University of Sudan for Science & Technology

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Almost Over Complete and Overtotal Sequences with Almost Square and Subprojectivity of Banach Spaces

المتتاليات فوق التامة تقريباً وفوق الكلية مع المربعة تقريباً وتحت الإسقاطية لفضاءات باناخ

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Detication

To my husband ...

To my children ...

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frist of all the praise and thanks be to Allah whom to scribed all perfection and majesty. Thanks after Allah must be to my supervisor Prof. Shawgy Hussein Abdall who supervised this research and guid me in patience until the results of this research are obtained.

Abstract

We provide information about the structure of a sequence in a separable Banach space. We prove that non-reflexive spaces which are M-ideals in their biduals are almost square. We show that every space containing a copy of c_0 can be renormed to be almost square. A local and a weak version of almost square spaces are also studied. We study superprojective Banach spaces. We show that they cannot contain copies of ℓ_1, which restricts the search for non-reflexive examples of these spaces. We examine the stability of subprojectivity of Banach spaces under various operations, such as direct or twisted sums, tensor products, and forming spaces of operators. Along the way, we obtain new classes of subprojective spaces.

الخلاصة

إشترطنا معلومة حول بناء متتالية في فضاء باناخ القابل للإنفصال . أوضحنا أن الفضاءات غير الإنعكاسية والتي هي مثاليات – M في ثنائياتها المزدوجة هي دائماً مربعة تقريباً . وتم إيضاح أن أي فضاء محتوياً نسخة إلى C_0 يمكن أن يعاد إنتظامها ليكون مربع تقريباً. تمت دراسة الإصدارة الضعيفة والموضوعية لفضاءات المربع التقريبية. درسنا فضاءات باناخ فوق الإسقاطية. وأوضحنا أنها لايمكن أن تحتوي نسخاً إلى ℓ_1 والتي تقصر البحث لأجل أمثلة غير إنعكاسية لهذه الفضاءات. إختبرنا الإستقرارية تحت الإسقاطية إلى فضاءات باناخ تحت عمليات متنوعة حيث هي كالمجاميع المباشرة والملتوية وضرب تنسروتكون فضاءات للمؤثرات. أعطينا عائلات إلى الفضاءات تحت الإسقاط.

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Chapter 1

Banach Spaces with Almost Overcomplete and Overtotal Sequences

A sequence in a separable Banach space $X\langle resp. in\ the\ dual\ space\ X^*\rangle$ is said to be overcomplete (OC in short) $\langle resp. overtotal\ (OT in\ short) on\ X\rangle$ whenever the linear span of each subsequence is dense in X ‹resp. each subsequence is total on X›. A sequence in a separable Banach space $X\langle resp. in\ the\ dual\ space\ X^*\rangle$ is said to be almost overcompletes (AOC in short) $\langle resp. almost\ overtotal\ (AOT\ in\ short) on\ X\rangle$ whenever the closed linear span of each subsequence has finite codimension in$\langle resp. the\ annihilator\ (in\ X) of\ each\ subsequence\ has\ finite\ dimension\rangle$. We provide information about the structure of such sequences. In particular it can happen that, an AOC $\langle resp. AOT\rangle$ given sequence admits countably many not nested subsequences such that the only subspace contained in the closed linear span of every of such subsequences is the trivial one ⟨resp. the closure of the linear span of the union of the annihilators in X of such subsequences is the whole X⟩. Moreover, any AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ contains some subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ that is OC in $[\{x_{n_j}\}_{j\in\mathbb{N}}]$; any AOT sequence $\{f_n\}_{n\in\mathbb{N}}$ contains some subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ that is OT on any subspace of X complemented to $\{f_{n_j}\}_{j\in\mathbb{N}}^{\top}$.

We use standard Geometry of Banach spaces. In particular:

(i) [S]stands for the closure of the linear span of the set S;

(ii) The annihilator in X^* of a subset Γ of the Banach space X is the subspace $\Gamma^\perp \subset X^*$ whose members are the bounded linear functionals on X that vanish on Γ;

(iii) The annihilator in X of a subset Γ of the dual space X^* is the subspace $\Gamma^\top \subset X, \Gamma^\top = \cap_{f \in \Gamma} \ker f$;

(iv) A set $\Gamma \subset X^*$ is called total over X whenever $\Gamma^\top = \{0\}$.

A sequence in a Banach space X is called overcomplete (*OC* in short) in X whenever the linear span of each of its subsequences is dense in X. It is a well-known fact that overcomplete sequences exist in any separable Banach space. On the basis of this notion, we introduced the following new notions.

(i) A sequence in a Banach space X is called almost overcomplete (AOC in short) whenever the closed linear span of each of its subsequences has finite codimension in X.

(ii) A sequence in the dual space X^* of the Banach space X is called overtotal on X (OT in short) whenever each of its subsequences is total over X.

(iii) A sequence in the dual space X^* of the Banach space X is called almost overtotal (AOT in short) on X whenever the annihilator (in X) of each of it's subsequences has finite dimension.

For instance, the fact that bounded *AOC* as well as *AOT* sequences must be strongly relatively compact makes it possible to answer quickly in the positive the following questions.

(i) Must any infinite-dimensional closed subspace of l_∞ contain infinitely many linearly independent elements with infinitely many zero-coordinates?

(ii) Let $X \subset C(K)$ be an infinite-dimensional subspace of *C(K)* where *K* is metric compact. Must an (infinite) sequence $\{t_k\}_{k\in\mathbb{N}}$ exist in *K* such that $x(t_k) = 0$ for infinitely many linearly independent $x \in X$?

Our first aim is to provide information about the structure of *AOC* and *AOT* sequences. In particular, for any separable Banach space *X* the following questions seem to be of interest.

(i) *Does an AOC sequence exist in X that admits countably many subsequences such that the intersection of their closed linear spans is the origin?*

(ii) *Does an AOT sequence exist on X that admits countably many subsequences such that the closure of the linear span of the union of their annihilators in X is the whole X?*

Our second aim is to give a possible explanation for the following fact. As a consequence of a theorem, by using strong relative compactness of bounded *AOT* sequences we get e.g., as a special case, that any infinite-dimensional closed subspace of l_p contains infinitely many elements with infinitely many zero-coordinates not only when $p = \infty$, as we mentioned at the beginning, but for any $p \geq 1$. However, the case $p < \infty$ looks much more complicated to be handled than the case $p = \infty$. we provide an example to show one possible reason for that.

Here we point out only the evident fact that, if $\{(x_n, x_n^*)\}$ is a countable biorthogonal system, then neither $\{x_n\}$ can be almost overcomplete in $[\{x_n\}]$, nor $\{x_n^*\}$can be almost overtotal on $[\{x_n\}]$.

We start by recalling a simple method, due to Ju. Lyubich, to get an overcomplete sequence in any separable Banach space *X*.

Fact (1.1)[1]: *Let* $\{e_k\}_{k\in\mathbb{N}}$ *be any bounded sequence such that* $[\{e_k\}_{k\in\mathbb{N}}] = X$. *Then the sequence*

$$\{y_m\}_{m=2}^{\infty} = \{\sum_{k=1}^{\infty} e_k m^{-k}\}_{m=2}^{\infty}$$

is OC in X.

Proof. Let $\{y_{m_i}\}_{j=1}^{\infty}$ be any subsequence of $\{y_m\}_{m=2}^{\infty} = \{\sum_{k=1}^{\infty} e_k m^{-k}\}_{m=2}^{\infty}$, let

$$f \in X^* \cap \{y_{m_j}\}^{\perp} \qquad (1.1)$$

and let D be the open unit disk in the complex field. Since the complex function $\emptyset: D \to \mathbb{C}$ defined by $\varphi(t) = \sum_{k=1}^{\infty} f(e_k)t^k$ is holomorphic, from $f(y_{m_j}) = \emptyset(1/m_j) = 0\ for\ j = 1, 2, \ldots,$ it follows $\emptyset \equiv 0$ that forces $f(e_k) = 0$ for every k ∈ ℕ. Since *f* in (1.1) was arbitrarily chosen, it follows $\left[\{y_{m_j}\}\right] = X$.

Proposition (1.2)[1]: *Any (infinite-dimensional) separable Banach space* X *contains an AOC sequence* $\{x_n\}_{n\in\mathbb{N}}$ *with the following property: for each* $i \in \mathbb{N}$, $\{x_n\}_{n\in\mathbb{N}}$ *admits a subsequence, that we denote by* $\{x_j^i\}_{j\in\mathbb{N}}$ *to lighten notation, such that both the following conditions are satisfied*

a) codim $X[\{x_j^i\}_{j\in\mathbb{N}}] = i$;

b) $\bigcap_{i\in\mathbb{N}}[\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$.

Proof. Let the biorthogonal system $\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$ provide anormalized M-basis for *X*. We recall that, by definition, the sequence $\{e_k^*\}_{k\in\mathbb{N}}$ must be total on *X*. Moreover, it is a well-known fact that, at least when A is a finite subset of $\mathbb{N}$, a (topological) complement in X to the subspace $[\{e_k\}_{k\in A}]$ is the subspace $[\{e_k\}_{k\in \mathbb{N}\setminus A}]$. For $i = 1, 2, \ldots$ put

$$Y_i = \left[\{e_k\}_{k\notin\{i,i+1,i+2,\ldots,2i-1\}}\right] \tag{1.2}$$

so $codim_X Y_i = i$. For each integer $i \in \mathbb{N}, Y_i$ is a Banach space itself so, by Fact (1.1), the sequence $\{y_m^i\}_{m\geq 2} \subset Y_i$ defined by

$$y_m^i = \sum_{k=1, k\notin\{i,i+1,i+2,\ldots,2i-1\}}^{\infty} m^{-ik} e_k \quad i = 1, 2, \ldots, m = 2, 3, \ldots \tag{1.3}$$

provides an OC sequence in Y_i.

Order in any way the countable set $\cup_{i\in\mathbb{N}, m\geq 2}\{y_{i_m}\}$ as a sequence $\{x_n\}_{n\in\mathbb{N}}$. For each i, select a subsequence $\{x_p^i\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ whose terms belong to $\{y_m^i\}_{m\geq 2}$: this last sequence being OC in Y_i, we have $codim_X[\{x_p^i\}_{p\in\mathbb{N}}] = codim_X Y_i = i$. Moreover, since the sequence $\{e_k^*\}_{k\in\mathbb{N}}$ is total on X, it is clear that $\cap_{i=1}^{\infty} Y_i = \{0\}$, so $\cap_{i=1}^{\infty}[\{x_p^i\}_{p\in\mathbb{N}}] = \{0\}$ too.

It remains to show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is AOC in X. Let $\{x_{n_j}\}_{j\in\mathbb{N}}$ be any of its subsequences. Two cases are possible.

A) For some $\bar{i}, \{x_{n_j}\}_{j\in\mathbb{N}}$ contains infinitely many terms from $\{y_m^{\bar{i}}\}_{m\geq 2}$: being $\{y_m^{\bar{i}}\}_{m\geq 2}$ OC in $Y_{\bar{i}}$, we have $codim_X[\{x_{n_j}\}_{j\in\mathbb{N}} \leq codim_X Y_{\bar{i}} = \bar{i}$ and we are done.

B) For each $i, \{x_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^i\}_{m\geq 2}$. Take any

$$f \in \{x_{n_j}\}_{j\in\mathbb{N}}^{\perp}. \tag{1.4}$$

We prove that $f(e_k) = 0$ for every $k \in \mathbb{N}$: it implies $f = 0$, that means that $\{x_{n_j}\}_{j\in\mathbb{N}}$ is complete in X.

Suppose by contradiction that $f(e_{\bar{k}}) \neq 0$ for some index $\bar{k}$: without loss of generality we may assume that $\bar{k}$ is the first of such indexes. For j ∈ ℕ, let

$$y_{m(j)}^{i(j)} = x_{n_j};$$

put

$$A = \{i : i = i(j), j \in \mathbb{N}, i(j) > \bar{k}\}.$$

Under our assumption $i(j)$ goes to infinity with j, so A is infinite and we have $e_{\bar{k}} \in Y_i$ for every $i \in A$. For $i \in A$, put

$$m_i = min\{m(j) : i(j) = i, y_{m(j)}^{i(j)} \in \{y_m^i\}_{m\geq 2}\}.$$

From (1.4) it follows that, for each $i \in A$, we have

$$f(e_{\bar{k}}) = -m_i^{i\bar{k}} \sum_{k>\bar{k},k\notin\{i,i+1,i+2,\dots,2i-1\}}^{\infty} m_i^{-ik} f(e_k) \tag{1.5}$$

hence

$$|f(e_{\bar{k}})| \leq m_i^{i\bar{k}} \|f\| \textstyle\sum_{k>\bar{k},k\notin\{i,i+1,i+2,\dots,2-1\}}^{\infty} m_i^{-ik} \leq \|f\| \sum_{k=\bar{k}+1}^{\infty} m_i^{i(\bar{k}-k)}$$

$$\leq 2\|f\| m_i^{-i} \to 0 \; as \; i \to \infty \tag{1.6}$$

that forces $f(e_{\bar{k}}) = 0$, so contradicting our assumption. We are done.

Our construction above can be modified by replacing (1.2) with

$$Y_i = [\{e_k\}_{k\neq i}] \tag{1.7}$$

and modifying (1.3), (1.5) and (1.6) according to that. In this case it is still true that $\bigcap[\{x_{n_j}\}_{j\in\mathbb{N}}] = \{0\}$ as $\{x_{n_j}\}_{j\in\mathbb{N}}$ ranges among all possible subsequences of the *AOC* sequence $\{x_n\}_{n\in\mathbb{N}}$, but actually the codimension of the closure of the linear span of any subsequence is at most1. We need, the following alternative version.

Proposition (1.3)[1]: *Any (infinite-dimensional) separable Banach space* X *contains an AOC sequence* $\{x_n\}_{n\in\mathbb{N}}$ *with the following property:* $\{x_n\}_{n\in\mathbb{N}}$ *admits countably many subsequences* $\{x_j^i\}_{j\in\mathbb{N}}$, $i = 1, 2, \ldots$, *such that both the following conditions are satisfied*

a) $codim_X[\{x_j^i\}_{j\in\mathbb{N}}] = 1$ *for each i;*

b) $\bigcap_{i\in\mathbb{N}}[\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$.

By the previous proposition, it is matter of evidence that actually the conclusion $\bigcap_{i\in\mathbb{N}}[\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$ is due to the fact that infinitely many pairwise "skew" subsequences can be found of $\{x_n\}_{n\in\mathbb{N}}$. This consideration is stressed by the following proposition.

Proposition (1.4)[1]: *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be any AOC sequence in any (infinite-dimensional) separable Banach space* X *and let* $\{x_j^1\}_{j\in\mathbb{N}} \supset \{x_j^2\}_{j\in\mathbb{N}} \supset \{x_j^3\}_{j\in\mathbb{N}} \supset \ldots$ *be any countable family of nested subsequences of* $\{x_n\}_{n\in\mathbb{N}}$. *Then the increasing sequence of integers* $\{codim_X[\{x_j^i\}_{j\in\mathbb{N}}]\}_{i\in\mathbb{N}}$ *is finite (so eventually constant).*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an *AOC* not *OC* sequence in X and let $\{x_j^1\}_{j\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X. Put

$$X_1 = \left[\{x_j^1\}_{j\in\mathbb{N}}\right], \qquad p_1 = codim_X X_1 \geq 1.$$

If $\{x_j^1\}_{j\in\mathbb{N}}$ is *OC* in X_1 we are done; otherwise, let $\{x_{jk}^1\}_{k\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X_1. Put

$$\{x^1_{jk}\}_{k\in\mathbb{N}} = \{x^2_j\}_{j\in\mathbb{N}}, \qquad X_2 = [\{x^2_j\}_{j\in\mathbb{N}}], \qquad p_2 = codim_X X_2 > p_1.$$

Now we can continue in this way. Let us prove that this process must stop after finitely many steps. Assume the contrary, i.e. that a nested infinite family

$$\{x^1_j\}_{j\in\mathbb{N}} \supset \{x^2_j\}_{j\in\mathbb{N}} \supset \ldots \supset \{x^i_j\}_{j\in\mathbb{N}} \supset \ldots$$

of subsequences of $\{x_n\}_{n\in\mathbb{N}}$ can be found such that $p_i \uparrow \infty$ as $i \uparrow \infty$, where $p_i = codim_X X_i \; with \; X_i = [\{x^i_j\}_{j\in\mathbb{N}}]$.

Under this assumption, we can construct a linearly independent sequence $\{f_i\}_{i=1}^{\infty} \subset X^*$ such that, for each $i, f_i \in X^{\perp}_{i+1} \backslash X^{\perp}_i$. For each i, let y_i be an element of the sequence $\{x^i_j\}_{j\in\mathbb{N}}$ not belonging to the sequence $\{x^{i+1}_j\}_{j\in\mathbb{N}}$ such that $f_i(y_i) \neq 0$ (of course such an element must exist): because of our construction we have $f_k(y_i) = 0$ for each $k < i$. Without loss of generality we may assume $f_i(y_i) = 1$.

Now, following a standard procedure due to Markushevich, put

$$g_1 = f_1, \qquad g_2 = f_2 - f_2(y_1)g_1, \qquad g_3 = f_3 - f_3(y_1)g_1 - f_3(y_2)g_2, \ldots\ldots,$$

$$g_k = f_k - \sum_{i=1}^{k-1} f_k(y_i)g_i \;, \ldots.$$

Clearly we have $g_k(y_i) = \delta_{k,i}$ for each $k, i \in \mathbb{N}$, so actually $\{y_k, g_k\}_{k\in\mathbb{N}}$ is a biorthogonal system with $\{y_k\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$. This is a contradiction since $\{x_n\}_{n\in\mathbb{N}}$ was an *AOC* sequence.

As an immediate consequence we get the following

Corollary (1.5)[1]: *Any AOC sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in a separable Banach space* X *contains some subsequence* $\{x_{n_j}\}_{j\in\mathbb{N}}$ *that is OC in* $[\{x_{n_j}\}_{j\in\mathbb{N}}]$ *(with, of course,* $[\{x_{n_j}\}_{j\in\mathbb{N}}]$ *of finite codimension in* X*).*

The results shown about *AOC* sequences have a dual restatement for *AOT* sequences.

Proposition (1.6)[1]: *Let* X *be any (infinite-dimensional) separable Banach space. Then there is a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ *that is AOT on* X *and, for each* $i \in \mathbb{N}$, *admits a subsequence* $\{f_j^i\}_{j\in\mathbb{N}}$ *such that both the following conditions are satisfied*

a) $dim\{f_j^i\}_{j\in\mathbb{N}}^{\top} = i;$

b) $[\ \cup_{i\in\mathbb{N}}\{f_j^i\}_{j\in\mathbb{N}}^{\top}] = X.$

Proof. The idea for the proof is the same as for the proof of Proposition (1.2), so we confine ourselves to sketch the fundamental steps.

Let the biorthogonal system $\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$ provide an M-basis for X with $\{e_k^*\}_{k\in\mathbb{N}}$ a norm-one sequence in X^*. For $i = 1, 2, \ldots$ put

$$Z_i = \left[\{e_k\}_{k=i}^{2i-1}\right], \qquad Y_i = \left[\{e_k\}_{k\neq\{i,i+1,i+2,\ldots,2i-1\}}\right], \qquad {}^*Y_i = [\{e_k^*\}_{k\neq\{i,i+1,i+2,\ldots,2i-1\}}].$$

Clearly $X = Z_i \oplus Y_i$ and ${}^*Y_i^{\top} = Z_i$, so $dim\, {}^*Y_i^{\top} = i$ for i =1, 2, ... For each integer $i \in \mathbb{N}$, the sequence $\{y_m^{*i}\}_{m\geq 2} \subset {}^*Y_i$ defined by

$$y_m^{*i} = \sum_{k=1,k\notin\{i,i+1,i+2,\ldots,2i-1\}}^{\infty} m^{-ik} e_k^* \qquad i = 1,2,\ldots, m = 2,3,\ldots$$

being overcomplete in the Banach space ${}^*Y_i$, is overtotal on Y_i.

Order in any way the countable set $\cup_{i\in\mathbb{N},m\geq 2}\{y_m^{*i}\}$ as a sequence $\{f_n\}_{n\in\mathbb{N}}$. For each i, select a subsequence $\{f_p^i\}_{p\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ whose terms belong to

$\{y_m^{*i}\}_{m\geq 2}$: since this last sequence is overtotal on Y_i, we have $\{f_p^i\}_{p\in\mathbb{N}}^{\top} = Z_i$ too, so $\dim\{f_p^i\}_{p\in\mathbb{N}}^{\top} = i$. Moreover, since the sequence $\{e_k\}_{k\in\mathbb{N}}$ is complete in *X*, we have $[\cup_{i=1}^{\infty} Z_i] = X$.

It remains to show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is *AOT* on *X*. Let $\{f_{n_j}\}_{j\in\mathbb{N}}$ be any of its subsequences. Two cases are possible.

a) For some $\bar{i}$, $\{f_{n_j}\}_{j\in\mathbb{N}}$ contains infinitely many terms from $\{y_m^{*\bar{i}}\}_{m\geq 2}$: being $\{y_m^{*\bar{i}}\}_{m\geq 2}$ *OT* on $Y_{\bar{i}}$, we have $\{f_{n_j}\}_{j\in\mathbb{N}}^{\top} \subset Z_{\bar{i}}$, $dim\{f_{n_j}\}_{j\in\mathbb{N}}^{\top} \leq \bar{i}$ and we are done.

b) For each i, $\{f_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^{*i}\}_{m\geq 2}$. Take any $x \in \{f_{n_j}\}_{j\in\mathbb{N}}^{\top}$: by proceeding exactly as in *B)* of the proof of Proposition (1.2), just interchanging the roles of points and functionals, we get $e_k^*(x) = 0$ for every $k \in \mathbb{N}$. $\{e_k^*\}_{k\in\mathbb{N}}$ being total on X, it follows x = 0. It means that $\{f_{n_j}\}_{j\in\mathbb{N}}$ too is total on *X* and again we are done.

The proof is complete.

As we did for *AOC* sequences, with obvious modifications in the previous proof we can obtain for *AOT* sequences the following alternative version to Proposition (1.6): it is the dual version to Proposition (1.3).

Proposition (1.7)[1]: *Let* X *be any (infinite-dimensional) separable Banach space. Then there is a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ *that is AOT on* X *and admits countably many subsequences* $\{f_j^i\}_{j\in\mathbb{N}}$, $i = 1, 2, \ldots$, *such that both the following conditions are satisfied*

a) $dim\{f_j^i\}_{j\in\mathbb{N}}^{\top} = 1$ *for each i;*

b) $[\bigcup_{i\in\mathbb{N}}\{f_j^i\}_{j\in\mathbb{N}}^{\top} = X$.

We point out that, though the existence of an *AOT* sequence on a Banach space *X* does not imply *X* to be separable (one of the significant applications of this concept we have shown was to the space l_∞), the results we have shown in Propositions(1.6) and (1.7), as they have been stated, must concern only separable spaces. In fact, the annihilator of any subsequence of any *AOT* sequence being finite-dimensional, the closed linear span of the union of countably many of such annihilators must be separable too.

Finally we notice that also Proposition (1.4) has its dual version that shows that the countably many subsequences in the statement of Proposition (1.7) cannot be assumed to be nested. The proof can be carried on exactly like the proof of Proposition (1.5), just interchanging the roles of points and functionals, so we omit it.

Proposition (1.8)[1]: *Let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence AOT on any (infinite-dimensional) Banach space* X *and let $\{f_j^1\}_{j\in\mathbb{N}} \supset \{f_j^2\}_{j\in\mathbb{N}} \supset \{f_j^3\}_{j\in\mathbb{N}} \supset \ldots$ be anycountable family of nested subsequences of $\{f_n\}_{n\in\mathbb{N}}$. Then the increasing sequence of integers $\{dim\{f_j^i\}_{j\in\mathbb{N}}^{\top}\}_{i\in\mathbb{N}}$ is finite (so eventually constant).*

As an immediate consequence of Proposition (1.8) we get the following

Corollary (1.9)[1]: *Any AOT sequence $\{f_n\}_{n\in\mathbb{N}}$ on a Banach space X contains some subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ that is OT on any subspace of X complemented to $\{f_j^n\}_{j\in\mathbb{N}}^{\top}$ (with, of course, $\{f_j^n\}_{j\in\mathbb{N}}^{\top}$ of finite dimension).*

We are devoted to provide an example that may be of interest in Operator theory. It was proved e.g. that any infinite-dimensional closed subspace of l_p contains infinitely many elements with infinitely many zero-coordinates not only when $p = \infty$, as we mentioned at the beginning, but for any $p \geq 1$. In fact the following much more general results have been proved there.

Theorem (1.10)[1]: *Let* X *be a separable infinite-dimensional Banach space and $T: X \to l_\infty$ be a one-to-one bounded non-compactlinear operator. Then there exist an infinite-dimensional subspace $Y \subset X$ and a strictly increasing sequence $\{n_k\}$ of integers such that $e_{n_k}(Ty) = 0$ for any $y \in Y$ and for any k (e_n the "*n*-coordinate functional" on l_∞).*

Theorem (1.11)[1]: *Let X,Y be infinite-dimensional Banach spaces. Let Y have an unconditional basis $\{u_i\}_{i=1}^{\infty}$ with $\{e_i\}_{i=1}^{\infty}$ as the sequence of the associated coordinate functionals. Let $T: X \to Y$ be a one-to-one bounded non-compact linear operator. Then there exist an infinite-dimensional subspace $Z \subset X$ and a strictly increasing sequence $\{k_l\}$ of integers such that $e_{k_l}(Tz) = 0$ for any $z \in Z$ and any $l \in \mathbb{N}$.*

To prove both the theorems, the fundamental tool was the fact that bounded *AOT* sequences are strongly relatively compact was then obtained as a quite easy consequence of the Ascoli–Arzelà Theorem: (if a sequence $\{f_n\}_1^{\infty}$ in *C(X)* is bounded and equicontinuous then it has a uniformly convergent subsequence.

In this statement,

(a) " $\mathcal{F}$ ½ $C(X)$ is bounded" means that there exists a positive constant $M < \infty$ such that $|f(x)|$. M for each $x \in X$ and each $f \in \mathcal{F}$ and

(b) " $\mathcal{F}$ ½ $C(X)$ is equicontinuous" means that: for every $\varepsilon > 0$ there exist $\delta > 0$ (which depends only on ε) such that for $x, y \in X$:

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F}$$

Where d is the metric on X)[5], the proof of Theorem (1.11) has required some additional delicate tools. One could expect that Theorem (1.11) should be proved in a simple way by the following argument.

"Under notation as in the statement of Theorem (1.11), assume by contradiction that for each sequence of integers $\{i_j\}$ we have $dim\left(\{T^*e_{i_j}\}^\top\right) < \infty$. Then the sequence $\{T^*e_i\} \subset X^*$ is almost overtotal on X, so $\{T^*e_i\}$ is relatively norm-compact in X^*. $\{e_i\}$ being the sequence of the coordinate functionals associated to the (unconditional) basis $\{u_i\}$ of Y, that forces T to be a compact operator, contradicting our assumption."

Example (1.12)[1]: There exist a Banach space Y with an unconditional basis $\{u_i\}_{i\in\mathbb{N}}$, $\{e_i\}_{i\in\mathbb{N}}$ being the sequence of the associated coordinate functionals, and a non-compact operator $T: c_0 \to Y$ such that $T^*e_i \to 0$ as $i \to \infty$ (so the sequence $\{T^*e_i\}$ is relatively norm compact).

Proof. Let $\{u_i^k\}_{i=1}^k$ be the natural (algebraic) basis of $\mathbb{R}^k$. For $k \in \mathbb{N}$, define $T_k: \mathbb{R}^k \to \mathbb{R}^k$ in the following way

$$T_k\left(\sum_{i=1}^k a_i u_i^k\right) = \sum_{i=1}^k \frac{a_i u_i^k}{k}, \qquad a_i \in \mathbb{R}\, for\ i = 1, \dots, k.$$

Let $l_\infty^k \langle resp.\, l_1^k\rangle$ be the k-dimensional space $\mathbb{R}^k$ endowed with the max-norm $\langle resp. the\ 1 - norm\rangle$. If we consider $T_k: l_\infty^k \to l_1^k$, we easily get $\|T_k\| = 1$ for every $k \in \mathbb{N}$.

For a sequence $\{X_k, \|\cdot\|_{X_k}\}_{k=1}^\infty$ of Banach spaces, consider the Banach space $(\bigoplus_{k=1}^\infty X_k)_{c_0}$ (the linear space, under the usual algebraic operations, whose elements are the sequences $\{x_k\}_{k=1}^{\infty}, x_k \in X_k$ for each k, such that $\|x_k\|_{X_k} \to 0$ as $k \to \infty$, endowed with the norm $\|\{x_k\}_{k=1}^{\infty}\| = max_k \|x_k\|_{X_k}$).

