.2.10):

The Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ is the smooth map near the identity defined by

$$\left\langle \mathbb{F}L\left(v_q\right), w_q \right\rangle := \left.\frac{d}{ds}\right|_{s=0} L\left(v_q + s w_q\right).$$

In the finite dimensional case , the local expression of $\mathbb{F}L$ is

$$\mathbb{F}L\left(q^i, \dot{q}^i\right) = \left(q^i, \frac{\partial L}{\partial \dot{q}^i}\right) = \left(q^i, p_i\left(q, \dot{q}\right)\right).$$

If the skew – symmetric matrix (2.20) is invertible , the Lagrangian L is said to be regular . In this case , by the implicit function theorem , $\mathbb{F}L$ is locally invertible . If $\mathbb{F}L$ is a diffeomorphism , L is called hyperregular .

Definition (2.2.11):

Given a Lagrangian L , the action of L is the map $A : TQ \to \Re$ given by

$$A(v) := \left\langle \mathbb{F}L(v), v \right\rangle \quad , \text{ or , in coordinates } A(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i \quad , \qquad (2.21)$$

and the energy of L is

$$E(v) := A(v) - L(v) \text{ , or , in coordinates } E(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(q, \dot{q}). \qquad (2.22)$$

Definition (2.2.12):

A vector field Z on TQ is called a Lagrangian vector field if

$$\Omega_L(v)\left(Z(v), w\right) = \left\langle \mathbf{d}E(v), w \right\rangle,$$

for all $v \in T_q Q$, $w \in T_v(TQ)$.

Proposition (2.2.13):

The energy is conserved along the flow of a Lagrangian vector field Z .

Proof :

Let $v(t) \in TQ$ be an integral curve of Z . Skew – symmetry of Ω_L implies

$$\frac{d}{dt} E(v(t)) = \langle \mathbf{d}E(v(t)), \dot{v}(t) \rangle = \langle \mathbf{d}E(v(t)), Z(v(t)) \rangle$$

$$= \Omega_L(v(t))\big(Z(v(t)), Z(v(t))\big) = 0 \,.$$

Thus , $E(v(t))$ is constant in t .

Recall that a Lagrangian L is said to be hyperregular if its Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ is a diffeomorphism .

The equivalence between the Lagrangian and Hamiltonian formulations for hyperregular Lagrangians and Hamiltonians is summarized below ,

(a) Let L be a hyperregular Lagrangian on TQ and $H = E \circ (\mathbb{F}L)^{-1}$, where E is the energy of L and $(\mathbb{F}L)^{-1} : T^*Q \to TQ$ is inverse of the Legendre transformation . Then the Lagrangian vector field Z on TQ and the Hamiltonian vector field X_H on T^*Q are related by the identity

$$(\mathbb{F}L)^* X_H = Z \,.$$

Furthermore , if $c(t)$ is an integral curve of Z and $d(t)$ an integral curve of X_H with $\mathbb{F}L(c(0)) = d(0)$, then $\mathbb{F}L(c(t)) = d(t)$ and their integral curves coincide on the manifold Q . That is , $\tau_Q(c(t)) = \pi_Q(d(t)) = \gamma(t)$, where $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ are the canonical bundle projections .
In particular , the pull back of the inverse Legendre transformation $\mathbb{F}L^{-1}$ induce a one – form Θ and a closed two – form Ω on T^*Q by

$$\Theta = \left(\mathbb{F}L^{-1}\right)^* \Theta_L \;\; , \Omega = -\mathbf{d}\Theta = \left(\mathbb{F}L^{-1}\right)^* \Omega_L \;\; .$$

In coordinates , these are the canonical presymplectic and symplectic forms , respectively ,

$$\Theta = p_i \mathbf{d}q^i \quad , \quad \Omega = -\mathbf{d}\Theta = \mathbf{d}p_i \wedge \mathbf{d}q^i \;\; .$$

(b) A Hamiltonian $H : T^*Q \to \Re$ is said to be hyperregular if the smooth map $\mathbb{F}H : T^*Q \to TQ$ defined by

$$\left\langle \mathbb{F}H(\alpha_q), \beta_q \right\rangle := \left.\frac{d}{ds}\right|_{s=0} H(\alpha_q, s\beta_q) \;\; , \quad \alpha_q, \beta_q \in T_q^*Q \;\; ,$$

is a diffeomorphism . Define the action of H by $G := \langle \Theta, X_H \rangle$. If H is a hyperregular Hamiltonian then the energies of L and H , and the actions of L and H are related by

$$E = H o (\mathbb{F}H)^{-1} \ , \qquad A = G o (\mathbb{F}H)^{-1} .$$

Also , the Lagrangian $L = A - E$ is hyperregular and $\mathbb{F}L = \mathbb{F}H^{-1}$.

(c) These construction define a bijective correspondence between hyperregular Lagrangians and Hamiltonians .

Remark (2.2.14):

For through discussions of many additional results arising from the Hamilton's principle for hyperregular Lagrangians .

Definition (2.2.15): (Cotangent Lift)

Given two manifolds Q and S related by a diffeomorphism $f : Q \to S$, the cotangent lift $T^*f : T^*S \mapsto T^*Q$ of f is defined by

$$\langle T^*f(\alpha), v \rangle = \langle \alpha, Tf(v) \rangle \tag{2.23}$$

where $\alpha \in T_s^*S$, $v \in T_q Q$, and $s = f(q)$.

Cotangent lifts preserve the action of the Lagrangian L , which we write as

$$\langle \mathbf{p}, \dot{\mathbf{q}} \rangle = \langle \alpha, \dot{\mathbf{s}} \rangle \tag{2.24}$$

where $\mathbf{p} = T^*f(\alpha)$ is the cotangent lift of α under the diffeomorphism f and $\dot{\mathbf{s}} = Tf(\dot{\mathbf{q}})$ is the tangent lift of $\dot{\mathbf{q}}$ under the function f , which is written in Euclidean coordinate components as $q^i \to s^i = f^i(\mathbf{q})$. Preservation of the action in (2.24) yields the coordinate relations ,

(Tangent lift in coordinates) $\quad \dot{s}^j = \dfrac{\partial f^j}{\partial q^i} \dot{q}^i \Rightarrow$

$$p_i = \alpha_k \frac{\partial f^k}{\partial q^i} \quad \text{(Cotangent lift in coordinates)}$$

Thus , in coordinates , the cotangent lift is the inverse transpose of the tangent lift.

Remark (2.2.16):

The cotangent lift of a function preserves the induced action one – form ,

$$\langle \mathbf{p}, \mathbf{dq} \rangle = \langle \alpha, \mathbf{ds} \rangle ,$$

so it is a source of (pre -) symmetric transformations .

An important example of a Lagrangian vector field is the geodesic spray of a Riemannian metric . A Riemannian manifold is a smooth manifold Q endowed with a symmetric nondegenerate covariant tensor g , which is positive define . Thus , on each tangent space $T_q Q$ there is nondegenerate definite inner product defined by pairing with $g(q)$.

If (Q, g) is a Riemannian manifold , there is a natural Lagrangian on it given by the kinetic energy K of the metric g , namely ,

$$K(v) := \tfrac{1}{2} g(q)(v_q, v_q) ,$$

for $q \in Q$ and $v_q \in T_q Q$. In finite dimensions , in a local chart ,

$$K(q, \dot{q}) = \tfrac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j \; .$$

The Legendre transformation is in this case $\mathbb{F}K(v_q) = g(q)(v_q, \cdot)$, for $v_q \in T_q Q$. In coordinates , this is

$$\mathbb{F}K(q, \dot{q}) = \left(q^i , \frac{\partial K}{\partial \dot{q}^i} \right) = \left(q^i , g_{ij}(q) \dot{q}^j \right) =: \left(q^i , p_i \right).$$

The Euler – Lagrange equations become the geodesic equations for the metric g , given (for finite dimensional Q in a local chart)

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0 \quad , \quad i = 1, \ldots\ldots, n \; ,$$

where the three – index quantities

$$\Gamma^h_{jk} = \tfrac{1}{2} g^{hl} \left(\frac{\partial g_{jl}}{\partial q^k} + \frac{\partial g_{kl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right) , \quad \text{with } \; g_{ih} g^{hl} = \delta^l_i \; ,$$

are the Christoffel symbols of the Levi – Civita connection on (Q, g).

Remark (2.2.17):

A classic problem is to determine the metric tensors $g_{ij}(q)$ for which these geodesic equations admit enough additional conservation laws to be integrable.

Remark (2.2.18):

The Lagrangian vector field associated to K is called the geodesic spray . Since the Legendre transformation is a diffeomorphism (in finite dimensions or in finite dimensions if the metric is assumed to be strong) ,the geodesic spray is always a second order equation .

The variation approach to geodesic recovers the classical formulation using covariant derivatives , as follows . Let $\mathfrak{X}(Q)$ denote the set of vector fields on the manifold Q . The covariant derivative

$$\nabla : \mathfrak{X}(Q)\times\mathfrak{X}(Q)\to\mathfrak{X}(Q), \quad (X, Y) \mapsto \nabla_X (Y),$$

of the Levi – Civita connection on (Q, g) is given in local charts by

$$\nabla_X (Y) = \Gamma_{ij}^k X^i Y^j \frac{\partial}{\partial q^k} + X^i \frac{\partial Y^k}{\partial q^i} \frac{\partial}{\partial q^k} .$$

If $c(t)$ is a curve on Q and $Y \in \mathfrak{X}(Q)$, the covariant derivative of Y along $c(t)$ is defined by

$$\frac{DY}{Dt} := \nabla_{\dot{c}} Y ,$$

or locally ,

$$\left(\frac{DY}{Dt}\right)^k = \Gamma_{ij}^k (c(t))\dot{c}^i(t) Y^j (c(t)) + \frac{d}{dt} Y^k (c(t)).$$

A vector field is said to be parallel transported along $c(t)$ if

$$\frac{DY}{Dt} = 0 .$$

Thus $\dot{c}(t)$ is parallel transported along $c(t)$ if and only if

$$\ddot{c}^i + \Gamma_{jk}^i \dot{c}^j \dot{c}^k = 0$$

In classical differential geometry a geodesic is defined to be a curve $c(t)$ in Q whose tangent vector $\dot{c}(t)$ is parallel transported along $c(t)$. As the expression above shows , geodesics are integral curves of the Lagrangian vector field defined by the kinetic energy of g .

Definition (2.2.19):

A classical mechanical system is given by a Lagrangian of the form $L(v_q) = K(v_q) - V(v_q)$, for $v_q \in T_q Q$. The smooth function $V : Q \to R$ is called the potential energy . The total energy of this system is given by $E = K + V$ and the Euler – Lagrange equations (which are always second order for a hyperregular Lagrangian) are

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k + g^{il} \frac{\partial V}{\partial q^l} = 0 \quad , \quad i = 1, \ldots\ldots\ldots, n \, ,$$

where g^{ij} are the entries of the inverse matrix of (g_{ij}).

Definition (2.2.20):

If $Q = \Re^3$ and the metric is given by $g_{ij} = \delta_{ij}$, these equations are Newton's equations of motion (2.3) of a partial in a potential field .

Remark (2.2.21): (Gauge Invariance)

The Euler – Lagrange equations are unchanged under

$$L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) \to L' = L + \frac{d}{dt} \gamma(\mathbf{q}(t), \dot{\mathbf{q}}(t)), \qquad (2.24)$$

for any function $\gamma : \Re^{6N} = \{(\mathbf{q}, \dot{\mathbf{q}}) : \mathbf{q}, \dot{\mathbf{q}} \in \Re^{3N}\} \to \Re$.

Remark (2.2.22): (Generalized Coordinate Theorem)

The Euler – Lagrange equations are unchanged in form under any smooth invertible mapping $f : \{\mathbf{q} \to \mathbf{s}\}$. That is , with

$$L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = \tilde{L}(\mathbf{s}(t), \dot{\mathbf{s}}(t)), \qquad (2.25)$$

Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)-\frac{\partial L}{\partial \mathbf{q}}=0 \Leftrightarrow \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\mathbf{s}}}\right)-\frac{\partial \tilde{L}}{\partial \mathbf{s}}=0 . \tag{2.26}$$

Example (2.2.23): (Charged Particle in Magnetic Field)

Consider a particle of charged e and mass m moving in a magnetic field $\mathbf{B}$, where $\mathbf{B}=\nabla\times\mathbf{A}$ is a given magnetic field on $\Re^3$ .The Lagrangian for the motion is given by the " minimal coupling " prescription (jay – dot – ay)

$$L(\mathbf{q},\dot{\mathbf{q}})=\frac{m}{2}\|\dot{\mathbf{q}}\|^2+\frac{e}{c}\mathbf{A}(\mathbf{q}).\dot{\mathbf{q}} \, ,$$

in which the constant c is the speed of light . The derivatives of this Lagrangian are

$$\frac{\partial L}{\partial \dot{\mathbf{q}}}=m\,\dot{\mathbf{q}}+\frac{e}{c}\mathbf{A}=:\mathbf{p} \quad \text{and} \quad \frac{\partial L}{\partial \mathbf{q}}=\frac{e}{c}\nabla\mathbf{A}^T.\dot{\mathbf{q}}$$

Hence , the Euler – Lagrange equations for this system are

$$m\,\ddot{\mathbf{q}}=\frac{e}{c}\left(\nabla\mathbf{A}^T.\dot{\mathbf{q}}-\nabla\mathbf{A}.\dot{\mathbf{q}}\right)=\frac{e}{c}\dot{\mathbf{q}}\times\mathbf{B}$$

(Newton's equations for the Lorentz force) . The Lagrangian L is hyperregular , because

$$\mathbf{p}=\mathbb{F}L(\mathbf{q},\dot{\mathbf{q}})=m\,\dot{\mathbf{q}}+\frac{e}{c}\mathbf{A}(\mathbf{q})$$

has the inverse

$$\dot{\mathbf{q}}=\mathbb{F}H(\mathbf{q},\mathbf{p})=\frac{1}{m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{q})\right).$$

The corresponding Hamiltonian is given by the invertible change of variables ,

$$\mathrm{H}(\mathbf{q},\mathbf{p})=\mathbf{p}.\dot{\mathbf{q}}-L(\mathbf{q},\dot{\mathbf{q}})=\frac{1}{2m}\left\|\mathbf{p}-\frac{e}{c}\mathbf{A}\right\|^2 . \tag{2.27}$$

The Hamiltonian H is hyperregular since

$$\dot{\mathbf{q}}=\mathbb{F}H(\mathbf{q},\mathbf{p})=\frac{1}{m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right) \quad \text{has the inverse} \; \mathbf{p}=\mathbb{F}L(\mathbf{q},\dot{\mathbf{q}})=m\,\dot{\mathbf{q}}+\frac{e}{c}\mathbf{A} \; .$$

The canonical equations for this Hamiltonian recover Newton's equations for the Lorentz force law .

Example (2.2.24): (Changed Particle in a Magnetic Field by The Kaluza – Klein Construction)

Although the minimal – coupling Lagrangian is not expressed as the kinetic energy of a metric , Newton's equations for the Lorentz force law may still be obtained as geodesic equations . This is accomplished by suspending them in a higher dimensional space via the Kaluza – Klein construction , which proceeds as follows .

Let Q_{KK} be the manifold $\Re^3 \times S^1$ with variables $(\mathbf{q},\theta)$. On Q_{KK} introduce the one – form $A+\mathbf{d}\theta$ (which defines a connection one – form on the trivial circle bundle $\Re^3 \times S^1 \to \Re^3$) and introduce the Kaluza – Klein Lagrangian $L_{KK} : TQ_{KK} = T\,\Re^3 \times TS^1 \mapsto \Re$ as

$$L_{KK}\left(\mathbf{q},\theta,\dot{\mathbf{q}},\dot{\theta}\right) = \frac{1}{2} m \left\|\dot{\mathbf{q}}\right\|^2 + \frac{1}{2}\left\|\left\langle \mathbf{A}+\mathbf{d}\theta,\left(\mathbf{q},\dot{\mathbf{q}},\theta,\dot{\theta}\right)\right\rangle\right\|^2$$

$$= \frac{1}{2} m \left\|\dot{\mathbf{q}}\right\|^2 + \frac{1}{2}\left(\mathbf{A}.\dot{\mathbf{q}}+\dot{\theta}\right)^2 .$$

The Lagrangian L_{KK} is positive definite in $\left(\dot{\mathbf{q}},\dot{\theta}\right)$; so it may be regarded as the kinetic energy of a metric , the Kaluza – Klein metric on TQ_{KK} . (This construction fits the idea of $U(1)$ gauge symmetry for electromagnetic fields in $\Re^3$. It can be generalized to a principle bundle with compact structure group endowed with a connection . The Kaluza – Klein Lagrangian in this generalization leads to Wong's equations for a color – charged particle moving in a classical Yong – Mills field .) The Legendre transformation for L_{KK} gives the momenta

$$\mathbf{p} = m\dot{\mathbf{q}} + \left(\mathbf{A}.\dot{\mathbf{q}}+\dot{\theta}\right)\mathbf{A} \quad \text{and} \quad \pi = \mathbf{A}.\dot{\mathbf{q}}+\dot{\theta} . \tag{2.28}$$

Since L_{KK} does not depend on θ, the Euler – Lagrange equation

$$\frac{d}{dt}\frac{\partial L_{KK}}{\partial \dot{\theta}} = \frac{\partial L_{KK}}{\partial \theta} = 0 ,$$

where $\pi = \partial L_{KK} / \partial \dot{\theta}$ is conserved . The charge is now defined by $e := c\pi$. The Hamiltonian H_{KK} associated to L_{KK} by the Legendre transformation (2.28) is

$$H_{KK}\left(\mathbf{q},\theta,\mathbf{p},\pi\right)=\mathbf{p}.\dot{\mathbf{q}}+\pi\dot{\theta}-L_{KK}\left(\mathbf{q},\dot{\mathbf{q}}.\theta,\dot{\theta}\right)$$

$$=\mathbf{p}.\frac{1}{m}\left(\mathbf{p}-\pi\mathbf{A}\right)+\pi\left(\pi-\mathbf{A}.\dot{\mathbf{q}}\right)-\frac{1}{2}m\left\|\dot{\mathbf{q}}\right\|^{2}-\frac{1}{2}\pi^{2}$$

$$=\mathbf{p}.\frac{1}{m}\left(\mathbf{p}-\pi\mathbf{A}\right)+\frac{1}{2}\pi^{2}-\pi\mathbf{A}.\frac{1}{m}\left(\mathbf{p}-\pi\mathbf{A}\right)-\frac{1}{2m}\left\|\mathbf{p}-\pi\mathbf{A}\right\|^{2}$$

$$=\frac{1}{2m}\left\|\mathbf{p}-\pi\mathbf{A}\right\|^{2}+\frac{1}{2}\pi^{2} \qquad (2.29)$$

On the constant level set $\pi=e/c$, the Kaluza – Klein Hamiltonian H_{KK} is a function of only the variables $(\mathbf{q},\mathbf{p})$ and is equal to the Hamiltonian (2.27) for charged particle motion under the Lorentz force up to an additive constant . This example provides an easy but fundamental illustration of the geometry of (Lagrangian) reduction by symmetry . The canonical equations for the Kaluza – Klein Hamiltonian H_{KK} now reproduce Newton's equations for the Lorentz force law .

Chapter (3)

The Rigid Body Equations on $SO(n)$, Heavy Top Equations and Euler – Poincare' (EP) Reduction Theorem

Section (3.1): The Rigid Body in three Dimensions, Momentum Maps and Rigid Body Equations on $SO(n)$

We start with some the rigid body in three dimensions .

In the absence of external torques , Euler's equations for rigid body motion are :

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3 , \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1 , \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2 , \end{aligned} \tag{3.1}$$

or , equivalently ,

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega} \quad ,$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity vector and I_1, I_2, I_3 are the moments of intertia of the rigid body .

Answer (3.1.1): (Lagrangian Formulation)

The Lagrangian answer is this : These equations may be expressed in Euler – Poincare' form on the Lie algebra $\Re^3$ using the Lagrangian

$$l(\boldsymbol{\Omega}) = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) = \frac{1}{2}\boldsymbol{\Omega}^T . \mathbb{I}\boldsymbol{\Omega} \ , \tag{3.2}$$

which is the (rotational) kinetic energy of the rigid body .

Proposition (3.1.2):

The Euler rigid body equations are equivalent to the rigid body action principle for a reduced action

$$\delta S_{red} = \delta\int_a^b l(\boldsymbol{\Omega})dt = 0 , \tag{3.3}$$

where variations $\boldsymbol{\Omega}$ are restricted to be of the form

$$\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega} \times \boldsymbol{\Sigma}, \tag{3.4}$$

in which $\boldsymbol{\Sigma}(t)$ is a curve in $\Re^3$ that vanishes at the endpoints in time .

Proof:

Since $l(\boldsymbol{\Omega}) = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle$, and $\mathbb{I}$ is symmetric, we obtain

$$\delta\int_a^b l(\boldsymbol{\Omega})dt = \int_a^b \langle \mathbb{I}\boldsymbol{\Omega}, \delta\boldsymbol{\Omega}\rangle dt$$

$$= \int_a^b \left\langle \mathbb{I}\boldsymbol{\Omega}, \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega}\times\boldsymbol{\Sigma}\right\rangle dt$$

$$= \int_a^b \left[\left\langle -\frac{d}{dt}\mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Sigma}\right\rangle + \langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\times\boldsymbol{\Sigma}\rangle\right] dt$$

$$= \int_a^b \left\langle -\frac{d}{dt}\mathbb{I}\boldsymbol{\Omega} + \mathbb{I}\boldsymbol{\Omega}\times\boldsymbol{\Omega}, \boldsymbol{\Sigma}\right\rangle dt \ ,$$

upon integrating by parts and using the endpoint conditions, $\boldsymbol{\Sigma}(b) = \boldsymbol{\Sigma}(a) = 0$.
Since $\boldsymbol{\Sigma}$ is otherwise arbitrary, then

$$-\frac{d}{dt}(\mathbb{I}\boldsymbol{\Omega}) + \mathbb{I}\boldsymbol{\Omega}\times\boldsymbol{\Omega} = 0\,,$$

which are Euler's equations (3.1).

Let's derive this variational principle from the standard Hamilton's principle.

An element $\boldsymbol{R} \in SO(3)$ gives the configuration of the body as a map of a reference configuration $\mathcal{B} \in \Re^3$ to the current configuration $\boldsymbol{R}(\mathcal{B})$; the map $\boldsymbol{R}$ takes a reference or label point $X \in \mathcal{B}$ to a current point $x = \boldsymbol{R}(X) \in \boldsymbol{R}(\mathcal{B})$.

When the rigid body is in motion, the matrix $\boldsymbol{R}$ is time – dependent . Thus,

$$x(t) = \boldsymbol{R}(t)X$$

with $\boldsymbol{R}(t)$ a curve parametrized by time in $SO(3)$. The velocity of a point of the body is

$$\dot{x}(t) = \dot{\boldsymbol{R}}(t)X = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1}(t)x(t).$$

Since $\boldsymbol{R}$ is an orthogonal matrix, $\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}$ and $\dot{\boldsymbol{R}}\boldsymbol{R}^{-1}$ are skew matrices. Consequently,

$$\dot{x} = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1}x = \omega \times x \ . \tag{3.5}$$

This formula defines the spatial angular velocity vector ω. Thus, ω is essentially given by right translation of $\dot{\boldsymbol{R}}$ to the identity. That is, the vector

$$\omega = \left(\dot{\boldsymbol{R}}\boldsymbol{R}^{-1}\right)^{\wedge}.$$

The corresponding body angular velocity is defined by

$$\boldsymbol{\Omega} = R^{(-1)}\omega \ , \tag{3.6}$$

so that $\boldsymbol{\Omega}$ is the angular velocity relative to a body fixed frame. Notice that

$$\begin{aligned} \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\ X &= \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\boldsymbol{R}^{-1}x = \boldsymbol{R}^{-1}(\omega \times x) \\ &= \boldsymbol{R}^{-1}\omega \times \boldsymbol{R}^{-1}x = \boldsymbol{\Omega} \times X \ , \end{aligned} \tag{3.7}$$

so that $\boldsymbol{\Omega}$ is given by left translation of $\dot{\boldsymbol{R}}$ to the identity. That is, the vector

$$\boldsymbol{\Omega} = \left(\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right)^{\wedge}.$$

The kinetic energy is obtained by summing up $|\dot{x}|^2/2$ (where $|.|$ denotes the Euclidean norm) over the body. This yields

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(X)\left|\dot{\boldsymbol{R}}X\right|^2 d^3X \ , \tag{3.8}$$

in which ρ is a given mass density in the reference configuration. Since

$$\left|\dot{\boldsymbol{R}}X\right| = |\omega \times x| = \left|\boldsymbol{R}^{-1}(\omega \times x)\right| = |\boldsymbol{\Omega} \times X|,$$

K is a quadratic function of $\boldsymbol{\Omega}$. Writing

$$K = \frac{1}{2}\boldsymbol{\Omega}^T . \mathbb{I}\boldsymbol{\Omega} \tag{3.9}$$

defines the moment of inertia tensor $\mathbb{I}$, which, provided the body does not degenerate to a line, is a positive – define (3×3) matrix, or better, a quadratic form. This quadratic form can be diagonalized by a change of basis; thereby defining the principle axes and moments of inertia. In this basis, we write $\mathbb{I} = \operatorname{diag}(I_1, I_2, I_3)$.

The function K is taken to be the Lagrangian of the system on $TSO(3)$ (and by means of the Legendre transformation we obtain the corresponding Hamiltonian description on $T^*SO(3)$). Notice that K in equation (3.8) is left (not right) invariant on $TSO(3)$, since

$$\boldsymbol{\Omega} = \left(\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right)^{\wedge}.$$

It follows that the corresponding Hamiltonian will also be left invariant.

In the framework of Hamilton's principle, the relation between motion in $\boldsymbol{R}$ space and motion in body angular velocity (or $\boldsymbol{\Omega}$) space is follows.

Proposition (3.1.3):

The curve $\boldsymbol{R}(t) \in SO(3)$ satisfies the Euler – Lagrange equations for

$$L\left(\boldsymbol{R}, \dot{\boldsymbol{R}}\right) = \frac{1}{2}\int_{\mathcal{B}} \rho(X)\left|\dot{\boldsymbol{R}}X\right|^2 d^3X \ , \tag{3.10}$$

if and only if $\boldsymbol{\Omega}(t)$ defined by $\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{v} \in \Re^3$ satisfies Euler's equations

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}\,. \tag{3.11}$$

The proof of this relation will illustrate how to reduce variational principles using their symmetry groups. By Hamilton's principle, $\boldsymbol{R}(t)$ satisfies the Euler – Lagrange equations, if and only if

$$\delta\int L\left(\boldsymbol{R}, \dot{\boldsymbol{R}}\right)dt = 0\,.$$

Let $l(\boldsymbol{\Omega}) = \frac{1}{2}(\mathbb{I}\boldsymbol{\Omega}).\boldsymbol{\Omega}$, so that $l(\boldsymbol{\Omega}) = L\left(\boldsymbol{R}, \dot{\boldsymbol{R}}\right)$ where the matrix $\boldsymbol{R}$ and the vector $\boldsymbol{\Omega}$ are related by the hat map, $\boldsymbol{\Omega} = \left(\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right)^{\wedge}$. Thus, the Lagrangian L is left $SO(3)$– invariant. That is,

$$l(\boldsymbol{\Omega}) = L\left(\boldsymbol{R}, \dot{\boldsymbol{R}}\right) = L\left(\mathbf{e}, \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right).$$

To see how we should use this left – invariance to transform Hamilton's principle, define the skew matrix $\hat{\boldsymbol{\Omega}}$ by $\hat{\boldsymbol{\Omega}}\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{v}$ for any $\mathbf{v} \in \Re^3$.

