Chapter 2

Real Analytic Families of Harmonic Functions and a domain with Small Hole

We show a general result on continuation propeties of some particular assumptions real analytic familes of harmonic functions in domains with a small hole and we prove that the validity of the equality $u_\varepsilon(p) = U_p[\varepsilon]$ for negative depends on the parity of the dimension.

Section(2.1): Main Results for Real Analytic Families of Harmonic Function on Ω (ε)

We fix once for all $n \epsilon \mathbb{N}$, $n \geq 3$, [0, 1]. Here $\mathbb{N}$ denotes the set of natural numbers including 0. Then we fix two sets and Ω^0 in the n-dimensional Euclidean space $\mathbb{R}^n$.The letter 'i' stands for 'inner domain' and the letter '0' stands for 'outer domain'. We assume that i and 0 satisfy the following condition Ω^i and Ω^0 are open bounded connected subsets of $\mathbb{R}^n$ of (13) class $C^{0,\alpha}$ such that $\mathbb{R}^n \setminus cI\Omega^i$ and $\mathbb{R}^n \setminus cI\Omega^0$ are connected, and such that the origin 0 of $\mathbb{R}^n$ belongs both to Ω^i and Ω^0.

Here $cI\Omega$ denotes the closure of Ω for all $\Omega \subseteq \mathbb{R}$. For the definition of functions and sets of the usual Schauder class $C^{0,\alpha}$ and $C^{1,\alpha}$, we refer for example to Gilbarg and Trudinger .

We note that condition (13) implies that Ω^i and Ω^0 have no holes and that there exists a real number ε_0 such that

$$\varepsilon_0 > 0 \text{ and } \varepsilon cI\Omega^i \subseteq \Omega^0 \text{ for all } \varepsilon \in [-\varepsilon_{0,0}] . \qquad (14)$$

Then we denote by (ε) the perforated domain defined by

$$\Omega(\varepsilon) \equiv \Omega^0 \setminus (\varepsilon cI\Omega^i) \qquad \forall \varepsilon \in [-\varepsilon_0, \varepsilon_0].$$

A simple topological argument shows that (ε) is an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,}$ for all $\varepsilon \in [-\varepsilon_{0,0}] \setminus \{0\}$. Moreover, the boundary (ε) has exactly the two connected components $\partial\Omega^0$ and $\varepsilon\partial\Omega^i$, for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We also note that (0) = $\Omega^0 \setminus \{0\}$.

Now let $f^i \in C^{1,}(\partial\Omega^i)$ and $f^0 \in C^{1,\alpha}(\partial\Omega^0)$. Let $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$. We consider the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega(\varepsilon)\,, \\ u(x) = f^i(x/\varepsilon) & \text{for } x \in \varepsilon\partial\Omega^i\,, \\ u(x) = f^0(x) & \text{for } x \in \partial\Omega^0. \end{cases} \tag{15}$$

As is well known, the problem in (3) has a unique solution in $C^{1,}(cI\Omega((\varepsilon))$. We denote such a solution by u_ε.Then we fix a point p in $\Omega^0 \setminus \{0\}$ and we take $\varepsilon_p \in [0,\varepsilon_0]$, such that $p \in \Omega(\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_p]$. In particular, it makes sense to consider $u_\varepsilon(p)$ for all $\varepsilon \in [0,\varepsilon_p]$. Thus we can ask the following question. What can be said of the map from $[0, \varepsilon_p]$ to $\mathbb{R}$ which takes ε to (p)?

Questions of this type have been largely investigated by the so called Asymptotic Analysis. We mention here as an example the work of Maz'ya, Nazarov, and Plamenevskij]. The techniques of Asymptotic Analysis aim at representing the behavior of (p) as $\varepsilon \to 0^+$ in terms of regular functions of ε plus a remainder which is smaller than a known infinitesimal function of ε. Instead, by the different approach proposed by Lanza de Cristoforis and by possibly shrinking ε_p, we can represent the function which takes ε to $u_\varepsilon(p)$ as the restriction to $[0,\varepsilon_p]$ of a real analytic map defined on $[-\varepsilon_p,\varepsilon_p]$.

We can consider what we call the 'macroscopic' behaviour of the family $\{u_\varepsilon\}_{\in[0,\varepsilon_0]}$. Indeed, if $\Omega_M \subseteq \Omega^0$ is open, and $0 \notin cI\Omega_M$, and $\varepsilon_M \in [0,\mathcal{E}_0]$ is such that $cI\Omega_M \cap (\varepsilon cI\Omega^i) = \emptyset$ for all $\varepsilon \in [-\varepsilon_M,\varepsilon_M]$, then $cI\Omega_M \subseteq cI\Omega(\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_M]$. Thus it makes sense to consider the restriction $u_{\varepsilon \setminus cI\Omega M}$ for all $\varepsilon \in [0,\varepsilon_M]$. In particular, it makes sense to consider the map from $[0,\varepsilon_M]$ to $C^{1,\alpha}(cI\Omega_M)$ which takes ε to $u_{\varepsilon\setminus cI\Omega M}$. Then we prove in Proposition (2.2.1)[2] that there exists a real number $\varepsilon_1 \in [0, \varepsilon_0]$ such that the following statement holds.

(a_1) Let $\Omega_M \subseteq \Omega^0$ be open and such that $0 \notin cI\Omega_M$. Let $\varepsilon_M \in [0, \varepsilon_1]$ be such that $cl\Omega_M \cap (\varepsilon cl\Omega^i) = \emptyset$ for all $\varepsilon \in [-\varepsilon_M, \varepsilon_M]$ .Then there exists a real analytic operator U_M from $[-\varepsilon_M, \varepsilon_M]$ to $C^{1,\alpha}(cl\Omega_M)$ such that

$$u_{\varepsilon \setminus cl\Omega M} = [\varepsilon] \quad \forall \varepsilon \in [0, \varepsilon_M]. \tag{16}$$

Here the letter 'M' stands for 'macroscopic'. But we can also consider the 'microscopic' behavior of the family $\{u_\varepsilon\}_{\in [0,_0]}$ in proximity of the boundary of the hole. To do so we denote by $(\varepsilon.)$ the rescaled function which takes $x \in (1/\varepsilon) cl\Omega(\varepsilon)$ to $u_\varepsilon(\varepsilon x)$, for all $\varepsilon \in [0, \varepsilon_0]$.

If $\Omega_m \subseteq \mathbb{R}^n \setminus cl\Omega^i$ is open, and $\varepsilon_m \in [0, \varepsilon_1]$ is such that $\varepsilon cl\Omega_m \subseteq \Omega^0$ for all $\varepsilon \in [-\varepsilon_{m,}]$,then $cl\Omega_m \subseteq (1/\varepsilon)\Omega(\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_m]$ and it makes sense to consider the map from $[0, \varepsilon_m]$

to $C^{1,\alpha}(cl\Omega_m)$ which takes ε to $u_\varepsilon(\varepsilon.)$. In Proposition (2.2.1)[2] we prove that there exists $\varepsilon_1 \in [0, \varepsilon_0)$ such that the following statement holds.

(a$_2$) Let $\Omega_m \subseteq \mathbb{R}^n \setminus$ be open and bounded. Let $\varepsilon_m \in [0, \varepsilon_1]$ be such that $\varepsilon cl\Omega_m \subseteq \Omega^0$ for all $\varepsilon \in [-\varepsilon_{m,}]$.Then there exists a real analytic operator U_m from $[-\varepsilon_m, \varepsilon_m]$. to $C^{1,\alpha}(cl\Omega_m)$ such that

$$u_\varepsilon(\varepsilon.)_{|cl\Omega m} = [\varepsilon] \quad \forall \varepsilon \in [0, \varepsilon_m]. \tag{17}$$

Here the letter 'm' stands for 'microscopic'.

We now observe that Proposition (2.2.1)[2] states that the equalities in (16) and (17) hold in general only for ε positive, but the functions $u_{\varepsilon|cl\Omega M}$, $U_M[\varepsilon]$, $u_\varepsilon(\varepsilon.)_|$ and $U_m[\varepsilon]$ are defined also for ε negative. Thus, it is natural to formulate the following question.

What happens to the equalities in (16) and (17) for ε negative? (18)

The purpose is to answer to the question formulated here above. In particular, we prove in Theorem (2.1.8)[2] that the equalities in (16) and (17) hold also for ε negative if the dimension n is even. Instead, if the dimension n is odd we show in Proposition (2.2.3)[2] that the equalities in (16) and (17) hold for ε negative only if there exists a real constant c such that $f^i = c$ and $f^0 = c$ identically (so that

$(x) = c$ for all $x \in cI\Omega(\varepsilon)$ and $\varepsilon \in [-\varepsilon_{0,0}] \setminus \{0\}$.)

However, we note that the conditions expressed in (a_1) and (a_2) are not related to the particular boundary value problem in (3). Indeed, we could prove the validity of (a_1) and (a_2) for families of functions $\{u_\varepsilon\}_{\varepsilon \in [0,\varepsilon 0]}$ which are solutions of problems with different boundary conditions, such as those considered in Lanza de Cristoforis. For this reason, we investigate the properties of families of functions $\{u_\varepsilon\}_{\varepsilon \in [0,\varepsilon 1]}$ such that

(a0) $u_\varepsilon \in C^{1,}(cI\Omega(\varepsilon))$ and $\Delta u_\varepsilon = 0$ in (ε) for all $\varepsilon \in [0,\varepsilon_1]$

and which satisfy the conditions in (a_1) and (a_2), but which are not required to satisfy any specific boundary condition on $\partial\Omega(\varepsilon_1)$. To do so, we introduce the following terminology.

Let $\varepsilon_1 \in [0, \varepsilon_0]$. We say that $\{u_\varepsilon\}_{\varepsilon \in [0, \varepsilon_1]}$ is a right real analytic family of harmonic functions on (ε_1) if it satisfies the conditions in (a_0), (a_1),(a_2). We say that $\{u_\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$ is a real analytic family of harmonic functions on (ε_1) if it satisfies the following conditions (b_0)-(b_1).

(b_0) $v_0 \in C^{1,\alpha}(cI\Omega^0)$ and $\Delta v_0 = 0$ in Ω^0, $\in C^{1,\alpha}(cI\Omega(\varepsilon))$ and $\Delta v_\varepsilon = 0$ in $\Omega(\varepsilon)$ for all $\varepsilon \in [-\varepsilon_1,\varepsilon_1] \setminus \{0\}$.