Clearly we have

$$c_0 = (\oplus_{k=1}^{\infty} l_{\infty}^k)_{c_0} . \qquad (1.8)$$

Put

$$Y = (\oplus_{k=1}^{\infty} l_1^k)_{c_0} .$$

Order the set $\cup_{k=1}^{\infty} \{u_i^k\}_{i=1}^k$ in the natural way and rename it as

$$\{u_1^1, u_1^2, u_2^2, \ldots, u_1^k, \ldots, u_k^k, \ldots\} = \{u_1, u_2, u_3, \ldots\}. \qquad (1.9)$$

Of course $\{u_i\}_{i=1}^{\infty}$is an unconditional basis both for c_0 and for Y. Call P_k the natural norm-one projection of c_0 onto l_{∞}^k suggested by (1.8) and define $T: c_0 \to Y$in the following way

$$Tx = \sum_{i=0}^{\infty} T_k P_k x , \qquad x \in c_0.$$

T is a (linear) non-compact operator, since $\left\|T(\sum_{i=1}^k u_i^k)\right\| = 1$ and $\sum_{i=1}^k u_i^k$ is weakly null as $k \to \infty$. However, if we denote by $\{e_i\}_{i=1}^{\infty}$ the sequence of the coordinate functionals associated to the basis $\{u_i\}_{i=1}^{\infty}$of Y, it is true that $T^* e_i \to 0$ in X^* as $i \to \infty$. In fact, for $x = \sum_{k=1}^{\infty} \sum_{j=1}^k x_j^k u_j^k \in B_{c_0}$ the following holds

$$\left|x_j^k\right| \leq 1 \quad 1 \leq j \leq k, \qquad k = 1, 2, \ldots$$

so, if we denote by $u_{j_i}^{k_i}$ the element u_i as identified by (9), we have

$$|(T^* e_i)(x)| = |e_i(Tx)| = |e_i(\sum_{k=1}^{\infty} \sum_{j=1}^{k} x_j^k u_j^k / k)| = |x_{j_i}^{k_i}| / k_i \leq 1/k_i.$$

Since $k_i \to \infty$ with i, we are done.

Chapter 2

Almost Square Banach Spaces

We single out and study a natural class of Banach spaces – almost square Banach spaces. In an almost square space we can find, given a finite set $x_1, x_2, \ldots, x_N$ in the unit sphere, a unit vector y such that $\|x_i - y\|$ is almost one. These spaces have duals that are octahedral and finite convex combinations of slices of the unit ball of an almost square space have diameter

Section (2.1): Examples and Characterizations.

Let X be a Banach space with unit ball B_X, unit sphere S_X, and dual space X^*.

Definition (2.1.1)[2]: We will say that a Banach space X is

(i) *locally almost square* (lasq) if for every $x \in S_X$ there exists a sequence

$$(y_n) \subset B_X \text{ such that } \|x \pm y_n\| \to 1 \text{ and } \|y_n\| \to 1.$$

(ii) *weakly almost square* (wasq) if for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly.

(iii) *almost square* (asq) if for every finite subset $(x_i)_{i=1}^N \subset S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x_i \pm y_n\| \to 1$ for every $i = 1, 2, \ldots, N$ and $\|y_n\| \to 1$.

Obviously (wasq) implies (lasq), but it is not completely obvious that (asq) implies (wasq). This will be shown in Theorem (2.1.24).

In the language of Schäefer a Banach space X is (lasq) if and only if no $x \in S_X$ is *uniformly non-square*. Gao and Lau considered the following parameter

$$G(X) = sup\{inf\{max\{\|x + y\|, \|x - y\|\}, y \in S_X\}, x \in S_Y\}.$$

We see that X is (lasq) if and only if *G(X)* = 1. Gao and Lau showed that L_1 is (lasq) while $L_p, 1 < p \leq \infty$, and $\ell_p, 1 \leq p \leq \infty$, are not.

A separable Banach space X has Kalton and Werner's *property* (m_∞) if

$$\lim_n sup\|x + y_n\| = max(\|x\|, \lim_n sup\| y_n\|)$$

for every $x \in X$ whenever $y_n \to 0$ weakly. From Rosenthal's ℓ_1 theorem: (Let (x_n) be a bounded sequence in a Banach space X. Either there is a subsequence which is equivalent to the ℓ^1-basis or there is a subsequence (x_{n_k}) which is weakly Cauchy (i.e. $(x'(x_{n_k}))$ converges for every $x' \in X'$))[6] it is clear that such spaces must be (asq) if they do not contain a copy of ℓ_1. However, if X does not contain a copy of ℓ_1, then X has propery (m_∞) if and only if, for any $\varepsilon > 0$, X is ε-isometric to a subspace of c_0. We will see that this is much stronger than (asq), see for example in Corollary (2.2.7).

Our main interest in the (*asq) properties come from their relation to diameter two properties. Recall that a *slice* of B_X is a set of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$. we find the following definition.

Definition (2.1.2)[2]: A Banach space X has the

(i) *local diameter* 2 *property (LD2P)* if every slice of B_X has diameter 2.

(ii) *diameter* 2 *property (D2P)* if every nonempty relatively weakly open subset of B_X has diameter 2.

(iii) *strong diameter* 2 *property (SD2P)* if every finite convex combination of slices of B_X has diameter 2. (i.e. $\sum_{i=1}^{n} \lambda_i S_i$ has diameter 2 whenever $\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1$, and $S_1, \dots, S_n$ are slices of B_X.)

The starting point was the observation by Kubiak that if *X* is (lasq) then *X* has the LD2P and similarly if *X* is (wasq) then *X* has the D2P. The basic idea from Kubiak's proof works also for (asq):

Proposition (2.1.3)[2]: *If a Banach space X is (asq) then X has the SD2P.*

Proof. Let $S_i = S(x_i^*, \varepsilon_i)$, $i = 1, \dots, N$, be slices of B_X with $x_i^* \in SX^*$ and $0 < \varepsilon_i < 1$.

Let $\varepsilon = min\{\varepsilon_i\}/4$. Find $x_i \in S_X$ with $x_i^* (x_i) > 1 - \varepsilon$. Find sequence (y_n) with $\|x_i \pm y_n\| \to 1$ and $\|y_n\| \to 1$. Choose n_0 such that $\|x_i \pm y_{n_0}\| < 1 + \varepsilon$ for i = 1, 2, . . . ,N and $\|y_{n_0}\| > 1 - \varepsilon$. Then

$$\pm x_i^* (y_{n_0}) = x_i^*(x_i \pm y_{n_0}) - x_i^* (x_i) < 1 + \varepsilon + \varepsilon - 1 = 2\varepsilon$$

and

$$x_i^* (\frac{x_i \pm y_{n_0}}{1 + \varepsilon}) = \frac{1}{1 + \varepsilon}(x_i^*(x_i) \pm x_i^*(y_{n_0})) \geq \frac{1}{1 + \varepsilon}(1 - \varepsilon - 2\varepsilon) > 1 - \varepsilon_i.$$

This means that $(x_i \pm y_{n_0})/(1 + \varepsilon) \in S_i$ and $\|y_{n_0}\| > 1 - \varepsilon$ and hence, by Lemma (2.2.1), *X has* the SD2P.

It is known that the three diameter 2 properties are different. That the LD2P and the D2P are different was shown. That the D2P and the SD2P are different was shown . A natural question is whether (lasq), (wasq), and (asq) are different properties. We will show that L_1 is a (wasq) space which is not (asq) in Corollary (2.1.25).

Haller, Langemets, and Põldvere considered the following versions of octahedral norms.

Definition (2.1.4)[2]: A Banach space X is said to be

(i) *locally ctahedral* if for every $x \in S_X$ and every $\varepsilon > 0$ there is a $y \in S_X$ such that $\|x \pm y\| \geq 2 - \varepsilon$.

(ii) *weakly octahedral* if for every finite subset $(x_i)_{i=1}^{N} \subset S_X$, every $x^* \in B_{X^*}$, and every $\varepsilon > 0$ there is a $y \in S_X$ such that
$\|x_i + ty\| \geq (1 - \varepsilon)(|x^*(x_i)| + t)$ for all $i = 1, 2, \dots, N$ and $t > 0$.

(iii) *octahedral* if for every finite subset $(x_i)_{i=1}^{N} \subset S_X$ and every $\varepsilon > 0$ there is a $y \in S_X$ such that $\|x_i \pm y\| \geq 2 - \varepsilon$ for all $i = 1, 2, \dots, N$.

We have the following theorem .

Theorem (2.1.5)[2]: *Let X be a Banach space. Then*

(i) X has the LD2P if and only if X^* is locally octahedral.

(ii) X has the D2P if and only if X^* is weakly octahedral.

(iii) X has the SD2P if and only if X^* is octahedral.

This theorem shows that the ℓ_1 structure of the norm of X^* is connected to diameter two properties of the space. The connection between the SD2P and octahedrality has also been studied. We give characterizations of (lasq) and (asq) as the corresponding ℓ_∞ structure. (See Corollary (2.1.20) and Theorem (2.1.21))

We will give examples of spaces which are (lasq), (wasq), and (asq). We start with a few characterizations of (lasq) and (asq). In particular, we show in Theorem (2.1.21) that if X is (asq) then for every finite-dimensional subspace E of X and every $\varepsilon > 0$ there is a $y \in S_Y$ such that

$$(1 - \varepsilon) \max(\|x\|, |\lambda|) \leq \|x + \lambda x\| \leq (1 + \varepsilon) \max(\|x\|, |\lambda|)$$

for all $x \in E$ and all scalars λ. Using this we show, in Lemma (2.1.23), that (asq) spaces have to contain almost isometric copies of c_0. This in turn gives the second main result, Theorem (2.1.24), which shows that (asq) implies (wasq). The final main result is Theorem (2.1.28), where we show that every Banach space that contains a copy of c_0 can be equivalently renormed to be (asq).

We return to more examples. The result is that spaces which are M-ideals in their biduals are (asq) (see Theorem (2.2.6)). However, the class of (asq) spaces is much bigger than the class of spaces that are M-ideals in their biduals (see Examples (2.1.6) and (2.2.17)).

We study the stability of both (local/weak) octahedrality and (*asq) when forming absolute sums of Banach spaces. We show that local and weak octahedral, (lasq), and (wasq) spaces have nice stability properties but the situation is different for (asq). For $1 \leq p < \infty$ the ℓp-sum of two Banach spaces is never (asq). Note that an ℓ_p-sum of two Banach spaces can only be octahedral if $p = 1$ or $p = \infty$.

We connect (asq) with the intersection property of Behrends and Harmand. We show that (asq) spaces fail the intersection property and give a quantitative version of this fact in Theorem (2.2.16).We also give an example of a space that fails the intersection property and is not (lasq).

We follow standard Banach space notation. We consider real Banach spaces only.

We will provide examples of Banach spaces which are (lasq), (wasq), and (asq) and spaces which are not. Let us start with the prototype of an (asq) space - the space, c_0, of null-sequences.

Example (2.1.6)[2]: Let $(x_i)_{i=1}^N \subset Sc_0$ and e_n the n'th standard basis vector in c_0. Then it is clear that $\|x_i \pm e_n\| \to 1$ as $n \to \infty$ for every $i = 1, 2, \dots, N$, so c_0 is (asq). Also, as $e_n \to 0$ weakly in c_0, it follows that c_0 is (wasq). Given a sequence of Banach spaces (X_i) it is clear that actually the c_0-sum, $c_0(X_i)$ is (asq) (and (wasq)). In contrast to c_0 being (asq), the space, c, of convergent sequences is not even (lasq).

Example (2.1.7)[2]: Let $x = (1, 1, \dots, 1, \dots) \in S_c$. Now, if $\|x \pm y_n\| \to 1$, then $\|y_n\| \nrightarrow 1$. Because, if the value of one coordinate of y_n was close to ±1, then the maximum of that coordinate of $x \pm y_n$ would be close to 2. And so c is not (lasq).

Recall that a point x in the unit-ball B_X of a Banach space X is an *extreme point* in B_X if for every $y \in B_X$ with $\|x \pm y\| = 1$ we have $\|y\| = 0$. If for every sequence $(y_n) \subset B_X$ with $\|x \pm y_n\| \to 1$ we have $\|y_n\| \to 0, x$ is said to be a *strong extreme point*.

Note that arguing similarly as in Example (2.1.7) we get that the sequence (y_n) in this example must converge in norm to 0. Thus, by definition, $x = (1, 1, \dots, 1, \dots)$ is a strong extreme point in B_c.

Straight from the definition of a strong extreme point, we actually have the following general fact.

Fact (2.1.8)[2]: The unit ball of (lasq) spaces cannot have strong extreme points. The constant 1 function in $\ell_\infty, C[0,1]$, and $L_\infty[0,1]$ is a strong extreme point in the unit ball of these spaces, so neither $\ell_\infty, C[0,1]$ nor $L_\infty[0,1]$ are (lasq). We noted in the introduction that Gao and Lau have shown that $L_1[0,1]$ is (lasq).

Example (2.1.9)[2]: $L_1[0,1]$ is (wasq). Let $f \in SL_1$ and define $f_n = f r_n$ where (r_n) are the Rademacher functions. Shows that $(f_n) \subset SL_1$ is weakly null and that

$$\|f \pm f_n\|_1 = \int |f(t)|(1 \pm r_n(t))dt = \int |f(t)|dt \pm \int |f(t)|r_n(t)dt \to 1$$

Next we will present a whole class of spaces which are (wasq). This is the class of Cesàro function spaces.

For an interval $I \subset R$ by $L_0(I)$ we denote the set of all (equivalence classes of) real valued Lebesgue measurable functions on I. Any Banach space $E = E(I) \subset L_0(I)$ with a norm $\|\cdot\|$ satisfying the condition that $f \in E$ and $\|f\| \leq \|g\|$ whenever $0 \leq f \leq g$ a.e., $f \in L_0(I)$, and $g \in E$ is called a *Banach function lattice*.

The Köthe dual of a Banach function lattice E on I is the space E' of all $f \in L_0(I)$ such that the associate norm $\|f\| := \sup_{g \in B_E} \int_I |f(x)g(x)|dx$ is finite. The Köthe dual is again a Banach function lattice.

Let $I = (0, l)$ where $0 < l \leq \infty$ is fixed and let $0 < \omega \in L_0(I)$ be a weight. The *weighted Cesàro function space* on I is defined for $1 \leq p < \infty$ as

$$C_{p,\omega}(I) := \left\{ f \in L_0(I): \|f\|_{C_{p,\omega}} := \left(\int_I \left(\omega(x) \int_0^x |f(t)|dt \right)^p dx \right)^{1/p} < \infty \right\}.$$

It is known that $C_{p,\omega} = C_{p,\omega}(I)$ in the natural pointwise order is a separable order continuous Banach function lattice and hence order isometric to a Köthe function space.

Kubiak proved the following result.

Theorem (2.1.10)[2]: *The space $C_{p,\omega}$ is (wasq).*

Later we will show that $C_{p,\omega}$ is not (asq).

Note that the space $C_{1,1/x}[0,1]$ is isometrically isomorphic to $L_1[0,1]$. Also, it is worth noting that for $1 < p < \infty$ and every weight ω every point on the unit sphere of $C_{p,\omega}$ is extreme, i.e. $C_{p,\omega}$ is strictly convex. Thus contrary to strong extreme points, extreme points do not seem to have anything to do with being (lasq), (wasq), or (asq).

We provide examples of (asq), (lasq), and non-(lasq) from the class of Lindenstrauss spaces (i.e. the Banach spaces with duals isometric to $L_1(\mu)$ for some positive measure μ).

Definition (2.1.11)[2]: X is a *G-space* if there are a compact Hausdorff space K and $(s_i, t_i, \lambda_i) \in K \times K \times \mathbb{R}, i$ in some index set I, such that X is isometric to

$$\{f \in C(K) : f(s_i) = \lambda_i f(t_i) \; for \; all \; t \in K\}.$$

X is a C_σ*-space* if there are a compact Hausdorff space K and an involutory homeomorphism $\sigma : K \to K$ $(i.e. \sigma^2 = id_K)$ such that X is isometric to

$$\{f \in C(K) : f(t) = -f(\sigma(t)) \; for \; all \; t \in K\}.$$

X is a C_Σ*-space* if it is a C_σ-space for some fixed point free involution σ on some K.

Note that every G-space is Lindenstrauss. It is also nown that an extreme point in a Lindenstrauss space is actually strongly extreme. So only the Lindenstrauss spaces without extreme points have the chance to be (lasq).

Example (2.1.12)[2]: Lazar and Lindenstrauss considered the following example:

$$X = \left\{ f \in C[0,1]: 2f(0) = -f\left(\frac{1}{3}\right), \quad 2f(1) = -f\left(\frac{2}{3}\right) \right\}.$$

Then ext $B_X = \emptyset$ and X is a G-space of codimension 2 in C[0,1].

The space X is (lasq). Indeed, let $f \in S_X$ and $\varepsilon > 0$. If $f(0) \neq 0$, then since $f(0)f\left(\frac{1}{3}\right) < 0$ there exists $x_0 \in (0, \frac{1}{3})$ such that $f(x_0) = 0$. Let (a,b) be neighborhood of x_0 such that $|f(x)| < \varepsilon$ on (a,b) (and $0, \frac{1}{3} \notin (a,b)$).

Define $g(x_0) = 1$ and g(0) = 0 outside (a,b). Then $2g(0) = 0 = -g\left(\frac{1}{3}\right)$ and $2g(1) = 0 = -g\left(\frac{2}{3}\right)$ so $g \in S_X$. Clearly $\|f \pm g\| < 1 + \varepsilon$.

If $f(0) = 0$, then $f\left(\frac{1}{3}\right) = 0$ and we can find neighborhoods A and B of 0 and $\frac{1}{3}$ where $|f(x)| < \varepsilon$. Define $\left(\frac{1}{3}\right) = -1$, $g(0) = \frac{1}{2}$, and g(x) = 0 outside A and B. Then $\|f \pm g\| < 1 + \varepsilon$ and $g \in S_X$.

The space X is not (asq). To this end, consider the two functions f_1 and f_2 in S_X that look like this:

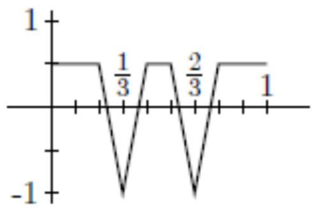

Fig (2.1)

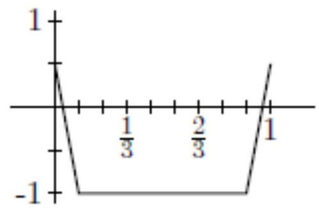

Fig (2.2)

If $g \in C[0,1]$ with $\|f_i \pm g\| < 1+\varepsilon$ then $|g| < \varepsilon$ on $\left[\frac{1}{9}, \frac{8}{9}\right]$ from the second function and $|g| < \frac{1}{2} + \varepsilon$ on $\left[0, \frac{1}{9}\right]$ and $\left[\frac{8}{9}, 1\right]$ from the first. Thus no such g can exist in B_X with $\|g\| > 1 - \varepsilon$.

Example (2.1.13)[2]: Let $K = [0,1]$ and $\sigma(t) = 1 - t$. Then σ is a homeomorphism of K with $\sigma^2 = Id$. $\frac{1}{2}$ is a fixed point. Let

$$X = \{f \in C[0,1]: f(t) = -f(\sigma(t)) \text{ for all } t \in [0,1]\}$$

The space X is (asq). To see this, let $f_1, f_2, \ldots, f_N \in S_X$. Since $f_i\left(\frac{1}{2}\right) = -f_i\left(\sigma\left(\frac{1}{2}\right)\right) = -f_i\left(\frac{1}{2}\right)$ we must have $f_i\left(\frac{1}{2}\right) = 0$. Now, find an interval (a,b) around $\frac{1}{2}$ where $|f_i(x)| < \varepsilon \; for \; i = 1, 2, \ldots, N$. Let $g \in S_X$ have its support on (a,b). Then $\|f_i \pm g\| < 1 + \varepsilon$ and $\|g\| = 1$. F.ex. g could look something like this:

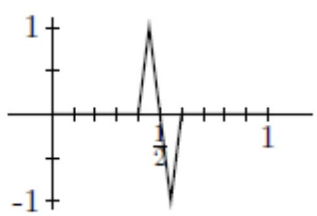

Fig (2.3)

Proposition (2.1.14)[2]: *C_σ spaces are (asq) when σ has a non-isolated fixed point.*

Proof. Let X be a C_σ space. If x_0 is a fixed point for σ, then $f(x_0) = -f(\sigma(x_0)) = -f(x_0)$ for all $f \in X$. Hence $f(x_0) = 0$ for all $f \in X$. With a common non-isolated zero we can use the same idea as in the example above.

Proposition (2.1.15)[2]: *If K is a locally compact Hausdorff space, then $C_0(K)$ is (asq).*

Proof. Let $f_1, f_2, \ldots, f_N \in S_{C_0}(K)$ Find compact $L \subset K$ such that $|f_i(x)| < \varepsilon$ outside L. Let g be a norm one function with support on $K \setminus L$. Then $\|f_i \pm g\| < 1 + \varepsilon$.

We get the following characterization of Banach spaces that are (lasq).

Proposition (2.1.16)[2]: *Let X be a Banach space. The following are equivalent.*

(i) *X is (lasq).*

(ii) *For every $x \in S_X$ there exists a sequence $(y_n) \subset X$ such that $\|y_n\| \to 1$ and $\|x \pm y_n\| \to 1$ as $n \to \infty$.*

(iii) *For every $x \in S_X$ there exists a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$ as $n \to \infty$.*

(iv) *For every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|y_n\| \to 1$ and $\|\lambda_{x_i} \pm y_n\| \to 1$ as $n \to \infty$ for all $\lambda \in [0,1]$.*

It is clear that we also have the following characterization of Banach spaces that are (asq).

Proposition (2.1.17)[2]: *Let X be a Banach space. The following are equivalent.*

(i) *X is (asq).*

(ii) *For every finite subset $(x_i)_{i=1}^{N} \subset S_X$ there exists a sequence $(y_n) \subset X$ such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ as $n \to \infty$ for every $i = 1, 2, \ldots, N$.*

(iii) *For every finite subset* $(x_i)_{i=1}^{N} \subset S_X$ *there exists a sequence* $(y_n) \subset S_X$ *such that* $\|x_i \pm y_n\| \to 1$ *as* $n \to \infty$ *for every* $i = 1, 2, \ldots, N$.

(iv) *For every finite subset* $(x_i)_{i=1}^{N} \subset S_X$ *there exists a sequence* $(y_n) \subset B_X$ *such that* $\|y_n\| \to 1$ *and* $\|\lambda_{x_i} \pm y_n\| \to 1$ *as* $n \to \infty$ *for every* $i = 1, 2, \ldots, N$ *and all* $\lambda \in [0,1]$.

Note that since we have finitely many vectors to play within the definition of (asq) we may drop the plusminus sign.

Proposition (2.1.18)[2]: *Let X be a Banach space.*

X is (lasq) if and only if for every $x \in S_X$ *and* $\varepsilon > 0$ *there exists* $y \in S_X$ *such that* $\|x \pm y\| \le 1 + \varepsilon$.

X is (asq) if and only if for every finite subset $(x_i)_{i=1}^{N} \subset S_X$ *and* $\varepsilon > 0$ *there exists* $y \in S_X$ *such that* $\|x_i + y\| \le 1 + \varepsilon$.

We have the following lemma reference.

Lemma (2.1.19)[2]: *Assume* $x, y \in S_X$ *such that* $1 - \varepsilon \le \|x \pm y\| \le 1 + \varepsilon$, *then*

$$(1 - \varepsilon)max(|\alpha|, |\beta|) \le \|\alpha x + \beta y\| \le (1 + \varepsilon)max(|\alpha|, |\beta|)$$

for all scalars α *and* β.

Proof. Let $M = max(|\alpha|, |\beta|)$. We need to show that

$$(1 - \varepsilon) \le \left\|\frac{\alpha}{M}x + \frac{\beta}{M}y\right\| \le (1 + \varepsilon).$$

It is enough to show

$$(1 - \varepsilon) \le \|\lambda x + y\| \le (1 + \varepsilon)$$

for all $0 < \lambda \le 1$. We have

$$\|\lambda^{-1}y + x\| = \|(1 + \lambda^{-1})y - (y - x)\| \geq (1 + \lambda^{-1}) - \|x - y\| \geq \lambda^{-1} - \varepsilon$$

since $\|x - y\| \leq 1 + \varepsilon$. Hence $\|\lambda x + y\| \geq 1 - \varepsilon\lambda \geq 1 - \varepsilon$.

Also

$$\|\lambda^{-1}y + x\| = \|(\lambda^{-1} - 1)y + (y + x)\| \leq (\lambda^{-1} - 1) + 1 + \varepsilon = \lambda^{-1} + \varepsilon$$

and hence $\|\lambda x + y\| \leq 1 + \varepsilon\lambda \leq 1 + \varepsilon$.

Corollary (2.1.20)[2]: *If X is (lasq), then X contains almost isometric copies of* ℓ_∞^2.

For (asq) Banach spaces we can say even more.

Theorem (2.1.21)[2]: *Let X be a Banach space. If X is (asq) then for every finite dimensional subspace* $E \subset X$ *and* $\varepsilon > 0$ *there exists* $y \in S_X$ *such that*

$$(1 - \varepsilon)max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon)max(\|x\|, |\lambda|)$$

for all scalars λ *and all x* □ *E.*

Moreover, given a finite dimensional subspace $F \subset X^*$ *we may choose the above y so that* $|f(y)| < \varepsilon\|f\|$ *for every* $f \in F$.

It is clear from Proposition (2.1.18) that the above theorem is actually a characterization of (asq).

Proof. Let E be a finite dimensional subspace of X and let $\varepsilon > 0$. Find $\varepsilon/2$-net $(x_i)_{i=1}^N$ for S_E. Choose $y \in S_X$ such that $\|x_i \pm y\| < 1 + \varepsilon/2$. Assume that $\|x_i \pm y\| \leq 1 - \varepsilon/2$, then

$$1 = \|x_i\| \leq \frac{1}{2}\|x_i + y\| + \frac{1}{2}\|x_i - y\| < \frac{1}{2}(1 + \varepsilon/2 + 1 - \varepsilon/2) = 1.$$

Contradiction. So $\|x_i \pm y\| > 1 - \varepsilon/2$.

Let $x \in S_E$. Find i such that $\|x_i - x\| < \varepsilon/2$. Then

$$\|x \pm y\| \leq \|x_i \pm y\| + \|x - x_i\| < 1 + \varepsilon$$

and

$$\|x \pm y\| = \|x - x_i + x_i \pm y\| \geq \|x_i \pm y\| - \|x_i - x\| > 1 - \varepsilon.$$

Hence by using Lemma (2.1.19) we get

$$(1 - \varepsilon)max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon)max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E$.

For the moreover part let $F \subset X^*$ be a finite dimensional subspace and let $(f_i)_{i=1}^M \subset S_F$ be an $\varepsilon/2$-net. For each i choose $z_i \in S_X$ with $f_i(z_i) > 1 - \varepsilon/4$. Let $E' = span\{E, (z_i)_{i=1}^M\}$ and use the first part of the proof to find $y \in S_X$ such that

$$(1 - \varepsilon/4)max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon/4)max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E'$.

Since $|f_i(z_i \pm y)| \leq \|z_i \pm y\| \leq 1 + \varepsilon/4$ we get

$$-\varepsilon/2 = 1 - \varepsilon/4 - (1 + \varepsilon/4) \leq f_i(z_i) - f_i(z_i - y) = f_i(y) \leq f_i(z_i + y) - f_i(z_i)$$
$$\leq 1 + \varepsilon/4 - 1 + \varepsilon/4 = \varepsilon/2.$$

so that $|f_i(y)| < \varepsilon/2$. Thus for every $f \in S_F$ we have $|f(y)| \leq |(f - f_i)(y)| + |f_i(y)| \leq \varepsilon$.

Let us note the following corollary.

Corollary (2.1.22)[2]: *If a Banach space X is (asq), then* $0 \in \overline{ext^{w^*}} B_{X^*}$.

Repeated use of the theorem gives the following lemma.

Lemma (2.1.23)[2] *If X is (asq), then for every finite dimensional subspace E of X and every* $\varepsilon > 0$ *there exists a subspace Y of X which is* ε*-isometric to* c_0 *such that* $E \oplus Y$ *is* ε*-isometric to* $E \oplus_\infty c_0$.

Proof. Find sequence $(\varepsilon_n) \subset \mathbb{R}^+$ such that $\prod_{n=1}^{\infty}(1 + \varepsilon_n) < 1 + \varepsilon$ and $\prod_{n=1}^{\infty}(1 - \varepsilon_n) > 1 - \varepsilon$. Using Theorem (2.1.21) we inductively choose a sequence $(y_n) \subset S_X$ such that

$$(1 - \varepsilon_n)max\{\|e\|, |\lambda|\} \leq \left\|e + \lambda_{y_n}\right\| \leq (1 + \varepsilon_n)max\{\|e\|, |\lambda|\}$$

for every $e \in span\{E, (y_i)_{i=1}^{n-1}\}$ and every $\lambda \in \mathbb{R}$. Then $Y = \overline{span\{(y_n)\}}$ is ε-isometric to c_0 and defining $S: E \oplus_\infty c_0 \to E \oplus Y$ by $S(e, a) = e + Ta$ where $T : c_0 \to Y$ is the ε-isometry. We have

$$\left\|S(e, \sum_{n=1}^{N} a_n e_n)\right\| = \left\|e + \sum_{n=1}^{N} a_n y_n\right\| \leq (1 + \varepsilon_N)max\left\{\left\|e + \sum_{n=1}^{N-1} a_n y_n\right\|, |aN|\right\} \leq \cdots$$

$$\leq \prod_{n=1}^{N}(1 + \varepsilon_n)\, max\{\|e\|, |a_1|, |a_2|, \ldots, |a_N|\} < (1 + \varepsilon)\left\|(e, \sum_{n=1}^{N} a_n e_n)\right\|,$$

and similarly $\|S(e, \sum_{n=1}^{N} a_n e_n)\| > (1 - \varepsilon)\left\|(e, \sum_{n=1}^{N} a_n e_n)\right\|$. Thus S must be an ε-isometry onto $E \oplus Y$ since T is onto Y.

