University of Sudan for Science & Technology

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Isomorphically and Best Approximation in Polyhedral Banach Space

الأيزومورفكلي والتقريب الأفضل لفضاءات باناخ متعددة الأوجه

A thesis Submitted in Partial Fulfillment of the Requirements of the M.Sc. Degree in Mathematics

By

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Detication

Alhamdulillah firstly and lastly, I would like to dedicate this thesis to my beloved parents and everyone who stood by my side.

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Abstract

We classify the isomorphically polyhedral $\mathcal{L}_\infty$_space, direct sum of Banach space, generalized centers of finite sets, best approximation and smoothness in polyhedral Banach Spaces.

الخلاصة

صنفنا فضاءات _ $\mathcal{L}_\infty$ متعددة الأوجه . الايزومورفكليه والجمع المباشر لفضاءات باناخ والمراكز المعممه للفئات المنتهيه والتقريب الأفضل والملسان في فضاءات باناخ متعددة الأوجه.

The Contents

Chapter 1

On Isomorphically Polyhedral $\mathcal{L}_\infty$-Spaces

We show that there exist $\mathcal{L}_\infty$-subspaces of ageneralized.

Section (1.1): Isomorphically Polyhedral Spaces

A Banach space is said to be polyhedral if the closed unit ball of every finite dimensional subspace is the closed convex hull of a finite number of points. Polyhedrality is a geometrical notion: c_0 is polyhedral while c is not. It is also an hereditary notion: every subspace of a polyhedral space is polyhedral. The isomorphic notion associated with polyhedrality is: A Banach space is said to be isomorphically polyhedral if it admits a polyhedral renorming. The simplest examples of isomorphically polyhedral spaces are the $C(\alpha)$ spaces for α an ordinal, and their subspaces. In [5] we surveyed what is known about polyhedral $\mathcal{L}_\infty$-spaces, which can be summarized as follows:

(i) There are polyhedral spaces which are not $\mathcal{L}_\infty$: indeed, any non $\mathcal{L}_\infty$ subspace of $c_0(\Gamma)$ — recall that subspaces of $c_0(\Gamma)$ are $\mathcal{L}_\infty$-spaces if and only if they are isomorphic to $c_0(I)$.

(ii) There are Lindenstrauss spaces not polyhedral: $C[0,1]$.

(iii) A result of Fonf [8] asserts that preduals of ℓ_1 are isomorphically polyhedral.

(iv) Fonf informed us that the result fails for $\ell_1(\Gamma)$: Kunen's compact $\mathcal{K}$ provides, under CH, a scattered, non metrizable, compact so that $C(\mathcal{K})$ space has the rare property that every uncountable set of elements contains one that belongs to the closure of the convex hull of the others. And this property was used by Jiménez and Moreno to show that every equivalent renorming of $C(\mathcal{K})$ has only a countable number of weak*-strongly exposed points. Thus, no equivalent renorming can be polyhedral. At the same time $C(\mathcal{K})^* = \ell_1(\Gamma)$ since $\mathcal{K}$ is scattered.

(v) The trees T for which $C(T)$ is isomorphically polyhedral are characterized. Thus, there are scattered compact K (not depending on CH as it occurs with Kunen's compact) such that $C(K)$ is not isomorphically polyhedral.

Whether isomorphically polyhedral L_∞-spaces are isomorphically Lindenstrauss. The purpose of this note is to show that the answer is no.

A Banach space X is said to be an $\mathcal{L}_{\infty,\lambda}$-space if every finite dimensional subspace F of X is contained in another finite dimensional subspace of X whose Banach-Mazur distance to the corresponding space ℓ_∞^n is at most λ . The space X is said to be an $\mathcal{L}_\infty$-space if it is an $\mathcal{L}_{\infty,\lambda}$-space for some λ. The basic theory and examples of $\mathcal{L}_\infty$-spaces. A Banach space X is said to be a Lindenstrauss space if it is an isometric predual of some space $L_1(\mu)$. Lindenstrauss spaces correspond to $\mathcal{L}_{\infty,1^+}$-spaces. A Lindenstrauss space is an $\mathcal{L}_{\infty,1}$-space if and only if it is polyhedral (i.e., the unit ball of every finite dimensional subspace is a polytope).

A Banach space X is said to have Pelczyński's property (V) if each operator defined on X is either weakly compact or an isomorphism on a subspace isomorphic to c_0. Pelczyński shows that *C(K)*-spaces enjoy property (V), and Johnson and Zippin that Lindenstrauss spaces also have (V).

Let $\alpha: A \to Z$ and $\beta: B \to Z$ be operators acting between Banach spaces. the pull-back space PB is defined as $PB = PB(\alpha, \beta) = \{(a, b) \in A \oplus_\infty B: \alpha(a) = \beta(b)\}$. It has the property of yielding a commutative diagram

$$\begin{array}{ccc} PB & \xrightarrow{\;'\beta\;} & A \\ {\scriptstyle '\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\;\beta\;} & Z \end{array} \qquad (1)$$

in which the arrows after primes are the restriction of the projections onto the corresponding factor. Needless to say (1) is minimally commutative in the sense that if the operators $''\beta : C \to A$ and $''\alpha : C \to B$ satisfy $\alpha \circ {''\beta} = \beta \circ {''\alpha}$, then there is a unique operator $\gamma : C \to PB$ such that $''\beta = {'\beta}\gamma$ and $''\beta = {'\beta}\gamma$. Clearly, $\gamma(c) = (''\beta(c), ''\alpha(c))$ and $\|\gamma\| \leq max\{\|''\alpha\|, \|''\beta\|\}$. Quite clearly $'\alpha$ is onto if α is. As a consequence of this, if one has an exact sequence

$$0 \longrightarrow Y \xrightarrow{\;\iota\;} X \xrightarrow{\;\pi\;} Z \longrightarrow 0 \qquad (2)$$

and an operator $u : A \to Z$ then one can form the pull-back diagram of the couple (π, u):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\;\iota\;} & X & \xrightarrow{\;\pi\;} & Z & \longrightarrow & 0 \\ & & & & {\scriptstyle 'u}\uparrow & & \uparrow{\scriptstyle u} & & \\ & & & & PB & \xrightarrow{\;'\pi\;} & A & & \end{array} \qquad (3)$$

Recalling that $'\pi$ is onto and taking $j(y) = (0, \iota(y))$, it is easily seen that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \Vert & & {}'u \uparrow & & \uparrow u & & \\ 0 & \longrightarrow & Y & \xrightarrow{j} & \mathrm{PB} & \xrightarrow{'\pi} & A & \longrightarrow & 0 \end{array} \qquad (4)$$

Thus, the lower sequence is exact, and we shall refer to it as the pull-back sequence. The well-known splitting criterion is: the pull-back sequence splits if and only if u lifts to X; i.e., there is an operator $U: A \to X$ such that $\pi U = u$.

Theorem (1.1.1)[1]: There is a separable isomorphically polyhedral $\mathcal{L}_\infty$ space that is not isomorphically Lindenstrauss. Moreover, it is a subspace of an isomorphically polyhedral Lindenstrauss space.

Proof. We need to recall from [1] the existence of nontrivial exact sequences

$$0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega \xrightarrow{q} c_0 \longrightarrow 0$$

in which the quotient map q is strictly singular. This fact makes Ω fail Pe lczy´nski's property (V). Since Lindenstrauss spaces share with $C(K)$-spaces Pe lczyński's property (V), the space Ω is not isomorphic to a Lindenstrauss space. Of course it is an $\mathcal{L}_\infty$-space since this is a 3-space property. Thus, our purpose is to show that there is an Ω as above that is isomorphically polyhedral.

We recall from [1] the parameter $\rho_N(c_0)$, defined as the the least constant such that if $T: c_0 \to \ell_\infty(\omega^N)$ is a bounded linear operator such that

$dist(Tx, C(\omega^N)) \leq \|x\| \; for \; all \; x \in c_0$ then there is a linear map $L: c_0 \to C(\omega^N)$ with $\|T - L\| \leq \rho_N(c_0)$.

We show that $lim \; \rho_N(c_0) = +\infty$. Now we need a specific choice for each N: there is a bounded operator $T_N : c_0 \to \ell_\infty(\omega^N)$ so that $dist(T_N x, C(\omega^N)) \leq \|x\|$ for all $x \in c_0$ but such that if $E \subset c_0$ is a subspace of c_0 almost isometric to c_0 then $\rho_N(c_0) \leq 2\|T_N - L\|$ for any linear map $L: c_0 \to C(\omega^N)$.

Let, for each N, a linear continuous operator $T_N : c_0 \to \ell_\infty(\omega^N)$ as above. We form the twisted sum space

$$C(\omega^N) \oplus_{T_N} c_0 = (C(\omega^N) \times c_0, \|\cdot\| T_N)$$

endowed with the norm $\|(h, x)\|_{T_N} = max\{\|h - T_N x\|, \|x\|\}$. This yields an exact sequence

$$0 \longrightarrow C(\omega^N) \xrightarrow{i_N} C(\omega^N) \oplus_{T_N} c_0 \xrightarrow{q_N} c_0 \longrightarrow 0$$

with embedding $i_N(f) = (f, 0)$ and quotient map $q_N(f, x) = x$. The identity map $id: C(\omega^N) \oplus_{T_N} c_0 \longrightarrow C(\omega^N) \oplus_\infty c_0$ is an isomorphism since

$$\|T_N\|^{-1} \|(f, x)\|_{T_N} \leq \|(f, x)\|_\infty \leq \|T_N\| \|(f, x)\|_{T_N}$$

and therefore the space $C(\omega^N) \oplus_{T_N} c_0$ is isomorphically polyhedral. We need now to use the main result in [1] asserting that in a separable isomorphically polyhedral space every norm can be approximated by a polyhedral norm. Let $\|\cdot\|_{P_N}$ be a polyhedral norm in $C(\omega^N) \oplus_{T_N} c_0$ that is 2-equivalent to $\|\cdot\|_{P_N}$.

The sequence (4) splits, but the norm of the projection goes to infinity with N: Indeed, if

$$P : C(\omega^N) \oplus_{T_N} c_0 \to C(\omega^N)$$

is a linear continuous projection then P has to have the form $P(f, x) = (f - Lx, 0)$, where $L: c_0 \to C(\omega^N)$ is a certain linear map. Thus, if $x \in c_0$ is a norm one element, one gets $P(T_N x, x) = (T_N x - Lx, 0)$ and thus $T_N x - Lx \leq \|P\|\|x\|$, hence $\|T_N - L\| \leq \|P\|$. The choice of T_N forces $\lim_{N\to\infty} inf \|P\| = +\infty$. Therefore, the c_0-sum

$$0 \longrightarrow c_0(C(\omega^N)) \longrightarrow c_0(C(\omega^N)) \oplus_{P_N} c_0 \xrightarrow{(q_N)} c_0(c_0) \longrightarrow 0$$

cannot split. The space $c_0(C(\omega^N) \oplus_{P_N} c_0)$ is isomorphically polyhedral as any c_0-sum of polyhedral spaces. We now define a suitable operator Δ so that when making the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0(C(\omega^N)) & \longrightarrow & c_0(C(\omega^N)) \oplus_{P_N} c_0 & \xrightarrow{(q_N)} & c_0(c_0) & \longrightarrow & 0 \\ & & & & \uparrow \delta & & \uparrow \Delta & & \\ 0 & \longrightarrow & c_0(C(\omega^N)) & \longrightarrow & \Omega & \xrightarrow[q]{} & c_0 & \longrightarrow & 0 \end{array}$$

the map q is strictly singular. That prevents Ω from being Lindenstrauss under any equivalent renorming.

Pick as Δ the diagonal operator $c_0 \to c_0(c_0)$ induced by the scalar sequence $(\rho_N(c_0)^{-1/2}) \in c_0$; i.e.,

$$\Delta(x) = (\rho_N(c_0)^{-1/2} x)_N.$$

Assume that q is not strictly singular. Then, there is a subspace E of c_0 and a linear bounded map $V\colon E \to \Omega$ so that $qV = \Delta_{|E}$. By the c_0 saturation and the distortion properties of c0, there is no loss of generality assuming that E is an almost isometric copy of c_0. By the commutativity of the diagram $(q_N)\delta V = \Delta_{|E}$, which in particular means that $q_N \delta V(e) = \rho_N(c_0)^{-1/2} e$ for all $e \in E$. This means that the map δV has on E the form $(L_N e, \rho_N(c_0)^{-1/2} e)_N$ where $L_N : E \to C(\omega^N)$ is a linear map; by continuity, there is a constant M so that $(L_N e, \rho_N(c_0)^{-1/2} e) \leq M\|e\|$, which means

$$\left\| L_N e - T_N \rho_N(c_0)^{-1/2} e \right\| \leq M\|e\|$$

and thus

$$\rho_N(c_0)^{1/2} L_N - T_N \leq M \rho_N(c_0)^{1/2}.$$

This contradicts the fact that $E = c_0$, the definition of $\rho_N(c_0)$ and the choice of T_N.

To conclude the proof, the definition of pull-back space implies that Ω is actually a subspace of $c_0(C(\omega^N) \oplus_{P_N} c_0) \oplus_\infty c_0$, hence isomorphically polyhedral.

Since $c_0(C(\omega^N)) \simeq C(\omega^{\mathbb{N}})$, the space Ω above yields a twisted sum

$$0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega \overset{q}{\longrightarrow} c_0 \longrightarrow 0$$

in which q is strictly singular. The dual sequence

$$0 \longrightarrow \ell_1 \longrightarrow \Omega^* \longrightarrow \ell_1 \longrightarrow 0$$

necessarily splits and thus Ω^* can be renormed to be ℓ_1, although Ω cannot be endowed with an equivalent norm $|\cdot|$ so that $(\Omega, |\cdot|)^* = \ell_1$. Moreover,

Ω is actually a subspace of the isomorphically polyhedral Lindenstrauss space $c_0(C(\omega^N) \oplus_{P_N} c_0) \oplus c0$.

We show now that one can produce an $\mathcal{L}_\infty$-variation of Ω still farther from Lindenstrauss spaces. Lazar and Lindenstrauss showed that Lindenstrauss polyhedral spaces X enjoy the property that compact X-valued operator admit equal norm extensions. We introduce the Lindenstrauss-Pelczyński spaces (in short $\mathcal{L}_P$-spaces) as those Banach spaces E such that all operators from subspaces of c_0 into E can be extended to c_0. The spaces are so named because Lindenstrauss and Pelczyński first proved in [1] that $C(K)$-spaces have this property. Lindenstrauss spaces have also the property (as well as $\mathcal{L}_\infty$-spaces not containing c_0 and, of course, all their complemented subspaces. The construction of the space Ω above has been modified in [1] to show that for every subspace $H \subset c_0$ there is an exact sequence

$$0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega_H \longrightarrow c_0 \longrightarrow 0$$

in which the space Ω_H is not a Lindenstrauss-Pelczyński space [1]; more precisely, there is an operator $H \to \Omega_H$ that cannot be extended to the whole c_0.

Proposition (1.1.2)[1]: There is an isomorphically polyhedral $\mathcal{L}_\infty$-space that is not an LPspace.

Proof. Consider the exact sequence $0 \to C(\omega^\omega) \to \Omega \to c_0 \to 0$ with strictly singular quotient constructed above. Since every quotient of c0 is isomorphic to a subspace of c_0, we can consider that there is an embedding $u_H : c_0/H \to c_0$. The pull-back sequence $0 \to C(\omega^\omega) \to P_H \xrightarrow{p} c_0/H \to 0$ also has strictly singular quotient map. We form the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & = & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & P_H & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow t & & \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & \Omega_H & \xrightarrow[Q]{} & c_0 & \longrightarrow & 0 \\
 & & & & j\uparrow & & \uparrow i & & \\
 & & & & H & = & H & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad (5)
$$

to show, exactly as in [1] that Ω_H is not an $\mathcal{LP}$-space since j cannot be extended to c_0 through i. The space Ω_H has been obtained from a pull-back diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0(C(\omega^N)) & \longrightarrow & \Omega & \xrightarrow[Q]{} & c_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow u_H & & \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & P_H & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow t & & \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & \Omega_H & \xrightarrow[Q]{} & c_0 & \longrightarrow & 0
\end{array}
\qquad (6)
$$

and thus it is a subspace of $\Omega \oplus c_0$, hence isomorphically polyhedral.

Chapter 2

Polyhedral Direct Sums of Banach sSpaces, and Generalized Centers of Finite Sets

A Banach space X is said to satisfy (GC) if the set $E_f\ (a)$ of minimizers of the function $X \ni x \longmapsto f(\|x-a_1\|,\ldots,\|x-a_n\|)$ is nonempty for each integer $n \geq 1$, each $a \in X^n$ and each continuous nondecreasing coercive real-valued function f on $\mathbb{R}_+^n$. We study stability of certain polyhedrality properties under making direct sums, in order to be able to use results of, Lindenstrauss and an appropriate for every topological space.

Section (2.1): Finite Polyhedral Sums and Arbitrary c_0-Sums:

In Approximation Theory and Mathematical Economy, one often looks for a point in a Banach space X that would approximate (in an appropriate sense) a given bounded set $A \subset X$. Such problems, sometimes called *one-point location problems*, consist in minimizing a function depending on the distances from the elements of A.

Given a real-valued nondecreasing function f on $\mathbb{R}_+^n := [0,\infty)^n$, we are interested in the set $E_f\ (a)$ of minimizers of the function

$$\varphi(x) = f(\|x-a_1\|,\ldots,\|x-a_n\|) \quad (x \in X).$$

The Banach space X is said to *satisfy* (GC) if $E_f\ (a)$ is nonempty whenever n is a positive integer, $a \in X^n$, and f is continuous, nondecreasing and coercive. This property was introduced and studied in [2]. For instance, every dual Banach space satisfies (GC). One of the results in [2] states that if X is a finite-dimensional polyhedral Banach space and T is any topological space then the function space $C_b(T,X)$ (of bounded continuous functions of T into X) satisfies (GC).

We always consider $\mathbb{R}^n$ partially ordered by the coordinate-wise ordering. By $\mathbb{R}^n_+$ we denote the corresponding positivecone (i.e., the cone of all nonnegative elements).

Given a Banach space X, we denote by B_X and B^0_X its closed and open unit ball. Then $S_X = \partial B_X$ is the unit sphere. For $E \subset X^*$ and $x \in X$, we use the notation $\langle E, x\rangle = \{ f(x): f \in E\}$.

For $x \in X$ and $r \geq 0$, we define $B(x,r) = x + rB_X$ and $B^0(x,r) = int\, B(x,r)$ $(= x + rB^0_X \; if \; r > 0)$. A *boundary* of X is a set $B \subset B_{X^*}$ such that for each $x \in X$ there exists $f \in$ ß such that $\|x\| = f(x)$ (in other words, $\|x\| = max\, \langle ß, x\rangle$ for each $x \in X$).

By the Krein–Milman theorem (let K be a compact convex subset of X. Then, the theorem states K that is the closed convex hull of itsextreme points.

The closed convex hull above is defined as the intersection of all closed convex subsets of X that contain K)[5], the set $ext(B_{X^*})$ (extreme points of B_{X^*}) is always a boundary.

It is well known that the subdifferential (in the sense of Convex Analysis) of the norm ||·|| at *x* is exactly the set

$$\partial\|\cdot\|(x) = \{f \in B_X : f(x) = \|x\|\}.$$

Observe that $\partial\|\cdot\|(x) = \partial\|\cdot\|(x/\|x\|) \subset S_{X^*}$ for $x \neq 0$, and $\partial\|\cdot\|(0) = B_{X^*}$.

A Banach space X is called polyhedral if the unit ball of each of its finite-dimensional subspaces is a polytope; in this case we say that X satisfies (P). Many important results on polyhedral Banach spaces are due to V.P.

Definition (2.1.1)[2]: A Banach space is said to satisfy:

(a) $(P\Delta)$ if X is polyhedral and there exists a boundary ß for X such that $\partial\|\cdot\|(x) \cap ß$ is finite for each $x \in S_X$.

(b) $(*)$ if there exists a boundary B for X such that

$f(x) < 1$ whenever $x \in S_X$ and f is a w^*-cluster point of ß

(that is, if f is a w^*-cluster point of ß then either $\|f\| < 1$ or f does not attain its norm).

Fact (2.1.2)[2]: Let X be a Banach space.

(a) The properties (P), $(P\Delta)$ and $(*)$ are hereditary to closed subspaces.

(b) One has the implications $(*) \Rightarrow (P\Delta) \Rightarrow (P)$, and no other implication holds true.

(c) In Definition 1.2(a), (b), one can equivalently consider the particular boundary $ß = ext(B_{X^*})$.

(d) If X satisfies $(P\Delta)$ then each point $x \in S_X$ has a neighborhood V in which B_X coincides with a finite intersection of

closed halfspaces having x as a boundary point, that is,

$$B_X \cap V = \bigcap_{i=1}^{m} (x + H_i) \cap V$$

where each H_i is of the form $H_i = \{y \in X : h_i(y) \geq 0\}$ with $h_i \in S_{X^*}$.

(e) For finite-dimensional X, the conditions $(*)$,$(P\Delta)$ and (P) are equivalent.

Proof. For (c) and (d), see [2]; (a) easily follows from (c) and (b) can be found in [2]. Finally, if X is a finite-dimensional polyhedral space then $ext(B_{X^*})$ is a finite boundary for X, hence X satisfies $(*)$. This, together with (b), gives (e).

A function $\pi\colon \mathbb{R}_+^n \to \mathbb{R}_+$ is a *norm on* $\mathbb{R}_+^n$ if it is subadditive and positively homogeneous, and $\pi(t) = 0 \Leftrightarrow t = 0$.

A norm π on $\mathbb{R}_+^n$ is *polyhedral* if it is of the form $\pi(t) = max_{1\le j\le m}\, g_j(t)$ where $g_1, \ldots, g_m \in (\mathbb{R}^n)^*$. In this case we say that the family $\{g_1, \ldots, g_m\}$ *generates* π.

Given $\pi : \mathbb{R}_+^n \to \mathbb{R}_+$, we consider the following two canonical extensions of π to the whole $\mathbb{R}^n$:

$$\hat{\pi}(t) = \pi(|t|) \quad and \quad \tilde{\pi}(t) = \pi(t \vee 0)$$

(as usual, we denote $|t| = t \vee (-t) = (|t_1|, \ldots, |t_n|)$).

It is not difficult to show the following basic properties.

Lemma (2.1.3)[2]: Let π be a polyhedral nondecreasing norm on $\mathbb{R}_+^n$. Then every minimal family $\{g_1, \ldots, g_m\} \subset (\mathbb{R}^n)^*$ generating π is contained in $(\mathbb{R}^n)_+^*$ (i.e., each g j is nondecreasing on $\mathbb{R}^n$, or equivalently, its coordinates in the canonical identification $(\mathbb{R}^n)^* = \mathbb{R}^n$ are all nonnegative).

Proof. Fix $k \in \{1, \ldots, m\}$. By minimality, there exists $t \in \mathbb{R}_+^n$ with $\pi(t) = g_k(t) > max\{g_j(t)\colon j \in \{1, \ldots, m\}, j \neq k\}$. By continuity, $g_k = \pi$ in an open ball contained in $\mathbb{R}_+^n$. The fact that g_k is nondecreasing in an open ball easily implies that g_k is nondecreasing on the whole $\mathbb{R}^n$.

Let n be a positive integer, X a Banach space. Given $f: \mathbb{R}_+^n \to \mathbb{R}$ and $a = (a_1, \ldots, a_n) \in X^n$, consider the function $\varphi : X \to R$, given by

$$\varphi(x) = f(\|x - a_1\|, \ldots, \|x - a_n\|).$$

We define

$$r_f(a) = inf \varphi(X) \qquad (f - radius\ of\ a),$$

$$E_f\ (a) = x \in X: \varphi(x) = r_f(a) \quad (\text{the set of } f - centers \text{ of } a).$$

Moreover, if $E\ E_f \neq \emptyset$, we say that a admits f -centers.

