CHAPTER 1

Introduction

1.1 Standard Model:

Standard model (SM) is the electroweak model which unifies electromagnetic and weak interactions. The SM is proved experimentally to be successful in explaining a number of elementary particles interactions [1,2], recently the discovery of Higgs particles come as an ultimate reward confirming the SM [3].

The nature of elementary particles interactions plays a central role in modern technology; this is because it shows how different fields and energies are correlated. This helps in producing new techniques that be utilized to solve the energy problem and promoting the present techniques.

The Standard Model, which physicists have populated since the 1950s with quarks, leptons, and force-carrying particles, does not hold the answers. But major neutrino experiments in the US, Japan, and Europe are collecting data while undergoing expansion and construction, and they are gearing up to address these problems. These initiatives could not only unravel the mysteries of the ghostly particles, but the research might lead into larger questions about the nature of all things [4].

1.2 Research problem:

The present standard model (SM) mathematical model is complex [4].Moreover the standard model needs to be unified with gravity and strong interaction [5].

1.3 Literature review:

Different attempts were made to develop (modify) perturbation theory and momentum concept to solve some quantum mechanics problems [6,7,8,9,10,11] , they are mainly based on Schrödinger or Heisenberg or quantum field theory [12,13,14,15,16,17].

In the work done by Dirar [18] , the ordinary Heisenberg equation of motion describes the time evolution of the quantum system. But ignores the spatial evolution. In this paper, we carried a derivation with a new Heisenberg form, which clearly describes the spatial evolution of the quantum system. The derivation of the spatial Heisenberg representation is carried in the ordinary space and the abstract space, where the two expressions coincide. This new treatment can be utilized to deduce the x and p_x commutation relation, beside the classical relation between the force and the potential.

In the work done by Lutfi [19], a new quantum model that accounts for the medium friction is derived. The first advantage is the addition of natural oscillation of particle. The second advantage is the incorporation of frication effect in the Hamiltonian operator. This means that both Schrödinger equation and energy Eigen equation are affected by friction. The Eigen energy is not affected by friction, which is in direct conflict with experiment and common sense.

For the work done by Rasha [20], a relation of the Josephson current density equation is successfully derived; this is done through a new derivation of the equation of quantum by neglecting kinetic Newtonian term in the energy expression.

In the work done by Sawsan [21], Transverse relaxation time T_2 , plays an important role in MRI. It represents one of the important factors affecting image quality and the relaxation time. The aim of this work is to utilize the laws of quantum mechanics and electromagnetic theory to obtain new expressions for T_2.

I the work done by Dirar [22], Special relativity is isolated from the main stream of physics, because the equation of motion and energy does not stem from a single Lagrangian. More over the expression of time, length, mass and energy does not include the effect of fields. In this work a generalized version of GR which cure these defects is presented.

1.4 Aim of the work:

The aim of this work is to construct a new mathematical model based on a momentum perturbation in the generalized coordinates; this may bridge the gap between the Standard Model and Gravity beside strong interaction.

1.5 Research Methodology:

1. Constant Lagrangian of the SM having new terms to do further unifications and simplifications.

2. To find the new momentum perturbation.

3. To see how this new Lagrangian respond to the gauge invariance.

4. To see how mass can be generated.

1.6 Thesis Layout:

This thesis consists of 4 Chapters. Chapter 1 is the introduction .Chapters 2 and 3 are devoted for Quantum field theory and literature review. The contribution is done in chapter 4.

CHAPTER 2

Quantum Mechanics & Quantum Field Theory

2.1 Introduction:

Quantum field theory is the basic mathematical language that is used to describe and analyze the physics of elementary particles [23, 24, 25,].

The ordinary Lagrangian is dependent on coordinate variables; beside generalized coordinates and their first derivatives unfortunately this Lagrangian is found to be unable to describe the generalized Einstein generalized general relativity (EGGR) without adding to it a second derivative in the generalized coordinate [26,27,28,29].

This chapter is devoted to extend this notion to describe the general fields besides investigating its direct impact on Higgs field and its role in generating mass [30].

2.2 The Principle of Least Action:

It state the physical system moves in a trajectory, such that the action.

$$I = \int L(q, \dot{q}, x)dx \tag{2.2.1}$$

Is extreme (minimum or maximum)

$$\delta I = \int \delta L(q, \dot{q}, x)dx = 0 \tag{2.2.2}$$

$$\delta y = \bar{y}(x) - y(x) \tag{2.2.3}$$

$$\delta q = \bar{q}(x) - q(x) \tag{2.2.4}$$

$$\delta \dot{q} = \bar{\dot{q}}(x) - \dot{q}(x) \tag{2.2.5}$$

$$\delta \dot{q} = \frac{d\bar{q}(x)}{dx} - \frac{dq(x)}{dx} \tag{2.2.6}$$

$$\delta \dot{q} = \frac{d}{dx}[\bar{q}(x) - q(x)] = \frac{d}{dx}\delta q(x) = \frac{d}{dx}\delta q \qquad (2.2.7)$$

Using the laws of partial differentiator and since $L = L(q, \dot{q}, x)$ it follows that.

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial x}\delta x \qquad (2.2.8)$$

But using equation (2.2.3) it follows that:

$$\delta x = \bar{x}(x) - x(x) = x - x = 0 \qquad (2.2.9)$$

Also from (2.2.7) and (2.2.8) one can get:

$$\frac{\partial L}{\partial q}\delta \dot{q} = \frac{\partial L}{\partial \dot{q}}\frac{d}{dx}\delta q \qquad (2.2.10)$$

But,

$$\frac{d}{dx}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] = \frac{d}{dx}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dx}\delta q \qquad (2.2.11)$$

$$\therefore \frac{\partial L}{\partial \dot{q}}\delta \dot{q} = \frac{\partial}{\partial \dot{q}}\frac{d}{dx}\delta q = \frac{d}{dx}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] - \frac{d}{dx}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q \qquad (2.2.12)$$

Substitute (2.2.12) and (2.2.7) in (2.2.8) to get:

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{d}{dx}\left[\frac{\partial L}{d\dot{q}}\delta q\right] - \frac{d}{dx}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q \qquad (2.2.13)$$

$$\delta L = \left[\frac{\partial L}{\partial q} - \frac{d}{dx}\left(\frac{\partial L}{d\dot{q}}\right)\right]\delta q + \frac{d}{dx}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] \qquad (2.2.14)$$

Substitute (2.2.14) in (2.2.2) to obtain:

$$\delta I = \int_a^b \delta L\, dx = 0 \qquad (2.2.15)$$

$$\delta I = \int_a^b \left[\frac{\partial L}{\partial q} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q dx + \int_a^b \frac{d}{dx}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] dx = 0 \qquad (2.2.16)$$

Put $f(x) = \frac{\partial L}{\partial \dot{q}}\delta q$

Hence the second integral (2.2.16) reduces to:

$$\int_a^b \frac{df(x)}{dx}dx = \int_a^b df(x) = [f(x)]_a^b = f(b) - f(a) \qquad (2.2.17)$$

$$= \frac{\partial L}{\partial \dot{q}}\bigg|_{x=b} \delta q(b) - \frac{\partial L}{\partial q}\bigg|_{x=a} \delta q(a) = 0 \qquad (2.2.18)$$

$$\delta q(a) = \bar{q}(a) - q(a) = 0 \qquad (2.2.19)$$

$$\delta q(b) = \bar{q}(b) - q(b) = 0 \qquad (2.2.20)$$

$$\int_a^b \left[\frac{\partial L}{\partial q} - \frac{d}{dx}\frac{\partial L}{\partial \dot{q}}\right]\delta q = 0 \qquad (2.2.21)$$

$$\frac{\partial L}{\partial q} - \frac{d}{dx}\frac{\partial L}{\partial \dot{q}} = 0 \qquad (2.2.22)$$

This equation is called Euler-Lagrange equation [31].

2.3 symmetries and conservation laws:

One of the most interesting properties of the Lagrangian is the invariance of the Lagrangian which lead naturally to conservation laws. These are many types of transformations: continues or discrete, geometrical or internal, global or local. The link between continues symmetry transformations under which the Lagrangian is invariant is provided by Noether's theorem, since space and time appear on an equal forting in special relativity (SR),thus the invariance of launder space-time translation will lead to momentum-energy conservation [32,33,34,35].

Conservation of momentum results from invariance under space translation, while energy conservation results from time translation. To obtain energy momentum tensor considers the translation of the space time variables of the form; $x_\mu \rightarrow \acute{x}_\mu = x_\mu + \delta x^\mu$

2.3.1 Energy-Momentum Conservation:

To obtain the energy-momentum tensor considers translations the space-time variables of the form [36,37].

$$x_\mu \rightarrow \acute{x}_\mu = x_\mu + \delta x^\mu \qquad (2.3.1.1)$$

Where $\delta_{x\mu}$is independent of x_μ.

If the Lagrangian is invariant in form under this transformation it follows that;

$$\delta L = L(\acute{x}) - L(x) = \frac{\partial L}{\partial x^\mu}\delta x^\mu$$

$$\delta L = \partial_\mu L \delta x^\mu = (\partial_\mu L)\delta_\nu^\mu \delta x^\nu = \partial_\mu[\delta_\nu^\mu L]\delta x^\nu \qquad (2.3.1.2)$$

If L has no explicit dependence onx^μ, it follows that

$$L = L(\phi, \partial_\mu \phi) \qquad (2.3.1.3)$$

Thus

$$\delta L = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \partial_\mu}\delta(\partial_\mu \phi) \qquad (2.3.1.4)$$

Where

$$\delta\phi = \phi(\acute{x}) - \phi(x) = \phi(x_\mu + \delta x_\mu) - \phi x_\mu$$

$$\frac{\partial \phi}{\partial x^\mu}dx^\mu = (\partial_\mu \phi)\delta x^\mu = \delta x^\mu \partial_\mu \phi \qquad (2.3.1.5)$$

$$\delta(\partial_\mu \phi) = \partial_\mu \phi(x)$$

$$\frac{\partial \partial_\mu \phi}{\partial x^\nu}\delta x^\nu = \delta\mu^\nu \partial_\nu \partial_\mu \phi \qquad (2.3.1.6)$$

From Euler-Lagrange equation:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \; \frac{\partial L}{\partial \partial_\mu \phi} = 0$$

$$\frac{\partial L}{\partial \phi} = \partial_\mu \; \left[\frac{\partial L}{\partial(\partial_\mu \phi)}\right] \qquad (2.3.1.7)$$

Thus inserting (2.3.1.5, 6, 7) in (2.3.1.4) yields:

$$\delta L = \left[\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}\right] \delta x^\mu \partial_\mu \phi + \frac{\partial L}{\partial(\partial_\mu \phi)} \delta x^\nu \partial_{\nu\mu} \phi \qquad (2.3.1.8)$$

Since ν, μ are dummy indices it follows that:

$$\delta x^\mu \; \partial_\mu \phi = \; \delta x^\nu \partial_\nu \phi$$

Hence (2.3.1.8) becomes;

$$\delta L = \left[\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}\right] \delta x^\nu \partial_\nu \phi + \frac{\partial L}{\partial \partial_\mu \phi} \delta x^\nu \partial_\mu \partial_\nu \phi$$

Therefore;

$$\delta L = \left\{\partial_\mu \left[\frac{\partial L}{\partial \partial_\mu \phi}\right] \partial_\nu \phi + \frac{\partial L}{\partial \partial_\mu \phi} \partial_\mu \partial_\nu \phi \right\} \delta x^\nu$$

$$= \left[(\partial_\mu A) B + A \partial_\mu B\right] \delta x^\nu$$

$$= \; \left[(\partial_\mu A) B\right] \delta x^\nu = \left[\partial_\mu \left[\frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi\right]\right] \delta x^\nu \qquad (2.3.1.9)$$

Where;

$$A = \frac{\partial L}{\partial \partial_\mu \phi} \; , B = \; \partial_\nu \phi \qquad (2.3.1.10)$$

In view of (2.3.1.10) and (2.3.1.2);

$$\partial_\mu \left[\frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi\right] \delta x^\nu = \; \partial_\mu \left[\delta^\mu_\nu L\right] \delta x^\nu$$

Thus,

$$\partial_\mu \left[\frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta_\nu^\mu L \right] = 0 \qquad (2.3.1.11)$$

Dividing both sides by δx^ν,it follows that;

$$\partial_\mu \left[\frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta_\nu^\mu L \right] \delta x^\nu = 0 \qquad (2.3.1.12)$$

Denoting

$$T_\nu^\mu = \frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta_\nu^\mu L \qquad (2.3.1.13)$$

Thus;

$$\partial_\mu T_\nu^\mu = T_{\nu,\mu}^\mu = 0 \qquad (2.3.1.14)$$

Hence the tensor T_ν^μ is conserved and it is called the energy momentum tensor.

The Hamiltonian of the system is given by zero-zero component of the energy momentum tensor, i.e.

$$H = T_0^0 = \frac{\partial L}{\partial \partial_0 \phi} \partial_0 \phi - \delta_0^0 L \qquad (2.3.1.15)$$

Which stands for the Hamiltonian density, where $\delta_0^0 = 1$ and

$$H = \frac{\partial L}{\partial \partial_0 \phi} \partial_0 \phi - L \qquad (2.3.1.16)$$

The momentum density P_i is given by;

$$P_i = T_i^0 = \frac{\partial L}{\partial \partial_0 \phi} \partial_i \phi - \delta_i^0 L = \frac{\partial L}{\partial \partial_0 \phi} \partial_i \phi \qquad (2.3.1.17)$$

Where; $\delta_i^0 = 0 \,, \; 0 \neq i \qquad (2.3.1.18)$

2.4 Classical Lagrangian and Hamiltonian Equation of Motion:

We review briefly the structure of classical Hamiltonian theory; the equation of motion of a conservative dynamical system that has f degree of freedom may be derived from a Lagrangian function, $L(q_1, \ldots . q_f, \dot{q}_1, \ldots \dot{q}_f, t)$ of the coordinatesq_i.

The velocities $\dot{q}_i = \frac{dq_i}{dt}$ and time, by means of a variational principle [38,39]

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{2.4.1}$$

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \tag{2.4.2}$$

The resulting Lagrangian equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2.4.3}$$

$$i = 1, \ldots . f$$

If now we define a momentum canonically conjugate to q_i as $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ and a Hamiltonian function of the coordinates and momenta as:

$$H(q_1, \ldots . q_{f,} p_1, \ldots . p_f, t) = \sum_{i=1}^{f} p_i \dot{q}_i - L \tag{2.4.4}$$

Variation of H leads to the Hamiltonian equation of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}, i = 1, \ldots . f \tag{2.4.5}$$

2.5 Second Order Field Dependent Lagrangian:

The Lagrangian of (EGGR) is in the form.

$$L = L\,(x_\gamma\,,\phi\,,\partial_{\mu\phi}\,,\partial_{\mu\upsilon\phi}) \tag{2.5.1}$$

Where:

$$X_\gamma = x_o\,,x_1\,,x_2\,,x_3$$

$$X_o = t\ ,x_1 = x\,,x_2 = y\,,x_3 = z \tag{2.5.2}$$

Thus the Lagrangian variation takes the form:

$$\delta L = \frac{\partial L \partial_{x\mu}}{\partial_{x\mu}} + \frac{\partial L \delta\phi}{\partial\phi} + \frac{\partial L \delta\partial_{\mu\phi}}{\partial\partial_\mu\phi} + \frac{\partial L \delta\partial_{\mu\nu}\phi}{\partial\partial_{\mu\nu}\phi} \tag{2.5.3}$$

Where:

$$\delta x\mu = 0 \quad \delta\partial_\mu\phi = \partial_\mu\phi(x) - \partial_\mu\phi(x)$$

$$= \partial_\mu[\phi(x) - \phi(x)]$$

$$= \partial_\mu\partial\phi$$

$$\delta\partial_{\mu\upsilon}\phi = \partial_{\mu\upsilon}\phi(x1) - \partial_{\mu\upsilon}\phi(x) = \partial_{\mu\upsilon}[\phi(x) - \phi(x)]$$

$$= \partial_{\mu\upsilon}\delta\phi$$

Thus:

$$\delta L\ \frac{\delta\partial_\mu\phi}{\partial\partial_\mu\phi} = \ \partial L \frac{\partial_{x\mu}\partial_\mu\delta\phi}{\partial\partial_\mu\phi} = \partial_\mu\,\frac{[\partial L\delta\phi]}{\partial\partial_\mu\phi} - \partial_\mu\,\frac{[\partial L]\delta\phi}{\partial\partial_\mu\phi}$$

Similarly:

$$\delta L\ \frac{\delta\partial_{\mu\nu}\phi}{\partial\partial_{\mu\nu}\phi} = \ \partial L \frac{\partial_{\mu\nu}\delta\phi}{\partial\partial_{\mu\nu}\phi} = \frac{\partial L\delta\mu - \partial_\mu(\partial_\nu\delta\phi)}{\partial\partial_{\mu\nu}\phi}$$

$$\partial_\mu\,[\partial L\partial_\nu\partial\phi] - \frac{\partial_\mu[\partial L]\delta\phi}{\partial\partial_{\mu\nu}\phi} - \frac{\partial_\mu[\partial L]\partial_\nu\partial_\nu\delta\phi}{\partial\partial_{\mu\nu}\phi} \qquad (2.5.4)$$

Thus The Lagrangian of the electroweak field takes the form:

$$L = i\gamma\Psi\partial_\mu\Psi - m\Psi\Psi - j\mu A\mu - \frac{1}{4}F\mu\upsilon F \qquad (2.5.5)$$

2.5.1 Disappearance of Mass Term in the Lagrangian:

The second term in the Lagrangian is given by:

$$m_{\Psi\Psi} = \rho \qquad (2.5.1.1)$$

According to poison equation:

$$\Phi = \partial_{\mu\upsilon} = -c_1\rho \qquad (2.5.1.2)$$

Thus the mass term in L can be replaced by (2.5.5) to get:

$$L = i\gamma\mu\Psi\partial_\mu\Psi + C_o\partial_{\mu\upsilon}\phi - j_\mu A_\mu - \frac{1}{4}F_{\mu\upsilon}F$$

$$C_o = \frac{1}{C_1} \qquad (2.5.1.3)$$

It is clear that the mass term which prevents invariance disappear. According to equation (2.5.1.3) the mass term appears to be.

$$\delta L = i\gamma\mu\Psi\partial_\mu\Psi + C_o\partial_{\mu\upsilon}\phi \qquad (2.5.1.4)$$

Thus the need to Higgs fields variables to generate mass need to be revised [40].