A consequence of Lemma (2.1.23) is that the sequence (y_n) in the definition of (asq) may be chosen to be weakly null. This enables us to connect the (asq) and (wasq) properties.

Theorem (2.1.24)[2]: *If a Banach space X is (asq) then for every* $x_1, x_2, \ldots, x_N \in S_X$ *there exists* $(y_n) \subset B_X$ *such that* $\|x_i \pm y_n\| \to 1$ *for all* i, $y_n \to 0$ *weakly, and* $\|y_n\| \to 1$.

In particular, (asq) implies (wasq).

Proof. Let $x_1, x_2, \ldots, x_N \in S_X$ and $E = span\{(x_i)_{i=1}^{N}\}$, and choose a sequence $(y_n) \subset S_X$ as in Lemma (2.1.23) Let $Z = E \oplus_\infty c_0$ and $z_i = (x_i, 0) \in Z$. Since the standard basis $(e_n)_{n=1}^{\infty} \subset S_{c_0}$ is weakly null so is

$w_n = (0, e_n)$ in Z. By Lemma (2.1.23) there exists an ε-isometry S from Z onto $E \oplus Y$ where $Y = span\{(y_n)\}$. The weak-weak continuity of S shows that $y_n \to 0$ weakly in $F \oplus Y$ and hence also in X.

By definition $S(e, \pm e_n) = e \pm y_n$ for every $e \in E$. Since

$$(1 - \varepsilon_n) max\{\|e\|, 1\} \le \|e \pm y_n\| \le (1 + \varepsilon_n) max\{\|e\|, 1\}$$

for every $e \in E$, we in particular have $(1 - \varepsilon_n) \le \|x_i \pm y_n\| \le (1 + \varepsilon_n)$, so $x_i \pm y_n \to 1$.

Corollary (2.1.25)[2]: *(asq) is strictly stronger than (wasq).*

Proof. From the theorem we have that all (asq) spaces are (wasq). By Example (2.1.9) $L_1[0,1]$ is (wasq), but $L_1[0,1]$ does not contain c_0 so it is not (asq).

Question (2.1.26)[2]: *Is (wasq) strictly stronger than (lasq)?*

In Lemma (2.1.23) we proved that if X is (asq) then X contains almost isometric copies of c_0. Accordingly a Banach space X contains an *asymptotically isometric copy of* c_0 if for every null-sequence $(\varepsilon_n)_{n=1}^{\infty} \subset (0,1)$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in X such that

$$\max_{n \in F}(1 - \varepsilon_n)\,|a_n| \le \left\|\sum_{n \in F} a_n x_n\right\| \le \max_{n \in F}(1 + \varepsilon_n)\,|a_n|$$

for all choices of scalars (a_n) and all finite subsets F of $\mathbb{N}$. Pfitzner showed that *M*-embedded spaces contain an asymptotically isometric copy of c_0 using the local characterization of *M*-ideals. If we instead use Theorem (2.1.21) in Pfitzner's proof we get the following.

Proposition (2.1.27)[2]: *If X is (asq), then X contains an asymptotically isometric copy of* c_0. *Moreover,* X^* *contains an asymptotically isometric* *copy of* ℓ_1.

We know that every (asq) space contains c_0. Next we will show that any Banach space containing c_0 can be equivalently renormed to be (asq). This improves the Proposition which says that any Banach space containing c_0 can be equivalently renormed to have the SD2P. The proof of the following result is based on a renorming technique.

Theorem (2.1.28)[2]: *A Banach space can be equivalently renormed to be (asq) if and only if it contains a copy of* c_0.

Proof. As an (asq)-space contains c_0, the "only if part" is clear.

For the "if" part, first renorm X to contain c_0 isometrically. Denote by $\|.\|$ the new norm on X. Let

$$A = \{Y \subset X : c_0 \subset Y, Y \; separable\},$$

and order A by inclusion, i.e. $Y_2 \leq Y_1 \; if \; Y_2 \subset Y_1$. For every $Y \in A$ there exists by Sobczyk's theorem: (Let X be a separable Banach space and Y a closed subspace of X. If $T_o: Y \longrightarrow C_o$ is a linear operator of norm λ, there exists an extension $T: X \longrightarrow C_o$ of norm at most 2λ)[7] a projection P_Y onto c_0 with norm 2 or less.

Let P_Y be such a projection and for each $Y \in A$ and $x \in Y$ let

$$\|x\|_Y := max\{\|P_Y(x)\|, \|x - P_Y(x)\|\}.$$

Further let $L_Y : X \rightarrow [0, 3\|x\|]$ be defined by $L_Y(x) = \|x\|_Y$ if $x \in Y$ and 0 if $\notin Y$. We can consider L_Y as an element in the product space $\Pi = \prod_{x \in X}[0, 3\|x\|]$. As Π is compact by Tychonoff's theorem: ((Tychonoff) For each $i \in I$, let X_i be a nonempty topological space, and let $X = \prod_{i \in I} X_i$, endowed with the product topology.

a) The following are equivalent:

(i) Each X_i is quasi-compact.

(ii) X is quasi-compact.

b) The following are equivalent:

(i) Each X_i is compact.

(ii) X is compact.

Every implication except (i) ⇒ (ii) in part a) is straightforward: let $i \in I$. Then $X_i = \pi_i(X)$, so if X is quasi-compact, so is X_i. Moreover, by the Slice Lemma, X_i is homeomorphic to a subspace of X, so if X is Hausdorff, so is X_i. Finally the product of Hausdorff spaces is Hausdorff. Henceforth by "Tychonoff's Theorem")[8] , the net $(L_Y) \subset \Pi$ has a convergent subnet also denoted by (L_Y). Finally define

$$|||x||| = \lim_{Y} \|x\|_Y .$$

It is straightforward to show that $||| \cdot |||$ is a norm on X which satisfies

$\frac{1}{2}\|x\| \leq |||x||| \leq 3\|x\|$. Also $||| \cdot |||$ extends the max norm $\|\cdot\|$ on c_0.

Finally we show that $(X, ||| \cdot |||)$ is (asq). Let $(\varepsilon_n)_{n=1}^{\infty}$ be a strictly decreasing null sequence of positive reals, $(x_i)_{i=1}^{N} \subset S_{(X,|||\cdot|||)}$, $(e_n)_{n=1}^{\infty}$ the sequence of standard basis vectors in c_0, and e_0 the zero vector. The goal is to show that for all $i = 1, \ldots, N$ we have $|||x_i + e_k||| \to 1$ as $k \to \infty$.

Let $Y_0 = span\{(x_i)_{i=1}^{N}, c_0\}$ and choose $Y_1 \in A$ with $Y_1 \supset Y_0$ such that for all $i = 1, \ldots, N$ we have

$$\left| |||x_i + e_0||| - \|x_i + e_0\|_{Y_1} \right| < \varepsilon_1.$$

Then for $n \geq 1$ inductively choose $Y_{n+1} \in A$ with $Y_{n+1} \supset Y_n$ such that for all $i = 1, \ldots, N$ we have

$$\left| |||x_i + e_k||| - \|x_i + e_k\|_{Y_n} \right| < \varepsilon_n \text{ for every } k \le n.$$

(Note that the inequality above holds also for every $Y \in A$ with $Y \supset Y_n$.)

Put $Y = \overline{\cup_{n=1}^{\infty} Y_n}$. Note that $Y \in A$ as $c_0 \subset Y$ and Y is separable. Observe that for all $i = 1, \ldots, N$ and $n \geq k$ we have

$$\left| |||x_i + e_k||| - \|x_i + e_k\|_{Y_n} \right| \le \left| |||x_i + e_k||| - \|x_i + e_k\|_{Y_n} \right| < \varepsilon_n.$$

so $|||x_i + e_k||| = \|x_i + e_k\|_Y$ *as* $\varepsilon_n \downarrow 0$. In particular, we have

$$\|x_i - P_Y(x_i)\| \le \|x_i\|_Y = \|x_i + e_0\|_Y = |||x_i + e_0||| = 1.$$

We now get for all $i = 1, \ldots, N$

$$\|x_i + e_k\|_Y = max\{\|P_Y(x_i) + e_k\|, \|x_i - P_Y(x_i)\|\} \le max\{\|P_Y(x_i) + e_k\|, 1\} \to 1$$

as $k \to \infty$ since $P_Y(x_i) \in c_0$ and c_0 is (asq).

Section (2.2): Stability and Connection with the I_p

Let us start this section by proving that the Cesàro function space $C_{p,\omega}$, for $1 \le p < \infty$, is not (asq) though it is (wasq).

First we recall some definitions. An element f in a Banach function lattice

E is called *order continuous* if for every $0 \le f_n \le |f|$ a.e. such that $f_n \downarrow 0$ a.e. we have that $\|f_n\| \downarrow 0$. We say that E is *order continuous* if every element in E is order continuous. A Banach function lattice $(E, \|\cdot\|)$ has the *Fatou property* if for any sequence $(f_n) \subset E$ and any $f \in L_0(I)$ such that $0 \le f_n \le f$ a.e., $f_n \uparrow f$ a.e., and $\sup_n \|f_n\| < \infty$ we have that $f \in E$ and $\|f\| = \lim_n \|f_n\|$. We know that $C_{p,\omega}$ is order continuous and has the Fatou-property.

Proposition (2.2.1)[2]: *The space $C_{p,\omega}$ does not contain an isomorphic copy of c_0.*

Proof. Let (f_n) be an increasing norm bounded sequence in $C_{p,\omega}$. It is enough to show that (f_n) has a norm limit. If (f_n) has a pointwise a.e. limit *f*, then it follows from the Fatou property that *f* is in $C_{p,\omega}$. Moreover, put $g_n = f - f_n$. Then $0 \le g_n \le f - f_1$ and $g_n \downarrow 0$. By order continuity we get that $\|f - f_n\| = \| g_n\| \to 0$ as wanted.

It only remains to prove that the pointwise limit exists. (f_n) increasing means that $f_n(x) \le f_{n+1}(x)$ for a.e. *x*. By completeness it is enough to show that $(f_n(x))$ is a bounded sequence for a.e. *x*. Assume not, i.e. that $\sup_n f_n(x) = \infty$ on a compact *A* of positive Lebesgue measure $\lambda(A) > 0$.

Split *A* into two parts A_1 and A_2 with $\lambda(A_1) > 0$ and $\lambda(A_2) > 0$ such that $x \le y$ for all $x \in A_1$ and $y \in A_2$.

We know that

$$K = \int_{A_2} w(x)^p \, dx > 0.$$

Let $S = \sup_n \|f_n\| < \infty$. Choose *k* such that $S^p < M^p K$ where

$$M = \int_{A_1} |f_k(t)| \, dt .$$

Then

$$S^p \ge \|f_k\|^p = \int_I \left(w(x) \int_0^x |f(t)| dt \right)^p dx \ge \int_{A_2} \left(w(x) \int_0^x |f(t)| dt \right)^p dx$$

$$\ge \int_{A_2} \left(w(x) \int_{A_1} |f(t)| dt \right)^p dx = \int_{A_2} (w(x) M)^p \, dx = M^p K$$

and we have a contradiction.

From Lemma (2.1.23) we now obtain the following.

Corollary (2.2.2)[2]: *The Cesàro function space $C_{p,\omega}$ is not (asq).*

A closed subspace X of a Banach space Y is said to be a *u-summand* in Y if there is a subspace Z of Y so that $Y = X \bigoplus Z$ and if $x \in X \; and \; z \in Z \; then \; \|x + z\| = \|x - z\|$.

Corollary (2.2.3)[2]: *The space $C_{p,\omega}$ is a u-summand in its bidual.*

Proof. We know that an order continuous Banach lattice not containing a copy of c_0 is a u-summand in its bidual.

It was proved that $C_{p,1/x}$ contains an asymptotically isometric copy of ℓ_1. This was further extended to $C_{p,\omega}$. We obtain the following result.

Proposition (2.2.4)[2]: *The space $C_{p,\omega}$ contains a complemented sublattice isomorphic to ℓ_1.*

We will now present a new class of (asq) spaces. For this we need to introduce some concepts.

Recall that a subspace X in a Banach space Y is an *ideal* in Y if the annihilator $X^{\perp}$ is the kernel of a norm one projection on Y^* . The subspace X is called *locally 1-complemented* in Y if for every finite dimensional subspace E of Y and every $\varepsilon > 0$ there exists a linear operator $u : E \to X$ such that $u(e) = e$ for all $e \in E \cap X$ and $\|u\| \leq 1 + \varepsilon$. Fakhoury proved that X is an ideal in Y precisely when it is locally 1-complementedin Y .

We say that X is an *almost isometric ideal (ai-ideal)* in Y if X is locally 1-complemented in Y in such a way that the operator $u : E \to X$ is an almost isometry, i.e. in addition to the above we have $(1+\varepsilon)^{-1}\|e\| \leq \|u(e)\| \leq (1+\varepsilon)\|e\|$ for all $e \in E$. The fact that X is an ai-ideal in its bidual is commonly referred to as *the Principle of Local Reflexivity (PLR).*

Lemma (2.2.5)[2]: *If X is (asq) and Y is an ai-ideal in X then Y is (asq). In particular (lasq) is inherited by ai-ideals.*

Proof. Let $y_1, y_2, \ldots, y_N \in S_Y$ and $1 > \varepsilon > 0$. Find $x \in S_X$ such that $\|y_i + x\| \leq 1 + \frac{\varepsilon}{4}$. Now, choose an $\frac{\varepsilon}{4}$ - isometry $u : E \to Y$ such that u is the identity on $E \cap Y$ where $E = span\{(y_j)N_j = 1, x\}$. Define $z = u(x)/\|u(x)\|$. Then $z \in S_Y$ and $\|u(x) - z\| = |\|u(x)\| - 1| \leq \frac{\varepsilon}{4}$ and

$$\|y_i + z\| \leq \|u(y_i + x)\| + \|u(x) - z\| \leq \left(1 + \frac{\varepsilon}{4}\right)\left(1 + \frac{\varepsilon}{4}\right) + \frac{\varepsilon}{4} \leq 1 + \varepsilon.$$

so Y is (asq) by Proposition (2.1.18).

If X is an ideal in Y with an ideal projection P on Y^* which for every $y^* \in Y^*$ satisfies $\|y^*\| = \|Py^*\| + \|y^* - Py^*\|$, then X is said to be an *M-ideal* in Y (P is called the M-ideal projection on Y^*). If X is an M-ideal in X^{**}, then X is said to be *M-embedded*. For M-ideals we often get (asq) for free.

Theorem (2.2.6)[2]: *Let Y be a proper subspace of a non-reflexive Banach space X. If Y is both an M-ideal and an ai-ideal in X, then Y is (asq).*

Proof. Let $\varepsilon > 0$ and choose $0 < \delta < 1$ with $(1+\delta)^2(1 + 3\delta(1+\delta)^2) < 1 + \varepsilon$. Write $X^{**} = (PX^*)^{\perp} \oplus_{\infty} Y^{\perp\perp}$. This is possible as Y is an M-ideal in X and thus $X^* = P(X^*) \oplus_1 Y^{\perp}$ (P denotes here the M-ideal projection on X^*). Let

$y_1, y_2, \ldots, y_N \in S_Y$ and $z \in S_{(PX^*)^\perp}$, and put $E = span\{(y_i)_{i=1}^N, z\} \subset X^{**}$. Use the PLR to find a δ-isometry $v : E \to X$ which is the identity on $E \cap X$. Further, put $F = v(E) \subset X$ and use that Y is an ai-ideal in X to find a δ-isometry $u : F \to Y$ which is the identity on $F \cap Y$. Now with $y = uv(z)/\|uv(z)\| \in S_Y$ *we use* $uv(y_i) = y_i$ to get

$$\|y_i + y\| = \left\|y_i + \frac{uv(z)}{\|uv(z)\|}\right\| \le (1 + \delta)^2 \left\|y_i + \frac{z}{\|uv(z)\|}\right\|$$
$$\le (1 + \delta)^2 \left(\|y_i + z\| + \left\|z - \frac{z}{\|uv(z)\|}\right\|\right) < 1 + \varepsilon$$

since

$$\left\|z - \frac{z}{\|uv(z)\|}\right\| = \frac{1}{\|uv(z)\|} |1 - \|uv(z)\||$$
$$\le (1 + \delta)^2 (|1 - \|v(z)\|| + |\|v(z)\| - \|uv(z)\||)$$
$$\le (1 + \delta)^2 (\delta + \delta(1 + \delta)) \le 3\delta(1 + \delta)^2.$$

Using Proposition (2.1.18) we are done.

Since every Banach space is an ai-ideal in its bidual by the PLR we immediately have the following corollary.

Corollary (2.2.7)[2]: *Non-reflexive M-embedded spaces are (asq).*

The following spaces are examples of M-embedded spaces: $c_0(\Gamma)$ (for any set Γ), K(*H*) of compact operators on a Hilbert space *H*, and *C*(T)/*A* where T denotes the unit circle and *A* the disk algebra. From Example (2.1.6) the space $c_0(\ell_1)$ is (asq). However, this space contains a copy of ℓ_1 and therefore can not be M-embedded. Thus the class of (asq) spaces properly contains the class of M-embedded spaces.

From Theorem (2.2.6) we also obtain the following result.

Corollary (2.2.8)[2]: *Let X be a non-reflexive Banach space. Let Y be both an M-ideal and an ai-ideal in X. Then both X and Y have the SD2P.*

We start by introducing the notion of a general absolute sum of a family of Banach spaces. Our goal is to show that (lasq) and (wasq) spaces are stable under absolute sums and it turns out that locally and weakly octahedral Banach spaces are stable by forming absolute sums too.

Let I be a non-empty set and let E be a R-linear subspace of R^I (the space of all functions from I to $\mathbb{R}$).

Definition (2.2.9)[2]: An *absolute norm* on E is a complete norm $||\cdot||_E$ satisfying

(i) Given $(a_i)_{i\in I}, (b_i)_{i\in I} \in \mathbb{R}^I$ *with* $|a_i| = |b_i|$ for every $i \in I$, if $(a_i)_{i\in I} \in E$, then $(b_i)_{i\in I} \in E$ with $\|(a_i)_{i\in I}\|_E = \|(b_i)_{i\in I}\|_E$.

(ii) For every $i \in I$, the function $e_i: I \to \mathbb{R}$ given by $e_i(j) = \delta_{ij}$ for $j \in I$, belongs to I and $\|e_i\|_E = 1$.

We have the following lemma on absolute norms.

Lemma (2.2.10)[2]: *Let E be as above with an absolute norm. Then*

(iii) $\ell_1(I) \subseteq E \subseteq \ell_\infty(I)$ *with contractive inclusions. Equivalently,*

$$sup\{|a_i| : i \in I\} \leq \|(a_i)_{i\in I}\|_E \leq \sum_{i\in I} |a_i|$$

for all $(a_i)_{i\in I} \in E$.

(iv) Given $(a_i)_{i\in I}, (b_i)_{i\in I} \in \mathbb{R}^I$ *with* $|b_i| \leq |a_i|$ *for every i ∈ I, if* $(a_i)_{i\in I} \in E$, *then* $(b_i)_{i\in I} \in E$ *with* $\|(b_i)_{i\in I}\|_E \leq \|(a_i)_{i\in I}\|_E$.

Note that $E \subset \mathbb{R}^I$ can be viewed as a Köthe function: (In particular, going back to the original three-space problem for the Hilbert spaces $X = Y = \ell_2$, Kalton and Peck choose

$$F\left(\sum_i x_i e_i\right) = \sum_i (log\|x\| - log|x_i|)x_i e_i$$

This produces the the Kalton-Peck's space $Z = Z_2$,)[9] space (and hence a Banach lattice) on the space $(I, P(I), \mu)$, where $P(I)$ is the power set of I and μ is the counting measure on I. It is known that E is order continuous if and only if E does not contain an isomorphic copy of ℓ_∞ if and only if span $\{e_i : i \in I\}$ is dense in E.

The Köthe dual E' of a Banach space $E \subset \mathbb{R}^I$ with absolute norm is the linear subspace of $\mathbb{R}^I$ defined by

$$E' := \left\{(a_i)_{i\in I} \in R^I : sup \sum_{i\in I} |a_i b_i| < \infty, (b_i)_{i\in I} \in B_E)\right\}.$$

It is not hard to see that

$$\|(a_i)_{i\in I}\|_{E'} := sup\left\{\sum_{i\in I} |a_i b_i| : (b_i)_{i\in I} \in B_E\right\}$$

defines an absolute norm on E'. Every $(b_i)_{i\in I} \in E'$ defines a functional on E by

$$(a_i)_{i\in I} \to \sum_{i\in I} b_i a_i .$$

This induces an embedding $E' \to E^*$ which is easily seen to be linear and isometric. If span$\{e_i : i \in I\}$ is dense in E then the embedding $E' \to E^*$ is surjective, and so E' and E^* can be identified.

Now, if $(X_i)_{i\in I}$ is a family of Banach spaces we put

$$[\bigoplus_{i\in I} X_i]_E := \{(x_i)_{i\in I} \in \prod_{i\in I} X_i : (\|x_i\|)_{i\in I} \in E\}.$$

It is clear that this defines a subspace of the product space $[\bigoplus_{i\in I} X_i]_E$ which becomes a Banach space when given the norm

$$\|(x_i)_{i\in I}\| := \|(\|x_i\|)_{i\in I}\|_E, \qquad (x_i)_{i\in I} \in [\bigoplus_{i\in I} X_i]_E.$$

This Banach space is said to be the *absolute sum of the family* $(X_i)_{i\in I}$ *with respect to E.* Every $(x_i^*)_{i\in I} \in [\bigoplus_{i\in I} X_i^*]_{E'}$ defines a functional on $[\bigoplus_{i\in I} X_i]_E$ by

$$(x_i)_{i\in I} \to \sum_{i\in I} x_i^* (x_i)$$

This embedding is isometric and is onto if span $\{e_i : i \in I\}$ is dense in E.

Putting $I = \mathbb{N}$ and $E = \ell_p(I)$ it is clear that for $1 \le p \le \infty$ the ℓ_p sum (c_0 *sum if* $p = \infty$) of a family of Banach spaces $(X_i)_{i\in I}$ is an absolute sum with respect to E (for which $[\bigoplus_{i\in I} X_i^*]_{E'} = [\bigoplus_{i\in I} X_i]_{E^*}$ as span$\{e_i : i \in I\}$ is dense in $\ell_p(I)$ in this case). It was proved that locally and weakly octahedral spaces are stable by taking ℓ_p sums of two Banach spaces. A closer look at the argument reveals that it extends to general absolute sums as well. This can also be obtained from Propositions (2.2.11) and (2.2.13) below.

Proposition (2.2.11)[2]: *Let I be a set, E a subspace of* $\mathbb{R}^I$ *with an absolute norm, and* $(X_i)_{i\in I}$ *a family of Banach spaces which are locally octahedral (resp. (lasq)). Then their absolute sum* $X = (\bigoplus_{i\in I} X_i)_E$ *is locally octahedral (resp. (lasq)).*

Proof. Let $\varepsilon > 0$ and consider an $x = (x_i)_{i\in I} \in X$ with norm 1. In both cases we want to find $y \in S_X$ that satisfies

$$\alpha + \varepsilon \geq \|x \pm y\|_E \geq \alpha - \varepsilon,$$

with α = 2 in the locally octahedral case (see Definition (2.1.4)) and α = 1 in the (lasq) case (see Proposition (2.1.18)). By ignoring coordinates where $x_i = 0$ we may (and do) assume that $x_i \neq 0$ for all $i \in I$. By assumption, for every $i \in I$, there exists $y_i \in S_{X_i}$ such that

$$\alpha + \varepsilon \geq \left\| \frac{x_i}{\|x_i\|} \pm y_i \right\| \geq \alpha - \varepsilon.$$

We may take $y = (\|x_i\|_{y_i})_{i \in I}$. Indeed,

$$\|y\|_E = \|(\|x_i\|\|yi\|)_{i\in I}\|_E = \|(\|x_i\|)_{i\in I}\|_E = 1$$

and

$$\|x \pm y\|_E = \left\|(\left\|x_i \pm \|x_i\|y_i\right\|)_{i\in I}\right\|_E \geq (\alpha - \varepsilon)\|(\|x_i\|)_{i\in I}\|_E = \alpha - \varepsilon.$$

Similarly one has that $\|x \pm y\|_E \leq \alpha + \varepsilon$.

The same idea works for absolute sums of (wasq) spaces as long as we have some control over the dual.

Proposition (2.2.12)[2]: *Let E be a subspace of* R^I *with an absolute norm such that* span$\{e_i : i \in I\}$ *is dense in E and* span $\{e_i^* : i \in I\}$ *is dense in* E^*. *If* $(X_i)_{i\in I}$ *is a family of Banach spaces which are (wasq), then* $X = (\bigoplus_{i\in I} X_i)_E$ *is (wasq).*

Proof. Let $x = (x_i)_{i\in I} \in S_X$. Our task is to find a weakly-null sequence

$(y_n) \subset S_X$ such that

$$\|x \pm y_n\|_E \to 1.$$

We may (and do) assume that $x_i \neq 0$ for all $i \in I$. By assumption, for every $i \in I$, there exist weakly-null sequences $(y_n^i) \subset S_{X_i}$ such that

$$\left\| \frac{x_i}{\|x_i\|} \pm y_n^i \right\| \to 1.$$

Just like in Proposition (2.2.11), we let $y_n = (\|x_i\| y_n^i)_{i\in I}$ and get $\|x \pm y_n\|_E \to 1$. Note that $\|y_n\|_E = 1$. Finally, let $x^* = (x_i^*)_{i\in I} \in X^*$ and $\varepsilon > 0$. Since $\text{span}\{e_i^* : i \in I\}$ is dense in E^* there is a finite set of indices $F \subset I$ such that $\left\|(\|x_i^*\|)_{i\in I\setminus F}\right\|_{E^*} < \varepsilon/2$. Let

$$x_F^* = \sum_{i\in F} x_i^* e_i^* \ .$$

Find $n_0 \in \mathbb{N}$ such that $|x_i^*(\|x_i\| y_n^i)| < \varepsilon/(2|F|)$ for all $i \in F$, whenever $n > n_0$. (Possible since (y_n^i) is weakly-null for every $i \in I$). We get

$$|x^*(y_n)| \leq |x_F^*(y_n)| + \|x^* - x_F^*\|_{E^*}\|y_n\|_E \leq \left|\sum_{i\in F} x_i^* \ (\|x_i\| y_n^i)\right| + \left\|(\|x_i^*\|)_{i\in I\setminus F}\right\|_{E^*} < \varepsilon$$

whenever $n > n_0$. Thus (y_n) is weakly-null.

For absolute sums of weakly octahedral spaces we have to work a bit harder.

Proposition (2.2.13)[2]: *Let I be a set, E a subspace of $\mathbb{R}^I$ with an absolute norm such that* $\text{span}\{e_i : i \in I\}$ *is dense in E, and $(X_i)_{i\in I}$ a family of Banach spaces which are weakly octahedral. Then their absolute sum $X = (\bigoplus_{i\in I} X_i)_E$ is weakly octahedral.*

Proof. Let $\varepsilon > 0$, let $x^1 = (x_i^1)_{i\in I}, \ldots, x^N = (x_i^N)_{i\in I} \in S_X$, and $x^* = (x_i^*)_{i\in I} \in B_{X^*}$ Our task here is to find $y \in S_X$ such that

$\|x^k + ty\|_E \geq (1 - \varepsilon)(|\sum_{i\in I} x_i^* \ (x_i^k)| + t)$ for all $t > 0$ and $k = 1, 2, \ldots, N$.

Let $z_i^* = \frac{x_i^*}{\|x_i^*\|}$ if $x_i^* \neq 0$ and $z_i^* = 0$ otherwise. By the weak octahedrality of X_i, for every $i \in I$, there exists a $y_i \in S_{X_i}$ such that

(2.1)

$$\left\|\frac{x_i^k}{\|x_i^k\|} + ty_i\right\|_E \geq (1 - \varepsilon/2)(\frac{|z_i^*(x_i^k)|}{\|x_i^k\|} + t) \; for\; all\; t > 0\; and\; k = 1,2,...,N.$$

If $x_i^k = 0$ for some $i \in I$, then take y_i to be any element from S_{X_i} . Now (2.1) implies that

$$\|x_i^k + ty_i\|_E \geq (1 - \varepsilon/2)(\frac{|x_i^*(x_i^k)|}{\|x_i^k\|} + t) \; for\; all\; t > 0\; and\; k = 1,2,...,N.$$

Since $\|x^*\| = \|(\|x_i^*\|)_{i\in I}\|_{E^*} \leq 1$, there is a list of reals $(\alpha_i)_{i\in I} \subset \mathbb{R}$ such that $\|(\alpha_i)_{i\in I}\|_E = \|(|\alpha_i|)_{i\in I}\|_E = 1$ and

$$\sum_{i\in I} \|x_i^*\| \cdot |\alpha_i| > \|x^*\|(1 - \frac{\varepsilon/2}{1 - \varepsilon/2}).$$

We take $y = (|\alpha_i|y_i)_{i\in I} \in S_X$ to get

$$\begin{aligned}
\|x^*\|\|x^k + ty\|_E &\geq \sum_{i\in I} \|x_i^*\| \cdot \left\|x_i^k + |\alpha_i|ty_i\right\| \\
&\geq (1-\varepsilon/2)\sum_{i\in I}\|x_i^*\|\left(\frac{|x_i^*(x_i^k)|}{\|x_i^*\|} + |\alpha_i|t\right) \\
&\geq (1-\varepsilon/2)\left(|\sum_{i\in I} x_i^*(x_i^k)| + \|x^*\|(1-\frac{\varepsilon/2}{1-\varepsilon/2})t\right) \\
&\geq (1-\frac{\varepsilon/2}{1-\varepsilon/2})(1-\varepsilon/2)\|x^*\|\left(|\sum_{i\in I} x_i^*(x_i^k)| + t\right) \\
&= \|x^*\|(1-\varepsilon)\left(|\sum_{i\in I} x_i^*(x_i^k)| + t.\right)
\end{aligned}$$

Dividing both sides by $\|x^*\|$ we get the desired inequality.