We differentiate the relation $\boldsymbol{R}^{-1}\dot{\boldsymbol{R}} = \hat{\boldsymbol{\Omega}}$ with respect to $\boldsymbol{R}$ to get

$$-\mathbf{R}(\delta\mathbf{R})\mathbf{R}^{-1}\dot{\mathbf{R}} + \mathbf{R}^{-1}\left(\delta\dot{\mathbf{R}}\right) = \delta\boldsymbol{\Omega}\,. \tag{3.12}$$

Let the skew matrix $\hat{\boldsymbol{\Sigma}}$ be defined by

$$\hat{\boldsymbol{\Sigma}} = \boldsymbol{R}^{-1}\delta\boldsymbol{R}\,, \tag{3.13}$$

And define the vector $\boldsymbol{\Sigma}$ by

$$\hat{\boldsymbol{\Sigma}}\mathbf{v} = \boldsymbol{\Sigma}\times\mathbf{v}\,. \tag{3.14}$$

Note that

$$\dot{\hat{\boldsymbol{\Sigma}}} = -\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\boldsymbol{R}^{-1}\delta\boldsymbol{R} + \boldsymbol{R}^{-1}\delta\dot{\boldsymbol{R}}\,,$$

so

$$\boldsymbol{R}^{-1}\delta\dot{\boldsymbol{R}} = \dot{\hat{\boldsymbol{\Sigma}}} + \boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\,\hat{\boldsymbol{\Sigma}}\,. \tag{3.15}$$

Substituting (3.15) and (3.13) into (3.12) gives

$$-\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Omega}} + \dot{\hat{\boldsymbol{\Sigma}}} + \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Sigma}} = \delta\boldsymbol{\Omega}\,,$$

that is,

$$\delta\boldsymbol{\Omega} = \dot{\hat{\boldsymbol{\Sigma}}} + \left[\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\Sigma}}\right]. \tag{3.16}$$

The identity $\left[\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\Sigma}}\right] = (\boldsymbol{\Omega}\times\boldsymbol{\Sigma})^{\wedge}$ holds by Jacobi's identity for the cross product and so

$$\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega}\times\boldsymbol{\Sigma}\,. \tag{3.17}$$

This calculations prove the following:

Theorem (3.1.4):

For a Lagrangian which is left – invariant under $SO(3)$, Hamilton's variational principle

$$\delta S = \delta\int_a^b L\left(\boldsymbol{R}, \dot{\boldsymbol{R}}\right)dt = 0 \tag{3.18}$$

on $TSO(3)$ is equivalent to the reduced variational principle

$$\delta S_{\text{red}} = \delta \int_a^b l(\mathbf{\Omega})dt = 0 \tag{3.19}$$

with $\mathbf{\Omega} = \left(\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right)^{\wedge}$ on $\Re^3$ where the variations $\delta\mathbf{\Omega}$ are of the form

$$\delta\mathbf{\Omega} = \dot{\mathbf{\Sigma}} + \mathbf{\Omega} \times \mathbf{\Sigma},$$

With $\mathbf{\Sigma}(a) = \mathbf{\Sigma}(b) = 0$.

Reconstruction of $\boldsymbol{R}(t) \in SO(3)$, in Theorem (3.1.4), Euler's equations for the rigid body

$$\mathbb{I}\dot{\mathbf{\Omega}} = \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega} ,$$

follow from the reduced variational principle (3.19) for the Lagrangian

$$l(\mathbf{\Omega}) = \frac{1}{2}(\mathbb{I}\mathbf{\Omega}).\mathbf{\Omega} , \tag{3.20}$$

which is expressed in terms of the left – invariant time – dependent angular velocity in the body, $\mathbf{\Omega} \in so(3)$. The body angular velocity $\mathbf{\Omega}(t)$ yields the tangent vector $\dot{\boldsymbol{R}}(t) \in T_{\boldsymbol{R}(t)}SO(3)$ along the integral curve in the relation group $\boldsymbol{R}(t) \in SO(3)$ by the relation,

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{R}(t)\mathbf{\Omega}(t) .$$

This relation provides the reconstruction formula. It's solution as a linear differential equation with time – dependent coefficients yields the integral curve $\boldsymbol{R}(t) \in SO(3)$ for the orientation of the rigid body, once the time dependence of $\mathbf{\Omega}(t)$ is determined from the Euler equations.

A dynamical system on a manifold M

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in \mathrm{M}$$

is said to be in Hamiltonian form, if it can be expressed as

$$\dot{\mathbf{x}}(t) = \{\mathbf{x}, H\}, \qquad \text{for} \quad \mathrm{H{:}M} \mapsto \Re ,$$

in terms of a Poisson bracket operation,

$$\{.,.\}: \mathcal{F}(M)\times\mathcal{F}(M)\mapsto\mathcal{F}(M),$$

which is bilinear, skew – symmetric and satisfies the Jacobi identity and (usually) the Leibniz rule.

As we shall explain, reduced equations arising from group – invariant Hamilton's principles on Lie groups are naturally Hamiltonian. If we Legendre transform our reduced Lagrangian for the $SO(3)$ left invariant variational principle (3.19) for rigid body dynamics, then its simple, beautiful and well – known Hamiltonian formulation emerges.

Definition (3.1.5):

The Legendre transformation $\mathbb{F}l : \mathrm{so}(3)\to so(3)^*$ is defined by

$$\mathbb{F}l(\Omega)=\frac{\delta l}{\delta\Omega}=\Pi .$$

The Legendre transformation defines the body angular momentum by the variations of the rigid – body's reduced Lagrangian with respect to the body angular velocity. For the Lagrangian in (3.20), the $\Re^3$ components of the body angular momentum are

$$\Pi_i = I_i\Omega_i = \frac{\partial l}{\partial\Omega_i}, \qquad i=1,2,3 . \tag{3.21}$$

Let

$$h(\Pi):=\langle\Pi,\Omega\rangle - l(\Omega),$$

where the pairing $\langle .,.\rangle : \mathrm{so}(3)^*\times so(3)\to\Re$ is understood in components as the vector dot product on $\Re^3$

$$\langle\Pi,\Omega\rangle := \mathbf{\Pi}.\mathbf{\Omega} .$$

Hence , are finds the expected expression for the rigid body Hamiltonian

$$h=\frac{1}{2}\mathbf{\Pi}.\mathbb{I}^{-1}\mathbf{\Pi} := \frac{\Pi_1^2}{2I_1}+\frac{\Pi_2^2}{2I_2}+\frac{\Pi_3^2}{2I_3} . \tag{3.22}$$

The Legendre transform $\mathbb{F}l$ for this case is a diffeomorphism, so we may solve for

$$\frac{\partial h}{\partial \Pi} = \Omega + \left\langle \Pi, \frac{\partial \Omega}{\partial \Pi} \right\rangle - \left\langle \frac{\partial l}{\partial \Omega}, \frac{\partial \Omega}{\partial \Pi} \right\rangle = \Omega .$$

In $\Re^3$ coordinates, this relation expresses the body angular velocity as the derivative of the reduced Hamiltonian with respect to the body angular momentum, namely (introducing grad – notation),

$$\nabla_\Pi h := \frac{\partial h}{\partial \mathbf{\Pi}} = \mathbf{\Omega} .$$

Hence ,the reduced Euler – Lagrange equations for l may be expressed equivalently in angular momentum vector components in $\Re^3$ and Hamiltonian h as:

$$\frac{d}{dt}(\mathbb{I}\mathbf{\Omega}) = \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega} \Leftrightarrow \dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \nabla_\Pi h := \{\mathbf{\Pi}, h\} .$$

This expression suggests we introduce the following rigid body Poisson bracket on functions of the $\mathbf{\Pi}$'s:

$$\{f , h\}(\mathbf{\Pi}) := -\mathbf{\Pi}.(\nabla_\Pi f \times \nabla_\Pi h) . \tag{3.23}$$

For the Hamiltonian (3.22), one checks that the Euler equations in terms of the rigid – body angular momenta,

$$\dot{\Pi}_1 = \frac{I_2 - I_3}{I_2 I_3} \Pi_2 \Pi_3 \quad , \quad \dot{\Pi}_2 = \frac{I_3 - I_1}{I_3 I_1} \Pi_3 \Pi_1 \quad , \quad \dot{\Pi}_3 = \frac{I_1 - I_2}{I_1 I_2} \Pi_1 \Pi_2 , \tag{3.24}$$

that is,

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega} . \tag{3.25}$$

are equivalent to

$$\dot{f} = \{f , h\}, \quad \text{with } f = \mathbf{\Pi} .$$

The Poisson bracket proposed (3.23) is example of a Lie Poisson bracket, which we will show separately satisfies the defining relations to be a Poisson bracket.

The rigid body Poisson bracket (3.23) is a special case of the Poisson bracket for functions on $\Re^3$,

$$\{f , h\} = -\nabla c . \nabla f \times \nabla h \tag{3.26}$$

This bracket generates the motion

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h \qquad (3.27)$$

For this bracket the motion takes place along the intersections of level surfaces of the functions c and h in $\Re^3$.In particular, for the rigid body, the motion takes place along intersections of angular momentum spheres $c = \|\mathbf{x}\|^2 / 2$ and energy ellipsoids $h = \mathbf{x}.\mathbb{I}\mathbf{x}$.

Remark (3.1.6):

Consider the $\Re^3$ Poisson bracket

$$\{f, h\} = -\nabla c . \nabla f \times \nabla h \qquad (3.28)$$

Let $c = \mathbf{x}^T . \mathbb{C}\mathbf{x}$ be a quadratic form on $\Re^3$, and let $\mathbb{C}$ be the associated symmetric 3×3 matrix. Determine the conditions on the quadratic function $c(\mathbf{x})$ so that this Poisson bracket will satisfy the Jacobi identity.

Now we discuss the momentum maps.

Symmetries are often associated with conserved quantities. For example, the flow of any $SO(3)$–invariant Hamiltonian vector field on $T^*\Re^3$ conserves angular momentum, $\mathbf{q}\times\mathbf{p}$. More generally, given a Hamiltonian H on a phase space P , and a group action of G on P that conserves H , there is often an associated "momentum map" $J : P \to \mathcal{G}^*$ that is conserved by the flow of the Hamiltonian vector field.

Definition (3.1.7):

A Poisson bracket on a manifold P is a skew – symmetric bilinear operation on

$$\mathcal{F}(P) := C^\infty(P, \Re)$$

The pair $(P, \{.,.\})$ is called a Poisson manifold .

Remark (3.1.8):

The Leibniz identity is sometimes not included in the definition. Note that bilinearity, skew – symmetry and the Jacobi identity are the axioms of a Lie algebra.

In what follows, a Poisson bracket is a binary operation that makes $\mathcal{F}(P)$ into a Lie algebra and also satisfies the Leibniz identity.

Definition (3.1.9):

A Poisson map between two Poisson manifolds is a map

$$\varphi:\left(P_1,\{.,.\}_1\right)\to\left(P_2,\{.,.\}_2\right)$$

the preserves the brackets, meaning

$$\{F o\varphi, G o\varphi\}_1=\{F,G\}_2\, o\varphi \text{, for all } F,G\in\mathcal{F}(P_2).$$

Definition (3.1.10):

An action Φ of G on a Poisson manifold $(P,\{.,.\})$ is canonical if Φ_{g} is a Poisson map for every g, that is,

$$\left\{F o\Phi_{\mathrm{g}}, K o\Phi_{\mathrm{g}}\right\}=\{F,G\}\, o\Phi_{\mathrm{g}}$$

for every $F,K\in\mathcal{F}(P)$.

Definition (3.1.11):

Let $(P,\{.,.\})$ be a Poisson manifold, and let $H:P\to\Re$ be differentiable. The Hamiltonian vector field for H is the vector field X_H defined by

$$X_H(F)=\{F,H\},\ \text{ for all } F\in\mathcal{F}(P)$$

Remark (3.1.12):

X_H is well – defined because of the Leibniz identity and the correspondence between vector fields and derivations.

Remark (3.1.13):

$X_H(F)=\pounds_{X_H}F=\dot{F}$, the Lie derivative of F along the flow of X_H . The equations

$$\dot{F}=\{F,H\}$$

Called "Hamilton's equations" , have already appeared in Theorem(2.1.), and equivalent definition of X_H .

Let G act smoothly on P , and let $\xi \in \vartheta$. Recall that the infinitesimal generator ξ_P is the vector field on P defined by

$$\xi_P(x) = \frac{d}{dt} g(t)x \Big|_{t=0}$$

for some path $g(t)$ in G such that $g(0)=e$ and $g'(0)=\xi$.

Remark (3.1.14):

For matrix groups, we can take $g(t)=\exp(t\xi)$. This works in general for the exponential map of an arbitrary Lie group. For matrix groups,

$$\xi_P(\mathbf{x}) = \frac{d}{dt} \exp(t\xi)\mathbf{x} \Big|_{t=0} = \xi \mathbf{x} \qquad \text{(matrix multiplication).}$$

Example (3.1.15): (The Momentum Map for the Relation Group)

Consider the cotangent bundle of ordinary Euclidean space $\Re^3$. This is the Poisson (symplectic) manifold with coordinates $(\mathbf{q},\mathbf{p}) \in T^*\Re \cong \Re^6$, equipped with the canonical Poisson bracket. An element g of the rotation group $SO(3)$ acts on $T^*\Re^3$ according to

$$g(\mathbf{q},\mathbf{p}) = (g\,\mathbf{q}, g\,\mathbf{p})$$

Set $g(t)=\exp(tA)$, so that $\frac{d}{dt}\Big|_{t=0} g(t) = A$ and the corresponding Hamiltonian vector field is

$$X_A = (\dot{\mathbf{q}},\dot{\mathbf{p}}) = (A\mathbf{q}, A\mathbf{p})$$

where $A \in so(3)$ is a skew – symmetric matrix . The corresponding Hamiltonian equations read

$$\dot{\mathbf{q}} = A\mathbf{q} = \frac{\partial J_A}{\partial \mathbf{p}} \quad , \qquad \dot{\mathbf{p}} = A\mathbf{p} = -\frac{\partial J_A}{\partial \mathbf{q}} .$$

Hence ,

$$J_A(\mathbf{q},\mathbf{p}) = -A\mathbf{p}.\mathbf{q} = a_i \varepsilon_{ijk} p_k q_j = \mathbf{a}.\mathbf{q}\times\mathbf{p} .$$

for a vector $\mathbf{a} \in \Re^3$ with components a_i , $i = 1, 2, 3$. So the momentum map for the rotation group is the angular momentum $J = \mathbf{q} \times \mathbf{p}$.

Example (3.1.16):

Consider angular momentum $J = \mathbf{q} \times \mathbf{p}$, defined on $p = T^*\Re^3$.For every $\xi \in \Re^3$, define

$$J_\xi(\mathbf{q}, \mathbf{p}) := \xi.(\mathbf{q} \times \mathbf{p}) = \mathbf{p}.(\xi \times \mathbf{q})$$

Using Remark (3.1.14),

$$X_{J_\xi}(\mathbf{q}, \mathbf{p}) = \left(\frac{\partial J_\xi}{\partial \mathbf{p}}, \frac{\partial J_\xi}{\partial \mathbf{q}} \right) = (\xi \times \mathbf{q}, \xi \times \mathbf{p}) = \hat{\xi}_{\mathbf{p}}(\mathbf{q}, \mathbf{p}),$$

where the last line is the infinitesimal generator corresponding to $\hat{\xi} \in so(3)$. Now suppose $H : P \to \Re$ is $SO(3)$ - invariant.

We have $\pounds_{\hat{\xi}} H = 0$. It follows that

$$\pounds_{X_H} J_\xi = \{J_\xi, H\} = -\{H, J_\xi\} = -\pounds_{X_{J_\xi}} H = -\pounds_{\xi_p} H = 0 .$$

Since this holds for all ξ, we have shown that J is conserved by the Hamiltonian flow.

In order to generalise this example, we recast it using the hat map $\wedge : \Re^3 \to so(3)$ and the associated map $\sim : (\Re^3)^* \to so(3)^*$, and the standard identification $(\Re^3)^* \cong \Re^3$ via the Euclidean dot product. We consider J as a function from P to $so(3)^*$ given by $J(\mathbf{q}, \mathbf{p}) = (\mathbf{q} \times \mathbf{p})^\sim$. For any $\xi = \hat{\mathbf{v}}$, we define $J_\xi(\mathbf{q}, \mathbf{p}) = \left\langle (\mathbf{q} \times \mathbf{p})^\sim, \hat{\mathbf{v}} \right\rangle = (\mathbf{q} \times \mathbf{p}).\mathbf{v}$. As before, we find that $X_{J_\xi} = \xi_p$ for every ξ, and J is conserved by the Hamiltonian flow. We take the first property, $X_{J_\xi} = \xi_p$, as the general definition of a momentum map. The conservation of J follows by the same Poisson bracket calculation as in the example; the result is Noether's Theorem.

Definition (3.1.17):

A momentum map for a canonical action of G on P is a map $J : P \to \vartheta^*$ such that, for every $\xi \in \vartheta$, the map $J_\xi : P \to \Re$ defined by $J_\xi(P) = \langle J(P), \xi \rangle$ satisfies

$$X_{J_\xi} = \xi_p$$

Theorem (3.1.18): (Noether's Theorem)

Let G act canonically on $(P, \{.,.\})$ with momentum map J. If H is G – invariant , then J is conserved by the flow of X_H.

Proof :

For every $\xi \in \vartheta$,

$$\pounds_{X_H} J_\xi = \{J_\xi, H\} = -\{H, J_\xi\} = -\pounds_{X_{J_\xi}} H = -\pounds_{\xi_p} H = 0 .$$

Theorem (3.1.19):

Let G act on Q , and by cotangent lifts on T^*Q . Then $J : T^*Q \to \vartheta^*$ defined by, for every $\xi \in \vartheta$,

$$J_\xi(\alpha_q) = \langle \alpha_q, \xi_Q(q) \rangle, \text{ for every } \alpha_q \in T_q^*Q ,$$

is a momentum map (the "standard one") for the G action with respect to the classical Poisson bracket.

Proof :

We need to show that $X_{J_\xi} = \xi_{T^*Q}$, for every $\xi \in \vartheta$. From the definition of Hamiltonian vector fields, this is equivalent to showing that $\xi_{T^*Q}[F] = \{F, J_\xi\}$ for every $F \in \mathcal{F}(T^*Q)$. We verify this for finite – dimensional Q by using cotangent – lifted local coordinates.

$$\frac{\partial J_\xi}{\partial p}(q, p) = \xi_Q(q)$$

$$\frac{\partial J_\xi}{\partial q^i}(q,p)=\left\langle p,\frac{\partial}{q^i}\left(\xi_Q(q)\right)\right\rangle$$

$$=\left\langle p,\frac{\partial}{\partial q^i}\left(\frac{\partial}{\partial t}\Phi_{(\exp(t\xi))}(q)\Big|_{t-0}\right)\right\rangle=\left\langle p,\frac{\partial}{\partial t}\left(\frac{\partial}{\partial q^i}\Phi_{(\exp(t\xi))}(q)\right)\Big|_{t=0}\right\rangle$$

$$=\frac{\partial}{\partial t}\left\langle p,T\,\Phi_{(\exp(t\xi))}\frac{\partial}{\partial q^i}(q)\right\rangle\Big|_{t=0}=\frac{\partial}{\partial t}\left\langle T^*\Phi_{(\exp(t\xi))}p,\frac{\partial}{\partial q^i}(q)\right\rangle\Big|_{t=0}$$

$$=\left\langle -\xi_{T^*Q}(q,p),\frac{\partial}{\partial q^i}(q)\right\rangle$$

$$\frac{\partial J_\xi}{\partial q}(q,p)=-\xi_{T^*Q}(q,p)$$

So for every $F\in\mathcal{F}\left(T^*Q\right)$,

$$\xi_{T^*Q}[F]=\frac{\partial}{\partial t}F\left(\exp(t\xi)q,\exp(t\xi p)\right)\Big|_{t=0}$$

$$=\frac{\partial F}{\partial q}\xi_Q(q)+\frac{\partial F}{\partial p}\xi_{T^*Q}(q,p)=\frac{\partial F}{\partial q}\frac{\partial J_\xi}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial J_\xi}{\partial q}$$

$$=\left\{F,J_\xi\right\}$$

which completes the proof.

Example (3.1.20):

Let $G\subset M_n(\Re)$ be a matrix group, with cotangent – lifted action on $(q,p)\in T^*\Re^n$. For every $g\subset M_n(\Re)$, $q\mapsto gq$. The cotangent – lifted action is $(q,p)\mapsto\left(gq,g^{-1}p\right)$. Thus, writing $g=\exp(t\xi)$, the linearization of this group action yields the vector field

$$X_\xi=\left(\xi q,-\xi^T p\right)$$

The corresponding Hamiltonian equations read

$$\xi q=\frac{\partial J_\xi}{\partial p}\quad,\quad -\xi^T p=-\frac{\partial J_\xi}{\partial q}$$

This yields the momentum map $J(q,p)$ given by

$$J_{\xi}(q,p)=\langle J(q,p),\xi\rangle=p^{T}\xi_{Q}(q)=p^{T}\xi q \ .$$

In coordinates, $p^{T}\xi q=p_{i}\xi_{j}^{i}q^{j}$, so $J(q,p)=q^{i}p_{j}$.

Example (3.1.21):

Calculate the momentum map of the cotangent lifted action of the group of translations of $\Re^{3}$.

Solution:

The element $x\in\Re^{3}$ acts on $\mathbf{q}\in\Re^{3}$ by addition of vectors,

$$x.(\mathbf{q})=\mathbf{q}+x \ .$$

The infinitesimal generator is $\lim_{x\to 0}\frac{d}{dx}(\mathbf{q}+x)=Id$. Thus, $\xi_{\mathbf{q}}=Id$ and

$$\langle J_{k},\xi\rangle=\langle(\mathbf{q},\mathbf{p}),\xi_{\mathbf{q}}\rangle=\langle\mathbf{p},Id\rangle=p_{i}\delta_{k}^{i}=p_{k}$$

This is also Hamiltonian with $J_{\xi}=\mathbf{p}$, so that $\{\mathbf{p},J_{\xi}\}=0$ and $\{\mathbf{q},J_{\xi}\}=Id$.

Example (3.1.22):

Let G act on itself by left multiplication, and by cotangent lifts on $T^{*}G$. We first note that the infinitesimal action on G is

$$\xi_{G}(g)=\frac{d}{dt}\exp(t\xi)g\,\Big|_{t=0}=TR_{g}\xi \ .$$

Let J_{L} be momentum map for this action. Let $\alpha_{g}\in T_{g}^{*}G$. For every $\xi\in\vartheta$, we have

$$\langle J_{L}(\alpha_{g}),\xi\rangle=\langle\alpha_{g},\xi_{G}(g)\rangle=\langle\alpha_{g},TR_{g}\xi\rangle$$

$$=\langle TR_{g}^{*}\alpha_{g},\xi\rangle$$

so $J_{L}(\alpha_{g})=TR_{g}^{*}\alpha_{g}$. Alternatively, writing $\alpha_{g}=T^{*}L_{g^{-1}}\mu$ for some $\mu\in\vartheta^{*}$ we have

$$J_{L}(T^{*}L_{g^{-1}}\mu)=TR_{g}^{*}T^{*}L_{g^{-1}}\mu=Ad_{g^{-1}}^{*}\mu \ .$$

Example (3.1.23):

Consider the previous example for a matrix group G. For any $Q \in G$, the pairing given above allows use to consider any element $P \in T_Q^*G$ as a matrix. The natural pairing of T_Q^*G with T_QG now has the formula,

$$\langle P, A\rangle = -\frac{1}{2} tr\left(P^T A\right), \text{ for all } A \in T_Q G .$$

We compute the cotangent – lifts of the left and right multiplication actions:

$$\left\langle T^*L_Q(P), A\right\rangle = \left\langle P, TL_Q(A)\right\rangle = \langle P, QA\rangle$$

$$= -\frac{1}{2} tr\left(P^T QA\right) = -\frac{1}{2} tr\left(\left(Q^T P\right)^T A\right) = \left\langle Q^T P, A\right\rangle$$

$$\left\langle T^*R_Q(P), A\right\rangle = \left\langle P, TR_Q(A)\right\rangle = \langle P, AQ\rangle$$

$$= -\frac{1}{2} tr\left(P(AQ)^T\right) = -\frac{1}{2} tr\left(PQ^T A^T\right) = \left\langle PQ^T, A\right\rangle$$

In summary,

$$T^*L_Q(P) = Q^T P \quad \text{and} \quad T^*R_Q(P) = PQ^T$$

We thus compute the momentum maps as

$$J_L(Q, P) = T^*R_Q P = PQ^T$$

$$J_R(Q, P) = T^*L_Q P = Q^T P$$

In the special case of $G = SO(3)$, these matrices PQ^T and $Q^T P$ are skew – symmetric, since they are elements of $so(3)$. Therefore,

$$J_L(Q, P) = T^*R_Q P = \frac{1}{2}\left(PQ^T - QP^T\right)$$

$$J_R(Q, P) = T^*L_Q P = \frac{1}{2}\left(Q^T P - P^T Q\right)$$

Definition (3.1.24):

A momentum map is said to be equivalent when it is equivalent with respect to the given action on P and the coadjoint action on ϑ^*. That is,

$$J(g,p) = Ad^*_{g^{-1}} J(p)$$

for every $g \in G$, $p \in P$, where $g.p$ denotes the action of g on the point p and where Ad denotes the adjoint action .

Example (3.1.25): (Momentum Map for Symplectic Representations)

Let (V,Ω) be a symplectic vector space and let G be Lie group acting linearly and symplectically on V . This action admits an equivalent momentum map $\mathbf{J}: V \to \vartheta$ given by

$$J^{\xi}(v) = \langle \mathbf{J}(v), \xi \rangle = \frac{1}{2}\Omega(\xi v, v),$$

where ξv denotes the Lie algebra representation of the element $\xi \in \vartheta$ on the vector $v \in V$. To verify this, note that the infinitesimal generator $\xi_V(v) = \xi v$, by the definition of the Lie algebra representation induced by the given Lie group representation, and that $\Omega(\xi u, v) = -\Omega(u, \xi v)$ for all $u, v \in V$. Therefore

$$\mathbf{d}J^{\xi}(u)(v) = \tfrac{1}{2}\Omega(\xi u, v) + \tfrac{1}{2}\Omega(\xi v, u) = \Omega(\xi u, v).$$

Equivariance of $\mathbf{J}$ follows from the obvious relation $g^{-1}.\xi.g.v = \left(Ad_{g^{-1}}\xi\right)v$ for any $g \in G$, $\xi \in \vartheta$, and $v \in V$.

Example (3.1.26): (Cayley – Klein Parameters and The Hopf Fibration)

Consider the natural action of $SU(2)$ on $\mathbb{C}^2$. Since this action is by isometries of the Hermition metric, it is automatically symplectic and therefore has a momentum map $\mathbf{J}: \mathbb{C}^2 \to su(2)^*$ given in Example (3.1.24), that is,

$$\langle \mathbf{J}(z,w), \xi \rangle = \tfrac{1}{2}\Omega\big(\xi.(z,w),(z,w)\big),$$

where $z, w \in \mathbb{C}$ and $\xi \in su(2)$. Now the symplectic form on $\mathbb{C}^2$ is given by minus the imaginary part of the Hermitian inner product. That is, $\mathbb{C}^n$ has Hermitian inner product given by $\mathbf{z}.\mathbf{w} := \sum_{j=1}^{n} z_j \bar{w}_j$,where $\mathbf{z} = (z_1,........,z_n)$, $\mathbf{w} = (w_1,.......,w_n) \in \mathbb{C}^n$. The symplectic form is thus given by $\Omega(\mathbf{z},\mathbf{w}) := -\operatorname{Im}(\mathbf{z}.\mathbf{w})$ and it is identical to the one given before on $\Re^{2n}$ by identifying $\mathbf{z}=\mathbf{u}+i\mathbf{v}\in\mathbb{C}^n$ with $(\mathbf{u},\mathbf{v})\in\Re^{2n}$ and $\mathbf{w}=\mathbf{u}'+i\mathbf{v}'\in\mathbb{C}^n$ with $(\mathbf{u}',\mathbf{v}')\in\Re^{2n}$.

The Lie algebra $su(2)$ of $SU(2)$ consists of 2×2 skew Hermitian matrices of trace zero. This Lie algebra is isomorphic to $so(3)$ and therefore to $(\Re^3,\times)$ by the isomorphism given by

$$\mathbf{x}=\left(x^1,x^2,x^3\right)\in\Re^3\mapsto\tilde{\mathbf{x}}:=\frac{1}{2}\begin{bmatrix}-ix^3 & -ix^1-x^2\\ -ix^1+x^2 & ix^3\end{bmatrix}\in su(2).$$

Thus we have $[\tilde{\mathbf{x}},\tilde{\mathbf{y}}]=(\mathbf{x}\times\mathbf{y})^{\sim}$ for any $\mathbf{x},\mathbf{y}\in\Re^3$. Other useful relations are $\det(2\tilde{\mathbf{x}})=\|\mathbf{x}\|^2$ and trace $(\tilde{\mathbf{x}}\tilde{\mathbf{y}})=-\frac{1}{2}\mathbf{x}.\mathbf{y}$. Identify $su(2)^*$ with $\Re^3$ by the map $\mu\in su(2)^*\mapsto\breve{\mu}\in\Re^3$ defined by

$$\breve{\mu}.\mathbf{x}:=-2\langle\mu,\tilde{\mathbf{x}}\rangle$$

for any $\mathbf{x}\in\Re^3$. With these notations, the momentum map $\breve{\mathbf{J}}:\mathbb{C}^2\to\Re^3$ can be explicitly computed in coordinates: for any $\mathbf{x}\in\Re^3$ we have

$$\begin{aligned}\breve{\mathbf{J}}(z,w).\mathbf{x}&=-2\langle\mathbf{J}(z,w),\tilde{\mathbf{x}}\rangle\\ &=\tfrac{1}{2}\operatorname{Im}\left(\begin{bmatrix}-ix^3 & -ix^1-x^2\\ -ix^1+x^2 & ix^3\end{bmatrix}\begin{bmatrix}z\\ w\end{bmatrix}.\begin{bmatrix}z\\ w\end{bmatrix}\right)\\ &=-\tfrac{1}{2}\left(2\operatorname{Re}(w\bar{z}),2\operatorname{Im}(w\bar{z}),|z|^2-|w|^2\right).\mathbf{x}.\end{aligned}$$

Therefore

$$\breve{\mathbf{J}}(z,w)=-\tfrac{1}{2}\left(2w\bar{z},|z|^2-|w|^2\right)\in\Re^3.$$

Thus , $\breve{\mathbf{J}}$ is a Poisson map from $\mathbb{C}^2$, endowed with the canonical symplectic structure, to $\Re^3$, endowed with the +Lie – Poisson structure. Therefore , $-\breve{\mathbf{J}}:\mathbb{C}^2\to\Re^3$ is a canonical map, if $\Re^3$ has the – Lie – Poisson bracket relative to which the free rigid body equations are Hamiltonian . Pulling back the Hamiltonian $H(\mathbf{\Pi})=\mathbf{\Pi}.\mathbb{I}^{-1}\mathbf{\Pi}/2$ to $\mathbb{C}^2$ gives a Hamiltonian function (called collective) on $\mathbb{C}^2$. The classical Hamilton equations for this function are therefore projected by $-\breve{\mathbf{J}}$ to the rigid body equations $\dot{\mathbf{\Pi}}=\mathbf{\Pi}\times\mathbb{I}^{-1}\mathbf{\Pi}$. In this context, the variables (z,w) are called the Cayley – Klein parameters.

