Chapter 1

Composition as an integral operator

Let S be the unit sphere and B the unit ball in C^{n}, and denote by L^1(S)the usual Lebesgue space of integrable functions onS. We define four "composition operators" acting onL^1(S)and associated with a Borel function ϕ :S →B, by first taking one of four natural extensions of f∈ L^1(S) to a function on B, then composing with $\phi$and taking radial limits.

Section (1.1):Norm Estimates for the Reproducing Kernels and Carleson Measures with Boundednesss

Composition operators acting on a space X of functions holomorphic on the unit disk **D** in **C**, or more generally the unit ball **B** = **B**$_n$ in C^n, have been the subject of a great deal of research. In this setting, a holomorphic self-map $\varphi$of **B** induces the composition operator C_ϕ, defined for $f$holomorphic on **B** by $C_\phi f$=fo φ. The basic problem is to relate function theoretic properties of $\varphi$to operator theoretic properties of C_φ. On many of the classical Banach spaces of holomorphic functions(where function F:C→C is said to be analytic in an open set $A_C$if it is different s_t each point of the set A .the function f:C→C is said to be hoiomorphic if it has power series representation)[5] on D, including the Hardy spaces H^p(D)and Bergman spaces L_a^p(D)(let D be n open sub set the complex plan C andL_a^2(D)denote the collection of all analytic function F: D→C complex modulus is square integrable with respect to area measure .the L_a^2(D)somtime also denoted A^2 (D)(is colled the Bergman space can also generalized toL_a^p(D) where 0<P<∞)[6], every composition operator is bounded and their study involves other properties, such as when a composition operator is compact. In higher dimensions, when n ≥2, boundedness of a composition operator is not auto-matic, even on H^p(**B**)or L_a^p(**B**).In 1990, D. Sarasonintroduced the viewpoint of composition operators as integral operators acting on spaces of functions defined on the unit circle ∂D. For φa holomorphic self-map of **D** and f∈L^1(∂D), $C_\phi f$was defined on ∂**D**by taking the harmonic extension of f to D, composing with φ, and then taking radial limits. As in the classical setting of composition operators acting on H^p(D), every such operator is bounded, and problems such as characterizing when the operator is compact were studied by Sarason. In the present chapter, we generalize Sarason's approach in two significant ways to define composition operators acting on L^1(**S**) $=L^1$(S, dσ), where S $=\partial$**B** is the unit sphere in C^n and dσ denotes the normalized surface area measure on **S**. First, we do not assume that the symbol ϕ of the operator is holomorphic on **B**; we only assume that $\boldsymbol{\phi} : S \to \overline{B}$ is Borel measurable. Section, we compose φ with four natural extensions of $f \in L^1(S)$ to a function on $\overline{B}$, resulting in four different "composition operators". Not surprisingly, not all such operators are bounded, even in dimension one. Our main results provide characterizations of when these operators are bounded or compact. We begin with some background needed to define the operators .By a reproducing kernel K for the function space X on **B** we mean that **K** is a continuous function on **B**×**S** such that

$$f(z) = \int_{\mathbf{S}} f(\zeta) K(z,\zeta)\, d\sigma(\zeta), \quad z \in \mathbf{B}$$

for all **f**∈**X**∩**C**($\overline{\mathbf{B}}$). On **B** we have several reproducing kernels: the Cauchy kernel K^c, Poisson kernel K^h, and Poisson**S**zego kernel K^m given by

$$K^c(z,\zeta) := \frac{1}{(1-\langle z,\zeta\rangle)^n},$$
$$K^h(z,\zeta) := \frac{1-|z|^2}{|z-\zeta|^{2n}},$$
$$K^m(z,\zeta) := \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}$$

For **z**∈**B** and **ζ**∈**S.** Here, and throughout the chapter, $\langle 0|0\rangle$ denotes the Hermitian inner product on C^n, i.e.,$\langle \mathbf{z}|\mathbf{w}\rangle = \sum_{\mathbf{j}}^{\mathbf{n}} \mathbf{z_j} \mathbf{w_j}$ for z=(z_1, ..., z_n)and w=(w_1, ..., w_n). Also, we define the pluriharmonic(function U=U(z) of n complex spaces C^n n≥ 1 that has continuous domain D of the complex spaces of the coordinateness $X_u, Y_u . Z_{u=} X_u + i Y_u$,U=1,……n.in D up to the second order the following system of n^2 equations in D)[7]Poisson kernel $\mathbf{K^p}$as

$$K^p(z,\zeta) := K^c(z,\zeta) + \overline{K^c(z,\zeta)} - 1.$$

Note that $\boldsymbol{K^c}$, is a reproducing kernel for the holomorphic functions, $\boldsymbol{K^h}$for the harmonic functions,$\boldsymbol{K^m}$for the invariant harmonic functions and $K^p$for the pluriharmonic functions. We note for later use an easy but useful fact that

$$K^x(r\eta,\zeta) = \overline{K^x(r\zeta,\eta)}, \quad \eta,\zeta \in \mathbf{S},\ 0 < r < 1 \tag{1}$$

for each x ∊{c, h, m, p}.

Let **φ:S→$\overline{\boldsymbol{B}}$**be a Borel function. We say that $\varphi$is holomorphicif it is σ-almost everywhere given by the boundary function (i.e. the radial limit function) of a holomorphic selfmap of **B**. In case φ is holomorphic, we will identify $\varphi$with its holomorphic extension. For each x ∊{c, h, m, p}we wish to define a “composition operator”C_ϕ^xon L^1(**S**), i.e. a linear operator that takes f∊L^1(**S**)to another function defined on **S** that comes from composition of *f* with φ. Since functions in L^1(**S**)are only defined on **S**modulo sets of σ-measure 0, a problem with the definition of these operators arises if φtakes a subset of **S** of positive σ-measure to a set in **S** of σ-measure 0. This difficulty does not come up in the classical setting where n =1and $\varphi$is holomorphic, but it is present in dimension $n \geq 2$even if it is assumed that $\varphi$is holomorphic. The example below illustrates such difficulty.

Example (1): (n ≥2). There exists *f*∊**BMOA**(**B**)and a holomorphic φsuch that $\lim_{r\to 1^-} f(r\varphi(\zeta))$does not exist at any ζ∊**S**.

Proof: Let *I* be an inner function on **B**. Namely, let *I*:**B**→**D**be a holomorphic function such that $I(\eta) = \lim_{r\to -1} I(r\eta) \in \partial D$for almost every $\eta \in S$; for the existence of such an inner function. Define ϕ=(I, 0, ..., 0). It is known that there exists f∊**BMOA**(**B**)such that $\lim_{r\to -1} f(re^{i\theta}, 0, \ldots, 0)$

does not exist for any $\theta \in [0, 2\pi)$. The pair f and $\varphi$is the desired example.An additional assumption about $\varphi$is required to deal with the problem. The pullback measure $\sigma o \varphi^{-1}$is the Borel measure defined for a Borel set **E**$\subset \overline{B}$by σo φ^{-1}(**E**) $=\sigma\{\zeta \in S : \varphi(\zeta) \in E\}$. For the rest of the chapter we reserve the letter $\varphi$to denote functions satisfying thatφ : S $\rightarrow \overline{B}$ is aBorel function and

$$\left(\sigma \circ \varphi^{-1}\right)\big|_{\mathbf{S}} \ll \sigma \tag{2}$$

where $(\sigma \circ \varphi^{-1})|_s$is the restriction of the measure $\sigma o \varphi^{-1}$to **S**. We will see below that this assumption is required for the operators $C_\varphi^x$to be well-defined.Integration against one of the kernels K^x, x $\in$\{c, h, m, p\}, gives an extension of a function $f \in L^1$(**S**)to a function f^xon **B** that is respectively holomorphic, harmonic, invariant harmonic, or pluriharmonic. That is,

$$f^x(z) = \int\limits_{\mathbf{S}} f(\zeta) K^x(z, \zeta)\, d\sigma(\zeta), \quad z \in \mathbf{B}. \tag{3}$$

We then use radial limits (which exist σ-a.e. on S; to extend the definition of f^x from **B** to $\overline{B}$; that is

$$f^x(w) = \lim_{r \to -1} f^x(rw) \quad w \in \overline{\mathbf{B}},\ x \in \{c, h, m, p\}. \tag{4}$$

This $f^x$is naturally referred to as the x-extension of $f \in L^1$(**S**).It is well known that in some, but not all, settings the function $f^x|_s$recovers fσ-a.e. as in the next proposition In what follows, $\boldsymbol{H^t}$**(S)**,$1 \le t < \infty$, denotes the closed subspace of $\boldsymbol{L^t}$**(S)** =$\boldsymbol{L^t}$**(S, dσ)**, the usual Lebesgue space with norm $\|0\|_t$, consisting of all boundary functions of H^t(B) functions. As is well known, $\boldsymbol{H^t}$**(S)**is isometrically identified with $\boldsymbol{H^t}$**(B)**.

Proposition (1.1.1)[1]: *The following relations hold*:

(a) *If $x \in \{c,p\}$ and $f \in H^1(\mathbf{S})$, then $f^x|_{\mathbf{S}} = f$ σ-a.e.;*
(b) *If $x \in \{h,m\}$ and $f \in L^1(\mathbf{S})$, then $f^x|_{\mathbf{S}} = f$ σ-a.e.;*
(c) *If $f \in C(\mathbf{S})$ in addition to the hypothesis of (a) or (b), then $f^x \in C(\overline{\mathbf{B}})$;*
(d) *If $x \in \{c,p\}$ and $f \in L^1(\mathbf{S})$, then in general $f^x|_{\mathbf{S}} \neq f$;*
(e) *If $x \in \{c,h,m,p\}$, the transform $f \to f^x|_{\mathbf{S}}$ is L^t-bounded for each $1 < t$*

For $f \in L^1(\mathbf{S})$ and $x \in \{c, h, m, p\}$, we define the function $C_\varphi^x f$ on **S** by

$$C_\varphi^x f = f^x \circ \varphi.$$

Clearly, this is well defined, because f^x remains the same even if f is altered on a set of σ-measure 0. Also, it should be remarked that this defines $C_\varphi^x f$ off a set of σ-measure 0 on S. To see this, we have

$$C_\varphi^x f(\zeta) = \lim_{r \to 1^-} f^x(r\varphi(\zeta)), \quad \zeta \in \mathbf{S},$$

and this limit exists precisely when f^x has a radial limit at $\varphi(\zeta)$. Thus $C_\varphi^x f$ has been defined at points $\zeta \in S \setminus \varphi^{-1}(E)$, where $\mathbf{E} \subset \mathbf{S}$ is the set of σ-measure 0 where f^x fails to have a radial limit. Since $\sigma[\varphi^{-1}(\mathbf{E})] = 0$ by the assumption (1.1.1), $C_\varphi^x f$ has been defined σ-a.e. on **S** .In general, C_φ^x is a linear operator from L^1**(S)** to the vector space of (equivalence classes of) measurable functions on **S**. From Proposition (1.1.1)(a)–(b), the restriction of $C_\varphi^x f$ for each $x \in \{c, h, m, p\}$ to H^1(**S**) is the usual composition operator:

$$C_\varphi^x f = f \circ \varphi, \quad f \in H^1(\mathbf{S}) \tag{5}$$

where the f in the right-hand side denotes the holomorphic extension of $f \in H^1$(**S**). Similarly, Proposition (1.1.1)(b) shows that the restriction of C_φ^x to L^1(**S**) is the usual composition operator when $x \in \{h, m\}$.A basic problem in the study of composition operators is to characterize those symbols φ for which the restriction of the composition operator C_φ to a Banach space X is bounded or compact. Before stating our main result, which provides such characterizations for the operators C_φ^x acting on L^t(**S**), we introduce some notation. We first introduce the extended kernels. Given $x \in \{c, h, m, p\}$ and $w \in B$, we denote by $K^x(\cdot, w)$ the x-extension of $\overline{K^x(w, \cdot)}$, i.e.,

$$\mathscr{K}^x(\cdot, w) = \left[\overline{K^x(w, \cdot)}\right]^x.$$

Note that each $K^x(\cdot\ , \mathrm{w})$is continuous on the whole **B** by Proposition(1.1.1) (c). More explicitly, we have by (3), (4)and Proposition(1.1.1) (c)

$$\mathscr{K}^x(z, w) = \begin{cases} \int_{\mathbf{S}} K^x(z, \zeta)\overline{K^x(w, \zeta)}\, d\sigma(\zeta) & \text{if } z \in \mathbf{B} \\ \overline{K^x(w, z)} & \text{if } z \in \mathbf{S}. \end{cases}$$

Except for the PoissonSzego kernel, the extended kernels have explicit formulae for $z \in B$ and w$\in\overline{B}$:

$$\begin{aligned} \mathscr{K}^c(z, w) &= \frac{1}{(1 - \langle z, w\rangle)^n}, \\ \mathscr{K}^h(z, w) &= \frac{1 - |z|^2|w|^2}{[z, w]^{2n}}, \\ \mathscr{K}^p(z, w) &= \frac{1}{(1 - \langle z, w\rangle)^n} + \frac{1}{(1 - \langle w, z\rangle)^n} - 1 \end{aligned}$$

where [z, w] = $\sqrt{1 - 2\Re(z \cdot \overline{w}) + |z|^2|w|^2}$. The formulae for $\chi^c$and $\chi^b$are easily ver-ified. The formula for $\chi^h$is also well known; Note that the right-hand sides of the formulae above continuously extend to $\overline{\boldsymbol{B}}$×$\overline{\boldsymbol{B}}$\Δ where Δ denotes the diagonal of **S ×S**. Such extensions are still denoted by $\chi^x$for x ∈{c, h, p}. Note that

$$\mathscr{K}^x(rz, w) = \mathscr{K}^x(z, rw), \quad x \in \{c, h, p\} \tag{6}$$

For $X = M$when $n \geq 2$, no explicit formula of closed form is available; the main difficulty is the fact that the invariant harmonicity is not dilation invariant. Nevertheless, we have natural growth estimate

$$\mathscr{K}^m(z, w) \approx \frac{(1 - |z|^2|w|^2)^n}{|1 - \langle z, w\rangle|^{2n}} \tag{7}$$

for z, w∈B for the lower estimate and the remark after for the upper estimate. Also, we can still naturally extend $\chi^m$to $\overline{\boldsymbol{B}}$×$\overline{\boldsymbol{B}}$\Δ as follows. First, noting that $\chi^m$is symmetric on **B ×B**, we extend $\chi^m$to $\overline{\boldsymbol{B}}$×**B** by symmetry. So, χ^m(ζ, w) =χ^m(w, ζ)for ζ∈S and w∈B. Next, noting that χ^m(w, η)continuously extends to the zero function on **S** \{η}, we simply define K^m(ζ, η) =0for ζ, η ∈ $S$with ζ≠η. Now, one can check that such an extension, still denoted by K^m, is also symmetric on $\overline{\boldsymbol{B}}$×$\overline{\boldsymbol{B}}$\Δ and continuous in each variable separately. Although not needed in this chapter, we remark that $K^m$is actually continuous on $\overline{\boldsymbol{B}}$×$\overline{\boldsymbol{B}}$\Δ. We remark that the dilation commuting true for χ^m.Given φ as in (1.1.1)and x ∈{c, h, m, p}, using the extended kernels introduced above, we now define the functions

$$\Lambda^x_{\varphi,1}(z) := \left\|\mathscr{K}^x(\varphi(\cdot), z)\right\|_1, \quad z \in \overline{\mathrm{B}},$$

and, for $1 < t < \infty$,

$$\Lambda^x_{\varphi,t}(z) := \begin{cases} \frac{\|\mathscr{K}^x(\varphi(\cdot),z)\|_t}{\|K^x(z,\cdot)\|_t} & \text{if } z \in \mathrm{B} \\ 0 & \text{if } z \in \mathrm{S}. \end{cases}$$

Note that these functions are well defined, because each χ^m(φ(·), z)with**z∈S**is a Borel function defined on Soff the set φ^{-1}{z}of σ-measure 0. The definition of $A^x_{\varphi,t}$for 1 <t <∞requiring to be 0 on the boundary may seem peculiar. We define it in this way only for the purpose of stating the next theorem in a unified way.

Theorem (1.1.2)[1]:. Let x ∈{c, h, m, p}, 1 ≤t <∞, and assume φ satisfies (1.1.1). Then the following statements hold:

(a)$C^x_\varphi$is bounded on L^t(S)if and only if $A^x_{\varphi,t}$is bounded on B;

(b)$C^x_\varphi$is compact on L^t(S)if and only if $A^x_{\varphi,t}$∈C($\overline{\boldsymbol{B}}$).

Note that the restriction of $A^x_{\varphi,t}$to Bis always continuous and hence, when 1 <t <∞, the statement that $A^x_{\varphi,t}$∈C(B)is equivalent to Λ^x_ϕ,t(z) →0as |z| →-1. Thus Condition (b) in Theorem (1.1.2) can be viewed as a “littleoh” version of Condition (a). Our proof for the boundedness characterization in the theorem above actually yields norm estimates. Also, one can easily recover Sarason’s one-variable characterization for L^1-compactness.When $1 < t < \infty$, Carleson measure methods are available and provide alternate characterizations of when $C_\varphi^x$is bounded or compact on L^t(**S**) Carleson measure methods do not provide a characterization of when $C_\varphi^x$to be bounded or compact on L^1(**S**), but can be used to establish relationships with the operators on L^t(**S**).

Theorem (1.1.3)[1]: Let x ∈{c, h, m, p}and φbe as in (1.1.1). Then the following statements hold:

(a)If $C_\varphi^x$is bounded (respectively compact) on L^1(**S**), then $C_\varphi^x$is bounded (compact) on L^t(S)for all t ∈(1, ∞);

(b)For each x ∈{c, h, m, p}there exists φ such that $C_\varphi^x$is compact on L^t(**S**), 1 <t <∞, but $C_\varphi^x$is not bounded onL^t(**S**).Finally, we remark that the Poisson kernel $\chi^h$for the unit ball of the Euclidean space of real dimension dis given by

$$K^h(\xi,\eta)=\frac{1-|\xi|^2}{|\xi-\eta|^d}$$

For ξ and η in R^d with |ξ| <1and |η| =1. Our results for the operator C_φ^h have natural formulations in this setting, and remain valid with the same proofs. This comment does not extend to the operators C_φ^x for x ∈{c, m, p}, due to the appearance of the Hermitian inner product⟨· |· ⟩in the corresponding kernels.Throughout the chapter we use the same letter C to denote various positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants C will be sometimes specified in parentheses. For nonnegative quantities X and Y the notation $X \lesssim Y$or Y≳Xmeans $X \leq CY$for some inessential constant C. Similarly, we write X≈Yif both X≲Y and Y≲Xhold.In this part we collect some basic notions and related facts to be used in our proofs .We first recall the well known integral estimates related to the reproducing kernels under consideration. Given α real, put

$$I_\alpha(z)=\int\limits_{S}\frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+\alpha}}\quad\text{and}\quad J_\alpha(z)=\int\limits_{S}\frac{d\sigma(\zeta)}{|z-\zeta|^{2n-1+\alpha}}$$

for $z\in\boldsymbol{B}$. The growth estimates for these integrals are well known:

$$I_\alpha(z)\approx J_\alpha(z)\approx\begin{cases}(1-|z|^2)^{-\alpha} & \text{if } \alpha>0\\ 1+\log(1-|z|^2)^{-1} & \text{if } \alpha=0\\ 1 & \text{if } \alpha<0\end{cases}\tag{8}$$

for $z\in B$. Proofs can be found, for example, in for $I_\alpha$and J_α, respectively.As an immediate consequence of (8), we have the following norm estimates for reproducing kernels for $1<t<\infty$:

$$\left\|K^x(z,\cdot)\right\|_t^t\approx\begin{cases}(1-|z|^2)^{n(1-t)} & \text{if } x=c,m\\ (1-|z|^2)^{(2n-1)(1-t)} & \text{if } x=h\end{cases}\tag{9}$$

for$z\in B$. Also, we have

$$\left\|K^c(z,\cdot)\right\|_1\approx 1+\log\left(1-|z|^2\right)^{-1},$$

but$K^x(z,\cdot)_1=1$ for $x=m,h$. For $x=p$, since $|K^p(z,\cdot)|\le|K^c(z,\cdot)|$, we have, for each **1 ≤t <∞,**

$$\left\|K^p(z,\cdot)\right\|_t\lesssim\left\|K^c(z,\cdot)\right\|_t.\tag{10}$$

When **1 <t <∞,** by the **K** or anyiVagi Theorem asserting that the Cauchy transform followed by the **K** or any**i** maximal function is $\boldsymbol{L^t}$-bounded, there is a constant **C=C(t, n) >0**such that

$$\|f\|_t\le C\|\operatorname{Re} f\|_t\tag{11}$$

for functions **f∈$\boldsymbol{H^t}$(S)**with $\mathbf{lim}\,\boldsymbol{f}(\mathbf{0})=\mathbf{0}$. So, the estimate in (10)can be reversed for **1 <t <∞.** We remark in passing that the reverse estimate of (10)is also valid when **t =1**and **n ≥2**, as can be seen by using (10)to convert integration over the sphere to a weighted integral over the unit disk, and then that harmonic conjugation is **L**¹bounded on the standard weighted

Bergman spaces of the unit disk .The term normal family refers to a family of functions with the property that every sequence in the family contains a subsequence converging uniformly on compact subsets of the domain. As is well known, a family of holomorphic functions that is uniformly bounded on each compact subset of the domain is a normal family. An argument using that result is often called anormal family argument. Such a normal family argument textends to harmonic functions and hence to pluriharmonic functions, The cases x =c, p, hare also included in the statement for easier reference later.

Lemma (1.1.4)[1]: Let x ∈{c, h, m, p}. Given a bounded set **F**in **L**^{1}**(S),** let F^X={f^x:f∈F}. Then $F^X$is a normal family on **B**.

Proof: The cases x =c, p, hare easily seen from the remark above. To treat the case x=m, we first introduce some notation. Given $z \in B$, let $\tau_z$be the in n volatileautomorphism of **B** that exchanges 0and z. It is known that

$$f^m \circ \tau_z = (f \circ \tau_z)^m, \quad f \in L^1(\mathrm{S}); \tag{12}$$

Also, ρ(z, w) **=|τ_a(b)|**is known to be a metric, called the pseudo hyperbolic metric, on **B** .Let E⊂B be a compact set. We claim there is a constant **C=C(E) >0**such that

$$\left|f^m(a) - f^m(b)\right| \le C\rho(a,b)\|f\|_1, \quad a, b \in E \tag{13}$$

for all f∈L^1(**S**). With this granted, we see that $F^m$is equicontinuous on each compact subset, which is the key to the proof; the lemma follows then from the standard argument using the Arzela Ascoli Theorem and the diagonal process. Let **f∈L^1(S).** Since

$$\left|f^m(0) - f^m(z)\right| \le \|f\|_1 \sup_{\eta \in \mathrm{S}} \left|1 - \left(\frac{1-|z|^2}{|1-\langle z,\eta\rangle|^2}\right)^n\right| \lesssim \frac{|z|}{(1-|z|)^n}\|f\|_1$$

for z∈B, we see that

$$\left|f^m(0) - f^m(z)\right| \le C_1|z|\|f\|_1, \quad z \in E$$

for some constant $C_1=C_1(E)>0$. Now, given a, b $\in$E, we have by (12)

$$\left|f^m(a)-f^m(b)\right|=|(f\circ\tau_a)^m(0)-(f\circ\tau_a)^m\left(\tau_a(b)\right)|\le C_1\rho(a,b)\|f\circ\tau_a\|_1$$

Meanwhile, we have again by (12)

$$\|f\circ\tau_a\|_1=|f\circ\tau_a|^m(0)=|f|^m(a)\le\left(\frac{1+|a|}{1-|a|}\right)^n\|f\|_1\le C_2\|f\|_1$$

for some constant $C_2=C_2(E)>0$. Combining these observations, we conclude (13), as claimed. The proof is complete .We recall the notions of Carleson measures that are needed in our work. Let $1<t<\infty$and x $\in$\{c, h, m, p\}. Let μ be a positive finite Borel measure on $\overline{\boldsymbol{B}}$. We say that **μ** is an **x**-Carleson measure for L^t(**S**)if there exists some constant **C>0**such that

$$\int_{\overline{\mathbf{B}}}\left|f^x\right|^t d\mu\le C\int_{\mathbf{S}}\left|f^x\right|^t d\sigma,\quad f\in L^t(\mathrm{S}).\tag{14}$$

That is, μ is an ***x***-Carleson measure for L^t(**S**)if and only if the mapping $f(x)|_{\overline{S}}\to f(x)$is continuous from L^t(**S**)toL^t(μ). We write N^x(μ)for the infimum of the constants C for which inequality (14)holds, so $[N^x(\mu)]^{\frac{1}{t}}$is the norm of this mapping. If, in addition, this mapping is compact, then μ is said to be a compact Carleson measure for L^t(**S**). Characterizations for (compact) x Carleson measures for L^t(**S**)are given in terms of Carleson sets that are balls defined using a metric appropriate for the kernel K^X.For ζ$\in$**S**and$0<\delta<1$, let

$$S^x(\zeta,\delta)=\left\{z\in\overline{\mathbf{B}}\,:\,\left|1-\langle z,\zeta\rangle\right|<\delta\right\},\quad x\in\{c,m,p\},$$

and

$$S^h(\zeta,\delta)=\left\{z\in\overline{\mathbf{B}}\,:\,|z-\zeta|<\delta\right\}.$$

Now, we put

$$M_\delta^x(\mu):=\sup_\zeta\frac{\mu[S^x(\zeta,\delta)]}{\sigma[S^x(\zeta,\delta)\cap\mathrm{S}]}.$$

Note that c-Carleson measures for $L^t(\mathbf{S})$are precisely the well-known Carleson measures for $H^t(\mathbf{B})$and that the two notions of x-Carleson measures for x ∈{c, p}coincide by (11). We have the following characterizations for each **1 <t <∞**:

(i) M is an X Carleson measure for $L^t(\mathbf{S})$ ⇐⇒supδm_{φ}^{x}(μ) <∞;

(ii)M is a compact X Carleson measure for $L^t(S)$ ⇐⇒M_xδ(μ) →0as δ→0+.

A reference for the case x ∈{c, p}, i.e. for Carleson measures for $H^t(\mathbf{B})$, While we have not been able to find a reference for the characterization of M Carleson measures, it should be well known that they also coincide with the Hardy space Carleson measures. Indeed, in all cases the necessity of the characterizing condition is established using natural test functions and simple estimates of the kernel. The proof of sufficiency in the Hardy space case given in goes through for $x = m$with almost no change. A comment is needed regarding just one part of the proof the point wise estimate of the K or any imaximal function of $f \in H^t(\mathbf{B})$by the Hardy–Little wood maximal function of f associated with non-isotropic balls. That this estimate remains valid when $x = m$ is the content of The characterization when x =h for measures supported on **B**; the extension to measure supported on $\overline{\boldsymbol{B}}$ is standard. Alternatively, it can be observed that Euclidean (rather than non-isotropic) versions of the key ingredients of the proof in the Hardy space case are well known. Moreover, setting m^x(μ) =supδM_xδ(μ), we have

$$m^x((\mu) \approx N^x((\mu).$$

Of particular importance is that the characterization of (compact) X Carleson measures for $L^t(\mathbf{S})$is independent of t >1, and that the characterization is the same for x ∈{c, m, p}. But the characterization differs for x =h^when n >1, since

$$\sigma\left[S^x(\zeta,\delta)\cap \mathbf{S}\right] \approx \begin{cases} \delta^n & \text{if } x \in \{c,m,p\} \\ \delta^{2n-1} & \text{if } x = h. \end{cases}$$

When x =h this is elementary. When x ∈{c, m, p} Finally, we note that the restriction t >1for x ∈{h, m, p}comes from the same restriction in the L^tboundedness of the HardyLittlewood maximal function as well as in the KoranyiVagi Theorem mentioned after (10), when proving the sufficiency of the characterizing conditions. On the other hand, one may remove the restriction **t >1**when **x =c**, considering f∈H^t(B)in (14)instead of f^x, and the characterization remains the same for **0 <t ≤1**.The relevance of Carleson measures to composition operators comes from the idea of pullback measure. Associated with φ as in (2)is the pullback measure σ∘φ^{-1}, which is the Borel measure defined for a Borel set E⊂$\overline{\boldsymbol{B}}$by σ∘$\varphi^{-1}$(**E**) =σ{ζ∈S: φ(ζ) ∈E}. Use of a change of variable formula from measure theory shows that

$$\|C_\varphi^x f\|_t^t = \int_{\mathbf{S}} |f^x \circ \varphi|^t \, d\sigma = \int_{\overline{\mathbf{B}}} |f^x|^t \, d(\sigma \circ \varphi^{-1}) \tag{15}$$

for any f∈L^t(**S**), **1 ≤t <**∞and x ∈{c, h, m, p}. This gives the following proposition. In what follows, $C_\varphi^x L^t$(**S**)denotes the operator norm of C_φ^xacting on L^t(**S**).

Lemma (1.1.5)[1]: Let x ∈{c, h, m, p}, 1 <t <∞, and φbe as in (2). Then $C_\varphi^x$is bounded (respectively compact) on L^t(**S**). if and only if σ∘$\varphi^{-1}$is a (compact) X Carleson measure for L^t(**S**). Moreover, the operator norm satisfies

$$\|C_\varphi^x\|_{L^t(\mathbf{S})}^t \approx N^x(\sigma \circ \varphi^{-1}) \approx M^x(\sigma \circ \varphi^{-1});$$

the constants suppressed above depend on x and t, but are independent of φ.

Proof : If σ∘$\varphi^{-1}$is a Carleson measure, then use of (15), (12)and Proposition(1.1.1) (e) shows that $C_\varphi^x$is bounded on L^t(**S**). with$\|C_\varphi^x\|_{L^t(\mathbf{S}).}^t \leq N^x$(σ∘$\phi^{-1}$). Conversely, if $C_\varphi^x$is bounded on L^t(**S**). and f∈L^t(**S**). then

$$\int_{\overline{\mathbf{B}}} |f^x|^t \, d(\sigma \circ \varphi^{-1}) = \|C_\varphi^x f^x\|_t^t \leq \|C_\varphi^x\|^t \|f^x\|_t^t,$$

and so $\sigma\circ\varphi^{-1}$is an X Carleson measure for $L^t(\mathbf{S})$and $N^x(\sigma\circ\phi^{-1}) \leq \|C_\varphi^x\|_{L^t(\mathbf{S})}^t$. This, together, completes the proof for C_φ^xbounded. We note that the dependence of the constants on xand tcomes from the application of Proposition (1.1.1) (e). The proof for C_φ^xcompact is similar and so is omitted .We mention some immediate consequences*. Let φ be as in (1.1.1).*

Then the following statements hold:

(a)If x ∈{c, h, m, p}and**1 <t_1, t_2<∞,** then C_φ^x is bounded (respectively compact) on $L^{t_1}(\mathbf{S})$if and only if $C_\varphi^x$is bounded (compact) on $L^{t_2}(\mathbf{S})$.

(b)If x, y∈{c, m, p}and**1 <t <∞,** then $C_\varphi^x$is bounded (respectively compact) on $L^t(\mathbf{S})$. if and only if $C_\varphi^y$is bounded (compact) on $L^t(\mathbf{S})$.

(c)If x ∈{c, m, p}and**1 <t <∞,** then C_φ^x:$H^1(\mathbf{S}) \to L^1(\mathbf{S})$. is bounded (respectively compact) if and only if $C_\varphi^x$is bounded (compact) on $L^t(\mathbf{S})$.

(d)If x ∈{c, m, p}, **1 <t <∞,** and $C_\varphi^h$is bounded (respectively compact) on $L^t(\mathbf{S})$, then $C_\varphi^x$is bounded (compact) on $L^t(\mathbf{S})$.

For (a) note that the characterizations of (compact) X Carleson measures are independent of $1 < t < \infty$. For (b) note that (compact) X Carleson measures for x ∈{c, m, p}, precisely being the same as those for the Hardy spaces, coincide. For (c) note from (5)and (17)that C_φ^x:$H^1(S) \to L^1(S)$is bounded (respectively compact) if and only if $\sigma\circ\varphi^{-1}$is a (compact) Carleson measure for H^1(S). In case of (c) note

$$\|C_\varphi^x\|_{H^1(\mathrm{S})\to L^1(\mathrm{S})} \approx N^x(\sigma\circ\varphi^{-1}) \approx \|C_\varphi^x\|_{L^t(\mathrm{S})}^t$$

where$\|C_\varphi^x\|_{\mathbf{H^1(S)\to L^1(S)}}$denotes the operator norm of C_φ^x:$H^1(\mathbf{S}) \to L^1(\mathbf{S})$. For (d) note that $C_\varphi^h$is bounded (compact) on $L^t((\mathbf{S})$if and only if $\sigma o\varphi^{-1}$is a (compact) Carleson measure for the harmonic Hardy space $h^t(\mathbf{B})$, which is isometrically isomorphic to $L^t(\mathbf{S})$when t >1. Since $H^t(\mathbf{B})$is isometrically isomorphic to H^t(S) ⊂$L^t(\mathbf{S})$, we see that $\sigma\circ\varphi^{-1}$is a (compact) Carleson measure for H^t(**S**), and the result follows as in (b). An example will be presented in Part5that shows the converse to (d) fails badly for $t \geq 1$, there exists φsuch that $C_\varphi^x$is compact on L^t(**S**)for x ∈{c, m, p}, but C_φ^h is not bounded.We first mention some remarks for holomorphic symbols. So, assume φ is holomorphic in the following three remarks.

(i) In conjunction we note that if the standard composition operator $C_\varphi$maps H^1(**B**)into H^t(**B**)for some 0 <t <∞, then each C_φ^x, when restricted to H^1(**S**), is precisely the same as $C_\varphi$if a Hardy function is identified with its boundary function. To see this, let f∈H^1(**S**)(or H^1(**B**)) and put f_r(z) =f(rz)for 0 <r<1. Then, as r→-1, we have f_r→$f$in H^1(**B**)and thus $C_{\varphi J_r} \to C_\varphi f$ in H^t(**B**). Also, note that the boundary function of $C_\varphi f_r$is f_xo φ, which is obvious by the continuity of fron **S**. Thus, by Fatou's Lemma and (5), we obtainJ_r

$$0 = \lim_{r \to 1^-} \int_{\mathbf{S}} |C_\varphi f_r - C_\varphi f|^t \, d\sigma \geq \int_{\mathbf{S}} \left|C_\varphi^x f - C_\varphi f\right|^t d\sigma,$$

which shows that $C_\varphi^x f$is the boundary function of $C_\varphi f$, as asserted.

(ii) The absolute continuity hypothesis is satisfied if $C_\varphi$is bounded on H^t(**B**)for some/all **0 <t**. Thus is satisfied for all holomorphic self-maps of D, which is not the case on multidimensional balls.

(iii) that when x ∈{c, m, p}, $C_\varphi^x$is bounded (respectively compact) on L^t(S)for some/all **1 <t <**∞if and only if $C_\varphi$is bounded (compact) on H^t(**B**)for some/all 0 <t <∞.Finally, we mention an elementary result from real analysis that will be used repeatedly, following the approach of Sarason.

Lemma (1.1.6)[1]: Let f∈L^1(**S**)and {f_j}be a sequence of functions in L^1(**S**)such that f_j→ Fσ-a.e. on S. Then $\|f_j\|_1 \|f\|_1$→if and only if $\left\| \|f_j\|_1 - \|f\|_1 \right\|$ →0.

In this part we prove the boundedness parts of our results stated in the Introduction. Proof for the boundedness part of Theorem(1.1.2) is split in the next two propositions, since they differ when t =1or $1 < t < \infty$. We first characterizeboundedness for the case t =1.

Proposition (1.1.7)[1]: Let x ∈{c, h, m, p}and φ .Then $C_\varphi^x$is bounded on L^1(**S**)if and only ifA_φ^x,1is bounded on **B**. Moreover, the operator norm satisfies $\|C_\varphi^x\|_{L^1(S)}=\sup_{z\in B} A_{\varphi,1}^x(z)$.

Proof: Let z∈B. For x ∈{c, h, p}we choose $K_z=k^h$(z, ·)as a test function. Note that $\chi^x$is harmonic on **B**in each variable separately. So, from the reproducing property ofK_z, we see that

$$\begin{aligned}(k_z)^x(w) &= \int_S K^h(z,\eta)K^x(w,\eta)\,d\sigma(\eta), \quad w\in\mathbf{B}\\ &= \int_S K^h(z,\eta)\mathscr{K}^x(w,\eta)\,d\sigma(\eta)\\ &= \mathscr{K}^x(w,z),\end{aligned}$$

and so

$$C_\varphi^x k_z = (k_z)^x\circ\varphi = \mathscr{K}^x\big(\varphi(\cdot),z\big).$$

Since$\|k_z\|$=1, integration on Sshows that $\|C_\varphi^x\|_{L^1(S)} \geq \|\chi x(\varphi(\cdot),z)\|_1, z$∈**B**. Thus we conclude

$$\|C_\varphi^x\|_{L^1(\mathbf{S})} \geq \sup_{z\in\mathbf{B}} A_{\varphi,1}^x(z).$$

This inequality also holds for x =m, with the same proof except for choosing $k^z=k^m$(z, ·)as a test function in this case. We now prove the reverse inequality. Let f∈L^1(**S**)and assume $f^x$is defined at φ(ζ), ζ∈S. For x ≠m

$$\big(f^x\circ\varphi\big)(\zeta) = \lim_{r\to 1^-}\int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(\varphi(\zeta),r\eta\big)\,d\sigma(\eta). \tag{16}$$

This remains valid for x =m, even h tough is no longer true in that case. In fact, when φ(ζ) ∈S, the above is certainly true by (1). On the other hand, when φ(ζ) ∈B, we have

$$\begin{aligned}\big(f^x\circ\varphi\big)(\zeta) &= \lim_{r\to 1^-}\int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(r\varphi(\zeta),\eta\big)\,d\sigma(\eta)\\ &= \int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(\varphi(\zeta),\eta\big)\,d\sigma(\eta)\\ &= \int_{\mathbf{S}} f(\eta)\lim_{r\to 1^-}\mathscr{K}^x\big(\varphi(\zeta),r\eta\big)\,d\sigma(\eta)\\ &= \lim_{r\to 1^-}\int_{\mathbf{S}} f(\eta)\mathscr{K}^x\big(\varphi(\zeta),r\eta\big)\,d\sigma(\eta);\end{aligned}$$

the second and the last equalities hold by the Dominated Convergence Theorem and the third equality holds by the continuity of $K^{x}(\varphi\,(\zeta),\cdot\ )on\ \overline{\boldsymbol{B}}$.So, for any x ∈{c, h, m, p}, we have by, Fatou's Lemma and Fubini's Theorem.

$$\begin{aligned}\|C_\varphi^x f\|_1 &= \int\limits_{\mathbf{S}} \left|(f^x \circ \varphi)(\zeta)\right| d\sigma(\zeta) \\ &\leq \liminf_{r\to 1^-} \int\limits_{\mathbf{S}} \int\limits_{\mathbf{S}} |f(\eta)| \left|\mathscr{K}^x(\varphi(\zeta), r\eta)\right| d\sigma(\eta)\, d\sigma(\zeta) \\ &\leq \|f\|_1 \sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^x(z)\end{aligned}$$

and thus conclude

$$\|C_\varphi^x\|_{L^1(\mathbf{S})} \leq \sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^x(z),$$

which completes the proof.

For the proof below (and later use), we recall the following slice integration formula for n >1

$$\int\limits_{\mathbf{S}} \psi(\langle \eta, \xi\rangle)\, d\sigma(\eta) = \frac{n-1}{\pi} \int\limits_{\mathbf{D}} \psi(\lambda)\left(1-|\lambda|^2\right)^{n-2} dA(\lambda) \tag{17}$$

for any positive measurable function ψon **D** and ξ∈**S**. Here, A denotes the area measure on **D**.

Corollary (1.1.8)[1]: If n ≥1and C_φ^c is bounded on L^1(s), then σ[φ^{-1}(**S**)] =0. If n ≥2and C_φ^p is bounded on L^1(s)then σ[φ^{-1}(**S**)] =0.We remark that the statement for C_φ^b does not extend to n =1. For example, with iddenoting the identity map of S, note that C_{id}^p =C_{id}^h is the identity operator on L^1(s)in the one-dimensional case.

Proof: It is easily seen from Fatou's Lemma that A_{id}^c (η)≤sup$_{z\in B}$$C_{id,1}^p$ (z)for all η∈S. Thus we have

$$\begin{aligned}\sup_{z\in\mathbf{B}} \Lambda_{\varphi,1}^c(z) &\geq \int\limits_{\mathbf{S}} \Lambda_{\varphi,1}^c(\eta)\, d\sigma(\eta) \\ &= \int\limits_{\mathbf{S}} \int\limits_{\mathbf{S}} \frac{d\sigma(\eta)}{|1-\langle\varphi(\zeta),\eta\rangle|^n}\, d\sigma(\zeta) \\ &\geq \int\limits_{\varphi^{-1}(\mathbf{S})} \int\limits_{\mathbf{S}} \frac{d\sigma(\eta)}{|1-\langle\varphi(\zeta),\eta\rangle|^n}\, d\sigma(\zeta).\end{aligned}$$

Note that the inner integral of the above diverges for each $\zeta\in\varphi^{-1}$(**S**). This is elementary when n =1; when n ≥2it is easily seen using. So, the result for x =c is a consequence of Proposition(1.1.7).

The proof for $x = p$is similar:

$$\sup_{z\in\mathbf{B}} \Lambda^p_{\varphi,1}(z)+1 \geq \int_{\mathbf{S}} \left(\Lambda^p_{\varphi,1}(\eta)+1\right) d\sigma(\eta)$$
$$\geq \int_{\varphi^{-1}(\mathbf{S})} \int_{\mathbf{S}} \frac{|2\operatorname{Re}(1-\langle\varphi(\zeta),\eta\rangle)^n|}{|1-\langle\varphi(\zeta),\eta\rangle|^{2n}} \, d\sigma(\eta)\, d\sigma(\zeta).$$

Since n ≥2and $\varphi(\zeta)\in$S, (17)is available to compute the inner integral to be

$$\int_{\mathbf{S}} \frac{|2\operatorname{Re}(1-\langle\varphi(\zeta),\eta\rangle)^n|}{|1-\langle\varphi(\zeta),\eta\rangle|^{2n}} \, d\sigma(\eta) = \frac{2(n-1)}{\pi} \int_{\mathbf{D}} \frac{|\operatorname{Re}(1-\lambda)^n|}{|1-\lambda|^{2n}} \left(1-|\lambda|^2\right)^{n-2} dA(\lambda).$$

This integral can be seen to diverge by using polar coordinates centered at λ =1and integrating over a small sector, and the result again holds by Proposition (1.1.7).Now, we turn to the case 1 <t <∞, where some auxiliary estimates are needed. First, we need the following estimate as to how the kernels grow on certain Carleson sets.

Lemma (1.1.9)[1]: Let $\zeta_0\in$S, δ∈(0, 1)and put z=(1 −δ)ζ_0. Then there are constants C_1=C_1(n) >0and C_2=C_2(n) >0such that

$$\left|\mathscr{K}^x(z,w)\right| \geq C_1 \times \begin{cases} \delta^{-n} & \text{if } \ x = c, m, p \\ \delta^{-(2n-1)} & \text{if } x = h \end{cases}$$

Before the proof, we remark that we can take c_2=1when $x \in \{c, h, m\}$. It is when $x = p$ and $n > 1$ that $0 < C_2 < 1$ is necessary.

Proof: For $x \neq p$ the proof is a straightforward estimate using the explicit formula for χ^x(w, z), and will be omitted.For $x = p$, assume first that δ∈(0, 1/16). Choose C_2∈(0, 1), depending only on n, so small that Re(a_n) ≥1/2 for all allying in the disk with center at 1and radius C_2. Let w∈S^p(ζ_0 , c_2δ)and put λ≠ζ_0, w. Note

$$\left|1-\frac{1-(1-\delta)\lambda}{\delta}\right| = \frac{(1-\delta)|1-\lambda|}{\delta} < c_2,$$

which means that 1−(1−δ)λ/δ lies in the disk with center at 1and radius c_2. Now, since $|1 - (1 - \delta)\lambda| \leq 2\delta$and $\delta < \frac{1}{16}$, we obtain

$$\mathscr{K}^p(z,w) = \frac{2\,\mathrm{Re}[1-(1-\delta)\lambda]^n}{|1-(1-\delta)\lambda|^{2n}} - 1 \geq \frac{1}{(2\delta)^{2n}} \cdot \frac{2\delta^n}{2} - 1 \geq \frac{\delta^{-n}}{4^{n+1}},$$

which completes the proof when δ∈(0, 1/16). The extension to δ∈(0, 1)can be accomplished by replacing $C_2$by $C_{2/16}$. The proof is complete. We have the optimal norm estimate (9)for the reproducing kernels except for the pluriharmonic case. In the pluriharmonic case, we have an upper estimate (10)for$x = p$. What is needed here is the lower estimate for 1 <t <∞. We do not know a reference and thus a proof is provided below. Other cases are restated for easier reference.

Lemma (1.1.10)[1]: Given $\mathbf{1} < t < \infty$, the estimate

$$\left\|K^x(z,\cdot)\right\|_t^t \approx \begin{cases} (1-|z|^2)^{n(1-t)} & \mathit{if}\ x = c,p,m \\ (1-|z|^2)^{(2n-1)(1-t)} & \mathit{if}\ x = h \end{cases}$$

holds for $z \in \mathbf{B}$.

Proof: We only need to establish the lower estimate for x =p. Let $z \in B$, z_=0, put ζ_0=z/|z| and set $E_z=S^p(z/|z|, c_2(1-|z|)) \cap S$, where $c_2$is the constant provided by Lemma(1.1.9) Note that z=(1 −δ)ζ0where δ=1 −|z|. Thus by Lemma (1.1.9) we have

$$\left|K^p(z,\zeta)\right| \gtrsim \left(1-|z|^2\right)^{-n}, \quad \zeta \in E_z$$

so that

$$\|K^p(z,\cdot)\|_t^t \geq \int_{E_z} \left|K^p(z,\zeta)\right|^t d\sigma(\zeta) \gtrsim \frac{\sigma(E_z)}{(1-|z|^2)^{nt}} \approx \left(1-|z|^2\right)^{n(1-t)},$$

which completes the proof. We are now ready to characterize boundedness for the case 1 <t <∞.

Proposition (1.1.11)[1]: *Let* $x \in \{c,h,m,p\}$, 1 <t <∞and φ be as in (2). Then $C_\varphi^x$is bounded on $L^t(\mathbf{S})$if and only if$A_{\varphi,t}^x$,is bounded on **B**. Moreover, the operator norm satisfies

$$\left\|C_\varphi^x\right\|_{L^t(\mathbf{S})} \approx \sup_{z\in\mathbf{B}} \Lambda_{\varphi,t}^x(z);$$

the constants suppressed above depend on x and t, but are independent of φ.

Proof: Fix *any* $x \in \{c,h,m,p\}$, let $z \in B$and choose $k_z=\overline{K^x(z,\cdot)}$as a test function. Then

$$(k_z)^x(w) = \int_S \overline{K^x(z,\eta)} K^x(w,\eta)\, d\sigma(\eta) = \mathscr{K}^x(w,z), \quad w \in \mathbf{B}$$

and so

$$C_\varphi^x k_z = (k_z)^x \circ \varphi = \mathscr{K}^x\big(\varphi(\cdot), z\big).$$

Hence

$$\left\|C_\varphi^x\right\|_{L^t(\mathbf{S})} \geq \frac{\left\|\mathscr{K}^x(\varphi(\cdot),z)\right\|_t}{\left\|K^x(z,\cdot)\right\|_t} = \Lambda_{\varphi,t}^x(z)$$

and this is true for any $z \in B$. Taking the supremum over $z \in B$, we obtain

$$\left\|C_\varphi^x\right\|_{L^t(\mathbf{S})} \geq \sup_{z\in\mathbf{B}} \Lambda_{\varphi,t}^x(z).$$

For the reverse inequality, let $\zeta_0 \in \mathbf{S}$, $\delta \in (0, 1)$and put $w=(1-\delta)\zeta_0$. First, consider the case $x \in \{c,m,p\}$. see that

$$\left[\Lambda_{\varphi,t}^x(w)\right]^t \approx (1-|w|)^{n(t-1)} \int_{\mathbf{S}} \left|\mathscr{K}^x(\varphi(\cdot),w)\right|^t d\sigma(\zeta)$$

$$\gtrsim \delta^{n(t-1)} \int_{\varphi^{-1}[S^x(\zeta_0,c_2\delta)]} \delta^{-nt}\, d\sigma(\zeta)$$
$$= \frac{(\sigma\circ\varphi^{-1})[S^x(\zeta_0,c_2\delta)]}{\delta^n}.$$

This estimate is independent of $\zeta_0$and δ, so taking the supremum yields.

$$\sup_{z\in\mathbf{B}} \left[\Lambda_{\varphi,t}^x(z)\right]^t \gtrsim M^x\big(\sigma\circ\varphi^{-1}\big).$$

Hence $\sigma\circ\varphi^{-1}$is an X Carleson measure with the norm estimate $N^x(\sigma\circ\varphi^{-1}) \approx M^x(\sigma\circ\varphi^{-1}) \leq \|A^x_{\varphi,t}\|^t_{L^t(s)}$ Since $\|C^x_{\varphi,t}\|^t_{L^{t(s)}} \approx N(\sigma\circ\varphi^{-1})$from Lemma (1.1.4) , we conclude$\|C^x_{\varphi}\|_{L^t(S)}$ $\sup_{z\in B}A^x_{\varphi,t}(z)$, which completes the proof for $x \in \{c, m, p\}$.The proof when x =his similar, using the norm estimate for $\|K^h(z,\cdot\ )\|^t_t$from (1.1.4)and the lower bound for $\|K^h(w,z)\|$ w∈ $S^h(\zeta_0, c_2\delta)$.

As an application we now show that L^1boundedness implies L^tboundedness for each $1 < t < \infty$ which is the content of the boundedness part of Theorem(1.1.3) (a). In view of this result, one may wonder whether its converse would hold.

Proposition (1.1.12)[1]: Let $x \in \{c, h, m, p\}$, $1 < t < \infty$and φ be as in (2). If $C^x_\varphi$is boundedon $L^1(\mathbf{S})$, then $C^x_\varphi$is bounded on $L^t(\mathbf{S})$. Moreover, the operator norms satisfy$\|C^x_\varphi\|^t_{L^1(s)} \leq C\,\|C^x_\varphi\|_{L^1(s)}$for some constant $C = C(x, t) > 0$.

Proof: Since L^1-boundedness of C^x_φimplies the boundedness of $C^x_\phi: H^1(S) \to L^1(S)$, the case x ≠his contained in Theorem (1.1.5) (c). So, let x =h. Suppose $C^h_\varphi$is bounded on $L^1(S)$. By Lemma (1.1.4), to show $C^h_\varphi$is bounded on $L^t(\mathbf{S})$, it suffices to show that $\sigma\circ\varphi^{-1}$is an h Carleson measure. Given $\zeta_0\in S$and $\delta\in(0, 1)$, put $w=(1-\delta)\zeta_0$.

$$A^h_{\varphi,1}(w) \geq \int_{\varphi^{-1}[S^h(\zeta_0,c_2\delta)]} \left|\mathscr{K}^h(\varphi(\zeta), w)\right| d\sigma(\zeta) \gtrsim \frac{(\sigma\circ\varphi^{-1})[S^h(\zeta_0, c_2\delta)]}{\delta^{2n-1}},$$

and this estimate is independent of $\zeta_0$and $\delta \in (0, 1)$. Taking the supremum over all ζ0and δyields $M^h(\sigma\circ\varphi-1) \geq \sup_{z\in B}A^h_{\varphi,1},1(z).$ Now, since $C^h_\varphi(\sigma\circ\varphi-1)$from Lemma(1.1.4), and $\sup_{z\in\mathbf{B}}A^h_{\varphi,1}(z) = \|C^h_\varphi\|_{L^1(s)}$from Proposition (1.1.7), the proof is complete.

We now close the part with the following remarks:

(i) When $x \in \{h, m\}$, the RieszThorin Interpolation Theorem could also be used to prove Proposition (1.1.11) with norm estimate

$$\left\|C_{\varphi}^{x}\right\|_{L^{t}(\mathrm{S})}^{t} \leq \left\|C_{\varphi}^{x}\right\|_{L^{1}(\mathrm{S})} \tag{18}$$

since in these cases $C_{\varphi}^{x}$is bounded on L∞(S)with operator norm 1. When x ∈{c, p}this method does not work, as $C_{\varphi}^{x}$is not bounded on L^{∞}(S)in general. To see examples of $C_{\varphi}^{c}$and C_{φ}^{p}which are not bounded on L^{∞}(S), simply consider φ =id. Note that Ccidis the Cauchy transform. As is well known, the Cauchy transform (and hence Cpidas well) is not bounded on L^{∞}(S). In fact the Cauchy transform takes L^{∞}(S)into the space of functions of bounded mean oscillation with respect to nonisotropic balls.