We have seen that for a sequence of non-trivial Banach spaces (X_i) the space $c_0(X_i)$ is always (asq). Similarly $\ell_1(X_i)$ is always octahedral.

Note that $X \bigoplus_p Y, 1 < p < \infty$, can never be (asq), because it fails the SD2P. But even though the SD2P property is stable by forming ℓ_1 sums, it turns out that the ℓ_1 sum of Banach spaces can never be (asq).

Lemma (2.2.14)[2]: *Let X and Y be nontrivial Banach spaces. Then $X \bigoplus_1 Y$ is never (asq).*

Proof. Let $Z = X \bigoplus_1 Y$, $x \in S_X$, and $y \in S_Y$. Consider norm 1 elements $z_1 = \left(-\frac{1}{3}x, \frac{2}{3}y\right)$ and $z_2 = \left(\frac{2}{3}x, -\frac{1}{3}y\right)$. Assume on the contrary that there is $a\ w = (w_x, w_y) \in S_Z$ with $\|z_i \pm w\| \leq 1 + \frac{1}{9}$. Then

$$\|w_x\| + \left\|\frac{2}{3}y\right\| \leq \frac{1}{2}\left(\left\|-\frac{1}{3}x + w_x\right\| + \left\|\frac{2}{3}y + w_y\right\| + \left\|\frac{1}{3}x + w_x\right\|) + \left\|\frac{2}{3}y - w_y\right\|\right)$$
$$\leq max\{\|z_1 + w\|, \|z_1 - w\|\} \leq 1 + \frac{1}{9}$$

so that $\|w_x\| \leq \frac{1}{3} + \frac{1}{9}$. Similarly $\|w_y\| \leq \frac{1}{3} + \frac{1}{9}$. We get $\|w\| < 1$ which is a contradiction.

Proposition (2.2.15)[2]: *Let X and Y be nontrivial Banach spaces and* $1 \leq p < \infty$.

(i) *If $X \bigoplus_p Y$ is (lasq), then X is (lasq).*

(ii) *If $X \bigoplus_p Y$ is (wasq), then X is (wasq).*

Proof. (i). The function $f(x) = x^{1/p}$ is uniformly continuous on [0, 2] so given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$.

Also the function $g(x) = x^p$ is continuous at $x = 1$ so there exists $\eta > 0$ such that $|g(1) - g(y)| \leq \delta$ whenever $|1 - y| \leq \eta$.

Let $X \bigoplus_p Y$ be (lasq). Assume $x \in S_X$. Then there is $(u, v) \in S_{X \bigoplus_p Y}$ such that

$$\|(x,0) \pm (u,v)\|_p = (\|x \pm u\|^p + \| v\|^p)^{1/p} \leq 1 + \eta.$$

(Note that $u \neq 0$, else $\|(x,v)\| = 2^{1/p} > 1 + \varepsilon$.) We have (since $t \mapsto t^p$ is increasing)

$$\|x \pm u\|^p + \|v\|^p \leq (1+\eta)^p \leq 1^p + \delta = 1 + \delta$$

hence

$$\|x \pm u\|^p \leq 1 + \delta - \|v\|^p = \|u\|^p + \|v\|^p - \|v\|^p + \delta = \|u\|^p + \delta.$$

Taking *p*-th roots we get

$$\|x \pm u\| \leq \|u\| + \varepsilon$$

since $|\|u\|^p + \delta - \|u\|^p| = \delta$. Let $z = u/\|u\|$. Then

$$\|x \pm z\| \leq \|x \pm u\| + \|z - u\| \leq \|u\| + \varepsilon + 1 - \|u\| = 1 + \varepsilon.$$

(ii). The proof is similar to (i). Indeed, for $\varepsilon_n = \frac{1}{n}$ find the sequence η_n and observe that if a sequence (u_n, v_n) converges weakly to (0, 0) in $\bigoplus_p Y$, then u_n converges weakly to 0 in *X*.

We end this section by showing that for finite ℓ_∞ sums we only need to assume that only one of the spaces is (lasq), (wasq) or (asq).

Proposition (2.2.16)[2]: *Let X and Y be nontrivial Banach spaces.*

(i) $X \bigoplus_\infty Y$ *is (lasq) if and only if either X or Y is (lasq).*

(ii) $X \bigoplus_\infty Y$ *is (wasq) if and only if either X or Y is (wasq).*

(iii) $X \bigoplus_\infty Y$ *is (asq) if and only if either X or Y is (asq).*

Proof. We will prove it only for (asq) spaces – others will follow similarly.

Suppose that $Z = X \bigoplus_\infty Y$ is (asq). Let $x_1, x_2, \ldots, x_N \in S_X$ and $y_1, y_2, \ldots, y_N \in S_Y$. Then (x_i, y_i) is in S_Z for every $i = 1, 2, \ldots, N$ and by our assumption there is a sequence $z_n = (u_n, v_n)$ in B_Z such that $\|(x_i, y_i) \pm (u_n, v_n)\| \to 1$ for every $i = 1,2,\ldots,N$ and $||z_n|| \to 1$. Since $||z_n|| \to 1$there is a subsequence such that either $||u_n|| \to 1$ or $||v_n|| \to 1$.

Thus one of the spaces X or Y must be (asq).

Suppose now that X is (asq). Let $z_i = (x_i, y_i) \in S_Z$ for $i = 1,2,\ldots,N$.

Using Proposition (2.1.17), we can find a sequence $(u_n) \subset B_X$ such that $||u_n|| \to 1$ and $\|x_i \pm u_n\| \to 1$ for every $i = 1,2,\ldots,N$. Put $z_n = (u_n, 0)$. Then $\|z_n\| = \|u_n\| \to 1$ and $\|z_i \pm z_n\| = max\{\|x_i \pm u_n\|, y_i\} \to max\{1, \|y_i\|\} = 1$ for every $i = 1,2,\ldots,N$. Thus Z is (asq).

We explore the connection between (asq) spaces and the intersection property introduced.

A Banach space X has the *intersection property* (IP) if for every $\varepsilon > 0$ there exist $x_1, x_2, \ldots, x_N$ in X with $\|x_i\| < 1, i = 1,2,\ldots,N$, such that if $y \in X$ with $\|x_i \pm y\| \leq 1$, for every $i = 1,2,\ldots,N$, then $\|y\| \leq \varepsilon$.

We will say that X *ε-fails the IP*, $0 < \varepsilon < 1$, if for all $x_1, x_2, \ldots, x_N$ in X with $\|x_i\| < 1, i = 1,2,\ldots,N$, there exists a $y \in X$ such that $\|x_i \pm y\| \leq 1$ and $\|y\| > \varepsilon$.

Theorem (2.2.17)[2]: *A Banach space X is (asq) if and only if X ε-fails the IP for all* $0 < \varepsilon < 1$.

Proof. Assume X is (asq) and let $0 < \varepsilon < 1$ be fixed.

Assume $(x_i)_{i=1}^{N} \subset \mathring{B}_X$. Choose $\delta > 0$ such that $(1 + \delta)^2\varepsilon \leq 1$ and $(1 + \delta)\|x_i\| \leq 1 \; for \; i = 1,2,\ldots,N$.

Let $E = span\{(x_i)_{i=1}^{N}\}$. By Theorem (2.1.21), there exists $y \in S_X$ such that

$$\|x + ry\| \leq (1 + \delta) max(\|x\|, |r|)$$

for all $x \in E$ and all scalars r. In particular,

$$\|x_i \pm (1 + \delta)\varepsilon y\| \le (1 + \delta) max(\|x_i\|, (1 + \delta)\varepsilon) \le 1$$

and $\|(1 + \delta)\varepsilon y\| = (1 + \delta)\varepsilon > \varepsilon$.

Conversely, assume X ε-fails the IP for $0 < \varepsilon < 1$ and let $x_1, x_2, \ldots, x_N \in S_X$. Let $\varepsilon > 0$. Since $z_i = \frac{x_i}{1+\varepsilon} \in \mathring{B}_X$ there exists a $y \in X$ with $\|y\| > 1 - \varepsilon$ such that $\|z_i \pm y\| \le 1$. Note that y is the midpoint of the line segment $[y + z_i, y - z_i]$ hence $\|y\| \le 1$. We get

$$\|x_i + y\| \le \|x_i - z_i\| + \|z_i + y\| \le 1 - \frac{1}{1 + \varepsilon} + 1 = 1 + \frac{\varepsilon}{1 + \varepsilon} < 1 + \varepsilon$$

and

$$\left\|x_i + \frac{y}{\|y\|}\right\| \le \|x_i + y\| + \left\|y - \frac{y}{\|y\|}\right\| \le 1 + 2\varepsilon.$$

From Proposition (2.1.18) we conclude that X is (asq).

Example (2.2.18)[2]: The space $X = \ell_\infty(C_\Sigma(S^m))$ is (asq) but not a proper M-ideal in any superspace.

Here S_m is the Euclidean sphere in R^{m+1} and

$$C_\Sigma(S^m) = \{f \in C(S^m) : f(s) = -f(-s)\ \forall s \in S^m, \}$$

where $C(S^m)$ is the space of continuous functions on S^m.

It is proved that this X not a proper M-ideal in any superspace. A small adjustment to the proof shows that X ε-fails the IP for every $0 < \varepsilon < 1$.

Example (2.2.19)[2]: For every $0 < \varepsilon < 1$ there exists a Banach space which is not (lasq), but ε-fails the IP.

Let $r = 3/(1 - \varepsilon)$ and consider the following G-space:

$$X = \{f \in C[0,1]: f(0) = rf(1)\}.$$

X is not (lasq): Let $f(x) = 1$ on $[0,\frac{1}{r}]$ and $f(x) = \frac{r+1}{r} - x$ on $[\frac{1}{r}, 1]$. If $g \in B_X$ with $|f(x) \pm g(x)| < 1 + \delta$, then $|g(x)| < (1 - \frac{1}{r}) + \delta$ everywhere. We cannot have $\|f \pm g\| < 1 + \delta$ and $\|g\| > 1 - \delta$ when $\delta \leq \frac{1}{2r}$.

X ε-fails the IP: First we note that if $f \in X$ and $\|f\| \leq 1$, then $|f(1)| \leq \frac{1}{r}$. If not then $|f(0)| = |rf(1)| = r|f(1)| > 1$.

Let $f_1, f_2, \dots, f_N \in B_X$ and $0 < \varepsilon < 1$. Since $|f_i(1)| \leq \frac{1}{r}$, there exists neighborhood of 1, say $(a,1]$, where $|f_i(x)| < \frac{2}{r}$. Define g such that supp $g \subset (a, 1)$ and $\|g\| = \varepsilon + \frac{1}{r}$. For $x \in (a, 1)$ we have

$$|f_i(x) \pm g(x)| < \frac{2}{r} + \varepsilon + \frac{1}{r} = \varepsilon + 1 - \varepsilon = 1.$$

Hence $\|f_i \pm g\| \leq 1$.

Next we will show that every (asq) space contains a separable subspace which is (asq).

Proposition (2.2.20)[2]: *If X is (asq), then for every separable subspace Y of X there exists a separable subspace Z with $Y \subset Z \subset X$ and Z is (asq).*

Proof. Let $Y \subset X$ and let $\varepsilon_n = 2 - n$.

Let A_1 be a countable dense set in S_Y. For each finite family G in A_1 find y_G in S_X such that $\|x \pm y_G\| < 1 + \varepsilon_1$ for all $x \in G$. Let Y_1 be the closure of span$\{Y, (y_G)\}$. Y_1 is separable.

Let A_2 be a countable dense set in S_{Y_1}. For each finite family G in A_2 find y_G in S_X such that $\|x \pm y_G\| < 1 + \varepsilon_2$ for all $x \in G$. Let Y_2 be the closure of span$\{Y_1, (y_G)\}$. Y_2 is separable.

We continue in the same fashion and let $Z = \overline{\cup Y_n}$.

Let $z_1, z_2, \ldots, z_N \in S_Z$ and $\varepsilon > 0$. Choose k such that $\varepsilon_k < \varepsilon/2$ and find $x_1, x_2, \ldots, x_N$ in A_k with $\|x_i - z_i\| < \varepsilon/2$. Then there exists a y in $S_{Y_{k+1}} \subset S_Z$ with $\|z_i \pm y\| < 1 + \varepsilon$ for $i = 1, 2, \ldots, N$.

Proposition (2.2.21)[2]: *If X is (lasq), then for every separable subspace Y of X there exists a separable subspace Z with $Y \subset Z \subset X$ and Z is (lasq).*

Proof. Same idea as for (asq), but we only need to consider single parent families.

Chapter 3

Banach Spaces and Superprojectivity

We show that the class of superprojective spaces is stable under finite products, certain unconditional sums, certain tensor products, and other operations, providing new examples.

Section (3.1): Some Properties of Superprojective Spaces

A Banach space X is called *subprojective* if every (closed) infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X, and X is called *superprojective* if every infinite-codimensional subspace of X is contained in an infinite-codimensional subspace complemented in X. These two classes of Banach spaces were introduced by Whitley

There are many examples of subprojective spaces, like ℓ_p for $1 \leq p < \infty$, $L_p(0,1)$for $2 \leq p < \infty$, C(K) with K a scattered compact and some Lorentz and Orlicz spaces. It is not difficult to show that subspaces of subprojective spaces are subprojective, and quotients of superprojective spaces are superprojective and, as a consequence of the duality relations between subspaces and quotients, a reflexive space is sub-projective (superprojective) if and only if its dual space is superprojective (subprojective), which provides many examples of reflexive superprojective spaces. However, the only examples of non-reflexive superprojective spaces previously known are the C(K) spaces with K a scattered compact and their infinite-dimensional quotients.

Some of the duality relations between subprojective and superprojective spaces are known to fail in general:

a) X being subprojective does not imply that X^* is superprojective, for instance for $X = c_0$and $X^* = \ell_1$.

b) X^* being subprojective does not imply that X is superprojective, for instance for the hereditarily indecomposable space obtained whose dual is isomorphic to ℓ_1.

However we do not know if the remaining relations are valid:

(a') Does X being superprojective imply that X^* is subprojective?

(b') Does X^* being superprojective imply that X is subprojective?

The answer to these two questions is likely negative, but we know of few examples of non-reflexive super-projective spaces to check, and none of them is a dual space.

Oikhberg and Spinu have studied the stability properties of subprojective spaces under vector sums, tensor products and other operations, obtaining plenty of new examples of subprojective spaces.

We will begin with some auxiliary results shows some properties of subprojective and superprojective spaces, such as the fact that superprojective spaces cannot contain copies of ℓ_1, which restricts the search for non-reflexive examples of these spaces, and we also characterise the superprojectivity of some projective tensor products. We show several stability results for the class of superprojective spaces under finite products, certain unconditional sums and certain tensor products, and we provide new examples of superprojective spaces.

The dual space of a Banach space X is X^*, and the action of $x^* \in X^*$ on $x \in X$ is written as $\langle x^*, x\rangle$. Given a subset M of a Banach space X, its annihilator in X^* will be denoted by $M^\perp$ if M is a subset of X^*, its annihilator in X will be denoted by $M_\perp$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X, then $[x_n : n \in \mathbb{N}]$will denote the closed linear span of $(x_n)_{n\in\mathbb{N}}$ in X. The injective and projective tensor products of X and Y are respectively denoted by $X \hat{\otimes}_\varepsilon Y$ and $X \hat{\otimes}_\pi Y$.

Operators will always be bounded. The identity operator on X is denoted by I_X. Given an operator $T: X \to Y$, $N(T)$ and $R(T)$ denote the kernel and the range of T, and $T^*: Y^* \to X^*$ denotes its conjugate operator. An operator $T: X \to Y$ is strictly singular if $T|_M$ is an isomorphism only if M is finite-dimensional; and T is strictly cosingular if there is no operator $Q: Y \to Z$ with Z infinite-dimensional such that QT is surjective or, equivalently, if there is no infinite-codimensional (closed) subspace N of Y such that $R(T) + N = Y$.

The way that superprojective Banach spaces are defined means that we will be dealing with infinite-codimensional subspaces and their induced quotients often, so we will adopt the following definition.

Definition (3.1.1)[3]: We will say that an operator $T: X \to Y$ is a surjection if T is surjective and Y is infinite-dimensional.

The following results will be useful when dealing with complemented subspaces, surjections and superprojective spaces. Similar results were given to study improjective operators.

Proposition (3.1.2)[3]: *For a Banach space X, the following are equivalent:*

(i) *X is superprojective;*

(ii) *For any surjection $T: X \to Y$, there exists another surjection $S: Y \to Z$ such that N(ST)is complemented in X.*

Proof. For the direct implication, let $T: X \to Y$ be a surjection, so that $N(T)$ is infinite-codimensional in *X*. By the superprojectivity of *X*, $N(T)$ is contained in a complemented, infinite-codimensional subspace *M* of *X*, and clearly *T(M)* is closed in *Y*. Thus the quotient map *Q* from *Y* onto $Y/T(M)$ is a surjection such that $N(QT) = M$ is complemented in *X*.

For the converse implication, let *M* be an infinite-codimensional subspace of *X*, so that $Q_M: X \to X/M$ is a surjection. Then there exists another surjection $S: X/M \to Z$ such that $N(SQ_M)$ is infinite-codimensional and complemented in *X*, and contains *M*.

The next result allows to push the complementation of a subspace through an operator under certain conditions.

Proposition (3.1.3)[3]: *Let X,Y and Z be Banach spaces and let $T: X \to Y$ and $S: Y \to Z$ be operators such that ST is a surjection and N(ST) is complemented in X. Then N(S) is complemented in Y.*

Proof. Let *H* be a subspace of *X* such that $X = N(ST) \oplus H$. Since $ST: X \to Z$ is a surjection, $ST|_H$ must be an isomorphism onto *Z*; in particular, $T|_H$ is an isomorphism and $Y = N(S) \oplus T(H)$, as proved by the projection $T(ST|_H)^{-1}S: Y \to Y$.

A simple consequence of Propositions (3.1.2) and (3.1.3) is the fact that the class of superprojective spaces is stable under quotients.

Proposition (3.1.4)[3]: *Let X be a superprojective Banach space and let $T: X \to Y$ be a surjection. Then Y is superprojective.*

Proof. Let $S: Y \to Z$ be a surjection; then ST is a surjection and, by Proposition (3.1.2), there exists another surjection $R: Z \to W$ such that $N(RST)$ is complemented in X. By Proposition (3.1.3), $N(RS)$ is complemented in Y, which means, again by Proposition (3.1.2), that Y is superprojective.

We will state a technical observation on the behaviour of surjections on spaces that have a complemented superprojective subspace.

Proposition (3.1.5)[3]: *Let X be a Banach space, let $P: X \to X$ be a projection with $P(X)$ superprojective and let $S: X \to Y$ be a surjection such that* SP *is not strictly cosingular. Then there exists another surjection $R: Y \to Z$ such that $N(RS)$ is complemented in X.*

Proof. Let $J: P(X) \to X$ be the natural inclusion; then $SP = SJP$ is not strictly cosingular, so neither is $SJ: P(X) \to Y$. Therefore, there exists a quotient map $Q: Y \to W$ such that QSJ is a surjection, and Proposition (3.1.2) provides another surjection $R: W \to Z$ such that $N(RQSJ)$ is complemented in $P(X)$; by Proposition (3.1.3), $N(RQS)$ is complemented in X, where $RQ: Y \to Z$ is a surjection.

The following result gives some simple but useful necessary conditions for a Banach space X to be subprojective or superprojective.

Proposition (3.1.6)[3]: Let X and Z be infinite-dimensional Banach spaces.

(i) If $J: Z \to X$ is a strictly cosingular embedding, then X is not subprojective.

(ii) If $Q: X \to Z$ is a strictly singular surjection, then X is not superprojective.

Proof. (i) If $X = M \oplus H$ with $M \subseteq J(Z)$, then $Q_H J$ is surjective. Since J is strictly cosingular, H is finite-codimensional and M is finite-dimensional.

(ii) If $X = M \oplus H$ with $N(Q) \subseteq M$, then $Q|_H$ is an embedding. Since Q is strictly singular, H is finite-dimensional.

Proposition (3.1.6) has several straightforward consequences. Proposition (3.1.7) was proved for subprojective spaces with the same example but a different argument. Here we extend it to superprojective spaces. Recall that a class C of Banach spaces satisfies the *three-space property*if a Banach space X belongs to C whenever M and X/M belong to C for some subspace M of X.

Proposition (3.1.7)[3]: *The classes of subprojective and superprojective spaces do not satisfy the three-space property.*

Proof. Let $1 < p < \infty$ and recall that ℓ_p is both subprojective and superprojective. Let Z_p be introduced by the Kalton–Peck space Then there exists an exact sequence

$$0 \to \ell_p \overset{i}{\to} Z_p \overset{q}{\to} \ell_p \to 0$$

in which i is strictly cosingular and q is strictly singular. By Proposition (3.1.6), Z_p is neither subprojective nor superprojective.

Proposition (3.1.8)[3]: *Let X be a Banach space containing a subspace isomorphic to ℓ_1. Then X is not superprojective and X^* is not subprojective.*

Proof. If X contains a subspace isomorphic to ℓ_1, then there exists a surjective operator $Q: X \to \ell_2$which is 2-summing, therefore weakly compact and completely continuous, therefore strictly singular: Indeed, if $Q|_M$ is an isomorphism, then *M* is reflexive and weakly convergent sequences in *M* are convergent, so *M* is finite-dimensional. By Proposition (3.1.6), *X* is not superprojective.

For the second part, observe that $Q^{**}: X^{**} \to \ell_2$ is also 2-summing. Then Q^{**} is strictly singular, hence $Q^*: \ell_2 \to X^*$ is a strictly cosingular embedding.

Proposition (3.1.8) allows to fully characterise the superprojectivity of *C(K)* spaces. Recall that a compact space is called *scattered* if each of its non-empty subsets has an isolated point.

Corollary (3.1.9)[3]: *Let K be a compact set. Then C(K) is superprojective if and only if K is scattered*

Proof. If *K* is scattered, then *C(K)* is superprojective. On the other hand, if *K* is not scattered, then *C(K)* contains a copy of $\ell_1$and cannot be superprojective by Proposition (3.1.8).

It also follows immediately that certain tensor products cannot be superprojective.

Corollary (3.1.10)[3]: *Let X and Y be Banach spaces and suppose that X admits an unconditional finite-dimensional decomposition and* $L(X, Y^*) \neq K(X, Y^*)$. *Then* $X \hat{\otimes}_\pi Y$ *is not superprojective.*

Proof. Note that $(X \hat{\otimes}_\pi Y)^* \equiv L(X, Y^*)$. Since $L(X, Y^*) \neq K(X, Y^*)$, we have that $L(X, Y^*)$contains ℓ_∞, hence $X \hat{\otimes}_\pi Y$ contains a (complemented) copy of ℓ_1.

Since the spaces ℓ_p have an unconditional basis and are subprojective and superprojective for $1 < p < \infty$, we can now characterise the superprojectivity of the tensor products $\ell_p \hat{\otimes}_\pi \ell_q$.

Corollary (3.1.11)[3]: *Let* $1 < p, q < \infty$. *Then the following are equivalent:*

(i) $\ell_p \hat{\otimes}_\pi \ell_q$ *is superprojective;*

(ii) $\ell_p \hat{\otimes}_\pi \ell_q$*is reflexive;*

(iii) $L(\ell_p, \ell_q^*) = K(\ell_p, \ell_q^*)$*;*

(iv) $p > q/(q-1)$.

Proof. We have that $\ell_p \hat{\otimes}_\pi \ell_q$ is reflexive if and only if $L(\ell_p, \ell_q^*) = K(\ell_p, \ell_q^*)$ if and only if $p > q/(q-1)$. If $L(\ell_p, \ell_q^*) \neq K(\ell_p, \ell_q^*)$, then $\ell_p \hat{\otimes}_\pi \ell_q$ is not superprojective by Corollary (3.1.10); otherwise, $\ell_p \hat{\otimes}_\pi \ell_q$is reflexive and $\ell_p \hat{\otimes}_\pi \ell_q = (\ell_p^* \hat{\otimes}_\varepsilon \ell_q^*)^*$, so $\ell_p^* \hat{\otimes}_\varepsilon \ell_q^*$ is reflexive and subprojective and $\ell_p \hat{\otimes}_\pi \ell_q$ is superprojective.

Corollary (3.1.12)[3]: $\ell_p \hat{\otimes}_\pi \ell_q$ *is not superprojective for any* $1 \leq p, q \leq \infty$.

Proof. If p is either 1or strictly greater than 2, then L_p itself is not superprojective, so neither is $\ell_p \hat{\otimes}_\pi \ell_q$, and similarly for q. Thus, we are only concerned with the case $1 < p, q \leq 2$, but then both L_p and L_q^* contain complemented copies of ℓ_2, so $L(L_p, L_q^*) \neq K(L_p, L_q^*) and$ $\ell_p \hat{\otimes}_\pi \ell_q$ is not superprojective by Corollary (3.1.10).

Section (3.2): Stability Results for Superprojective.

We will show some stability results for the class of superprojective spaces and proves that the direct sum of two superprojective Banach spaces is again superprojective.

Proposition (3.2.1)[3]: *Let X and Y be Banach spaces. Then $X \oplus Y$ is superprojective if and only if both X and Y are superprojective.*

Proof. X and Y are quotients of $X \oplus Y$; if $X \oplus Y$ is superprojective, then so are X and Y by Proposition (3.1.4).

Conversely, assume that X are Y are both superprojective, and define the projections $P_X: X \oplus Y \to X \oplus Y$, with range X and kernel Y, and $P_Y: X \oplus Y \to X \oplus Y$, with range Y and kernel X. Take surjection $S: X \oplus Y \to Z$. Then $S = SP_X + SP_Y$ is not strictly cosingular, so either SP_X or SP_Y is not strictly cosingular; without loss of generality, we will assume that it is SP_X. By Proposition (3.1.5), there exists another surjection $R: Z \to W$ such that *N(RS)* is complemented in X⊕Y, which finishes the proof by Proposition (3.1.2).

Recall that an operator $T: X \to Y$ is *upper semi-Fredholm* if *N(T)* is finite-dimensional and *R(T)* is closed, and T is *lower semi-Fredholm* if *R(T)* is finite-codimensional (hence closed). Note that T is lower semi-Fredholm if and only if T^* is upper semi-Fredholm.

Theorem (3.2.2)[3]: *Let X be a Banach space, let Λ be a well-ordered set and let $(P_\lambda)_{\lambda \in \Lambda}$ and $(Q_\lambda)_{\lambda \in \Lambda}$ be bounded families of projections on X such that:*

(i) $P_\lambda^* x^* \underset{\lambda}{\longrightarrow} x^*$ *for every* $x^* \in X^*$;

(ii) *$P_\mu P_\nu = P_{min\{\mu,\nu\}}$ and $Q_\mu Q_\nu = Q_{\min\{\mu,\nu\}}$ for every $\mu, \nu \in \Lambda$;*

(iii) *$Q_\mu P_\nu = P_\nu Q_\mu$ for every $\mu, \nu \in \Lambda$, and $Q_\mu P_\nu = P_\nu$ if $\mu \geq \nu$;*

(iv) *$Q_\lambda(X)$ is superprojective for every $\lambda \in \Lambda$;*

(v) *For every unbounded strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of elements in Λ and every sequence $(x_k^*)_{k\in\mathbb{N}}$ of non-null elements in X^* such that $x_1^* \in R(P_{\lambda_1}^*)$ and $x_k^* \in R\big(P_{\lambda_k}^*(I - Q_{\lambda_{k-1}}^*)\big)$ for $k > 1$, the subspace $[x_k^*: k \in \mathbb{N}]_\perp$ is contained in a complemented infinite-codimensional subspace of X.*

Then X is superprojective.

Here, an unbounded sequence in Λ is one that does not have an upper bound within Λ. Also, this result is only really interesting if Λ does not have a maximum element; otherwise, if λ is the maximum of Λ, then $P_\lambda = I_X$ by condition (i) and $Q_\lambda = Q_\lambda P_\lambda = P_\lambda = I_X$ by condition (iii), so $X = Q_\lambda(X)$ is already superprojective by condition (iv).

Proof. Let *M* be an infinite-codimensional subspace of *X* and let us denote its natural quotient map by $S: X \to X/M$. If there exists $\lambda \in \Lambda$ such that SQ_λ is not strictly cosingular, then Proposition (3.1.5) provides another surjection $R: X/M \to Z$ such that *N(RS)* is complemented in *X*. Since *N(RS)* is infinite-codimensional and contains *M* we are done.