Definition (3.1.27):

A momentum map is Ad^* – equivariant iff

$$J(g.x) = Ad^*_{g^{-1}} J(x)$$

for all $g \in G$, $x \in P$.

Proposition (3.1.28):

All cotangent – lifted actions are Ad^* – equivariant .

Proposition (3.1.29):

Every Ad^* – equivariant momentum map $J : P \to \vartheta^*$ is a Poisson map, with respect to the '+' Lie Poisson bracket on ϑ^*.

Now we discuss the Quick summary for momentum maps .

Let G be a Lie group , ϑ its Lie algebra , and let ϑ^* be its dual . Suppose that G acts symplectically on a symplectic manifold P with symplectic form denoted by Ω. Denote the infinitesimal generator associated with the Lie algebra element ξ by ξ_P and let the Hamiltonian vector field associated to a function $f : P \to \Re$ be denoted X_f , so that $df = X_f \rfloor \Omega$.

Definition (3.1.30): (History and Overview)

A momentum map $J : P \to \vartheta^*$ defined by the condition relating the infinitesimal generator ξ_P of a symmetry to the vector field of its corresponding conservation law, $\langle J , \xi \rangle$,

$$\xi_P = X_{\langle J , \xi \rangle}$$

for all $\xi \in \vartheta$. Here $\langle J , \xi \rangle : P \to \Re$ is defined by the natural pointwise pairing .

A momentum map is said to be equivariant when it is equivariant with respect to the given action on P and the coadjoint action on ϑ^*. That is,

$$J(g.p) = Ad^*_{g^{-1}} J(p)$$

for every $g \in G$, $p \in P$, where $g.p$ denotes the action of g on the point p and where Ad denotes the adjoint action .

The links with mechanics were developed in the work of Lagrange , Poisson , Jacobi and , later , Noether . In particular , Noether showed that a momentum map for the action of a group G that is a symmetry of the Hamiltonian for a given system is a conservation law for that system .

Now we discuss the rigid body equations on $SO(n)$.

The left invariant generalized rigid body equations on $SO(n)$ may be written as

$$\dot{Q} = Q\,\Omega\,,$$

$$\dot{M} = M\,\Omega - \Omega M =: [M,\Omega], \qquad (RB_n)$$

where $Q \in SO(n)$ denotes the configuration space variable (the attitude of the body) , $\Omega = Q^{-1}\dot{Q} \in so(n)$ is the body angular velocity , and

$$M := J(\Omega) = D^2\Omega + \Omega D^2 \in so^*(n),$$

Is the body angular momentum . Here $J : so(n) \to so(n)^*$ is the symmetric (with respect to the above inner product) positive definite operator defined by

$$J(\Omega) = D^2\Omega + \Omega D^2,$$

where D^2 is the square of the constant diagonal matrix $D = diag\{d_1, d_2, d_3\}$ satisfying $d_i^2 + d_j^2 > 0$ for all $i \neq j$. For $n = 3$ the elements of d_i^2 are related to the standard diagonal moment of inertia tensor I by

$$I = diag\{I_1, I_2, I_3\}\ ,\ I_1 = d_2^2 + d_3^2\ ,\ I_2 = d_3^2 + d_1^2\ ,\ I_3 = d_1^2 + d_2^2\,.$$

The Euler equations for the $SO(n)$ rigid body $\dot{M} = [M,\Omega]$ are readily checked to be the Euler – Lagrange equations on $so(n)$ for the Lagrangian

$$L(Q,\dot{Q}) = l(\Omega) = \tfrac{1}{2}\langle \Omega, J(\Omega)\rangle,\ \text{with}\ \Omega = Q^T\dot{Q}\,.$$

The momentum is found via the Legendre transformation to be

$$\frac{\partial l}{\partial \Omega} = J(\Omega) = M\ ,$$

And the corresponding Hamiltonian is

$$H(M)=\frac{\partial l}{\partial \Omega}.\Omega - l(\Omega)=\tfrac{1}{2}\left\langle M, J^{-1}(M)\right\rangle .$$

The quantity M is the angular momentum in the body frame . The corresponding angular momentum in space,

$$m = QMQ^T \ , \text{ is conserved } \dot{m}=0 .$$

Indeed , conservation of spatial angular momentum m implies Euler's equations for the body angular momentum $M = Q^T m Q = Ad_Q^* m$.

This Hamiltonian $H(M)$ is invariant under the action of $SO(n)$ from the left . The corresponding conserved momentum map under this symmetry is

$$J_L : T^* SO(n) \mapsto so(n)^* \text{ is } J_L(Q,P)=PQ^T$$

On the other hand , we know that the momentum map for right action is

$$J_R : T^* SO(n) \mapsto so(n)^* \ , \ J_R(Q,P)=Q^T P$$

Hence $M = Q^T P = J_R$. Therefore , one computes

$$H(Q,P)=H(Q,Q.M)=H(Id,M) \qquad \text{(by left invariance)}$$

$$=H(M)=\frac{1}{2}\left\langle M, J^{-1}(M)\right\rangle$$

$$=\frac{1}{2}\left\langle Q^T P, J^{-1}\left(Q^T P\right)\right\rangle$$

Hence , we may write the $SO(n)$ rigid body Hamiltonian as

$$H(Q,P)=\frac{1}{2}\left\langle Q^T P, \Omega(Q,P)\right\rangle$$

Consequently , the variational derivatives of $H(Q,P)=\frac{1}{2}\left\langle Q^T P, \Omega(Q,P)\right\rangle$ are

$$\delta H = \left\langle Q^T \delta P + \delta Q^T P, \Omega(Q,P)\right\rangle$$

$$= tr\left(\delta P^T Q \Omega\right) + tr\left(P^T \delta Q \Omega\right)$$

$$= tr\left(\delta P^T Q \Omega\right) + tr\left(\delta Q \Omega P^T\right)$$

$$= tr\left(\delta P^T Q\Omega\right) + tr\left(\delta Q^T P\Omega^T\right)$$

$$= \langle \delta P, Q\Omega \rangle - \langle \delta Q, P\Omega \rangle$$

where skew symmetry of Ω is used in the last step , that is , $\Omega^T = -\Omega$. Thus , Hamilton's canonical equations take the form ,

$$\dot{Q} = \frac{\delta H}{\delta P} = Q\Omega \ , \quad \dot{P} = -\frac{\delta H}{\delta Q} = P\Omega . \qquad (3.29)$$

Equations (3.29) are the symmetric generalized rigid body equations , from viewpoint of optimal control . Combining them yields ,

$$Q^{-1}\dot{Q} = \Omega = P^{-1}\dot{P} \Leftrightarrow \left(PQ^T\right)^{\cdot} = 0 ,$$

in agreement with conservation of the momentum map $J_L(Q,P) = PQ^T$ corresponding to symmetry of the Hamiltonian under left action of $SO(n)$.This momentum map is the angular momentum in space , which is related to the angular momentum in the body by $PQ^T = m = QmQ^T$. Thus , we recognize the canonical momentum as $P = QM$ (see Example (3.1.22) , and the momentum maps for left and right actions as

$$J_L = m = PQ \qquad \text{(spatial angular momentum)}$$

$$J_R = M = Q^T P \qquad \text{(body angular momentum)}$$

Thus , momentum maps $TG^* \mapsto \vartheta^*$ corresponding to symmetries of the Hamiltonian produce conservation laws ; while momentum maps $TG^* \mapsto \vartheta^*$ which do not correspond to symmetries may be used to re – express the equations on ϑ^*, in terms of variables on TG^* .

Section (3.2): Free Ellipsoidal Motion on $GL(n)$, Heavy Top Equations and Euler – Poincare' (EP) Reduction Theorem

We start by studying the Manakov's formulation of the $SO(4)$ rigid body .

The Euler equations on $SO(4)$ are

$$\frac{dM}{dt} = M\Omega - \Omega M = [M, \Omega], \qquad (RB_n)$$

where Ω and M are skew symmetric 4×4 matrices . The angular frequency Ω is a linear function of the angular momentum , M . Manakov[41] "deformed" these equations into

$$\frac{d}{dt}(M + \lambda A) = [(M + \lambda A), (\Omega + \lambda B)],$$

where A, B are also skew symmetric 4×4 matrices and λ is a scalar constant parameter . For these equations to hold for any value of λ, the coefficient of each power must vanish .

(1) The coefficient of λ^2 is

$$0 = [A, B]$$

So A and B must commute . So , let them be constant and diagonal :

$$A_{ij} = diag(a_i)\delta_{ij} \ , \ B_{ij} = diag(b_i)\delta_{ij} \qquad \text{(no sum)}$$

(2) The coefficient of λ is

$$0 = \frac{dA}{dt} = [A, \Omega] + [M, B]$$

Therefore , by antisymmetry of M and Ω,

$$(a_i - a_j)\Omega_{ij} = (b_i - b_j)M_{ij} \Leftrightarrow \Omega_{ij} = \frac{b_i - b_j}{a_i - a_j}M_{ij} \qquad \text{(no sum)}$$

(3) Finally , the coefficient of λ^0 is the Euler equation ,

$$\frac{dM}{dt} = [M, \Omega],$$

but now with the restriction that the moments of inertia are of the form ,

$$\Omega_{ij} = \frac{b_i - b_j}{a_i - a_j}M_{ij} \qquad \text{(no sum)}$$

which turns out to posses only 5 free parameters .

With these conditions , Manakov's deformation of the $SO(4)$ rigid body implies for every power n that

$$\frac{d}{dt}(M+\lambda A)^n = \left[(M+\lambda A)^n, (\Omega+\lambda B)\right].$$

Since the commutator is antisymmetric , its trace vanishes and one has

$$\frac{d}{dt} trace\,(M+\lambda A)^n = 0$$

after commuting the trace operation with time derivative . Consequently ,

$$trace\,(M+\lambda A)^n = \text{constant}$$

for each power of λ. That is , all the coefficients of each power of λ are constant in time for the $SO(4)$ rigid body .

Answer (3.2.1):

The traces of the powers $trace\,(M+\lambda A)^n$ are given by

$n=2: trM^2 + 2\lambda(AM) + \lambda^2 trA^2$

$n=3: trM^3 + 3\lambda tr\left(AM^2\right) + 3\lambda^2 trA^2M + \lambda^3 trA^3$

$n=4: trM^4 + 4\lambda tr\left(AM^3\right) + \lambda^2\left(2trA^2M^2 + 4trAMAM\right) + \lambda^3 trA^3M + \lambda^4 trA^4$

The number of conserved quantities for $n=2,3,4$ are , respectively , one $\left(C_1 = trM^2\right)$,one $\left(I_1 = trAM^2\right)$ and two ($C_2 = trM^4$ and $I_2 = 2trA^2M^2 + 4trAMAM$). The quantities $C_1$and C_2 are Casimirs for the Lie – Poisson bracket for the rigid body . Thus , $\{C_1, H\} = 0 = \{C_2, H\}$ for any Hamiltonian $H(M)$; so of course C_1 and C_2 are conserved . However , each Casimir only reduce the dimension of the system by one . The dimension of the original phase space is $\dim T^*SO(n) = n(n-1)$. This is reduced in half by left invariance of the Hamiltonian to the dimension of the dual Lie algebra $\dim so(n)^* = n(n-1)/2$. For $n=4$, $\dim so(4)^* = 6$. One then subtracts the number of Casimirs (two) by passing to their level surfaces , which leaves four dimensions remaining in this case . The other two constants of motion I_1 and I_2 turn out to be sufficient for integrability

, because they are in involution $\{I_1, I_2\} = 0$ and because the level surfaces of the Casimirs are symplectic manifolds .

Now we discuss the free ellipsoidal motion on $GL(n)$.

Consider the deformation of a body in $\Re^n$ given by

$$x(t, x_0) = Q(t)x_0, \tag{3.30}$$

with $x, x_0 \in \Re^n$, $Q(t) \in GL_+(n, \Re)$ and $x(t_0, x_0) = x_0$, so that $Q(t_0) = Id$. (The subscript + in $GL_+(n, \Re)$ means $n \times n$ matrices with positive determiniant .) Thus , $x(t, x_0)$ is the current (Eulerian) position at time t of a material parcel that was at (Lagrangian) position x_0 at time t_0. The "deformation gradient" that is , the Jacobian matrix $Q = \partial x / \partial x_0$ of this "Lagrange – to – Euler map," is a function of only time , t ,

$$\partial x / \partial x_0 = Q(t), \quad \text{with} \quad \det Q > 0 .$$

The velocity of such a motion is given by

$$\dot{x}(t, x_0) = \dot{Q}(t)x_0 = \dot{Q}(t)Q^{-1}(t)x = u(t, x). \tag{3.31}$$

The kinetic energy for such a body occupying a reference volume $\mathcal{B}$ defines the quadratic form ,

$$L = \tfrac{1}{2}\int_{\mathcal{B}} \rho(x_0)\left|\dot{x}(t, x_0)\right|^2 d^3x_0 = \tfrac{1}{2}tr\left(\dot{Q}(t)^T I\dot{Q}(t)\right) = \tfrac{1}{2}\dot{Q}^i_A I^{AB}\dot{Q}^i_B .$$

Here I is the constant symmetric tensor ,

$$I^{AB} = \int_{\mathcal{B}} \rho(x_0)x_0^A x_0^B d^3x_0 ,$$

which we will take as being proportional to the identity $I^{AB} = c_0^2\delta^{AB}$ for the remainder of these considerations . This corresponds to taking an initially spherical reference configuration for the fluid . Hence , we are dealing with the Lagrangian consisting only of kinetic energy ,

$$L = \tfrac{1}{2}tr\left(\dot{Q}(t)^T \dot{Q}(t)\right).$$

The Euler – Lagrange equations for this Lagrangian simply represent free motion on the group $GL_+(n, \Re)$,

$$\ddot{Q}(t)=0,$$

which is immediately integrable as

$$Q(t)=Q(0)+\dot{Q}(0)t,$$

where $Q(0)$ and $\dot{Q}(0)$ are the values at the initial time $t=0$. Legendre transforming this Lagrangian for free motion yields

$$P=\frac{\partial L}{\partial \dot{Q}^T}=\dot{Q}.$$

The corresponding Hamiltonian is expressed as

$$H(Q,P)=\tfrac{1}{2}tr\left(P^T P\right)=\tfrac{1}{2}\|P\|^2.$$

The canonical equations for this Hamiltonian are simply

$$\dot{Q}=P \quad , \quad \text{with} \quad \dot{P}=0.$$

The deformation tensor $Q(t)\in GL_+(n,\Re)$ for such a body may be decomposed as

$$Q(t)=R^{-1}(t)D(t)S(t). \tag{3.32}$$

This is the polar decomposition of a matrix in $GL_+(n,\Re)$. The interpretations of the various components of the motion can be seen from equation (3.30). Namely,

(1) $R\in SO(n)$ rotates the x –coordinates ,
(2) $S\in SO(n)$ rotates the x_0 – coordinates in the reference configuration ,and
(3) D is a diagonal matrix which represents stretching deformations along the principal axes of the body .

The two $SO(n)$ rotations lead to their corresponding angular frequencies , defined by

$$\Omega=\dot{R}R^{-1} \quad , \quad \Lambda=\dot{S}S^{-1}. \tag{3.33}$$

Rigid body motion will result , when S restricts to the identity matrix and D is a constant diagonal matrix .

Remark (3.2.2):

The combined motion of a set of fluid parcels governed by (3.30) along the curve $Q(t) \in GL_{+}(n,\Re)$ is called "ellipsoidal," because it can be envisioned in three dimensions as a fluid ellipsoid whose orientation in space is governed by $R \in SO(n)$, whose shape is determined by D consisting of its instantaneous principle axes lengths and whose internal circulation of material is described by $S \in SO(n)$. In addition , fluid parcels initially arranged along a straight line within the allipse will remain on a straight line .

In Hamilton's principle , $\delta\int Ldt = 0$, we chose a Lagrangian $L : TGL_{+}(n,\Re) \to \Re$ in the form

$$L\left(Q,\dot{Q}\right) = T\left(\Omega, \Lambda, \mathrm{D}, \dot{\mathrm{D}}\right), \tag{3.34}$$

in which the kinetic energy T is given by using polar decomposition $Q(t) = R^{-1}(t)D(t)S(t)$ in (3.32), as follows .

$$\dot{Q} = R^{-1}\left(-\Omega D + \dot{D} + D\Lambda\right)S \ . \tag{3.35}$$

Consequently , the kinetic energy for ellipsoidal motion becomes

$$T = \frac{1}{2}trace\left[-\Omega D^{2}\Omega - \Omega D\dot{D} + \Omega D\Lambda D + \dot{D}D\Omega + \dot{D}^{2} - D\Lambda^{2}D - \dot{D}\Lambda D + D\Lambda D\Omega + D\Lambda\dot{D}\right]$$

$$= \frac{1}{2}trace\left[-\Omega^{2}D^{2} - \Lambda^{2}D^{2} + \underbrace{2\Omega D\Lambda D}_{Coriolis\,coupling} + \dot{D}^{2}\right]. \tag{3.36}$$

Remark (3.2.3):

Note the discrete exchange symmetry of the kinetic energy : T is invariant under $\Omega \leftrightarrow \Lambda$.

For $\Lambda = 0$ and D constant expression (3.36) for T reduces to the usual kinetic energy for the rigid – body ,

$$T\ |_{\Lambda=0, D=const} = -\frac{1}{4}trace\left[\Omega\left(D\Omega + \Omega D\right)\right]. \tag{3.37}$$

This Lagrangian (3.34) is invariant under the right action, $R \to R_{g}$ and $S \to S_{g}$, for $g \in SO(n)$. In taking variations we shall use the formulas

$$\delta\Omega = \dot{\Sigma} + \left[\Sigma, \Omega\right] \equiv \dot{\Sigma} - ad_{\Omega}\Sigma \,, \qquad \Sigma \equiv \delta R R^{-1}, \tag{3.38}$$

$$\delta\Lambda = \dot{\Xi} + \left[\Xi, \Lambda\right] \equiv \dot{\Xi} - ad_{\Lambda}\Xi \,, \qquad \Xi \equiv \delta S S^{-1}, \tag{3.39}$$

in which the ad –operation is defined in terms of the Lie – algebra (matrix) commutator $[.,.]$ as , for example , $ad_{\Omega}\Sigma \equiv [\Omega, \Sigma]$. Substituting these formulas into Hamilton's principle gives

$$0 = \delta \int L dt = \int dt \, \frac{\partial L}{\partial \Omega} . \delta\Omega + \frac{\partial L}{\partial \Lambda} . \delta\Lambda + \frac{\partial L}{\partial D} \delta D + \frac{\partial L}{\partial \dot{D}} \delta \dot{D} \,,$$

$$= \int dt \, \frac{\partial L}{\partial \Omega} . \left[\dot{\Sigma} - ad_{\Omega}\Sigma\right] + \frac{\partial L}{\partial \Lambda} . \left[\dot{\Xi} - ad_{\Lambda}\Xi\right] + \left[\frac{\partial L}{\partial D} - \frac{d}{dt}\frac{\partial L}{\partial \dot{D}}\right] \delta D$$

$$= - \int dt \left[\frac{d}{dt}\frac{\partial L}{\partial \Omega} - ad^{*}_{\Omega}\frac{\delta L}{\delta \Omega}\right] . \Sigma + \left[\frac{d}{dt}\frac{\partial L}{\partial \Lambda} - ad^{*}_{\Lambda}\frac{\partial L}{\partial \Lambda}\right] . \Xi + \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{D}} - \frac{\partial L}{\partial D}\right] \delta D \tag{3.40}$$

where the operation ad^{*}_{Ω}, for example , is defined by

$$ad^{*}_{\Omega}\frac{\partial L}{\partial \Omega} . \Sigma = - \frac{\partial L}{\partial \Omega} . ad_{\Omega}\Sigma = - \frac{\partial L}{\partial \Omega} . [\Omega, \Sigma], \tag{3.41}$$

and the dot '.' denotes pairing between the Lie algebra and its dual . This could also have been written in the notation using $\langle .,. \rangle : \vartheta^{*} \times \vartheta \rightarrow \Re$, as

$$\left\langle ad^{*}_{\Omega}\frac{\partial L}{\partial \Omega}, \Sigma \right\rangle = - \left\langle \frac{\partial L}{\partial \Omega}, ad_{\Omega}\Sigma \right\rangle = - \left\langle \frac{\partial L}{\partial \Omega}, [\Omega, \Sigma] \right\rangle . \tag{3.42}$$

The Euler – Poincare' dynamics is given by the stationarity conditions for Hamilton's principle ,

$$\Sigma : \frac{d}{dt}\frac{\partial L}{\partial \Omega} - ad^{*}_{\Omega}\frac{\partial L}{\partial \Omega} = 0 \,, \tag{3.43}$$

$$\Xi : \frac{d}{dt}\frac{\partial L}{\partial \Lambda} - ad^{*}_{\Lambda}\frac{\partial L}{\partial \Lambda} = 0 \,, \tag{3.44}$$

$$\delta D : \frac{d}{dt}\frac{\partial L}{\partial \dot{D}} - \frac{\partial L}{\partial D} = 0 \,. \tag{3.45}$$

These are the Euler – Poincare' equations for the ellipsoidal motions generated by Lagrangians of the form given in equation (3.34) . For example , such Lagrangians determine the dynamics of the Riemann ellipsoids – circulating , rotating , self – gravitating fluid flows at constant density within an ellipsoidal boundary .

The Euler – Poincare' equations ((3.43) – (3.45)) involve angular momenta defined in terms of the angular velocities Ω, Λ and the shape D by

$$M = \frac{\partial T}{\partial \Omega} = -\Omega D^2 - D^2\Omega + 2D\Lambda D \,, \tag{3.46}$$

$$N = \frac{\partial T}{\partial \Lambda} = -\Lambda D^2 - D^2\Lambda + 2D\Omega D \,. \tag{3.47}$$

These angular momenta are related to the original deformation gradient $Q = R^{-1}DS$ in equation (3.30) by the two momentum maps from Example (3.1.22)

$$PQ^T - QP^T = \dot{Q}Q^T - Q\dot{Q}^T = R^{-1}MR \,, \tag{3.48}$$

$$P^TQ - Q^TP = \dot{Q}^TQ - Q^T\dot{Q} = S^{-1}NS \,. \tag{3.49}$$

To see that N is related to the vorticity , we consider the exterior derivative of the circulation one – form $\mathbf{u}.d\mathbf{x}$ defined as

$$d\left(\mathbf{u}.d\mathbf{x}\right) = curl\mathbf{u}.d\,\mathbf{S}$$

$$= \tfrac{1}{2}\left(\dot{Q}^TQ - Q^T\dot{Q}\right)_{jk} dx_0^j \wedge dx_0^k = \left(S^{-1}NS\right)_{jk} dx_0^j \wedge dx_0^k \,. \tag{3.50}$$

Thus , $S^{-1}NS$ is the fluid vorticity , referred to the Lagrangian coordinate frame . For Euler's fluid equation , Kelvin's circulation theorem implies $\left(S^{-1}NS\right)^{\cdot} = 0$.

Likewise , M is related to the angular momentum by considering

$$u_i x_j - u_j x_i = \dot{Q}_{ik} x_0^k x_0^l Q_{lj}^T - Q_{ik} x_0^k x_0^l \dot{Q}_{lj}^T \,. \tag{3.51}$$

For spherical symmetry , we may choose $x_0^k x_0^l = \delta^{kl}$ and , in this case , the previous expression becomes

$$u_i x_j - u_j x_i = \left[\dot{Q}Q^T - Q\dot{Q}^T\right]_{ij} = \left[R^{-1}MR\right]_{ij} \,. \tag{3.52}$$

Thus , $R^{-1}MR$ is the angular momentum of the motion , referred to the Lagrangian coordinate frame for spherical symmetry . In this case , the angular momentum is conserved , so that $\left(R^{-1}MR\right)^{\cdot} = 0$.

In terms of these angular momenta , the Euler – Poincare' – Lagrange equations ((3.43) – (3.45)) are expressed as

$$\dot{M} = [\Omega, M], \tag{3.53}$$

$$\dot{N} = [\Lambda, N], \tag{3.54}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{D}}\right) = \frac{\partial L}{\partial D} . \tag{3.55}$$

Perhaps not unexpectedly , because of the combined symmetries of the kinetic – energy Lagrangian (3.34) under both left and right actions of $SO(n)$, the first two equations are consistent with the conservation laws ,

$$\left(R^{-1}MR\right)^{\cdot} = 0 \qquad \text{and} \qquad \left(S^{-1}NS\right)^{\cdot} = 0 ,$$

respectively . Thus , equation (3.53) is the angular momentum equation while (3.54) is the vorticity equation . (Fluids have both types of circulatory motions.) The remaining equation (3.55) for the diagonal matrix D determines the shape of the ellipsoid undergoing free motion on $GL(n,\Re)$.

In three dimensions these expressions may be written in vector form by using the hat map , written now using upper and lower case Greek letters as

$$\Omega_{ij} = \varepsilon_{ijk}\,\omega_k \ , \qquad \Lambda_{ij} = \varepsilon_{ijk}\,\lambda_k \ ,$$

with $\varepsilon_{123} = 1$, and $D = diag\,\{d_1, d_2, d_3\}$.

Remark (3.2.4):

Locally the Lie algebra $so(4)$ is isomorphic to $so(3)\times so(3)$.

Hence , the angular – motion terms in the kinetic energy may be rewritten as

$$-\tfrac{1}{2} trace\left(\Omega^2 D^2\right) = \tfrac{1}{2}\left[\left(d_1^2 + d_2^2\right)\omega_3^2 + \left(d_2^2 + d_3^2\right)\omega_1^2 + \left(d_3^2 + d_1^2\right)\omega_2^2\right], \tag{3.56}$$

$$-\tfrac{1}{2} trace\left(\Lambda^2 D^2\right) = \tfrac{1}{2}\left[\left(d_1^2 + d_2^2\right)\lambda_3^2 + \left(d_2^2 + d_3^2\right)\lambda_1^2 + \left(d_3^2 + d_1^2\right)\lambda_2^2\right], \tag{3.57}$$

$$-\tfrac{1}{2} trace\left(\Omega D \Lambda D\right) = \left[d_1 d_2\left(\omega_3\lambda_3\right) + d_2 d_3\left(\omega_1\lambda_1\right) + d_3 d_1\left(\omega_2\lambda_2\right)\right]. \tag{3.58}$$

On comparing equations (3.37) and (3.56) for the kinetic energy of the rigid body part of the motion , we identify the usual moments of inertia as

$$I_k = d_i^2 + d_j^2 \ , \qquad \text{with} \quad i, j, k \ \text{ cyclic .}$$

The antisymmetric matrices M and N have vector representations in $3D$ given by

$$M_k = \frac{\partial T}{\partial \omega_k} = \left(d_i^2 + d_j^2\right)\omega_k - 2d_i d_j \lambda_k \ , \tag{3.59}$$

$$N_k = \frac{\partial T}{\partial \lambda_k} = \left(d_i^2 + d_j^2\right)\lambda_k - 2d_i d_j \omega_k \ , \tag{3.60}$$

again with i, j, k cyclic permutations of $\{1,2,3\}$.

In terms of their $3D$ vector representations of the angular momenta in equations (3.59) and (3.60) , the two equations (3.53) and (3.54) become

$$\dot{\mathbf{M}} = \left(\dot{R}R^{-1}\right)\mathbf{M} = \Omega\mathbf{M} = \omega \times \mathbf{M}, \quad \dot{\mathbf{N}} = \left(\dot{S}S^{-1}\right)\mathbf{N} = \Lambda\mathbf{N} = \lambda \times \mathbf{N}. \tag{3.61}$$

Relative to the Lagrangian fluid frame of reference , these equations become

$$\left(R^{-1}\mathbf{M}\right)^{\cdot} = R^{-1}\left(\dot{\mathbf{M}} - \omega \times \mathbf{M}\right) = 0 \, , \tag{3.62}$$

$$\left(S^{-1}\mathbf{N}\right)^{\cdot} = S^{-1}\left(\dot{\mathbf{N}} - \lambda \times \mathbf{N}\right) = 0 \, . \tag{3.63}$$

So each of these degrees of freedom represents a rotating , deforming body , whose ellipsoidal shape is governed by the Euler – Lagrange equations (3.55) for the lengths of its three principal axes .