(ii) One may also derive. In fact, when x ∈{h, m}, note that χ^{x}(φ(ζ), rη) dσ(η)is a probability measure for each 0 <r<1. So, given f∈L^{1}(S), applications of (1.1.8), Fatou's Lemma and Jensen's Inequality yield

$$\left\|C_{\varphi}^{x} f\right\|_{t}^{t} \leq \liminf_{r\to 1^{-}} \int\limits_{\mathrm{S}} \int\limits_{\mathrm{S}} \left|f(\eta)\right|^{t} \mathscr{K}^{x}\left(\varphi(\zeta), r\eta\right) d\sigma(\eta)\, d\sigma(\zeta).$$

Now, computing the ζ-integration first, we obtain

$$\left\|C_{\varphi}^{x} f\right\|_{t}^{t} \leq \|f\|_{t}^{t} \sup_{z\in \mathbf{B}} \Lambda_{\varphi,1}^{x}(z) = \|f\|_{t}^{t} \left\|C_{\varphi}^{x}\right\|_{L^{1}(\mathrm{S})},$$

which yields (1.1.11).

section (1.2): Compactness and Examples

Recall that a linear operator on a Banach space X is said to be compact if any bounded sequence {x_j}in X contains a subsequence {x_{jk}}for which $T_x n_k$converges in X. As in the case of boundedness, proof for the compactness part of Theorem (1.1.2) is split in the two Propositions (1.2.1) and (1.2.6) below. This time the case $1 < t < \infty$ is easier to handle and so we first characterize compactness for that case.

Proposition (1.2.1)[1]: $Let\ x \in \{c, h, m, p\}, 1 < t < \infty$and φbe as in (1.1.1). Then $C_{\varphi}^{X}$is compact on L^t(**S**)if and only if $C_{\varphi,t}^{X}$,t∈C($\overline{B}$).

Proof: We first prove the necessity. Fix $any\ x\ \in \{c,h,m,p\}$and suppose $C_{\varphi}^{x}$is compact on L^t(**S**). Note that $A_{\varphi,t}^{x}$is clearly continuous on Band was defined to be 0 on S. S$_o$, in order to see $A_{\varphi,t}^{x}$∈C($\overline{\mathbf{B}}$), it suffices to show that

$$A_{\varphi,t}^{x}(z) \to 0 \quad \text{as } |z| \to \text{-}1 \qquad (19)$$

Given z∈**B**, put $f_z := \mathscr{K}^x(\cdot,z)/\|K^x(z,\cdot)\|_t$ so that $\|f_z\|_t = 1$ and $A_{\varphi,t}^{x}(z) = \|C_{\varphi}^{x} f_z\|_t$.

Now, suppose that (1.2.1)fails to hold. Then one can find an ϵ>0and a sequence $\{z^j\}$ ⊂**B** such that z^jconvergent to a boundary point, say $\eta_0 \in S$, and

$$\left\|C_{\varphi}^{x} f_{z^j}\right\|_t \geq \epsilon > 0 \qquad (20)$$

for all j. Since $C_{\varphi}^{c}$is compact, we may assume, by passing to a subsequence if necessary, that $\{C_{\varphi}^{h} f_{zj}\}$is norm convergent in L^t(**S**). On the other hand, note from Lemma (1.1.10)that$C_{\varphi}^{x} f_{zj}$=$\chi^x(\varphi(\cdot\ ),$ z^j)/||k^x(zj,・)||$_t$→0pointwise as j→+∞on S \$\varphi^{-1}(\eta_0)$, and hence σ-a.e. on S by (1.1.1). Hence $C_{\varphi}^{x} f_{zj}$→0in norm, which contradicts (20). Hence (19)holds, and the proof of the necessity is complete.

Now, to prove the sufficiency, let $x\ \in \{c,p,m\}$and assume $A_{\varphi,t}^{x}$∈C($\overline{\mathbf{B}}$). Given ζ0∈Sand δ∈(0, 1), put z=(1 −δ)ζ$_0$so that 1 −|z| =δ. Then

$$\begin{aligned}\frac{\|\mathscr{K}^x(\varphi(\cdot),z)\|_t^t}{\|K^x(z,\cdot)\|_t^t} &\geq \frac{1}{\|K^x(z,\cdot)\|_t^t} \int\limits_{\varphi^{-1}(S^x(\zeta_0,c_2\delta))} \left|\mathscr{K}^x(\varphi(\zeta),z)\right|^t d\sigma(\zeta) \\ &\gtrsim \frac{\delta^{-nt}\sigma[\varphi^{-1}(S^x(\zeta_0,c_2\delta))]}{\delta^{-nt+n}}\end{aligned}$$

where the last inequality holds by Lemma (1.1.9) and (1.1.6). Since

$$\frac{\|\mathscr{K}^x(\varphi(\cdot),z)\|_t}{\|K^x(z,\cdot)\|_t} = A_{\varphi,t}^{x}(z) \to 0$$

as |z| → -1by continuity of $A_{\varphi,t}^{x}$ we conclude by Lemma (1.1.6) that $C_{\varphi}^{x}$is compact on L^t(**S**). The argument for $x\ = h$, using the alternate lower bounds provided by Lemmas (1.1.9) and (1.1.10), is similar. This completes the proof of the sufficiency and thus of the proposition.

We now turn to the compactness characterization for the case $t = 1$. We need some preliminary lemmas.

Lemma (1.2.2)[1]: $Let\ x \in \{c, h, m, p\}$and φ be as in (1.1.1). Assume, in addition, φ takes Sinto a compact subset of B. Then $C_\varphi^x$is compact on L^t(**S**)for each $1 \leq t < \infty$.

Proof: $Fix\ x \in \{c, h, m, p\}$and $1 \leq t < \infty$Since φ(**S**)is contained in a compact subset of **B**, it is easily that $A_{\varphi,t}^x$is bounded on **B**. So, $C_\varphi^x$is bounded on L^t(S)by Propositions (1.1.4) and (1.1.11). Now, using Lemma (1.1.2) , the rest of the proof is a standard normal family argument.

Lemma (1.2.3)[1]: $Let\ x \in \{c, h, m, p\}$and φ be as in (1.1.1), and define the function G^xon $\overline{\mathbf{B}}\{0\}by$

$$G_\varphi^x(\nu\eta) := \int_{\mathrm{S}} \left| \mathscr{K}^x\left(\nu\varphi(\zeta), \eta\right) - \mathscr{K}^x\left(\varphi(\zeta), \eta\right) \right| d\sigma(\zeta) \tag{21}$$

for$0 < \nu \leq 1\ and\ \eta \in S. Then, for\ 0 < s < 1,$

$$\left\|C_{s\varphi}^x - C_\varphi^x\right\|_{L^1(\mathrm{S})} \leq \limsup_{\mathrm{r}\to\text{-}1} \Big(\sup_{\eta\in\mathrm{S}} \left[G_\varphi^x(rs\eta) + G_\varphi^x(r\eta)\right] \Big).$$

Proof:$Fix\ 0 < s < 1.$Note that sφsatisfies (1.1.1), because $\sigma_0(s_\varphi)^{-1}|_s$is the zero measure. Given f∈L^1(**S**), using Fatou's Lemma, we have

$$\begin{aligned} \left\|\left(C_{s\varphi}^x - C_\varphi^x\right) f\right\|_1 &\leq \liminf_{\mathrm{r}\to\text{-}1} \int_{\mathrm{S}} \left| \int_{\mathrm{S}} f(\eta)\left[K^x\left(rs\varphi(\zeta), \eta\right) - \mathscr{K}^x\left(r\varphi(\zeta), \eta\right)\right. \right. \\ &\leq \|f\|_1 \limsup_{\mathrm{r}\to\text{-}1} \Big(\sup_{\eta\in\mathrm{S}} \left[G_\varphi^x(rs\eta) + G_\varphi^x(r\eta)\right] \Big), \end{aligned}$$

as required.

We remark that if $x \in \{c, h, p\}$, then (1.1.6)is available to give

$$G_\varphi^x(\nu\eta) = \int_{\mathrm{S}} \left| \mathscr{K}^x\left(\varphi(\zeta), \nu\eta\right) - \mathscr{K}^x\left(\varphi(\zeta), \eta\right) \right| d\sigma(\zeta). \tag{22}$$

Now shows that if $A^x_{\varphi,1}$∈C($\overline{\boldsymbol{B}}$), then G^x_{φ}∈C($\overline{\boldsymbol{B}}$/{0})and vanishes on S. Hence, for $x \in \{c,h,p\}$, $A^x_{\varphi,1} \in C$ ($\overline{\mathbf{B}}$)implies $\|C^c_{s\varphi} - C^c_{\varphi}\|_{L^1}$(S)→0as s →-1,Since each $C^x_{s_\varphi}$is compact on L^1(**S**), it follows that $C^x_{\Phi}$is as well.

These remarks do not extend to $x = m$, since (1.1.5)is not available in that case. the extension to$x = m$ is valid, though the proof is much more involved. The next two lemmas will be used in that proof.

Lemma (1.2.4)[1]: Let φ be as in (1.1.1). If $A^m_{\varphi,1}$∈C($\overline{\mathbf{B}}$), then C^c_{φ}:H^1(**S**) →L^1(**S**)is compact.

Proof: .As in the proof of Lemma (1.2.3), each sφ, $0 < s < 1$, satisfies (1.1.1)and $C^m_{s\varphi}$:H^1(S) →L^1(**S**)is compact by Lemma(1.2.4). So it suffices to show

$$\left\|C^m_{s\varphi} - C^m_{\varphi}\right\| \to 0 \quad \text{as } s \to \text{-}1$$

where$\|C^m_{s\varphi} - C^m_{s\varphi}\|$denote the operator norm acting from H^1(S)into L^1(S).

Let Q^mbe the function defined on $\overline{\mathbf{B}}$\{0}by

$$Q^m(\nu\eta) := \int_{\mathrm{S}} \left|\mathscr{K}^m(\varphi(\zeta),\nu\eta) - \mathscr{K}^m(\varphi(\zeta),\eta)\right| d\sigma(\zeta),$$

for**0 <v≤1**and $\eta \in S$. From the hypothesis that $A^m_{\varphi,1}$∈C($\overline{\boldsymbol{B}}$)and Lemma (1.1.8) , we see that $Q^m$is a continuous function vanishing on S. So, given ≥ 0, we can fix a v∈(0, 1)such that

$$\sup_{\eta\in\mathrm{S}} Q^m(\nu\eta) < \epsilon. \tag{23}$$

Let f∈H^1(**S**)and identify it with *f*∈H^1(**B**). *Let 0 <s <1*and put f_s(z) =f(sz).

Note $C^m_{s\varphi}f$=*f*(sφ) =$f_s(\varphi)$ =$C^m_{\varphi}f_s$for each $0 < s < 1$, because *f* is holomorphic. It follows from (1.1.7)that

$$\begin{aligned}(C^m_{s\varphi} - C^m_{\varphi})f(\zeta) &= C^m_{\varphi}(f_s - f)(\zeta)\\ &= \lim_{r\to\text{-}1} \int_{\mathrm{S}} \mathscr{K}^m(\varphi(\zeta), r\eta)\left[f_s(\eta) - f(\eta)\right] d\sigma(\eta)\end{aligned}$$

For σ-almost every ζ∈S. Thus, by Fatou's Lemma, we have

$$\|(C^m_{s\varphi} - C^m_{\varphi})f\|_{\substack{\\ r\to 1}} < \liminf_{\to 1^-} \int_{\mathbf{S}} \left| \int_{\mathbf{S}} \mathscr{K}^m(\varphi(\zeta), r\eta)\,[f_s(\eta) - f(\eta)]\, d\sigma(\eta) \right| d\sigma(\zeta). \tag{24}$$

Meanwhile, note from Fubini's Theorem

$$\begin{aligned}
&\int_{\mathbf{S}} \int_{\mathbf{S}} \left| \mathscr{K}^m(\varphi(\zeta), r\eta) - \mathscr{K}^m(\varphi(\zeta), \eta) \right| \left| f_s(\eta) - f(\eta) \right| d\sigma(\eta)\, d\sigma(\zeta) \\
&\qquad \le \|f_s - f\|_1 \left[\sup_{\eta \in \mathbf{S}} Q^m(r\eta) \right] \\
&\qquad \le 2\|f\|_1 \left[\sup_{\eta \in \mathbf{S}} Q^m(r\eta) \right] \to 0 \quad \text{as } r \to 1
\end{aligned}$$

Therefore, by the triangle inequality and Fubini's Theorem, for the v fixed above, we obtain from (1.2.6)

$$\begin{aligned}
\left\|(C^m_{s\varphi} - C^m_{\varphi})f\right\|_1 &\le \int_{\mathbf{S}} \left| \int_{\mathbf{S}} \mathscr{K}^m(\varphi(\zeta), \eta)\,[f_s(\eta) - f(\eta)]\, d\sigma(\eta) \right| d\sigma(\zeta) \\
&\le I + II,
\end{aligned} \tag{25}$$

where

$$I := \int_{\mathbf{S}} Q^m(\nu\eta) \left| f_s(\eta) - f(\eta) \right| d\sigma(\eta) < 2\epsilon \|f\|_1,$$

by (24), and

$$II := \int_{\mathbf{S}} \left| \int_{\mathbf{S}} \mathscr{K}^m(\varphi(\zeta), \nu\eta)\,[f_s(\eta) - f(\eta)]\, d\sigma(\eta) \right| d\sigma(\zeta).$$

To estimate II, we first note from the reproducing property

$$f(\nu s z) = \int_{\mathbf{S}} \mathscr{K}^m(z, \nu\eta) f_s(\eta)\, d\sigma(\eta), \quad z \in \mathbf{S},$$

because Km(z, vη) $=\overline{K^m(\nu\eta, z)}$ $=K^m$(vz, η)by (1). This also remains valid for **z**∈B. To see it, note from Fubini's Theorem, (1.1.1), and the reproducing property of the kernel that

$$\begin{aligned}
\int_{\mathbf{S}} \mathscr{K}^m(z, \nu\eta) f_s(\eta)\, d\sigma(\eta) &= \int_{\mathbf{S}} f_s(\eta) \left\{ \int_{\mathbf{S}} K^m(z, \xi) \overline{K^m(\nu\eta, \xi)}\, d\sigma(\xi) \right\} d\sigma(\eta) \\
&= \int_{\mathbf{S}} \left\{ \int_{\mathbf{S}} K^m(\nu\xi, \eta) f_s(\eta)\, d\sigma(\eta) \right\} K^m(z, \xi)\, d\sigma(\xi) \\
&= \int_{\mathbf{S}} f_{\nu s}(\xi) K^m(z, \xi)\, d\sigma(\xi) \\
&= f(\nu s z).
\end{aligned}$$

A similar argument yields

$$f(\nu sz) = \int_{\mathbf{S}} \mathscr{K}^m(sz, \nu\eta) f(\eta)\, d\sigma(\eta), \quad z \in \overline{\mathbf{B}}.$$

Therefore, we have

$$\int_{\mathbf{S}} \mathscr{K}^m\big(\varphi(\zeta), \nu\eta\big) f_s(\eta)\, d\sigma(\eta) = \int_{\mathbf{S}} \mathscr{K}^m\big(s\varphi(\zeta), \nu\eta\big) f(\eta)\, d\sigma(\eta)$$

atσ-almost every $\zeta \in S$and thus by Fubini's Theorem

$$\begin{aligned} II &\le \int_{\mathbf{S}}\int_{\mathbf{S}} \big|\mathscr{K}^m\big(s\varphi(\zeta), \nu\eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \nu\eta\big)\big| \cdot \big|f(\eta)\big|\, d\sigma(\eta)\, d\sigma(\zeta) \\ &\le \|f\|_1 \sup_{\eta\in\mathbf{S}} \int_{\mathbf{S}} \big|\mathscr{K}^m\big(s\varphi(\zeta), \nu\eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \nu\eta\big)\big|\, d\sigma(\zeta). \end{aligned}$$

Now, since 0 <v<1, the Dominated Convergence Theorem can be used to see the (uniform) continuity of the mapping (s, η) →K^m(sφ(・), vη)from [0, 1] ×S to L^1(**S**), yielding

$$II < \epsilon\|f\|_1,$$

provided sis sufficiently close to 1. Along with (25), this shows that $C_{s\varphi}^m - C_{\varphi}^m$boundedly takes H^1(**S**)into L^1(**S**)and, moreover, that (4)holds. The proof is complete.

Lemma (1.2.5)[1]: Let φ be as in (1.1.1)and put

$$\widetilde{G}^m(r\eta) := \int_{\mathbf{S}} \left| \frac{(1 - |\varphi(\zeta)|^2)^n}{|1 - \langle\varphi(\zeta), r\eta\rangle|^{2n}} - \mathscr{K}^m\big(\varphi(\zeta), \eta\big) \right| d\sigma(\zeta)$$

for$\mathbf{0} < r \le \mathbf{1}$and η∈S. If $A_{\varphi,1}^c$∈C($\overline{\boldsymbol{B}}$), then G^m∈C($\overline{B}$\{0}).

Proof: Assume A_{φ}^m∈C(**B**). Since $G^m$is clearly continuous on **B** \{0}by the Dominated Convergence Theorem, it suffices to prove $G^m$is continuous at every point in **S**, whereG^mvanishes. Given s ∈(0, 1)and $\eta \in S$, we use temporary notation

$$E_s(\eta) := \varphi^{-1}\big[S^m(\eta, s)\big] \cap \mathbf{S}$$

For short. Given$\eta \in S \text{ and } 0 < r < 1$, note

$$\frac{(1-|z|^2)^n}{|1-\langle z,r\eta\rangle|^{2n}} \le \frac{2^n}{(1-r)^n}, \quad \text{for } z\in\overline{\mathbb{B}},$$

and

$$\left|\frac{1-\langle z,r\eta\rangle}{1-\langle z,\eta\rangle}\right| \ge 1-\left|\frac{(1-r)\langle z,\eta\rangle}{1-\langle z,\eta\rangle}\right| \ge \frac{1}{2}, \quad \text{for } z\notin S^m\big(\eta,2(1-r)\big).$$

Therefore, given arbitrary η_0∈Sand 0 <δ<1, we have

$$\begin{aligned}\int\limits_{E_\delta(\eta_0)} \frac{(1-|\varphi(\zeta)|^2)^n}{|1-\langle\varphi(\zeta),r\eta\rangle|^{2n}}d\sigma(\zeta) &= \int\limits_{E_\delta(\eta_0)\setminus E_{2(1-r)(\eta)}} + \int\limits_{E_\delta(\eta_0)\cap E_{2(1-r)(\eta)}} \\ &\lesssim \int\limits_{E_\delta(\eta_0)} \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\,d\sigma(\zeta) + \frac{\sigma[E_{2(1-r)}(\eta)]}{(1-r)^n}.\end{aligned}$$

Thus, setting

$$I(\eta) := \int\limits_{\mathrm{S}} \left|\mathscr{K}^m\big(\varphi(\zeta),\eta\big) - \mathscr{K}^m\big(\varphi(\zeta),\eta_0\big)\right| d\sigma(\zeta)$$

we obtain

$$\begin{aligned}&\int\limits_{E_\delta(\eta_0)} \left|\frac{(1-|\varphi(\zeta)|^2)^n}{|1-\langle\varphi(\zeta),r\eta\rangle|^{2n}} - \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\right| d\sigma(\zeta) \\ &\lesssim \int\limits_{E_\delta(\eta_0)} \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\,d\sigma(\zeta) + \frac{\sigma[E_{2(1-r)}(\eta)]}{(1-r)^n} \\ &\lesssim \int\limits_{E_\delta(\eta_0)} \mathscr{K}^m\big(\varphi(\zeta),\eta_0\big)\,d\sigma(\zeta) + I(\eta) + M^m_{2(1-r)}\big(\sigma\circ\varphi^{-1}\big).\end{aligned}$$

Note that σ∘$\varphi^{-1}$is a compact m-Carleson measure, sinceC_φ^m:H[1](**S**) →L[1](**S**)is compact by Lemma (1.2.4). Thus the last term of the above tends to 0 uniformly in η as $r\to -1$. Meanwhile, since

$$\lim_{\eta\to\eta_0}\int\limits_{\mathrm{S}} \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\,d\sigma(\zeta) = \int\limits_{\mathrm{S}} \mathscr{K}^m\big(\varphi(\zeta),\eta_0\big)\,d\sigma(\zeta)$$

by the continuity of $A^m_{\varphi,1}$on S, we have I(η) →0as η→$\eta_0$by Lemma (1.1.8). Also, note

$$\lim_{r\eta\to\eta_0}\int\limits_{\mathrm{S}\setminus E_\delta(\eta_0)} \left|\frac{(1-|\varphi(\zeta)|^2)^n}{|1-\langle\varphi(\zeta),r\eta\rangle|^{2n}} - \mathscr{K}^m\big(\varphi(\zeta),\eta\big)\right| d\sigma(\zeta) = 0$$

by the Dominated Convergence Theorem. It follows from these observations that

$$\limsup_{r\eta\to\eta_0, 0<r<1} \widetilde{G}^m(r\eta) \lesssim \int_{E_\delta(\eta_0)} \mathscr{K}^m(\varphi(\zeta),\eta_0)\, d\sigma(\zeta).$$

In the display above note that the right-hand tends to 0 as δ→0^+, because $\|\chi^m(\varphi(\cdot)\eta 0\|$=$A^m_{\varphi,1}(\eta_0)$ <∞. Since the left-hand side is independent of δ, we conclude that

$$\limsup_{r\eta\to\eta_0, 0<r<1} \widetilde{G}^m(r\eta) = 0 = \widetilde{G}^m(\eta_0).$$

Hence $G^m$is continuous at every point in **S** as required, and the proof is complete. We are now ready to characterize the compactness for the case t =1.

Proposition (1.2.6)[1]: $Let\ x\ \in \{c,h,m,p\}$and φ be as in (1.1.1). Then $c^x_\varphi$is compact on L^1(**S**)if and only if $A^m_{\varphi,1}$∈C(**B**).

Proof: We first prove the necessity. So, suppose $c^x_\varphi$is compact on L^1(**S**)and let $\{w_j\}$be a sequence of points in B with $w_j \to w_0$. To show $A^m_{\varphi,1}$∈C(**B**), it suffices to show that there is a subsequence $\{w_{j_k}\}$such that $A^m_{\varphi,1}(w_{j_k})$→$A^m_{\varphi,1}(w_0)$. Let k_w=K^h(w, ·)for x ∈{c, p, h}and k_w=K^m(w, ·)for x =m. Then $\|k^x{}_{w1}\|$=1for w∈**B**and $\|k_{w1}\|$=0for w∈S. Thus $\{w_{j_k}\}$is a bounded sequence in L^1(S), and since $c^x_\varphi$is compact it follows that $\{c^x_\varphi w_{j_k}\}$ =$\{\chi^x(\varphi(\cdot), w_j)\}$has a subsequence $\{\chi^x(\varphi(\cdot), w_{j_k})\}$that converges in norm. Since $k^x(\varphi(\cdot), w_j) \to \chi^x(\varphi(\cdot), w_0)$σ-a.e., it follows that $\chi^x(\varphi(\cdot), w_{j_k}) \to \chi^x(\varphi(\cdot), w_0)$in norm. Hence,

$$A^x_{\varphi,1}(w^{j_k}) = \left\|\mathscr{K}^x(\varphi(\cdot), w^{j_k})\right\|_1 \to \left\|\mathscr{K}^x(\varphi(\cdot), w_0)\right\|_1 = A^x_{\varphi,1}(w_0)$$

as required, and this completes the proof that $A^m_{\varphi,1}$∈C($\overline{\boldsymbol{B}}$).Sufficiency has already been proved for $x\ \in \{c,h,p\}$in the remarks following the proof of Lemma(1.2.3). So for the rest of the proof we put x =m, and assume $A^m_{\varphi,1}$∈C($\overline{\boldsymbol{B}}$). As in those remarks, it suffices to show that

$$\left\|C^m_{s\varphi} - C^m_{\varphi}\right\|_{L^1(S)} \to 0 \quad \text{as } s \to \text{-}1$$

From Lemma (1.2.3) , it suffices to show

$$\lim_{s \to -1} \left(\sup_{\eta \in \mathbf{S}} G_{\varphi}^{m}(s\eta) \right) = 0, \tag{26}$$

where$G_{\varphi}^{m}$was defined in (4). Let z∈B and η∈S. When s <1, $\chi^{m}(s_z$, η) =$K_m(s_z$, η), so the explicit formula of the kernel can be used. For s ∈[0, 1)we have

$$\begin{aligned} K^m(sz,\eta) &= \frac{(1-|sz|^2)^n}{|1-\langle sz,\eta\rangle|^{2n}} \\ &= \frac{[(1-|z|^2)+|z|^2(1-s^2)]^n}{|1-\langle z,s\eta\rangle|^{2n}} \\ &= \frac{(1-|z|^2)^n}{|1-\langle z,s\eta\rangle|^{2n}} + \mathcal{O}(1)\cdot\sum_{k=1}^{n}\frac{(1-s)^k(1-|z|^2)^{n-k}}{|1-\langle z,s\eta\rangle|^{2n}} \end{aligned}$$

where O(1)is uniform in z and *s*. Thus

$$\left| K^m(sz,\eta) - \frac{(1-|z|^2)^n}{|1-\langle z,s\eta\rangle|^{2n}} \right| \lesssim \sum_{k=1}^{n} \frac{(1-s)^k}{|1-\langle z,s\eta\rangle|^{n+k}}, \quad z \in \overline{\mathbf{B}}.$$

Hence, setting

$$g_{a,k} := \frac{(1-|a|)^k}{(1-\langle\,\cdot\,,a\rangle)^{n+k}}, \quad a \in \mathbf{B},\ k = 1,2,\ldots,n$$

we obtain

$$G_{\varphi}^{m}(s\eta) \lesssim \widetilde{G}_{\varphi}^{m}(s\eta) + \sum_{k=1}^{n} \left\| C_{\varphi}^{m} g_{s\eta,k} \right\|_1.$$

The function $G_{\varphi}^{m}$was introduced in Lemma (1.2.5), where it was shown that $G_{\varphi}^{m}(s_\eta) \to 0$ as $s \to -1$uniformly in η. So, to complete the proof, it suffices to show the sum in the right-hand side of the display above converges to 0uniformly in η as $s \to -1$.Note that {$g_{s\eta}$,k}is a bounded set in H^1(**S**), note that, given $a^j \to \eta_0 \in S$, {$C_{\varphi}^{m}$$g_{aj}$,k} ={$g_{aj}$,k$\circ \phi$}(with kfixed) converges pointwise to 0on S \$\varphi^{-1}${η0}and hence σ-a.e. by (1.1.1). Since C_{φ}^{m}:H^1(S) →L^1(S)is compact by Lemma (1.2.4), a sub sequence converges to 0 in norm. It follows that

$$\lim_{s \to 1^-} \sup_{\eta \in \mathbf{S}} \sum_{k=1}^{n} \left\| C_{\varphi}^{m} g_{s\eta,k} \right\|_1 = 0,$$

as desired. This completes the proof. Having proved the compactness characterizations, we now prove the following, which is the content of the compactness.

Proposition (1.2.7)[1]: *Let* $x \in \{c, h, m, p\}, 1 < t < \infty$ and φ be as in (1.1.1). If C_φ^x is compacton $L^1(\mathbf{S})$, then C_φ^x is compact on $L^t(\mathbf{S})$.

Proof: As in the proof of Proposition(1.1.9), the case $x \neq h$ is contained in Theorem (1.1.10) (c). For $x = h$, from Proposition (1.2.6) and the hypothesis that C_φ^h is compact on $L^1(\mathbf{S})$ we havethat $A_{\varphi,1}^h \in C(\overline{\boldsymbol{B}})$. So by Lemma (1.1.7) it follows that G_φ^h, defined in (1.2.3) , is continuous and vanishes on **S**. Thus Lemma (1.2.3) shows that given $\epsilon>0$, there exists $\delta > 0$ such that

$$\left\|C_{s\varphi}^h - C_\varphi^h\right\|_{L^1(\mathrm{S})} \leq \epsilon, \quad \text{for all } s \in (1-\delta, 1).$$

Also, the operator $C_{s\varphi}^h - C_\varphi^h$ clearly acts boundedly on $L^\infty(\mathbf{S})$ with

$$\left\|C_{s\varphi}^h - C_\varphi^h\right\|_{L^\infty(\mathrm{S})} \leq 2.$$

It now follows from the RieszThorin Interpolation Theorem that

$$\left\|C_{s\varphi}^h - C_\varphi^h\right\|_{L^t(\mathrm{S})} \leq 2^{1-1/t}\epsilon^{1/t}, \quad \text{for all } s \in (1-\delta, 1);$$

the slightly different norm estimate implicit in the proof of the Marcinkiewicz Interpolation could also be used. (An alternate approach, available since C_φ^h is an integral operator, is to use Schur's test Hence

$$\lim_{s \to 1^-} \left\|C_{s\varphi}^h - C_\varphi^h\right\|_{L^t(\mathrm{S})} = 0.$$

Since each $C_{s\varphi}^h$, 0 <s <1is compact on L^t(S), C_φ^h is also. The proof is complete.

Applying our compactness characterization, we can show by explicit examples that L^t compactness for each $1 < t < \infty$ may not imply L^1 boundedness. This, in particular, shows that the converse of Proposition (1.1.6) does not hold.

Proposition (1.2.8)[1]: For each $x \in \{c,h,m,p\}$there exists ϕ^xsuch that $C^h_{\varphi x}$is compact on $L^t(\mathbf{S})$, $1 < t < \infty$, but $C^h_{\varphi x}$is not bounded on $L^1(\mathbf{S})$.We fix as a standard reference point

$$\mathrm{e} := (1,0,\ldots,0) \in \mathbf{S}. \tag{27}$$

Proof: Put

$$d_x(\zeta) := \begin{cases} |1-\langle \mathrm{e},\zeta\rangle| & \text{if } x = c,p,m \\ |\mathrm{e}-\zeta| & \text{if } x = h, \end{cases} \qquad \zeta \in \mathbf{S}$$

and

$$h_x(s) := \begin{cases} s|\log s|^{1/n} & \text{if } x = c,p,m \\ s|\log s|^{1/(2n-1)} & \text{if } x = h, \end{cases} \qquad 0 \le s < \epsilon$$

where∈>0is a sufficiently small number chosen so that $0 \le h_x(s) \le 1$. Given $x \in \{c,h,m,p\}$, put $V_x = S^x(e, \in) \cap \mathbf{S}$and define

$$\varphi_x(\zeta) := \big(1-(h_x \circ d_x)(\zeta),\, 0,\ldots,0\big)\chi_{V_x}(\zeta)$$

Where V_x denotes the characteristic function of V_x. Clearly, this function satisfies (1.1.1).We show that $C^h_{\varphi x}$is not bounded on $L^1(\mathbf{S})$, only for the case x =c; the proofs for other cases are similar and thus omitted. By Fatou's Lemma $A^c_{\varphi c,1}$(e) $\le lim_{r\to -1}$inf$A^c_{\varphi c,1}$(re), and so by Proposition(1.1.7).it suffices to prove that $A^c_{\varphi c,1}$(e) =∞. Note

$$\left|\mathscr{K}^c(\varphi_c(\zeta),\mathrm{e})\right| = \frac{1}{|1-\langle \mathrm{e},\zeta\rangle|^n}\left(\log\frac{1}{|1-\langle \mathrm{e},\zeta\rangle|}\right)^{-1}, \quad \zeta \in V_c$$

and so

$$A^c_{\varphi_c,1}(\mathrm{e}) = \int_{V_c} \frac{1}{|1-\langle \mathrm{e},\zeta\rangle|^n}\left(\log\frac{1}{|1-\langle \mathrm{e},\zeta\rangle|}\right)^{-1} d\sigma(\zeta).$$

It is elementary to see that this integral diverges when n =1., So, $C^c_{\varphi c}$is not bounded on $L^1(\mathbf{S})$.

Now, we $let\ 1 < t < \infty$and show that each $C^x_{\varphi x}$is compact on $L^t(\mathbf{S})$. By Lemma (1.1.6) it suffices to show that $\sigma \circ \varphi_x^{-1}$is a compact X Carleson measure. It is easy to see

$$\left|1-\left\langle \mathbf{e},\varphi_x(\zeta)\right\rangle\right| \le \left|1-\left\langle \eta,\varphi_x(\zeta)\right\rangle\right|$$

for any ζ, η∈S. Thus we have

$$\varphi_x^{-1}\left(S^x(\eta,\delta)\right) \subset \varphi_x^{-1}\left(S^x(\mathbf{e},\delta)\right), \quad \eta \in \mathbf{S},\ 0<\delta<1$$

and consequently it suffices to consider the Carleson sets S^x(e, δ).Continuing under the assumption that x =c, the other cases being similar, note that $h_c$is invertible (when) is sufficiently small). Thus, for $\zeta \in V_c$, we see that φ_c(ζ) ∈S^c(e, δ)if and only if dc(ζ) <h_c^{-1}(δ). Thus ϕ_c^{-1}[S^c(e, δ)] =S^c(e, h_c^{-1}(δ))for all δ sufficiently small. Hence

$$\frac{(\sigma\circ\varphi_c^{-1})[S^c(\mathbf{e},\delta)]}{\sigma[S^c(\mathbf{e},\delta)]} \approx \left(\frac{h_c^{-1}(\delta)}{\delta}\right)^n \to 0 \quad \text{as } \delta \to 0^+,$$

and so σ∘$\varphi^{-1}$is a compact C Carleson measure. This completes the proof. As another consequence of our compactness characterization, we can easily recover Sarason's result. Recall that (1.1.1)is satisfied by all holomorphic self-maps of **D**.

Corollary (1.2.10)[1]: Let φ be a holomorphic self-map of Dsuch that φ(0) =0. Then $C_\varphi^h$is compact on L^1(∂D)if and only if $A_{\varphi,1}^h$(η) =1for all η∈∂D.

Proof: Note that χ^h(φ(·), z)is a bounded harmonic function on **D** for each z∈D. We thus have

$$\Lambda_{\varphi,1}^h(z) = \mathscr{K}^h\left(\varphi(0),z\right) = \mathscr{K}^h(0,z) = 1$$

for all z∈D. So, the corollary is immediate from Proposition (1.2.6).if x, y∈{c, p, m}, t >1, and $C_\varphi^x$is bounded (respectively compact) on L^t(S), then $C_\varphi^y$is bounded (compact) on L^t(S). Also, if $C_\varphi^h$is bounded (respectively compact) on L^t(S), **1 <t <∞,** then $C_\varphi^x$is bounded (compact) on L^t(S)for$x \in \{c,p,m\}$. Example (2) below shows that the converse fails badly: C_φ^x, x ∈ $\{c,p,m\}$, can be compact while $C_\varphi^h$is not bounded.Carleson measure methods were used to get these results for t >1. When t =1, Carleson measure methods are not available and the situation is quite different. In this part we consider the 12 implications of the type If $C_\varphi^h$is boundedon L1(S), then $C_\varphi^y$is boundedon L1(S),where$x \neq y$. We show that 2 of these implications hold, while 9 of the remaining 10 fail. In fact we show that 7 of these fail badly, in that C_φ^xcan be compact while $C_\varphi^y$is not bounded. The last cases that we were not able to resolve will be stated as questions at the end of this part. We begin with the two implications that do hold:

Proposition (1.2.11)[1]: Let φbe as in (1.1.1). If $C_\varphi^c$is bounded (respectively compact) on L1(S), then $C_\varphi^x$is bounded (compact) on L1(S)for$x = p, m$.

Proof: We first consider $x = p$. We have χ^p=2 Re$\chi^{c-1}$and so the boundedness. For compactness, if $z, w \in \overline{\boldsymbol{B}}$, then

$$\left\|\mathscr{K}^p(\varphi(\cdot), z) - \mathscr{K}^p(\varphi(\cdot), w)\right\|_1 \leq 2\left\|\mathscr{K}^c(\varphi(\cdot), z) - \mathscr{K}^c(\varphi(\cdot), w)\right\|_1.$$

Proposition(1.2.7)[1]now shows that $A_{\varphi,1}^c$∈C($\overline{\boldsymbol{B}}$)implies $A_{\varphi,1}^p$∈C($\overline{\boldsymbol{B}}$), and so the statement for compactness follows from Proposition (1.2.6).We now consider the boundedness for x =m. Let z∈B. Since χ^m(φ(ζ), η) ≤2^n|K^c(φ(ζ), η)|, we have by Fubini's Theorem and Fatou's Lemma

$$\begin{aligned}\left\|\mathscr{K}^m(\varphi(\cdot), z)\right\|_1 &= \int\limits_{\mathrm{S}}\left\{\int\limits_{\mathrm{S}} K^m(\varphi(\zeta), \eta)\, d\sigma(\zeta)\right\} K^m(z, \eta)\, d\sigma(\eta) \\ &\leq 2^n \liminf_{r\to 1} \int\limits_{\mathrm{S}}\left\{\int\limits_{\mathrm{S}} \left|\mathscr{K}^c(\varphi(\zeta), r\eta)\right| d\sigma(\zeta)\right\} K^m(z, \eta)\, d\sigma(\eta \\ &\leq 2^n \sup_{w\in \mathrm{B}} \left\|\mathscr{K}^c(\varphi(\cdot), w)\right\|_1 \int\limits_{\mathrm{S}} K^m(z, \eta)\, d\sigma(\eta).\end{aligned}$$

Since the last integral above is equal to 1, this shows that $A_{\varphi,1}^m$is bounded by 2$^n A_{\varphi,1}^c$. Hence the result for boundedness follows from Proposition (1.2.6)Finally, we consider the compactness for $x = m$. Assume that $c_\varphi^c$is compact on L1(**S**), or equivalently by Proposition (1.2.4) that $A_{\varphi,1}^c$is continuous on **B**. Now by Proposition (1.2.6) again, to see that $C_\varphi^m$is compact, it suffices to show that $A_{\varphi,1}^m$is continuous on **B**. Since $A_{\varphi,1}^m$is continuous on B, it suffices to show that it is radially uniformly continuous. Let ≥0and η∈S. there is $\delta >$ 0independent of $\eta \in S$such that for all $\xi \in S^m(\eta, \delta)$

$$\int_{\mathbf{S}} \left| \left| K^c\big(\varphi(\zeta), \eta\big) \right| - \left| K^c\big(\varphi(\zeta), \xi\big) \right| \right| d\sigma(\zeta) < \epsilon, \tag{28}$$

since$A^m_{\varphi,1}$is (uniformly) continuous on $\overline{\boldsymbol{B}}$.

Note

$$\begin{aligned} |K^m(z,\eta) - K^m(z,\xi)| &= (1-|z|^2)^n \left| |K^c(z,\eta)|^2 - |K^c(z,\xi)|^2 \right| \\ &\leq 2^{n+1} \left| |K^c(z,\eta)| - |K^c(z,\xi)| \right| \end{aligned}$$

For ζ∈S and z∈B. With K^mreplaced by χ^mthis estimate extends to all z∈S, since χ^m(z, η) =0when z∈S.

$$\int_{\mathbf{S}} \left| \mathscr{K}^m\big(\varphi(\zeta), \eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \xi\big) \right| d\sigma(\zeta) < 2^{n+1}\epsilon \tag{29}$$

For ξ∈S^m(η, δ). $Let\ 0\ < r < 1$. Since $f_s\ \chi^m$(rη, ξ) dσ(ξ) =1

$$\int_{\mathbf{S}} \left| \mathscr{K}^m\big(\varphi(\zeta), \eta\big) - \mathscr{K}^m\big(\varphi(\zeta), \xi\big) \right| d\sigma(\zeta) < 2^{n+1}\epsilon \tag{30}$$

Thus, setting M=sup$_{z\in B}A^m_{\varphi,1}$ (z), we obtain by Fubini's Theorem and(29)

$$\begin{aligned} &\left| A^m_{\varphi,1}(r\eta) - A^m_{\varphi,1}(\eta) \right| \\ &\quad \leq \int_{\mathbf{S}} \left\{ \int_{\mathbf{S}} \left| \mathscr{K}^m\big(\varphi(\zeta), \xi\big) - \mathscr{K}^m\big(\varphi(\zeta), \eta\big) \right| d\sigma(\zeta) \right\} K^m(r\eta, \xi)\, d\sigma(\xi) \\ &\quad \leq 2^{n+1}\epsilon \int_{S^m(\eta,\delta)} K^m(r\eta,\xi)\, d\sigma(\xi) + 2M \int_{\mathbf{S}\setminus S^m(\eta,\delta)} K^m(r\eta,\xi)\, d\sigma(\xi) \\ &\quad \leq 2^{n+1}\epsilon + 2M \frac{(1-r^2)^n}{\delta^{2n}}. \end{aligned}$$

Therefore we have

$$\left| A^m_{\varphi,1}(r\eta) - A^m_{\varphi,1}(\eta) \right| < \big(2^{n+1}+1\big)\epsilon, \quad \eta \in \mathbf{S}$$

for all r sufficiently close to 1. The proof is complete.

We now turn to the examples that demonstrate the failures of the implications dis-cussed at the beginning of this part. Recall C^p_φ=C^h_φin the one-dimensional case. So, the restriction $n\ \geq$

2in the next example is required, by Theorem (1.1.1) (b) for $1 < t < \infty$, and by Proposition (1.2.10) for $t = 1$. In what follows, $e \in S$denotes the point specified in (1.1.7).

Example (2): $Let\ 1 \leq t < \infty, x \in \{c, p, m\}$, and $n \geq 2$. Then there exists φ such that $C_\varphi^x$is compact on L^t(S), but $C_\varphi^h$is not bounded on L^t(S).

Proof:First, consider the case $1 < t < \infty$. In this case we use the Carleson measure characterizations. $Fix\ \frac{n}{n-1} < a < 1$and let

$$\varphi(\zeta) := \left(L_a(\zeta_1), 0, \ldots, 0\right)$$

where $L_a(\lambda) = 1 - (1 - \lambda)^a$, $\lambda \in D$. We remark that $L_a$is a conformal map of Donto a teardrop-shaped region in D with vertex . Since $|\varphi(\zeta) - e| = |1 - \zeta_1|^a = |1\ \langle \varphi(\zeta),, e\rangle|$, we have

$$\begin{aligned} \left(\sigma \circ \varphi^{-1}\right)\left[S^x(\mathbf{e}, \delta)\right] &= \left(\sigma \circ \varphi^{-1}\right)\left[S^h(\mathbf{e}, \delta)\right] \\ &= \sigma\left\{\zeta \in \mathbf{S} : |1 - \zeta_1| < \delta^{1/a}\right\} \\ &\approx \delta^{n/a} \end{aligned}$$

Accordingly, using (), we obtain

$$M_\delta^h\left(\sigma \circ \varphi^{-1}\right) \gtrsim \frac{\delta^{n/a}}{\delta^{2n-1}} \to \infty \quad \text{as } \delta \to 0^+,$$

which, together with Lemma (1.2.7), shows that $C_\varphi^h$is not bounded on L^t(S). On the other hand, using ,we

$$\frac{(\sigma \circ \varphi^{-1})[S^x(\mathbf{e}, \delta)]}{\sigma[S^x(\mathbf{e}, \delta)]} \approx \frac{\delta^{n/a}}{\delta^n} \to 0 \quad \text{as } \delta \to 0^+.$$

As in the proof of Proposition (1.2.8), this shows that $C_\varphi^x$is compact on L^t(S). This completes the proof for t >1.

Now, we consider the case t =1. This time we let

$$\varphi(\zeta) := \left(1 - h(\operatorname{Re} \zeta_1), 0, \ldots, 0\right)$$

where $h(s) = (1 - s)^{1/2}$for $0 \leq s \leq 1$and $h(s) = 1$otherwise. Again, φclearly satisfies (1.1.2).

$$A^h_{\varphi,1}(\mathrm{e}) \approx \int_{-1}^{1} \frac{(1-s^2)^{n-3/2}}{h^{2n-1}(s)}\, ds \approx 1 + \int_{0}^{1} \frac{ds}{1-s} = \infty.$$

Thus we see from Proposition(1.2.1)and Fatou's Lemma that $C^h_\varphi$is not bounded on L[1](S).

To prove that $C^x_\varphi$is compact on L[1](**S**), we only need to consider for x =c by Proposition (1.1.3). Since φ(S)touches Sonly at e, it suffices to show that $A^c_{\varphi,1}$is continuous at e by Proposition (1.2.6). Let $w \in \overline{B}$. Since $Re(1 - \overline{w}_1) \geq 0$, we have

$$\left|1 - \langle \varphi(\zeta), w\rangle\right| = \left|h(\operatorname{Re}\zeta_1) + (1-\overline{w}_1)\big(1 - h(\operatorname{Re}\zeta_1)\big)\right| \geq h(\operatorname{Re}\zeta_1)$$

So that

$$\left|\mathscr{K}^c\big(\varphi(\zeta), w\big)\right| \leq \frac{1}{|1-\operatorname{Re}\zeta_1|^{n/2}}, \qquad \zeta \in \mathbf{S},\ w \in \overline{\mathbf{B}}.$$

Note

$$\int_{\mathrm{S}} \frac{d\sigma(\zeta)}{|1-\operatorname{Re}\zeta_1|^{n/2}} \approx 1 + \int_{0}^{1} \frac{(1-s)^{n-3/2}}{(1-s)^{n/2}}\, ds < \infty;$$

this is where the restriction $n \geq 2$ comes into play. Thus we conclude via the Dom-inated Convergence Theorem that $A^c_{\varphi,1}$is continuous at e, as required. The proof is complete.

Next we give a simple example that shows and C^m_φmay be bounded on L[1](S), while $C^p_\varphi$is not. Recall that iddenotes the identity map of S.

Example (3): For $n \geq 2$, Chidand Cmidare bounded on L[1](S), but C cid and C pit are not bounded on L[1](S).

Proof:$C^h_{id}$and $C^m_{id}$are simply the identity operator on L[1](S)by Proposition (1.1.1) (b), and so bounded. That $C^p_{id}$is not bounded is a consequence , and it follows from Proposition (1.1.1)that C_1disc not bounded. The restriction $n \geq 2$in the next two examples is required, since C^p_φ=$C^m_\varphi$when $n = 1$. Also, when $n = 1$, Cpidis bounded on L[1](S)but $C^c_{id}$is not, as discussed above. But a different example is required to differentiate between the behavior of C_φ^p and C_φ^c on L¹(**S**)when $n \geq 2$, since in that case C_{id}^p is not bounded.

Example (4): Let n ≥2. Then there is φ satisfying (1.1.1)such that C_φ^m is compact on L¹(S), but C_φ^c and C_φ^p are not bounded on L¹(S).In the proof below we will use the non-isotropic triangle inequality.

$$\left|1-\langle z^1,z^2\rangle\right|^{1/2} \leq \left|1-\langle z^1,z^3\rangle\right|^{1/2} + \left|1-\langle z^3,z^2\rangle\right|^{1/2} \tag{31}$$

valid for all z¹, z², z³∈B.

Proof: Define a sequence $\{q_k\} \subset \overline{\boldsymbol{B}}$ by

$$q_k = \left(1-\frac{1}{2^k}, \sqrt{\frac{2}{2^k}-\frac{1}{2^{2k}}-\frac{1}{2^k(\log k)^2}},\ 0,\ldots,0\right), \quad k=2,3,\ldots$$

so that

$$\left|1-\langle q_k,\mathrm{e}\rangle\right| = \frac{1}{2^k} \quad \text{and} \quad 1-|q_k|^2 = \frac{1}{2^k(\log k)^2}. \tag{32}$$

Let $\{E_k\}_{k=2}^{\infty}$ be a partition of Sinto Borel sets E_k such that σ(E_k) $=\frac{\text{cn}}{k2^{nk}}$, where (cn)⁻¹=$\sum_{k=2}^{\infty}\frac{1}{k2^{nk}}$. Denote by x_k the characteristic function of E_k and define $:S \rightarrow \boldsymbol{B}$ by $\varphi = \sum_{k=2}^{\infty}$ qkχx. Then φ is a Borel function and

$$\sigma \circ \varphi^{-1} = c_n \sum_{k=2}^{\infty} \frac{\delta_k}{k2^{nk}},$$

where δ_k is the point mass at q_k. Clearly φ satisfies (1.1.1)since φ^{-1}(**S**) =∅. Also, note

$$\Lambda_{\varphi,1}^{x}(w) = c_n \sum_{k=2}^{\infty} \frac{1}{k2^{nk}} \left|\mathscr{K}^x(q_k,w)\right|, \quad w \in \overline{\mathrm{B}} \tag{33}$$

for any x.

$$\Lambda_{\varphi,1}^{p}(\mathrm{e}) + 1 \geq c_n \sum_{k=2}^{\infty} \frac{1}{k2^{nk}} \frac{2}{|1-\langle q_k,\mathrm{e}\rangle|^n} = c_n \sum_{k=2}^{\infty} \frac{2}{k} = \infty. \tag{34}$$

Thus, from Fatou's Lemma and Proposition (1.2.6), $C_{\varphi}^{p}$is not bounded on L[1](S)which in turn implies$C_{\varphi}^{c}$is not bounded on L[1](**S**)by Proposition (1.2.7).We now turn to the proof that$C_{\varphi}^{m}$is compact on L[1](S). Since $q_{k\to e}$, it follows from (1.1.7)that the sequence $\{\chi^{m}(q_k, \cdot)\}_k$is uniformly bounded on each compact subset of $\overline{\mathbf{B}}$\{e}. Accordingly, for $x = m$ converges uniformly on each compact subset of $\overline{\mathbf{B}}$\{\{e}. So,$A_{\varphi,1}^{m}$is continuous on $\overline{B}$\{e}, because each χ^{m}(q$_k$, $\cdot$)is continuous on B. For this it is enough to show that there is a constant $C = C(n) > 0$ such that

$$C\left|\Lambda_{\varphi,1}^{m}(w)-\Lambda_{\varphi,1}^{m}(\mathrm{e})\right| \leq \frac{1}{\log M}+\sum_{k=2}^{M-1} \frac{\left|\mathscr{K}^{m}(q_k, w)-\mathscr{K}^{m}(q_k, \mathrm{e})\right|}{k 2^{nk}} \tag{35}$$

for any integers M, N with $N-2 \geq M \geq 3$and $w \in S_m$(e, 2$_{-N}$).Let $M \geq 3$be a given positive integer. As a preliminary step towards (34), we need certain estimate for the series$\sum_{k=M}^{\infty} \frac{\chi^{m}(qk.w)}{k2^{nk}}$. First, we show that there is a constant $C = C(n) > 0$such that

$$\sum_{k=M}^{\ell-2} \frac{\mathscr{K}^{m}(q_k, w)}{k 2^{nk}} \leq \frac{C}{\log M}, \quad w \in S^{m}\left(\mathrm{e}, 2^{-\ell}\right) \tag{36}$$

for $\ell \geq M+2$. To see this, for $M \leq k \leq \ell-2$and $w \in S_m$(e, $2^{-\ell}$), note by

$$\left|1-\langle q_k, w\rangle\right| \geq \frac{1}{2}\left|1-\langle q_k, \mathrm{e}\rangle\right|-\left|1-\langle w, \mathrm{e}\rangle\right| \geq \frac{1}{2^{k+1}}-\frac{1}{2^{\ell}} \geq \frac{1}{2^{k+2}}$$

So we get that

$$\mathscr{K}^{m}(q_k, w) \lesssim \frac{(1-|q_k|^2)^n+(1-|w|^2)^n}{|1-\langle q_k, w\rangle|^{2n}} \lesssim 2^{2kn}\left[\frac{1}{2^{kn}(\log k)^{2n}}+\frac{1}{2^{\ell n}}\right].$$

there is a constant $C = C(n) > 0$such that

$$\mathscr{K}^{m}(q_k, w) \lesssim \frac{1}{(1-|q_k|^2)^n}=2^{nk}(\log k)^{2n},$$

for all integers $\ell \geq 2$. Finally, we show that there is a constant $C = C(n) > 0$such that

$$\sum_{k=\ell-1}^{\ell+2} \frac{\mathscr{K}^m(q_k, w)}{k2^{nk}} \leq C\frac{(\log \ell)^{2n}}{\ell}, \quad w \in \overline{\mathbf{B}} \tag{37}$$

for positive integers. To see this, for $k \geq \lambda + 3 and\ w/\in S_m(e, 2^{\ell-1})$, note by (37)

$$\sum_{k=\ell+3}^{\infty} \frac{\mathscr{K}^m(q_k, w)}{k2^{nk}} \leq \frac{C}{\ell}, \quad w \notin S^m(\mathbf{e}, 2^{-\ell-1})$$

so that, again using (17),

$$\mathscr{K}^m(q_k, w) \lesssim \frac{2^n}{|1 - \langle q_k, w\rangle|^n} \leq 2^{4n+\ell n}.$$

Since $\sum_{k=\ell 3}^{\infty} 2^{nk} / (k2^{nk}) \leq 1/\ell$ this yields (1).