Otherwise, assume that SQ_λ is strictly cosingular for every $\lambda \in \Lambda$. Let $C \geq 1$ be such that $\|P_\lambda\| \leq C$ and $\|Q_\lambda\| \leq C$ for every $\lambda \in \Lambda$, and let $\varepsilon = 1/8C^3 > 0$. We will construct a strictly increasing sequence $\lambda_1 < \lambda_2 < \ldots$ of elements in Λ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of norm-one elements in $M^\perp \subseteq X^*$ such that $\|Q_{\lambda_{k-1}}^* x_k^*\| < 2^{-k}\varepsilon$ and $\|P_{\lambda_k}^* x_k^* - x_k^*\| < 2^{-k}\varepsilon$ for every $k \in \mathbb{N}$, where we write $Q_{\lambda_0} = 0$ for convenience. To this end, let $k \in \mathbb{N}$, and assume that λ_{k-1}

has already been obtained. By hypothesis, $Q^*_{\lambda_{k-1}}S^* = (SQ_{\lambda_{k-1}})^*$ is not an isomorphism, where $S^*: (X/M)^* \to X^*$ is an isometric embedding with range $M^\perp$, so there exists $x^*_k \in M^\perp$ such that $\|x^*_k\| = 1$ and $\|Q^*_{\lambda_{k-1}}x^*_k\| < 2^{-k}\varepsilon$, and then there is $\lambda_k > \lambda_{k-1}$such that $\|P^*_{\lambda_k}x^*_k - x^*_k\| < 2^{-k}\varepsilon$ by condition (i), which finishes the inductive construction process. Let $H = [x^*_k : k \in \mathbb{N}] \subseteq X^*$; then $H_\perp$ is infinite-codimensional and contains M.

It is easy to check that the operators $T_k := (I - Q_{\lambda_k} - 1)P_{\lambda_k}$ are projections with norm $\|T_k\| \leq (1 + C)C \leq 2C^2$, and that $T_iT_j = 0\, if\ i \neq j$.

Let now $z^*_k = T^*_k(x^*_k) = P^*_{\lambda_k}(I - Q^*_{\lambda_{k-1}})x^*_k$ for each $k \in \mathbb{N}$; then

$$\|z^*_k - x^*_k\| \leq \|P^*_{\lambda_k} x^*_k - x^*_k\| + \|P^*_{\lambda_k} Q^*_{\lambda_{k-1}}x^*_k\| < 2^{-k}\varepsilon + 2^{-k}\varepsilon C \leq 2^{1-k}\varepsilon C < 1/2,$$

so $1/2 < \|z^*_k\| < 3/2$ for every $k \in N$. If we take $x_k \in X$ such that $\|x_k\| < 2$ and $\langle z^*_k, x_k\rangle = 1$ for each $k \in N$, and define $z_k = T_k x_k$, it follows that

$$\langle z^*_k, z_k\rangle = \langle z^*_k, T_k x_k\rangle = \langle T^*_k\ z^*_k, x_k\rangle = \langle z^*_k, x_k\rangle = 1$$

for every $k \in \mathbb{N}$ and

$$\langle z^*_i, z_j\rangle = \langle T^*_i z^*_i, T_j z_j\rangle = \langle z^*_i, T_i T_j z_j\rangle = 0$$

If $i \neq j$, which makes $(z^*_k, z_k)_{n\in\mathbb{N}}$ a biorthogonal sequence in (X^*, X). In the spirit of the principle of small perturbations, let $K: X \to X$ be the operator defined as $K(x) = \sum_{n=1}^{\infty}\langle x^*_n - z^*_n, x\rangle z_n$; then

$$\sum_{n=1}^{\infty}\|x^*_n - z^*_n\|\|z_n\| < \sum_{n=1}^{\infty}(2^{1-n}\varepsilon C)(4C^2) = \sum_{n=1}^{\infty}2^{-n} = 1,$$

so K is well defined and $U = I + K$ is an isomorphism on X. Moreover, $K^*: X^* \to X^*$ is defined as $K^*(x^*) = \sum_{n=1}^{\infty}\langle x^*, \mathrm{z_n}\rangle(\mathrm{x^*_n} - \mathrm{z^*_n})$, so $K^*(z^*_k) = x^*_k - z^*_k$

and $U^*(z_k^*) = x_k^*$ for every $k \in N$. Let $Z = [z_k^*: k \in \mathbb{N}]$; then $U^*(Z) = H$ and $U(H_\perp) = Z_\perp$.

Next we will show that Z is weak* closed in X^*. Note first that $T_j P_{\lambda_i} = \left(I - Q_{\lambda_{j-1}}\right) P_{\lambda_j} P_{\lambda_i} = \left(I - Q_{\lambda_{j-1}}\right) P_{\lambda_j} = T_j \ if\ i \geq j$, and $T_j P_{\lambda_i} = (I - Q_{\lambda_{j-1}}) P_{\lambda_j} P_{\lambda_i} = (I - Q_{\lambda_{j-1}}) P_{\lambda_i} = 0$ otherwise. Given that $z_k^* \in R(T_k^*)$ for every $k \in \mathbb{N}$, this means that $P_{\lambda_i}^* z_j^* = z_j^* \ if\ i \geq j$ and $P_{\lambda_i}^* z_j^* = 0$ otherwise, so $P_{\lambda_k}^*(Z) = [z_1^*, \ldots, z_k^*]$, which is finite-dimensional, for every $k \in N$. Let x^* be a weak* cluster point of Z; then $P_{\lambda_k}^* x^* \in P_{\lambda_k}^*(Z) \subseteq Z$ and $P_{\lambda_k}^* x^* \underset{k}{\rightarrow} x^*$ by condition (i), so $x^* \in Z$ and Z is indeed weak* closed. The fact that $H = U^*(Z)$ implies that H is weak* closed, as well.

This means, in turn, that no Q_λ^* can be an isomorphism on H for any $\lambda \in \Lambda$. To see this, consider the natural quotient $Q_{H_\perp}: X \to X/H_\perp$, where $X/H_\perp$ is infinite-dimensional. Since $M \subseteq H_\perp$, the operator $Q_{H_\perp}$ factors through $S = Q_M: X \to X/M$ and, since SQ_λ is strictly cosingular for every $\lambda \in \Lambda$ by our initial hypothesis, it follows that $Q_{H_\perp} Q_\lambda$ cannot be surjective for any $\lambda \in \Lambda$, or even lower semi-Fredholm; equivalently, Q_λ^* cannot be upper semi-Fredholm on ${H_\perp}^\perp$ for any $\lambda \in \Lambda$, where ${H_\perp}^\perp = H$ because H is weak* closed.

Finally, we will check that the sequence $(\lambda_k)_{k \in \mathbb{N}}$ is unbounded. Assume, to the contrary, that there existed some $\lambda \in \Lambda$ such that $\lambda_k \leq \lambda$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we would have $T_k Q_\lambda = (I - Q_{\lambda k-1}) P_{\lambda k} Q_\lambda = (I - Q_{\lambda k-1}) P_{\lambda_k} = T_k$, so $Q_\lambda^* z_k^* = z_k^*$ and Q_λ^* would be an isomorphism on Z. But then $Q_\lambda^* U^{-1^*}$ would be an isomorphism on H, where $U^{-1} = I - U^{-1}K$ is a compact perturbation of the identity, so Q_λ^* would be upper semi-Fredholm on H, a contradiction.

Now that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is known to be unbounded, condition (v) states that $Z_\perp$ is contained in a complemented infinite-codimensional subspace of X, and then so is $H_\perp = U^{-1}(Z_\perp)$.

Note that any sequence $(P_n)_{n\in\mathbb{N}}$ of projections in X satisfying the conditions of Theorem (3.2.2) effectively defines a Schauder decomposition for X, where the components are the ranges of each operator $P_n(I - P_{n-1}) = P_n - P_{n-1}$; equivalently, each P_n is the projection onto the sum of the first ncomponents. For the purposes of Theorem (3.2.2), these components need not be finite-dimensional.

Regarding condition (v), a further remark is in order. It may very well be the case that there are no unbounded strictly increasing sequences in Λ, for instance if $\Lambda = [0, \omega_1)$, where ω_1 is the first uncountable ordinal, in which case condition(v) is trivially satisfied and does not impose any additional restriction on X or the projections. In terms of the proof of Theorem (3.2.2), this means that SQ_λ must be eventually not strictly cosingular for some $\lambda \in \Lambda$, and this is so because Q_λ^* is an isomorphism on Z for any λ greater than the supremum of $(\lambda_k)_{k\in\mathbb{N}}$, so Q_λ^* is upper semi-Fredholm on H and SQ_λ is not strictly cosingular, as per the last paragraphs of the proof of Theorem (3.2.2).

We will not need the full strength of Theorem (3.2.2), projections $(P_\lambda)_{\lambda\in\Lambda} = (Q_\lambda)_{\lambda\in\Lambda}$ is involved.

Theorem (3.2.3)[3]: *Let X be a Banach space, let Λ be a well-ordered set and let $(P_\lambda)_{\lambda\in\Lambda}$ be a bounded family of projections on X such that:*

(i) $P_\lambda^* x^* \underset{\lambda}{\to} x^*$ *for every* $x^* \in X^*$*;*

(ii) $P_\mu P_\nu = P_{min\{\mu,\nu\}}$ *for every* μ, $\nu \in \Lambda$*;*

(iii) $P_\lambda(X)$ *is superprojective for every* $\lambda \in \Lambda$*;*

(iv) *For every unbounded strictly increasing sequence* $(\lambda_k)_{k\in\mathbb{N}}$ *of elements in* Λ *and every sequence* $(x_k^*)_{k\in\mathbb{N}}$ *of non-null elements in* X^* *such that* $x_1^* \in R(P_{\lambda_1}^*)$ *and* $x_k^* \in R(P_{\lambda_k}^* - P_{\lambda_{k-1}}^*)$ *for* $k > 1$*, the subspace* $[x_k^*: k \in \mathbb{N}]_\perp$ *is contained in a complemented infinite-codimensional subspace of X.*

Then X is superprojective.

Our first use of Theorems (3.2.2) and (3.2.3) will be to prove that the (infinite) sum of superprojective spaces, such as $\ell_p(X_n) or\ c_0(X_n)$, is also superprojective, if the sum is done in a "superprojective" way.

Definition (3.2.4)[3]: We will say that a Banach space $E \subseteq \mathbb{R}^{\mathbb{N}}$ is a solid sequence space if, for every $(\alpha_n)_{n\in\mathbb{N}} \in E$ and $(\beta_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $|\beta_n| \le |\alpha_n|$ for every $n \in \mathbb{N}$, it holds that $(\beta_n)_{n\in\mathbb{N}} \in E$ and $\|(\beta_n)_{n\in\mathbb{N}}\| \le \|(\alpha_n)_{n\in\mathbb{N}}\|$.

We will say that E is an unconditional sequence space if it is a solid sequence space and the sequence of canonical vectors $(e_i)_{i\in\mathbb{N}}$ is a normalised basis for *E*, where $e_i = (\delta_{ij})_{j\in\mathbb{N}}$.

If E is an unconditional sequence space, then its canonical basis $(e_n)_{n\in\mathbb{N}}$ is actually 1-unconditional, and its conjugate E^* can be identified with a solid sequence space itself in the usual way, where the action of $\beta = (\beta_n)_{n\in\mathbb{N}} \in E^*$ on $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in E$ is $\langle \beta, \alpha \rangle = \sum_{n=1}^{\infty} \beta_n \alpha_n$. If the canonical basis $(e_n)_{n\in\mathbb{N}}$ is shrinking in *E*, then E^* is additionally unconditional (the coordinate functionals are a basis for E^*).

We have the following construction.

Definition (3.2.5)[3]: Let *E* be a solid sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We will write $E(X_n)$ for the Banach space of all sequences $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ for which $(\|x_n\|)_{n\in\mathbb{N}} \in E$, with the norm $\|(x_n)_{n\in\mathbb{N}}\| = \|(\|x_n\|)_{n\in\mathbb{N}}\|_E$.

The identification of the dual of an unconditional sequence space with another solid sequence space can be carried up to the sum of spaces.

Proposition (3.2.6)[3]: *Let E be an unconditional sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then $E(X_n)^* \equiv E^*(X_n^*)$.*

Proof. Each $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$ clearly defines an element of $E(X_n)^*$, so we only have to show the converse identification.

Let $z^* \in E(X_n)^*$, let $J_n: X_n \to E(X_n)$ be the canonical inclusion of X_n into $E(X_n)$ for each $n \in N$ and let $x_n^* = J_n^*(z^*) \in X_n^*$ for each $n \in \mathbb{N}$; we will prove that $z^* = (x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$.

To prove that $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$, choose $x_n \in X_n$ such that $\|x_n\| = 1$ and $\langle x_n^*, x_n\rangle \geq \frac{1}{2}\|x_n^*\|$ for each $n \in \mathbb{N}$, and take any non-negative $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in E$. By the definition of $E(X_n)$, we have that $(\alpha_n x_n)_{n\in\mathbb{N}} \in E(X_n)$, so

$$\sum_{n=1}^{\infty}\|x_n^*\|\alpha_n \leq \sum_{n=1}^{\infty} 2\langle x_n^*, x_n\rangle\alpha_n = 2\sum_{n=1}^{\infty}\langle x_n^*, \alpha_n x_n\rangle = 2\sum_{n=1}^{\infty}\langle J_n^*(z^*), \alpha_n x_n\rangle$$

$$= 2\sum_{n=1}^{\infty}\|z^*, J_n(\alpha_n x_n)\| = 2\langle z^*, (\alpha_n x_n)_{n\in\mathbb{N}}\rangle \leq 2\|z^*\|\|(\alpha_n x_n)_{n\in\mathbb{N}}\|$$

$$= 2\|z^*\|\|\alpha\|.$$

This proves that $(\|x_n^*\|) \in E^*$ and, as a consequence, $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$.

Now, given $i \in \mathbb{N}$ and $x_i \in X_i$, we have $\langle (x_n^*)_{n\in\mathbb{N}}, J_i(x_i)\rangle = \langle x_i^*, x_i\rangle = \langle z^*, \mathrm{J_i}(\mathrm{x_i}),\rangle$ so $(x_n^*)_{n\in\mathbb{N}}$ and z^* coincide over the finitely non-null sequences of $E(X_n)$ and therefore $z^* = (x_n^*)_{n\in\mathbb{N}}$.

We will prove that the sum of superprojective spaces is also superprojective, if the sum is done in a superprojective way, which translates to the requirement that the space E governing the sum must be superprojective itself. This excludes ℓ_1 and, more generally, imposes that any unconditional basis in E be shrinking, for the same reasons that ℓ_1 is not superprojective.

Proposition (3.2.7)[3]: *Let X be a superprojective Banach space and let $(x_n)_{n\in\mathbb{N}}$ be an unconditional basis of X. Then $(x_n)_{n\in\mathbb{N}}$ is shrinking.*

Proof . If $(x_n)_{n\in\mathbb{N}}$ is unconditional but not shrinking, then X contains a (complemented) copy of ℓ_1 and cannot be superprojective by Proposition (3.1.6).

Theorem (3.2.8)[3]: *Let E be an unconditional sequence space and let $(x_n)_{n\in\mathbb{N}}$be a sequence of Banach spaces. Then $E(X_n)$ is superprojective if and only if all of* E *and X_n are superprojective.*

Proof. Let $X = E(X_n)$. All of E and X_n are quotients of X; if X is superprojective, then so are E and each X_n.

Assume now that E and each X_n are superprojective, and define the projections $P_n: X \to X$ as $P_n((x_n)_{n\in\mathbb{N}}) = (x_1, \ldots, x_n, 0, \ldots)$ for each $n \in \mathbb{N}$. We will prove that the sequence $(P_n)_{n\in\mathbb{N}}$ meets the criteria of Theorem (3.2.2). The fact that $(P_n)_{n\in\mathbb{N}}$ is associated with the natural Schauder decomposition of $X = E(X_n)$ is enough for condition (ii) to hold. For condition (iii), note that $P_n(X)$ is isometric to $\bigoplus_{i=1}^{n} X_i$, which is superprojective by Proposition (3.2.1). As for condition (i), E is superprojective and its canonical basis $(e_n)_{n\in\mathbb{N}}$ is unconditional, therefore shrinking by Proposition (3.2.1), so E^* is unconditional and $(P_n^*)_{n\in\mathbb{N}}$ is the sequence of projections associated with the natural Schauder decomposition of $E(X_n)^* \equiv E^*(X_n^*)$.

To prove condition (iv), let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of integers, let $T_1 = P_{n_1}$ and $T_k = P_{n_k} - P_{n_{k-1}}$ for $k > 1$, and let $x_k^* \in R(T_k^*)$ be non-null for each $k \in \mathbb{N}$, as in Theorem (3.2.1). Define $M = [x_k^* : k \in \mathbb{N}]_\perp$, which is infinite-codimensional. Then $x_k^* \in X^* \equiv E^*(X_n^*)$, so

$$x_k^* = (0, \dots, 0, z_{n_{k-1}+1}^*, \dots, z_{n_k}^*, 0, \dots),$$

where $z_i^* \in X_i^*$. Pick a normalised $z_i \in X_i$ such that $\langle z_i^*, z_i\rangle \geq \|z_i^*\|/2$ for each $i \in \mathbb{N}$, and consider the operator $J: E \to X$ defined as $J((\alpha_n)_{n\in\mathbb{N}}) = (\alpha_n z_n)_{n\in\mathbb{N}}$, which is an isometric embedding by the definition of $X = E(X_n)$.

We claim that $Q_M J: E \to X/M$ is a surjection. Indeed, given $x = (x_n)_{n\in\mathbb{N}} \in X$, with each $x_n \in X_n$, let $\alpha_n = \langle z_n^*, x_n\rangle / \langle z_n^*, z_n\rangle \; if \; z_n^* \neq 0$, else $\alpha_n = 0$, for each $n \in \mathbb{N}$, and define $\alpha = (\alpha_n)_{n\in\mathbb{N}}$. Then $|\alpha_n| \leq 2\|x_n\|$ for every $n \in \mathbb{N}$, so $\alpha \in E$, and $\langle x_k^*, x - J(\alpha)\rangle = \sum_{i=n_{k-1}-1}^{n_k} \langle z_i^*, x_i - \alpha_i z_i\rangle = 0$ for every $k \in \mathbb{N}$, so $x - J(\alpha) \in M$ and $Q_M(x) = Q_M J(\alpha) \in R(Q_M J)$.

Now, by the superprojectivity of E and Proposition (3.1.2), there exists another surjection $S: X/M \to Z$ such that $N(SQ_M J)$ is complemented in E; by Proposition (3.1.3), $N(SQ_M)$ is complemented in X, where $M \subseteq N(SQ_M)$ and $R(SQ_M) = Z$, which is infinite-dimensional.

We have the following result.

Lemma (3.2.9)[3]: *Let* X *be a Banach space, let* E *be an unconditional sequence space and let* T, $(T_k)_{k\in\mathbb{N}}$ *be projections in* X *such that*

(i) $T_i T_j = 0 \; if \; i \neq j;$

(ii) $T_k T = TT_k = T_k$ *for every* $k \in \mathbb{N}$*;*

(iii) R(T) *embeds into* $E(R(T_k))$ *via the mapping that takes* $x \in R(T)$ *to* $(T_k(x))_{k\in\mathbb{N}}$.

Let $x_k^* \in R(T_k^*)$ *be non-null for each* $k \in \mathbb{N}$*. Then* $[x_k^* : k \in \mathbb{N}]_\perp$*is complemented in X.*

Proof. We will assume without loss of generality that $\|x_k^*\| = 1$ for every $k \in \mathbb{N}$. Let $Z = E(R(T_k))$ and let $U: R(T) \to Z$ be the isomorphism into Z defined as $U(x) = (T_k(x))_{k\in\mathbb{N}}$.

Note that, in fact, $(T_k(x))_{k\in\mathbb{N}} = (T_k(T(x)))_{k\in\mathbb{N}} = U(T(x)) \in Z$ for every $x \in X$, so $(\|T_k(x)\|)_{k\in\mathbb{N}} \in E$ and $\|(\|T_k(x)\|)_{k\in\mathbb{N}}\|_E = \|U(T(x))\|Z$ for every $x \in X$. Define $Q: X \to E$ as $Q(x) = (\langle x_k^*, x\rangle)_{k\in\mathbb{N}}$; then

$$|\langle x_k^*, x\rangle| = |\langle T_k^* (x_k^*), x\rangle| = |\langle x_k^*, T_k(x)\rangle| \leq \|T_k(x)\|$$

for every $x \in X$, so Q is well defined and $\|Q\| \leq \|UT\|$. Also, $(T_k(x))_{k\in\mathbb{N}} \in E$ implies that $T_k x \underset{k}{\to} 0$ for every $x \in X$, so there exists a constant C such that $\|T_k\| \leq C$ for every $k \in \mathbb{N}$.

Take now $x_k \in X$ such that $\langle x_k^*, x_k\rangle = 1$ and $\|x_k\| \leq 2$ for each $k \in \mathbb{N}$, so that $\langle x_i^*, T_j x_j\rangle = \langle T_j^* x_i^*, x_j\rangle = \delta_{ij}$ for every $i, j \in N$, and define $J: E \to X$ as $J((\alpha_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \alpha_n T_n(x_n)$. Then $U(J((\alpha_n)_{n\in\mathbb{N}})) = (\alpha_k T_k(x_k))_{k\in\mathbb{N}}$, as seen by considering the action of UJ over the finitely non-null sequences of E, where $1 \leq \|T_k(x_k)\| \leq C$ for every $k \in \mathbb{N}$, so $UJ: E \to Z$ is an isomorphism, and so must be J. Finally,

$$QJ((\alpha_n)_{n\in\mathbb{N}}) = \left(\langle x_k^*, \sum_{n=1}^{\infty} \alpha_n T_n(x_n)\rangle\right)_{k\in\mathbb{N}} = (\alpha_k)_{k\in\mathbb{N}},$$

so $QJ = I_E$ and JQ is a projection in X with kernel $[x_k^* : k \in \mathbb{N}]_\perp$.

Theorem (3.2.10)[3]: *Let X and Y be c_0 or ℓ_p for $1 < p < \infty$. Then $X \hat{\otimes}_\varepsilon Y$ is superprojective.*

Proof. Let $R_n: X \to X$ be the projection given by $R_n((\alpha_k)_{k\in\mathbb{N}}) = (\alpha_1, \dots, \alpha_n, 0, \dots)$ for each $n \in \mathbb{N}$, and similarly for Y. (We are abusing the notation here for the sake of simplicity in that R_n is really a different operator on each of X and Y unless they are the same space.) Define the projections

$$P_n = R_n \otimes R_n$$

$$\begin{aligned} Q_n &= I_X \hat{\otimes}_\varepsilon Y - (I_X - R_n) \otimes (I_Y - R_n) \\ &= R_n \otimes R_n + (I_X - R_n) \otimes R_n + R_n \otimes (I_Y - R_n) \end{aligned}$$

We will prove that the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in N}$ meet the criteria of Theorem (3.2.2).

Conditions (ii) and (iii) are readily satisfied, because they clearly hold for the elementary tensors $e_i \otimes e_j$. For condition (i), both X^* and Y^* are ℓ_q spaces for some $1 \le q < \infty$, so $R_n^*(x^*) \underset{n}{\to} x^*$ for every $x^* \in X^*$, and similarly for Y^*, so $P_n^*(z^*) = (R_n^* \otimes R_n^*)(z^*) \underset{n}{\to} z^*$ for every $z^* \in (X \hat{\otimes}_\varepsilon Y)^* = X^* \hat{\otimes}_\pi Y^*$, again because it holds for the elementary tensors. For condition (iv), note that the range of Q_n is the direct sum of the ranges of $R_n \otimes R_n$, $(I_X - R_n) \otimes R_n$ and $R_n \otimes (I_Y - R_n)$, where the first one is finite-dimensional and the other two are the sum of finitely many copies of $N(R_n)$ in X and Y, respectively, which are finite-codimensional subspaces of X and Y, respectively, hence superprojective.

To prove condition (v), let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of integers and let $T_1 = P_{n_1}$ and $T_k = (I - Q_{n_{k-1}})P_{n_k}$ for $k > 1$, as in Theorem

(3.2.2). Note that, for $k > 1$, Tk is the projection $T_k = (R_{n_k} - R_{n_{k-1}}) \otimes (R_{n_k} - R_{n_{k-1}})$, so $T_i T_j = 0 \; if \; i \neq j$. Using Tong's result on diagonal submatrices, the operator $T = \sum_{k=1}^{\infty} T_k$ is a norm-one projection in $X \hat{\otimes}_\varepsilon Y$, with $T_k T = T T_k = T_k$ for every $k \in \mathbb{N}$, and *R(T)* embeds into $c_0(R(T_k))$ or $\ell_s(R(T_k))$ for suitable $1 < s < \infty$ in the natural way, so Lemma (3.2.9) ensures that $[x_k^*: k \in \mathbb{N}]_\perp$ is complemented in $X \hat{\otimes}_\varepsilon Y$ for any choice of non-null elements $x_k^* \in R(T_k^*)$.

Theorem (3.2.10) can actually be extended to injective tensor products of finitely many copies of c_0 and $\ell_p (1 < p < \infty)$ inductively in the obvious way with only minor modifications.

We will show that *C(K,X)* is superprojective whenever so is *X* at least if *K* is an interval of ordinals, which includes the case where *K* is scattered and metrisable .

Theorem (3.2.11)[3]: *Let X be a superprojective Banach space and let λ be an ordinal. Then $C_0([0,\lambda],X)$ and $C([0,\lambda],X)$ are superprojective.*

Proof. The proof will proceed by induction in λ. Assume that $C_0([0,\mu],X)$ and $C([0,\mu],X)$ are indeed superprojective for all $\mu < \lambda$; we will first prove that $C_0([0,\lambda],X)$ is superprojective too. If λ is not a limit ordinal, then $\lambda = \mu + 1$ for some μ and $C_0([0,\lambda],X) \equiv C([0,\mu],X)$, which is superprojective by the induction hypothesis.

Otherwise, if λ is a limit ordinal, define the projections

$$P_\mu : C_0([0,\lambda],X) \to C_0([0,\lambda],X)$$

as $P_\mu(f) = f\chi_{[0,\mu]}$ for each $\mu < \lambda$. We will prove that the family $(P_\mu)_{\mu<\lambda}$ meets the criteria of Theorem (3.2.3). Condition (ii) is immediate to check.

For condition (iii), $P_\mu(C_0([0,\lambda],X))$ is isometric to $C([0,\mu],X)$, which is superprojective by the induction hypothesis.

For condition (i), we have $C_0([0,\lambda])^* = \ell_1\left([0,\lambda)\right)$ and $C_0([0,\lambda],X)^* = (C_0([0,\lambda]) \hat{\otimes}_\varepsilon X)^* = C_0([0,\lambda])^* \hat{\otimes}_\pi X^*$,so $C_0([0,\lambda],X)^* = \ell_1([0,\lambda)) \hat{\otimes}_\pi X^* = \ell_1([0,\lambda),X^*)$ and $P_\mu^*(z) = z\chi_{[0,\mu]} \underset{\mu}{\to} z$ for every $z \in \ell_1([0,\lambda),X^*)$.

As for condition (iv), let $(\lambda_k)_{k\in\mathbb{N}}$ be an unbounded strictly increasing sequence of elements in $[0,\lambda)$, should it exist, and let $T_1 = P_{\lambda_1}$ and $T_k = P_{\lambda_k} - P_{\lambda_{k-1}}$ for $k > 1$, as in Theorem (3.2.3). Then T_k is the projection given by $T_k(f) = f\chi_{[\lambda_{k-1}+1,\lambda_k]}$ for $k > 1$, so $T_iT_j = 0\, if\ i \neq j$. Since $(\lambda_k)_{k\in\mathbb{N}}$ is unbounded in $[0,\lambda)$, its supremum must be λ itself, so $C_0([0,\lambda],X) = c_0(R(T_k)) = c_0(C([\lambda_{k-1}+1,\lambda_k],X))$ and Lemma (3.2.9), with $T = I$, ensures that $[x_k^*: k \in \mathbb{N}]_\perp$ is complemented in $C_0([0,\lambda],X)$ for any choice of non-null elements $x_k^* \in R(T_k^*)$.

Finally, $C([0,\lambda],X) = C_0([0,\lambda],X) \oplus X$, which is superprojective by Proposition (3.2.1).

Note that unbounded strictly increasing sequences in [0, λ) may not exist for certain λ, in which case the remark after Theorem (3.2.2) applies and P_μ cannot be strictly cosingular for all $\mu < \lambda$.

Chapter 4

Banach Spaces and Subprojectivity

A Banach space X is called subprojective if any of its infinite dimensional subspaces Y contains a further infinite dimensional subspace complemented in X. We are devoted to systematic study of subprojectivity.

Section (4.1): General Facts about Subproectivity of Tensor Products and Spaces of Operators.

A Banach space X is called *subprojective* if every subspace $Y \subset X$ contains a further subspace $Z \subset Y$, complemented in X. This notion was introduced, in order to study the (pre)adjoints of strictly singular operators. Recall that an operator $T \in B(X, Y)$ is *strictly singular* $(T \in SS(X, Y))$ if T is not an isomorphism on any subspace of X. In particular, it was shown that, if Y is subprojective, and, for $T \in B(X, Y), T^* \in SS(Y^*, X^*)$, then $T \in SS(X, Y)$.

Later, connections between subprojectivity and perturbation classes were discovered. More specifically, denote by $\Phi_+(X, Y)$ the set of *upper semi-Fredholm operators* that is, operators with closed range, and finite dimensional kernel. If $\Phi_+(X, Y) \neq \emptyset$, we define the *perturbation class*

$$P\Phi_+(X, Y) = \{S \in B(X, Y): T + S \in \Phi_+(X, Y) \; whenever \; T \in \Phi_+(X, Y)\}.$$

It is known that $SS(X, Y) \subset P\Phi + (X, Y)$. In general, this inclusion is proper. However, we get $SS(X, Y) = P\Phi + (X, Y)$ if Y is subprojective .

Several classes of subprojective spaces are described. Common examples of non-subprojective space are $L_1(0, 1)$ (since all Hilbertian subspaces of L_1 are not complemented), $C(\Delta)$, where Δ is the Cantor set, or ℓ_∞ (for the same reason). The disc algebra is not subprojective, it contains a copy of $C(\Delta)$,.