2.6 Schrödinger Equation for free particle:

For Schrödinger ordinary equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V_0\psi = E\psi \tag{2.6.1}$$

For free particle the potential (V) = zero so that the equation become,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V_0 A\, e^{ikx} = E\psi \tag{2.6.2}$$

Where for solution of this equation let:

$$\psi = A\, e^{ikx} \tag{2.6.3}$$

Insert equation (2.6.3) in (2.6.2) get:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}A\, e^{ikx}+V_0 A\, e^{ikx} = E.A\, e^{ikx} \tag{2.6.4}$$

$$-\frac{\hbar^2}{2m}Ak^2\, e^{ikx}+V_0 A\, e^{ikx} = E.A\, e^{ikx}$$

$$-\frac{\hbar^2}{2m}Ak^2\, e^{ikx} = (E-V_0)A\, e^{ikx} \tag{2.6.5}$$

Where V = 0,

$$\frac{2mE}{\hbar^2} = k^2$$

$$\therefore k = \sqrt{\frac{2m\,E}{\hbar^2}} \tag{2.6.6}$$

2.6.1 Schrödinger Equation from Lagrangian:

Schrodinger equation can be obtained from the Lagrangian form [41, 42]

$$L = i\hbar\dot{\psi}^*\psi + \frac{\hbar^2}{2m}\,\nabla\psi^*\nabla\psi + V(r,t)\psi^*\psi \qquad (2.6.1.1)$$

$$L = i\hbar\frac{d\psi^*}{dt}\psi + \frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x} + V\psi^*\psi \qquad (2.6.1.2)$$

$$\nabla = \frac{\partial}{\partial x}\ ,\psi = \frac{\partial\psi^*}{\partial t}\,,\quad \cdot = \frac{\partial}{\partial t}$$

Where the generalized Euler-Lagrangian equation takes the form:

$$\frac{\partial L}{\partial q} - \sum_{i=0}^{3}\frac{\partial}{\partial x_i}\left[\frac{\partial L}{\partial\partial_i q}\right] = 0 \qquad (2.6.1.3)$$

With $x_0 = ict\ ,\ \ x_1 = x\ ,\ \ \ x_2 = y\ , x_3 = z\ and$

$$\frac{\partial}{\partial x_0} = \frac{\partial}{ic\partial t}\quad ,\partial_0 q = \frac{\partial}{\partial x_0}q = \frac{\partial}{ic\partial t}q\quad ,\frac{\partial}{\partial x_0} = \frac{1}{ic\partial t}$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}\quad ,\qquad \partial_1 q = \frac{\partial q}{\partial x_1} = \frac{\partial q}{\partial x}$$

$$\therefore\ \frac{\partial}{\partial x_0}\left[\frac{\partial L}{\partial\partial_0 q}\right] = \frac{\partial}{ic\partial t}\left[\frac{\partial L}{\partial\partial q/ic\partial t}\right] = \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial\partial q/\partial t}\right] \qquad (2.6.1.4)$$

$$\frac{\partial}{\partial x_1}\left[\frac{\partial L}{\partial\partial_1 q}\right] = \frac{\partial}{\partial x_1}\left[\frac{\partial L}{\partial\partial q/\partial x_1}\right] = \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial\partial q/\partial x}\right]$$

If we consider one dimension only equation (2.6.1.3) reduced to ($let\ q\ = \psi^*$) for ($i\ =\ 0{,}1$)

$$\frac{\partial L}{\partial\psi^*} - \frac{\partial}{\partial x_0}\left[\frac{\partial L}{\partial\partial_0\psi^*}\right] - \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial\partial\psi^*/\partial x}\right] = 0 \qquad (2.6.2.5)$$

Substitute equation (2.6.1.4) in (2.6.1.5) we get:

$$\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial\left(\frac{\partial \psi^*}{\partial t}\right)}\right] - \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial\left(\frac{\partial \psi^*}{\partial t}\right)}\right] = 0 \qquad (2.6.1.6)$$

In view of (2.6.1.1):

$$\frac{\partial L}{\partial \psi^*} = V\psi \qquad (2.6.1.7)$$

$$\frac{\partial L}{\partial(\partial\psi^*/\partial t)} = \frac{\partial L}{\partial \psi^*} = i\hbar\psi$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial(\partial\psi^*/\partial t)}\right] = \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \psi^*}\right] = i\hbar\frac{\partial \psi}{\partial t} \qquad (2.6.1.8)$$

$$\frac{\partial L}{\partial\left(\frac{\partial \psi^*}{\partial x}\right)} = -\frac{\hbar^2}{2m}\frac{\partial \psi}{\partial x}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial L}{\partial\left(\frac{\partial \psi^*}{\partial x}\right)}\right] = \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \qquad (2.6.1.9)$$

Substitute equations (2.6.1.7) (2.6.1.8) (2.6.1.9) yields:

$$V\psi - i\hbar\frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = 0$$

$$\therefore -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar\frac{\partial \psi}{\partial t} \qquad (2.6.1.10)$$

This is the ordinary Schrödinger equation [43].

2.7 Time independent Perturbation Theory:

Time-independent perturbation theory is an approximation scheme that applies in the following context, we know the solution to the Eigenvalue problem of the Hamiltonian H°, and we want the solution to $H° + H_1$, where H_1 is small compared to $H°$ in a sense to be made precise shortly. For instance, $H°$ can be the Coulomb Hamiltonian for an electron bound to proton and $H_1$the addition due to an external electric field that is weak compared to the proton's field at the (average) location of the electron. One refers to $H°$ as the unperturbed Hamiltonian and H_1as the perturbing Hamiltonian or perturbation [44].

$$\widehat{H}_0 u_k = E_k u_k \tag{2.7.1}$$

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_1 \tag{2.7.2}$$

$$\widehat{H}\psi = E\psi \tag{2.7.3}$$

$$\widehat{H} = \widehat{H}_o + \lambda\widehat{H} = \widehat{H}_o + \lambda\widehat{H}_1 \tag{2.7.4}$$

$$\psi = \psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \lambda^3\psi_3 + \cdots \tag{2.7.5}$$

$$E = E_0 + \lambda E_1 + \lambda^2 E_2 + \lambda^3 E_3)$$

Substitute equation (2.7.4), (2.7.5), and (2.7.6) in (2.7.3), when $\lambda \rightarrow 0$

$$\widehat{H}\psi = E\psi \tag{2.7.7}$$

$$\widehat{H}_0 = E_0\psi_0 \tag{2.7.8}$$

$$\psi_o = u_k , E_o = E_k \tag{2.7.9}$$

$$\left(\widehat{H}_0 + \lambda\widehat{H}_1 + \cdots\right)(\psi_0 + \lambda\psi_1 + \lambda^2\psi + \cdots) \tag{2.7.10}$$

$$(E_0 + \lambda E_1 + \cdots)(\psi_0 + \lambda\psi_1 + \lambda^2\psi + \cdots) \tag{2.7.11}$$

$$\widehat{H}_0\psi_0 + \lambda\left(\widehat{H}_1\psi_0 + \widehat{H}_0\psi_1\right) + \lambda^2\left(E_1\psi_1 + \widehat{H}_0\psi_2\right) =$$

$$E_0\psi_0 + \lambda(E_1\psi_0 + E_0\psi_1) + \lambda^2(E_1\psi_1 + E_0\psi_2 + E_2\psi_0 + \cdots) \quad (2.7.12)$$

$$\widehat{H}_0\psi_0 + E_0\psi_0 \quad (2.7.13)$$

$$\widehat{H}_1\psi_0 + \widehat{H}_0\psi_1 = E_1\psi_0 + E_1\psi_0 \quad (2.7.14)$$

$$\widehat{H}_1\psi_1 + \widehat{H}_0\psi_2 = E_1\psi_1 + E_0\psi + E_2\psi_0 \quad (2.7.15)$$

$$\psi \approx \psi_0 + \psi_1$$

$$E \approx E_0 + E_1$$

Where $\lambda = 1$

$$\psi_1 = \sum_N c_n u_n \quad (2.7.16)$$

Substitute (2.7.14), (2.7.15) in (2.7.16) yields:

$$\widehat{H}_1 u_k + \widehat{H}_0 \sum_n c_n u_n = E_o \sum_n c_n u_n + E_1 u_k \quad (2.7.17)$$

$$\widehat{H}_0 \sum_n c_n u_n = \sum_n c_n \widehat{H}_o u_n = \sum_n c_n E_n u_n \quad (2.7.18)$$

$$\widehat{H}_1 u_k + \sum_n c_n E_n u_n = E_1 u_k + E_k \sum_n c_n u_n \quad (2.7.19)$$

Multiply both sides by u_j and integrate over dr yields;

$$\int \bar{u}_j \widehat{H}_1 u_k dr + \sum_n E_n c_n \int \bar{u}_j u_n dr \quad (2.7.20)$$

$$E_1 \int \bar{u}_j u_k dr + E_k \sum_n c_n \int u_j u_n dr \quad (2.7.21)$$

$$(u_j, \widehat{H}_1 u_k) + \sum_n c_n E_n \delta_{jn} = E_1 \delta_{jk} + E_k \sum_n c_n \delta_{jn} \quad (2.7.22)$$

So;

$$\left(\widehat{H}_1\right)_{jk} + E_j c_j = E_1 \delta_{jk} + E_k c_j \qquad (2.7.23)$$

$$\left(\widehat{H}_1\right)_{jk} = \left(u_j, \widehat{H}_1 u_k\right) \qquad (2.7.24)$$

Where j = k

$$\left(\widehat{H}_1\right)_{kk} = E_k c_k = E_1 + E_k c_k \qquad (2.7.25)$$

$$E_1 = \left(\widehat{H}\right)_{kk} = \left(u_k, \widehat{H}_1 u_k\right) = \int \bar{u}_k \widehat{H}_1 u_k dr \qquad (2.7.26)$$

$$E_1 = \left(\widehat{H}_1\right)_{kk} = \left(u_k, \widehat{H}_1 u_k\right) \qquad (2.7.27)$$

Where $j \neq k$

$$\left(\widehat{H}_1\right)_{kk} = E_j c_j + E_k c_j \qquad (2.7.28)$$

$$c_j = \frac{\left(\widehat{H}_1\right)_{kk}}{E_k - E_j} \qquad (2.7.29)$$

$$E \approx E_0 + E_1$$

$$E = E_k + \left(\widehat{H}\right)_{kk} \qquad (2.7.30)$$

$$\psi \approx \psi_0 + \psi_1 = u_k + \sum_{n \neq k} \frac{\left(\widehat{H}\right)_{nk}}{E_k - E_n} u_n \qquad (2.7.31)$$

2.8 Time dependent Perturbation Theory:

Consider unperturbed Hamiltonian $\hat{H}_0$ satisfying Schrödinger equation,

$$\hat{H}_o \left|\psi_n^o\right\rangle = i\hbar \frac{d}{dt} |\psi_n^0\rangle \tag{2.8.1}$$

One can split $|\psi_n^0\rangle$ to time dependant and independent part in the form

$$|\psi_n^0\rangle = |p(t)\big|u(\underline{r})\rangle \tag{2.8.2}$$

Substitute (2.8.2) in (2.8.1) to get:

$$\hat{H}_o |p(t)|u(r)\rangle = i\hbar \frac{d|p(t)\rangle}{dt}\bigg| u_n(r)\rangle$$

We can split this equation into two parts to be

$$\frac{\hat{H}_o |u_n\rangle}{|u_n\rangle} = \frac{i\hbar}{|p\rangle}\frac{d|p\rangle}{dt} = E_n$$

To get

$$\hat{H}_o \big|u_n\rangle = E_n |u_n\rangle \tag{2.8.3}$$

$$\int \frac{d|p\rangle}{|p\rangle} = \frac{E_n}{i\hbar}\int dt$$

$$ln\bigg|p\rangle = \frac{\hbar w_n}{i\hbar} t = \frac{\hbar w_n t}{i\hbar} = \frac{\hbar w_n t}{i^2\hbar} = -iw_n t$$

Where;

$$\hbar w_n = E_n \tag{2.8.4}$$

Thus

$$|p\rangle = e^{-iw_n t} \tag{2.8.5}$$

Inserting (2.8.5) in (2.8.1):

$$|\psi_n^o\rangle = e^{-iw_n t}|u_n\rangle \qquad (2.8.6)$$

Assume that an unperturbed system described by (2.8.1) is subjected to a time dependant perturbation, like subjecting atoms to electromagnetic radiation such that the Hamiltonian becomes.

$$\hat{H} = \hat{H}_o + \hat{H}_1 \qquad (2.8.7)$$

Where H_1 stands for time dependant perturbation, in this case Schrödinger equation takes the form.

$$\hat{H}|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle$$

$$(\hat{H}_0 + \hat{H}_1)|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle \qquad (2.8.8)$$

Since $|\psi_n^o\rangle$ are energy Eigen states, one can expand in term of them i.e.

$$\Big|\psi\rangle = \sum_k c_k(t)|\psi_k^0\rangle = \sum_k c_k(t)e^{-iw_k t}|u_k\rangle \qquad (2.8.9)$$

Multiplying both sides by $\langle u_j|$ one gets,

$$\langle u_j|\hat{H}|\psi\rangle = i\hbar\langle u_j|\frac{d}{dt}|\psi\rangle$$

$$\langle u_j|(\hat{H}_0 + \hat{H}_1)\sum_k c_k(t)e^{-iw_k t}|u_k\rangle$$

$$= i\hbar\langle u_j|\frac{d}{dt}\sum_k c_k(t)e^{-iw_k t}|u_k\rangle \qquad (2.8.10)$$

Thus,

$$\sum_k c_k e^{-iw_k t}\langle u_j|\hat{H}_o|u_k\rangle + \langle u_j|\hat{H}_1|u_k\rangle$$

$$= i\hbar\sum_k\left[-iw_k c_k + \frac{dc_k}{dt}\right]e^{-iw_k t}\langle u_j|u_k\rangle$$

But:

$$\langle u_j|\hat{H}_o|u_k\rangle = \langle u_j|E_k|u_k\rangle = E_k\langle u_j|u_k\rangle = E_k\delta_{jk} \qquad (2.8.11)$$

$$\left(\hat{H}_1\right)_{jk} = \langle u_j|\hat{H}_1|u_k\rangle, -i^2\hbar w_k = E_k \qquad (2.8.12)$$

Hence,

$$\sum_k c_k e^{-iw_k t}\left[E_k\delta_{jk} + \left(\hat{H}_1\right)_{jk}\right] = \sum_k e^{-iw_k t}\left[c_k E_k \delta_{jk} + i\hbar\frac{dc_k}{dt}\delta_{jk}\right]$$

$$c_j e^{-iw_j t}E_j + \sum_k \left(\hat{H}_1\right)_{jk} c_k e^{-iw_k t} = c_j e^{-iw_j t}E_j + i\hbar e^{-iw_j t}\frac{dc_j}{dt}$$

Thus the time evolution of the time dependant wave function in the energy space is given by:

$$i\hbar\frac{dc_j}{dt} = \sum_k \left(\hat{H}_1\right)_{jk} e^{-iw_{jk}t}c_k \qquad (2.8.13)$$

Where,

$$w_{jk} = w_j - w_k \qquad (2.8.14)$$

2.9 Heisenberg picture:

The ordinary Schrödinger equation is given by,

$$i\hbar\frac{d|\psi\rangle_s}{dt} = \hat{H}|\psi_s\rangle \qquad (2.9.1)$$

The solution is,

$$\frac{d|\psi\rangle_s}{|\psi\rangle_s} = \frac{\hat{H}_o}{i\hbar}\int dt$$

One can define Heisenberg state vector to be,

$$|\psi_s\rangle = e^{\frac{\hat{H}_o}{i\hbar}t}\,|\psi(o)\rangle_s = e^{\frac{\hat{H}_o}{i\hbar}t}|\psi\rangle_H \qquad (2.9.2)$$

Where, Heisenberg state vector takes the form,

$$|\psi(o)\rangle_s = |\psi\rangle_H \qquad (2.9.3)$$

Expectation values are the same in all pictures; therefore

$$Q_{class} = \langle\psi_s|Q_s|\psi_s\rangle = \langle\psi_H|Q_H|\psi_H\rangle$$

Thus,

$$\langle\psi|_s = e^{\frac{\hat{H}_o}{i\hbar}t}\langle\psi|_H = \langle\psi|_H\, e^{\frac{\hat{H}_o}{i\hbar}t} \qquad (2.9.4)$$

Hence,

$$\langle\psi|_H\, e^{\frac{\hat{H}_o}{i\hbar}t} Q_s e^{\frac{\hat{H}_o}{i\hbar}t}|\psi\rangle_H = \langle\psi|_H\, Q_H|\psi\rangle_H \qquad (2.9.5)$$

Therefore Heisenberg operator is given by,

$$Q_H = e^{\frac{\hat{H}_o}{i\hbar}t} Q_s e^{\frac{\hat{H}_o}{i\hbar}t} \qquad (2.9.6)$$

$$\frac{dQ_H}{dt} = \frac{iH_o}{\hbar} Q_S e^{-i\frac{\hat{H}_o}{i\hbar}t} + e^{\frac{iH_o t}{\hbar}} Q_s e^{\frac{iH_o t}{\hbar}} + e^{\frac{\widehat{iH_o}}{i\hbar}t} + e^{\frac{iH_o t}{\hbar}} \frac{dQ_s}{dt} e^{\frac{iH_o t}{\hbar}}$$

$$= \frac{iH_o}{\hbar} Q_H \frac{i}{\hbar}\, Q_H H_o + \left(\frac{\partial Q}{\partial t}\right)_H \qquad (2.9.7)$$

$$= \frac{i}{\hbar}[H_o QH - Q_H H_o] + \left(\frac{\partial Q}{\partial t}\right)_H \qquad (2.9.8)$$

$$\frac{dQ_H}{dt} = \frac{i}{\hbar}\left[\hat{H}_o, Q_H\right] + \left(\frac{\partial Q}{\partial t}\right)_H \qquad (2.9.9)$$

Which is the Heisenberg equation.