. Since

$$\sum_{k=M}^{\infty} \frac{\mathscr{K}^m(q_k, \mathrm{e})}{k2^{nk}} = \sum_{k=M}^{\infty} \frac{1}{k(\log k)^{2n}} \lesssim \frac{1}{\log M},$$

we have

$$\begin{aligned}
&\left|\Lambda^m_{\varphi,1}(w) - \Lambda^m_{\varphi,1}(\mathrm{e})\right| \\
&\lesssim \sum_{k=2}^{\infty} \frac{|\mathscr{K}^m(q_k, w) - \mathscr{K}^m(q_k, \mathrm{e})|}{k2^{nk}} \\
&= \sum_{k=2}^{M-1} \frac{|\mathscr{K}^m(q_k, w) - \mathscr{K}^m(q_k, \mathrm{e})|}{k2^{nk}} + \sum_{k=M}^{\infty} \frac{\mathscr{K}^m(q_k, w)}{k2^{nk}} + \frac{1}{\log M}
\end{aligned} \tag{38}$$

for $w \in B$. Let Nbe a given positive integer with N≥M+2and fix w∈S_m(e, 2^{-N}). Choose ℓ≥N such that w∈S^m(e, - ℓ) \Sm(e, 2 ℓ1). Then we see from (35), (36)and (37)that the section and term of the above is dominated by some constant (depending only on n) times

$$\frac{1}{\log M} + \frac{(\log \ell)^{2n}}{\ell} + \frac{1}{\ell} \lesssim \frac{1}{\log M} + \frac{(\log M)^{2n}}{M} + \frac{1}{M} \lesssim \frac{1}{\log M};$$

the constants suppressed in these estimates are independent of M and N. From this and (37)we conclude (36), as asserted. The proof is complete.

Example (6): Let n ≥2. Then there is φ such that C_φ^p is compact on L[1](S) but C_φ^c and C_φ^m are not bounded on L[1](**S**).

Proof: Define a sequence $\{a_k\}$ of complex numbers by

$$a_k = 1 - \frac{e^{i\pi/2n}}{2^k}, \quad k = 1, 2, \ldots$$

Also, since

$$|1 - a_k| = \frac{1}{2^k} \quad \text{and} \quad \mathrm{Re}(1 - a_k)^n = 0.$$

we have

$$\frac{1 - |a_k|^2}{2} = \frac{1}{2^k}\left(\cos\frac{\pi}{2n} - \frac{1}{2^{k+1}}\right),$$

We have

$$1 - |a_k| \approx \frac{1}{2^k}, \quad k = 1, 2, \ldots; \tag{39}$$

it is this step where the restriction $n \geq 2$ comes into play. This in particular shows $\{a_k\} \subset$D.

Now, as in the proof of Example(4), take a Borel function φ :S →B such that

$$A_{\varphi,1}^x(w) = c_n \sum_{k=1}^{\infty} \frac{1}{k2^{nk}} \left|\mathscr{K}^x(a_k\mathbf{e}, w)\right|, \quad w \in \overline{\mathbf{B}} \tag{40}$$

for any x; this time we take $c_n = \sum_{k=1}^{\infty} \frac{1}{k2^{nk}}^{-1}$. It is easily checked that $A_{\varphi,1}^m(e)$ =∞. So, as in the proof of Example(6) C_φ^m is not bounded on L[1](S), which in turn implies C_φ^c is not bounded on L[1](**S**).

Turning to the proof that C_φ^p is compact on L[1](**S**), first note that an argument similar to the one used in the previous example will show that $A_{\varphi,1}^m$ is continuous on $\overline{\boldsymbol{B}}\backslash\{e\}$. Thus.6it suffices to show that $A_{\varphi,1}^p$ is continuous at e. To this end we will prove

$$\left|\Lambda^p_{\varphi,1}(w)-\Lambda^p_{\varphi,1}(\mathbf{e})\right|\leq\frac{C}{N} \tag{41}$$

For w∈S^c(e, 2^{-N}) \S^c(e, 2^{-N-1})and for some constant C>0independent of N and w. Here, and in the rest of the proof, N denotes an arbitrary positive integer.

To begin with, let $w\in\overline{\boldsymbol{B}}$. Note

$$\mathscr{K}^p(a_k\mathbf{e},w)=2\frac{\mathrm{Re}(1-a_k\overline{w_1})^n}{|1-a_k\overline{w_1}|^{2n}}-1$$

and, in particular,

$$\mathscr{K}^p(a_k\mathbf{e},\mathbf{e})=2\frac{\mathrm{Re}(1-a_k)^n}{|1-a_k|^{2n}}-1=-1$$

for each k. Hence we have by

$$\begin{aligned}\left|\Lambda^p_{\varphi,1}(w)-\Lambda^p_{\varphi,1}(\mathrm{e})\right|&\leq c_n\sum_{k=1}^{\infty}\frac{1}{k2^{nk}}\left|\left|\mathscr{K}^p(a_k\mathrm{e},w)\right|-1\right|\\&\lesssim\sum_{k=1}^{\infty}\frac{1}{k2^{nk}}\frac{|\operatorname{Re}(1-a_k\overline{w_1})^n|}{|1-a_k\overline{w_1}|^{2n}}\\&=\sum_{k\leq N+1}+\sum_{k>N+1}.\end{aligned}$$

We now restrict w∈S^c(e, 2^{-N}) \Sc(e, 2^{-N-1})and estimate each sum of the above separately. For the second term, since

$$|1-a_k\overline{w_1}|\geq|1-\overline{w_1}|-|1-a_k|\geq\frac{1}{2^{N+1}}-\frac{1}{2^k}$$

for each k, we have

$$\sum_{k>N+1}\lesssim\frac{1}{N}. \tag{42}$$

To estimate the first sum, note

$$|1-w_1|\approx 2^{-N}\leq 2|1-a_k|,\quad k\leq N+1.$$

Hence, using that $\mathrm{Re}(1-a_k)^n=0$, we have

$$\begin{aligned}\left|\mathrm{Re}(1-a_k\overline{w_1})^n\right| &= \left|\mathrm{Re}\big[(1-a_k)+a_k(1-\overline{w_1})\big]^n\right|\\ &\lesssim |1-a_k|^{n-1}|1-w_1|\\ &\approx \frac{1}{2^{k(n-1)}2^N}, \quad k\leq N+1.\end{aligned}$$

This, together with (42), yields

$$\begin{aligned}\left|\mathrm{Re}(1-a_k\overline{w_1})^n\right| &= \left|\mathrm{Re}\big[(1-a_k)+a_k(1-\overline{w_1})\big]^n\right|\\ &\lesssim |1-a_k|^{n-1}|1-w_1|\\ &\approx \frac{1}{2^{k(n-1)}2^N}, \quad k\leq N+1.\end{aligned}$$

This, together with (41), yields

$$\sum_{k\leq N+1} \lesssim \sum_{k=1}^{N+1}\frac{1}{k2^{nk}}\frac{2^{2nk}}{2^{k(n-1)}2^N}=\frac{1}{2^N}\sum_{k=1}^{N+1}\frac{2^k}{k}\lesssim\frac{1}{N}. \tag{43}$$

Now, we conclude (41)by (42)and (43). The proof is complete.

Chapter2

Highly MultistableComposite Surfaces

The concept is then extended to surfaces composed of three and by extension more identical bistable shells connected in series in order to achieve additional stable states. The multistable behaviour of these surfaces is investigated by finite element analysis and verified by experimental work.

Section (2.1):Tripled Bistable Structures and Connected Bistable Composite Shells

Due to their multiple discrete stable configurations, compliant multistable surfaces have been considered for use in many adaptiveapplications. They offer several advantages when used as structures requiring shape variation, including the reduction in required components and an increase in their potential operational environments The increasingly high demand for adaptive structures across many fields of engineering, but in particular aerospace, including flow control and adaptive optics, makes research into extending the degree of multistability, and hence the adaptively ,timely. It has long been known composite laminates with un symmetric layups may present multiple stable configurations at room temperature Due to the mismatch of coefficients of thermal expansion in the directions axial and transverse to the fiber, residual thermal stresses build up during the curing process. These residual stresses cause the plates to curve into one of two possible stable cylindrical shapes after curing. In addition, each cylindrical shape can transition to the other by means of an applied external actuation.However, an individual bistable composite laminate normally cannot fulfill the requirements of real world applications. On the one hand, bistable composite shells are required to be connected with other components, on the other hand, adaptive applications may need more than two stable configurations. Therefore, the extension of previous studies of bistable plates to achieve multistable structures composed of multiple bitable composite shells is a subject of interest. For example, Mattioni et al. connected one edge of a bistable composite shell to a symmetric, i.e. monostable, laminate to demonstrate the use of bistable composites is feasible in morphing structures. Although the movement of one edge of the bistable laminate is restrained, the compound surface demonstrates two discrete stable configurations. However, when two edges of a bistable composite laminate are clamped by monostable laminates, the plate only demonstrates one stable configuration. To regain multistability, a designed surface consisting of symmetric and unsymmetric laminate parts may be introduced, for example, the shell demonstrated by Arietta et al. The embedded composite shell with variable stiffness can demonstrate bistability and avoid the conventional connections which may increase the risk of laminate failure. However, these embedding designs cannot increase the number of stable configurations. To achieve high degrees of multistability, Dai et al. fabricated tristable composite lattices by connecting four bistable rectangular laminates with discrete joints which were then assembled n lattice cells by bolts. The lattice structure can present 2n stable shapes. A similar design is demonstrated in. These attempts to achieve highly multistable structures have two main disadvantages: first, large numbers of components are used – this negates one of the principle advantages

of the use of compliant mechanism in adaptive system; second bolted connections may reduce the performance and lifetime of composites. In this chapter, novel multistable composite surfaces are constructed by connecting several identical bistable composite shells in series. As previously stated, if n bistable components are connected, the resulting system may exhibit up to 2n discrete stable configurations. This has

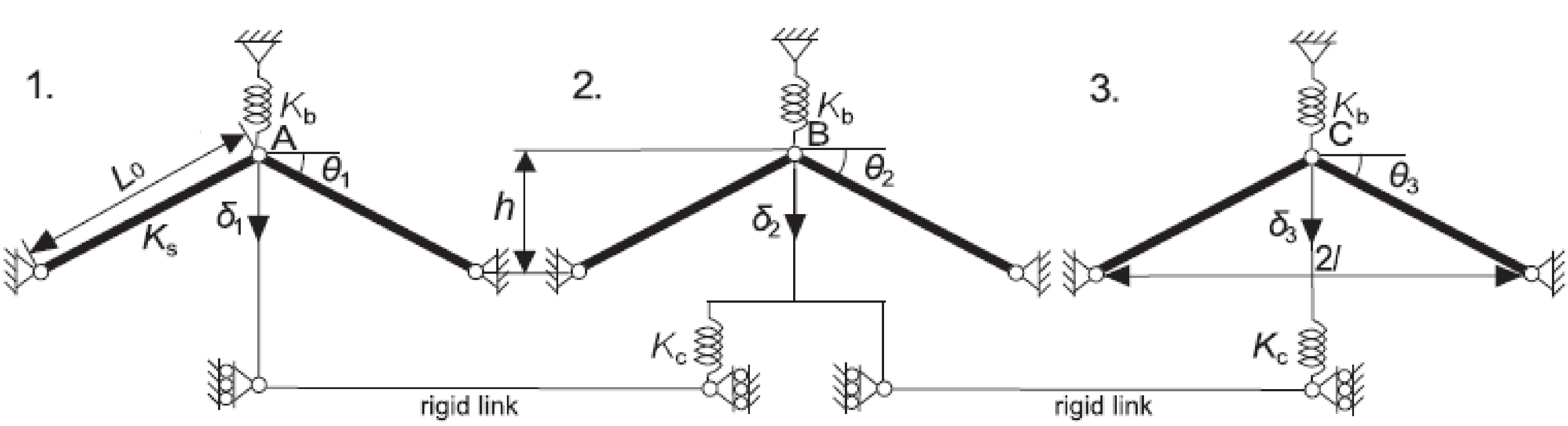


Fig. 1. Three connected biased von Mises truss structures.

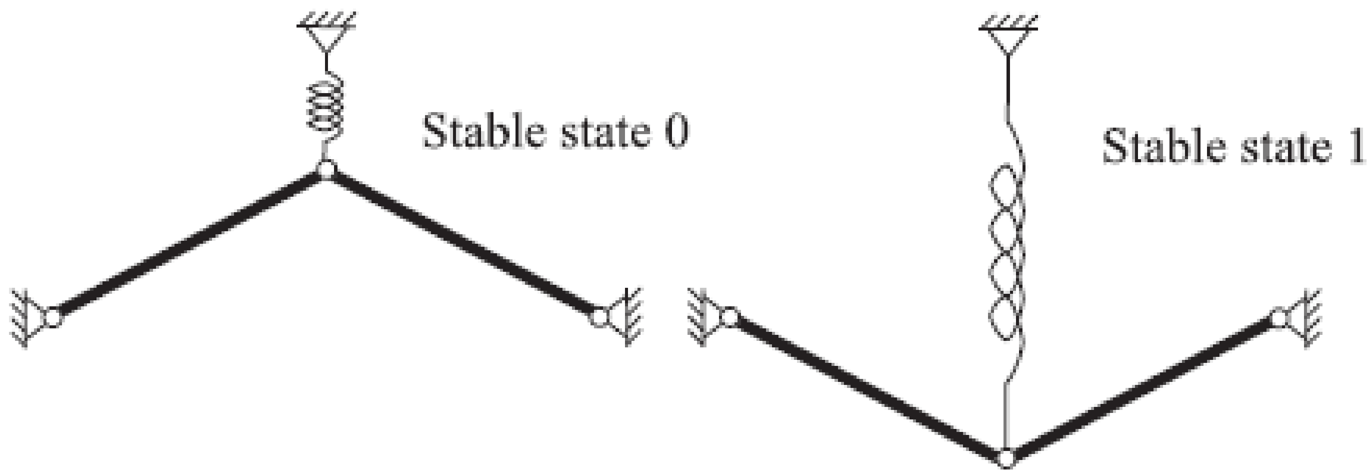


Fig. 2. Two stable configurations of a biased von Mises truss.

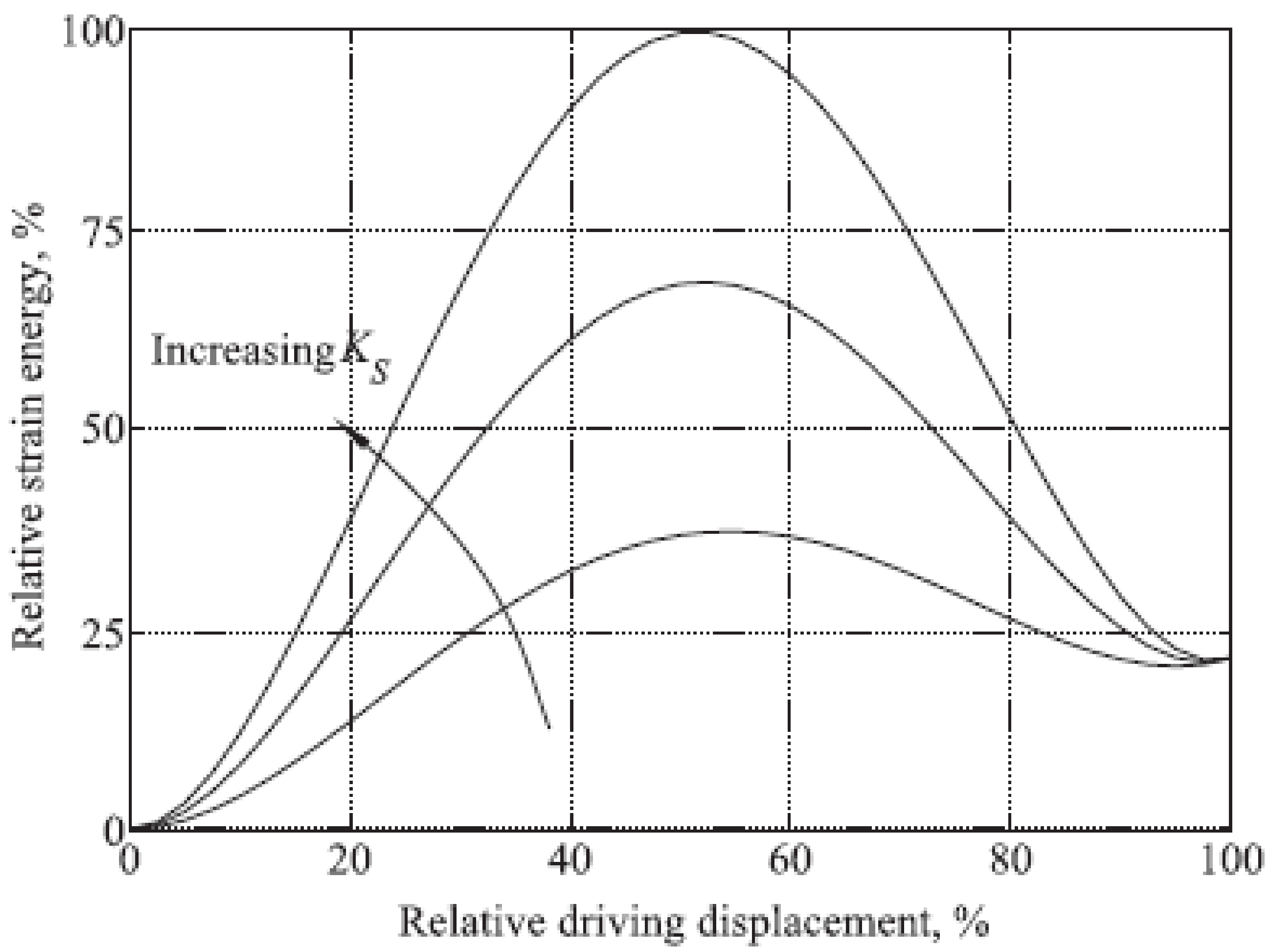


Fig. 4. Relative strain energy vs. relative displacement plots for a single asymmetrically bistable ($K^b \neq 0$) von Mises truss with variable K_s .

previously been demonstrated for systems where the individual bistable components are decoupled and may be independently actuated via the design of statically and kinetically determinate systems. When components are connected in a continuous sense, however, it is essential to consider their interaction by means of the coupling along common boundaries. In this chapter we begin this investigation by considering the interaction between two bistable plates and then extend this to the study of three connected bistableplates.

Furthermore, in order to interpret the multistable behaviour of these newly designed surfaces, a two-dimensional multistable an alogmodel will be introduced first and a parametric study will be carried out to characterise the controlling parameters which determine multistability of the analog model. Inspired by the understanding gained from this simplified study, a design method is developed to assist the compound surface achieving higher multistability. Since each of these shells possesses two stable states, the compound structure can theoretically present up to a maximum of eight stable configurations. However, due to the interaction between the connected shells, it will be shown that a maximum of seven discrete stable states can be achieved in a surface consisting of three connected square bistable composite shells.

Before analysing the behaviour of connected bistable composite shells, it is instructive to consider an analog model consisting of three biased von Mises truss systems to understand the general behaviour of bistable elements connected in series. The three von Mises trusses are connected in series by rigid bars and coupling springs. By varying key parameters, the multistable behaviour of the truss structure is controlled and a maximum eight discrete stable states, as expected, are found to exist. As illustrated in Fig. 1, the three identical von Mises trusses consist of elastic rods of initial length L_0. The span width of each truss.

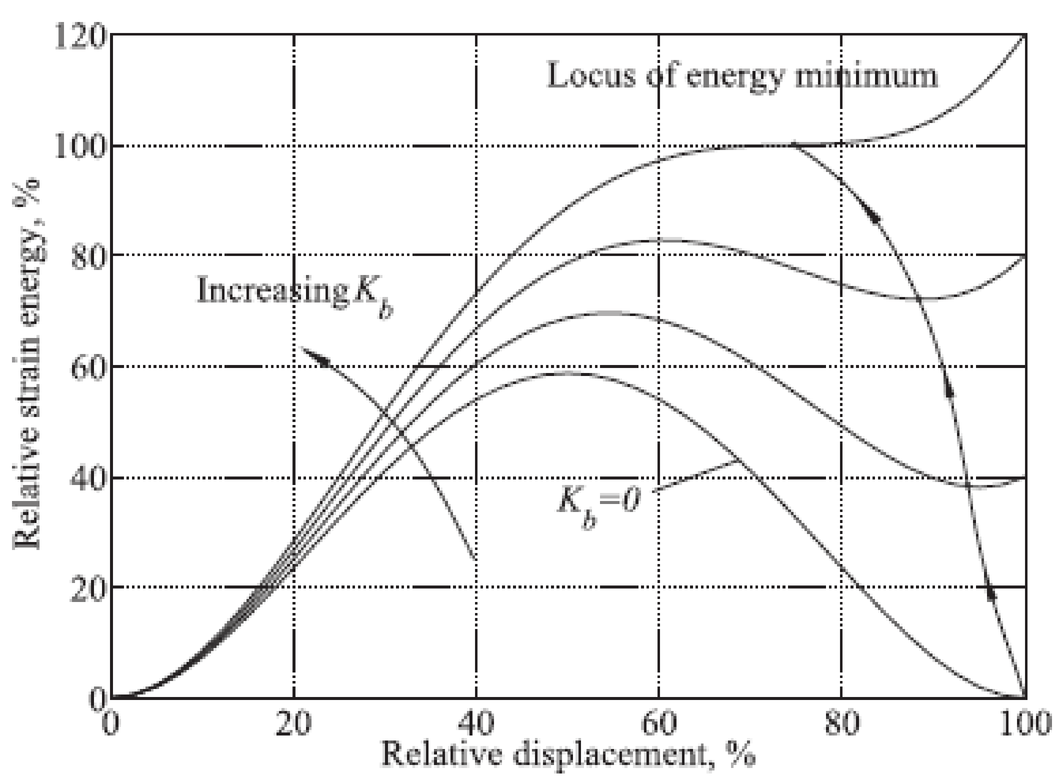


Fig. 3. Relative strain energy vs. relative displacement plots for a single von Mises truss with variable bias stiffness K_b showing the transition from symmetric to asymmetric bistability

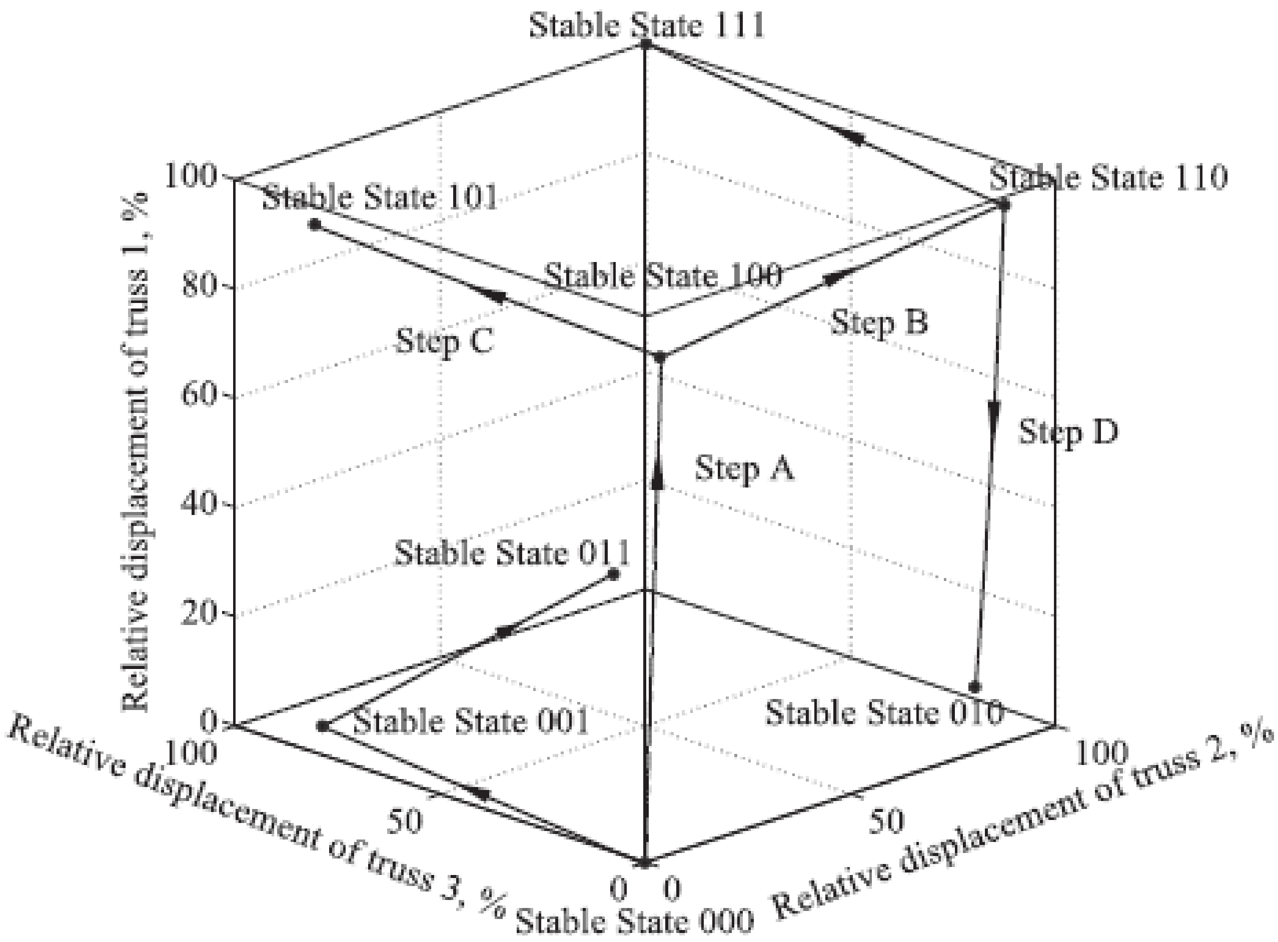
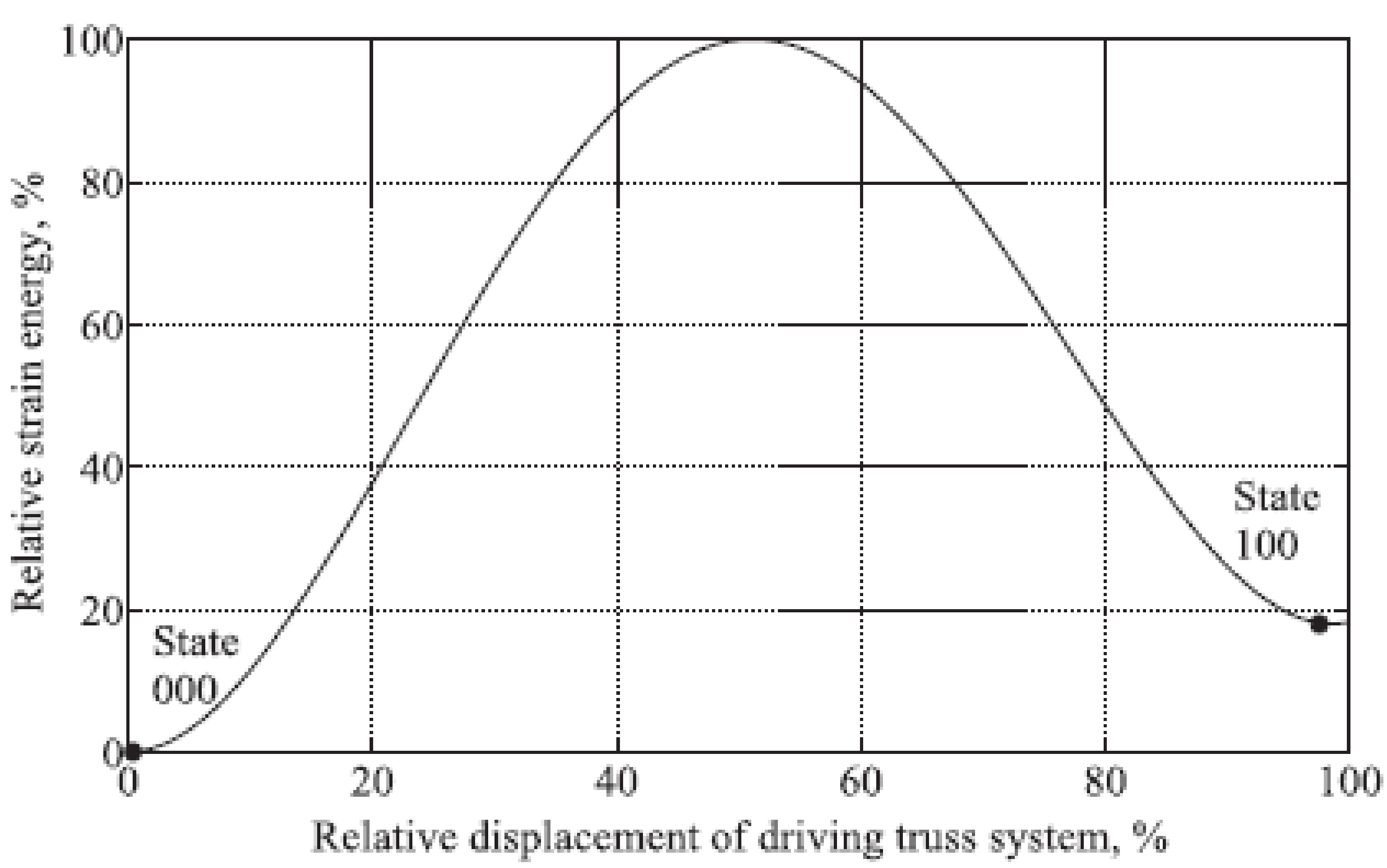
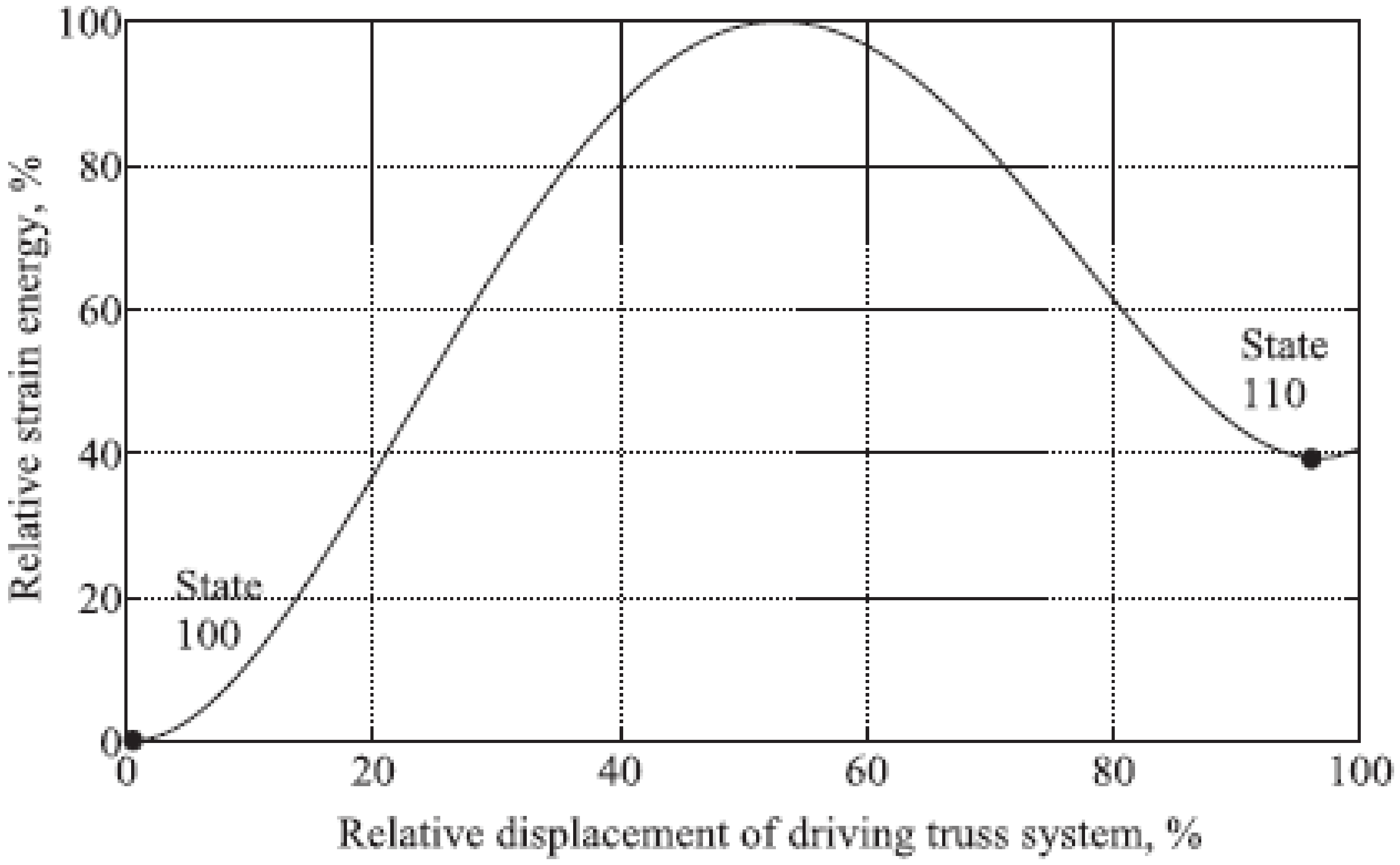


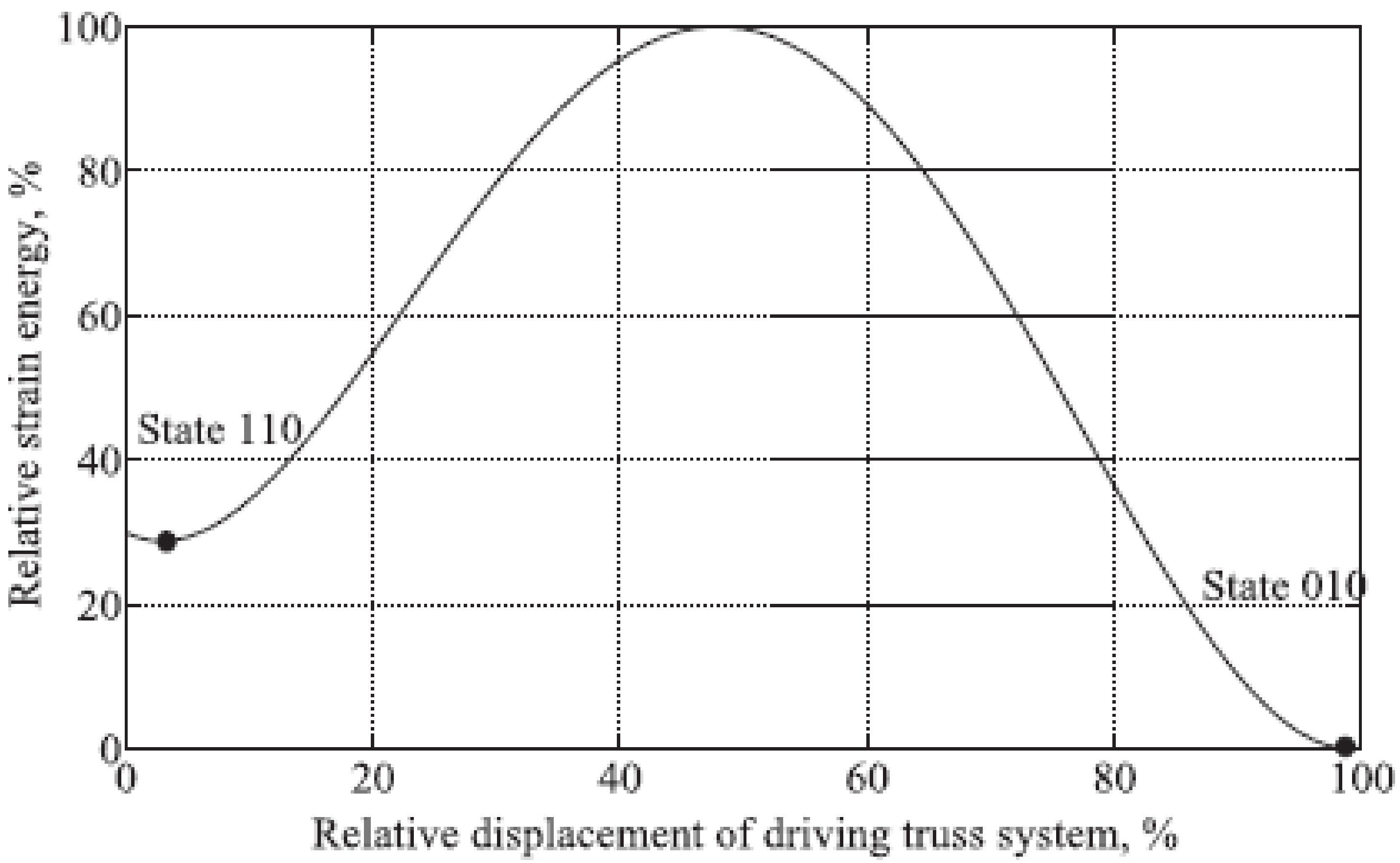
Fig. 5. Plots of Relative displacement of trusses vs. relative strain energy in different actuation steps for the octo-stable truss structure. system is 21 The axial stiffness of the rods is K_s. Biasing springs K_b are attached to the centrally-located hinges A, B and C. Any two adjacent truss systems are connected as shown by a rigid bar.



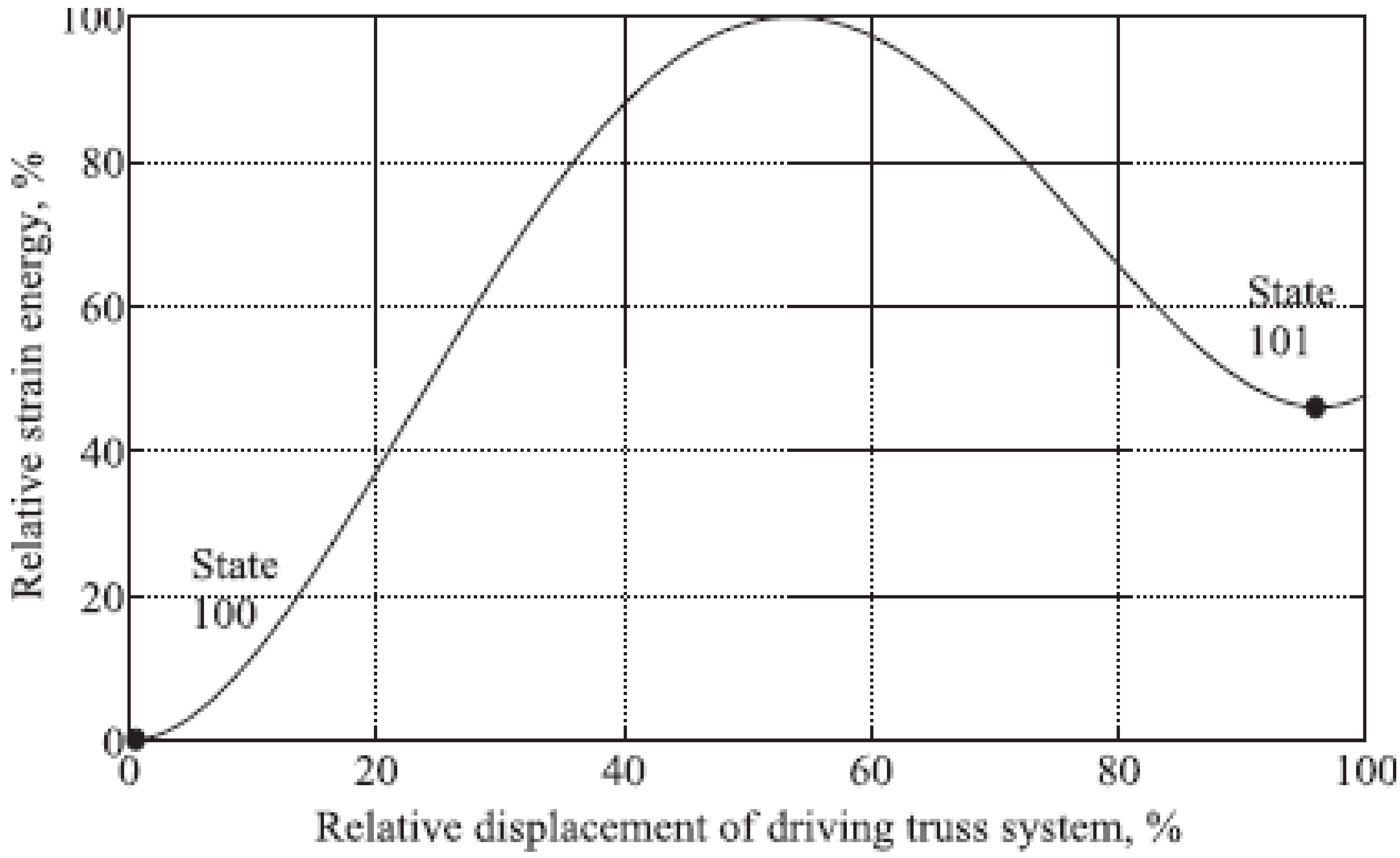
(a) Step A



(b) Step B



(c) Step C



(d) Step D

Fig. 6. Plots of Relative displacement of trusses in different actuation steps for the tripled trusses.

and coupling spring K_c All of the trusses are shown in their initial strain free stable configurations. The initial height of the truss members is defined by the vertical distance h and the maximum displacement of a truss is 2h.We first investigate the bistability of a single truss. When a vertical is placement d is applied, the rods will be compressed and the truss system will move downward. Meanwhile the strain energy stored by the truss will rise. When the displacement reaches a critical value, the strain energy will reach a maximum value and will drop as the displacement is increased. As the truss is displacement controlled there is no dynamic jump to the second stable state. This truss system then continues to a new stable configuration corresponding to a second energy minimum (see Fig. 2). The relationship between the displacement relative to double height 2h and the stored strain energy relative to the largest local strain energy minimum of a bistable truss achieved Fig. 3 demonstrates the relationship of the relative displacement and relative strain energy of a von Mises truss with increasing K_b, in which 100% relative displacement indicates the vertical displacement of the truss is the maximum displacement, 2h. The two local strain energy minima correspond to two stable states. When K_b ¼ 0, the two local minimum values are identical, and the system is symmetrically bistable. With K_b increasing, the actuation energy of one stable state rises and the energy gap between two stable states increases. The stable state possessing lower potential energy becomes the preferred stable state. When K_b reaches a critical value, the second energy minimum disappears and the system becomes monostable. In other words, to ensure that each individual truss is bistable, the stiffness of the bias spring K_b must be lower than a critical value. Besides the stiffness of biasing spring K_b, the bistability of thevon Mises truss is also determined by K_s. Fig. 4 illustrates the influenceof K_s on the stored strain energy of the bistable truss. With K_s increasing, higher energy is required for actuation. This means s_tIfferrors cause the stable configurations of the truss to be more.

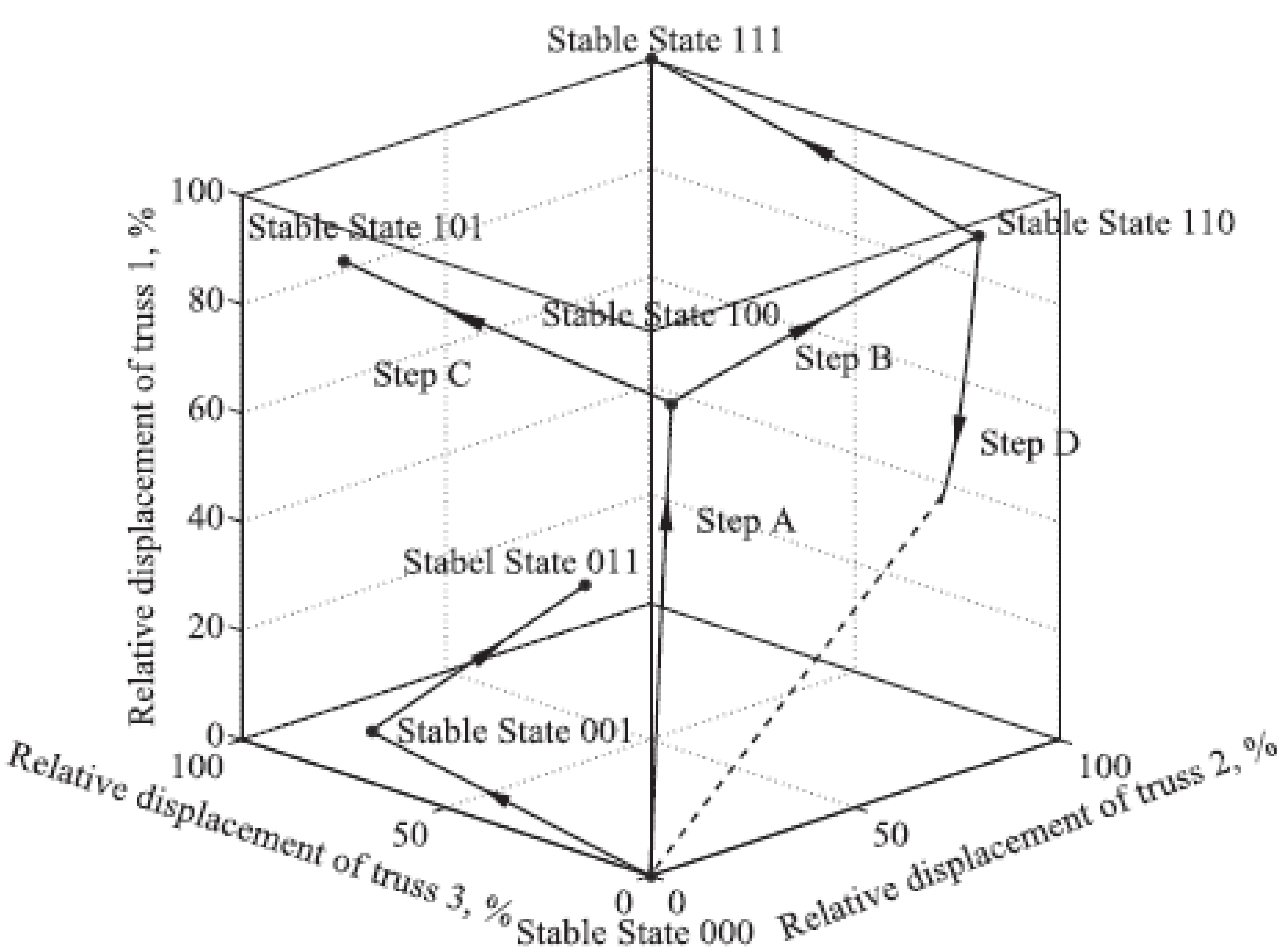


Fig. 7. Plots of Relative displacement of trusses v_s relative strain energy for the hepta-stable truss structure.

Table 1

Properties of IM7/8552 carbon fiber composites.

E_{11} (GP	E_{22} (GP	G_{12} (GP	V_{1}	α_1 (10-6/oC	α_2 (10-6/oC	Thickness (mm)	Side leng (mm
164	12	5.3	0.3	0.02	31.2	0.131	100

stable in the sense that greater energy input is required to transition between the states. After understanding the bistability of an individual von Misestruss, the next step is to study the influences of these key parameters on the multistable behaviour of connected truss systems. When one truss of this connected structure is actuated by displacement control, the motion will be transmitted to other driven trusses by the rigid bars and connecting springs. If the vertical displacement of point A; B; C is denoted by δ_1; δ_2; δ_3, the loads appliedby the connecting spring on point A; B; C are expressed as

$$\begin{aligned} P_1 &= K_c(\delta_2 - \delta_1); \\ P_2 &= K_c(\delta_1 - \delta_2) + K_c(\delta_3 - \delta_2); \\ P_3 &= K_c(\delta_2 - \delta_3). \end{aligned} \quad (1)$$

The equilibrium path P_i , d_i of these three systems must also be

equal to

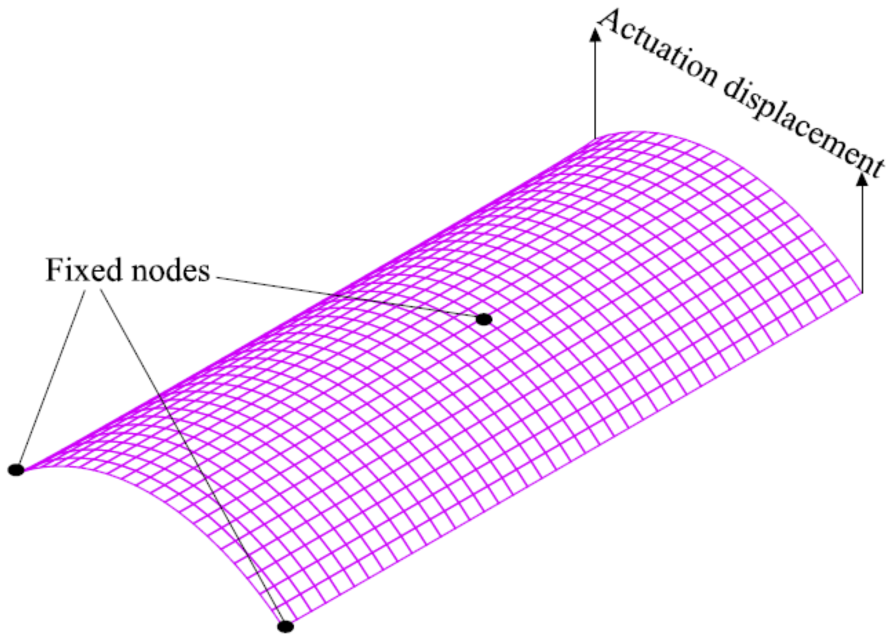
$$P_i = K_b\delta_i - 2K_sX_i \sin\theta_i, \quad i = 1, 2, 3, \quad (2)$$

in which X_i are the compressive deformations of the diagonal members and θ_i are the angles between a hinge and the horizontal direction.

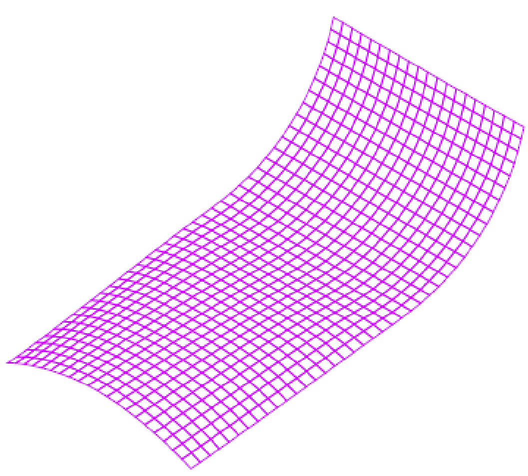
They can be obtained by

$$\begin{aligned} X_i &= L_0 - \sqrt{l^2 + (h - \delta_i)^2}, \quad i = 1, 2, 3, \\ \theta_i &= \arctan\left(\frac{h - \delta_i}{l}\right), \quad i = 1, 2, 3. \end{aligned} \quad (3)$$

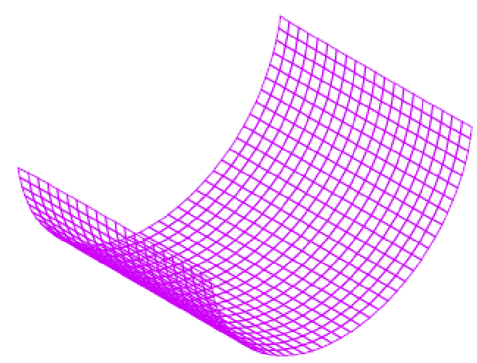
Combining Eqs (1), (2) and (3), it may be seen that the force-controlled displacement of the two driven trusses may be expressed as a function of the imposed displacement δ on the driving truss.



(a) Initial primary stable state (00)



(b) Intermediate state (01)



(c) Final primary stable state (11).

Fig. 8. Actuation procedure for coupled bistable composite shells away from a primary stable state.