If f is of the form $f(t_1, \ldots, t_n) = max_{1 \le i \le n}\ \rho_i t_i$ where $\rho = (\rho_1, \ldots, \rho_n) \in (0, \infty)^n$, we denote

$$r_\rho(a) = r_f(a), \qquad E_\rho(a) = E_f(a).$$

If $E_\rho(a) \neq \emptyset$, we say that a admits weighted Chebyshev centers for the weight ρ.

Definition (2.1.4)[2]: A Banach space X is said to satisfy (GC) if for each positive integer n, every $a \in X^n$ admits f–centers whenever $f: \mathbb{R}_+^n \to \mathbb{R}$ is a continuous, nondecreasing and coercive function.

Theorem (2.1.5)[2]: A Banach space X satisfies (GC) if and only if, for each positive integer n and each $\rho \in (0, \infty)^{\mathrm{n}}$, every $\mathrm{a} \in \mathrm{X}^{\mathrm{n}}$ admits weighted Chebyshev centers for the weight ρ.

Let us recall some (semi)continuity notions for multivalued mappings.

Definition (2.1.6)[2]: Let T be a Hausdorff topological space, X a normed linear space, $: T \to 2^X, t_0 \in T$.

(a) F is *l.s.c.* (lower semicontinuous) at t_0 if for each open set $A \subset X$ such that $A \cap F(t_0) \neq \emptyset$ there exists a neighborhood $V \subset T$ of t_0 such that $A \cap F\ (t) \neq \emptyset$ whenever $\in V$.

(b) F is *u.s.c.* (upper semicontinuous) at t_0 if for each open set $A \subset X$ such that $F(t_0) \subset A$ there exists a neighborhood $V \subset T$ of t_0 such that $F(t) \subset A$ whenever $\in V$.

(c) F is *H-l.s.c.* (Hausdorff lower semicontinuous) at t_0 if for each $\varepsilon > 0$ there exists a neighborhood $V \subset T$ of t_0 such that $F(t_0) \subset F(t) + \varepsilon B_M$ whenever $\in V$.

(d) F is *H-u.s.c.* (Hausdorff upper semicontinuous) at t_0 if for each $\varepsilon > 0$ there exists a neighborhood $V \subset T$ of t_0 such that $F(t) \subset F(t_0) + \varepsilon B_M$ whenever $\in V$.

(e) Let "s.c." denote one of the four semicontinuity properties defined in (a)–(d). We say that F is *s.c. on a set* $E \subset T$ if the restriction $F\,|_E$ is s.c. at each point of E.

(f) The *effective domain* of F is the set $dom(F) = \{x \in T: F(x) \neq \emptyset\}$.

It is easy to see that one always has the implications H-l.s.c.⇒l.s.c., and u.s.c.⇒H-u.s.c. Moreover, F is both H-l.s.c. and H-u.s.c. at t_0 if and only if F is continuous at t_0 with respect to the Hausdorff pseudometric

$$d_H(A, B) = max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

on 2^M. (Note that d_H, restricted to the closed elements of 2^M, is a metric with values in [0,∞].) In this case we say that F is *H-continuous* at t_0.

We shall need the following two lemmas. The first one, Lemma (2.1.7).

Lemma (2.1.7)[2]: Let T, S be Hausdorff topological spaces, X, Y normed linear spaces.

(a) Let $F: T \to 2^X$ be Hausdorff lower (upper) semicontinuous on $\mathrm{dom}(F)$, $\phi: S \to T$ continuous, $\psi: X \to Y$ uniformly continuous. Then also the mappings

$F_1 : S \to 2^X, \qquad F_1(s) = F(\varphi(s)),$

$F_2 : T \to 2^Y, \qquad F_2(t) = \psi(F(t)),$

are Hausdorff lower (upper) semicontinuous on their effective domains.

(b) Let $F: T \to 2^X$ be l.s.c., and $f: T \to X$ continuous. Then the multivalued mappings $t \longmapsto F(t) + f(t)$ and $t \longmapsto \overline{F(t)}$ are l.s.c.

Lemma (2.1.8)[2]: Let K be a compact Hausdorff topological space, X a Banach space. Let $\Phi : K \to 2^X$ be l.s.c. with nonempty closed convex values. Let $\varepsilon > 0$ and a continuous $v: K \to X$ be such that

$$\Phi(t) \cap B^0(v(t), \varepsilon) \neq \emptyset \qquad (t \in K).$$

Then Φ admits a continuous selection $u: K \to X$ such that $|u(t) - v(t)| \leq \varepsilon$ for each $\in K$.

Proof. Consider the mappings $\Phi_1(t) = \Phi(t) \cap B^0(v(t), \varepsilon)$ and $\Phi_2(t) = \overline{\Phi_1(t)}$. We can write

$$\Phi_1(t) = v(t) + ([\Phi(t) - v(t)] \cap B^0(0, \varepsilon)).$$

By Lemma (2.1.7)(b), $\Phi - v$ is l.s.c.; and this easily implies that $[\Phi - v] \cap B^0(0, \varepsilon)$ is l.s.c., too. By Lemma (2.1.7) (b) again, Φ_1 and Φ_2 are l.s.c., and hence, by Michael's selection theorem:

(Let E be a Banach space, X a paracompact space and $F : X \to E$ a lower hemicontinuous multivalued map with nonempty convexclosed values. Then there exists a continuous selection $f : X \to E$ of F.

Conversely, if any lower semicontinuous multimap from topological space X to a Banach space, with nonempty convex closed values admits continuous selection, then X is paracompact. This provides another characterization forparacompactness)[6], there exists a continuous $u: K \to X$ such that $u(t) \in \Phi_2(t) \subset \Phi(t) \cap B(0, \varepsilon)$ $(t \in K)$.

Corollary (2.1.9)[2]: Let K be a compact Hausdorff topological space, X a Banach space, and $G_1, G_2: K \to 2^X$ two bounded l.s.c. multivalued mappings with nonempty closed convex values. For $i = 1, 2$, let $\Sigma_i \subset C(K, X)$ be the set of all continuous selections of G_i .

Then

$$\mathrm{dist}_H(\Sigma_1, \Sigma_2) \leq \sup_{t\in K} d_H\big(G_1(t), G_2(t)\big) \qquad (1)$$

(where dist_H and d_H denote the Hausdorff distance in $C(K, X)$ and X, respectively).

Proof. Consider an arbitrary $\varepsilon > \sup_{t\in K} d_H\,(G_1(t), G_2(t))$. Given $v \in \Sigma_1$, we have $v(t) \in G_2(t) + \varepsilon B_X^0$, that is, $G_2(t) \cap B_0(v(t), \varepsilon) \neq \emptyset$, for each $t \in K$. By Lemma (2.1.8), there exists $w \in C(K, X)$ such that $w(t) \in G_2(t) \cap B(v(t), \varepsilon)$ $(t \in K)$. Since $w \in \Sigma_2$, we have $v \in \Sigma_2 + \varepsilon B_{C(K,X)}$. This proves $\Sigma_1 \subset \Sigma_2 + \varepsilon B_{C(K,X)}$. Interchanging the role of Σ_1 and Σ_2, we get that $dist_H(\Sigma_1, \Sigma_2) \leq \varepsilon$. Now, (1) follows by passing to limit for $\varepsilon \searrow sup_{t\in K}\, d_H(G_1(t), G_2(t))$.

We are interested in stability of the properties (GC), (P), $(P\Delta)$ and $(*)$ under making finite "polyhedral direct sums" and arbitrary c_0-sums. The case of (GC) has been already done in [2], while the other cases, though simple, are new.

Definition (2.1.10)[2]: We say that a Banach space X is a *polyhedral direct sum* of Banach spaces $X_1, \ldots, X_n$ if $X = X_1 \oplus \cdots \oplus X_n$ and the norm on X is of the form

$$\|x\|_X = \pi\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n}, \qquad x = (x_1, \ldots, x_n) \in X,$$

where π is a polyhedral nondecreasing norm on $\mathbb{R}^n_+$. In this case, we shall write

$$X = (X_1 \oplus \cdots \oplus X_n)_\pi .$$

Let us recall the definition of the c_0-sum of Banach spaces X_γ $(\gamma \in \Gamma)$, where Γ is an arbitrary (nonempty) set. It is the Banach space

$$X \equiv \left(\bigoplus_{\gamma\in\Gamma} X_\gamma\right)_{c_0} = \left\{x = (x_\gamma)_{\gamma\in\Gamma} : \ x_\gamma \in X_\gamma \ for\ \gamma \in \Gamma, \left(\left\|x_\gamma\right\|_{X_\gamma}\right)_{\gamma\in\Gamma} \in c_0(\Gamma)\right\}$$

in the norm $\|x\|_X = max_{\gamma\in\Gamma} \left\|x_\gamma\right\|_{X_\gamma}$.

Theorem (2.1.11)[2]: Let $X = (X_1 \oplus \cdots \oplus X_n)_\pi$ be a polyhedral direct sum of Banach spaces $X_i (1 \le i \le n)$. Let $\mathcal{P}$ be one of the properties (GC), (P). Then the following conditions are equivalent:

(i) X satisfies $\mathcal{P}$;

(ii) each X_i satisfies $\mathcal{P}$.

Proof. The equivalence for $\mathcal{P} = (GC)$ is a very particular case of [2]. Let us consider the case $\mathcal{P} = (P)$. The implication (i) ⇒ (ii) is obvious since, for each i, the mapping $X_i \ni y \longmapsto \frac{1}{\pi(e_i)}(0,\ldots,0,y,0,\ldots,0) \in X$ (where y is at the i-th position) is an isometric linear embedding of X_i into X, and the property (P) is hereditary.

(ii) ⇒ (i) for $\mathcal{P} = (P)$. Let $\{g_1,\ldots,g_m\} \subset (\mathbb{R}^n)^*$ be a minimal family that generates π. Then, by Lemma (2.1.3), we can identify each g_j with a vector $g_j = (g_j^1,\ldots,g_j^n) \in \mathbb{R}^n$ such that $g_j^i \geq 0\ (1 \leq i \leq n)$. For $1 \leq i \leq n$, let $P_i : X \to X_i$ be the canonical projection $x \longmapsto x_i$. Let Y be a finite-dimensional subspace of X. Then each $Y_i = P_i(Y)$, being a finite-dimensional subspace of X_i , is polyhedral and hence there exists a finite set $F_i \subset Y_i^*$ such that $\|\cdot\|_{X_i} = max\langle F_i,\cdot\rangle\ on\ Y_i$. Now, for each $\in Y$, we can write

$$\|y\|_X = \pi\|P_1 y\|_{X_1},\ldots,\|P_n\, y\|_{X_n} = \max_{1\leq j\leq m}\sum_{i=1}^{n} g_j^i\|P_i\, y\|_{X_i}$$

$$= \max_{1\leq j\leq m}\sum_{i=1}^{n} g_j^i\, max\langle F_i\,, P_i\, y\rangle.$$

Denoting $E_j^i = \{g_j^i(f \circ P_i) : f \in F_i\}\ (\subset Y^*)$, we have

$$\|y\|_X = \max_{1\leq j\leq m} \max\langle\sum_{i=1}^{n} E_j^i, y\rangle.$$

Since the set $\bigcup_{j=1}^{m}(\sum_{i=1}^{n} E_j^i)$ is finite, Y is polyhedral by [2].

The following fact is a particular case of a general chain rule formula for subdifferentials of convex functions.

Proposition (2.1.12)[2]: Let $X = (X_1 \oplus \cdots \oplus X_n)_\pi$ be a finite polyhedral direct sum of Banach spaces, $\tilde{\pi}$ defined as in [2], $x \in X$. Then the following chain rule formula holds

$$\partial\|\cdot\|_X(x) = \left\{x^* \in X^*: x^* = \sum_{i=1}^{n} \xi_i^*(u_i^* \circ P_i), \xi^* \in \partial\tilde{\pi}(\|x_1\|, \dots, \|x_n\|), u_i^* \in \partial\|\cdot\|_{X_i}(x_i)\right\},$$

where P_i is the canonical projection of X onto X_i, and $\xi^* = (\xi_1^*, \dots, \xi_n^*) \in \mathbb{R}^n$ $(= (\mathbb{R}^n)^*)$.

Symbolically, $\partial\|\cdot\|_X(x) = \sum_{i=1}^{n}[\partial\tilde{\pi}(\|x_1\|, \dots, \|x_n\|)]_i \cdot [\partial\|\cdot\|_{X_i}(x_i)] \circ P_i$.

Lemma (2.1.13)[2]: Let X be a real vector space, $A, B \subset X$ and $[a, b] \subset \mathbb{R}_+$. Then

$$\text{ext}([a, b] \cdot A) \subset a\,\text{ext}(A) \cup b\,\text{ext}(A), \qquad \text{ext}(A + B) \subset \text{ext}(A) + \text{ext}(B).$$

The equivalence in Theorem (2.1.11) does not hold for $\mathcal{P} = (P\Delta)$ or $\mathcal{P} = (*)$. For these properties, we have to consider only particular polyhedral direct sums. To this end, we introduce the following notion.

Definition (2.1.14)[2]: Given $i \in \{1, \dots, n\}$, a norm π on $\mathbb{R}_+^n$ is said to be *handy in the i-th coordinate* if for each $\bar{t} \in \mathbb{R}_+^n \setminus \{0\}$ with $\bar{t}_i = 0$ we have $\pi(\bar{t} + \tau\, e_i) = \pi(\bar{t})$ whenever $\tau > 0$ is sufficiently small (where e_i denotes the *i*-th canonical unit vector of $\mathbb{R}^n$).

Example (2.1.15)[2]: Let $\rho_1, \dots, \rho_n > 0$. The polyhedral nondecreasing norm $\pi(t) = max_{1\le i\le n}\, \rho_i t_i$ on $\mathbb{R}_+^n$ is handy in each coordinate. On the other hand, the polyhedral nondecreasing norm $\pi(t) = \sum_{i=1}^{n} \rho_i t_i$ on $\mathbb{R}_+^n$ is handy in no coordinate.

Theorem (2.1.16)[2]: Let $X = (X_1 \oplus \cdots \oplus X_n)_\pi$ be a polyhedral direct sum of Banach spaces X_i $(1 \le i \le n)$. Let $\mathcal{P}$ be one of the properties $(P\Delta)$, $(*)$. Then the following conditions are equivalent:

(i) X satisfies $\mathcal{P}$;

(ii) for each $i \in \{1, \dots, n\}$ one has:

(a) X_i satisfies $\mathcal{P}$, and

(b) either $\dim(X_i) < \infty$ or π is handy in the i-th coordinate.

Proof. Let $\{g_1, \ldots, g_m\} \subset (\mathbb{R}^n)^* \cong \mathbb{R}^n$ be a minimal family that generates π. Write $g_j = (g_j^1, \ldots, g_j^n)$ $(1 \leq j \leq m)$.

(i)⇒(ii). Since X_i is isometric with a closed subspace of X and $\mathcal{P}$ is hereditary, (a) is satisfied. Assume that (b) fails. Then, for some $i \in \{1, \ldots, n\}$, we have $dim(X_i) = \infty$ and, for some $\bar{t} \in \mathbb{R}_+^n \setminus \{0\}$ with $\bar{t}_i = 0$, we have $\pi(\bar{t} + \tau\, e_i) > \pi(t)$ for each $\tau > 0$. Since π is the maximum of finitely many linear functionals, the function $\tau \mapsto \pi(\bar{t} + \tau\, e_i)$ is the maximum of finitely many affine functions. Consequently, $(\frac{\partial \pi}{\partial t_i}) + (\bar{t})$ (the i-th right partial derivative of π at $\bar{t}$) is positive. This implies (via the Hahn–Banach theorem) (If $p: V \to R$ is a sublinear function, and $\varphi: U \to R$ is a linear functional on a linear subspace $U \subseteq V$ which is dominated by pon U, i.e.

$$\phi(x) \leq (x) \qquad for\ all\ \ x \in U$$

then there exists a linear extension $\psi: V \to R$ of φ to the whole space V, i.e., there exists a linear functional ψ such that

$\psi(x) \leq \varphi(x) \qquad for\ all\ \ x \in U$

$\psi(x) \leq p(x) \qquad for\ all\ x \in V$)[7] that there exists $h = (h_1, \ldots, h_n) \in \partial\tilde{\pi}(\bar{t})$ with $h_i > 0$. Fix $x = (x_1, \ldots, x_n) \in X$ such that $\|x_j\|_{X_j} = \bar{t}_j (1 \leq j \leq n)$. In particular, $x \neq 0$ and $x_i = 0$. For $j \neq i$, fix any $u_j^* \in \partial\|\cdot\|_{X_j}(x_j)$. Then, by Proposition (2.1.16), $\partial\|\cdot\|_X(x)$ contains the (w^*-compact) set

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} h_j \cdot \left(u_j^* \circ P_j\right) + h_i \cdot (B_{X_i^*} \circ P_i)$$

which is infinite-dimensional and is contained in S_{X^*} . By the Krein–Milman theorem [5], this set has infinitely many extreme points. Now, Fact (2.1.2) easily implies that this is in contradiction with (i).

(ii) ⇒ (i) for $\mathcal{P} = (P\Delta)$. First observe that, in the case that $dim(X_i) = \infty$, the condition (b) implies that $(\frac{\partial \tilde{\pi}}{\partial t_i}) + (t) = (\frac{\partial \pi}{\partial t_i}) + (t) = 0$ whenever $t \in \mathbb{R}_+^n \setminus \{0\}, t_i = 0$; and for such t, since $(\frac{\partial \tilde{\pi}}{\partial t_i}) - (t) = 0$ by definition of $\tilde{\pi}$, we must have $h_i = 0$ whenever $h = (h_1, \ldots, h_n) \in \partial\tilde{\pi}(t)$. In other words, if $[\partial\tilde{\pi}(t)]_i \neq \{0\}$ then either $t_i \neq 0$ or $dim(X_i) < \infty$.

Second, by Theorem (2.1.11) and Fact (2.1.2) X is polyhedral. Now, given $x \in S\,X$, the point $t = (\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n})$ is nonzero, and the chain rule (Proposition (2.1.12) implies that

$$\partial\|\cdot\|_X (x) = \sum_{i=1}^{n} [\partial\tilde{\pi}(t)]_i \cdot \left[\partial\|\cdot\|_{X_i}(x_i) \circ P_i\right] = \sum_{\substack{j=1 \\ [\partial\tilde{\pi}(t)]_i \neq\{0\}}}^{n} [\partial\tilde{\pi}(t)]_i \cdot P_i^*\left(\partial\|\cdot\|_{X_i}(x_i)\right).$$

Each summand of the last sum is the algebraic product of a compact subinterval of $\mathbb{R}_+$ with a finite-dimensional polytope. By Lemma (2.1.13), $\partial\|\cdot\|_X(x)$ has only finitely many extreme points. Using Fact (2.1.2), we conclude that (i) holds.

(ii) ⇒ (i) for $\mathcal{P} = (*)$. By Fact (2.1.2)(c), each X_i satisfies the condition in Definition (2.1.1) with the particular boundary $ß_i = ext(B_{X_i^*})$. We shall show that the set

$$ß = \bigcup_{j=1}^{m} \left(g_j^1 ß_1 \times \cdots \times g_j^n ß_n\right)$$

is a boundary for *X*, satisfying the condition from the definition .

Given $b \in B$, there exist $j \in \{1, \ldots, m\}$ and $b_i \in ß_i (1 \leq i \leq n)$ such that $b = (g_j^1 b_1, \ldots, g_j^n b_n)$. Then $b(x) = \sum_{i=1}^{n} g_j^i b_i(x_i) \leq \sum_{i=1}^{n} g_j^i \|x_i\|_{X_i} \leq \|x\| (x \in X)$. Thus $ß \subset B_{X^*}$.

Given $x \in X \setminus \{0\}$, put $t = (\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n})$ and take $j \in \{1, \ldots, m\}$ such that $\pi(t) = g_j(t)$. For each $i \in \{1, \ldots, n\}$ there exists $b_i \in ß_i$ such that $\|x_i\|_{X_i} = b_i(x_i)$. Then $\|x\|_X = \pi(t) = g_j(t) = \sum_{i=1}^{n} g_j^i \|x_i\|_{X_i} = \sum_{i=1}^{n} g_j^i b_i(x_i) = b(x)$ where $b := (g_j^1 b_1, \ldots, g_j^n b_n) \in ß$. Thus ß is a boundary for *X*.

It remains to show $(*)$. Let $f = (f_1, \ldots, f_n)$ be a *w**-cluster point of ß, and $x = (x_1, \ldots, x_n) \in X \setminus \{0\}$. There exists $k \in \{1, \ldots, m\}$ such that f is a *w**-cluster point of $C := g_k^1 ß_1 \times \cdots \times g_k^n ß_n$. Denote $t = (\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n})$ and consider two cases.

Case 1: $g_k(t) < \pi(t)$. For every choice of $(b_1, \ldots, b_n) \in ß_1 \times \cdots \times ß_n$, the functional $b := (g_k^1 b_1, \ldots, g_k^n b_n) \in C$ satisfies $b(x) = \sum_{i=1}^{n} g_k^i b_i(x_i) \leq \sum_{i=1}^{n} g_k^i \|x_i\|_{X_i} = g_k(t)$. Since f is a *w**-cluster point of *C*, we have $f(x) \leq g_k(t) < \pi(t) = \|x\|$.

Case 2: $g_k(t) = \pi(t)$. The fact that f is a *w**-cluster point of *C* means that for each $i \in \{1, \ldots, n\}$ one has $f_i \in \overline{g_k^i ß_i}^{w^*}$, and there exists $i \in \{1, \ldots, n\}$ for which f_i is a *w**-cluster point of $g_k^i ß_i$. Thus $g_k^i > 0$ and $ß_i \equiv ext(B_{X_i^*})$ is infinite, hence $dim(X_i = \infty$ (otherwise X_i^* would be a finite-dimensional polyhedral space by [2]. We claim that $\|x_i\|_{X_i} \equiv t_i > 0$. (Indeed, if $t_i = 0$ then (b) implies that for some (small) $\tau > 0$ we have $g_k(t + \tau\, e_i) \leq \pi(t + \tau\, e_i) = \pi(t) = g_k(t)$, and this leads to $0 < \tau\, g_k^i = \tau\, g_k(e_i) = g_k(t + \tau\, e_i) - g_k(t) \leq 0$, a contradiction.)

For $g_k^i > 0$ put $b_i := \frac{f_i}{g_k^i}$, and for $g_k^i = 0$ put $b_i = 0$. Observe that *bi* is a *w**-cluster point of $ß_i$, and hence $\|x_i\|_{X_i} > b_i(x_i)$. Consequently, $\|x\|_X = \pi(t) = g_j(t) = \sum_{i=1}^{n} g_k^i \|x_i\|_{X_i} > \sum_{i=1}^{n} g_k^i b_i(x_i) = f(x)$. The proof is complete.

Finally, we prove a theorem about arbitrary *c*0-sums. Its proof is easy and standard.

Theorem (2.1.17)[2]: Let $\{X_\gamma : \gamma \in \Gamma\}$ be an arbitrary (nonempty) family of Banach spaces, p one of the properties (GC), (P), $(P\Delta)$, $(*)$. Then the following conditions are equivalent:

(i) $X := (\bigoplus_{\gamma\in\Gamma} X_\gamma)c_0$ satisfies p;

(ii) each X_γ satisfies p.

Proof. The case $p = (GC)$ is contained in [2]. Now let $p \in \{(P), (P\Delta), (*)\}$. The implication (i) ⇒ (ii) is immediate from the fact that each X_γ is isometric to a subspace of *X*. The implication (ii) ⇒ (i) for $p = (P)$ is well known (see [1, Section 3, (ℓ)]).