$L_p(0,1)$ is subprojective if and only if $2 \leq p < \infty$. Consequently, the Hardy space H_p on the disc is subprojective for exactly the same values of p. Indeed, H_∞ contains the disc algebra. For $1 < p < \infty$, H_p is isomorphic to L_p. The space H_1 contains isomorphic copies of L_p for $1 < p \leq 2$. On the other hand, VMO is subprojective.

We prove that subprojectivity is stable under suitable direct sums. However, subprojectivity is not a 3-space property. Consequently, subprojectivity is not stable under the gap metric. Considering the place of subprojective spaces in Gowers dichotomy, we observe that each subprojective space has a subspace with an unconditional basis. However, we exhibit a space with an unconditional basis, but with no subprojective subspaces .

We investigate the subprojectivity of tensor products, and of spaces of operators. A general result on tensor products yields the subprojectivity on $\ell_p \check{\otimes} \ell_q$ and $\ell_p \hat{\otimes} \ell_q$ for $1 \leq p, q < \infty$, as well as of $K(L_p, L_q)$ for $1 < p \leq 2 \leq q < \infty$. We also prove that the space B(X) is never subprojective, and give an example of non-subprojective tensor product $\ell_2 \otimes_\alpha \ell_2$.

We work with *C(K)* spaces, with *K* compact metrizable. We begin by observing that *C(K)* is subprojective if and only if K is scattered. Then we prove that *C(K,X)* is subprojective if and only if both *C(K)* and *X* are. Turning to spaces of operators, we show that, for *K* scattered, $\prod_{qp}(C(K), \ell_q)$ is subprojective. Then we study continuous fields on a scattered base space, proving that any scattered separable *CCR* C^*-algebra is subprojective.

We show that, in many cases, subprojectivity passes from a sequence space to the associated Schatten spaces.

Proceeding to Banach lattices, we prove that p-disjointly homogeneous p-convex lattices $(2 \leq p < \infty)$ are subprojective. We show that the lattice $\widetilde{X(\ell_p)}$ is subprojective whenever *X* is.

Consequently, if *X* is a subprojective space with an unconditional basis and non-trivial cotype, then *Rad(X)* is subprojective.

We use the standard Banach space results and notation. By *B(X, Y)* and *K(X,Y)* we denote the sets of linear bounded and compact operators, respectively, acting between Banach spaces *X* and *Y*. *B(X)* refers to the closed unit ball of *X*. For $p \in [1, \infty]$, we denote by p' the "adjoint" of *p* (that is, $1/p + 1/p' = 1$).

We showing that subprojectivity passes to direct sums.

Proposition (4.1.1)[4]: *(a) Suppose X and Y are Banach spaces. Then the following are equivalent:*

(i) Both X and Y are subprojective.

(ii)(2) X $\oplus$ Y is subprojective.

(b) Suppose $X_1, X_2, \ldots$ are Banach spaces, and ε is a space with a 1*-unconditional basis. Then the following are equivalent:*

(i) The spaces $\varepsilon, X_1, X_2, \ldots$ are subprojective.

(ii) $(\sum n \ X_n)_\varepsilon$ is subprojective.

In (b), we view ε as a space of sequences of scalars, equipped with the norm $\|\cdot\|_\varepsilon$. $(\sum n \ X_n)_\varepsilon$ refers to the space of all sequences $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, endowed with the norm $\|(x_n)_{n\in\mathbb{N}}\| = \|(\|x_n\|X_n)\|_\varepsilon$. Due to the

1-unconditionality (actually,1-suppression unconditionality suffices), $(\sum n\ X_n)_\varepsilon$ is a Banach space.

We begin by making two simple observations.

Proposition (4.1.2)[4]: *Consider Banach spaces X and X′, and $T \in B(X, X')$. Suppose Y is a subspace of X, $T|_Y$ is an isomorphism, and T(Y) is complemented in X′. Then Y is complemented in X.*

Proof. If Q is a projection from X' to *T(Y)*, then $T^{-1}QT$ is a projection from *X* onto *Y* .

This immediately yields:

Corollary (4.1.3)[4]: *Suppose X and X′ are Banach spaces, and X′ is subprojective. Suppose, furthermore, that Y is a subspace of X, and there exists $T \in B(X, X')$ so that $T|_Y$ is an isomorphism. Then Y contains a subspace complemented in X.*

We have the following version of "Principle of Small Perturbations". We include the proof for the sake of completeness.

Proposition (4.1.4)[4]: *Suppose (x_k) is a seminormalized basic sequence in a Banach space X, and (y_k) is a sequence so that $lim_k \|x_k - y_k\| = 0$. Suppose, furthermore, that every subspace of $span[y_k : k \in \mathbb{N}]$ contains a subspace complemented in X. Then $span[x_k : k \in \mathbb{N}]$ contains a subspace complemented in X.*

Proof. Replacing x_k by $x_k/\|x_k\|$, we can assume that (x_k) normalized. Denote the biorthogonal functionals by x_k^*, and set $K = sup_k \|x_k^*\|$. Passing to a subsequence, we can assume that $\sum_k \|x_k - y_k\| < 1/(2K)$. Define the operator *U∈ B(X)* by setting $Ux = \sum_k x_k^*(x)(y_k - x_k)$. Clearly $\|U\| < 1/2$, and therefore, $V = I_X + U$ is invertible. Furthermore, $Vx_k = y_k$. If Q is a projection from X onto a subspace $W \subset span[y_k : k \in \mathbb{N}]$, then $P = V^{-1}QV$ is a projection from X onto a subspace $Z \subset span[x_k : k \in \mathbb{N}]$.

(a) Throughout the proof, P_X and P_Y stand for the coordinate projections from $X \oplus Y$ onto X and Y, respectively. We have to show that any subspace E of $X \oplus Y$ contains a further subspace G, complemented in $X \oplus Y$.

Show first that E contains a subspace F so that either $P_X|_F$ or $P_Y|_F$ is an isomorphism. Indeed, suppose $P_X|_F$ is not an isomorphism, for any such F. Then $P_X|_E$ is strictly singular, hence there exists a subspace $F \subset E$, so that $P_X|_F$ has norm less than 1/2. But $P_X + P_Y = I_{X \oplus Y}$, hence, by the triangle inequality, $\|P_Y f\| \geq \|f\| - \|P_X f\| > \|f\|/2$ for any $f \in F$. Consequently, $P_Y|_F$ is an isomorphism.

Thus, by passing to a subspace, and relabeling if necessary, we can assume that E contains a subspace F, so that $P_X|F$ is an isomorphism. By Corollary (4.1.3), F contains a subspace G, complemented in X.

Set $F' = P_X(F)$, and let V be the inverse of $P_X : F \to F'$. By the subprojectivity of X, F' contains a subspace G', complemented in X via a projection Q. Then $P = VQP_X$ gives a projection onto $G = V(G') \subset F$.

(b) Here, we denote by P_n the coordinate projection from $X = (\sum_k X_k)_\varepsilon$ onto X_n. Furthermore, we set $Q_n = \sum_{k=1}^{n} P_k$, and $Q_n^\perp = 1 - Q_n$. We have to show that any subspace $Y \subset X$ contains a subspace Y_0, complemented in X. To this end, consider two cases.

(i) For some n, and some subspace $Z \subset Y$, $Q_n|_Z$ is an isomorphism. By part (a), $X_1 \oplus \ldots \oplus X_n = Q_n(X)$ is subprojective. Apply Corollary (4.1.3) to obtain Y_0.

(ii) For every $n, Q_n|_Y$ is not an isomorphism – that is, for every $n \in \mathbb{N}$, and every $\varepsilon > 0$, there exists a norm one $y \in Y$ so that $\|Q_n y\| < \varepsilon$. Therefore, for every sequence of positive numbers (ε_i), we can find $0 = N_0 < N_1 < N_2 < \ldots$, and a sequence of norm one vectors $y_i \in Y$, so that, for every $i, \|Q_{N_i} y_i\|, \|Q^{\perp}_{N_{i+1}} y_i\| < \varepsilon_i$.

By a small perturbation principle, we can assume that Y contains norm one vectors (y'_i) so that $Q_{N_i} y'_i = Q^{\perp}_{N_{i+1}} y'_i = 0$ for every i. Write $y'_i = (z_j)_j^{N_{i+1}} = N_{i+1}$, with $z_j \in X_j$.

Then $Z = span[(0, \ldots, 0, z_j, 0, \ldots): j \in \mathbb{N}]$ (z_j is in j-th position) is complemented in X. Indeed, if $z_j \neq 0$, find $z_j^* \in X_j^*$ so that $\|z_j^*\| = \|z_j\|^{-1}$, and $\langle z_j^*, z_j \rangle = 1. If z_j = 0$, set $z_j^* = 0$. For $x = (x_j)_{j \in \mathbb{N}} \in X$, define $Rx = (\langle z_j^*, x_j \rangle z_j)_{j \in \mathbb{N}.}$ It is easy to see that R is a projection onto Z, and $\|R\|$ does not exceed the unconditionality constant of ε.

Now note that $J: Z \to E: (\alpha_1 z_1, \alpha_2 z_2, \ldots) \mapsto (\alpha_1 \|z_1\|, \alpha_2 \|z_2\|, \ldots)$ is an isometry. Let$Y' = span[y'_i : i \in \mathbb{N}]$, and $Y_\varepsilon = J(Y')$. By the subprojectivity of $\varepsilon, Y_\varepsilon$ contains a subspace W, which is complemented in ε via a projection R_1. Then $J^{-1} R_1 J R$ is a projection from X onto $Y_0 = J^{-1}(W) \subset X$.

Recall that if X is a Banach space, then

$$\ell_p^{weak}(X) = \{x = (x_n)_{n=1}^{\infty} \in X \times X \times X \ldots : \sup_{x^* \in X^*} (\sum |x^*(x_n)|^p)^{\frac{1}{p}} < \infty\}.$$

It is known that $\ell_p^{weak}(X)$ is isomorphic to $B(\ell_{p'}, X)(\frac{1}{p} + \frac{1}{p'} = 1)$. We show that, for $X = \ell_r (r \geq p'), B(\ell_{p'}, X)$ contains a copy of ℓ_∞, and therefore, is not subprojective. To this end, denote by (e_i) and (f_i) the canonical bases in ℓ_r and $\ell_{p'}$ respectively. For $\alpha = (\alpha_i) \in \ell\infty$, define $B(\ell_{p'}, X) \ni U\alpha : e_i \mapsto \alpha_i f_i$. Clearly, U is an isomorphism.

Note that the situation is different for $r < p'$. Then, by Pitt's Theorem, $B(\ell_{p'}, \ell_r) = K(\ell_{p'}, \ell_r)$. In the next section we prove that the latter space is subprojective.

We show that subprojectivity is not a 3-space property.

Proposition (4.1.5)[4]: *For $1 < p < \infty$ there exists a non-subprojective Banach space Z_p, containing a subspace X_p, so that X_p and Z_p/X_p are isomorphic to ℓ_p.*

It is easy to see that subprojectivity is stable under isomorphisms. However, it is not stable under a rougher measure of "closeness" of Banach spaces – the gap measure. If Y and Z are subspaces of a Banach space X, we define the *gap* (or *opening*)

$$\Theta_X(Y, Z) = max \left\{ \sup_{y \in Y, \|y\|=1} dist(y, Z), \sup_{z \in Z, \|z\|=1} dist(z, Y) \right\}.$$

We refer to the comprehensive survey for more information. Here, we note that Θ_X satisfies a "weak triangle inequality", hence it can be viewed as a measure of closeness of subspaces. The following shows that subprojectivity is not stable under Θ_X.

Proposition (4.1.6)[4]: *There exists a Banach space X with a subprojective subspace Y so that, for every $\varepsilon > 0$, X contains a non-subprojective space Z with $\Theta_X(Y, Z) \leq \varepsilon$.*

Proof. Our Y will be isomorphic to ℓ_p, where $p \in (1, \infty)$ is fixed. By Proposition (4.1.5), there exists a non-subprojective Banach space W, containing a subspace W_0, so that both W_0 and $W' = W/W_0$ are isomorphic to ℓ_p. Denote the quotient map $W \to W'$ by q. Consider $X = W \oplus_1 W'$ and

$Y = W_0 \oplus_1 W' \subset E$. Furthermore, for $\varepsilon > 0$, define $Z_\varepsilon = \{\varepsilon w \oplus_1 qw : w \in W\}$. Clearly, Y is isomorphic to $\ell_p \oplus \ell_p \sim \ell_p$, hence subprojective, while Z_ε is isomorphic to W, hence not subprojective. We have $\Theta_X(Y, Z_\varepsilon) \le \varepsilon$.

Looking at subprojectivity through the lens of Gowers dichotomy and observing that a subprojective Banach space does not contain hereditarily indecomposable subspaces, we immediately obtain the following.

Proposition (4.1.7)[4]: *Every subprojective space has a subspace with an unconditional basis.*

The converse to the above proposition is false.

Proposition (4.1.8)[4]: *There exists a Banach space with an unconditional basis, without subprojective subspaces.*

Proof. T. Gowers and B. Maurey construct a Banach space X with a 1-unconditional basis, so that any operator on X is a strictly singular perturbation of a diagonal operator. We prove that X has no subprojective subspaces. In doing so, we are re-using the notation of that paper. In particular, for $n \in \mathbb{N}$ and $x \in X$,

We define $\|x\|_{(n)}$ as the supremum of $\sum_{i=1}^{n} \|x_i\|$, where $x_1, \ldots, x_n$ are successive vectors so that $x = \sum_i x_i$. It is known that, for every block ubspace Y in X, every $c > 1$, and every $n \in N$, there exists $y \in Y$ so that $1 = \|y\| \le \|y\|_{(n)} < c$. This technical result can be used to establish a remarkable property of X: suppose Y is a subspace of X, with a normalized block basis (y_k). Then any zero-diagonal (relative to the basis (y_k)) operator on Y is strictly singular. Consequently, any $T \in B(Y)$ can be written as $T = \Lambda + S$, where Λ is diagonal, and S is zero-diagonal, hence strictly singular. This result is proved for $Y = X$, but an inspection yields the generalization described above.

Suppose, for the sake of contradiction, that X contains a subprojective subspace Y. A small perturbation argument shows we can assume Y to be a block subspace. Blocking further, we can assume that Y is spanned by a block basis (y_j), so that $1 = \|y_j\| \le \|y_j\|_{(j)} < 1 + 2^{-j}$. We achieve the desired contradiction by showing that no subspace of $Z = span[y_1 + y_2, y_3 + y_4, \ldots]$ is complemented in Y.

Suppose P is an infinite rank projection from Y onto a subspace of Z. Write $P = \Lambda + S$, where S is a strictly singular operator with zeroes on the main diagonal, and $\Lambda = (\lambda_j)_{j=1}^{\infty}$ is a diagonal operator (that is, $\Lambda y_j = \lambda_j y_j$ for any j). As $\sup_j \|y_j\|_{(j)} < \infty$, we have $\lim_j S y_j = 0$. Note that $(\Lambda + S)^2 = \Lambda + S$, hence $diag\,(\lambda_j^2 - \lambda_j) = \Lambda^2 - \Lambda = S - \Lambda S - S\Lambda - S^2$ is strictly singular, or equivalently, $\lim_j \lambda_j(1 - \lambda_j) = 0$. Therefore, there exists a $0 - 1$ sequence (λ'_j) so that $\Lambda' - \Lambda$ is compact (equivalenty, $\lim_j(\lambda_j - \lambda'_j) = 0$), where $\Lambda' = diag\,(\lambda'_j)$ is a diagonal projection. Then $P = \Lambda' + S'$, where $S' = S + (\Lambda - \Lambda')$ is strictly singular, and satisfies $\lim_j S' y_j = 0$. The projection P is not strictly singular (since it is of infinite rank), hence $\Lambda' = P - S'$ is not strictly singular. Consequently, the set $J = \{j \in \mathbb{N} : \lambda'_j = 1\}$ is infinite.

Now note that, for any j, $\|Py_j - y_j\| > 1/2$. Indeed, $Py_j \in Z$, hence we can write $Py_j = \sum_k \alpha_k (y_{2k-1} + y_{2k})$. Let $\ell = [j/2]$. By the 1-unconditionality of our basis, $\|y_j - Py_j\| \geq \|y_j - \alpha_\ell(y_{2\ell-1} + y_{2\ell})\| \geq max\{|1-\alpha_\ell|, |\alpha_\ell|\} \geq 1/2$. For $j \in J, S'y_j = Py_j - y_j$, hence $\|S'y_j\| \geq 1/2$, which contradicts $\lim_j \|S'y_j\| = 0$.

Finally, one might ask whether, in the definition of subprojectivity, the projections from X onto Z can be uniformly bounded. More precisely, we call a Banach space X *uniformly subprojective* (*with constant C*) if, for every subspace $Y \subset X$, there exists a subspace $Z \subset Y$ and a projection P: $X \to Z$ with $\|P\| \leq C$. The proof essentially shows that the following spaces are uniformly subprojective: (i) ℓ_p $(1 \leq p < \infty)$ and c_0; (ii) the Lorentz sequence spaces $l_{p,w}$; (iii) the Schreier space; (iv) the Tsirelson space; (v) the James space. Additionally, $L_p(0,1)$ is uniformly subprojective for $2 \leq p < \infty$. This can be proved by combining Kadets-Pelczynski dichotomy with the results about the existence of "nicely complemented" copies of ℓ_2. Moreover, any c_0-saturated separable space is uniformly subprojective, since any isomorphic copy of c_0 contains a λ-isomorphic copy of c_0, for any $\lambda > 1$. By Sobczyk's Theorem, a λ-isomorphic copy of c_0 is 2λ-complemented in every separable superspace. In particular, if K is a countable metric space, then *C(K)* is uniformly subprojective.

However, in general, subprojectivity need not be uniform. Indeed, suppose $2 < p_1 < p_2 < \ldots < \infty$, and $\lim_n p_n = \infty$. By Proposition (4.1.1) (b), $X = (\sum_n L_{p_n}(0,1))_2$ is subprojective. The span of independent Gaussian random variables in L_p (which we denote by G_p) is isometric to ℓ_2. Therefore, any projection from L_p onto G_p has norm at least $c_0\sqrt{p}$, where c_0 is a universal constant. Thus, X is not uniformly subprojective.

Section (4.2): Subprojectivity of Schatten Spaces and with lattice Valued ℓ_P Spaces.

Suppose $X_1, X_2, \ldots, X_k$ are Banach spaces with unconditional FDD, implemented by finite rank projections $(P'_{1n}), (P'_{2n}), \ldots, (P'_{k\,n})$, respectively. That is, $P'_{in}P'_{im} = 0$ unless $n = m, \lim_N \sum_{n=1}^{N} P'_{in} = I_{X_i}$ point-norm, and $\sup_{N,\pm} \|\sum_{n=1}^{N} \pm P'_{in}\| < \infty$ (this quantity is sometimes referred to as *the FDD constant of* X_i). Let $E_{in} = ran\,(P'_{in})$.

We say that a sequence $(w_j)_{j=1}^{\infty} \subset X_1 \otimes X_2 \otimes \ldots \otimes X_k$ is *block-diagonal* if there exists a sequence $0 = N_1 < N_2 < \ldots$ so that

$$w_j \in \left(\sum_{n=N_j+1}^{N_{j+1}} E_{1n}\right) \otimes \left(\sum_{n=N_j+1}^{N_{j+1}} E_{2n}\right) \otimes \ldots \otimes \left(\sum_{n=N_j+1}^{N_{j+1}} E_{kn}\right).$$

Suppose ε is an unconditional sequence space, and $\widetilde{\otimes}$ is a tensor product of Banach spaces. The Banach space $X_1 \widetilde{\otimes} X_2 \widetilde{\otimes} \ldots \widetilde{\otimes} X_k$ is said to *satisfy the ε-estimate* if there exists a constant $C \geq 1$ so that, for any block diagonal sequence $(w_j)_{j \in N}$ *in* $X_1 \widetilde{\otimes} X_2 \widetilde{\otimes} \ldots \widetilde{\otimes} X_k$, we have

$$C^{-1} \left\| (\|w_j\|)_{j \in N} \right\|_E \leq \left\| \sum_j w_j \right\| \leq C \left\| (\|w_j\|)_{j \in N} \right\|_E \qquad (4.1)$$

Corollary (4.2.1)[4]: *Suppose the Banach spaces X_1 and X_2 have unconditional FDD, co-type* 2 *and type* 2 *respectively, and both X_1^* and X_2 are subprojective. Then $K(X_1, X_2)$ is subprojective.*

This happens, for instance, if $X_1 = L_p(\mu)$ or $\mathfrak{C}_p$ $(1 < p \leq 2)$ and $X_2 = L_q(\mu)$ *or* $\mathfrak{C}_q$ $(2 \leq q < \infty)$. Indeed, the type and cotype of these spaces are well known. The Haar system provides an unconditional basis for L_p. The existence of unconditional FDD of $\mathfrak{C}_p$ spaces is given.

Theorem (4.2.2)[4]: *Suppose $X_1, X_2, \ldots, X_k$ are subprojective Banach spaces with un-conditional FDD, and $\tilde{\otimes}$ is a tensor product. Suppose, furthermore, that for any finite increasing sequence $i = [1 \leq i_1 < \ldots < \ldots i_\ell \leq k]$, there exists an uncondi-tional sequence space ε_i, so that $X_{i_1} \tilde{\otimes} X_{i_2} \tilde{\otimes} \ldots \tilde{\otimes} X_{i_k}$ satisfies the ε_i-estimate. Then $X_1 \tilde{\otimes} X_2 \tilde{\otimes} \ldots \tilde{\otimes} X_k$ is subprojective.*

A similar result for ideals of operators holds as well. We keep the notation for projections implementing the FDD in Banach spaces X_1 and X_2. We say that a Banach operator ideal *A* is suitable (for the pair (X_1, X_2)) if the finite rank operators are dense in $A(X_1, X_2)$ (in its ideal norm). We say that a sequence $(w_j)_{j \in N} \subset A(X_1, X_2)$ is *block diagonal* if there exists a sequence $0 = N_1 < N_2 < \ldots$ so that, for any $j, w_j = (P_{2,N_j} - P_{2,N_{j-1}}) w_j (P_{1,N_j} - P_{1,N_{j-1}})$. If ε is an unconditional sequence space, we say that $K(X_1, X_2)$ *satisfies the* ε-*estimate* if, for some constant *C*,

$$C^{-1} \left\| \left(\|w_j\| \right)_j \right\|_E \leq \left\| \sum_j w_j \right\|_A \leq C \left\| \left(\|w_j\| \right)_j \right\|_E \qquad (4.2)$$

holds for any finite block-diagonal sequence (w_j).

Proof. We will prove the theorem by induction on *k*. Clearly, we can take $k = 1$ as the basic case. Suppose the statement of the theorem holds for a tensor product of any $k - 1$ subprojective Banach spaces that satisfy ε-estimate. We will show that the statement holds for the tensor product of *k* Banach spaces $X = X_1 \tilde{\otimes} X_2 \tilde{\otimes} \ldots \tilde{\otimes} X_k$.

For notational convenience, let $P_{in} = \sum_{k=1}^{n} P'_{ik}$, and $I_i = I_{X_i}$. If $A \in B(X)$ is a projection, we use the notation $A^{\perp}$ for $I_X - A$. Furthermore, define the projections $Q_n = P_{1n} \otimes P_{2n} \otimes \ldots \otimes P_{kn}$ and $R_n = P_{1n}^{\perp} \otimes P_{2n}^{\perp} \ldots \otimes P_{kn}^{\perp}$. Renorming all X_i's if necessary, we can assume that their unconditional FDD constants equal 1.

First show that, for any $n, ranR_n^{\perp}$ is subprojective. To this end, write $R_n^{\perp} = \sum_{i=1}^{k} P^{(i)}$, where the projections $P^{(i)}$ are defined by

$$P(1) = P_{1n} \otimes I_2 \otimes \ldots \otimes I_k,$$

$$P(2) = P_{1n}^{\perp} \otimes P_{2n} \otimes I3 \otimes \ldots \otimes I_k,$$

$$P(3) = P_{1n}^{\perp} \otimes P_{2n}^{\perp} \otimes P3n \otimes I4 \otimes \ldots \otimes I_k, \ldots\ldots\ldots$$

$$P(k) = P_{1n}^{\perp} \otimes P_{2n}^{\perp} \otimes \ldots \otimes P_{k-1,n}^{\perp} \otimes P_{kn}$$

(note also that $P^{(i)}P^{(j)} = 0$ unless *i* = *j*). Thus, there exists *i* so that $P^{(i)}$ is an isomorphism on a subspace $Y' \subset Y$. Now observe that the range of $P^{(i)}$ is isomorphic to a subspace of $\ell_{\infty}^{N}(X^{(i)})$, where $N = rank\ P_{in}$, and

$$X^{(i)} = X_1 \widetilde{\otimes} X_2 \widetilde{\otimes} \ldots \widetilde{\otimes} X_{i-1} \widetilde{\otimes} X_{i+1} \widetilde{\otimes} \ldots \widetilde{\otimes} X_k.$$

By the induction hypothesis, $X^{(i)}$ is subprojective. By Proposition (4.1.1), ran $P^{(i)}$ is subprojective for every *i*, hence so is $R_n^{\perp}$.

Now suppose *Y* is an infinite dimensional subspace of *X*. We have to show that *Y* contains a subspace *Z*, complemented in *X*. If there exists $n \in \mathbb{N}$ so that $R_n^{\perp}|_Y$ is not strictly singular, then, by Corollary (4.1.3), *Z* contains a subspace complemented in *X*.

Now suppose $R_n^{\perp}|_Z$ is strictly singular for any n. It is easy to see that, for any sequence of positive numbers (ε_m), one can find $0 = n_0 < n_1 < n_2 < \ldots$, and norm one elements $x_m \in Y$, so that, for any $m, \|R_{n_{m-1}}^{\perp} x_m\| + \|x_m -$

$Q_{n_m} x_m\| < \varepsilon_m$. By a small perturbation, we can assume that $x_m = R^{\perp}_{n_{m-1}} Q_{n_m} x_m$. That is,

$$x_m \in ran\left((P_{1,n_m} - P_{1,n_{m-1}}) \otimes (P_{2,n_m} - P_{2,n_{m-1}}) \otimes \ldots \otimes (P_{k,n_m} - P_{k,n_{m-1}})\right).$$

Let $E_{im} = ran\,(P_{i,n_m} - P_{i,n_{m-1}})$, and $W = span[E_{1m} \otimes E_{2m} \otimes \ldots \otimes E_{km} : m \in \mathbb{N}] \subset X$. Applying "Tong's trick", and taking the 1-unconditionality of our FDDs into account, we see that

$$U : X \to W : x \mapsto \sum_m \left((P_{1,n_m} - P_{1,n_{m-1}}) \otimes \ldots \otimes (P_{k,n_m} - P_{k,n_{m-1}})\right)x$$

defines a contractive projection onto W. Furthermore, $Z = span[x_m : m \in \mathbb{N}]$ is complemented in *W*. Indeed, the projection $P_{i,n_m} - P_{i,n_{m-1}} (i, m \in \mathbb{N})$ is contractive, hence we can identify $E_{1m} \tilde{\otimes} \ldots \tilde{\otimes} E_{km}$ *with* $(E_{1m} \otimes \ldots \otimes E_{km}) \cap X$. By the Hahn-Banach Theorem, for each *m* there exists a contractive projection U_m on $E_{1m} \tilde{\otimes} \ldots \tilde{\otimes} E_{2m}$, with range $span[x_m]$. By our assumption, there exists an unconditional sequence space ε so that $X_1 \tilde{\otimes} \ldots \tilde{\otimes} X_k$ satisfies the ε-estimate. Then, for any finite sequence $w_m \in E_{1m} \tilde{\otimes} \ldots \tilde{\otimes} E_{km}$, (4.2.5) yields

$$\left\|\sum_k U_k w_k\right\| \le C\|(\|U_k w_k\|)\|_\varepsilon \le C\|(\|w_k\|)\|_\varepsilon \le C^2 \left\|\sum_k U_k w_k\right\|.$$

Thus, *Z* is complemented in *X*.

Theorem (4.2.3)[4]: *Suppose X_1 and X_2 are Banach spaces with unconditional FDD, so that X_1^* and X_2 are subprojective. Suppose, furthermore,that the ideal* A *is suitable for (X_1, X_2), and* A(X_1, X_2) *satisfies the* ε*-estimate for some unconditional sequence* ε. *Then $A(X_1, X_2)$ is subprojective.*

Have the following consequences.

Sketch of the proof of Theorem (4.2.3). On $A(X_1, X_2)$ we define the projection $R_n : A(X_1, X_2) \to A(X_1, X_2) : w \mapsto P_{2n}^{\perp} w P_{1n}$. Then the range of $R_n^{\perp}$ is isomorphic to $X_1^* \oplus \ldots \oplus X_1^* \oplus X_2 \oplus \ldots \oplus X_2$. Then proceed as in the the proof of Theorem (4.2.2) (with k = 2).

To prove Corollary (4.2.6), we need two auxiliary results.

Lemma (4.2.4)[4]: *Suppose* $1 < p_i < \infty$ $(1 \le i \le n)$ *and* $X = \check{\otimes}_{i=1}^{n} \ell_{p_i}$.

(i) If $\sum 1/p_i > n-1$, *then X satisfies the* ℓ_s*-estimate with* $1/s = \sum 1/p_i - (n-1)$.