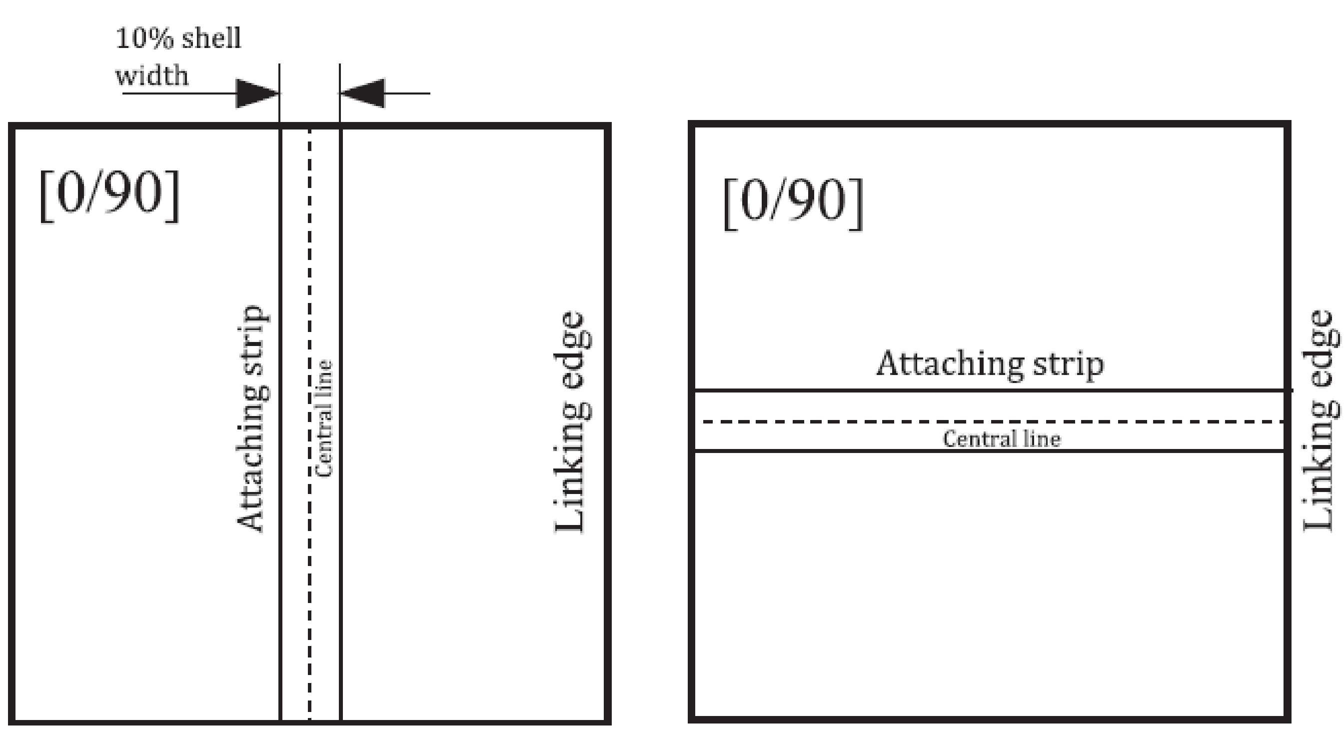


Fig. 9. Two different approaches of attaching a biasing strip on a single shell.

In order to identify each state clearly, we use a binary notation. The original stable state of the truss is denoted as 0 and the actuated stable state is denoted as 1. Stable state 000 and 111 are two primary stable states since there is no interaction between the connected trusses. When not all connected trusses are in the same state, the tripled truss structure is in an intermediate configuration. In an intermediate state, the interaction between two adjacent trusses within different states may be large enough to trigger one of the trusses to its other state; in other words, this intermediate state is not stable. To determine the stability of intermediate states, four actuation steps are applied. Specifically:

(i) step A: actuate from state 000 to state 100;

(ii)step B: actuate from state 100 to state 110;

(iii) step C: actuate from state 110 to state 010;

(iv) step D: actuate from state 100 to state 101.

States 001 and 011 are not investigated as their behaviour maybe inferred due to the symmetry of the system. In all cases the actuated truss is subjected to displacement control, and the other two trusses are in a force-controlled regime. The tripled von Mises truss structure possesses eight stable states if and only if the actuation of the driving truss does not actuate the other two trusses in every actuation step. Fig. 5 demonstrates the relative displacements of the trusses of an$o^c$to-stable truss structure by the four actuation steps. It is clear that, in each actuation step, the relative displacements of the driven trusses are small then the structure reaches a new stable state. It indicates that no automatic snapthrough occurs and the expected stable state can be achieved. The energy graphs of the tripled truss structure (see Fig. 6) verify the stability of these new states. It is clear that the structure reaches a local energy minimum at the newstate.The analytical results show the multistability of the tripled von Mises trusses is determined by the connection between the trusses. If the value of$K_c$rises, the interaction between connected trusses will also increase. If K_c exceeds a critical value, the tripled truss systems will not demonstrate eight stable states. Fig. 7 illustrates the relative displacements of the trusses of a hepta-stable truss structure by the four actuation steps. In the actuation step C, the movement of truss 1 results in increasing external strain energy introduced to truss 2. Before truss 1 is fully actuated, the introduced energy to truss 2 already exceeds the actuation energy and the truss snaps back to state 0. In other words, the intermediate state 010 of this structure is no longer a stable state. By actuating truss 1, the structure will jump through from state 110 to state 000 directly. Thus, this structure only possesses seven stable states. As K_c increases, more intermediate states become switched off via a similar mechanism.T_0avoid the unwanted automatic actuation, the actuationenergy of the truss needs to be raised. According to the investigation of the multistability of the single truss, the actuation energy is determined by the relative stiffness of the bar, K_s. By increasing the value of K_s, the stable state 010 is achievable again and the tripled truss structure regains $o^c$to stability successfully. In other words, higher K_s can help the tripled trusses with strong connection to achieve higher multistability. Although the bias stiffness K_b can affect the degree of asymmetric bistability of a single truss system, the influence of the value of K_b on achieving the o^ctostability of this tripled structure is limited. This is because, despite resulting in a more stable state 0, the increasing K_b will lead the actuated state 1 to be less stable. Therefore, in order to ensure the tripled von Mises trusses possesses more stable states, these bistable trusses should not be highly asymmetrically bistable. In summary, the response of the tripled bistable trusses shows that its multistability is determined by the relative stiffness of the bias and coupling springs and the bars. Specifically:

(i) as the value of bias stiffness K_b increases from 0, the single truss changes from being symmetrically bistable to being asymmetrically bistable. If K_b is over a critical value, the truss will be monostable;

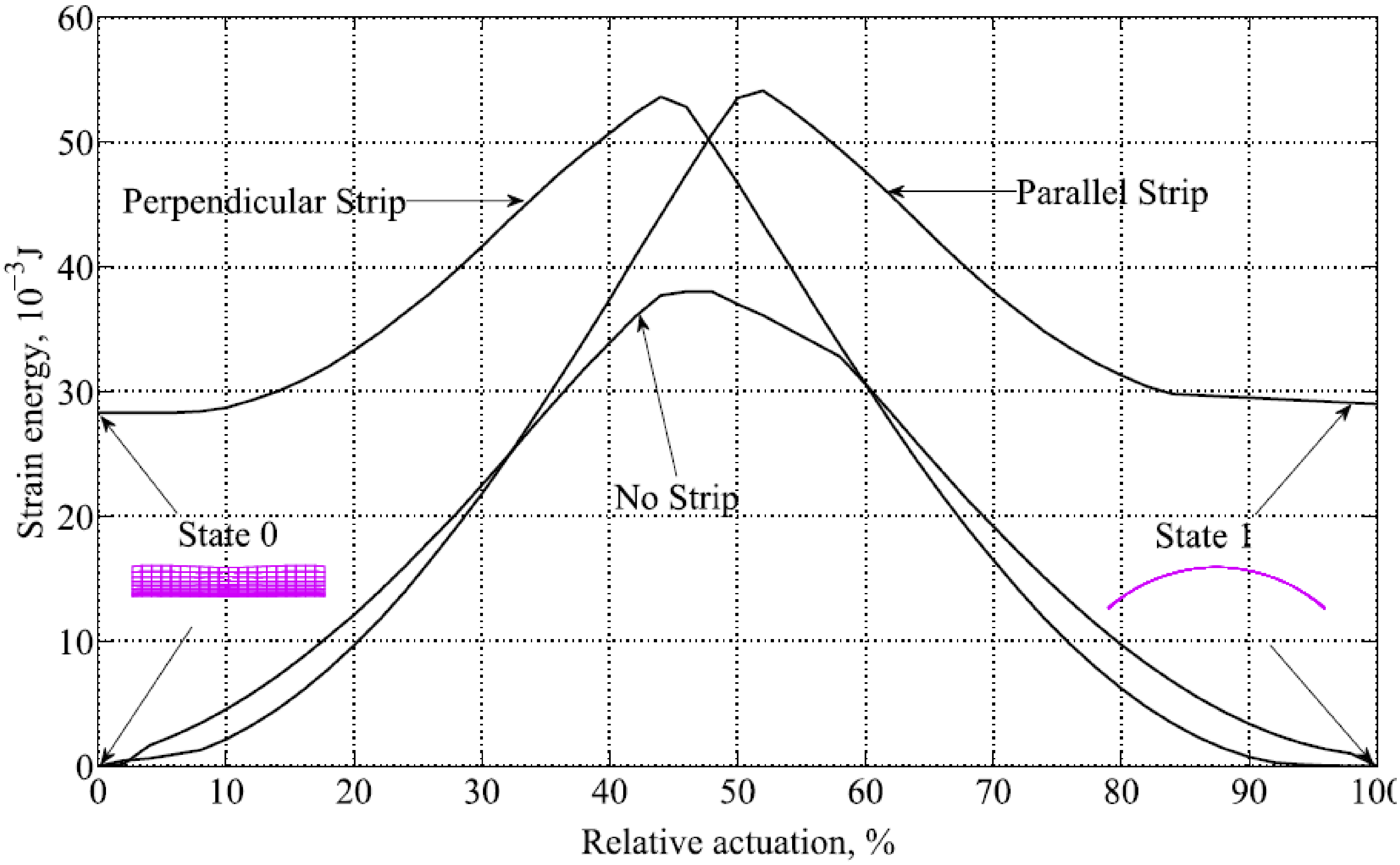


Fig. 10. Strain energy vs. relative actuation percentage plots for a symmetric bistable shell and two identical shells with biasing strips attached.

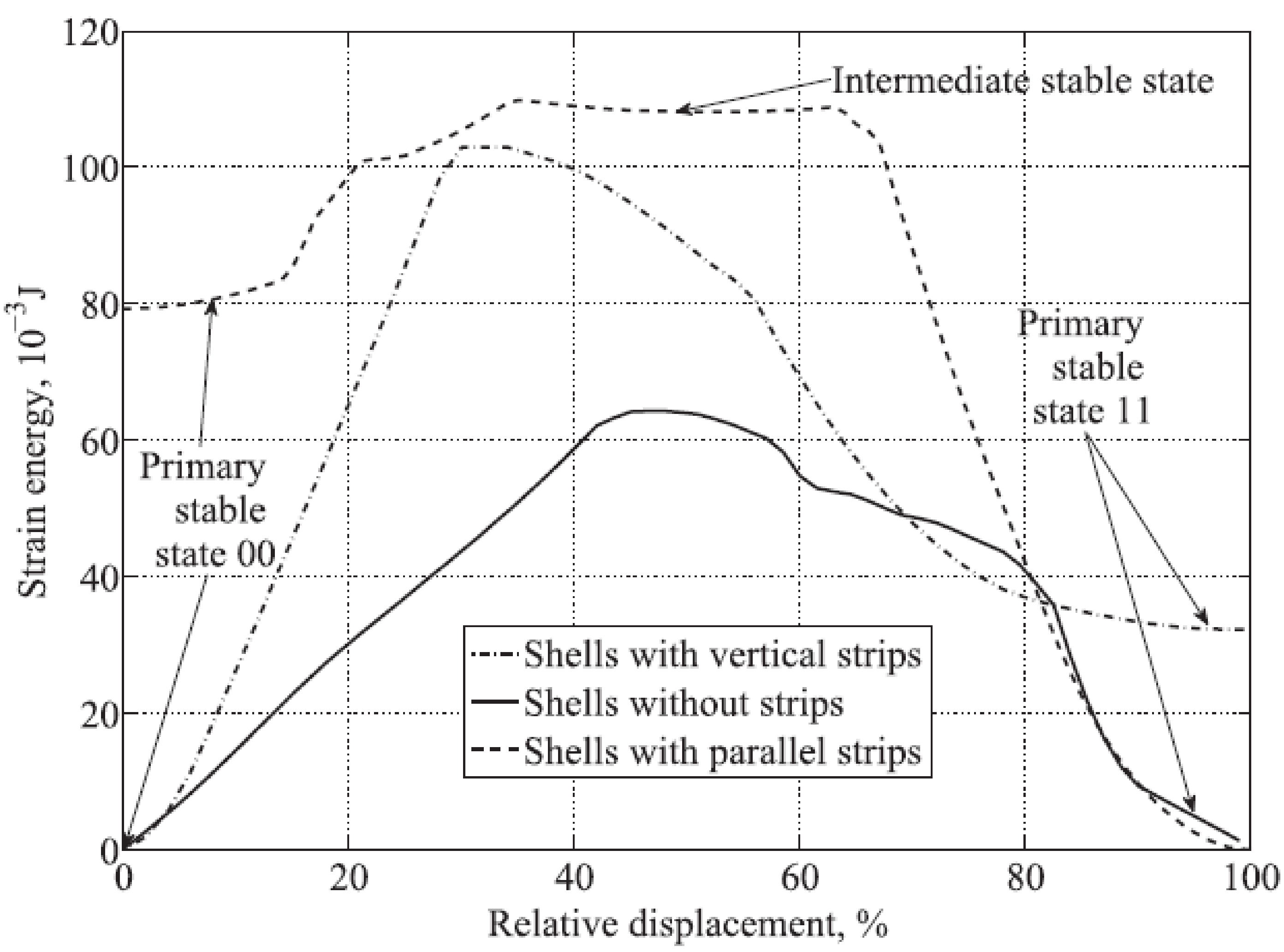


Fig. 11. Strain energy vs. relative actuation displacement percentage plots for three different coupled bistable shells.

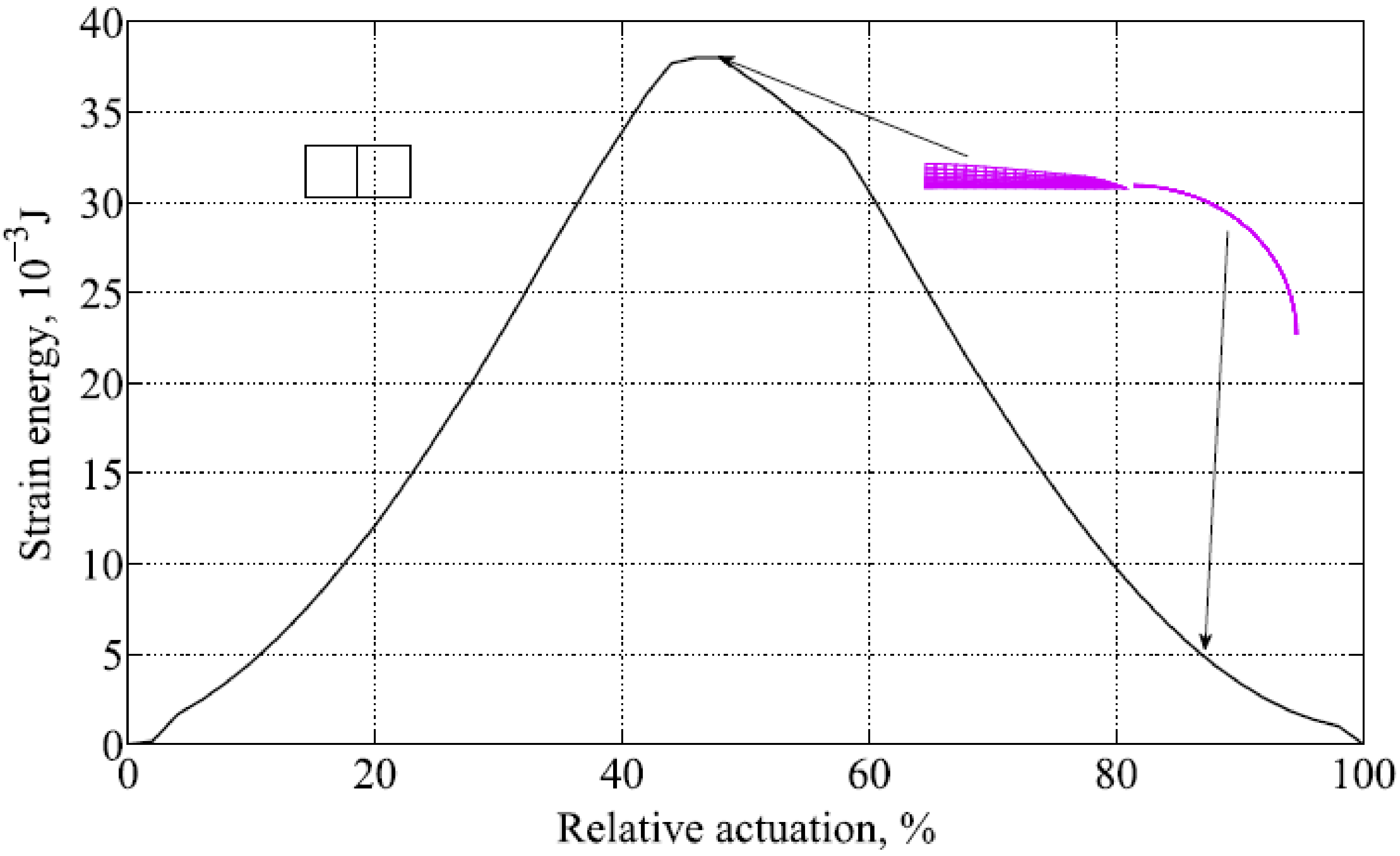


Fig. 12. Configurations of two symmetrically bistable shells in an intermediate displacement-controlled state and their corresponding strain energy: this coupled system is bistable.

(ii) higher coupling stiffness $K_c$will lead to higher actuation energy of the individual truss;

(iii) to avoid the actuation of the driven truss caused by the driving truss, in other words, to ensure the system achieves o^ctostability, the relative stiffness of the connecting spring K_c must be lower than a critical value;

(iv) the critical value of $K_c$is dependent on the value of K_s i.e. the symmetry of bistability of each individual truss. The critical value of $K_c$can be raised by increasing the relative stiffness of the rods, K_s

(v) a structure composed of three symmetrically bistable trusses is more likely to achieve octostability than those composed of asymmetrically bistable trusses.

The understanding gained from the previous part can assist in interpreting the response of the novel designed surfaces. We restrict the number of bistable units of the multistable structure to be up to three in this chapter, however, this concept can be easily extended to a multistable surface consisting of many more bistable shells connected by the same approach. The mechanical properties are taken from the manufacture's data sheet and listed in Table 1.Whilst restricting the design space, this enables the behavior to be validated by experimental models. The stacking sequence is [0=90] to ensure that the laminates have a moderate out-of-plane displacement which would help the connected shells to demonstrate more stable states and avoiding the generation of

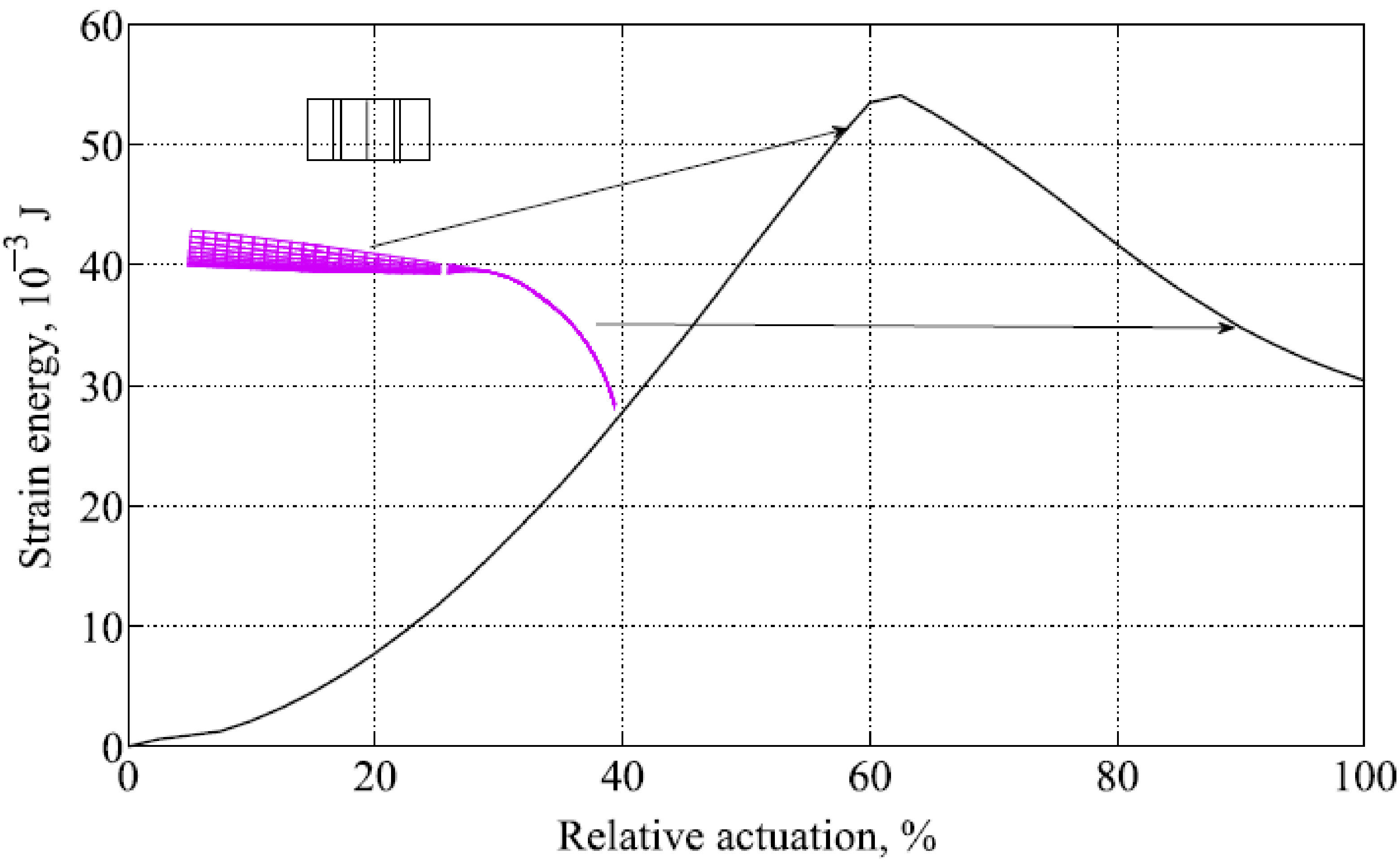


Fig. 13. Configurations of two asymmetrically bistable shells in an intermediate displacement-controlled state and their corresponding strain energy: this coupled system is quadstable.

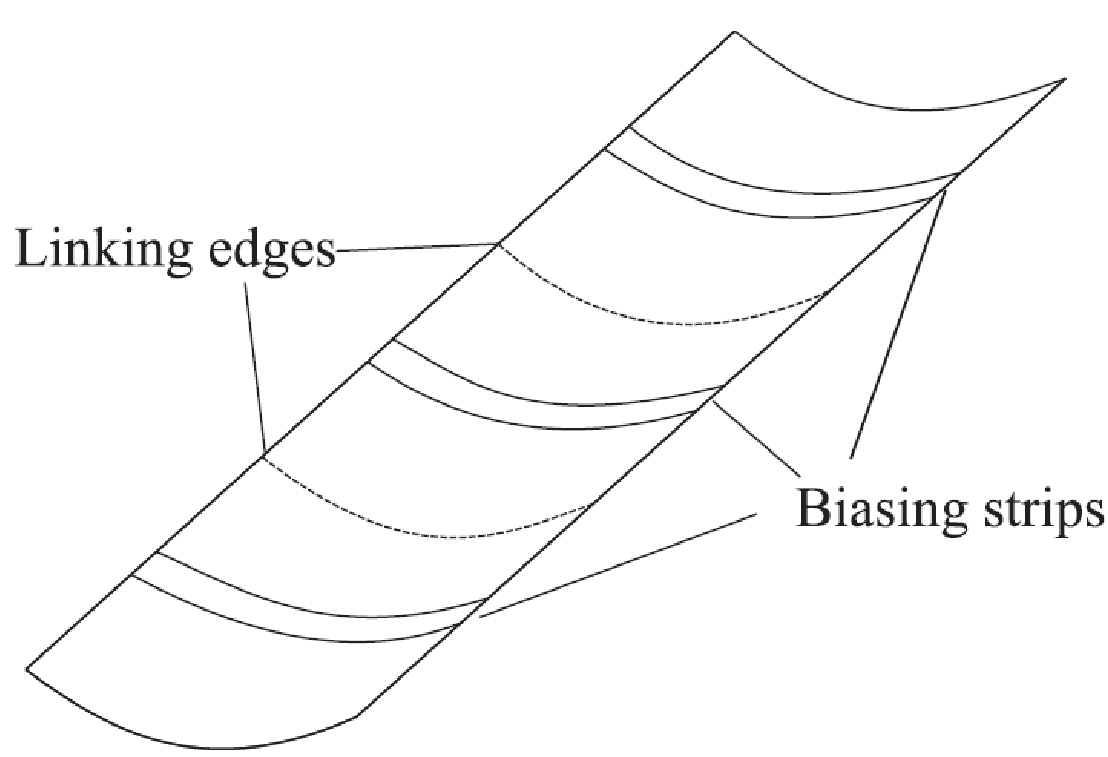
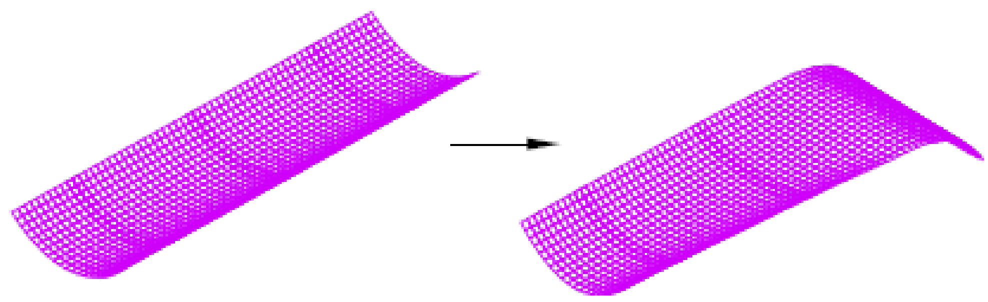
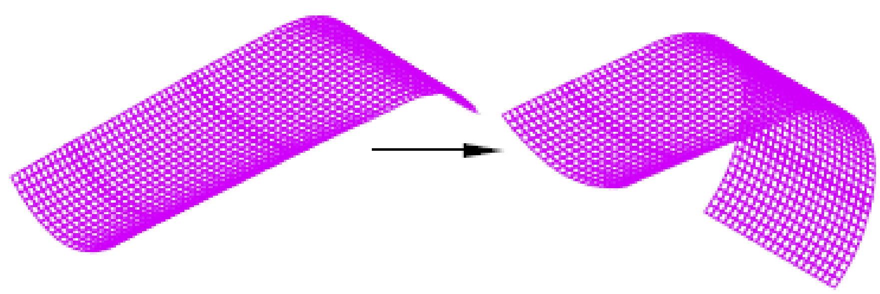


Fig. 14. Three connected square shells shown here with biasing strips attached in their parallel orientation .twisting curvature after curing. For the sake of easy identification, we continue to use binary notation to represent the states. If a shell curves along the linking edge direction, the shell is defined as being in state 0; otherwise, it is in state 1.It is noted that a Ritz energy analysis is often considered a fast and reliable approach when investigating the multistable behavior of unsymmetric composite laminates. However, this approach is not adopted in this chapter due to the large number of terms in the approximating polynomials required to adequately represent the coupled shells. The significantly increased calculation time and the presence of many local minima make a Ritz approach no longer a feasible way of investigating connected bistable composite shells. Therefore, Finite Element Analysis (FEA) is exclusively used as the simulation method in this research.Before considering the behaviour of the tripled connected bistable shells, we first investigate the multistable behaviour of coupled bistable shells. This is because, although the multistable behavior of the tripled bistable truss structure has been understood, the nature of the interaction between fully connected shells is significantly more complex than the one dimension connection which is used in the analog model. For a coupled bistable shell structure, a maximum of four stable states may be present. Among these four stable states, the two primary states exist in which both of the shells have the same stable configuration. Besides two primary stable states, the connected plates may achieve stability during intermediate states in which the two shells individually present different configurations. Due to the continuous connection, the two plates will be subject to considerable deformations which may trigger the surface jump to the primary stable state during an attempted transition to an intermediate state. Thus, the key to achieving highly multistable surfaces is the stability of intermediate states.FEA is performed with the commercial software SAMCEF V13.1.

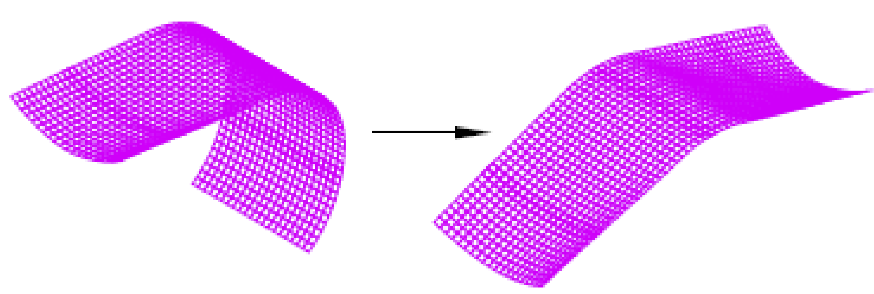
Following a mesh refinement study, a single plate is simulated by 400 8-node square shell elements. This level of discretisation



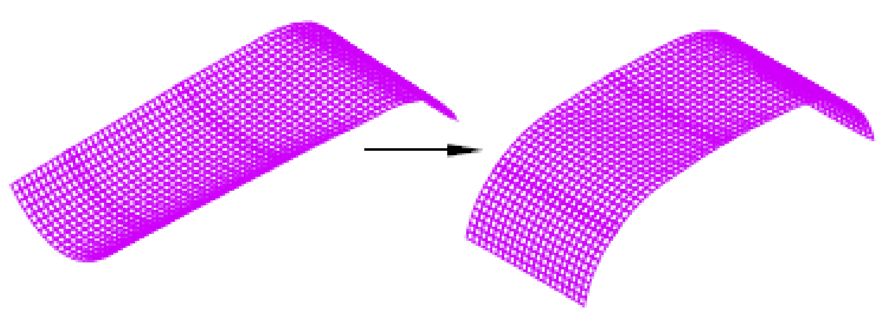
(a) Step A



(b) Step B



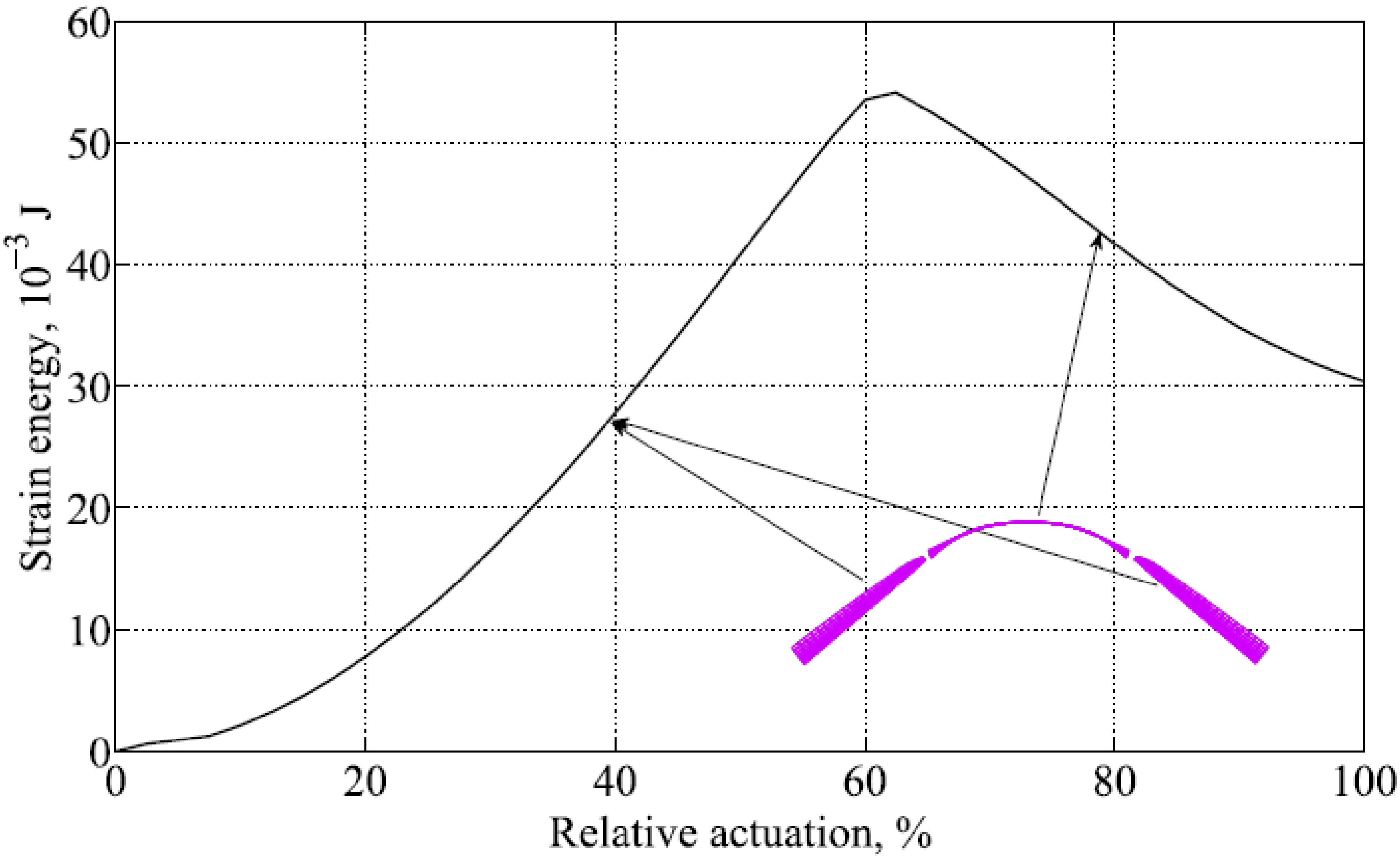
(c) Step C



(d) Step D

Fig. 15. Finite element predictions of the stable configurations of three connected bistable plates and the transitions between these states resulting from actuation. has been shown to provide mesh independent solutions. Inertial phenomena are neglected in this study; a geometrically nonlinear static analysis strategy is therefore adopted. The two shells are connected continuously along a common boundary. The manufacturing process is simulated first by imposing a ramped temperature increase of 160°C. After manufacturing, the actuation process is implemented by applying a controlled displacement to the driving shell until the shell snaps through to another stable state. Meanwhile, the central point and two free vertices of the driven shell have to be fixed for avoiding rotation (see Fig. 8(a)). The

deformation will transmit to the driven shell through the common edge. After the actuation, the fixed vertices are released and if the driven shell does not snap to a new stable configuration and the driving shell maintains its new shape (see Fig. 8(b)), the intermediate state is deemed stable and the connected shells determined to have four stable states. Otherwise, the intermediate state is not stable and the structure will jump to a new primary stable state (see Fig. 8(c)). In this case the compound surface remains bistable. According to the analog model, the degree of bistable asymmetry influences the multistable behaviour of the connected trusses. Considering the corresponding effect for coupled composite shells, biasing strips made by symmetric {0≠0} laminates are attached on the center of bistable composite shells to vary the asymmetric bistability of the composite shell. To investigate the influence of attaching biasing strips on the bistability of composite shells, the strips may be attached parallel or perpendicularly to the linking edges respectively (see Fig. 9). The influence of biasing strips on the bistable behaviour of a single shell is shown in Fig. 10. As expected, by attaching the biasing strip, the two stable states no longer exist at the same potential energy level. The degree of asymmetric bistability of the single shell increases and the stable state possessing lower potential energy becomes the preferred stable state. It is also noted that the selection of one of the two possibleor ientations of biasing strip enables either of the stable states to be made preferential. We consider three cases when the unbiased shell and the two differently biased shells are connected to an identical partner. To demonstrate the stability of intermediate states, the surfaces begin from the primary stable state 00 and are subjected to controlled displacements as the actuation is applied to the two free vertices of one shell. During the actuation procedure, deformations will transmit to the driven shell through the linking edges. With the controlled displacement increasing, the driving shell will snap through to the new stable state at a critical point, and the strain energy of the driven shell will rise during this actuation procedure. If a compound surface has stable intermediate states, the driven shell will not snap through with the driving shell and the strain energy of the whole structure will reach a local minimum. It should be noted, due to symmetry, that only one intermediate state is investigated in this study.Fig. 11 presents the potential energy graph of three different coupled bistable shells during the actuation from the primary stable state 00 to the primary stable state 11. Since some nodes on the surface need to be temporarily fixed during the actuation to avoid unwanted rotations and buckling, the potential energy curves show some asymmetries, but the local minima in each case are clearly visible. The coupled symmetrically bistable shells and the coupled bistable shells with perpendicular biasing strips only demonstrate two potential energy minima which represent two primary stable states, whereas the coupled bistable shells with parallel biasing strips demonstrate three local energy minima. Besides the two primary stable states, this surface can achieve two stable intermediate states and possesses quadstability. By tailoring the asymmetric bistability, coupled bistable composite shells can successfully achieve a higher degree of multistability. This behaviour may be explained by consideration of the strain energy of the individual bistable units in the coupled system. Fig. 12 corresponds to the case when the two connected bistable plates are symmetrically bistable. The right hand shell has been transitioned via displacement control into state 1 and the left hand shell has deformed to accommodate this transition. It can be seen that the deformation imparted to the driven shell introduces sufficient strain energy to cause it to exit the energy well corresponding to state 0 and upon release will dynamically jump to state 1. Therefore the only stable equilibrium configuration are states 00 and 11 and it can be seen that there are no stable intermediate configurations. This coupled system is therefore bistable. We now consider the effect of causing state 1 to become energetically preferential by means of the addition of parallel biasing strips. As the degree of bistable asymmetry increases a critical point is reached at which the displacement controlled transition of one shell from state 0 to state 1 causes the other shell to reach the limit point corresponding to the peak of the energy hill. As the bistable asymmetry is further increased, as shown in Fig. 13, the driven shell no longer exits the energy well of state 0 and consequently when displacement control is released the coupled system



ig. 16. Configurations of three asymmetrically bistable shells in the intermediate displacement-controlled state 010 and their corresponding strain energy: indicates state 010 is stable.

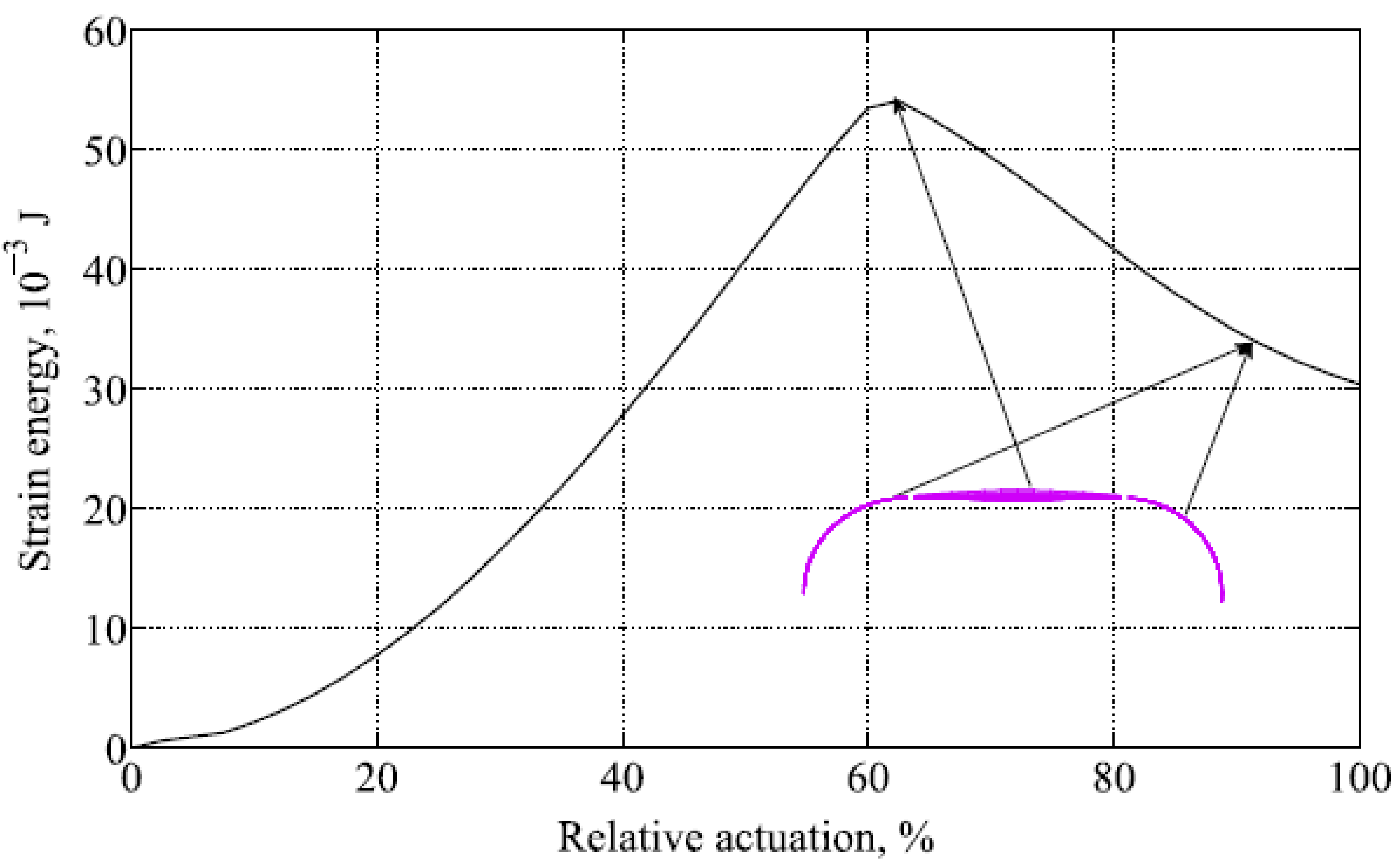


Fig. 17. Configurations of three asymmetrically bistable shells in the intermediate displacement-controlled state 101 and their corresponding strain energy: indicates state 101 is unstable. may adopt a stable intermediate state in which both shells adopt a configuration corresponding to the average strain energy of the two displacement-controlled configurations. This coupled system is therefore quadstable. Having understood the multistable behaviour of two connected bistable shells, we now generalise to a multistable surface composed of a series of coupled bistable composite shells. We focus on three connected square bistable shells (see Fig. 14) but the evaluated behaviour is readily extended to surfaces consisting of many more shells connected in series. Theoretically, a surface composed of three bistable shells may present a maximum of eight stable states. Based on the study of the coupled bistable shells in the previous part, all these shells are made asymmetrically bistable via a biasing strip parallel attached to the linking edges in order to obtain more stable states. In addition to the two primary stable states 000 and 111, this surface has up to a six intermediate stable states. Four actuation steps are applied to verify the stability of these intermediate states, specifically:

(vi) step A: actuate from state 000 to state 100.

(vii) step B: actuate from state 100 to state 110.

(viii) step C: actuate from state 110 to state 010.

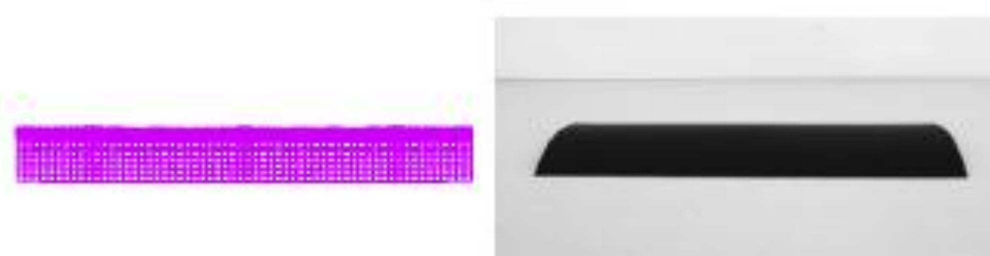
(ix) step D: actuate from state 100 to state 101.

In each actuation step, a controlled displacement is only applied to the driving shell as the actuation load and the other two driven shells are unconstrained. If and only if no driven shell is actuated automatically, the intermediate state obtained is deemed to be stable. Fig. 15 illustrates the configuration transitions in these steps. Due to symmetry, the shapes of state 001 and state 011 are not illustrated. The existence of intermediate stable states in two connected asymmetric bistable shells with parallel biasing strips has clearly been demonstrated. The existence of stable intermediate state 100 and 110 of tripled bistable shells may be directly inferred as it involves connecting an identical shell to one of the free edges of coupled bistable shells. Since the newly connected shell is in the same stable state as its adjacent shell, no additional strain energy is introduced. The FE results prove this assumption: the two intermediate states are stable. The two remaining possible states that must be considered are states 010 and 101. In both these cases the central shell is connected to two other shells which are in a different state. Fig. 16 shows the case when the compound surface is in state 010. In this case the energy imparted to the two end shells is insufficient to cause them to exit the energy well corresponding to state 0. Consequently when displacement control is removed the system will adopt the stable equilibrium configuration 010.The case where the surface is in state 101 is shown in Fig. 17. As a result sufficient energy is imparted to the central shell to exit the energy well corresponding to state 0. When the displacement control is removed the system will dynamically jump to state 111 and consequently state 101 does not correspond to a stable equilibrium configuration. We now consider the reason behind the instability of state 101 ingreater detail. It can be observed, with reference to Figs. 12 and 13 that when a shell in state 0 is connected to a shell in state 1, on release from displacement control a much greater proportion of the strain energy is transferred to the shell with the initially-curved common edge (state 0). This means that in state 101 a large proportion of the strain energy is transferred to the middle shell which is always sufficient to trigger snapthrough. The resistance of the middle shell to snapthrough may be increased through the imposition of asymmetric bistability, however, the second stable state is always annihilated before the dynamic snapthrough is overcome.

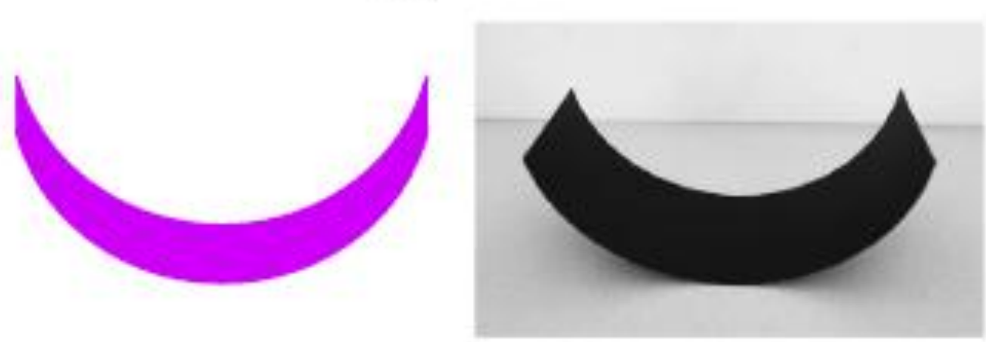
Section (2.2):Experimental Investigation and Conclusions

An experimental investigation is carried out to verify the above conclusions. Both composite shells with and without biasing strips are fabricated. To ensure the shells have continuous, robust and smooth connections, the compound shells are fabricated directly as a whole rectangular shell. Because this research investigates only the number of stable configurations of composite shells, the comparison between the numerical results and the experimental results focuses on the qualitative bistability behaviour. Differences in geometry are the result of manufacturing imperfections and thickness variations.

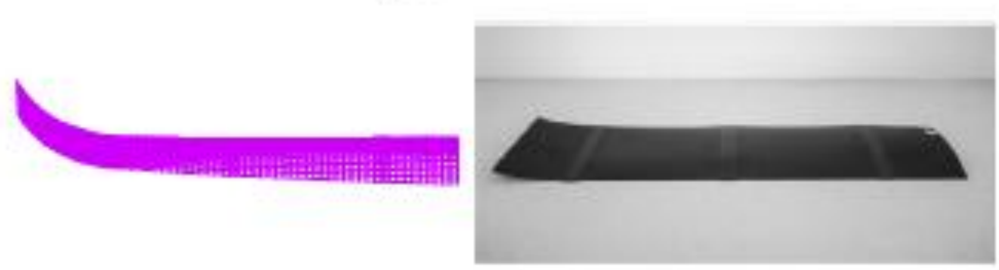
Fig. 18 illustrates the experimental results of the stable configurations of compound composite surface. The physical composite surface shows the same number of stable states as expected. The tripled composite shells with parallel strips shows seven stable configurations in total. A highly multistable surface fabricated



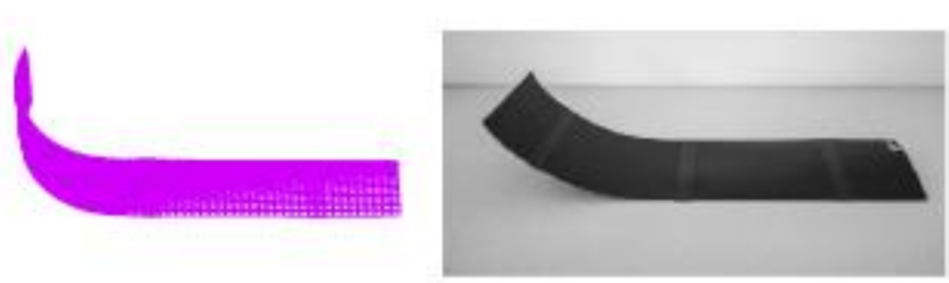
(a) State 000



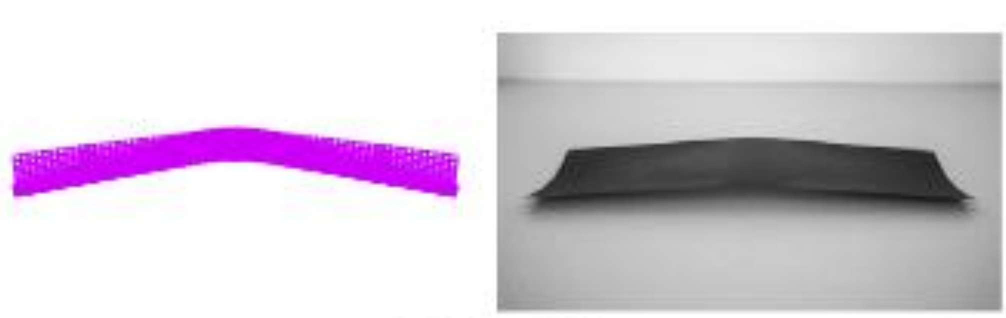
(b) State 111



(c) State 001 (also 100)



(d) State 011 (also 110)



(e) State 010

Fig. 18. The maximum seven stable configurations of tripled bistable shells and the corresponding FEA results.

without conventional fixation and presenting smooth curvature changes has been successfully demonstrated.Composite surfaces possessing highly stability show high potential for use in adaptive applications. In this chapter, inspired by an analog model composed by three bistable von Mises truss systems, surfaces consisting of series connected bistable composite shells are presented. Since the asymmetric bistability of bistable elements is proved to determine the multistability of the whole analog model, the asymmetric bistability of individual shell is tailored by attaching a biasing strip. For coupled bistable composite shells, a quadstable surface is achieved as expected. The multistable surface design is also extended to tripled bistable composite shells which is the basic case of series connected bistable composite shells. By attaching biasing strips parallel to the linking edges, the composite surface demonstrates a higher degree of multistability, specifically seven discrete stable configurations. The investigation into the multistable behaviour of tripled bistable composite shells can be developed to design longer composite surfaces composed of more bistable units. The primary conclusions of this chapter are:

(x) the tripled biased von Mises truss systems can demonstrate a maximum of eight stable states by varying the degree of asymmetric bistability of the individual units;

(xi) coupled bistable composite shells are demonstrated to possess bistable or quadstable behaviour by tailoring the bistable asymmetry of the individual shell;

(xii) three series-connected bistable composite shells with parallel biasing strips may achieve a composite surface possessing seven discrete stable configurations. The theoretically-possible eighth stable state is shown not to exist for square shells.Continuing work will build on the understanding gained in this research to design and construct surfaces where the individual bistable units are connected to form fully three-dimensional adaptive multistable surfaces.