(b_1) Let $\Omega_M \subseteq \Omega^0$ be open and such that $0 \notin cI\Omega_M$. Let $\varepsilon_M \in [0, \varepsilon_1]$ be such that $cI\Omega_M \cap \varepsilon cI\Omega^i = \emptyset$ for all $\varepsilon \in [-\varepsilon_M,]$. Then there exists a real analytic operator V_M from $[-\varepsilon_M, \varepsilon_M]$ to $C^{1,}(cI\Omega_M)$ such that

$$v_{\varepsilon|cI\Omega M} = V_M[\varepsilon] \quad \forall \varepsilon \in [-\varepsilon_M,\varepsilon_M].$$

(b2) Let $\Omega_M \subseteq \mathbb{R}^n \setminus cI\Omega^i$ be an open and bounded subset. Let $\varepsilon_m \in [0, \varepsilon_1]$ be such that $\varepsilon cI\Omega_m \subseteq \Omega^0$ for all $\varepsilon \in [-\varepsilon_m, \varepsilon_m]$. Then there exists a real analytic operator V_m from $[-\varepsilon_m, \varepsilon_m]$ to $C^{1,}(cI\Omega_m)$ such that

$$v_\varepsilon(\varepsilon.)_{|cI\Omega m} = [\varepsilon] \quad \forall \varepsilon \in [-\varepsilon_m, \varepsilon_m] \setminus \{0\} \ . \qquad (19)$$

Here $(\varepsilon.)$ denotes the map which takes $x \in (1/\varepsilon)cI\Omega(\varepsilon)$ to $v_\varepsilon(\varepsilon x)$, for all $\varepsilon \in [-\varepsilon_1,\varepsilon_1] \setminus \{0\}$.

We also note that we do not ask in condition (b_2) that the equality in

(7) holds forε=0. In particular, $v_0(0.)|_{cI\Omega m}$ is necessarily a constant function on $cI\Omega_m$, while V_m [0] may be nonconstant. Finally, we say that $\{\omega_\varepsilon\}_{\varepsilon \in [-\varepsilon_1,\varepsilon_1]}$ is a real analytic family of harmonic functions on Ω^0 if it satisfies the following conditions (c_0),(c_1).

(c_0) $\omega_\varepsilon \in C^{1,}(cI\Omega^0)$ and $\Delta\omega_\varepsilon = 0$ in Ω^0 for all $\varepsilon \in [-\varepsilon_1,\varepsilon_1]$.

(c_1) The map from $[-\varepsilon_1, \varepsilon_1]$ to $C^{1,}(cI\Omega^0)$ which takes ε to ω_ε is real analytic.

We state the results in Theorems (2.1.8) and (2.1.9), where we consider separately the case of dimension n even and of dimension n odd, respectively. In particular, by Theorems (2.1.8) and (2.1.9) we can deduce the validity of the following statements (j) and (jj).

(j) If the dimension n is even and $\{u_\varepsilon\}_{\varepsilon \in [0, \varepsilon 0]}$ is a right real analytic family of harmonic functions on (ε) then there exists a real analytic family of harmonic functions

$$\{v_\varepsilon\}_{\varepsilon\in[-\varepsilon 1,\varepsilon 1]} \text{ on } \Omega(\varepsilon) \text{ such that } u_\varepsilon = v_\varepsilon \text{ for all } \varepsilon \in [0,\varepsilon_1].$$

(jj) If the dimension n is odd and $\{v_\varepsilon\}_{\varepsilon \in [-\varepsilon 1, \varepsilon 1]}$ is a real analytic family of harmonic functions on (ε) then there exists a real analytic family of harmonic functions

$$\{\omega_\varepsilon\}_{\varepsilon\in[-\varepsilon 1,\varepsilon 1]} \text{ on } \Omega^0 \text{ such that } v_\varepsilon = \omega_{\varepsilon|cI\Omega(\varepsilon)} \text{ for all } \varepsilon \in [-\varepsilon_1,\varepsilon_1].$$

In particular we note that for n odd statement (jj) implies that for each $\varepsilon \in [-\varepsilon_{1,1}]$ the function v_ε can be extended inside the hole $\varepsilon\Omega^i$ to aharmonic function defined on the whole of Ω^0. As is well known, the condition of existence of an extension of a harmonic function defined on (ε) to is quite restrictive. Hence, case (jj) has to be considered, in a sense, as exceptional.

We introduce some known results of Potential Theory. In particular, we adopt the approach proposed by Lanza de Cristoforis for the analysis of elliptic boundary value problems in domains with a small hole. Accordingly, we show that the boundary value problem in

(3) is equivalent to a suitable functional equation $\Lambda = 0$, where Λ is a real analytic operator between Banach spaces. Then we analyze equation $\Lambda = 0$ by exploiting the Implicit Function Theorem for real analytic functions . We prove theorems (2.1.8) and (2.1.9), where we consider separately case n even and n odd, respectively. Then in Examples (2.1.10), (2.1.11) and (2.1.12) we show that the the assumptions in Theorems (2.1.8) and (2.1.9) cannot be weakened in a sense which we clarify below. In particular, by Examples (2.1.11) and (2.1.12) we deduce that analogs of statements (j) and (jj) do not hold if we replace the assumption that $u_\varepsilon, v_\varepsilon, \omega_\varepsilon$ are harmonic with the weaker assumption that u_ε ,v_ε, ω_ε are real analytic. In the last Section 4 we consider some particular cases and we show some applications of Theorems (2.1.8) and (2.1.9).

We consider the family $\{u_\varepsilon\}_{\varepsilon \in [0, \varepsilon 0\}}$ of the solutions $C^{1,}(cI\Omega(\varepsilon))$ in of (3).

We show that there exists $\varepsilon \in [0, \varepsilon_1]$ such that $\{u_\varepsilon\}_{\varepsilon \in [0, \varepsilon 0\}}$ satisfies the conditions in (a_1) and (a_2) we also prove that we can take $\varepsilon_1 = \varepsilon_0$ if the dimension n is even. In Proposition (2.2.1)[2] we assume that n is even and we consider a right real analytic family $\{u_\varepsilon\}_{\in[0,\varepsilon 0\}}$ of harmonic function on $\Omega(\varepsilon)$. Then, conditions (a1) and (a2) imply that $u_{\varepsilon|cI\Omega m}$ and can be represented by means of convergent power series of ε for ε small and positive. Under the condition that either $\Omega^i = -\Omega^i$ or that $\Omega^0 = -\Omega^0$ satisfies some suitable symmetry assumptions, and we obtain some additional information on the power series expansion of $u_{\varepsilon|cI\Omega M}$ and $(\varepsilon.)_{|cI\Omega m}$ for ε small and postive.

We answer to the question in (6) by exploiting Theorem (2.1.9).

We denote by S_n the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \frac{|x|^{2^{-n}}}{(2-n)S_n} \qquad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Here S_n denotes the $(n$-1) dimensional measure of the unit sphere in $\mathbb{R}$

. As is well known S_n is the fundamental solution of the Laplace operator in $\mathbb{R}$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ .Let $\mu \in C^{0,}(\partial\Omega)$. Then we denote by $[\mu]$ the single layer potential of density μ. Namely $[\mu]$ is the function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$v[\mu](x) \equiv \int_{\partial\Omega} S_n(x-y)\mu(y)d\sigma_y \quad \forall x \in \mathbb{R}^n.$$

Then we have the following well known Lemma.

Lemma (2.1.1.)[2]: Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,}$. Let $\widetilde{\Omega}$ be an open bounded subset of $\mathbb{R}^n \setminus cl\Omega$.Then the map from $C^{0,\alpha}(\partial\Omega)$.to $C^{1,\alpha}(cl\Omega)$ which takes μ to] $[\mu]_{|cl\Omega}$ is linear and continuous, and the map from $C^{0,\alpha}(\partial\Omega)$.to $C^{1,\alpha}(cl\Omega)$ which takes μ to $v[\mu]_{|cl\Omega}$ is linear and continuous. Moreover, the map from $C^{0,}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ which takes μ to $[\mu]_{|\partial\Omega}$ is a linear homeomorphism .We observe that the last sentence of Lemma (2.1.1) holds only if the dimension n is greater or equal than 3. Indeed, in the planar case the map which takes μ to $[\mu]_{|\partial\Omega}$ is not in general a homeomorphism from $C^{0,\alpha}(\partial\Omega)$.to $C^{1,\alpha}(\partial\Omega)$,we have assumed that $n \geq 3$ and thus we can exploit Lemma (2.1.1) to convert a Dirichlet boundary value problem for the Laplace operator into a system of integral equations. In order to study the integral equations corresponding to the Dirichlet problem in the perforated domain (ε), with $\varepsilon \in [-\varepsilon_{0,0}]\setminus\{0\}$ we now introduce the operators Λ_1 and Λ_{-1} . Let θ {-1,-1} then we denote by $\Lambda_\theta \equiv (\Lambda^{i}_{\theta}, \Lambda^{0}_{\theta})$ the operator from

$[-\varepsilon_0, \varepsilon_0] \times C^{1,}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0) \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ to $C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$

Defined by

$$\Lambda^{i}_{\theta}[\varepsilon, f^i, f^0, \mu^i, \mu^0](x) \equiv \theta\int_{\partial\Omega^i} S_n(x-y)\mu^i(y)\, d\sigma_y$$
$$+\int_{\partial\Omega^i} S_n(\varepsilon x-y)\mu^0(y)\, d\sigma_y \ - f^i(x) \quad \forall x \in \partial\Omega^i,$$

$$\Lambda^{0}_{\theta}[\varepsilon, f^i, f^0, \mu^i, \mu^0](x) \equiv \varepsilon^{i-2}\int_{\partial\Omega^i} S_n(x-\varepsilon y)\mu^i(y)\, d\sigma_y$$
$$+\int_{\partial\Omega^0} S_n(x-y)\mu^0(y)\, d\sigma_y \ - f^0(x) \quad \forall x \in \partial\Omega^0$$

for all $(\varepsilon, f^i, f^0, \mu^i, \mu^0) \in [-\varepsilon_0,\varepsilon_0] \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0) \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ then, by Lemma (2.1.1) we deduce the validity of the following Proposition (2.1.2).

Proposition (2.1.2) [2]: Let Ω^i, Ω^0. be as in (1). Let ε_0 be as in (2). Let $\varepsilon \in [-\varepsilon_{0,0}] \setminus \{0\}$. Let $(f^i, f^0) \in C^{1,}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$ Let $\theta \equiv (\operatorname{sgn} \varepsilon)^n$. Then there exists a unique pair of functions $(\mu^i, \mu^0) \in C^{0,}(\partial\Omega^i) \times C^{0,}(\partial\Omega^0)$ such that

$$\Lambda_\theta [\varepsilon, f^i, f^0, \mu^i, \mu^0] = (0, 0). \tag{20}$$

Moreover, the function u from (ε) to $\mathbb{R}$ defined by

$$(x) \equiv \varepsilon^{n-2} \int_{\partial\Omega^i} S_n(x-\varepsilon y)\, d\sigma_y + \int_{\partial\Omega^0} S_n(x-y)\mu^0(y)\, d\sigma_y \quad \forall x \in cI\Omega\,(\varepsilon)$$

is the unique solution in $C^{1,\alpha}(cI\Omega(\varepsilon))$ of the boundary value problem in (3).