(ii)(2) If $\sum 1/p_i \le n-1$, *then* X *satisfies the* c_0*-estimate.*

Proof. Suppose (w_j) is a finite block-diagonal sequence in *X*. We shall show that $\left\|\sum_j w_j\right\| = \left\|\left(\left\|w_j\right\|\right)\right\|_s$, with s as in the statement of the lemma. To this end, let (U_{ij}) be coordinate projections on ℓ_{p_i} for every $1 \le i \le n$, such that $w_j = U_{1j} \otimes \ldots \otimes U_{nj} w_j$, and for each *i*, $U_{ik} U_{im} = 0$ unless $k = m$. Letting $p_i' = p_i/(p_{i-1})$, we see that

$$\left\|\sum_j w_j\right\| = \sup_{\xi_i \in \ell_{p_i'}, \|\xi_i\| \le 1} \left|\langle \sum_j w_j, \otimes_i \xi_i \rangle\right|.$$

Choose $\otimes_i \xi_i$ with $\|\xi_i\| \le 1$, and let $\xi_{ij} = U_{ij} \xi_i$. Then $\sum_j \left\|\xi_{ij}\right\|^{p_i'} \le 1$, and

$$\left|\langle \sum_j w_j, \otimes_i \xi_i \rangle\right| \le \sum_j \left|\langle w_j, \otimes_i \xi_i \rangle\right| = \sum_j \left|\langle w_j, \otimes_i \xi_{ij} \rangle\right| \le \sum_j \|w_j\| \prod_{i=1}^{n} \left\|\xi_{ij}\right\|.$$

Now let $1/r = \sum 1/p_i' = n - \sum 1/p_i$. By Hölder's Inequality,

$$\left(\sum_j \left(\prod_{i=1}^{n} \left\|\xi_{ij}\right\|\right)^r\right)^{1/r} \le \prod_{i=1}^{n} \left(\sum_j \left\|\xi_{ij}\right\|^{p_i'}\right)^{1/p_i'} \le 1.$$

If $\sum 1/p_i \le n-1$, then r $\le$ 1, hence $\sum_j \prod_{i=1}^{n} \|\xi_{ij}\| \le 1$. Therefore, $\|\sum_j w_j\| \le max_j \|w_j\| = (\|w_j\|)_{c_0}$. Otherwise, r > 1, and

$$\left\|\sum_j w_j\right\| \le \left(\sum_j \|w_j\|^s\right)^{1/s} \left(\sum_j (\prod_{i=1}^{n} \|\xi_{ij}\|)^r\right)^{1/r} \le \left(\sum_j \|w_j\|^s\right)^{1/s} = (\|w_j\|)_s,$$

where $1/s = 1 - 1/r = \sum 1/p_i - n + 1$.

In a similar fashion, we show that $\|\sum_j w_j\| > (\|w_j\|)_s$. For s = ∞, the inequality $\|\sum_j w_j\| > max_j \|w_j\|$ is trivial. If s is finite, assume $\sum_j \|w_j\|^s = 1$ (we are allowed to do so by scaling). Find norm one vectors $\xi_{ij} \in \ell_{p'_I}$ so that $\xi_{ij} = U_{ij}\xi_i$, and $\|w_j\| = \langle w_j, \otimes_i \xi_{ij}\rangle$. Let $\gamma_j = \|w_j\|^{s/r}$. Then $\sum_j \gamma_j^r = 1 = \sum_j \gamma_j \|w_j\|$. Further, set $\alpha_{ij} = \gamma_j^{\prod_{l\neq i} p'_l / (\sum_{m=1}^{n} \prod_{l\neq m} p'_l)}$. An elementary calculation shows that $\gamma_j = \prod_{i=1}^{n} \alpha_{ij}$, and $\sum_j \alpha_{ij}^{p'_i} = 1$. Let $\xi_i = \sum_j \alpha_{ij}\xi_{ij}$. Then $\|\xi\|_{p'} =$, and therefore,

$$\left\|\sum_j w_j\right\| \ge \langle \sum_j w_j, \otimes_i \xi_i\rangle = \sum_j \prod_{i=1}^{n} \alpha_{ij} \langle w_j, \otimes_i \xi_{ij}\rangle = \sum_j \gamma_j \|w_j\| = 1.$$

This establishes the desired lower estimate.

Lemma (4.2.5)[4]: *For $1 \le p_i \le \infty$, $X = \ell_{p_1} \hat{\otimes} \ell_{p_2} \hat{\otimes} \ldots \hat{\otimes} \ell_{p_n}$ satisfies the ℓ_r-estimate, where $1/r = \sum 1/p_i$ if $\sum 1/p_i < 1$, and* r = 1 *otherwise. Here, we interpret ℓ_∞ as c_0.*

Proof. The spaces involved all have the Contractive Projection Property (the identity can be approximated by contractive finite rank projections). Thus, the duality between injective and projective tensor products of finite dimensional spaces shows that, for $w \in X$,

$$\|w\| = sup\{|\langle x, w\rangle| : x \in \ell_{p'_1} \check{\otimes} \ldots \check{\otimes} \ell_{p'_n}, \|x\| \le 1\}$$

(here, as before, $1/p_i' + 1/p_i = 1$). Abusing the notation somewhat, we denote by P_{im} the projection on the span of the first *m* basis vectors of both ℓ_{p_i} and $\ell_{p_i'}$. Suppose a finite sequence $(w_k)_{k=1}^{N} \in X$ is block-diagonal, or more precisely, $w_k = \big((P_{1,m_k} - P_{1,m_{k-1}}) \otimes \ldots \otimes (P_{n,m_k} - P_{n,m_{k-1}})\big)w_k$ for every k. Define the operator U on X by setting $Ux = \sum_{k=1}^{N}\big((P_{1,m_k} - P_{1,m_{k-1}}) \otimes \ldots \otimes (P_{n,m_k} - P_{n,m_{k-1}})\big)x$. We also use U_0 to denote the similarly defined operator on X^*. By "Tong's trick", since X and X^* has an unconditional basis, $U(U_0)$ is a contractive projection onto its range $W(W_0)$. Then

$$\left\|\sum_k w_k\right\| = sup\left\{|\langle\sum_k w_k, x\rangle| : \|x\|_{X^*} \le 1\right\} = sup\left\{|\langle U(\sum_k w_k), x\rangle| : \|x\|_{X^*} \le 1\right\}$$
$$= sup\left\{|\sum_k w_k, U_0 x_i| : \|x\|_{X^*} \le 1\right\}.$$

Write $U_0 x = \sum_{k=1}^{N} x_k$. By Lemma (4.2.4) there is an s (either $1/s = \sum 1/p_i' - (n-1) = 1 - \sum 1/p_i$ or $s = \infty$) $\|(\|x_k\|)\|_s = \|U_0 x\| \le \|x\| \le 1$. Moreover,

$$\langle\sum_k w_k, U_0 x\rangle = \langle\sum_k w_k, \sum_k x_k\rangle = \sum_k \langle w_k, x_k\rangle,$$

and therefore,

$$\left\|\sum_k w_k\right\| = sup\left\{\sum_k |\langle w_k, x_k\rangle| : \|(\|x_k\|)\|_s \le 1\right\} = \|(\|w_k\|)\|_r.$$

Corollary (4.2.6)[4]: *The spaces $X_1 \check{\otimes} \ldots \check{\otimes} X_n$ and $X_1 \hat{\otimes} \ldots \hat{\otimes} X_n$ are subprojective where X_i is ether isomorphic to ℓ_{p_i} $(1 \le p_i < \infty)$ or c_0 for every $1 \le i \le n$.* For n = 2, this result goes back to (the injective and projective cases, respectively).

Suppose a Banach space X has an FDD implemented by projections (P'_n)– that is, $P'_n P'_m = 0$ unless n = m, $sup_{N,\pm} \|\sum_{n=1}^{N} \pm P_n\| < \infty$, and $\lim_N \sum_{n=1}^{N} P_n = I_X$ point-norm. We say that X satisfies *the lower* p-*estimate* if there exists a constant C so that, for any finite sequence $\xi_j \in ran\, P_j$, $\left\|\sum_j \xi_j\right\|^p \geq C \sum_j \left\|\xi_j\right\|^p$. The smallest C for which the above inequality holds is called *the lower* p-*estimate constant*. *The upper* p-*estimate,* and *the upper* p-*estimate constant*, are defined in a similar manner. Note that; if X is an unconditional sequence space, then the above definitions coincide with the standard one.

Proof. Combine Theorem (4.2.2) with Lemma (4.2.4) and (4.2.5).

Corollary (4.2.7)[4]: *Suppose the Banach spaces X_1 and X_2 have unconditional FDD, satisfy the lower and upper* p-*estimates respectively, and both X_1^* and X_2 are subprojective. Then $K(X_1, X_2)$ is subprojective.* Before proceeding, we mention several instances where the above corollary is applicable. Note that, if X has type 2 (cotype 2), then X satisfies the upper (resp.lower) 2-estimate. Indeed, suppose X has type 2, and $w_1, \ldots, w_n$ are such that $w_j = P_j w_j$ for any j. Then

$$\left\|\sum_j w_j\right\| \leq CAve_{\pm} \left\|\sum_j \pm w_j\right\| \leq CT_2(X) \left(\sum_j \left\|w_j\right\|^2\right)^{1/2}$$

($T_2(X)$ is the type 2 constant of X). The cotype case is handled similarly. Thus, we can state:

Proof . To apply Theorem (3.2.3), we have to show that $K(X_1, X_2)$ satisfies the c_0-estimate. By renorming, we can assume that the FDD constants of X_1 and X_2 equal 1. Suppose $(w_k)_{k=1}^N$ is a block-diagonal sequence, with $w_k = (P_{2,n_k} - P_{2,n_{k-1}}) w_k (P_{1,n_k} - P_{1,n_{k-1}})$. Let $w = \sum_k w_k$. Then $\|w\| \geq$

$\left\|(P_{2,n_k} - P_{2,n_{k-1}})w(P_{1,n_k} - P_{1,n_{k-1}})\right\| = \|w_k\|$, hence $\|w\| \geq max_k \|w_k\|$. To prove the reverse inequality (with some constant), pick a norm one $\xi \in X_1$, and let $\xi_k = (P_{1,n_k} - P_{1,n_{k-1}})x$. Then $\eta_k = w\xi_k$ satisfies $(P_{2,n_k} - P_{2,n_{k-1}})\eta_k = \eta_k$. Set $\eta = w\xi = \sum_k \eta_k$. Denote by $C_1(C_2)$ lower (upper) p-estimate constants of X_1 (resp. X_2). Then

$$\|w\xi\|^p = \|\eta\|^p \leq C_1 \sum_k \|\eta_k\|^p \leq C_2 \sum_k \|w_k\|^p \|\xi_k\|^p \leq \max_k \|w_k\|_p \ C_2 \sum_k \|\xi_k\|^p$$
$$\leq \max_k \|w_k\|^p \ C_2 C_1 \left\|\sum_k \xi_k\right\|^p = C_2 C_1 \|\xi\|^p.$$

Taking the supremum over all $\xi \in B(X_1)$, $\|w\| \leq (C_1 C_2)^{1/p} max_k \|w_k\|$.

In general, a tensor product of subprojective spaces (in fact, of Hilbert spaces) need not be subprojective.

Proposition (4.2.8)[4]: *There exists a tensor norm $\otimes_\alpha$, so that, for every Banach spaces X and Y , $X \otimes_\alpha Y$ is a Banach space, and $\ell_2 \otimes_\alpha \ell_2$ is not subprojective.*

Proof. Note first that there exists a separable symmetric sequence space ε which is not subprojective. Indeed, let *U* be the space with an unconditional basis which is complementably universal for all spaces with unconditional bases. As noted, this space has a symmetric basis (in fact, uncountably many non-equivalent symmetric bases). On the other hand, *U* is not subprojective, since it contains a (complemented) copy of L_p for $1 < p < 2$. Renorming *U* to make its basis 1-symmetric, we obtain ε.

Now suppose *X* and *Y* are Banach spaces. For $\in X \otimes Y$, we set $\|a\|_\alpha = sup\{\|(u \otimes v)(a)\|_{\varepsilon(H,K)}\}$, where the supremum is taken over all contractions $u : X \to H$ and $v : Y \to K$ (*H* and *K* are Hilbert spaces). Clearly

$\otimes_\alpha$ is a norm on $\otimes Y$. It is easy to see that, for any $a \in X \otimes Y, T_X \in B(X, X_0)$, and $T_Y \in B(Y, Y_0)$, $\|(T_X \otimes T_Y)(a)\|_\alpha \leq \|T_X\|\|T_Y\|\|a\|_\alpha$. Consequently, $\|x \otimes y\|_\alpha = \|x\|\|y\|$. Thus, $\|\cdot\|_\alpha$ is indeed a tensor norm. We denote by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ in this norm.

If X and Y are Hilbert spaces, then for $a \in X \otimes Y$ we have $\|a\|_\alpha = \|a\|_{\varepsilon(X^*,Y)}$. Identifying ℓ_2 with its adjoint, we see that ε embeds into $\ell_2 \otimes_\alpha \ell_2$ as the space of diagonal operators. As ε is not subprojective, neither is $\ell_2 \otimes_\alpha \ell_2$.

Here is another wide class of non-subprojective spaces.

Theorem (4.2.9)[4]: *Let* X *be an infinite dimensional Banach space. Then* B(X) *is not subprojective.*

Proof. Suppose, for the sake of contradiction, that *B(X)* is subprojective. Fix a norm one element $x^* \in X^*$. For $x \in X$ define $T_x \in B(X) : y \mapsto \langle x^*, y\rangle x$. Clearly $M = \{T_x : x \in X\}$ is a closed subspace of *B(X),* isomorphic to *X*. Therefore, *X* is subprojective. By Proposition (4.1.7), we can find a subspace $N \subset M$ with an unconditional basis. We shall deduce that *B(X)* contains a copy of ℓ_∞, which is not subprojective.

If *N* is not reflexive, then *N* contains either a copy of c_0 or a copy of ℓ_1, any subspace of $\ell_p(c_0)$ contains a further subspace isomorphic to ℓ_p (resp. c_0) and complemented in ℓ_p (resp. c_0), hence we can pass from *N* to a further subspace *W*, isomorphic to ℓ_1 or c_0, and complemented in *X* by a projection P. Embed *B(W)* isomorphically into *B(X)* by sending $T \in B(W)$ to PTP ∈ *B(X),* where P is a projection from *X* onto *W*. It is easy to see that *B(W)* contain subspaces isomorphic to ℓ_∞, thus, $B(X)$ is not subprojective.

There is only one option left: N is reflexive. Pick a subspace $W \subset N$, complemented in X. It has the Bounded Approximation Property. As in the previous paragraph, $B(W)$ embeds isomorphically into $B(X)$. Since$B(W) \neq K(W)$, shows that $B(W)$ contains an isomorphic copy of ℓ_∞. This rule out the subprojectivity of *B(X).*

We deal with spaces of functions on scattered spaces. Recall that a topological space is *scattered* if every compact subset has an isolated point. It is known that a compact set is scattered and metrizable if and only if it is countable (in this case, *C(K)*, and even its dual, are separable). It is well known that, if *K* is a compact Hausdorff set, then *C(K)* is separable if and only if *K* is metrizable.

If K is countable, then $C(K)$ is c_0-saturated, and the copies of c_0 are complemented, by Sobczyk's Theorem. Otherwise, by Milutin's Theorem, $C(K)$ is isomorphic to $C([0,1])$. Thus, a separable space $C(K)$ is subprojective if and only if *K* is scattered.

Furthermore, it is known that *K* is scattered if and only if it supports no non-zero atomic measures. Then $C(K)^*$ is isometric to ℓ_1(K). Otherwise, $C(K)^*$ contains a copy of $L_1(0,1)$. Thus, $C(K)^*$ is subprojective if and only if *K* is scattered.

We study the subprojectivity of projective and injective tensor products of *C(K)*. We have the following:

Theorem (4.2.10)[4]: *Suppose* K *is a compact metrizable space, and* X *is a Banach space. Then the following are equivalent:*

(i) K is scattered, and X is subprojective.

(ii) C(K,X) is subprojective.

Proof. The implication (ii) ⇒ (i) is easy. The space *C(K,X)* contains copies of *C(K)* and of *X*, hence the last two spaces are subprojective. By the preceding paragraph, *K* must be scattered.

To prove (i) ⇒ (ii), first fix some notation. Suppose λ is a countable ordinal. We consider the interval $[0,\lambda]$ with the order topology – that is, the topology generated by the open intervals (α,β), as well as $[0,\beta)$ and $(\alpha,\lambda]$. Abusing the notation slightly, we write $C(\lambda,X)$ for $C([0,\lambda],X)$.

Suppose *K* is scattered. *K* is isomorphic to $[0,\lambda]$, for some countable limit ordinal λ. Fix a subprojective space *X*. We use induction on λ to show that, for any countable ordinal λ,

$$C(\lambda, X)\ is\ subprojective. \tag{4.3}$$

By Proposition (4.1.1), (4.2.10) holds for $\lambda \leq \omega$ (indeed, c is isomorphic to c_0, hence $c(X) = c\check{\otimes}X$ is isomorphic to $c_0(X) = c_0\check{\otimes} X)$. Let F denote the set of all countable ordinals for which (4.2.10) fails. If F is non-empty, then it contains a minimal element, which we denote by μ. Note that μ is a limit ordinal. Indeed, otherwise it has an immediate predecessor $\mu - 1$. It is easy to see that $C(\mu, X)$ is isomorphic to $C(\mu - 1, X) \oplus X$, hence, by Proposition (4.1.11),$C(\mu - 1, X)$ is not subprojective. Let $C_0(\mu, X) = \{f \in C(\mu, X): \lim_{\nu\to\mu} f(\nu) = 0\}$. Clearly $C(\mu, X)$ is isomorphic to $C_0(\mu, X) \oplus X$, hence we obtain the desired contradiction by showing that $C_0(\mu, X)$ is subprojective.

To do this, suppose Y is a subspace of $C_0(\mu, X)$, so that no subspace of *Y* is complemented in $C_0(\mu, X)$. For $\nu < \mu$, define the projection $P_\nu: C(\mu, X) \to C(\nu, X): f \mapsto f_1[0, \nu]$. If, for some $\nu < \mu$ and some subspace $Z \subset Y, P_\nu|_Z$ is an isomorphism, then *Z* contains a subspace complemented in *X*, by the induction hypothesis and Corollary (4.1.13). Now suppose $P_\nu|_Y$ is strictly singular for any v. We construct a sequence of "almost disjoint" elements of Y. To do this, take an arbitrary y_1 from the unit sphere of Y. Pick $\nu_1 < \mu$ so that $\|y_1 - P_{\nu_1} y_1\| < 10^{-1}$. Now find a norm one $y_2 \in Y$ so that $\|P_{\nu_1} y_2\| < 10^{-2}/2$. Proceeding further in the same manner, we find a sequence of ordinals $0 = \nu_0 < \nu_1 < \nu_2 < \ldots$, and a sequence of norm one elements $y_1, y_2, \ldots \in Y$, so that $\|y_k - z_k\| < 10^{-k}$, where $z_k = (P_{\nu_k} - P_{\nu_{k-1}}) y_k$. The sequence (z_k) is equivalent to the c_0 basis, and the same is true for the sequence (y_k).

Moreover, span $[z_k: k \in \mathbb{N}]$ is complemented in $C(\mu, X)$. Indeed, let $\nu = \sup_k \nu_k$. We claim that $\mu = \nu$. If $\nu < \mu$, then P_ν is an isomorphism on span$[y_k: k \in \mathbb{N}]$, contradicting our assumption. Let $W_k = (P_{\nu_k} - P_{\nu_{k-1}})(C_0(X))$, and find a norm one linear functional wk so that $w_k(z_k) = \|z_k\|$. Define

$$Q: C_0(\mu, X) \to C_0(\mu, X): f \mapsto \sum_k w_k\big((P_{\nu_k} - P_{\nu_{k-1}})f\big)\, z_k.$$

Note that limk $\|(P_{\nu_k} - P_{\nu_{k-1}})f\| = 0$, hence the range of Q is precisely the span of the elements z_k. By Small Perturbation Principle, Y contains a subspace complemented in $C_0(\mu, X)$,.

The above theorem shows that $C(K)\check{\otimes} X$ is subprojective if and only if both *C(K)* and *X* are. We do not know whether a similar result holds for other tensor products. We have:

Proposition (4.2.11)[4]: *Suppose* K *is a compact metrizable space, and* W *is either* $\ell_p (1 \le p < \infty)$ *or* c_0. *Then* $C(K) \hat{\otimes} W$ *is subprojective if and only if* K *is scattered.*

Proof. Clearly, if K is not scattered, then $C(K)$ is not subprojective. So suppose K is scattered. We deal with the case of $W = \ell_p$, as the c_0 case is handled similarly. As before, we can assume that $K = [0, \lambda]$, where λ is a countable ordinal. We use transfinite induction on λ. The base case is easy: if λ is a finite ordinal, then $C(\lambda) \hat{\otimes} \ell_p = \ell_\infty^N \hat{\otimes} \ell_p$ is subprojective. Furthermore the same is true for $\lambda = \omega$ (then $C(\lambda) = c$).

Suppose, for the sake of contradiction, that λ is the smallest countable ordinal so that $C(\lambda) \hat{\otimes} \ell_p$ is not subprojective. Reasoning as before, we conclude that λ is a limit ordinal. Furthermore, $C(\lambda) \sim C_0(\lambda)$, hence $C_0(\lambda) \hat{\otimes} \ell_p$ is not subprojective.

Denote by $Q_n : \ell_p \to \ell_p$ the projection on the first n basis vectors in ℓ_p, and let $Q_n^\perp = I - Q_n$. For $f \in C_0(\lambda)$ and an ordinal $\nu < \lambda$, define $P_\nu f = \chi_{[0,\nu]} f$, and $P_\nu^\perp = I - P_\nu$.

Suppose X is a subspace of $C_0(\lambda) \hat{\otimes} \ell_p$ which has no subspaces complemented in $C_0(\lambda) \hat{\otimes} \ell_p$. By the induction hypothesis, $(P_\nu \otimes I_{\ell_p})|_Y$ is strictly singular for any $\nu < \lambda$. Furthermore, $(I_{C_0(\lambda)} \otimes Q_n)|_Y$ must be strictly singular. Indeed, otherwise Y has a subspace Z so that $(I_{C_0(\lambda)} \otimes Q_n)|_Z$ is an isomorphism, whose range is subprojective (the range of $I_{C_0(\lambda)} \otimes Q_n$ is isomorphic to the sum of n copies of $C(\lambda)$, hence subprojective). Therefore, for any $\nu < \lambda$ and $n \in N$, $(I - P_\nu^\perp \otimes Q_n^\perp)|_Y$ is strictly singular. Therefore we can find a normalized basis (x_i) in Y, and sequences $0 = \nu_0 < \nu_1 < \ldots < \lambda$, and $0 = n_0 < n_1 < \ldots$, so that $\|x_i - (P_{\nu_{i-1}}^\perp \otimes Q_{n_{i-1}}^\perp) x_i\| < 10^{-3i}/2$. By passing to a further subsequence, we can assume that $\|(P_{\nu_i} \otimes Q_{n_i}) x_i\| < 10^{-3i}/2$.Thus, by the Small Perturbation

Principle, it suffices to show the following statement: If (y_i) is a normalized sequence is $C_0(\lambda)\hat{\otimes}\ell_p$, so that there exist non-negative integers $0 = n_0 < n_1 < n_2 < \ldots$,, and ordinals $0 = \nu_0 < \nu_1 < \nu_2 < \ldots < \lambda$, with the property that $y_i = \big((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}})\big)y_i$ for any i, then $Y = span[y_i : i \in \mathbb{N}]$ is contractively complemented in $C(K)\hat{\otimes}\ell_p$.

Denote by X the span of all x's for which there exists an i so that $x = \big((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}})\big)x$. Then Y is contractively complemented in $(K)\hat{\otimes}\ell_p$. In fact, we can define a contractive projection onto X as follows. Suppose first $= \sum_{j=1}^{N} a_j \otimes b_j$, with b$_i$'s having finite support in ℓ_p. Then set $Pu = \sum_{i=1}^{\infty}\big((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}})\big)u$. Due to our assumption on the b$_i$'s, there exists M so that $Pu = \sum_{i=1}^{M}\big((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}})\big)u$. To show that $\|Pu\| \leq \|u\|$, define, for $\varepsilon = (\varepsilon_i)_{i=1}^{M} \in \{-1,1\}^M$, the operator of multiplication by $\sum_{i=1} M\varepsilon_i\chi_{[\nu_{i-1}+1,\nu_i]}$ on $C_0(\lambda)$. The operator $V_\varepsilon \in B(\ell_p)$ is defined similarly. Bot U_ε and V_ε are contractive. Furthermore, $Pu = Ave_\varepsilon(U_\varepsilon \otimes V_\varepsilon)u$. Therefore, we can use continuity to extend P to a contractive projection from $C_0(\lambda)\hat{\otimes}\ell_p$ onto X.

It to construct a contractive projection from X onto Y, we need to show that the blocks of X satisfy the ℓ_p-estimate. That is, if $x_i = \big(P_{\nu_i} - P_{\nu_{i-1}}\big) \otimes \big(Q_{n_i} - Q_{n_{i-1}}\big)x_i$ for each i, then $\|\sum_i x_i\|^p = \sum_i \|x_i\|^p$. To this end, use trace duality to identify $(C_0(\lambda)\hat{\otimes}\ell_p)^*$ with $B(\ell_p, \ell_1([0,\lambda)),)$. P^* is the "block" projection onto the space of "block diagonal" operators which map the elements of ℓ_p supported on $(n_{i-1}, n_i]$ onto the vectors in ℓ_1 supported on $(\nu_{i-1}, \nu_i]$. If T_i' s are the blocks of such an operator, then $\|\sum_i T_i\|^{p'} = \sum_i \|T_i\|^{p'}$, where $1/p + 1/p' = 1$. By duality, $\|\sum_i x_i\|^p = \sum_i \|x_i\|^p$.

Lemma (4.2.12)[4]: *Suppose* X *is a Banach space,* K *is a compact metrizable scattered space, and* $1 \le p \le q < \infty$. *Then, for any* $T \in \prod_{qp}(C(K), X)$, *and any* $\varepsilon > 0$, *there exists a finite rank operator* $S \in \prod_{qp}(C(K), X)$ *with* $\pi_{pq}(T - S) < \varepsilon$.

In proving Proposition (4.2.13) and Lemma (4.2.12), we consider the cases of $p = q$ and $p < q$ separately. If $p = q$, we are dealing with q-summing operators. By Pietsch Factorization Theorem, $T \in B(C(K), X)$ is q-summing if and only if there exists a probability measure μ on K so that T factors as $\tilde{T}\circ j$, where $j: C(K) \to L_q(\mu)$ is the formal identity, and $\|T\| \le \pi_q(T)$. Moreover, μ and $\tilde{T}$ can be selected in such a way that $\|\tilde{T}\| = \pi_q(T)$. As K is scattered, there exist distinct points $k_1, k_2, \ldots \in K$, and non-negative scalars $\alpha_1, \alpha_2, \ldots$, so that $\sum_i \alpha_i = 1$, and $\mu = \sum_i \alpha_i \delta_{k_i}$.

Now suppose $T \in B(C(K), X)$ satisfies $\pi_q(T) = 1$. Keeping the above notation, find $N \in \mathbb{N}$ so that $(\sum_{i=N+1}^{\infty} \alpha_i)^{\frac{1}{q}} < \varepsilon$. Denote by u and v the operators of multiplication by $\chi_{\{k_1,\ldots,k_N\}}$ and $\chi_{\{k_{N+1},k_{N+2},\ldots\}}$, respectively, acting on $L_q(\mu)$. It is easy to see that rank $u \le N$, and $\|v_j\| < \varepsilon$. Then $S = \tilde{T}uj$ works in Lemma (4.2.12). If $1 \le p < q$, then $\prod_{qp}(C(K), X) = \prod_{q1}(C(K), X)$, with equivalent norms. Henceforth, we set $p = 1$. We have a probability measure μ on K, and a factorization $T = \tilde{T}j$, where $j: C(K) \to L_{q1}(\mu)$ is the formal identity, and $\tilde{T}: L_{q1}(\mu) \to X$ satisfies $\|\tilde{T}\| \le c\pi_{q1}(T)$ (c is a constant depending on q).

In this case, the proof of Lemma (4.2.12) proceeds as for q-summing operators, except that now, we need to select N so that $c(\sum_{i=N+1}^{\infty} \alpha_i)^{1/q} < \varepsilon$.

Proposition (4.2.13)[4]: *Suppose* K *is a scattered compact metrizable space, and* $1 \leq p \leq q < \infty$. *Then the space* $\prod_{qp}(C(K), \ell_q)$ *is subprojective.*

Recall that $\prod_{qp}(X, Y)$ stands for the space of (*q*,*p*)-summing operators – that is, the operators for which there exists a constant *C* so that, for any $x_1, \ldots, x_n \in X$,

$$\left(\sum_i \|Tx_i\|^q\right)^{1/q} \leq C \sup_{x^* \in B(X^*)} \left(\sum_i |x^*(x_i)|^p\right)^{1/p}.$$

The smallest value of *C* is denoted by $\pi_{pq}(T)$.

Note that, if a compact Hausdorff space *K* is not scattered, then $C(K)^*$ contains L_1, hence $\prod_{qp}(C(K), \ell_q)$ is not subprojective.