When a vertical displacement d is applied to a single biased von Mises truss of the type shown in Fig. 1, the corresponding vertical load P applied at the apex in conditions of static equilibrium is given by.

$$P = K_b\delta - 2K_s\left(L_0 - \sqrt{l^2 + (h-\delta)^2}\right)\sin\theta, \tag{5}$$

in which θ is the angle between a hinge and the horizontal direction;

K_b is the biasing spring stiffness; K_s is the axial stiffness of the rod; L_0 is the initial length of the rod; l is the half of span width of the truss; and h is the initial height of the truss. The rotation angle θ is evaluated according to

$$\theta = \arctan\left(\frac{h-\delta}{l}\right). \tag{6}$$

The stored strain energy U in the von Mises truss resulting from

the imparted displacement is given by

$$U = \frac{1}{2}P\delta = \frac{1}{2}\left(K_b\delta - 2K_s\left(L_0 - \sqrt{l^2 + (h-\delta)^2}\right)\sin\theta\right)\delta. \tag{7}$$

The strain energy is a uniquely determined function of the vertical displacement δ as the system is energetically conservative. The valuesof K_b and K_s determine the relationship between the strain energy U and the vertical displacement δ.When three identical von Mises trusses are connected by couplingsprings K_c, the loads applied to each truss system from the connecting spring are given by

$$\begin{aligned} P_1 &= K_c(\delta_2 - \delta_1); \\ P_2 &= K_c(\delta_1 - \delta_2) + K_c(\delta_3 - \delta_2); \\ P_3 &= K_c(\delta_2 - \delta_3). \end{aligned} \tag{8}$$

Application of Eq. (4) shows that the equilibrium paths followed

by each truss are also equal to

$$P_i = K_b\delta_i - 2\left(L_0 - \sqrt{l^2 + (h-\delta_i)^2}\right)\sin\theta_i, \quad i = 1,2,3, \tag{9}$$

where the angles between a hinge and the horizontal direction h are

$$\theta_i = \arctan\left(\frac{h - \delta_i}{l}\right), \quad i = 1, 2, 3 \tag{10}$$

Combining these equations, it may be seen that the force-controlled displacement of the two driven trusses may be expressed as a function of the imposed displacement d on the driving truss. Thus the stored strain energy of the complete system is given by

$$U = \sum_{i=1}^{3} \frac{1}{2} P_i \delta_i. \tag{11}$$

Chapter (3)

A Spectral Theory of Linear Operators on Rigged Hilbert Spaces under Analyticity Conditions

It is shown that there exists a dense subspace X of H such that the resolvent$(\lambda - \mathrm{T})^{-1}\varphi$ of the operator T has an analytic continuation from the lower half plane to the upper half plane as an X' valued holomorphic function for any φ ∈X, even when T has a continuous spectrum on R, where $\mathrm{X}'$is a dual space of X. The rigged Hilbert space consists of three spaces X⊂H⊂ X' .Ageneralized Eigen values and a generalized eigenfunction in X' are defined by using the analytic continuation of the resolvent as an operator from X into X' . Other basic tools of the usual spectral theory, such as a spectrum, resolvent, Riesz projection and semi group are also studied in terms of a rigged Hilbert space. They prove to have the same properties as those of the usual spectral theory. The results are applied to estimate asymptotic behavior of solutions of evolution equations.

Section (3.1):Spectral Theory on a Hilbert Space and Gelfand Triplet

A spectral theory of linear operators on topological vector spaces is one of the central issues in functional analysis. Spectra of linear operators provide us with much information about the operators. However, there are phenomena that are not explained by spectra. Consider a linear evolution equation $\frac{dx}{dt} = Tx$defined by some linear operator *T*. It is known that if the spectrum of *T*is included in the left half plane, any solutions *x*(*t*)decay to zero as $t \to \infty$ with an exponential rate, while if there is a point of the spectrum on the right half plane, there are solutions that diverge as $t \to \infty$ On the other hand, if the spectrum set is included in the imaginary axis, the asymptotic behavior of solutions is far from trivial; for a finite dimensional problem, a solution *x*(*t*)is a polynomial in *t*, however, for an infinite dimensional case, a solution can decay exponentially even if the spectrum does not lie on the left half plane. In this sense, the spectrum set does not determine the asymptotic behavior of solutions. Such an exponential decay of a solution is known as Landau damping in plasma physics , and is often observed for Schrodinger operators. Now it is known that such an exponential decay can be induced by resonance poles or generalized Eigen values s.Eigen values s of a linear operator $T$are singularities of the resolvent$(\lambda - T)^{-1}$. Resonance poles are obtained as singularities of a continuation of the resolvent in some sense. In the literature, resonance poles are defined in several ways: Let Tbe a selfadjoint operator (for simplicity) on a Hilbert space H with the inner product $(\cdot,\cdot)$. Suppose that Thas the continuous spectrum $\sigma_c(T)$ on the real axis. For Schrodinger operators, spectral deformation (complex distortion) technique is often employed to define resonance poles. Agiven operator $T$is deformed by some transformation so that the continuous spectrum$\sigma_c(T)$moves to the upper (or lower) half plane. Then, resonance poles are defined as Eigen values s of the deformed operator. One of the advantages of the method is that studies of resonance poles are reduced to the usual spectral theory of the deformed operator on a Hilbert space. Another way to define resonance poles is to use analytic continuations of matrix elements of the resolvent. By the definition of the spectrum, the resolvent $(\lambda - T)^{-1}$diverges in norm when $\lambda \in \sigma_c(T)$. However, the matrix element $((\lambda - T)^{-1}\varphi, \varphi)$for some "good" function $\varphi \in H$may exist for $\lambda \in \sigma_c(T)$, and the function $f(\lambda) = ((\lambda - T)^{-1}\varphi, \varphi)$may have an analytic continuation from the lower half plane to the upper half plane through an interval on $\sigma_c(T)$. Then, the analytic continuation may have poles on the upper half plane, which is called a resonance pole or a generalized Eigen values . In the study of reaction diffusion equations, the Evans function is often used, whose zeros give Eigen values s of a given differential operator. Resonance poles can be defined as zeros of an analytic continuation of the Evans function for other definitions of resonance poles.

Although these methods work well for some special classes of Schrodinger operators, an abstract spectral theory of resonance poles has not been developed well. In particular, a precise definition of an Eigen function associated with a resonance pole is not obviousin general. Clearly a pole of a matrix element or the Evans function does not provide an Eigen function. In Chiba , a definition of the eigenfunction associated with a resonance pole is suggested for a certain operator obtained from the Kuramotomodel. It is shown that the Eigen function is a distribution, not a usual function. This suggests that an abstract theory of topological vector spaces should be employed for the study of a resonance pole and its Eigen function of an abstract linear operator. Our approach based on rigged Hilbert spaces allows one to develop a spectral theory of resonance poles in a parallel way to "standard course of functional analysis". To explain our idea based on rigged Hilbert spaces, let us consider the multiplication operator $M : \varphi(\omega) \to \omega_{\varphi}(\omega)$on the Lebesgue space $L^2(\mathbf{R})$. The resolvent is given as

$$\left((\lambda-\mathcal{M})^{-1}\phi,\psi^*\right)=\int_{\mathbf{R}}\frac{1}{\lambda-\omega}\phi(\omega)\psi(\omega)d\omega,$$

Where $\psi^*=\overline{\psi(\omega)}$, which is employed to avoid the complex conjugate of $\psi(\omega)$in the right hand side. This function of λ is holomorphic on the lower half plane, and it does not exist for $\lambda \in \mathbf{R}$; the continuous spectrum of M is the whole real axis. However, if φ and ψ have analytic continuations near the real axis, the right hand side has an analytic continuation from the lower half plane to the upper half plane, which is given by

$$\int_{\mathbf{R}}\frac{1}{\lambda-\omega}\phi(\omega)\psi(\omega)d\omega+2\pi\mathrm{i}\phi(\lambda)\psi(\lambda),$$

where $i = \sqrt{-1}$. Let X be a dense subspace of $L^2(\mathbf{R})$consisting of functions having analytic continuations near the real axis. A mapping, which maps $\varphi \in X$ to the above value, defines a continuous linear functional on X, that is, an element of the dual space X', if X is equipped with a suitable topology. Motivated by this idea, we define the linear operator $A(\lambda) : X \to X'$ to be

$$\langle A(\lambda)\psi \mid \phi\rangle=\begin{cases}\int_{\mathbf{R}}\frac{1}{\lambda-\omega}\psi(\omega)\phi(\omega)d\omega+2\pi\mathrm{i}\psi(\lambda)\phi(\lambda) & (\mathrm{Im}(\lambda)>0),\\ \lim_{y\to -0}\int_{\mathbf{R}}\frac{1}{x+\mathrm{i}y-\omega}\psi(\omega)\phi(\omega)d\omega & (x=\lambda\in\mathbf{R}),\\ \int_{\mathbf{R}}\frac{1}{\lambda-\omega}\psi(\omega)\phi(\omega)d\omega & (\mathrm{Im}(\lambda)<0),\end{cases} \quad \mathbf{(1)}$$

for $\psi, \varphi \in X$, where $\langle \cdot \mid \cdot \rangle$is a paring for (X', X). When $Im(\lambda) < 0$, $A(\lambda) = (\lambda - M)^{-1}$, while when $\mathrm{Im}(\lambda) \geq 0$, $A(\lambda)\psi$ is not included in $L^2(\mathbf{R})$but an element of X'. In this sense, $A(\lambda)$ is called the analytic continuation of the resolvent of Min the generalized sense. In this manner, the triplet $X \subset L^2(\mathbf{R}) \subset X'$, which is called the rigged Hilbert space or the Gelfandtriplet ,in introduced.In this chapter, a spectral theory on a rigged Hilbert space is proposed for an operator of the form $T = H + K$, where H is a selfadjoint operator on a Hilbert space H, whose spectral measure has an analytic continuation near the real axis, when the domain is restricted to some dense subspace $X$of H, as above. K is an operator densely defined on Xsatisfying certain boundedness conditions. Our purpose is to investigate spectral properties of the operator $T = H + K$. At first, the analytic continuation $A(\lambda)$of the resolvent$(\lambda - H)^{-1}$is defined as an operator from X intoX'in the same way as Eq.(1). In general, $A(\lambda) : X \to X'$is defined on a nontrivial Riemann surface of λso that when λlies on the original complex plane, it coincides with the usual resolvent $(\lambda - H)^{-1}$. The usual eigeequation $(\lambda - T)v = 0$ is rewritten as

$$(\lambda - H) \circ \left(id - (\lambda - H)^{-1}K\right)v = 0.$$

By neglecting the first factor and replacing $(\lambda - H)^{-1}$ by its analytic continuation $A(\lambda)$, we arrive at the following definition: If the equation

$$\left(id - A(\lambda)K^{\times}\right)\mu = 0 \tag{2}$$

has a nonzero solution μ in X', such a λ is called a generalized Eigen values (resonance pole) and μ is called a generalized Eigen function, where $K^{\times}: X' \to X'$is a dual operator of K. When λlies on the original complex plane, the above equation is reduced to the usual eigenequation. In this manner, resonance poles and corresponding Eigen functions are naturally obtained without using spectral deformation technique or poles of matrix elements.Similarly, the resolvent in the usual sense is given by

$$(\lambda - T)^{-1} = (\lambda - H)^{-1} \circ \left(id - K(\lambda - H)^{-1}\right)^{-1}.$$

an analytic continuation of the resolvent of Tin the generalized sense is defined to be

$$\mathcal{R}_\lambda = A(\lambda) \circ \left(id - K^\times A(\lambda)\right)^{-1} : X \to X' \tag{3}$$

When λ lies on the original complex plane, this is reduced to the usual resolvent $(\lambda - T)^{-1}$. With the aid of the generalized resolvent R_λ, basic concepts in the usual spectral theory, such as eigenspaces, algebraic multiplicities, point, continuous, residual spectra, Riesz projections are extended to those defined on a rigged Hilbert space. It is shown that they have the same properties as the usual theory. For example, the generalized Riesz projection Π_0 for an isolated resonance pole λ_0 is defined by the contour integral of the generalized resolvent.

$$\Pi_0 = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathcal{R}_\lambda d\lambda : X \to X'. \tag{4}$$

Properties of the generalized Riesz projection Π_0 areinvestigated in detail. Note that in the most literature, the eigenspace associated with a resonance pole is *defined* to be the range of the Riesz projection. In this chapter, the eigenspace of a resonance pole is defined as the set of solutions of the Eigen equation, and it is *proved* that it coincides with the range of the Riesz projection as the standard functional analysis. Any function $\varphi \in X$proves to be uniquely decomposed as $\varphi = \mu_1 + \mu_2$, where $\mu_1 \in \Pi_0 X$and $\mu_2 = (id - \Pi_0)X$, both of which are elements of X'. These results play an important role when applying the theory to dynamical systems. The generalized Riesz projection around a resonance pole λ_0 on the left half plane defines a stable subspace in the generalized sense, both of which are subspaces of X'. Then, the standard idea of the dynamical systems theory may be applied to investigate the asymptotic behavior and bifurcations of an infinite dimensional dynamical system. Such a dynamics induced by a resonance pole is not captured by the usual eigenvalues s. Many properties of the generalized spectrum will be shown. In general, the generalized spectrum consists of the generalized point spectrum , the generalized continuous spectrum and the generalized residual spectrum If the operator K satisfies a certain compactness condition, the RieszSchauder theory on a rigged Hilbert space applies to conclude that the generalized

(5)

spectrum consists only of a countable number of resonance poles having finite multiplicities. It is remarkable that even if the operator T has the continuous spectrum , the generalized spectrum consists only of a countable number of resonance poles when K satisfies the compactness condition. Since the topology on the dual space $X'$is weaker than that on the Hilbert space H, the continuous spectrum of T disappears, while eigenvalues s remain to exist as the generalized spectrum. This fact is useful to estimate embedded eigenvalues s. eigenvalues s embedded in the continuous spectrum is no longer embedded in our spectral theory. Thus, the Riesz projection is applicable to obtain eigenspaces of them .Although resonance poles have been well studied for Schrodinger operators, a spectral theory in this chapter is motivated by establishing bifurcation theory of infinite dimensional dynamical systems, for which spectral deformation technique is not applied. In Chiba, a bifurcation structure of an infinite dimensional coupled oscillators (Kuramotomodel) is investigated by means of rigged Hilbert spaces. It is shown that when a resonance pole of a certain linear operator, which is obtained by the linearization of the system around a steady state, gets across the imaginary axis as a parameter of the system varies, then a bifurcation occurs.This part is devoted to a review of the spectral theory of a perturbed selfadjoint operator on a Hilbert space to compare the spectral theory. Let H be a Hilbert space over **C**. The inner product is defined so that

$$(a\varphi,\psi) = (\varphi,\overline{a}\psi) = a(\varphi,\psi), \tag{6}$$

where$\overline{a}$is the complex conjugate of $a \in \boldsymbol{C}$. Let us consider an operator $T = H + K$defined on a dense subspace of H, where H is a selfadjoint operator, and K is a compact operator on H which need not be selfadjoint. Let λ and $v=v_\lambda$ be an Eigen values and an Eigen function, respectively, of the operator T defined by the equation $\lambda_v = H_v + K_v$. This is rearranged as

$$(\lambda - H)(id - (\lambda - H)^{-1}K_v = 0,$$

Where *id* denotes the identity on *H*. In particular, when *λ* is not an Eigen values of *H*, it is an Eigen values of *T* if and only if *id*−(*λ* −*H*)−1*K* is not injective in *H*. Since the essential spectrum is stable under compact perturbations, the essential spectrum σ_c(*T*) of *T* is the same as that of *H*, which lies on the real axis. Since *K* is a compact perturbation, the Riesz–Schauder theory shows that the spectrum outside the real axis consists of the discrete spectrum; for any $\delta > 0$, the number of Eigen values s satisfying $|Im(\lambda)| \geq \delta$ is finite, and their algebraic multiplicities are finite. Eigen values s may accumulate only on the real axis. To find Eigen values s embedded in the essential spectrum $\sigma_e(T)$is a difficult and important problem. In this chapter, a new spectral theory on rigged Hilbert spaces will be developed to obtain such embedded Eigen values s and corresponding eigenspaces. Let R_λ=(*λ* −*T*)−1 be the resolvent. Let $\lambda_j$be an Eigen values of *T* outside the real axis, and $\gamma_j$be a simple closed curve enclosing separated from the rest of the spectrum. The projection to the generalized eigenspace V_j=U_n≥1Ker(λ_j −*T*)^n is given by

$$\Pi_j = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_j} R_\lambda d\lambda. \tag{7}$$

Let us consider the semigroup$e^{\mathrm{i}Tt}$ generated by iT. Since iHgenerates the C^0 semigroup e^{iHt} and $K$is compact, i_T. also generates the C^0 semigroup It is known that $e^{iTt}$is obtained by the Laplace inversion formula (Hille and Phillips).

$$e^{\mathrm{i}Tt}\phi = \frac{1}{2\pi \mathrm{i}} \lim_{x\to\infty} \int_{-x-\mathrm{i}y}^{x-\mathrm{i}y} e^{\mathrm{i}\lambda t}(\lambda - T)^{-1}\phi d\lambda, \tag{8}$$

$for\ t\ > 0 and\ \varphi\ \in D(T)$, where $y > 0$is chosen so that all Eigen values s $\lambda$of T satisfy Im$(\lambda)\ > -y$, and the limit $x\ \to \infty$ exists with respect to the topology of *H*. Thus the contour is the horizontal line on the lower half plane. $Let\ \varepsilon\ > 0$ be a small number and $\lambda_0,\ ...,\ \lambda_N$Eigen values s of *T* satisfying Im$(\lambda_j)\ \leq -\varepsilon, j = 0, \ldots, N$. The residue theorem provides

$$e^{\mathrm{i}Tt}\phi = \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} e^{\mathrm{i}xt+\varepsilon t}(x - \mathrm{i}\varepsilon - T)^{-1}\phi dx + \frac{1}{2\pi \mathrm{i}} \sum_{j=0}^{N} \int_{\gamma_j} e^{\mathrm{i}\lambda t}(\lambda - T)^{-1}$$

where$\gamma_j$is a sufficiently small closed curve enclosing λ_j. Let M_j be the smallest integer such that $(\lambda_j - T)^{M_j}\prod_j = 0$. This is less thanor equal to the algebraic multiplicity of λ_j. Then, e^{iTt} is calculated as

$$e^{\mathrm{i}Tt}\phi = \frac{1}{2\pi\mathrm{i}}\int_{\mathbf{R}} e^{\mathrm{i}xt+\varepsilon t}(x - \mathrm{i}\varepsilon - T)^{-1}\phi dx + \sum_{j=0}^{N} e^{\mathrm{i}\lambda_j t}\sum_{k=0}^{M_j-1}\frac{(-\mathrm{i}t)^k}{k!}(\lambda_j - T)^k \Pi_j\phi.$$

The sectionond term above diverges as $t \to \infty$because $Re(i\lambda_j) \geq \varepsilon$. On the other hand, if there are no Eigen values s on the lower half plane, we obtain

$$e^{\mathrm{i}Tt}\phi = \frac{1}{2\pi\mathrm{i}}\int_{\mathbf{R}} e^{\mathrm{i}xt+\varepsilon t}(x - \mathrm{i}\varepsilon - T)^{-1}\phi dx,$$

for any small $\varepsilon > 0$. In such a case, the asymptotic behavior of e^{iTt} is quite nontrivial. One of the purposes in this chapter is to give a further decomposition of the first term above under certain analyticity conditions to determine the dynamics of e^{iTt}. In the previous part, we give the review of the spectral theory of the operator $T = H + K$on *H*. In this part, the notion of spectra, Eigen functions, resolvents and projections are extended by means of a rigged Hilbert space. It will be shown that they have similar properties to those on *H*. They are used to estimate the asymptotic behavior of the semigroup e^{iTt} and to find embedded Eigen values s.Let X'be a locally convex Hausdorff topological vector space over **C** and X'its dual space. X' is a set of continuous anti linear functional on *X*. For $\mu \in X' and\ \varphi \in X, \mu(\varphi)$is denoted by $\langle\mu\,|\varphi\rangle$.For any $a, b \in \boldsymbol{C}, \varphi, \psi \in X and\ \mu, \xi \in X'$the equalities

$$\langle\mu \mid a\varphi + b\psi\rangle = \overline{a}\langle\mu \mid \varphi\rangle + \overline{b}\langle\mu \mid \psi\rangle, \tag{9}$$

$$\langle a\mu + b\xi \mid \varphi\rangle = a\langle\mu \mid \varphi\rangle + b\langle\xi \mid \varphi\rangle, \tag{10}$$

hold. In this chapter, an element of $X'$is called a generalized function . Several topologies can be defined on the dual space X'. Two of the most usual topologies are the weak dual topology (weak $*$topology) and the strong dual topology (strong $*$ topology). A sequence $\{\mu_j\} \subset X'$is said to be weakly convergent to $\mu \in X if\ \langle\mu_j|\varphi\rangle \to \langle\mu\,|\varphi\rangle$for each $\varphi \in X$; a sequence $\{\mu_j\} \subset X'$is said to be strongly convergent to μ $\in X'$if $\langle\mu_j|\varphi\rangle \to \langle\mu\,|\varphi\rangle$uniformly on any bounded subset of X.Let *H* be a Hilbert space with the inner product $(\cdot,\cdot)$ such that *X* is a dense subspace of *H*. Since a Hilbert space is isomorphic to its dual space, we obtain $H \subset X'$through $H \neq H'$.

Definition (3.1.1)[3]: If a locally convex Hausdorff topological vector space *X* is a dense sub-space of a Hilbert space *H*and a topology of *X* is stronger than that of *H*, the triplet

$$X \subset H \subset X' (11)$$

is called the *rigged Hilbert space*or the *Gelfand triplet*. The *canonical inclusion* *i*: *X*→$X'$is defined as follows:for $\psi \in X$, we denote *i*$(\psi) by \mid \varphi|$, which is defined to be

$$i(\psi)(\varphi) = \langle \psi \mid \varphi \rangle = (\psi, \varphi), \qquad (12)$$

for any $\varphi \in X$ (note that we also use $i = \sqrt{-1}$). The inclusion from *H* into $X'$is also defined as above. It is easy to show that the canonical inclusion is injective if and only if *X* is a dense subspace of *H*, and the canonical inclusion is continuous if and only if a topology of *X* is stronger than that of *H*.A topological vector space *X* is called Mantel if it is barreled and every bounded set of *X*is relatively compact. A Mantel space has a convenient property that on a bounded set *A* of a dual space of a Mantel space, the weak dual topology coincides with the strong dual topology. In particular, a weakly convergent series in a dual of a Mantel space also converges with respect to the strong dual topology. Furthermore, a linear map from a topological vector space to a Mantel space is a compact operator if and only if it is a bounded operator. It is known that the theory of rigged Hilbert spaces works best when the space *X*is a Mantel or a nuclear space .

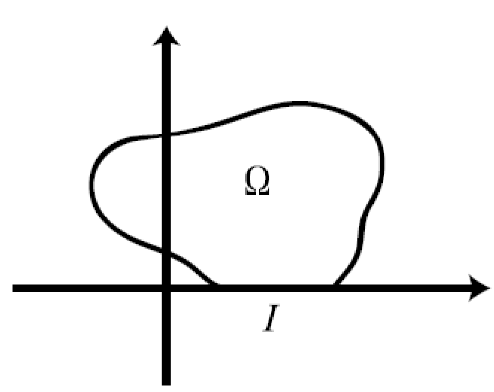


Fig.1.Adomainonwhich $E[\psi, \varphi](\omega)$is holomorphic

and Komatsu for sufficient conditions for a topological vector space to be a Mantel space or a nuclear space. Let *H* be a Hilbert space over **C** and *H* a selfadjoint operator densely defined on *H* with the spectral measure $\{E(B)\}B \in B$; that is, *H* is expressed as $H = \int_R \omega dE(\omega)$. Let *K* be some linear operator densely defined on *H*. Our purpose is to investigate spectral properties of the operator $T = H + K. Let\ \Omega \subset \boldsymbol{C}$ be a simply connected open domain in the upper half plane such that the interpart of the real axis and the closure of $\Omega$is a connected interval $\tilde{I}. Let\ I = \tilde{I} \backslash \partial \tilde{I}$ be an open interval (see Fig.1). For a given $T = H + K$, we suppose that there exists a locally convex Hausdorff vector space $X(\Omega)$ over **C** satisfying the following conditions.

(X_1) $X(\Omega)$ is a dense subspace of *H*.

(X_2) A topology on $X(\Omega)$ is stronger than that on *H*.

(X_3) $X(\Omega)$is a quasi-complete barreled space.

(X_4) For any $\varphi \in X(\Omega)$, the spectral measure $(E(B)\varphi, \varphi)$is absolutely continuous on the interval *I*. Its density function, denoted by $E[\varphi, \varphi](\omega)$, has an analytic continuation to $\Omega \cup I$.

$(X_5$X5) For each $\lambda \in I \cup \Omega$, the bilinear form $E[\cdot,\cdot](\lambda) : X(\Omega) \times X(\Omega) \to \boldsymbol{C}$is separately continuous $(i.e. E[\cdot, \varphi](\lambda) : X(\Omega) \to$**C** and $E[\varphi,\cdot](\lambda): X(\Omega) \to \boldsymbol{C}$are continuous for fixed $\varphi \in X(\Omega))$.

Because of (X_1) and (X_2), the rigged Hilbert space $X(\Omega) \subset H \subset X(\Omega)'$is well defined, where$X(\Omega)'$is a space of continuous *anti*-linear functional and the canonical inclusion *I* is defined by Eq.(12). Sometimes we denote $i(\psi)$ by ψ for simplicity by identifying if $X(\Omega)$ with $X(\Omega)$. The assumption (X_3) is used to define Pettis integrals and Taylor expansions of $X(\Omega)'$valued holomorphic functions in Part3.5(refer to Treves for basic terminology of topological vector spaces such as quasi-complete and barreled space. In this chapter, to understand precise definitions of them is not so important; it is sufficient to know that an integral and holomorphy of $X(\Omega)'$valued functions are well defined if $X(\Omega)$is quasi-complete barreled. For example, Mantel spaces, Fréchet spaces, Banach spaces and Hilbert spaces are barreled. Due to the assumption (X4) with the aid of the polarization identity, we can show that $(E(B)\varphi,\psi)$is absolutelycontinuous on I for any $\varphi,\psi \in X(\Omega)$. $Let\ E[\varphi,\psi](\omega)$be the density function;

$$d\big(E(\omega)\phi,\psi\big) = E[\phi,\psi](\omega)d\omega, \quad \omega \in I. \tag{13}$$

Then, $E[\varphi,\ \psi](\omega)$is holomorphic in $\omega \in I \cup \Omega$. We will use the above notation for any $\omega \in \boldsymbol{R}$ for simplicity, although the absolute continuity is assumed only on I. Since $E[\varphi,\psi](\omega)$is absolutely continuous on I, $H$is assumed not to have eigenvalues s on$I$. (X_5) is used to prove the continuity of a certain operator .

Let A be a linear operator densely defined on $X(\Omega)$. Then, the dual operator $A'$is defined as follows: the domain $D(A)'$ is the set of elements $\mu \in X(\Omega)'$such that the mapping $\varphi \to \langle\mu\,|A\varphi\rangle$from $D(A) \subset X(\Omega)$into **C**is continuous. Then, $A': D(A') \to X(\Omega)'$is defined by

$$\langle A'\mu \mid \phi\rangle = \langle\mu \mid A\phi\rangle, \quad \phi \in \mathrm{D}(A),\ \mu \in \mathrm{D}\big(A'\big). \tag{14}$$

If $A$is continuous on $X(\Omega)$, then $A'$is continuous on $X(\Omega)'$for both of the weak dual topology and the strong dual topology. The (Hilbert) adjoint $A^*$of A is defined through $(A\varphi,\psi) = (\varphi,A^*\psi)$as usual when A is densely defined on H.

Lemma (3.1.2)[3]:*Let Abe a linear operator densely defined on H. Suppose that there exists a dense subspace Y of $X(\Omega)$such thA^*at $Y \subset X(\Omega)$so that the dual $(A^*)'$is defined. Then,* $((A^*)'$ *is an extension of A and* $i\ o\ A = (A\ *)'o\ i$ /D(A)*. In particular,* $D((A\ *)')$⊃/D(A)*.*

Proof: By the definition of the canonical inclusion i, we have

$$i(A\psi)(\phi) = (A\psi,\phi) = \big(\psi,A^*\phi\big) = \langle\psi \mid A^*\phi\rangle = \big\langle\big(A^*\big)'\psi \mid \phi\big\rangle, \tag{15}$$

for any $\psi \in D(A)$ and $\varphi \in Y$.

In what follows, we denote $(A^*)'$by $A^\times$. Thus Eq.(15)means $i\ o\ A\ = A \times oi_{D(A)}$. Note that $A^\times$=$A'$when A is self-adjoint. For the operators H and K, we suppose that

(X_6) there exists a dense subspace $Y\ of\ X(\Omega)\ such\ that\ HY \subset X(\Omega)$.

(X_7) $K$is Hbounded and $K^*Y \subset X(\Omega)$.

(X_8) $K^{\times}A(\lambda)iX(\Omega) \subset iX(\Omega) for\ any\ \lambda \in \{Im(\lambda) < 0\} \cup I \cup \Omega.$

The operator $A(\lambda){:}\, iX(\Omega) \to X(\Omega)'$will be defined later. Recall that when *K* is *H*-bounded ,D(*T*) =D(*H*)and $K(\lambda - H)^{-1}$is bounded on *H* for $\lambda \notin \boldsymbol{R}$. In some sense, (X_8) is a "dual version" of this condition because *A*(*λ*)proves to be an extension of $(\lambda - H)^{-1}$. In particular, we will show that *K*ˣ*A*(*λ*)*i*=*I* (*K*$(\lambda - H)^{-1}$when $Im(\lambda) < 0$. Our purpose is to investigate the operator$T = H + K$with these conditions. Due to (X_6) and (X_7), the dual operator $T^{\times}$of T^*=*H*+K^* is well defined. It follows that D($T^{\times}$)=D($H^{\times}$) ∩D($K^{\times}$)and

$$\mathrm{D}(T^{\times}) \supset i\mathrm{D}(T) = i\mathrm{D}(H) \supset iY.$$

In particular, the domain of *T*ˣ is dense in $X(\Omega)'$.To define the operator *A*(*λ*), we need the next lemma.

Lemma (3.1.3)[3]:*Suppose that a function q*(*ω*)*is integral on* R *and holomorphic on ΩUI. Then, the function*

$$Q(\lambda) = \begin{cases} \int_{\mathbf{R}} \frac{1}{\lambda-\omega} q(\omega)d\omega & (\mathrm{Im}(\lambda) < 0), \\ \int_{\mathbf{R}} \frac{1}{\lambda-\omega} q(\omega)d\omega + 2\pi \mathrm{i} q(\lambda) & (\lambda \in \Omega), \end{cases} \tag{16}$$

is holomorphic on $\{\lambda \mid Im(\lambda) < 0\} \cup \Omega \cup I$.

Proof: Putting $\lambda = x + iy$ with $x, y \in \boldsymbol{R}$yields

$$\int_{\mathbf{R}} \frac{1}{\lambda - \omega} q(\omega)d\omega = \int_{\mathbf{R}} \frac{x-\omega}{(x-\omega)^2 + y^2} q(\omega)d\omega - \mathrm{i} \int_{\mathbf{R}} \frac{y}{(x-\omega)^2 + y^2} q(\omega)d\omega.$$

Due to the formula of the Poisson kernel, the equalities

$$\lim_{y\to+0} \int_{\mathbf{R}} \frac{y}{(x-\omega)^2 + y^2} q(\omega)d\omega = \pi q(x), \qquad \lim_{y\to-0} \int_{\mathbf{R}} \frac{y}{(x-\omega)^2 + y^2} q(\omega)d\omega = -\pi q(x),$$

hold when q is continuous at $x \in I$ Thus we obtain

$$\lim_{y\to-0} \int_{\mathbf{R}} \frac{1}{\lambda - \omega} q(\omega)d\omega = \lim_{y\to+0} \Big(\int_{\mathbf{R}} \frac{1}{\lambda-\omega} q(\omega)d\omega + 2\pi \mathrm{i} q(\lambda) \Big) = \pi V(x) + \pi \mathrm{i} q(x),$$

where

$$V(x) := \lim_{y\to 0} \frac{1}{\pi} \int_{\mathbf{R}} \frac{x-\omega}{(x-\omega)^2 + y^2} q(\omega)d\omega$$

is the Hilbert transform of *q*. It is known that *V*(*x*) is Lipschitz continuous on *I* if *q*(*x*) is Therefore, two holomorphic functions in Eq.(16)coincide with one another on *I* and they are continuous on *I*. This proves that *Q*(*λ*) is holomorphic on $\{\lambda \mid Im(\lambda) < 0\} \cup \Omega \cup I$.Put $u_\lambda = (\lambda - H)^{-1}\psi$for $\psi \in H$. In general, $u_\lambda$is not included in *H* when $\lambda \in I$because of the continuous spectrum of *H*. Thus $u_\lambda$does not have an analytic continuationfrom the lower half plane to *Ω* with respect to λas an *H*-valued function. To define an analytic continuation of u_λ, we regard it as a generalized function in $X(\Omega)'$ by the canonical inclusion. Then, the action of $i((\lambda - H)^{-1}\psi)$is given by

$$i((\lambda - H)^{-1}\psi)(\phi) = ((\lambda - H)^{-1}\psi, \phi) = \int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega, \quad \mathrm{Im}(\lambda) < 0.$$

Because of the assumption (X4), this quantity has an analytic continuation to $\Omega \cup I$as

$$\int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega + 2\pi \mathrm{i} E[\psi, \phi](\lambda), \quad \lambda \in \Omega.$$

Motivated by this observation, define the operator $A(\lambda) : iX(\Omega) \to X(\Omega)'$to be

$$\langle A(\lambda)\psi \mid \phi \rangle = \begin{cases} \int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega + 2\pi \mathrm{i} E[\psi, \phi](\lambda) & (\lambda \in \Omega), \\ \lim_{y \to -0} \int_{\mathbf{R}} \frac{1}{x + \mathrm{i}y - \omega} E[\psi, \phi](\omega) d\omega & (\lambda = x \in I), \\ \int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega & (\mathrm{Im}(\lambda) < 0), \end{cases} \tag{17}$$

for any $\psi \in iX(\Omega), \varphi \in X(\Omega)$. Indeed, we can prove by using (X_5) that $A(\lambda)\psi$is a continuous functional, $\langle A(\lambda)\psi|\varphi\rangle$is holomorphic on $\{Im(\lambda) < 0\} \cup \Omega \cup I$. When $Im(\lambda) < 0$, we have $\langle A(\lambda)\psi|\varphi\rangle = ((\lambda - H)^{-1}\psi, \varphi)$. In this sense, the oper-ator *A*(*λ*) is called the analytic continuation of the resolvent$(\lambda - H)^{-1}$as a generalized function. By using it, we extend the notion of eigenvalues andeigenfunctions. Recall that the equation for eigenfunctions of *T*is given by $(id - (\lambda - H)^{-1}K)_v = 0$. Since the analytic continuation of $(\lambda - H)^{-1} in\, X(\Omega)'$is *A*(*λ*), we make the following definition.

Definition (3.1.4)[3]: Let $R(A(\lambda))$be the range of $A(\lambda)$. If the equation

$$(id - A(\lambda)K^{\times})\mu = 0 \tag{18}$$

has a nonzero solution $\mu\ in\ R(A(\lambda))$for some $\lambda \in \Omega \cup I \cup \{\lambda \mid Im(\lambda) < 0\}, \lambda$ is called a *generalized eigen value* of T and μ is called a *generalized eigenfunction* associated with λ. A generalized eigenvalue on Ω is called a *resonance pole.* Note that the assumption (X8) is used to define $A(\lambda)K^{\times}\mu$for $\mu \in R(A(\lambda))$ because the domain of $A(\lambda)$is $iX(\Omega)$. Applied by $K^{\times}$, is rewritten as

$$\big(id - K^{\times} A(\lambda)\big) K^{\times}\mu = 0. \tag{19}$$

If $K^{x}\mu=0$, Eq.(18)shows μ =0. This means that if $\mu \neq 0$ is a generalized eigenfunc-tion, $K^{x}\mu\neq0$ and $id-K^{x}A(\lambda)$ is not injective on $iX(\Omega)$. Conversely, if $id-K^{\times}A(\lambda)$ is not injective on $iX(\Omega)$, there is a function $\varphi \in iX(\Omega)$such that $(id-K^{x}A(\lambda))\varphi$ =0. Applying $A(\lambda)$from the left, we see that $A(\lambda)\varphi$ is a generalized Eigen function. Hence, λ is a generalized Eigen value if and only if $id-K^{x}A(\lambda)$is not injective on $iX(\Omega)$.

Theorem (3.1.5)[3]:*Let λ be a generalized Eigen value of T and μa generalized eigenfunction associated with λ. Then the equality*

$$T^{\times}\mu = \lambda\mu \tag{20}$$

holds.

Proof: At first, let us show $D(\lambda - H^{x}) \supset R(A(\lambda))$. By the operational calculus, we have $E[\psi, ('\lambda - H)\varphi](\omega) = (\lambda - \omega)E[\psi, \varphi](\omega)$. When$\lambda \in \Omega$, this gives

For

$$\begin{aligned}\langle A(\lambda)\psi \mid (\bar{\lambda} - H)\phi\rangle &= \int_{\mathbf{R}} \frac{1}{\lambda - \omega} E\big[\psi, (\bar{\lambda} - H)\phi\big](\omega)d\omega + 2\pi\mathrm{i}E\big[\psi, (\bar{\lambda} - H)\phi\big](\lambda) \\ &= \int_{\mathbf{R}} E[\psi, \phi](\omega)d\omega + 2\pi\mathrm{i}(\lambda - \omega)|_{\omega=\lambda} E[\psi, \phi](\lambda) \\ &= \langle\psi \mid \phi\rangle,\end{aligned}$$

For $\psi \in X(\Omega)$ and $\varphi \in Y$. It is obvious that $\langle\psi|\varphi\rangle$is continuous in $\varphi$with respect to the topology of $X(\Omega)$. This proves that $D(\lambda - H^x) \supset R(A(\lambda))$and $(\lambda - H^x)A(\lambda) = id: iX(\Omega) \to iX(\Omega)$. When μ is a generalized Eigen function, $\mu \in D(\lambda - H^x)$because $\mu = A(\lambda)K^x\mu$. Then, Eq.(18)provides

$$(\lambda - H^{\times})(id - A(\lambda)K^{\times})\mu = (\lambda - H^{\times} - K^{\times})\mu = (\lambda - T^{\times})\mu = 0.$$

The proofs for the cases $\lambda \in I$and $Im(\lambda) < 0$ are done in the same way.This theorem means that λ is indeed an Eigen value of the dual operator T^x. In general, the set of generalized Eigen values is a proper subset of the set of Eigen values of T^x. Since the dual space $X(\Omega)'$ is "too large", typically every point on $\Omega$is an Eigen value of T. In this sense, generalized eigenvalues are wider concept than Eigen values of T, while narrower concept than eigenvalues of T^x. In the literature, resonance poles are defined as poles of an analytic continuation of a matrix element of the resolvent. Our definition is based on a straightforward extension of the usual Eigen equation and it is suitable for systematic studies of resonance poles. Before defining a multiplicity of a generalized Eigen value, it is convenient to investigate properties of the operator $A(\lambda)$. $For\ n = 1,2, \ldots$ let us define the linear operator $A(n)(\lambda) : iX(\Omega) \to X(\Omega)'$to be

$$\langle A^{(n)}(\lambda)\psi \mid \phi\rangle = \begin{cases} \int_{\mathbf{R}} \frac{1}{(\lambda-\omega)^n} E[\psi,\phi](\omega)d\omega + 2\pi \mathrm{i} \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}|_{z=\lambda} E[\psi,\phi](z) \\ \quad (\lambda \in \Omega), \\ \lim_{y\to -0} \int_{\mathbf{R}} \frac{1}{(x+\mathrm{i}y-\omega)^n} E[\psi,\phi](\omega)d\omega, \quad (\lambda = x \in I), \\ \int_{\mathbf{R}} \frac{1}{(\lambda-\omega)^n} E[\psi,\phi](\omega)d\omega, \quad (\mathrm{Im}(\lambda) < 0). \end{cases} \quad \textbf{(21)}$$

It is easy to show by integration by parts that $\langle A^{(n)}(\lambda)\psi|\varphi\rangle$is an analytic continuation of $((\lambda - H)^{-1}n\psi, \varphi)$from the lower half plane to $\Omega. A^{(1)}(\lambda)$is also denoted by $A(\lambda)$ as before. The next proposition will be often used to calculate the generalized resolvent and projections.

Proposition(3.1.6)[3]:*For any integers* $j \geq n \geq 0$, *the operator* $A(j)(\lambda)$ *satisfies*

(i) $(\lambda - H^{\times})^n A^{(j)}(\lambda) = A^{(j-n)}(\lambda)$, *where* $A^{(0)}(\lambda) := id$.

(ii) $A^{(j)}(\lambda)(\lambda - H^{\times})^n|_{iX(\Omega)\cap D(A^{(j)}(\lambda)(\lambda-H^{\times})^n)} = A^{(j-n)}(\lambda)|_{iX(\Omega)\cap D(A^{(j)}(\lambda)(\lambda-H^{\times})^n)}$.

In particular, $A(\lambda)(\lambda - H^{\times})\mu = \mu$ *when* $(\lambda - H^{\times})\mu \in iX(\Omega)$.

(iii) $\frac{d^j}{d\lambda^j}\langle A(\lambda)\psi \mid \phi\rangle = (-1)^j j! \langle A^{(j+1)}(\lambda)\psi \mid \phi\rangle$, $j = 0, 1, \ldots$.

(iv) *For each* $\psi \in X(\Omega)$, $A(\lambda)\psi$ *is expanded as*

$$A(\lambda)\psi = \sum_{j=0}^{\infty}(\lambda_0 - \lambda)^j A^{(j+1)}(\lambda_0)\psi, \tag{22}$$

where the right hand side converges with respect to the strong dual topology.

Proof: (i) Let us show $(\lambda - H^{\times})A^{(j)}(\lambda) = A^{(j-1)}(\lambda)$. We have to prove that $\mathrm{D}(\lambda - \mathrm{H}^{x}) \supset \mathrm{R}(\mathrm{A}^{(j)}(\lambda))$. For this purpose, put $\mu_{\lambda}(y) = A^{(j)}(\lambda)\psi|(\bar{\lambda} - H)y$ for $\psi \in X(\Omega)$ and $y \in Y$. It is sufficient to show that the mapping $y \to \mu_{\lambda}(y)$ from *Y* into **C** is continuous with respect to the topology on $X(\Omega)$. Suppose that $Im(\lambda) > 0$. By the operational calculus, we obtain

$$\begin{aligned}\mu_\lambda(y) &= \int_{\mathbf{R}} \frac{1}{(\lambda - \omega)^j} E[\psi, (\bar{\lambda} - H)y](\omega)d\omega + 2\pi \mathrm{i}\frac{(-1)^{j-1}}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\Big|_{z=\lambda} E[\psi, (\bar{\lambda} - H)y](z) \\ &= \int_{\mathbf{R}} \frac{\lambda - \omega}{(\lambda - \omega)^j} E[\psi, y](\omega)d\omega + 2\pi \mathrm{i}\frac{(-1)^{j-1}}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\Big|_{z=\lambda} (\lambda - z)E[\psi, y](z) \\ &= \left((\lambda - H)^{1-j}\psi, y\right) + 2\pi \mathrm{i}\frac{(-1)^{j-2}}{(j-2)!}\frac{d^{j-2}}{dz^{j-2}}\Big|_{z=\lambda} E[\psi, y](z).\end{aligned} \tag{23}$$

Since $E[\psi, y](z)$ is continuous in $y \in X(\Omega)$ (the assumption (X_5) and $E[\psi, y](z)$ is holomorphic in *z*, for any $\varepsilon > 0$, there exists a neighborhood U_1 of zero in *X*(*Ω*) such that $\left|\left(\frac{d^{j-2}}{dz^{j-2}}\right) E[\psi, y](z)\right| < \varepsilon \ at\ z = \lambda$ for $y \in U_1 \cap Y$. Let U_2 be a neighborhood of zero in *H* such that $\|y\|_H < \varepsilon\ for\ y \in U_2$. Since the topology on *X*(*Ω*) is stronger than that on *H*, $U_2 \cap$*X*(*Ω*) is a neighborhood of zero in *X*(*Ω*). If $y \in U_1 \cap U_2 \cap Y$, we obtain

$$\left|\mu_\lambda(y)\right| \le \left\|(\lambda - H)^{1-j}\psi\right\|\varepsilon + 2\pi \mathrm{i}\frac{(-1)^{j-2}}{(j-2)!}\varepsilon.$$

Note that $(\lambda - H)^{1-j}$ is bounded when $\lambda \notin$**R** and $1 - j \le 0$ because H is selfadjoint. This proves that μ_λ is continuous, so that $\mu\lambda = (\lambda - H^{\times})A^{(j)}(\lambda)\psi \in X(\Omega)'$. The proof of the continuity for the case Im(*λ*) <0 is done in the same way. When $\lambda \in I$, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ in the lower half plane such that $\mu_\lambda(y) = \lim_{j\to\infty} \mu_{\lambda_j}(y)$ Since *X*(*Ω*) is barreled, Banach -Steinhaus theorem is applicable to conclude that the limit μ_λ of continuous linear mappings is also continuous. This proves D(*λ* −*H*^{x}) ⊃R(*A*$^{(j)}$(*λ*)) and (*λ* −*H*^{x})*A*$^{(j)}$(*λ*) is well defined for any $\lambda \in \{Im(\lambda) < 0\} \cup I \cup \Omega$. Then, the above calculation immediately shows that $(\lambda - H^{\times})A^{(j)}(\lambda) = A^{(j-1)}(\lambda)$. By the induction, we obtain (i).(ii) is also proved by the operational calculus as above, and (iii) is easily obtained by induction.

For (iv), since $\langle A(\lambda)\psi|\varphi\rangle$is holomorphic, it is expanded in a Taylor series as

$$\langle A(\lambda)\psi \mid \phi\rangle = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^j}{d\lambda^j}\bigg|_{\lambda=\lambda_0} \langle A(\lambda)\psi \mid \phi\rangle(\lambda-\lambda_0)^j$$

$$= \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j \langle A^{(j+1)}(\lambda_0)\psi \mid \phi\rangle, \quad \textbf{(24)}$$

for each $\varphi, \psi \in X(\Omega)$. This means that the functional $A(\lambda)\psi$is weakly holomorphic in λ. Then, $A(\lambda)\psi$ turns out to be strongly holomorphic and expanded as Eq.(22) in which basic facts on $X(\Omega)'$valued holomorphic functions are given.Unfortunately, the operator $A(\lambda) : iX(\Omega) \rightarrow X(\Omega)'$is not continuous if $iX(\Omega)$is equipped with the relative topology from $X(\Omega)'$Even if $\langle\psi| \rightarrow 0 \; in \;\; iX(\Omega) \subset X(\Omega)'\rangle$, the value $E[\psi,\varphi](\lambda)$does not tend to zero in general because the topology on $X(\Omega)'$is weaker than that on $X(\Omega)$. However, $A(\lambda)$proves to be continuous if $iX(\Omega)$is equipped with the topology induced from $X(\Omega)$ by the canonical inclusion.

Proposition (3.1.7)[3]: *$A(\lambda) \circ i.X(\Omega) \rightarrow X(\Omega)'$ is continuous if $X(\Omega)'$ is equipped with the weak dual topology.*

Proof: Suppose $\lambda \in \Omega$and fix $\varphi \in X(\Omega)$. Because of the assumption $(X5)$, for any $\varepsilon > 0$, there exists a neighborhood U_1 of zero in $X(\Omega)$such that $|E[\psi,\varphi](\lambda)| < \varepsilon$for $\psi \in U_1$. Let U_2 be a neighborhood of zero in H such that $\|\psi\|_H < \varepsilon$ for $\psi \in U_2$. Since the topology on $X(\Omega)$ is stronger than that on H, $U_2 \cap X(\Omega)$ is a neighborhood of zero in $X(\Omega)$. If $\psi \in U = U_1 \cap U_2$,

$$\begin{aligned}\left|\langle A(\lambda)\psi \mid \phi\rangle\right| &\leq \left\|(\lambda-H)^{-1}\right\|_{\mathcal{H}} \cdot \|\phi\|_{\mathcal{H}} \cdot \|\psi\|_{\mathcal{H}} + 2\pi\left|E[\psi,\phi](\lambda)\right| \\ &= \left(\left\|(\lambda-H)^{-1}\right\|_{\mathcal{H}} \cdot \|\phi\|_{\mathcal{H}} + 2\pi\right)\varepsilon.\end{aligned}$$

This proves that $A(\lambda)\,oI$is continuous in the weak dual topology. The proof for the case $Im(\lambda) < 0$is done in a similar manner. When $\lambda \in I$, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$in the lower half plane such that $A(\lambda)\,oi = \lim_{j\to\infty} A(\lambda_j)oi$. Since $X(\Omega)$is barreled, Banach–Steinhaus theorem is applicable to conclude that the limit $A(\lambda)oi$ of continuous linear mappings is also continuous.Now we are in a position to define an algebraic multiplicity and a generalized Eigen space of generalized Eigen values. Usually, an Eigen space is defined as a set of solutions of the equation$(\lambda - T)^n v = 0$. For example, when *n* =2, we rewrite it as

$$(\lambda - H - K)(\lambda - H - K)v$$

$$= (\lambda - H)^2 \left(id - (\lambda - H)^{-2} K(\lambda - H)\right) \circ \left(id - (\lambda - H)^{-1} K\right) v = 0.$$

Dividing by $(\lambda - H)^2$ yields

$$\left(id - (\lambda - H)^{-2} K(\lambda - H)\right) \circ \left(id - (\lambda - H)^{-1} K\right) v = 0.$$

Since the analytic continuation of $(\lambda - H)^{-n}$in $X(\Omega)'$is $A^{(n)}(\lambda)$, we consider the equation

$$\left(id - A^{(2)}(\lambda) K^{\times} (\lambda - H^{\times})\right) \circ \left(id - A(\lambda) K^{\times}\right) \mu = 0.$$

Motivated by this observation, we define the operator $B^{(n)}(\lambda) : D(B^{(n)}(\lambda)) \subset X(\Omega)' \to X(\Omega)'$to be

$$B^{(n)}(\lambda) = id - A^{(n)}(\lambda) K^{\times} \left(\lambda - H^{\times}\right)^{n-1}. \quad \textbf{(25)}$$

Then, the above equation is rewritten as $B^{(2)}(\lambda) B^{(1)}(\lambda) \mu = 0$. The domain of $B^{(n)}(\lambda)$is the domain of $A^{(n)}(\lambda) K^{\times} (\lambda - H \times)^{n-1}$. The following equality is easily proved.

$$\left(\lambda - H^{\times}\right)^k B^{(j)}(\lambda) = B^{(j-k)}(\lambda) \left(\lambda - H^{\times}\right)^k \big|_{D(B^{(j)}(\lambda))}, \quad j > k. \quad \textbf{(26)}$$

Definition (3.1.8)[3]: Let *λ* be a generalized Eigen value of the operator *T*. The generalized eigen space of *λ* is defined by

$$V_\lambda = \bigcup_{m \geq 1} \operatorname{Ker} B^{(m)}(\lambda) \circ B^{(m-1)}(\lambda) \circ \cdots \circ B^{(1)}(\lambda). \quad \textbf{(27)}$$

We call dim $V_\lambda$the algebraic multiplicity of the generalized Eigen values λ.

Theorem (3.1.9)[3]: *For any* $\in V_\lambda$*, there exists an integer* M *such that* $(\lambda - T \times)^M_\mu = 0$.

Proof: Suppose that $B^{(M)}(\lambda) o \cdots o B^{(1)}(\lambda)\mu = 0. Put\ \xi = B^{(M-1)}(\lambda)\ o \cdots o B^{(1)}(\lambda)\mu.$

$$0 = \left(\lambda - H^{\times}\right)^{M-1} B^{(M)}(\lambda)\xi$$

$$= B^{(1)}(\lambda)\left(\lambda - H^{\times}\right)^{M-1}\xi = \left(id - A(\lambda)K^{\times}\right)\left(\lambda - H^{\times}\right)^{M-1}\xi.$$

Since $D(\lambda - H^{\times}) \supset R(A(\lambda))$, it turns out that $(\lambda - H^{\times})^{M-1}\xi \in D(\lambda - H^{\times})$. Then, we obtain

$$\begin{aligned} 0 &= (\lambda - H^{\times})(id - A(\lambda)K^{\times})(\lambda - H^{\times})^{M-1}\xi \\ &= (\lambda - H^{\times} - K^{\times})(\lambda - H^{\times})^{M-1}\xi = (\lambda - T^{\times})(\lambda - H^{\times})^{M-1}\xi. \end{aligned}$$

By induction, we obtain $(\lambda - T^{\times})_{\mu}^{M} = 0$In general, the space $V_{\lambda}$is a proper subspace of the usual Eigen space$U_{m\geq1}Ker(\lambda - T \times)^{m}$of T^{x}. Typically $U_{m\geq1}Ker(\lambda - T^{\times})^{m}$ becomes of infinite dimensional because the dual space $X(\Omega)'$is "too large", however, $V_{\lambda}$is a finite dimensional space in many cases.In this subpart, we define a generalized resolvent. As the usual theory, it will be used to construct projections and semi groups. Let R_{λ} =$(\lambda - T)^{-1}$be the resolvent of Tas an operator on $H.A$ simple calculation shows

$$R_{\lambda}\psi = (\lambda - H)^{-1}\left(id - K(\lambda - H)^{-1}\right)^{-1}\psi. \tag{28}$$

Since the analytic continuation of $(\lambda - T)^{-1}$in the dual space is $A(\lambda)$, we make the following definition. In what follows, put $\tilde{\Omega} = \Omega \cup I \cup \{\lambda \mid Im(\lambda) < 0\}$.