(ii) ⇒(i) for $p = (P\Delta)$. Fix $\in S_X$. It is easy to see that, in a neighborhood *U* of *x*, we have $\|y\|_X = max_{\gamma\in\Gamma_0} |y_\gamma| > 0$ where $\Gamma_0 \subset \Gamma$ is a finite set. Thus $\|y\|_X = \left\|P_{\Gamma_0}\, y\right\|_Z$ where $Z = (\bigoplus_{\gamma\in\Gamma_0} X_\gamma)_\infty$ and $P_{\Gamma_0} : X \to Z$ is the canonical projection.

By Theorem (2.1.15) and Example (2.1.16), Z satisfies $(P\Delta)$. It follows that $\partial\|\cdot\|_X(x) = P^*_{\Gamma_0}(\partial\|\cdot\|_Z(P_{\Gamma_0}x))$ has finitely many extreme points, and we are done.

(ii) ⇒(i) for $p = (*)$. For $\in \Gamma$, let $ß_\gamma$ be a boundary for X_γ , satisfying the condition from the definition of (∗). Let $\{e_\gamma\}_\gamma \in \Gamma$ be the canonical Schauder basis of $c_0(\Gamma)$. Consider the set

$$B := \{be_\gamma : \gamma \in \Gamma, b \in ß_\gamma\}$$

(where $be\ \gamma \in X^* \cong (\bigoplus_{\gamma\in\Gamma} X^*_\gamma)_1$ has value b at γ and is null at all other points). It is easy to see that this set is a boundary for X. If f is a w^*-cluster point of $ß$ then either $f = 0$ or $f = he_\gamma$ where $\gamma \in \Gamma$ and h is a w^*-cluster point of $ß_\gamma$. In the second case, we have $f(x) = h(x_\gamma) < \|x\|_X$ whenever $x \in X \setminus \{0\}$. The proof is complete.

Example (2.1.18)[2]: The Banach space

$$\left(\bigoplus_{n=1}^{\infty} \ell_1(n)\right)_{c_0},$$

the c_0-direct sum of the $\ell_1(n)$ (i.e., n-dimensional ℓ_1) spaces, satisfies (GC) and (∗) (and hence also $(P\Delta)$).

Section (2.2): Direct Applications to Generalized Centers and The X -center Map for $C(K, X)$ Spaces:

It is not a new idea that Chebyshev and similar centers of finite sets can be viewed as best approximations in the direct sum X_n, equipped with an appropriate norm, by elements of a certain subspace, namely the "diagonal" $D_n = \{(x_1, \ldots, x_n) \in X_n : x_1 = \cdots = x_n\}$. Thus, under appropriate assumptions, we may be able to deduce results about centers from known results about nearest points. And this is what we are going to do in this section.

Let n be a positive integer, π a nondecreasing norm on $\mathbb{R}^n_+, X$ a Banach space. Consider the π-direct sum

$$(X^n)_\pi = \underbrace{(X \oplus \cdots \oplus X)}_{n} \pi$$

with its norm $\|u\|_\pi = \pi(\|u_1\|_X, \ldots, \|u_n\|_X)$. Let $d_n : X \to D_n$ be the canonical identification, given by $d_n(x) = (x, \ldots, x)$.

Since $\|d_n(x)\|_\pi = \|x\|_\pi(1, \ldots, 1)$ $(x \in X), d_n$ is a positive multiple of an isometry.

Now, given $a = (a_1, \ldots, a_n) \in X_n$, the π-centers of a are the minimizers of the function $\varphi(x) = \pi(\|x - a_1\|, \ldots, \|x - a_n\|) = \|d_n(x) - a\|_\pi$. Thus the π-centers of a correspond (in the identification d_n) to the nearest points to a in the diagonal D_n.

Corollary (2.2.1)[2]: Let X, n and π be as above. Let $P_{D_n} : (X^n)_\pi \to 2^{D_n}$ be the metric projection (i.e., the "nearest point map"), given by

$$P_{D_n}(u) = \{v \in D_n : \|u - v\|_\pi = \mathrm{dist}_\pi(u, D_n)\}.$$

Then $E_\pi(a) = d_n^{-1}(P_{D_n}(a))$ for each $\in (X_n)_\pi$.

As usual, the support of a vector $t \in \mathbb{R}_+^n$ is the set $\mathrm{spt}(t) = \{i: t_i > 0\}$.

Definition (2.2.2)[2]: We shall say that a norm π on $\mathbb{R}_+^n$ is *handy* if π is handy in each coordinate.

Theorem (2.2.3)[2]: Let X be a Banach space satisfying (GC) and $(P\Delta)$, and let π be a polyhedral nondecreasing norm on $\mathbb{R}_+^n$. Assume that either $\dim(X) < \infty$ or π is handy.

(a) The π-center map E_π on X_n is H-l.s.c. (in particular, E_π admits a continuous selection).

(b) If, in addition, X satisfies $(*)$ then E_π is H-continuous on X_n.

Proof. Since X is (GC), Corollary (2.2.1) implies that D_n is proximinal in $(X^n)_\pi$, that is, the metric projection P_{D_n} has nonempty values. By Theorem (2.1.16), $(X^n)_\pi$ satisfies $(P\Delta)$ [and $(*)$ whenever X satisfies $(*)$] By [2], P_{D_n} is H-l.s.c. [and H-continuous if $(X^n)_\pi$ has $(*)$]. The rest follows from Corollary (2.2.1) (existence of a continuous selection for E_π is guaranteed by Michael's selection theorem [6]).

By $C_b(T, X)$ we mean the Banach space of all bounded continuous X-valued functions on a topological space T, equipped with the supremum norm. For T compact, we write just $C(T, X)$ (instead of $C_b(T, X)$).

Every $a \in C_b(T, X)^n$ can be considered as a continuous function $a: T \to X^n$, defined by $a(t) = (a_1(t), \dots, a_n(t))$.

Observation (2.2.4)[2]: Let T be a topological space, X a Banach space, $f: \mathbb{R}_+^n \to \mathbb{R}$ a nondecreasing function. Then

$$r_f(a(t)) \le r_f(a) \qquad (a \in C_b(T, X)^n, t \in T).$$

Proof.

$$\begin{aligned} r_f(a(t)) &= \inf_{x\in X} f(\|x-a_1(t)\|,\ldots,\|x-a_n(t)\|) \\ &= \inf_{v\in C_b(T,X)} f(\|v(t)-a_1(t)\|,\ldots,\|v(t)-a_n(t)\|) \\ &\le \inf_{v\in C_b(T,X)} f(\|v-a_1\|_\infty,\ldots,\|v-a_n\|_\infty) = r_f(a). \end{aligned}$$

Now, we prove the first main result of the present paper. It generalizes [2] saying that $C_b(T,X)$ satisfies (GC) whenever X is a finite-dimensional polyhedral space. Observe that every finite-dimensional polyhedral space satisfies (GC) (by compactness) and $(P\Delta)$ (by Fact (2.1.2)).

Theorem (2.2.5)[2]: Let X be a Banach space satisfying (GC) and $(P\Delta)$. Then $C_b(T,X)$ satisfies (GC) for every topological space T. (In particular, each nonempty finite subset of $C_b(T,X)$ has a Chebyshev center.)

Proof. Fix $\rho \in (0,\infty)^n$. Since the polyhedral nondecreasing norm $\pi(t) = max_{1\le i\le n}\, \rho_i t_i$ $(t \in \mathbb{R}_+^n)$ is handy, E_ρ admits a continuous selection $e: X^n \to X$ (Theorem (2.2.3)). Given $a \in C_b(T,X)n = C_b(T,X^n), put\ u = e \circ a: T \to X$. For each $t \in T, u(t) \in E_\rho(a(t))$ and hence, by Observation (2.2.4),

$$\begin{aligned} r_\rho(a) &\ge \sup_{t\in T} r_\rho(a(t)) = \sup_{t\in T} \max_{1\le i\le n} \rho_i \|u(t)-a_i(t)\| \\ &= \max_{1\le i\le n} \rho_i \|u-a_i\|_\infty . \end{aligned}$$

This implies that $u \in E_\rho(a)$. We have proved that each $a \in C_b(T,X)^n$ admits weighted Chebyshev centers for all weights. Apply Theorem (2.1.5).

We study the multivalued π-center map

$$E_\pi : C(K,X)^n \to 2^{C(K,X)},$$

where π is a nondecreasing norm on $\mathbb{R}^n_+, K$ is a compact Hausdorff topological space, and X is a Banach space.

Notation (2.2.6)[2]: Let K, X be as above.

(a) For $a \in C(K,X)^n$, we define $\hat{a} \in C_b(K^n, X^n)$ by

$\hat{a}(t_1,\ldots,t_n) = (a_1(t_1),\ldots,a_n(t_n))$.

(b) We define $\Theta: C(K,X) \to C(K^n,X^n)$ by

$\Theta u(t_1,\ldots,t_n) = (u(t_1),\ldots,u(t_n))$,

that is, $\Theta u = \widehat{d_n \circ u}$ (where d_n is the canonical map of X onto the diagonal D_n of X^n).

Obviously, for every direct-sum norm on X^n, Θ is a linear isomorphism of $C(K,X)$ into $C(K^n,X^n)$.

The following simple lemma gives possibility to represent π-centers in the space $C(K,X)$ as continuous selections of certain multivalued mappings on K^n with values in X^n.

Lemma (2.2.7)[2]: Let π, K and X be as above, $a \in C(K,X)^n$. Then $\Theta(E_\pi(a))$ is exactly the set

$$\hat{E}_\pi(\hat{a}) := v \in C(K^n,X^n): v(t) \in \Psi(t) \text{ for all } t = (t_1,\ldots,t_n) \in K^n,$$

where

$$\Psi(t) = \begin{cases} B_\pi(\hat{a}(t_1,\ldots,t_n), r_\pi(a)) \cap D_n & \text{if } t_1 = \cdots = t_n, \\ B_\pi(\hat{a}(t_1,\ldots,t_n), r_\pi(a)) & \text{otherwise.} \end{cases}$$

Proof. Given $u \in C(K,X)$, we have the following chain of obvious equivalences

$u \in E_\pi(a) iff\ \pi(\|u-a_1\|_\infty,\ldots,\|u-a_n\|_\infty) \le r_\pi(a)$

$iff\ \forall t_1,\dots,t_n \in K$: $\pi(\|u(t_1)-a_1(t_1)\|,\dots,\|u(t_n)-a_n(t_n)\|) \le r_\pi(a)$

$iff\ \forall t_1,\dots,t_n \in K$: $\Theta_u(t_1,\dots,t_n) \in B_\pi(\hat{a}(t_1,\dots,t_n), r_\pi(a))$.

Now, the inclusion $\Theta(E_\pi(a)) \subset \hat{E}_\pi(\hat{a})$ is clear. To show the other inclusion, assume $v \in \hat{E}_\pi(\hat{a})$. Then, for each $\tau \in K, v_1(\tau) =\cdots= v_n(\tau) =: u(\tau)$, and $\Theta u(t_1,\dots,t_n) = v(t_1,\dots,t_n) \in B_\pi(\hat{a}(t_1,\dots,t_n), r_\pi(a))$ whenever $t_1,\dots,t_n \in K$. That is, $\Theta u = v$ and $u \in E_\pi(a)$.

Lemma (2.2.7) suggests to study the set of continuous selections of mappings of type. The main such result, Theorem (2.2.10), needs two preliminary steps.

The following proposition can be easily proved by adapting methods from [2].

Proposition (2.2.8)[2]: Let X be a Banach space, $Y \subset X$ a closed subspace. Consider the multivalued mapping

$$G: X \to 2^X, \qquad G(x) = B(x,1) \cap Y .$$

(a) If X satisfies $(P\Delta)$, then G is H-l.s.c. on its effective domain.

(b) If X satisfies $(*)$, then G is H-continuous on its effective domain.

Proof. In this section we are going to prove Proposition (2.2.11). Our proof is just an easy extension of analogous results in [2] concerning metric projection, since $P_Y(x) = G(x)$ whenever $dist(x,Y) = 1$ (Y is a closed subspace of X, G is as in Proposition (2.2.11). Lemma(2.2.9) , (i) ⇒ (ii), was essentially claimed at the beginning of the proof of [2, Proposition (2.2.6)]; Proposition (2.2.11) corresponds to [2, Proposition (2.2.6)], while Proposition (2.2.11) is based on [2], Theorem (2.2.4).

Lemma(2.2.9)[2]: Let X be a Banach space, $x_0 \in S_X$. The following assertions are equivalent:

(i) x_0 is a (QP)-point for X, that is, there exists a neighborhood U of x_0 such that $[x_0, y] \subset S_X$ whenever $y \in U \cap S_X$;

(ii) There exists $r > 0$ such that $x_0 + \frac{2r(x-x_0)}{\|x-x_0\|} \in B_X$ whenever $x \in B_X$ and $0 < \|x - x_0\| < r$.

Proof. (i) ⇒ (ii). Let U be as (i). Choose $r \in (0,1)$ such that $B(x_0, 3r) \subset U$. Let $\in B_X, 0 < \|x - x_0\| < r, u = x_0 + \frac{2r(x-x_0)}{\|x-x0\|}$.

Assume that $\|u\| > 1$. Since $x \in (x_0, u) \cap B_X$, we have $[x_0, u] \cap B_X = [x_0, x_1]$ for some $x_1 \in (x_0, u)$. Obviously $x_1 \in S_X$. Fix $v \in (x_1, u)$ so close to x_1 to have $1 < \|v\| < 1 + r$. Since $\left\|\frac{v}{\|v\|} - x_0\right\| = v \left\|(\frac{1}{\|v\|} - 1) + (v - x_0)\right\| \le (\|v\| - 1) + \|u - x_0\| < r + 2r = 3r$, we must have $[\frac{v}{\|v\|}, x_0] \subset S_X$. Since $x_1 \in (v, x_0)$, we can write $x_1 = tv + (1 - t)x_0$ where $t \in (0,1)$. But

$$S_X \ni x_1 = (t\|v\| + (1-t)) \cdot \left[\frac{t\|v\|}{t\|v\| + (1-t)} \frac{v}{\|v\|} + \frac{1-t}{t\|v\| + (1-t)} x_0\right]$$

leads to a contradiction since the point in square brackets is of norm one and $t\|v\| + (1 - t) > t + (1 - t) = 1$.

(ii) ⇒ (i). Let $r > 0$ be as in (ii). We claim that $U = B_0(x_0, r)$ satisfies (i). If this is not the case, there exists $y \in B_0(x_0, r) \cap S_X$ such that $[y, x_0] \cap B_X^0 \neq \emptyset$. This clearly implies that $aff\{y, x_0\} \cap B_X = [y, x_0]$ ("aff" denotes the affine hull). By (ii), $:= x_0 + \frac{2r(y-x_0)}{\|y-x_0\|} \in B_X$. On the other hand, since $\|u - x_0\| = 2r > \|y - x_0\|$, we have $u \in aff\{y, x_0\} \setminus [y, x_0] = aff\{y, x_0\} \setminus B_X$, a contradiction.

Lemma(2.2.10)[2]: Let Y be a closed subspace of a Banach space X, $q: X \to X/Y$ the corresponding quotient map. Let G be as in Proposition (2.2.9), and

$$R: X/Y \to 2^X, \qquad R(\xi) = q^{-1}(\xi) \cap B_X.$$

Then $dom(R) = q(B_X)$, and

$$G(x) = x - R(q(x)) \text{ and } R(\xi) = \sigma(\xi) - G(\sigma(\xi))(x \in X, \xi \in X/Y),$$

where $\sigma: X/Y \to X$ is a continuous selection of q^{-1} (it exists by Michael's selection theorem).

Proof. Everything follows easily from the following chain of obvious equivalences.

$$z \in R(q(x)) \quad \text{if and only if } q(z) = q(x), \qquad \|z\| \leq 1$$

$$\text{if and only if } x - z =: y \in Y, \qquad \|x - y\| \leq 1$$

$$\text{if and only if } z = x - y, \qquad y \in G(x).$$

Let us recall the statement of Proposition (2.2.9).

Corollary (2.2.11)[2]: Let X be a Banach space, $Y \subset X$ a closed subspace. Consider the multivalued mapping

$$F: X \times \mathbb{R}_+ \to 2^X, \qquad F(x, r) = B(x, r) \cap Y.$$

(a) If X satisfies $(P\Delta)$, then F is H-l.s.c. on its effective domain.

(b) If X satisfies $(*)$, then F is H-continuous on its effective domain.

Proof. Fix $(\bar{x}, \bar{r}) \in dom(F)$,. If $\bar{r} > 0$, then for all $(x, r) \in dom(F)$ that are sufficiently close to $(\bar{x}, \bar{r})$ we have $F(x, r) = r[B(\frac{x}{r}, 1) \cap Y] = rG(\frac{x}{r})$ where G is as in Proposition (2.2.8). By that theorem and Lemma (2.1.7), F

has the required (semi)continuity property at $(\bar{x}, \bar{r})$. Now, assume that $\bar{r} = 0$. Then $(\bar{x}, \bar{r}) = F(\bar{x}, 0) = \{\bar{x}\} \subset Y$. If $y \in F(x, r)$, then $\|y - \bar{x}\| \leq \|y - x\| + \|x - \bar{x}\| \leq r + \|x - \bar{x}\|$. This shows that $F|_{dom(F)}$ is *H*-continuous at $(\bar{x}, 0)$.

Theorem (2.2.12)[2]: Let K be a compact Hausdorff topological space, $K_0 \subset K$ a closed set, $\mathcal{X}$ a Banach space satisfying $(*)$, $Y \subset \mathcal{X}$ a closed subspace, $r: C(K, \mathcal{X}) \to [0, \infty)$ a continuous function. For $a \in C(K, \mathcal{X})$, denote

$$\Phi_a(t) = \begin{cases} B(a(t), r(a)) \cap Y & \text{for } t \in K_0, \\ B(a(t), r(a)) & \text{for } t \in K \setminus K_0, \end{cases}$$

and

$$\Sigma(a) = \{v \in C(K, \mathcal{X}) : v(t) \in \Phi_a(t) \text{ for all } t \in K\}.$$

Consider the set $D = \{a \in C(K, \mathcal{X}) : \Phi_a(t) \neq \emptyset \text{ for each } t \in K\}$.

Then:

(a) $\Sigma(a) \neq \emptyset$ for each $a \in D$;

(b) the multivalued mapping $\Sigma: D \to 2^{C(K, \mathcal{X})}$ is H-continuous.

Proof.

(a) Fix $a \in D$. The map Φ_a is obviously H-continuous on $K \setminus K_0$. We claim that Φ_a is H-l.s.c. (and hence l.s.c.) at the points of K_0. By Corollary (2.2.9), $\Phi_a | K_0$ is H-continuous. Given $\bar{t} \in K_0$ and $\varepsilon > 0$, there exists a neighborhood U of $\bar{t}$ such that:

a. $\Phi_a(\bar{t}) \subset \Phi_a(t) + \varepsilon B_{\mathcal{X}}^0$ whenever $t \in U \cap K_0$;

b. $B(a(\bar{t}), r(a)) \subset B(a(t), r(a)) + \varepsilon B_{\mathcal{X}}^0$ whenever $t \in U \setminus K_0$.

Thus, also for $t \in U \setminus \mathcal{K}_0$, we have $\Phi_a(\bar{t}) \subset B(a(\bar{t}), r(a)) \subset \Phi_a(t) + \varepsilon B_{\mathcal{X}}^0$. This proves our claim. Now, (a) follows by Michael's selection theorem [6].

(b) Fix $\bar{a} \in D$ and $\varepsilon > 0$. Then $C := \bar{a}(\mathcal{K}_0) \times \{r(\bar{a})\}$ is a compact set contained in $dom(F)$, where $F(x, r) = B(x, r) \cap Y$. Since F is H-continuous on $dom(F)$ (Corollary (2.2.9)), a standard compactness argument shows that there exists $\delta > 0$ such that $d_H(F(x, r), F(y, s)) < \varepsilon$ whenever $(x, r) \in C$, $(y, s) \in dom(F)$, $\|x - y\| < \delta$ and $|t - s < \delta$. Let $\eta \in (0, \delta)$ be such that $|r(a) - r(\bar{a})| < \delta$ whenever $a \in D$, $\|a - \bar{a}\|_\infty < \eta$. Now, it is obvious that

$$d_H(\Phi_a(t), \Phi_{\bar{a}}(t)) < \varepsilon \quad whenever\ t \in \mathcal{K}, a \in D, \|a - \bar{a}\|_\infty < \eta.$$

By Corollary (2.1.12), $dist_H(\Sigma(a), \Sigma(\bar{a})) \leq \varepsilon$ whenever $a \in D, \|a - \bar{a}\|_\infty < \eta$.

Now, we can easily deduce the second main result of the present paper. It seems to be new even for finite-dimensional polyhedral spaces X.

Theorem (2.2.13)[2]: Let K be a compact Hausdorff topological space, X a Banach space satisfying (GC) and (∗), n a positive integer, and π a polyhedral nondecreasing norm on $\mathbb{R}_+^n$. Assume that either X is finite-dimensional or π is handy (see Definition (2.2.2)). Then the π-center map

$$E_\pi: C(K, X)^n \to 2^{C(K,X)}$$

is nonempty-valued and H-continuous on $C(K, X)^n$.

Proof. In the notation of Lemma (2.2.7), $E_\pi(a) = \Theta^{-1} \circ \hat{E}_\pi(\hat{a})$ $(a \in C(K, X)^n)$. Thus it suffices to show that $\hat{E}_\pi(\cdot)$ is H-continuous on the set

$$\mathcal{D} = \{\hat{a} : a \in C(K, X)^n\}.$$

Recall that $C(K,X)$ satisfies (GC) by Corollary (2.2.5) and Fact (2.1.2).
In particular, the mapping Ψ from Lemma (2.2.7) has nonempty values for every $\hat{a} \in D$. Now, it suffices to apply Theorem (2.2.12) with $= K^n$ and $\mathcal{X} = (X^n)_\pi$. (Notice that $\mathcal{D} \subset D$ in this case.)

Recall that, given a finite set $A = \{a_1, \ldots, a_n\}$ in a Banach space X, the set $E(A)$ of all Chebyshev centers of A coincides with the set $E_{\mathbf{1}}(a)$ of weighted Chebyshev centers of the *n*-tuple $a = (a_1, \ldots, a_n) \in X^n$ for the constant weight $\mathbf{1} = (1, \ldots, 1)$.

Let us write $\mathbb{N} = \{1, 2, \ldots\}$ for the set of positive integers. Given $m \in \mathbb{N}$, we denote

$$p\mathcal{P}_m(X) = A \in 2^X : 1 \leq \ card\ A \ \leq m.$$

Let us state the following quite natural lemma. The main technical "difficulty" stays in the fact that an element of $\mathcal{P}_m(X)$ can have a cardinality *k* smaller than *m*, and it is not an ordered *k*-tuple.

Lemma (2.2.14)[2]: Let $\mathrm{m} \in \mathbb{N}$, and X a Banach space in which every nonempty set of at most m elements admits a Chebyshev center. Assume that the mapping $\mathrm{E}_{\mathbf{1}} : \mathrm{X}^{\mathrm{m}} \to 2^{\mathrm{X}}$ is H-continuous. Then the Chebyshev-center map $\mathrm{A} \longmapsto \mathrm{E}(\mathrm{A})$ is continuous in the Hausdorff metric on $\mathcal{P}_{\mathrm{m}}(\mathrm{X})$.

Proof. First observe that the continuity of the $E_{\mathbf{1}}$-map on X^m implies that the $E_{\mathbf{1}}$-maps on X^k $(1 \leq k \leq m)$ are all Hcontinuous. This follows immediately from the fact that, for the max-norms on X^k and on X^m, the embedding $\zeta : X^k (x_1, \ldots, x_k) \to (x_1, \ldots, x_k, x_k, \ldots, x_k) \in X^m$ is an isometry and $E_{\mathbf{1}}(x) = E_{\mathbf{1}}(\zeta(x))$ for each $x \in X^k$.