Now we discuss the Heavy top equations .

A top is a rigid body of mass m rotating with a fixed point of support in a constant gravitational field of acceleration $-g\,\hat{\mathbf{z}}$ pointing vertically downward . The orientation of the body relative to the vertical axis $\hat{\mathbf{z}}$ is defined by the unit vector $\mathbf{\Gamma} = \boldsymbol{R}^{-1}(t)\hat{\mathbf{z}}$ for a curve $\boldsymbol{R}(t) \in SO(3)$. According to its definition , the unit vector $\mathbf{\Gamma}$ represents the motion of the vertical direction as seen from the rotating body . Consequently , it satisfies the auxiliary motion equation ,

$$\dot{\mathbf{\Gamma}} = -\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}(t)\mathbf{\Gamma} = \mathbf{\Gamma} \times \mathbf{\Omega} \, .$$

Here the rotation matrix $\boldsymbol{R}(t) \in SO(3)$, the skew matrix $\hat{\boldsymbol{\Omega}} = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} \in so(3)$ and the body angular frequency vector $\boldsymbol{\Omega} \in \Re^3$ are related by the hat map , $\boldsymbol{\Omega} = \left(\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right)^{\wedge}$, where $\wedge : \left(so(3), [.,.]\right) \to \left(\Re^3, \times\right)$ with $\hat{\boldsymbol{\Omega}}\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{v}$ for any $\mathbf{v} \in \Re^3$.

The motion of a top is determined from Euler's equations in vector form ,

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega} + mg\,\boldsymbol{\Gamma} \times \boldsymbol{\chi} , \tag{3.64}$$

$$\dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega} , \tag{3.65}$$

where $\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \boldsymbol{\chi} \in \Re^3$ are vectors in the rotating body frame . Here

(1) $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity vector ,

(2) $\mathbb{I} = diag\,(I_1, I_2, I_3)$ is the moment of inertia tensor , diagonalized in the body principle axes ,

(3) $\boldsymbol{\Gamma} = R^{-1}(t)\hat{\mathbf{z}}$ represents the motion of the unit vector along the vertical axis , as seen from the body ,

(4) χ is the constant vector in the body from the point of support to the body's center of mass ,

(5) m is the total mass of the body and g is the constant acceleration of gravity .

Proposition(3.2.5):

The heavy top equations are equivalent to the heavy top action principle for a reduced action

$$\delta S_{red} = 0, \text{ with } S_{red} = \int_a^b l\,(\boldsymbol{\Omega}, \boldsymbol{\Gamma})dt = \int_a^b \tfrac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle - \langle mg\boldsymbol{\chi}, \boldsymbol{\Gamma}\rangle dt , \tag{3.66}$$

where variations of $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ are restricted to be of the form

$$\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega} \times \boldsymbol{\Sigma} \quad \text{and} \quad \delta\boldsymbol{\Gamma} = \boldsymbol{\Gamma} \times \boldsymbol{\Sigma}, \tag{3.67}$$

arising from variations of the definitions $\boldsymbol{\Omega} = \left(\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right)^{\wedge}$ and $\boldsymbol{\Gamma} = \boldsymbol{R}^{-1}(t)\hat{\mathbf{z}}$ in which $\boldsymbol{\Sigma}(t) = \left(\boldsymbol{R}^{-1}\delta\boldsymbol{R}\right)^{\wedge}$ is a curve in $\Re^3$ that vanishes at the endpoints in time .

Proof :

Since $\mathbb{I}$ is symmetric and χ is constant , we obtain the variation

$$\delta\int_a^b l\left(\mathbf{\Omega},\mathbf{\Gamma}\right)dt=\int_a^b\langle\mathbb{I}\mathbf{\Omega},\delta\mathbf{\Omega}\rangle-\langle mg\,\chi,\delta\mathbf{\Gamma}\rangle dt$$

$$=\int_a^b\left\langle\mathbb{I}\mathbf{\Omega},\dot{\mathbf{\Sigma}}+\mathbf{\Omega}\times\mathbf{\Sigma}\right\rangle-\langle mg\,\chi,\mathbf{\Gamma}\times\mathbf{\Sigma}\rangle dt$$

$$=\int_a^b\left\langle-\frac{d}{dt}\mathbb{I}\mathbf{\Omega},\mathbf{\Sigma}\right\rangle+\langle\mathbb{I}\mathbf{\Omega},\mathbf{\Omega}\times\mathbf{\Sigma}\rangle-\langle mg\,\chi,\mathbf{\Gamma}\times\mathbf{\Sigma}\rangle dt$$

$$=\int_a^b\left\langle-\frac{d}{dt}\mathbb{I}\mathbf{\Omega}+\mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega}+mg\mathbf{\Gamma}\times\chi,\mathbf{\Sigma}\right\rangle dt\ ,$$

upon integrating by parts and using the endpoint conditions , $\mathbf{\Sigma}(b)=\mathbf{\Sigma}(a)=0$. Since $\mathbf{\Sigma}$ is otherwise arbitrary , (3.66) is equivalent to

$$-\frac{d}{dt}\mathbb{I}\mathbf{\Omega}+\mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega}+mg\mathbf{\Gamma}\times\chi=0\,,$$

which is Euler's motion equation for the heavy top (3.64) . This motion equation is completed by the auxiliary equation $\dot{\mathbf{\Gamma}}=\mathbf{\Gamma}\times\mathbf{\Omega}$ in (3.65) arising from the definition of $\mathbf{\Gamma}$.

The Legendre transformation for $l\left(\mathbf{\Omega},\mathbf{\Gamma}\right)$ gives the body angular momentum

$$\mathbf{\Pi}=\frac{\partial l}{\partial\mathbf{\Omega}}=\mathbb{I}\mathbf{\Omega}\,.$$

The well known energy Hamiltonian for the heavy top then emerges as

$$h\left(\mathbf{\Pi},\mathbf{\Gamma}\right)=\mathbf{\Pi}.\mathbf{\Omega}-l\left(\mathbf{\Omega},\mathbf{\Gamma}\right)=\tfrac{1}{2}\left\langle\mathbf{\Pi},\mathbb{I}^{-1}\mathbf{\Pi}\right\rangle+\langle mg\,\chi,\mathbf{\Gamma}\rangle\,, \tag{3.68}$$

which is the sum of the kinetic and potential energies of the top .

Let $f\,,h:\vartheta^*\to\Re$ be real – valued functions on the dual space ϑ^*. Denoting elements of ϑ^* by μ, the functional derivative of f at μ is defined as the unique element $\delta f/\delta\mu$ of ϑ defined by

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left[f\left(\mu+\varepsilon\delta\mu\right)-f\left(\mu\right)\right]=\left\langle\delta\mu,\frac{\delta f}{\delta\mu}\right\rangle, \tag{3.69}$$

for all $\delta\mu\in\vartheta^*$, where $\langle.,.\rangle$ denotes the pairing between ϑ^* and ϑ.

Definition (3.2.6): (Lie – Poisson Brackets and Lie Poisson Equations)

The $(\pm)$ Lie – Poisson brackets are defined by

$$\{f,h\}_{\pm}(\mu)=\pm\left\langle \mu,\left[\frac{\delta f}{\delta\mu},\frac{\delta h}{\delta\mu}\right]\right\rangle=\mp\left\langle \mu,ad_{\delta h/\delta\mu}\frac{\delta f}{\delta\mu}\right\rangle. \qquad (3.70)$$

The corresponding Lie – Poisson equations , determined by $\dot{f}=\{f,h\}$ read

$$\dot{\mu}=\{\mu,h\}=\mp ad^{*}_{\delta h/\delta\mu}\mu\,, \qquad (3.71)$$

where one defines the ad^{*} operation in terms of the pairing $\langle .,.\rangle$, by

$$\{f,h\}=\left\langle \mu,ad_{\delta h/\delta\mu}\frac{\delta f}{\delta\mu}\right\rangle=\left\langle ad^{*}_{\delta h/\delta\mu}\mu,\frac{\delta f}{\delta\mu}\right\rangle.$$

The Lie – Poisson setting of mechanics is a special case of the general theory of systems on Poisson manifolds , for which there is now an extensive theoretical development .

An important feature of the rigid body bracket carries over to general Lie algebras . Namely , Lie – Poisson brackets on ϑ^{*} arise from canonical brackets on the cotangent bundle (phase space) $T^{*}G$ associated with a Lie group G which has ϑ as its associated Lie algebra . Thus , the process by which the Lie – Poisson brackets arise is the momentum map

$$T^{*}G \mapsto \vartheta^{*}.$$

For example , a rigid body is free to rotate about its center of mass and G is the (proper) rotation group $SO(3)$. The choice of $T^{*}G$ as the primitive phase space is made according to the classical procedures of mechanics described earlier . For the description using Lagrangian mechanics , one forms the velocity phase space TG . The Hamiltonian description on $T^{*}G$ is then obtained by standard procedures: Legendre transforms , etc.

The passage from $T^{*}G$ to the space of $\mathbf{\Pi}$'s (body angular momentum space) is determined by left translation on the group . This mapping is an example of a momentum map ; that is , a mapping whose components are the "Noether quantities" associated with a symmetry group . The map from $T^{*}G$ to ϑ^{*} being a

Poisson map is a general fact about momentum maps . The Hamiltonian point of view of all this is a standard subject .

Remark (3.2.7): (Lie – Poisson Description of the Heavy Top)

As it turns out , the underlying Lie algebra for the Lie – Poisson description of the heavy top consists of the Lie algebra $se(3,\Re)$ of infinitesimal Euclidean motions in $\Re^3$. This is a bit surprising , because heavy top motion itself does not actually arise through actions of the Euclidean group of rotations and translations on the body , since the body has a fixed point ! Instead , the Lie algebra $se(3,\Re)$ arises for another reason associated with the breaking of the $SO(3)$ isotropy by the presence of the gravitational field . This symmetry breaking introduces a semidirect – product Lie – Poisson structure which happens to coincide with the dual of the Lie algebra $se(3,\Re)$ in the case of the heavy top . As we shall see later , a close parallel exists between this case and the Lie – Poisson structure for compressible fluids .

The Lie algebra of the special Euclidean group in $3D$ is $se(3)=\Re^3\times\Re^3$ with the Lie bracket

$$\left[(\xi,\mathbf{u}),(\eta,\mathbf{v})\right]=(\xi\times\eta,\xi\times\mathbf{v}-\eta\times\mathbf{u}). \tag{3.72}$$

We identify the dual space with pairs $(\mathbf{\Pi},\mathbf{\Gamma})$; the corresponding $(-)$ Lie – Poisson bracket called the heavy top bracket is

$$\{f\ ,h\}(\mathbf{\Pi},\mathbf{\Gamma})=-\mathbf{\Pi}.\nabla_{\Pi}f\ \times\nabla_{\Pi}h-\mathbf{\Gamma}.(\nabla_{\Pi}f\ \times\nabla_{\Gamma}h-\nabla_{\Pi}h\times\nabla_{\Gamma}f\ ). \tag{3.73}$$

The Lie – Poisson bracket and the Hamiltonian (3.68) recover the equations (3.64) and (3.65) for the heavy top , as

$$\dot{\mathbf{\Pi}}=\{\mathbf{\Pi},h\}=\mathbf{\Pi}\times\nabla_{\Pi}h+\mathbf{\Gamma}\times\nabla_{\Gamma}h=\mathbf{\Pi}\times\mathbb{I}^{-1}\mathbf{\Pi}+\mathbf{\Gamma}\times mg\,\chi\,,$$

$$\dot{\mathbf{\Gamma}}=\{\mathbf{\Gamma},h\}=\mathbf{\Gamma}\times\nabla_{\Pi}h=\mathbf{\Gamma}\times\mathbb{I}^{-1}\mathbf{\Pi}\,.$$

Remark (3.2.8): (Semidirect Products and Symmetry Breaking)

The Lie algebra of the Euclidean group has a structure which is a special case of what is called a semidirect product . Here , it is the semidirect product action $so(3)\oplus\Re^3$ of the Lie algebra of rotations $so(3)$ acting on the infinitesimal translations $\Re^3$, which happens to coincide with $se(3,\Re)$. In general , the Lie bracket for semidirect product action $\vartheta \oplus V$ of a Lie algebra ϑ on a vector space V is given by

$$\left[(X,a),(\bar{X},\bar{a})\right]=\left(\left[X,\bar{X}\right],\bar{X}(a)-X(\bar{a})\right)$$

in which $X,\bar{X}\in\vartheta$ and $a,\bar{a}\in V$. Here , the action of the Lie algebra on the vector space is denoted , for example , $X(\bar{a})$. Usually , this action would be the Lie derivative .

Lie – Poisson brackets defined on the dual spaces of semidirect product Lie algebras tend to occur under rather general circumstances when the symmetry in T^*G is broken , for example , reduced to an isotropy subgroup of a set of parameters . In particular , there are similarities in structure between the Poisson bracket for compressible flow and that for the heavy top . In the latter case , the vertical direction of gravity breaks isotropy of $\mathbf{R}^3$ from $SO(3)$ to $SO(2)$. The general theory for semidirect products is reviewed in a variety of places , including Marsden . Many interesting examples of Lie – Poisson brackets on semidirect products exist range from simple fluids , to changed fluid plasmas , to magnetized fluids , to multiphase fluids , to super fluids , to Yang – Mills fluids , relativistic , or not , and to liquid crystals .

The Lagrangian in the heavy top action principle (3.66) may be transformed into a quadratic form . This is accomplished by suspending the system in a higher dimensional space via the Kaluza – Klein construction . This construction proceeds for the heavy top as a slight modification of the well – known Kaluza – Klein construction for a changed particle in a prescribed magnetic field .

Let Q_{KK} be the manifold $SO(3)\times\Re^3$ with variables $(\mathbf{R},\mathbf{q})$. On Q_{KK} introduce the Kaluza – Klein Lagrangian $L_{KK}:TQ_{KK}\approx TSO(3)\times T\Re^3\mapsto\Re$ as

$$L_{KK}\left(\boldsymbol{R},\boldsymbol{q},\dot{\boldsymbol{R}},\dot{\boldsymbol{q}};\hat{\boldsymbol{z}}\right)=L_{KK}\left(\boldsymbol{\Omega},\boldsymbol{\Gamma},\boldsymbol{q},\dot{\boldsymbol{q}}\right)=\tfrac{1}{2}\langle\mathbb{I}\boldsymbol{\Omega},\boldsymbol{\Omega}\rangle+\tfrac{1}{2}\left|\boldsymbol{\Gamma}+\dot{\boldsymbol{q}}\right|^2, \qquad (3.74)$$

with $\boldsymbol{\Omega}=\left(\boldsymbol{R}^{-1}\dot{\boldsymbol{R}}\right)^{\wedge}$ and $\boldsymbol{\Gamma}=\boldsymbol{R}^{-1}\hat{\boldsymbol{z}}$. The Lagrangian L_{KK} is positive definite in $(\boldsymbol{\Omega},\boldsymbol{\Gamma},\dot{\boldsymbol{q}})$; so it may be regarded as the kinetic energy of a metric , the Kaluza – Klein metric on TQ_{KK} .

The Legendre transformation for L_{KK} gives the momenta

$$\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega} \quad \text{and} \quad \mathbf{p} = \mathbf{\Gamma} + \dot{\mathbf{q}} \,. \tag{3.75}$$

Since L_{KK} does not depend on $\mathbf{q}$, the Euler – Lagrange equation

$$\frac{d}{dt}\frac{\partial L_{KK}}{\partial \dot{\mathbf{q}}} = \frac{\partial L_{KK}}{\partial \mathbf{q}} = 0 \,,$$

shows that $\mathbf{p} = \partial L_{KK} / \partial \dot{\mathbf{q}}$ is conserved . The constant vector $\mathbf{p}$ is now identified as the vector in the body ,

$$\mathbf{p} = \mathbf{\Gamma} + \dot{\mathbf{q}} = -mg\,\chi \,.$$

After this identification , the heavy top action principle in Proposition (3.2.5) with the Kaluza – Klein Lagrangian returns Euler's motion equation for the heavy top (3.64) .

The Hamiltonian H_{KK} associated to L_{KK} by the Legendre transformation (3.75) is

$$\begin{aligned} H_{KK}\left(\mathbf{\Pi}, \mathbf{\Gamma}, \mathbf{q}, \mathbf{p}\right) &= \mathbf{\Pi}.\mathbf{\Omega} + \mathbf{p}.\dot{\mathbf{q}} - L_{KK}\left(\mathbf{\Omega}, \mathbf{\Gamma}, \mathbf{q}, \dot{\mathbf{q}}\right) \\ &= \tfrac{1}{2}\mathbf{\Pi}.\mathbb{I}^{-1}\mathbf{\Pi} - \mathbf{p}.\mathbf{\Gamma} + \tfrac{1}{2}\left|\mathbf{p}\right|^2 \\ &= \tfrac{1}{2}\mathbf{\Pi}.\mathbb{I}^{-1}\mathbf{\Pi} + \tfrac{1}{2}\left|\mathbf{p} - \mathbf{\Gamma}\right|^2 - \tfrac{1}{2}\left|\mathbf{\Gamma}\right|^2 . \end{aligned}$$

Recall that $\mathbf{\Gamma}$ is a unit vector . On the constant level set $\left|\mathbf{\Gamma}\right|^2 = 1$, the Kaluza – Klein Hamiltonian H_{KK} is a positive quadratic function , shifted by a constant . Likewise , on the constant level set $\mathbf{p} = -mg\,\chi$, the Kaluza – Klein Hamiltonian H_{KK} is a function of only the variables $\left(\mathbf{\Pi}, \mathbf{\Gamma}\right)$ and is equal to the Hamiltonian (3.68) for the heavy top up to an additive constant . Consequently , the Lie – Poisson equations for the Kaluza – Klein Hamiltonian H_{KK} now reproduce Euler's motion equation for the heavy top (3.64) .

Now we discuss the Euler – Poincare' (EP) reduction theorem .

Remark (3.2.9): (Geodesic Motion)

In many interesting cases , the Euler – Poincare' equations on the dual of a Lie algebra ϑ^* correspond to geodesic motion on the corresponding group G . The relationship between the equations on ϑ^* and on G is the content of the basic Euler – Poincare' theorem discussed later . Similarly , on the Hamiltonian side , the preceding paragraphs described the relation between the Hamiltonian equations on $T^{*}G$ and the Lie – Poisson equations on ϑ^{*}. The issue of geodesic motion is specially simple: if either the Lagrangian on ϑ or the Hamiltonian on ϑ^{*} is purely quadratic , then the corresponding motion on the group is geodesic motion .

Many of our previous considerations may be recast immediately as Euler – Poincare' equations .

(i) Rigid bodies $\cong\left(EPSO(n)\right)$,

(ii) Deforming bodies $\cong\left(EPGL_{+}(n,\Re)\right)$,

(iii) Heavy tops $\cong\left(EPSO(3)\times\Re^{3}\right)$,

(iv) EPDiff

This work applies reduction by symmetry to Hamilton's principle . For a G – invariant Lagrangian defined on TG , this reduction takes Hamilton's principle from TG to $TG/G\cong\vartheta$. Stationarity of the symmetry – reduced Hamilton's principle yields the Euler – Poincare' equations on ϑ^{*}. The corresponding reduced Legendre transformation yields the Lie – Poisson Hamiltonian formulation of these equations .

Euler – Poincare' Reduction starts with a right (respectively , left) invariant Lagrangian $L:TG\to\Re$ on the tangent bundle of a Lie group G . This means that $L\left(T_{g}R_{h}(v)\right)=L(v)$, respectively $L\left(T_{g}L_{h}(v)\right)=L(v)$, for all $g,h\in G$ and all $v\in T_{g}G$. In shorter notation , right invariance of the Lagrangian may be written as

$$L\left(g(t),\dot{g}(t)\right)=L\left(g(t)h,\dot{g}(t)h\right),$$

for all $h\in G$.

Theorem (3.2.10): (Euler – Poincare' Reduction)

Let G be a Lie group , $L:TG\to\Re$ a right – invariant Lagrangian , and $l:=L|_{\vartheta}:\vartheta\to\Re$ be its restriction to ϑ. For a curve $g(t)\in G$, let

$$\xi(t)=\dot{g}(t).g(t)^{-1}:=T_{g(t)}R_{g(t)^{-1}}\dot{g}(t)\in\vartheta .$$

Then the following four statements are equivalent :

(i) $g(t)$ satisfies the Euler – Lagrange equations for Lagrangian L defined on G .

(ii) The variational principle

$$\delta\int_a^b L\left(g(t),\dot{g}(t)\right)dt = 0$$

holds , for variations with fixed endpoints .

(iii) The (right – invariant) Euler – Poincare' equations hold:

$$\frac{d}{dt}\frac{\delta l}{\delta\xi} = -ad_\xi^*\frac{\delta l}{\delta\xi}.$$

(iv) The variational principle

$$\delta\int_a^b l\left(\xi(t)\right)dt = 0$$

holds on ϑ, using variations of the form $\delta\xi = \dot{\eta} - [\xi,\eta]$, where $\eta(t)$ is an arbitrary path in ϑ which vanishes at the endpoints , that is , $\eta(a) = \eta(b) = 0$.

Proof :

The proof consists of three steps .

Step (I): Proof that (i) $\Leftrightarrow$ (ii)

This is Hamilton's principle: the Euler – Lagrange equations follow from stationary action for variations δg which vanish at the endpoints .

Step (II): Proof that (ii) $\Leftrightarrow$ (iv)

Proving equivalence of the variational principles (ii) on TG and (iv) on ϑ for a right – invariant Lagrangian requires calculation of the variations $\delta\xi$ of $\xi = \dot{g}g^{-1}$ induced by δg . To simplify the exposition , the calculation will be done first for matrix Lie groups , then generalized to arbitrary Lie groups .

Step (IIA): Proof that (ii) $\Leftrightarrow$ (iv)

For $\xi = \dot{g}g^{-1}$, define $g_\varepsilon(t)$ to be a family of curves in G such that $g_0(t) = g(t)$ and denote

$$\delta g := \left.\frac{dg_\varepsilon(t)}{d\varepsilon}\right|_{\varepsilon=0}.$$

The variation of ξ is computed in terms of δg as

$$\delta\xi = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(\dot{g}_{\varepsilon}g_{\varepsilon}^{-1}\right) = \frac{d^2 g}{dtd\varepsilon}\Big|_{\varepsilon=0} g^{-1} - \dot{g}g^{-1}(\delta g)g^{-1}. \tag{3.76}$$

Set $\eta := g^{-1}\delta g$. That is , $\eta(t)$ is an arbitrary curve in ϑ which vanishes at the endpoints . The time derivative of η is computed as

$$\dot{\eta} = \frac{d\eta}{dt} = \frac{d}{dt}\left(\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_{\varepsilon}\right)g^{-1}\right) = \frac{d^2 g}{dtd\varepsilon}\Big|_{\varepsilon=0} g^{-1} - (\delta g)g^{-1}\dot{g}g^{-1}. \tag{3.77}$$

Taking the difference of (3.76) and (3.77) implies

$$\delta\xi - \dot{\eta} = -\dot{g}g^{-1}(\delta g)g^{-1} + (\delta g)g^{-1}\dot{g}g^{-1} = -\xi\eta + \eta\xi = -[\xi,\eta].$$

That is , for matrix Lie algebras ,

$$\delta\xi = \dot{\eta} - [\xi,\eta],$$

where $[\xi,\eta]$ is the matrix commutator . Next , we notice that right invariance of L allows one to change variables in the Lagrangian by applying $g^{-1}(t)$ from the right , as

$$L\left(g(t),\dot{g}(t)\right) = L\left(e,\dot{g}(t)g^{-1}(t)\right) =: l\left(\xi(t)\right).$$

Combining this definition of the symmetry – reduced Lagrangian $l:\vartheta\to\Re$ together with the formula for variations $\delta\xi$ just deduced proves the equivalence of (ii) and (iv) for matrix Lie groups .

Step (IIB): Proof that (ii)$\Leftrightarrow$(iv)

The same proof extends to any Lie group G by using the following lemma .

Lemma (3.2.11):

Let $g : U \subset \Re^2 \to G$ be a smooth map and denote its partial derivatives by

$$\begin{aligned}\xi(t,\varepsilon) &:= T_{g(t,\varepsilon)}R_{g(t,\varepsilon)^{-1}}\frac{\partial g(t,\varepsilon)}{\partial t},\\ \eta(t,\varepsilon) &:= T_{g(t,\varepsilon)}R_{g(t,\varepsilon)^{-1}}\frac{\partial g(t,\varepsilon)}{\partial\varepsilon}.\end{aligned} \tag{3.78}$$

Then

$$\frac{\partial \xi}{\partial \varepsilon}-\frac{\partial \eta}{\partial t}=-[\xi,\eta], \tag{3.79}$$

where $[\xi,\eta]$ is the Lie algebra bracket on ϑ. Conversely , if $U \subset \Re^2$ is simply connected and $\xi,\eta : U \to \vartheta$ are smooth functions satisfying (3.79) , then there exists a smooth function $g : U \to G$ such that (3.78) holds .

Proof of Lemma (3.2.11):

Write $\xi = \dot{g}g^{-1}$ and $\eta = g'g^{-1}$ in natural notation and express the partial derivatives $\dot{g} = \partial g/\partial t$ and $g' = \partial g/\partial \varepsilon$ using the right translations as

$$\dot{g} = \xi o g \quad \text{and} \quad g' = \eta o g .$$

By the chain rule , these definitions have mixed partial derivatives

$$\dot{g}' = \xi' = \nabla \xi.\eta \quad \text{and} \quad \dot{g}' = \dot{\eta} = \nabla \eta.\xi .$$

The difference of the mixed partial derivatives implies the desired formula (3.79),

$$\xi' - \dot{\eta} = \nabla \xi.\eta - \nabla \eta.\xi = -[\xi,\eta] = -ad_{\xi\eta} .$$

(Note the minus sign in the last two terms .)

Step (III): Proof of equivalence (iii)$\Leftrightarrow$(iv)

Let us show that the reduced variational principle produces the Euler – Poincare' equations . We write the functional derivative of the reduced action $S_{red} = \int_a^b l(\xi)dt$ with Lagrangian $l(\xi)$ in terms of the natural pairing $\langle .,. \rangle$ between ϑ^* and ϑ as

$$\delta\int_a^b l(\xi(t))dt = \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \delta\xi \right\rangle dt = \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} - ad_{\xi\eta} \right\rangle dt$$

$$= \int_a^b \left\langle \frac{\delta l}{\delta \xi}, \dot{\eta} \right\rangle dt - \int_a^b \left\langle \frac{\delta l}{\delta \xi}, ad_{\xi\eta} \right\rangle dt$$

$$= -\int_a^b \left\langle \frac{d}{dt}\frac{\delta l}{\delta \xi} + ad_\xi^* \frac{\delta l}{\delta \xi}, \eta \right\rangle dt .$$

The last equality follows from integration by parts and vanishing of the variation $\eta(t)$ at the endpoints . Thus , stationarity $\delta\int_a^b l\left(\xi(t)\right)dt = 0$ for any $\eta(t)$ that vanishes at the endpoints is equivalent to

$$\frac{d}{dt}\frac{\delta l}{\delta\xi} = -ad_{\xi}^{*}\frac{\delta l}{\delta\xi},$$

which are the Euler – Poincare' equations .

Remark (3.2.12): (Left – Invariant Euler – Poincare' Equations)

The same theorem holds for left invariant Lagrangians on TG , except for a sign in the Euler – Poincare' equations ,

$$\frac{d}{dt}\frac{\delta l}{\delta\xi} = +ad_{\xi}^{*}\frac{\delta l}{\delta\xi},$$

which arises because left – invariant variations satisfy $\delta\xi = \dot{\eta} + [\xi,\eta]$ (with the opposite sign) .

The procedure for reconstructing the solution $v(t) \in T_{g(t)}G$ of the Euler – Lagrange equations with initial conditions $g(0) = g_0$ and $\dot{g}(0) = v_0$ starting from the solution of the Euler – Poincare' equations is as follows . First , solve the initial value problem for the right – invariant Euler – Poincare' equations:

$$\frac{d}{dt}\frac{\delta l}{\delta\xi} = -ad_{\xi}^{*}\frac{\delta l}{\delta\xi} \quad \text{with} \quad \xi(0) = \xi_0 := v_0 g_0^{-1}$$

Then from the solution for $\xi(t)$ reconstruct the curve $g(t)$ on the group by solving the "linear differential equation with time – dependent coefficients "

$$\dot{g}(t) = \xi(t)g(t) \quad \text{with} \quad g(0) = g_0 .$$

The Euler – Poincare' reduction theorem guarantees then that $v(t) = \dot{g}(t) = \xi(t).g(t)$ is a solution of the Euler – Lagrange equations with initial condition $v_0 = \xi_0 g_0$.

Remark (3.2.13):

A similar statement holds , with obvious changes for left – invariant Lagrangian systems on TG .

As in the equivalence relation between the Lagrangian and Hamiltonian formulations discussed earlier , the relationship between symmetry – reduced Euler – Poincare' and Lie – Poisson formulations is determined by the Legendre transformation .