Definition (3.1.10)[3]: If the inverse $(id - K^{\times}A(\lambda))^{-1}$exists, define the generalized resolvent $R_{\lambda}: iX(\Omega) \to X(\Omega)'$to be

$$\mathcal{R}_{\lambda} = A(\lambda) \circ \left(id - K^{\times}A(\lambda)\right)^{-1} = \left(id - A(\lambda)K^{\times}\right)^{-1} \circ A(\lambda), \quad \lambda \in \hat{\Omega}. \tag{29}$$

The second equality follows from $(id - A(\lambda)K^{\times})A(\lambda) = A(\lambda)(id - K^{\times}A(\lambda))$. Recall that $id - K^{\times}A(\lambda)$is injective on $iX(\Omega)$if and only if $id - A(\lambda)K^{\times}$is injective on $R(A(\lambda))$.Since $A(\lambda)$ is not continuous, $R_{\lambda}$is not a continuous operator in general. However, it is natural to ask whether $R_{\lambda}\, o\, i : X(\Omega) \to X(\Omega)'$is continuous or not because $A(\lambda)oI$ is continuous.

Definition (3.1.11)[3]: The generalized resolvent set $\hat{\varrho}(T)$is defined to be the set of points λ $\hat{\Omega}$.satisfying the following: there is a neighborhood $V_{\lambda} \subset \hat{\Omega}$. of$\lambda$ such that for any $\lambda' \in V_{\lambda}$, $R_{\lambda} \circ i$is a densely defined continuous operator from $X(\Omega)$ into $X(\Omega)'$, where $X(\Omega)'$, is equipped with the weak dual topology, and the set $\{R_{\lambda oi(\psi)}\}_{\lambda \in V\lambda}$is bounded in $X(\Omega)'$for each $\psi \in X(\Omega)$. The set $\hat{\varrho}(T)$ =$\tilde{\Omega} \backslash \hat{\varrho}(T)$ is called the *generalized spectrum* of T. The *generalized point spectrum* $\hat{\varrho}_{p}(T)$ is the set of points $\lambda \in \hat{\varrho}(T)$at which $id - K^{\times}A(\lambda)$is not injective. The *generalized* *residual spectrum* $\hat{\varrho}_r(\mathrm{T})$ is the set of points $\lambda \in \hat{\varrho}(T)$ such that the domain of $R_\lambda o\ i$ is not dense in *X*(*Ω*). The *generalized continuous spectrum* is defined to be $\hat{\sigma}c(T) = \hat{\sigma}(T)\backslash(\hat{\sigma}p(T) \cup \hat{\sigma}_r(T))$. By the definition, $\hat{\varrho}$ (*T*) is an open set. To require the existence of the neighborhood V_λ in the above definition is introduced by Waelbroeck for the spectral theory on locally convex spaces. If $\hat{\varrho}(T)$ were simply defined to be the set of points such that $R_\lambda o\ i$ is a densely defined continuous operator as in the Banach space theory, $\hat{\varrho}(T)$ is not an open set in general. If *X*(*Ω*) is a Banach space and the operator id-$i^{-1}K^\times A(\lambda)i$ is continuous on $X(\Omega)$ for each $\lambda \in \widehat{\Omega}$, we can show that $\lambda \in \hat{\varrho}(T)$ if and only if id-$i^{-1}K^\times A(\lambda)i$ has a continuous inverse on *X*(*Ω*).

Theorem(3.1.12)[3]:

(i) *For each* $\psi \in X(\Omega), R_{\lambda i\psi}$ *is an* $X(\Omega)'$*-valued holomorphic function in* $\lambda \in \hat{\varrho}(T)$.

(ii) *Suppose* $Im(\lambda) < 0 and\ \lambda \in \hat{\varrho}(T) \cap \hat{\varrho}(T)$, *where* $\hat{\varrho}(T)$ *is the resolventset of T in H-sense. Then,* $\langle R_{\lambda\psi} | \varphi\rangle = ((\lambda - T)^{-1}\psi, \varphi) for\ any\ \psi, \varphi \in X(\Omega)$.

This theorem means that $\langle R_\lambda\psi|\varphi\rangle$ is an analytic continuation of $(\lambda - T)^{-1}\psi, \varphi)$ from the lower half plane to $\hat{\varrho}(T)$ through the interval *I*. We always suppose that the domain of $R_\lambda o\ i$ is continuously extended to the whole *X*(*Ω*) when $\lambda \notin \hat{\sigma}(T)$. The significant point to be emphasized is that to prove the *strong* holomorphy of $R_\lambda o\ i(\psi)$, it is sufficient to assume that $R_\lambda o\ i$:*X*(*Ω*) $\to X(\Omega)'$ is continuous in the *weak* dual topology on $X(\Omega)'$

Proof: Since $\hat{\varrho}(T)$ is open, when $\lambda \in \hat{\varrho}(T)$, $R_{\lambda+h}$ exists for sufficiently small $h \in \boldsymbol{C}$. Put $\psi_\lambda = i^{-1}(id - K^\times A(\lambda))^{-1}i(\psi) for\ \psi \in X(\Omega)$. It is easy to verify the equality

$$\begin{aligned}&\mathcal{R}_{\lambda+h}i(\psi) - \mathcal{R}_\lambda i(\psi)\\&\quad= \big(A(\lambda+h) - A(\lambda)\big)i(\psi_\lambda) + \mathcal{R}_{\lambda+h}i \circ i^{-1}K^\times\big(A(\lambda+h) - A(\lambda)\big)i(\psi_\lambda).\end{aligned}$$

Let us show that $i^{-1}K^\times A(\lambda)i(\psi) \in X(\Omega)$ is holomorphic in λ. For any $\psi, \varphi \in X(\Omega)$, we obtain

$$\begin{aligned}\langle \phi \mid i^{-1}K^\times A(\lambda)i\psi\rangle &= (\phi, i^{-1}K^\times A(\lambda)i\psi) = \overline{(i^{-1}K^\times A(\lambda)i\psi, \phi)}\\&= \overline{\langle K^\times A(\lambda)i\psi \mid \phi\rangle} = \overline{\langle A(\lambda)i\psi \mid K^*\phi\rangle}.\end{aligned}$$

From the definition of $A(\lambda)$, it follows that $\langle\varphi\,|i^{-1}K^{\times}A(\lambda)i\psi\rangle$is holomorphic in $\overline{\lambda}$. Since $X(\Omega)$is dense in $X(\Omega)'$, $\langle\mu\,|i^{-1}K^{\times}A(\lambda)i\psi\rangle$is holomorphic in λ for any $\mu\in X(\Omega)'$ by Montel's theorem. This means that $i^{-1}K^{\times}A(\lambda)i\psi$ is weakly holomorphic. Since $X(\Omega)$ is a quasi-complete locally convex space, any weakly holomorphic function is holomorphic with respect to the original topology. This proves that $i^{-1}K^{\times}A(\lambda)i\psi$ is holomorphic in λ(note that the weak holomorphy in λ implies the strong holomorphy in λbecause functional in $X(\Omega)'$are *anti*-linear. Next, the definition of $\hat{\varrho}(T)$ implies that the family $\{R_\mu\circ i\}_{\mu\in V\lambda}$ of continuous operators is bounded in the point wise convergence topology. Due to Banach–Steinhaus theorem. the family is equicontinuous. This fact and the holomorphy of $A(\lambda)$ and $i^{-1}K^{\times}A(\lambda)i\psi$ prove that $R_{\lambda+h}i(\psi)$converges to $R_\lambda i(\psi)$ as $h\to 0$with respect to the weak dual topology. In particular, we obtain

$$\lim_{h\to 0}\frac{\mathcal{R}_{\lambda+h}i-\mathcal{R}_\lambda i}{h}(\psi)=\frac{dA}{d\lambda}(\lambda)i(\psi_\lambda)+\mathcal{R}_\lambda i\circ\frac{d}{d\lambda}\big(i^{-1}K^{\times}A(\lambda)i\big)(\psi_\lambda), \tag{30}$$

which proves that $R_\lambda i(\psi)$ is holomorphic in $\lambda$with respect to the weak dual topology on $X(\Omega)'$. Since $X(\Omega)$is barreled, the weak dual holomorphy implies the strong dual holomorphy.Let us prove (ii). Suppose $Im(\lambda)<0$. Note that $R_\lambda o\ i$is written as $R\lambda\ o\ i\ =A(\lambda)\ o(id-i^{-1}K^{\times}A(\lambda)i)^{-1}$. We can show the equality

$$\big(id-i^{-1}K^{\times}A(\lambda)i\big)f=\big(id-K(\lambda-H)^{-1}\big)f\in X(\Omega). \tag{31}$$

Indeed, for any$f,\psi\in X(\Omega)$, we obtain

$$\begin{aligned}\big\langle\big(i-K^{\times}A(\lambda)i\big)f\mid\psi\big\rangle&=\langle if\mid\psi\rangle-\big\langle A(\lambda)if\mid K^{*}\psi\big\rangle\\&=\langle if\mid\psi\rangle-\big\langle i\circ(\lambda-H)^{-1}f\mid K^{*}\psi\big\rangle\\&=(f,\psi)-\big(K(\lambda-H)^{-1}f,\psi\big)=\big(\big(id-K(\lambda-H)^{-1}\big)f,\psi\big)\end{aligned}$$

Thus, R_λ satisfies for $\varphi=(id-i^{-1}K^{\times}A(\lambda)i)f$that

$$\begin{aligned}\mathcal{R}_\lambda i\phi&=A(\lambda)i\circ\big(id-i^{-1}K^{\times}A(\lambda)i\big)^{-1}\phi\\&=i(\lambda-H)^{-1}\circ\big(id-K(\lambda-H)^{-1}\big)^{-1}\phi=i(\lambda-T)^{-1}\phi.\end{aligned}$$

Since $\lambda \in \hat{}\rho(T), (id - i^{-1}K^{\times}A(\lambda)i)X(\Omega)$is dense in $X(\Omega)$ and $R_{\lambda}i : X(\Omega) \to X(\Omega)'$is continuous. Since $\lambda \in \rho(T)$, $i(\lambda - T)^{-1}: H \to X(\Omega)'$is continuous. Therefore, taking the limit proves that$R_{\lambda}i\varphi$=$i(\lambda - T)^{-1}\varphi$ holds for any $\varphi \in X(\Omega)$.

Proposition (3.1.13)[3]: *The generalized resolvent satisfies*

(i) $(\lambda - T^{\times}) \circ R_{\lambda} = id|_{iX(\Omega)}$

(ii)If $\mu \in X(\Omega)'$ *satisfies* $(\lambda - T^{\times})\mu \in iX(\Omega)$, *then* $R_{\lambda} \circ (\lambda - T^{\times})\mu = \mu$.

(iii) $T^{\times}\, o\, \mathrm{R}_{\lambda}|\mathrm{iY} = \mathrm{R}_{\lambda} o\, T^{\times}|_{\mathrm{iY}}$

Proof: gives $id = (\lambda - H^{\times})A(\lambda) = (\lambda - T^{\times} + K^{\times})A(\lambda)$. This proves

$$\begin{aligned} &(\lambda - T^{\times}) \circ A(\lambda) = id - K^{\times}A(\lambda) \\ &\Rightarrow \quad (\lambda - T^{\times}) \circ A(\lambda) \circ (id - K^{\times}A(\lambda))^{-1} = (\lambda - T^{\times}) \circ \mathcal{R}_{\lambda} = id. \end{aligned}$$

Next, when $(\lambda - T^{\times})\mu \in iX(\Omega), A(\lambda)(\lambda - T^{\times})\mu$ is well defined and gives.

$$A(\lambda)(\lambda - T^{\times})\mu = A(\lambda)(\lambda - H^{\times} - K^{\times})\mu = (id - A(\lambda)K^{\times})\mu.$$

This proves$\mu = (id - A(\lambda)K^{\times})^{-1}A(\lambda)(\lambda - K^{\times})\mu = R_{\lambda}(\lambda - T^{\times})\mu$. Finally, note that $(\lambda - T^{\times})iY = i(\lambda - T)Y \subset iX(\Omega)$because of the assumptions (X_6), (X_7). Thus part(iii) of the proposition immediately follows from (i), (ii).Let $\Sigma \subset \hat{\sigma}(T)$ be a bounded subset of the generalized spectrum, which is separated from the rest of the spectrum by a simple closed curve $\gamma \subset \Omega \cup I \cup \{\lambda \mid Im(\lambda) < 0\}$. Define the operator $\Pi\Sigma: iX(\Omega) \to X(\Omega)'$to be

$$\Pi_{\Sigma}\phi = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \mathcal{R}_{\lambda}\phi \, d\lambda, \quad \phi \in iX(\Omega), \tag{32}$$

where the integral is defined as the Pettis integral. Since $X(\Omega)$ is assumed to be barreled by (X_3), $X(\Omega)'$is quasi-complete and satisfies the convex envelope property. Since $R_{\lambda}\varphi$ is strongly holomorphic in λ, the Pettis integral of $R_{\lambda}\varphi$ exists for the definition and the existence .theorem of Pettis integrals. Since $R_{\lambda}o\ i : X(\Omega) \to X(\Omega)'$is continuous, proves that $\Pi_{\Sigma}o\ i$ is a continuous operator from $X(\Omega)$ into $X(\Omega)'$equipped with the weak dual topology. Note that the equality

$$T^{\times}\int_{\gamma}\mathcal{R}_{\lambda}\phi\,d\lambda=\int_{\gamma}T^{\times}\mathcal{R}_{\lambda}\phi\,d\lambda, \tag{33}$$

holds. To see this, it is sufficient to show that the set $\{\langle T^{\times}R_{\lambda}\varphi\ |\psi\rangle\}\lambda\in\gamma$ is bounded for each $\psi\in X(\Omega)$due to yields $T\times R_{\lambda}\varphi\ =\lambda R_{\lambda}\varphi\ -\varphi$. Since $R_{\lambda}\lambda$is holomorphic and $\gamma$is compact, $\{\langle T^{\times}R_{\lambda}\varphi\ |\psi\rangle\}_{\lambda\in\gamma}$is bounded

.Although $\Pi_{\Sigma}\ o\ \Pi_{\Sigma}$is not defined, we call $\Pi_{\Sigma}$the *generalized Riesz projection*for Σbecause of the next proposition.

Proposition (3.1.14)[3]: $\mathbf{\Pi}_{\Sigma}\big(iX(\Omega)\big)\cap(id-\Pi_{\Sigma})\big(iX(\Omega)\big)=\{0\}$*and the direct sum satisfies*

$$iX(\Omega)\subset\Pi_{\Sigma}\big(iX(\Omega)\big)\oplus(id-\Pi_{\Sigma})\big(iX(\Omega)\big)\subset X(\Omega)'. \tag{34}$$

In particular, for any $\varphi\ \in X(\Omega)$*, there exist* μ_1,μ_2 *such that* φ *is uniquely decomposed as*

$$i(\phi)=\langle\phi|=\mu_1+\mu_2,\quad \mu_1\in\Pi_{\Sigma}\big(iX(\Omega)\big),\ \mu_2\in(id-\Pi_{\Sigma})\big(iX(\Omega)\big). \tag{35}$$

Proof: We simply denote $\langle\varphi|as\ \varphi$. It is sufficient to show that Π_{Σ} $(iX(\Omega))\cap(id-\Pi_{\Sigma})(iX(\Omega))=\{0\}$. Suppose that there exist $\varphi,\psi\in iX(\Omega)$such that $\Pi_{\Sigma}\varphi\ =\psi-\Pi_{\Sigma}\psi$. Since $\Pi_{\Sigma}(\varphi\ +\psi)\ =\psi\in iX(\Omega)$, we can again apply the projection to the both sides as $\Pi_{\Sigma}o\ \Pi\Sigma(\varphi\ +\psi)=\Pi_{\Sigma}\psi$. Let γ'be a closed curve which is slightly larger than γ. Then,

$$\begin{aligned}\Pi_{\Sigma}\circ\Pi_{\Sigma}(\phi+\psi)&=\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{\gamma'}\mathcal{R}_{\lambda'}\left(\int_{\gamma}\mathcal{R}_{\lambda}(\phi+\psi)d\lambda\right)d\lambda'\\&=\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{\gamma'}\mathcal{R}_{\lambda'}\left(\int_{\gamma}\frac{(\lambda-\lambda')+(\lambda'-T^{\times})}{\lambda-\lambda'}\mathcal{R}_{\lambda}(\phi+\psi)d\lambda\right)d\lambda'\\&\quad-\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{\gamma'}\mathcal{R}_{\lambda'}\left(\int_{\gamma}\frac{\lambda'-T^{\times}}{\lambda-\lambda'}\mathcal{R}_{\lambda}(\phi+\psi)d\lambda\right)d\lambda'.\end{aligned}$$

Eq(33)shows

$$\begin{aligned}\Pi_{\Sigma}\circ\Pi_{\Sigma}(\phi+\psi)&=\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{\gamma'}\mathcal{R}_{\lambda'}\left(\int_{\gamma}\frac{\lambda-T^{\times}}{\lambda-\lambda'}\mathcal{R}_{\lambda}(\phi+\psi)d\lambda\right)d\lambda'\\&\quad-\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{\gamma'}\mathcal{R}_{\lambda'}\circ(\lambda'-T^{\times})\left(\int_{\gamma}\frac{\mathcal{R}_{\lambda}}{\lambda-\lambda'}(\phi+\psi)d\lambda\right)d\lambda'.\end{aligned}$$

Proposition(3.1.13) show

$$= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\gamma'} \mathcal{R}_{\lambda'} \left(\int_{\gamma} \frac{\phi+\psi}{\lambda-\lambda'} d\lambda \right) d\lambda' - \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\gamma'} \left(\int_{\gamma} \frac{\mathcal{R}_\lambda}{\lambda-\lambda'} (\phi+\psi) d\lambda \right) d\lambda'$$

$$= 0 - \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\gamma} \mathcal{R}_\lambda (\phi+\psi) \cdot \int_{\gamma'} \frac{1}{\lambda-\lambda'} d\lambda' \cdot d\lambda$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \mathcal{R}_\lambda (\phi+\psi) d\lambda = \Pi_\Sigma (\phi+\psi).$$

This proves that $\Pi_\Sigma\varphi = 0$The above proof also shows that as long as$\Pi_\Sigma\varphi \in iX(\Omega), \Pi_\Sigma\ o\ \Pi_\Sigma$is defined and $\Pi_\Sigma\ o\ \Pi\Sigma\varphi = \Pi_\Sigma\varphi$.

Proposition (3.1.15)[3]: $\mathbf{\Pi_\Sigma}|iY\ is\ T^\times$ *invariant:* $\Pi_\Sigma\ o\ T^\times|iy = T^\times o\ \Pi_\Sigma|iy.$

Proof: Let λ_0 be an isolated generalized Eigen values , which is separated from the rest of the generalized spectrum by a simple closed curve $\gamma \subset \Omega \cup I \cup \{\lambda \mid Im(\lambda) < 0\}$. Let

$$\Pi_0 = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_0} \mathcal{R}_\lambda d\lambda, \qquad \textbf{(36)}$$

be a projection for λ_0 and $V_0 = U_{m\geq 1}$Ker$B^{(m)}(\lambda_0) \circ \cdots \circ B(1)(\lambda_0)$ a generalized eigenspace of λ_0. The main theorem in this chapter is stated as follows.

Theorem (3.1.16)[3]:*If* $\Pi_0 iX(\Omega)$ *is finite dimensional, then* $\Pi_0 iX(\Omega) = V_0$.In the usual spectral theory, this theorem is easily proved by using the resolvent equation. In our theory, the composition $R_{\lambda_} \circ R_\lambda$is not defined because $R_\lambda$is an operator from $iX(\Omega)$into $X(\Omega)'$. As a result, the resolvent equation does not hold and the proof of the above theorem is rather technical.

Proof: Let $R_\lambda = \sum_{j=-\infty}^{\infty}(\lambda_0 - \lambda)^j E_j$be a Laurent series of R_λ, which converges in the strong dual topology Since

$$id = \left(\lambda - T^\times\right) \circ \mathcal{R}_\lambda = \left(\lambda_0 - T^\times - (\lambda_0 - \lambda)\right) \circ \sum_{j=-\infty}^{\infty} (\lambda_0 - \lambda)^j E_j,$$

we obtain $E_{-n-1} = (\lambda_0 - T^\times)E_{-n}forn = 1, 2, ...$ Thus the equality

$$E_{-n-1} = \left(\lambda_0 - T^\times\right)^n E_{-1} \qquad \textbf{(37)}$$

holds. Similarly, $id|iY = R_\lambda\ o\ (\lambda - T)|_{iy}$ provides $E_{-n-1}|_{iY} = E_{-n} o\ (\lambda_0 - T^\times)|_{iY}$. Thus we obtain $R(E_{-n-1}|_{iY}) \subseteq R(E_{-n})$for any $n \geq 1$. Since $Y$is dense in $X(\Omega)$ and the range of $E_{-1} = -\Pi_0$ is finite dimensional, it turns out that $R(E_{-n}|_{iY}) \subseteq R(E_{-n})$ and $R(E_{-n-1}|_{iY}) \subseteq R(E_{-n})$ for any $n \geq 1$. This implies that the prin-ciple part $\sum_{-\infty}^{-1}(\lambda_0 - \lambda)^j E_j$of the Laurent series is a finite dimensional operator. Hence, there exists an integer $M \geq 1$such that $E_{-M-1} = 0$. This means that λ_0 is a pole of R_λ:

$$\mathcal{R}_\lambda = \sum_{j=-M}^{\infty} (\lambda_0 - \lambda)^j E_j. \tag{38}$$

Next, from the equality $(id - A(\lambda)K^{\times})o\ R_\lambda = A(\lambda)$, we have

$$\left(id - \sum_{k=0}^{\infty}(\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0)K^{\times}\right) \circ \sum_{j=-M}^{\infty} (\lambda_0 - \lambda)^j E_j = \sum_{k=0}^{\infty}(\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0).$$

Comparing the coefficients of $(\lambda_0 - \lambda)^{-1}$ on both sides, we obtain

$$\left(id - A(\lambda_0)K^{\times}\right)E_{-1} - \sum_{j=2}^{M} A^{(j)}(\lambda_0)K^{\times}E_{-j} = 0. \tag{39}$$

Substituting Eq.(35)and $E_{-1}=-\Pi_0$ provides

$$B^{(1)}(\lambda_0)\Pi_0 - \sum_{j=2}^{M} A^{(j)}(\lambda_0)K^{\times}\left(\lambda_0 - T^{\times}\right)^{j-1}\Pi_0 = 0. \tag{40}$$

In particular, this implies R(Π_0) ⊂D($B^{(1)}(\lambda_0)$). Hence, $(\lambda_0 - T^{\times})\Pi_0$ can be rewritten as

$$\left(\lambda_0 - T^{\times}\right)\Pi_0 = \left(\lambda_0 - H^{\times}\right) \circ \left(id - A(\lambda_0)K^{\times}\right)\Pi_0 = \left(\lambda_0 - H^{\times}\right)B^{(1)}(\lambda_0)\Pi_0.$$

Then, by using the definition of $B^{(2)}(\lambda_0)$,Eq(39) is rear anyged as

$$B^{(2)}(\lambda_0)B^{(1)}(\lambda_0)\Pi_0 - \sum_{j=3}^{M} A^{(j)}(\lambda_0)K^{\times}\left(\lambda_0 - T^{\times}\right)^{j-1}\Pi_0 = 0.$$

Repeating similar calculations, we obtain

$$B^{(M)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\Pi_0 = 0. \tag{41}$$

This proves $\Pi_0 iX(\Omega) \subset V0$.Let us show $\Pi_0 iX(\Omega) \supset V_0\Pi_0 iX(\Omega)$. From the equality $R_\lambda\ o\ (id - K^{\times}A(\lambda)) = A(\lambda)$, we have

$$\begin{aligned}\sum_{j=-M}^{\infty} (\lambda_0 - \lambda)^j E_j \circ \left(id - K^{\times}\sum_{k=0}^{\infty}(\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0)\right) \\ = \sum_{k=0}^{\infty}(\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0).\end{aligned} \tag{42}$$

Comparing the coefficients of $(\lambda_0 - \lambda)^k$on both sides for $k = 1, 2, \ldots$, we obtain

$$E_k\big(id - K^{\times}A(\lambda_0)\big)\phi - \sum_{j=1}^{\infty} E_{-j+k}K^{\times}A^{(j+1)}(\lambda_0)\phi = A^{(k+1)}(\lambda_0)\phi, \quad (43)$$

for any $\varphi \in iX(\Omega)$, where the left hand side is a finite sum. Note that $K^{\times}A(j)(\lambda_0)iX(\Omega) \subset iX(\Omega) for\ any\ j = 1, 2, \ldots$because $K^{\times}A(\lambda)iX(\Omega) \subset iX(\Omega)$for any λ(the assumption(X8)).Now suppose that $\mu \in V_0$is a generalized Eigen function satisfying $B(M)(\lambda_0)\ o \cdots oB^{(1)}(\lambda_0)\mu = 0$. For this μ, we need the following lemma.

Lemma(3.1.17)[3]: *For any k*=0, 1, *...,* M^{-1}

(i) $(\lambda_0 - T^{\times})^k\mu = (\lambda_0 - H^{\times})^k B^{(k)}(\lambda_0)\ o \cdots oB^{(1)}(\lambda_0)\mu$.

(ii) $K^{\times}(\lambda_0 - T^{\times})^k\mu \in iX(\Omega)$.

Proof: μ is included in the domain of $(\lambda_0 - T^{\times})^k$. Thus the left hand side of (i) indeed exists. Then, we have

$$\begin{aligned}(\lambda_0 - H^{\times})^k B^{(k)}(\lambda_0) &= (\lambda_0 - H^{\times})^k\big(id - A^{(k)}(\lambda_0)K^{\times}(\lambda_0 - H^{\times})^{k-1}\big)\\ &= (\lambda_0 - H^{\times} - K^{\times})(\lambda_0 - H^{\times})^{k-1} = (\lambda_0 - T^{\times})(\lambda_0 - H^{\times})^{k-1}.\end{aligned}$$

Repeating this procedure yields (i). To prove (ii), let us calculate

$$0 = K^{\times}\big(\lambda_0 - H^{\times}\big)^k B^{(M)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu.$$

the part (i) of this lemma give

$$\begin{aligned}0 &= K^{\times}B^{(M-k)}(\lambda_0) \circ \cdots \circ B^{(k+1)}(\lambda_0) \circ \big(\lambda_0 - H^{\times}\big)^k \circ B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu\\ &= K^{\times}B^{(M-k)}(\lambda_0) \circ \cdots \circ B^{(k+1)}(\lambda_0) \circ \big(\lambda_0 - T^{\times}\big)^k \mu.\end{aligned}$$

For example, when $k = M^{-1}$, this is reduced to

$$0 = K^{\times}\big(id - A(\lambda_0)K^{\times}\big) \circ \big(\lambda_0 - T^{\times}\big)^{M-1}\mu.$$

This proves $K^{\times}(\lambda_0 - T^{\times})M^{-1}, \mu = K^{\times}A(\lambda_0)K^{\times}(\lambda_0 - T^{\times})^{M-1}\mu \in iX(\Omega)$. This is true for any $k = 0, 1, \ldots, M^{-1}$; it follows from the definition of $B^j(\lambda_0)$'s that $K^{\times}(\lambda 0 - T^{\times})^k\mu$is expressed as a linear combination of elements of the form $K^{\times}A(j)(\lambda_0)\xi_j, \xi_j \in iX(\Omega)$. Since $K^{\times}A(j)(\lambda_0)iX(\Omega) \subset iX(\Omega)$, we obtain $K^{\times}(\lambda_0 - T\times)^k\mu \in iX(\Omega)$.Since $K^{\times}(\lambda_0 - T^{\times})^k\mu \in iX(\Omega)$, we can substitute$\varphi = K^{\times}(\lambda_0 - T^{\times})^k\mu$ The resultant equation is rearranged as

$$E_k K^\times \left(id - A(\lambda_0)K^\times\right)\left(\lambda_0 - T^\times\right)^k \mu$$
$$- \left(id + \sum_{j=1}^{k} E_{-j+k} K^\times \left(\lambda_0 - H^\times\right)^{k-j}\right) A^{(k+1)}(\lambda_0) K^\times \left(\lambda_0 - T^\times\right)^k \mu$$
$$= \sum_{j=k+1}^{\infty} E_{-j+k} K^\times A^{(j+1)}(\lambda_0) K^\times \left(\lambda_0 - T^\times\right)^k \mu.$$

Further, $(\lambda_0 - \mathrm{T}^\times)^{\mathrm{k}} = ((\lambda_0 - H^\times)^{\mathrm{k}} \mathrm{B}^{(\mathrm{k})}(\lambda 0)\ o \cdots o\ \mathrm{B}^{(1)}(\lambda_0)$provides

$$E_k K^\times \left(\lambda_0 - H^\times\right)^k B^{(k+1)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu - \left(id + \sum_{j=1}^{k} E_{-j+k} K^\times \left(\lambda_0 - H^\times\right)^{k-j}\right)$$
$$\times A^{(k+1)}(\lambda_0) K^\times \left(\lambda_0 - H^\times\right)^k B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu$$

$$= \sum_{j=k+1}^{\infty} E_{-j+k} K^\times A^{(j+1)}(\lambda_0) K^\times \left(\lambda_0 - T^\times\right)^k \mu. \qquad \mathbf{(44)}$$

On the other hand, comparing the coefficients of $(\lambda_0 - \lambda)^0$of provides

$$E_0\left(id - K^\times A(\lambda_0)\right)\phi - \sum_{j=1}^{\infty} E_{-j} K^\times A^{(j+1)}(\lambda_0)\phi = A(\lambda_0)\phi,$$

for any $\varphi\ \in iX(\Omega)$. Substituting $\varphi\ = K^\times\mu\ \in iX(\Omega)$provides

$$\left(id + E_0 K^\times\right) B^{(1)}(\lambda_0)\mu = \mu + \sum_{j=1}^{\infty} E_{-j} K^\times A^{(j+1)}(\lambda_0) K^\times \mu. \qquad \mathbf{(45)}$$

By adding Eq(43)toEq(44) for $k = 1, \ldots, M^{-1}$, we obtain

$$\left(id + E_0 K^\times\right) B^{(1)}(\lambda_0)\mu - \sum_{k=1}^{M-1} \left(id + \sum_{j=1}^{k} E_{-j+k} K^\times \left(\lambda_0 - H^\times\right)^{k-j}\right)$$
$$\times A^{(k+1)}(\lambda_0) K^\times \left(\lambda_0 - H^\times\right)^k B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu$$

The left hand side above is rewritten as

$$+ \sum_{k=1}^{M-1} E_k K^\times \left(\lambda_0 - H^\times\right)^k B^{(k+1)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu$$
$$= \mu + \sum_{k=0}^{M-1} \sum_{j=1}^{\infty} E_{-j} K^\times A^{(j+k+1)}(\lambda_0) K^\times \left(\lambda_0 - T^\times\right)^k \mu. \qquad \mathbf{(46)}$$

The left hand side above is rewritten as

$$
\begin{aligned}
&\left(id + E_0K^{\times} + E_1K^{\times}\left(\lambda_0 - H^{\times}\right)\right)B^{(2)}(\lambda_0)B^{(1)}\\
&-\sum_{k=2}^{M-1}\left(id + \sum_{j=1}^{k} E_{-j+k}K^{\times}\left(\lambda_0 - H^{\times}\right)^{k-j}\right)\\
&\times A^{(k+1)}(\lambda_0)K^{\times}\left(\lambda_0 - H^{\times}\right)^{k}B^{(k)}(\lambda_0)\circ\cdots\\
&+\sum_{k=2}^{M-1} E_kK^{\times}\left(\lambda_0 - H^{\times}\right)^{k}B^{(k+1)}(\lambda_0)\circ\cdots\circ
\end{aligned}
$$

Repeating similar calculations, we can verify that is rewritten as

$$
\begin{aligned}
&\left(id + \sum_{j=0}^{M-1} E_jK^{\times}\left(\lambda_0 - H^{\times}\right)^{j}\right)B^{(M)}(\lambda_0)\circ\cdots\circ B^{(1)}(\lambda_0)\mu\\
&= \mu - \sum_{k=0}^{M-1}\sum_{j=1}^{\infty} E_{-j}K^{\times}A^{(j+k+1)}(\lambda_0)K^{\times}\left(\lambda_0 - T^{\times}\right)^{k}\mu. \qquad \textbf{(47)}
\end{aligned}
$$

Since $B(M)(\lambda_0)0\cdots 0B^{(1)}(\lambda_0)\mu = 0$, we obtain

$$
\mu = \sum_{k=0}^{M-1}\sum_{j=1}^{\infty} E_{-j}K^{\times}A^{(j+k+1)}(\lambda_0)K^{\times}\left(\lambda_0 - T^{\times}\right)^{k}\mu.
$$

Since $R(E_{-j}) \subset R(E_{-1}) = R(\Pi_0)$, this proves $\Pi_0 iX(\Omega) \supset V_0$. Thus the proof of $\Pi_0 iX(\Omega) = V_0$is completed.

We show a few criteria to estimate the generalized spectrum. Recall that$\hat{\sigma}p(T) \subset \sigma p(T^{\times})$because of. The relation between $\hat{\sigma}(T)$and $\sigma(T)$ is given as follows.

Proposition (3.1.18)[3]: *Let $\boldsymbol{C}- = \{Im(\lambda) < 0\}$be an open lower half plane. Let $\sigma p(T)$and $\sigma(T)$be the point spectrum and the spectrum in the usual sense, respectively. Then, the following relations hold.*

(i)$\hat{\sigma}(T) \cap \mathbf{C}- \subset \sigma(T) \cap \mathbf{C}-$. *In particular,* $\hat{\sigma}p(T) \cap \boldsymbol{C} \subset \sigma p(T) \cap \boldsymbol{C}$

(ii)*Let $\Sigma \subset \boldsymbol{C}-$be a bounded subset of $\sigma(T)$which is separated from the rest of the spectrum by a simple closed curve γ. Then, there exists as point of$\hat{}\sigma(T)$inside γ. In particular, if$\lambda \in \boldsymbol{C}$is an isolated point of $\sigma(T)$, then $\lambda \in \hat{\sigma}(T)$.*

Proof: Note that when $\lambda \in \boldsymbol{C}$, the generalized resolvent satisfies $R\lambda \, o \, i = i \, o \, (\lambda - T)^{-1}$.

(i) Suppose that $\lambda \in \_(T) \cap \boldsymbol{C}$, where $\varrho(T)$is the resolvent set of T in the usual sense. Since H is a Hilbert space, there is a neighborhood $V_\lambda \subset \varrho(T) \cap \boldsymbol{C}$of λsuch that $(\lambda' - T)^{-1}$is continuous on H for any $\lambda' \in V\lambda$and the set $\{(\lambda' - T) - 1\psi\}_{\lambda' \in V\lambda}$is bounded in H for each $\psi \in X(\Omega)$. Since $i : H \to X(\Omega)'$is continuous and since the topology of $X(\Omega)$is stronger than that of $H, R_\lambda \ o \ i \ = i \ o \ (\lambda' - T)^{-1}$is a continuous operator from $X(\Omega)$into$X(\Omega)'$for any $\lambda' \in V\lambda$, and the set $\{R_\lambda, o \ i\psi\}\lambda' \in V\lambda$is bounded in $X(\Omega)$'. This proves that $\lambda \in \hat{}\varrho(T) \cap \boldsymbol{C}$.Next, suppose that $\lambda \in \boldsymbol{C}$ is a generalized Eigen values satisfying $(id - K^\times A(\lambda))i(\psi) = 0 for \ \psi \in X(\Omega)$. Since $\lambda - H$is invertible on H when $\lambda \in \boldsymbol{C}$, putting $\varphi = (\lambda - H)^{-1}\psi$ provides

$$\big(id - K^\times A(\lambda)\big)i(\lambda - H)\phi = \big(i(\lambda - H) - K^\times i\big)\phi = i(\lambda - T)\phi = 0,$$

and thus $\lambda \in \sigma p(T)$.

(ii) Let P be the Riesz projection for $\Sigma \subset \sigma(T) \cap \boldsymbol{C}$, which is defined as $P = (2\pi i)^{-1} \int (\lambda - T)^{-1} d\lambda$. Since γ encloses a point of $\sigma(T), PH \neq \emptyset$. Since $X(\Omega)$is dense in $H, PX(\Omega) \neq \emptyset$. This fact and $R_\lambda \ o \ i \ = i \ o \ (\lambda - T)^{-1}$prove that the range of the generalized Riesz projection defined by is not zero. Hence, the closed curve γ encloses a point of $\hat{}\sigma(T)$.A few remarks are in order. If the spectrum of Ton the lower half plane consists of discrete Eigen values, (i) and (ii) show that $\sigma p(T) \cap \boldsymbol{C} = \sigma(T) \cap \boldsymbol{C} = \hat{}\sigma(T) \cap \boldsymbol{C}$. However, it is possible that a generalized Eigen value on $I$is not an Eigen value in the usual sense. For such an example. In most cases, the continuous spectrum on the lower half plane is not included in the generalized spectrum because the topology on $X(\Omega)'$is weaker than that on H, although the point spectrum and the residual spectrum may remain to exist as the generalized spectrum. Note that the continuous spectrum on the interval I also disappears; for the resolvent$(\lambda - T)^{-1} = (\lambda - H)^{-1}(id - K(\lambda - H) - 1)^{-1}$ in the usual sense, the factor $(\lambda - H)^{-1}$induces the continuous spectrum on the real axis because $H$is selfadjoint. For the generalized resolvent, $(\lambda - H)^{-1}$is replaced by $A(\lambda)$, which has no singularities. This suggests that obstructions when calculating the Laplace inversion formula by using the residue theorem may disappear .Recall that a linear operator L from a topological vector space X_1 to another topolog-ical vector space X_2 is said to be bounded if there exists a neighborhood $U \subset X_1$ such that $LU \subset X_1$is a bounded set. When $L = L(\lambda)$is parameterized by λ, it is said to be bounded uniformly in λ if such a neighborhood $U$is independent of λ. When the domain X_1 is a Banach space, $L(\lambda)$is bounded uniformly in λif and only if $L(\lambda)$is continuous for each λ. Similarly, L is called compact if there exists a neighborhood $U \subset X_1$such that $LU \subset X_2$ is relatively compact. When $L = L(\lambda)$is parameterized by λ, it is said to be compact uniformly in λ_i f such a neighborhood $U$is independent of λ. When the domain X_1 is a Banach space, $L(\lambda)$is compact uniformly in λif and only if $L(\lambda)$is compact for each λ. When the range X_2 is a Montel space, a (uniformly) bounded operator is (uniformly) compact because every bounded set in a Montel space is relatively compact. Put ˆ$\Omega = \{Im(\lambda) < 0\} \cup I \cup \Omega$as before. In many applications, $i^{-1}K^{\times}A(\lambda)i$ is a bounded operator. In such a case, the following proposition is useful to estimate the generalized spectrum.

Proposition (3.1.18)[3]: *Suppose that for* $\lambda \in \hat{\Omega}$, *there exists a neighborhood* $U_\lambda \subset \hat{\Omega}$ *of* λ *such that* $i^{-1}K^{\times}A(\lambda')i : X(\Omega) \to X(\Omega)$ *is a bounded operator uniformly in* $\lambda' \in U_\lambda$*.If* $id - i^{-1}K^{\times}A(\lambda)i$ *has a continuous inverse on* $X(\Omega)$*, then* $\lambda \notin \hat{\sigma}(T)$.

Proof: Note that $R_\lambda \circ I$ is rewritten as $R_\lambda o\, i = A(\lambda) oi\, o\, (id - i^{-1}K^{\times}A(\lambda)i)^{-1}$. Since $A(\lambda)oi$ is continuous, it is sufficient to prove that there exists a neighborhood $V_\lambda$of λsuch that the set $(id - i^{-1}K^{\times}A(\lambda')i)^{-1}\psi\lambda' \in V\lambda$is bounded in $X(\Omega)$ for each $\psi \in X(\Omega)$. For this purpose, it is sufficient to prove that the mapping $\lambda' \to (id - i^{-1}K^{\times}A(\lambda')i)^{-1}\psi$is continuous in $\lambda' \in V_\lambda$. Since $i^{-1}K^{\times}A(\lambda)i$ is holomorphic, there is an operator $D(\lambda, h)$ on $X(\Omega)$such that

$$id - i^{-1}K^{\times}A(\lambda + h)i = id - i^{-1}K^{\times}A(\lambda)i - hD(\lambda, h)$$

$$= \left(id - hD(\lambda, h)\left(id - i^{-1}K^{\times}A(\lambda)i\right)^{-1}\right) \circ \left(id - i^{-1}K^{\times}A(\lambda)i\right)$$

Since$i^{-1}K \times A(\lambda)i$is a bounded operator uniformly in$\lambda \in U\lambda, D(\lambda, h)$is a bounded operator when h is sufficiently small. Since $(id–i^{-1}K^{\times}A(\lambda)i)^{-1}$is continuous by the assumption, $D(\lambda, h)(id - i^{-1}K^{\times}A(\lambda)i)^{-1}$is a bounded operator. Then, Bruyn's theorem shows that $id -$

$h_D(\lambda, h)(id - i^{-1}K^{\times}A(\lambda)i)^{-1}$ has a continuous inverse for sufficiently small *h*and the inverse is continuous in *h*(when $X(\Omega)$ is a Banach space, Bruyn's theorem is reduced to the existence of the Neumann series). This proves that $(id - i^{-1}K^{\times}A(\lambda + h)i)^{-1}\psi$ exists and continuous in *h* for each *ψ*.As a corollary, if *X*(*Ω*)is a Banach space and $i^{-1}K^{\times}A(\lambda)i$ is a continuous operator on $X(\Omega)$for each *λ*, then $\lambda \in \hat{\Omega}(T)$if and only if $id - i^{-1}K^{\times}A(\lambda)i$has a continuous inverse on $X(\Omega)$. Because of this proposition, we can apply the spectral theory on locally convex spaces to the operator $id - i^{-1}K^{\times}A(\lambda)i$to estimate the generalized spectrum. In particular, like as Riesz–Schauder theory in Banach spaces, we can prove the next theorem.

Theorem (3.1.19)[3]:*In addition to*$(X1)$– $(X8)$*, suppose that* $i^{-1}K^{\times}A(\lambda)i : X(\Omega) \to X(\Omega) \textit{is a compact operator uniformly in } \lambda \in \hat{\Omega} = \{Im(\lambda) < 0\} \cup I \cup \Omega$*. Then, the following statements are true.*

(i) *For any compact set* $D \subset \hat{\Omega}$*, the number of generalized eigenvalues in Dis finite (thus*$\hat{\sigma}p(T)$*consists of a countable number of generalized Eigen values s and they may accumulate only on the boundary of* $\hat{\Omega}$ *or infinity).*

(ii)For each $\lambda 0 \in \hat{\sigma}p(T)$*, the generalized eigenspace* V_0*is of finite dimensional and* $\Pi_0\, iX(\Omega) = V_0$.

(iii) $\hat{\sigma}_c(T) = \hat{\sigma}_r(T) = \emptyset$.

If $X(\Omega)$ is a Banach space, the above theorem follows from well known Riesz–Schauder theory. Even if $X(\Omega)$ is not a Banach space, we can prove the same result is useful to find embedded Eigen values s of *T*:

Corollary (3.1.20)[3]:*Suppose that Tis selfadjoint. Under the assumptions in, the number of Eigen values of* $T = H + K(in\ H - sense)$ *in any compact set* $D \subset I$*is finite. Their algebraic multiplicities* dimKer$(\lambda - T)$*are finite.*

Proof: Let $\lambda_0 \in I$be an Eigen values of *T*. It is known that the projection *P*0 to the corresponding Eigen space is given by

$$\mathcal{P}_0\phi = \lim_{\varepsilon\to -0} \mathrm{i}\varepsilon\cdot(\lambda_0+\mathrm{i}\varepsilon - T)^{-1}\phi, \quad \phi\in\mathcal{H}, \tag{48}$$

where the limit is taken with respect to the topology on *H*. When $Im(\lambda) < 0$, we have $R_\lambda i(\varphi) = i(\lambda - T)^{-1}\varphi$ for $\varphi \in X(\Omega)$. This shows

$$i\circ\mathcal{P}_0\phi = \lim_{\varepsilon\to -0} \mathrm{i}\varepsilon\cdot\mathcal{R}_{\lambda_0+\mathrm{i}\varepsilon}\circ i(\phi), \quad \phi\in X(\Omega).$$

Let $R_\lambda = \sum_{j-\infty}^{\infty}(\lambda 0 - \lambda)^j\, Ej$ be the Laurent expansion of R_λ, which converges around λ_0. This provides

$$i\circ\mathcal{P}_0 = \lim_{\varepsilon\to -0} \mathrm{i}\varepsilon \sum_{j=-\infty}^{\infty}(-\mathrm{i}\varepsilon)^j E_j\circ i.$$

Since the right hand side converges with respect to the topology on $X(\Omega)'$, we obtain

$$i\circ\mathcal{P}_0 = -E_{-1}\circ i = \Pi_0\circ i, \qquad E_{-2} = E_{-3} = \cdots = 0, \tag{49}$$

where Π_0 is the generalized Riesz projection for λ_0. Since $\lambda_0$is an eigenvalues , $P_0 H \neq \emptyset$. Since $X(\Omega)$ is a dense subspace of $H, P_0 X(\Omega)' = \emptyset$. Hence, we obtain $\Pi_0 iX(\Omega)' = \emptyset$, which implies that λ_0 is a generalized Eigen values ; $\hat{\sigma}_c(T) = \hat{}\sigma_p(T)$ Since$\hat{\sigma}(T)$ is countable, so is $\hat{\sigma}_p(T)$. Since $\Pi_0 iX(\Omega)$is a finite dimensional space, so is $P_0 X(\Omega)$. Then, $P_0 H = P_0 X(\Omega)$ proves to be finite dimensional because $P_0 H$is the closure of $P_0 X(\Omega)$.Our results are also useful to calculate eigenvectors for embedded Eigen values. In the usual Hilbert space theory, if an Eigen value λ is embedded in the continuous spectrum of T, we cannot apply the Riesz projection for *λ* because there are no closed curves in **C**which separate *λ* from the rest of the spectrum. In our theory, $\hat{\sigma}c(T) = \hat{\sigma}r(T) = \emptyset$. Hence, the generalized Eigen values s are indeed isolated and the Riesz projection Π_0 is applied to yield $\Pi_0 iX(\Omega) = V_0$. Then, the Eigen space in *H*sense is obtained as $V_0 \cap D(T)$.

Proof of Theorem (3.1.19)The theorem follows from Riesz–Schauder theory on locally convex spaces developed in Ringrose .Here, we give a simple review of the argument in. We denote $X(\Omega) = X and\ i^{-1}K^{\times}A(\lambda)i = C(\lambda)$for simplicity. A pairing for $(X', X)$is denoted by $\langle\cdot \mid \cdot\rangle X$.

Since $C(\lambda): X \to X$is compact uniformly in λ, there exists a neighborhood V_0 of zero in X, which is independent of λ, such that $C(\lambda)V \subset X$ is relatively compact. Put $p(x) = inf\{|\lambda|;\ x \in \lambda V\}$. Then, p is a continuous semi-norm on X and $V = \{x \mid p(x) < 1\}$. Define a closed subspace M in X to be

$$M = \{x \in X \mid p(x) = 0\} \subset V. \tag{50}$$

Let us consider the quotient space X/M, whose elements are denoted by [x]. The semi-norm p induces a norm P on X/M by $P([x]) = p(x)$. If X/M is equipped with the norm topology induced by P, we denote the space as B. The completion of B, which is a Banach space, is denoted by B_0. The dual space $B_0'$of B_0 is a Banach space with the norm

$$\|\mu\|_{\mathcal{B}_0'} := \sup_{P([x])<1} \left|\langle \mu \mid [x] \rangle_{\mathcal{B}_0}\right|, \tag{51}$$

where $\langle \cdot \mid \cdot \rangle B0$is a pairing for ($B_0'$, B_0). Define a subspace $S \subset X$to be

$$S = \left\{\mu \in X' \;\middle|\; \sup_{x \in V} \left|\langle \mu \mid x \rangle_X\right| < \infty\right\}. \tag{52}$$

The linear mapping $\hat{}: S \to B_0'(\mu \mapsto \hat{\mu})$ defined through $\langle \hat{}\mu | [x] \rangle B_0 = \langle \mu \,| x \rangle X$ is bijective. Define the operator $Q(\lambda) : B \to B$ to be $Q(\lambda)[x] = [C(\lambda)x]$. Then, the equality

$$\langle \hat{\mu} \mid Q(\lambda)[x] \rangle_{\mathcal{B}_0} = \langle \mu \mid C(\lambda)x \rangle_X \tag{53}$$

holds for $\mu \in S$and $x \in X$. Let $Q_0(\lambda) : B_0 \to B_0$ be a continuous extension of $Q(\lambda)$. Then, $Q_0(\lambda)$is a compact operator on a Banach space, and thus the usual Riesz–Schauder theory is applied. By using , it is proved that $z \in \boldsymbol{C}$is an Eigen values of $C(\lambda)$if and only if it is an Eigen values of $Q_0(\lambda)$. In this manner, we can prove that *The number of Eigen values of the operator* $C(\lambda) : X \to X$*is at most countable, which can accumulate only at the origin. The Eigen spaces*$\cup_{m \geq 1} Ker(z - C(\lambda))^m$*of nonzero eigenvalues zare finite dimensional. If* $Z \neq 0$ *is not an Eigen values ,* $z - C(\lambda)$*has a continuous inverse on X. for the complete proof.*Now we are in a position to prove. Suppose that $\lambda$is not a generalized Eigen values . Then, 1is not an

Eigen values of $C(\lambda) = i^{-1}K^{\times}A(\lambda)i$. The above theorem con-cludes that $id - C(\lambda)$ has a continuous inverse on $X(\Omega)$. Since $C(\lambda)$is compact uniformly in λ, implies $\lambda \notin \hat{\sigma}(T)$. This proves the part (iii) of.Let us show the part (i) of the theorem. $Let\ z = z(\lambda)$be an Eigen values of $C(\lambda)$. We suppose that $z(\lambda_0)$ =1so that λ_0 is a generalized Eigen values . As was proved in the proof of, $\langle \mu \,|C(\lambda)x\rangle X$is holomorphic in λ. Eq.(53)shows that $\langle \hat{}\mu|Q(\lambda)[x]\rangle$ B0is holomorphic for any $\hat{}\langle \mu \in B_0'\rangle$and $[x] \in B$. Since B0is a Banach space and B is dense in $B_0, Q_0(\lambda)$is a holomorphic family of operators. Recall that the Eigen values $z(\lambda)$of $C(\lambda)$is also an Eigen values of $Q_0(\lambda)$satisfying $z(\lambda_0) = 1$. Then, the analytic perturbation theory of operators shows that there exists a natural number p such that $z(\lambda)$is holomorphic as a function of $(\lambda - \lambda_0)^{1/p}$. Let us show that $z(\lambda)$is not a constant function. If $z(\lambda) \equiv 1$, every point in $\hat{}\hat{\Omega}$ is a generalized Eigen value. Due to, the open lower half plane is included in the point spectrum of T. Hence, there exists $f = f_\lambda$in H such that $f = K(\lambda - H)^{-1}f$ for any $\lambda \in \boldsymbol{C}$. However, since K is H bounded, there exist nonnegative numbers a and b such that

$$\begin{aligned}&\|K(\lambda - H)^{-1}\|\\ &\quad \leq a\|(\lambda - H)^{-1}\| + b\|H(\lambda - H)^{-1}\| = a\|(\lambda - H)^{-1}\| + b\|\lambda(\lambda - H)^{-1} - id\|\end{aligned}$$

which tends to zero as $|\lambda| \to \infty$ outside the real axis. Therefore,$\|f\| \leq \|K(\lambda - H)^{-1}\| \cdot \|f\| \to 0$, which contradicts with the assumption. Since $z(\lambda)$is not a constant, there exists a neighborhood $U \subset \boldsymbol{C}$of λ_0 such that $z(\lambda)' = 1$when $\lambda \in U$ and $\lambda' = \lambda_0$. This implies that $\lambda \in U\backslash\{\lambda_0\}$is not a generalized eigenvalues and proved. finally, let us prove the part (ii) of. Put $\tilde{}C(z) = (z-1) \cdot id + C(z)$and $\tilde{}Q(z) = (z-1) \cdot id + Q(z)$. They satisfy $\langle \mu|\tilde{}Q(\lambda)[x]\rangle$

$$\langle \hat{\mu} \mid \left(\lambda - \tilde{Q}(z)\right)^{-1}[x]\rangle_{\mathcal{B}_0} = \langle \mu \mid \left(\lambda - \tilde{C}(z)\right)^{-1}x\rangle_X.$$

Since an Eigen space of $Q(z)$ is finite dimensional, an eigenspace of $\tilde{}Q(z)$ is also finite dimensional. Thus the resolvent $(\lambda - \tilde{Q}(z))^{-1}$is meromorphic in $\lambda \in \hat{\Omega}$. Since $\tilde{Q}(z)$is holomorphic, $(\lambda - \tilde{Q}(\lambda))^{-1}$is also meromorphic. The above equality shows that $\langle \mu\,|(\lambda - \tilde{C}(\lambda))^{-1}x\rangle X$ is meromorphic for any $\mu \in S$. Since $S$is dense in X', it turns out that $(\lambda -$

$\tilde{C}(\lambda))^{-1}x$ is meromorphic with respect to the topology on *X*. Therefore, the generalized resolvent

$$\mathcal{R}_\lambda \circ i = A(\lambda) \circ i \circ \left(id - i^{-1}K^\times A(\lambda)i\right)^{-1} = A(\lambda) \circ i \circ \left(\lambda - \tilde{C}(\lambda)\right)^{-1} \quad \textbf{(54)}$$

is meromorphic on $\hat{\Omega}$. Now we have shown that the Laurent expansion of $R_\lambda$is for some $M \geq 0$. Then, we can prove by the same way as the proof of. To prove that $\Pi_0 iX(\Omega)$is of finite dimensional, we need the next lemma.