2.10 Quantum Field theory interaction picture:

In standard quantum mechanics, we're taught to take the classical degrees of freedom and promote them to operators acting on a Hilbert space. The rules for quantizing a field are no different. Thus the basic degrees of freedom in quantum field theory are operator valued functions of space and time. This means that we are dealing with an infinite number of degrees of freedom at least one for every point in space. This infinity will come back to bite on several occasions. It will turn out that the possible interactions in quantum field theory are governed by a few basic principles: locality, symmetry and renormalization group flow (the decoupling of short distance phenomena from physics at larger scales). These ideas make QFT a very robust framework: given a set of fields there is very often an almost unique way to couple them together. The answer is: almost everything. As I have stressed above, for any relativistic system it is a necessity. But it is also a very useful tool in non-relativistic systems with many particles. Quantum field theory has had a major impact in condensed matter, high energy physics, cosmology, quantum gravity and pure mathematics. It is literally the language in which the laws of Nature are written [45].

2.11 Free Field Equations:

In quantum mechanics, canonical quantization is a recipe that takes us from the Hamiltonian formalism of classical dynamics to the quantum theory. The recipe tells us to take the generalized coordinates q_a and their conjugate momenta p^a and promote them to operators. The Poisson bracket structure of classical mechanics morphs into the structure of commutation relations between operators, so that, in units with $\hbar = 1$,

$$[q_a, q_b] = [p^a, p^b] = 0$$

$$[q_a, p^b] = i\delta_a^b$$

In field theory we do the same, now for the field $Q_a(\vec{x})$ and its momentum conjugate $\pi^b(\vec{x})$. Thus a quantum field is an operator valued function of space obeying the commutation relations;

$$[\phi_a(\vec{x}), \phi_b(\vec{y})] = [\pi^a(\vec{x}), \pi^b(\vec{y})] = 0$$

$$[\phi_a(\vec{x}), \phi_b(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})\delta_a^b$$

Note that we've lost all track of Lorentz invariance since we have separated space $\vec{x}$ and time t. We are working in the Schrodinger picture so that the operators $\phi_a(\vec{x})$ and $\pi^a(\vec{x})$ do not depend on time at all - only on space. All time dependence sits in the states $|\psi\rangle$ which evolve by the usual Schrodinger equation.

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

We aren't doing anything different from usual quantum mechanics; we're merely applying the old formalism to fields. Be warned however that the notation $|\psi\rangle$ for the state is deceptively simple; if you were to write the wave function in quantum field theory, it would be a functional, and that is a function of every possible configuration of the field ϕ.

The typical information we want to know about a quantum theory is the spectrum of the Hamiltonian H. In quantum field theories, this is usually very hard. One reason for this is that we have an infinite number of degrees of freedom at least one for every point $\vec{x}$ in space. However, for certain theories known as free theories we can find a way to write the dynamics such that each degree of freedom evolves independently [46] from all the others. Free field theories typically have Lagrangian which are quadratic in the fields, so that the equations of motion are linear. For example, the simplest relativistic free theory is the classical Klein-Gordon (KG) equation for a real scalar field $\phi(\vec{x}, t)$,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

To exhibit the coordinates in which the degrees of freedom decouple from each other, we need only take the Fourier transform,

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)} e^{i\vec{p}.\vec{x}} \phi(\vec{p},t)$$

Then $\phi(\vec{p};t)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + (p^2 + m^2)\right)\phi(\vec{p},t) = 0$$

Thus, for each value of $\vec{p}, \phi(\vec{p},t)$ solves the equation of a harmonic oscillator vibrating at frequency;

$$\omega_{\vec{p}} = +\sqrt{\vec{p}^2 + m^2}$$

We learn that the most general solution to the KG equation is a linear superposition of simple harmonic oscillators, each vibrating at a different frequency with different amplitude.

To quantize $\phi(\vec{x},t)$ we must simply quantize this infinite number of harmonic oscillators. Let's recall how to do this.

2.11.1 The Simple Harmonic Oscillator:

Consider the quantum mechanical Hamiltonian with the canonical commutation relations$[q;p] = i$. To _nd the spectrum we define the creation and annihilation operators (also known as raising/lowering operators, or sometimes ladder operators) [47]

$$a = \sqrt{\frac{\omega}{2}}q + \frac{i}{\sqrt{2\omega}}p \text{ , } \acute{a} = \sqrt{\frac{\omega}{2}}q - \frac{i}{\sqrt{2\omega}}p$$

Which can be easily inverted to give

$$q = \frac{1}{\sqrt{2\omega}}(a + \acute{a}), p = -i\frac{1}{\sqrt{2\omega}}(a - \acute{a})$$

Substituting into the above expressions we find

$$[a, \acute{a}] = 1$$

While the Hamiltonian is given by

$$H = \frac{1}{2}\omega(a\acute{a} + \acute{a}a) = \omega\left(\acute{a}a + \frac{1}{2}\right)$$

One can easily confirm that the commutators between the Hamiltonian and the creation and annihilation operators are given by

$$[H, \acute{a}] = \omega\acute{a} \; and \; [H, a] = -\omega a$$

These relations ensure that a and $\acute{a}$ take us between energy eigenstates. Let $|E\rangle$ be an eigenstate with energy E, so that$H\,|E\rangle = E\,|E\rangle$. Then we can construct more eigenstates by acting with a and $\acute{a}$,

$$H\acute{a}|E\rangle = (E+\omega)\acute{a}|E\rangle \,, Ha|E\rangle = (E-\omega)a|E\rangle$$

So we find that the system has a ladder of states with energies

$$\ldots., E-\omega, E, E+\omega, E+2\omega, \ldots.$$

If the energy is bounded below, there must be a ground state $|0\rangle$ which satisfies a$|0\rangle$ = 0

This has ground state energy (also known as zero point energy),

$$H|0\rangle = \frac{1}{2}\omega|0\rangle$$

Excited states then arise from repeated application of $\acute{a}$,

$$|n\rangle = (\acute{a})^n|0\rangle \;\; with \;\; H|n\rangle = \left(n+\frac{1}{2}\right)\omega|n\rangle$$

Where I've ignored the normalization of these states so, $\langle n|n\rangle \neq 1$.

2.12 Interaction Picture Equation:

There's a useful viewpoint in quantum mechanics to describe situations where we have small perturbations to a well-understood Hamiltonian. Let's return to the familiar ground of quantum mechanics with a finite number of degrees of freedom for a moment. In the Schrodinger picture, the states evolve as [48].

$$i\frac{d|\psi\rangle_S}{dt} = H|\psi\rangle_S$$

While the operators Q_S are independent of time.

In contrast, in the Heisenberg picture the states are fixed and the operators change in time;

$$Q_H(t) = e^{iHt}Q_S e^{-iHt}$$

$$|\psi\rangle_H = e^{iHt}|\psi\rangle_S$$

The interaction picture is a hybrid of the two. We split the Hamiltonian up as

$$H = H_0 + H_{int}$$

The time dependence of operators is governed by H_o, while the time dependence of states is governed by H_{int}. Although the split into H_o and H_{int} is arbitrary, it's useful when H_o is soluble (for example, when H_o is the Hamiltonian for a free field theory).

The states and operators in the interaction picture will be denoted by a subscript I and are given by,

$$|\psi(t)\rangle_I = e^{iH_o t}|\psi(t)\rangle_S$$

$$Q_I(t) = e^{iH_o t}Q_S e^{-iH_o t}$$

This last equation also applies to H_{int}, which is time dependent. The interaction Hamiltonian in the interaction picture is,

$$H_I \equiv (H_{int})_I = e^{iH_o t}(H_{int})_S e^{-iH_o t}$$

The Schrodinger equation for states in the interaction picture can be derived starting from the Schrodinger picture.

$$i\frac{d|\psi\rangle_S}{dt} = H_S|\psi\rangle_S \Rightarrow i\ \frac{d}{dt}\left(e^{-iH_o t}|\psi\rangle_I\right) = (H_o + H_{int})_S e^{-iH_o t}|\psi\rangle_I$$

$$\Rightarrow\ i\frac{d|\psi\rangle_I}{dt} = e^{iH_o t}(H_{int})e^{-iH_o t}|\psi\rangle_I$$

So we learn that

$$i\frac{d|\psi\rangle_I}{dt} = H_I(t)|\psi\rangle_I$$

2.12.1 Dyson's Formula:

We want to solve (3.14). Let's write the solution as

$$|\psi(t)\rangle_I = U(t, t_0)|\psi(t_0)\rangle_I$$

Where $U(t, t_0)$ is a unitary time evolution operator such that $U(t_1, t_2)\ U(t_2, t_3) = U(t_1, t_3)$ and $U(t, t) = 1$. Then the interaction picture Schrodinger equation (3.14) requires that

$$i\frac{dU}{dt} = H_I(t)U$$

If H_I were a function, then we could simply solve this by

$$U(t,t_0) = exp\left(-i\int_{t_o}^{t} H_I(\acute{t})d\acute{t}\right)$$

But there's a problem. Our Hamiltonian H_I is an operator, and we have ordering issues. Let's see why this causes trouble. The exponential of an operator is defined in terms of the expansion,

$$exp\left(-i\int_{t_o}^{t} H_I(\acute{t})d\acute{t}\right) = 1 - i\int_{t_o}^{t} H_I(\acute{t})d\acute{t} + \frac{(-i)^2}{2}\left(\int_{t_o}^{t} H_I(\acute{t})d\acute{t}\right)^2 +$$

But when we try to differentiate this with respect to t, we find that the quadratic term gives us

$$-\frac{1}{2}\left(\int_{t_o}^{t} H_I(\acute{t})d\acute{t}\right) H_I(t) - \frac{1}{2} H_I(t) \left(\int_{t_o}^{t} H_I(\acute{t})d\acute{t}\right)$$

Now the second term here looks good, since it will give part of the $H_I(t)U$ that we need on the right-hand side of (3.16). But the first term is no good since the $H_I(t)$ sits the wrong side of the integral term, and we can't commute it through because $H_I(\acute{t}) \, , H_I(t) \neq 0$ when$\acute{t} \neq t$.

CHAPTER 3

Literature Review

3.1 Introduction:

Perturbation theory is important since it simplify solving quantum equations. Different attempts were made to develop (modify) perturbation theory to solve some quantum mechanics problems in this chapter we mention some of them [49, 50, 51, 52, 53].

3.2 Ordinary Time Evolution Heisenberg Equation:

The ordinary Heisenberg picture form describes [49] the time evolution of any quantum system through the operator $\hat{Q}$. The matrix representation of the operator $\hat{Q}$ in the $\{u_i\}$ space is given by,

$$\hat{Q}_{ij} = \int \bar{u}_i \hat{Q} u_j \;\; \underline{dr} \qquad (3.2.1)$$

The differentiation of both sides with respect to time yields

$$\frac{dQ_{ij}}{dt} = \int \frac{d\bar{u}_i}{dt} \hat{Q}\, u_j \; d\underline{r} + \int u_i \frac{d\hat{Q}}{dt} u_j d\underline{r} + \int \bar{u}_i \hat{Q} \frac{du_j}{dt}\, dr$$

$$\hat{H} = i\hbar \frac{\partial}{\partial t}\, , \frac{\partial}{\partial t} = \frac{\hat{H}}{i\hbar}\, , \qquad (3.2.2)$$

$$\frac{dQ_{ij}}{dt} = \frac{-1}{i\hbar} \int \overline{Hu_i} \hat{Q}\, u_j \; \underline{dr} + \frac{1}{i\hbar} \int u_i\, \hat{Q} \hat{H} u_j \underline{dr} + \left(\frac{dQ}{dt}\right)_{ij} \qquad (3.2.3)$$

Since $\hat{H}$ is hermition, it follows that,

$$\int \overline{Hu_i} \hat{Q}\, u_j \; \underline{dr} = \int u_i \hat{H}\, \hat{Q} u_j \underline{dr} \qquad (3.2.4)$$

$$\frac{dQ_{ij}}{dt} = \frac{i}{\hbar} \int \bar{u}_i [\hat{H}\hat{Q} - \hat{Q}\hat{H}] u_j \; \underline{dr} + \left(\frac{d\hat{Q}}{dt}\right)_{ij}$$

$$\frac{dQ_{ij}}{dt} = \frac{i}{\hbar} \left[\hat{H}, \hat{Q}\right] + \left(\frac{d\hat{Q}}{dt}\right)_{ij} \qquad (3.2.5)$$

This Heisenberg equation describes non localized system as far as the integration is taken over $\underline{r}$ .The physical meaning of matrix element Q_{ij} can be know j by using the definition of the matrix element.

$$Q_{ij} = \int \bar{\psi}_i \, \hat{Q} \, \psi_j \, dr \qquad (3.2.6)$$

The Eigen functions ψ_i and ψ_j can be expressed in terms of the Eigen function of $\hat{Q}$, which satisfies:

$$\hat{Q} u_n = \; q n u_n \qquad (3.2.7)$$

This is not surprising since ordinary unit vectors in polar coordinates $\hat{\theta}$ and $\hat{r}$ can be expressed in terms of $\hat{i}$ and $\hat{j}$ in Cartesian coordinates in the form

$$\hat{r} \; \cos\theta \, \hat{\imath} + \; sin\theta \hat{\jmath} \;\; = \; \sin\theta \, \hat{\imath} \; + \cos\theta \, \hat{\jmath} \, , \qquad (3.2.8)$$

Thus one can express ψ_i 's in terms of u_n ' s , in the form,

$$\psi_i = \sum_n c_n u_n \;\; , \psi_j = \sum_m c_m u_m$$

As a result equation (3.2.6) can be written in the form

$$Q_{ij} = \sum_{n,m} \overline{c_n} c_m \int \overline{u_n} \, \hat{Q} u_m dr = \sum_{n,m} \overline{c_n} c_m q_m \delta_{nm} = \sum_n |c_n|^2 \, q_n$$

$$= \sum_n |c_n(t)|^2 q_n = \; \langle \hat{Q} \rangle \, (t) \qquad (3.2.9)$$

This is average of the physical quantity. This average can change with time according to the change of the probability

$$|c_n(t)|^2$$

Thus

$$\frac{\mathrm{dQ_{ij}}}{\mathrm{dt}} = \text{ Rate of change of the average value } \langle \hat{\mathrm{Q}} \rangle \qquad (3.2.10)$$

The change of matrix elements in the coordinate space can be found by using the definition of the wave function, it is well known that wave function ψ of a free particle is a function of space and momentum where it takes in one dimension form

$$\psi = A e^{\frac{i}{\hbar}(p_x - Et)} \qquad (3.2.11)$$

The same holds for the momentum eigen function in one dimension, where it is given by

$$u(p) = Ae^{\frac{i}{\hbar}(p_x)} \tag{3.2.12}$$

Thus one expects the average of any physical quantity Q to be spatially dependent if it is integrated over the momentum p, i.e.

$$\langle Q \rangle = \int_0^\infty \bar{\psi}\, \hat{Q}\, u(p)dp = Q(x)$$

$$\langle Q \rangle = \int_0^\infty \bar{\psi}\, \hat{Q}\, \psi\, dp = f(x,t) = Q(x,t) \tag{3.2.13}$$

These expectation values stands for the average values for systems localized at *x*.

Thus it is different from the expectation value of a system having a momentum *p*, described mathematically by

$$\langle Q \rangle = \int_0^\infty \bar{\psi}\,(p,x)\, \hat{Q}\, \psi\,(p,x)\, dx = Q(p,x) \tag{3.2.14}$$

It seems that the former expression (3.2.3) describes the particle nature where the system is spatially localized while the latter one (3.2.4) is suitable for the wave nature where the matrix element

$$Q_{ij} = \int \bar{u}(p_i,x)\, \hat{Q}\, u(p_j,x)\, dx = Q(p_i,p_j) \tag{3.2.15}$$

Describes a function of momentum, while

$$Q_{ij} = \int \bar{u}(p_i,x)\, \hat{Q}\, u(p,x)\, dx = Q(x_i,x_j) \tag{3.2.16}$$

Describes a function of x_i , x_j . The space here is discrete, which is not surprising as far as *x* for harmonic oscillator is given from

$$\langle V \rangle = \frac{1}{2}kx^2 \approx \frac{1}{2}\left(n + \frac{1}{2}\right)\hbar\omega$$

Thus:

$$x \approx \sqrt{n + \frac{1}{2}} \sqrt{\frac{\hbar\omega}{k}} \qquad (3.2.17)$$

$$\langle x \rangle = \int_0^\infty \bar{u}_n x u_m \, dx = \frac{1}{\alpha}\sqrt{\frac{n}{2}} \qquad m = n - 1 \qquad (3.2.18)$$

For hydrogen Atom the Bohr Radius is given by

$$r = a_o n^2 \qquad n = 1,2,3, \dots\dots\dots\dots \qquad (3.2.19)$$

Although x_i represents a discrete values for atomic particles or systems, but x can be treated continuous as well. This is true when x_i , s are very close to each other that one can consider them as continuous, where

$$x_i = x \qquad x_{i+1} = x_i + dx_i = x_i \qquad (3.2.20)$$

This is true as well as the dimension of the atom is in the range of

$$1\,A \approx 10^{-10} m \qquad (3.2.21)$$

Thus one expects the distance between atomic orbits to be much smaller.

