Sudan University of Science & Technology
College of studies

Differential forms & deRham Groups with some applications

الصيغ التفاضلية و زمر درهام مع بعض التطبيقات

A thesis Submitted in Partial Fulfillment,For the Degree of M.Sc. in Mathematics

By

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بسم الله الرحمن الرحيم

الآيـــــــة

بسم الله الرحمن الرحيم

قال تعالي :

(وزكريا إذ نادى ربه رب لا تذرني فردا وأنت خير الوارثين)

صدق الله العظيم

الأنبياء 89

Dedication

To my dears father and mother for their kindness and help.

To my brothers and sisters for their encouragement .

To all faithful teachers who paved the way of Knowledge

And education for me

Acknowledgement

Iwould like to express my gratitude to my supervisor **Dr. Imad eddin Abdulla Abduraheem** for his generous, instruction, advice and encouragements during the preparation of this thesis.

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For their great help and assistance ,

In the end I wish to thank all those who help or encourage me to accomplish this work.

Abstract

Throughout this research we have dealt with submanifolds of Euclidean space and with forms defined in open subsets of Euclidean space.

This approach has the advantage of conceptual simplicity ,one tends to be more comfortable dealing with supspaces of R^n than with arbitrary metric spaces.

Also we discussed some important ideas that are sometimes obscured by the familiar surroundings

الملخص

في هذا البحث تعاملنا مع الفضاءات الجزئية (Submanifolds) لفضاء إقليدس وتعريفات بعض الصيغ للمجموعات المفتوحة في هذا الفضاء وهذا يقودنا للمييزات والمفاهيم البسيطة في الفضاء R^n والفضاء المتري.

أيضا تمت مناقشة بعض الأفكار المهمة التي قد لا تظهر في الأحوال المألوفة

Table of Contents

Introduction

In this research we deal with submanifolds of Euclidean space and with forms defines in Euclidean Space and it is orgenized as follows:

Firestlly ,we present the k-dimensional analogues of curves and surfaces ,Also ,we discuss the notation of K-dimensional volume of several objects.we study the integral of scalar functions over a K- manifold with respect to a k-volume with some applications.

In chapter 2 ,we introduced a product operation into the set of all tensors on Linear spaces,and we derive some properities of the alternative tensors. Also we define the concept of permutations and the product operation in the set of alternative tensors.

In chapter 3 ,we study tensors algebra in R^n, and introduce the concept of tensor field and differential forms,with some applications.

Finally ,we discuss additional conditions for the K-form to be exact ,and we illustrated that condition for a K-form W to be exact is the codition that W be closed is not in general sufficient.

Also we discuss differentiability of maps between differentiable manifolds with some applications.

Chapter (1)

Introduction to manifold

Section (1-1):- the Volume of aparallelopiped and parametrized manifold

In the following we will discuss the k −dimensional analogues of curves and surfaces; they are called k-manifolds in R^n . And also we define a noti-on of k-dimensional volume for such objects.

Lemma (1-1-1):- Let W be a linear subspace of R^n of dimension k .Then there is an orthonormal basis for R^n whose first k elements form a basis for W .

Proof

there is a basis $a_1 \ldots a_n$ for R^n whose first k elements form abasis for W .There is a standard procedure for forming from these vectors an orthogonal set of vectors $b_1 \ldots b_n$ such that for each i ,the vectors $b_1,\ldots,b_i$ span the same space as the vectors $a_1 \ldots a_i$. It is called the Gram-Schmidt process .

we recall it here

Given $a_1 \ldots a_n$, we set

$b_1 = a_1$, $b_2 = a_2 - \lambda_{21} b_1$ and for general i

$$b_i = a_i - \lambda_{i1} b_1 - \lambda_{i2} b_2 - \ldots \lambda_{i,i-1} b_{i-1} \qquad (1-1)$$

where the λ_{ij} are scalars yet to be specified. No matter what these scalars are, however , we note that for each j the vector a_j equals a linear combination of the vectors $b_1,\ldots,b_j$. Furthermore, for each j the vector b_j can be written as a linear combination of the vectors $a_1 \ldots a_j$. These two facts imply that, for each i , $a_1,\ldots,a_i$ and $b_1,\ldots,b_i$; span the same subspace of R^n . It also follows that the vectors $b_1,\ldots,b_n$ are independent, for there are n of them, and they span R^n as we have just noted. In particular, none of the b_i can equal 0.

Theorem (1-1-2) :- Let W be a k - dimensional linear subspace of R^n There is an orthogonal transformation h : $R^n \rightarrow R^n$ that carries W onto the subspace $R^k \times 0$ of R^n .

Proof

Choose an orthonormal basis $b_1,\ldots,b_n$ for R^n such that the first k basis elements $b_1,\ldots,b_k$ form a basis for W Let g: $R^n \to R^n$ be the linear transformation $g(x) = \mathcal{B} \cdot x$, where $\mathcal{B}$ is the matrix with successive columns $b_1,\ldots,b_n$ · Then g is an orthogonal transformation, and $g(e_i) = b_i$; for all i. In particular, g carries $R^k \ x\ 0$, which has basis $e_1,\ldots,e_k$ onto w .

Theorem (1 – 1 -3):- There is au nique function V that assigns, to each $k - tuple$ $x_1 \ldots x_k$ of elements of R^n, a non-negative number such that:

(1) If $h : R^n \to R^n$ is an orthogonal transformation, then

$$V(h(x_1),\ldots,h(x_k)) = V(x_1,\ldots,x_k) \qquad (1-2)$$

(2) If $y_1 \cdots y_k$ belong to the subspace R^k x 0 , of $\mathcal{R}^n$, so that

$$y_i = \begin{bmatrix} z_i \\ 0 \end{bmatrix} \quad \text{for} \quad z_i \in R^k, \text{ then} \qquad (1-3)$$

$$V(y_1\ldots,y_k) = [\det[z_1,\ldots,z_k] \qquad (1-4)$$

The function V vanishes if and only if the vectors $x_1 \ldots x_k$ are dependent. It satisfies the equation

$$V(x_1,\ldots x_k) = [\det(X^{tr}.X)]^{1/2} \qquad (1-5)$$

where X is the n by k matrix $X =$ $[x_1 \ldots x_k]$ .We often denote V($x_1 \ldots x_k$)simply by V(X) .

Proof

Given $X = [x_1 \ldots x_k]$, define $F(X) = det(X^{tr} \cdot X)$. (1 - 6)

Step1. If $h : R^n \to R^n$ is an orthogonal transformation, given by the equation $h(x) : A.x$ where A is an orthogonal matrix, then

$$F(A \cdot X) = det\ ((A.X)^{tr}.(A.X)) = det(X^{tr} \cdot X) = F(X). \qquad (1-7)$$

Furthermore, if Z $is\ a\ k\ by\ k\ matrix, and\ if\ Y\ is\ the$ n $by\ k\ matrix.$

$$y_i = \begin{bmatrix} z \\ 0 \end{bmatrix}$$

$$F(\mathrm{Y}) = \det([z^{tr}\ \ 0].\begin{bmatrix} z \\ \hline 0 \end{bmatrix}) = \det(z^{tr}.\ z) = \det{}^2 z \qquad (1-8)$$

Step 2:- It follows that F is non-negative. For given $x_1 \ldots x_k$ in R^n

let W be a k -dimensional subspace of R^n containing them Let $h(x) = A \cdot X$ be an orthogonal transformation of R^n carrying W onto the subspace $R^k x 0$. Then $A \cdot X$ has the form

$$A.X = \binom{z}{0} \quad (1-9)$$

so that $F(X) = F(A.X) = \det^2 Z \geq 0.$ Note that $F(X) = 0$ if and only if the columns of Z are dependent, and this occurs if and only if the vectors $x_1 ..., x_k$ are dependent.

Step 3. Now we define $V(x) = (F(x))^{1/2}$ It follows from the computations of Step 1 that V satisfies conditions (1) and (2). And it follows from the computation of Step 2 that V is uniquely characterized by these two conditions.

Definition.(1 -1 -4):- If $x_1 ... x_k$ are independent vectors in R^n , we define the k-dimensional volume of the parallelopiped $P = P(x_1 ... x_k)$ to be the number $V(x_1 ... x_k)$, which is positive.

EXAMPLE(1-1-5) :- Consider two independent vectors a and b in $\mathcal{R}^3$; let X be the matrix $X = [a\ b]$. Then $V(X)$ is the area of the parallelogram with edges a and b. Let θ be the angle between a and b, defined by the equation then

$$V((X))^2 = \det(X^{tr}.\ X) = \|a\|^2 \|b\|^2 (1 - \cos^2 \theta) = \|a\|^2 \|b\|^2 \sin^2 \theta \quad (1-10)$$

Definition(1 -1 – 6):- Let $x_1 ... x_k$ be vectors in R^n with $k \leq n$. Let X be the matrix $X = [x_1 ... x_k]$ · If $I = (i_1 ... i_k)$ is a k −tuple of integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we call I an ascending k-tuple from the set $\{1, \ldots, n\}$, and we denote by

$$x_1 \text{ or } or\ by \quad X(i_1 ... i_k)$$

the k by k submatrix of X consisting of rows $i_1 ... i_k$ of X .

Theorem (1 - 1 -7):- Let X be an n by k matrix with $k \leq n$. Then

$$V(x) = \sum_{[I]} det^2 x_1]^{1/2} \quad (1-11)$$

where the symbol $[I]$ indicates that the summation extends over all ascending k-tuples from the set $\{1, \ldots, n\}$. This theorem may be thought of as a Pythagorean theorem for k −volume. It states that the square of the volume of a k-

parallelopiped P in R^n is equal to the sum of the squares of the volumes of the k-parallelopipeds obtained by projecting P onto the various coordinate k-planes of R^n.

Proof

Let X have size n by k. Let

$$F(X) = det(X^{tr} \cdot X) \quad \text{and} \quad G(x) = \sum_{[i]} \det^2 x_1 \qquad (1 - 12)$$

Proving the theorem is equivalent to showing that $F(X) = G(X)$ for all X.

Step 1. The theorem holds when $k = 1$ *or* $k = n$. *If* $k = 1$, then X is a column matrix with entries $\lambda_1,\ldots\lambda_n$, say. Then

$$F(X) = \sum(\lambda_\iota)^2 = G(x) \qquad (1 - 13)$$

If $k = n$, the summation in the definition of G has only one term, and $F(X) = det\,^2 X = G(X)$. (1 - 14) Step 2. If $X : [x_1 \ldots x_k]$ and the x_i are orthogonal, then

$$F(X) = \|x_1\|^2 \|x_2\|^2 \ldots \|x_k\|^2 \qquad (1 - 15)$$

Step 3. Consider the following two elementary column operations, where j≠ℓ:

(1) Exchange columns j and ℓ.
(2) Replace column j by itself plus c times column ℓ.

We show that applying either of these operations to X does not change the values of F or G.

Given an elementary row operation, with corresponding elementary matrix E, then $E \cdot X$ equals the matrix obtained by applying this elementary row operation to X. One can compute the effect of applying the corresponding elementary column operation to X by transposing X, premultiplying by E, and then transposing back. Thus the matrix obtained by applying an elementary column operation to X is the matrix

$$(E\,.x^{tr})^{tr} = X\,.E^{tr} \qquad (1 - 16)$$

It follows that these two operations do not change the value of F. For

$$F(x.E^{tr}) = \det(E.x^{tr}.x.E^{tr}) = (\det E)(\det(x^{tr}.x))(\det E^{tr}) = F(x) \qquad (1 - 17)$$

since det $E = \pm 1$ for these two elementary operations.

Nor do these operations change the value of G. Note that if one applies one of these elementary column operations to X and then deletes all rows but $i_1 \ldots i_k$ the result is the same as if one had first deleted all rows but $i_1 \dots i_k$ and then applied the elementary column operation. This means that

$(x.E^{tr})_{\mathrm{I}} = x_{\mathrm{I}} \, . E^{tr}$

We then compute

$G(X.E^{tr}) = \sum_{[\mathrm{I}]} det^2(x \, .E^{tr})_{\mathrm{I}} = \sum_{[\mathrm{I}]} det^2(x_i \, .E^{tr})_{\mathrm{I}} = G(x)$(1-18

Step 4:- In order to prove the theorem for all matrices of a given size, we show that it suffices to prove it in the special case where all the entries of the bottom row are zero except possibly for the last entry, and the columns form an orthogonal set.

Given X , if the last row of X has a non-zero entry, we may by elementary operations of the specified types bring the matrix to the form

$$D = \begin{bmatrix} * \\ 0 \dots 0 \lambda \end{bmatrix} \qquad (1 - 19)$$

where $\lambda \neq 0$. If the last row of X has no non-zero entry, it is already of this form, with $\lambda = 0$. One now applies the Gram-Schmidt process to the columns of this matrix. The first column is left as is. At the general step, the j^{th} column is replaced by itself minus scalar multiples of the earlier columns. The Gram-Schmidt process thus involves only elementary column operations of type (2). And the zeros in the last row remain unchanged during the process. At the end of the process, the columns are orthogonal, and the matrix still has the form of D.

Step 5:- by induction on n . If $n = 1$, then $k = 1$ and Step 1 applies. If $n = 2$, then $k = 1$ or $k = 2$, and Step 1 applies. Now suppose the theorem holds for matrices having fewer than n rows. We prove it for matrices of size n by k. In view of Step 1 , we need only consider the case $1 < k < n$. In view of Step 4, we may assume that all entries in the bottom row of X , except possibly for the last, are zero, and that the columns of X are orthogonal. Then X has the form

$$X = \left[\begin{array}{ccc} b_1 \dots & b_{k-1} & b_k \\ \hline 0 \dots & 0 & \lambda \end{array} \right] \qquad (1 - 20)$$

the vectors b_i of R^{n-I} are orthogonal because the columns of X are orthogonal vectors in R^n . For convenience in notation, let B and C denote the matrices

$$B = [b_1 \dots, b_k] \quad \text{and} \quad C = [b_1 \dots, b_{k-1}] \qquad (1 - 21)$$

We compute $F(X)$ in terms of B and C as follows:

$$F(X) = \|b_1\|^2 \dots \|b_{k-1}\|^2 (\|b_k\|^2 + \lambda^2)$$

by step (2) $F(B) + \lambda^2 F(C)$ (1 - 21)

To compute $G(X)$, we break the sum mation in the definition of $G(X)$ into two parts, according to the value of i_k . We have

$$G(X)=\sum_{i_k<n}\det^2 X_1+\sum_{i_k=n}\det^2 X_1 \qquad (1-22)$$

Now if $i=(i_1...i_k)$ is an ascending k-tuple with $i_k < n$, then $X_1 = B_1$. Hence the first summation in (1 - 22) equals $G(B)$. On the other hand, if $i_k = n$, one computes

$$\det X((i_1,....,i_{k-1},n)=\pm\lambda\det C(i_1,...,i_{k-1})$$

It follows that the second summation in (1-22) equals . $\lambda^2 G(C)$. Then

$$G(X) = G(B) + \lambda^2 G(C) . \qquad (1-23)$$

The induction hypothesis tells us that $F(B) = G(B)$ and $F(C) = G(C)$. It follows that $F(X) = G(X)$.

Definition:- (1 -1 - 8):- Let $k \leq n$. Let A be open in $\mathcal{R}^k$, and let $\alpha : A\rightarrow R^n$ be a map of class $C^r (r \geq 1)$. The set $Y = \propto (A)$, together with the map $\propto$, constitute what is called parametrized-manifold, of dimension k. We denote this parametrized-manifold by $Y_\propto$; and we define the (k-dimensional) volume of $Y_\propto$ by the equation

$$v(\mathrm{Y}_\alpha)=\int_A V(D_\alpha), \qquad (1-24)$$

provided the integral exists.

Let us give aplausibility argument to justify this definition of volume. Suppose A is the interior of a rectangle Q in R^k, and suppose $\alpha : A\rightarrow R^n$ can be extended to be of class C^r in a neighborhood of Q . Let $Y = \propto (A)$.