Proposition (2.1.3): Let Ω^i, Ω^0 be as in (1). Let $\theta \in \{-1, 1\}$. Let $(f^i, f^0) \in C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$. Then, there exists a unique pair of functions $(\mu^i, \mu^0) \in C^{0,}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ such that

$$\Lambda_\theta [0, \ , f^0, \mu^i, \mu^0] = (0,0). \tag{21}$$

Moreover, the function $u \equiv [\mu^0]_{|cI\Omega 0}$ is the unique solution in $C^{1,\alpha}(\partial\Omega^0)$ of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \partial\Omega^0, \\ u = f^0 & \text{on } \partial\Omega^0. \end{cases}$$

Proof: We observe that the equation in (9) is equivalent to the following system of equations

$$\begin{cases} \theta_v[\mu^i]_{|\partial\Omega^i} + v[\mu^0](0) = f^i & on\ \partial\Omega^i, \\ v[\mu^0]_{|\partial\Omega^0} = f^0 & on\ \partial\Omega^0 . \end{cases}$$

Then the validity of the Lemma can be deduced by Lemma (2.1.1).

In the following Propositions (2.1.5),(2.1.6) and (2.1.7) we exploit the Implicit Function Theorem for real analytic maps to investigate the dependence of the solution (μ^i, μ^0) of the Equations in (8) and (9) upon (ε, f^i, f^0) in particular, in Proposition (2.1.5) we study what happens for ε small, while in Propositions (2.1.6) and (2.1.7) we consider the case of dimension n even and odd, respectively. To prove Propositions (2.1.5),(2.1.6) and (2.1.7) we need to analyze the regularity of the operator Λ_θ. The definition of Λ_θ involves the single layer $[\mu]$ and also integral operators which display no singularity. To analyze their regularity we need the following Lemma (2.1.4).

Lemma (2.1.4)[2]: Let Ω, $\tilde{\Omega}$ be open bounded subsets of $\mathbb{R}^n$ of class $C^{1,}$. Then the following statements hold.

(i) The map G from $\{(\psi, \phi, \mu) \in C^{1,\alpha}(\partial\widetilde{\Omega}, \mathbb{R}_n) \times C^{1,\alpha}(\partial\Omega, \mathbb{R}n) \times C^{0,\alpha}(\partial\Omega) : \psi(\partial\Omega) \cap \phi(\partial\Omega) = \emptyset\}$ to $C^{1,\alpha}(\partial\Omega)$ which takes (ψ, ϕ, μ) to the function $G[\psi, \phi, \mu]$ defined by

$$G[\psi, \phi, \mu] \equiv \int_{\partial\Omega} S_n(\psi(x) - \phi(y))\mu(y)\, d\sigma_y \quad \forall x \in \partial\tilde{\Omega},$$

is real analytic.

(ii)The map H from $\{(\Phi, \phi, \mu) \in C^{1,\alpha}(cl\widetilde{\Omega}\ \mathbb{R}_n) \times C^{1,\alpha}(\partial\Omega, \mathbb{R}^n) \times C^{0,\alpha}(\partial\Omega) : \Phi(cl\Omega) \cap \phi(\partial\Omega) = \emptyset\}$ to $C^{1,\alpha}(cl\widetilde{\Omega})$ which takes (Φ, ϕ, μ) to the function $H[\Phi, \phi, \mu]$ defined by

$$H[\Phi, \phi, \mu] \equiv \int_{\partial\Omega} S_n(\Phi(x) - \phi(y))\mu(y)\, d\sigma_y \quad \forall x \in cl\tilde{\Omega},$$

is real analytic.

Proposition (2.1.5)[2]: Let Ω^i, Ω^0 be as in (1).Let ε_0 be as in (2) Let $\theta \in \{-1,1\}$. Let $(\tilde{f}^i, \tilde{f}^0) \in C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$. Let the pair $(\tilde{\mu}^i, \tilde{\mu}^0)$ be the unique solution in $C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ of $\Lambda_\theta[0, \tilde{f}^i, \tilde{f}^0, \tilde{\mu}^i, \tilde{\mu}^0]=0$. Then there exist $\tilde{\varepsilon}$ in $[0, \varepsilon_0]$and an open neighborhood $\mathcal{U}$ of $(\tilde{f}^i, \tilde{f}^0)$ in $C^{1,\alpha}(\partial\Omega^i) \times C^{0,\ \alpha}(\partial\Omega^0)$, and an open neighborhood $\mathcal{V}$ of $(\tilde{\mu}^i, \tilde{\mu}^0)$ in $C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$, and a real analytic operator $\widetilde{M}_\theta \equiv (\widetilde{M}^i{}_\theta, \widetilde{M}^0{}_\theta)$ from $[-\tilde{\varepsilon}, \tilde{\varepsilon}] \times \mathcal{U}$ to $\mathcal{V}$ such that the set of zeros of Λ_θ in $[-\tilde{\varepsilon}, \tilde{\varepsilon}] \times \mathcal{U} \times \mathcal{V}$ coincides with the graph of M_θ. In particular,

$$[\varepsilon, f^i, f^0, \widetilde{M}[\varepsilon,,^0]] = (0,0) \quad \forall(\varepsilon, f^i, f^0) \in [-\tilde{\varepsilon}, \tilde{\varepsilon}] \times \mathcal{U}. \quad (22)$$

Proof: We note that the existence and uniqueness of the solution $(\tilde{\mu}$, $\tilde{\mu}^0)$follows by Proposition (2.1.3). We now prove the statement by applying the Implicit Function Theorem for real analytic maps to the equation in (10) around $(0, \tilde{f}^i, \tilde{f}^0, \tilde{\mu}^i, \tilde{\mu}^0)$. To do so, we first show that Λ_θ is real analytic from $[-\varepsilon_0,\varepsilon_0]\times C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^0) \times C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^0)$ to $C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^0)$. By Lemma (2.1.4)(i) the map from $[-\varepsilon_0,\varepsilon_0]\times C^{0,\alpha}(\partial\Omega^0)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes (ε, μ^0) to the function $\int_{\partial\Omega^0} S_n(\varepsilon x - y)\mu^0(y)\, d\sigma_y$ of $x \in \partial\Omega^i$ is real analytic Lemma (2.1.1) implies that the map from $C^{0,}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes μ^i to the function $\int_{\partial\Omega^i} S_n(x-y)\, \mu^i(y) d\sigma_y$ of $x \in \partial\Omega^i$ is real analytic. Then, by standard calculus in Banach space we deduce that $\Lambda^i{}_\theta$ is real analytic from $[-\varepsilon_0, \varepsilon_0]\times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0) \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ to $C^{1,\alpha}(\partial\Omega^i)$. By a similar argument we can show that $\Lambda^0{}_\theta$ is real analytic from $[-\varepsilon_0, \varepsilon_0]\times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0) \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ to $C^{1,\alpha}(\partial\Omega^0)$. Hence $\Lambda^0{}_\theta$ is real analytic.

Now we observe that the partial differential of Λ_θ at $(0, \tilde{f}^i, \tilde{f}^0, \tilde{\mu}^i, \tilde{\mu}^0)$ with respect to the variables (μ^i, μ^0) is delivered by the following formulas

$$\partial_{(\mu,\, \mu 0)}\, \Lambda^{\mathrm{i}}_{\theta}\, [0, \tilde{f}^i, \tilde{f}^0, \tilde{\mu}^i, \tilde{\mu}^0](\mu^{-}, \mu^{-0})(x) \qquad (23)$$

$$= \theta \int_{\partial\Omega^i} S_n(x-y)\, \mu^i(y) d\sigma_y + \int_{\partial\Omega^0} S_n(y)\mu^0(y)\, d\sigma_y \qquad \forall x \in \partial\Omega,$$

$$\partial_{(\mu,\, \mu 0)}\, \Lambda^{0}_{\theta}\, [0, \tilde{f}^i, \tilde{f}^0, \tilde{\mu}^i, \tilde{\mu}^0](\mu^{-i}, \mu^{-0})(x) = \int_{\partial\Omega^0} S_n(x-y)\, \mu^0(y) d\sigma_y \qquad \forall x \in \partial\Omega^0$$

for all $(\mu^{-i}, \mu^{-0}) \in C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$.We have to show that the differential ${}_{(\mu,\, \mu 0)}\Lambda_\theta\, [0, \tilde{f}^i, \tilde{f}^0, \tilde{\mu}^i, \tilde{\mu}^0]$ is a linear homeomorphism. By the Open Mapping Theorem, it suffices to show that it is a bijection from $C^{0,}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ to $C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$.

Let $(\tilde{f}^i, \tilde{f}^0) \in C^{1,}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$. By the equalities in (11) and by Lemma (2.1.1) we deduce that there exists a unique pair $(\mu^{-}, \mu^{-0}) \in C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ such that

$${}_{(\mu,\, \mu 0)}\Lambda_\theta\, [0, \tilde{f}^i, \tilde{f}^0, \tilde{\mu}^i, \tilde{\mu}^0](\bar{\mu}^i, \bar{\mu}^0) = (\bar{f}^i, \bar{f}^0)$$

(see also the proof of Lemma (2.1.3)) Hence we can invoke the Implicit Function Theorem for real analytic maps in Banach spaces and deduce the existence of $\tilde{}$, $\mathcal{U}$, $\mathcal{V}$, M^{θ} as in the statement.

Proposition (2.1.6)[2]: Let Ω^i, Ω^0 be as in (1). Let ε_0 be as in (2). If the dimension n is even, then there exists a real analytic map $M \equiv (M^i, M^0)$ from $[-\varepsilon_0,\varepsilon_0]\times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$ to $C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ such that

$$\Lambda_1 [\varepsilon, f^i, f^0, M [\varepsilon, f^i, f^0]] = (0, 0) \tag{24}$$

for all $(\varepsilon, f^i, f^0) \in [-\varepsilon_0,\varepsilon_0]\times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$.

Proof: By Propositions (2.1.2) and (2.1.3) we deduce that there exists a unique map M from $[-\varepsilon_0, \varepsilon_0] \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$ to $C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ which satisfies (12) We show that M is real analytic by exploiting the Implicit Function Theorem for real analytic maps.