Definition (3.2.14):

The Legendre transformation $\mathbb{F}l : \vartheta \to \vartheta^*$ is defined by

$$\mathbb{F}l\left(\xi\right) = \frac{\delta l}{\delta \xi} = \mu .$$

Let $h\left(\mu\right) := \left\langle \mu, \xi \right\rangle - l\left(\xi\right)$. Assuming that $\mathbb{F}l$ is a diffeomorphism yields

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi .$$

So the Euler – Poincare' equations for l are equivalent to the Lie – Poisson equations for h :

$$\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) = -ad_{\xi}^{*} \frac{\delta l}{\delta \xi} \Leftrightarrow \dot{\mu} = -ad_{\delta h/\delta \mu}^{*} \mu .$$

The Lie – Poisson equations may be written in the Poisson bracket form

$$\dot{f} = \left\{ f , h \right\}, \tag{3.80}$$

where $f : \vartheta^* \to \Re$ is an arbitrary smooth function and the bracket is the (right) Lie – Poisson bracket given by

$$\left\{ f , h \right\}\left(\mu\right) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle = -\left\langle \mu, ad_{\delta h/\delta \mu} \frac{\delta f}{\delta \mu} \right\rangle = -\left\langle ad_{\delta h/\delta \mu}^{*} \mu, \frac{\delta f}{\delta \mu} \right\rangle . \tag{3.81}$$

In the important case when ℓ is quadratic , the Lagrangian L is the quadratic form associated to a right invariant Riemannian metric on G . In this case , the Euler – Lagrange equations for L on G describe geodesic motion relative to this metric and these geodesics are then equivalently described by either the Euler – Poincare' , or the Lie – Poisson equations .

Chapter (4)

Diffeons – Singular Momentum, Hamilton – Poincare' Reduction and Lie – Poisson Equations

Section (4.1): Diffeons – Singular Momentum Solutions of the EPDiff Equation for Geodesic Motion in Higher Dimensions

We start with some discuss the Euler – Poincare' equation on the diffeomorphisms (EPDiff).

Eulerian geodesic motion of a fluid in n dimensions is generated as on EP equation via Hamilton's principle , when the Lagrangian is given by the kinetic energy. The kinetic energy defines a norm $\|\mathbf{u}\|^2$ for the Eulerian fluid velocity , taken as $\mathbf{u}(\mathbf{x},\mathrm{t}): \mathrm{R}^n \times R^1 \rightarrow R^n$. The choice of the kinetic energy as a positive functional of fluid velocity $\mathbf{u}$ is a modeling step that depends upon the physics of the problem being studied . We shall choose the Lagrangian ,

$$\|\mathbf{u}\|^2 = \int \mathbf{u}.Q_{op}\mathbf{u}d^n x = \int \mathbf{u}.\mathbf{m}d^n x \ , \qquad (4.1)$$

so that the positive – definite , symmetric , operator Q_{op} defines the norm $\|\mathbf{u}\|$, for appropriate (homogeneous , say , or periodic) boundary conditions . The EPDiff equation is the Euler – Poincare' equation for this Eulerian geodesic motion of a fluid . Namely ,

$$\frac{d}{dt}\frac{\delta \ell}{\delta \mathbf{u}} + ad_{\mathbf{u}}^* \frac{\delta \ell}{\delta \mathbf{u}} = 0 , \quad \text{with} \quad \ell[\mathbf{u}] = \tfrac{1}{2}\|\mathbf{u}\|^2 . \qquad (4.2)$$

Here ad^* is the dual of the vector – field ad –operation (the commutator) under the natural L^2 pairing $\langle .,. \rangle$ induced by the variational derivative $\delta\ell[\mathbf{u}] = \langle \delta\ell/\delta\mathbf{u}, \delta\mathbf{u} \rangle$. This pairing provides the definition of ad^*,

$$\langle ad_{\mathbf{u}}^* \mathbf{m}, \mathbf{v} \rangle = -\langle \mathbf{m}, ad_{\mathbf{u}}\mathbf{v} \rangle , \qquad (4.3)$$

where $\mathbf{u}$ and $\mathbf{v}$ are vector fields , $ad_{\mathbf{u}}\mathbf{v} = [\mathbf{u},\mathbf{v}]$ is the commutator , that is , the Lie bracket given in components by (summing on repeated indices)

$$[\mathbf{u},\mathbf{v}]^i = u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j} , \quad \text{or} \quad [\mathbf{u},\mathbf{v}] = \mathbf{u}.\nabla\mathbf{v} - \mathbf{v}.\nabla\mathbf{u} . \qquad (4.4)$$

The notation $ad_{\mathbf{u}}\mathbf{v} := [\mathbf{u},\mathbf{v}]$ formally denotes the adjoint action of the right Lie algebra of $\mathrm{Diff}(D)$ on itself , and $\mathbf{m} = \delta\ell/\delta\mathbf{u}$ is the fluid momentum , a one – form density whose co – vector components are also denoted as $\mathbf{m}$.

If $\mathbf{u} = u^j\,\partial/\partial x^j$, $\mathbf{m} = m_i dx^i \otimes dV$, then the preceding formula for $ad^*_{\mathbf{u}}(\mathbf{m}\otimes dV)$ has the coordinate expression in $\Re^n$,

$$\left(ad^*_{\mathbf{u}}\mathbf{m}\right)_i dx^i \otimes dV = \left(\frac{\partial}{\partial x^j}\left(u^j m_i\right) + m_j \frac{\partial u^j}{\partial x^i}\right) dx^i \otimes dV \ . \tag{4.5}$$

In this notation , the abstract EPDiff equation (4.2) may be written explicitly in Eulerian coordinates as a partial differential equation for a co – vector function

$$\mathbf{m}(\mathbf{x},\mathrm{t}) : \mathrm{R}^n \times R^1 \to R^n \ .$$

Namely ,

$$\frac{\partial}{\partial t}\mathbf{m} + \underbrace{\mathbf{u}.\nabla\mathbf{m}}_{Convection} + \underbrace{\nabla\mathbf{u}^T.\mathbf{m}}_{Stretching} + \underbrace{\mathbf{m}\left(div\ \mathbf{u}\right)}_{Expansion} = 0, \quad \text{with} \quad \mathbf{m} = \frac{\delta\ell}{\delta\mathbf{u}} = Q_{op}\mathbf{u} \ . \tag{4.6}$$

To explain the terms in underbraces , we rewrite EPDiff as preservation of the one – form density of momentum along the characteristic curves of the velocity . Namely,

$$\frac{d}{dt}\left(\mathbf{m}.d\,\mathbf{x}\otimes dV\right) = 0 \quad \text{along} \quad \frac{d\,\mathbf{x}}{dt} = \mathbf{u} = G * \mathbf{m} \, . \tag{4.7}$$

This form of the EPDiff equation also emphasizes its nonlocality , since the velocity is obtained from the momentum density by convolution against the Green's function G of the operator Q_{op} . Thus , $\mathbf{u} = G * \mathbf{m}$ with $Q_{op}G = \delta(\mathbf{x})$, the Dirac measure . We may check that this "characteristic form" of EPDiff recovers its Euclidean form by computing directly,

$$\frac{d}{dt}\left(\mathbf{m}.d\,\mathbf{x}\otimes dV\right) = \frac{d\,\mathbf{m}}{dt}.d\,\mathbf{x}\otimes dV + \mathbf{m}.d\,\frac{d\,\mathbf{x}}{dt}\otimes dV + \mathbf{m}.d\,\mathbf{x}\otimes\left(\frac{d}{dt}dV\right)$$

$$\text{along} \quad \frac{d\,\mathbf{x}}{dt} = \mathbf{u} = G * \mathbf{m}$$

$$= \left(\frac{\partial}{\partial t}\mathbf{m} + \mathbf{u}.\nabla\mathbf{m} + \nabla\mathbf{u}^T.\mathbf{m} + \mathbf{m}\left(div\,\mathbf{u}\right)\right).d\,\mathbf{x}\otimes dV = 0 \, .$$

Remark (4.1.1):

The EPDiff may be written as

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_{\mathbf{u}}\right)\left(\mathbf{m}.d\,\mathbf{x}\otimes dV\right)=0\,, \tag{4.8}$$

where $\mathcal{L}_{\mathbf{u}}$ is the Lie derivative with respect to the vector field with components $\mathbf{u}=G*\mathbf{m}$. And the EPDiff may also be written equivalently in terms of the operators div , grad and curl in $2D$ and $3D$ as

$$\frac{\partial}{\partial t}\mathbf{m}-\mathbf{u}\times curl\,\mathbf{m}+\nabla\left(\mathbf{u}.\mathbf{m}\right)+\mathbf{m}\left(div\,\mathbf{u}\right)=0\,. \tag{4.9}$$

Thus , for example , its numerical solution would require an algorithm which has the capability to deal with the distinctions and relationships among the operators div , grad and curl .

Let's derive the EPDiff equation (4.6) by following the proof of the EP reduction theorem leading the Euler – Poincare' equations for right invariance in the form (4.2) . Following this calculation for the present case yields

$$\delta\int_a^b l\left(\mathbf{u}\right)dt=\int_a^b\left\langle\frac{\delta l}{\delta\mathbf{u}},\delta\mathbf{u}\right\rangle dt=\int_a^b\left\langle\frac{\delta l}{\delta\mathbf{u}},\dot{\mathbf{v}}-ad_{\mathbf{u}}\mathbf{v}\right\rangle dt$$

$$=\int_a^b\left\langle\frac{\delta l}{\delta\mathbf{u}},\dot{\mathbf{v}}\right\rangle dt-\int_a^b\left\langle\frac{\delta l}{\delta\mathbf{u}},ad_{\mathbf{u}}\mathbf{v}\right\rangle dt=-\int_a^b\left\langle\frac{d}{dt}\frac{\delta l}{\delta\mathbf{u}}+ad_{\mathbf{u}}^{*}\frac{\delta l}{\delta\mathbf{u}},\mathbf{v}\right\rangle dt\,,$$

where $\langle .,.\rangle$ is the pairing between elements of the Lie algebra and its dual . In our case , this is the L^2 pairing , for example ,

$$\left\langle\frac{\delta l}{\delta\mathbf{u}},\delta\mathbf{u}\right\rangle=\int\frac{\delta l}{\delta u^i}\delta u^i d^n x$$

This pairing allows us to compute the coordinate form EPDiff equation explicitly , as

$$\int_a^b\left\langle\frac{\delta l}{\delta\mathbf{u}},\delta\mathbf{u}\right\rangle dt=\int_a^b dt\int\frac{\delta l}{\delta u^i}\left(\frac{\partial v^i}{\partial t}+u^j\frac{\partial v^i}{\partial x^j}-v^j\frac{\partial u^i}{\partial x^j}\right)d^n x$$

$$=-\int_a^b dt\int\left\{\frac{\partial}{\partial t}\frac{\delta l}{\delta u^i}+\frac{\partial}{\partial x^j}\left(\frac{\delta l}{\delta u^i}u^j\right)+\frac{\delta l}{\delta u^j}\frac{\partial u^j}{\partial x^i}\right\}v^i d^n x$$

Substituting $\mathbf{m} = \delta l / \delta \mathbf{u}$ now recovers the coordinate forms for the coadjoint action of vector fields in (4.5) and the EPDiff equation itself in (4.6) . When $\ell[\mathbf{u}] = \frac{1}{2}\|\mathbf{u}\|^2$, EPDiff describes geodesic motion on the diffeomorphisms with respect to the norm $\|\mathbf{u}\|$.

Lemma (4.1.2):

In Step (IIB) of the proof of the Euler – Poincare' reduction theorem (that (ii) $\Leftrightarrow$ (iv) for an arbitrary Lie group) a certain formula for the variations for time – dependent vector fields was employed . That formula was employed again in the calculation above as

$$\delta \mathbf{u} = \dot{\mathbf{v}} - ad_{\mathbf{u}} \mathbf{v} . \tag{4.10}$$

This formula may be rederived as follows in the present context . We write $\mathbf{u} = \dot{g} g^{-1}$ and $\mathbf{v} = g' g^{-1}$ in natural notation and express the partial derivatives $\dot{g} = \partial g / \partial t$ and $g' = \partial g / \partial \varepsilon$ using the right translations as

$$\dot{g} = \mathbf{u} o g \quad \text{and} \quad g' = \mathbf{v} o g .$$

To compute the mixed partials , consider the chain rule for say $\mathbf{u}(g(t,\varepsilon)\mathbf{x}_0)$ and set $\mathbf{x}(t,\varepsilon) = g(t,\varepsilon).\mathbf{x}_0$. Then ,

$$\mathbf{u}' = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} . \frac{\partial \mathbf{x}}{\partial \varepsilon} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} . g'(t,\varepsilon)\mathbf{x}_0 = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} . g' g^{-1} \mathbf{x} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} . \mathbf{v}(\mathbf{x}) .$$

The chain rule $\dot{\mathbf{v}}$ gives a similar formula with $\mathbf{u}$ and $\mathbf{v}$ exchanged . Thus , the chain rule gives two expressions for the mixed partial derivative $\dot{g}'$ as

$$\dot{g}' = \mathbf{u}' = \nabla \mathbf{u}.\mathbf{v} \quad \text{and} \quad \dot{g}' = \dot{\mathbf{v}} = \nabla \mathbf{v}.\mathbf{u} .$$

The difference of the mixed partial derivatives then implies the desired formula (4.10) , since

$$\mathbf{u}' - \dot{\mathbf{v}} = \nabla \mathbf{u}.\mathbf{v} - \nabla \mathbf{v}.\mathbf{u} = -[\mathbf{u}, \mathbf{v}] = -ad_{\mathbf{u}} \mathbf{v} .$$

We shall discuss the solutions of EPDiff for pressureless compressible geodesic motion in one spatial dimension . This is the EPDiff equation in $1D$,

$$\partial_t m + ad_u^* m = 0, \quad \text{or, equivalently,} \tag{4.11}$$

$$\partial_t m + u m_x + 2 u_x m = 0 , \quad \text{with } m = Q_{op} u . \tag{4.12}$$

1- The EPDiff equation describes geodesic motion on the diffeomorphism group with respect to a family of metrics for the fluid velocity $u(t,x)$, with notation ,

$$m = \frac{\delta \ell}{\delta u} = Q_{op} u \quad \text{for a kinetic – energy Lagrangian} \tag{4.13}$$

$$\ell(u) = \tfrac{1}{2}\int u Q_{op} u dx = \tfrac{1}{2}\|u\|^2 . \tag{4.14}$$

2- In one dimension , Q_{op} in equation (4.13) is a positive , symmetric operator that defines the kinetic energy metric for the velocity.

3- The EPDiff equation (4.12) is written in terms of the variable $m = \delta\ell/\delta u$. It is appropriate to call this variational derivative m , because it is the momentum density associated with the fluid velocity u .

4- Physically , the first nonlinear term in the EPDiff equation (4.12) is fluid transport .

5- The coefficient 2 arises in the second nonlinear term , because , in one dimension , two of the summands in $ad_u^* m = u m_x + 2 u_x m$ are the same , cf. equation (4.5) .

6- The momentum is expressed in terms of the velocity by $m = \delta\ell/\delta u = Q_{op} u$. Equivalently , for solutions that vanish at spatial infinity , one may think of the velocity as being obtained from the convolution ,

$$u(x) = G * m(x) = \int G(x-y) m(y) dy , \tag{4.15}$$

where G is the Green's function for the operator Q_{op} on the real line .

7- The operator Q_{op} and its Green's function G are chosen to be even under reflection , $G(-x) = G(x)$, so that u and m have the same parity . Moreover , the EPDiff equation (4.12) conserves the total momentum $M = \int m(y) dy$, for any even Green's function .

The EPDiff equation (4.12) on the real line has the remarkable property that its solutions collectivize into the finite dimensional solutions of the "N – pulson " form that was discovered for a special form of G , then was extended for any even G ,

$$u(x,t) = \sum_{i=1}^{N} p_i(t) G(x - q_i(t)) . \tag{4.16}$$

Since $G(x)$ is the Green's function for the operator Q_{op}, the corresponding solution for the momentum $m = Q_{op}u$ is given by a sum of delta functions ,

$$m(x,t) = \sum_{i=1}^{N} p_i(t)\delta\big(x - q_i(t)\big). \tag{4.17}$$

Thus , the time – dependent "collective coordinates" $q_i(t)$ and $p_i(t)$ are the positions and velocities of the N pulses in this solution . These parameters satisfy the finite dimensional geodesic motion equations obtained as canonical Hamiltonian equations

$$\dot{q}_i = \frac{\partial H_N}{\partial p_i} = \sum_{j=1}^{N} p_j G\big(q_i - q_j\big), \tag{4.18}$$

$$\dot{p}_i = -\frac{\partial H_N}{\partial q_i} = -p_i \sum_{j=1}^{N} p_j G'\big(q_i - q_j\big), \tag{4.19}$$

in which the Hamiltonian is given by the quadratic form ,

$$H_N = \tfrac{1}{2} \sum_{i,j=1}^{N} p_i p_j G\big(q_i - q_j\big). \tag{4.20}$$

Remark (4.1.3):

In certain sense, equations (4.18) – (4.19) comprise the analog for the peakon momentum relation (4.17) of the "symmetric generalized rigid body equations" in (3.29) .

Thus, the canonical equations for the Hamiltonian H_N describe the nonlinear collective interactions of the N – pulson solutions of the EPDiff equation (4.12) as finite – dimensional geodesic motion of a particle on N – dimensional surface whose co – metric is

$$G^{ij}(q) = G\big(q_i - q_j\big). \tag{4.21}$$

The case $G(x) = e^{-|x|/\alpha}$ with a constant length scale α is the Green's function for which the operator in the kinetic energy Lagrangian (4.13) is $Q_{op} = 1 - \alpha^2\partial_x^2$. For this (Helmholtz) operator Q_{op}, the Lagrangian and corresponding kinetic energy norm are given by,

$$\ell[u] = \tfrac{1}{2}\|u\|^2 = \tfrac{1}{2}\int uQ_{op}u\,dx = \tfrac{1}{2}\int u^2 + \alpha^2 u_x^2\,dx\,, \quad \text{for} \quad \lim_{|x|\to\infty} u = 0\,.$$

This Lagrangian is the H^1 norm of the velocity in one dimension . In this case, the EPDiff equation (4.12) is also the zero – dispersion limit of the completely integrable CH equation for unidirectional shallow water waves first derived,

$$m_t + um_x + 2mu_x = -c_0 u_x + \gamma u_{xxx}\,, \quad m = u - \alpha^2 u_{xx}\,. \tag{4.22}$$

This equation describes shallow water dynamics as completely integrable solition motion at quadratic order in the asymptotic expansion for unidirectional shallow water waves on a free surface under gravity . For more details and explanations of this asymptotic expansion for unidirectional shallow water waves to quadratic order.

Because of the relation $m = u - \alpha^2 u_{xx}$, equation (4.22) is nonlocal. In other words, it an integral – partial differential equation . In fact, after writing equation (4.22) in the equivalent form,

$$\left(1-\alpha^2\partial^2\right)\left(u_t + uu_x\right) = -\partial\left(u^2 + \tfrac{1}{2}\alpha^2 u_x^2\right) - c_0 u_x + \gamma u_{xxx}\,, \tag{4.23}$$

one sees the interplay between local and nonlocal linear dispersion in its phase velocity relation,

$$\frac{\omega}{k} = \frac{c_0 - \gamma k^2}{1+\alpha^2 k^2}\,, \tag{4.24}$$

for waves with frequency ω and wave number k linearized around $u = 0$. For $\gamma/c_0 < 0$, short waves and long waves travel in the same direction . Long waves travel faster then short ones (as required in shallow water) provided $\gamma/c_0 > -\alpha^2$. Then the phase velocity lies in the interval $\omega/k \in \left(-\gamma/\alpha^2, c_0\right]$.

The famous Korteweg – de Vries (KdV) solition equation,

$$u_t + 3uu_x = -c_0 u_x + \gamma u_{xxx}\,, \tag{4.25}$$

emerges at linear order in the asymptotic expansion for shallow water waves, in which one takes $\alpha^2 \to 0$ in (4.23) and (4.24) . In KdV, the parameters c_0 and γ are seen as deformation of the Riemann equation,

$$u_t + 3uu_{xx} = 0\,.$$

The parameters c_0 and γ represent linear wave dispersion , which modifies and eventually balances the tendency for nonlinear waves to steepen and break . The parameter α , which introduces nonlocality , also regularizes this nonlinear tendency , even in the absence of c_0 and γ .

Now we discuss the Diffeons – singular momentum solutions of the EPDiff equation for geodesic motion in higher dimensions .

As an example of the EP theory in higher dimensions , we shall generalize the one – dimensional pulson solutions of the previous section to n – dimensions . The corresponding singular momentum solutions of the EPDiff equation in higher dimensions are called "diffeons" .

Eulerian geodesic motion of a fluid in n – dimensions is generated as an EP equation via Hamilton's principle , when the Lagrangian is given by the kinetic energy . The kinetic energy defines a norm $\|\mathbf{u}\|^2$ for the Eulerian fluid velocity , $\mathbf{u}(\mathbf{x},t): R^n \times R^1 \to R^n$. As mentioned earlier , the choice of the kinetic energy as a positive functional of fluid velocity $\mathbf{u}$ is a modeling step that depends upon the physics of the problem being studied . Following our earlier procedure , as in equation (4.1) and (4.2) , we shall choose the Lagrangian ,

$$\|\mathbf{u}\|^2 = \int \mathbf{u}.Q_{op}\mathbf{u}\, d^n x = \int \mathbf{u}.\mathbf{m}\, d^n x \ . \qquad (4.26)$$

so that the positive – definite , symmetric , operator Q_{op} defines the norm $\|\mathbf{u}\|$, for appropriate boundary conditions and the EPDiff equation for Eulerian geodesic motion of a fluid emerges ,

$$\frac{d}{dt}\frac{\delta\ell}{\delta\mathbf{u}} + ad^*_{\mathbf{u}}\frac{\delta\ell}{\delta\mathbf{u}} = 0 , \qquad \text{with} \quad \ell[\mathbf{u}] = \tfrac{1}{2}\|\mathbf{u}\|^2 . \qquad (4.27)$$

The corresponding Legendre transform yields the following invertible relations between momentum and velocity ,

$$\mathbf{m} = Q_{op}\mathbf{u} \qquad \text{and} \quad \mathbf{u} = G * \mathbf{m}, \qquad (4.28)$$

where G is the Green's function for the operator Q_{op} , assuming appropriate boundary (on $\mathbf{u}$) that allow inversion of the operator Q_{op} to determine $\mathbf{u}$ from $\mathbf{m}$.

The corresponding Hamiltonian is ,

$$h[\mathbf{m}] = \langle \mathbf{m}, \mathbf{u} \rangle - \tfrac{1}{2}\|\mathbf{u}\|^2 = \tfrac{1}{2}\int \mathbf{m}.G * \mathbf{m}\, d^n x \equiv \tfrac{1}{2}\|\mathbf{m}\|^2, \qquad (4.29)$$

which also defines a norm $\|\mathbf{m}\|$ via a convolution kernel G that is symmetric and positive , when the Lagrangian $\ell[\mathbf{u}]$ is a norm . As expected , the norm $\|\mathbf{m}\|$ given by the Hamiltonian $h[\mathbf{m}]$ specifies the velocity $\mathbf{u}$ in terms of its Legendre – dual momentum $\mathbf{m}$ by the variational operation ,

$$\mathbf{u} = \frac{\delta h}{\delta \mathbf{m}} = G * \mathbf{m} \equiv \int G(\mathbf{x}-\mathbf{y})\mathbf{m}(\mathbf{y})\, d^n y \ . \qquad (4.30)$$

We shall choose the kernel $G(\mathbf{x}-\mathbf{y})$ to be translation – invariant (so Noether's theorem implies that total momentum $\mathbf{M} = \int \mathbf{m}\, d^n x$ is conserved) and symmetric under spatial reflections (so that $\mathbf{u}$ and $\mathbf{m}$ have the same parity) .

After the Legendre transformation (4.29) , the EPDiff equation (4.27) appears in its equivalent Lie – Poisson Hamiltonian form ,

$$\frac{\partial}{\partial t}\mathbf{m} = \{\mathbf{m}, h\} = -ad^*_{\delta h/\delta \mathbf{m}}\mathbf{m}\, . \qquad (4.31)$$

Here the operation $\{.,.\}$ denotes the Lie – Poisson bracket dual to the (right) action of vector fields amongst themselves by vector – field commutation

$$\{f, h\} = -\left\langle \mathbf{m}, \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta h}{\delta \mathbf{m}}\right]\right\rangle$$

For more details and additional background concerning the relation of classical EP theory to Lie – Poisson Hamiltonian equations .

In a moment we will also consider the momentum maps for EPDiff .

The momentum for the one dimensional pulson solutions (4.17) on the real line is supported at points via the Dirac delta measures in its solution ansatz ,

$$m(x,t) = \sum_{i=1}^{N} p_i(t)\,\delta\big(x - q_i(t)\big), \qquad m \in R^1 . \qquad (4.32)$$

We shall develop n – dimensional analogs of these one – dimensional pulson solutions for the Euler – Poincare' equation (4.9) by generalizing this solution ansatz to allow measure – valued n – dimensional vector solutions $\mathbf{m} \in R^n$ for which the Euler – Poincare' momentum is supported on co – dimension – k subspaces R^{n-k} with integer $k \in [1,n]$. For example , one may consider the two – dimensional vector momentum $\mathbf{m} \in R^2$ in the plane that is supported on one – dimensional curves (momentum fronts) . Likewise , in three dimensions , one could consider two – dimensional momentum surfaces , etc . The corresponding vector momentum ansatz that we shall use is the following , cf. the pulson solutions (4.32) ,

$$\mathbf{m}(\mathbf{x},t) = \sum_{i=1}^{N} \int \mathbf{P}_i\,(s,t)\,\delta\left(\mathbf{x} - \mathbf{Q}_i\,(s,t)\right) ds\ , \qquad \mathbf{m} \in R^n\,. \tag{4.33}$$

Here , $\mathbf{P}_i, \mathbf{Q}_i \in R^n$ for $i = 1, 2, ..., N$. For example , when $n - k = 1$, so that $s \in R^1$ is one – dimensional , the delta function in solution (4.33) supports an evolving family of vector – valued curves , called momentum filaments . (For simplicity of notation , we suppress the implied subscript i in the arclength s for each $\mathbf{P}_i$ and $\mathbf{Q}_i$.) The Legendre – dual relations (4.28) imply that the velocity corresponding to the momentum filament ansatz (4.33) is ,

$$\mathbf{u}(\mathbf{x},t) = G * \mathbf{m} = \sum_{j=1}^{N} \int \mathbf{P}_j\,(s',t)\,G\left(\mathbf{x} - \mathbf{Q}_j\,(s',t)\right) ds'\,. \tag{4.34}$$

Just as the $1D$ case of the pulsons , we shall show that substitution of the nD solution ansatz (4.33) and (4.34) into the EPDiff equation (4.6) produces canonical geodesic Hamiltonian equations for the n – dimensional vector parameters $\mathbf{Q}_i\,(s,t)$ and $\mathbf{P}_i\,(s,t)$, $i = 1, 2, ..., N$.

For definiteness in what follows , we shall consider the example of momentum filaments $\mathbf{m} \in R^n$ supported on one – dimensional space curves in R^n, so $s \in R^1$ is the arclength parameter of one of these curves . This solution ansatz is reminiscent of the Biot – Savart Law for vortex filaments , although the flow is not incompressible . The dynamics of momentum surfaces , for $s \in R^k$ with $k \in n$, follow a similar analysis .

Substituting the momentum filament ansatz (4.33) for $s \in R^1$ and its corresponding velocity (4.34) into the Euler – Poincare' equation (4.6) , then integrating against a smooth test function $\phi(\mathbf{x})$ implies the following canonical equations (denoting explicit summation on $i, j \in 1, 2, ..., N$),

$$\frac{\partial}{\partial t}\mathbf{Q}_i\,(s,t) = \sum_{j=1}^{N} \int \mathbf{P}_j\,(s',t)\,G\left(\mathbf{Q}_i\,(s,t) - \mathbf{Q}_j\,(s',t)\right) ds' = \frac{\delta H_N}{\delta \mathbf{P}_i}\,, \tag{4.35}$$

$$\frac{\partial}{\partial t}\mathbf{P}_i(s,t) = -\sum_{j=1}^{N}\int\left(\mathbf{P}_i(s,t).\mathbf{P}_j(s',t)\right)\frac{\partial}{\partial \mathbf{Q}_i(s,t)}G\left(\mathbf{Q}_i(s,t)-\mathbf{Q}_j(s',t)\right)ds' = -\frac{\delta H_N}{\delta \mathbf{Q}_i},$$

$$(\text{sum on } j \text{ , no sum on } i \text{) .} \tag{4.36}$$

The dot product $\mathbf{P}_i.\mathbf{P}_j$ denotes the inner , or scalar , product of the two vectors $\mathbf{P}_i$ and $\mathbf{P}_j$ in R^n . Thus , the solution ansatz (4.33) yields a closed set of integro – partial – differential equations (IPDEs) given by (4.35) and (4.36) for the vector parameters $\mathbf{Q}_i(s,t)$ and $\mathbf{P}_i(s,t)$ with $i=1,2,.......,N$. These equations are generated canonically by the following Hamiltonian function $H_N : \left(R^n \times R^n\right)^{\otimes N} \to R$,

$$H_N = \tfrac{1}{2}\iint\sum_{i,j=1}^{N}\left(\mathbf{P}_i(s,t).\mathbf{P}_j(s',t)\right)G\left(\mathbf{Q}_i(s,t)-\mathbf{Q}_j(s',t)\right)dsds' . \tag{4.37}$$

This Hamiltonian arises by substituting the momentum ansatz (4.33) into the Hamiltonian (4.29) obtained from the Legendre transformation of the Lagrangian corresponding to the kinetic energy norm of the fluid velocity . Thus , the evolutionary IPDE system (4.35) and (4.36) represents canonically Hamiltonian geodesic motion on the space of curves in R^n with respect to the co – metric given on these curves in (4.37) . The Hamiltonian $H_N = \tfrac{1}{2}\|\mathbf{P}\|^2$ in (4.37) defines the norm $\|\mathbf{P}\|$ in terms of this co – metric that combines convolution using the Green's function G and sum over filaments with the scalar product of momentum vectors in R^n .