Fix $A \in \mathcal{P}_m(X)$ and $\varepsilon > 0$. Denote k = card A, $A = \{a_1, \ldots, a_k\}$ and $\Delta = \frac{1}{2}$

$min\{\|a_i - a_j\| : i \neq j\}$. Clearly $1 \leq k \leq m$ and $\Delta > 0$. Put

$$S = \{s = (s_1, \ldots, s_k) \in \mathbb{N}^k : |s| := s_1 + \cdots + s_k \leq m\}.$$

For each $s \in S$, define $a(s) \in X^{|s|}$ by

$$a(s) = (\underbrace{a_1, \ldots, a_1}_{s_1}, \ldots, \underbrace{a_k, \ldots, a_k}_{s_k})$$

Since the set S is finite and the $E_{\mathbf{1}}$-maps on X^k $(1 \leq k \leq m)$ are H-continuous, there exists $\delta \in (0, \Delta)$ such that

$$d_H(E_{\mathbf{1}}(b), E_{\mathbf{1}}(a)) < \varepsilon \qquad whenever\ s \in S, b \in X^{|s|},\ \|b - a(s)\|_\infty < \delta.$$

Now, let $B \in \mathcal{P}_m(A)$ be such that $d_H(B, A) < \delta$. Since $\delta < \Delta$, the sets $B_j := B \cap B_0(a_j, \delta)$ $(1 \leq j \leq k)$ are nonempty and pairwise disjoint, and $B = \bigcup_{j=1}^k B_j$. Denote $s_j = card\, B_j$ and $B_j = \{b_1^{(j)}, \ldots, b_{s_j}^{(j)}\}$ $(1 \leq j \leq k)$. Then the point

$$b = \left(b_1^{(1)}, \ldots, b_{s_1}^{(1)}, \ldots, b_1^{(k)}, \ldots, b_{s_k}^{(k)}\right)$$

belongs to $X^{|s|}$ where $s = (s_1, \ldots, s_k)$, and $\|b - a(s)\|_\infty < \delta$. Thus

$$d_H(E(B), E(A)) = d_H\big(E_{\mathbf{1}}(b), E_{\mathbf{1}}(a(s))\big) < \varepsilon.$$

Corollary (2.2.15)[2]: Let K be a compact Hausdorff topological space and X a Banach space satisfying (GC) and (∗). Then each nonempty finite set in $C_b(K, X)$ admits a Chebyshev center, and for each positive integer m the Chebyshev-center map $A \mapsto E(A)$ is continuous in the Hausdorff metric on $\mathcal{P}_m(C_b(K, X))$.

Proof. The assertion follows from Theorem (2.2.11) and Lemma (2. 2.12) by observing that the maximum norm $\pi(t) = max_{1 \leq i \leq n}\, t_i$ on $\mathbb{R}_+^n$ is handy $(n \in \mathbb{N})$.

Proposition (2.2.16)[2]: Let X be a Banach space, $Y \subset X$ a closed subspace. Consider the multivalued mapping

$$G: X \to 2^X, G(x) = B(x, 1) \cap Y.$$

(a) If X satisfies $(P\Delta)$, then G is H-l.s.c. on its effective domain.

(b) If X satisfies (∗), then G is H-continuous on its effective domain.

Proof. By Lemma A.2, it suffices to prove the same properties for the multivalued mapping *R* (in place of *G*).

(a) Let *X* satisfy $(P\Delta)$. By [2, Theorems (2.2.4) and (2.1.14)], *R* is l.s.c. on $dom(R) = q(B_X)$, and H-continuous on $B^0_{X/Y} = q(B^0_X)$.

We have to show that $R|_{q(B_X)}$ is H-l.s.c. at each point of $q(B_X) \cap S_{X/Y}$.

Fix $\xi_0 \in q(B_X) \cap S_{X/Y}$ and $x_0 \in R(\xi_0)$. By [2, Theorem (2.1.13)] (see also [2, Fact (2.1.14)] or our Fact (2(d).1.12)), each point of S_X is a (QP)-point for *X*, a notion defined in Lemma A.1. Hence there exists $r > 0$ as in Lemma A.1(ii). Fix an arbitrary $\varepsilon \in (0, r)$.

Since $R|_{q(B_X)}$ is l.s.c. at ξ0, there exists a relative neighborhood *V* of ξ_0 in $q(B_X)$ such that, for each $\xi \in V \setminus \{\xi_0\}$, there exists $x_\xi \in R(\xi) \cap B^0(x_0, \varepsilon). Now, for\ such\ \xi$,

$$x_\xi + \frac{r(x_\xi - x_0)}{\|x_\xi - x_0\|} = x_0 + \left(\|x_\xi - x_0\| + r\right) \frac{x_\xi - x_0}{\|x_\xi - x_0\|} \in B_X$$

since $r + \|x_\xi - x_0\| < 2r$. So we have the following situation: $\xi_0, \xi \in q(B_X), \xi \neq \xi_0, x_0 \in R(\xi_0), x\xi \in R(\xi), r > 0, x_\xi + \frac{r(x_\xi - x_0)}{\|x_\xi - x_0\|} \in B_X$. By direct application of [2, Lemma (2.1.16)], we get

$$\sup_{z_0 \in R(\xi_0)} dist(z_0, R(\xi)) \leq \frac{2\|x_\xi - x_0\|}{r} < \frac{2\varepsilon}{r} \quad (\xi \in V \setminus \{\xi_0\}).$$

This proves that $R|_{q(B_X)}$ is H-l.s.c. at ξ_0.

(c) If X satisfies $(*)$, R is H-u.s.c. on $q(B_X)$ by [2, Theorem (2.2.7)].

Chapter 3

Best Approximation in Polyhédral Banach Spaces

We study conditions under which the metric projection of a polyhedral Banach space X onto a closed subspace is Hausdorff lower or upper semicontinuous.

We show contains examples illustrating the importance of some hypotheses in the main results.

Section (3.1):Polyhedral Banach Spaces and Matric Projections with Hausdorff Lower and Upper semicontncity:

X denotes a real Banach space such that $dimX \geq 2$, with closed unit ball B_X , open unit ball B_X^0 and unit sphere S_X , and X^* is the dual of X. The set of all nonempty bounded closed convex subsets of X is denoted by $ßCC(X)$, and $[x, y] = conv\{x, y\}$ is the closed segment with endpoints x and y. We shall use the following further notations.

By $extC$ we denote the set of the extreme points of a convex set C. By riC we mean the relative interior of C in the sense of convex analysis, that is, the relative interior of C in its affine hull $affC$.

For $x \in S_X, D(x)$ is the image of x by the (multivalued) duality mapping, i.e.

$$D(x) = D_X(x) = \{ f \in S_{X^*} : f(x) = 1\}.$$

Observe that $extD(x) = D(x) \cap extB_{X^*}$ by the Krein–Milman theorem.

If A is a set in X^*, then A' denotes the set of all w*-accumulation points (called also w*-limit points or w*-cluster points) of A:

$$A' = \{ f \in X^* : f \in \overline{A \setminus \{ f \}}^{w^*} \}.$$

Recall also that a set ß $\subset B_{X^*}$ is 1-norming if

$$\|x\| = \sup_{f \in B} f(x). \tag{3.1}$$

A boundary for X is a 1-norming set ß $\subset B_{X^*}$ such that the supremum in (3.1) is in fact a maximum for each $x \in X$. The set $extB_{X^*}$ is an example of a boundary.

Definition (3.1.1)[3]: A set $P \in BCC(X)$ is a polytope if the intersection of P with any finitedimensional affine set is a (finite-dimensional) polytope.

A Banach space X is said to be polyhedral if B_X is a polytope.

Let us recall that X is polyhedral iff each two-dimensional subspace of X is polyhedral [3]

If X is polyhedral, then the set w*- exp B_{X^*} (of all w*-exposed points of B_{X^*}) coincides with the set w*-strexpB_{X^*} (of all w*-strongly exposed points of B_{X^*}); moreover, this set is a boundary which is contained in any other boundary, and for each of its elements f , the face $f^{-1}(1) \cap S_X$ has nonempty relative interior in S_X .

A finite-dimensional space X is polyhedral iff X^* is polyhedral. On the other hand, an infinite-dimensional dual Banach space is never polyhedral [3] (even it is not isomorphic to any polyhedral space [4]).

Fact (3.1.2)[3]: ([6]). If P is a separable polytope in a Banach space, then aff$P$ is closed and ri$P \neq \emptyset$.

We shall deal with the following three geometric properties, two of them already defined in Introduction.

Definition (3.1.3)[3]: Let X be a Banach space. We say that X satisfies $(*)$ if there exists a boundary $ß \subset S_{X^*}$ such that

$$f(x) < 1 \quad whenever\ x \in S_X\ and\ f \in ß'. \tag{3.2}$$

We say that X satisfies (Δ) if there exists a boundary $ß \subset S_{X^*}$ such that

$$D(x) \cap ß\ is\ finite\ for\ each\ x \in S_X\ . \tag{3.3}$$

We say that X is (QP) ("quasi-polyhedral" [1]) if each $x \in S_X$ has a neighborhood V such that $[x, y] \subset S_X$ whenever $y \in V \cap S_X$.

Lemma (3.1.4)[3]: Let X be a polyhedral Banach space, $ß \subset S_{X^*}$ a boundary for $X, x \in S_X$. Then

$$D(x) = \overline{conv}^{w^*}[D(x) \cap ß].$$

In particular, $D(x) = conv[D(x) \cap ß]$ whenever $D(x) \cap ß$ is finite.

Proof. Denote $B_0 = D(x) \cap ß$. If the assertion is not true, there exists $f \in D(x) \setminus \overline{conv}^{w^*} B_0$. By the Hahn–Banach theorem, there exists $y \in X$ such that $f(y) > sup_{g \in B_0} g(y)$. Note that y cannot be a multiple of x since all the involved functionals have value 1 at x. Consider the two-dimensional subspace $Y = span\{x, y\}$.

Since B_Y is a polygon, a part of S_Y consists of two nondegenerate line segments $[x, v_1]$ and $[x, v_2]$, where v_1, v_2 are two of the vertices of B_Y . For $i = 1,2$, fix an arbitrary $w_i \in (x, v_i)$ and choose $g_i \in ß$ such that $g_i(w_i) = 1$. This implies that $[x, v_i] \subset g_i^{-1}(1)$ and hence $g_i \in B_0$. It is easy to see that $f|_Y \in D_Y(x) = [g_1|_Y, g_2|_Y]$. But then we get $f(y) \leq max\{g_1(y), g_2(y)\} \leq sup_{g \in B_0}\, g(y)$, a contradiction.

It is well known that the properties defined in Definition (3.1.3) are hereditary and, moreover, they are satisfied by any finite-dimensional polyhedral space X; for this and the following fact see.

Fact (3.1.5)[3]: The following implications hold:

(a) $(*) \Rightarrow (QP)$ with $(\Delta) \Leftrightarrow polyhedral\ with\ (\Delta)$;

(b) $(QP) \Rightarrow polyhedral.$

Moreover, none of the simple implications " $\Rightarrow$ " can be reversed.

Observation (3.1.6)[3]: A Banach space X is polyhedral with (Δ) if and only if for each $x \in S_X$ there exist a neighborhood V of x and finitely many closed halfspaces $H_1, \dots, H_n$, each containing B_X , such that $B_X \cap V = (H_1 \cap \dots \cap H_n) \cap V$ (that is, roughly speaking, each $x \in S_X$ has a neighborhood in which B_X coincides with a finite intersection of closed halfspaces containing B_X).

Proof. Let X be polyhedral with (Δ). By Fact (3.1.5), X is *(QP)*. It follows easily (see also [2]) that there exists a neighborhood U of x such that $D(y) \subset D(x)$ whenever $\in U_1 := U \cap S_X$. The set $B_0 := D(x) \cap ß$ is finite and, by Lemma (3.1.4), $D(x) = convB_0$. Thus, for any $y \in U_1, \|y\| = 1 = sup_{f \in D(x)} f(y) = max_{f \in B_0}\, f(y)$. The open set $V := \bigcup_{\lambda > 0} \lambda U_1$ contains x and satisfies $V \cap B_X = V \cap \bigcap_{f \in B_0} Hf$ where $H_f = \{z \in X : f(x) \leq 1\}$.

On the other hand, if X satisfies the condition with halfspaces, it is (QP) and hence polyhedral. Moreover, the norm-one functionals that define all involved halfspaces form a boundary ß that satisfies (4) in Definition (3.1.3).

The following fact is an easy consequence of the definition of property $(*)$.

Fact (3.1.7)[3]: Let X be polyhedral with $(*)$, $x \in S_X$. Then

$$sup\{h(x) \;\; h \in B \setminus D(x)\} < 1,$$

where ß is any boundary satisfying (3) in Definition (3.1.3).

Lemma (3.1.8)[3]: Let X be a polyhedral Banach space, ß $\subset S_{X^*}$ a boundary for $X, x, y \in X$ such that $[x, y] \cap B_X = \{x\}$. Then there exists $h \in$ ß such that $h(x) = 1$ and $h(y) > 1$.

Proof. The assumptions imply that $x \in S_X$ and $\notin B_X$. If y is a (necessarily positive) multiple of x, then any $h \in D(x) \cap$ ß works. Now, assume that $Z :$ $= span\{x, y\}$ has dimension two. Then B_Z is a polygon. If $\notin extB_Z$, then x is an interior point of one of the faces of B_Z . Then any $h \in D(x) \cap$ ß works since $\|z\| = h(z)$ whenever $z \in Z$ is sufficiently near to x. If $\in extB_X$, then two distinct faces F_1, F_2 of B_Z meet at x. Since ß is a boundary, there exist $h_1, h_2 \in$ ß such that $F_i \subset h_i^{-1}(1)$ $(i = 1, 2)$. Then $\|z\| =$ $max\{h_1(z), h_2(z)\}$ whenever $z \in Z$ is sufficiently near to x. It follows that, for some $i \in \{1, 2\}, h = h_i$ works.

Lemma (3.1.9)[3]: Let X be a polyhedral Banach space, ß $\subset S_{X^*}$ a boundary for $X, x_0 \in S_X$. Consider the sets

$$B_0 = D(x_0) \cap ß, \qquad A = \bigcap_{h \in B_0} h^{-1}(1), \qquad F = A \cap S_X = A \cap B_X.$$

Then $A = affF$ and $x_0 \in riF$.

Proof. Obviously, the affine set A and the convex set F are closed. If $A = \{x_0\}$, we have also $F = \{x_0\}$ and the assertion is satisfied. Now, suppose $A \neq \{x_0\}$. Fix an arbitrary $x \in A\backslash\{x_0\}$ and observe that $Y := span\{x_0, x\}$ has dimension two. If $x_0 \in extB_Y$ then two distinct faces of the polygon B_Y meet at x_0. Denote by C one of these two faces that does not contain x. Since ß is a boundary for X, there exists $h \in$ ß such that $C \subset h^{-1}(1)$. But in this case we have $h(x_0) = 1$ *and* $h(x) < 1$, a contradiction with the fact that $x \in A$. Hence x_0 is an interior point of a face of B_Y.

In fact, we have proved that each line in A containing x_0 intersects F in a nondegenerate segment with x_0 in its relative interior, that is, x_0 is an algebraic interior point of F in A. A standard Baire category argument implies that $x_0 \in int_A F$, which completes the proof.

In what follows, Y is a closed subspace of a Banach space X, and $q: X \to X/Y$ is the corresponding quotient map. Recall that the metric projection onto Y is the multivalued mapping

$$P_Y: X \to 2^Y, \qquad P_Y(x) = \{y \in Y: \|x - y\| = d(x, Y)\},$$

where $d(x, Y) = dist(x, Y) = inf_{y\in Y}\|x - y\|$. We say that Y is proximinal if $P_Y(x) \neq \emptyset$ for each $x \in X$; and Y is strongly proximinal [11] if $P_Y(x) \neq \emptyset$ and $d(y_n, P_Y(x)) \to 0$ whenever $x \in X, \{y_n\} \subset Y$, $\|x - y_n\| \to d(x, Y)$.

The following definition weakens the notion of strong proximinality by considering only the points $x \in X$ for which $P_Y(x)$ is nonempty.

Definition (3.1.10)[3]: We shall say that Y is relatively strongly proximinal if

$$d(y_n, P_Y(x)) \to 0$$

whenever $x \in X, P_Y(x) \neq \emptyset, \{y_n\} \subset Y, \|x - y_n\| \to d(x, Y)$.

Let us recall basic definitions about multivalued mappings. For our purposes it suffices to remain within the framework of normed linear spaces.

Definition (3.1.11)[3]: Let L, M be normed linear spaces, $F: L \to 2^M, x_0 \in L$.

(a) F is l.s.c. (lower semicontinuous) at x_0 if for each open set $A \subset M$ such that $A \cap F(x_0) \neq \emptyset$ there exists a neighborhood $V \subset L$ of x_0 such that $A \cap F(x) \neq \emptyset$ whenever $x \in V$.

(b) F is u.s.c. (upper semicontinuous) at x_0 if for each open set $A \subset M$ such that $F(x_0) \subset A$ there exists a neighborhood $V \subset L$ of x_0 such that $F(x) \subset A$ whenever $x \in V$.

(c) F is H-l.s.c. (Hausdorff lower semicontinuous) at x_0 if for each $\varepsilon > 0$ there exists a neighborhood $V \subset L$ of x_0 such that $F(x_0) \subset F(x) + \varepsilon B_M$ whenever $x \in V$.

(d) F is H-u.s.c. (Hausdorff upper semicontinuous) at x_0 if for each $\varepsilon > 0$ there exists a neighborhood $V \subset L$ of x_0 such that $F(x) \subset F(x_0) + \varepsilon B_M$ whenever $x \in V$.

(e) Let "s.c." denote one of the four semicontinuity properties defined in (a)–(d). We say that F is s.c. on a set $E \subset L$ if the restriction $F|_E$ is s.c. at each point of E.

(f) The effective domain of F is the set $domF = \{x \in L: F(x) \neq \emptyset\}$.

1. It is easy to see that one always has the implications H-l.s.c. ⇒ l.s.c., and u.s.c. ⇒ H-u.s.c..

Moreover, F is both H-l.s.c. and H-u.s.c. at x_0 if and only if F is continuous at x_0 with respect to the Hausdorff pseudometric

$$d_H(A,B) = max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

on 2^M. (Note that d_H, restricted to the closed elements of 2^M, is a metric with values in $[0,\infty]$.)

Definition (3.1.12)[3]: Given a closed subspace $Y \subset X$, we define the multivalued mapping

$$R_Y: X/Y \to 2^X, \qquad R_Y(\xi) = q^{-1}(\xi) \cap B_X,$$

where $q: X \to X/Y$ is the quotient map.

Observe that $domR_Y = q(B_X)$ and this set contains $B^0_{X/Y} = q(B^0_X)$. It is easy to see that Y is proximinal if and only if $(B_X) = B_{X/Y}$.

Appropriate versions of the following technical lemma and its corollary (Corollary (3.1.14)) are true for bounded closed convex sets. However, for simplicity of formulation, we state them just for B_X.

Lemma (3.1.13)[3]: Suppose that

$$\xi_0, \xi \in q(B_X), \qquad \xi \neq \xi_0, \qquad x_0 \in R_Y(\xi_0), \qquad x \in R_Y(\xi), \qquad r > 0,$$
$$x + r(x - x_0)\|x - x_0\| \in B_X.$$

Then

$$\sup_{z_0\in R_Y(\xi_0)} d(z_0, R_Y(\xi)) \leq \frac{2\|x - x_0\|}{r}$$

Proof. Fix an arbitrary $z_0 \in R_Y(\xi_0)$. Define $z = x + \frac{r}{\|x-x_0\|+r}(z_0 - x_0)$, and observe that $z \in q^{-1}(\xi)$. An easy calculation shows that, for $u_x = x + r\frac{(x-x_0)}{\|x-x_0\|}$, we have

$$z = \frac{\|x - x_0\|}{\|x - x_0\| + r} u_x + \frac{r}{\|x - x_0\| + r} z_0.$$

Consequently, $z \in B_X$ since $u_x, z_0 \in B_X$. It follows that $z \in R_Y(\xi)$, and hence $d(z_0, R_Y(\xi)) \le \|z - z_0\| = \frac{\|x-x_0\|}{\|x-x_0\|+r} \|u_x - z_0\| \le \frac{2\|x-x_0\|}{\|x-x0\|+r} \le \frac{2\|x-x0\|}{r}$.

Corollary (3.1.14)[3]: The multivalued mapping RY is locally Lipschitz (in the Hausdorff metric) on $B^0_{X/Y}$.

Proof. Given $\xi_0 \in B^0_{X/Y}$, fix an arbitrary $x_0 \in q^{-1}(\xi_0) \cap B^0_X$. Let $r > 0$ be such that $x_0 + 5r B_X \subset B_X$. Consider, in $/Y$, arbitrary two distinct points $\xi, \eta \in \xi_0 + r B^0_{X/Y}$. There exists $x \in q^{-1}(\xi)$ such that $\|x - x_0\| < r$. Then $x \in B_X$ implies that $x \in R_Y(\xi)$. There exists $y \in q^{-1}(\eta)$ such that $\|x - y\| < 2\|\xi - \eta\|$. Since $\|\xi - \eta\| < 2r$, we have $y \in x + 4rB_X \subset x_0 + 5rB_X \subset B_X$; hence $y \in R_Y(\eta)$. Moreover, $u_x := x + r(x - y)\|x - y\| \in (x_0 + rB_X) + rB_X \subset B_X$. By Lemma (3.1.13), $sup_{z \in R_Y}(\eta) d(z, R_Y(\xi)) \le \frac{2}{r}\|x - y\| \le \frac{4}{r}\|\xi - \eta\|$. By interchanging ξ and η, we conclude that $d_H(R_Y(\xi), R_Y(\eta)) \le \frac{4}{r}\|\xi - \eta\|$ whenever $\xi, \eta \in \xi_0 + r B^0_{X/Y}$.

The next lemma gives a link between semicontinuity properties of the metric projection P_Y and those of R_Y. It is based on the following simple observation.

Observation (3.1.15)[3]: If $x \in X, d(x, Y) = 1$ and $\xi = q(x)$, then

$$R_Y(\xi) = x - P_Y(x).$$

Proof. The formula follows from the following chain of obvious equivalences.

$$
\begin{aligned}
z \in R_Y(\xi) \quad & iff q(z) = \xi, \|z\| \leq 1 \\
& iff\ x - z = y \in Y, \|x - y\| \leq 1 \\
& iff\ z = x - y, y \in P_Y(x).
\end{aligned}
$$

Lemma (3.1.16)[3]: Let "s.c." denote one of the properties l.s.c., u.s.c., H-l.s.c., H-u.s.c. Then P_Y is s.c. on its effective domain if and only if RY is s.c. on the set $\Sigma = (domR_Y) \cap S_X/Y = q(B_X) \cap S_X/Y$.

Proof. First, notice that P_Y is semi-linear with respect to Y in the sense that $P_Y(t\,x) = t\,P_Y(x)$ and $P_Y(x + y) = P_Y(x) + y$ whenever $x \in X, y \in Y$ and $t \in \mathbb{R}$. Moreover, the restriction $(P_Y)|_{domP_Y}$ is obviously s.c. at each point of *Y*. It follows easily by homogeneity that P_Y is s.c. on its effective domain if and only if P_Y is s.c. on the set

$$
S = q^{-1}(S_X/Y) \cap domP_Y = \{x \in domP_Y : d(x, Y) = 1\}.
$$

For $x \in S$, Observation (3.1.15) implies that $P_Y(x) = x - R_Y(q(x))$ and $q(x) \in \Sigma$. It follows that P_Y is s.c. on S whenever R_Y is s.c. on Σ.