By Lemmas (2.1.1) and (2.1.4) and by standard calculus in Banach space we verify that Λ_1 is real analytic from $[-\varepsilon_0, \varepsilon_0] \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0) \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ to $C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$ (see also the proof of Proposition (2.1.5)) By the Implicit Function Theorem for real analytic maps, it clearly suffices to prove that if $(\varepsilon, f^i, f^0)$is in $[-\varepsilon_0,\varepsilon_0] \times C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^0)$ then the partial differential of Λ_1 at $(\varepsilon, f^i, f^0, M [\varepsilon, f^i, f^0])$ with respect to the variables (μ^i, μ^0) is a linear homeomorphism from $C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ onto $C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$By Proposition(2.1.5), we can confine ourselves to consider (ε, f^i, f^0) in $([-\varepsilon_0, \varepsilon_0] \setminus \{0\}) \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$. By standard calculus in Banach space, the partial differential ${}_{(\mu_i\mu_0)}\Lambda_1 [\varepsilon, f^i, f^0, M [\varepsilon, f^i, f^0]]$ is delivered by the following formulas

$$_{(\mu,\mu 0)}\,\Lambda_1^{\mathrm{i}}\,[0, f^i, f^0, \bar{\mu}^i, \bar{\mu}^0](\bar{\mu}^i, \bar{\mu}^0)(x)$$
$$=\int_{\partial\Omega^i} S_n(x-y)\,\bar{\mu}^i(y)d\sigma_y + \int_{\partial\Omega^0} S_n(\varepsilon x-y)\bar{\mu}^0(y)\,d\sigma_y \qquad \forall x \in \partial\Omega^i,$$
$$_{(\mu,\mu 0)}\,\Lambda_1^{0}\,[0, f^i, f^0, \bar{\mu}^i, \bar{\mu}^0](\bar{\mu}^i, \bar{\mu}^0)(x)$$
$$= \varepsilon^{n-2}\int_{\partial\Omega^i} S_n(x-\varepsilon y)\,\bar{\mu}^i(y)d\sigma_y + \int_{\partial\Omega^0} S_n(x-y)\bar{\mu}^0(y)\,d\sigma_y \;\forall x\in \partial\Omega^0,$$

for all $(\bar{\mu}^i, \bar{\mu}^0) \in C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^0)$ then by Lemma (2.1.1) and by the

Open Mapping Theorem, we deduce that $_{(\mu i,\, \mu 0)}\ \Lambda_1 [\varepsilon, f^i, f^0, M [\varepsilon, f^i, f^0]]$ is a linear homeomorphism from $C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^0)$ onto $C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$ the proof of the Proposition is now complete.

Proposition (2.1.7)[2]: Let Ω^i, Ω^0 be as in (1). Let ε_0 be as in (2). If the dimension n is odd, then there exist real analytic maps $\mathrm{M}_+ \equiv (\mathrm{M}_+^{\mathrm{i}}, \mathrm{M}_+^{0})$ from $[0,\varepsilon 0] \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$ to $C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^0)$ and $\mathrm{M}_- \equiv (\mathrm{M}_-^{\mathrm{i}}, \mathrm{M}_-^{0})$ from to $[-\varepsilon_0, 0] \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$ to $C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^0)$ such that

$$\Lambda_1 [\varepsilon, f^i, f^0, \mathrm{M}_+ [\varepsilon, f^i, f^0]] = (0, 0)$$

for all $(\varepsilon, f^i, f^0) \in [0, \varepsilon_0] \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$, and such that

$$\Lambda_{-1} [\varepsilon, f^i, f^0, \mathrm{M}_+ [\varepsilon, f^i, f^0]] = (0, 0)$$

for all $(\varepsilon, f^i, f^0) \in [-\varepsilon_0, 0] \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$.

We prove theorems (2.1.8) and (2.1.9). In theorem (2.1.8) we consider the case of dimension n even. We note that theorem (2.1.8) implies the validity of statement (j).

Theorem (2.1.8)[2]: Assume that the dimension n is even. Let , Ω^0 be as in (1). Let ε_0 be as in (2). Let $\varepsilon_1 \in [0, \varepsilon_0]$. let $\{\mathcal{U}_\varepsilon\}_{\varepsilon\in[0,\varepsilon 1]}$ be a family of functions which satisfies the condition in (a0) and such that

(i) there exists a real analytic operator B^0 from $[-\varepsilon_1, \varepsilon_1]$ to $C^{1,\alpha}(\partial\Omega^0)$ such that

$$u_\varepsilon (x) = B^0 [\varepsilon] (x) \text{ for all } x \in \partial\Omega^0 \text{ and all } \varepsilon \in [0, \varepsilon_1],$$

(ii) there exists a real analytic operator B^i from $[-\varepsilon_1, \varepsilon_1]$ to $C^{1,\alpha}(\partial\Omega^i)$ such that

$$u_\varepsilon (\varepsilon x) = B^i [\varepsilon] (x) \text{ for all } x \in \partial\Omega^i \text{ and all } \varepsilon \in [0, \varepsilon_1].$$

Then there exists a family of functions $\{v_\varepsilon\}_{\varepsilon\in[-\varepsilon_1, \varepsilon_1]}$ which satisfies the conditions in (b_0) - (b_2) and such that $u_\varepsilon = v_\varepsilon$ for all $\varepsilon \in [0, \varepsilon_1]$.

Proof: Let $M \equiv (M^i, M^0)$ be the map in Proposition (2.1.6) we set

$$v_\varepsilon^i(x) \equiv \varepsilon^{n-2}\int_{\partial\Omega^i} S_n(x-\varepsilon y)\, M^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)\,d\sigma_y \qquad \forall x \in (\varepsilon)$$

for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]\backslash\{0\}$, and $v_\varepsilon^i(x) \equiv 0$ for all $x \in cI\Omega^0$. Then we set

$$v_{\varepsilon}^{0}(x) \equiv \varepsilon^{n-2}\int_{\partial\Omega^{0}} S_n(x-y)\, M^{0}[\varepsilon, B^{i}[\varepsilon], B^{0}[\varepsilon]](y)\,d\sigma_y,$$

$$v_{\varepsilon}(x) \equiv v^{i}_{\varepsilon}(x) + v^{0}(x), \qquad \forall x \in cI\Omega(\varepsilon)\,,$$

for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$.By classical Potential Theory, we deduce that $\{v_\varepsilon\}_{\varepsilon\in[-\varepsilon1,\varepsilon1]}$ satisfies the condition in (b_0). Now let Ω_M, ε_M be as in (b_1). Let $V_M \equiv v_{\varepsilon\backslash cI\Omega M}$ for all $\varepsilon \in [-\varepsilon_M, \varepsilon_M]$. We show that V_M is real analytic and hence $\{\mathcal{V}_\varepsilon\}_{\in[-\varepsilon1,\, \varepsilon1]}$ satisfies the condition in $(b1)$. To do so we prove that V_M is real analytic in a neighborhood of a fixed point ε^* of $[-\varepsilon_M, \varepsilon_M]$ We note that $cI\Omega_M \cap \varepsilon^* cI\Omega^i = \emptyset$.Then, by a standard argument based on the existence of smooth partitions of unity and on Sard's Theorem we can show that there exists an open bounded set $\tilde{\Omega}$of class $C^{1,\alpha}$ such that $\Omega_M \subseteq \tilde{\Omega} \subseteq \Omega^0$ and $cI\tilde{\Omega} \cap \varepsilon^* I\Omega^i = \emptyset$. Then, by the continuity of the real function which takes ε to dist $(\varepsilon cI, cI\tilde{\Omega}) \equiv \inf\{|x\text{-}y| : x \in \varepsilon cI\Omega^i, y \in cI\tilde{\Omega}\}$ we deduce that there exists $\delta > 0$ such that $\varepsilon cI\Omega^i \cap cI\tilde{\Omega} = \emptyset$ for all $\varepsilon \in [\varepsilon^*\text{-}\delta, \varepsilon^*+\delta]$. Possibly shrinking δ we can assume that $[\varepsilon^*\text{-}\delta, \varepsilon^*+\delta] \subseteq [-\varepsilon_1, \varepsilon_1]$. Then, by Lemma (2.1.4)(ii) and by the real analyticity of M , B^i , B^0 we verify that the map from $[\varepsilon^*\text{-}\delta, \varepsilon^*+\delta]$ to $C^{1,\alpha}(cI\tilde{\Omega})$ which takes ε to $v^i_{\varepsilon\backslash cI\tilde{\Omega}}$ is real analytic. Then, by the boundedness of the restriction operator from $C^{1,}(cI\tilde{\Omega})$ to $C^{1,\alpha}(cI\Omega_M)$ and by standard calculus in Banach space, the map from $[\varepsilon^*\text{-}\delta, \varepsilon^*+\delta]$ to $C^{1,\alpha}(cI\Omega_M)$ which takes ε to $v^i_{\varepsilon\backslash cI\Omega M}$ is real analytic. By Lemma (2.1.1) and by the real analyticity of M , B^i , B^0 and by the boundedness of the restriction operator from $C^{1,\alpha}(cI\Omega^0)$ to $C^{1,\alpha}(cI\Omega_M)$ we deduce that the map from $[\varepsilon^*\text{-}\delta, \varepsilon^*+\delta]$ to $C^{1,\alpha}(cI\Omega_M)$ which takes ε to $v^0_{\varepsilon\backslash cI\Omega M}$ is real analytic. Then the map from $[\varepsilon^*\text{-}\delta, \varepsilon^*+\delta]$ to $C^{1,}(cI\Omega_M)$ which takes ε to $V_M\ [\varepsilon] = v^i_{\varepsilon|cI\Omega M} + v^0_{\varepsilon|cI\Omega M}$ is real analytic. Thus, $\{v_\varepsilon\}_{\varepsilon\,\in\,[-\varepsilon_1,\,\varepsilon_{1,}]}$ satisfies the conditions in $(b1)$. Now we prove that $\{v_\varepsilon\}_{\varepsilon\in\,[-\varepsilon_1,\,\varepsilon_1]}$ satisfies the conditions in $(b2)$. Let Ω_m and ε_m be as in $(b2)$. Let $[\varepsilon]$ be defined by

$$V_m[\varepsilon](x) \equiv \int_{\partial\Omega^{i}} S_n(x-y)\, M^{i}[\varepsilon, B^{i}[\varepsilon], B^{0}[\varepsilon]](y)\,d\sigma_y$$

$$+\int_{\partial\Omega^{0}} S_n(\varepsilon x-y)\, M^{0}[\varepsilon, B^{i}[\varepsilon], B^{0}[\varepsilon]](y)\,d\sigma_y \qquad \forall x \in cI\Omega_m$$

for all $\varepsilon \in [-\varepsilon_m, \varepsilon_m]$ Clearly,

$$v_\varepsilon(\varepsilon x) = [\varepsilon](x) \quad \forall x \in cl\Omega_m, \ \varepsilon \in [-\varepsilon_m, \varepsilon_m] \setminus \{0\}. \qquad (25)$$

We prove that the map from $[-\varepsilon_m, \varepsilon_m]$ to $C^{1,}(cl\Omega_m)$ which takes ε to $V_m[\varepsilon]$ is real analytic.