Remark (4.1.4):

Note the Lagrangian property of the s coordinate , since

$$\frac{\partial}{\partial t}\mathbf{Q}_i(s,t) = \mathbf{u}\left(\mathbf{Q}_i(s,t),t\right).$$

Now we discuss the singular solution momentum map $\mathbf{J}_{Sing}$ for diffeons .

The diffeon momentum filament ansatz (4.33) reduces , and collectivizes the solution of the geodesic EP PDE (4.6) in $n+1$ dimensions into the system (4.35) and (4.36) of $2N$ canonical evolutionary IPDEs . One can summarize the mechanism by which this process occurs , by saying that the map that implements the canonical $(\mathbf{Q},\mathbf{P})$ variables in terms of singular solutions is a (cotangent bundle) momentum map . Such momentum maps are Poisson maps; so the canonical Hamiltonian nature of the dynamical equations for $(\mathbf{Q},\mathbf{P})$ fits into a general theory which also provides a framework for suggesting other avenues of investigation .

Theorem (4.1.5):

The momentum ansatz (4.33) for measure – valued solutions of the EPDiff equation (4.6) , defines an equivariant momentum map

$$\mathbf{J}_{Sing} : T^{*}Emb\left(S,\Re^{n}\right) \to \mathfrak{X}\left(\Re^{n}\right)^{*}$$

that is called the singular solution momentum map .

We shall explain the notation used in the theorem's statement in the course of its proof . Right away , however , we note that the sense of "defines " is that the momentum solution ansatz (4.33) expressing $\mathbf{m}$ (which are functions of s) can be regarded as a map from the space of $\left(\mathbf{Q}(s),\mathbf{P}(s)\right)$ to the space of $\mathbf{m}$'s . This will turn out to be the Lagrange – to – Euler map for the fluid description of the singular solutions .

We shall give two proofs of this result from two rather different viewpoints . The first proof below uses the formula for a momentum map for a cotangent lifted action , while the second proof focuses on a Poisson bracket computation. Each proof also explains the context in which one has a momentum map .

First proof:

For simplicity and without loss of generality , let us take $N=1$ and so suppress the index a . That is , we shall take the case of an isolated singular solution . As the proof will show , this is not a real restriction .

To set the notation , fix a k – dimensional manifold S with a given volume element and whose points are denoted $s \in S$. Let $Emb\left(S,\Re^{n}\right)$ denote the set of smooth embeddings $\mathbf{Q}: S \to \Re^{n}$. (If the EPDiff equations are taken on a manifold M , replace $\Re^{n}$ with M .) Under appropriate technical conditions , which we shall just treat formally here , $Emb\left(S,\Re^{n}\right)$ is a smooth manifold .

The tangent space $T_{\mathbf{Q}}Emb\left(S,\Re^{n}\right)$ to $Emb\left(S,\Re^{n}\right)$ at the point $\mathbf{Q} \in Emb\left(S,\Re^{n}\right)$ is given by the space of material velocity fields , namely the linear space of maps

$\mathbf{V}: S \rightarrow \Re^n$ that are vector fields over the map $\mathbf{Q}$. The dual space to this space will be identified with the space of one – form densities over $\mathbf{Q}$, which we shall regard as maps $\mathbf{P}: S \rightarrow \left(\Re^n\right)^*$. In summary , the cotangent bundle $T^*Emb\left(S,\Re^n\right)$ is identified with the space of pairs of maps $(\mathbf{Q},\mathbf{P})$.

These give us the domain space for the singular solution momentum map . Now we consider the action of the symmetry group . Consider the group $\mathfrak{G} = Diff$ of diffeomorphisms of the space $\mathfrak{S}$ in which the EPDiff equations are operating , concretely on our case $\Re^n$. Let it act on $\mathfrak{S}$ by composition on the left. Namely for $\eta \in Diff\ \left(\Re^n\right)$, we let

$$\eta.\mathbf{Q} = \eta o \mathbf{Q}\,. \tag{4.38}$$

Now lift this action to the cotangent bundle $T^*Emb\left(S,\Re^n\right)$ in the standard way . This lifted action is a symplectic (and hence Poisson) action and has an equivariant momentum map . We claim that this momentum map is precisely given by the ansatz (4.33).

To see this , one only needs to recall and then apply the general formula for the momentum map associated with an action of general Lie group $\mathfrak{G}$ on a configuration manifold Q and cotangent lifted to T^*Q .

First let us recall the general formula . Namely , the momentum map is the map $\mathbf{J}: T^*Q \rightarrow \vartheta^*$ (ϑ^* denotes the dual of the Lie algebra ϑ of $\mathfrak{G}$) defined by

$$\mathbf{J}\left(\alpha_q\right).\xi = \left\langle \alpha_q, \xi_Q(q)\right\rangle, \tag{4.39}$$

where $\alpha_q \in T_q^*Q$ and $\xi \in \vartheta$, where ξ_Q is the infinitesimal generator of the action of $\mathfrak{G}$ on Q associated to the Lie algebra element ξ, and where $\left\langle \alpha_q, \xi_Q(q)\right\rangle$ is the natural pairing of an element of T_q^*Q with an element of T_qQ .

Now we apply this formula to the special case in which the group $\mathfrak{G}$ is the diffeomorphism group $Diff\ \left(\Re^n\right)$, the manifold Q is $Emb\left(S,\Re^n\right)$ and where the action of the group on $Emb\left(S,\Re^n\right)$ is given by (4.38) . The sense in which the Lie algebra of $\mathfrak{G} = Diff$ is the space $\vartheta = \mathfrak{X}$ of vector fields is well – understood . Hence , its dual is naturally regarded as the space of one – form densities . The momentum map is thus a map $\mathbf{J}: T^*Emb\left(S,\Re^n\right) \rightarrow \mathfrak{X}^*$.

With $\mathbf{J}$ given by (4.39) , we only need to work out this formula . First , we shall work out the infinitesimal generators . Let $X \in \mathfrak{X}$ be a Lie algebra element . By differentiating the action (4.38) with respect to η in the direction of X at the identity element we find that the infinitesimal generator is given by

$$X_{Emb(S,\mathfrak{R}^n)}(\mathbf{Q}) = X \, o \mathbf{Q} .$$

Thus , taking α_q to be the cotangent vector $(\mathbf{Q},\mathbf{P})$, equation (4.39) gives

$$\langle \boldsymbol{J}(\mathbf{Q},\mathbf{P}), X \rangle = \langle (\mathbf{Q},\mathbf{P}), X \, o\mathbf{Q} \rangle = \int_S P_i(s) X^i(\mathbf{Q}(s)) d^k s .$$

On the other hand , note that the right hand side of (4.33) (again with the index a suppressed , and with t suppressed as well) , when paired with the Lie algebra element X is

$$\left\langle \int_S \mathbf{P}(s)\delta(\mathbf{x}-\mathbf{Q}(s)) d^k s, X \right\rangle = \int_{\mathfrak{R}^n}\int_S \left(P_i(s)\delta(\mathbf{x}-\mathbf{Q}(s)) d^k s\right) X^i(\mathbf{x}) d^n x$$

$$= \int_S P_i(s) X^i(\mathbf{Q}(s)) d^k s .$$

This shows the expression given by (4.33) is equal to $\boldsymbol{J}$ and so the result is proved .

Second proof:

As is standard , one can characterize momentum maps by means of the following relation , required to hold for all functions F on $T^*Emb(S,\mathfrak{R}^n)$; that is , functions of $\mathbf{Q}$ and $\mathbf{P}$:

$$\{F, \langle \boldsymbol{J}, \xi \rangle\} = \xi_P[F]. \tag{4.40}$$

In our case , we shall take $\boldsymbol{J}$ to be given by the solution ansatz and verify that it satisfies this relation . To do so , let $\xi \in \mathfrak{X}$ so that the left side of (4.40) becomes

$$\left\{F, \int_S P_i(s)\xi^i(\mathbf{Q}(s)) d^k s\right\} = \int_S \left[\frac{\delta F}{\delta Q^i}\xi^i(\mathbf{Q}(s)) - P_i(s)\frac{\delta F}{\delta P_j}\frac{\delta}{\delta Q^j}\xi^i(\mathbf{Q}(s))\right] d^k s .$$

On the other hand , one can directly compute from the definitions that the infinitesimal generator of the action on the space $T^*Emb\left(S,\Re^n\right)$ corresponding to the vector field $\xi^i(\mathbf{x})\frac{\partial}{\partial Q^i}$ (a Lie algebra element) , is given by

$$\delta\mathbf{Q} = \xi o\mathbf{Q}, \qquad \delta\mathbf{P} = -P_i(s)\frac{\partial}{\partial\mathbf{Q}}\xi^i\left(\mathbf{Q}(s)\right).$$

An important element left out in this proof so far is that it does not make clear that the momentum map is equivariant , a condition needed for the momentum map to be Poisson . The first proof took care of this automatically since momentum maps for cotangent lifted actions are always equivariant and hence are Poisson .

Thus , to complete the second proof , we need to check directly that the momentum map is equivariant . Actually , we shall only check that it is infinitesimally invariant by showing that it is a Poisson map from $T^*Emb\left(S,\Re^n\right)$ to the space of $\mathbf{m}$'s (the dual of the Lie algebra of $\mathfrak{X}$) with its Lie – Poisson bracket . This sort of approach to characterize equivariant momentum maps is discussed in an interesting .

The following direct computation shows that the singular solution momentum map (4.33) is Poisson . This is accomplished by using the canonical Poisson brackets for $\{\mathbf{P}\}$, $\{\mathbf{Q}\}$ and applying the chain rule to compute $\left\{m_i(\mathbf{x}), m_j(\mathbf{y})\right\}$, with notation $\delta_k'(\mathbf{y}) \equiv \partial\delta(\mathbf{y})/\partial y^k$.

We get

$$\left\{m_i(\mathbf{x}), m_j(\mathbf{y})\right\} = \left\{\sum_{a=1}^{N}\int ds\, P_i^a(s,t)\,\delta\left(\mathbf{x}-\mathbf{Q}^a(s,t)\right), \sum_{b=1}^{N}\int ds' P_j^b(s',t)\,\delta\left(\mathbf{y}-\mathbf{Q}^b(s',t)\right)\right\}$$

$$= \sum_{a,b=1}^{N}\iint ds\, ds' \Big[\left\{P_i^a(s), P_j^b(s')\right\}\delta\left(\mathbf{x}-\mathbf{Q}^a(s)\right)\delta\left(\mathbf{y}-\mathbf{Q}^b(s')\right)$$

$$-\left\{P_i^a(s), Q_k^b(s')\right\}P_j^b(s')\,\delta\left(\mathbf{x}-\mathbf{Q}^a(s)\right)\delta_k'\left(\mathbf{y}-\mathbf{Q}^b(s')\right)$$

$$-\left\{Q_k^a(s), P_j^b(s')\right\}P_i^a(s)\,\delta_k'\left(\mathbf{x}-\mathbf{Q}^a(s)\right)\delta\left(\mathbf{y}-\mathbf{Q}^b(s')\right)$$

$$+\left\{Q_k^a(s), Q_\ell^b(s')\right\}P_i^a(s)P_j^b(s')\,\delta_k'\left(\mathbf{x}-\mathbf{Q}^a(s)\right)\delta_\ell'\left(\mathbf{y}-\mathbf{Q}^b(s')\right)\Big].$$

Substituting the canonical Poisson bracket relations

$$\left\{P_i^a\left(s\right),P_j^b\left(s'\right)\right\}=0$$

$$\left\{Q_k^a\left(s\right),Q_\ell^b\left(s'\right)\right\}=0\,,$$

and $$\left\{Q_k^a\left(s\right),P_j^b\left(s'\right)\right\}=\delta^{ab}\delta_{kj}\delta\left(s-s'\right)$$

into the preceding computation yields

$$\left\{m_i\left(\mathbf{x}\right),m_j\left(\mathbf{y}\right)\right\}=\left\{\sum_{a=1}^{N}\int ds\,P_i^a\left(s,t\right)\delta\left(\mathbf{x}-\mathbf{Q}^a\left(s,t\right)\right),\sum_{b=1}^{N}ds'P_j^b\left(s',t\right)\delta\left(\mathbf{y}-\mathbf{Q}^b\left(s',t\right)\right)\right\}$$

$$=\sum_{a=1}^{N}ds\,P_j^a\left(s\right)\delta\left(\mathbf{x}-\mathbf{Q}^a\left(s\right)\right)\delta_i'\left(\mathbf{y}-\mathbf{Q}^a\left(s\right)\right)$$

$$-\sum_{a=1}^{N}\int ds\,P_i^a\left(s\right)\delta_j'\left(\mathbf{x}-\mathbf{Q}^a\left(s\right)\right)\delta\left(\mathbf{y}-\mathbf{Q}^a\left(s\right)\right)$$

$$=-\left(m_j\left(\mathbf{x}\right)\frac{\partial}{\partial x^i}+\frac{\partial}{\partial x^j}m_i\left(\mathbf{x}\right)\right)\delta\left(\mathbf{x}-\mathbf{y}\right).$$

Thus ,

$$\left\{m_i\left(\mathbf{x}\right),m_j\left(\mathbf{y}\right)\right\}=-\left(m_j\left(\mathbf{x}\right)\frac{\partial}{\partial x^i}+\frac{\partial}{\partial x^j}m_i\left(\mathbf{x}\right)\right)\delta\left(\mathbf{x}-\mathbf{y}\right),\qquad(4.41)$$

which is readily checked to be the Lie – Poisson bracket on the space of $\mathbf{m}$'s , restricted to their singular support . This completes the second proof of theorem .

Corollary (4.1.6):

The singular solution momentum map defined by the singular solution ansatz (4.33) , namely ,

$$\mathbf{J}_{Sing}:T^*Emb\left(S,\Re^n\right)\to\mathfrak{X}\left(\Re^n\right)^*$$

is a Poisson map from the canonical Poisson structure on $T^*Emb\left(S,\Re^n\right)$ to the Lie – Poisson structure on $\mathfrak{X}\left(\Re^n\right)^*$.

This is perhaps the most basic property of the singular solution momentum map.

Since the solution ansatz (4.33) has been shown in the preceding Corollary to be a Poisson map , the pull back of the Hamiltonian from $\mathfrak{X}^*$ to $T^*Emb(S,\Re^n)$ gives equations of motion on the latter space that project to the equations on $\mathfrak{X}^*$.

Thus , the basic fact that the momentum map $\mathbf{J}_{Sing}$ is Poisson explains why the functions $\mathbf{Q}^a(s,t)$ and $\mathbf{P}^a(s,t)$ satisfy canonical Hamiltonian equations .

Note that the coordinate $s\in\Re^k$ that labels these functions is a "Lagrangian coordinate" in the sense that it does not evolve in time but rather labels the solution .

In terms of the pairing

$$\langle .,. \rangle : \vartheta^* \times \vartheta \to \Re, \qquad (4.42)$$

between the Lie algebra ϑ (vector fields in $\Re^n$) and its dual ϑ^* (one – form densities in $\Re^n$) , the following relation holds for measure – valued solutions under the momentum map (4.33),

$$\begin{aligned}
\langle \mathbf{m},\mathbf{u}\rangle &= \int \mathbf{m}.\mathbf{u}\, d^n\mathbf{x}, \quad L^2 \text{ pairing for } \mathbf{m},\mathbf{u}\in\Re^n \\
&= \iint \sum_{a,b=1}^{N} \left(\mathbf{P}^a(s,t).\mathbf{P}^b(s',t)\right) G\left(\mathbf{Q}^a(s,t)-\mathbf{Q}^b(s',t)\right) ds\, ds' \\
&= \int \sum_{a=1}^{N} \mathbf{P}^a(s,t).\frac{\partial \mathbf{Q}^a(s,t)}{\partial t} ds \\
&= \left\langle\left\langle \mathbf{P},\dot{\mathbf{Q}}\right\rangle\right\rangle, \qquad (4.43)
\end{aligned}$$

which is the natural pairing between the points $(\mathbf{Q},\mathbf{P})\in T^*Emb(S,\Re^n)$ and $(\mathbf{Q},\dot{\mathbf{Q}})\in T\,Emb(S,\Re^n)$. This corresponds to preservation of the action of the Lagrangian $\ell[\mathbf{u}]$ under cotangent lift of $Diff\,(\Re^n)$.

The pull – back of the Hamiltonian $H[\mathbf{m}]$ defined on the dual of the Lie algebra ϑ^*, to $T^*Emb(S,\Re^n)$ is easily seen to be consistent with what we had before:

$$H[\mathbf{m}] = \tfrac{1}{2}\langle \mathbf{m}, G*\mathbf{m}\rangle = \tfrac{1}{2}\langle\langle \mathbf{P}, G*\mathbf{P}\rangle\rangle \equiv H_N[\mathbf{P},\mathbf{Q}]. \qquad (4.44)$$

In summary , in concert with the Poisson nature of the singular solution momentum map , we see that the singular solutions in terms of $\mathbf{Q}$ and $\mathbf{P}$ satisfy

Hamiltonian equations and also define an invariant solution set for the EPDiff equations . In fact:

This invariant solution set is a special coadjoint orbit for the diffeomorphism group.

Now we discuss the geometry of the momentum map.

In this section we explore the geometry of the singular solution momentum map discussed earlier in a little more detail . The treatment is formal , in the sense that there are a number of technical issues in the infinite dimensional case that will be left open . We will mention a few of these as we proceed .

We claim that the image of the singular solution momentum map is a coadjoint orbit in $\mathfrak{X}^*$. This means that (modulo some issues of connectedness and smoothness , which we do not consider here) the solution ansatz by (4.33) defines a coadjoint orbit in the space of all one – form densities , regarded as the dual of the Lie algebra of the diffeomorphism group . These coadjoint orbits should be thought of as singular orbits – that is, due to their special nature , they are not generatic.

Recognizing them as coadjoint orbits is one way of gaining further insight into why the singular solutions form dynamically invariant sets – it is a general fact that coadjoint orbits in ϑ^* are symolectic submanifolds of the Lie – Poisson manifold ϑ^* (in our case $\mathfrak{X}\left(\Re^n\right)^*$) and , correspondingly , are dynamically invariant for any Hamiltonian system on ϑ^*.

The idea of the proof of our claim is simply this: whenever one has an equivariant momentum map $\mathbf{J}: P \to \vartheta^*$ for the action of a group G on a symplectic or Poisson manifold P , and that action is transitive , then the image of $\mathbf{J}$ is an orbit (or at least a piece of an orbit). Roughly speaking , the reason that transitivity holds in our case is because one can "more the images of the manifolds S around at will with arbitrary velocity fields " using diffeomorphisms of $\Re^n$.

The momentum map $\mathbf{J}_{Sing}$ involves $Diff\left(\Re^n\right)$, the action of the diffeomorphism group on the space of embeddings $Emb\left(S,\Re^n\right)$ by smooth maps of the target space $\Re^n$, namely,

$$Diff\left(\Re^n\right): \mathbf{Q}.\eta = \eta o \mathbf{Q}, \tag{4.45}$$

where , recall, $\mathbf{Q}:S\to\Re^n$. As above , the cotangent bundle $T^*Emb\left(S,\Re^n\right)$ is identified with the space of pairs of maps $(\mathbf{Q},\mathbf{P})$, with $\mathbf{Q}:S\to\Re^n$ and $\mathbf{P}:S\to T^*\Re^n$.

However , there is another momentum map $\mathbf{J}_S$ associated with the right action of the diffeomorphism group of S on the embeddings $Emb\left(S,\Re^n\right)$ by smooth maps of the "Lagrangian labels" S (fluid particle relabeling by $\eta:S\to S$). This action is given by

$$Diff\ (S):\mathbf{Q}.\eta=\mathbf{Q}o\eta\ . \tag{4.46}$$

The infinitesimal generator of this right action is

$$X_{Emb\left(S,\Re^n\right)}(\mathbf{Q})=\frac{d}{dt}\Big|_{t=0}\mathbf{Q}o\eta_t=T\mathbf{Q}oX\ . \tag{4.47}$$

where $X\in\mathfrak{X}$ is tangent to the curve η_t at $t=0$. Thus , again taking $N=1$ (so we suppress index a) and also letting α_q in the momentum map formula (4.39) be the cotangent vector $(\mathbf{Q},\mathbf{P})$, one computes $\mathbf{J}_S$:

$$\langle\mathbf{J}_S(\mathbf{Q},\mathbf{P}),X\rangle=\langle(\mathbf{Q},\mathbf{P}),T\mathbf{Q}.X\rangle$$

$$=\int_S P_i(s)\frac{\partial Q^i(s)}{\partial s^m}X^m(s)d^ks$$

$$=\int_S X\ \left(\mathbf{P}(s).d\ \mathbf{Q}(s)\right)d^ks$$

$$=\left(\int_S \mathbf{P}(s).d\ \mathbf{Q}(s)\otimes d^ks,X\ (s)\right)$$

$$=\langle\mathbf{P}.d\ \mathbf{Q},X\rangle\ .$$

Consequently , the momentum map formula (4.39) yields

$$\mathbf{J}_S\left(\mathbf{Q},\mathbf{P}\right)=\mathbf{P}.d\ \mathbf{Q}\ , \tag{4.48}$$

with the indicated pairing of the one – form density $\mathbf{P}.d\ \mathbf{Q}$ with the vector field X .

We have set things up so that the following is true.

Proposition (4.1.7):

The momentum map $\mathbf{J}_S$ is preserved by the evolution equations (4.35) – (4.36) for $\mathbf{Q}$ and $\mathbf{P}$.

Proof :

It is enough to notice that the Hamiltonian H_N in equation (4.37) is invariant under the cotangent lift of the action of $Diff\ (S)$; it merely amounts to the invariance of the integral over S under reparametrization ; that is , the change of variables formula ; keep in mind that $\mathbf{P}$ includes a density factor .

Remark (4.1.8):

(1) This result is similar to the Kelvin – Noether theorem for circulation Γ of an ideal fluid , which may be written as $\Gamma = \oint_{c(s)} D(s)^{-1}\mathbf{P}(s).d\mathbf{Q}(s)$ for each Lagrangian circuit $c(s)$, where D is the mass density and $\mathbf{P}$ is again the canonical momentum density. This similarity should come as no surprise , because the Kelvin – Noether theorem for ideal fluids arises from invariance of Hamilton's principle under fluid parcel relabeling by the same right action of the diffeomorphism group , as in (4.45).

(2) Note that , being an equivariant momentum map, the map $\mathbf{J}_S$, as with $\mathbf{J}_{Sing}$, is also a Poisson map . That is , substituting the canonical Poisson bracket into relation (4.48) ; that is , the relation $\mathbf{M}(\mathbf{x}) = \sum_i P_i(\mathbf{x})\nabla Q^i(\mathbf{x})$ yields the Lie – Poisson bracket on the space of $\mathbf{M}$'s. We use the different notations $\mathbf{m}$ and $\mathbf{M}$ because these quantities are analogous to the body and spatial angular momentum for rigid body mechanics. In fact, the quantity $\mathbf{m}$ given by the solution Ansatz ; specifically , $\mathbf{m} = \mathbf{J}_{Sing}(\mathbf{Q},\mathbf{P})$ gives the singular solutions of the EPDiff equations , while $\mathbf{M}(\mathbf{x}) = \mathbf{J}_S(\mathbf{Q},\mathbf{P}) = \sum_i P_i(\mathbf{x})\nabla Q^i(\mathbf{x})$ is a conserved quantity.

(3) In the Lagrange of fluid mechanics , the expression of $\mathbf{m}$ in terms of $(\mathbf{Q},\mathbf{P})$ is an example of a Clebsch representation , which expresses the solution of the EPDiff equations in terms of canonical variables that evolve by standard canonical Hamilton equations. This has been known in the case of fluid mechanics for more than 100 years. For modern discussions of the Clebsch representation for ideal fluids.

(4) One more remark is in order ; namely the special case in which $S = M$ is of course allowed. In this case , Q corresponds to the map η itself and $\mathbf{P}$ just corresponds to conjugate momentum . The quantity $\mathbf{m}$ corresponds to the spatial (dynamic) momentum density (that is, right translation of $\mathbf{P}$ to the identity) , while $\mathbf{M}$ corresponds to the conserved "body" momentum density (that is, left translation of $\mathbf{P}$ to the identity) .

$Emb\left(S,\Re^n\right)$ admits two group actions . These are: the group $Diff\ (S)$ of diffeomorphisms of S , which acts by composition on the right ; and the group $Diff\ \left(\Re^n\right)$ which acts by composition on the left. The group $Diff\ \left(\Re^n\right)$ acting from the left produces the singular solution momentum map, $\mathbf{J}_{Sing}$. The action of $Diff\ (S)$ from the right produces the conserved momentum map $\mathbf{J}_S : T^*Emb\left(S,\Re^n\right) \rightarrow \mathfrak{X}(S)^*$. We now assemble both momentum maps into one figure as follows:

$$
\begin{tikzcd}
 & T^*Emb\left(S,M\right) \arrow[dl, "\mathbf{J}_{Sing}"'] \arrow[dr, "\mathbf{J}_S"] & \\
\mathfrak{X}(M)^* & & \mathfrak{X}(S)^*
\end{tikzcd}
$$

Section (4.2): Hamilton – Poincare' Reduction and Lie – Poisson Equations with Applications

We start with some discuss the fluid a' la Holm, Marsden and Ratiu[31].

Almost all fluid models of interest admit the following general assumptions . These assumptions form the basic of the Euler – Poincare' theorem for Continua that we shall state later in this section , after introducing the notation necessary for dealing geometrically with the reduction of Hamilton's principle from the material (or Lagrangian) picture of fluid dynamics , to the spatial (or Eulerian) picture . This theorem was first stated and proved by Holm , Marsden and Ratiu[31] , to which we refer for additional details , as well as for abstract definitions and proofs .

Basic assumptions underlying the Euler – Poincare' theorem for continua

(1) There is a right representation of a Lie group G on the vector space V and G acts in the natural way on the right on $TG\times V^*:(U_g,a)h=(U_gh,ah)$.

(2) The Lagrangian function $L:TG\times V^*\to\Re$ is right $G-$invariant under the isotropy group of $a_0\in V^*$.

(3) In particular , if $a_0\in V^*$, define the Lagrangian $L_{a_0}:TG\to\Re$ by $L_{a_0}(U_g)=L(U_g,a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G , where G_{a_0} is the isotropy group of a_0 .

(4) Right $G-$invariance of L permits one to define the Lagrangian on the Lie algebra ϑ of the group G . Namely , $\ell:\vartheta\times V^*\to\Re$ is defined by ,

$$\ell(u,a)=L\left(U_g g^{-1}(t),a_0g^{-1}(t)\right)=L\left(U_g,a_0\right),$$

where $u=U_g g^{-1}(t)$ and $a=a_0g^{-1}(t)$. Conversely , this relation defines for any $\ell:\vartheta\times V^*\to\Re$ a right $G-$invariant function $L:TG\times V^*\to\Re$.

(5) For a curve $g(t)\in G$, let $u(t):=\dot g(t)g(t)^{-1}$ and define the curve $a(t)$ as the unique solution of the linear differential equation with time dependent coefficients $\dot a(t)=-a(t)u(t)$, where the action of an element of the Lie algebra $u\in\vartheta$ on an advected quantity $a\in V^*$ is denoted by concatenation from the right . The solution with initial condition $a(0)=a_0\in V^*$ can be written as $a(t)=a_0g(t)^{-1}$.

Remark (4.2.1):

(1) Let $\vartheta(D)$ denote the space of vector fields on D of some fixed differentiability class . These vector fields are endowed with the Lie bracket given in components by (summing on repeated indices)

$$[\mathbf{u},\mathbf{v}]^i = u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j} \tag{4.49}$$

The notation $ad_{\mathbf{u}}\mathbf{v} := [\mathbf{u},\mathbf{v}]$ formally denotes the adjoint action of the right Lie algebra of $Diff\ (D)$ on itself .