Let P be apartition of Q. Consider one of the subrectangles

$$R=[a_1,a_1+h_1]\times,...,\times[a_k,a_k+h_k] \qquad (1-25)$$

determined by P. Now R is mapped by $\propto$ onto a "curved rectangle" contained in Y. The edge of R having endpoints a and $a + h_i e_i$ is mapped by $\propto$ into acurve in R^n ; the vector joining the initial point of this curve to the final point is the vector $\alpha(a+h_i e_i-\alpha(a)$

Afirst-order approximation to this vector is, as we know,thevector

$$v_i=D_\alpha(a).h_i e_i=(\partial\alpha/\partial x_i).h_i \qquad (1-26)$$

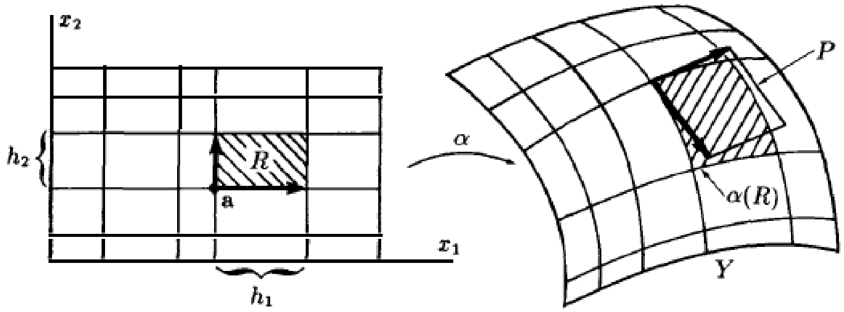


figure (1-1)

It is plausible therefore to consider the k-dimensional parallelopiped P whose edges are the vectors V_i to be in some sense afirst-order approximation to the "curved rectangle" $\propto (R)$. See Figure (1-1) . The k-dimensional volume of P is the number

$$V(v_1,...v_k) = V(\delta\alpha/\partial x_1),...,\delta\alpha/\partial x_k).(h_1...h_k) = V(D_\alpha(a).v(R) \quad (1-27)$$

When we sum this expression over all subrectangles $\mathcal{R}$, we obtain a number which lies between the lower and upper sums for the function $V(D_\propto)$ relative to the partition P. Hence this sum is an approximation to the integral

$$\int_A V(D_\alpha),$$

the approximation may be made as close as we wish by choosing an appropriate partition P.

Definition(1 -1 - 9):- Let A be open in $\mathcal{R}^k$; let $\alpha : A \rightarrow R^n$ be of class C^r ; letY = $\propto$ (A). Let f be a real-valued continuous function defined at each point of Y. We define the integral of f over $Y_\propto$, with respect to volume, by the equation

$$\int_{Y_\alpha} f dv = \int_A (f \circ \alpha) V(D_\alpha) \quad (1-28)$$

provided this integral exists.

Here we are reverting to "calculus notation" in using the meaningless symbol dV to denote the "integral with respect to volume." Note that in this notation ,

$$v(Y_\alpha) = \int_{Y_\alpha} dV \quad (1-29)$$

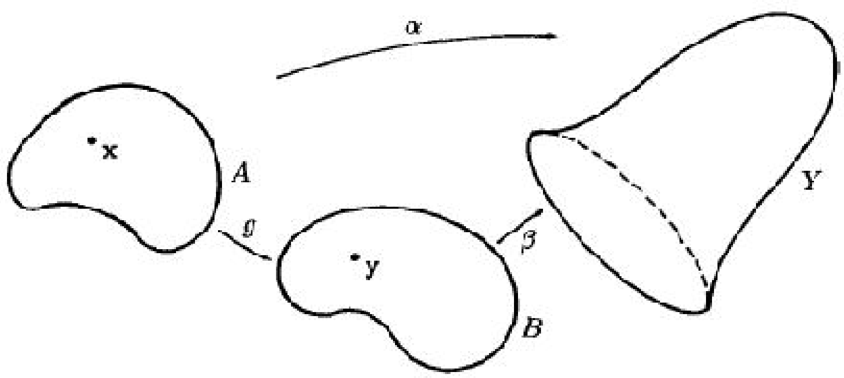
We show that this integral is "invariant under reparametrization."

Theorem (1 -1 - 10):- Let $g : A \rightarrow B$ be a diffeomorphism of open sets in $\mathcal{R}^k$. Let $\beta : B \rightarrow R^n$ be a map of class C^r; let $Y = \beta(B)$ Let $\alpha = \beta \circ g$

then $\alpha : A \to R^n$ and Y = ∝(A) . If f : Y → R is a continuous function, then f is integrable over Y_β Y_α if and only if it is integrable over Y_α ; in this case

$$\int_{Y_\alpha} f\,dv = \int_{Y_\beta} f\,dv \qquad (1 - 30)$$

In particular, $v(Y_\alpha) = v(Y_\beta)$



figuer(1-2)

Proof

We must show that

$$B\int (f\circ\beta)V(D\beta) = \int_A (f\circ\alpha)(V(D_\alpha) \qquad (1 - 31)$$

where one integral exists if the other does . See Figure (1-2).The change of variables theorem tells us that

$$\int_\beta (f\circ\beta)V(D\beta) = \int_A ((f\circ\beta)\circ g)(V(D\beta)\circ g|\det Dg|) \qquad (1 - 32)$$

We show that $(V(D\beta)\circ g|\det Dg|) = V(D_\alpha)$ (1 - 33)

EXAMPLE (1 -1 - 11):- Let A be an open interval in R^1, and let $\alpha : A \to R^n$ be a map of class C^r. Let $Y = \propto (A)$. Then $Y_\propto$ is called a parametrized-curve in $\mathcal{R}^n$ and its
1 - dimensional volume is often called its length. This length is given by the formula

$$v(Y_\alpha) = \int_A V(D\alpha) = \int_A \left[\left(\frac{d\alpha_1}{dt}\right)^2 + \cdots + \left(\frac{d\alpha_n}{dt}\right)^2\right]^{1/2} \qquad (1 - 34)$$

since $D_\propto$ is the column matrix whose entries are the functions $d\alpha_i/dt$

Theorem(1 -1 - 12):- Let Q be a rectangle in $\mathcal{R}^2$ and let $\alpha : A \rightarrow R^n$ be a map of class C^r defined in an open set containing Q. Given $\varepsilon > o$, there is a $\delta > 0$ such that for every partition P of Q of mesh less than δ,

$$\left| A(p) - \int_Q V(D_\alpha) \right| < \varepsilon$$

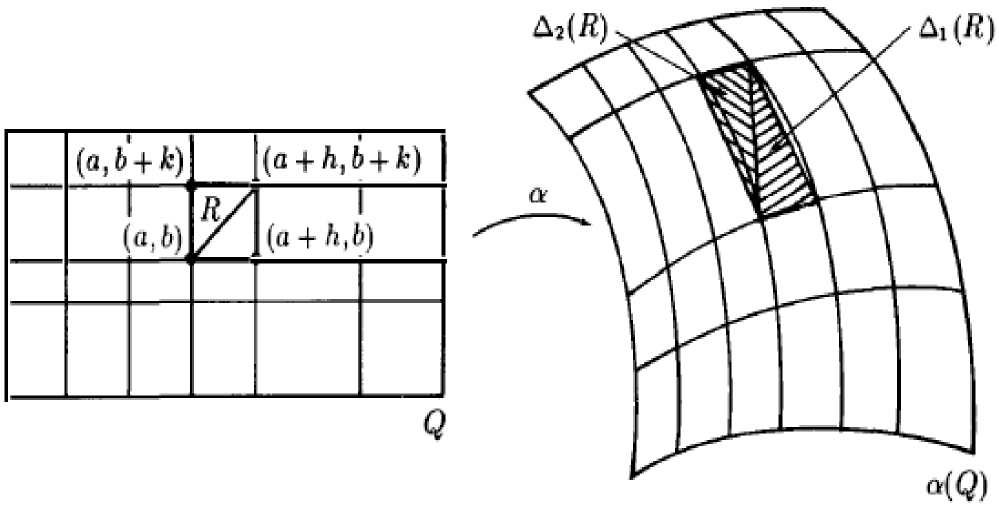


figure (1-3)

Proof

(a) Given points $x_1 \ldots x_6$ of Q, let

$$D\alpha(x_1 \ldots x_6) = \begin{bmatrix} D_1\alpha_1(x_1) & D_2\alpha_1(x_4) \\ D_1\alpha_2(x_2) & D_2\alpha_2(x_5) \\ D_1\alpha_3(x_3) & D_2\alpha_3(x_6) \end{bmatrix} \qquad (1 - 35)$$

Then $D_\propto$ is just the matrix $D_\propto$ with its entries evaluated at different points of Q. Show that if $\mathcal{R}$ is a subrectangle determined by P, then there are points $x_1 \ldots x_6$ of $\mathcal{R}$ such that

$$v(\Delta_1(R) = \frac{1}{2} V(D\alpha(x_1 \ldots x_6)).v(R) \qquad (1 - 36)$$

(b) Given $\varepsilon > 0$, show one can choose $\delta > 0$ so that if X_i, y_i ε Q with $|x_i - y_i| < \delta$ for $i = 1,\ldots,6$ then

$$V(D\alpha(x_1 \ldots x_6)). - V(D\alpha(y_1 \ldots y_6)). < \varepsilon|$$

Definition (1 -1 - 13) :-Let $\kappa > 0$. Suppose that M is a subspace of $\mathcal{R}^n$. having the following property: For each $p\varepsilon$M , there is a set V containing p that is open in M , a set U that is open in $\mathcal{R}^n$. and a continuous map $\propto: U \to V$ carrying U onto V in a one-to-one fashion, such that:

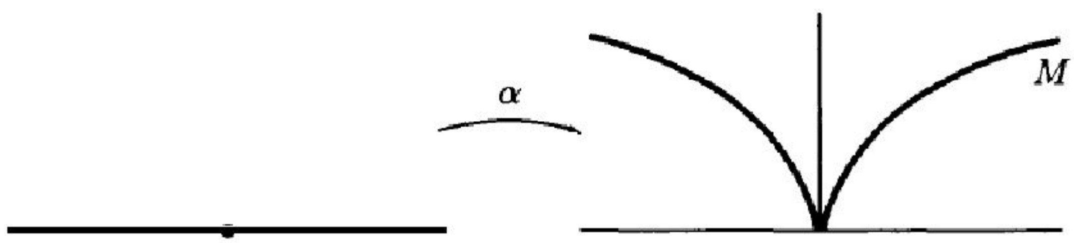
(1) a is of class C^r .

(2) $\alpha^{-1} : U \to v$ is continuous.

(3) $Da(x)$ has rank k for each $x \in U$.

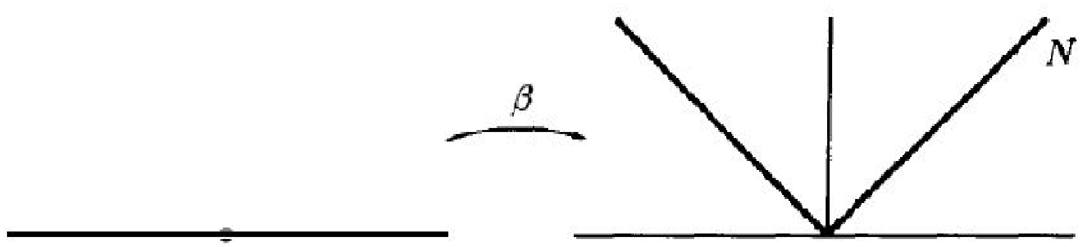
Then M is called a k-manifold without boundary in $\mathcal{R}^n$. , of class C^r. The map $\propto$ is called a coordinate patch on M about p.

EXAMPLE (1 -1 - 14):-Consider the case $k = 1$. If $\propto$ is a coordinate patch on M, the condition that $D_\propto$ have rank 1 means merely that $D_\propto \neq 0$. This condition rules out the possibility that M could have "cusps" and "corners." For example, let $\alpha{:}R \to R^2$ be given by the equation $\propto(t) = (t^3, t^2)$, and let M be the image set of $\propto$. Then M has a cusp at the origin. (See Figure (1-4) Here $\propto$ is of classC^∞ and $\propto^{-1}$ is continuous, but $D_\propto$ does not have rank 1 at $t = o$.



Figure(1-4)

Similarly, let $\beta: R \to R^2$ be given by $\beta(t) = (t^3, |t^3|)$, and let N be the image set of β. Then N has a corner at the origin. (See Figure (1-5) Here



Figure(1-5)

β is of class C^2 (as you can check) and β^{-1} is continuous , but $D\beta$ does not have rank 1 at $t = o$.

Definition(1 -1 - 15):- Let S be a subset of R^k ; let $f: S \to R^n$. We say that f is of class C^r on S if f may be extended to a function $g : U \to \mathcal{R}^n$. that is of class C^r on an open set U of R^k containing S.

Lemma (1 -1 - 16):-Let S be asubset of R^k; let $f : S \to R^n$ If for each $x \epsilon S$, there is a neighborhood U_x of x and a function $g_x : U_x \to R^n$. of class C^r that agrees with$f$ on $U_x \cap S$, then$f$ is of class C^r on S.

Proof

Cover S by the neighborhoods U_x ; let A be the union of these neighborhoods ; let $\{ \emptyset_i \}$ be a partition of unity on A of class C^r dominated by the collection $\{U_x\}$ · For each i, choose one of the neighborhoods U_x containing the support of $\emptyset_i$;, and let g_i denote the C^r function$g_x: U_x \to R^n$. The C^r function $\emptyset_i g_i: U_x \to R^n$ vanishes outside a closed subset of U_x; we extend it to a C^r function h_i; on all of A by letting it vanish outside U_x. Then we define

$$g(x) = \sum_{i=1}^{\infty} h_i(x) \qquad (1 - 37)$$

for each $x \epsilon A$. Each point of A has a neighborhood on which g equals afinite sum of functions h_i; thus g is of class C^r on this neighborhood and hence on all of A. Furthermore, if $x \epsilon S$, then

$$h_i(x) = \emptyset_i(x) g_{i(x)} = \emptyset_i f(x) f(x) \qquad (1 - 38)$$

for each i for which $\emptyset_i(x) \neq o$. Hence if $x \epsilon S$,

$$g(x) = \sum_{i=1}^{\infty} \phi_i(x) f(x) = f(x) \qquad (1 - 39)$$

Definition(1 -1 - 17):- Let H^k denote upper half-space in R^k, consisting of those $x \epsilon R^k$ for which x_k>o . Let H_k^+ denote the open upper half-space,consisting of those x for which $X_k > o$ We shall be particularly interested in functions defined on sets that are open in H^k but not open in R^k . In this situation , we have the following useful result:

Lemma (1 -1 - 18):-Let U be open in H^k but not in R^k ; $\alpha: U \to R^n$ be of class C^r . Let $\beta: U' \to R^n$ be a C^r extension of a defined on an open set U' of R^k . Then for $x \epsilon U$, the derivative $D\beta(x)$ depends only on the function $\propto$ and is independent of the extension β. It follows that we may denote this derivative by $D \propto (x)$ without ambiguity.

Proof

Note that to calculate the partial derivative $\frac{\delta\beta_i}{\delta x_j}$ at x, we form the difference quotient $[\beta(x+h_{ej}) - \beta(x)]/h$ and take the limit as h approaches 0. For calculation purposes, it suffices to let h approach 0 through positive values. In that case, if x is in H^k then so is $x + he_j$. Since the functions $\beta$and $\propto$ agree at points of H^k , the value of $D\beta(x)$ depends only on a. See Figure (1-6)

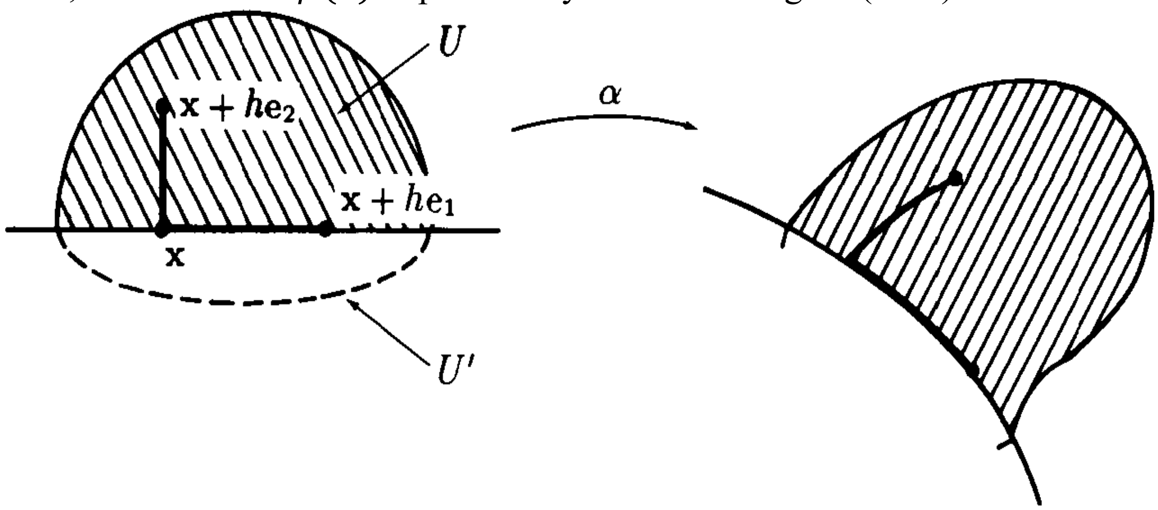


Figure (1-6)

Definition(1 -1 - 19):- Let $k > O$. A k-manifold in R^n of class C^r is a subspace M of R^n having the following property: For each $p \in M$, there is an open set V of M containing p, a set U that is open in either R^k or H^k , and a continuous map $\alpha: U \to V$ carrying U onto V in a one-to-one fashion, such that:

(1)$\propto$ is of class C^r .

(2) $\propto^{-1}: V \to U$ is continuous.