To do so we prove that V_m is real analytic in a neighborhood of a fixed point ε^* of $[-\varepsilon_m, \varepsilon_m]$. By a standard argument based on the existence of smooth partitions of unity, and on Sard's Theorem, and on the continuity of the distance function, we verify that there exist $\delta > 0$ and an open bounded subset $\tilde{\Omega}$ of $\mathbb{R}^n \setminus cl\Omega^i$ of class $C^{1,\alpha}$ such that $\Omega_m \subseteq \tilde{\Omega}$and $\varepsilon cl\tilde{\Omega} \subseteq \Omega^0$ for all $[\varepsilon^* - \delta, \varepsilon^* + \delta]$ possibly shrinking δ we can assume that $[\varepsilon^* - \delta, \varepsilon^* + \delta] \subseteq [-\varepsilon_m, \varepsilon_m]$.

Then we set

$$\tilde{V}_m[\varepsilon](x) \equiv \int_{\partial\Omega^i} S_n(x-y)\, M^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y$$

$$+ \int_{\partial\Omega^0} S_n(\varepsilon x - y)\, M^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y \qquad \forall x \in cl\tilde{\Omega}$$

for all $\varepsilon \in [\varepsilon^* - \delta, \varepsilon^* + \delta]$. So that $V_m[\varepsilon] = \tilde{V}_m[\varepsilon]_{|cl\Omega m}$ for all $\varepsilon \in [\varepsilon^* - \delta, \varepsilon^* + \delta]$. Then, by Lemma (2.1.1) and by Lemma (2.1.4)(ii), and by the real analyticity of M, and by standard calculus in Banach space, we deduce that V_m is real analytic from $[\varepsilon^* - \delta, \varepsilon^* + \delta]$ to $C^{1,\alpha}(cl\tilde{\Omega})$. Then, by the boundedness of the restriction operator from $C^{1,}(cl\tilde{\Omega})$ to $C^{1,\alpha}(cl\Omega_m)$, V_m is real analytic from $[\varepsilon^* - \delta, \varepsilon^* + \delta]$ to $C^{1,\alpha}(cl\Omega_m)$. Thus, the validity of (b_2) follows. Moreover, by Proposition (2.1.2) and by the uniqueness of the solution of the Dirichlet boundary value problem in (ε) we deduce that $u_\varepsilon = v_\varepsilon$ for $\varepsilon \in [0, \varepsilon_1]$. The validity of the theorem is now verified.

We now consider the case of dimension n odd and we prove Theorem (2.1.9).

Theorem (2.1.9)[2]: Assume that the dimension n is odd. Let Ω^i, Ω^0be as in (1) Let ε_0 be as in (2). Let $\varepsilon_0 \in [0, \varepsilon_0]$. Let $\{v_\varepsilon\}_{\varepsilon \in [-\varepsilon 1, \varepsilon 1]}$ be a family of functions which satisfies the condition in (b_0) and such that

(i) there exists a real analytic operator B^0 from $[-\varepsilon_1, \varepsilon_1]$ to s $C^{1,\alpha}(cl\Omega^0)$

such that

$$u_\varepsilon(x) = B^0[\varepsilon](x) \text{ for all } x \in \partial\Omega^0 \text{ and all } \varepsilon \in [-\varepsilon_1, \varepsilon_1],$$

(ii) there exists a real analytic operator B^i from $[-\varepsilon_1, \varepsilon_1]$ to $C^{1,\alpha}(cl\Omega^i)$ such that

$$u_\varepsilon(\varepsilon x) = B^i[\varepsilon](x) \text{ for all } x \in \partial\Omega^i \text{ and all } \varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}.$$

Assume that the family $\{v_\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$ satisfies at least one of the following conditions (iii) and (iv).

(iii)There exist an open non-empty subset Ω_M of $\Omega^0 \setminus \{0\}$ and a real number $\varepsilon_M \in [0, \varepsilon_1]$ such that $cl\Omega_M \cap \varepsilon cl\Omega^i = \emptyset$ for all $\varepsilon \in [-\varepsilon_M, \varepsilon_M]$, and a real analytic operator V_M from $[-\varepsilon_M, \varepsilon_M]$ to $C^{1,}(cl\Omega_M)$ such that

$$v_{\varepsilon|cl\Omega M} = V_M[\varepsilon] \quad \forall \varepsilon \in [-\varepsilon_M, \varepsilon_M].$$

(iv)There exist a bounded open non-empty subset Ω_m of, $\mathbb{R}^n \setminus cl\Omega^i$, and a real number $\varepsilon_m \in [0, \varepsilon_1]$ such that $\varepsilon cl\Omega_m \subseteq \Omega^0$ for all $\varepsilon \in [-\varepsilon_m, \varepsilon_m]$ and a real analytic operator V_m from $[-\varepsilon_m, \varepsilon_m]$ to $C^{1,\alpha}(cl\Omega_m)$ such that

$$v_{\varepsilon(\varepsilon.)|cl\Omega M} = V_m[\varepsilon] \quad \forall \varepsilon \in [-\varepsilon_M, \varepsilon_M] \setminus \{0\}.$$

Then there exists a family of functions $\{\omega_\varepsilon\}_{\varepsilon \in [-\varepsilon 1, \varepsilon 1]}$ which satisfies the conditions in (c_0), (c_1) and such that $v_\varepsilon = \omega_{\varepsilon|cl\Omega M}$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$.

Proof: Let $\widetilde{M}_1 \equiv (\widetilde{M}_1^i, \widetilde{M}_1^0)$, $\tilde{\varepsilon}$, $\mathcal{U}$ be as in Propositions (2.1.5) with $\theta \equiv 1$, and $\tilde{f}^i \equiv B^i[0]$ and $\tilde{f}^0 \equiv B^0[0]$ We show that $\widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]] = 0$ for ε in an open neighborhood of 0.

To do so we first prove that both conditions (iii) and (iv) imply that there exists $\tilde{\varepsilon}_* \in [0, \tilde{\varepsilon}]$ such that

$$\int_{\partial\Omega^i} S_n(x-y)\widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y$$
$$= 0 \quad \forall x \in \partial\Omega^0, \varepsilon \in [-\tilde{\varepsilon}_*, 0]. \tag{26}$$

Assume that $\{v_\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$ satisfies the condition in (iii). We can take $\tilde{\varepsilon}_M \in [0, \inf\{\varepsilon_M, \tilde{\varepsilon}\}]$ such that $([\varepsilon], B^0[\varepsilon]) \in \mathcal{U}$ for all $\varepsilon \in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]$. Then we set

$$\tilde{v}_\varepsilon(x) \equiv \varepsilon^{n-2} \int_{\partial\Omega^i} S_n(x-\varepsilon y)\widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y \tag{27}$$

$+\int_{\partial\Omega^0} S_n(x-y)\,\widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y \qquad \forall x\in cl\Omega(\varepsilon)$,

for all $\varepsilon \in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]\setminus\{0\}$, and

$$\tilde{v}_0(x) \equiv \int_{\partial\Omega^i} S_n(x-\varepsilon y)\,\widetilde{M}_1^0[0, B^i[0], B^0[0]](y)d\sigma_y \quad \forall x\in cl\Omega^0. \qquad (28)$$

Then Propositions (2.1.2) and (2.1.3) imply that $\tilde{v}_\varepsilon = v_\varepsilon$ for all $\varepsilon\in [0, \tilde{\varepsilon}_M]$. So that $\tilde{v}_{\varepsilon|cl\Omega M} = v_{\varepsilon|cl\Omega M}$ for all $\varepsilon\in [0, \tilde{\varepsilon}_M]$. We observe that the map from $\varepsilon\in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]$ to $C^{1,}(cl\Omega_M)$ which takes ε to $\tilde{v}_{\varepsilon|cl\Omega M}$ is real analytic (see also the argument developed in the proof of Theorem (2.1.8) for V_M). Then, by the assumption in (iii) and by the Identity Principle for real analytic maps, we have $\tilde{v}_{\varepsilon|cl\Omega M} = v_{\varepsilon|cl\Omega M}$ for all $\varepsilon\in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]$. We note that $\tilde{v}_\varepsilon$ is harmonic on $\Omega(\varepsilon)$ for all $\varepsilon\in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]$ .Thus, the equality $\tilde{v}_{\varepsilon|cl\Omega M} = v_{\varepsilon|cl\Omega M}$ implies that $\tilde{v}_\varepsilon = v_\varepsilon$ on the whole of $cl\Omega_M(\varepsilon)$ for all $\varepsilon\in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]$. In particular,

$$\tilde{v}(\varepsilon x) = v_\varepsilon(\varepsilon x) \qquad \forall x\in \partial\Omega^i,\ \varepsilon \in [-\tilde{\varepsilon}_M, 0],$$

which in turn implies that

$$-\int_{\partial\Omega^i} S_n(x-y)\widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y \qquad (29)$$
$$+\int_{\partial\Omega^0} S_n(\varepsilon x-y)\widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y = B^i[\varepsilon](x)$$

for all $x \in \partial\Omega^i$, $\varepsilon \in [-\tilde{\varepsilon}_M, 0]$. By the definition of $\widetilde{M}_1$ in Proposition (2.1.5) we have $\Lambda_1[\varepsilon, B^i[\varepsilon], B^0[\varepsilon], \widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]]] = 0$ for all $\varepsilon\in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]$. In particular , for $\varepsilon\in [-\tilde{\varepsilon}_M, 0]$ we have

$$\Lambda_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon], \widetilde{M}_1[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]]](x) \qquad (30)$$
$$= \int_{\partial\Omega^i} S_n(x-y)\widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y$$
$$+\int_{\partial\Omega^0} S_n(\varepsilon x-y)\,\widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y - B^i[\varepsilon](x) = 0 \quad \forall x \in \partial\Omega^i.$$

Then, by (17) and (18) we deduce the validity of (14) in case (iii) with $\tilde{\varepsilon}_* = \tilde{\varepsilon}_M$.