(2) Identify the Lie algebra of vector fields ϑ with its dual ϑ^* by using the L^2 pairing

$$\langle \mathbf{u},\mathbf{v}\rangle = \int_D \mathbf{u}.\mathbf{v}dV \ . \tag{4.50}$$

(3) Let $\vartheta(D)^*$ denote the geometric dual space of $\vartheta(D)$, that is , $\vartheta(D)^* := \Lambda^1(D)\otimes Den(D)$. This is the space of one – form densities on D . If $\mathbf{m}\otimes dV \in \Lambda^1(D)\otimes Den(D)$, then the pairing of $\mathbf{m}\otimes dV$ with $\mathbf{u}\in\vartheta(D)$ is given by the L^2 pairing ,

$$\langle \mathbf{m}\otimes dV\ ,\mathbf{u}\rangle = \int_D \mathbf{m}.\mathbf{u}dV \tag{4.51}$$

where $\mathbf{m}.\mathbf{u}$ is the standard contraction of a one – form $\mathbf{m}$ with a vector field $\mathbf{u}$.

(4) For $\mathbf{u}\in\vartheta(D)$ and $\mathbf{m}\otimes dV \in \vartheta(D)^*$, the dual of the adjoint representation is defined by

$$\langle ad^*_{\mathbf{u}}(\mathbf{m}\otimes dV\ ),\mathbf{v}\rangle = -\int_D \mathbf{m}.ad_{\mathbf{u}}\mathbf{v}dV \ = -\int_D \mathbf{m}.[\mathbf{u},\mathbf{v}]dV \tag{4.52}$$

and its expression is

$$ad^*_{\mathbf{u}}(\mathbf{m}\otimes dV\ ) = \left(£_{\mathbf{u}}\mathbf{m} + (div_{dV}\mathbf{u})\mathbf{m}\right)\otimes dV \ = £_{\mathbf{u}}(\mathbf{m}\otimes dV\ )\ , \tag{4.53}$$

where $div_{dV}\mathbf{u}$ is the divergence of $\mathbf{u}$ relative to the measure dV , that is , $£_{\mathbf{u}}dV \ = (div_{dV}\mathbf{u})dV$. Hence , $ad^*_{\mathbf{u}}$ coincides with the Lie – derivative $£_{\mathbf{u}}$ for one – form densities.

(5) If $\mathbf{u} = u^j\partial/\partial x^j$, $\mathbf{m} = m_i dx^i$, then the one – form factor in the preceding formula for $ad^*_{\mathbf{u}}(\mathbf{m}\otimes dV\ )$ has the coordinate expression

$$\left(ad_{\mathbf{u}}^{*}\mathbf{m}\right)_i dx^i = \left(u^j \frac{\partial m_i}{\partial x^j} + m_j \frac{\partial u^j}{\partial x^i} + \left(div_{dV}\,\mathbf{u}\right)m_i\right)dx^i$$

$$= \left(\frac{\partial}{\partial x^j}\left(u^j m_i\right) + m_j \frac{\partial u^j}{\partial x^i}\right)dx^i \,. \qquad (4.54)$$

The last equality assumes that the divergence is taken relative to the standard measure $dV = d^n\mathbf{x}$ in $\Re^n$. (On a Riemannian manifold the metric divergence needs to be used.)

Definition (4.2.2):

Elements of D representing the material particles of the system are denoted by X; their coordinates $X^A, A = 1,\dots\dots,n$ may thus be regarded as the particle labels.

(i) A configuration , which we typically denote by η, or g, is an element of $Diff\ (D)$.

(ii) A motion , denoted as η_t or alternatively as $g(t)$, is a time dependent curve in $Diff\ (D)$.

Definition (4.2.3):

The Lagrangian , or material velocity $\mathbf{U}(X,t)$ of the continuum along the motion η_t or $g(t)$ is defined by taking the time derivative of the motion keeping the particle labels X fixed:

$$\mathbf{U}(X,t) := \frac{d\eta_t(X)}{dt} := \frac{\partial}{\partial t}\Big|_X \eta_t(X) := \dot{g}(t).X \;.$$

These are convenient shorthand notations for the time derivative at fixed Lagrangian coordinate X .

Consistent with this definition of material velocity , the tangent space to $Diff\ (D)$ at $\eta \in Diff\ (D)$ is given by

$$T_\eta Diff\ (D) = \left\{\mathbf{U}_\eta : D \to TD : \mathbf{U}_\eta(X) \in T_{\eta(X)}D\right\}.$$

Elements of $T_\eta Diff\ (D)$ are usually thought of as vector fields on D covering η. The tangent lift of right translations on $T\ Diff\ (D)$ by $\varphi \in Diff\ (D)$ is given by

$$\mathbf{U}_\eta \varphi := T_\eta R_\varphi\left(\mathbf{U}_\eta\right) = \mathbf{U}_\eta o \varphi \,.$$

Definition (4.2.4):

During a motion η_t or $g(t)$, the particle labeled by X describes a path in D, whose points

$$x(X,t) := \eta_t(X) := g(t).X \ ,$$

are called the Eulerian or spatial points of this path , which is also called the Lagrangian trajectory , because a Lagrangian fluid parcel follows this path in space . The derivative $\mathbf{u}(x,t)$ of this path , evaluated at fixed Eulerian point x , is called the Eulerian or spatial velocity of the system:

$$\mathbf{u}(x,t) := \mathbf{u}(\eta_t(X),t) := \mathbf{U}(X,t) := \frac{\partial}{\partial t}\Big|_X \eta_t(X) := \dot{g}(t).X := \dot{g}(t)g^{-1}(t).x \ .$$

Thus the Eulerian velocity $\mathbf{u}$ is a time dependent vector field on D, denoted as $\mathbf{u}_t \in \vartheta(D)$, where $\mathbf{u}_t(x) := \mathbf{u}(x,t)$. We also have the fundamental relationships

$$\mathbf{U}_t = \mathbf{u}_t o \eta_t \quad \text{and} \quad \mathbf{u}_t = \dot{g}(t)g^{-1}(t),$$

where we denote $\mathbf{U}_t(X) := \mathbf{U}(X,t)$.

Definition (4.2.5):

The representation space V^* of $Diff\,(D)$ in continuum mechanics is often some subspace of the tensor field densities on D, denoted as $\mathfrak{T}(D)\otimes Den(D)$, and the representation is given by pull back . It is thus a right representation of $Diff\,(D)$ on $\mathfrak{T}(D)\otimes Den(D)$. The right action of the Lie algebra $\vartheta(D)$ on V^* is denoted as concatenation from the right . That is , we denote

$$a\mathbf{u} := \pounds_{\mathbf{u}} a \ ,$$

which is the Lie derivative of the tensor field density a along the vector field $\mathbf{u}$.

Definition (4.2.6):

The Lagrangian of a continuum mechanical system is a function

$$L : T\ Diff\ (D) \times V^* \to \Re,$$

which is right invariant relative to the tangent lift of right translation of $Diff\ (D)$ on itself and pull back on the tensor field densities. Invariance of the Lagrangian L induces a function $\ell : \vartheta(D) \times V^* \to \Re$ given by

$$\ell(\mathbf{u}, a) = L\left(\mathbf{u} o \eta, \eta^* a\right) = L\left(\mathbf{U}, a_0\right),$$

where $\mathbf{u} \in \vartheta(D)$ and $a \in V^* \subset \mathfrak{T}(D) \otimes Den(D)$, and where $\eta^* a$ denotes the pull back of a by the diffeomorphism η and $\mathbf{u}$ is the Eulerian velocity . That is ,

$$\mathbf{U} = \mathbf{u} o \eta \quad \text{and} \quad a_0 = \eta^* a. \tag{4.55}$$

The evolution of a is by right action , given by the equation

$$\dot{a} = -£_{\mathbf{u}} a = -a\mathbf{u}. \tag{4.56}$$

The solution of this equation , for the initial condition a_0 , is

$$a(t) = \eta_{t^*} a_0 = a_0 g^{-1}(t), \tag{4.57}$$

where the lower star denotes the push forward operation and η_t is the flow of $\mathbf{u} = \dot{g} g^{-1}(t)$.

Definition (4.2.7):

Advected Eulerian quantities are defined in continuum mechanics to be those variables which are Lie transported by the flow of the Eulerian velocity field . Using this standard terminology , equation (4.56) , or its solution (4.57) states that the tensor field density $a(t)$ (which may include mass density and other Eulerian quantities) is advected .

Remark (4.2.8): (Dual Tensor)

As we mentioned , typically $V^* \subset \mathfrak{T}(D) \otimes Den(D)$ for continuum mechanics . On a general manifold , tensors of a given type have natural duals . For example , symmetric covariant tensors are dual to symmetric contravariant tensor densities , the pairing being given by the integration of the natural contraction of these tensors . Likewise , k – forms are naturally dual to $(n-k)$ – forms , the pairing being given by taking the integral of their wedge product .

Definition (4.2.9):

The diamond operation $\Diamond$ between elements of V and V^* produces an element of the dual Lie algebra $\vartheta(D)^*$ and is defined as

$$\langle b \Diamond a, \mathbf{w} \rangle = -\int_D b.\pounds_{\mathbf{w}} a \,, \tag{4.58}$$

where $b.\pounds_{\mathbf{w}} a$ denotes the contraction , as described above , of elements of V and elements of V^* and $\mathbf{w} \in \vartheta(D)$. (These operations do not depend on a Riemannian structure.)

For a path $\eta_t \in Diff\ (D)$, let $\mathbf{u}(x,t)$ be its Eulerian velocity and consider the curve $a(t)$ with initial condition a_0 given by the equation

$$\dot{a} + \pounds_{\mathbf{u}} a = 0\,. \tag{4.59}$$

Let the Lagrangian $L_{a_0}(\mathbf{U}) := L(\mathbf{U}, a_0)$ be right – invariant under $Diff\ (D)$. We can now state the Euler – Poincare' Theorem for Continua of , Marsden and Ratiu[31] .

Theorem (4.2.10): (Euler – Poincare' Theorem for Continua)

Given a path η_t in $Diff\ (D)$ with Lagrangian velocity $\mathbf{U}$ and Eulerian velocity $\mathbf{u}$, the following are equivalent:

(i) Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L\left(X, \mathbf{U}_t(X), a_0(X)\right) dt = 0 \tag{4.60}$$

Holds , for variations $\delta\eta_t$ vanishing at the endpoints.

(ii) η_t satisfies the Euler – Lagrange equations for L_{a_0} on $Diff\ (D)$.

(iii) The constrained variational principle in Eulerian coordinates

$$\delta \int_{t_1}^{t_2} \ell(\mathbf{u}, a) dt = 0 \tag{4.61}$$

holds on $\vartheta(D) \times V^*$, using variations of the form

$$\delta\mathbf{u}=\frac{\partial\mathbf{w}}{\partial t}+[\mathbf{u},\mathbf{w}]=\frac{\partial\mathbf{w}}{\partial t}+ad_{\mathbf{u}}\mathbf{w}\,,\qquad \delta a=-£_{\mathbf{w}}a \tag{4.62}$$

where $\mathbf{w}_t=\delta\eta_t o\eta_t^{-1}$ vanishes at the endpoints.

(iv) The Euler – Poincare' equations for continua

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\mathbf{u}}=-ad_{\mathbf{u}}^{*}\frac{\delta\ell}{\delta\mathbf{u}}+\frac{\delta\ell}{\delta a}\Diamond a=-£_{\mathbf{u}}\frac{\delta\ell}{\delta\mathbf{u}}+\frac{\delta\ell}{\delta a}\Diamond a\,, \tag{4.63}$$

hold , with auxiliary equations $(\partial_t+£_{\mathbf{u}})=0$ for each advected quantity $a(t)$. The $\Diamond$ operation defined in (4.58) needs to be determined on a case by case basic , depending on the nature of the tensor $a(t)$. The variation $\mathbf{m}=\delta\ell/\delta\mathbf{u}$ is a one – form density and we have used relation (4.52) in the last step of equation (4.63).

We refer to Holm , Marsden and Ratiu[31] for the proof of this theorem in the abstract setting . We shall see some of the features of this result in the concrete setting of continuum mechanics shortly.

The following string of quantities shows directly that (iii) is equivalent to (iv):

$$0=\delta\int_{t_1}^{t_2}l\,(\mathbf{u},a)dt=\int_{t_1}^{t_2}\left(\frac{\delta l}{\delta\mathbf{u}}.\delta\mathbf{u}+\frac{\delta l}{\delta a}.\delta a\right)dt$$

$$=\int_{t_1}^{t_2}\left[\frac{\delta l}{\delta\mathbf{u}}.\left(\frac{\partial\mathbf{w}}{\partial t}-ad_{\mathbf{u}}\mathbf{w}\right)-\frac{\delta l}{\delta a}.£_{\mathbf{w}}a\right]dt$$

$$=\int_{t_1}^{t_2}\mathbf{w}.\left[-\frac{\partial}{\partial t}\frac{\delta l}{\delta\mathbf{u}}-ad_{\mathbf{u}}^{*}\frac{\delta l}{\delta\mathbf{u}}+\frac{\delta l}{\delta a}\Diamond a\right]dt\ . \tag{4.64}$$

The rest of the proof follows essentially the same track as the proof of the pure Euler – Poincare' theorem , modulo slight changes to accommodate the advected quantities.

In the absence of dissipation , most Eulerian fluid equations can be written in the EP form in equation (4.63) ,

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\mathbf{u}}+ad_{\mathbf{u}}^{*}\frac{\delta\ell}{\delta\mathbf{u}}=\frac{\delta\ell}{\delta a}\Diamond a,\quad\text{with}\ \ (\partial_t+£_{\mathbf{u}})a=0\,. \tag{4.65}$$

Equation (4.65) is Newton's Law: The Eulerian time derivative of the momentum density $\mathbf{m}=\delta\ell/\delta\mathbf{u}$ (a one – form density dual to the velocity $\mathbf{u}$) is equal to the force density $(\delta\ell/\delta a)\Diamond a$, with the $\Diamond$ operation defined in (4.58) . Thus , Newton's Law is written in the Eulerian fluid representation as

$$\frac{d}{dt}\bigg|_{Lag} \mathbf{m} := (\partial_t + £_{\mathbf{u}})\mathbf{m} = \frac{\delta\ell}{\delta a}\diamond a, \quad \text{with} \quad \frac{d}{dt}\bigg|_{Lag} a := (\partial_t + £_{\mathbf{u}})a = 0. \qquad (4.66)$$

(i) The left side of the EP equation in (4.66) describes the fluid's dynamics due to its kinetic energy. A fluid's kinetic energy typically defines a norm for the Eulerian fluid velocity , $KE = \frac{1}{2}\|\mathbf{u}\|^2$. The left side of the EP equation is the geodesic part of its evolution , with respect to this norm . However , in a gravitational field , for example, there will also be dynamics due to potential energy . And this dynamics will by governed by the right side of the EP equation .

(ii) The right side of EP equation in (4.66) modifies the geodesic motion is also a geometrical quantity . The diamond operation $\diamond$ represents the dual of the Lie algebra action of vectors fields on the tensor a. Here $\delta\ell/\delta a$ is the dual tensors , under the natural pairing (usually , L^2 pairing) $\langle .,. \rangle$ that is induced by the variational derivative of the Lagrangian $\ell(\mathbf{u}, \mathrm{a})$. The diamond operation $\diamond$ is defined in terms of this pairing in (4.58) . For the L^2 pairing , this is integration by parts of (minus) the Lie derivative in (4.58).

(iii) The quantity a is typically a tensor (for example , a density , a scalar, or a differential form) and we shall sum over the various types of tensors a that are involved in the fluid description . The second equation in (4.66) states that each tensor a is carried along by the Eulerian fluid velocity $\mathbf{u}$. Thus , a is for fluid "attribute" and its Eulerian evolution is given by minus its Lie derivative , $-£_{\mathbf{u}}a$. That is, a stands for the set of fluid attributes that each Lagrangian fluid parcel carries around (advects) , such as its buoyancy , which is determined by its individual salt , or heat content , in ocean circulation .

(iv) Many examples of how equation (4.66) arises in the dynamics of continuous media are given by Holm , Marsden and Ratiu[31] . The EP form of the Eulerian fluid description in (4.66) is analogous to the classical dynamics of right bodies (and tops , under gravity) in body coordinates . Rigid bodies and tops are also governed by Euler – Poincare' equations , as Poincare' showed in a two – page paper with no references.

Corollary (4.2.11): (Kelvin – Noether Circulation Theorem)

Assume $\mathbf{u}(x,t)$ satisfies the Euler – Poincare' equations for continua:

$$\frac{\partial}{\partial t}\left(\frac{\delta \ell}{\delta \mathbf{u}}\right) = -£_{\mathbf{u}}\left(\frac{\delta \ell}{\delta \mathbf{u}}\right) + \frac{\delta \ell}{\delta a}\Diamond a$$

and the quantity a satisfies the advection relation

$$\frac{\partial a}{\partial t} + £_{\mathbf{u}} a = 0. \tag{4.67}$$

Let η_t be the flow of the Eulerian velocity field $\mathbf{u}$, that is , $\mathbf{u} = (d\eta_t/dt) o\eta_t^{-1}$.
Define the advected fluid loop $\gamma_t := \eta_t o\gamma_0$ and the circulation map $I(t)$ by

$$I(t) = \oint_{\gamma_t} \frac{1}{D}\frac{\delta \ell}{\delta \mathbf{u}}. \tag{4.68}$$

In the circulation map $I(t)$ the advected mass density D_t satisfies the push forward relation $D_t = \eta * D_0$. This implies the advection relation (4.67) with $a = D$, namely , the continuity equation ,

$$\partial_t D + divD\mathbf{u} = 0.$$

Then the map $I(t)$ satisfies the Kelvin circulation relation ,

$$\frac{d}{dt} I(t) = \oint_{\gamma_t} \frac{1}{D}\frac{\delta \ell}{\delta a}\Diamond a. \tag{4.69}$$

Both an abstract proof of the Kelvin – Noether Circulation Theorem and a proof tailored for the case of continuum mechanical systems are given in Holm, Marsden and Ratiu[31]. We provide a version of the latter below .

Proof :

First we change variables in the expression for $I(t)$:

$$I(t) = \oint_{\gamma_t} \frac{1}{D_t}\frac{\delta l}{\delta \mathbf{u}} = \oint_{\gamma_0} \eta_t^* \left[\frac{1}{D_t}\frac{\delta l}{\delta \mathbf{u}}\right] = \oint_{\gamma_0} \frac{1}{D_0}\eta_t^*\left[\frac{\delta l}{\delta \mathbf{u}}\right].$$

Next , we use the Lie derivative formula , namely

$$\frac{d}{dt}\left(\eta_t^* \alpha_t\right) = \eta_t^*\left(\frac{\partial}{\partial t}\alpha_t + £_{\mathbf{u}}\alpha_t\right),$$

applied to a one – form density α_t . This formula gives

$$\frac{d}{dt}I\left(t\right) = \frac{d}{dt}\oint_{\gamma_0}\frac{1}{D_0}\eta_t^*\left[\frac{\delta l}{\delta \mathbf{u}}\right] = \oint_{\gamma_0}\frac{1}{D_0}\frac{d}{dt}\left(\eta_t^*\left[\frac{\delta l}{\delta \mathbf{u}}\right]\right)$$

$$= \oint_{\gamma_0}\frac{1}{D_0}\eta_t^*\left[\frac{\partial}{\partial t}\left(\frac{\delta l}{\delta \mathbf{u}}\right) + £_{\mathbf{u}}\left(\frac{\delta l}{\delta \mathbf{u}}\right)\right].$$

By the Euler – Poincare' equations (4.63) , this becomes

$$\frac{d}{dt}I\left(t\right) = \oint_{\gamma_0}\frac{1}{D_0}\eta_t^*\left[\frac{\delta l}{\delta a}\Diamond a\right] = \oint_{\gamma_t}\frac{1}{D_t}\left[\frac{\delta l}{\delta a}\Diamond a\right],$$

again by the change of variables formula .

Corollary (4.2.12):

Since the last expression holds for every loop γ_t, we may write it as

$$\left(\frac{\partial}{\partial t} + £_{\mathbf{u}}\right)\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} = \frac{1}{D}\frac{\delta l}{\delta a}\Diamond a\ . \tag{4.70}$$

Remark (4.2.13):

The Kelvin – Noether theorem is called so here because its derivation relies on the invariance of the Lagrangian L under the particle relabeling symmetry, and Noether's theorem is associated with this symmetry . However , the result (4.69) is the Kelvin circulation theorem: the circulation integral $I\left(t\right)$ around any fluid loop (γ_t, moving with the velocity of the fluid parcels $\mathbf{u}$) is invariant under the fluid motion .

These two statements are equivalent . We note that two velocities appear in the integrand $I\left(t\right)$: the fluid velocity $\mathbf{u}$ and $D^{-1}\,\delta\ell/\delta\mathbf{u}$. The latter velocity is the momentum density $\mathbf{m} = \delta\ell/\delta\mathbf{u}$ divided by the mass density D . These two velocities are the basic ingredients for performing modeling and analysis in any ideal fluid problem . One simply needs to put these ingredients together in the Euler – Poincare' theorem and its corollary , the Kelvin – Noether theorem .

Now we discuss the Euler – Poincare' theorem and GFD (geophysical fluid dynamics) .

We compute explicit formulae for the variations δa in the cases that the set of tensors a is drawn from a set of scalar fields and densities on $\Re^3$. We shall denote this symbolically by writing

$$a \in \{b, Dd^3x\} . \tag{4.71}$$

We have seen that invariance of the set a in the Lagrangian picture under the dynamics of $\mathbf{u}$ implies in the Eulerian picture that

$$\left(\frac{\partial}{\partial t} + £_{\mathbf{u}}\right) a = 0 ,$$

where $£_{\mathbf{u}}$ denotes Lie derivative with respect to the velocity vector field $\mathbf{u}$. Hence , for a fluid dynamical Eulerian action $\mathfrak{S} = \int dt\, \ell(\mathbf{u}; b, D)$, the advected variables b and D satisfy the following Lie – derivative relations ,

$$\left(\frac{\partial}{\partial t} + £_{\mathbf{u}}\right) b = 0 , \quad \text{or} \quad \frac{\partial b}{\partial t} = -\mathbf{u}.\nabla b , \tag{4.72}$$

$$\left(\frac{\partial}{\partial t} + £_{\mathbf{u}}\right) Dd^3x = 0 , \quad \text{or} \quad \frac{\partial D}{\partial t} = -\nabla.(D\,\mathbf{u}) . \tag{4.73}$$

In fluid dynamical applications , the advected Eulerian variables b and Dd^3x represent the buoyancy b (or specific entropy , for the compressible case) and volume element (or mass density) Dd^3x , respectively . According to Theorem (4.2.10) , equation (4.61) , the variations of the tensor functions a at fixed $\mathbf{x}$ and t are also given by Lie derivatives , namely $\delta a = -£_{\mathbf{w}} a$, or

$$\delta b = -£_{\mathbf{w}} b = -\mathbf{w}.\nabla b ,$$

$$\delta D d^3x = -£_{\mathbf{w}} \left(Dd^3x\right) = -\nabla.(D\,\mathbf{w}) d^3x . \tag{4.74}$$

Hence , Hamilton's principle (4.61) with this dependence yields

$$0 = \delta \int dt\, \ell(\mathbf{u}; b, D)$$

$$= \int dt \left[\frac{\delta \ell}{\delta \mathbf{u}} . \delta \mathbf{u} + \frac{\delta \ell}{\delta b} \delta b + \frac{\delta \ell}{\delta D} \delta D \right]$$

$$= \int dt \left[\frac{\delta \ell}{\delta \mathbf{u}} . \left(\frac{\partial \mathbf{w}}{\partial t} - ad_{\mathbf{u}} \mathbf{w} \right) - \frac{\delta \ell}{\delta b} \mathbf{w} . \nabla b - \frac{\delta \ell}{\delta D} \left(\nabla . (D\, \mathbf{w}) \right) \right]$$

$$= \int dt \; \mathbf{w} . \left[- \frac{\partial}{\partial t} \frac{\delta \ell}{\delta \mathbf{u}} - ad^{*}_{\mathbf{u}} \frac{\delta \ell}{\delta \mathbf{u}} - \frac{\delta \ell}{\delta b} \nabla b + D \nabla \frac{\delta \ell}{\delta D} \right]$$

$$= - \int dt \; \mathbf{w} . \left[\left(\frac{\partial}{\partial t} + £_{\mathbf{u}} \right) \frac{\delta \ell}{\delta \mathbf{u}} + \frac{\delta \ell}{\delta b} \nabla b - D \nabla \frac{\delta \ell}{\delta D} \right], \qquad (4.75)$$

where we have consistently dropped boundary terms arising from integrations by parts , by invoking natural boundary conditions . Specifically , we may impose $\hat{\mathbf{n}} . \mathbf{w} = 0$ on the boundary , where $\hat{\mathbf{n}}$ is the boundary's outward unit normal vector and $\mathbf{w} = \delta \eta_t o \eta_t^{-1}$ vanishes at the endpoints .

The Euler – Poincare' equations for continua (4.63) may now be summarized in vector form for adveced Eulerian variables a in the set (4.71) . We adopt the notational convention of the circulation map I in equations (4.68) and (4.69) that a one form density can be made into a one form (no longer the a density) by dividing it by the mass density D and we use the Lie – derivative relation for the continuity equation $\left(\partial / \partial t + £_{\mathbf{u}} \right) D d^3 x = 0$. Then , the Euclidean components of the Euler – Poincare' equations for continua in equation (4.75) are expressed in Kelvin theorem form (4.70) with a slight abuse of notation as

$$\left(\frac{\partial}{\partial t} + £_{\mathbf{u}} \right) \left(\frac{1}{D} \frac{\delta \ell}{\delta \mathbf{u}} . d\, \mathbf{x} \right) + \frac{1}{D} \frac{\delta \ell}{\delta b} \nabla b . d\, \mathbf{x} - \nabla \left(\frac{\delta \ell}{\delta D} \right) . d\, \mathbf{x} = 0 , \qquad (4.76)$$

in which the variational derivatives of the Lagrangian ℓ are to be computed according to the usual physical conventions , that is , as Fre'chet derivatives . Formula (4.76) is the Kelvin – Noether form of the equation of motion for ideal continua . Hence , we have the explicit Kelvin theorem expression , cf. equations (4.68) and (4.69),

$$\frac{d}{dt} \oint_{\gamma_t(\mathbf{u})} \frac{1}{D} \frac{\delta \ell}{\delta \mathbf{u}} . d\, \mathbf{x} = - \oint_{\gamma_t(\mathbf{u})} \frac{1}{D} \frac{\delta \ell}{\delta b} \nabla b . d\, \mathbf{x} , \qquad (4.77)$$

where the curve $\gamma_t(\mathbf{u})$ moves with the fluid velocity $\mathbf{u}$. Then , by Stokes' theorem , the Euler equations generate circulation of $\mathbf{v} := \left(D^{-1} \delta l / \delta \mathbf{u} \right)$ whenever the gradients ∇b and $\nabla \left(D^{-1} \delta l / \delta b \right)$ are not collinear . The corresponding conservation of potential vorticity q on fluid parcels is given by

$$\frac{\partial q}{\partial t}+\mathbf{u}.\nabla q=0\,, \quad \text{where} \;\; q=\frac{1}{D}\nabla b.curl\left(\frac{1}{D}\frac{\delta \ell}{\delta \mathbf{u}}\right). \tag{4.78}$$

This is also called PV convection . Equations (4.75) – (4.77) embody most of the panoply of equations for GFD . The vector form of equation (4.75) is ,

$$\underbrace{\left(\frac{\partial}{\partial t}+\mathbf{u}.\nabla\right)\left(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\right)+\frac{1}{D}\frac{\delta l}{\delta u^{j}}\nabla u^{j}}_{\textit{Geodesic Nonlinearity :Kinetic energy}}=\underbrace{\nabla\frac{\delta l}{\delta D}-\frac{1}{D}\frac{\delta l}{\delta b}\nabla b}_{\textit{Potential energy}} \tag{4.79}$$

In geophysical applications , the Eulerian variable D represents the frozen – in volume element and b is the buoyancy . In this case , Kelvin's theorem is

$$\frac{dI}{dt}=\iint_{S(t)}\nabla\left(\frac{1}{D}\frac{\delta l}{\delta b}\right)\times\nabla b.d\,\mathbf{S}\,,$$

with circulation integral

$$I=\oint_{\gamma(t)}\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}.d\,\mathbf{x}\,.$$

In the Eulerian velocity representation , we consider Hamilton's principle for fluid motion in a three dimensional domain with action functional $S=\int l dt$ and Lagrangian $l\left(\mathbf{u},\mathrm{b},\mathrm{D}\right)$ given by

$$l\left(\mathbf{u},\mathrm{b},\mathrm{D}\right)=\int \rho o D\left(1+b\right)\left(\tfrac{1}{2}\left|\mathbf{u}\right|^{2}+\mathbf{u}.\mathbf{R}\left(\mathbf{x}\right)-gz\right)-p\left(D-1\right)d^{3}x\;, \tag{4.80}$$

where $\rho_{tot}=\rho_0 D\left(1+b\right)$ is the total mass density , ρ_0 is a dimensional constant and $\mathbf{R}$ is a given function of $\mathbf{x}$. This variations at fixed $\mathbf{x}$ and t of this Lagrangian are the following ,

$$\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}=\rho_0\left(1+b\right)\left(\mathbf{u}+\mathbf{R}\right), \qquad \frac{\delta l}{\delta b}=\rho_0 D\left(\tfrac{1}{2}\left|\mathbf{u}\right|^{2}+\mathbf{u}.\mathbf{R}-gz\right),$$

$$\frac{\delta l}{\delta D}=\rho_0\left(1+b\right)\left(\tfrac{1}{2}\left|\mathbf{u}\right|^{2}+\mathbf{u}.\mathbf{R}-gz\right)-p\;, \quad \frac{\delta l}{\delta p}=-\left(D-1\right). \tag{4.81}$$

Hence , from the Euclidean component formula (4.79) for Hamilton principles of this type and fundamental vector identity ,

$$\left(\mathbf{b}.\nabla\right)\mathbf{a}+a_j\nabla b^{j}=-\mathbf{b}\times\left(\nabla\times\mathbf{a}\right)+\nabla\left(\mathbf{b}.\mathbf{a}\right), \tag{4.82}$$

we find the motion equation for an Euler fluid in three dimensions ,

$$\frac{d\mathbf{u}}{dt}-\mathbf{u}\times curl\ \mathbf{R}+g\,\hat{\mathbf{z}}+\frac{1}{\rho_0(1+b)}\nabla p=0\,, \qquad (4.83)$$

where $curl\ \mathbf{R}=2\mathbf{\Omega}(\mathbf{x})$ is the Coriolis parameter (that is , twice the local angular rotation frequency) . In writing this equation , we have used advection of buoyancy ,

$$\frac{\partial b}{\partial t}+\mathbf{u}.\nabla b=0\,,$$

from equation (4.72) . The pressure p is determined by requiring preservation of the constraint $D=1$, for which the continuity equation (4.73) implies $div\ \mathbf{u}=0$. The Euler motion equation (4.83) is Newton's Law for the acceleration of a fluid due to three forces : Coriolis , gravity and pressure gradient . The dynamic balance among these three forces produce the many circulatory flows of geophysical fluid dynamics . The conservation of potential vorticity q on fluid parcels for these Euler GFD flows is given by

$$\frac{\partial q}{\partial t}+\mathbf{u}.\nabla q=0\,, \quad \text{where , on using } D=1, \quad q=\nabla b.curl\left(\mathbf{u}+\mathbf{R}\right). \qquad (4.84)$$

Now we discuss the Hamilton – Poincare' reduction and Lie – Poisson equations.