Lemma (3.1.22)[3]: dimKer $B^{(n)}$ (λ) ≤dimKer $(id - K^\times A(\lambda))\ for\ any\ n\ \geq 1.$

Proof: Suppose that $B^{(n)}(\lambda)\mu\ = 0$with $\mu\ \neq 0$. Then, we have

$$\begin{aligned} K^\times\left(\lambda - H^\times\right)^{n-1}B^{(n)}(\lambda)\mu &= K^\times\left(\lambda - H^\times\right)^{n-1}\left(id - A^{(n)}(\lambda)K^\times\left(\lambda - H^\times\right)^{n-1}\right)\mu \\ &= \left(id - K^\times A(\lambda)\right) \circ K^\times\left(\lambda - H^\times\right)^{n-1}\mu = 0. \end{aligned}$$

If $K^\times(\lambda\ - H^\times)^{n-1}\mu\ = 0, B(n)(\lambda)\mu\ = 0$yields $\mu = A(n)(\lambda)K^\times(\lambda\ - H^\times)^{n-1}\mu\ = 0,$ which contradicts with the assumption $\mu\ \neq 0$. Thus we obtain $K^\times(\lambda\ - H^\times)^{n-1}\mu\ \in Ker\big(id - K^\times A(\lambda)\big)$and the mapping $\mu\ \rightarrow K^\times(\lambda\ - H^\times)^{n-1}\mu$ is one-to-one.Due to, Ker(*id*−*K*×*A*(*λ*)) is finite dimensional. Hence, $KerB^{(n)}(\lambda)$is also finite dimensional for any $n\ \geq 1$. This and prove that $\Pi_0 iX(\Omega)$is a finite dimensional space., $\Pi_0 iX(\Omega)\ = V_0$, which completes.

In this subpart, we suppose that

(S1)The operator $iT = i(H + K)$generates a *C*0-semigroup $e^{\mathrm{i}Tt}$on *H*(recall $i\ = \sqrt{-1}$).

For example, this is true when *K* is bounded on *H* or *T* is selfadjoint. By the Laplace inversion, the semi group is given as

$$\left(e^{\mathrm{i}Tt}\psi, \phi\right) = \frac{1}{2\pi\mathrm{i}} \lim_{x\to\infty} \int_{-x-\mathrm{i}y}^{x-\mathrm{i}y} e^{\mathrm{i}\lambda t}\left((\lambda - T)^{-1}\psi, \phi\right)d\lambda, \quad x, y \in \mathbf{R}, \quad \textbf{(55)}$$

where the contour is a horizontal line in the lower half plane below the spectrum of *T*. we have shown that if there is an Eigen value of *T*on the lower half plane, $e^{\mathrm{i}Tt}$diverges as $t \to\infty$, while if there are no Eigen values, to investigate the asymptotic behavior of $e^{\mathrm{i}Tt}$is difficult in general. Let us show that resonance poles induce an exponential decay of the semigroup.We use the residue theorem to calculate Let $\lambda_0 \in \Omega$be an isolated resonance pole of finite multiplicity. Suppose that the contour γ is deformed to the contourγ', which lies above λ_0, without passing the generalized spectrum $\hat{\sigma}(T)$ except for λ_0,

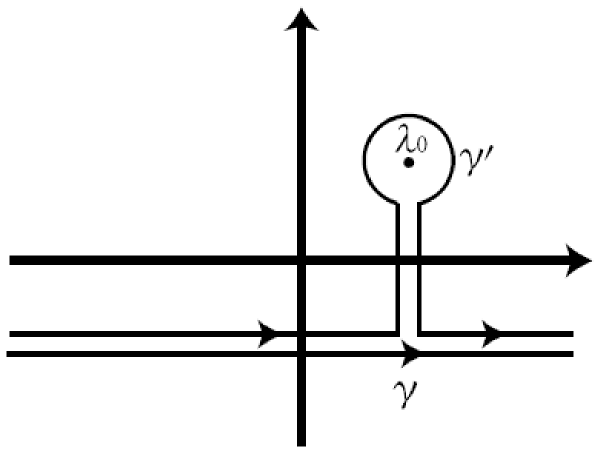


Fig.2.Deformationofthecontour

.Fig.2. For example, it is possible under the assumptions of. Recall that if $\psi, \varphi \in X(\Omega)$, $((\lambda - T)^{-1}\psi, \varphi)$defined on the lower half plane has an analytic continuation $\langle R_\lambda \psi | \varphi \rangle$defined on $\Omega \cup I \cup \{\lambda \mid Im(\lambda) < 0$Thus we obtain

$$\left(e^{\mathrm{i}Tt}\psi, \phi\right) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma'} e^{\mathrm{i}\lambda t} \langle \mathcal{R}_\lambda \psi \mid \phi \rangle d\lambda - \frac{1}{2\pi\mathrm{i}} \int_{\gamma_0} e^{\mathrm{i}\lambda t} \langle \mathcal{R}_\lambda \psi \mid \phi \rangle d\lambda, \qquad \textbf{(56)}$$

where γ_0 is a sufficiently small simple closed curve enclosing γ_0. Let $R_\lambda = \sum_{j=-M}^{\infty} (\lambda_0 - \lambda) jEj = 0$ be a Laurent series of $R_\lambda$as inthe proof of and $E^{-1} = -\Pi_0$, we obtain

$$\frac{1}{2\pi\mathrm{i}} \int_{\gamma_0} e^{\mathrm{i}\lambda t} \langle \mathcal{R}_\lambda \psi \mid \phi \rangle d\lambda = \sum_{k=0}^{M-1} e^{\mathrm{i}\lambda_0 t} \frac{(-\mathrm{i}t)^k}{k!} \left\langle \left(\lambda_0 - T^\times\right)^k \Pi_0 \psi \mid \phi \right\rangle,$$

where Π_0 is the generalized projection to the generalized eigenspace of λ_0. Since $Im(\lambda_0) > 0$, this proves that the second term in the right hand side of decays to zero as $t \to \infty$. Such an exponential decay of (a part of)the semigroup induced by res-onance poles is known as Landau damping in plasma physics , and is often observed for Schrodinger operators . A similar calculation is possible without defining the generalizedresolvent and the generalized spectrum as long as the quantity $((\lambda - T)^{-1}\psi, \varphi)$has an analytic continuation for some $\psi$and φ. Indeed, this has been done in the literature.Let us reformulate it by using the dual space to find a decaying state corresponding to λ_0. For this purpose, we suppose that (S2)the semi group$\{(e^{iTt})^*\}t \geq 0$ is an equicontinuous C_0semigroup on $X(\Omega)$.

Then, by the theorem in IX-13 of Yosida , the dual semi group $(e^{iTt})^{\times}=((e^{iTt})^{*})'$is also an equicontinuous C_0semi group generated by $iT^{\times}$. A convenient sufficient condition for (S2) is that:(S_2)'$K^*|X(\Omega)$is bounded and $\{e^{iHt}\}_t \geq 0$is an equicontinuous C_0 semi group on $X(\Omega)$.bIndeed, the perturbation theory of equicontinuousC_0semi groups shows that $(S2)'$implies $(S2)$. By using the dual semigroup, is rewritten as

$$\left(e^{\mathrm{i}Tt}\right)^{\times}\psi=\frac{1}{2\pi\mathrm{i}}\lim_{x\to\infty}\int_{-x-\mathrm{i}y}^{x-\mathrm{i}y} e^{\mathrm{i}\,\lambda t}\mathcal{R}_{\lambda}\psi d\lambda \tag{57}$$

for any $\psi \in iX(\Omega)$. Similarly, yields

$$\left(e^{\mathrm{i}Tt}\right)^{\times}\psi=\frac{1}{2\pi\mathrm{i}}\int_{\gamma'} e^{\mathrm{i}\,\lambda t}\mathcal{R}_{\lambda}\psi d\lambda-\sum_{k=0}^{M-1} e^{\mathrm{i}\,\lambda_0 t}\frac{(\mathrm{i}t)^k}{k!}\left(\lambda_0-T^{\times}\right)^k \Pi_0\psi \tag{58}$$

when λ_0 is a generalized eigenvalues of finite multiplicity. For the dual semigroup, the following statements hold.

Proposition (3.1.24)[3]:*Suppose*(S1) *and*(S2).

(i)*A solution of the initial value problem*

$$\frac{d}{dt}\xi=\mathrm{i}T^{\times}\xi,\quad \xi(0)=\mu\in \mathsf{D}\left(T^{\times}\right), \tag{59}$$

in $X(\Omega)'$ is uniquely given by $\xi(t) = (e^{iTt})^{\times}\mu$.

(ii)*Let $\lambda_0$be a generalized Eigen values and μ_{0a} corresponding generalized Eigen function. Then,*

$(e^{\mathrm{i}Tt})^{\times}\mu_0 = e^{\mathrm{i}T_0 t}\mu_0.$

(iii)*Let $\Pi_0$be a generalized projection for λ_0. The space $\Pi_0 iX(\Omega)$is $(e^{\mathrm{i}Tt})^{\times}$invariant:* $(e^{\mathrm{i}Tt})^{\times}\Pi_0 = \Pi_0(e^{\mathrm{i}Tt})^{\times}|iX(\Omega)$.

Proof: Since $\{(e^{\mathrm{i}Tt})^{\times}\}t\geq 0$ is an equicontinuous C_0 semigroup generated by i$T^{\times}$, (i) follows from the usual semi group theory Because of we have $iT^{\times}\mu_0 = i\lambda_0\mu_0$. Then,

$$\frac{d}{dt}e^{\mathrm{i}\lambda_0 t}\mu_0=\mathrm{i}\lambda_0 e^{\mathrm{i}\lambda_0 t}\mu_0=\mathrm{i}T^{\times}\left(e^{\mathrm{i}\lambda_0 t}\mu_0\right).$$

Thus $\xi(t) = e^{i\lambda 0t}\mu_0$is a solution By the uniqueness of a solution, we obtain(ii). Because of, we have

$$\frac{d}{dt}\left(e^{\mathrm{i}Tt}\right)^{\times}\mathcal{R}_{\lambda} = \mathrm{i}T^{\times}\left(\left(e^{\mathrm{i}Tt}\right)^{\times}\mathcal{R}_{\lambda}\right),$$
$$\frac{d}{dt}\mathcal{R}_{\lambda}\left(e^{\mathrm{i}Tt}\right)^{\times}\Big|_{iY} = \mathcal{R}_{\lambda}\cdot\left(e^{\mathrm{i}Tt}\right)^{\times}\mathrm{i}T^{\times}\Big|_{iY} = \mathrm{i}T^{\times}\left(\mathcal{R}_{\lambda}\left(e^{\mathrm{i}Tt}\right)^{\times}\right)\Big|_{iY}.$$

Hence, both of $(e^{\mathrm{i}Tt})^{\times}R_{\lambda} and\ R_{\lambda}(e^{\mathrm{i}Tt})^{\times}$are solutions of. By the uniqueness, we obtain $(e^{\mathrm{i}Tt})^{\times}R_{\lambda}|_{\mathrm{iY}} = (e^{\mathrm{i}Tt})^{\times}R_{\lambda}|_{\mathrm{iY}}$. Then, the definition of the projection $\Pi_0$proves$(e^{\mathrm{i}Tt})^{\times}\Pi_0|\mathrm{iY} = \Pi_0(e^{\mathrm{i}Tt})^{\times}|\mathrm{iY}$ with the aid of Since *Y* is dense in *X*(*Ω*)and both operators $(e^{\mathrm{i}Tt})^{\times}\Pi_0\ o\ \mathrm{i}$ and $\Pi_0(e^{\mathrm{i}Tt})^{\times}o\ \mathrm{i}\ = \Pi_0 o\ \mathrm{i}\ o\ e^{\mathrm{i}Tt}$are continuous on $X(\Omega)$.the equality is true on$iX(\Omega)$, any usual function $\varphi\ \in X(\Omega)$is decomposed $a\ s\ |\varphi = \mu_1 + \mu_2$with $\mu_1 \in \Pi_0 iX(\Omega)$and $\mu_2 \in (id - \Pi 0)iX(\Omega)$in the dual space., this decomposition is $(e^{\mathrm{i}Tt})^{\times}$invariant. When $\lambda_0 \in \Omega$, $(e^{\mathrm{i}Tt})^{\times}\mu_1 \in \Pi_0 iX(\Omega)$decays to zero exponentially as $t\ \to\infty$.gives the decomposition explicitly. Such an exponential decay can be well observed if we choose a function, which is sufficiently close to the generalized Eigen functionμ_0, as an initial state. Since $X(\Omega)$is dense in $X(\Omega)'$and since $(e^{\mathrm{i}Tt})^{\times}$is continuous, for any $T > 0$and $\varepsilon\ > 0$, there exists a function $\varphi_0 in\ X(\Omega)$such that

$$\left|\left\langle\left(e^{\mathrm{i}Tt}\right)^{\times}\phi_0 \mid \psi\right\rangle - \left\langle\left(e^{\mathrm{i}Tt}\right)^{\times}\mu_0 \mid \psi\right\rangle\right| < \varepsilon,$$

for $0\ \le t\ \le T$and $\psi \in X(\Omega)$. This implies that

$$\left(e^{\mathrm{i}Tt}\phi_0, \psi\right) \sim \left\langle\left(e^{\mathrm{i}Tt}\right)^{\times}\mu_0 \mid \psi\right\rangle = e^{\mathrm{i}\lambda_0 t}\langle\mu_0 \mid \psi\rangle, \qquad \textbf{(60)}$$

for the interval $0\ \le t\ \le T$. Thus generalized Eigen values describe the transient behavior of solutions.

section (3.2): An Application and Pettis Integrals and Vector Valued Holomorphic Functions on the Dual Space

Let us apply the present theory to the dynamics of an infinite dimensional coupled oscillators. The results in this part are partially obtained. Coupled oscillators are often used as models of collective synchronization phenomena. One of the important models for synchronization is the Kuramoto model defined by

$$\frac{d\theta_i}{dt} = \omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \dots, N, \tag{61}$$

where $\theta_i = \theta_i(t) \in [0, 2\pi)$denotes the phase of an $i-th$ oscillator rotating on a circle, $\omega i \in \boldsymbol{R}$is a constant called a natural frequency, $k \geq 0$ is a coupling strength, and where $N$is the number of oscillators When $k > 0$, there are interactions between oscillators and collective behavior may appear. For this system, the order parameter $\eta(t)$, which gives the centroid of oscillators, is defined to be

$$\eta(t) := \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j(t)}. \tag{62}$$

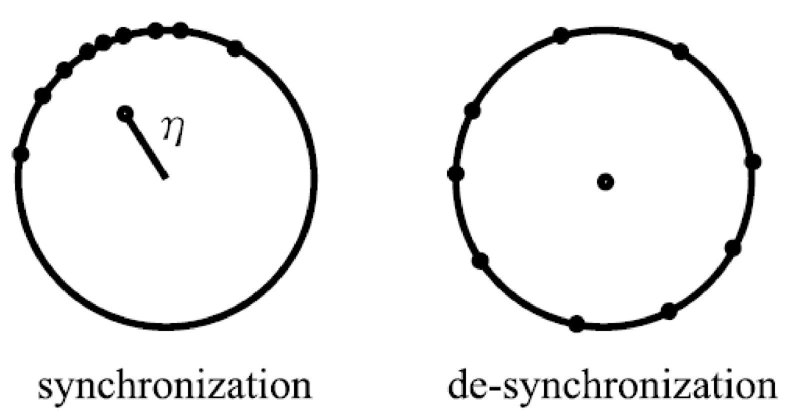


Fig.3.The order parameter of the Kuramoto model.

If $|\eta(t)|$takes a positive number, synchronous state is formed, while if $|\eta(t)|$is zero on time average, de-synchronization is stable.For many applications, N is too large so that statistical–mechanical description is applied. In such a case, the continuous limit of the Kuramoto model is often employed: At first, note that Eq (61) can be written as

$$\frac{d\theta_i}{dt} = \omega_i + \frac{k}{2\mathrm{i}}\left(\eta(t)e^{-\mathrm{i}\theta_i} - \overline{\eta(t)}e^{\mathrm{i}\theta_i}\right).$$

Keeping it in mind, the continuous model is defined as the equation of continuity of the form

$$\begin{cases} \dfrac{\partial \rho_t}{\partial t} + \dfrac{\partial}{\partial \theta}(v\rho_t) = 0, \\ v := \omega + \dfrac{k}{2\mathrm{i}}\left(\eta(t)e^{-\mathrm{i}\theta} - \overline{\eta(t)}e^{\mathrm{i}\theta}\right), \\ \eta(t) := \displaystyle\int_{\mathbf{R}} \int_0^{2\pi} e^{\mathrm{i}\theta}\rho_t(\theta,\omega)g(\omega)d\theta d\omega. \end{cases} \quad \textbf{(63)}$$

This is an evolution equation of a probability measure $\rho_t = \rho_t(\theta,\omega)$on S_1=[0, 2π) parameterized by $t \in \boldsymbol{R}$ and $\omega \in \boldsymbol{R}$. Roughly speaking, $\rho t(\theta,\omega)$denotes a probability that an oscillator having a natural frequency $\omega$is placed at a position θ. The ηabove is the continuous version of (62) , which is also called the order parameter, and $g(\omega)$is a given probability density function for natural frequencies. This system is regarded as a Fokker Planck equation of (61) . Indeed, it is known that the order parameter for the finite Eq(61) of dimensional system converges to that of the continuous model as $N \to \infty$in some probabilistic sense. To investigate the stability and bifurcations of solutions of the system is a famous difficult problem in this field . It is numerically observed that when $k > 0$is sufficiently small, then the de-synchronous state $/\eta/$ =0is asymptotically stable, while if $k$exceeds a certain value k_c, a nontrivial solution corresponding to the synchronous state $|\eta| > 0$ bifurcates from the desynchronous state. Indeed, Kuramoto conjectured thatKuramoto conjecture. *Suppose that natural frequencies ω_i are distributed according to a probability density function $g(\omega)$. If $g(\omega)$ is an even and unimodal function*

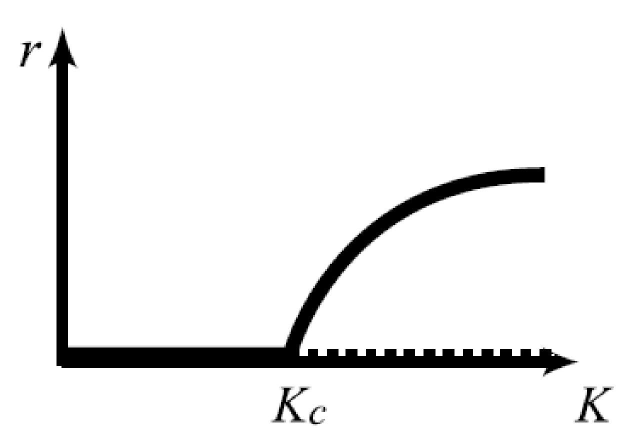


Fig.4.A bifurcation diagram of the order parameter. Solid lines denote stable solutions and dotted lines denote unstable solutions.

*such that $g''(0) \neq 0$, then the bifurcation diagram of $r = |\eta|$ is given as that is, if the coupling strength k is smaller than $k_c = \frac{2}{\pi g(0)}$, then $r \equiv 0$ is asymptotically stable. On the other hand, if k is larger than k_c, the synchronous state emerges; there exists a positive constant rcsuch that $r = r_c$ is asymptotically stable. Near the transition point K_c, R_c is of order $O((k-k_c)^{\frac{1}{2}})$.*A function $g(\omega)$is called unimodal $(at\ \omega = 0)$ if $g(\omega_1) > g(\omega_2)$for $0 \leq \omega_1 < \omega_2$and $g(\omega_1) < g(\omega_2) for\ \omega_1 < \omega_2 \leq 0$.for Kuramoto's discussion. The purpose here is to prove the linear stability of the de-synchronous state $|\eta| = r = 0 for\ 0 < k < k_c$ by applying our spectral theory when $g(\omega) = e^{-\omega_2/2}/\sqrt{2\pi}$ is assumed to be the Gaussian distribution as in the most literature. for nonlinear analysis and the proof of the bifurcation at $k = k_c$At first, let us observe that the difficulty of the conjecture is caused by the continuous spectrum. Let

$$Z_j(t,\omega) := \int_0^{2\pi} e^{\mathrm{i}j\theta} \rho_t(\theta,\omega) d\theta \tag{64}$$

be the Fourier coefficient of $\rho_t(\theta,\omega)$. Then, $Z_0(t,\omega) = 1 and\ Z_j$satisfy the differential equations

$$\frac{dZ_1}{dt} = \mathrm{i}\omega Z_1 + \frac{k}{2}\eta(t) - \frac{k}{2}\overline{\eta(t)} Z_2, \tag{65}$$

and

$$\frac{dZ_j}{dt} = j\mathrm{i}\omega Z_j + \frac{jk}{2}\left(\eta(t) Z_{j-1} - \overline{\eta(t)} Z_{j+1}\right), \tag{66}$$

for$j = 2, 3, \ldots . Let\ L^2(\boldsymbol{R}, g(\omega)d\omega)$be the weighted Lebesgue space and put $P_0(\omega) = 1 \in L^2(\boldsymbol{R}, g(\omega)d\omega)$. Then, the order parameter is written as $\eta(t)$ =$(Z_1,\ P_0)$by using the inner product on $L^2(\boldsymbol{R}, g(\omega)d\omega)$. Since our purpose is to investigate the dynamics of the order parameter, let us consider the linearized system of Z_1 given by

$$\frac{dZ_1}{dt} = \left(\mathrm{i}\mathcal{M} + \frac{k}{2}\mathcal{P}\right) Z_1, \tag{67}$$

Where$M : \varphi(\omega) \rightarrow \omega\varphi(\omega)$is the multiplicationoperator on $L^2(\boldsymbol{R}, g(\omega)d\omega) and\ P$ is the projection on $L^2(\boldsymbol{R}, g(\omega)d\omega)$defined to be

$$\mathcal{P}\phi(\omega) = \int_{\mathbf{R}} \phi(\omega)g(\omega)d\omega = (\phi, P_0)P_0. \quad \textbf{(68)}$$

To determine the linear stability of the de-synchronous state $\eta = 0$, we have to investigate the spectrum and the semigroup of the operator $T_1 = iM + \frac{k}{2}P$.The domain of$T_1 = iM + \frac{k}{2}P$ is given by $D(M) \cap D(P) = D(M)$, which is dense in L^2(**R**, $g(\omega)d\omega$). Since M is selfadjoint and since P is bounded, $T1$is a closed operator . Let $Q(T_1)$be the resolvent set of $T_1$and $\sigma(T_1)$ =**C** $/Q()T_1 the$e spectrum. Let $\sigma_p(T_1)$and $\sigma_c(T_1)$be the point spectrum (the set of Eigen values s) and the continuous spectrum of T_1, respectively.

Lemma (3.2.1)[3]:

(i) *Eigenvalues* λ *of* T_1 *are given as roots of*

$$\int_{\mathbf{R}} \frac{1}{\lambda - \mathrm{i}\omega} g(\omega)d\omega = \frac{2}{k}. \quad \textbf{(69)}$$

(ii) *The continuous spectrum of* T_1 *is given by*

$$\sigma_c(T_1) = \sigma(iM) = i\boldsymbol{R}. \quad \textbf{(70)}$$

Proof: Part (i) follows from a straightforward calculation of the equation $\lambda v = T_1 v$. Indeed, this equation yields

$$(\lambda - \mathrm{i}\omega)v = \frac{k}{2}\mathcal{P}v = \frac{k}{2} \cdot (v, P_0)P_0.$$

This is rewritten as $v = k/2 \cdot (v, P_0)(\lambda - i\omega)^{-1}P_0$. Taking the inner product with P_0, we obtain

$$1 = \frac{k}{2}\big((\lambda - \mathrm{i}\omega)^{-1}P_0, P_0\big),$$

which gives the desired result. Part (ii) follows from the fact that the essential spectrum is stable under the bounded perturbation. The essential spectrum of $T_1$is the same as $\sigma(iM)$.

Since M is defined on the weighted Lebesgue space and the weight $g(\omega)$is the Gaussian, $\sigma(iM) = i\cdot\text{supp}(g)$ =i**R**. Our next task is to calculate roots to obtain Eigen values of T_1. Put $k_c = \frac{2}{\pi g(0)}$, which is called Kuramoto's transition point.

Lemma (3.2.2)[3]: *When $k$is larger than k_c, there exists a unique Eigen values $\lambda(k)$of T_1 on the positive real axis. As kdecreases, the Eigen values $\lambda(k)$approaches to the imaginary axis, and at $k = k_c$, it is absorbed into the continuous spectrum and disappears. When $0 < k < k_c$, there are no Eigen values s.*

Proof: Put $\lambda = x + iy$ with $x, y \in \boldsymbol{R}$,is rewritten as

$$\begin{cases} \displaystyle\int_{\mathbf{R}} \frac{x}{x^2 + (\omega - y)^2} g(\omega) d\omega = \frac{2}{k}, \\ \displaystyle\int_{\mathbf{R}} \frac{\omega - y}{x^2 + (\omega - y)^2} g(\omega) d\omega = 0. \end{cases} \tag{71}$$

The first equation implies that if there is an eigenvalues $x + iy$ for $k > 0$, then $x > 0$. Next, the second equation is calculated as

$$0 = \int_{\mathbf{R}} \frac{\omega - y}{x^2 + (\omega - y)^2} g(\omega) d\omega = \int_0^{\infty} \frac{\omega}{x^2 + \omega^2} \big(g(y + \omega) - g(y - \omega)\big) d\omega.$$

Since $g$is an even function, y=0is a root of this equation. Since g is unmoral, $g(y + \omega) - g(y - \omega) > 0$when $y < 0$, $\omega > 0$ and $g(y + \omega) - g(y - \omega) < 0$ when $y > 0, \omega > 0$. Hence, $y = 0$ is a unique root. This proves that an eigenvalues should be on the positive real axis, if it exists. Let us show the existence. If $|\lambda|$is large, (79) is expanded as

$$\frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) = \frac{2}{k}.$$

Thus Rouche's theorem proves that (79) has a root$\lambda \sim \frac{k}{2} if\ k > 0$is sufficiently large. Its position $\lambda(k)$is continuous (actually analytic) in k as long as it exists. The eigenvalues disappears only when $\lambda \to +0$as $k \to k_c$for some value k_c. Substituting y=0and taking the limit $x \to +0$, $k \to k_c$, we have

$$\lim_{x\to+0}\int_{\mathbf{R}}\frac{x}{x^2+\omega^2}g(\omega)d\omega=\frac{2}{k_c}$$

The well-known formula

$$\lim_{x\to+0}\int_{\mathbf{R}}\frac{x}{x^2+\omega^2}g(\omega)d\omega=\pi g(0)$$

provides $k_c=\frac{2}{\pi g(0)}$. Since $k_c$is uniquely determined, the Eigen values $\lambda(k)$exists for $k>k_c$, disappears at $k=k_c$and there are no Eigen values s for $0<k<k_c$.This lemma shows that when k is larger than k_c, $Z_1=0$ of is unstable because of the Eigen values with a positive real part. However, when $0<k<k_c$, there are no Eigen values s and the spectrum of T_1consists of the continuous spectrum on the imaginary axis. Hence, the usual spectral theory does not provide the stability of solutions. To handle this difficulty, let us introduce a rigged Hilbert space.

To apply our theory, let us define a test function space $X(\Omega)$. Let $Exp_+(\beta,n)$ be the set of holomorphic functions on the region $\boldsymbol{C}_n=\{z\in\boldsymbol{C}\mid Im(z)\geq-1/n\}$such that the norm

$$\|\phi\|_{\beta,n}:=\sup_{\mathrm{Im}(z)\geq-1/n}e^{-\beta|z|}\left|\phi(z)\right| \tag{72}$$

is finite. With this norm, $Exp_+(\beta,n)$is a Banach space. Let $Exp_+(\beta)$be their inductive limit with respect to$n$ =1, 2, ...

$$\mathrm{Exp}_+(\beta)=\varinjlim_{n\geq1}\mathrm{Exp}_+(\beta,n)=\bigcup_{n\geq1}\mathrm{Exp}_+(\beta,n) \tag{73}$$

Next, define Exp_+ to be their inductive limit with respect to $\beta=0,1,2,\ldots$

$$\mathrm{Exp}_+=\varinjlim_{\beta\geq0}\mathrm{Exp}_+(\beta)=\bigcup_{\beta\geq0}\mathrm{Exp}_+(\beta). \tag{74}$$

Thus $Exp_+$is the set of holomorphic functions near the upper half plane that can grow at most exponentially. Then, we can prove the next proposition.

Proposition (3.2.3)[3]: Exp_+ *is a topological vector space satisfying*

(i) Exp_+ *is a complete Montel space* Exp_+ *is a dense subspace of* $L^2 \quad (\boldsymbol{R}, g(\omega)d\omega)$.

(ii) *Thetopology of* Exp_+ *is stronger than that of* $L^2(\boldsymbol{R}, g(\omega)d\omega)$

(iii) *Theoperators* M *and Pare continuous on* Exp_+. *In particular,* $T_1: Exp_+ \to Exp_+$ *is continuous (note that it is not continuous on* $L^2(\boldsymbol{R}, g(\omega)d\omega)$*).*

for the proof. Thus, $X(\Omega) = Exp_+$ satisfies (X1) to (X3) and the rigged Hilbert space

$$\mathrm{Exp}_+ \subset L^2\big(\mathbf{R}, g(\omega)d\omega\big) \subset \mathrm{Exp}'_+ \tag{75}$$

is well-defined. Furthermore, the operator

$$T := T_1/\mathrm{i} = \mathcal{M} + \frac{k}{2\mathrm{i}}\mathcal{P} \tag{76}$$

satisfies the assumptions (X4) to (X8) with $H = M$ and $K = \frac{k}{2i}P$. Indeed, the analytic continuation $A(\lambda)$ of the resolvent $(\lambda - M)^{-1}$ is given by

$$\big\langle A(\lambda)\psi \mid \phi\big\rangle = \begin{cases} \int_{\mathrm{R}} \frac{1}{\lambda-\omega}\psi(\omega)\phi(\omega)g(\omega)d\omega + 2\pi\mathrm{i}\psi(\lambda)\phi(\lambda)g(\lambda) & (\mathrm{Im}(\lambda) > 0), \\ \lim_{y\to -0}\int_{\mathrm{R}} \frac{1}{x+\mathrm{i}y-\omega}\psi(\omega)\phi(\omega)g(\omega)d\omega & (x = \lambda \in \mathbf{R}), \\ \int_{\mathrm{R}} \frac{1}{\lambda-\omega}\psi(\omega)\phi(\omega)g(\omega)d\omega & (\mathrm{Im}(\lambda) < 0), \end{cases} \tag{77}$$

for $\psi, \varphi \in Exp_+$. Since functions in Exp_+ are holomorphic near the upper half plane, (X4) and (X5) are satisfied with $I = \boldsymbol{R}$ and Ω=(the upper half plane). Since *M* and *P* are continuous on Exp_+, (X6) and (X7) are satisfied with *Y*=Exp_+. For (X8), note that the dual operator $K^\times$ of *K* is given as

$$K^\times \mu = \frac{k}{2\mathrm{i}}\langle \mu \mid P_0\rangle\langle P_0| \in i\,\mathrm{Exp}_+ = iX(\Omega). \tag{78}$$

Since the range of $K^\times$ is included in $iX(\Omega)$, (X8) is satisfied. Therefore, all assumptions in Part are verified and we can apply our spectral theory to the operator T_1/i.

Remark(4.2.4)[3]: T_1 is not continuous on $Exp_+(\beta, n)$ for fixed $\beta > 0$ because of the multi plication $M : \varphi \to \omega\varphi$. The inductive limit in *β* is introduced so that it becomes continuous. The proof of Lemma (3.2.1)[3] shows that the Eigen function of T_1 associated with *λ* is given by

$$v_\lambda = \frac{1}{\lambda - i\omega}, \quad \lambda > 0.$$

If $\lambda > 0$ is small, v_λ is not included in $Exp_+(\beta, n)$ for fixed n. The inductive limit in n is introduced so that any Eigen functions are elements of Exp_+. Furthermore, the topology of Exp_+ is carefully defined so that the strong dual Exp_+ becomes a Fréchet Montel space. It is known that the strong dual of a Montel space is also Montel. Since Exp_+ is defined as the inductive limit of Banach spaces, its dual is realized as a projective limit of Banach spaces $Exp_+(\beta, n)'$, which is Fréchet by the definition. Hence, the contraction principle is applicable on Exp_+ which allows one to prove the existence of center manifolds of the system though nonlinear problems are not treated in this chapter.

Proposition (3.2.5)[3]:

(i) *The generalized continuous and the generalized residual spectra are empty.*

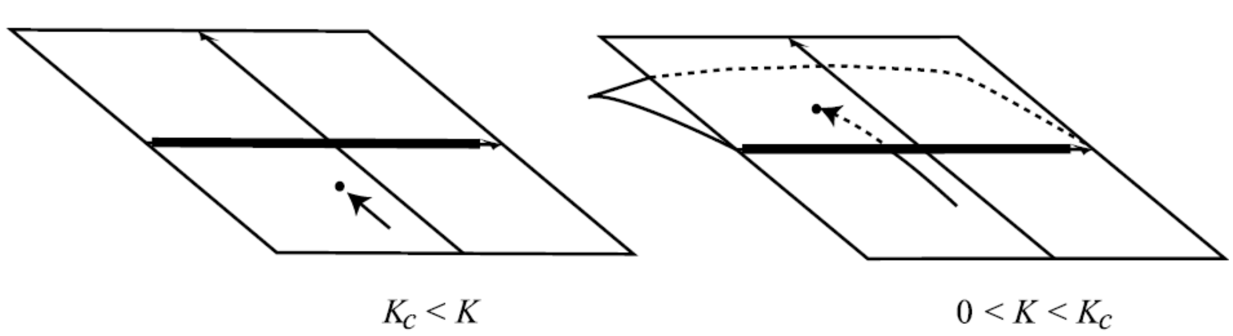


Fig.5.As k decreases, the Eigen values of $\frac{T1}{i}$ disappears from the original complex plane by absorbed into the continuous spectrum on the real axis. However, it still exists as a resonance pole on the sectionond Riemann sheet of the generalized resolvent.

(iii) *For any k>0, there exist infinitely many generalized Eigen values s on the upper half plane.*

(iv) *For $k > k_c$, there exists a unique generalized Eigen values λ(k) on the lower half plane, which is an Eigen values of T_1/i in $L_2(\boldsymbol{R}, g(\omega)d\omega)'$ sense. As k decreases, $\lambda(k)$ goes upward and at $k = k_c$, $\lambda(k)$ gets across the real axis and it becomes a resonance pole. When*

$0 < k < k_c, \lambda(k)$*lies on the upper half plane and there are no generalized Eigen values s on the lower half plane.*

Proof.(i) Since $K^{\times}$is a onedimensional operator, it is easy to verify the assumption Hence, the generalized continuous and the generalized residual spectra are empty.

(iii) Let $\lambda$and μ be a generalized Eigen values and a generalized Eigen function, $\lambda$and μsatisfy $(id - K^{\times}A(\lambda))K^{\times}\mu = 0$. In our case,

$$\langle K^{\times}\mu \mid \phi\rangle = \frac{k}{2\mathrm{i}}\langle \mu \mid P_0\rangle\langle P_0 \mid \phi\rangle$$

And

$$\langle K^{\times}A(\lambda)K^{\times}\mu \mid \phi\rangle = \langle A(\lambda)K^{\times}\mu \mid K^{*}\phi\rangle = \left(\frac{k}{2\mathrm{i}}\right)^2\langle \mu \mid P_0\rangle\langle P_0 \mid \phi\rangle\langle A(\lambda)P_0 \mid P_0\rangle.$$

for any $\varphi \in Exp_+$. Hence, generalized Eigen values s are given as roots of the equation

$$\frac{2\mathrm{i}}{k} = \langle A(\lambda)P_0 \mid P_0\rangle = \begin{cases} \int_{\mathbf{R}} \frac{1}{\lambda-\omega}g(\omega)d\omega + 2\pi\mathrm{i}g(\lambda) & (\mathrm{Im}(\lambda) > 0), \\ \int_{\mathbf{R}} \frac{1}{\lambda-\omega}g(\omega)d\omega & (\mathrm{Im}(\lambda) < 0). \end{cases} \quad \textbf{(79)}$$

Since $g$is the Gaussian, it is easy to verify that for $Im(\lambda) > 0$has infinitely many roots $\{\lambda_n\}_{n=0}^{\infty}$ $such\ that\ Im(\lambda_n) \to \infty$ and they approach to the rays arg$(z) = \frac{\pi}{4}, \frac{3\pi}{4}$ $as\ n \to \infty$.

(iv) When $Im(\lambda) < 0$, in which λ is replaced by $i\lambda$. Thus Lemma(shows that when $k > k_c$, there exists a root $\lambda(k)$on the lower half plane. k decreases, $\lambda(k)$ goes upward and for $0 < k_c < k$, it becomes a root of the first equation of because the right hand side of is holomorphic in λ.

shows that a generalized Eigen function associated with $\lambda$is given by $\mu = A(\lambda)K^{\times}\mu = \frac{k}{2i}\langle\mu\,|P_0\rangle \cdot A(\lambda)\_P_0|$. We can choose a constant $\mu\langle|P_0\rangle as\ \langle\mu\,|P_0\rangle = \frac{2i}{k}$. Then, $\mu = A(\lambda)'P_0| = A(\lambda)i(P_0)$. When Im($\lambda$) <0, μ is a usual function written as $\mu = (\lambda - \omega)^{-1} \in Exp_+$, although when $Im(\lambda) \geq 0$, $\mu$is not included in $L_2(\boldsymbol{R}, g(\omega)d\omega)$but an element of the dual space Exp_+. In what follows, we denote generalizedEigen values s by $\{\lambda_n\}_{n=0}^{\infty} such\ that\ Im\,|\lambda_n| \leq |\lambda_n + 1|\,for\ n = 0, 1, \ldots$, and a corresponding generalized Eigen function by $\mu n = A(\lambda_n)\langle P_0|$. proves that they satisfy $T_1^{\times}\mu_n = i\lambda_n\mu_n$. Note that when $0 < k < k_c$, all generalized Eigen values s satisfy $Im(\lambda_n) > 0$.Next, let us calculate the generalized resolvent of T_1/i. yields

$$\mathcal{R}_\lambda\phi - A(\lambda)K^{\times}\mathcal{R}_\lambda\phi = A(\lambda)\phi \quad \Longrightarrow \quad \mathcal{R}_\lambda\phi = A(\lambda)\phi + \frac{k}{2i}\langle\mathcal{R}_\lambda\phi \mid P_0\rangle A(\lambda)\langle P_0|, \tag{80}$$

for any $\varphi \in Exp_+$. Taking the inner product with P_0, we obtain

$$\langle\mathcal{R}_\lambda\phi \mid P_0\rangle = \frac{\langle A(\lambda)\phi \mid P_0\rangle}{1 - \frac{k}{2i}\langle A(\lambda)P_0 \mid P_0\rangle} = \frac{\langle A(\lambda)P_0 \mid \phi\rangle}{1 - \frac{k}{2i}\langle A(\lambda)P_0 \mid P_0\rangle}.$$

Substituting this into (62) we obtain

$$\mathcal{R}_\lambda\phi = A(\lambda)\phi + \left(2i/k - \langle A(\lambda)P_0 \mid P_0\rangle\right)^{-1}\langle A(\lambda)P_0 \mid \phi\rangle \cdot A(\lambda)\langle P_0|. \tag{81}$$

Then, the generalized Riesz projection for the generalized eigenvalues $\lambda_n$is given by

$$\Pi_n\phi = \frac{1}{2\pi i}\int_\gamma \mathcal{R}_\lambda\phi d\lambda = D_n\langle A(\lambda_n)P_0 \mid \phi\rangle \cdot A(\lambda_n)\langle P_0| = D_n\langle\mu_n \mid \phi\rangle \cdot \mu_n, \tag{82}$$

Or

$$\langle\Pi_n\phi \mid \psi\rangle = D_n\langle\mu_n \mid \phi\rangle \cdot \langle\mu_n \mid \psi\rangle, \tag{83}$$

where $D_n$is a constant defined by

$$D_n = \lim_{\lambda\to\lambda_n}(\lambda - \lambda_n) \cdot \left(2i/k - \langle A(\lambda)P_0 \mid P_0\rangle\right)^{-1}.$$

the range of $\Pi_n$is spanned by the generalized eigen-function μ_n.Now we are in a position to give a spectral decomposition theorem of the semi group generated by $T_1 = iM + \frac{k}{2}P$. Since iMgenerates theC^0semigroup on$K_2(\boldsymbol{R}, g(\omega)d\omega)$

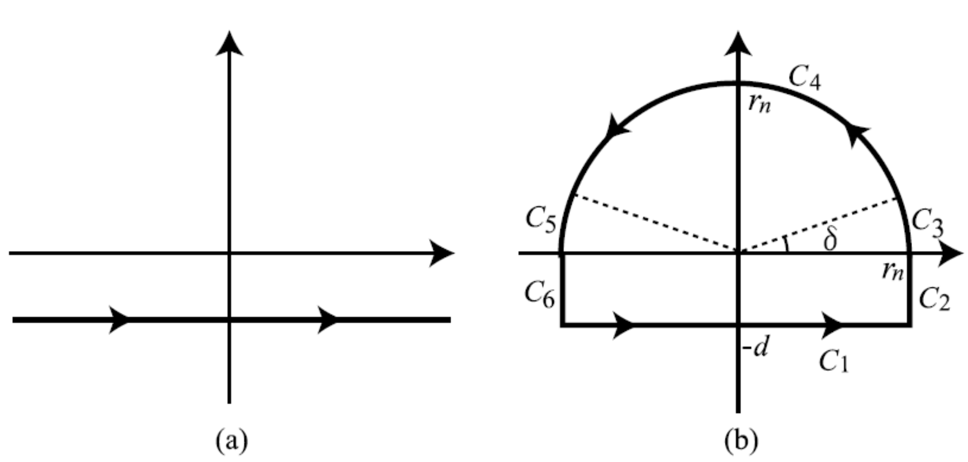


Fig.6.The contour for the Laplacein version formula.

and $P$is bounded, T_1 also generates the C^0semigroup given by

$$e^{T_1 t}\phi = \lim_{y\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{x-\mathrm{i}y}^{x+\mathrm{i}y} e^{\lambda t}(\lambda - T_1)^{-1}\phi d\lambda, \tag{84}$$

for $t > 0$, where x is a sufficiently large number. In $L^2(\mathbf{R}, g(\omega)d\omega)$ theory, we cannot deform the contour from the right half plane to the left half plane because T_1 has the continuous spectrum on the imaginary axis. Let us use the generalized resolvent $R_\lambda of\ T_1/i$. For this purpose, we rewrite the above as

$$e^{T_1 t}\phi = \lim_{y\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{-y-\mathrm{i}x}^{y-\mathrm{i}x} e^{\mathrm{i}\lambda t}(\lambda - T_1/\mathrm{i})^{-1}\phi d\lambda \tag{85}$$

whose contour is the horizontal line on the lower half plane (Fig.6(a)). Recall that when $Im(\lambda) < 0, ((\lambda - T_1/i)^{-1}\varphi, \psi) = \langle R\lambda\varphi\ |\psi\rangle$ for $\varphi, \psi \in Exp_+$ because of Thus we have.

$$\langle e^{T_1 t}\phi \mid \psi\rangle = \lim_{y\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{-y-\mathrm{i}x}^{y-\mathrm{i}x} e^{\mathrm{i}\lambda t}\langle \mathcal{R}_\lambda \phi \mid \psi\rangle d\lambda. \tag{86}$$

Since $\langle R\lambda\varphi\ |\psi\rangle$ is a meromorphic function whose poles are generalized eigenvalues s $\{\lambda n\}_{n=0}^{\infty}$, we can deform the contour from the lower half plane to the upper half plane. With the aid of the residue theorem, we can prove the next theorems.

Theorem (3.2.5)[3]: *(Spectral decomposition). For any $\varphi, \psi \in Exp_+$, there exists $t_0 > 0$ such that the equality*

$$\langle e^{T_1 t}\phi \mid \psi\rangle = \sum_{n=0}^{\infty} D_n e^{\mathrm{i}\lambda_n t}\langle \mu_n \mid \phi\rangle \cdot \langle \mu_n \mid \psi\rangle \tag{87}$$

holds for $t > t_0$. Similarly, the dual semigroup $(e^{T_1 t})^\times$ satisfies

$$\left(e^{T_1 t}\right)^\times \phi = \sum_{n=0}^{\infty} D_n e^{\mathrm{i}\lambda_n t}\langle \mu_n \mid \phi\rangle \cdot \mu_n \tag{88}$$

for $\varphi \in Exp_+$ and $t > t_0$, where the right hand side converges with respect to the strong dual topology on Exp_+'

Theorem (3.2.6)[3]: *(Completeness).*

(i) *A system of generalized Eigen functions* $\{\mu_n\}_{n=0}^{\infty}$ *is complete in the sense that If* $\langle\mu_n|\psi\rangle =$ 0 *for* $n = 0, 1, \ldots$, *then* ψ=0.

(ii)$\mu_0, \mu_1, \ldots$*are linearly independent of each other: if* $\sum_{n\neq 0}^{\infty} a_n\mu_n = 0$.*with* $an \in \boldsymbol{C}$, *then* $a_n = 0$ *for every n.*

(iii) *The decomposition of* $(e^{T_1 t})^{\times}$ *using* $\{\mu_n\}_{n=0}^{\infty}$ *is uniquely expressed as.*

Corollary (3.2.7)[3]:*(Linear stability). When* $0 < k < k_c$, *the order parameter* $\eta(t) =$ (Z_1, P_0) *for the linearized system decays exponentially to zero* as $t \to \infty$ *if the initial con-dition is an element of* Exp_+

Proof: When an initial condition of the system is given by $\varphi \in Exp^+$, the order parameter is given by $\eta(t) = (Z_1, P_0) = (e^{T_1 t}\varphi, P_0). If\ 0 < k < k_c$, all generalized Eigen values s lie on the upper half plane, so that $Re[i\lambda_n] < 0 for\ n = 0, 1, \ldots$. Then the corollary follows from

(i) If $\langle\mu_n|\psi\rangle$=0 for all *n*, provides for any Exp^+. Since $Exp^+$is dense in L^2 $(\boldsymbol{R}, g(\omega)d\omega)$, we obtain $(e^{T_1 t})^{*}\psi = 0$ for any $t > t_0$, which proves $\psi = 0$.

(ii) Suppose that $\sum_{n=0}^{\infty} a_n\mu_n$ = 0

$$0 = \left(e^{T_1 t}\right)^{\times} \sum_{n=0}^{\infty} a_n\mu_n = \sum_{n=0}^{\infty} a_n \left(e^{T_1 t}\right)^{\times} \mu_n = \sum_{n=0}^{\infty} a_n e^{\mathrm{i}\lambda_n t}\mu_n.$$

Changing the label if necessary, we can assume that

$$Re[i\lambda_0] \geq Re[i\lambda_1] \geq Re[i\lambda_2] \geq \cdots,$$

without loss of generality. Suppose that $Re[i\lambda_0] = \cdots = Re[i\lambda_k]$and $Re[i\lambda_k] > Re[i\lambda_{k+1}]$. Then, the above equality provides

$$\mathrm{Re}[\mathrm{i}\lambda_0] \geq \mathrm{Re}[\mathrm{i}\lambda_1] \geq \mathrm{Re}[\mathrm{i}\lambda_2] \geq \cdots,$$

Taking the limit $t \to \infty$yields

$$0 = \sum_{n=0}^{k} a_n e^{\mathrm{i}\mathrm{Im}[\mathrm{i}\lambda_n]t}\mu_n + \sum_{n=k+1}^{\infty} a_n e^{(\mathrm{i}\lambda_n - \mathrm{Re}[\mathrm{i}\lambda_0])t}\mu_n.$$

Taking the limit $t \to \infty$ yields

$$0=\lim_{t\to\infty}\sum_{n=0}^{k}a_n e^{\mathrm{iIm}[\mathrm{i}\lambda_n]t}\mu_n.$$

Since the finite set $\mu_0,\ldots,\mu_k$of eigenvectors are linearly independent as in a finite dimensional case, we obtain $an=0$for $n=0,\ldots,k$. The same procedure is repeated to prove $a_n=0$ for every *n*.

(iii) This immediately follows from part(ii) of the theorem.