We now assume that (iv) holds. Then there exists $\tilde{\varepsilon}_m \in [0, \inf\{\varepsilon_m, \tilde{\varepsilon}\}]$ such that

$$(B^i[\varepsilon], B^0[\varepsilon]) \in \mathcal{U} \text{ for all } \varepsilon\in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m].$$

We set

$\bar{v}(x) \equiv {}^{n\text{-}2}(\mathrm{sgn}\,\varepsilon) \int_{\partial\Omega^i} S_n(x-\varepsilon y)\, \widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y$

$$+\int_{\partial\Omega^0} S_n(x-y)\, \widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y \qquad \forall x \in cI\Omega(\varepsilon),$$

for all $\varepsilon \in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m] \backslash \{0\}$ and

$$\bar{v}_0(x) \equiv \int_{\partial\Omega^i} S_n(x-\varepsilon y)\, \widetilde{M}_1^0[0, B^i[0], B^0[0]](y)d\sigma_y \qquad \forall x \in cI\Omega^0.$$

By Propositions (2.1.2) and (2.1.3) we deduce that $\bar{v}_\varepsilon = v_\varepsilon$ for all $\varepsilon \in [0, \tilde{\varepsilon}m]$. So that

$$\bar{v}_\varepsilon(\varepsilon x) = (\varepsilon x) \qquad \forall x \in , \qquad \varepsilon \in [0, \tilde{\varepsilon}_m].$$

Then we set

$\bar{V}_m[\varepsilon](x) \equiv \int_{\partial\Omega^i} S_n(x-y)\, \widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y$

$$+\int_{\partial\Omega^0} S_n(\varepsilon x-y)\, \widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y \qquad \forall x \in cI\Omega_m$$

for all $\varepsilon \in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]$. We observe that $\bar{V}$ is a real analytic map from $[-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]$ to $C^{1,\alpha}(cI\Omega_m)$ and that $\tilde{v}_\varepsilon(x) = \bar{V}_m[\varepsilon](x)$ for all $x \in cI\Omega_m$ and for all $\varepsilon \in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m] \backslash \{0\}$ (see also the argument developed in the proof of Theorem (2.1.8) for then, by the assumption in (iv) and by the Identity Principle for real analytic maps we have $\bar{V}[\varepsilon] = Vm[\varepsilon]$ for all $\varepsilon \in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]$, and thus $\bar{v}_\varepsilon(\varepsilon.)_| = v_\varepsilon(\varepsilon)_{|cI\Omega m}$ for all $\varepsilon \in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m] \backslash \{0\}$. We now note that $\bar{v}_\varepsilon$ is harmonic on (ε) for all $\varepsilon \in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]$. Thus, the equality $\tilde{v}(\varepsilon.)_{|cI\Omega m} = v_\varepsilon(\varepsilon.)_{|cI\Omega m}$ implies that $\tilde{v}_\varepsilon = v_\varepsilon$ on the whole of $c(\varepsilon)$ for all $\varepsilon \in [-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m] \backslash \{0\}$. In particular,

$$\bar{v}(x) = v_\varepsilon(x) \qquad \forall x \in \partial\Omega^0 \quad, \varepsilon \in [-\tilde{\varepsilon}_m, 0],$$

which in turn implies that

$-\varepsilon^{n-2}\int_{\partial\Omega^i} S_n(x-\varepsilon y)\, \widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y$

$$+\int_{\partial\Omega^0} S_n(x-y)\, \widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y - B^0[\varepsilon](x) \qquad (31)$$

for all $x \in \partial\Omega^0$, $\varepsilon \in [-\tilde{\varepsilon}_m, 0]$. By the definition of $\widetilde{M}_1$ in Proposition (2.1.5) we have $\Lambda_1[\varepsilon, B^i[\varepsilon], B^0[\varepsilon], \widetilde{M}_1[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]]] = 0$ for all $\varepsilon \in [-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]$. In particular, for $\varepsilon \in [-\tilde{\varepsilon}_M, 0]$ we have

$$\Lambda_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon], \widetilde{M}_1[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]]](x) \qquad (32)$$

$$= \varepsilon^{n-2} \int_{\partial\Omega^i} S_n(x-\varepsilon y)\, \widetilde{M}_1^i\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right](y)d\sigma_y$$

$$+\int_{\partial\Omega^0} S_n(x-y)\,\widetilde{M}_1^0[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y - B^0[\varepsilon](x)$$

$$= 0 \quad \forall x \in \partial\Omega^0.$$

Then, by (19) and (20) we deduce that

$$\int_{\partial\Omega^i} S_n(x-\varepsilon y)\,\widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y = 0$$

$$\forall x \in \partial\Omega^0 \quad \varepsilon \in [-\tilde{\varepsilon}_M, 0]. \tag{33}$$

Now let $\varepsilon \in [-\tilde{\varepsilon}_M, 0]$. Let $v_\varepsilon^\#$ be the function from $\mathbb{R}^n \setminus \varepsilon\Omega^i$ to $\mathbb{R}$ defined by

$$v_\varepsilon^\#(x) \equiv \int_{\partial\Omega^i} S_n(x-\varepsilon y)\,\widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]](y)d\sigma_y \quad \forall x \in \mathbb{R}^n \setminus \varepsilon\Omega^i.$$

Then we have $v_\varepsilon^\#(x) = 0$ for all $x \in \mathbb{R}^n \setminus cI\Omega^0$ and equality (21) implies that $v_\varepsilon^\#(x) = 0$

For all $x \in \partial\Omega^0$. Moreover, by the decay properties at infinity of S_n we have $\lim_{|x|\to\infty} v_\varepsilon^\#(x) = 0$.Thus $v_\varepsilon^\#|_{\mathbb{R}_n \setminus cI\Omega 0}$ coincides with the unique solution of the exterior homogeneous Dirichlet problem in $\mathbb{R}^n \setminus cI\Omega^0$ which vanishes at infinity. Accordingly

$$v_\varepsilon^\#(x) = 0 \text{ for all } x \in \mathbb{R}^n \setminus cI\Omega^0.$$

We now observe that $\Delta\, v_\varepsilon^\#(x) = 0$ for all $x \in \mathbb{R}^n \setminus \varepsilon cI\Omega^i$. Thus, by the Identity Principle for real analytic functions we have $v_\varepsilon^\#(x) = 0$ for all $x \in \mathbb{R}^n \setminus \varepsilon\Omega^i$. In particular, $v_\varepsilon^\#(\varepsilon x) = 0$ for all $x \in \partial\Omega^i$. Then by a straightforward calculation we deduce the validity of (14) in case

(iv) with $\tilde{\varepsilon}_* \equiv \tilde{\varepsilon}_m$.

Hence, the equality in (14) holds both in case (iii) and (iv) with $\tilde{\varepsilon}_* \in [0, \tilde{\varepsilon}]$. Then Lemma (2.1.1) implies that $\widetilde{M}_1^i\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] = 0$ for all $\varepsilon \in [-\tilde{\varepsilon}_*, 0]$. Thus, by a standard argument based on the Identity Principle for real analytic functions we deduce that

$$\widetilde{M}_1^i\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] = 0 \qquad \forall x \in [-\tilde{\varepsilon}_*, \tilde{\varepsilon}_*]. \tag{34}$$

We now observe that the equality in (22) implies that

$$\Lambda_\theta[\varepsilon, B^i[\varepsilon], B^0[\varepsilon], 0, \widetilde{M}_1^i[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]]] = (0, 0)$$

$$\forall x \in [-\tilde{\varepsilon}_*, \tilde{\varepsilon}_*], \quad \theta \in \{-1, 1\}. \tag{35}$$

Let M_+ and M_- be as in Proposition (2.1.7). Then by equality (23), and by Lemma (2.1.1), and by Propositions (2.1.2), (2.1.7), and by a standard argument based on the Identity Principle for real analytic functions we verify that $M_+^i\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] = 0$ for all $\varepsilon \in [0, \varepsilon_1]$, and that $M_-^i\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] = 0$ for all $\varepsilon \in [-\varepsilon_1, 0]$, and that $M_+^0\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] = \widetilde{M}_1^0\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right]$ for $\varepsilon \in [0, \tilde{\varepsilon}_*]$, and that $M_-^0\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] = \widetilde{M}_1^0\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right]$ for $\varepsilon \in [-\tilde{\varepsilon}_*, 0]$,

So, if we set

$$m^0[\varepsilon] \equiv \begin{cases} M_+^0\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] & if \ \varepsilon \in [\tilde{\varepsilon}_*, \varepsilon_1], \\ \widetilde{M}_1^0\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] & if \ \varepsilon \in [-\tilde{\varepsilon}_*, \tilde{\varepsilon}_*], \\ M_-^0\left[\varepsilon, B^i[\varepsilon], B^0[\varepsilon]\right] & if \ \varepsilon \in [-\varepsilon_1, -\tilde{\varepsilon}_*] \end{cases}$$

and we define

$$\omega(x) \equiv \int_{\partial\Omega^0} S_n(x-y) m^0[\varepsilon](y) d\sigma_y \quad \forall x \in cI\Omega^0, \ \varepsilon \in [-\varepsilon_1, \varepsilon_1],$$

then $\{\omega_\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$ satisfies the conditions in (c_0), (c_1) and $v_\varepsilon = \omega_{\varepsilon|cI\Omega(\varepsilon)}$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$ (see also Propositions (2.1.2) and (2.1.3)) The validity of the theorem is now verified. We now show that in Theorem (2.1.8) it is necessary to require the validity of condition (iii) or of condition (iv). To do so, we construct for n odd a family of functions $\{v_\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$ which satisfies the conditions in (b_0) , (i) , (ii) but not the conditions in (iii) and (iv) (see Example (2.1.10) here below) In particular, for such a family it is not possible to find $\{\omega\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$ which satisfies the conditions in (c_0), (c_1) and such that $v_\varepsilon = \omega_{\varepsilon|cI\Omega(\varepsilon)}$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$.

Example (2.1.10)[2]: Assume that the dimension n is odd. Assume that Ω^0 and Ω^i coincide with the set $\{x \in \mathbb{R}^n : |x| < 1\}$, Let $\Omega(\varepsilon) \equiv \{x \in \mathbb{R}^n : |\varepsilon| < |x| < 1\}$ for all $\varepsilon \in [-1, 1]$. Let v_ε be the function from (ε) to $\mathbb{R}$ defined by

$$(x) \equiv \frac{\varepsilon|\varepsilon|^{n-2}}{1-|\varepsilon|^{n-2}}(|x|^{2-n} - 1) \qquad \forall x \in cI\Omega(\varepsilon)$$

for all $\varepsilon \in [-1, 1]\setminus\{0\}$. Let $v_0(x) \equiv 0$ for all $x \in cI\Omega^0$. Then $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$ satisfies the condition in (b_0) and the conditions in (i), (ii) of Theorem (2.1.9) but not the conditions in (iii) and (iv).

Proof: Let $\varepsilon \in [-1, 1]\setminus\{0\}$. Then $v_\varepsilon \in C^{1,}(cI\Omega(\varepsilon))$ and we have $\Delta v_\varepsilon = 0$ in $\Omega(\varepsilon)$. Further $(x) = 0$ if $|x| = 1$, and $v_\varepsilon(x) = \varepsilon$ if $|x| = |\varepsilon|$ for all $\varepsilon \in [-1, 1]$.Thus $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$ satisfies the condition in (b_0) and the conditions in (i), (ii) of Theorem(2.1.9). Now let x_0 be a point of $\mathbb{R}^n$ with $0 < |x| < 1$. We show that the map which takes ε to (x_0) is not real analytic in a neighborhood of $\varepsilon = 0$. In particular $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$ does not satisfy the condition in (iii).