(3) $D \propto (x)$ has rank k for each $x \in U$.

The map a is called a coordinate patch on M *about* p. We extend the definition to the case $k = o$ by declaring a discrete collection of points in R^n to be a o-manifold in R^n .Note that a manifold without boundary is simply the special case of amanifold where all the coordinate patches have domains that are open in R^k .Figure (1-7) illustrates a 2-manifold in R^3 . Indicated are two coordinate patches on M , one whose domain is open in R^2 and the other whose domain is open in H^2 but not in R^n

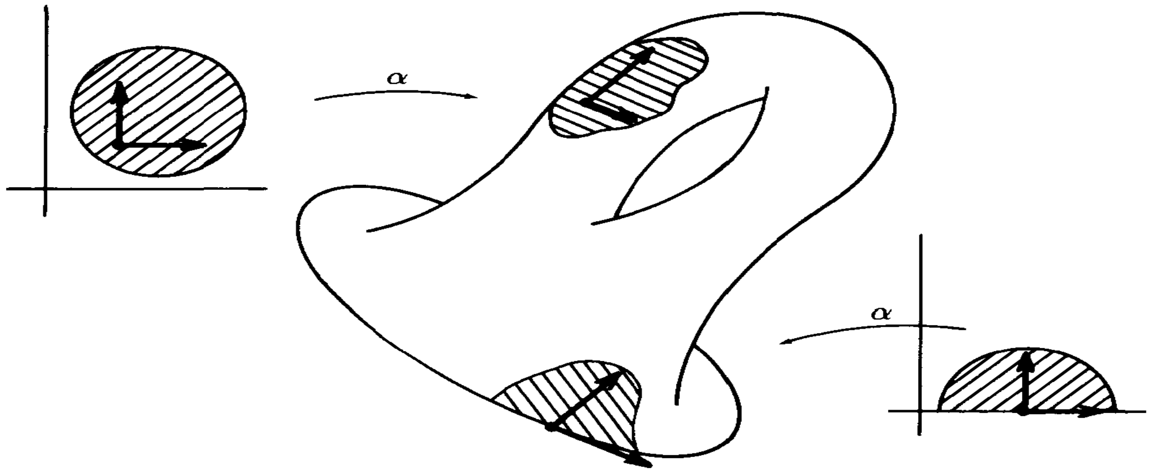


Figure (1-7)

It seems clear from this figure that in a k-manifold , there are two kinds of points, those that have neighborhoods that look like $open\ k$ −balls, and those that do not but instead have neighborhoods that look like open half-balls of dimension k. The latter points constitute what we shall call the boundary of M. Making this definition precise, however, requires a certain amount of effort. We shall deal with this question in the next section

Lemma(1 -1 - 20):- Let M be a manifold in R^n , and let $\alpha: U \to V$ be acoordinate patch on M. If U_0 is a subset of U that is open in U, then the restriction of a to U_o is also a coordinate patch on M.

Proof

The fact that U_0 is open in U and a^{-1} is continuous implies that the set $V_0 = \alpha(U_o)$ is open in V. Then U_0 is open in R^k or H^k (according as U is open in R^k or H^k), and V_0 is open in M . Then the map α/U_0 is a coordinate patch on M : it carries U_o onto V_0 in a one-to-one fashion; it is of class C^r because a is; its inverse is continuous being simply a restriction of α^{-1} ; and its derivative has rank k becauseD_α does.

Section (1-2):- The Manifold with boundary and Integrating

In the following we make precise what we mean by the boundary of a manifold;and also we prove a theorem that is useful in practice for constructing manifolds.

Theorem (1 -2 - 21):- Let M be a k-manifold in R^n of class C^r . Let $\propto_0: U_0 \to V_0$ and $\propto_1: U_1 \to V_1$ be coordinate patches on M, with $W = V_0 \cap V_1$ non-empty. Let $W_i = \propto_i^{-1}(W)$. Then the map
$\propto_1^{-1} \circ \propto_0: W_0 \to W_1$ is of class C^r ' and its derivative is non-singular.
Typical cases are pictured in Figure (1 -8). We often call $\propto_1^{-1} \circ \propto_0$ the transition function between the coordinate patches $\propto_0$ and $\propto_1$

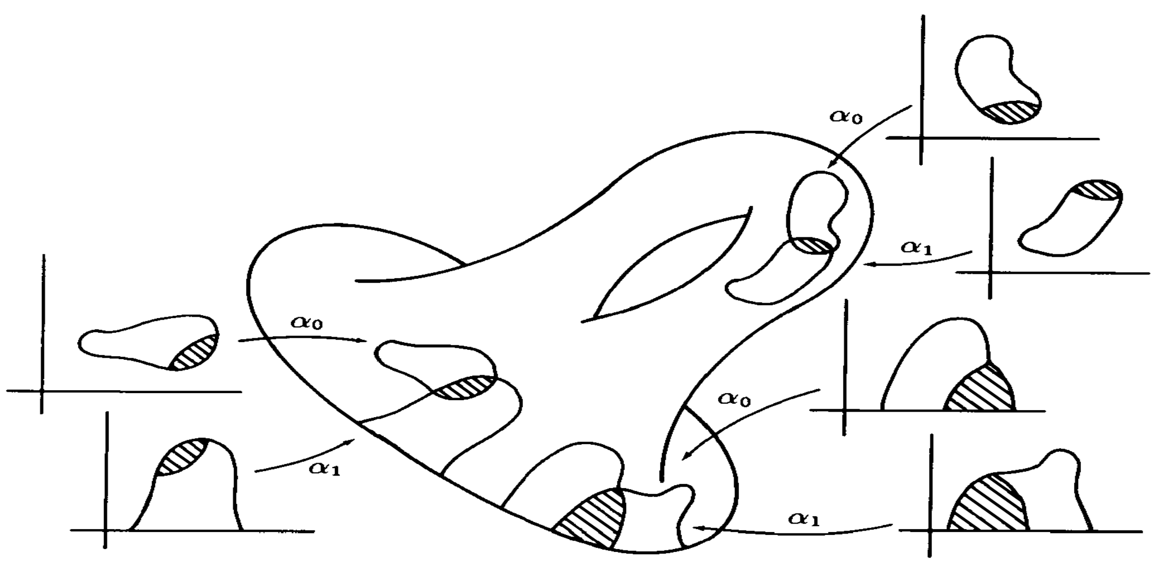


Figure (1-8).

Proof

It suffices to show that if $\propto: U \to V$ is a coordinate patch on M ,then $\propto^{-1}: V \to R^k$ is of class C^r , as a map of the subset V of R^n into R^k For then it follows that, since $\propto_0$ and $\propto_1^{-1}$ are of class C^r , so is their composite $\propto_1^{-1} \circ \propto_0$. The same argument applies to show $\propto_0^{-1} O \propto_0$ is of class C^r; then the chain rule implies that both these transition functions have non-singular derivatives.

To prove that $\propto^{-1}$ is of class C^r , it suffices (by Lemma (1-1-16) to show that it is locally of class C^r. Let P_0 be a point of V; let $\propto^{-1}(p_0) = x_0$. We show $\propto^{-1}$extends to a C^rfunction defined in a neighborhood of p_0 in $\mathcal{R}^n$ .Let us first consider the case where U is open in H^k but not in R^k . By assumption , we can extend a to a C^r map β of an open set U' of R^k into $\mathcal{R}^n$ Now $D\alpha(x_0)$ has rank k, so some k rows of this matrix are independent; assume for convenience the first k rows are independent. Let $\Pi : \mathcal{R}^n \to R^k$ project R^n onto its first k coordinates. Then the map $g = \pi o \beta$ maps U' into R^k, and $Dg(x_0)$ is non-singular. By the inverse function theorem, g is a C^r diffeomorphism of an open set W of R^k about x_0 with an open set in R^k. See Figure (1-9)

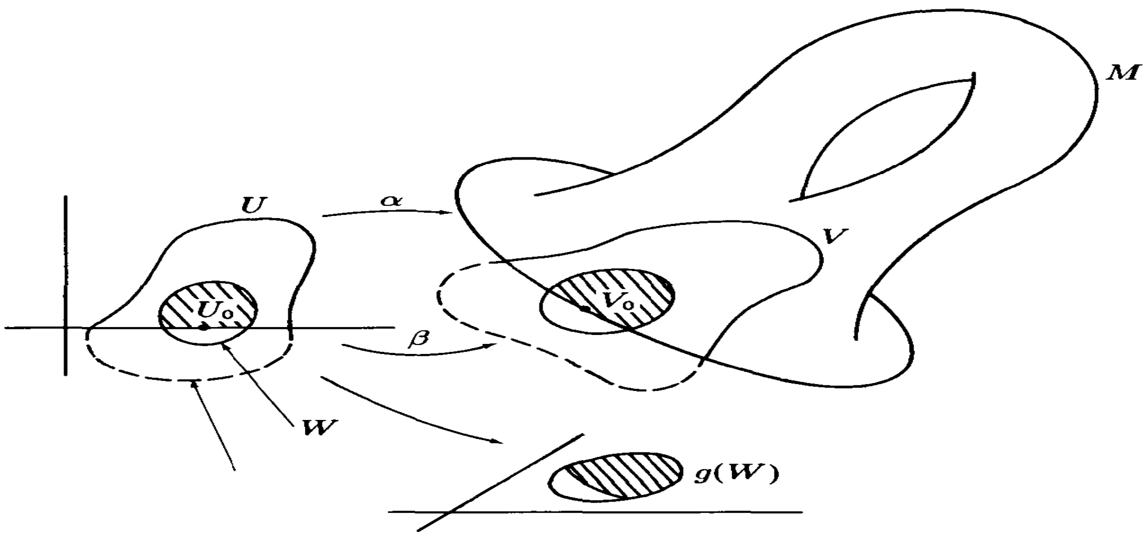


Figure (1-9)

We show that the map $h = g^{-1} o \, \pi$,which is of class C^r, is the desired extension of $\propto^{-1}$ to a neighborhood A of p_0. To begin, note that the set $U_0 = W \cap U$ is open in U, so that the set $V_0 = \propto (U_o)$ is open in V; this means there is an open set A of R^n such that $A \cap V = V_o$. We can choose A so it is contained in the domain of h (by intersecting with $\pi^{-1}(g(W))$if necessary). Then $h : A \to R^k$ is of class C^r; and if $p \in A \cap V = V_0$, then we let
$x = \propto^{-1} (p)$ and compute

$$h(p) = h\left(\propto (x)\right) = (g^{-1}(\pi(\propto (x))) = g^{-1} (g(x)) = x = a^{-1} (p), \qquad (1 - 40)$$

Definition(1 -2 - 22):- Let M be a k-manifold in R^n; let $p \in M$. If there is acoordinate patch $\propto: U \to V$ on M about p such that U is open in R^k, we say p is an interior point of M. Otherwise , we say p is a boundary point of M. We denote the set of boundary points of M by M , and call this set the boundary of M .

Lemma (1 -2 - 23):- Let M be a k-manifold in R^n ; let $\propto: U \to V$ be acoordinate patch about the point p of M .

(a) If U is open in R^k , then p is an interior point of M.

(b) If U is open in H^k and if $p = \propto (x_0)$ for $x_0 \in H^k_+$, then p is an interior point of M.

(c) If U is open in H^k and $p = \propto (x_0))$ for $x_0 \epsilon R^{k-1} \times 0$, then p is boundary point of M.

Proof

(a) is immediate from the definition. (b) is almost as easy. Given$\propto$: $U \to V$ as in (b), let $U_0 = U \cap U_+^k$ and let $V_0 = \propto (U_0)$. Then $\propto / U_0$ mapping U_0 onto V_0, is a coordinate patch about p, with U_0 open in $\mathcal{R}^k$.

(c). Let α: $U_0 \to V_0$ be a coordinate patch about p , with U_0 open in H^k and $p = \alpha o(x_0)\, for\, x_0 \in R^{k-1} \times 0$. We assume there is a coordinate patch $\alpha_1 : U_1 \to V_1$ about p with U_1 open in $\mathcal{R}^k$. and derive a contradiction .

Since V_0 and V_1 are open in M, the set $W = V_0 \cap V_1$ is also open in M.Let $W_i = \alpha_i^{-1}$ (W) for $i = 0 , 1$; then W_0 is open in H^k and contains x_0 , and W_1 is open in $\mathcal{R}^k$. The preceding theorem tells us that the transition function

$$\alpha_0^{-1} \circ \alpha_1 : W_1 \to W_0$$

is a map of class C^r carrying W_1 onto W_0 in a one-to-one fashion, with non-singular derivative. But W_0 is contained in H^k and contains the point x_0 of $R^{k-1} \times 0$, so it is not open in $\mathcal{R}^k$! See Figure (1-10)

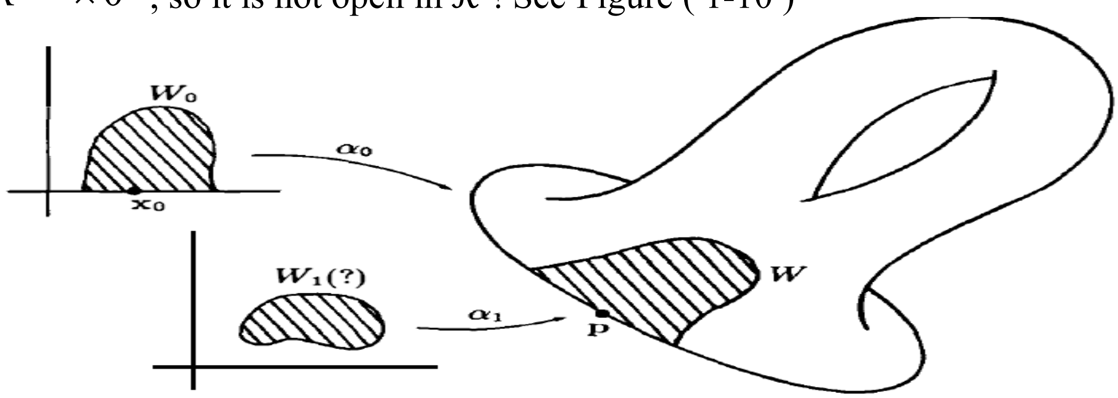


Figure (1-10)

Theorem (1 -2 - 24):-Let $M\ be\ a\ k - manifold\ in$ R^n , of class C^r. If ∂M is non-empty, then ∂M is a k-1 manifold without boundary in R^n of class C^r.

Proof

Let $p \in \partial\, M. Let\, \alpha : U \to V$ be a coordinate patch on M about p. Then U is open in H^k $and\, p = \alpha(x_0)$ for some $x_0 \in \partial H^k$. By the preceding lemma, each point of $U \cap H_+^k$ is mapped by α to an interior point of M, and each point of $U \cap (\partial H^k)$) is mapped to a point of ∂M. Thus the restriction of α to $U \cap (\partial H^k)$ carries this set in a one-to-one fashion onto the open set $V_0 = V \cap \partial M$ of ∂M . Let U_0 be the open set of R^{k-1} such that

$U_0 \times 0 = U \cap \partial H^k$; $if\ x \in U_0$, define $\alpha(x) = \alpha_0(x)$ .Then $\alpha_0 : U_0 \to V_0$ is a coordinate patch on ∂M. It is of class C^r because α is, and its derivative has rank $k - 1$ because $D\alpha_0(x)$ consists simply of the first $k - 1$ columns of the matrix $D\alpha(x, 0)$. The inverse α_0^{-1} is continuous because it equals the restriction to V_0 of the continuous function α^{-1}, followed by projection of R^k ×l: onto its first k −1 coordinates.

Theorem (1 -2 - 25):-Let φ be open in R^n ; $let\ f : \varphi \to R$ be of class C^rLet M be the set of points x for which $f(x) = 0$ let N be the set of pointsfor which $f(x) \geq 0$. Suppose M is non-empty and $D\ f(x)$ has rank 1 at each point of M. Then N is an $n - manifold$ in R^n and $\partial N = M$.

Proof

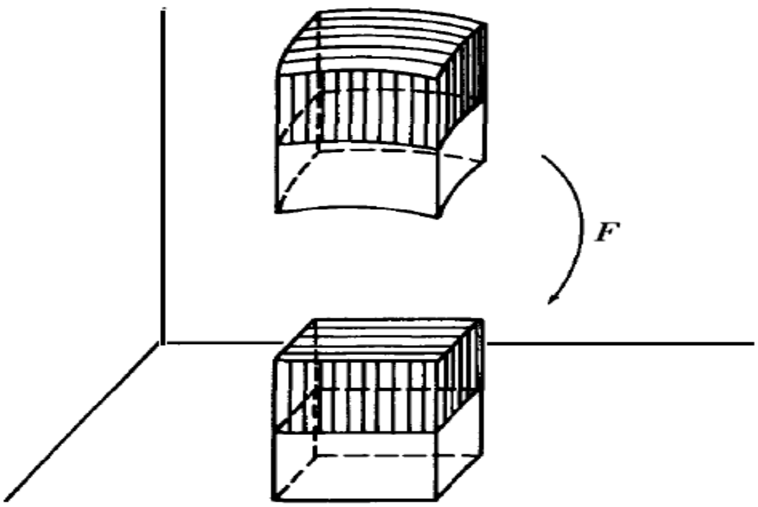
Suppose first that p is a point of N such that $f\ (p) > 0$. Let U be the open set in R^n consisting of all points x for which $f(x) > 0$; let $\alpha : U \to U$ be the identity map. Then a is (trivially) a coordinate patch on N about p whose domain is open in R^n.