In the Euler – Poincare' framework one starts with a Lagrangian defined on the tandent bundle of a Lie group G

$$L:TG\rightarrow\Re$$

and the dynamics is given by Euler – Lagrange equations arising from the variational principle

$$\delta\int_{t_0}^{t_1}L\left(g,\dot{g}\right)dt=0$$

The Lagrangian L is taken left\ right invariant and because of this property one can reduce the problem obtaining a now system which is defined on the Lie algebra ϑ of G , obtaining a now set of equations , the Euler – Poincare' equations , arising from reduced variational principle

$$\delta\int_{t_0}^{t_0}l\left(\xi\right)dt=0$$

where $l\left(\xi\right)$ is the reduced Lagrangian and $\xi\in\vartheta$.

Problem (4.2.14):

Is there a similar procedure for Hamiltonian system?

More precisely : given a Hamiltonian function

$$H : T^{*}G \to \Re$$

defined on the cotangent bundle $T^{*}G$, one wants to perform a similar procedure of reduction and derive the equations of motion on the dual of the Lie algebra ϑ^{*} , provided the Hamiltonian is again left\ right invariant .

Hamilton – Poincare' reduction gives a positive answer to this problem , in the context of variational principles as it is done in the Euler – Poincare' framework: we are going to explain how procedure is performed .

More in general , we will also consider advected quantities belonging to a vector space V on which G acts , so that the Hamiltonian is written in this case as

$$H : T^{*}G \times V^{*} \to \Re$$

The space V is regarded here exactly the same as in the Euler – Poincare' theory .

The equations of motion , that is , Hamilton's equations , may be derived from the following variational principle

$$\delta \int_{t_0}^{t_1} \left\{ \left\langle p(t), \dot{g}(t) \right\rangle - H_{a_0}\left(g(t), p(t) \right) \right\} dt = 0$$

as it is well know from ordinary classical mechanics ($\dot{g}(t)$ has to be considered as the tangent vector to the curve $g(t)$, so that $\dot{g}(t) \in T_{g(t)}G$).

Problem (4.2.15):

What happens if H_{a_0} is left\ right invariant ?

It turns out that in this case the whole function

$$F(g, \dot{g}, p) = \langle p, \dot{g} \rangle - H_{a_0}(g, p)$$

is also invariant . The proof is straightforward once the action is specified (from now on we consider only left invariance):

$$h(g, \dot{g}, p) = \left(hg, T_g L_h \dot{g}, T_{hg}^{*} L_{h^{-1}} p \right)$$

where $T_g L_h : T_g G \to T_{hg} G$ is the tangent of the left translation map $L_{hg} = hg \in G$ at the point g and $T^*_{hg} L_{h^{-1}} : T^*_g G \to T^*_{hg} G$ is the dual of the map $T_{hg} L_{h^{-1}} : T_{hg} G \to T_g G$.

We now check that

$$\langle hp, h\dot{g} \rangle = \left\langle T^*_{hg} L_{h^{-1}} p, T_g L_h \dot{g} \right\rangle$$

$$= \left\langle p, T_{hg} L_{h^{-1}} o T_g L_h \dot{g} \right\rangle$$

$$= \left\langle p, T_g \left(L_{h^{-1}} o L_h \right) \dot{g} \right\rangle = \langle p, \dot{g} \rangle$$

where the chain rule for the tangent map has been used . The same result holds for the right action .

Due to this invariance property , one can write the variational principle as

$$\delta \int_{t_0}^{t_1} \left\{ \langle \mu, \xi \rangle - h(\mu, a) \right\} dt = 0$$

with

$$\mu(t) = g^{-1}(t) p(t) \in \vartheta^*, \quad \xi(t) = g^{-1}(t) \dot{g}(t) \in \vartheta, \quad a(t) = g^{-1}(t) a_0 \in V^*$$

In particular $a(t)$ is the solution of

$$\dot{a}(t) = -\xi(t) a_0 .$$

where a Lie algebra action of ϑ on V^* is implicitly defined . In order to find the equations of motion one calculates the variations

$$\delta \int_{t_0}^{t_1} \left\{ \langle \mu, \xi \rangle - h(\mu, a) \right\} dt = \int_{t_0}^{t_1} \left\{ \langle \delta\mu, \xi \rangle + \langle \mu, \delta\xi \rangle - \left\langle \delta\mu, \frac{\delta h}{\delta\mu} \right\rangle - \left\langle \delta a, \frac{\delta h}{\delta a} \right\rangle \right\} dt$$

As in the Euler – Poincare' theorem , we use the following expressions for the variations

$$\delta\xi = \dot{\eta} + [\xi, \eta], \qquad \delta a = -\eta a$$

and using the definition of the diamond operator we find

$$\int_{t_0}^{t_1} \left\{ \langle \delta\mu, \xi \rangle + \langle \mu, \delta\xi \rangle - \left\langle \delta\mu, \frac{\delta h}{\delta\mu} \right\rangle - \left\langle \delta a, \frac{\delta h}{\delta a} \right\rangle \right\} dt$$

$$= \int_{t_0}^{t_1} \left\{ \left\langle \delta\mu, \xi - \frac{\delta h}{\delta\mu} \right\rangle + \left\langle \mu, \dot{\eta} + ad_{\xi\eta} \right\rangle + \left\langle \eta a, \frac{\delta h}{\delta a} \right\rangle \right\} dt$$

$$= \int_{t_0}^{t_1} \left\{ \left\langle \delta\mu, \xi - \frac{\delta h}{\delta\mu} \right\rangle + \left\langle -\dot{\mu} + ad_{\xi}^{*}\mu, \eta \right\rangle - \left\langle \frac{\delta h}{\delta a} \Diamond a, \eta \right\rangle \right\} dt$$

so that

$$\xi = \frac{\delta h}{\delta\mu}$$

and the equations of motion are

$$\dot{\mu} = ad_{\xi}^{*}\mu - \frac{\delta h}{\delta a} \Diamond a$$

together with

$$\dot{a} = -\frac{\delta h}{\delta\mu} a \text{ .}$$

This equations of motion written on the dual Lie algebra ϑ are called Lie – Poisson equations . We have now proven the following:

Theorem (4.2.16): (Hamilton – Poincare' Reduction Theorem)

With the preceding notation , the following statements are equivalent:

(1) With a_0 hold fixed , the variational principle

$$\delta \int_{t_0}^{t_1} \left\{ \left\langle p(t), \dot{g}(t) \right\rangle - H_{a_0}\left(g(t), p(t)\right) \right\} dt = 0$$

Holds , for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints .

(2) $\left(g(t), p(t)\right)$ satisfies Hamilton's equations for H_{a_0} on G .

(3) The constrained variational principle

$$\delta \int_{t_0}^{t_1} \left\{ \left\langle \mu(t), \xi(t) \right\rangle - h\left(\mu(t), a(t)\right) \right\} dt = 0$$

holds for $\vartheta \times V^{*}$, using variations of ξ and a of the form

$$\delta\xi = \dot{\eta} + [\xi, \eta], \qquad \delta a = -\eta a$$

where $\eta(t) \in \vartheta$ vanishes at the endpoints .

(4) The Lie – Poisson equations hold on $\vartheta \times V^{*}$

$$(\dot{\mu}, \dot{a}) = \left(ad_{\xi}^{*}\mu - \frac{\delta h}{\delta a} \Diamond a, -\frac{\delta h}{\delta\mu} a \right)$$

Remark (4.2.17):

More exactly one should start with an invariant Hamiltonian defined on

$$T^*(G \times V) = T^*G \times V \times V^*$$

However , as mentioned by Holm , Marsden and Ratiu[31], such an approach turns out to be equivalent to the treatment presented here .

Remark (4.2.18): (Legendre Transform)

Lie – Poisson equations may arise from the Euler – Poincare' setting by Legendre transform

$$\mu = \frac{\delta l}{\delta \xi}.$$

If this is a diffeomorphim , then the Hamilton – Poincare' theorem is equivalent to the Euler – Poincare' theorem .

Remark (4.2.19): (Lie – Poincare' Structure)

One shows that $\vartheta^* \times V^*$ is a Poisson manifold:

$$\begin{aligned}
\dot{F}(\mu,a) &= \left\langle \dot{\mu}, \frac{\delta F}{\delta \mu} \right\rangle + \left\langle \dot{a}, \frac{\delta F}{\delta a} \right\rangle \\
&= \left\langle ad^*_{\delta H/\delta\mu}\mu - \frac{\delta H}{\delta a} \Diamond a, \frac{\delta F}{\delta \mu} \right\rangle - \left\langle \frac{\delta H}{\delta \mu} a, \frac{\delta F}{\delta a} \right\rangle \\
&= \left\langle \mu, \left[\frac{\delta H}{\delta \mu}, \frac{\delta F}{\delta \mu} \right] \right\rangle - \left\langle \frac{\delta H}{\delta a} \Diamond a, \frac{\delta F}{\delta \mu} \right\rangle - \left\langle \frac{\delta H}{\delta \mu} a, \frac{\delta F}{\delta a} \right\rangle \\
&= -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle - \left\langle \frac{\delta H}{\delta a} \Diamond a, \right\rangle - \left\langle \frac{\delta H}{\delta \mu} a, \frac{\delta F}{\delta a} \right\rangle \\
&= -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle - \left\langle a, \frac{\delta F}{\delta \mu} \frac{\delta H}{\delta a} - \frac{\delta H}{\delta \mu} \frac{\delta F}{\delta a} \right\rangle
\end{aligned}$$

In fact it can be easily shown that this structure

$$\{F,H\}(\mu,a) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle - \left\langle a, \frac{\delta F}{\delta \mu} \frac{\delta H}{\delta a} - \frac{\delta H}{\delta \mu} \frac{\delta F}{\delta a} \right\rangle$$

satisfies the definition of a Poisson structure . In particular one finds that any dual Lie algebra ϑ is a Poisson manifold .

Remark (4.2.20): (Right Invariance)

It can be shown that for a right invariant Hamiltonian one has

$$\{F,H\}(\mu,a)=+\left\langle \mu,\left[\frac{\delta F}{\delta\mu},\frac{\delta H}{\delta\mu}\right]\right\rangle+\left\langle a,\frac{\delta F}{\delta\mu}\frac{\delta H}{\delta a}-\frac{\delta H}{\delta\mu}\frac{\delta F}{\delta a}\right\rangle$$

$$(\dot{\mu},\dot{a})=-\left(ad_{\xi}^{*}\mu-\frac{\delta h}{\delta a}\Diamond a,-\frac{\delta h}{\delta\mu}a\right)$$

with all signs changed respect to the case of left invariance .

Now we discuss the two applications .

In plasma physics a main topic is collisionless particle dynamics , whose main equation , the Vlasov equation , will be heuristically derived here . In this context a central role is held by the distribution function on phase space $f\ (\mathbf{q},\mathbf{p},t)$, basically expressing the particle density on phase space . Intended as a density one defines $F:=f\ (\mathbf{q},\mathbf{p},t)d\,\mathbf{q}d\,\mathbf{p}$: because of the conservation of particles , one writes the continuity equation just as one does as in the context of fluid dynamics

$$\dot{F}+\nabla.(\mathbf{u}F)=0$$

where $\mathbf{u}$ is a "velocity " vector field on phase space , which is given by the single particle motion

$$\mathbf{u}=(\dot{\mathbf{q}},\dot{\mathbf{p}})\in\mathfrak{X}\left(T^{*}\Re^{N}\right)$$

if we now assume that the generic single particle undergoes a Hamiltonian motion , the Hamiltonian function $h(\mathbf{q},\mathbf{p},t)$ can be introduced directly by means of the single particle Hamilton's equations

$$(\dot{\mathbf{q}},\dot{\mathbf{p}})=\left(\frac{\partial h}{\partial\mathbf{p}},-\frac{\partial h}{\partial\mathbf{q}}\right)$$

which shows that $\mathbf{u}$ has zero divergence , assuming the Hessian of h is symmetric . Therefore , the Vlasov equation written in terms of the distribution function $f\ (\mathbf{q},\mathbf{p},t)$ is

$$\dot{f} + \mathbf{u}.\nabla f = 0$$

Expanding now the Hamiltonian h as the total single particle energy

$$h(\mathbf{q},\mathbf{p},t) = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{q},\mathbf{p},t)$$

one obtains the more common form

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}.\frac{\partial f}{\partial \mathbf{q}} - \frac{\partial V}{\partial \mathbf{q}}.\frac{\partial f}{\partial \mathbf{p}} = 0$$

Problem (4.2.21):

Can the Vlasov equation be cast in Lie – Poisson form ?

We show here why the answer is yes . First we write the Vlasov equation in terms of a generic single particle Hamiltonian h as

$$\dot{f} + \{f, h\} = 0$$

where we recall the canonical Poisson bracket

$$\{f, h\} = \frac{\partial f}{\partial \mathbf{q}}.\frac{\partial h}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}}.\frac{\partial h}{\partial \mathbf{q}}$$

The main point , of this discussion is that the canonical Poisson bracket provides the set $\mathcal{F}(T^*\Re^N)$ of the functions on the phase space with a Lie algebra structure

$$[k, h] = \{k, h\}$$

At this point , in order to look for a Lie – Poisson equation , one calculates the coadjoint operator such that

$$\langle f, \{h, k\}\rangle = \langle f, ad_h k\rangle = \langle ad_h^* f, k\rangle = \langle -\{h, f\}, k\rangle$$

where the last equality is justified by Leibniz property of the Poisson bracket , with the pairing defined as

$$\langle f, g\rangle = \int f \; g \, d\mathbf{q} d\mathbf{p} \; .$$

In conclusion , the argument above shows that the Vlasov equation can in fact be written in the Lie – Poisson form

$$\dot{f} + ad_h^* f = 0$$

The reduced Lagrangian for ideal compressible fluids is written as

$$l(\mathbf{u}, D) = \int \tfrac{D}{2}|\mathbf{u}|^2 - De(D)\, d\,\mathbf{x}$$

where $\mathbf{u} \in \mathfrak{X}(M \subset \Re^3)$ is tangential on the boundary ∂M and D is the advected density , which satisfies the continuity equation

$$\partial_t D + \mathcal{L}_{\mathbf{u}} D = 0\,.$$

Moreover , the internal energy satisfies the barotropic First Law of Thermodynamics

$$de = -p(D)\,d\left(D^{-1}\right) = \frac{p(D)}{D^2} dD$$

for the pressure $p(D)$. The "reduced" Legendre transform on this Lie algebra $\mathfrak{X}(\Re^3)$ is given by

$$\mathbf{m} = D\mathbf{u}$$

and the Hamiltonian is then written as

$$h(\mathbf{m}, D) = \langle \mathbf{m}, \mathbf{u} \rangle - l(\mathbf{u}, D)$$

that is

$$h(\mathbf{m}, D) = \int \tfrac{1}{2D}|\mathbf{m}|^2 + De(D)\, d\,\mathbf{x}$$

The Lie – Poisson equations in this case are as from the general theory

$$\partial_t \mathbf{m} = -ad^*_{\delta h/\delta \mathbf{m}} \mathbf{m} - \frac{\delta h}{\delta D} \Diamond D$$

$$\partial_t D = -\mathcal{L}_{\delta h/\delta \mathbf{m}} D$$

Earlier we found that the coadjoint action is given by the Lie derivative . On the other hand we may calculate the expression of the diamond operation from its definition

$$\left\langle \frac{\delta h}{\delta D}, -£_\eta D \right\rangle = \left\langle \frac{\delta h}{\delta D} \Diamond D, \eta \right\rangle$$

to be

$$\left\langle \frac{\delta h}{\delta D}, -divD\eta \right\rangle = \left\langle D\nabla \frac{\delta h}{\delta D}, \eta \right\rangle$$

Therefore , we have

$$\frac{\delta h}{\delta D} \Diamond D = D\nabla \frac{\delta h}{\delta D}$$

where

$$\delta h / \delta D = -\frac{|\mathbf{m}|^2}{2D^2} + \left(e + \frac{p}{D} \right)$$

Substituting into the momentum equation and using the First Law to find $d\left(e + p/D\right) = \left(1/D\right)dp$ yields

$$\partial_t \mathbf{m} = -\mathcal{L}_{\mathbf{u}}\mathbf{m} - \nabla p$$

Upon expanding the Lie derivative for the momentum density $\mathbf{m}$ and using the continuity equation for the density , this quickly becomes

$$\partial_t \mathbf{u} = -\mathbf{u}.\nabla \mathbf{u} - \frac{1}{D}\nabla p$$

which is Euler's equation for a barotropic fluid .

The barotropic equations recover Euler's equations for ideal incompressible fluid motion when the internal energy in the reduced Lagrangian for ideal compressible fluids is replaced by the constraint $D = 1$,as

$$l\left(\mathbf{u}, D\right) = \int \tfrac{D}{2}|\mathbf{u}|^2 - p\left(D - 1\right)d\,\mathbf{x}$$

where again $\mathbf{u} \in \mathfrak{X}\left(M \subset \Re^3\right)$ is tangential on the boundary ∂M and the advected density D satisfies the continuity equation ,

$$\partial_t D + divD\mathbf{u} = 0$$

This equation enforces incompressibility $div\,\mathbf{u} = 0$ when evaluated on the constraint $D = 1$. The pressure p is now a Lagrange multiplier , which is determined by the condition that incompressibility be preserved by the dynamics .

References:

1) AMBloch, Nonholonomic mechanics and control, Interdisciplinary Applied Mathematics24, Springer, New York (2003) MR1978379 With the collaboration of J Baillieul, P Crouch and J Marsden. With scientific input from P S Krishnaprasad, RM Murray and D Zenkov
2) AM Bloch, RWBrockett, P E Crouch, Double bracket equations and geodesic flowson symmetric spaces, Comm. Math. Phys. 187 (1997) 357–373 MR1463833
3) AM Bloch, P E Crouch, Optimal control and geodesic flows, Systems Control Lett.28 (1996) 65–72 MR1404249
4) AS Mišˇcenko, AT Fomenko, Euler equation on finite-dimensional Lie groups, Izv. Akad. Nauk SSSR Ser. Mat. 42 (1978) 396–415, 471 MR0482832
5) AWeinstein, Sophus Lie and symplectic geometry, Exposition. Math. 1 (1983) 95–96 MR693812
6) AWeinstein, Groupoids: unifying internal and external symmetry. A tour through someexamples, from: "Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)",Contemp. Math. 282, Amer. Math. Soc., Providence, RI (2001) 1–19 MR1855239
7) B Riemann, Untersuchungen über die Bewegungen eines flüssigen gleichartigen Ellipsoides, Abh. d. Königl. Gesell. der Wis. zu Göttingen 9 (1860) 3–36
8) B Kostant, Orbits, symplectic structures and representation theory, from: "Proc. U.S.- Japan Seminar in Differential Geometry (Kyoto, 1965)"Nippon Hyoronsha, Tokyo (1966) p. 71 MR0213476
9) CH Kimberling, Emmy Noether, Amer. Math. Monthly 79 (1972) 136–149 MR0299446
10) D McDuff, D Salamon, Introduction to symplectic topology, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York (1995) MR1373431
11) DD Holm, T Schmah, C Stoica, Geometric Mechanics and symmetry: From Finite to Infinite Dimensions, Oxford Texts in Applied and Engineering Mathematics 12, Oxford University Press, Oxford (2009) MR2548736 With solutions to selected exercises by David C P Ellis
12) DD Holm, J E Marsden, T S Ratiu, The Euler–Poincaré equations and semidirect products with applications to continuum theories, Adv. Math. 137 (1998) 1–81 MR1627802
13) DD Holm, J E Marsden, T S Ratiu, Euler–Poincaré models of ideal fluids with nonlinear dispersion, Phys. Rev. Lett. 80 (1998) 4173–4177
14) DD Holm, Euler–Poincaré dynamics of perfect complex fluids, from: "Geometry, mechanics, and dynamics", Springer, New York (2002) 113–167 MR1919828

15) D Holm, The Euler–Poincaré variational framework for modeling fluid dynamics, from: "Geometric mechanics and symmetry", London Math. Soc. Lecture Note Ser. 306, Cambridge Univ. Press, Cambridge (2005) 157–209 MR2172979
16) DD Holm, Geometric Mechanics Part I: Dynamics and Symmetry, Imperial College Press, London (2008) MR2419209
17) DD Holm, Geometric Mechanics Part II: Rotating, Translating and Rolling, Imperial College Press, London (2008) MR2419210
18) DD Holm, BA Kupershmidt, Poisson structures of superfluids, Phys. Lett. A 91 (1982) 425–430 MR677628
19) DD Holm, BA Kupershmidt, Poisson brackets and Clebsch representations for magnetohydrodynamics, multifluid plasmas, and elasticity, Phys. D 6 (1983) 347–363 MR709394
20) DD Holm, BA Kupershmidt, The analogy between spin glasses and Yang–Mills fluids, J. Math. Phys. 29 (1988) 21–30 MR921760
21) DD Holm, J E Marsden, Momentum maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation, from: "The breadth of symplectic and Poisson geometry", Progr. Math. 232, Birkhäuser, Boston (2005) 203–235 MR2103008
22) DG Ebin, J Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid., Ann. of Math. .2/ 92 (1970) 102–163 MR0271984
23) EA Coddington, N Levinson, Theory of Ordinary Differential Equations, Robert E Krieger (1984)
24) F Calogero, An integrable Hamiltonian system, Phys. Lett. A 201 (1995) 306–310 MR1331829
25) F Calogero, J-P Françoise, A completely integrable Hamiltonian system, J. Math. Phys. 37 (1996) 2863–2871 MR1390240
26) H Ertel, Ein neuer hydrodynamischer Wirbelsatz, Meteorol. Z. 59 (1942) 271–281
27) H Dullin, G Gottwald, DD Holm, An integrable shallow water equation with linear and nonlinear dispersion, Phys. Rev. Lett. 87 (2001) 194501–04
28) HR Dullin, GA Gottwald, DD Holm, Camassa–Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves, Fluid Dynam. Res. 33 (2003) 73–95 MR1995028 In memoriam Prof. Philip Gerald Drazin 1934–2002
29) HR Dullin, GA Gottwald, DD Holm, On asymptotically equivalent shallow water wave equations, Phys. D 190 (2004) 1–14 MR2043789
30) J E Marsden, Lectures on mechanics, London Mathematical Society Lecture Note Series 174, Cambridge University Press, Cambridge (1992) MR1171218
31) J E Marsden, T JR Hughes, Mathematical foundations of elasticity, Dover Publications, New York (1994) MR1262126 Corrected reprint of the 1983 original

32) J E Marsden, T S Ratiu, Introduction to mechanics and symmetry, Texts in Applied Mathematics 17, Springer, New York (1994) MR1304682 A basic exposition of classical mechanical systems
33) J E Marsden, T S Ratiu, Mechanics and Symmetry: Reduction Theory, in preparation (2003)
34) J E Marsden, T Ratiu, A Weinstein, Reduction and Hamiltonian structures on duals of semidirect product Lie algebras, from: "Fluids and plasmas: geometry and dynamics (Boulder, Colo., 1983)", Contemp. Math. 28, Amer. Math. Soc., Providence, RI (1984) 55–100 MR751975
35) J E Marsden, T Ra¸tiu, A Weinstein, Semidirect products and reduction in mechanics, Trans. Amer. Math. Soc. 281 (1984) 147–177 MR719663
36) J Marsden, A Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Mathematical Phys. 5 (1974) 121–130 MR0402819
37) J Marsden, A Weinstein, Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids, Phys. D 7 (1983) 305–323 MR719058 Order in chaos (Los Alamos, N.M., 1982)
38) JM Lee, Introduction to smooth manifolds, Graduate Texts in Mathematics 218, Springer, New York (2003) MR1930091
39) JK Hunter, YX Zheng, On a completely integrable nonlinear hyperbolic variational equation, Phys. D 79 (1994) 361–386 MR1306466
40) JV José, E J Saletan, Classical dynamics: A contemporary approach, Cambridge University Press, Cambridge (1998) MR1640663
41) J Gibbons, DD Holm, B Kupershmidt, Gauge-invariant Poisson brackets for chromohydrodynamics, Phys. Lett. A 90 (1982) 281–283 MR668288
42) K Ohkitani, Eigenvalue problems in three-dimensional Euler flows, Phys. Fluids A 5 (1993) 2570–2572
43) L Haine, The algebraic complete integrability of geodesic flow on SO.N/, Comm. Math. Phys. 94 (1984) 271–287 MR761797
44) OB Fringer, DD Holm, Integrable vs. nonintegrable geodesic soliton behavior, Phys. D 150 (2001) 237–263 MR1820736
45) P Libermann, C-M Marle, Symplectic geometry and analytical mechanics, Mathematics and its Applications 35, D. Reidel Publishing Co., Dordrecht (1987) MR882548 Translated from the French by Bertram Eugene Schwarzbach
46) P J Olver, Applications of Lie groups to differential equations, second edition, Graduate Texts in Mathematics 107, Springer, New York (1993) MR1240056
47) R Abraham, J E Marsden, Foundations of mechanics, Benjamin/Cummings Publishing Co. Advanced Book Program, Reading, MA (1978) MR515141 Second edition, revised and enlarged, With the assistance of Tudor Ra¸tiu and Richard Cushman
48) R Camassa, DD Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664 MR1234453

49) R Dedekind, Zusatz zu der vorstehenden Abhandlung, J. Reine Angew. Math. 58 (1860)
50) S Chandrasekhar, Ellipsoidal Figures of Equilibrium, Yale University Press (1969)
51) S Lie, Theorie der Transformationsgruppen. Zweiter Abschnitt, Teubner, Leipzig (1890)
52) SV Manakov, A remark on the integration of the Eulerian equations of the dynamics of an n–dimensional rigid body, Funkcional. Anal. i Priložen. 10 (1976) 93–94 MR0455031
53) S Smale, Topology and mechanics I, Invent. Math. 10 (1970) 305–331 MR0309153
54) S Smale, Topology and mechanics II: The planar n–body problem, Invent. Math. 11 (1970) 45–64 MR0321138
55) T Ra¸tiu, The motion of the free n–dimensional rigid body, Indiana Univ. Math. J. 29 (1980) 609–629 MR578210
56) T S Ratiu, R Tudoran, L Sbano, E Sousa Dias, G Terra, A crash course in geometric mechanics, from: "Geometric mechanics and symmetry", London Math. Soc. Lecture Note Ser. 306, Cambridge Univ. Press, Cambridge (2005) 23–156 MR2172978 Notes of the courses given by Ratiu
57) V Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier .Grenoble/ 16 (1966) 319–361 MR0202082
58) VI Arnold, Mathematical methods of classical mechanics, second edition, Graduate Texts in Mathematics 60, Springer, New York (1989) MR997295 Translated from the Russian by K Vogtmann and A Weinstein
59) VI Arnold, BA Khesin, Topological methods in hydrodynamics, Applied Mathematical Sciences 125, Springer, New York (1998) MR1612569
60) V Guillemin, S Sternberg, Symplectic techniques in physics, Cambridge University Press, Cambridge (1984) MR770935