To do so, we prove that the map which takes ε to $\varepsilon|\varepsilon|^{n-2} / (1 - |\varepsilon|^{n-2})$ is not in C^{n-1} for ε in a neighborhood of 0. We note that

$$\frac{\varepsilon|\varepsilon|^{n-2}}{1-|\varepsilon|^{n-2}} = \varepsilon|\varepsilon|^{n-2}\psi_1(\varepsilon)+\psi_2(\varepsilon) \qquad \forall \varepsilon \in [-1, 1]$$

with $\psi_1(\varepsilon) \equiv (1-\varepsilon^{2(n-2)})^{-1}$ and $\psi_2(\varepsilon) \equiv \varepsilon^{2(n-2)+1}(1-\varepsilon^{2(n-2)})^{-1}$ .The maps ψ_1 and ψ_2 are real analytic from [-1, 1] to $\mathbb{R}$ and we have $\psi_1(0) = 1$. We observe that $\left(\frac{d}{d\varepsilon}\right)^{(n-1)}(\varepsilon|\varepsilon|^{n-2}) = (n-1)!\ \operatorname{sgn} \varepsilon$. Then we deduce that

$$\left(\frac{d}{d\varepsilon}\right)^{(n-1)}\left(\frac{\varepsilon|\varepsilon|^{n-2}}{1-|\varepsilon|^{n-2}}\right) = (n-1)!\,(\operatorname{sgn} \varepsilon)\,\psi_1(\varepsilon)+\psi_3(\varepsilon) \quad \forall \varepsilon \in [-1, 1]\setminus\{0\} \qquad (36)$$

where ψ_3 is a continuous map from [-1, 1] to $\mathbb{R}$. The function on the right hand side of (24) has no continuous extension on [-1, 1] and our proof is complete. The proof that $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$ does not satisfy (iv) is similar and is accordingly omitted.

We show in the following Example (2.1.11) that analogues of Theorem (2.1.8) and statement (j) do not hold if we replace the assumption that $\mathcal{U}_\varepsilon$ is harmonic on $\Omega(\varepsilon)$ for $\varepsilon \in [0, \varepsilon_1]$ with the weaker assumption that u_ε is real analytic on $\Omega(\varepsilon)$.Similarly, we show in Example (2.1.12) that analogs of Theorem (2.1.9) and statement (jj) are not true if we replace the assumption that and ω_ε are harmonic on $\Omega(\varepsilon)$ and Ω^0, respectively, with the weaker assumption that v_ε and ω_ε are real analytic on (ε) and Ω^0 respectively.

Example (2.1.11)[2]: Let Ω^0 and Ω^i be equal to $\{x \in \mathbb{R}^n : |x| < 1\}$. Let $(\varepsilon) \equiv \{ x \in \mathbb{R}^n : |\varepsilon| < |x| < 1\}$ for all $\varepsilon \in [-1, 1]$.

Let u_ε be the function of $C^{1,\alpha}((\varepsilon))$ defined by

$$u_\varepsilon(x) \equiv |x| \qquad \forall x \in cI\Omega(\varepsilon), \quad \varepsilon \in [0, 1].$$

Then U_ε is real analytic on (ε) for all $\varepsilon \in [0, 1]$ and the family $\{u_\varepsilon\}_{\varepsilon \in [0, 1]}$ satisfies the conditions in (a_1), (a_2), but there exists no family of functions $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$ on $\Omega(\varepsilon)$ which satisfies the conditions in (b_1), (b_2) and such that $u_\varepsilon = v_\varepsilon$ for all $\varepsilon \in [0, 1]$.

Proof: Clearly u_ε belongs to $C^{1,\alpha}(cI\Omega(\varepsilon))$ and is real analytic on $\Omega(\varepsilon)$ for all $\varepsilon \in [0, 1]$.

Moreover, a straightforward calculation shows that $\{u_\varepsilon\}_{\varepsilon \in [0, 1]}$ satisfies the conditions in (a_1), (a_2), Assume by contradiction that there exists a family $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$ of functions on (ε) which satisfies the conditions in (b_1), (b_2) and such that $u_\varepsilon = v_\varepsilon$ for all $\varepsilon \in [0, 1]$. Then condition (b_1) and the Identity Principle for real analytic maps imply that we have $v_\varepsilon(x) = |x|$ for all $x \in \mathbb{R}$ with $1/2 \leq |x| \leq 1$ and for all $\varepsilon \in [-1/2, 1/2]$. Condition ($b_2$) and the Identity Principle for real analytic maps imply that we have $v_\varepsilon(\varepsilon x) = \varepsilon|x|$ for all $x \in \mathbb{R}^n$ with $1 \leq |x| \leq 2$ and for all $\varepsilon \in [-1/2, 1/2]\backslash\{0\}$. Let $^* \in [-1/2, -1/4]$. Let $x^* \in \mathbb{R}$ with $1/2 < |x^*| < 2|\varepsilon^*|$. So that $1 < |x^*/\varepsilon^*| < 2$. Then $v_{\varepsilon^*}(x^*) = |x^*|$ and $v_{\varepsilon^*}(x^*) = v_{\varepsilon^*}(\varepsilon^*(x^*/\varepsilon^*)) = \varepsilon^*|x^*/\varepsilon^*| = -|x^*|$. Acontradiction.

Example (2.1.12)[2]: Let Ω^0 and Ω^i be equal to $\{x \in \mathbb{R}^n : |x| < 1\}$. Let $(\varepsilon) \equiv \{ x \in \mathbb{R}^n : |\varepsilon| < |x| < 1\}$ for all $\varepsilon \in [-1, 1]$. Let v_ε be the function of $C^{1,}(cI\Omega(\varepsilon))$ defined by

$$(x) \equiv \varepsilon^2/|x|^2 \qquad \forall x \in (\varepsilon), \quad \varepsilon \in [-1, 1]\backslash\{0\}.$$

Let $v_0(x) = 0$ for all $x \in cI\Omega^0$. Then v_0 is real analytic on Ω^0, and v_ε is real analytic on $\Omega(\varepsilon)$ for all $\varepsilon \in [-1, 1]\backslash\{0\}$, and the family $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$

satisfies the conditions in (b_1), (b_2), but for any fixed $\varepsilon^* \in$ [-1, 1]\{0} there exists no function ω_{ε^*} real analytic on Ω^0 which satisfies the equality $v_{\varepsilon^*} = \omega_{\varepsilon^*|cI\Omega(\varepsilon^*)}$.

Proof: Clearly v_ε belongs to $C^{1,}(cI\Omega(\varepsilon))$ and is real analytic on $\Omega(\varepsilon)$ for all $\varepsilon \in$ [-1, 1]\{0}.

Moreover, a straightforward calculation that $\{v_\varepsilon\}_{\varepsilon \in [-1, 1]}$ satisfies the conditions in (b_1), (b_2). Now let $\varepsilon^* \in$ [-1, 1]\{0}. Let $\tilde{\omega}_{\varepsilon^*}$ be a real analytic map on Ω^0 \ {0} such that $v\varepsilon^* = \tilde{\omega}_{\varepsilon^*|(\varepsilon^*)}$. By the Identity Principle for real analytic maps we deduce that $\tilde{\omega}_{\varepsilon^*}(x) = (\varepsilon^*)^2/|x|^2$ for all $x \in \Omega^0$\{0}. Thus $\tilde{\omega}_{\varepsilon^*}$ has no continuous extension on Ω^0 and the validity of the statement follows.

Section (2.2): Some Particular Cases:

We consider some particular cases and we show some consequences of Theorems (2.1.8) and (2.1.9). In the following Proposition (2.2.1) we show that the family $\{u\}_{\varepsilon \in [0, \varepsilon 0]}$ of the solutions of the boundary value problem in (3) satisfies the conditions in (a_1) and (a_2) for some $\varepsilon_1 \in [0, \varepsilon_0]$.

Proposition (2.2.1)[2]: Let Ω^i, Ω^0 be as in (1). Let ε_0 be as in (2). Let (f^i, f^0) $\in C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^0)$. Let u_ε denote the unique solution in $C^{1,\alpha}(cI\Omega(\varepsilon))$ of the boundary value problem in (3) for all $\varepsilon \in [0, \varepsilon_0]$. Then there exists $\varepsilon_1 \in [0, \varepsilon_0]$. such that the family $\{ u_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$ satisfies the conditions in (a_1) and (a_2) If the dimension n is even, then we can take $\varepsilon_1 = \varepsilon_0$.

Proof: If the dimension n is even, then the validity of the Proposition follows by Theorem (2.1.8) with $\varepsilon_1 \equiv \varepsilon_0$ and $[\varepsilon] \equiv f^i$, $B^0[\varepsilon] \equiv f^0$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. So let n be odd. Let $\widetilde{M}_1 \equiv (\widetilde{M}_1^i ,\widetilde{M}_1^0)$, $\tilde{\varepsilon}$, be as in Propositions (2.1.5) with $\theta \equiv 1$, $\tilde{f}^i \equiv f^i$ and $\tilde{f}^0 \equiv f^0$. We set $\varepsilon_1 \equiv \tilde{\varepsilon}$. Let Ω_M and ε_M be as in (a1) . Let $\tilde{v}_\varepsilon$ be defined as in (15), (16) with $B^i[\varepsilon] \equiv f^i$ and $B^0[\varepsilon] \equiv f^0$ for all $\varepsilon \in [-\tilde{\varepsilon}, \tilde{\varepsilon}]$,then we set $U_M[\varepsilon] \equiv \tilde{v}_{\varepsilon|cI\Omega M}$ for all $\varepsilon \in [-\varepsilon_M, \varepsilon_M]$.Then we show that U_M is real analytic from [-ε_M, ε_M] to $C^{1,\alpha}(cl\Omega_M)$ (see also the argument exploited in the proof of Theorem (2.1.8) for V_M.) The validity of (a_1) is thus proved.

Now let and ε_m be as in (a_2). Let [ε] be defined by

$$U_m[\varepsilon](x) \equiv \int_{\partial\Omega^i} S_n(x-y)\, \widetilde{M}_1^i[\varepsilon, f^i, f^0](y)\, d\sigma_y$$

$$+ \int_{\partial\Omega^0} S_n(\varepsilon x-y)\, \widetilde{M}_1^0[\varepsilon, f^i, f^0](y)\, d\sigma_y \qquad \forall x \in cl\Omega_m$$

for all $\varepsilon \in$ [-ε_m, ε_m]. Clearly,

$$u_\varepsilon(\varepsilon x) = U_m[\varepsilon](x) \qquad \forall x \in cl\Omega_m$$

for all $\varepsilon \in$ [0, ε_m]. We verify that U_m is real analytic from [-ε_m, ε_m]. to $C^{1,\alpha}(cl\Omega_m)$ (see also the argument exploited in the proof of Theorem (2.1.8) for V_m.) Accordingly the validity of (a2) follows.