In view of equations (3.2.13), the matrix element of a system in states i and j is in general a function of x if it is integrated over p , i.e.

$$Q_{ij} = \int_0^\infty \bar{u}_i(p,x)\, \hat{Q}\, u_j\,(p,x)\, dp = Q_{ij}(x) \qquad (3.2.22)$$

Thus one can describe their spatial evolution by using a derivation similar to that done by Heisenberg to describe the time evolution of the quantum systems, but before doing this derivation it is important to confirm the dependence of the wave vectors on p and x in equation (3.2.22) by utilizing the wave function

$$\psi = Ae^{\frac{i}{\hbar}(p_x - E_n t)} \qquad (3.2.23)$$

This can describe a free particle in the energy state n. The wave function of a particle which resembles a free electron in a certain crystal can be described by

$$\psi = Ae^{\frac{i}{\hbar}E_n t} \sin \alpha x \qquad (3.2.24)$$

$$\alpha = \frac{\sqrt{2mE_n}}{\hbar}$$

This again is the state of a particle in the energy state n.

For harmonic oscillator the ground state is given by

$$\psi(x) = Ae^{-\alpha^2 x^2} = Ae^{-k^2x^2} = Ae^{-\frac{p^2x^2}{\hbar^2}} \quad (3.2.25)$$

Where:

$$\alpha^2 = \frac{m\omega}{\hbar} = \frac{(mc^2)\omega}{c^2\hbar} = \frac{\hbar\omega^2}{c^2\hbar} = \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda f}\right)^2 = k^2 = \frac{p^2}{\hbar^2}$$

Thus s ψ_s again has momentum and spatial dependence Spatial changed matrix element.

According to equation (3.2.22) $Q_{ij}(x)$ is a spatially dependent matrix element. Thus one can see how it changes spatially w.r.t.x to get

$$\frac{dQ_{ij}}{dx} = \int_0^\infty \frac{du_i}{dx} \hat{Q} u_j dp + \int_0^\infty u_i \frac{d\hat{Q}}{dx} u_j dp + \int_0^\infty \bar{u}_j \hat{Q} \frac{du_j}{dx} dp \quad (3.2.26)$$

But since

$$\frac{d}{dx} = \frac{i}{\hbar}\hat{p} \quad (3.2.27)$$

It follows that:

$$\frac{dQ_{ij}}{dx} = \frac{-i}{\hbar}\int_0^\infty \overline{pu_\iota}\,\hat{Q}u_j\,dp + \frac{i}{\hbar}\int_0^\infty \overline{u_\iota}\,\hat{Q}\hat{p}u_j\,dp + \left(\frac{dQ}{dx}\right)_{ij} \quad (3.2.28)$$

But since $\hat{p}$ is a hermetical operator it follows that

$$\int \overline{pu_\iota}\,\hat{Q}u_j dp = \int \overline{u_\iota}\hat{p}\hat{Q}u_j\,dp \quad (3.2.29)$$

Inserting (3.29) in (3.28) yields

$$\frac{dQ_{ij}}{dx} = \frac{-i}{\hbar}\int \overline{u_\iota}[\hat{P}\hat{Q} - \hat{Q}\hat{P}]u_j\,dr + \left(\frac{dQ}{dx}\right)_{ij}$$

$$\frac{\hbar}{i}\frac{dQ_{ij}}{dx} = [\hat{Q},\hat{P}]_{ij} + \frac{\hbar}{i}\left(\frac{dQ}{dx}\right)_{ij}$$

Or:

$$i\hbar\frac{dQ_{ij}}{dx} = \left[\hat{P},\hat{Q}\right]_{ij} + i\hbar\left(\frac{dQ}{dx}\right)_{ij} \qquad (3.2.30)$$

This is the Heisenberg spatial equation of motion. Thus in view of equation (3.2.26), it is clear that the matrix elements of Q_{ij} describe a system of independent momentum as far as the integration is taken over p from 0 to ∞. Thus the momentum p and the wavelength λ where

$$p = \frac{h}{\lambda} \qquad (3.2.31)$$

are indefinite, According to uncertainty relation

$$\Delta x = \frac{h}{\Delta p} \qquad (3.2.32)$$

Thus equation (3.2.30) describes a highly localized system which exhibits particle nature.

Spatial changed operator in the abstract space the abstract vector space picture can be utilized to deduce the Heisenberg spatial evolution using the form of $\hat{p}$ in the coordinate space

$$\frac{\hbar}{i}\frac{d}{dx}|\psi\rangle = \hat{p}|\psi\rangle \qquad (3.2.33)$$

Where $|\psi\rangle$ stands for a certain state vector. Rearranging (3.2.33) yields

$$\frac{d|\psi\rangle}{|\psi\rangle} = \frac{i}{\hbar}\int \hat{p}dx + c \quad , ln|\psi\rangle = \frac{i\hat{p}}{\hbar}x + c$$

$$|\psi(x)\rangle = |\psi\rangle = e^{\frac{i\hat{p}_x}{\hbar}}e^{c} \qquad (3.2.34)$$

At x = 0

$$|\psi(0)\rangle = |.\psi(x=0)\rangle = e^{c} \qquad (3.2.35)$$

Thus

$$|\psi\rangle = e^{\frac{ip_x}{\hbar}}|\psi_0\rangle \quad , \quad |\psi_s\rangle = e^{\frac{ip_x}{\hbar}}|\psi_H\rangle \qquad (3.2.36)$$

The spatial dependence of ψ can be removed in Heisenberg representation ψ_H by choosing ψ_H to be ψ_0 , hence

$$|\psi\rangle = |\psi_s\rangle \quad , \quad |\psi_H\rangle = |\psi_0\rangle \qquad (3.2.37)$$

Where ψ_s represents Schrödinger spatially dependent wave vector. As far as the expectation value is independent of representation it follows that

$$\langle\hat{Q}\rangle = \langle\psi_s|\hat{Q}_s|\psi_s\rangle = \langle\psi_H|\hat{Q}_H|\psi_H\rangle \tag{3.2.38}$$

In view of equations (36) and (38) one gets

$$\langle\hat{Q}\rangle = \langle\psi_H|\, e^{\frac{-i\hat{p}_x}{\hbar}}\hat{Q}_s e^{\frac{i\hat{p}_x}{\hbar}}|\psi_H\rangle = \langle\psi_H|\hat{Q}_H|\psi_H\rangle \tag{3.2.39}$$

Thus the operator $\hat{Q}_H$ in the Heisenberg picture takes

$$\hat{Q}_H = e^{\frac{-i\hat{p}_x}{\hbar}}\hat{Q}_s e^{\frac{i\hat{p}_x}{\hbar}} \tag{3.2.40}$$

Where $\hat{Q}_s$ represents the operator in the Schrödinger picture.

Differentiating both sides with respect to, yields

$$\frac{d\hat{Q}_H}{dx} = \frac{-i}{\hbar}\,\hat{p}\hat{Q}_H + e^{-\frac{i\hat{p}_x}{\hbar}}\frac{d\hat{Q}_s}{dx}e^{\frac{i\hat{p}_x}{\hbar}} + e^{-\frac{i\hat{p}_x}{\hbar}}Q_s e^{\frac{i\hat{p}_x}{\hbar}}\left[\frac{i\hat{p}}{\hbar}\right]$$

$$= \frac{-i}{\hbar}\,\hat{p}\hat{Q}_H + \frac{-i}{\hbar}\,\hat{Q}_H\hat{p} + \left(\frac{d\hat{Q}}{dx}\right)_H$$

Hence, the spatial evolution of any operator in the Heisenberg picture can be described by the equation

$$\frac{d\hat{Q}_H}{dx} = \frac{1}{i\hbar}\left[\hat{p}, \hat{Q}_H\right] + \left(\frac{d\hat{Q}}{dx}\right)_H \tag{3.2.41}$$

This expression is typical to equation (3.2.30)

If $\hat{Q}_H = \hat{x}$ one gets

$$i\hbar\frac{d\hat{x}}{dx} = [\hat{p}, \hat{x}] = i\hbar \tag{3.2.42}$$

Which is the ordinary commutation relation for $\hat{p}$ and $\hat{x}$. Again by setting $\hat{Q}_H$ to be

$$i\hbar\frac{dV}{dx}\psi = [\hat{p}(V\psi) - V\hat{p}\psi] = V\hat{p}\psi + \psi\hat{p}V - V\hat{p}\psi = \psi\hat{p}V = i\hbar\frac{\partial V}{\partial x}\psi$$

But when V depends on x only

$$F = -\frac{\partial V}{\partial X} = -\frac{dV}{dx} = -\frac{\partial V}{\partial x} \tag{3.2.43}$$

This is typical to the classical relation between the potential and the field force.

3.2.1 Discussion:

The rate of change of the matrix element $\hat{Q}_{ij}$ is clear from equation (3.2.10) to be related to the average quantities time evolution. Equation (3.2.10) states that this rate of change is related to the rate of change of the average value of $\langle Q \rangle$ in the space $\{u_i\}$. The displacement of harmonic oscillator in equations (3.2.17, 3.2.18), beside the Bohr radius in (3.2.19) shows that the ordinary space is quantized on the micro scale, hence x_i and x_j are discrete, thus the matrix elements $\hat{Q}_{ij}$are well defined. The wave function for the free particle and the harmonic oscillator depends on both p and x . Thus the wave functions in general are dependent on p and x , which make integration over $x \; or \; p$ physically meaning full. Hence, one can obtain spatially dependent matrix elements $\hat{Q}_{ij}$ by integrating over.

The spatial change of the matrix elements is required to see how average physical quantities evolve with the derivations made in section (3.2.5) for the coordinate space shows that such change is related to the momentum operator $\hat{p}$ as shown in equation (3.2.30). A similar Heisenberg spatial equation is obtained in the abstract space as shown by equation (3.2.41).

Equations (3.2.42) and (3.2.43) confirm the validity of this spatial picture, where the ordinary commutation relation for p and x and the classical relation between F and V are obtained with the formula of the Heisenberg spatial picture. The representation of the operator matrix elements in the $\{ u_i\}$ space shows that Heisenberg form is related the time rate of change of the expectation value of any operator. The Heisenberg spatial form, in which the change of operators matrix elements in space, is derived in both ordinary and abstract space. The spatial evolution of the matrix elements is shown to be related to the momentum operator. This spatial picture is shown to be capable of obtaining the ordinary.

3.3 Quantum Equations for Frictional Medium:

For resistive medium one can write the wave function ψ similar to E.

This is justifiable as far as [51]

$E^2 \alpha$ Number of photon

$\psi^2 \alpha$ Number of particles

Thus, one can write ψ to be:

$$\psi = Ae^{\frac{i}{\hbar}\left(E - \frac{i\hbar}{\tau}\right)t} \tag{3.3.1}$$

$$\frac{\partial \psi}{\partial t} = \frac{-i}{\hbar}\left(E - \frac{i\hbar}{\tau}\right)\psi$$

$$i\hbar\frac{\partial \psi}{\partial t} + \frac{i\hbar}{\tau}\psi = E\psi$$

$$i\hbar\left[\frac{\partial}{\partial t} + \frac{1}{\tau}\right]\psi = E\psi \tag{3.3.2}$$

The energy operator takes the form:

$$\hat{H}\,\psi = E\psi \tag{3.3.3}$$

Using:

$$E = \frac{P^2}{2m} + V \tag{3.3.4}$$

$$E\psi = \frac{P^2}{2m}\psi + V\psi \tag{3.3.5}$$

$$\frac{\partial \psi}{\partial x} = \frac{i}{\hbar}P\psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{-P}{\hbar^2}\psi$$

In 3 dimensions:

$$-\hbar\nabla^2\psi = P^2\psi \tag{3.3.6}$$

$$i\hbar\left[\frac{\partial}{\partial t} + \frac{1}{\tau}\right]\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$i\hbar\frac{\partial \psi}{\partial x} = -\frac{\hbar^2}{2m}$$

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi - \frac{i\hbar}{\tau}\psi \tag{3.3.7}$$

3.3.1 String Oscillatory Solution:

To find solution for harmonic oscillator, it is important to separate variables. Thus one can write the wave function [51]

$$\psi(r,t) = f(t)u(r) \tag{3.3.8}$$

Inserting of equation (3.3.8) in (3.3.7) yields:

$$i\hbar\frac{\partial f}{f\partial t} = \frac{\hbar^2}{2mu}\nabla^2 u + V = \frac{i\hbar}{\tau} + E_0 \tag{3.3.9}$$

Therefore:

$$i\hbar\frac{\partial f}{\partial t} = E_0 f \tag{3.3.10}$$

The solution of this equation is:

$$f = A_0 e^{-i\beta_0 t} \tag{3.3.11}$$

Substituting (3.3.11) in (3.3.10) yields:

$$\hbar\beta_o = E_o \tag{3.3.12}$$

For Harmonic Oscillator the potential is given by:

$$V = \frac{1}{2}kx^2 \tag{3.3.13}$$

This equation (3.3.9) reduces to:

$$\frac{-\hbar^2}{2m}\nabla^2 u + \frac{1}{2}kx^2 u = \left(E_o + \frac{i\hbar}{\tau}\right)u = Eu \tag{3.3.14}$$

But the energy of harmonic Oscillator is given by:

$$E = E_0 + \frac{i\hbar}{\tau} = \left(n + \frac{1}{2}\right)\hbar\omega \tag{3.3.15}$$

$$n = 1,2,3,\ldots$$

The harmonic Oscillator satisfies periodicity condition,1 .e:

$$f(t+T) = f(t) \tag{3.3.16}$$

In view of equation (3.3.11) this requires:

$$e^{-i\beta_o t} = cos\beta_0 T - i\,sin\beta_0 T = 1$$

This means that:

$$cos\beta_0 T = 1 \quad , \quad sin\beta_0 T$$

Hence:

$$\beta_0 T = 2\pi s \quad ,$$

$$s = 1,2,3,\ldots$$

$$\beta_0 = \frac{2\pi}{T} s = s\omega \qquad (3.3.17)$$

Inserting (3.3.17) in (3.3.12), the energy is given by:

$$E_0 = \hbar\,\omega s \qquad (3.3.18)$$

This energy lost by friction is thus gives according to eqn (3.3.14, 3.3.15) and (3.3.18) given by:

$$E_f = E - E_0 = \hbar\omega\left(n - s + \frac{1}{2}\right) \qquad (3.3.19)$$

Maxwell equation for electric field is used to find a useful expression for absorption coefficient was found. The solution suggested includes damping term. The relation between polarization and displacement, together with the expression of conductivity for direct current, in addition to the eqn of electron motion in the presence of electric field and frictions. It is found also that α is equal to the reciprocal of relaxation time τ . Using the electron equation of motion, together with conductivity relation of alternating current, beside Maxwell equation solution for travelling an attenuated wave in section (5) a useful expression for absorption coefficient is found also. According to eqn (3.3.13) velocity relation is found and is inserted in eqn (3.3.14) to find complex conductivity in eqn (3.3.15). This eqn is substituted in Maxwell solution in eqn (3.3.16). By assuming the wave number K to be equal to that of free space a useful expression for friction energy eqn (3.3.15) is found by using Plank quantum hypothesis, for non polarized and polarized medium. It is very striking to note that this frictional energy expression is similar to that of eqn (3.3.1).Quantum Equation for frictional medium, which is reduced to ordinary Schrödinger equation, is found in section 4 eqn (3.3.14). By treating particles as vibrating strings or moving in a circular crbit, a useful expression of friction energy is found. The friction energy is shown to be quantized.

Maxwell Equations for damping or non-damping electromagnetic wave, in the presence of friction can be used to derive new Schrodinger Equation; this equation reduces to ordinary Schrodinger Equation and shows quantized friction energy.

3.4 New Quantum Equation:

Newton laws of motion are used to describe macroscopic objects. The Newtonian energy E is a sum of kinetic and potential energy V, i.e.:

$$E = \frac{1}{2}mv^2 + V = \frac{p^2}{2m} + V \tag{3.4.1}$$

Where m, v, p are the mass, velocity and momentum respectively.

According to a theorem of Bloch's [52], in such superconductors the momentum p is zero.

$$p = \frac{mv^2}{2} + \frac{qA}{c}$$

Thus (3.4.1) becomes:

$$E = V \tag{3.4.2}$$

This is related to the fact that in Josephson Effect the tunneling potential is considered to be larger than kinetic term squaring both sides' yields:

$$E^2 = V^2 \tag{3.4.3}$$

Multiplying both sides by ψ , one gets:

$$E^2\Psi = V^2\,\Psi \tag{3.4.4}$$

The wave function of a free particle is given by:

$$\psi = Ae^{\frac{i}{\hbar}(px - Et)} \tag{3.4.5}$$

Differentiating both sides with respect x and t wee

$$\frac{\partial\psi}{\partial t} = \frac{-i}{\hbar}E\psi$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{-i}{\hbar}E\frac{\partial}{\partial t}\psi = \frac{i^2}{\hbar^2}E^2\,\psi = \frac{-E^2}{\hbar^2}\psi$$

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = E^2\,\psi \tag{3.4.6}$$

Similarly:

$$\frac{\partial\psi}{\partial x} = \frac{i}{\hbar}P\psi$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} = \frac{i}{\hbar}P\frac{\partial\psi}{\partial x} = \frac{ip}{\hbar}\left(\frac{ip}{\hbar}\right)\psi = \frac{i^2p^2}{\hbar^2}$$

$$-\hbar^2\nabla^2\psi = P^2\psi \tag{3.4.7}$$

Substitute (3.4.6) in (3.4.4) to get

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = V^2\psi \tag{3.4.8}$$

3.4.1 Josephson Effect Equation:

In Josephson Effect electrons are considered as having small kinetic energy compared to the potential. Thus

Schrödinger Equation (3.4.8), in which kinetic term is neglected is suitable for describing the Josephson Effect. To derive Josephson Effect equation, consider the solution:

$$\psi = D\,sin(\alpha\,t + \phi) \tag{3.4.1.1}$$

The tunneling potential is constant inside a superconductor, thus

$$V = V_0 \tag{3.4.1.2}$$

From (3.4.1.1), one can differentiate Ψ with respect to time twice to get:

$$\frac{\partial\psi}{\partial t} = \alpha D cos(\alpha t + \phi)$$

$$\frac{\partial^2\psi}{\partial t^2} = -\alpha^2 D\,sin(\alpha\,t + \phi) = -\alpha^2\psi \tag{3.4.1.3}$$

Substitute (3.4.1.2) and (3.4.1.3) in (3.4.1.1) to obtain:

$$\hbar^2\alpha^2\psi = V_0^2\psi$$

$$\alpha^2 = \frac{V_0^2}{\hbar^2}$$

$$\alpha = \pm\frac{V_0}{\hbar} \tag{3.4.1.4}$$

By Substituting (3.4.1.4) in (3.4.1.1) and choosing a negative sign, that is in dealing with the change in potential energy one gets

$$\psi = Dsin\left(\frac{-eV_0}{\hbar}t + \phi\right) \quad (3.4.1.5)$$

But the energy density J is given by:

$$J = e\frac{\partial n}{\partial t} = e\frac{|\psi|^2}{\partial t} = 2e|\psi|\frac{d|\psi|}{dt}$$

$$J = 2\ e\ Dsin(\alpha t + \phi)\left(\frac{-e}{\hbar}V_0\right)cos(\alpha t + \phi) \quad (3.4.1.6)$$

$$J = -2\ e^2\frac{DV_0}{\hbar}sin\theta cos\theta$$

$$\theta = \phi - \frac{eV_0 t}{\hbar}$$

By using mathematical identity

$$Sin2\theta = 2sin\theta cos\theta$$

One can rewrite Equation (3.4.1.6) to be

$$J = -\frac{e^2 DV_0}{\hbar}sin\left(2\phi - \frac{2eV_0 t}{\hbar}\right) = Asin\left(2\phi - \frac{2eV_0 t}{\hbar}\right) \quad (3.4.1.7)$$

Setting:

$$2\phi = \delta(0)$$

The current density is given by:

$$J = J_0 \sin\left(\delta(0) - \frac{2eVt}{\hbar}\right) \quad (3.4.1.8)$$

Which is the Josephson Effect equation.