We have the following lemma:

Proof. It is well known that, for any $T, \pi_{qp}(T) = \pi_{qp}(T^{**})$. Moreover, by Lemma (4.2.12), any (*q*,*p*)-summing operator on *C(K)* can be approximated by a finite rank operator. Then we can identify $\prod_{qp}(C(K), X)$ with the completion of the algebraic tensor product $C(K)^* \otimes X$ in the appropriate tensor norm which we denote by α. Recalling that $C(K)^* = \ell_1$ (the canonical basis in ℓ_1 corresponds to the point evaluation functionals), we can describe α in more detail: for $u = \sum_i a_i \otimes x_i \in \ell_1 \otimes X, \|u\|_\alpha = \pi_{qp}(\bar{u})$, where $\bar{u}: \ell_\infty \to X$ is defined by $\bar{u}b = \sum_i b(a_i)x_i$. Furthermore, by the injectivity of the ideal $\prod_{qp}$, $\pi_{qp}(\bar{u}) = \pi_{qp}(\kappa_X \circ \bar{u})$, where $\kappa_X: X \to X^{**}$ is the canonical embedding. Finally, $\kappa_X \circ \bar{u} = \tilde{u}^{**}$, with $\tilde{u} : c_0 \to X$ defined via $\tilde{u}b = \sum_i b(a_i)x_i$.

To finish the proof, we need to show that $\ell_1 \otimes_\alpha \ell_q$ satisfies the ℓ_q estimate. To this end, suppose we have a block-diagonal sequence $(u_i)_{i=1}^n$, and show that $\|\sum_i u_i\|_\alpha^q \sim \sum_i \|u_i\|_\alpha^q$. Abusing the notation slightly, we identify u_i with an operator from ℓ_∞^N *to* ℓ_q^N (where N is large enough), and identify $\|\cdot\|_\alpha$ with $\pi_{qp}(\cdot)$.

First show that $\|\sum_i u_i\|_\alpha^q \le c^q \sum_i \|u_i\|_\alpha^q$, where c is a constant (depending on q). We have disjoint sets $(S_i)_{i=1}^n$ *in* $\{1, \ldots, N\}$ so that $u_i e_j = 0$ for $j \notin S_i$.

Therefore there exists a probability measure μ_i, supported on S_i, so that

$$\|u_i f\|^q \le c_1^q \pi_{qp}(u_i)^q \|f\|_\infty^{q-p} \|f\|_{L_p(\mu_i)}^p$$

for any $f \in \ell_\infty^N$ (c_1 is a constant). Now define the probability measure μ on $\{1, \ldots, N\}$:

$$\mu = \left(\sum_i \pi_{qp}(u_i)^q\right)^{-1} \sum_i \pi_{qp}(u_i)^q \mu_i .$$

For $f \in \ell_\infty^N$, set $f_i = f\chi_{S_i}$. Then the vectors $u_i f_i$ are disjointly supported in ℓ_q, and therefore,

$$\left\|(\sum_i u_i) f\right\|^q = \sum_i \|u_i f_i\|^q \le c_1^q \sum_i \pi_{qp}(u_i)^q \|f_i\|_\infty^{q-p} \|f_i\|_{L_p(\mu_i)}^p \le c_1^q \|f\|_\infty^{q-p} \sum_i \pi_{qp}(u_i)^q \|f_i\|_{L_p(\mu_i)}^p .$$

An easy calculation shows that

$$\|f_i\|_{L_p(\mu_i)}^p = \left(\sum_i \pi_{qp}(u_i)^q\right)^{-1} \sum_i \pi_{qp}(u_i)^q \|f_i\|_{L_p(\mu)}^p ,$$

hence

$$\left\|(\sum_i u_i) f\right\|^q \le c_1^q \left(\sum_i \pi_{qp}(u_i)^q\right) \|f\|_\infty^{q-p} \sum_i \|f_i\|_{L_p(\mu_i)}^p = c_1^q \left(\sum_i \pi_{qp}(u_i)^q\right) \|f\|_\infty^{q-p} \|f\|_{L_p(\mu)}^p .$$

Therefore, $\pi_{qp}(\sum_i u_i) \le c\left(\sum_i \pi_{qp}(u_i)^q\right)^{1/q}$, for some universal constant c.

Next show that $\|\sum_i u_i\|_\alpha^q \geq c'^q \sum_i \|u_i\|_\alpha^q$, where c' is a constant. There exists a probability measure μ on $\{1, \ldots, N\}$ so that, for any $f \in \ell_\infty^N$,

$$\left\|\left(\sum_u u_i f\right)\right\|^q \geq c_2^q \pi_{qp}\left(\sum_i u_i\right)^q \|f\|_\infty^{q-p} \|f\|_{L_p(\mu)}^p$$

For each i let $\alpha_i = \left\|\mu|_{S_i}\right\|_{\ell_1^N}$, and $\mu_i = \mu_i / \alpha_i$ (if $\alpha_i = 0$, then clearly $u_i = 0$). Then for any i, and any $f \in \ell_\infty^N$,

$$\|u_i f\|^q = \left\|\left(\sum_i u_i\right)(\chi_{S_i} f)\right\|^q \leq c_2^q \pi_{qp}\left(\sum_i u_i\right)^q \alpha_i \|f\|_\infty^{q-p} \|f\|_{L_p(\mu_i)}^p,$$

hence $\pi_{qp}(u_i) \leq c' \alpha_i^{1/q} \pi_{qp}(\sum_i u_i)$ (c' is a constant). As $\sum_i \alpha_i = 1$, we conclude that $\sum_i \pi_{qp}(u_i)^q \leq c'^q \pi_{qp}(\sum_i u_i)$.

We refer for an introduction into continuous fields of Banach spaces. To set the stage, suppose K is a locally compact Hausdorff space (the *base space*), and $(X_t)_{t \in K}$ is a family of Banach spaces (the spaces X_t are called (*fibers*). A *vector field* is an element of $\prod_{t \in K} X_t$. A linear subspace X of $\prod_{t \in K} X_t$ is called a *continuous field* if the following conditions hold:

(i) For any $t \in K$, the set $\{x(t) : x \in X\}$ is dense in X_t.

(ii) For any $x \in X$, the map $t \mapsto \|x(t)\|$ is continuous, and vanishes at infinity.

(iii) Suppose x is a vector field so that, for any $\varepsilon > 0$ and any $t \in K$, there exist an open neighborhood $U \ni t$ and $y \in X$ for which $\|x(s) - y(s)\| < \varepsilon$ for any $s \in U$. Then $x \in X$.

Equipping X with the norm $\|x\| = max_t \|x(t)\|$, we turn it into a Banach space.

We prove:

Proposition (4.2.14)[4]: *Suppose* K *is a scattered metrizable space,* X *is a separable con-tinuous vector filed on* K, *so that, for every* $t \in K$, *the fiber* X_t *is subprojective. Then* X *is subprojective.*

Proof. Using one-point compactification if necessary, we can assume that K is compact. As before, we assume that $K = [0, \lambda]$ (λ is a countable ordinal). We denote by $X_{(0)}$ the set of all $x \in X$ which vanish at λ. If $\nu \leq \lambda$, we denote by $X_{[\nu]}$ the set of all $x \in X_\lambda$ which vanish outside of $[0, \nu]$. $x\chi_{[0,\nu]} \in X$ for any $x \in X$, hence $X_{[\nu]}$ is a Banach space. We then define the restriction operator $P_\nu : X \to X_{[\nu]}$. We denote by $Q_\nu : X \to X_\nu$ the operator of evaluation at ν.

We say that a countable ordinal λ has Property P if, whenever X is a continuous separable vector field whose fibers are subprojective, then X is subprojective. Using transfinite induction, we prove that any countable ordinal has this property.

The base of induction is easy to handle. Indeed, when λ is finite, then X embeds into a direct sum of (finitely many) subprojective spaces X_ν. Now suppose, for the sake of contradiction, that λ is the smallest ideal failing Property P. Note that λ is a limit ordinal. Indeed, otherwise it has an immediate predecessor λ_-, and X embeds into a direct sum of two subprojective spaces – namely, $X_{[\lambda_-]}$ and X_λ.

Suppose Y is a subspace of X, so that no subspace of Y is complemented in X. We shall achieve a contradiction once we show that Y contains a copy of c_0.

By Proposition (4.1.2), Q_λ is strictly singular on Y. Passing to a smaller subsequence if necessary, we can assume that, Y has a basis $(y_i)_{i\in\mathbb{N}}$, so that

(i) for any finite sequence (α_i), $\|\sum_i \alpha_i y_i\| > \max_i |\alpha_i|/2$, and (ii) for any i, $\|Q_\lambda y_i\| < 10^{-4i}$. Consequently, for any $y \in span[y_j : j > i]$, $\|Q_\lambda y\| < 10^{-4i}$. Indeed, we can assume that y is a norm one vector with finite support, and write y as a finite $sume\ y = \sum_j \alpha_j y_j$.

By the above, $|\alpha_i| \leq 2$ for every i. Consequently,

$$\|Q_\lambda y\| \leq \sum_j |\alpha_j| \|Q_\lambda y_j\| \leq 2\sum_{j>i} 10^{-4j} < 10^{-4i}.$$

Now construct a sequence $\nu_1 < \nu_2 < \ldots < \lambda$ of ordinals, a sequence $1 = n_1 < n_2 < \ldots$ or positive integers, and a sequence $x_1, x_2, \ldots$ of norm one vectors, so that (i) $x_j \in span[y_i : n_j \leq i < n_{j+1}]$, (ii) $\|P_{\nu_i} x_i\| < 10^{-4i}$, and (iii) $\|P_{\nu_{i+1}} x_i\| < 10^{-4i}$. To this end, recall that, by Proposition (4.1.2) again, $P_\nu|_Y$ is strictly singular for any $\nu < \lambda$. Pick an arbitrary $\nu_1 < \lambda$, and find a norm 1 vector $x_1 \in span[y_1, \ldots, y_{n_2 - 1}]$ so that $\|P_{\nu_1} x_1\| < 10^{-4}$. We have $\|Q_\lambda x_1\| < 10^{-4}$. By continuity, we can find $\nu_2 > \nu_1$ so that $\|P_{\nu_2} x_1\| < 10^{-4}$. Next find a norm one $x_2 \in span[y_{n_2}, \ldots, y_{n_3 - 1}]$ so that $\|P_{\nu_2} x_1\| < 10^{-8}$. Proceed further in the same manner.

We claim that the sequence (x_i) is equivalent to the canonical basis in c_0. Indeed, for each i let $x_i'' = P_{\nu_i} x_i + P_{\nu_{i+1}} x_i$, and $x_i' = x_i - x_i''$. Since we are working with the sup norm, $\|x_i'\| = \|x_i\| = 1$ for any i. Furthermore, the elements x_i' are disjointly supported, hence, for any (α_i) finite sequence of scalars (α_i), $\sum_i \|\alpha_i x_i'\| = \max_i |\alpha_i|$.

By the triangle inequality,

$$\left| \left\| \sum_i \alpha_i x_i \right\| - \left\| \sum_i \alpha_i x_i' \right\| \right| \leq \sum_i |\alpha_i| \, \|x_i''\| < \max_i |\alpha_i| \sum_{i=1}^{\infty} 2 \cdot 20^{-4i} < 10^{-3} \max_i |\alpha_i|,$$

which yields the desired result.

To state a corollary of Proposition (4.2.14), recall that a C^*-algebra A is *CCR* (or *liminal*) if, for any irreducible representation π of A on a Hilbert space $H, \pi(A) = K(H)$. A C^*-algebra A is *scattered* if every positive linear functional on A is a sum of pure linear functionals ($f \in A^*$ is called *pure* if it belongs to an extreme ray of the positive cone of A^*). For equivalent descriptions of scattered C^*-algebras.

Corollary (4.2.15)[4]: *Any separable scattered CCR C^*-algebra is subprojective.*

Proof. Suppose A is a separable scattered CCR C^*-algebra. As shown, the spectrum of a separable CCR algebra is a locally compact Hausdorff space. If, in addition, the algebra is scattered, then its spectrum $\hat{A}$ is scattered as well. In fact, $\hat{A}$ is separable. It is easy to see that any separable locally compact Hausdorff space is metrizable. We have A can be represented as a vector field over $\widehat{A}$, with fibers of the form $\pi(A)$, for irreducible representations π. As A is CCR, the spaces $\pi(A) = K(H_\pi)$ (H_π being a separable Hilbert space) are subprojective. To finish the proof, apply Proposition (4.2.14).

The last corollary leads us to

Conjecture (4.2.16)[4]: A separable C^*-algebra is scattered if and only if it is subprojective.

It is known that a scattered C^*-algebra is GCR. However, it need not be CCR (consider the unitization of $K(\ell_2)$).

We establish:

Proposition (4.2.17)[4]: *Suppose $\mathfrak{C}_\varepsilon$ is a symmetric sequence space, not* *containing c_0.*

Suppose, furthermore, that $(z_n) \subset \mathfrak{C}_\varepsilon$ is a normalized sequence, so that, for every k, $lim_n \|Q_k z_n\| = 0$. Then, for any ε > 0, $\mathfrak{C}_\varepsilon$ contains sequences $(\tilde{z}_n)$ and (z'_n), so that:

- *(i)* $(\tilde{z}_n)$ *is a subsequence of* (z_n).
- *(ii)* $\sum_n \|\tilde{z}_n - z'_n\| < \varepsilon$.
- *(iii)* (z'_n) *lies in the subspace Z of* $\mathfrak{C}_\varepsilon$ *, with the property that* (i) Z *is* 3-*isomorphic to either* $\ell_2, \varepsilon,$ *or* $\ell_2 \oplus \varepsilon$*, and* (ii) Z *is the range of a projection of norm not exceeding* 3.

Proof. We implie the existence of $(\tilde{z}_n)$ and (z'_n), so that (i) and (ii) are satisfied, and $z'_k = a \otimes E_{1k} + b \otimes E_{1k} + c_k \otimes E_{kk}$ $(k \geq 2)$. Thus, $z'_n \subset Z = Z_r + Z_c + Z_d$, where $Z_r = span[a \otimes E_{1k} : k \geq 2]$ (the row component), $Z_c = span[b \otimes E_{1k} : k \geq 2]$ (the column component), and Z_d (the diagonal component) contains $c_k \otimes E_{kk}$, for any k. More precisely, we can write $c_k = u_k d_k v_k$, where u_k and v_k are unitaries, and d_k is diagonal. Then we set $Z_d = span[u_k E_{ii} v_k \otimes E_{kk} : i \in \mathbb{N}, k \geq 2]$.

It remains to build contractive projections P_r, P_c, and P_d onto Z_r, Z_c, and Z_d, respectively, so that $Z_c \cup Z_d \subset ker\, P_r, Z_r \cup Z_d \subset ker\, P_c$, and $Z_r \cup Z_c \subset ker\, P_d$. Indeed, then $P = P_r + P_c + P_d$ is a projection onto $Z_r + Z_c + Z_d$, and the latter space is completely isomorphic to $Z_0 = Z_r \oplus Z_c \oplus Z_d$. The spaces Z_r, Z_c, and Z_d are either trivial (zero-dimensional), or isomorphic to ℓ_2, ℓ_2, and ε, respectively.

P_d is nothing but a coordinate projection, in the appropriate basis:

$$P_d(u_k E_{ij} v_\ell \otimes E_{k\ell}) = \begin{cases} u_k E_{ii} v_k \otimes E_{kk} & k = \ell \geq 2, i = j \\ 0 & otherwise \end{cases}$$

(for the sake of convenience, we set $u_1 = v_1 = I_{\ell_2}$). Next construct P_r (P_c is dealt with similarly). If $a = 0$, just take $P_r = 0$. Otherwise, let $a' = a/\|a\|$, and find $f \in \mathfrak{C}_\varepsilon^*$ so that $\|f\| = 1 = \langle f, a'\rangle$. For $x = \sum_{k,\ell} x_{k\ell} \otimes E_{k\ell}$, define

$$P_r x = a' \otimes \sum_{\ell \geq 2} \langle f, x_{1\ell}\rangle E_{1\ell} ,$$

hence $\|P_r x\|_\varepsilon^2 = \sum_{\ell > 2} |\langle f, x_{1\ell}\rangle|^2$. It remains to show $\|P_r x\| \leq \|x\|$. This inequality is obvious when $P_r x = 0$. Otherwise, set, for $\ell \geq 2$,

$$\alpha_\ell = \frac{\overline{\langle f, x_{1\ell}\rangle}}{(\sum_{\ell > 2} |\langle f, x_{1\ell}\rangle|^2)^{1/2}} ,$$

$y = I_{\ell_2} \otimes \sum_{\ell \geq 2} \alpha_\ell E_{\ell 1}$, and$z = I_{\ell 2} \otimes E_{11}$. Then$\|y\|_\infty = (\sum_{\ell \geq 2} |\alpha_\ell|^2)^{1/2} = 1 = \|z\|_\infty$, and $zxy = \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \otimes E_{11}$. Therefore,

$$\|P_r x\|_\varepsilon = \left(f, \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \right) \leq \left\| \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \right\|_\varepsilon = \left\| \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \otimes E_{11} \right\|_\varepsilon = \|zxy\|_\varepsilon$$
$$\leq \|z\|_\infty \|x\|_\varepsilon \|y\|_\infty = \|x\|_\varepsilon ,$$

which is what we need.

Proposition (4.2.18)[4]: *Suppose ε is a symmetric sequence space, not containing c_0. Then $\mathfrak{C}_\varepsilon$ is subprojective if and only if ε is subprojective.*

The assumptions of this proposition are satisfied, for instance, if $\varepsilon = \ell_p$ $(1 \leq p < \infty)$, or if ε is the Lorentz space $\mathrm{l}(w, p)$: (Given $p, r \in (0, \infty]$, the Lorentz space $L^{p,r}(\Omega)$ is defined by $L^{p,r}(\Omega) = \{f \in M(\Omega);\ \|f\|_{p,r}\ \ \|f\|_{p,r;\Omega} < \infty\}$)[10]. However, not every symmetric sequence space is subprojective. Indeed, suppose ε is Pelczynski's universal space: it has an unconditional basis (u_i) so that any other unconditional basis is equivalent to its subsequence. As explained in, ε has a symmetric basis. Fix $1 < p < q < 2$. Then the Haar basis in $L_p(0,1)$

is unconditional, hence $L_p(0,1)$ is isomorphic to a complemented subspace X of ε. It is well known that ℓ_q is contained in $L_p(0,1)$. Call the corresponding subspace of ε by X'.

Then no subspace of X' is complemented in E: otherwise, $L_p(0,1)$ would contain a complemented copy of ℓ_q, which is impossible.

For the proof, we need a technical result.

Proof. The space $\mathfrak{C}_\varepsilon$ contains an isometric copy of ε, hence thesubprojectivity of $\mathfrak{C}_\varepsilon$ implies that of ε. To prove the converse, suppose ε is subprojective, and Z_0 is a subspace of $\mathfrak{C}_\varepsilon$, and show that it contains a further subspace Z, complemented in $\mathfrak{C}_\varepsilon$. To this end, find a normalized sequence $(z_n) \subset Z_0$, so that $lim_n \|Q_k z_n\| = 0$ for every k. By Proposition (4.2.18), (z_n) has a subsequence (z'_n), contained in a subspace Z_1, which is complemented in $\mathfrak{C}_\varepsilon$, and isomorphic either to ε, ℓ_2, or $\varepsilon \oplus \ell_2$. By Proposition (4.1.1), Z_1 is subprojective, hence $span[z'_n : n \in \mathbb{N}]$ contains a subspace complemented in Z_1, hence also in $\mathfrak{C}_\varepsilon$.

As a consequence we obtain:

Proposition (4.2.19)[4]: *The predual of a von Neumann algebra* A *is subprojective if and only if* A *is purely atomic.*

We say that A is *purely atomic* if any projection in it has an atomic subprojection. It is easy to see that this happens if and only if $A = (\sum_i B(H_i))_\infty$. The "if" direction is easy. Conversely, if A is purely atomic, denote by $(e_i)_{i\in I}$ a maximal collection of mutually non-equivalent atomic projections in A. Denote by *z(p)* the central cover of *p*. Then $z(e_i)z(e_j) = 0$ if $i \neq j$, and $\sum_i z(e_i) = 1$. Consequently, $A = \sum_i z(e_i)A$.

For a fixed i, let $(f_j)_{j\in J(i)}$ be a maximal family of mutually orthogonal atomic projections, so that e_i is one of these projections. The f_j 's have the same central cover (namely, $z(e_i)$), hence they are all equivalent to e_i. Furthermore, $(e_i) = \sum_{j\in J(i)} f_j$, hence $z(e_i)A$ is isomorphic to $B(\ell_2(J(i)))$.

Proof. If a von Neumann algebra A is not purely atomic, then, as explained, A_* contains a (complemented) copy of $L_1(0,1)$. This establishes the "only if" implication of Proposition (4.2.19). Conversely, if A is purely atomic, then A_* is isometric to a (contractively complemented) subspace of $\mathfrak{C}_1(H)$, and the latter is subprojective.

We say that X is p-*disjointly homogeneous* (p-*DH* for short) if every disjoint normalized sequence contains a subsequence equivalent to the standard basis of ℓ_p.

For the sake of completeness we have

Proposition (4.2.20)[4]: *Let* X *be a* p-*convex. Then every subspace, spanned by a disjoint sequence equivalent to the canonical basis of ℓ_p, is complemented.*

Proof. Let $(x_k) \subset X$ be a disjoint normalized sequence. Since X is DH, by passing to a subsequence, (x_k) is an ℓp basic sequence. Then, in the p-concavification $X_{(p)}$ the disjoint sequence $({x_k}^p)$ is an ℓ_1 basic sequence. Therefore, there exists a functional $x^* \in [({x_k}^p)]$ such that $x^*({x_k}^p) = 1$ for all k. By the Hahn-Banach Theorem x^* can be extended to a positive functional in ${X_{(p)}}^*$. Define a seminorm $\|x\|_p = (x^*(|x^p|))^{\frac{1}{p}}$ on X. Denote by N the subset of X on which this seminorm is equal to zero. Clearly, N is an ideal, therefore, the quotient space $\tilde{X} = X/\mathcal{N}$ is a Banach lattice, and the quotient map $Q: X \to \tilde{X}$ is an orthomorphism. With the defined seminorm $\tilde{X}$ is an abstract L_p-space, and the disjoint sequence $Q(x_k)$ is normalized. Therefore it is an ℓ_p basic sequence that spans a complemented subspace (in particular, Q is an isomorphism when restricted to $[x_k]$). Let $\tilde{P}$ be a projection from $\tilde{X}$ onto $[Q(x_k)]$.

Then $P = Q^{-1}\tilde{P}Q$ is a projection from X onto $[x_k]$.

Proposition (4.2.21)[4]: *Let* X *be a* p-*convex,* p-*disjointly homogeneous Banach lattice* ($p \geq 2$). *Then any subspace of* X *contains a complemented copy of either* ℓ_p *or* ℓ_2. *Consequently,* X *is subprojective.*

Proof. First, note that X is order continuous. Let $M \subseteq X$ be an infinite dimensional separable subspace. Then there exists a complemented order ideal in X with a weak unit that contains M. Therefore, without loss of generality, we may assume that X has a weak unit. Then there exists a probability measure μ such that we have continuous embeddings

$$L_\infty(\mu) \subseteq X \subseteq L_p(\mu) \subseteq L_2(\mu) \subseteq L_1(\mu).$$

Consequently, there exists a constant $c_1 > 0$ so that $c_1\|x\|_p \leq \|x\|$ for any $x \in X$.

We have the following:

Case 1. M contains an almost disjoint bounded sequence. By Proposition (4.2.20) M contains a copy of ℓ_p complemented in X.

Case 2. The norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent on M. Thus, there exists $c_2 > 0$ so that, for any $y \in M$,

$$c_2\|y\|_2 \geq c_2\|y\|_1 \geq \|y\| \geq c_1\|y\|_p \geq c_1\|y\|_2.$$

In particular, M is embedded into $L_2(\mu)$ as a closed subspace. The orthogonal projection from $L_2(\mu)$ onto M then defines a bounded projection from X onto M.

The preceding result implies that Lorentz space $\Lambda_p, W(0,1)$ is subprojective since it is p-DH and p-convex ($p \geq 1$).

If X is a Banach lattice, and $1 \leq p < \infty$, denote by $\widetilde{X(\ell_p)}$ the completion of the space of all finite sequences $(x_1, \ldots, x_n)$ (with $x_i \in X$), equipped with the norm $\|(x_1, \ldots, x_n)\| = \left\|(\sum_i |x_i|^p)^{1/p}\right\|$, where

$$(\sum_i |x_i|^p)^{1/p} = sup\left\{|\sum_i \alpha_i x_i| : \sum_i |\alpha_i|^{p'} \leq 1\right\}, with \frac{1}{p} + \frac{1}{p'} = 1.$$

We have:

Proposition (4.2.22)[4]: *Suppose* X *is a subprojective separable space, with the lattice structure given by an unconditional basis, and* $1 \leq p < \infty$. *Then* $\widetilde{X(\ell_p)}$ *is subprojective.*

Proof. To show that any subspace $Y \subset \widetilde{X(\ell_p)}$ has a further subspace Z, complemented in $\widetilde{X(\ell_p)}, let\ x_1, x_2, \ldots$ and $e_1, e_2, \ldots$ be the canonical bases in X and ℓ_p, respectively. Then the elements $u_{ij} = x_i \otimes e_j$ form an unconditional basis in $\widetilde{X(\ell_p)}$, with

$$\left\|\sum a_{ij} u_{ij}\right\| = \left\|\sum_i \left(\sum_j |a_{ij}|^p\right)^{1/p} x_i\right\|_X = \left\|\sum_i \left(\sup_{\sum_j |\alpha_j|^{p'} \leq 1} |\sum_j \alpha_{ij} a_{ij}|\right) x_i\right\|_X \quad (4)$$

Let P_n be the canonical projection onto $span[u_{ij}: 0 \leq i \leq n, j \in \mathbb{N}]$, and set $P_n^{\perp} = I - P_n$. The range of P_n is isomorphic to ℓ_p, hence, if $P_n|_Y$ is not strictly singular for some n, we are done, by Corollary (4.1.3). If $P_n|Y$ is strictly singular for every n, find a normalized sequence (y_i) in Y, and $1 =$

$n_1 < n_2 < \ldots$, so that $\|P_{n_i} y_i\|, \|P_{n_{i+1}}^{\perp} y_i\| < 100^{-i}/2$. By small perturbation, it remains to prove the following: if $y_i = P_{n_i}^{\perp} P_{n_{i+1}} y_i$, then $span[y_i : i \in \mathbb{N}]$ contains a subspace, complemented in $\widetilde{X(\ell_p)}$. Further, we may assume that for each *i* there exists M_i so that we can write

$$y_i = \sum_{n_i < k \leq n_{i+1}, 1 \leq j \leq M_i} a_{k_j} u_{k_j} \ .$$

For each $k \in [n_i + 1, n_{i+1}]$ (and arbitrary $i \in N$) find a finite sequence $(\alpha_{k_j})_{ij}^M = 1$ so that $\sum_j |\alpha_{k_j}|^{p'} = 1$, and $|\sum_j \alpha_{k_j} a_{k_j}| = (\sum_j |a_{k_j}|^p)^{1/p}$. Define $U : \widetilde{X(\ell_p)} \to X : u_{k_j} \mapsto \alpha_{k_j} a_{k_j} x_k$. By (4.2.22), U is a contraction, and $U|_{span[y_i : i \in N]}$ is an isometry. To finish the proof.

Recall that X is subprojective, and apply Corollary (4.1.3).

Recall that, for a Banach space X, we denote by Rad(X) the completion of the finite sums $\sum_n r_n x_n$ ($r_1, r_2, \ldots$ are Rademacher functions, and $x_1, x_2, \ldots \in X$) in the norm of $L_1(X)$ (equivalently, by Khintchine-Kahane Inequality, in the norm of $L_p(X)$). If X has a unconditional basis (x_i) and finite cotype, then Rad(X) is isomorphic to $\widetilde{X(\ell_2)}$ (here we can view X as a Banach lattice, with the order induced by the basis (x_i)). Indeed, X is q-concave, forsome q. An array (amn) can be identified both with an element of Rad(X) (with the norm $\int_0^1 \|\sum_m \sum_n a_{mn} r_n x_m\|$), and with an element of $\widetilde{X(\ell_2)}$ (with the norm $\|\sum_m (\sum_n |a_{mn}|^2)^{1/2} x_m\|$). Then

$$D\left\|\sum_m\left(\sum_n|a_{mn}|^2\right)^{\frac{1}{2}}x_m\right\| \le \left\|\sum_m\int_0^1|\sum_n a_{mn}r_n|x_m\right\| = \left\|\int_0^1|\sum_m\sum_n a_{mn}r_nx_m|\right\|$$

$$\le \left\|\int_0^1\sum_m\sum_n a_{mn}r_nx_m\right\| \le \left(\int_0^1\left\|\sum_m\sum_n a_{mn}r_nx_m\right\|^q\right)^{\frac{1}{q}}$$

$$\le M_q\left\|\left(\int_0^1\left|\sum_m\sum_n a_{mn}r_nx_m\right|^q\right)^{\frac{1}{q}}\right\| \le M_q\left\|\sum_m\left(\int_0^1\left|\sum_n a_{mn}r_n\right|^q\right)^{\frac{1}{q}}x_m\right\| \le$$

$$CM_q\left\|\sum_m\left(\sum_n|a_{mn}|^2\right)^{1/2}x_m\right\|,$$

where M_q is a q-concavity constant, while *D* and *C* come from Khintchine's inequality. Thus, we have proved:

Proposition (4.2.23)[4]: *If* X *is a subprojective space with an unconditional basis andnon-trivial cotype, then* Rad(*X*) *is subprojective.*

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