In the following Proposition (2.2.2)we assume that n is even and we consider a family $\{u_\varepsilon\}_{\varepsilon \in [0, \varepsilon 1]}$ of harmonic functions on $\Omega(\varepsilon)$ which satisfies the conditions in Theorem (2.1.8). Then we investigate the power series that describe $u_{\varepsilon|cl\Omega M}$ and $(\varepsilon.)_{cl\Omega m}$ for small and positive under suitable symmetry assumptions on B^i , B^0, Ω^i and Ω^0.

Proposition (2.2.2.)[2]: Assume that n is even. Let Ω^i, Ω^0 be as in (1). Let ε^0 be as in (2). Let $\varepsilon_1 \in$ [0, ε_0]. Let $\{u_\varepsilon\}_{\varepsilon \in [0, \varepsilon 1]}$, B^i and B^0 be as in Theorem (2.1.8). Let Ω_M, ε_M be as in (b_1). Let Ω_m, ε_m be as in (b_2). Let $\zeta \in$ { -1, 1}. Then the following statements hold.

(i) If Ω^i = -Ω^i and

$$B^i[\varepsilon](x) = \zeta B^i[-\varepsilon](-x), \qquad B^0[\varepsilon](y) = \zeta B^0[-\varepsilon](y)$$

for all $x \in \partial\Omega^i$, $y \in \partial\Omega^0$, $\varepsilon \in$ [-ε_1, ε_1], then there exist $\tilde{\varepsilon}_M \in$ [0, ε_M] and a sequence $\{u_{M,j}\}_{j\in\mathbb{N}}$ in $C^{1,\alpha}(cl\Omega_M)$ such that

$$u_{\varepsilon|cl} = \varepsilon^{(1-\zeta)/2} \sum_{j=0}^{\infty} u_{M,j}\varepsilon^{2j} \qquad \forall \varepsilon \in [0, \tilde{\varepsilon}_M],$$

where the series converges in $C^{1,\alpha}(cl\Omega_M)$.

(ii) If Ω^0 = -Ω^0 and

$$B^i[\varepsilon](x) = \zeta B^i[-\varepsilon](x), \qquad B^0[\varepsilon](y) = \zeta B^0[-\varepsilon](-y)$$

for all $x \in \partial\Omega^i$, $y \in \partial\Omega^0$, $\varepsilon \in$ [-ε_1, ε_1], then there exist $\tilde{\varepsilon}_m \in$ [0, ε_m] and a sequence $\{u_{m,j}\}_{j\in\mathbb{N}}$ in $C^{1,\alpha}(cI\Omega_m)$ such that

$$u_\varepsilon(\varepsilon.)_{|cI\Omega m} = {}^{(1-\zeta)/2} \sum_{j=0}^{\infty} u_{m,j}\varepsilon^{2j} \quad \forall\varepsilon \in [0, \tilde{\varepsilon}_m],$$

where the series converges in $C^{1,\alpha}(cI\Omega_m)$.

Proof: Let $\{v_\varepsilon\}_{\varepsilon\in[-\varepsilon,\varepsilon]}$ be as in Theorem (2.1.8). Then $\{v_\varepsilon\}_{\varepsilon\in[-\varepsilon,\varepsilon]}$ satisfies the conditions in (b_1), (b_2) and we deduce that there exist $\tilde{\varepsilon}_M \in [0, \varepsilon_M]$, $\tilde{\varepsilon}_m \in [0, \varepsilon_m]$ and sequences in $C^{1,}(cI\Omega_M)$ and $\{v_{m,j}\}_{j\in\mathbb{N}}$ in $C^{1,\alpha}(cI\Omega_m)$ such that

$$v_{\varepsilon|cI\Omega M} = \sum_{j=0}^{\infty} v_{M,j}\,\varepsilon^j \quad \forall\varepsilon\in[-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M],$$

$$v_\varepsilon(\varepsilon.)_{|cI\Omega m} = \sum_{j=0}^{\infty} v_{m,j}\,\varepsilon^j \quad \forall\varepsilon\in[-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]\backslash\{0\},$$

where the first and second series converge in $C^{1,\alpha}(cI\Omega_M)$ and $C^{1,\alpha}(cI\Omega_m)$, respectively.

Then, by the assumptions in (i) and by Proposition (2.1.2), and by the uniqueness of the solution of the Dirichlet problem in (ε) for all $\varepsilon\in[-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]\backslash\{0\}$. We deduce that $(\varepsilon) = (-\varepsilon)$ and that $v_\varepsilon = \zeta v_{-\varepsilon}$ for all $\varepsilon\in[-\tilde{\varepsilon}_M, \tilde{\varepsilon}_M]\backslash\{0\}$. Thus we have $\sum_{j=0}^{\infty} v_{M,j}\,(-\varepsilon) = \zeta\sum_{j=0}^{\infty} v_{M,j}\,\varepsilon^j$ for all $\varepsilon\in[-\tilde{\varepsilon}_M, \tilde{\varepsilon}\,]$, which implies that $v_{M,2j+(1+\zeta)/2} = 0$ for all $j\in\mathbb{N}$.

If we now set $u_{M,} \equiv v_{M,2j+(1-\zeta)/2}$ for all $j\in\mathbb{N}$, then the validity of statement (i) follows.

Similarly, by the assumptions in (ii) and by Proposition (2.1.2), and by the uniqueness of the solution of the Dirichlet problem in (ε) for $\varepsilon\in[-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]\backslash\{0\}$, we deduce that $\Omega(\varepsilon) = -\Omega(-\varepsilon)$ and that $v_\varepsilon = \zeta v_{-\varepsilon}(-x)$ for all $x \in cI\Omega_m$ and all $\varepsilon\in[-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]\backslash\{0\}$. In particular $(\varepsilon x) = \zeta v_{-\varepsilon}(-\varepsilon x)$ for all $x \in$ and all $\varepsilon\in[-\tilde{\varepsilon}_m, \tilde{\varepsilon}_m]\backslash\{0\}$. We deduce that $\sum_{j=0}^{\infty} v_{M,j}\,(-\varepsilon) = \zeta\sum_{j=0}^{\infty} v_{m,j}\,\varepsilon^j$ for all $\varepsilon\in[-\tilde{\varepsilon}_m, \tilde{\varepsilon}\,]$, which in turn implies that $v_{m,2j+(1+\zeta)/2} = 0$ for all $j\in\mathbb{N}$. If we now set $u_{m,} \equiv v_{m,2j+(1-\zeta)/2}$ for all $j\in\mathbb{N}$, then the validity of statement (ii) follows. Now let n be odd. Let $\{u_\varepsilon\}_{\varepsilon\in[0,\varepsilon 0]}$ denote the family of the solutions of (3). As an immediate consequence of the following Proposition (2.2.3) one can verify that the equalities in (4) and (5) hold for ε negative only if there exists c $\in\mathbb{R}$ such that $u_\varepsilon(x)$ = c for all $x \in cl\Omega(\varepsilon)$ and $\varepsilon \in [0, \varepsilon_0]$.

Proposition (2.2.3)[2]: Assume that n is odd. Let Ω^i, Ω^0 be as in (1). Let ε_0 be as in (2). Let $\varepsilon_1 \in [0, \varepsilon_0]$. Let $\{v_\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$, and B^0 be as in Theorem (2.1.9). Then the following statements are equivalent.

(i)There exist functions $\in C^{1,}(\partial\Omega^i)$ and $f^0 \in C^{1,\alpha}(\partial\Omega^0)$ such that $B^i[\varepsilon] = f^i$ and $B^0[\varepsilon] = f^0$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$.

(ii) There exists a constant c ∈ ℝ such that (x) = c for all $x \in (\varepsilon)$ and all $\varepsilon \in [-\varepsilon 1, \varepsilon 1]$.

Proof: Clearly statement (ii) implies (i). So we have to show that (i) implies (ii) By Theorem (2.1.9) there exists a family $\{\omega_\varepsilon\}_{\varepsilon \in [-\varepsilon_1, \varepsilon_1]}$ of harmonic functions on Ω^0 such that $v_\varepsilon = \omega_{\varepsilon|cl\Omega(\varepsilon)}$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. In particular we have $\omega_{\varepsilon|\partial}\Omega^0 = B^0[\varepsilon] = f^0$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$ and $(\varepsilon.)_{|\partial\Omega i} = B^i[\varepsilon] = f^i$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}$. By the uniqueness of the solution of the Dirichlet problem in Ω^0 and by Lemma (2.1.1) we deduce that $\omega_\varepsilon = \omega_0$ for all $\in [-\varepsilon_1, \varepsilon_1]$ and that there exists $\mu^0 \in C^{0,\alpha}(\partial\Omega^0)$ such that

$$(x) = \int_{\partial\Omega^0} S_n\,(x-y)^0(y)\,d\sigma_y \quad \forall x \in cl\Omega^0,\ \varepsilon \in [-\varepsilon_1, \varepsilon_1]. \qquad (37)$$

We now prove that f^i is constant on $\partial\Omega^i$. Indeed, equality $(\varepsilon.)_| = f^i$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}$ and (25) imply that

$$(x) = \int_{\partial\Omega^0} S_n\,(\varepsilon x-y)^0(y)\,d\sigma_y \quad \forall x \in \partial\Omega^i,\ \varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}. \qquad (38)$$

Since the map from $[-\varepsilon_1, \varepsilon_1]$ to $C^{1,}(\partial\Omega^i)$ which takes ε to the function $\int_{\partial\Omega^0} S_n\,(\varepsilon x-y)^0(y)\,d\sigma_y$ of $x \in \partial\Omega^i$ is real analytic, we can take the limit as $\varepsilon \to 0$ in (26) and we obtain

$$(x) = \int_{\partial\Omega^0} S_n\,(y)^0(y)\,d\sigma_y = \omega_0(0) \quad \forall x \in \partial\Omega^i$$

Now let $\varepsilon^* \in [0, \varepsilon 1]$ be fixed. Then we have $\omega_0(x) = \omega_{\varepsilon^*}(x) = f^i(x/\varepsilon^*) = \omega_0(0)$ for all $x \in \varepsilon^*\partial\Omega^i$. Since ω_0 is harmonic in $\varepsilon^*\Omega^i$ we deduce that $\omega_0(x) = \omega_0(0)$ for all $x \in \varepsilon^* cl\Omega^i$.

Then, by the Identity Principle for real analytic functions $\omega_0(x) = \omega_0(0)$ for all $x \in cl\Omega^0$. By defining c ≡ $\omega_0(0)$ the validity of statement (ii) follows.