3.4.2 Discussion

Equation (3.4.1.2) shows a new energy equation based on Newtonian mechanics, with the neglected kinetic term. This equation is used to derive a new quantum Equation in (3.4.1.8). This new equation is based on Newtonian energy with no kinetic term beside the wave equation of a free particle. This derivation resembles simple derivations of Schrodinger equation except the fact that the kinetic term is neglected this equation is used to derive simple Josephson current density equation. This Equation (3.4.1.8) is the same as the old one, but derived using simple arguments.

Neglecting kinetic Newtonian term in the energy expression, one can easily derive a new quantum equation.

This equation is shown to be successful in deriving simple Josephson current density equation.

3.5 Transverse relaxation time from time dependent perturbations theory:

The time dependent perturbation theory provides scientists with powerful tool to find the probability that a certain energy level, for two energy levels system can be occupied at a certain time (t). This probability is obtained with the aid of the wave function, $(C_m\,(t))$, in the energy space the probability that the energy level in this occupation is given by [53]

$$|c_m(t)|^2 = \frac{4\pi^2}{\hbar^2}\mu_{mn}^2(E_{0x})^2 t$$

Where μ_{mn} stands for the matrix element of the electric dipole moment P_{mn} which is given by:

$$P_{mn} = e\mu_{mn} = e\int \mu_m x \mu_n dx \qquad (3.5.1)$$

It is straight forward to observe that at $t = 0$ the state m is completely empty, there for equation (1) reads:

$$|c_m(t=0)|^2 = 0 \qquad (3.5.2\ )$$

Where:

E = electric field intensity

$\hbar$ = Planck's Constant

If immediately after t = 0 the state m is occupied till $t = T_2$, in this case equation (1) reads:

$$|c_m(t)|^2 = \frac{4\pi^2}{\hbar^2}\mu_{mn}^2(E_{0x})^2 T_2 = 1 \qquad (3.5.3)$$

Thus the relaxation time is given by:

$$T_2 = \frac{\hbar^2}{4\pi^2}\frac{1}{E_{ox}^2\mu_{mn}} \qquad (3.5.4)$$

μ_{mn} Refers to electric polarization from state m to state n.

This expression is valid when the stimulating photons frequencies range from $0 - \infty$, if a photon of single frequency f which has energy approximate equal to the energy difference between the states m and n such that:

$$E_m - E_n \approx hf = \hbar\omega$$

$$\hbar\omega_m - \hbar\omega_n \approx \hbar\omega$$

$$\omega_m - \omega_n \approx 2\pi f$$

ω = angular frequency

$$sin\frac{1}{2}(\omega_m - \omega_n - 2\pi f) \approx (\omega_m - \omega_n - 2\pi f) \qquad (3.5.5)$$

$$\frac{1}{2}(\omega_m - \omega_n - 2\pi f) \approx a\ small\ angle.$$

Thus:

$$|c_m(t)|^2 = \frac{4\pi^2}{\hbar^2}\mu_{mn}^2 E_{ox}^2 sin^2\frac{{}^1\!/_2\,(\omega_m - \omega_n - 2\pi f)t}{(\omega_m - \omega_n - 2\pi f)^2} = \frac{4\pi_{mn}^2 E_{ox}^2(\omega_m - \omega_n - 2\pi f)^2}{4(\omega_m - \omega_n - 2\pi f)^2}$$

$$|c_m(t)|^2 = \mu_{mn}^2 E_{ox}^2 t \qquad (3.5.6)$$

Again at $t = 0$

$$|c_m(t=0)|^2 = 0 \qquad (3.7.7)$$

$i.e.$ the state m is empty. But at $t = T_2$ if it is fully occupied, i.e.

$$|c_m(t=0)|^2 = \mu_{mn}^2 E_{ox}^2 T_2 = 1 \qquad (3.5.8)$$

The relaxation time can thus be given by:

$$\tau = T_2 = \frac{1}{\mu_{mn}E_{ox}^2} \qquad (3.5.9)$$

But the square of electric field intensity is related to the photon density (n) and the photon energy $\hbar\omega$ via the relation

$$E_{ox}^2 = n\hbar\omega$$

$$T_2 = \frac{1}{\mu_{mn}n\hbar\omega} \qquad (3.5.10)$$

If the energy of the photon excites protons between two states spitted by the interval due to the effect of the internal magnetic field

B. Then:

$$\hbar\omega = g\mu_B M_s B_i \quad (3.5.11)$$

Hence:

$$T_2 \alpha \frac{1}{B_i} \quad , T_2 \alpha \frac{1}{\omega} \quad (3.5.12)$$

g = constant (g factor)

μ_B = Bohar magnetron

m_s = Spin magnetic quantum number

B_i = internal magnetic field intensity

Thus the relaxation time depends on internal field as well as free frequencies.

3.6 Transverse Relaxation Time and Conductivity in the Presence of Internal Local Field

When a photon which oscillates in electric and magnetic fields E and B are applied on a proton of mass (m), in the presence of a resistive force$F_r = \frac{mv}{\tau}$, $\tau = T_2$ and internal magnetic field B_i , the equation of motion thus becomes [3.7.6,3.7.7]:

$$m\frac{dv}{dt} = eE - \frac{mv}{T_2} + B_i ev + Bev \quad (3.6.1)$$

Where:

$E = Electric\ field\ strength$

$e = Electron\ charge$

$v = velocity$

$m = mass$

Assume the solution:

$$v = v_o e^{i\omega t} \quad , \quad E = E_o e^{i\omega t} \quad (3.6.2)$$

By putting equation (3.6.2) into equation (3.6.1) one gets:

$$im\omega v_o = eE_o - \frac{mv_o}{T_o} + B_i ev_o + B_e v_o$$

$$\left[\frac{m}{T_2} - B_i e - B_e + im\omega\right] v_o = eE_o$$

$$v = v_o e^{i\omega t} = \frac{e}{\frac{m}{T^2} - B_i e + B_e + im\omega} E$$

The conductivity is thus given by:

$$J = ev = \frac{e}{\frac{m}{T^2} - B_i e - B_e + im\omega} E = \sigma E$$

Where:

$J = Current\ density$

$\sigma = Electrical\ Conductivity$

The conductivity can be spitted in to imaginary part and real part:

$$\sigma = \sigma_1 + i\sigma_2 = \frac{e^2\left[\frac{m}{T^2} - B_i e - B_e - im\omega\right]}{\left[\frac{m}{T^2} - B_i e - B_e\right]^2 + [m\omega]^2}$$

The conductivity is thus given by the real part:

$$\sigma_1 = e^2 \frac{\left[\frac{m}{T^2} - B_i e - B_e - im\omega\right]}{\left[\frac{m}{T^2} - B_i e - B_e\right]^2 + [m\omega]^2} \qquad (3.6.3)$$

When the conductivity is very low as in the case of the human body

$$\frac{me^2}{T^2} = (B_i - B)e^3$$

Thus the relaxation time is given by:

$$T_2 = \frac{m}{(B_i + B)e} \qquad (3.6.4)$$

$$T_2 \alpha \frac{1}{B_i}$$

Reviewing equation (3.6.3) and equation (3.6.4) we found the value of the quantum transverse relaxation time, is thus depend on external and internal magnetic fields.

3.7 generalized special relativity:

The behavior of matter in the presence of the gravitational field was discussed in many standard texts [53.54.55]. In these text the equation of motion of matter in gravitational and the matter energy momentum tensor are treated separately. The equation of motion of matter is obtained either by expressing the equation of motion of straight line in curvilinear coordinate system [51] or by minimizing the proper time [52] or even by using Euler-Lagrange equations [53].The energy momentum tensor of matter was found by generalizing its special relativistic form in a curved space . This situation is not inconformity with the classical field theories, where the equation of motion and the expression for the energy momentum tensor stem from only one action and from the same Lagrangian. On the other hand the physical properties of matter; like time, length, and mass; in special relativity (SR) are incomplete for not recognizing the effect of fields on them.

Many attempts were made to modify SR to include the effect of gravity and other fields. The attempts concentrate on the notion of mass and energy without accounting the influence of both fields and motion on time and length. Using the ordinary classical Euler-Lagrange equations [3.7.1.5] a full expression for the equation of motion of matter in an arbitrary gravitational field and the energy momentum tensor are obtained from the same Lagrangian in section (3). Stemming from General Relativity (GR) the effect of gravitation and other fields on time, length and mass are obtained in section (3).

3.7.1 The Equation of motion and the Energy-Momentum Tensor for Matter:

In this section a full expression for the energy-momentum tensor of matter in the form of a perfect fluid as well as the equation of motion of matter, in particle form, in the gravitational field can be obtained from the action principle. By variation of the matter action the equation of motion and the energy-momentum tensor can be derived.

Taking the field variables to be x^k and assuming the Lagrangian to depend only on x^k and its first derivates it follows that:

$$\mathcal{L} = \mathcal{L}(x^k, U^\mu) \tag{3.7.1.1}$$

With

$$U^\mu = \frac{dx^\mu}{dt}$$

Being the four-velocity and t the proper time.

To obtain the energy-momentum tensor of a perfect fluid we choose the Lagrangian of matter to have the form:

$$\mathcal{L} = A_1 + A_2 g_{\mu\nu} U^\mu U^\nu \tag{3.7.1.2}$$

Where the parameters A_1 and A_2 are independent of the metric $g_{\mu\nu}$ and the velocity U^μ .

The energy-momentum tensor of matter is given to be

$$T_{\rho\sigma} = g_{\rho\sigma}\mathcal{L} - g_{\lambda\sigma} \frac{\partial\mathcal{L}}{\partial\partial_\lambda x^k} \partial_\rho x^k$$

$$= g_{\rho\sigma}\mathcal{L} - g_{\lambda\sigma} \frac{\partial\mathcal{L}}{\delta^\mu_\lambda \partial U^k} \delta^\mu_\rho U^k$$

$$= g_{\rho\sigma}\mathcal{L} - g_{\lambda\sigma} \frac{\partial\mathcal{L}}{\partial U^\rho} U^\lambda \tag{3.7.1.3}$$

According to formula

$$g_{\mu\nu} U^\mu U^\nu = -1$$

Then the Lagrangian becomes

$$\mathcal{L} = A_1 - A_2 \tag{3.7.1.4}$$

Using this equation and inserting (3.7.1.4) in equation (3.7.1.3) yields.

$$T_{\rho\sigma} = g_{\rho\sigma}(A_1 - A_2) - \ 2A_2 U_\rho U_\sigma \tag{3.7.1.5}$$

If we set

$$A_1 - A_2 \ \equiv p \ , A_1 + A_2 = \ -\rho \tag{3.7.1.6}$$

Then

$$T_{\rho\sigma} = g_{\rho\sigma} p + \ (\rho + p) U_\rho U_\sigma \tag{3.7.1.7}$$

Which is the expression for the energy-momentum tensor of matter in a perfect fluid form [53].

The equation of motion can be obtained by using Euler-Lagrange equation, where

$$\frac{\partial \mathcal{L}}{\partial x^k} - \ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial U^k}\right) = 0 \tag{3.7.1.8}$$

Using equations (3.7.1.6) and (3.7.1.2) the various terms in the equation of motion are given by,

$$\frac{\partial \mathcal{L}}{\partial x^k} = \ \frac{\partial A_1}{\partial x^k} + g_{\mu\nu} U^\mu U^\nu \frac{\partial A_2}{\partial x^k} + \ A_2 U^\mu U^\nu \frac{\partial g_{\mu\nu}}{\partial x^k}$$

$$= \partial \frac{(A_1 - A_2)}{\partial x^k} + \ A_2 U^\mu U^\nu \frac{\partial g_{\mu\nu}}{\partial x^k}$$

$$= \frac{\partial p}{\partial x^k} + A_2 U^\mu U^\nu \ \frac{\partial g_{\mu\nu}}{\partial x^k} \tag{3.7.1.9}$$

And

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial U^k}\right) = 2\frac{d}{dt}\left[A_2 g_{\mu k}\, U^\mu\right]$$

$$= 2U^\lambda U_k \frac{dA_2}{dx^\lambda} + \ 2A_2 \frac{\partial g_{\mu k}}{\partial x^\lambda} \ U^\mu U^\lambda + 2A_2 g_{\mu k} \frac{dU^\mu}{dt}$$

$$\lambda \rightarrow \nu, \lambda \rightarrow \mu \ and \ \mu \rightarrow \nu$$

We get

$$U^{\mu}U^{\lambda}\frac{\partial g_{k\mu}}{\partial x^{\lambda}} = \frac{1}{2}\left[U^{\mu}U^{\nu}\frac{\partial g_{k\mu}}{\partial x^{\nu}} + U^{\mu}U^{\nu}\frac{\partial g_{k\nu}}{\partial x^{\mu}}\right]$$

Hence

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial U^{k}}\right) = A_2 U^{\mu}U^{\nu}\left[\frac{\partial g_{k\mu}}{\partial x^{\nu}} + \frac{\partial g_{k\nu}}{\partial x^{\mu}}\right] + 2A_2 g_{\mu k}\frac{dU^{\mu}}{dt} + 2U^{\lambda}U_{k}\frac{dA_2}{dx^{\lambda}} \qquad (3.7.10)$$

The equation of motion is then given by substituting equation (3.7.1.9) and (3.7.1.10) in equation (3.7.1.8) and by multiplying both sides by $g^{k\lambda}$ to get

$$-2A_2 U^{\mu}U^{\nu}\frac{g^{k\lambda}}{2}\left[\frac{\partial g_{k\mu}}{\partial x^{\nu}} + \frac{\partial g_{k\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{k}}\right]$$

$$-2A_2\delta_{\mu}^{\lambda}\frac{dU^{\mu}}{dt} - 2g^{k\lambda}U_{\lambda}U_{k}\frac{dA_2}{dx^{\lambda}} + g^{k\lambda}\frac{dp}{dx^{k}} = 0$$

When we consider the motion of a point test particle of small mass, the pressure p vanishes and the density variation is negligible. Therefore by equation (3.7.6) we get

$$P = 0, \rho = constant, A_1 = A_2 = \frac{-\rho}{2} \qquad (3.7.1.11)$$

The equation of motion of matter in a gravitational field is then given by [3.7.1.15, 16, and 17].

$$\Gamma_{\mu\nu}^{\lambda}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt} + \frac{d^2x^{\lambda}}{dt^2} = 0 \qquad (3.7.1.12)$$

It is very interesting to note that this expression obtained from the matter action represents an alternative derivation of the geodesic equation.