On the other hand, the multivalued mapping $q^{-1}: X/Y \to 2^X$ is l.s.c. (since q is open) and hence admits a continuous selection σ by Michael's selection theorem. Now, for $\xi \in \Sigma$, we have $d(\sigma(\xi), Y) = \|\xi\|X/Y = 1$ and $R_Y(\xi) = \sigma(\xi) - P_Y(\sigma(\xi))$ (Observation (3.1.15)), and hence $\sigma(\xi) \in S$. It follows that R_Y is s.c. on Σ whenever P_Y is s.c. on S.

Lemma (3.1.17)[3]: (Separable Reduction). Assume that our multivalued mapping R_Y is not H-u.s.c. on $q(B_X)$. Then X contains a separable closed subspace X_0 such that, for $Y_0 = Y \cap X_0$, the corresponding mapping

$$
R_{Y_0} : X_0/Y_0 \to 2^{X_0}, \qquad R_{Y_0}(\eta) = q^{-10}(\eta) \cap B_{X_0}
$$

(where $q_0: X_0 \to X_0/Y_0$ is the quotient map) is not H-u.s.c. on q(B_{X_0}).

Proof. Assume that R_Y is not H-u.s.c. at some $\xi_0 \in q(B_X)$. There exist $\{\xi_n\} \subset q(B_X), x_n \in R_Y(\xi_n)$ and $a > 0$ such that $d(x_n, R_Y(\xi_0)) \geq a$. Fix an arbitrary $x_0 \in R_Y(\xi_0)$ and, for each $n \geq 1$, find $z_n \in q^{-1}(\xi_n)$ such that $\|z_n - x_0\| < \|\xi_n - \xi_0\| + \frac{1}{n}$. Define

$$X_0 = \overline{span}[\{x_n\}_{n\geq 0} \cup \{z_n\}_{n\geq 1}].$$

The subspace $Y_0 = Y \cap X_0$ contains all the points $z_n - x_n (n \geq 1)$. Put $\eta_n = q_0(z_n)$ and $\eta_0 = q_0(x_0)$, and observe that $\eta_n \to \eta_0$ since $z_n \to x_0$. For $n \geq 0$, we have $x_n \in q^{-10}(\eta_n) \cap B_{X_0} = R_{Y_0}(\eta_n)$. Since $R_{Y_0}(\eta_0) = (x_0 + Y_0) \cap B_{X_0} \subset (x_0 + Y) \cap B_X = R_Y(\xi_0)$, we have

$$d(x_n, R_{Y_0}(\eta_0)) \geq d(x_n, R_Y(\xi_0)) \geq a \quad (n \geq 1)$$

which shows that R_{Y_0} is not H-u.s.c. at η_0. _

As a starting point, we shall prove a result about lower semicontinuity (rather than Hausdorff lower semicontinuity) of P_Y (Theorem (3.1.22)). The main tool is the following proposition.

Proposition (3.1.18)[3]: Let Y be a closed subspace of a Banach space X. Let $H_1, \ldots, H_n$ be closed halfspaces in X. Then the mapping $: X_n \to 2_Y$, given by

$$F(x_1, \ldots, x_n) = Y \cap \bigcap_{i=1}^{n} (x_i + H_i)$$

is lower semicontinuous on its effective domain.

Proof. If $(x_1, \ldots, x_n) \in dom\, F$ and some translate of Y belongs to H_i for some i , then necessarily $Y \subset x_i + H_i$. Hence we can (and do) suppose that

Y is not parallel to any H_i , the topological boundary of $H_i (i = 1, \ldots, n)$. By Lemma (3.1.22),

$$F(x_1, \ldots, x_n) = \bigcap_{i=1}^{n} (r_i\, (x_i)) + \widetilde{H}_i$$

where $r_i: X \to Y$ is a continuous retraction and $\widetilde{H}_i = Y \cap H_i$ is a closed halfspace in $Y (i = 1, \ldots, n)$. By Lemma (3.1.20), the mapping $Y_n \to 2^Y, (y_1, \ldots, y_n) \longmapsto \bigcap_{i=1}^{n} (y_i + \widetilde{H}_i)$, is lower semicontinuous on its effective domain; hence also F is.

Lemma (3.1.19)[3]: Let $H_1, \ldots, H_n$ be closed halfspaces in a normed linear space X. Then the mapping $: X^n \to 2^X$, given by

$$F(x_1, \ldots, x_n) = \bigcap_{i=1}^{n} (x_i + H_i)$$

is lower semicontinuous on dom F.

Proof. The case of dim $X < \infty$ was proved in [3]. Indeed, if $H_i = \{x \in X: f_i\, (x) \geq t_i\}, L = \bigcap_{i=1}^{n} f_i^{-1}\, (0)$ and $q: X \to X/L$ is the quotient map, the sets $\widetilde{H}_i = q(H_i)$ are hyperplanes in the (finite-dimensional) space $/Y$. Hence the mapping $\tilde{F}: (X/L)^n \to 2^{X/L}, \tilde{F}(\xi_1, \ldots, \xi_n) = \bigcap_{i=1}^{n} (\xi_i + \widetilde{H}_i)$, is lower semicontinuous on its effective domain. The rest follows from the fact that $F = q^{-1} \circ \tilde{F} \circ Q$ where $Q(x_1, \ldots, x_n) = (q(x_1), \ldots, q(x_n))$, since Q is continuous and q is open.

Fact (3.1.20)[3]: Let Y be a closed subspace of a Banach space X. Then there exists a continuous retraction p of X onto Y.

Proof. Let $q: X \to X/Y$ be the quotient map and G be a positively homogeneous continuous selection of q^{-1} (the so-called Bartle–Graves mapping). Then $p(x) = x - G(q(x))$ defines the desired retraction.

Lemma (3.1.21)[3]: Let *Y* be a closed subspace of a Banach space *X*. Let *H* be a closed halfspace in *X* that contains no translate of *Y*. Then $\widetilde{H} = Y \cap H$ is a closed halfspace in Y, and there exists a continuous retraction r of *X* onto *Y* such that

$$Y \cap (x + H) = r(x) + \widetilde{H} \qquad for\ each\ x \in X.$$

Proof. Let $f \in X^* \setminus Y^{\perp}$ and $t \in \mathbb{R}$ be such that $H = \{x \in : f(x) \geq t\}$. Obviously $\widetilde{H}$ is a closed halfspace in *Y* since f is not constant on *Y*. Fix $y_0 \in Y$ such that $f(y_0) = 1$. By Fact (3.1.20), there exists a continuous retraction p of $f^{-1}(0)$ onto $Y \cap f^{-1}(0)$. Then the mapping $r(x) = f(x)y_0\ p(x - f(x)y_0)$is a continuous retraction onto *Y* such that $f(r(x)) = f(x)$ for all $x \in X$. This easily implies the assertion.

Theorem (3.1.22)[3]: Let *X* be a polyhedral Banach space with (Δ), $Y \subset X$ *a* closed subspace. Then the corresponding mapping R_Y is l.s.c. on its effective domain $q(B_X)$.

Proof. We want to prove that the restriction $R_Y|_{q(B_X)}$ is l.s.c. at each $\xi_0 \in q(B_X)$. This is certainly true for $\xi_0 \in B^0_{X/Y}$ by Corollary (3.1.17).

Now, let $\xi_0 \in q(B_X) \cap S_{X/Y}$. Fix $x_0 \in R_Y(\xi_0)$ and an open neighborhood V of x_0. Since $x_0 \in S_X$, we can apply Observation (3.1.7): by taking a smaller neighborhood we can suppose that there exist finitely many closed halfspaces $H_i \subset X (i = 1, \ldots, n)$ such that

$$B_X \subset \bigcap_{i=1}^{n} H_i \quad and \quad V \cap B_X = V \cap \bigcap_{i=1}^{n} H_i \ .$$

Observe that $x_0 \in R_Y(\xi_0) \cap V = (x_0 + Y) \cap B_X \cap V = (x_0 + Y) \cap \bigcap_{i=1}^{n} H_i \cap V = x_0 + [Y \cap \bigcap_{i=1}^{n}(H_i - x_0) \cap (V - x_0)]$. Thus $0 \in \Phi(x_0)$, where the multivalued mapping

$$\Phi(x) := Y \cap \bigcap_{i=1}^{n}(H_i - x)$$

is l.s.c. on its effective domain (Proposition (3.1.18)). Choose $\varepsilon > 0$ and an open neighborhood W of x_0 so that $W + \varepsilon B_X \subset V$. By the lower semicontinuity of Φ, there exists an open neighborhood U of x_0 such that

$$\|x - x_0\| < \varepsilon \quad and \quad \Phi(x) \cap (W - x_0) \neq \emptyset \quad whenever \quad x \in U, \Phi(x) \neq \emptyset.$$

Notice that $q(U)$ is an open set in $/Y$. For $\xi \in q(U) \cap q(B_X)$ choose $x \in q^{-1}(\xi) \cap U$ and observe that

$$\Phi(x) \supset Y \cap (BX - x) = [(x + Y) \cap B_X] - x = R_Y(\xi) - x \neq \emptyset.$$

Consequently,

$$\emptyset \neq \Phi(x) \cap (W - x_0) \subset \Phi(x) \cap (V - x) = \left[(x + Y) \cap \bigcap_{i=1}^{n} H_i \cap V\right] - x$$
$$= [R_Y(\xi) \cap V] - x,$$

which implies that $R_Y(\xi) \cap V \neq \emptyset$. The proof is complete.

The step from "l.s.c." to "H-l.s.c." is now guaranteed by the following easy consequence of Lemma (3.1.13).

Proposition (3.1.23)[3]: Let X be $(QP), Y \subset X$ a closed subspace. If P_Y is l.s.c. on its effective domain, then P_Y is H-l.s.c. on its effective domain.

Proof. By Lemma (3.1.16), we have to show that RY is H-l.s.c. on $E := q(B_X) \cap S_{X/Y}$ whenever it is just l.s.c. on E. Given $\xi_0 \in E$, choose an arbitrary $x_0 \in R_Y(\xi_0)$. The fact that X is *(QP)* easily implies that there exists $r > 0$ such that

$$x_0 + \frac{2r\,(x - x_0)}{\|x - x_0\|} \in B_X \; whenever \; x \in S_X, 0 < \|x - x_0\| < r. \tag{3.4}$$

Let $\varepsilon \in (0, r)$ be given. Since $R_Y|_E$ is l.s.c. at ξ_0, there exists a neighborhood $U \subset S_{X/Y}$ of ξ_0 such that for each $\xi \in U \cap E$ there exists $x_\xi \in R_Y(\xi) \cap B_0(x_0, \varepsilon)$. Now, for $\xi \in U \cap E, \xi \neq \xi_0$, (5) implies that

$$u_{x_\xi} := x_\xi + \frac{r\,(x_\xi - x_0)}{\|x_\xi - x_0\|} = x_0 + \left(r + \|x_\xi - x_0\|\right)\frac{x_\xi - x_0}{\|x_\xi - x_0\|} \in B_X$$

since $r + \|x_\xi - x_0\| < 2r$ and $x_\xi \in S_X$. By Lemma (3.1.13), we have the estimate $sup_{z_0 \in R_Y(\xi_0)}\, d(z_0, R_Y(\xi)) \leq \frac{2\|x_\xi - x_0\|}{r} < \frac{2\varepsilon}{r}$, which completes the proof.

Theorem (3.1.24)[3]: Let X be a polyhedral Banach space with $(\Delta), Y \subset X$ a closed subspace. Then P_Y is H-l.s.c. on its effective domain.

Proof. By Theorem (3.1.22) and Lemma (3.1.16), P_Y is l.s.c. on its effective domain. Now, Fact [1] and Proposition (3.1.23) conclude the proof.

Property (Δ) of a polyhedral Banach space is not sufficient for Hausdorff upper semicontinuity of P_Y, even if Y is proximinal and of codimension two. In Theorem (3.1.26), we give a positive result under the stronger assumption that X is a Banach space with $(*)$. Let us start with the following simple

Observation (3.1.25)[3]: Let M, Y be subspaces of a vector space X. If M has finite codimension in X, then $M \cap Y$ has finite codimension in Y.

Proof. Put $= M \cap Y$. Let Y_1 be an algebraic complement of N in Y. Then $M \cap Y_1 = (M \cap Y) \cap Y_1 = N \cap Y_1 = \{0\}$. Consequently, $codim_Y N = dim Y_1 \leq codim_X M < \infty$.

Recall that, given a closed subspace $Y\ of\ X, q: X \to X/Y$ denotes the quotient map, and $R_Y: X/Y \to 2^X$ is defined by $R_Y(\xi) = q^{-1}(\xi) \cap B_X$.

Theorem (3.1.26)[3]: Let X be a polyhedral Banach space with $(*), Y \subset X\ a$ closed subspace. Then the corresponding mapping R_Y is H-u.s.c. on its effective domain $q(B_X)$.

Proof. By separable reduction (Lemma (3.1.17)), we may assume that X is separable. Suppose that R_Y is not H-u.s.c. at some $\xi_0 \in q(B_X)$. There exist $\{\xi_n\} \subset q(B_X), z_n \in R_Y(\xi_n)$ and $a > 0$ such that $\xi_n \to \xi_0$ and $d(z_n, R_Y(\xi_0)) > a$.

By Corollary (3.1.14), we must have $\xi_0 \in S_{X/Y}$. Since $R_Y(\xi_0)$ is a separable polytope, Fact (3.1.2) assures that $L := aff RY\ (\xi_0)$ is closed and there exists $x_0 \in \mathrm{ri} R_Y(\xi_0)$ (the relative interior of $R_Y(\xi_0)$). Consider the sets

$$B_0 = D(x_0) \cap ß, \qquad A = \bigcap_{h \in B_0} h^{-1}(1), \qquad F = A \cap S_X = A \cap B_X .$$

By Lemma (3.1.9), $A = aff F$ and $x_0 \in ri F$. Let us denote $R_0 = R_Y(\xi_0) - x_0, L_0 = L - x_0, F_0 = F - x_0, A_0 = A - x_0$.

We claim that

$$L_0 = A_0 \cap Y. \tag{3.5}$$

To see this, notice that $R_Y(\xi_0) \subset S_X$ and $x_0 \in \mathrm{ri}F$ imply $R_Y(\xi_0) \subset A$. Then $F \cap (x_0 + Y) = A \cap B_X \cap (x_0 + Y) = A \cap R_Y(\xi 0) = R_Y(\xi 0)$, and hence $A_0 \cap Y = \mathbb{R}^+ F_0 \cap Y = \mathbb{R}^+(F_0 \cap Y) = \mathbb{R}^+ R_0 = L_0$ (where $\mathbb{R}^+ E$ denotes the set of all positive multiples of the elements of E), which is (3.6).

Since A_0 is a subspace of finite codimension in X, by Observation (3.1.25) we can write

$$Y = L_0 \oplus V \tag{3.6}$$

where V is a finite-dimensional subspace.

By Theorem (3.1.22), R_Y is l.s.c. on $q(B_X)$, hence there exist points $x_n \in R_Y(\xi_n)$ such that $x_n \to x_0$. Since $z_n - x_n \in Y$, (3.7) implies that we can write

$$z_n = x_n + y_n + v_n \quad where \quad y_n \in L_0, v_n \in V.$$

By passing to a subsequence, we can suppose that $v_n \to v \in V$.

We claim that $v = 0$. Indeed, if not, then $v \in Y \backslash L_0 = Y \backslash A_0$. Since $x_0 \in riF$, we must have $[x_0 + v, x_0] \cap B_X = \{x_0\}$. By Lemma (3.1.8), there exists $h \in B_0$ such that $h(x_0 + v) > 1$. Observe that $L_0 \subset A_0 \subset h^{-1}(0)$. Thus we have $1 < h(x_0 + v) = lim\, h(x_n + v_n) = lim\, h(z_n - y_n) = lim\, h(z_n) \leq 1$, a contradiction which proves that $v_n \to 0$.

Since $y_n \in L_0 \subset A_0$ and $x_0 \in int_A F$, the numbers

$$t_n := max\{t \geq 0 : x_0 + t\, y_n \in F\} = max\{t \geq 0 : x_0 + t\, y_n \in R_Y\,(\xi_0)\}$$

are positive and there exists $r > 0$ such that $r \leq \|t_n\, y_n\| \leq 2$ for each n. Moreover, $\|y_n\| = \|z_n - x_n - v_n\| \geq \|z_n - x_0\| - \|x_n - x_0\| - \|v_n\|$and $\|y_n\| \leq 2 + \|v_n\|$. Since $\|z_n - x_0\| > a$, we can suppose that $a < \|y_n\| <$

3 for each n. Then $\frac{r}{3} < t_n < \frac{2}{a}$. Passing to a subsequence, we can suppose that $t_n \to t_0 > 0$.

We claim that $t_0 < 1$. To see this, suppose the contrary, i.e., $t_0 \geq 1$. Then $t'_n := min\{t_n, 1\} \to 1$ and $x_0 + t'_n\, y_n \in R_Y\,(\xi_0)$. Consequently

$$a < \|z_n - x_0 - t'_n\, y_n\| = \|x_n + v_n + y_n - x_0 - t'_n\, y_n\|$$
$$\leq \|x_n - x_0\| + \|v_n\| + 3(1 - t'_n) \to 0.$$

This contradiction proves that $0 < t_0 < 1$.

We can suppose that $t_n < 1$ for each n. Then the definition of t_n implies that $[x_0 + t_n y_n, x_0 + y_n] \cap B_X = \{x_0 + t_n y_n\}$. By Lemma (3.1.8), there exist functionals $h_n \in D(x_0 + t_n y_n) \cap ß$ such that $h_n(x_0 + y_n) > 1$. It follows that $h_n \notin D(x_0)$. Hence, by Fact (3.1.7), $sup_n\, h_n(x_0) =: \sigma < 1$. Then

$$h_n(y_n) = \frac{1}{t_n}[h_n(x_0 + t_n\, y_n) - h_n(x_0)] \geq \frac{1 - \sigma}{t_n}.$$

But then we get

$$1 \geq lim\ sup\ h_n(z_n) = lim\ sup\ h_n(x_n + v_n + y_n) = lim\ sup\ h_n(x_0 + y_n)$$
$$= lim\ sup h_n[(x_0 + t_n y_n) + (1 - t_n) h_n(y_n)]$$
$$\geq 1 + lim\ sup \frac{(1 - t_n)(1 - \sigma)}{t_n} = \frac{1 + (1 - t_0)(1 - \sigma)}{t_0} > 1,$$

a contradiction which completes the proof.

Theorem (3.1.27)[3]: Let X be a polyhedral Banach space with $(*)$, $Y \subset X$ a closed subspace. Then Y is relatively strongly proximinal and P_Y is Hausdorff continuous on its effective domain.

Proof. By Theorem (3.1.26) and Lemma (3.1.16), $P_Y|_{domP_Y}$ is H-u.s.c. By Theorem (3.1.24) and Fact (3.1.5), it is also H-l.s.c. Finally, Y is relatively strongly proximinal by Theorem (3.2.1) proved in the next section (3.2).

Corollary (3.1.29)[3]: Let X satisfy $(*)$. Then every proximinal subspace of X is strongly proximinal and the corresponding metric projection is Hausdorff continuous.

Section (3.2): Proximinality of Subspaces and Polyhedrality of Quotients with Examples:

Let Y be a closed subspace of a Banach space X. Recall that $q: X \to X/Y$ denotes the quotient map, and $P_Y: X \to 2^Y$ is the metric projection onto Y. By $NA(X)$ we mean the set of all norm-attaining elements of X^*. For definitions of proximinality and strong proximinality.

In this section, we consider the following four properties, already introduced in Introduction:

(A) Y is strongly proximinal;

(B) Y is proximinal;

(C) $Y^\perp \subset NA(X)$;

(D) X/Y is polyhedral.

Y will be of finite codimension in X.

Obviously,(A) implies (B).

(a) for Y proximinal, (A) holds iff P_Y is H-u.s.c. (Theorem (3.2.1));

(b) for X/Y reflexive, (B) implies (C) (Observation (3.2.2));

(c) for X/Y finite-dimensional, [(C) and (D)] implies (B) (Lemma (3.2.3)).

The implication " $\Leftarrow$ " in (a) seems to be new. In its proof (proof of Theorem (3.2.1)), it is quite convenient to use our mapping R_Y (see Definition (3.1.13).

Theorem (3.2.1)[3]: Let Y be a closed subspace of a Banach space X. Then Y is relatively strongly proximinal if and only if the metric projection

P_Y is H-u.s.c. on its effective domain. (In particular, a proximinal subspace Y is strongly proximinal if and only if P_Y is H-u.s.c.)

Proof. The implication " $\Rightarrow$ " follows easily from definitions. (For Y proximinal, it has been observed in [1].) Let us show the other implication. Assume that Y is not relatively strongly proximinal. This means that there exist $x \in domP_Y$, $\{y_n\} \subset Y$ and $a > 0$ such that $\|x - y_n \to d(x, Y)$ and $d(y_n, P_Y(x)) > a$ for each n. Since obviously $\notin Y$, by homogeneity we can (and do) suppose that $d(x, Y) = 1$. Define

$$x_n = \frac{x}{\|x - y_n\|}, \qquad z_n = x_n - \frac{y_n}{\|x - y_n\|} = \frac{x - y_n}{\|x - y_n\|},$$
$$\xi_n = q(x_n) = q(z_n), \xi = q(x).$$

Then we have: $R_Y(\xi) = x - P_Y(x)$ (Observation (3.1.16)), $\xi \in q(B_X) \cap S_{X/Y}, \xi_n \in q(B_X)$ and $z_n \in q^{-1}(\xi_n) \cap B_X = R_Y(\xi_n)$ for each n; and $\xi_n \to \xi$ since $x_n \to x$. Now, $since$ $\|x - y_n\| \to 1$, we can write

$$\liminf_{n\to\infty} d(z_n, RY(\xi)) = \liminf_{n\to\infty} d\left(\frac{y_n}{\|x - y_n\|} - x_n, P_Y(x) - x\right)$$
$$= \liminf_{n\to\infty} d\left(\frac{y_n}{\|x - y_n\|} + (x - x_n), P_Y(x)\right)$$
$$= \liminf_{n\to\infty} d\left(\frac{y_n}{\|x - y_n\|}, P_Y(x)\right) = \liminf_{n\to\infty} d(y_n, P_Y(x)) \geq a.$$

It follows that $R_Y|_{q(B_X)}$ is not H-u.s.c. at ξ. By Lemma (3.1.17), P_Y is not H-u.s.c. on its effective domain.

Observation (3.2.2)[3]: (a) If $Y^\perp \subset NA(X)$, then X/Y is reflexive.

(b) If Y is proximinal and X/Y is reflexive, then $Y^\perp \subset NA(X)$.

Proof. (a) is an immediate consequence of the James theorem (In order that a bounded, closed and convex subset K of a Banach space be weakly compact, it suffices that every functional attain its supremum on K)[8]. To show (b), fix an arbitrary $f \in Y^{\perp} = (X/Y)^{*}$. There exists $\xi \in S_{X/Y}$ such that $f(\xi) = \|f\|$. Since Y is proximinal, there exists $\in R_Y(\xi) = q^{-1}(\xi) \cap S_X$. Then $f(x) = f(\xi) = \|f\|$ implies that $f \in NA(X)$.

Lemma (3.2.3)[3]: Let X be a Banach space, $Y \subset X$ a closed subspace of finite codimension. If $Y^{\perp} \subset NA(X)$ and X/Y is polyhedral, then Y is proximinal.