Now suppose that $f(p) = 0$. Since $Df(p)$ is non-zero, at least one of the partial derivatives $D_i f(p)$ is non-zero. Suppose $D_n f(p) \neq f$ 0. Define $F : \varphi \to R^n$ by the equation $F(x) = (x_1, \ldots, x_{n-1}, f(x))$. Then

$$DF = \begin{bmatrix} I_{n-1} & 0 \\ * & D_n\, f \end{bmatrix} \qquad (1 - 41)$$

so that $DF(p)$ is non-singular. It follows that F is a diffeomorphism of a neighborhood A of p in R^n with an open set B of R^n. Furthermore, F carries the open set $A \cap N$ of N onto the open $set\ B \cap H^n\ of\ H^n$, since $x \in N$ if and only if $f(x) \geq 0$. It also carries $A \cap M$ onto $B \cap \partial H^n$, since $x \in M$ if and only if $f(x) = 0$. Then $F^{-1} : B \cap H^n \to A \cap N$ is the required coordinate patch on N. See Figure (1-11)

Definition(1 -2 - 26):- Let $B^n(a)$ consist of all points $x\ of\ R^n$for which $\|x\| \leq a$,and let $S^{n-1}(a)$ consist of all x for which $\|x\| = a$. We call them the $n - ball$ and the $n - 1$ sphere, respectively, of radius a.



Figure(1-11)

Corollary (1 -2 - 27):- The $n - ball\ B^n(a)$ is an n-manifold in R^n of class C^∞ , $and\ S^{n-1}(a) = \partial B^n(a)$.

Proof

We apply the preceding theorem to the function $f(x) = a^2 - \|x\|^2$ Then

$Df(x) = [(-2x_1) \cdots (-2x_n)]$, (1 - 42)

which is non-zero at each point of $S^{n-1}(a)$.

Now we will discuss to define what we mean by the integral of a continuous scalar function f over a manifold M in R^n. For simplicity, we shall restrict ourselves to the case where M is compact. The extension to the general case can be carried out by methods analogous to those used in §16 in treating the extended integral.First we define the integral in the case where the support of f can be covered by a single coordinate patch.

Definition(1 -2 - 28):- Let M be a compact k −manifold in R^n, of class C^r . Let $f: M \to R^n$ be a continuous function. Let C = Support f; then C is compact. Suppose there is a coordinate patch $\alpha : U \to V$ on M such that$C \subset V$. Now $\alpha^{-1}(C)$ is compact. Therefore, by replacing U by a smaller open set if necessary, we can assume that U is bounded. We define the integral of f over M by the equation

$\int_M f dV = \int_{Int\ U}(fo\alpha\ )V(D\alpha)$ (1 - 43)

Here Int $U = U\ if\ U$ is open in R^k , and Int $U = U \cap H_+^k U$ is open in H^k but not in R^k .

It is easy to see this integral exists as an ordinary integral, and hence as an extended integral : The function $F = (f\ o\ \alpha)V(D\alpha)$ is continuous on U and vanishes outside the compact set a$^{-1}(C)$; hence F is bounded. If U is open in R^k , then F vanishes near each point x_0 of Bd $U.If\ U$ is not open in R^k , then F

vanishes near each point of $Bd\ U$ not in ∂H^k , a set that has measure zero in R^k . In either case, F is integrable over U and hence over Int U. See Figure (1-12).

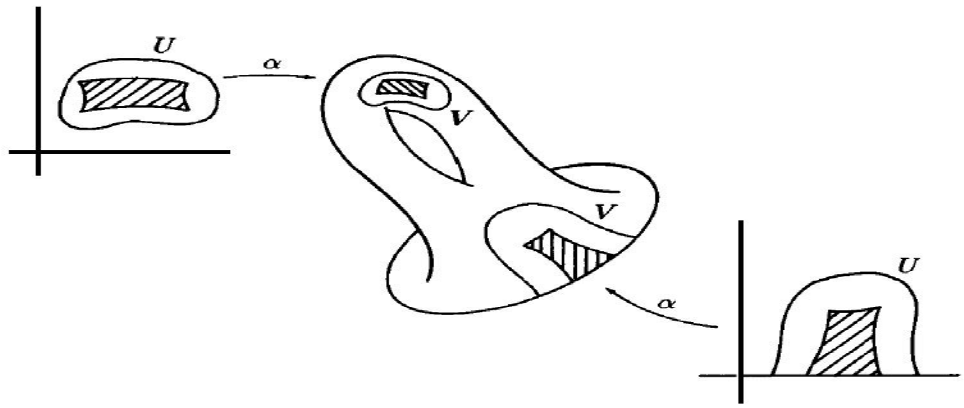


figure (1-12)

Lemma (1 -2 - 29):-If the support of f can be covered by a single coordinate patch, the integral $\int_M f\ dV$ is well-defined, independent of the choice of coordinate patch.

Proof

We prove a preliminary result. Let $\alpha : U \to V$ be a coordinate patch containing the support of f. Let W be an open set in U such that $\alpha(W)$ also contains the support of f. Then

$$\int_{\text{int} W} (fo\alpha)V(D\alpha) = \int_{\text{int} U} fo\alpha)V(D\alpha) \qquad (1 - 44)$$

the (ordinary) integrals over W and V are equal because the integrand vanishes outside W

$Let\ \alpha_i : U_i \to V_i\ for\ i = 0,1$ be coordinate patches on M such that both V_0 and V_1 contain the support of f. We wish to show that $\int_{Int\ U_0} (fo\ \alpha_0)V\ (D\alpha_0) = \int_{Int\ U_1} (fo\ \alpha_1)V\ (D\alpha_1)$ (1 - 45)

Let $W = V_0 \cap V_1$ and let $W_i = \alpha_i^{-1}(W)$. In view of the result of the preceding paragraph, it suffices to show that this equation holds with U_i replaced by $W_i\ , for\ i = 0,1$. Since $\alpha_i^{-1}\ o\ \alpha_0 : Int\ W_0 \to Int\ W_1$ is a diffeomorphism, this result follows at once from Theorem (1-1-10)To define $\int_M fdV$ in general, we use a partition of unity on M .

Lemma (1 -2 - 30):- Let M be a compact k-manifold in R^n, of class C^r.Given a covering of M by coordinate patches, there exists a finite collection of C^∞ functions $\emptyset_1, \dots, \emptyset_l$ mapping R^n into R such that:

(1) $\emptyset_i(x) \geq 0 \quad for\ all\ x.$

(2) Given i, the support of $\emptyset_i$ is compact and there is a coordinate patch $\alpha_i : U_i \to V_i$ belonging to the given covering such that $((Support\ \emptyset_i) \cap M) \subset V_i$.

(3) $\sum \emptyset_i(x) = 1 \qquad for\ x \in M.$

We call $\{\emptyset_1, .., \emptyset_l\}$ a partition of unity on M dominated by the given collection of coordinate patches.

Proof

For each coordinate patch $\alpha : U \to V$ belonging to the given collection , choose an open set A_V $of\ R^n$ such that $A_V \cap M = V$. Let A be the union of the sets A_V . Choose a partition of unity on A that is dominated by this open covering of A . Local finiteness guarantees that all but finitely many of the functions in the partition of unity vanish identically on M. Let $\{\emptyset_1, .., \emptyset_l\}$ be those that do not.

Definition(1 -2 - 31):- Let M be a compact k-manifold in R^n, of class C^r. Let $f : M \to R$ be a continuous function. Choose a partition of unity $\emptyset_1, .., \emptyset_l$ on M that is dominated by the collection of all coordinate patches on M. We define the integral of f over M by the equation

$$\int_M \mathrm{f\, dV} = \sum_{\mathrm{i}=1}^{\ell} \left[\int_{\mathrm{M}} (\emptyset_{\mathrm{i}}\ \mathrm{f})\ \mathrm{dv} \right]. \tag{1 - 46}$$

Then we define the (k-dimensional) volume of M by the equation

$$v(M) = \int_M I dV \tag{1 - 47}$$

If the support of f happens to lie in a single coordinate patch $\alpha : U \to V$, this definition agrees with the preceding definition. For in that case, letting $A = \text{Int}\ U$, we have

$$\left.\begin{aligned} &\textstyle\sum_{i=1}^{\ell} \left[\int_M (\emptyset_i f) dV\right] = \sum_{i=1}^{\ell} \left[\int_A (\emptyset_i o\alpha)(f\ o\alpha) D(\propto)\right] \text{ by definition,} \\ &= \textstyle\int_A \left[\sum_{i=1}^{\ell} (\emptyset_i o\alpha)(f\ o\alpha) V(D\alpha)\right] \text{ by linearity,} \\ &\textstyle\int_A (f\ o\alpha) V(D\alpha) \ \text{ because } \sum_{i=1}^{\ell} (\emptyset_i o\alpha) = 1\ on \quad A, \\ &= \textstyle\int_M f\ dV\ \ by\ definittion\ . \end{aligned}\right\} \tag{1-48}$$

We note also that this definition is independent of the choice of the partition of unity. Let $\psi_1, \dots\dots, \psi_m$ be another choice for the partition of unity.Because the support of $\psi_j\, f$ lies in a single coordinate patch, we can apply the computation just given (replacing f by $\psi_j\, f$) to conclude that

$$\sum_{i=1}^{\ell}\left[\int_M (\emptyset_i \psi_j\, f)\, dV\right] = \int_M (\psi_j\, f) dV.$$

(1 - 49)

Summing over j, we have

$$\sum_{j=1}^{m}\sum_{i=1}^{\ell}\left[\int_M (\emptyset_i \psi_j\, f)\, dV\right] = \sum_{j=1}^{m}\left[\int_M (\psi_j\, f) dV.\right]$$

(1 - 50)

Symmetry shows that this double summation also equals

$$\sum_{i=1}^{\ell}\left[\int_M (\emptyset_i\, f)\, dV\right]$$

(1 - 51)

Theorem (1 -2 - 32):-Let M be acompact $k-manifold$ in R^n,of class C^r Let $f, g : M \to R$ be continuous. Then

$$\int_M (af + bg) dV = a \int_M f dV + b \int_M g\, dV. \qquad (1 - 52)$$

This definition of the integral $\int_M$ f dV is satisfactory for theoretical purposes, but not for practical purposes. If one wishes actually to integrate a function over the $n-1$ sphere S^{n-1}, for example, what one does is to break S^{n-1} into suitable "pieces," integrate over each piece separately, and add the results together. We now prove a theorem that makes this procedure more precise.

Definition(1 -2 - 33):- Let M be a compact k-manifold in R^n, of class C^r. A subset D of M is said to have measure zero in M if it can be covered by countably many coordinate patches $\alpha_i : U_i \to V_i$ such that the set

$$D_i = \alpha^{-1} (D \cap V_i) \qquad (1 - 53)$$

has measure zero in R^k for each i.

An equivalent definition is to require that for any coordinate patch $\alpha : U \to V$ on M, the set $\alpha^{-1} (D \cap V_i)$ have measure zero in R^k. To verify this fact, it suffices to show that $\alpha^{-1}(D \cap V_i)$has measure zero for each i. And this follows from the fact that the set $\alpha_i^{-1}(D \cap V \cap V_i)$has measure zero because it is a subset of D_i and that $\alpha^{-1} o \alpha_i$ is of class C^r .

Theorem (1 -1 - 34):-Let M be a compact k-manifold in R^n, of class C^r . Let $f : M \to R$ be a continuous function. Suppose that $\alpha_i : A_i \to M_i$, $for\ i = 1, \dots, N$, is a coordinate patch on M, such that A_i is open in R^k and M is the disjoint union of the open sets $M_1, \dots, M_N\ of\ M$ and $a$set k of measure zero in M. Then

$$\int_M f dV = \sum_{i=1}^{N} \left[\int_A (f o \alpha_i) V\ (D\alpha_i) \right]. \tag{1 - 54}$$

This theorem says that $\int_M \mathrm{f\, dV}$ can be evaluated by breaking M up into pieces that are parametrized-manifolds and integrating f over each piece separately.

Proof.

Since both sides of (1-54) are linear in f, it suffices to prove the theorem in the case where the set C = Support f is covered by a single coordinate patch $\alpha : U \to V$. We can assume that U is bounded. Then

$$\int_M f dV = \int_{Int\ U} (f o \alpha) V\ (D\alpha), \tag{1-54a}$$

by definition.

Step 1. Let $W_i = \alpha^{-1}(M_i \cap V)$ and let $L = \alpha^{-1}(K \cap V)$. Then W_i; is open in R^k , and L has measure zero in R^k ; and U is the disjoint union of L and the sets W_i . See Figures (1-13a) and (1-13b) We show first that

$$\int_M f dV = \sum_i \left[\int_{W_i} (f o \alpha) V\ (D\alpha) \right]. \tag{1 - 55}$$

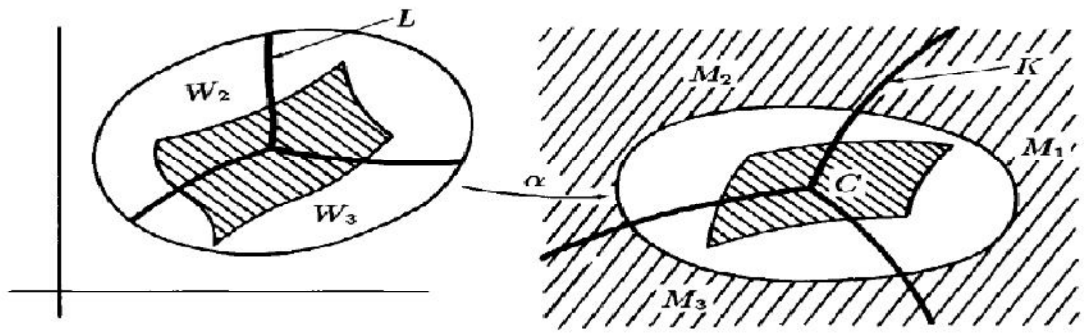
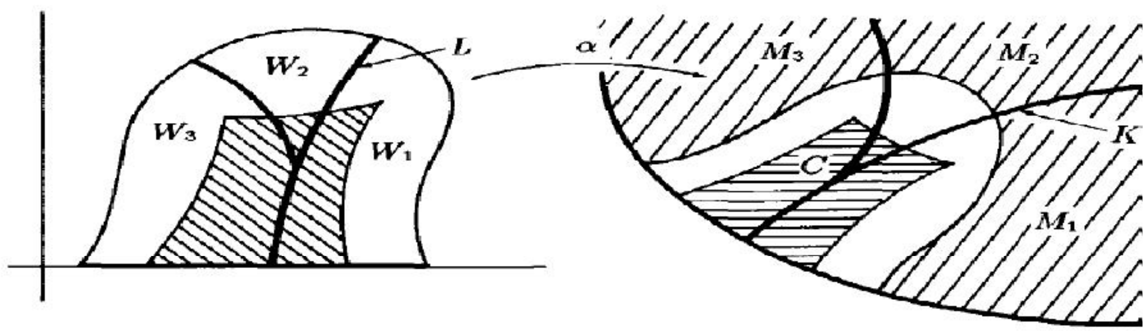


Figure 25.2



Figure(1-13)

Note that these integrals over W_i exist as ordinary integrals . For the function $F = (f \ o \ \alpha)V(D \propto)$ is bounded , and F vanishes near each point of $W_i \ Bd$ not in L. Then we note that

$$\sum_i \left[\int_{W_i} F \right] = \int_{(Int\ U\ )-L} F \quad by\ additivity, = \tag{1 - 56}$$

$$\int_{Int\ U} F \qquad since\ L\ has\ measure\ zero, \int_M F\ \ dV \qquad by\ additivity$$

Step 2. We complete the proof by showing that

$$\int_{W_i} F \ = \ \int_{A_i} F_i, \tag{1 - 57}$$

where $F_{i\,,} = (f \ o \ \alpha_i)V(D\alpha_i)$. See Figure (1-14)