3.7.2 Special Relativity in the presence of Gravitation:

In SR the time, length and mass can be obtained in any moving frame by either multiplying or dividing their values in the rest frame by a factor γ

$$\gamma = \sqrt{1-\frac{v^2}{c^2}}$$

To see how gravity affects these quantities it is convenient to re express γ in terms of the proper time [4]

$$c^2 d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \qquad (3.7.2.13)$$

Which is a common language to both SR and general relativity. We know that in SR (3.7.2.13) reduces to [10]

$$c^2 d\tau^2 = c^2 dt^2 - dx^i dx^i \,, x^0 = ct$$

$$\frac{d\tau}{dt} = \sqrt{1-\frac{1}{c^2}\frac{dx^i}{dt}\frac{dx^i}{dt}} = \sqrt{1-\frac{v^2}{c^2}} = \gamma \qquad (3.7.2.14)$$

Thus we can easily generalized γ to include the effect of gravitation by using (13) and adopting the weak field approximation where [11]

$$g_{11} = g_{22} = g_{33} = -1 \,, g_{00} = 1 + \frac{2\phi}{c^2} \qquad (3.7.2.15)$$

$$\gamma = \frac{d\tau}{dt} = \sqrt{g_{00} - \frac{1}{c^2}\frac{dx^i}{dt}\frac{dx^i}{dt}} = \sqrt{g_{00} - \frac{v^2}{c^2}} \qquad (3.7.2.16)$$

When the effect of motion only is considered, the expression for time in SR take the form [53]

$$dt = \frac{dt_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (3.7.2.17)$$

Where the subscript 0 stands for the quantity measured in a rest frame. While if gravity only affect time, its expression is given by [3.7.2.11].

$$dt = \frac{dt_0}{\sqrt{g_{00}}} \qquad (3.7.2.18)$$

In view of equation (3.7.2.17) , (18) and (16) the expression

$$dt = \frac{dt_0}{\gamma} \qquad (3.7.2.19)$$

Can be generalized to recognize the effect of motion as well as gravity on time, to get;

$$dt = \frac{dt_0}{\sqrt{g_{00-\frac{v^2}{c^2}}}} \qquad (3.7.2.20)$$

The same result can be obtained for the volume where the effect of motion and gravity respectively reads [52]

$$V = V_0 \sqrt{1 - \frac{v^2}{c^2}} \qquad (3.7.2.21)$$

$$V = \sqrt{g}\,\mathrm{V}_0 = \sqrt{g_{00}}\,\mathrm{V}_0 \qquad (3.7.2.22)$$

The generalization can be done by utilizing (3.7.2.14) and (16) to find that

$$V = \gamma V_0 = \sqrt{g_{00} - \frac{v^2}{c^2}} V_0 \qquad (3.7.2.23)$$

To generalized the concept of mass to include the effect of gravitation we use the expression for the Hamiltonian in general relativity, i.e. [53]

$$H = \rho c^2 = g_{00} T^{00} = g_{00} \rho_0 \left(\frac{dx^0}{dt}\right)^2 = g_{00} \frac{\rho_0 c^2}{\gamma^2}$$

$$= g_{00} \frac{\mathrm{m_0 c^2}}{\mathrm{v_0 \gamma^2}} \qquad (3.7.2.24)$$

Using equations (23) and (24) yields

$$\rho c^2 = \frac{mc^2}{V} = \frac{g_{00} m_0 c^2}{\gamma V} \qquad (3.7.2.25)$$

Therefore

$$m = \frac{g_{00} m_0}{\sqrt{g_{00} - \frac{v^2}{c^2}}} \qquad (3.7.2.26)$$

Which is the expression of mass in the presence of gravitation.

Using equations (3.7.2.15) and (3.7.2.26) when the field is weak and the speed is small, the energy E is given by

$$E = mc^2 = m_0 g_{00} \left(g_{00} - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \qquad (3.7.2.27)$$

In the weak field

$$E = m_0 \left(1 + \frac{2\phi}{c^2}\right)\left(1 + \frac{2\phi}{c^2} - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} C^2$$

$$\approx m_0 (1) \left(1 - \frac{2\phi}{c^2} + \frac{1}{2}\frac{v^2}{c^2}\right) c^2$$

$$E = m_0 c^2 + \frac{1}{2} m_0 v^2 - m_0 \phi$$

$$E = m_0 c^2 + T + V \qquad (3.7.2.28)$$

Unlike SR which doesn't include the potential energy, equation (3.7.2.28) shows that the energy is reduced to the classical expressions which include potential energy.

$$V = -m_0 \phi$$

3.8 Special Relativity in the presence of other fields:

According to general relativity (GR) and standard model (SM) the effect of the field on physical quantities manifests itself via the space. The space deformation in our model manifests itself through the ϕ which can be given with the aid of equation (3.7.1.13) and (16) to be.

$$\phi = \frac{dT}{dt} = \sqrt{\frac{g_{\mu\nu}}{c^2}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = \sqrt{\frac{g_{00}c^2}{c^2}\frac{dt^2}{dt^2} - \frac{1}{c^2}g_\alpha \beta \nu^\alpha \nu^\beta}$$

$$\gamma = \sqrt{g_{00} - \frac{g_{\alpha\beta}}{c^2}\nu^\alpha \nu^\beta} \qquad (3.8.1)$$

Where $\alpha = 1,2,3$

The effect of the field on gamma is incorporated in the deformation parameters $g_{\alpha\beta}$ andg_{00}. According to SM [52] the presence of the gauge fields W_μ and B_μ deform the space by changing the ordinary derivative ∂_μ to the covariant derivative D_μ.i.e

$$D_\mu = \partial_\mu + igI.W_\mu + i\left(\frac{g}{2}\right)YB_\mu \qquad (3.8.2)$$

Where the factors g , g , I and Y are parameters determining the nature of interaction. On the other hand the covariant derivative in GR [52] is given by

$$D_\mu = \partial_\mu - \Gamma^\lambda_{\mu\nu} \qquad (3.8.3)$$

Where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, g\left[\partial_\mu g_{k\nu} + \partial_\nu g_{k\mu} - \partial_k g_{\mu\nu}\right] \qquad (3.8.4)$$

The relation between the metric $g_{\mu\nu}$ and the field can be obtained from relations (3.8.2) and (3.8.3) with the aid of the relation

$$\partial_{\lambda} g_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}\, g_{\rho\mu} = \, g_{\mu\nu;\lambda} \qquad (3.8.5)$$

Where (3.8.2) and (3.8.3) gives:

$$\Gamma^{\lambda}_{\mu\nu} = \, -igI.W_{\mu} - \, \mathrm{i}\left(\frac{g}{2}\right)\lambda \mathrm{B}_{\mu} \qquad (3.8.6)$$

According to these relations the generalized expression for the time volume, mass and energy is given according to equations (3.8.1) ,(19),(23) and (25) by.

$$dt = \frac{dt_0}{\gamma} \qquad (3.8.7)$$

$$m = \frac{g_{00}\, m_{00}}{\gamma} \qquad (3.8.8)$$

$$\mathrm{E} = \mathrm{mc}^2 = \frac{\mathrm{g}_{00}\mathrm{m}_0\mathrm{c}^2}{\gamma} \qquad (3.8.9)$$

3.9 The origin of mass

It is well known that the mass is the origin of the gravitational field, therefore it is quite natural to expect the ground state of gravity to be related to the mass. The ground state energy can be found by minimizing the expression (3.7.2.25) energy in the presence of the gravitational field. Minimizing E yields

$$\frac{dE}{d\varphi} = \frac{dE}{dg_{00}}\frac{dg_{00}}{d\varphi}\,\frac{m_0 c^2\left[\frac{1}{2}g_{00} - \frac{v^2}{c^2}\right]}{(g_{00} - v^2 c^2)^{\frac{3}{2}}}\left[\frac{2}{c^2}\right] = 0 \qquad (3.9.1)$$

$$g_{00} = \frac{2v^2}{c^2} = \frac{2c^2}{c^2} = 2 \qquad (3.9.2)$$

Where the field particles moves with the speed of light, therefore equation (15) :

$$1 + \frac{2\phi}{c^2} = 2, \phi = \frac{c^2}{2}$$

There potential energy,

$$V = -m\varphi = -\frac{m_0 c^2}{2} \tag{3.9.3}$$

The energy E is given to be:

$$E = mc^2 = \frac{2m_0 c^2}{\sqrt{2-1}} = 2m_0 c^2 \tag{3.9.4}$$

Equation (3.9.3) indicates that the potential is negative which means that the particles are bounded in a negative state. The minimum energy is twice rest mass energy.

The matter Lagrangian is used to obtain the matter energy momentum tensor and the equation of motion. It is found that this matter tensor represents the energy-momentum tensor of a perfect fluid as in the case of GR. The equation of motion of matter indicates that a point-like particle moves on a geodesic curve in the free fall.

The Schwarzschild solution comes in the case of a weak static field. Thus the equation of motion and the energy stem from a single action, unlike SR, and in conformity with the principle of least action.

On the other hand the effect of gravity as well as motion on time, volume, and mass shows the dependence of them on the potential on the same footing as the velocity. Unlike SR, the expression of energy include the potential energy. When the classical limit is considered it is also very interesting to note that when the effect of gravity alone is considered on mass in equation (3.7.2.26) the mass increases which indicates that the field increases the mass. The generalized expression of time volume, mass, and energy in which the effect of fields on them is present through the metric is also exhibited. The expression for minimum energy and minimum potentials indicated that the ground state energy contains both the mass of both the particle and its antiparticle are present. Moreover this ground state have negative energy inconformity with Dirac relativistic quantum theory [53].

3.10 Summary and Critique:

The work done by authors to modify quantum theory make contribution in using momentum operator to derive the Heisenberg spatial evolution of the quantum system. Some of them are concerned with recognizing the effect of friction, while others derive new Schrödinger equation by ignoring the momentum term, to explain Josephson effect by using simple mathematics unfortunately all these attempts does not account for perturbation of momentum by potential field.

CHAPTER 4

New Momentum perturbation and string theory

4.1 Introduction:

The standard texts are can concerned the development of some (approximate) techniques of solving the time evolution of interacting fields. It is common to use the interaction picture of the time evolution in Quantum field theory.

The time evolution of operators in the interaction picture is quite simple, it is equal to the time evolution of the free fields. Indeed, both time evolutions (the one of the free fields and the one of the interacting fields in the interaction picture) are controlled by the same Hamiltonian H_o. Let us emphasize the similarities and the differences between the interacting fields in the Heisenberg and the interaction pictures.

Oscillations are a physical phenomenon seen in a wide variety of physical systems. They are especially important in that they describe the motion of a system when perturbed slightly from equilibrium. We work through the basic formalism of simple linear oscillations, both natural and driven, consider coupled oscillating systems and show how to decompose them in terms of normal coordinates, and apply the theory to oscillations of continuous systems to introduce wave phenomena.

The linear simple harmonic oscillator (SHO) is the foundation of the theory of oscillations.

The simple harmonic oscillator is one of the most important problems in quantum mechanics, from a pedagogical point of view it can be used to illustrate the basic concept and methods in quantum mechanics.

From a practical point of view it has applications in a variety of branches on modern physics, molecular, spectroscopy, solid state physics, nuclear structure, quantum field theory, quantum optics, quantum statistical mechanics and so forth.

Attempts were made to develop perturbation theory but are complex in their mathematics. This work is concerned time dependant perturbation based on generalized special relativity (GSR).

Quantum field theory (QFT) is a subject which has evolved considerably over the years and continues to do so. From its beginnings in elementary particle physics it has found applications in many other branches of science, in particular condensed matter physics but also as far afield as biology and economics. In this thesis we shall be adopting an approach (the perturbation theory) However, to set this in its context, it is useful to have some historical perspective on the development of the subject.

Quantum field theory is the basic mathematical language that is used to describe and analyze the physics of elementary particles.

The ordinary Lagrangian is dependent on coordinate variables; beside generalized coordinates and their first derivatives unfortunately this Lagrangian is found to be unable to describe the generalized Einstein generalized general relativity (EGGR) without adding to it a second derivative in the generalized coordinate.

Many attempts were made to develop quantum field theory to unify forces, but with no remarkable success.

4.2 Time Independent Momentum perturbation:

The motion of fast particles in special relativity (SR), can enable to simplify energy-momentum relation, where

$$P = \frac{m_0}{1 - \frac{v^2}{c^2}} \to \infty , \frac{v}{c} \to 1$$

Also For small rest mass m_o ,

$$E^2 = C^2P^2 + m_oc^4 \approx c^2p^2$$

Thus one can to a good approximation write:

$$E = CP \tag{4.2.1}$$

Consider a Hamiltonian which has this form:

$$H = H_o + H_1 \tag{4.2.2}$$

Where the second term represents perturbation which is given by:

$$H_1 = V_1 \tag{4.2.3}$$

With the r.h.s stands for the perturbation potential. According to equation (4.2.1) the Hamiltonian of the stable system is given by,

$$H_o = CP_o \tag{4.2.4}$$

The Hamiltonian of the perturbed system can be given according to equation (4.2.1) and (4.2.2) as:

$$H = CP = H_o + H_1 = CP_o + H_1 = CP_o + CP_1 \tag{4.2.5}$$

Where

$$P_1 = \frac{H_1}{C} \tag{4.2.6}$$

The perturbation of (p) momentum by afield is not surprising according to G.G.R:

$$p = m_0v\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = m_0v\left(1 + \frac{v^2}{c^2} - \frac{\varphi}{c^2}\right)$$

$$m_ov + \frac{m_0v^3}{c^2} - \frac{m_0v\varphi}{c^2} = p_0 + \frac{m_0v^3}{c^2} - \frac{m_0v\varphi}{c^2}$$

Consider particle of definite momentum Eigen equation of the form:

$$\hat{p}_0 u_k = \frac{\hbar}{i} \nabla u_k = p_k u_k \quad (4.2.7)$$

Perturbation of momentum is given by:

$$\hat{P} = \widehat{P_o} + P_1 = P_0 + \frac{H_1}{C} \quad (4.2.8)$$

Thus its action on an arbitrary wave function ψ yields:

$$\hat{P}\psi = \hat{P}_o\psi + \frac{H_1}{C}\,\Psi$$

$$\hat{P}\psi = \left(\frac{\hbar}{i}\nabla + \hat{P}_1\right)\psi \quad (4.2.9)$$

Let: $$\hat{P} = \hat{P}_0 + \lambda \hat{P}_1 \quad (4.2.10)$$

Consider the solution:

$$\psi = \psi_o + \lambda\psi_1 + \lambda^2\psi_2 + \dots\dots$$

$$P = P_o + \lambda P_1 + \lambda^2 P_2 + \dots\dots \quad (4.2.11)$$

Sub equation (4.2.10) in (4.2.11) to get:

$$\hat{P}\psi = P\psi \quad (4.2.12)$$

$$\left(\hat{P}_0 + \lambda\hat{P}_1\right)(\psi_o + \lambda\psi_1) = (p_o + \lambda P_1)(\psi_o + \lambda\psi_1) \quad (4.2.13)$$

The free terms satisfy,

$$\hat{P}_o\psi_o = P_o\psi_o$$

$$\hat{P}_o\psi_1 + \hat{P}_1\psi_o = P_o\psi_1 + P_1\psi_o \quad (4.2.14)$$

Comparing (4.2.14) and (4.2.7) gives,

$$\psi_o = u_k \;, P_o = P_k \quad (4.2.15)$$

Sub equation (4.2.15) in (4.2.14) gives,

$$\hat{P}_o\psi_1 + \hat{P}_1 u_k = P_k\psi_1 + P_1 u_k \quad (4.2.16)$$

One can write ψ_1 in terms of momentum Eigen value to get,

$$\psi_1 = \sum c_n u_n \tag{4.2.17}$$

Inserting (4.2.17) in (4.2.16) gives,

$$\sum c_n \hat{P}_o u_n + \hat{P}_1 u_k = P_k \sum c_n u_n + P_1 u_k$$

Multiply both sides by $\overline{u_j}$and integrate over dr to get,

$$\sum_n c_n P_n \int \overline{u_j} u_n dr + \int \overline{u_j} \hat{P}_1 u_k dr = P_k \sum c_n \int \overline{u_j} u_n dr + P_1 \int \overline{u_j} u_k dr$$

$$= \sum_n c_n P_n \delta_{jn} + \left(\hat{P}_1\right)_{jk} = P_k \sum_n c_n \delta_{jn} + P_1 \delta_{jk}$$

$$c_j P_j + \left(\hat{P}_1\right)_{jk} = P_k c_j + P_1 \delta_{jk}$$

Let j = k ,

$$p_1 = (\hat{p}_1)_{kk} \tag{4.2.18}$$

Thus from (4.2.8, 4.2.15, 4.2.18) ,

$$p = p_k + \frac{1}{c}\left(\hat{H}_1\right)_{kk} \tag{4.2.19}$$

$where : \ j \neq k$

$$c_j = \frac{(\hat{p}_1)_{jk}}{c_k - c_j} = \frac{1}{c}\frac{(H_1)_{jk}}{(c_k - c_j)} \tag{4.2.20}$$

Thus from (4.2.11, 4.2.15, 4.2.17, 4.2.20),

$$\psi = u_k + \sum_j c_j u_j$$

4.3 Time Independent perturbation for string Harmonic oscillator:

The harmonic oscillator potential is probably the most widely used potential in physics, because of its ability to represent physical potentials in the vicinity of stable equilibrium.