Proof. Since $B_{X/Y}$ is a finite-dimensional polytope, it is a convex hull of its extreme points (that are also exposed points, in this case). For $\in extB_{X/Y}$, take $f \in S_{Y^{\perp}}$ such that $f(\xi) = 1$ and $f(\eta) < 1$ whenever $\eta \in B_{X/Y} \setminus \{\xi\}$. Since $f \in NA(X)$, there exists $x \in S_X$ with $1 = f(x) = f(q(x))$. By the choice of , we must have $q(x) = \xi$. We have proved that $extB_{X/Y} \subset q(B_X)$. Consequently, $B_{X/Y} = conv(extB_{X/Y}) \subset q(B_X)$, which implies that $q(B_X) = B_{X/Y}$. And this is equivalent to proximinality of Y.

In the rest of this section, as well as in the following sections containing counterexamples, we consider the properties (A)–(D) in the case of a finite-codimensional subspace Y of X, under suitable assumptions on X, stronger than polyhedrality (namely, property $(*)$ or polyhedrality with (Δ)). See Definition (3.1.3) for properties $(*)$ and (Δ).

Theorem (3.2.4)[3]: Let X be a polyhedral Banach space with (Δ), $Y \subset X$ a closed subspace of finite codimension. If Y is proximinal then the quotient X/Y is polyhedral.

Proof. We have to prove that the finite-dimensional space $Y^{\perp}$ (the dual of X/Y) is polyhedral. Suppose this is not the case. Then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset extB_{Y^{\perp}}$ of pairwise distinct functionals. Let $\xi_n \in S_{X/Y}$ be such that $f_n(\xi_n) = 1 (n \geq 1)$. By compactness ($X/Y$ has finite dimension!), we can suppose that $\xi_n \to \xi_0$. By proximinality of Y and by Theorem (3.1.23), the mapping $R_Y(\xi) = q^{-1}(\xi) \cap B_X$ has nonempty values and is lower semicontinuous on $S_{X/Y}$; hence it admits a continuous selection (Michael's theorem)[5]. It follows that there exist points $x_n \in S_X$ such that $q(x_n) = \xi_n$ for all $n \geq 0$, and $x_n \to x_0$. Observe that $f_n \in D(x_n)$ for each $n \geq 1$.

By Fact (3.1.6), X is (QP); hence $D(z) \subset D(x_0)$ for each $z \in S_X$ sufficiently close to x_0 (cf. [2]). It follows that $f_n \in D(x_0)$ for each sufficiently large *n*. Observe that the duality mapping of X/Y satisfies $D_{X/Y}(\xi_0) = D(x_0) \cap Y^{\perp}$. For each sufficiently large n, we have

$$f_n \in D(x_0) \cap extB_{Y^{\perp}} = D_{X/Y}(\xi_0) \cap extB_{(X/Y)^*} = extD_{X/Y}(\xi_0)$$
$$= ext(D(x_0) \cap Y^{\perp}).$$

But this is a contradiction since the last set is finite (indeed, $D(x_0)$ is a finite-dimensional polytope by the property (Δ) and Lemma (3.1.5)).

Lemma (3.2.5)[3]: Let X be a Banach space with $(*)$, $B \subset S_{X^*}$ the corresponding boundary. Let a sequence $\{\lambda_n\} \subset \Lambda_1$ be such that the functionals

$$f_n = \sum_{h \in ß} \lambda_n(h) h \qquad (n \in \mathbb{N})$$

converge in the weak* topology to some $f \in S_{X^*} \cap NA(X)$. Then there exist $\lambda \in \Lambda_1$ and an increasing sequence $\{n_k\}$ of positive integers such that:

- λ has a finite support $supp(\lambda)$,
- $f = \sum_{h \in ß} \lambda(h)h$,
- $\| f_{n_k} - f \| \to 0$, $\|\lambda_{n_k} - \lambda\|_1 \to 0$.

Proof. Since $\bigcup_{n\geq 1} \mathrm{supp}(\lambda_n)$ is countable, a standard diagonal method gives a subsequence of $\{\lambda_n\}$ that converges pointwise to some $\lambda \in \Lambda$; for simplicity, let us denote it again by $\{\lambda_n\}$.

Let $x_0 \in S_X$ be such that $f(x_0) = 1$. Since X has $(*)$, the set $B_0 := D(x_0) \cap ß$ is finite. By Fact (3.1.8), $\sigma := sup_{h \in B \setminus B_0} h(x_0) < 1$. Now, we have

$$f_n(x_0) = \sum_{h \in ß} \lambda_n(h)h(x_0) \leq \sum_{h \in B_0} \lambda_n(h) + \sigma \sum_{h \in ß \setminus B_0} \lambda_n(h)$$

$$= (1 - \sigma) \sum_{h \in B_0} \lambda_n(h) + \sigma .$$

It follows that

$$\sum_{h \in B_0} \lambda_n(h) \geq \frac{f_n(x_0) - \sigma}{1 - \sigma}.$$

Passing to limits, we obtain $\sum_{h \in B_0} \lambda(h) \geq 1$. Consequently, $\|\lambda\|_1 = 1$ and $supp(\lambda) \subset B_0$. By the well-known fact that pointwise and norm convergence coincide on the unit sphere of $\ell_1(ß)$, we get that $\|\lambda_n - \lambda\|_1 \to 0$. And this easily implies that $\|f_n - f\| \to 0$.

As a consequence of Lemma (3.2.5), we get the following proposition. Notice that $S_{X^*} \cap NA(X) = D(S_X)$.

Proposition (3.2.6)[3]: Let X be a Banach space with $(*)$. Let $\{f_n\} \subset D(S_X)$ be a sequence converging in the weak* topology to a functional $f \in D(S_X)$. Then $D-1(f_n) \subset D^{-1}(f)$ for each sufficiently large n.

Proof. Assume the contrary. Passing to a subsequence, we can suppose that

$$D^{-1}(f_n) \not\subset D^{-1}(f) \quad for\ each\ n.$$

By Lemma (3.1.5), we have $f_n, f \in convB$, where $ß \subset S_{X^*}$ is a boundary satisfying (3) in Definition (3.1.3). By Lemma (3.2.5), passing to a further subsequence, we can suppose that f_n, f can be expressed as convex combinations

$$f_n = \sum_{h \in ß} \lambda_n(h)h, \qquad f = \sum_{h \in ß} \lambda(h)h,$$

where $\lambda_n, \lambda \in \Lambda_1$ have finite supports and $\lambda_n \to \lambda\ in\ \ell_1^+$ (ß). There exists an index n_0 such that

$$supp(\lambda) \subset supp(\lambda_n) \quad whenever\ n \geq n_0.$$

Now, let $n \geq n_0$ and $x \in D^{-1}(f_n)$. Since $1 = f_n(x) = \sum_{h \in ß} \lambda_n(h)h(x)$, we must have $h(x) = 1$ whenever $h \in supp(\lambda_n)$. It follows that

$$f(x) = \sum_{h \in supp(\lambda)} \lambda(h)h(x) = \sum_{h \in supp(\lambda)} \lambda(h) = 1,$$

that is, $x \in D^{-1}(f)$. We have proved that $D^{-1}(f_n) \subset D^{-1}(f)$, which is a contradiction.

Amir and Deutsch [1] defined the following notion: given a Banach space E, a point $x \in S_E$ is a (QP)-point of BE if there exists a neighborhood U of x such that

$$[y, x] \subset S_E \quad whenever \quad y \in U \cap S_E. \qquad (3.7)$$

Thus the space E is (QP) if and only if each point of its unit sphere is a (QP)-point of S_E. It is easy to see (cf. [11, Section 3]) that (3.8) in this definition can be equivalently replaced with any of the following two conditions:

$$D_E(y) \subset D_E(x) \quad whenever \quad y \in U \cap S_E\,; \tag{3.8}$$

$$\exists M \subset S_E \text{dense such that:} D_E(y) \cap D_E(x) \neq \emptyset \text{ whenever } y \in U \cap M. \tag{3.9}$$

Theorem (3.2.8)[3]: Let X be a Banach space with $(*)$. Then:

(a) weak* and norm convergence of sequences coincide in the set $D(S_X) = NA(X) \cap S_{X^*}$;

(b) every element of $D(S_X)$ is a (QP)-point of B_{X^*} .

Proof. (a) and (b) follow from Lemma (3.2.5) and Proposition (3.2.6), respectively. (For (b) use (10) with $E = X^*, M = D(S_X).$)

Theorem (3.2.8)[3]: Let X be a Banach space with $(*)$, $Y \subset X$ a closed subspace of finite codimension. If $Y^\perp \subset NA(X)$, then the quotient X/Y is polyhedral and the subspace Y is strongly proximinal.

Proof. By Corollary (3.1.29), it suffices to show that Y is proximinal. By Lemma (3.2.3), this will be proved once we show that X/Y is polyhedral, or equivalently, that $Y^\perp = (X/Y)^*$ is polyhedral. If $Y^\perp$ is not polyhedral, $Y^\perp$ is not (QP). Thus there exist $f, f_n \in S_{Y^\perp} (n \in \mathbb{N})$ such that $f_n \to f$ and $[f_n, f\,] \not\subset S_{Y^\perp}$. By Proposition (3.3.6), we can suppose that

$$D^{-1}(f_n) \subset D^{-1}(f)\ (n\ \in \mathbb{N}).$$

Choose $x_n \in D^{-1}(f_n)$. Then $f_n(x_n) = 1$ and also $f(x_n) = 1$, which implies that $f_n, f \in D_{X/Y}\,(q(x_n))$. Consequently, $[f_n, f\,] \subset D_{X/Y}(q(x_n)) \subset S_{Y^\perp}$, which is a contradiction.

Example (3.2.9)[3]: There exist a Banach space X, isomorphic to c_0, and a closed subspace $Y \subset X$ of codimension two such that:

(a) X is polyhedral with (Δ),

(b) Y is proximinal,

(c) Y is not strongly proximinal,

(d) P_Y is not H-u.s.c.

Proof. Let $\{e_n\}$ be the standard basis of c_0. For $x = \sum_{n=1}^{\infty} x_n e_n \in c_0$, define

$$|||x||| = max\left\{\|x\|_\infty, \sup_{n\geq 3}\left(\frac{n}{n+1}|x2| + \frac{2}{n+1}|x_n|\right)\right\}.$$

Clearly, $|||\cdot|||$ is an equivalent norm on c_0. Put $X = (c_0, |||\cdot|||)$.

To prove (a), fix $\in S_X$. Find an integer $n_0 \geq 3$ such that $|x_n| < \frac{1}{8}$ whenever $n \geq n_0$. Let $y = \sum_{n=1}^{\infty} y_n e_n \in S_X$ be such that $\|y - x\|_\infty \leq \frac{1}{8}$. Then, for $n \geq n_0$, we have $|y_n| \leq \frac{1}{4}$ and

$$\frac{n}{n+1}|y_2| + \frac{2}{n+1}|y_n| \leq \frac{n}{n+1} + \frac{1}{2(n+1)} = \frac{2n+1}{2n+2} < 1.$$

It easily follows that, in a certain neighborhood of x, B_X coincides with a finite intersection of closed halfspaces. Now, (a) follows from Observation (3.1.7).

Consider the canonical projection $\pi_2: X \to Z := span\{e_1, e_2\}$, defined by $\pi_2(\sum_{n=1}^{\infty} x_n e_n) = x_1 e_1 + x_2 e_2$. The norm of X is a lattice norm, that is, $|||x||| \leq |||y|||$ whenever $x, y \in X$ are such that $|x_n| \leq |y_n|$ for each n. Let $x \in X$. Define $Y = \overline{span}\{e_n\}_{n\geq 3}$ and observe that, for every $\in Y$, we have

$$|||x - y||| \geq |||\pi_2(x - y)||| = |||\pi_2(x)||| = |||x - (x - \pi 2(x))|||.$$

Since $-\pi_2(x) \in Y$, we have $x - \pi_2(x) \in P_Y(x)$, which proves that Y is proximinal.

By the last inequality, the quotient map $q: X \to X/Y$, restricted to Z, is an isometry between Z and X/Y . Thus we can consider our multivalued mapping R_Y (see Definition (3.1.13)) as a mapping $R_Y : Z \to 2^X$, $R_Y(z) = (z + Y) \cap B_X$. Since Y is proximinal, $R_Y = B_{X/Y}$. Consider the points

$$z_0 = e_1 + e_2, \qquad z_n = e_1 + \frac{n-1}{n} e_2,$$

$$x_n = e_1 + \frac{n-1}{n} e_2 + e_n \ (n \geq 3).$$

It is easy to see that $|||z_0||| = |||z_n||| = |||x_n||| = 1$. Thus we have $x_n \in R_Y(z_n)$ $(n \geq 3)$, and $z_n \to z_0$. Now, observe that every $x \in R_Y(z_0)$ is of the form $x = e_1 + e_2 + \sum_{n=3}^{\infty} t_n e_n$, where $\frac{n}{n+1} + \frac{2}{n+1} |t_n| \leq 1$. The last inequality easily implies that $|t_n| \leq \frac{1}{2}$ for every $n \geq 3$.We conclude that

$$d_{|||\cdot|||}(x_n, R_Y(z_0)) \geq d_{\|\cdot\|_\infty}(x_n, R_Y(z_0)) \geq \frac{1}{2} \quad (n \geq 3),$$

and the restriction $R_Y|_{S_Z}$ is not H-u.s.c. at z_0. By Lemma (3.1.17), P_Y is not H-u.s.c. By Theorem (3.2.1), Y is not strongly proximinal.

The aim of this section is to provide Example (3.2.13). Let us start with some preparatory facts. The criterion of polyhedrality in Proposition (3.2.10) is of independent interest.

For a set $A \subset X^*$, we use the following notation for its annihilators:

$$A^\top = \{x \in X: x|_A \equiv 0\}, \qquad A^\top = \{F \in X^{**} : F|_A \equiv 0\}.$$

Proposition (3.2.10)[3]: Let X be a Banach space and $ß \subset B_{X^*}$ a boundary for X. Assume that for each $f \in ß' \cap D(S_X)$ there exists a symmetric set $K \subset X^*$ such that $dim(K^\top) \le 1$ and $f + K \subset B_{X^*}$. Then X is polyhedral.

Proof. Consider an arbitrary two-dimensional subspace Y of X. Suppose that B_Y is not a polytope. Then $ß_{Y^*}$ has infinitely many extreme points. Since $extB_{X^*}$ is closed (hence compact), it contains pairwise distinct functionals $g_0, g_1, g_2, \ldots$ such that $g_n \to g_0$. For each $n \ge 1$, an easy application of the Krein–Milman theorem gives existence of $f_n \in extB_{X^*}$ such that $f_n|_Y = g_n$. Let f_0 be a w*-limit point of $\{f_n\}_{n\ge1}$. Then $f_0|_Y = g_0$ and $f_0 \in (extB_{X^*})' \subset ß'$, where the last inclusion follows from the Krein Milman theorem [5]. Moreover, for some $\in S_Y \subset S_X$, we have $f_0(y) = g_0(y) = 1$, which implies that $f_0 \in ß' \cap D(S_X)$. By our assumption, there exists a symmetric set $K \subset X^*$ such that $dimK^\top \le 1$ and $f_0 + K \subset B_{X^*}$. Since Y cannot be contained in $K^\top$, there exists $h \in K$ such that $h|_Y \ne 0$. Since $f_0 = \frac{1}{2}(f_0 + h) + \frac{1}{2}(f_0 - h)\ and\ f_0 \pm h \in B_{X^*}$, we have $g_0 = \frac{1}{2}(g_0 + h|_Y) + \frac{1}{2}(g_0 - h|_Y)$ and $g_0 \pm h|_Y \in B_{Y^*}$, a contradiction with the fact that $g_0 \in extB_{Y^*}$.

Let $I \subset R$ be an interval and $\varphi: I \to \mathbb{R}$ a convex function. Recall that the epigraph of ϕ is the set

$$epi(\varphi) = \{(t, s) \in I \times \mathbb{R}: s \ge f(t)\}.$$

We shall need the following simple lemma based on elementary properties of convex functions of one real variable.

Lemma (3.2.11)[3]: Let $\varphi : (-\delta, \delta) \to \mathbb{R}$ be a convex function with $\varphi(0) = 0, \varphi(t) > 0 \; for \; t \in (0, \delta)$, and $\varphi' + (0) = 0$. Then there exist points $p_n = (t_n, s_n) \in \mathbb{R}^2$ $(n \in \mathbb{N})$ such that:

(a) $\delta > t_1 > t_2 > \cdots > 0, s_n > 0 (n \in \mathbb{N}), t_n \to 0$;

(b) for each n, the line $\Lambda_n = aff\{p_n, p_{n+1}\}$ does not intersect the epigraph of φ;

(c) the slopes d_n of Λ_n $(n \in \mathbb{N})$ form a decreasing sequence.

proof. Take any decreasing sequence $\{\tau_n\} \subset (0, \delta)$ of smooth points of φ, such that $\tau_n \to 0$. Denoting $d_n = \frac{1}{2}\varphi'(\tau_n)$, we have $d_n \geq d_{n+1} > 0 (n \in \mathbb{N})$ and $d_n \to \frac{1}{2}\varphi' + (0) = 0$ $(since \; \varphi'_+$ is right continuous, see [18, p. 7]). By passing to a subsequence, we can suppose that $\{d_n\}$ is decreasing.

Let Λ_n be the tangent line to the graph of $\frac{1}{2}\varphi$ at the point of abscissa τ_n, that is the line of equation

$$s = \frac{1}{2}\varphi(\tau_n) + d_n(t - \tau_n).$$

Since $\varphi(t) \geq 0$ for $t \in (-\delta, 0)$, and Λ_n supports $epi(\frac{1}{2}\varphi)$ at the point of abscissa τ_n, it is easy to see that Λ_n does not intersect $epi(\varphi)$. For each n, let $p_n = (t_n, s_n)$ be the point of intersection of Λ_n and Λ_{n+1}. Since $\tau_{n+1} < t_n < \tau_n \; and \; \frac{1}{2}\varphi(\tau_n + 1) < s_n < \frac{1}{2}\varphi(\tau_n)$, the points p_n have the required properties.

Now we are ready for our second example. It shows that, the implications (C) ⇒ (B) and (C) ⇒ (D) fail in general polyhedral spaces. (We already know from Theorem (3.2.8) that they hold under the assumption that X satisfies $(*)$.

Example (3.2.12)[3]: There exists a polyhedral Banach space E, isomorphic to c_0, and a closed subspace $Y \subset E$ of codimension two, such that $Y^{\perp} \subset NA(E)$, Y is not proximinal, and E/Y is not polyhedral.

The proof of Example (3.2.13) will be done in several steps.

First step of construction. We consider the elements of the sequence spaces c_0, ℓ_1, ℓ_∞ to be of the form $a = (a_0, a_1, a_2, \ldots)$, that is, the indexing starts with 0. Let $\{u_i\}_{i\geq 0}$ and $\{e_i\}_{i\geq 0}$ be the canonical bases of c_0 and $\ell_1 = (c_0)^*$, respectively. Define

$$K = \overline{conv}\{\pm 4^{-i}\,(e_1 - e_i\,) : i \geq 2\},$$

$$V = \overline{conv}^{w^*}\left[\,B_{\ell_1} \cup \pm(e_0 + K)\right] = conv\left[B_{\ell_1} \cup \pm(e_0 + K)\right]$$

(the last equality holds since B_{ℓ_1} and K are w*-compact and convex). Then V is the dual unit ball of an equivalent norm $\|\cdot\|$ on c_0, given by

$$\|x\| = max\; x(V).$$

We define $X = (c_0, \|\cdot\|)$.

Let us define also $F_1, F_2 \in \ell_\infty, g \in \ell_1$ and $L \subset X^*$ by

$$F_1 = (1, 1, 1, \ldots), F_2 = (1, -1, -1, \ldots),$$

$$g = e_1 - \sum_{i\geq 2} 2^{-i}\, e_i\,,$$

$$L = span\{e_0, g\}.$$

It is easy to see that $u_0 = \frac{1}{2}(F_1 + F_2), K \subset Ker(F_1) \cap Ker(F_2), u_0 \in S_X\,, e_0 \in S_{X^*}$ and $F_i \in S_{X^{**}} (i = 1, 2)$. Note that $F_1(e_0) = F_2(e_0) =$

$1, F_1(g) = \frac{1}{2}, F_2(g) = \frac{-1}{2}$, and hence $F_1|_L$ and $F_2|_L$ are linearly independent $D_{X^*}(e_0) = [F_1, F_2]$.

Claim 1. (the closed segment with endpoints F_1, F_2). Consequently, $D_L(e_0) = [F_1|_L, F_2|_L]$ by the Hahn–Banach theorem.

Proof. First, let us show that $Ker(F_1) \cap Ker(F_2) = \overline{span}\{e_1 - e_i\}_{i\geq 2}$.

The inclusion " $\supset$ " follows from the fact $that\, Fk\, (e_1 - e_i) = 0\, (k = 1, 2, i \geq 1)$. The equality holds since both the left- and the right-hand side have codimension two (indeed, $\ell_1 = span\{e_1 - e_i\}_{i\geq 2} \oplus span\{e_0, e_1\}$).

Now, since $F_k(e_0) = 1 (k = 1, 2)$, we have the inclusion $[F_1, F_2] \subset D_{X^*}(e_0)$. On the other hand, if $G \in D_{X^*}(e_0)$, then $G(e_0) = 1$ and (by symmetry of K) $G|_K \equiv 0$. Thus $G \in [\overline{span}\{e_1 - e_i\}_{i\geq 2}]^\perp = [Ker(F_1) \cap Ker(F_2)]^\perp = span\{F_1, F_2\}$. Write $G = \lambda F_1 + \mu F_2$, where $\lambda, \mu \in \mathbb{R}$. Since $1 = G(e_0) = \lambda + \mu$, we have $G = \lambda F_1 + (1 - \lambda)F_2 = (1, 2\lambda - 1, 2\lambda - 1, \ldots)$. Now, $1 \geq |G(e_1)| = |2\lambda - 1|$ implies that $\lambda \in [0, 1]$, and hence $g \in [F_1 + F_2]$.

Claim 2. If $f = ae_0 + bg \in S_L$ satisfies $b > 0$, then $F_2(f) < F_1(f) < 1$.

Proof. The first inequality is clear: $F_1(f) = a + b_2 > a - b_2 = F_2(f)$. To prove the second inequality, assume the contrary, that is $F_1(f) = 1$. Since $f \in V$, we can write

$$f = t\, z + sv + rw,$$

where $, s, r \geq 0, t + s + r = 1, z \in e_0 + K, v \in -e_0 + K, w \in B_{\ell_1}$.

Since $F_1(z) = F_1 e_0) = 1, F_1(v) = F_2(e_0) = -1, F_1(w) \leq \|F_1\|_\infty \|w\|_1 \leq 1$, we have

$$1 = F_1(f) = t - s + r\,F_1(w) \leq t - s + r \leq t + s + r = 1.$$

Thus the above inequalities are in fact equalities. This means that $s = 0$, and either $r = 0$ or $F_1(w) = 1$. If $F_1(w) = 1$, we necessarily have $w = \sum_{i\geq 0} \alpha_i\, e_i$ with $\alpha_i \geq 0 (i \geq 0)$, and if $r = 0$ we can take $w = 0$. In both cases, for each $i \geq 2$, we have

$$-2^{-i}b = f(u_i) = tz(u_i) + (1-t)w(u_i) \geq -4^{-i}\,t \geq -4^{-i}\,.$$

It follows that $b \leq 2^{-i}$ for each $i \geq 2$, and hence $b \leq 0$, which is a contradiction that completes the proof.

Observation. Note that Claim 1 and the second part of Claim 2 imply that the line $F_1|_L = 1$ is tangent to the “half-sphere” $\{ae_0 + bg \in S_L\colon b \geq 0\}$ at e_0.

Second step of construction. For better understanding of the following geometric construction in L, the reader is invited to sketch a simple diagram.