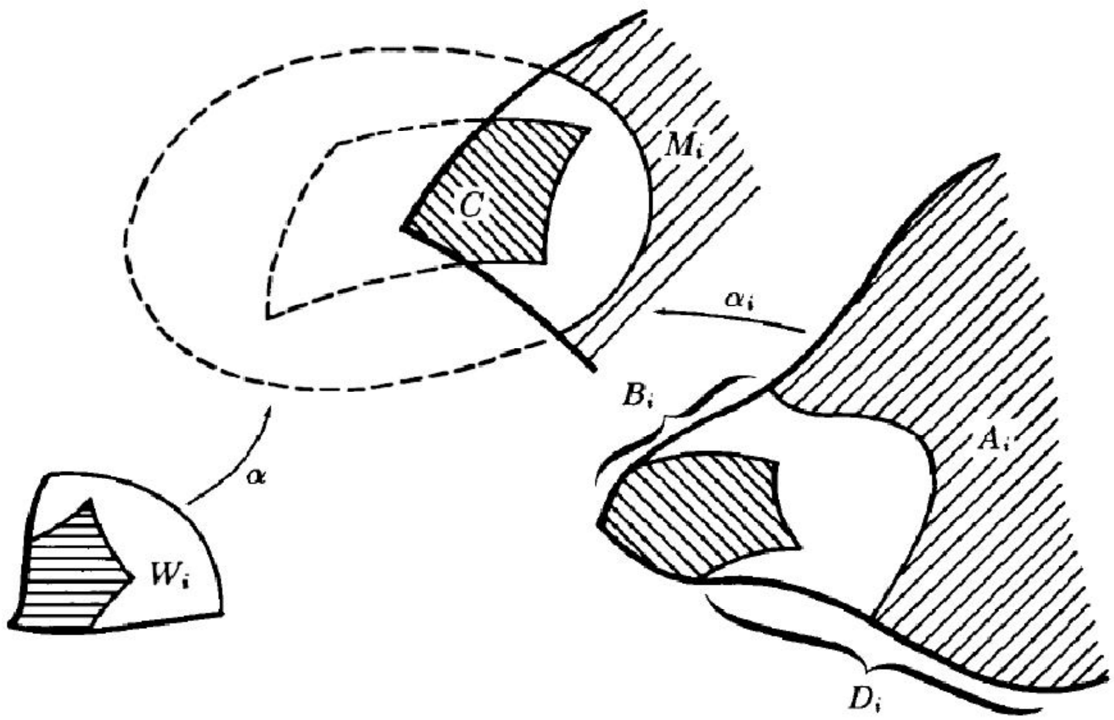


Figure (1-14)

The map $\alpha_i^{-1} o\ \alpha$ is a diffeomorphism carrying W_i onto the open set $B_i = \alpha_i^{-1}\ (M_i \cap V)$ (1 - 58)

of R^k . It follows from the change of variables theorem that

$\int_{W_i} F \ = \ \int_{B_i} F_i,$ (1 - 59)

just as in Theorem (1-1-10) To complete the proof, we show that

$\int_{B_i} F_i \ = \ \int_{A_i} F_i,$ (1 - 60)

These integrals may not be ordinary integrals, so some care is required .Since $C = Support\ f$ is closed in M, the set $\alpha_i^{-1}\ (C)$ is closed in A_i and its complement

$D_i \ = \ A_i \ - \ \alpha_i^{-1}\ (C)$ (1 - 61)

is open in A_i and thus in R^k . The function F_i vanishes on D_i . We apply additivity of the extended integral to conclude that

$\int_{A_i} F_i \ = \ \int_{B_i} F_i \ + \int_{D_i} F_i \ - \int_{B_i \cap D_i} F_i$ (1 - 62)

The last two integrals vanish.

Example (1 -2 - 35):- Consider the $2-$sphere $S^2(a)$ of radius a in R^3 . We computed the area of its open upper hemisphere as $2\pi a^2$ (See Example (1 -2-36) :-Since the reflection map $(x, y, z) \rightarrow (x, y, -z)$ is an isometry of R^3 , the open lower hemisphere constitute all of the sphere except for a set of measure zero in the sphere, it follows that S^2(a) has area $4\ \pi a^2$

Example (1 -2 - 37):- Here is an alternate method for computing the area of the 2-sphere; it involves no improper integrals.

Given $z_0 \in R$ with $|z_0| < a,$ the intersection of $S^2(a)$ with the plane $z = Z_0$ is the circle

$$z = z_0 ; \qquad x^2+y^2 = a^2 - (z_0)^2 \qquad (1 - 63)$$

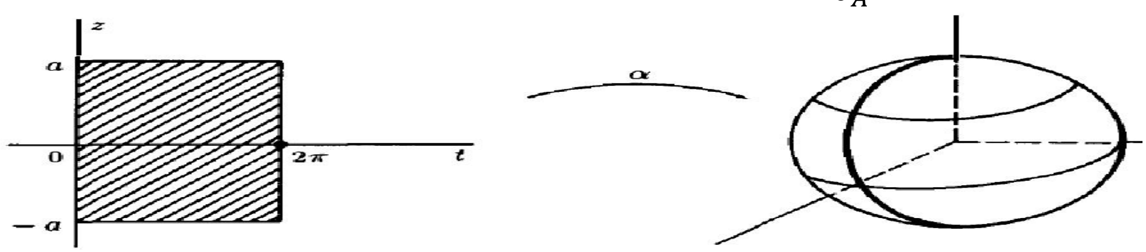
This fact suggests that we parametrize $S^2(a)$ by the function $\alpha : A \to R^3$ given by the equation

$$\alpha(t,z) = ((a^2 - z^2)^{1/2} \cos t, (a^2 - z^2)^{1/2} \sin t, z) \qquad (1 - 64)$$

where A is the set of all (t, z) for which $0 < t < 2\pi$ and $|z| < a$. It is easy to check that α is a coordinate patch that covers all of $S^2(a)$ except for agreat-circle arc, which h as measure zero in the sphere. See Figure (1-15) By the preceding theorem, we may use this coordinate patch to compute the area of $S^2(a)$. We have

$$D\alpha = \begin{bmatrix} -(a^2 - z^2)^{1/2} \sin t & (-z\cos t)/(a^2 - z^2)^{1/2} \\ (a^2 - z^2)^{1/2} \cos t & (-z\sin t)/(a^2 - z^2)^{1/2} \\ 0 & 1 \end{bmatrix} \qquad (1 - 65)$$

whence $V(D_\alpha) = a,$ as you can check. Then $v(S^2(a)) = \int_A a = 4\pi a^2$



Figure(1-15)

Chapter (2)

Tensors and wedge product

Section(2 - 1):- Multiliner Algebra

Definition :-(2- 1-1):- Let V be a vector space. Let $V^k = V \times \cdots \times V$ denote the set of all k-tuples $(v_1, \ldots, v_k)$ of vectors of V. A function $f : V^k \to R$ is said to be linear in the i^{th} variable if, given fixed vectors $v_j \; for \; j \neq i$, the function $T : V \to R$ defined by

$T(v) = f\,(v_1 \ldots, v_{i-1} \;, v, \ldots, v_{i+1} \ldots, v_k$ (2-1)

is linear. The function f is said to be multilinear if it is linear in the i^{th} variable for each i. Such a function f is also called a k-tensor, or atensor of order k, on V . We denote the set of all k-tensors on V by the symbol $\mathcal{L}^k(V)$. If $k = 1$, then $\mathcal{L}^1(V)$ is just the set of all linear transformations $f : V \to R$. It is sometimes called the dual space of V and denoted by V^* .

Theorem:-(2-1-2):- The set of all k-tensors on V constitutes a vector space if we define

$$(f + g)(v_1, \ldots, v_k) = f(v_1, \ldots, v_k) + g(v_1, \ldots, v_k),$$
$$(cf)(v_1, \ldots, v_k) = c(f(v_1, \ldots, v_k)). \qquad (2\text{-}2)$$

Lemma (2- 1 -3):- Let $a_1, \ldots, a_n$ *be a basis for* V. *If* f, g : $V^k \to R$ *are* $k-$ *tensors on* V, *and if*

$$f(a_{i_1}, \ldots, a_{ik}) = g(a_{i1}, \ldots, a_{ik}) \qquad (2\text{-}3)$$

for every $k-tupl \; I = (i_1, \ldots, i_k)$ *of integers from the set* $\{1, \ldots, n\}$, *then* $f = g$.

Proof

Given an arbitrary $k-$ *tuple* $(v_1, \ldots, v_k)$ *of vectors of* V, *let us* express each v_i in terms of the given basis, writing

$$v_i = \sum_{j=1}^{n} c_{ij} a_j \qquad (2\text{-}4)$$

Then we compute

$$f(v_1, \ldots, v_k) = \sum_{j_1=1}^{n} c_{1j_1} \, f(a_{j_1}, v_2 \ldots, v_k)$$

$$= \sum_{j_1}^{n} \sum_{j_2=1}^{n} c_{1j_1} c_{2j_2} f(a_{j_1} , a_{j_2}\ v_2, v_3 \dots , v_k) \tag{2-5}$$

and so on. Eventually we obtain the equation

$$f(v_1 , \dots . v_k) = \sum_{1\le j_1 .., j_k \le n}^{n} (c_{1j_1} c_{2j2} \dots . c_{kj_k} f(a_{j_1} ,, \dots . , a_{j_k}) \tag{2-6}$$

The same computation holds for g . It follows that f and g agree on all k-tuples of vectors if they agree on all k-tuples of basis elements. Just as a linear transformation from $V\ to\ W$ can be defined by specifying its values arbitrarily on k-tuples of basis elements .

Theorem (2 – 1 – 4) :- Let V be a vector space with basis $a_1, \dots, . a_n$ $Let\ I = (i_1, \dots, i_k)$ be a k-tuple of integers from the set $\{ 1 , \dots , n\}$. There is a unique $k - tensor$ $\emptyset_1 on\ V\ such\ that, for\ every\ k - tuple\ J\ (j_1 , \dots , j_k)$ $from\ the\ set \{ 1, \dots , n\}, = \emptyset_1 \left(a_{j1} , \dots . , a_{j_k} \right) = \begin{cases} 0 & if \quad I \neq J, \\ 1 & if \quad I = J, \end{cases}$ (2-7)

The tensors ϕ_1 are called the elementary k –tensors on V corresponding to the basis a_1 , . . . , a_n for V. Since they form a basis for . $\mathcal{L}^K$ (V) and since there are n^k distinct k-tuples from the set $\{ 1 , \dots , n\}$, the space . $\mathcal{L}^K$ (V) must have dimension n^k . When $k = 1$, the basis for $V*$ formed by the elementary tensors $\emptyset_1 , \dots , \emptyset_n$ is called the basis for V^* dual to the given basis for V.

Proof

First, consider the case $k = 1$. We know that we can determine a linear transformation $\emptyset_i : V \rightarrow R$ by specifying its values arbitrarily on basis elements. So we can define $\emptyset_i$ by the equation

$$(*) \qquad \phi_i(a_j) = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases} \tag{2-8}$$

These then are the desired 1-tensors. In the case $k > 1$, we define $\emptyset_I$ by the equation

$$\emptyset_I (v_I , \dots . , v_k) = [\emptyset_{i_1}(v_1)]. [\emptyset_{i_2}(v_2)] \dots . [\emptyset_{i_k}(v_k)] \tag{2-9}$$

It follows, from the facts that (1) each $\emptyset_i$ is linear and (2) multiplication is distributive, that $\emptyset_1$ is multilinear. One checks readily that it has the required value on(a_{j_1} , · · · , a_{j_k}).

We show that the tensors $\emptyset_1$ form a basis for $\mathcal{L}^K$(V). Given a k-tensor $f\ on\ V$, we show that it can be written uniquely as a linear combination of the tensors $\emptyset_I$. For each k-tuple $I = (i_1 , \dots , i_k)$, let d_1 be the scalar defined by the equation

$$d_1 = f(a_{i_1},, \dots , a_{i_k}) . \tag{2-10}$$

Then consider the k –tensor $g = \sum_j d_j \emptyset_j,$ (2-11)

where the summation extends over all k-tuples J of integers from the set{1,...,n}.The value of g on the $k - tuple$ $\left(a_{i_1}, \dots , a_{\ i_k}\right)$

equals d_1 , by (2-7),and the value of f on this k-tuple equals the same thing by definition. Then the preceding lemma implies that
$f = g$. Uniqueness of this representation of f follows from the preceding lemma.

Example (2-1-5) :- Consider the case $V = R^n. Let\ e_1 ,\dots, e_n$ be the usual basis for R^n; let $\emptyset_1 \dots, \emptyset_n$ be the dual basis for $\mathcal{L}^1(V)$ Then if x has components $x_1, \dots, x_n$ we have

$$\emptyset_i(x) = \emptyset(x_1\, e_1 + .. + \cdots + x_n\, e_n) = x_i \qquad (2\text{-}12)$$

Thus $\emptyset_i : R^n \to R$ equals projection onto the i^{th} coordinate.More generally, given $I = (i_1, \dots, i_k)$ the elementary tensor $\emptyset_I$ satisfies the equation

$$\emptyset_I\,(x_1 ,\dots, x_k) = \emptyset_{i_1}(x_1) \dots \emptyset_{i_k}(x_k). \qquad (2\text{-}13)$$

Let us write $X = [x_1 \cdots x_k]$, and let x_{ij} denote the entry of x in row i and column j. Then x_j is the vector having components $x_{1j} ,\dots, x_{nj}$ · In this notation,

$$\emptyset_I\,(x_1 , .. x_k) = x_{i_1 1}\, x_{i_2 2} \dots . x_{i_k k} \qquad (2\text{-}14)$$

Thus $\emptyset_I$ is just a monomial in the components of the vectors $x_1 , \dots , x_k$.; and the general k-tensor on R^n is a linear combination of such monomials.

It follows that the general 1-tensor on R^n is a function of the form

$$f(x) = d_1\, x_1 + \cdots + d_n x_n , \qquad (2\text{-}15)$$

for some scalars d_i . The general 2-tensor on R^n has the form

$$g(x,y) = \sum_{i,j=1}^{n} d_{ij} x_i y_j , \qquad (2\text{-}16)$$

Now we will discuss and introduce a product operation into the set of all tensors on V.The product of a k-tensor and an ℓ -tensor will be a k + ℓ tensor

Definition(2-1-6):- *Let* f be a k −tensor on V and let g be an ℓ-tensor on V.We define a$k + \ell$ *tensor* $f \otimes g$ *on* V by the equation

$$(f \otimes g)(v_1 ,\dots, v_{k+\ell}) = f(v_1 ,.., v_k).\, g(v_{k+1} ,\dots, v_{k+\ell}) \qquad (2\text{-}17)$$

It is easy to check that the function $f \otimes g$ is multilinear; it is called the tensor product of f and g .

Theorem(2-1-7) :-

Let J, g, h be tensors on V. Then the following properties hold:

(1) (*Associativity*). $f \otimes (g \otimes h) = (f \otimes g) \otimes h.$

(2) (*Homogeneity*). $(cf) \otimes g = c(f \otimes g) = f \otimes (cg)$.

(3) *Distributivity). Suppose f and g have the same order. Then*

$$(f + g) \otimes h = f \otimes h + g \otimes h,$$
$$h \otimes (f + g) = h \otimes f + h \otimes g.$$

(4) *Given a basis* $a_1 ,\dots, a_n$ *for V, the corresponding elementary*

$tensors\ \emptyset_I\ satisfy\ the\ equation$

$$\phi_1 = \phi_{i_1} \otimes \phi_{i_2} \ldots \otimes \phi_{i_k} \qquad where\ I\ =\ (i_1, \ldots, i_k) \qquad (2\text{-}18)$$

Proof

The proofs are straightforward. Associativity is proved, for instance , by noting that $if\ f, g, h\ have\ orders\ k, \ell, m, respectively$)

$(f\ \otimes\ (g\ \otimes\ h))\ (v_1, \ldots, V_{k+\mathcal{L}+m})$

$=f(v_i, \ldots,\ v_k) \cdot\ g(v_{k+I}, \ldots, v_{k+\mathcal{L}})h(v_{k+\mathcal{L}+I} \ldots, v_{k+\ell+m})$ (2-19)

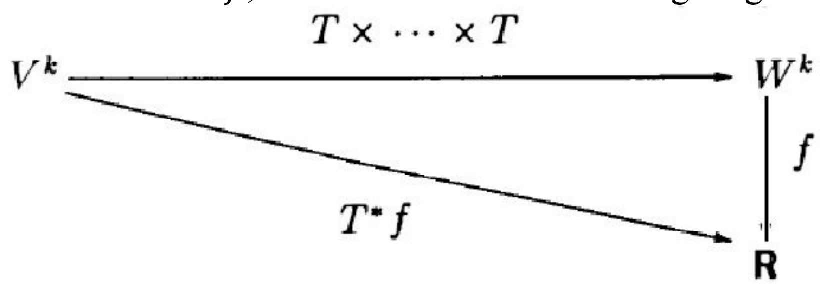
The value of $(f\ \otimes\ g)\ \otimes\ h$ on the given tuple is the same.

Now we will discuss to examine how tensors behave with respect to linear transformation of the underlying vector spaces.