Finally, let us prove. Recall that generalized Eigen values s are roots of Hence, there exist positive numbers *B*and $\{r_j\}_{j=1}^{\infty}$such that

$$\left|1-\frac{k}{2\mathrm{i}}\langle A(\lambda)P_0\mid P_0\rangle\right|\geq B \tag{89}$$

holds for $\lambda=rje^{i\theta}(0<\theta<\pi)$. Take a positive number *d* so that $Im(\lambda_n n)>-d\ \ for\ all\ n=0,1,\ldots$. Fix a small positive number δ and define a closed curve $C_{(j)}=C_1+\cdots+C_6$by

$$C_1=\{x-\mathrm{i}d\,|\,-r_j\leq x\leq r_j\}$$
$$C_2=\{r_j-\mathrm{i}y\,|\,0\leq y\leq d\}$$
$$C_3=\left\{r_je^{\mathrm{i}\theta}\;\middle|\;0\leq\theta\leq\delta\right\}$$
$$C_4=\left\{r_je^{\mathrm{i}\theta}\;\middle|\;\delta\leq\theta\leq\pi-\delta\right\},$$

and $C_5$and $C_6$are defined in a similar manner to $C_3$and C_2, respectively,

Let $\lambda_0,\lambda_1,\ldots,\lambda_N(j)$be generalized Eigen values s inside the closed curve *C*(*j*)., we have

$$\frac{1}{2\pi\mathrm{i}}\int_{C(j)}e^{\mathrm{i}\lambda t}\langle\mathcal{R}_\lambda\phi\mid\psi\rangle d\lambda=\sum_{n=1}^{N(j)}D_n e^{\mathrm{i}\lambda_n t}\langle\mu_n\mid\phi\rangle\langle\mu_n\mid\psi\rangle.$$

Taking the limit $j\to\infty(r_j\to\infty)$ provides

$$\langle e^{T_1 t}\phi\mid\psi\rangle=\lim_{j\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{C_2+\cdots+C_6}e^{\mathrm{i}\lambda t}\langle\mathcal{R}_\lambda\phi\mid\psi\rangle d\lambda$$

We can prove by the standard way that the integrals along C_2, C_3, $C_5$and C_6tend to zero as $j\to\infty$. The integral along $C_4$is estimated as

$$\left|\int_{C_4} e^{\mathrm{i}\lambda t}\langle \mathcal{R}_\lambda \phi \mid \psi\rangle d\lambda\right| \le \max_{\lambda\in C_4}\left|\langle \mathcal{R}_\lambda \phi \mid \psi\rangle\right| \cdot \int_\delta^{\pi/2} 2r_j e^{-r_j t\sin\theta} d\theta$$

$$\le \max_{\lambda\in C_4}\left|\langle \mathcal{R}_\lambda \phi \mid \psi\rangle\right| \cdot \int_\delta^{\pi/2} 2r_j e^{-2r_j t\theta/\pi} d\theta$$

$$\le \max_{\lambda\in C_4}\left|\langle \mathcal{R}_\lambda \phi \mid \psi\rangle\right| \cdot \frac{\pi}{t}\left(e^{-2r_j\delta t/\pi} - e^{-r_j t}\right).$$

It follows from that

$$\langle \mathcal{R}_\lambda \phi \mid \psi\rangle = \frac{1}{2\mathrm{i}/k - \langle A(\lambda)P_0 \mid P_0\rangle}$$

$$\left(\frac{2\mathrm{i}}{k}\langle A(\lambda)\phi \mid \psi\rangle - \langle A(\lambda)P_0 \mid P_0\rangle\langle A(\lambda)\phi \mid \psi\rangle + \langle A(\lambda)P_0 \mid \phi\rangle\langle A(\lambda)P_0 \mid \psi\rangle\right)$$

Since $\varphi, \psi \in Exp_+$, there exist positive constants C_1, C_2, β_1, β_2 such that

$$\left|\phi(\lambda)\right| \le C_1 e^{\beta_1|\lambda|}, \qquad \left|\psi(\lambda)\right| \le C_2 e^{\beta_2|\lambda|}.$$

Using the definition of $A(\lambda)$, we can show that there exist positive constants $D_0, \ldots, D_4$ such that

$$\left|\langle \mathcal{R}_\lambda \phi \mid \psi\rangle\right| \le \frac{D_0 + \left(D_1 + D_2 C_1 e^{\beta_1|\lambda|} + D_3 C_2 e^{\beta_2|\lambda|} + D_4 C_1 C_2 e^{(\beta_1+\beta_2)|\lambda|}\right)\cdot|g(\lambda)|}{|2\mathrm{i}/k - \langle A(\lambda)P_0 \mid P_0\rangle|}.$$

When $|g(\lambda)| \to \infty$ as $|\lambda| \to \infty$, this yields

$$\langle \mathcal{R}_\lambda \phi \mid \psi\rangle| \le \frac{D_0 + \left(D_1 + D_2 C_1 e^{\beta_1|\lambda|} + D_3 C_2 e^{\beta_2|\lambda|} + D_4 C_1 C_2 e^{(\beta_1+\beta_2)|\lambda|}\right)}{2\pi} + o(|\lambda|$$

When $|g(\lambda)|$ is bounded as $|\lambda| \to \infty$, is used to estimate . For both cases, we can show that there exists $D_5 > 0$ such that

$$\left|\langle \mathcal{R}_\lambda \phi \mid \psi\rangle\right| \le D_5 e^{(\beta_1+\beta_2)r_j} \quad \left(\lambda = r_j e^{\mathrm{i}\theta}\right).$$

Therefore, we obtain

$$\left|\int_{C_4} e^{\mathrm{i}\lambda t}\langle \mathcal{R}_\lambda \phi \mid \psi\rangle d\lambda\right| \le \frac{\pi D_5}{t}\left(e^{(\beta_1+\beta_2-2\delta t/\pi)r_j} - e^{(\beta_1+\beta_2-t)r_j}\right)$$

Thus if $t > t_0 = \frac{\pi(,\beta_1+\beta_2)}{2\delta}$, this integral tends to zero as $j \to \infty$, which proves . holds for each $\psi \in Exp +$, the right hand side of Eq.(88)converges with respect to the weak dual topology on Exp_+. Since $Exp_+$is a Montel space, a weakly convergent series also converges with respect to the strong dual topolog is to give the definition and the existence theorem of Pettis integrals. After that, a few results on vector-valued holomorphic functions are given. For the existence of Pettis integrals, the following property.(CE)for any compact set *K*, the closed convex hull of *K* is compact, which is sometimes called the convex envelope property, is essentially used. For the convenience of the reader, sufficient conditions for the property are listed below. We also give conditions for *X* to be barreled because it is assumed in (X3). Let *X* be a locally convex Hausdorff vector space, and X'its dual space.

- The closed convex hull *co*(*K*)of a compact set *K*in *X* is compact if and only if $\overline{co}(K)$is complete in the Mackey topology on *X*(Krein's theorem).
- *X* has the convex envelope property if *X* is quasi-complete.
- If *X* is born logical, the strong dual $X'$is complete. In particular, the strong dual of a metrizable space is complete.
- If *X* is barreled, the strong dual $X'$is quasi-complete. In particular, X'has the convex envelope property.
- Montel spaces, Fréchet spaces, Banach spaces and Hilbert spaces are barreled.
- The product, quotient, direct sum, (strict) inductive limit, completion of barreled spaces are barreled.

Let *X* be a topological vector space over **C** and (*S, μ*)a measure space. Let $f: S \to X$be a measurable *X*-valued function. If there exists a unique $I_f \in X$ such that $\langle \xi | I_f \rangle = \langle S\xi | f \rangle d\mu for\ any\ \xi \in X'$, $I_f$is called the *Pettis integral off*. It is known that if *X* is a locally convex Hausdorff vector space with the convex envelope property, *S*is a compact Hausdorff space with a finite Borel measure *μ*, and if $f: S \to X$is continuous, then the Pettis integral of *f* exists , we have defined the integral of the form $\int_{\gamma} R_{\lambda}\varphi d\lambda$, where $R_{\lambda}\varphi$ is an element of the dual $X(\Omega)'$. Thus our purpose here is to define a "dual version" of Pettis integrals.

In what follows, let X be a locally convex Hausdorff vector space over $\boldsymbol{C}, X'a$ strong dual with the convex envelope property, and let S be a compact Hausdorff space with a finite Borel measure μ, S is always a closed path on the complex plane. Let $f: S \to X'$ be a continuous function with respect to the strong dual topology on X'

Theorem (.3.2.8)[3]:

(i) *Under the assumptions above, there exists a unique* $I(f) \in X'$ *such that*

$$\langle I(f) \mid x\rangle = \int_{S} \langle f \mid x\rangle d\mu \qquad \textbf{(90)}$$

for any x $\in$*X. I(f) is denoted by I(f) =*$\int_S f d\mu$ *and called the Pettis integral of f.*

(ii) *The mapping* $f \to I(f)$ *is continuous in the following sense; for any neighborhood U of zero in* X' *equipped with the weak dual topology, there exists a neighborhood V of zero in* X' *such that if f(s)* $\in$ *V for any s* $\in$ *S, then* $I(f) \in U$.

(iii) *Furthermore, suppose that X is a barreled space. Let T be a linear operator densely defined on X and* T' *its dual operator with the domain* $D(T') \subset X'. If\ f(S) \subset D(T')$ *and the set* $\{T'f(s)|x\rangle\}_{s\in S}$ *is bounded for each x* $\in$*X, then,* $I(f) \in D(T')$ *and* $T'\, I(f) = I(T'f)$ *holds; that is,*

$$T' \int_{S} f d\mu = \int_{S} T' f d\mu \qquad \textbf{(91)}$$

holds.

The proof of (i) is done in a similar manner to that of the existence of Pettis integrals on *X*. Note that *T* is not assumed to be continuous for the part (iii). When *T* is continuous, the set $\{\langle T'f(s)\,|x\rangle\}_{s\in S}$ is bounded because T' and *f* are continuous.

Proof: At first, note that the mapping $\langle \cdot\ |x\rangle: X' \to \boldsymbol{C}$ is continuous because *X* can be canonically embedded into the dual of the strong dual X'. Thus $\langle f(\cdot)\,|x\rangle: S \to \boldsymbol{C}$ is continuous and it is integrable on the compact set *S* with respect to the Borel measure. Let us show the uniqueness. If there are two elements $I_1(f), I_2(f) \in X'$satisfying , we have $\langle I_1(f)\,|x\rangle = \langle I_2(f)\,|x\rangle$for any $x \in X$. By the definition of X', it follows $I_1(f) = I_2(f)$.Let us show the existence. We can assume without loss of generality that X is a vector space over Rand μ is a probability measure. Let $L \subset X$ be a finite set and put

$$V_L(f) = V_L := \left\{ x' \in X' \;\middle|\; \langle x' \mid x \rangle = \int_S \langle f \mid x \rangle d\mu,\ \forall x \in L \right\}. \qquad \textbf{(92)}$$

Since $\langle \cdot\ |x\rangle$is a continuous mapping, $V_l$is closed. Since $f$is continuous, $f(S)$ is compact in X'. Due to the convex envelope property, the closed convex hull $\overline{co}(f(S))$is compact. Hence, $W_L = V_L \cap \overline{co}(f(S))$is also compact. By the definition, it is obvious that $W_{L_1} \cap W_{L_2} = W_{L_1 L_2}$. Thus if we can prove that $W_L$is not empty for any finite set L, afamily $\{W_L\}_{L \in \{finiteset\}}$has the finite interpart property. Then, $\cap_L W_L$is not empty because $\overline{co}(f(S))$is compact. This implies that there exists $I(f) \in \cap_L W_L$such that $\langle I(f)\,|x\rangle = f_s \langle f|x\rangle d\mu$for any $x \in X$. Let us prove that $W_L$is not empty for any finite set $L = \{x_1, \ldots, x_n\} \subset X$. Define the mapping $L : X' \to \boldsymbol{R}_n\ to$ be

$$\mathcal{L}(x') = (\langle x' \mid x_1 \rangle, \ldots, \langle x' \mid x_n \rangle).$$

This is continuous and $L(f(S))$is compact in $\boldsymbol{R}^{\boldsymbol{n}}$. Let us show that the element

$$y := \left(\int_S \langle f \mid x_1 \rangle d\mu, \ldots, \int_S \langle f \mid x_n \rangle d\mu \right) \qquad \textbf{(93)}$$

is included in the convex hull $co(L(f(S)))of\ L(f(S))$. If otherwise, there exist real numbers $c_1, \ldots, c_n$such that for any $(z_1, \ldots, z_n) \in co(L(f(S)))$, the inequality

$$\sum_{i=1}^{n} c_i z_i < \sum_{i=1}^{n} c_i y_i, \quad y = (y_1, \ldots, y_n)$$

holds . In particular, since $L(f(S)) \subset co(L(f(S)))$,

$$\sum_{i=1}^{n} c_i \langle f \mid x_i \rangle < \sum_{i=1}^{n} c_i y_i.$$

Integrating both sides (in the usual sense) yields $\sum_{i=1}^{n} c_i y_i < \sum_{i=1}^{n} c_i y_i$. This is a contradiction, and therefore $y \in co(L(f(S)))$. Since $L$is linear, there exists $v \in co(f(S))$such that $y = L(v)$. This implies that $v \in VL \cap co(f(S))$, and thus$W_L$is not empty. By the uniqueness, $\cap_L W_L = \{I(f)\}$. Part (ii) of the theorem immediately follows form (90) and properties of the usual integral.Next, let us show Eq (91). When *X* is a barreled space, *I*(*f*)is included in $D(T')$so that $T'I(f)$is well defined. To prove this, it is sufficient to show that the mapping

$$x \mapsto \langle I(f) \mid Tx \rangle = \int_S \langle f \mid Tx \rangle d\mu = \int_S \langle T' f \mid x \rangle d\mu$$

from $D(T) \subset X$into **C** is continuous. By the assumption, the set $\{T'f(s) \,|x'\rangle\}_{s\in S}$ is bounded for each $x \in X$. Then, BanachSteinhaus theorem implies that the family $\{T'f(s)\}_{s\in S}$of continuous linear functionals are equicontinuous. Hence, for any $\varepsilon > 0$, there exists a neighborhood *U* of zero in *X*such that $|T'f(s) \,|x'|$<*ε*for any $s \in S$and $x \in U$. This proves that the above mapping is continuous, so that $I(f) \in D(T') and\ T'I(f) = T' \cap_L W_L$.

For a finite set $L \subset X$, put

$$V_L(T'f) = \left\{ x' \in X' \;\middle|\; \langle x' \mid x \rangle = \int_S \langle T' f \mid x \rangle d\mu, \ \forall x \in L \right\},$$

$$T'V_{TL}(f) = \left\{ T'x' \in X' \;\middle|\; x' \in \mathsf{D}(T'), \ \langle x' \mid x \rangle = \int_S \langle f \mid x \rangle d\mu, \ \forall x \in TL \right\}.$$

Put $W_L(f) = V_L(f) \cap \overline{co}(f(S))$ as before. It is obvious that $\cap_L W_L(f) \subset \cap_L W_L(f)$. Therefore,

$$\begin{aligned}\{T'I(f)\} = T' \bigcap_L W_L(f) &\subset T' \bigcap_L W_{TL}(f) \cap \mathsf{D}(T') \\ &\subset T' \bigcap_L \big(V_{TL}(f) \cap \overline{co}(f(S)) \cap \mathsf{D}(T')\big) \\ &\subset \bigcap_L \big(T'V_{TL}(f) \cap T'\overline{co}(f(S)) \cap \mathsf{R}(T')\big).\end{aligned}$$

On the other hand, if $y' \in T'V_{TL}(f)$, there exists $x' \in X'$such that $y' = T'x' and \ \langle x'|x\rangle = \int S \langle f|x\rangle d\mu$for any *x* ∈*TL*. Then, for any *x* ∈*L* ∩D(*T*),

$$\langle y' \mid x \rangle = \langle T'x' \mid x \rangle = \langle x' \mid Tx \rangle = \int_S \langle f \mid Tx \rangle d\mu = \int_S \langle T' f \mid x \rangle d\mu.$$

This implies that $y' \in V_{L\cap D(T)}(T\_f)$, and thus $T'V_{TL}(f) \subset VL \cap D(T)(T'f)$. Hence, we obtain

$$\{T'I(f)\} \subset \bigcap_L V_{L\cap D(T)}(T'f) \cap \overline{co}(T'f(S)) = \bigcap_L W_{L\cap D(T)}(T'f).$$

If $\langle x'|x\rangle = \langle f_S\langle f|x\rangle\rangle d\mu$ for dense subset of X, then it holds for any $x \in X$. Hence, we have

$$\{I(T'f)\} = \bigcap_L W_L(T'f) = \bigcap_L W_{L\cap D(T)}(T'f) \supset \{T'I(f)\}, \quad \textbf{(94)}$$

which proves $T'I(f) = I(T'f)$.Now that we can define the Pettis integral on the dual space, we can develop the "dual version" of the theory of holomorphic functions. Let $X$and X'. Let $f: D \to X'$be an X,-valued function on an open set $D \subset \boldsymbol{C}$.

Definition (3.2.9)[3]:

(i) f is called weakly holomorphic if $\langle f|x\rangle$is holomorphic on Din the classical sense for any $x \in X$(more exactly, it should be called weak-dual-holomorphic).

(ii) $f$is called strongly holomorphic if

$$\lim_{z_0\to z} \frac{1}{z_0 - z}\big(f(z_0) - f(z)\big) \quad \text{(the strong dual limit)} \quad \textbf{(95)}$$

exists in $X'$for any $z \in D$(more exactly, it should be called strong-dual-holomorphic).

Theorem (3.2.10)[3]: *Suppose that the strong dual X' satisfies the convex envelope property and $f: D \to X'$ is weakly holomorphic.*

(i) *If f is strongly continuous, Cauchy integral formula and Cauchy integral theorem hold:*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z_0)}{z_0 - z} dz_0, \qquad \int_\gamma f(z_0) dz_0 = 0,$$

where $\gamma \subset D$ is a closed curve enclosing $z \in D$.

(iii) If f is strongly continuous and if X' is quasi-complete, fis strongly holomorphic and is of C^∞ class.

(iii) *If X is barreled, the weak holomorphy implies the strong continuity. Thus*(i) *and*(ii) *above hold; f is strongly holomorphic and is expanded in a Taylor series as*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \quad (\textit{strong dual convergence}), \quad \textbf{(96)}$$

near $a \in D$. Similarly, a Laurent expansion and the residue theorem hold if f has an isolated singularity.

Proof: (i) Since f is continuous with respect to the strong dual topology, the Pettis integral

$$I(z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{f(z_0)}{z_0 - z} dz_0$$

exists. By the definition of the integral,

$$\langle I(z) \mid x \rangle = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\langle f(z_0) \mid x \rangle}{z_0 - z} dz_0$$

for any $x \in X$. Since $\langle f(z) \mid x \rangle$ is holomorphic in the usual sense, the right hand side above is equal to $\langle f(z) \mid x \rangle$. Thus we obtain $I(z) = f(z)$, which gives the Cauchy formula. The Cauchy theorem also follows from the classical one.

(ii) Let us prove that f is strongly holomorphic at z_0. Suppose that $z_0=0$ and $f(z_0)=0$ for simplicity. By the same way as above, we can verify that

$$\begin{aligned} \frac{f(z)}{z} &= \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{f(z_0)}{z_0(z_0 - z)} dz_0 \\ &= \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{f(z_0)}{z_0^2} dz_0 + \frac{z}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{f(z_0)}{z_0^2(z_0 - z)} dz_0. \end{aligned}$$

Since X' is quasi complete, the above converges as $z \to 0$ to yield

$$f'(0) := \lim_{z \to 0} \frac{f(z)}{z} = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{f(z_0)}{z_0^2} dz_0.$$

In a similar manner, we can verify that

$$f^{(n)}(z) := \frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{f(z_0)}{(z_0 - z)^{n+1}} dz_0 \qquad \textbf{(97)}$$

exists for any $n = 0, 1, 2, \ldots$.

(iii) If X is barreled, weakly bounded sets in X' are strongly bounded By using it, let us prove that a weakly holomorphic f is strongly continuous. Suppose that $f(0) = 0$ for simplicity. Since $\langle f(z) \mid x \rangle$ is holomorphic in the usual sense, Cauchy formula provides

$$\frac{\langle f(z) \mid x \rangle}{z} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{1}{z_0 - z} \frac{\langle f(z_0) \mid x \rangle}{z_0} dz_0.$$

Suppose that $|z| < \delta$ and γ is a circle of radius 2δ centered at the origin. Since $\langle f(\cdot) \mid x \rangle$ is holomorphic, there exists a positive number M such that $|f(z_0) \mid x'| < M$ forany $z_0 \in \gamma$. Then,

$$\left| \frac{\langle f(z) \mid x \rangle}{z} \right| \leq \frac{1}{2\pi} \cdot \frac{1}{\delta} \cdot \frac{M}{2\delta} \cdot 4\pi\delta = \frac{M}{\delta}.$$

This shows that the set $B = \left\{ \frac{f(z)}{z} \middle| |z| < \delta \right\}$ is weakly bounded in X' Since X is barreled, B is strongly bounded. By the definition of bounded sets, for any convex balanced neighborhood U of zero in X' equipped with the strong dual, there is a number $t > 0$ such that $tB \subset U$. This proves that

$$f(z) - f(0) = f(z) \in \frac{z}{t} U \subset \frac{\delta}{t} U$$

for $|z - 0| < \delta$, which implies the continuity of f with respect to the strong dual topology.

If X is barreled, X' is quasi-complete and has the convex envelope property. Thus the results in (i) and (ii) hold. Finally, let us show that $f(z)$ is expanded in a Taylor series around $a \in D$. Suppose $a = 0$ for simplicity. Let us prove that

$$S_m = \sum_{n=0}^{m} \frac{1}{n!} \frac{d^n f}{dz^n}(0) z^n$$

forms a Cauchy sequence with respect to the strong dual topology.

$$\frac{1}{n!} \langle f^{(n)}(0) \mid x \rangle = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\langle f(z_0) \mid x \rangle}{z_0^{n+1}} dz_0$$

for any $x \in X$. Suppose that $\gamma$is a circle of radius 2δcentered at the origin. There exists a constant $M_x > 0$ such that $|\langle f(z_0)\,|x\rangle| < Mx$ for any $z_0 \in \gamma$, which implies that the set $\{f(z_0) \mid z_0 \in \gamma\}$is weakly bounded. Because X is barreled, it is strongly bounded. Therefore, for any bounded set $B \subset X$, there is a positive number MBsuch that $|\langle f(z_0)\,|x\rangle| < MB for\ x \in B and\ z_0 \in \gamma$. Then, we obtain

$$\left|\frac{1}{n!}\langle f^{(n)}(0) \mid x\rangle\right| \leq \frac{1}{2\pi} \cdot \frac{M_B}{(2\delta)^{n+1}} \cdot 4\pi\delta = \frac{M_B}{(2\delta)^n}.$$

By using this, it is easy to verify that $\{\langle \mathrm{Sm}|\mathrm{x}\rangle\}_{\mathrm{m=0}}^{\infty}$is a Cauchy sequence uniformly in $x \in B$ when $|z| < \delta$. Since $X'$is quasi-complete, $S_m$converges as $m \to \infty$in the strong dual topology. By the Taylor expansion in the classical sense, we obtain

$$\langle f(z) \mid x\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dz_0^n}\bigg|_{z_0=0} \langle f(z_0) \mid x\rangle z^n = \sum_{n=0}^{\infty} \frac{1}{n!}\langle f^{(n)}(0) \mid x\rangle z^n.$$

Since $\lim_{m\to\infty} S_m$exists and$\langle \cdot\ \ |x\rangle: X' \to \boldsymbol{C}$is continuous, we have

$$\langle f(z) \mid x\rangle = \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n \;\middle|\; x \right\rangle,$$

For any $x \in X$. This proves for $a = 0$. The proof of a Laurent expansion, when f has an isolated singularity, is done in the same way. Then, the proof of the residue theorem immediately follows from the classical one.

In a well known theory of Pettis integrals on a space X, not a dual X', we need not assume that X is barreled because every locally convex space X has the property that any weakly bounded set is bounded with respect to the original topology. Since the dual X'does not have this property, we have to assume that X is barreled so that any weakly bounded set in $X'$is strongly bounded.

Chapter 4

Global Integral Criteria for Composition Operators

Let D_α denote the Dirichlet type space of functions analytic on the unit disk U and Q_α the conformal invariant version of this space. Any analytic self-map φ of U induces a composition operator C_φ acting on D_α, respectively, Q_α by C_φf = $f \circ$ φ, where f $\in D_\alpha$, respectively, f $\in Q_\alpha$.

Section (4.1):Dyadic Carleson Ceasures

Let U, ∂U and dm denote the unit disk, the unit circle and the two-dimensional Lebesgue measure on the complex plane C, respectively. In this chapter, we consider the class D_α, α ∈ (−1,∞), of functions f analytic on U for which

$$\|f\|_{D_\alpha}^2 := |f(0)|^2 + \int_U |f'(z)|^2 \left(1-|z|^2\right)^\alpha dm(z) < \infty.$$

Since D_0 is the classical Dirichlet space, these spaces are called Dirichlet type spaces or weighted Dirichlet spaces. Whereas there exists a lot of chapters on D_α, a relatively new concept was introduced in by the conformal invariant version of the space D_α, the spaces Q_α,α ∈ (−1,∞). A function f $\in D_\alpha$ belongs to if and only if

$$\|f\|_{Q_\alpha}^2 := |f(0)|^2 + \sup_{w\in U} \int_U |f'(z)|^2 \left(1-\left|\frac{w-z}{1-\overline{w}z}\right|^2\right)^\alpha dm(z) < \infty$$

Let Φ denote the set of non-constant analytic functions φ :U →U. Any such function defines a composition operator C_φ acting on a space of functions f analytic in U by the simple rule $C_\varphi f$ $= f\circ$ φ. There has been done much research on the relations between the function theoretic properties of φ and the topological properties of the operator C_φ in different circumstances, We want to characterize here by means of area integrals related to the function φ the boundedness and compactness of C_φ acting on D_α and Q_α. A central role in the proofs is played by a dyadic decomposition of U into Carleson windows or boxes and certain properties of positive measures on U defined with the help of such a decomposition, whereas the remaining parts are dedicated to the different types of integral criteria.

In this part, we consider sub arcs I $\subset \partial$U with arc length ℓ(I) and Carleson windows

$$S(I) := \{ r\zeta \mid (2\pi - \ell(I))/2\pi \leqslant r < 1,\ \zeta \in I \}$$

based on I and their top halves

$$R(I) := \{ r\zeta \mid (2\pi - \ell(I))/2\pi \leqslant r < (4\pi - \ell(I))/4\pi,\ \zeta \in I \}.$$

Now we consider the set of dyadic sub arcs

$$I_{n,k} := \{ \zeta \mid 2k\pi/2^n \leqslant \arg\zeta < 2(k+1)\pi/2^n \},$$
$$n \in \mathbf{N}_0,\ k = 0, 1, \ldots, 2^n - 1,$$

of ∂U and the decomposition of U by the windows R($I_{n,k}$). Obviously, they are pair wise disjoint and their union covers U. Further, the set

$$\left\{ a_{n,k} = \left(1 - \frac{\ell(I_{n,k})}{3\pi}\right) \exp\left(i\frac{(2k+1)\pi}{2^n}\right) \middle|\ n \in \mathbf{N}_0,\ k = 0, 1, \ldots, 2^n \right.$$

of the centers of R($I_{n,k}$) is separated. This means that the hyperbolic distance between different points is bounded away from zero. For further use we fix a numeration R_j= R(I_j), j $\in$ N, and denote a_j the center of R_j defined above. It is easily seen that the windows R_j have bounded hyperbolic diameter and that their linear dimensions are of the same order as $1 - |a_j|$. Thus, $\ell(I_j) \approx 1 - |a_j|$. Here and throughout this chapter the notation U ≈ V means that there exist positive constants C_1 and C_2 independent of U and V such that C_1V ≤ U ≤ C_2V . In addition, we will use the abbreviation U ≤ V for the fact that there exists a constant c independent of U and V such that U ≤ CV .

Lemma (4.1.1)[4]: *Let* $t, s+1 \in (1,\infty)$. Then for any $w \in U$ the approximative identities

$$|1 - \overline{w}z| \approx |1 - \overline{w}a_j|, z \in R_j, \qquad (\mathbf{1})$$

$$\int_U \frac{(1-|z|)^{t-2}}{|1-\overline{w}z|^{t+s}}\,dm(z) \approx \frac{1}{(1-|w|)^s} \qquad \textbf{(2)}$$

and

$$\sum_j \frac{(1-|a_j|)^t}{|1-\overline{w}a_j|^{t+s}} \approx \frac{1}{(1-|w|)^s} \qquad \textbf{(3)}$$

are valid.

As a preparation of our characterization theorems we now prove

Theorem (4.1.2)[4]: $Let\ \beta \in (1,\infty), p \in (0,\infty)$ and let μ be a finite positive measure on U. Then for a dyadic decomposition $R_j = R(I_j)$, $j \in N$, as above, the following equivalences are valid, wherein we use the abbreviations

$$\tau(w,z,\epsilon) := \frac{(1-|w|^2)^\epsilon}{|1-\overline{w}z|^{1+\epsilon}} \quad and \quad \nu_{j,\beta} := \frac{\mu(R_j)}{\ell(I_j)^\beta}:$$

$$\text{(i)} \qquad \sup_{w\in U} \int_U \tau(w,z,\epsilon)^\beta\,d\mu(z) < \infty \quad \Leftrightarrow \quad \|\mu\|_{C_\beta^d} := \sup_j \nu_{j,\beta} < \infty$$

for some (any) ∈>0.

$$\text{(ii)} \qquad \lim_{w\to\partial U} \int_U \tau(w,z,\epsilon)^\beta\,d\mu(z) = 0 \quad \Leftrightarrow \quad \lim_{j\to\infty} \nu_{j,\beta} = 0$$

for some (any) ∈>0.

$$\text{(iii)} \qquad \int_U \left(\int_U \tau(w,z,\epsilon)^\beta\,d\mu(z)\right)^p \frac{dm(w)}{(1-|w|^2)^2} < \infty \quad \Leftrightarrow$$
$$(\|\mu\|_{C_\beta^d,p})^p := \sum_j \nu_{j,\beta}^p < \infty$$

for some (any) ∈> $max\{\frac{1}{\beta}, \frac{1}{p\beta}\}$.

Proof: (i)⇒Taking w = a_j in any R_j we see that this direction is an immediate consequence of Lemma (4.1.1) (i)⇐Applying (1) and (3) to our dyadic composition of U we get

$$\int_U \tau(w,z,\epsilon)^\beta \, d\mu(z) \approx \sum_j \nu_{j,\beta}(1-|a_j|)^\beta \tau(w,a_j,\epsilon)^\beta .$$

This proves the second direction of (i) due to β >1 and $\nu_{j,\beta} \le \|\mu\|_{C_B^d}$(ii) This is similar to (i).

(iii)⇒Using (1) once again we find

$$\begin{aligned}&\int_U \left(\int_U \tau(w,z,\epsilon)^\beta \, d\mu(z)\right)^p \frac{dm(w)}{(1-|w|^2)^2}\\ &\quad\approx \sum_j \left(\sum_k \tau(a_j,a_k,\epsilon)^\beta (1-|a_k|)^\beta \nu_{k,\beta}\right)^p\end{aligned}$$

which implies the sufficiency of the left side of the equivalence (iii).(iii) ⇐ To prove the sufficiency of the right side we first recognize that$\|\mu\|_{C_{B,p}^d} < \infty$ implies $\|\mu\|_{C_B^d} < \infty$. Therefore, if $p \in (0,1]$, then (3) and the preceding estimate immediately yield the desired conclusion. If $p > 1$ we define a linear operator T acting on a space of sequences by

$$T\big(\{c_j \mid j \in \mathbf{N}\}\big) = \left\{\sum_k \tau(a_j,a_k,\epsilon)^\beta (1-|a_k|)^\beta c_k \;\middle|\; j \in \mathbf{N}\right\}.$$

Observe that T is bounded on $l^1$and l^∞owing to (3) and the case $p \in (0,1]$. An application of the Marcinkiewicz interpolation theorem yields that T is bounded on $l^p, p \in (1,\infty)$. This fact together with the above estimate for the double integral proves the rest of the assertion (iii).

Section (4.2):Integral Criterias

we repeat now some definitions and basic facts used in the sequel. A linear transformation T :X→Y between two Banach spaces X and Y is said to be bounded or compact if T maps bounded sets of X onto bounded or relatively compact sets of Y . If X = Y is a Hilbert space with inner product $\|\cdot,\cdot\|_x$, then for a bounded operator T on X we define its singular numbers as

$$s_n(T) = \inf\{\|T - K\| \mid K : X \to X \text{ has rank } n\}.$$

The compact operators are those bounded operators T for which $s_n(T) \to 0$ as$n \to \infty. For\ p\ \in\ (0,\infty)$ let

$$S_p(X) := \left\{ T : X \to X \mid |T|_p := \left(\sum_{n=0}^{\infty} (s_n(T))^p \right)^{1/p} < \infty \right\}$$

denote the class of all p-Schatten ideal operators on X. It is known that Tis bounded or compact on X if and only if $\boldsymbol{T^*}$**T** has this property and that T ∊ S_p(X) is equivalent to T^*T ∈ $S_{p/2}(X)$. Usually, the members of the classes S_1(X) and S_2(X) are called nuclear and Hilbert–Schmidt operators, respectively.

If p ≥1, the class S_p(X) is a Banach space relative to the norm | · |p. In the case p ∊ (0, 1) the class S_p(X) is a complete topological vector space relative to the metric $| \cdot |_p^p$ Furthermore, we use that for bounded operators T_1, T_2 and T ∊S_p(X) the inequality $|T_1 T\, T_2| \le$ $\|T_1\|\ |T\ |p\ \|T_2\|$ is valid.

To prove the desired characterizations for C_φ we use some known facts on Toeplitz operators on Bergman spaces. Recall that for α ∊ (−1,∞) the weightedBergman space A_α^2 consists of those functions f analytic in U which fulfill

$$\|f\|_{A_\alpha^2}^2 := \int_U |f(z)|^2 \left(1 - |z|^2\right)^\alpha dm(z) < \infty,$$

and that for a finite positive measure μ on U the Toeplitz operator T_μ^α on this space is defined by

$$\left(T_\mu^\alpha f\right)(z) := \int_U \frac{f(w)}{(1 - \overline{w}z)^{2+\alpha}}\, d\mu(w).$$

Lemma (4.2.1)[4]: *Let* $\alpha \in (-1, \infty), p \in (0, \infty)$ and let μ be a finite positive measure on U. Then T_μ^α is a bounded or vanishing or p-Schatten ideal operator on A_α^2if and only if μ is a bounded or vanishing or p-summing (α + 2)dyadic Carleson measure, respectively.

Proof: The definition of the inner product on A_α^2

$$\langle f, g\rangle_{A_\alpha^2} := \int_U f(z)\overline{g(z)}\left(1-|z|^2\right)^\alpha dm(z), \quad f, g \in A_\alpha^2,$$

together with the reproducing kernel formula of A_α^2

$$\left\langle T_\mu^\alpha f, g\right\rangle_{A_\alpha^2} = \frac{\alpha+1}{\pi}\int_U f(z)\overline{g(z)}\, d\mu(z), \quad f, g \in A_\alpha^2.$$

Thus $T_\mu^\alpha$is bounded or compact onA_α^2 if and only if the embedding mapE :A_α^2→$L^2(\mu)$ has the same property, respectively. This is the case if and only if μ is a bounded or vanishing (α + 2)Carleson measure Therefore the first two statements follow from Theorem (4.1.2) and the above remark. The third statement is just a by-product of Luecking's main.

Theorem (4.2.2)[4]: *Let* $\alpha \in (-1, \infty), p \in (0, \infty)$ and $\varphi \in \Phi$. Then the following equivalences, wherein we use $dm_\alpha(z) = (1-|z|^2)^\alpha$ dm(z) as an abbreviation, are valid:

(i) C_φ is bounded on D_α⇔

$$\sup_{w\in U}\int_U \tau(w, \phi(z), \epsilon)^{2+\alpha}|\phi'(z)|^2\, dm_\alpha(z) < \infty$$

for some (any) ∈>0.

(ii) C_φ is compact on D_α⇔

$$\lim_{w\to\partial U}\int_U \tau(w, \phi(z), \epsilon)^{2+\alpha}|\phi'(z)|^2\, dm_\alpha(z) = 0$$

for some (any) ∈<0.

(iii) C_φ is a pSchatten ideal operator on $D_\alpha \Leftrightarrow$

$$\int_U \left(\int_U \tau(w, \phi(z), \epsilon)^{2+\alpha} |\phi'(z)|^2 \, dm_\alpha(z) \right)^{p/2} \frac{dm(w)}{(1-|w|^2)^2} < \infty$$

for some (any) $\epsilon > \max\{\frac{1}{2+\alpha}, \frac{2}{2p+p\alpha}\}$.

Proof: Inserting the usual change-of-variable formula into the inner product $\langle \cdot,\cdot \rangle_{D_\alpha}$ of the Hilbert space D_α, we get

$$\langle C_\phi f, C_\phi g \rangle_{D_\alpha} = f(\phi(0))\overline{g(\phi(0))} + \int_U f'(w)\overline{g'(w)} M_\alpha(\phi, w) \, dm(w),$$

where

$$M_\alpha(\phi, w) = \begin{cases} \sum_{\phi(z_j)=w} (1-|z_j|^2)^\alpha, & w \in \phi(U), \\ 0, & w \in U \setminus \phi(U). \end{cases}$$

Using standard arguments it is easily shown that we may assume, without loss of generality, that φ(0) = 0 for the map φ ∈Φ under consideration. For this map, we define $B_\varphi = DC_\varphi D^{-1}$, where the differentiation operator D is defined by(Df)(z) =f(z) and its inverse by ($D^{-1}f$)(z) =$\int_0^z f(w)dw$ 0 f (w)dw. Both D and D^{-1} establish an isomorphism between D_α^0={f ∈ $D\alpha$ | f (0) = 0} and A_α^2 . Hence, we get for f, g ∈A_α^2 using again the changeofvariable formula,

$$\left\langle B_\phi^* B_\phi f, g \right\rangle_{A_\alpha^2} = \left\langle C_\phi D^{-1} f, C_\phi D^{-1} g \right\rangle_{D_\alpha} = \left\langle T_{\mu_\alpha}^\alpha f, g \right\rangle_{A_\alpha^2},$$

where $d_{\mu\alpha}(w)$ = (π/(α + 1))M_α(φ,w) $d_m(w)$ induces the Toeplitz operator$T_{u\alpha}^\alpha$T on A_α^2. This implies $B_\varphi{}^*B_\varphi$ = $T_{u\alpha}^\alpha$. From the previous analysis we see that C_φ :$D_\alpha^0 \rightarrow D_\alpha^0$is bounded, compact or in S_p if and only if $T_{u\alpha}^\alpha$ is bounded, compact or in $S_{p/2}(A_\alpha^2)$. Since Cφ|D_α and Cφ|D_0differ only by a one dimensional operator, a combination of Lemma (4.2.1) with Theorem (4.2.2) implies the desired assertions. From this theorem one may deduce simpler characterizations in special cases. We give two examples for this fact. The first one immediately follows from the third assertion of Theorem (4.2.2) in the case $p = 2$ using ∊ = 1 herein and combining the result with (2) and Fubini's theorem.

Corollary (4.2.3)[4]: $Let\ \alpha \in (-1,\infty)\ and\ \varphi \in \Phi$. Then

$$C_\phi \in S_2(D_\alpha) \quad \Leftrightarrow \quad \int_U \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^{2+\alpha}}\, dm_\alpha(z) < \infty.$$

This generalizes some results. Similar characterizations for $C_\varphi \in S_p(D_\alpha)$.

Corollary (4.2.4)[4]: $Let\ \alpha \in [0,\infty)\ and\ let\ \varphi \in \Phi$ be boundedly valent. Then C_φ is compact on $D_\alpha \Leftrightarrow$

$$C_\phi\ is\ compact\ on\ D_\alpha \quad \Leftrightarrow \quad \lim_{|z|\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2} = 0.$$

In the sequel we shall use the following abbreviations. The closed unit ball of a Banach space X will be denoted B_X and the characteristic function of a set E by 1_E. For $a \in U$ we consider the automorphism $\sigma a(z) = (a - z)/(1 - \bar{a}z)\ of\ U$and we put $dm_{a,\alpha}(z) = (1 - |\sigma_a(z)|^2)^\alpha\, dm(z)$. Now, we formulate a conformalinvariant version of the first two assertions of Theorem (4.2.2).

Theorem (4.2.5)[4]: $Let\ \alpha \in (0,\infty)\ and\ \varphi \in \Phi$. Then the following equivalences are valid:

(i) C_φ :$D_\alpha \to Q_\alpha$ is bounded ⇔

$$\sup_{a,w\in U} \int_U \tau(w,\phi(z),\epsilon)^{2+\alpha} |\phi'(z)|^2\, dm_{a,\alpha}(z) < \infty$$

for some (any) ∊>0.

(ii) C_φ :$D_\alpha \to Q_\alpha$ is compact ⇔

$$\lim_{|w|\to 1} \sup_{a\in U} \int_U \tau(w,\phi(z),\epsilon) |\phi'(z)|^2\, dm_{a,\alpha}(z) = 0$$

for some (any) ∈>0.

Proof: (i)⇒For α>0 and φ∈Φ let

$$N_\alpha(\phi,u,a)=\begin{cases}\sum_{\phi(z)=u}(1-|\sigma_a(z)|^2)^\alpha, & u\in\phi(U),\\ 0, & u\in U\setminus\phi(U)\end{cases}$$

If we put $d_{\mu\alpha,\varphi,a}(\cdot) = N_\alpha(\varphi, \cdot, a)dm(\cdot)$ we get by a change of variable

$$\int_U |(f\circ\phi)'(z)|^2\,dm_{a,\alpha}(z)=\int_U |f'(z)|^2\,d\mu_{\alpha,\phi,a}(z). \quad (4)$$

For w ∈ U \ {0}, we consider the function f_w defined by $\overline{w}f_w(z) = (1-\overline{w}z)^{-(2+\alpha)(1+\epsilon)/2}$ and we see that (2) implies $\|fw\|^2_{D\alpha} \approx (1-|w|)^{-(2+\alpha)(1+\epsilon)}$. Further, the assumption together with the Closed Graph Theorem(4.21.2) indicates that

$\|C_\phi f_w\|_{Q_\alpha}\preceq\|f_w\|_{D_\alpha}$. For $I=\{e^{i\theta}\mid|\theta-\theta_0|<2^{-1}\ell(I)\}$ and $w=(1-\ell(I)/(2\pi))e^{i\theta_0}$, we get using (4)

$$\left(\inf_{z\in S(I)}\left|f_w'(z)\right|^2\right)\mu_{\alpha,\phi,a}(S(I))\leqslant\int_{S(I)}\left|f_w'(z)\right|^2d\mu_{\alpha,\phi,a}(z)$$
$$\preceq(\ell(I))^{(2+\alpha)(1+\epsilon)}.$$

This estimate for the measures $\mu_{\alpha,\varphi,a}$ implies, due to Theorem (i), the desired assertion.

(i)⇐Again we use our dyadic decomposition $\{R_j \mid j\in N\}$ of U. For $f\in D\alpha$ let $a_j^*\in\overline{R}_j$ be such that $|f'(a_j^*)| = \sup\{|f'(z)| \mid z \in R_j\}$. Some elementary geometric considerations together with Lemma (4.1.1)and the submean value property of $|f|^2$ imply

$$\int_U |f'(z)|^2\,d\mu_{\alpha,\phi,a}(z)\preceq\sum_j\left|f'\left(a_j^*\right)\right|^2\left(1-|a_j|^2\right)^{\alpha+2}\preceq\|f\|^2_{D_\alpha} \quad (5)$$

Hence, the second part of (i) has been proved.(ii)⇒is similar to (i)⇒(ii)⇐It is easily seen that the global integral condition of (i) follows from the global integral condition of (ii). So, to prove the compactness of C_φ in our case it is sufficient to show that $\|C_\varphi f_n\|_{Q\alpha}\to 0$ for any sequence

$\{f_n\} \subset B_{D_\alpha}$ with $f_n \to 0$ uniformly on compact subsets of U. To this end, let $U_t = \{Z|\ |Z| \leq t\}$ for any $t \in (0, 1)$ and consider the inequality

$$\|C_\phi f_n\|^2_{Q_\alpha} - \left|f_n(\phi(0))\right|^2 \leqslant I_1(n,t) + I_2(n,t),$$

where

$$I_1(n,t) := \sup_{a\in U} \int_U \left|f_n'(z)\right|^2 \mathbf{1}_{U_t}(z)\, d\mu_{\alpha,\phi,a}(z)$$

and

$$I_2(n,t) := \sup_{a\in U} \int_U \left|f_n'(z)\right|^2 \mathbf{1}_{U\setminus U_t}(z)\, d\mu_{\alpha,\phi,a}(z)$$

It is clear that $\lim_{n\to\infty} I_1(n,t) = 0$ for any $t \in (0, 1)$. Now,we repeat the estimates in (5) for the measures $1_{U\setminus U_t} d\mu_{\alpha,\varphi,a}$ and see that $\lim_{t\to 1} sup_n I_2(n,t) = 0$. In the following theorem we consider the composition operators $C_\varphi : Q_\alpha \to D_\alpha$ and for $\varphi \in \Phi, t \in (0, 1)$ we define $U_{\varphi,t} = \{z \mid |\varphi(z)|\ n, t\}$.

Theorem (4.2.6)[4]: *Let* $\alpha \in (0,\infty)$ *and* $\varphi \in \Phi$. Then

(i) $C_\varphi : Q_\alpha \to D_\alpha$ is bounded⇔

$$\sup_{f\in B_{Q_\alpha}} \int_U \left|f'(\phi(z))\phi'(z)\right|^2 dm_\alpha(z) < \infty. \tag{6}$$

(ii) $C_\varphi : Q_\alpha \to D_\alpha$ is compact⇔φ satisfies (6) and

$$\lim_{t\to 1} \sup_{f\in B_{Q_\alpha}} \int_U \left|f'(\phi(z))\phi'(z)\right|^2 \mathbf{1}_{U\setminus U_{\phi,t}}(z)\, dm_\alpha(z) \tag{7}$$

Proof: (i) is just a reformulation of the definition of boundedness.

(ii)⇐ By the usual arguments we see that it is sufficient for our purpose to consider a sequence $\{f_n\} \subset B_{Q_\alpha}$ converging to 0 on compact subsets of U and to show that $\{C_\varphi f_n\}$

converges to 0 in the topology of the norm $\|\cdot\|_{D\alpha}$. Since (6) implies φ ∊ D_α and $\{f_n\}$ tends to 0 uniformly on compact subsets of U, we get for given ∊>0 and n big enough the estimate

$$\int_U \left| f_n'(\phi(z))\phi'(z) \right|^2 \mathbf{1}_{U_{\phi,t}}(z)\, dm_\alpha(z) < \epsilon \|\phi\|_{D_\alpha}^2 .$$

This estimate together with (7) immediately yields the assertion.

(ii) ⇒ Since (6) is implied by the boundedness of C_φ, we have to prove

only (7) and we know that, according to (6), φ ∊ D_α. Since $\{n^{-\frac{1}{2}} z_n\}$ is a norm bounded sequence in Q_α and converges to 0 uniformly on compact subsets of U, we see that $\{n^{-\frac{1}{2}}\|\varphi^n\|D_\alpha$α} tends to 0. With some additional arguing, we may conclude that for given (>0 and t ∊ (0, 1) big enough the estimate

$$\int_U |\phi'(z)|^2 \mathbf{1}_{U \setminus U_{\phi,t}}(z)\, dm_\alpha(z) < \epsilon$$

is valid. This implies that for f∈B_{Q_α}, fanalytic in U, we get

$$\int_U \left| f'(\phi(z))\phi'(z) \right|^2 \mathbf{1}_{U \setminus U_{\phi,t}}(z)\, dm_\alpha(z) < \epsilon \| f' \|_\infty^2$$

Now, we proceed as follows: we approximate f∈ B_{Q_α} by fs(z) = f (sz), s ∊(0, 1), s → 1, we use $\|f_s\|Q_\alpha \le \|f\|_{Q\alpha}$ and the compactness of C_φ to show that there exists a number t ∊ (0, 1) depending on f and (such that)

$$\int_U \left| f'(\phi(z))\phi'(z) \right|^2 \mathbf{1}_{U \setminus U_{\phi,t}}(z)\, dm_\alpha(z) < \epsilon.$$

The rest of the proof may easily be accomplished using the finite covering propertyof the set $C_\varphi(B_{Q_\alpha})$ which is relatively compact inD_α.

Corollary (4.2.7)[4]: Let α ∊ (0,∞) and φ∊Φ.

(i) If $C_\varphi : Q_\alpha \to D_\alpha$ is bounded, then

$$\sup_{\zeta\in\partial U}\int_U \frac{|\phi'(z)|^2}{|1-\bar{\zeta}\phi(z)|^2}\,dm_\alpha(z)<\infty. \quad \textbf{(8)}$$

(ii) If

$$\int_U \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2}\,dm_\alpha(z)<\infty, \quad \textbf{(9)}$$

then C_φφ :Q_α → is compact.

(iii) $If\ \varphi(U)$ lies in a polygon inscribed in ∂U, then (8) and (9) are equivalent.

Proof. (i) Since f_ζ (z) = log(1−ζz) ∈ Q_α for any ζ∈∂U, $\|f_\zeta\|_{Q\alpha}\leq 1$

we conclude that $\|f_\zeta\|$ /$\|f_\zeta\|_{Q\alpha}\in B_{Q_\alpha}$ and we see that Theorem (4.2.6)(i) implies (8).

(ii) Formula (9) together with the estimate

$$\left(1-|z|^2\right)|f'(z)|\preceq\|f\|_{Q_\alpha}\quad\text{for } f\in Q_\alpha$$

Thus, the assertion is a consequence of Theorem (4.2.6)(ii).

(iii) Obviously, we only have to prove the implication (8)⇒(9). To do so, we denote the vertices of the polygon in question by ζ_k, k = 1, . . .,n, say. Now we break the unit disk into pairwise disjoint pieces. One of them is a compact subset wherein the relation between the two integrals causes no troubles. The other ones are sets wherein the images of the points under the map φ come close to the points ζ_K . In these pieces we use $|\zeta_k$ −φ(z)| ≤ 1−|φ(z)|^2 to prove the rest of the equivalence. the following conformal invariant version of Theorem (4.2.6) may be proved in a similar way to verifying that theorem. Therefore, we leave the details for the interested reader.

Theorem(4.2.8)[4]:$Let\ \alpha\ \in\ (0,\infty)\ and\ \varphi\in\Phi$. Then

(i) C_φ is bounded on $Q_\alpha$if and only if

$$\sup_{a\in U,\, f\in B_{Q_\alpha}} \int_U \left|f'(\phi(z))\phi'(z)\right|^2 dm_{a,\alpha}(z) < \infty. \tag{10}$$

(ii) C_φ is compact on Q_α if and only if φ satisfies (10) and

$$\lim_{t\to 1} \sup_{a\in U,\, f\in B_{Q_\alpha}} \int_U \left|f'(\phi(z))\phi'(z)\right|^2 \mathbf{1}_{U\setminus U_{\phi,t}}(z)\, dm_{a,\alpha}(z) = 0. \tag{11}$$

In the following corollary, we give some simpler, but only necessary or sufficient conditions for the compactness or boundedness of C_φ on Q_α. In the proof, we use only Theorem (4.2.8) for the special functions f_ζ (z) = log(1 − ζz), ζ ∈∂U, as we have done above to prove Corollary (4.2.4). So, there is no need torepeat the arguments. To make a long story short we use the abbreviations

$$F_{\phi,\zeta}(z) := \frac{|\phi'(z)|^2}{|1-\bar{\zeta}\phi(z)|^2}, \qquad G_\phi(z) := \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2}$$

$for\ \zeta\ \in \partial U\ and\ \varphi \in \Phi.$

Corollary (4.2.9)[4]: Let α ∈ (0,∞) and φ∈Φ.

(i) If C_φ is bounded on Q_α, then

$$\sup_{a\in U,\, \zeta\in\partial U} \int_U F_{\phi,\zeta}(z)\, dm_{a,\alpha}(z) < \infty. \tag{12}$$

(ii) If φ satisfies

$$\sup_{a\in U} \int_U G_\phi(z)\, dm_{a,\alpha}(z) < \infty, \tag{13}$$

then C_φ is bounded on Q_α.

(iii) If C_φ is compact on Q_α then φ satisfies (12) and

$$\lim_{t\to 1} \sup_{a\in U,\, \zeta\in\partial U} \int_U F_{\phi,\zeta}(z) \mathbf{1}_{U\setminus U_{\phi,t}}(z)\, dm_{a,\alpha}(z) = 0. \tag{14}$$

(iv) If φ satisfies (13) and

$$\lim_{t\to 1} \sup_{a\in U} \int\limits_U G_\phi(z)\mathbf{1}_{U\setminus U_{\phi,t}}(z)\, dm_{a,\alpha}(z) = 0, \tag{15}$$

then C_φ is compact on Q_α.The following corollaries of this type possibly need some hints for the proofs.

Corollary (4.2.10)[4]: Let α ∈ (0,∞) and φ∈Φ.

(i) If φ is boundedvalet, then C_φ is bounded onQ_α.

(ii) $If\ f_U G_\varphi(z)\, dm(z) < \infty, then\ G_\varphi\ is\ compact\ on\ Q_\alpha.$

Proof: (i) We use that, according for f ∈Q_α

$$\|f\|^2_{Q_\alpha} - |f(0)|^2 \leqslant \alpha 2^\alpha \sup_{w\in U} \int\limits_0^1 \Bigg(\int\limits_{U_{\phi,r}} |f'(z)|^2\, dm(z) \Bigg) (1-r)^{\alpha-1}\, dr.$$

Therefore, for f ∈ $B_{Q\alpha}$ and φ boundedly valent the integral in (10) is less than a multiple of the integral in the above formula and (i) follows from Theorem (4.2.8)(i).

(ii) The integral condition in (ii) implies (13). Hence,

$$\lim_{t\to 1} \int\limits_U G_\phi(z)\mathbf{1}_{U\setminus U_{\phi,t}}(z)\, dm(z) = 0.$$

This implies that (15) holds and therefore (ii) is a consequence

Corollary(4.2.11)[4]:$Let\ \alpha\ \in\ (0,\infty)$ and let $\varphi \in \Phi$ be such that $\varphi(U)$ lies in a polygon inscribed in ∂U. Then

(i) C_φ is bounded on Q_α⇔(13) holds.

(ii) C_φ is compact on Q_α⇔ (13) and (15) hold.

Proof: It suffices to verify (ii). Of course, we only need to show that in our case (14) implies (15). This may be done using the proof ideas of Corollary. At the end, we want to mention without proof that it is possible to use the fact that all boundedly valent functions on U do not distinguish between the little Bloch spaces and the vanishing Q_α-spaces ($\alpha > 0$), as well as our present results to prove a conformal invariant version of Corollary (4.2.4).

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