From equation of momentum, let:

$$P = \frac{m_o v}{\sqrt{1 - \frac{2\varphi}{c^2} - \frac{v^2}{c^2}}} \tag{4.3.1}$$

For $\varphi \, , v^2 \ll c^2$

$$p = m_0 v \left(1 - \frac{2\varphi}{c^2} - \frac{v^2}{2c^2}\right)^{-\frac{1}{2}} \tag{4.3.2}$$

$$= m_0 v \left[1 - \frac{1}{2}\left(\frac{-2\varphi}{c^2} - \frac{v^2}{c^2}\right)\right] \tag{4.3.3}$$

$$\approx m_0 v \left(1 + \frac{\varphi}{c^2} + \frac{v^2}{2c^2}\right)$$

$$= p_0 \left(1 + \frac{\varphi}{c^2} + \frac{v^2}{2c^2}\right) \tag{4.3.4}$$

$$p_o + p_0 \frac{\varphi}{c^2} + \frac{p_o}{2} \frac{v^2}{c^2} = p_0 + p_1 \tag{4.3.5}$$

$$p_1 = p_0 \left(\frac{m_0 \varphi + \frac{1}{2} m_0 v^2}{m_0 c^2}\right) = \frac{p_o (V + T)}{m_o c^2} = \frac{p_o H}{m_0 c^2} \tag{4.3.6}$$

For harmonic oscillator:

$$H = H_{osc} = \frac{P^2}{2m} + V \tag{4.3.7}$$

This is since for harmonic oscillator,

$$x = x_0 e^{iwt} \tag{4.3.8}$$

$$v = \dot{x} = iwx$$

$$F = -kx = -mw^2 x \tag{4.3.9}$$

$$v = \frac{1}{2} kx^2 = \frac{1}{2} mw^2 x^2$$

$$T = \left|\frac{1}{2} mv^2\right| = \left|-\frac{1}{2} mw^2 x^2\right| = \frac{1}{2} mw^2 x^2 = V \tag{4.3.10}$$

There for: H = 2T = $\frac{p^2}{m}$

But: $\hat{p} u_k = p_k u_k \, , \hat{p}^2 u_k = \hat{p}^2{}_k$

$H u_k = \frac{p^2{}_k}{m} u_k = E_k u_k$

$$(p_1)_{kk} = \frac{p_o}{m_0 c^2} \int \overline{u_k} \widehat{H} u_k dr \tag{4.3.11}$$

$$P_1 = \frac{P_0}{m_0 c^2} \int \bar{u}_k E_k u_k \, dr$$

$$(p_1)_{kk} = \frac{P_0}{m_0 c^2} E \int \bar{u}_k u_k dr$$

$$= \frac{P_0}{m_0 c^2} \left[n + \frac{1}{2}\right] \hbar\omega \int \bar{u}_k u_k dr \tag{4.3.12}$$

Where:

$$\int \bar{u}_k u_k dr = 1$$

$$P_1 = \left[\left(n + \frac{1}{2}\right) \hbar w\right] \frac{P_0}{m_0 c^2} \tag{4.3.13}$$

$$H = T + V = 2V \ , \ T = V$$

$$\int \bar{u}_n H u_n dr = \langle H \rangle = 2\langle V \rangle = 2\int \bar{u}_n V u_n dr$$

Thus the perturbed term of momentum is given by:

$$\therefore (p_1)_{kk} = \left[\left(n+\frac{1}{2}\right)\hbar w\right]\frac{P_0}{m_0c^2} \quad (4.3.14)$$

Thus the total momentum of the system perturbed:

$$p = p_1 + p_0$$

$$\text{p} = \left\{\left[\frac{\left(n+\frac{1}{2}\right)}{m_0c^2}\hbar w\right] + 1\right\} p_0 \quad (4.3.15)$$

4.4 Particle in a crystal:

$$\psi(x+a) = \psi(x) \quad (4.4.1)$$

$$e^{ikx}.\, e^{ika} = e^{ikx}$$

Where $e^{ika} = 1$

$$Cos\ ka = 1$$

$$thus, \quad k = \frac{2n\pi}{a}$$

$$p_1 = \frac{p_o}{m_oc^2}\int e^{-ikx}\left(\frac{-\hbar^2}{2m}\nabla^2 + 0\right)e^{ikx}\, dx \quad (4.4.2)$$

$$\frac{p_o}{m_oc^2}\int e^{-ikx}\left(\frac{\hbar^2}{2m}k^2\right)e^{ikx}dx$$

$$p_1 = \frac{p_o}{m_oc^2}\left[\int (k)^2\psi\bar{\psi}\, dx\right]\left[\frac{\hbar^2}{2m}\right] \quad (4.4.3)$$

Where: $k = \frac{2n\pi}{a}$, V= 0 , there fore

$$P_1 = \frac{p_o}{m_o c^2}\frac{\hbar^2}{2m}\int\left(\frac{2n\pi}{a}\right)^2 \bar{\psi}\,\psi\,dx \qquad (4.4.4)$$

One uses normalization condition:

$$\int \psi\bar{\psi}\,dx = \int|\psi^2|dx = 1$$

Thus

$$\therefore P_1 = \left(\frac{2n\pi}{a}\right)^2 \frac{\hbar^2}{2m}\left(\frac{p_0}{m_0 c^2}\right) \qquad (4.4.5)$$

Thus the total momentum is thus given.

4.5 Particles in a box:

From the equation of energy in the perturbed system:

$$E = \hbar\omega \qquad (4.5.1)$$

Since:

$$\psi = Asin\alpha x + Bcos\alpha x$$

$$\bar{\psi} = Asin\alpha x + Bcos\alpha x$$

At : x = 0 , $\psi = 0$

$$\psi = Asin0 + Bcos0 = 0$$

$$0 + B = 0 , \therefore B = 0$$

$$\psi(x = l) = Asin\,\alpha\,l = 0$$

$$\sin\alpha\,x = 0 \qquad (4.5.2)$$

$$\alpha L = n\pi \qquad n = 0, 1, 2, 3, \ldots..$$

$$\alpha = \frac{n\pi}{L} \qquad (4.5.3)$$

From equation (4.5.1):

$$\nabla^2\psi = -\alpha^2\psi \tag{4.5.4}$$

$$\frac{\hbar^2}{2m}\alpha^2\psi = E_0\psi = \frac{\hbar^2}{2m}k^2\psi \tag{4.5.5}$$

$$\alpha = \pm k \tag{4.5.6}$$

Substitute α by K in equation (4.5.6) to get:

$$E_0 = E - V_0 = \left(\frac{p^2}{2m} + V_0\right) - V_0 = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{4.5.7}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V_0 = E\psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = (E - V_0)\psi = E_0\psi \tag{4.5.8}$$

$$K = \frac{n\pi}{L} \tag{4.5.9}$$

For momentum perturbation for particle in a box,

$$p_0 = \hbar k = \hbar\frac{n\pi}{L} \tag{4.5.10}$$

$$\hat{P}_1 = \left[\int \bar{\psi}\left(-\frac{\hbar^2}{2m}\nabla^2 + V_0\right)\psi dr\right]\frac{p_o}{m_o c^2}$$

$$\hat{P}_1 = \left[\int \bar{\psi}\left(\frac{\hbar^2}{2m}\mathrm{k}^2 + V_0\right)\psi dr\right]\frac{p_o}{m_o c^2} \tag{4.5.11}$$

$$\psi = A\sin\alpha x\,, \nabla\psi = -\alpha A\cos\alpha x$$

$$\nabla^2\psi = -\alpha^2 A\sin\alpha x = -\alpha^2\psi \tag{4.5.12}$$

$$\hat{p}_1 = \left[\frac{\hbar^2}{2m}\mathrm{k}^2 + V_0\right]\int \bar{\psi}\psi dr\left[\frac{p_o}{m_o c^2}\right] \tag{4.5.13}$$

$$= \frac{p_o}{m_o c^2}\left[\frac{\hbar^2}{2m}\mathrm{k}^2 + V_0\right]$$

Where:

$$K = \frac{n\pi}{L} \quad (4.5.14)$$

Thus:

$$p_1 = \frac{p_0}{m_0 c^2}\left[\frac{\hbar^2\pi^2}{2ml^2}n^2 + v_0\right] \quad (4.5.15)$$

$$p = p_0 + p_1 = p_0\left[1 + \frac{1}{m_0 c^2}\left(\frac{\hbar^2\pi^2}{2ml^2}n^2 + v_0\right)\right]$$

This means that the momentum is quantized.

4.6 Time dependant momentum perturbation:

Consider the Hamiltonian:

$$i\hbar\frac{d}{dt}\psi = H\psi = \left(\frac{\hat{p}^2}{2m} + V\right)\psi$$

If the system is perturbed such that,

$$i\hbar\frac{d}{dt}\psi = \left[\left(\frac{\hat{p}_0 + \hat{p}_1}{2m}\right) + V\right]\psi \quad (4.6.1)$$

Where the momentum perturbed is given by:

$$\hat{p} = \hat{p}_0 + \hat{p}_1 \quad (4.6.2)$$

With $\hat{p}_0$ representing non perturbed momentum satisfying.

$$\hat{p}_0 u_n = p_n u_n \quad (4.6.3)$$

To solve equation (2.1), one can write

$$\psi = \sum c_n u_n \quad (4.6.4)$$

Where,

$$c_n = c_n(p,t)\,, u_n = u(p,x) = e^{\frac{i}{\hbar}px} \qquad (4.6.5)$$

Inserting (4.6.4) in (4.6.1) yields:

$$i\hbar \sum c_n \frac{\partial c_n}{\partial t} = \sum_n \left[\frac{(\hat{p}_0^2 + \hat{p}_o\hat{p}_1 + \hat{p}_1\hat{p}_o + \hat{p}_1^2)}{2m} + V\right] u_n$$

$$\frac{1}{2m}\sum \hat{p}_n{}^2 u_n + \hat{p}_0\,\hat{p}_1\,u_n + \hat{p}_1\hat{p}_0 u_n + V u_n$$

$$\frac{1}{2m}\sum p_n^2\,u_n + \hat{p}_0\hat{p}_1\,u_n + p_n\hat{p}_1 u_n + V u_n$$

Multiplying both sides by $\overline{u_J}$ and integrating yields:

$$i\hbar \sum \int \overline{u_J}\,u_n dr\,\frac{\partial c_n}{\partial t}$$

$$= \frac{1}{2m}\sum_n p_n^2 \int \overline{u_J} u_n dr + \frac{1}{2m}\sum_n \overline{u_J}\hat{p}_o\hat{p}_1 u_n dr$$

$$+ \frac{1}{2m}\sum_n p_n c_n \int \bar{u}_j \hat{p}_1 u_n dr + \sum_n \int \bar{u}_j V u_n c_n dr$$

Thus:

$$i\hbar \sum_n \frac{\partial c_n}{\partial t}\delta_{jn}$$

$$= \frac{1}{2m}\sum_n \hat{p}_n^2 \delta_{jn}\,c_n$$

$$+ \frac{1}{2m}\sum_n (\hat{p}_0\hat{p}_1)_{jn} c_n + \frac{1}{2m}\sum_n p_n(\hat{p}_1)_{jn} + \sum_n (V)_{jn} c_n$$

$$i\hbar \frac{\partial c_j}{\partial t} = \frac{1}{2m} p_j^2 c_j + \frac{1}{2m}\sum [\hat{p}_0\hat{p}_1]_{jn} + p_n\,(\hat{p}_1)_{jn} c_n$$

$$+ \sum_n (V)_{jn}^{cn} \qquad (4.8.6)$$

$$i\hbar\frac{\partial c_j}{\partial t} = \frac{{p_j}^2}{2m}c_j + \sum_n V(\hat{r},t)\delta_{jn}^{cn} + \frac{1}{2m}\sum_n\left[(\hat{p}_0\hat{p}_1)_{jn} + p_n(\hat{p}_1)_{jn}\right]c_n$$

$$i\hbar\frac{\partial c_j}{\partial t} = \frac{{p_j}^2}{2m}c_j + V(\hat{r},t)c_j$$

$$+\frac{1}{2m}\sum_n\left[(\hat{p}_0\hat{p}_1)_{jn} + p_n(\hat{p}_1)_{jn}\right]c_n \qquad (4.6.7)$$

The potential term was found by using the matrix elements of V , where one uses the following relations:

$$\hat{r} = \frac{\hbar}{i}\frac{\partial}{\partial p} = \frac{\hbar}{i}\nabla p$$

$$r = \frac{\hbar}{i}\frac{\partial}{\partial p}e^{\frac{i}{\hbar}p_n r_n} = r_n u_n$$

$$\hat{r}^2 u_n = \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)u_n = r_n^2 u_n$$

$$\hat{r}^k u_n = r_n^k u_n \qquad (4.6.8)$$

$$\hat{r}^{-1}u_n = \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^{-1} e^{\frac{i}{\hbar}p_n r_n} = \left(\frac{\hbar}{i}\frac{\partial}{\partial p}e^{\frac{-i\,p_n r_n}{\hbar}}\right)^{-1}$$

$$= (-\,r_n u_n^{-1})^{-1} = (-r_n)^{-1}\,u_n$$

$$\hat{r}^2 u_n = \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^{-1}\left(\frac{\hbar}{i}\;e\frac{-i}{\hbar}p_n x\right)^{-1}$$

$$= \left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right)^{-1}(-p_n)^{-1}u_n = (-r_n)^{-2}u_n$$

Thus:

$$\hat{r}^{-k}u_n = (-r_n)^{-k}u_n \qquad (4.6.9)$$

Thus if :

$$V(\hat{r}) = \sum c_k \hat{r} k$$

$$V(\hat{r})u_j = \left(\sum c_k \hat{r} k\right) u_i = \left(\sum c_k r_i \, k\right) u_i \tag{4.6.10}$$

Also:

$$\hat{r} \, \overline{u}_\iota u_j = \hat{r} \left(e^{-\frac{ipr_i}{\hbar}} . e^{-\frac{ipr_j}{\hbar}} \right) = \frac{\hbar}{i} \frac{\partial}{\partial p} \left(e^{\frac{-ipr_i}{\hbar}} . e^{\frac{ipr_j}{\hbar}} \right)$$

$$= \left[-\hat{r}\overline{u}_\iota u_j + r_j \overline{u}_\iota u_j\right] = \left(r_j - r_i\right)\overline{u}_\iota u_j = r\overline{u}_\iota u_j$$

$$\hat{r}\overline{u}_\iota u_j = \hat{r}\left(r\overline{u}_\iota u_j\right) = \hat{r}\left(r\overline{u}_\iota u_j\right) = r\hat{r} u_i u_j = r^2 u_i u_j$$

$$\hat{r}^n \overline{u}_\iota u_j = r^n \overline{u}_\iota u_j \tag{4.6.11}$$

$$\hat{r}^{-1}\overline{u}_\iota u_j == \left(\frac{\hbar}{i} \frac{\partial}{\partial p} \, e^{i\frac{pr_i}{\hbar}} . e^{-\frac{ipr_i}{\hbar}} \right)^{-1}$$

$$= \left(r_i u_i \overline{u}_j - r_j u_i \overline{u}_j\right)^{-1} = \left(r_i - r_j\right)^{-1}\left(u_i \overline{u}_j\right)^{-1} = (-1)(r)^{-1} \, \overline{u}_i u_j$$

$$\hat{r}^{-2} u_i u_j = (-1) r^{-1}\left[\hat{r}^{-1} u_i u_j\right] = (-1)^2 \, r^{-2} \overline{u}_\iota u_j$$

$$r^{-n}\overline{u}_\iota u_j = (-1)^n \, r^{-n}\overline{u}_\iota u_j \tag{4.6.12}$$

Thus:

$$V(\hat{r}) \int \overline{u}_\iota u_j dr = \int \overline{u}_\iota V(r) u_j \, dr = (V)_{ij}$$

$$V_{ij} = V(\hat{r}) \int \overline{u}_\iota u_j dr = V(r)\delta_{ij} \tag{4.6.13}$$

It is very interesting to note that when no perturbation exists $\hat{p}_1 = 0$ and equation (4.6.1) reduced to:

$$i\hbar \, \frac{\partial c_j}{\partial t} = \frac{{p_j}^2}{2m} c_j + V(\hat{r}) c_j \tag{4.6.14}$$

Which is the ordinary Schrödinger equation in momentum space.

4.7 Harmonic oscillator momentum perturbation:

According to string theory particles can be considered as oscillating string. Also electrons moving in a circular orbit around the nucleus can also be treated as a harmonic oscillator. Thus the behavior of electrons and elementary particles can be explained by harmonic oscillator model.

The external field applied on these particles can be treated as a perturbation. The perturbing term can be found from the momentum equation in GSR to be,

$$p = \frac{m_0 v}{\sqrt{1 - \frac{2\varphi}{c^2} - \frac{v^2}{c^2}}}$$

For $\varphi\,, v^2 \ll c^2$

$$p = p_0 + p_1 \tag{4.7.1}$$

$$p = m_0 v \left(1 - \frac{2\varphi}{c^2} - \frac{v^2}{2c^2}\right)^{-\frac{1}{2}} \tag{4.7.2}$$

$$= m_0 v \left[1 - \frac{1}{2}\left(\frac{-2\varphi}{c^2} - \frac{v^2}{c^2}\right)\right] \tag{4.7.3}$$

$$\approx m_0 v \left(1 + \frac{\varphi}{c^2} + \frac{v^2}{2c^2}\right)$$

$$= p_0 \left(1 + \frac{\varphi}{c^2} + \frac{v^2}{2c^2}\right) \tag{4.7.4}$$

$$p_o + p_0 \frac{\varphi}{c^2} + \frac{p_o}{2}\frac{v^2}{c^2} = p_0 + p_1 \tag{4.7.5}$$

$$p_1 = p_0 \left(\frac{m_0 \varphi + \frac{1}{2} m_0 v^2}{m_0 c^2}\right) = \frac{p_o (V + T)}{m_o c^2} = \frac{p_o H}{m_0 c^2} \tag{4.7.6}$$

For harmonic oscillator:

$$H = H_{osc} = \frac{P^2}{2m} + V \tag{4.7.7}$$

$$T = V \tag{4.7.8}$$

$$Hu_k = \frac{p_k^2}{m} = E_k u_k \tag{4.7.9}$$

$$(\hat{p}_1)_{jn} = \frac{p_0}{mc^2} \int \bar{u}_j \hat{H} u_n dr$$

$$= \frac{p_0}{mc^2} \int \bar{u}_j E_n u_n dr$$

$$= \frac{p_0 E_n}{mc^2} \int \bar{u}_j u_n dr = \frac{p_0 E_n \delta_{jn}}{mc^2} \tag{4.7.10}$$

$$(\hat{p}_0 \hat{p}_1)_{jn} = \frac{p_0}{mc^2} \int \bar{u}_j \hat{p}_0 \hat{H} u_n dr$$

$$= \frac{p_0}{mc^2} \int \bar{u}_j \hat{p}_0 E_n u_n dr = \frac{p_0 E_n}{mc^2} \int \bar{u}_j \hat{p}_0 u_n dr$$

$$= \frac{p_0 E_n}{mc^2} \int \bar{u}_j p_k u_n dr = \frac{p_0 E_n p_n}{mc^2} \int \bar{u}_j u_n dr = \frac{p_0 E_n p_n}{mc^2} \delta_{jn} \tag{4.7.11}$$

Substituting (4.7.10), (4.7.11) in (4.6.7) to get:

$$i\hbar \frac{\partial c_j}{\partial t} = \frac{{p_j}^2}{2m} c_j + v\, c_j + \frac{p_0 E_j p_j c_j}{m^2 c^2}$$

Dropping j indices and assuming the solution:

$$c_j = c = u(p) v(t) \tag{4.7.12}$$

One gets:

$$i\hbar \frac{dv}{dt} = \frac{p^2}{2m} u + vu + \frac{p_0 E p\, u}{m^2 c^2} = Eu \tag{4.7.13}$$

$$\frac{p^2}{2m}u + vu = \left(1 + \frac{p_0 p.}{m^2 c^2}\right) Eu = \ c_0 Eu \qquad (4.7.14)$$