Definition.(2-1-8) Let $T : V \rightarrow W$ be a linear transformation. We define the dual transfor-Mation $T^* : \mathcal{L}^k\ (W)\mathcal{L}^k\ (V)$, which goes in the opposite direction) as follows: If f is in .$\mathcal{L}^k(W)$, and if $v_1, \ldots, v_k$ are vectors in V, then

$$(T^*\ f)(\ v_I, \ldots, v_k)\ =\ f(T(v_I), \ldots, T(v_k\ ))\ . \qquad (2\text{-}20)$$

The transformation $T^*$is the composite of the transformation $T\ x\ \cdots\ x\ T$and the transformation f, as indicated in the following diagram:



figu(2-1)

It is immediate from the definition that $T^*\ f$ is multilinear, since T is linear and f is multilinear.

Theorem (2-1-9) :-

$Let\ T : V \rightarrow Wbe$ a linear transformation; let $T^* : \mathcal{L}^k\ (W) \rightarrow \mathcal{L}^k\ (V)$ be the dual transformation. Then:

(1) $T^*\ is\ linear.$

(2) $T^*(f \otimes g)\ =\ T^*\ f \otimes T^*g.$

(3) $If\ S : W \rightarrow X$ is a linear transformation, then $(S\ o\ T)^*\ f\ = T^*(S^*\ f)$.

Proof

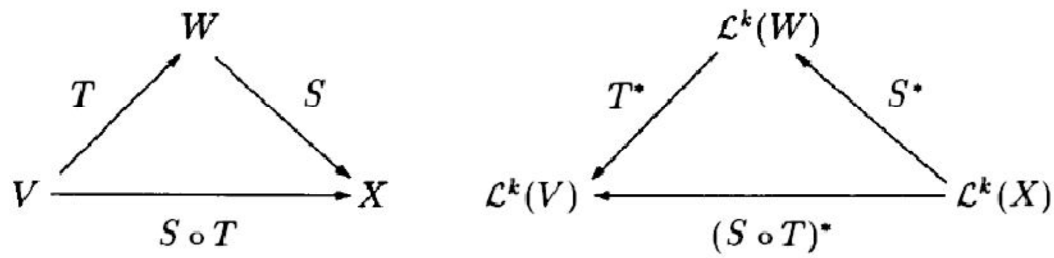
$(T^*\ (af +\ bg))\ (v_1, ..\ , v_k)\ =\ (af +\ bg)\ (T(v_1\ ), \ldots, T(v_k))$

$= a\ f\ (T(v_1), \ldots, T(v_K))\ +\ bg(T(v_1), \ldots, T(v_k))$

$$=\ a\ T^*\ f\ (\ v_1, \ldots, v_k\ )\ +\ bT^*g(\ v_1, \ldots, v_k) \qquad (2\text{-}21)$$

Whence $T^*(af\ +\ bg)\ =\ a\ T^*\ f\ +\ b\ T^*g.$

The following diagrams illustrate property (3)



Figure(2-2)

Section(2-2) :- Alternating tensors

In the following we will introduce the particular kind of tensors with which we shall be concerned-the alternating tensors-and derive some of their properties.In order to do this, we need some basic facts about permutations.

Definition(2-2-1) :- $Let\, k \geq 2$. a permutation of the set of integers $\{1, \ldots, k\}$is a one-to-one function a mapping this set onto itself. We denote the set of all such permutations by S_k . $If\ \sigma$ and T are elements of S_k , so are $\sigma\ o\ T$ and a^{-1} The set S_k thus forms a group , called the symmetric group permutation on the set $\{1, \ldots, k\}$. There are $k!$ elements in this group.

Definition(2-2-2):- Given $1 \leq i < k, let\ e_i$; be the element of S_k defined by $setting\ e_i(j) = j \quad for \quad j \neq i + 1$ and

$$e_i\,(i) = i + 1 \ \ and \ \ \mathrm{e_i(i + 1) = i}$$

We call e_i an elementary permutation. Note that $e_i\, oe_i$ equals the identity permutation, so that e_i is its own inverse.

Lemma (2-2-3):- $If\ \sigma \in S_k$, then σ equals a composite of elementary permutations. $Proof$

$Given\, 0 \leq i \leq k,$ we say that $\sigma\, fixes$ the first i integers if $\sigma(j) = j\, for\, 1 \leq j \leq i. If\ i = 0,$ then σ need not fix any integers at all. $If\ i = k,$ $then\ a\ fixes\ all\ the\ integers\ 1, \ldots, k,\ so\ that\ \sigma$ is the identity permutation . In this case the theorem holds , since the identity permutation $equals\ ei\ o\ ei\ for\ any\ j$.

We show that $if\ \sigma$ fixes the first $i - 1$ integers then σ can be written as the composite $\sigma = \pi\ o\ \sigma'$, where π is a composite of elementary permutations and σ' fixes the first i integers . The theorem then follows by induction.

The proof is easy. Since σ fixes the integers $1, \ldots, i - 1$, and since σ is one-to-one, the value of σ on i must be a number different from $1, \ldots, i - 1$. $If\ \sigma\,(i) = i, then\ we\ set\ \sigma' = \sigma\ and\ \ \pi$ equlal to the identity permutation , and our result holds. $If\ \sigma(\,i) = \ell > i, we\ set$

$$\sigma' = e_{\,i} o \cdots o\ e_{\iota - 1}\ o\ \sigma. \qquad (2\text{-}22)$$

Then σ' fixes the integers $1, \ldots, i - 1$ $because\ \sigma$ fixes these integers and so do $e_i, \ldots, e_{\ell - 1}$. And σ' also fixes $i, since\ \ \sigma(\,i) = \ell$ and

$$e_i(\cdots(e_{l-1}(\ell))\cdots) = i. \qquad (2\text{-}23)$$

We can rewrite the equation defining σ' in the form

$$e_{\ell-1}\, o \cdots o\, e_i\, o\, \sigma' = \sigma \qquad (2\text{-}24)$$

Definition.(2-2-4) :- *Let* $\sigma \in S_k$. Consider the set of all pairs of integers i, j from the set $\{1, \ldots, k\}$ for which $i < j$ *and* $\sigma(i) > \sigma(j)$. Each such pair is called an inversion in σ We define the sign of σ to be the number -1 if the number of inversions in σ is odd , and to be the number +1 if the number of inversions in σ is even .We call σ an odd or an even permutation according as the sign of σ equals -1 or +1 , respectively. Denote the sign of σ by sgn σ.

Lemma (2-2-5) :- *Let* $\sigma, T \in S_k$.

(a) *If* σ equals a composite of m elementary permutations, then sgn $\sigma = (-1)^m$.

(b) $sgn(\sigma\, o\, \tau) = (sgn\, \sigma) \cdot (sgn\, \tau)$.

(c) sgn $\sigma^{-1} = sgn\, \sigma$.

(d) *If* $p \neq q$, *and if* $\mathcal{T}$ *is the permutation that exchanges* p *and* q *and leaves all other integers fixed, then* $sgn\, \mathcal{T} = -1$.

Proof

Step 1. *We show that for any* σ

$$sgn(\sigma\, o\, e_l) = -sgn\, \sigma \qquad (2\text{-}25)$$

Given σ let us write down the values of σ in order as follows:

(*) $(\sigma(1), \sigma(2), \ldots, \sigma(l), \sigma(\ell+1), \ldots, \sigma(k))$.

Let $\mathcal{T} = \sigma o e_l$; then the corresponding sequence for $\mathcal{T}$is the $k-tuple$ of numbers $(\mathcal{T}(1), \mathcal{T}(2), \ldots, \mathcal{T}(l), \mathcal{T}(\ell+1), \ldots, \mathcal{T}(k))$

$= (\sigma(1), \sigma(2), \ldots, \sigma(\ell+1), \sigma(l), \ldots, \sigma(k))$. (**) (2 -26)

The number of inversions in σ and $\mathcal{T}$, respectively, are the number of pairs of integers that appear in the sequences (*) and (**), respectively, in the reverse of their natural order. We compare inversions in these two sequences. Let $p \neq q$; we compare the positions of $\sigma(p)$ and $\sigma(q)$ in these two sequences. If neither p nor q equals ℓ *or* $\ell + 1$, then $\sigma(p)$ and $\sigma(p)$appear in the same slots in both sequences, so they constitute an inversion in one sequence if and only if they constitute an inversion in the other. Now consider the case whereone, say p, equals either l *or* $l+1$, and the other q is different from both l and $\ell + 1$. Then $\sigma(p)$appears in the same slot in both sequences, but $\sigma(p)$ appears in the two sequences in adjacent slots. Nevertheless, it is still true that $\sigma(p)$and $\sigma(q)$constitute an inversion in one sequence if and only if they constitute an inversion in the other. So far the number of inversions in the two sequences are the same. But now we note that *if* $\sigma(\ell)$ *and* $\sigma(\ell+1)$ form an inversion in the first sequence, they do not form an inversion in the second ; and conversely. Hence sequence (**) has either one more inversion , or one fewer inversion, than (*) .

Step 2.We prove the theorem.The identity permutation has sign +1 ,and composing it successively with m elementary permutation changes its sign m times, by Step 1. Thus (a) holds. To prove (b), we write σ as the composite of m elementary permutations,and τ as the composite of n elementary permutations. Then σo τ is the composite of $m + n$ elementary permutations; and (b) follows from the equation $(-1)^{m+n}= (-1)^m (-1)^n$.

To check (c), we note that since $\sigma^{-1} o\, \sigma$ equals the identity permutation , $(sgn\ \sigma^{-1})(sgn\, \sigma) = 1$.

To prove (d), one simply counts inversions. Suppose that $p < q$. We can write the values of τ in order as

$(1, \ldots, p-1, q, p+1, \ldots, p+\ell-1, p, p+\ell+1, \ldots, k)$,

where $q = p + \ell$. Each of the pairs$\{q, p+1\}, .., \{q, p+\ell-1\}$ constitutes an inversion in this sequence, and so does each of the pairs $\{p+1, p\} \ldots$,$\{p+\ell$ - 1,p$\}$. Finally, $\{q, p\}$ is an inversion as well. Thus τ has $2\ell - 1$ inversions, so it is odd .

Definition(2-2-6):- $Let\, f$ be an arbitrary k-tensor on V. If $\sigma$is a permutation of $\{1, \ldots k\}$, we define f^{σ} by the equation

$$f^{\sigma}(v_1, \ldots, v_k) = f(v_{\sigma(1)} \ldots, v_{\sigma(k)}) \cdot \qquad (2\text{-}27)$$

Because f is linear in each of its variables, so is f^{σ} ; thus f^{σ} is a k −tensor on V. The tensor f is said to be symmetric if $f^e = f$ for each elementary permutation e, and it is said to be alternating if $f^e = -f$ for every elementary permutation e.Said differently, f is symmetric if

$$f(v_1, \ldots, v_{i+1}, v_i; \ldots, v_k) = f(v_1, \ldots, v_i, v_{i+1}; \ldots, v_k) \qquad (2\text{-}28)$$

for all i; and f is alternating if

$$f(v_1, \ldots, v_{i+1}, v_i; \ldots, v_k) = f(v_1, \ldots, v, v_{i+1}; \ldots, v_k) \qquad (2\text{-}29)$$

Definition.(2-2-7):- If V is a vector space, we denote the set of alternating k tensors on V by $A^k(V)$. It is easy to check that the sum of two alternating tensors is alternating, and that so is a scalar multiple of an alternating tensor. Then $A^k(V)$. is a linear subspace of the space .$\mathcal{L}^k(V)$ of all k −tensors on V. The condition that a 1-tensor be alternating is vacuous. Therefore we make the convention that $A^I(V) = \mathcal{L}^k(V)$.

EXAMPLE(2-2-8):- The elementary tensors of order $k > 1$ are not alternating, but certain linear combinations of them are alternating. For instance, the tensor $f = \emptyset_{i1j} - \emptyset_{j1i}$ is alternating, as you can check. Indeed, $if\ V = R^n$ and we use the usual basis for R^n and corresponding dual basis $\emptyset_{i,}$ the function f satisfies the equation

$$f(x, y) = xy_j - x_j y_i = det \begin{bmatrix} x_i & y_i \\ x_j & y_j \end{bmatrix} \qquad (2\text{-}30)$$

Here it is obvious that $f(y, x) = -f(x, y)$. Similarly, the function

$$g(x,y,z) = det\begin{bmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{bmatrix} \quad (2\text{-}31)$$

is an alternating 3-tensor on R^n; one can also write g in the form

$$g = \emptyset_{i,j,k} + \emptyset_{j,k,i} + \emptyset_{k,i,j} - \emptyset_{j,i,k} - \emptyset_{i,k,j} - \emptyset_{k,j,i} \quad (2\text{-}32)$$

In the following we now study the space $A^k(V)$; in particular, we find a basis for it

Lemma (2-2-9):- $Let\ f\ be\ a\ k-tensor\ on\ V;\ let\ \sigma,\tau\ \in\ S_k$.

(a) The transformation f $\rightarrow f^{\sigma}$ is a linear transformation of $.\mathcal{L}^k(V)$to $.\mathcal{L}^k(V)$. $It\ has\ the\ property\ that\ for\ all\ \sigma,\tau,$

$$(f^{\sigma})^{\tau} = f^{\tau O \sigma}.$$

(b) $The\ tensor\ f\ is\ alternating\ if\ and\ only\ if\ f^{\sigma}$ $=gn\ \sigma)f\ for$ all σ . $If\ f$ is alternating and $if\ V_p = V_q\ with\ p \neq q$, then $f(V_1,\ldots,V_k) = 0$.

$Proof.$

(a) The linearity property is straightforward; it states simply that

(b) $(af + bg)^{\sigma} = af^{\sigma} + bg^{\sigma}$. To complete the proof of (a), we compute

$$(f^{\sigma})^{\tau}(v_1,\ldots v_k) = f^{\sigma}\big(v\big)_{\tau(1)},\ldots v_{\tau(k)}\big) = f\big(W_{\sigma(1)},\ldots,W_{\sigma(k)}\big)$$
$$= f\left(V_{\tau(\sigma(1))},\ldots,V_{\tau(\sigma(k))}\right)$$
$$= f^{\tau\sigma\sigma}(v_1,\ldots,v_k) \quad (2\text{-}33)$$

(b) Given an arbitrary permutation σ, let us write it as the composite

$$\sigma = (\sigma_1\ o\ \sigma_2 \ldots .\ o\sigma_m) \quad (2\text{-}34)$$

Where each σ_i is an elementary permutation. Then

$f^{\sigma} = f^{\sigma_1 o \ldots o \sigma_m}$.

$=((\ldots(f^{\sigma m})\ldots)^{\sigma_2})^{\sigma_1}\ by\ (a)$, $\quad = (-1)^m f\ because\ f\ is\ alternating$,

$=(sgn\sigma)f$. (2- 35)

Now suppose $V_p = V_q\ for\ p \neq q. Let\ \tau\ be$ the permutation that exchanges p and q . Since $V_p = V_q$

$$f^{\tau}(v_1,\ldots,v_k) = f(v_1,\ldots,v_k). \quad (2\text{-}36)$$

On the other hand,

$$f^{\tau}(v_1,\ldots,v_k) = -f(v_1,\ldots,V_k). \quad (2\text{-}37)$$

Since sgn $\tau = -1$. It follows that $f(v_1,\ldots,v_k). = 0$

Lemma:- (2-2-10) :- Let $a_1,\ldots,a_n$be a basis for V. If f, g areternating$k - tensors\ on\ V, and\ if\ \ f(a_{i1}\ldots,a_{ik}.) = g(a_{i1}\ldots,a_{ik}.)$ $for\ every\ ascending\ k-tuple\ of\ integers\ I=$ $(i_1.,\ldots,i_k)$ $from\ the\ set\ \{1,\ldots,n\}, then\ f = g$.

Proof.

In view of Lemma (2-1-3), it suffices to prove that f and g have the same values on an arbitrary $k - tuple\ (a_{i1},\ldots,a_{jk}.)$ of basis elements. Let $J = (j_1.,\ldots,j_k)$

If two of the indices, say, j_p and j_q , are the same , then the values of f and g on this tuple are zero, by the preceding lemma. If all the indices are distinct, $let\ \sigma$ be the permutation of $\{1,\ldots,k\}$ such that the $k-tuple\ I = (j_{\sigma(1)},\ldots,j_{\sigma(k)})$ is ascending. Then

$f\ (a_{i1},\ldots,a_{ik}) = f^{\sigma}\ (a_{j1},\ldots a_{jk})$by definition of f^{σ} ,

$$f^{\sigma} = (sgn\ \sigma) f(a_{j1},\ldots a_{jk}) \quad (2\text{-}38)$$

because f is alternating. A similar equation holds for g. Since f and g agree on the $k-tuple\ (a_{j1},\ldots a_{jk})$they agree on the $k-tuple\ ((a_{j1},\ldots a_{jk})$

Theorem (2,2,11) :-

$Let\ V\ be\ a\ vector\ space\ with\ basis\ a_1,\ldots,a_n\ .Let\ I = (i_1,\ldots,i_k)$be an ascending $k-tuple$ from the set $\{1,\ldots,n\}.There$ is a unique alternating $k-tensor\ \varphi_1\ on\ V$ such that for every ascending k-tuple

$J = (j_1,\ldots,j_k)$ from the set $\{1,\ldots,n\}$,

$$\psi_I\ (a_{j1},\ldots,a_{jk}) = \begin{cases} 0 & if \quad I \neq J, \\ 1 & if \quad I = J, \end{cases} \quad (2-39)$$

The tensors$\psi_I\ form\ a\ basis\ for\ A^k\ (V)$.The tensor ψ_I in fact satisfies the formula $\psi_I = \sum_{\sigma}(sgn\sigma)(\emptyset_1)^{\sigma}$, (2- 40)

where the summation extends over all $\sigma \in S_k$.