The line $F_1|_L = 1$ supports BL at e_0. Hence, if we consider an appropriate coordinate system, centered at e_0 and with axis of abscissae on the line $F_1|_L = 1$, then the points of S_L that are sufficiently near to e_0 will form the graph of a convex function, defined in a neighborhood of the origin of the axis of abscissae. By Observation above, we can apply Lemma (3.2.11) to get pairwise distinct points $f_n = a_n e_0 + b_n g \in S_L (n \in \mathbb{N})$ such that an, $b_n > 0, b_n \searrow 0, a_n \to 1$, each line $\Lambda_n = aff\{f_n, f_{n+1}\}$ is disjoint from B_L , and the angle between Λ_n and the line $F_1|_L = 1$ tends decreasingly to 0.

Observe that the lines Λ_1 and $u_0 = 1$ are not parallel since their angle is greater than the one between Λ_1 and $F_1|_L = 1$. Let $h \in L$ be the common point of the lines Λ_1 and $u_0|_L = -1$. By our construction, the compact convex set

$$C = \overline{conv}\left[\{\pm f_j\}_{j\geq 2} \cup \{\pm h\}\right]$$

contains B_L , we have

$$extC = \{h, f_2, f_3, \ldots, e_0, -h, -f_2, -f_3, \ldots, -e_0\},$$

and $\partial_L C$ (the boundary of C in L) consists of the segments $[h, f_2], [f_2, f_3], [f_3, f_4], \ldots, [e0, -h], [-h, -f_2], [-f_2, -f_3], [-f_3, -f_4], \ldots, [-e_0, h]$ Define

$$W = \overline{conv}^{w^*}[V \cup C] = conv[V \cup C].$$

Then W is the dual unit ball of an equivalent norm $|||\cdot|||$ on c_0, given by

$$|||x||| := max\, x(W) = max\{\|x\|, max\, x(C)\}.$$

Denote $E = (c_0, |||\cdot|||)$.

Define $Y = L^{\top}$. Then Y is a subspace of codimension two in E, and $(E/Y)^* = Y^{\perp} = L$. Since, $B(L, |||\cdot|||) = C$ is not a polytope, the quotient E/Y is not polyhedral.

Claim 3. E is polyhedral.

Proof. Notice that $W = \overline{conv}^{w^*} B$, where

$$B = \{\pm e_i\}_{i\geq 0} \cup \left\{\pm e_0 \pm 4^{-i}(e_1 - e_i)\right\}_{i\geq 2} \cup \{\pm f_j\}_{j\geq 2} \cup \{\pm h\}. \qquad (3.10)$$

Moreover, B is a boundary for E (since $f_j \to e_0$ and $e_0 \pm 4^{-i}(e_1 - e_i) \to e_0$), and the only w*-limit points of B are the three points $0, \pm e_0$. Observe that $K^{\top} = \mathbb{R}u_0$. Thus E is polyhedral by Proposition (3.2.10).

Claim 4. $Y^{\perp} \subset NA(E)$.

Proof. We have to show that, for each $f \in S_E \cap Y^{\perp} = \partial W \cap L = \partial_L C$, there exists a nonzero $x \in E$ such that $f(x) = |||x|||$ $(= max\, x(W))$.

If $f \in [e_0, -h]$ or $f \in [-e_0, h]$, we can take $x = u_0 or\, x = -u_0$, respectively. If f belongs to any other of the segments that compose $\partial_L C$ (the boundary of C in L), then this segment is contained in one of the lines Λ_n. Moreover, this Λ_n is disjoint from V and supports C at f . Since V is w*-compact and Λ_n is w*-closed, the Hahn–Banach separation theorem [7] gives existence of some $x \in E\backslash\{0\}$ such that $max\, x(V) < inf\, x(\Lambda_n) =: \alpha$. Since x is necessarily constant on Λ_n, we have $max\, x(W) \leq \alpha = f(x)$.

Claim 5. *Y* is not proximinal in *E*.

Proof. We want to show that $(S_E) \neq S_{E/Y}$, where $q: E \rightarrow E/Y$ is the quotient map. Since (in canonical identifications) $L = (E/Y)^*$, we have $E/Y = (E/Y)^{**} = L^*$. Thus we can identify q with the restriction map

$$q: E \rightarrow L^*, \qquad x \mapsto x|L . \tag{3.11}$$

We have $F_1|_L \in S_{L^*}$ since $max\, F_1(C) = F_1(e_0) = 1$. Let us prove that $F_1|_L \notin (S_E)$. If this is not the case, there exists $x \in S_E$ with $x|_L = F_1|_L$. In particular, $e_0(x) = F_1(e_0) = 1$. Since $\|e_0\| = |||e_0||| = 1$, the inclusion $B_{E^*} \supset B_{X^*}$ and Claim 1 imply that $x \in D_{E^*}(e_0) \subset D_{E^*}(e_0) = F_1, F_2]$. But this implies that $x = u_0$ since $[F_1, F_2] \cap E = \{u_0\}$. Thus we get $F_1|_L = u_0|_L$, a contradiction since $F_1(g) \neq 0 = g(u_0)$.

The proof of Example (3.2.12) is complete.

In this section we provide the following example which shows that the implication (B) ⇒ (D) does not hold for general polyhedral spaces. (We already know from Theorem (3.2.4) that it holds under the additional assumption that *X* satisfies (Δ).)

Example (3.2.13)[3]: There exists a polyhedral Banach space E, isomorphic to c0, and a closed subspace $Y \subset E$ of codimension two, such that Y is proximinal and E/Y is not polyhedral.

The proof of Example (3.2.13) will go in a similar, but simpler, way as that of Example (3.2.12). First step of construction. Let $\{u_i\}_{i\geq 0}$ and $\{e_i\}_{i\geq 0}$ be the canonical bases (indices starting from zero!) of c_0 and $\ell 1 = (c_0)^*$, respectively. Define

$$K = \overline{conv}\left\{\pm \frac{1}{i} e_i : i \geq 1\right\},$$

$$V = \overline{conv}^{w^*}\left[B_{\ell_1} \cup \pm(e_0 + K)\right] = conv\left[B_{\ell_1} \cup \pm(e_0 + K)\right]. \quad (3.13)$$

Then V is the dual unit ball of an equivalent norm $\|\cdot\|$ on $c_0, given\ by\ \|x\| = max\ x(V)$. We define $X = (c_0, \|\cdot\|)$.

Observe that $spanK \subset Ker(u_0) \subset X^*, but\ spanK \neq Ker(u_0)$ by the Baire category theorem (indeed, $spanK = \bigcup_{n\geq 1} nK$ while K has empty relative interior in $Ker(u_0)$). Fix an arbitrary $g \in Ker(u_0) \setminus spanK$ and define $L \subset X^*$ by

$$L = span\{e_0, g\}.$$

Since u_0 attains its maximum over V at e_0, we have $e_0 \in S_{X^*}$.

Claim 1′. $D_{X^*}(e_0) = \{u_0\}$. Consequently, $D_L(e_0) = \{u_0|_L\}$ by the Hahn–Banach theorem.

Proof. If $F \in D_{X^*}(e_0)$ then $F|_K \equiv 0$ and $F(e_0) = 1$. Hence $F = u_0$. The other implication is obvious.

Claim 2′. If $f \in S_L$ and $f \neq e_0$, then $f(u_0) < 1$.

Proof. If $f \in S_L$ and $f(u_0) = 1$, then (3.13) implies that $f \in e_0 + K$. On the other hand, $f = e_0 + bg$ for some $b \in R$, since $f(u_0) = 1$ and $g(u_0) = 0$. Thus $bg \in K$, which is possible only if $b = 0$.

Second step of construction. By Claim 1′, the line u0|L = 1 is tangent to S_L at e_0; and by Claim 2′, e_0 is the unique common point of this line and S_L . As in the "Second step of construction" in the proof of Example (3.1.12), we can apply Lemma (3.2.11) to get pairwise distinct points $f_n = a_n e_0 + b_n g \in S_L$ $(n \in \mathbb{N})$ such that an, $b_n > 0, b_n \searrow 0, b_n \to 1$, each line $\Lambda_n = aff\{f_n, f_{n+1}\}$ is disjoint from B_L , and the angle between Λ_n and the line $u_0|_L = 1$ tends decreasingly to 0.

Let $h \in L$ be the common point of the lines Λ_1 and $u_0|_L = -1$. As in the proof of Example (3.2.12), the compact convex set

$$C = \overline{conv}[\{\pm f\ j\}\ j \geq 2 \cup \{\pm h\}]$$

contains B_L , its extreme points are the points $h, f_2, f_3, \ldots, e_0, -h -f_2, -f_3, \ldots, -e_0$, and its boundary (in L) consists of the segments $[h, f_2], [f_2, f_3], [f_3, f_4], \ldots, [e0, -h], [-h, -f_2], [-f_2, -f_3], [-f_3, -f_4], \ldots, [-e_0, h]$ Define

$$W = \overline{conv}^{w^*}[V \cup C] = conv[V \cup C].$$

Then W is the dual unit ball of an equivalent norm $|||\cdot|||$ on c_0, given by $|||x|||: = max\ x(W) = max\{\|x\|, max\ x(C)\}$. Denote $E = (c0, |||\cdot|||)$.

Define $Y = L^{\top}$. Then Y is a subspace of codimension two in E, and $(E/Y)^* = Y = L$. Since, $B(L, |||\cdot|||) = C$ is not a polytope, the quotient E/Y is not polyhedral.

Claim 3′. E is polyhedral.

Proof. The proof is identical to that of Claim 3 in the proof of Example (3.2.12).

Claim 4′. *Y* is proximinal in *E*.

Proof. As in Claim 5 (proof of Example (3.2.12)), we can canonically identify the quotient map $q: E \to X/E$ with the restriction map. We have to show that $(S_E) = S_{L^*}$.

Let $\ell \in S_{L^*}$. There exists $f \in S_L = \partial_L C$ such that the line $\ell = 1$ supports C at f . If $f = e_0$, then $\ell = u_0|L$ (Claim 1′), that is $\ell = q(u_0)$. Let $f \neq e_0$. Then the line $\ell = 1$ is disjoint from B_{X^*} . As in the proof of Claim 4 (proof of Example (3.1.12)), the Hahn–Banach separation theorem (applied to the sets $B_{X^*} and\ \ell^{-1}(1)$ in the w*-topology) gives a nonzero $x \in X$ such that $|||x||| = sup\ x(W) = 1\ and\ x|_L = \ell$. Then $x \in S_E$ and $\ell = q(x)$.

The proof of Example (3.2.13) is complete.

Chapter 4

Smooth and Polyhedral Approximation in Banach Spaces

We show that norms on certain Banach spaces X can be approximated uni-formly, on bounded subsets of X by C^∞ smooth norms and polyhedral norms. we show that this holds for any equivalent norm on $c_0(\Gamma)$, where Γ is an arbitrary set. We also give a necessary condition for the existence of a polyhedral norm on a weakly compactly generated Banach space.

Section (4.1): Approximation of Norms:

Given a Banach space $(X, \|\cdot\|)$ and $\varepsilon > 0$, we say that a new norm $|||\cdot|||$ is ε -equivalent to $\|\cdot\|$ if

$$|||x||| \leq \|x\| \leq (1+\varepsilon)|||x|||,$$

for all $x \in X$. Suppose that P is some geometric property of norms, such as smoothness or strict convexity. We shall say that a norm $\|\cdot\|$ can be approximated by norms having P if, given any $\varepsilon > 0$, there exists a norm having P that is ε -equivalent to $\|\cdot\|$. This is equivalent to the statement, often seen in the relevant literature, that $\|\cdot\|$ may be approximated uniformly, and with arbitrary precision, on bounded subsets of X by norms having P.

The question of whether all equivalent norms on a given Banach space can be approxi-mated by norms having P is a recurring theme in renorming theory. It is known to be true if P is the property of being strictly convex, or locally uniformly rotund.

Several works in the literature, such as [4,9], have addressed this question in the case of C^k smoothness or polyhedrality.

Definition (4.1.1)[4]: We say the norm $\|\cdot\|$ of a Banach space X is C^k smooth if its kth Fréchet derivative exists and is continuous at every point of $X\setminus\{0\}$. The norm said to be C^∞ smooth if this holds for all $k \in \mathbb{N}$.

For separable spaces, we have the following recent and conclusive result.

Theorem (4.1.2)[4]: Let X be a separable Banach space with a C^k smooth norm. Then any equivalent norm on X can be approximated by C^k smooth norms.

There is an analogous result to Theorem (4.1.2) for polyhedral norms.

Definition (4.1.3)[4]: We say a norm $\|\cdot\|$ on a Banach space X is polyhedral if, given any finite-dimensional subspace Y of X, the restriction of the unit ball of $\|\cdot\|$ to Y is a polytope.

Theorem (4.1.4)[4]: Let X be a separable Banach space with a polyhedral norm. Then any equivalent norm on X can be approximated by polyhedral norms.

Very little is known in the non-separable case. In this paper, we will focus much of our attention on the following class of spaces.

Definition (4.1.5.)[4]: Let Γ be a set. The set $c_0(\Gamma)$ consists of all functions $x : \Gamma \to \mathbb{R}$, with the property that $\{\gamma \in \Gamma : |x(\gamma)| \geq \varepsilon\}$ is finite whenever $\varepsilon > 0$. We equip $c_0(\Gamma)$ with the norm $\|\cdot\|_\infty$, where $\|x\|_\infty = max\,\{|x(\gamma)| : \gamma\ \in \Gamma\}$.

When Γ is uncountable, $c_0(\Gamma)$ is non-separable. The structure of $c_0(\Gamma)$ strongly promotes the existence of the sorts of norms under discussion in this paper. For example, it is well known that the canonical norm on $c_0(\Gamma)$ is polyhedral, and that it can be approximated by C^∞ smooth norms. In terms of finding positive non-separable analogues of Theorems (4.1.2) and (4.1.4), this class of spaces is a very plausible candidate.

The most general result concerning this class to date is given below. We shall call a norm $\|\cdot\|$ on $c_0(\Gamma)$ a lattice norm if $\|x\| \leq \|y\|$ whenever $x, y \in c_0(\Gamma)$ satisfy $|x(\gamma)| \leq |y(\gamma)|$ for each $\gamma \in \Gamma$.

Theorem (4.1.6)[4]: Every equivalent lattice norm on $c_0(\Gamma)$ can be approximated by C^∞ smooth norms.

The following result completely settles the approximation problem in the case of $c_0(\Gamma)$, from the point of view of C^∞ smooth norms and polyhedral norms. It solves a special case of [4].

Theorem (4.1.7)[4]: Let Γ be an arbitrary set, and let $\|\cdot\|$ be an arbitrary equivalent norm on $c_0(\Gamma)$. Then $\|\cdot\|$ can be approximated by both C^∞ norms and polyhedral norms.

Theorem (4.1.7) is a consequence of a more general result, Theorem (4.1.15), which involves spaces having Markushevich bases.

Definition (4.1.8.)[4]: Let $(X, \|\cdot\|)$ be a Banach space. A subset B of the closed unit ball B_{X^*} is a called a boundary of $\|\cdot\|$ if, for each x in the unit sphere S_X, there exists $f \in B$ such that $f(x) = 1$.

This is also known as a James boundary of X in the literature. The dual unit sphere S_{X^*} and the set $ext(B_{X^*})$ of extreme points of the dual unit ball B_{X^*} are always boundaries of $\|\cdot\|$, by the Hahn-Banach Theorem and (the proof of the) Krein-Milman Theorem, respectively. It is worth noting that the property of being a boundary is not preserved by isomorphisms in general: a boundary of $\|\cdot\|$ may not be a boundary of $|||\cdot|||$, where $|||\cdot|||$ is an equivalent norm. Since we will be changing norms in this paper, it will be necessary to bear this in mind.

Boundaries play a key role in the theory of both smooth norms and polyhedral norms. If $(X, \|\cdot\|)$ has a boundary that is countable or otherwise well-behaved, then X enjoys good geometric properties as a consequence

Recall that an element $f \in B_{X^*}$ is called a w*-strongly exposed point of B_{X^*} if there exists $x \in B_X$ such that $f(x) = 1$ and, moreover, $\|f - f_n\| \to 0$ whenever $(f_n) \subseteq B_{X^*}$ is a sequence satisfying $f_n(x) \to 1$. It is a simple matter to check that the (possibly empty) set w*-str $exp(B_{X^*})$ of w*-strongly exposed points of B_{X^*} is contained in any boundary of $\|\cdot\|$. We recall the following important result of Fonf, concerning polyhedral norms.

Theorem (4.1.9)[4]: Let $\|\cdot\|$ be a polyhedral norm on a Banach space X having density character κ. Then w*-str $exp(B_{X^*})$ has cardinality κ and is a boundary of $\|\cdot\|$ (so is the minimal boundary, with respect to inclusion). Moreover, given $f \in w^* - str\, exp(B_{X^*})$, the set $A_f \cap B_X$ has non-empty interior, relative to the affine hyperplane $A_f := \{x \in X: \ f(x) = 1\}$.

In particular, if X is separable and $\|\cdot\|$ is polyhedral, then w*-str $exp(B_{X^*})$ is a countable boundary. Conversely, according to [4], if $(X, \|\cdot\|)$ is a Banach space and $\|\cdot\|$ has a countable boundary B, then X admits equivalent polyhedral norms that approximate $\|\cdot\|$. Thus, in the separable case, the existence of equivalent polyhedral norms can be characterised purely in terms of the cardinality of the boundary.

In the non-separable case however, any analogous characterisations, if they exist, must generally rely on more than the cardinality of the boundary alone. There exist Banach spaces $(X, \|\cdot\|)$ having no equivalent polyhedral norms, yet X has density the continuum c, and $\|\cdot\|$ has boundary B of cardinality c. Such Banach spaces can take the form $X = C(T)$, where T is the 1-point compactification of a suitably chosen locally compact scattered tree.

Recall that a Banach space X is weakly compactly generated $(WCG)\ if\ X = \overline{span}^{\|\cdot\|}(K)$, where $K \subseteq X$ is weakly compact. Separable spaces and reflexive spaces are WCG. Exam-ples of WCG spaces that are neither include the $c_0(\Gamma)$ spaces above.

Theorem (4.1.10)[4]: Let X be WCG, and let the norm $\|\cdot\|$ on X be polyhedral. Then the boundary w*-str $exp(B_{X^*})$ of $\|\cdot\|$ may be written as

$$w^* - str\ exp(B_{X^*}) = \bigcup_{n=1}^{\infty} D_n\ ,$$

where each D_n is relatively discrete in the w*-topology.

The theorem above should be compared to the following sufficient condition: if the norm $\|\cdot\|$ on X admits a boundary B such that $B = \bigcup_{n=1}^{\infty} D_n$ and $B = \bigcup_{m=1}^{\infty} K_m$, where each D_n is relatively discrete in the w*-topology, and each Km is w*-compact, then $\|\cdot\|$ can be approximated by polyhedral norms. Thus Theorem (4.1.10) can be considered as a step towards a characterisation of the existence of polyhedral norms, in the WCG case.

Definition (4.1.11)[4]: We call an indexed set of pairs $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma} \subseteq X \times X^*$ a Markushevich basis (or M-basis) if

• $e_\alpha^*(e_\beta) = \delta_{\alpha\beta}$, (that is, $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma}$ is a biorthogonal system);

• $\overline{span}^{\|\cdot\|}(e_\gamma)_{\gamma\in\Gamma} = X$, and

• $(e_\gamma^*)_{\gamma\in\Gamma}$ separates the points of X.

Furthermore, an M-basis is called strong if $x \in \overline{span}^{\|\cdot\|}\{e_\gamma : e_\gamma^*(x) \neq 0\}$ for all $x \in X$, shrinking if $X^* = \overline{span}^{\|\cdot\|}(e*)_{\gamma\in\Gamma}$, and weakly compact if $\{e_\gamma : \gamma \in \Gamma\} \cup \{0\}$ is weakly compact.

The existence of an M-basis allows us to define supports of functionals in the dual space.

Definition (4.1.12)[4]: Let X be a Banach space with an M-basis $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma} and\ let\ f \in X^*$.

Define the support of f (with respect to the basis) to be the set

$$supp(f) = \{\gamma \in \Gamma : f(e_\gamma) \neq 0\}.$$

We say f has finite support if $supp(f)$ is finite.

The main result of this section, Theorem (4.1.15), states that if X has a strong M-basis then, given the right circumstances, the norm on X can be approximated by norms having boundaries that consist solely of elements having finite support. The following result illustrates the relevance of such boundaries to the current discussion. It amalgamates two theorems, both of which are stated with broader hypotheses in their original forms.

Theorem (4.1.13)[4]: Let a Banach space X have a strong M-basis, and suppose that the norm $\|\cdot\|$ has a boundary consisting solely of elements having finite support. Then $\|\cdot\|$ can be approximated by both C^∞ norms and polyhedral norms.

Now, for the rest of this section, we will assume that the Banach space X has a strong M-basis $(e_\gamma, e_\gamma^*)_{\gamma\in\Gamma}$, such that $\|e_\gamma\| = 1$ for all $\gamma \in \Gamma$. Furthermore, we will suppose that there is some fixed $L \geq 0$ satisfyinge $e_\gamma^* \leq L$ for all $\gamma \in \Gamma$.

Given $f \in X^*$, set $\|f\|_1 = \sum_{\gamma\in\Gamma} |f(e_\gamma)|$, whenever this quantity is finite, and set $\|f\|_1 = \infty$ otherwise. Observe that if $x = \sum_{\gamma\in F} e_\gamma^*(x)_{e_\gamma}$, for some finite $F \subseteq \Gamma$, then

$$|f(x)| \leq \sum_{\gamma\in F} |e_\gamma^*(x)|\,|f(e_\gamma)| \leq L\,\|x\| \sum_{\gamma\in F} |f(e_\gamma)| \leq L\,\|x\|\,\|f\|_1 ,$$

whence $\|f\| \leq L\,\|f\|_1$ for all $f \in X^*$. It is also easy to see that $\|\cdot\|_1$ is a w*-lower semicontinuous function on X^*, and that given $r > 0$, the norm-bounded set

$$W_r = \{f \in X^* : \|f\|_1 \leq\curlywedge\} ,$$

is symmetric, convex and w*-compact.

Let us consider the set $B = \{f \in S_{X^*} : \|f\|_1 < \infty\}$. Evidently, B is the countable union of the sets $S_{X^*} \cap W_r$, $r \in \mathbb{N}$, which are w*-closed in S_{X^*}. If $S_{X^*} \cap W_r$ contains a non-empty norm-open subset of S_{X^*} , for some $r \in \mathbb{N}$, then it is a straightforward matter to show that there exists $M \geq 0$ such that $\|f\|_1 \leq M\,\|f\|$ for all $f \in X^*$, whence $S_{X^*} \cap W_M = S_{X^*}$ and X is isomorphic to $c_0(\Gamma)$ via the map $x \mapsto (e_\gamma^*(x))_{\gamma\in\Gamma}$. If there is no such r, then of course B is of first category in S_{X^*} . If X is not isomorphic to any space of the form $c_0(\Gamma)$, then $B \neq S_{X^*}$, but B may still be a boundary of $\|\cdot\|$.

We shall be interested in cases where B is a boundary of $\|\cdot\|$.

The following lemma will be used in Theorem (4.1.15).

Lemma (4.1.14)[4]: Suppose that B as defined above is a boundary of $\|\cdot\|$. Then $X^* = \overline{span}^{\|\cdot\|}(e_\gamma^*)$,i.e., the M-basis of X is shrinking.