For harmonic oscillator:

$$v = \ \frac{1}{2} k\hat{x}^2 = \ \frac{1}{2} k\left(-\hbar^2 \nabla_{\mathrm{P}}{}^2\right) \qquad (4.7.15)$$

Thus:

$$-\frac{\hbar^2 k}{2} \nabla^2{}_k u + \frac{p^2}{2m}\ u = c_0 Eu \qquad (4.7.16)$$

$$-\frac{\hbar^2}{2} \nabla^2{}_p u + \frac{1}{2m} p^2\ u = \frac{c_0}{k} Eu \qquad (4.7.17)$$

Comparing this with ordinary harmonic oscillator equation:

$$-\frac{\hbar^2}{2} \nabla^2 u + \frac{1}{2} k_0 x^2\ u = E_0 u \qquad (4.7.18)$$

$$x\ \rightarrow p\ \ , \qquad k_0 = \ \frac{1}{m}, \qquad E_0 = \ \frac{c_0}{k} E \qquad (4.7.19)$$

Using the fact that:

$$E_0 = \left(n + \frac{1}{2}\right) \hbar\omega_0 \qquad (4.7.20)$$

$$k_0 = m_0 \omega_o^2\ , \qquad \omega_0 = \ \sqrt{\frac{k_0}{m}}$$

$$k = m\omega^2 \qquad (4.7.21)$$

Thus the energy is quantized and given by:

$$E = \ \frac{k}{c_0} E_0 = \ \frac{k}{c_0}\left(n + \frac{1}{2}\right) \hbar\omega_0 \qquad (4.7.22)$$

4.8 Heisenberg Interaction Picture:

$$\frac{dQ_{ij}}{dx} = \frac{d}{dx}\int \bar{u}_i Q\, u_j dr \tag{4.8.1}$$

$$\int \frac{d}{dx}\bar{u}_i Q u_j dr + \int \bar{u}_i \frac{dQ}{dx} u_j dr + \int \bar{u}_i Q \frac{d}{dx} u_j dr \tag{4.8.2}$$

First integral:

$$\int \overline{\frac{d}{dx} u_\imath} \hat{Q} u_j dr = \int \left(\overline{\frac{\imath}{\hbar} P u_\imath}\right) Q u_j dr \tag{4.8.3}$$

$$-\frac{i}{\hbar}\int \overline{\hat{P} u_\imath} \hat{Q} u_j dr = -\frac{i}{\hbar}\int \hat{P}\, u_i \psi_j dr \tag{4.8.4}$$

Where: $\hat{Q}\mathrm{u_j} = \psi_j$

Since (P) is hermition,

$$-\frac{i}{\hbar}\int \bar{u}_i \hat{P} \psi_j dr = -\frac{i}{\hbar}\int \bar{u}_i \hat{P}\hat{Q} u_j dr \tag{4.8.5}$$

Second integral:

$$\int \bar{u}_i \frac{dQ}{dx} u_j dr = \left(\frac{dQ}{dx}\right)_{ij} \tag{4.8.6}$$

$$\hat{p} = \frac{\hbar}{i}\frac{d}{dx} \;, \frac{d}{dx} = \frac{i\hat{p}}{\hbar} \tag{4.8.7}$$

Third integral:

$$\int \bar{u}_i \hat{Q} \frac{d}{dx} u_j dr = \frac{i}{\hbar}\int \bar{u}_i \hat{Q}\hat{P} u_j dr \tag{4.8.8}$$

Sub (4.8.6), (4.8.7), (4.8.8) in (4.8.2) to get:

$$\frac{dQ_{ij}}{dx} = -\frac{i}{\hbar}\int \bar{u}_i \hat{P}\hat{Q} u_j dr + \frac{i}{\hbar}\int \bar{u}_i \hat{Q}\hat{P} u_j dr + \left(\frac{dQ}{dx}\right)_{ij} \tag{4.8.9}$$

$$= \frac{i}{\hbar} \int \bar{u}_i [\hat{Q}\hat{P} - \hat{P}\hat{Q}] u_j dr + \left(\frac{dQ}{dx}\right)_{ij}$$

$$= \frac{i}{\hbar} [Q, P]_{ij} + \left(\frac{dQ}{dx}\right)_{ij}$$

$$[\hat{Q}, \hat{P}] = [\hat{Q}\hat{P} - \hat{P}\hat{Q}] \quad , \quad \hat{S} = [\hat{P}, \hat{Q}]$$

$$\frac{dQ}{dx} = -\frac{i}{\hbar}[\hat{P}, \hat{Q}] + \frac{dQ}{dx} = \frac{1}{i\hbar} \hat{S} + \frac{\partial \hat{Q}}{\partial x} = \hat{G} \tag{4.8.10}$$

$$\frac{d^2 Q_{ij}}{dx^2} = \frac{d^2 Q}{dx^2} = \frac{d}{dx}\left(\frac{dQ}{dx}\right) = \frac{d}{dx} G = \frac{d}{dx} G_{ij}$$

$$\frac{d}{dx} G_{ij} = \frac{i}{\hbar} \int \bar{u}_i [\hat{G}] u_j dr$$

$$\frac{d^2 Q_{ij}}{dx^2} = \int \frac{d\bar{u}_i}{dx} \hat{G} u_j dr + \frac{i}{\hbar} \int \bar{u}_i \frac{dG}{dx} u_j dr + \int \bar{u}_i G \frac{du_j}{dx} dr \tag{4.8.11}$$

The first term is given by,

$$\int \overline{\frac{du_\iota}{dx}} \hat{G} u_j dr = \int \left(\overline{\frac{\iota}{\hbar} \hat{P} \hat{u}_\iota}\right) \hat{G} u_j dr = -\frac{i}{\hbar} \int \overline{\hat{P} u_\iota} \psi_j dr$$

Since $\hat{p}$ is hermition.

$$-\frac{i}{\hbar} \int \overline{u_\iota} \hat{P} \psi_j dr = -\frac{i}{\hbar} \int \overline{u_\iota} \hat{P} G u_j dr \tag{4.8.12}$$

The second term can be defined to be,

$$\left(\frac{\partial G}{\partial t}\right)_{ij} = \int \bar{u}_i \frac{dG}{dt} u_j dr \tag{4.8.13}$$

The third term takes the form,

$$= \frac{i}{\hbar} \int \bar{u}_i G \hat{P} u_j dr \tag{4.8.14}$$

Thus,

$$\left(\frac{dG_{ij}}{dx}\right) = -\frac{i}{\hbar}\int u_i\left(\hat{P}\hat{G} - \hat{G}\hat{P}\right)u_j dr + \left(\frac{\partial G}{\partial x}\right)_{ij} \quad (4.8.15)$$

$$= \frac{1}{i\hbar}\left[\hat{P}, \hat{G}\right]_{ij} + \left(\frac{\partial G}{\partial x}\right)_{ij}$$

$$\frac{dG}{dx} = \frac{1}{i\hbar}\left[\hat{P}, \hat{G}\right] + \frac{\partial G}{\partial x} \quad (4.8.16)$$

Where,

$$G = \frac{d\hat{Q}}{dx} = \frac{1}{i\hbar}\left[\hat{P}, \hat{Q}\right] + \frac{\partial \hat{Q}}{\partial x}$$

Thus,

$$\frac{d^2\hat{Q}}{dx^2} = \frac{1}{i\hbar}\left[P, G\right] + \frac{\partial G}{\partial x} \quad (4.8.17)$$

$$i\hbar\frac{d^2Q}{dx^2} = \left[\hat{P}, G\right] + i\hbar\frac{\partial G}{\partial x}$$

But from Heisenberg interaction picture,

$$i\hbar\frac{d\hat{Q}}{dt} = \left[\hat{H}, \hat{Q}\right] + i\hbar\frac{\partial \hat{Q}}{\partial t} \quad (4.8.18)$$

4.9 Spatial Evolution in the Interaction Picture:

From the total spatial momentum:

$$\hat{P} = \hat{P}_o + \hat{P}_{int} \quad (4.9.1)$$

$$i\hbar\frac{d}{dx}|\psi\rangle_s = \hat{P}_o|\psi\rangle_s \quad (4.9.2)$$

Rearrange equation (4.9.2) and integrate over dx one get:

$$\int\frac{d|\psi\rangle_s}{|\psi\rangle_s} = \frac{\hat{p}_0}{i\hbar}\int dx$$

$$|\Psi(x)\rangle_s = |\psi\rangle_s = e^{\frac{\hat{p}_o x}{i\hbar}}|\psi\rangle_s \quad (4.9.3)$$

The average value is the same in all pictures; thus,

$$\langle\widehat{\mathrm{Q}}\rangle = \langle\Psi_{\mathrm{so}}|\widehat{\mathrm{Q}}_{\mathrm{s}}|\Psi_{\mathrm{so}}\rangle = \langle\Psi_{\mathrm{I}}|\widehat{\mathrm{Q}}_{\mathrm{I}}|\Psi_{\mathrm{I}}\rangle \quad (4.9.4)$$

But,

$$\langle\psi|\widehat{\mathrm{Q}}_{s}|\psi\rangle_{\mathrm{s}} = \langle\psi|e^{\frac{\hat{p}_0 x}{i\hbar}} Q_s e^{\frac{\hat{p}_0 x}{i\hbar}}|\psi\rangle \quad (4.9.5)$$

But since the interaction wave vector is given by,

$$|\Psi_{\mathrm{I}}\rangle = |\Psi_{\mathrm{so}}\rangle \quad (4.9.6)$$

Therefore,

$$\langle\Psi_s|\mathrm{Q}_s|\Psi_s|\rangle = \langle\Psi_{Io}|Q_I|\Psi_{Io}\rangle \quad (4.9.7)$$

Also,

$$\langle\Psi_{Io}| = \langle\Psi_{so}| = |\Psi_{Io}\rangle^{+} = |\Psi_{so}\rangle^{+} \quad (4.9.8)$$

Thus,

$$\langle\Psi_{Io}|e^{-\frac{iP_O x}{\hbar}} Q_s\, e^{-\frac{iP_O x}{\hbar}}|\Psi_{Io}\rangle = \langle\Psi_{Io}|Q_I|\Psi_{Io}\rangle \quad (4.9.9)$$

Hence,

$$Q_I = e^{-\frac{\hat{P}_O x}{i\hbar}} Q_s\, e^{-\frac{P_O x}{i\hbar}} \quad (4.9.10)$$

This is the operator in the interaction picture; let us see now the spatial evolution of the (Ψ) by assuming,

$$|\Psi_{\mathrm{I}}\rangle = U(x, x_o)|\Psi_{Io}\rangle \quad (4.9.11)$$

In view of equation (4.9.2):

$$\frac{\hbar}{i}\frac{d|\Psi_s\rangle}{dx} = \hat{P}|\Psi_{\mathrm{s}}\rangle \quad (4.9.12)$$

By using equation (4.9.3) one can assume also that,

$$|\Psi_s\rangle = e^{\frac{\hat{P}_O x}{i\hbar}}|\Psi_\mathrm{I}\rangle \tag{4.9.13}$$

Inserting equation (4.9.13) in the left hand side of (4.9.12) yields,

$$\frac{\hbar}{i}\frac{d|\Psi_s\rangle}{dx} = p_o \frac{i\hbar}{i\hbar} e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}\Big|\Psi_\mathrm{I}\rangle + i\hbar e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}\Big|\frac{d}{dx}|\Psi_\mathrm{I}\rangle$$

Thus,

$$\frac{\hbar}{i}\frac{d|\Psi_s\rangle}{dx} = e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}\left[P_o\Big|\Psi_\mathrm{I}\rangle + \frac{\hbar}{i}\frac{d|\Psi_I\rangle}{dx}\right] \tag{4.9.14}$$

But,

$$\frac{\hbar}{i}\frac{d|\Psi_s\rangle}{dx} = \hat{\mathrm{P}}|\Psi_s\rangle$$

From (4.9.1) and the right hand side of equation (4.9.12) one gets,

$$\frac{\hbar}{i}\frac{d|\Psi_s\rangle}{dx} = \left(\hat{P}_o + \hat{P}_{int}\right)|\Psi_\mathrm{s}\rangle = \left(\hat{P}_o + \hat{P}_{int}\right)e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}|\Psi_\mathrm{I}\rangle$$

$$= P_o e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}\Big|\Psi_\mathrm{I}\rangle + \hat{P}_{int} e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}|\Psi_\mathrm{I}\rangle \tag{4.9.15}$$

Comparing (4.9.14) and (4.9.15) yields,

$$e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}P_o|\Psi_\mathrm{I}\rangle + e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}\frac{\hbar}{i}\frac{d|\Psi_I\rangle}{dx} = P_o e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}\Big|\Psi_I\rangle + P_{int} e^{\frac{i\hat{\mathrm{P}}_O x}{\hbar}}|\Psi_\mathrm{I}\rangle$$

$$\frac{\hbar}{i}\frac{d|\Psi_I\rangle}{dx} = \hat{P}_{int}|\Psi_\mathrm{I}\rangle \tag{4.9.16}$$

Which is the spatial evolution of $|\Psi_\mathrm{I}\rangle$.

But from (4.9.11),

$$|\Psi_\mathrm{I}\rangle = U(x, x_o)|\Psi(x_o)_I\rangle \tag{4.9.17}$$

Also from equation (4.9.16),

$$\frac{\hbar}{i}\frac{d|\Psi_I\rangle}{dx} = \mathrm{P}_{int}|\Psi_\mathrm{I}\rangle$$

$$\frac{\hbar}{i}\frac{d}{dx}U|\Psi_{\mathrm{Io}}\rangle = \mathrm{P_I}\, U|\Psi_{\mathrm{Io}}\rangle \tag{4.9.18}$$

Since $|\Psi_{\mathrm{Io}}\rangle$ is constant, thus:

$$\frac{\hbar}{i}\frac{d}{dx}U \;=\; P_I U \tag{4.9.19}$$

This equation can be solved by interaction. Since,

$$|\psi(x)\rangle = U(x, x_o)|\psi(x_o)\rangle \tag{4.9.20}$$

at $x_o = x = 0$

$$|\psi(0)\rangle = U(0,0)|\psi(0)\rangle$$

$$U_0 = U(x_o, x_o) = U(o, o) = I$$

Consider the solution,

$$\frac{\hbar}{i}\int dU_1 = \int P_I\, U_O d_x \tag{4.9.21}$$

$$U_1 = \frac{i}{\hbar}\int_{x_0}^{x_1} P_I U_0 dx = \int_{x_0}^{x_1} P_I(x^1)dx^1 \tag{4.9.22}$$

Using equation (4.9.21) again,

$$\frac{\hbar}{i}dU_2 = P_I U_1 dx$$

$$U_2 = \frac{i}{\hbar}\int_{x_1}^{x_2} P_I(x^2)U_1 dx^2$$

$$= \left(\frac{i}{\hbar}\right)^2 \int_{x_1}^{x_2}\int_{x_0}^{x_1} P_I(x^1)P_I(x^2)dx^1\, dx^2 \tag{4.9.23}$$

Similarly,

$$\frac{\hbar}{i} dU_3 = P_I U_2 dx$$

$$U_3 = \frac{i}{\hbar} \int_{x_1}^{x_3} P_I(x^3)\, U_2 dx^3 \qquad (4.9.24)$$

$$\left(\frac{i}{\hbar}\right)^3 \int_{x_2}^{x_3} \int_{x_1}^{x_2} \int_{x_0}^{x_1} P_I\,(x^1) P_I(x^2) P_I(x^3)\; dx^1 dx^2 dx^3$$

On the other hand,

$$\frac{dU}{U} = \frac{iP_I}{\hbar} dx$$

Thus,

$$\int \frac{dU}{U} = \frac{i}{\hbar} \int P_I dx$$

$$ln\, U = \frac{i}{\hbar} \int P_I dx + c$$

Therefore,

$$U = e^c e^{\frac{i}{\hbar}} \int P_I dx$$

$$U = U_0 e^{\frac{i}{\hbar}} \int P_I dx \qquad (4.9.25)$$

4.10 Discussion:

In ordinary special relativity and Schrödinger equation, the effect of potential manifests itself through energy term. However the situation is different within the framework of GSR where the potential affect the momentum through the mass term. This means that momentum can be perturbed by the potential. The perturbation term p_1 is given by (4.2.6) to be related to the Hamiltonian. Expanding the momentum, wave function in terms of powers of λ (see equations (4.2.10, 4.2.11), one can find the new momentum and wave function in equations (4.2.20, 4.2.21) by using the momentum equation (4.2.12).

Treating elementary particles as oscillating strings the new momentum is found to be quantized as shown by equation (4.3.15). Inside the crystal having periodic structure, the total momentum is quantized as equations (4.2.19) and (4.4.5) indicates.

Treating the string as one dimensional box of length L, equation (4.5.15) shows that the string momentum is quantized.

In section (6) time dependent perturbed Schrödinger equation in momentum space is derived {see equation (4.6.7)}. This equation reduces to ordinary Schrödinger equation when perturbation term vanishes as shown by equation (4.6.14). By taking into account GSR momentum relation in equation (4.7.1), it is clear that P is affected by the potential through the mass term. The perturbation term P_1 is given by equation (4.7.1). Using string theory the particles can be treated as harmonic oscillator. The quantum perturbed equation equation is shown in equation (4.7.7).

The energy of perturbed system is given by equation (4.7.18) which shows that the energy is quantized.

In sections (8) and (9) the spatial evolutions of the system in terms of momentum operator are derived, the spatial evolution in the interaction picture leads to the equation (4.9.16). The solutions of this equation are shown by (4.9.20, 24, and 25).

These resembles that of time evolution for quantum field theory, this new version can describe spatial evolution of quantum system [54].

4.11 Conclusion:

Perturbation momentum quantum model based on GSR shows that the momentum is quantized it one treats particles as vibrating strings or finite length string.

The GSR opens a new horizon in developing spatial evolution of the quantum system on the bases of momentum operator.

4.12 Recommendation and outlook:

1. The perturbation theory need to be promoted construct new perturbative quantum field theory.

2. The perturbation of momentum need to be applied to hydrogen and other atoms.

3. Angular momentum perturbation theory needs to be developed.