The tensors $\psi_I$are called the elementary alternating k-tensors on V corresponding to the basis $a_1,\ldots,a_n for\ V$.

Proof.

$(\psi_I)^{\tau} = \sum_{\sigma}(sgn\ \sigma)((\emptyset_1)^{\sigma})^{\tau}$ by linearity, $=\sum_{\sigma}(sgn\ \sigma)((\emptyset_1)^{\tau o \sigma}$

$= (sgn\ \tau)\sum_{\sigma}(sgn(\tau o \sigma))(\emptyset_1)^{\tau o \sigma}$

$=(sgn\ \tau)\ \psi_I$ (2- 41)

the last equation follows from the fact that $\tau\ o\ \sigma$ ranges over S_k as σ does . We show $\varphi_I$has the desired values. Given J, we have

$$\psi_I(a_{j1},\ldots,a_{jk}) = \sum_{\sigma}\ (sgn\ \sigma)\emptyset_I(a_{j\sigma(1)}\ldots,a_{j\sigma(k)}). \quad (2\text{-}42)$$

Now at most one term of this summation can be non-zero, namely the term corresponding to the permutation σ for which $I = (j_{\sigma(1)}\ldots,j_{\sigma(k)})$ Since both I and J are ascending, this occurs only if $I = J$ and σ is the identity permutation , in which case the value is 1. If $I \neq J$, then all terms vanish . Now we show the ψ form a basis for $A^k(V).Let\ f$ be an alternating $k-tensor$on V. We show that f can be written uniquely as a linear combination of the tensors φ_I .

Given f , for each ascending $k-tuple\ I = (i_1.,\ldots,i_k)$ from the set $\{1,\ldots,n\}$, let d_I be the scalar

$$d_I = f(a_{i1},\ldots,a_{ik}). \quad (2\text{-}43)$$

Then consider the alternating $k-tensor$

$$g = \sum_{[J]} d_J \, \psi_J \, ,$$

(2-44)

Where the notation [J] indicates that the summation extends over all ascending $k - tuples$ from $\{1, \ldots, n\}$. If I is an ascending $k - tuple$, the the value of g on the $k - tuple$ $(a_{i1}, \ldots, a_{ik})$ equals d_I ; and the value of f on this $k - tuple$ is the same. Hence $f = g$. Uniqueness of this representation of f follows from the preceding lemma.

This theorem shows that once a basis $a_1, \ldots, a_n$ $for\ V$ has been chosen , an arbitrary alternating $k - tensor\ f$ can be written uniquely in the form

$$f = \sum_{[J]} d_J \, \psi_J$$

(2-45)

The numbers d_J are called components of f relative to the basis $\{\psi_J\}$. What is the dimension of the vector space $A^k(V)$? $If\ k = 1$, then $A^1(V)$ has dimension n, of course. In general, given $k > 1$ and given any subset of $\{1, \ldots, n\}$ having k elements, there is exactly one corresponding ascending $k - tuple$, and hence one corresponding elementary alternating $k - tensor$. Thus the number of basis elements for $A^k(V)$ equals the number of combinations of n objects , taken k at a time. This number is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (2\text{-}46)$$

Theorem (2-2-12) :- $Let\ T : V \longrightarrow W$ be a linear transforMation $.If\ f\ is\ an\ alternating\ tensor\ on\ W$, then $T^* f$ is an alternating $tensor\ on\ V$.

Definition(2-2-13):- Let $e_1, \ldots, e_n$ be the usual basis for R^n; let $\emptyset_1, \ldots, \emptyset_n$ denote the dual basis for $.\mathcal{L}^1(R^n)$The space $A^n R^n$ of alternating $n - tensors$ on R^n has dimension 1 ; the unique elementary alternating $n - tensor$ on R^n is the tensor $\psi_{1,\ldots,n}$ If $X = [x_1 \ldots . x_n]$ is an n by n matrix, we define the determinant of X by the equation

$$det\, X = \psi_{1,\ldots,n}(x_1 \ldots . x_n) \, . \qquad (2\text{-}47)$$

let us for the moment let g denote the fun ction

$$g(X) = \psi_1 \, (x_1 \ldots . x_n) \, , \qquad (2\text{-}48)$$

Where $I = (1, \ldots, n)$. The function g is multilinear and alternating as a function of the columns of X, because φ_1 is an alternating tensor. Therefore the function f defined by the equation $f(A) = g(A^{tr})$ is multilinear and alternating as a function of the rows of the matrix A. Furthermore,

$$f\ (I_n) = g\ (I_n) = \psi_1\ (e_1 , .. e_n\ ) = 1 \qquad (2\text{-}49)$$

Hence the function f satisfies the axioms for the determinant function. In particular, $f(A) = f(A^{tr})$. Then $f(A) = f(A^{tr}) = g(A^{tr})^{tr}. = g(A)$, so that g also satisfies the axioms for the determinant function , as desired

The formula for ψ_1 given in Theorem (2-2-11) gives rise to a formula for the determinant function. If $I = (1, \ldots, n)$, we have $det\, X = \sum_\sigma (sgn\ \sigma) \emptyset_1\ (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

$$= \sum_\sigma (sgn\ \sigma) x_{1,\sigma} . x_{2,\sigma(2)} \ldots x_{n,\sigma(n)}$$

(2-50)

as you can check. This formula is sometimes used as the definition of the determinant function.

We can now obtain a formula for expressing ψ_1 directly as a function of $k - tuples$ of vectors of R^n. It is the following:

Theorem (2-2-14):- Let $\psi_1$be an elementary alternating tensor on R^n corresponding to the usual basis for R^n, $where\ I = (i_1, \ldots, i_k) \cdot$ Given $vectors\ x_1, \ldots, x_k\ of\ R^n,\ \ let\ X\ be\ the\ matrix$

$X = [x_1 \ldots x_k] \cdot Then \qquad \psi_1(x_1, \ldots, x_k) = det\ X_1,$

where X_1 denotes the matrix whose successive rows are rows $i_1, \ldots, i_k\ of\ X.$

Proof.

We compute $\psi_1(x_1, \ldots, x_k) = \sum_\sigma (sgn\ \sigma) \emptyset_1\ (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$

$$=\sum_\sigma (sgn\ \sigma)\ (x_{i1,\sigma(1)}, \ldots, x_{i2,\sigma(2)} \ldots x_{ik,\sigma(k)} \qquad (2 - 51)$$

This is just the formula for det X_1

EXAMPLE (2-2-15):- Consider the space $A^3(R^4)$. The elementary alternating 3- tensors on $\mathbf{R}^4$, corresponding to the usual basis for R^4 , are the functions

$$\psi_{i,j,k}\ (x, y , z) = \det \begin{bmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{bmatrix} \qquad (2\text{-}52)$$

Where($i , j, k)\ equals$ (1 ,2,3) or (1 ,2,4) or (1 ,3,4) or (2,3,4). The general elemen t of $A^3(R^4)$ is a linear combination of these four functions.

section (2-3):- THE WEDGE PRODUCT

In the following we seek to define a product operation in the set of lternating tensors .

Theorem (2-3-1):-

Let V be a vector space. There is an operation that assigns, to each $f \in A^k(V)$ and each $g \in A^l(V)$, an element $f \wedge g \in A^{k+l} + l(V)$, such that the following properties hold:

(1) $\{Associativity). f \wedge (g \wedge h) = (f \wedge g) \wedge h$.

(2) $\{Homogeneity). (cf) \wedge g = c(f \wedge g) = f \wedge (cg)$.

(3) $(Distributivity). If\ f\ and\ g\ have\ the\ same\ order,$

$$(f + g) \wedge h = f \wedge h + g \wedge h,$$

$$h \wedge (f + g) = h \wedge f + h \wedge g.$$

$(Anticommutativity). If\ f\ and\ g\ have\ orders\ k\ and\ lrespectively, then$

$$g \wedge f = (-1)^{kl} f \wedge g$$

(5) Given a basis $a_1, \ldots, a_n\ for\ V, let\ \emptyset_i$ denote the dual basis for $V^*$and $let\ \psi_1$ denote the corresponding elementary alternating tensors. $If\ I = (i_1., \ldots, i_k)$ is an ascending k-tuple of integers$from\ the\ set\ \{1, \ldots, n\}, then$ $\psi_I = \emptyset_{i1} \cap \emptyset_{i2} \ldots \emptyset_{ik}$

These five properties characterize the product $\wedge$ uniquely for finitedimensional spaces V. Furthermore, it has the following additional property:

(6) $If\ T : V \rightarrow W$ is a linear transformation, and $if\ f$ and g are alternating tensors on W, then

$T^* (f \cap g) = T^* \mathrm{f} \cap \mathrm{T}^* \mathrm{g}$

The tensor $f \wedge g$ is called the wedge product of f and g. Note that property (4) implies that for an alternating tensor f of odd order, $f \wedge f = 0$.

Proof

Step 1. $Let\ F\ be\ a\ k - tensor\ on\ W$ (not necessarily alternating). For purposes of this proof, it is convenient to define a transformation $A : \mathcal{L}^k(V) \longrightarrow \mathcal{L}^k(V)$ by the formula

$$AF = \sum_{\sigma} (sgn\ \sigma) F^{\sigma}$$

(2-53)

where the summation extends over all $\sigma \in S_k$. (Sometimes a factor of $1/k!$ is included in this formula, but that is not necessary for our purposes.)note that in this notation, the definition of the elementary alternating tensors can be written as

$$\psi_I = A\emptyset_I \tag{2-54}$$

The transformation A has the following properties:

(i) $A\ is\ linear.$

(ii) $AF\ is\ an\ alternating\ tensor.$

(iii) $If\ F\ is\ already\ alternating, then\ AF = (k!)F.$

Let us check these properties. The fact that A is linear comes from the fact that the map $F \to F^{\sigma}$ is linear. The fact that $A F$ is alternating comes from the computation

$(AF)^{\tau} = \sum_{\sigma} (sgn\ \sigma)(F^{\sigma})^{\tau}$ $\quad by\ linearity,$

$= \sum_{\sigma}(sgn\ \sigma)F^{\tau o \sigma}$

$= (sgn\ \tau)\ \sum_{\tau}(sgn\ \tau o \sigma)F^{\tau o \sigma}$

$= (sgn\ \tau)AF.$ (2-55)

Step 2. If f is an alternating $k-tensor$ on V, and g is an alternating $\ell - tensor\ on\ V$, we define

$$f \wedge g = \frac{1}{k!\,l!} A\,(f \otimes g). \tag{2-56}$$

Then $f \cap g$ is an alternating tensor of order $k + \ell$. It is not entirely clear why the coefficient $1/k!\,\ell!$ appears in this formula.Some such coefficient is in fact necessary if the wedge product is to be associative. One way of motivating the particular choice of the coefficient $\frac{1}{k!l!}$ is the following: Let us rewrite the definition of $f \wedge g$ in the form

$$(f \wedge g)\,(V_1, \ldots, V_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma}(sgn\ \sigma)f(V_{\sigma(1)}, \ldots . V_{\sigma(k)}).\,g(\,V_{\sigma(k+1)}, \ldots . V_{\sigma(k+l)}) \tag{2-57}$$

Then let us consider a single term of the summation , say

$$(sgn\ \sigma)f(V_{\sigma(1)} \ldots, V_{\sigma(k)}) \cdot g(V_{\sigma(k+1)} \ldots, V_{\sigma(k+l)}) \cdot \tag{2-58}$$

A number of other terms of the summation can be obtained from this one by permuting the vectors $V_{\sigma(1)}, \ldots, V_{\sigma(k)}$ among themselves , and permuting the vectors $V_{\sigma(k+1)}, \ldots, V_{\sigma(k+l)}$ among themselves . Of course, the factor $(sgn\ \sigma)$ changes as we carry out these permutations, but because f and g are alternating, the values of f and g change by being multiplied by the same sign . Hence all these terms have precisely the same value. There are $k!\,l!$ such terms, so it is reasonable to divide the sum by this number to eliminate the effect of this redundancy.

Step 3. Associativity is the most difficult of the properties to verify, so we postpone it for the moment. To check homogeneity, we compute
$(cf)\wedge g = A((cf) \otimes g)/k!\,l!$
$= A(c(f \otimes g))/k!\,l!$ *by homogeneity of* $\otimes$,
$= cA(f \otimes g)/k!\,l!$ *by linearity of* A,
$$= c(f \wedge g). \qquad (2\text{-}59)$$
Asimilar computation verifies the other part of homogeneity. Distributivity follows similarly from distributivity of $\otimes$ and linearity of A.
Step 4. We verify anticommutativity. In fact, we prove something slightly more general: Let F and G be tensors of orders k *and* ℓ, respectively(not necessarily alternating). We show that
$$A(F \otimes G) = (-1)^{k\ell} A(G \otimes F). \qquad (2\text{-}60)$$
To begin , let π be the permutation *of* $(1,\ldots,k+\ell)$ such that
$$(\pi(1),\ldots\pi(k+\ell)) = (k+1, k+2, \ldots, k+\ell, 1,2,\ldots,k). \qquad (2\text{-}61)$$
Then sgn$\pi = (-1)^{k\ell}$. (Count inversions!) It is easy to see that $(G \circledR F)^{\pi} = F \otimes G$, *since*
$$(G \otimes F)^{\pi}(V_1,\ldots,V_{k+\ell}) = G(V_{k+1},\ldots,V_{k+\ell}) \cdot F(V_1,\ldots,V_k), \qquad (2\text{-}62)$$
$$(F \otimes G)(V_1,\ldots,V_{k+\ell}) = F(V_1,\ldots,V_k) \cdot G(V_{k+1},\ldots,V_{k+\ell}) \cdot \qquad (2\text{-}63)$$
We then compute
$$A(F \otimes G) = \sum_{\sigma}(sgn\,\sigma)(F \otimes G)^{\sigma}$$
$$= \sum_{\sigma}(sgn\,\sigma)((G \otimes F)^{\pi})^{\sigma}$$
$$= (sgn\,\pi)) = \sum_{\sigma}(sgn\,\sigma o \pi)(G \otimes F)^{\sigma o \pi}$$
$$= (sgn\,\pi))A(G \otimes F), \qquad (2\text{-}64)$$
since $\sigma o \pi$ runs over all elements of $S_{k+\ell}$ *as* σ *does* .
Step 5. Now we verify associativity. The proof requires several steps , of which the first is this

Let F and G be tensors (not necessarily alternating) of orders k and ℓ, respectively, such that $AF = 0$. *Then* $A(F \otimes G) = 0$ (2-65)
To prove that this result holds, let us consider one term of the expression for $A(F \otimes G)$, say the term
$$(sgn\,\sigma)F(V_{\sigma(1)},\ldots,V_{\sigma(k)}).G(V_{\sigma(k+1)},\ldots,V_{\sigma(k+\ell)})$$
Let us group together all the terms in the expression for $A(F \otimes G)$ that involve the same last factor as this one. These terms can be written in the form
$$(sgn\,\sigma)\left[\sum_{\tau}(sgn\,\tau\, F(V_{\sigma(\tau(1)},\ldots,V_{\sigma(\tau(k)})\right].G(V_{\sigma(k+1)},\ldots,V_{\sigma(k+l)})$$
where τ ranges over all permutations of $\{1,\ldots,k\}$. Now the expression in brackets is just
$AF(V_{\sigma(1)},\ldots,V_{\sigma(k)})$

which vanishes by hypothesis . Thus the terms in this group cancel one another.

The same argument applies to each group of terms that involve the same last factor. We conclude that $A(F \otimes G) = 0$.

Step 6 Let F be an arbitrary tensor and let h be an alternating tensor of order m . We show that

$$(AF) \wedge h = \frac{1}{m!} A(F \otimes h). \tag{2-66}$$

Let F have order k. Our desired equation can be written as

$$\frac{1}{k!\, m\,!} A\,((AF) \otimes h) = \frac{1}{m\,!} A(F \otimes h)\,. \tag{2-67}$$

Linearity of A and distributivity of $\otimes$ show this equation is equivalent to each of the equations

$$A\{(AF) \otimes h - (k!)F \otimes h\} = 0,$$
$$A\,\{\,[AF - (k!).F] \otimes h\} = 0. \tag{2-68}$$

In view of Step 5, this equation holds if we can show that

$$A\,[A\,F - (k!)F] = 0. \tag{2-69}$$

But this follows immediately from property (iii) of the transformation A, since AF is an alternating tensor of order k.