Proof. Let $F \subseteq \Gamma$ be finite, and define

$$X_F = \overline{span}^{\|\cdot\|}(e_\gamma)_{\gamma\in\Gamma\setminus F} \quad \text{and} \quad W_F = span(e_\gamma^*)_{\gamma\in F} .$$

Then $W_F = X_{\overline{F}}^{\perp}$ (the inclusion $X_{\overline{F}}^{\perp} \subseteq W_F$ follows from the fact that the basis is strong), and thus X^*/W_F naturally identifies with X_F^*, and $\left\|f \upharpoonright_{X_F}\right\| = d(f, W_F)$ for all $f \in X^*$, where

$$d(f, W_F) = inf\, \{\|f - g\| : g \in W_F\} .$$

Suppose, for a contradiction, that there exists $f \in X^*$ and $\varepsilon > 0$, such that $d(f, W_F) > \varepsilon$ for all finite $F \subseteq \Gamma$. Let F0 be empty. Since $\|f\| = d(f, W_{F_0}) > \varepsilon$, take a unit vector $x_0 \in X$ having finite support, such that $f(x_0) > \varepsilon$. Set $F_1 = supp\, x_0$. Since $\left\|f \upharpoonright_{X_{F_1}}\right\| = d(f, W_{F_1}) > \varepsilon$, there exists a unit vector $x_1 \in X$ having finite support in $\Gamma \setminus F1$, such that $f(x_1) > \varepsilon$. Define $F_2 = F_1 \cup supp\, x_1$. Continuing like this, we get a sequence of unit vectors (x_n) having finite, pairwise disjoint supports, such that $f(x_n) > \varepsilon$ for all i. Clearly, (x_n) is not weakly null.

On the other hand, if $f \in B$ and $y = \sum_{\gamma\in F} e_\gamma^*(y)e_\gamma$ is a unit vector, where $F \subseteq \Gamma$ is finite, then

$$|f(y)| \leq \sum_{\gamma\in\Gamma} |e_\gamma^*(y)|\,|f(e_\gamma)| \leq L \sum_{\gamma\in F} |f(e_\gamma)| .$$

It follows that $f(x_n) \to 0$ as $n \to \infty$. This holds for every element of B, which is a boundary, so $x_n \to 0$ weakly, by Rainwater's Theorem (Let X be a Banach space, let $\{x_n\}$ be a bounded sequence in X and $x \in X$. If $f(x_n) \to f(x)$ for every $f \in Ext(B_{X^*})$, then $x_n \xrightarrow{w} x$.

The symbol $x_n \xrightarrow{w} x$ denotes the convergence in weak topology. By B_{X^*} we denote the unit ball of the dual X^* and $Ext(B_{X^*})$ is the set of all extreme points of this set.)[9]. This is a contradiction.

We can now prove Theorem (4.1.15). The method of proof owes a debt to [4], although the approximation scheme used in that result fails in the case under consideration here, and substantial modifications must be made.

Theorem (4.1.15)[4]: Let a Banach space X have an M-basis as above, and suppose that B as above is a boundary. Given $\varepsilon > 0$, there exists an ε -approximation $|||\cdot|||$ of $\|\cdot\|$, which has a boundary consisting solely of elements having finite support. Consequently, by Theorem (4.1.13), $\|\cdot\|$ can be approximated by C^∞ smooth norms and polyhedral norms.

Proof. Fix $\varepsilon \in (0,1)$. Suppose $f \in X^*$ satisfies $\|f\|_1 < \infty$. We define a sequence of positive numbers and a sequence of subsets of Γ inductively. To begin, set

$$p(f,1) = max\,\{|f(e_\gamma)|: \gamma \in \Gamma\}\ and\ \ G(f,1) = \{\gamma \in \Gamma: |f(e_\gamma)| = p(z,1)\}\,.$$

Given $n \geq 2$, we define

$$p(f,n) = \begin{cases} (max\,\{|f(e_\gamma)| : \gamma \in \Gamma \backslash G(f,n-1)\} & if\ \Gamma \backslash G(z,n-1) \neq \emptyset \\ 0 & otherwise, \end{cases}$$

$$and\ G(f,n) = \{\gamma \in \Gamma : |f(e_\gamma)| \geq p(f,n)\}.$$

Observe that the set $G(f,n)$ is finite if and only if $p(f,n) \neq 0$ and, in this case, $\|f\|_1 \geq p(f,n)|G(f,n)|$. By induction, $|G(f,n)| \geq n$ for all n, so $p(f,n) \leq \|f\|_1\, n^{-1}$ and, in particular, $p(f,n) \to 0$. By construction, the sequence $(p(f,n))$ is decreasing, and strictly decreasing on the set of indices n at which it is non-zero. If $p(f,n) = 0$ for some $n \in \mathbb{N}$, then $f(e_\gamma) \neq 0$ for at most finitely many γ and hence f has finite support. Thus, when f has infinite support, we get a strictly decreasing sequence of positive numbers $p(f,n) \to 0$, and a strictly increasing sequence of finite sets $(G(f,n))$.

Provided $G(f,n)$ is finite, we define

$$w(f,n) = \sum_{\gamma \in G(f,n)} sgn\left(f(e_\gamma)\right) e_\gamma^*,$$

$$and\ h(f,n) = \sum_{i=1}^{n} (p(f,i) - p(f,i+1)) w(f,i).$$

Let $\gamma \in \Gamma$. If $\gamma \in \Gamma \setminus \bigcup_{n=1}^{\infty} G(f,n)$, then $h(f,m)(e_\gamma) = 0 = f(e_\gamma)$ for all m. Otherwise, let n be minimal, subject to the condition $\gamma \in G(f,n)$. By minimality, we have $p(f,n) = |f(e_\gamma)|$.

If $m < n$, then $h(f,m)(e_\gamma = 0$. If $m \geq n$, then we can see that

$$\begin{aligned} h(f,m)(e_\gamma) &= \sum_{i=n}^{m} (p(f,i) - p(f,i+1))\, sgn(f(\gamma)) \\ &= [p(f,n) - p(f,n+1) \\ &\quad + p(f,n+1) - p(f,n+2) \\ &\quad + \ldots - \ldots \\ &\quad + p(f,m) - p(f,m+1)] sgn(f(e_\gamma)) \\ &= |f(e_\gamma)| sgn\left(f(e_\gamma)\right) - p(f,m+1) sgn\left(f(e_\gamma)\right) \end{aligned}$$

$$= f(e_\gamma) - p(f,m+1)\, sgn(f(e_\gamma)).$$

From the calculation above and the fact that $p(f,m+1) < |f(e_\gamma)|$, we have

$$|h(f,m)(e_\gamma)| = |sgn(f(e_\gamma)(|f(e_\gamma)| - p(f,m+1))|$$
$$= |f(e_\gamma)| - p(f,m+1).$$

$Since\ p(f,m+1) \geq 0, we\ obtain\ |h(f,m)(e_\gamma)| \leq |f(e_\gamma)|.$

Therefore, for all $\gamma \in \Gamma, |h(f,m)(e_\gamma)| \leq |f(e_\gamma)|$ and $h(f,m)(e_\gamma) \to f(e_\gamma)$ as $m \to \infty$. We apply Lebesgue's Dominated Convergence Theorem to conclude that $\|f - h(f,m)\|_1 \to 0$. Since $\|\cdot\| \leq L\|\cdot\|_1$, we also get $\|f - h(f,m)\| \to 0$. Since the signs of $w(f,i)(e_\gamma)$ and $w(f,i')(e_\gamma)$ agree whenever they are non-zero,

$$\|h(f,n)\|_1 = \sum_{i=1}^{n}(p(f,i) - p(f,i+1))\, \|w(f,i)\|_1$$
$$= \sum_{i=1(}^{n} p(f,i) - p(f,i+1))|G(f,i)|.$$

Therefore, if f has infinite support, then $\|f\|_1 = \sum_{i=1}^{\infty}(p(f,i) - p(f,i+1))|G(f,i)|$.

Given $m > n$, define

$$g(f,n,m) = \begin{cases} \dfrac{\|f - h(f,n)\|_1}{|G(f,m)|} w(f,m) & if\ |G(f,m)| < \infty, \\ 0 & otherwise. \end{cases}$$

and $j(f,n,m) = h(f,n) + g(f,n,m), m > n$. Observe that $supp(j(f,n,m)) \subseteq G(f,m)$.

Let $B_r = B_{X^*} \cap W_r = \{f \in B_{X^*}: \|f\|_1 \leq r\}$. Of course, $B \subseteq \bigcup_{r=1}^{\infty} B_r$. We let

$$V_r = j(f,n,m): f \in B_r, m > n \text{ and } \|f - j(f,n,m)\| < 2^{-(r+2)}\varepsilon,$$

$$\text{and set } V = \bigcup_{r=1}^{\infty} (1 + 2^{-r}\varepsilon)V_r .$$

Define $|||x||| = sup\{f(x : f \in V\}$. This is the norm that we claim ε-approximates ‖·‖ and has a boundary consisting solely of elements having finite support.

First of all, we prove that $\|x\| < |||x||| \leq (1+\varepsilon)|||x|||$ whenever $x \neq 0$. Take $x \in X$ with $\|x\| = 1$ and let $f \in B$ such that $f(x) = 1$ (which is possible as B is a bound- ary of ‖·‖). Let r be minimal, such that $f \in B_r$. Since $\|f\| \leq L\|f\|_1$ for all $f \in X^*$, and $\|f - j(f,n,m)\|_1 \leq 2\,\|f - h(f,n)\|_1$, it follows that there exists n such that $\|f - j(f,n,m)\| < 2^{-(r+2)}\varepsilon$ whenever $m > n$. In particular,

$$\begin{aligned} |||x||| &\geq (1 + 2^{-r}\varepsilon)j(f,n,n+1)(x) \geq (1 + 2^{-r}\varepsilon)(1 - 2^{-(r+2)}\varepsilon) \\ &\geq 1 + 2^{-(r+1)}\varepsilon. \end{aligned}$$

To secure the other inequality, simply observe that if $f \in B_r$, $m > n$ and $\|f - j(f,n,m)\| < 2^{-(r+2)}\varepsilon$, then

$$\begin{aligned} (1 + 2^{-r}\varepsilon)j(f,n,m)(x) &\leq (1 + 2^{-r}\varepsilon)(1 + 2^{-(r+2)}\varepsilon) \\ &\leq 1 + (2^{-r} + 2^{-(r+2)} + 2^{-(2r+2)})\varepsilon \leq 1 + \varepsilon. \end{aligned}$$

This means that $|||x||| \leq 1 + \varepsilon$. By homogeneity, $\|x\| < |||x||| \leq (1 + \varepsilon)\,\|x\|$ whenever $x \neq 0$.

Now we show that $|||\cdot|||$ has a boundary consisting solely of elements having finite support. By Krein Milman's Theorem [5], we know that $ext(B_{(X,|||\cdot|||)^*}) \subseteq \bar{V}^{w^*}$. Define

$$D = \bigcap_{r=1}^{\infty}\left(\overline{\bigcup_{s=r}^{\infty}(1+2^{-s}\varepsilon)V_s}^{w^*}\right),$$

and let $d \in D$. For each $r \in \mathbb{N}$, $\|d\| \leq (1+2^{-r}\varepsilon)(1+2^{-(r+2)}\varepsilon)$, and hence $\|d\| \leq 1$.

Therefore, if $|||x||| = 1$, then

$$d(x) \leq \|d\|\ \|x\| \leq \|x\| < 1.$$

It follows that, with respect to $|||\cdot|||$, none of the elements of D are norm-attaining. Consequently, $\tilde{B} = ext(B_{(X,|||\cdot|||)^*})\backslash D$ is a boundary of $|||\cdot|||$. We claim that every element of $\tilde{B}$ has finite support.

Given $f \in B$, we have $f \in (1+2^{-r}\varepsilon)\overline{V_r}^{w^*}$ for some $r \in \mathbb{N}$. For a contradiction, we will assume that f has infinite support. According to Lemma (3.1.14), our M-basis is shrinking. It follows that supp g is countable for all $g \in X^*$. Thus, $\overline{V_r}^{w^*}$is Corson compact in the w*-topology, which implies that it is a Fréchet-Urysohn space. In particular, there exist sequences $(f_k) \subseteq B_r$, and $(n_k), (m_k) \subseteq \mathbb{N}$, with $n_k < m_k$ for all $k \in \mathbb{N}$, such that $(j(fk, nk, mk)) \subseteq V_r$ and $j(fk, nk, mk) \overset{w^*}{\to} l$, where $l = (1+2^{-r}\varepsilon)^{-1}f$.

We claim that, in fact, $f_k \overset{w^*}{\to} l$. First, we show that $h(f_k, n_k) \overset{w^*}{\to} l$. To this end, suppose that $|G(f_k, m_k)| \nrightarrow \infty$. Then by taking a subsequence if necessary, there exists $N \in \mathbb{N}$ such that $|supp(j(f_k, n_k, m_k))| \leq |G(f_k, m_k)| \leq N$ for all k. But as $j(f_k, n_k, m_k) \overset{w^*}{\to} l$, this would force $|supp(l)| \leq N < \infty$, which is not the case. Thus we must have $|G(f_k, m_k)| \to \infty$. Therefore, for all $\gamma \in \Gamma$, $g(f_k, n_k, m_k)(e_\gamma) \to 0$ as $k \to$

∞. Since $\|\cdot\| \leq L\|\cdot\|_1$, the sequence $(g(f_k, n_k, m_k))$ is bounded. Therefore, $g(f_k, n_k, m_k) \xrightarrow{w^*} 0$ and hence $h(f_k, n_k) \xrightarrow{w^*} l$.

We will now show that $f_k - h(f_k, n_k) \xrightarrow{w^*} 0$. For each $\gamma \in \Gamma$, $|f_k(\gamma) - h(f_k, n_k)(e_\gamma)| \leq |f_k(e_\gamma)|$, so $\|f_k - h(f_k, n_k)\|_1 \leq \|f_k\|_1$. Therefore, $(f_k - h(f_k, n_k))$ is a bounded sequence.

Given $\gamma \in \Gamma$,

$$|(f_k - h(f_k, n_k))(e_\gamma)| \leq p(f_k, n_k + 1) \leq \frac{\|f_k\|_1}{|G(f_k, n_k + 1)|} \leq \frac{r}{|G(f_k, n_k + 1)|}.$$

Since $h(f_k, n_k) \xrightarrow{w^*} l$, as above, the infinite support of l ensures that $|G(f_k, n_k)| \to \infty$. Therefore, $(f_k - h(f_k, n_k))(e_\gamma) \to 0$ and hence $f_k - h(f_k, n_k) \xrightarrow{w^*} 0$ as $k \to \infty$. It follows that $f_k \xrightarrow{w^*} l$ as claimed, and hence $l \in B_r$.

Fix $n \in \mathbb{N}$ such that $\|l - h(l, n)\|_1 < L^{-1}2^{-(r+3)}\varepsilon$. Then for all $m > n$,

$\|l - j(l, n, m)\| \leq L\| l - j(l, n, m)\|_1 \leq 2L \|l - h(l, n)\|_1 < 2^{-(r+2)}\varepsilon$.

So $j(l, n, m) \in V_r$ for all $m > n$. Let

$$\lambda_m = \frac{(p(l, m) - p(l, m + 1))|G(l, m)|}{\|l - h(l, n)\|_1}.$$

Note that $\lambda_m > 0$ whenever $m > n$. Since $\|l - h(l, n)\|_1 = \sum_{i=n+1}^{\infty}(p(l, i) - p(l, i + 1))|G(l, i)|$, we get $\sum_{m=n+1}^{\infty}\lambda_m = 1$.

$$\sum_{m=n+1}^{\infty} \lambda_m j(l, n, m) = \sum_{m=n+1}^{\infty} \lambda_m h(l, n) + \sum_{m=n+1}^{\infty} \lambda_m g(l, n, m)$$

$$= h(l, n) + \sum_{m=n+1}^{\infty} (p(l, i) - p(l, i + 1))w(l, i) = l.$$

Therefore, f is a nontrivial convex combination of elements of $(1+2^{-r}\varepsilon)V_r \subseteq B_{(X,|||\cdot|||)^*}$, so $f \notin ext(B_{(X,|||\cdot|||)^*})$, and hence $f \notin \tilde{B}$. This gives us our desired contradiction. In conclusion, $\tilde{B}$ is a boundary of $|||\cdot|||$ consisting solely of functionals having finite support.

Theorem (4.1.7) becomes a trivial consequence of Theorem (4.1.15).

Proof of Theorem (4.1.7). In this case $B = S_{(c_0(\Gamma),\|\cdot\|)^*}$, so it is a boundary of $\|\cdot\|$.

It is worth remarking that the implication (d) ⇒ (c) of [12, Theorem (4.1.15)] is essentially Theorem (4.1.15), but with the additional assumption that the M-basis is countable. The method of proof in that case is completely different from the one presented here.

Section (4.2): A Necessary Condition for Polyhedrality in WCG Spaces:

We begin this section with a lemma. It is based on straightforward geometry and is probably folklore, but is included for completeness since we have no direct reference for it.

Lemma (4.2.1)[4]: Suppose that $D \subseteq B_{X^*}$ has the property that for all $f \in D$, there exists $x_f \in X$ and $r_f > 0$ such that $\|x_f + z\| = f(x_f + z)$ whenever $\|z\| < r_f$. Then

(1) $r_f \leq \|x_f\|$, and

(2) $\|z\| < r_f \; and \; g \in D \setminus \{f\}$ implies $g(x_f + z) < \|x_f + z\|$.

In particular, if $f, g \in D$ are distinct then $\|x_g - x_f\| \geq r_f$.

Proof.

(1) Suppose that $\|x_f\| < r_f$. Let $y \in X$ satisfy $\|y\| < r_f - \|x_f\|$. Then $\|\pm y - x_f\| < r_f$ and so

$$f(y) = \|y\| = \|-y\| = f(-y) = -f(y),$$

meaning that $y \in ker\, f$. It follows that $f = 0$, which is impossible.

(2) Suppose $\|z\| < r_f, g \in D \setminus \{f\}$ and $g(x_f + z) = \|x_f + z\|$. Since $g \neq f$ we can find $y \in ker\, f$ such that $g(y) > 0$ and $\|y\| < r_f - \|z\|$. Otherwise we would have $ker\, f \subseteq ker\, g$, so $g = \alpha f$ for some α, and since $f(x_f + z) = \|x_f + z\| = g(x_f + z) = \alpha f(x_f + z)$, and $\|x_f + z\| > 0$ by (1), we conclude that $g = f$, which is not the case. Thus $\|y + z\| < r_f$ and so

$$\|x_f + y + z\| = f(x_f + y + z) = f(x_f + z).$$

On the other hand,

$$\|x_f + y + z\| \geq g(x_f + y + z) > g(x_f + z) = \|x_f + z\| = f(x_f + z).$$

Finally, if $f, g \in D$ are distinct and $\|x_g - x_f\| < r_f$, then by (2) we would have

$$\|x_g\| = g(x_g) = g\left(x_f + (x_g - x_f)\right) < \|x_f + (x_g - x_f)\| = \|x_g\| .$$

Armed with this lemma, we can give the proof of Theorem (4.1.10).

Proof of Theorem (4.1.10). Since X is WCG, we can find a weakly compact M-basis $(e_\gamma, e_\gamma^*) \in \Gamma$of X. Let E_n be the set of $x \in X$ that can be written as a linear combination of at most n elements of $(e_\gamma) \in \Gamma$. Let us define $B := w^* - str\, exp(B_{X^*})$. According to Theorem (4.1.9), for each $f \in B$, we can find a point $x \in span(e_\gamma) \in \Gamma$ that lies in the interior of $A_f \cap B_X$, where A_f is the supporting hyperplane as defined in that theorem. By a straightforward argument, it follows that there exists $r > 0$ such that $\|x + z\| = f(x + z)$ whenever $\|z\| < r$. Any such x belongs to some E_n.

Therefore, given $f \in B$, we can define n_f to be the minimal $n \in \mathbb{N}$ for which we can find an x and r as above, with $x \in E_n$.

Define $D_{n,m}$ to be the set of all $f \in B$ such that $n_f = n$, and there exist x and r, as described above, which in addition satisfy $r \geq 2^{-m}$ and

$$x = \sum_{\gamma \in F} a_\gamma e_\gamma ,$$

where $F \subseteq \Gamma$ has cardinality n and $|a_\gamma| \leq m$ for all $\gamma \in F$. Any such pair (x, r) will be called a witness for $f \in D_{n,m}$.

Evidently, $B = \bigcup_{n,m=1}^{\infty} D_{n,m}$. We claim that each $D_{n,m}$ is relatively discrete in the norm topology. For a contradiction, suppose otherwise and let $f, f_k \in D_{n,m}$ such that $\|f - f_k\| \to 0$. For each $k \in \mathbb{N}$, select a witness (x_k, r_k) for f_k. The set

$$L = \left\{ \sum_{\gamma \in F} a_\gamma e_\gamma : F \subseteq \Gamma \text{ has cardinality } n \text{ and } |a_\gamma| \leq m \text{ for all } \gamma \in F \right\},$$

is weakly compact, being a natural continuous image of $[-m, m]^n \times (\{e_\gamma : \gamma \in \Gamma\} \cup \{0\})^n$.

Thus, by the Eberlein-Ŝmulyan Theorem (Let X be a Banach space. For subset K the following are equivalent. a) K is relatively $\sigma(X, X^*)$ compact, i.e. $K\sigma(X, X^*)$ is compact. b) Every sequence in K contains a $\sigma(X, X^*)$-convergent subsequence. c) Every sequence in K has a $\sigma(X, X^*)$-accumulation point. We will need the following Lemma.)[10], and by taking a subsequence of (x_k) if necessary, we can assume that the x_k tend weakly to some $y \in L$. We claim that $y \in E_j$ for some $j < n$. Indeed, if

$$y = \sum_{\gamma \in F} a_\gamma e_\gamma ,$$

where $F \subseteq \Gamma$ has cardinality n and $\neq 0$ for all $\gamma \in F$, then there exists a K for which $e_\gamma^*(x_k) \neq 0$ for all $\gamma \in F$ and all $k \geq K$. Because each x_k can be expressed as a linear combination of n elements of $(e_\gamma)_{\gamma \in \Gamma}$, it follows that $x_k \in span(e_\gamma)_{\gamma \in F}$ whenever $k \geq K$.

Indeed, if

$$w = \sum_{\gamma \in G} b_\gamma e_\gamma ,$$

where $G \subseteq \Gamma$ has cardinality n, and if $e_\gamma^*(w) \neq 0$ for all $\gamma \in F$, then necessarily $F \subseteq G$, and equality of these sets follows since their cardinalities agree. Because the $x_k, k \geq K$,belong to a finite-dimensional space, it follows that $\|y - x_k\| \to 0$. However, by Lemma (4.2.1), we know that the x_k are uniformly separated in norm by 2^{-m} $(\leq r_k)$, so they cannot converge in norm to anything.

Thus $y \in E_j$ for some $j < n$, as claimed. Now fix $z \in X$ such that $\|z\| < 2^{-m}$. We have $\|x_k + z\| = f_k(x_k + z)$ for all k, because $2^{-m} \leq r_k$. As $\|f - f_k\| \to 0$ and $x_k + z \to y + z$ weakly, we get $\|x_k + z\| \to f(y + z) \leq y + z$. On the other hand, by w-lower semicontinuity of the norm, $\|y + z\| \leq f(y + z)$. So the equality $\|y + z\| = f(y + z)$ holds whenever $\|z\| < 2^{-m}$. In particular, $1 = \|x_k\| \to \|y\|$. However $y \in E_j$ and $j < n$, and this contradicts the minimal choice of $n_f = n$.

Thus each $D_{n,m}$ is relatively discrete in the norm topology. Since $D_{n,m} \subseteq B$ and since the norm and w*-topologies agree on B, it follows that Dn,m is relatively discrete in the w*-topology as well.

Finally, we recall that a Banach space X is called weakly Lindel¨of determined (WLD) if B_{X^*} is Corson compact in the w*- topology. The class of WLD spaces includes all WCG spaces. Any polyhedral Banach space is an Asplund space (this follows, for example, from [4]), and any WLD Asplund space is WCG. Therefore Theorem (4.1.10) extends to all WLD polyhedral spaces.

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