Step 7. Let f, g, h be alternating tensors of orders k, l, m respectively. We show that

$$(f \wedge g) \wedge h = \frac{1}{k!l!m!} A((f \otimes g) \otimes h)\,. \tag{2-70}$$

$Let\ F = f \otimes g, for\ convenience. We\ have$

$$f \wedge g = \frac{1}{k!l!} A\,F \tag{2-71}$$

by definition, so that

$$\left.\begin{aligned} (f \wedge g) \wedge h &= \frac{1}{k!l!} (AF) \wedge h \\ &\frac{1}{k!\,l!\,m!} A\,(F \otimes h)\ by\ \ Step\ 6, \\ &= \frac{1}{k!l!m!} A\,((f \otimes g) \otimes h)\,. \end{aligned}\right\} \tag{2-72}$$

Step 8. Finally, we verify associativity. Let f, g, h be as in Step 7.Then

$$\left.\begin{aligned} (k\,!\,l!\,m!)(f \wedge g) \wedge h &= A\big((f \otimes g) \otimes h\big) by\ Step\ 7, \\ &= A\,(f \otimes (g \otimes h)) by\ associativity\ of \otimes, \\ &= (-1)^{k(l+m)} A\,((g \otimes h) \otimes f)\ by\ Step\ 4, \\ &= (-1)^{k\,(l\,+\,m)}\,(l!\,m!\,k\,!)(g \wedge h) \wedge f\ by\ Step\ 7, \end{aligned}\right\}$$
$$= (k!\,l!\,m!)f \wedge (g \wedge h)\ by\ anticommutativity. \tag{2-73}$$

Step 9. We verify property (5). In fact , we prove something slightly more general. We show that for any collection $f ,\dots, f_k$ of $1-tensors,\ A(f_1\ \otimes\dots\otimes f_k) = f_1\ \wedge\ \dots\wedge f_k$. (2-74)

Property (5) is an immediate consequence, since

$$\psi_1 = A\phi_1 = A(\phi_{i1},\otimes \dots\otimes\ \phi_{ik}) \cdot \qquad (2\text{-}75)$$

Formula (*) is trivial for $k = 1$. Supposing it true for $k-1$, we prove it $for\ k. Set\ F = f_1 \otimes \dots\dots\otimes\ f_{k-1} \cdot Then$

$$A(F \otimes f_{\ k}) = (1!)(AF)\wedge f_k\ by\ Step\ 6 = (f \wedge \dots\wedge\ f_{k-1}) \wedge f_k \qquad (2\text{-}76\ )$$

by the induction hypothesis.

Step 10. We verify uniqueness ; indeed , we show how one can calculate wedge products , in the case of a finite-dimensional space V, using only properties (1)-(5). Let $\emptyset_i$ and ψ_1 be as in property (5). Given alternating tensors f and g, we can write them uniquely in terms of the elementary alternating tensors as

$$f = \sum_{[I]} d_1\psi_1\ \ and\ \ g = \sum_{[J]} c_J\psi_J$$

(Here I is an ascending $k-tuple$, and J is an ascending $\ell - tuple$, from the set$\{1 ,\dots, n\}$.) Distributivity and homogeneity imply that

$$f \wedge g\ = \sum_{[I]}\sum_{[J]} b_1\, c_J\, \psi_1 \wedge\ \psi_J. \qquad (2\text{-}77)$$

Therefore, to compute $f\ \wedge\ g$ we need only know how to compute wedge products of the form

$$\psi_I \wedge\ \psi_J = (\emptyset_{i1} \wedge\ \dots\dots\wedge \emptyset_{\ ik}) \wedge \left(\emptyset_{\ j1}\wedge\ \dots\ \wedge\emptyset_{jl}\right) \qquad (2\text{-}78)$$

For that, we use associativity and the simple rules

$$\emptyset_i \wedge\ \emptyset_j\ =\ \emptyset_j\wedge\emptyset_i\ \ and\ \ \emptyset_i \wedge\ \emptyset_i = 0 \qquad (2\text{-}79)$$

which follow from anticommu tativity. It follows that the product $\psi_I \wedge\ \psi_J$ equals zero if two indices are the same. Otherwise it equals $(sgn\ \pi)$ times theelementary alternating $k\ +\ \ell$ tensor $\psi_k$whose index is obtained by rearranging the indices in the $k\ +\ \ell$ tuple (I, J) in ascending order, where π is the permutation required to carry out this rearrangement.

Step 11. $Let\ T : V\ \rightarrow\ W$ be a linear transformation, and F be an arbitrary tensor on W (not necessarily alternating). It is easy to verify that

$T^*(F^\sigma\ )\ =\ (T^*\ F)^\sigma\ \ \ Sin\ ce\ T^*$ is linear, it then follows that

$T^*(AF)\ =\ A(T^*\ F)$ Now let f and g be alternating tensors on W of orders $k\ and\ l,$ respectively. We compute

$$T^*(f \wedge g)\ =\ \frac{1}{k!l!}\ T^*\ (A(f \otimes g)) = \frac{1}{k!l!}\ A(T^*(f\ \otimes\ g))$$

$$= \frac{1}{k!l!}\ A((T^*f) \otimes\ (T^*g))\ by\ Theorem$$

$$= (T^* f) \wedge (T^* g). \quad (2\text{- }80)$$

With this theorem, we complete our study of multilinear algebra.

Chapter(3):-
Application on tangent vectors and scalar fields :-

Section(3-1):- tangent vectors and differential forms and operators

In the following we will studied tensor algebra in R^n −tensor addition, alternating tensors, wedge products, and the like. Now we introduce the concept of a tensor field; more specifically, that of an alternating tensor field, which is called a "differential form." In the succeeding section, we shall introduce a certain operator on differential forms, called the "differential operator"d, which is the analogue of the operators grad, curl, and div.

Definition. (3-1-1):- Given $x \in R^n$, we define a tangent vector to R^n at x to be a pair $(x;\ v)$, where $v \in R^n$. The set of all tangent vectors to R^nat x forms a vector space if we define

$$(x;\ V) + (x\ ;\ w) = (x\ ;\ v + w),$$
$$c(x;\ v) = (x;\ cv)\ . \qquad (3\text{-}1)$$

It is called the tangent space to $R^n\ at\ x,$ and is denoted $T_x(R^n)$.

Although both x and v are elements of R^n in this definition, they play different roles. We think of x as a point of the metric space R^n and picture it as a "dot." We think of v as an element of the vector space R^n and picture it as an "arrow." We picture (x ; v) as an arrow with its initial point at x. The set T_x (R^n) is pictured as the set of all arrows with their initial points at x ; it is, of course, just the set x x R^n .

We do not attempt to form the sum $(x;\ v) + (y;\ w)\ if\ x \neq y$.

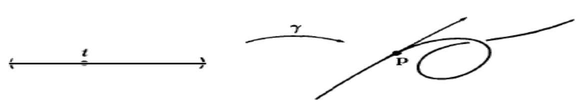
Definition (3-1-2) Let (a, b) be an open interval in R ; $let\ \gamma : (a, b) \longrightarrow R^n$ be a map of class $C.^r$ We define the velocity vector of γ , corresponding to the parameter value t, to be the $vector$ $(\gamma(t);\ D\gamma\ (t))$.

This vector is pictured as an arrow in R^n with its initial point at the point $p = \gamma(t)$. See Figure (3-1) . This notion of a velocity vector is of course a reformulation of a familiar notion from calculus. If

$$\gamma(t) = x(t)e_1 + y(t)e_2 + z(t)e_3 \qquad (3\text{-}2)$$

is a parametrized-curve in R^3, then the velocity vector of γ is defined in calculus as the vector

$$D\gamma\ (t) = \frac{dx}{dt}e_1 + \frac{dy}{dt}\ e_2 + \frac{dz}{dt}\ e_3 \qquad (3\text{-}3)$$



Figure(3-1)

Definition(3-1-3):- Let A be open in R^k or H^k ; let $a : A \to . R^n$ be of class C^r . Let $x \in A$, and let $p = \propto (x)$. We define a linear transformation

$$\alpha_* : T_x (R^k) \to T_p (R^n).$$

by the equation

$$\alpha_* (x; v) = (p ; D\alpha (x) \cdot v). \qquad (3\text{-} 4)$$

It is said to be the transformation induced by the differentiable map $\propto$

α Given $(x; v)$, the chain rule implies that the vector $\alpha_* (x; v)$ is in fact the velocity vector of the curve $\gamma(t) = \alpha(x + tv)$, corresponding to the parameter value $t = 0$. See Figure (3-2).

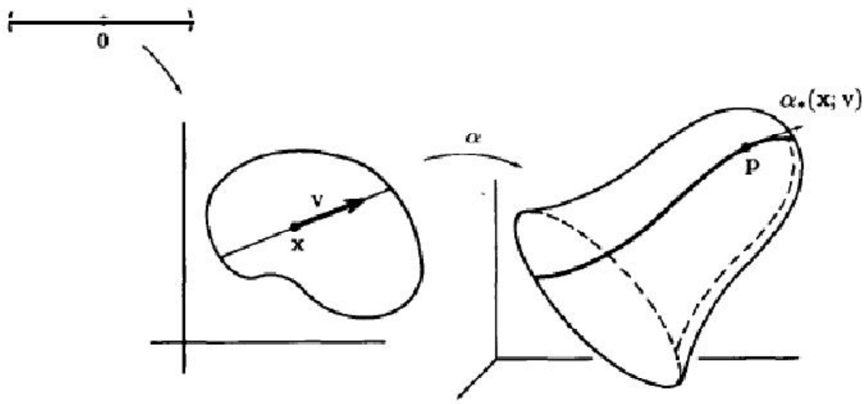


Figure (3-2).

Lemma (3-1-4) :- Let A be open in R^k or H^k ; let $\alpha \to R^m$ be of class C^r. Let B be an open set of R^m or H^m containing $\alpha(A)$; let $(J : B \to R^n$ be of class crC^r . Then $(\beta o\ \alpha)_* = \beta_* o\ \alpha_*$ (3-5)

$Proof$

This formula is just the chain rule. Let $y = \alpha(x)$ and let $z = \beta(y)$. We compute

$$(\beta\ o\ \alpha). (x; v) = (\beta(\alpha(x)) ; D(\beta\ o\ \alpha)(x) \cdot v)$$
$$= (\beta(y) ; D\beta(y) \cdot D\alpha(x) \cdot v)$$
$$= (\beta_* (\alpha_* (x; v)) . \qquad (3\text{-}6)$$

These maps and their induced transformations are indicated in the fowlloing

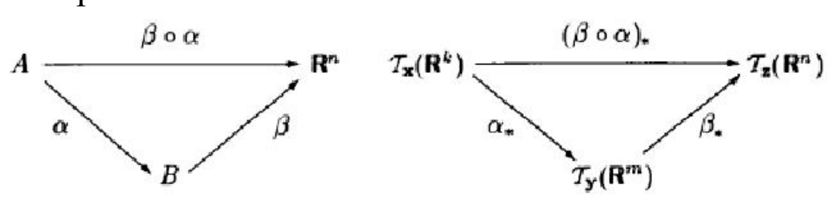


Figure (3-3).

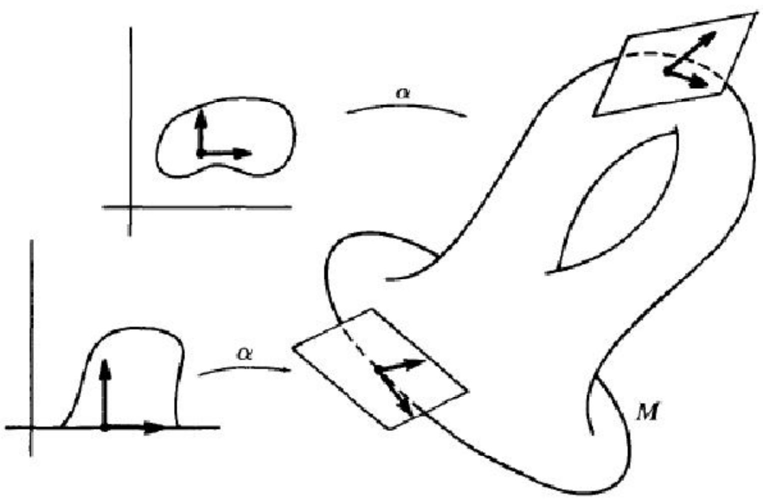
Definition(3-1-5):- If A is an open set in R^n, a tangent vector field in A is a continuous function $F : A \to R^n \times R^n$ such that $F(x) \in T_x (R^n)$, for each

$x \in A$. Then F has the form $F(x) = (x; f(x))$, where $f : A \to R^{n}$. If F is of class C^r,we say that it is a tangent vector field of class C^r .

Definition(3-1-6):- Let M be $a\ k - manifold$ of class C^r in R^{n}. If $p \in M$, choose a coordinate patch $\alpha : U \to V$ about p, where U is open in R^n or H^k. Let x be the point of U such that $\alpha(x) = p$. The set of all vectors of the form $\alpha_*(x; v)$, where v is a vector in R^k , is called the tangent space to M at p, and is denoted $T_p(M)$. Said differently,

$$T_p(M) = \alpha_* (T_x(R^k)) . \qquad (3\text{-}7)$$

It is not hard to show that $T_p(M)$ is a linear subspace of $T_p(R^n)$that is welldefined, independent of the choice of α. Because R^k is spanned by the vectors $e_1, \dots, e_k$: , the space $T_p(M)$ is spanned by the vectors $(p; D\alpha(x).e_j = \left(p ; \frac{\partial \alpha}{\partial x_j}\right)$, (3- 8)



(3-4)

for $j = 1, \dots, k$. Since $D\alpha$ has rank k, these vectors are independent; hence they form a basis for $T_p(M)$. Typical cases are pictured in Figure (3-4)

We denote the union of the tangent spaces $T_p(M), for\ p \in M, by\ T(M)$; and we call it the tangent bundle of $M. A$ tangent vector field to M is a continuous function $F : M \to T(M)$ such that $F(p) \in Tp(M)$ for each $p \in M$.

Now we will discuss tensor fields

Definition (3-1-7):- Let A be an open set in R^n . $A\ k - tensor$ field in A is afunction w ssigning, to each $x \in A$, a k-tensor defined on the vector space $T_x(R^n)$. That is, $\omega(x) \in \mathcal{L}^k(T_x(R^n))$

for each x. Thus $\omega(x)$ is a function mapping $k - tuples$ of tangent vectors to R^nat x into R ; as such, its value on a given $k - tuple$ can be written in the form

$$\omega(x)\ ((x; v_1), \dots \dots, (x; v_k))$$

We require this function to be continuous as a function of $(x, v_1, \dots, v_k)$ if it is of class C^r we say that w is a tensor field of class C^r. If it happens that $\omega(x)$ is an alternating $k-tensor$ for each x, then ω is called a differential form (or simply, a form) of order k, on A

More generally, if M is an m-manifold in R^n , then we define a $k-tensor$field on M to be a function w assigning to each $p \in M$ an element of $\mathcal{L}^k\,(T_p(M))$. If in fact $\omega(p)$ is alternating for each p, then w is called a differential form on M .

If ω is a tensor field defined on an open set of R^n containing M , then ω of course restricts to a tensor field defined on M, since every tangent vector to M is also a tangent vector to R^n . Conversely, any tensor field on M can be extended to a tensor field defined on an open set of R^n containing M; the proof, however, is decidedly non-trivial. For simplicity, we shall restrict ourselves in this book to tensor fields that are defined on open sets of R^n .

Definition. (3-1-8):- $Let\ e_1, \dots, e_n$be the usual basis for R^n . Then $(x;\ e_1, \dots, (x;\ e_n)$ is called the usual basis for $T_x\,(R^n)$. We define a $1-form\ \widetilde{\emptyset_\iota}\ on\ R^n$ by the equation

$$\widetilde{\emptyset_\iota}(x)(\,x; e_j\,) = \begin{cases} 0 & if \quad i \neq j, \\ 1 & if \quad i = j, \end{cases} \tag{3-9}$$

The forms $\widetilde{\emptyset_1}, \dots, \widetilde{\emptyset_n}$ are called the elementary $1-forms$ on R^n. Similarly, given an as-cending $k-tuple\ I\ =\ (i_1, \dots, i_k)$ from the set $\{1, \dots, n\}$, we define a k-form $\psi_{\,1}$by the equation

$$\widetilde{\psi_{\,1}}\,(x) = \widetilde{\phi_{\iota 1}}(x) \wedge \dots \wedge \tilde{\phi}_{ik}(x) \tag{3-10}$$

The forms $\psi_{\,1}$ are called the elementary $k-forms$ on R^n .

Note that for each x, the 1-tensors $\widetilde{\phi_1}(x), \dots, \widetilde{\phi_n}(x)$ constitute the basis for $\mathcal{L}^1\,(\,T_x\,(R^n\,))$ dual to the usual basis for $T_x\,(R^n)$, and the $k-tensor\ \widetilde{\psi_{\,1}}\,(x)$ is the corresponding elementary alternating tensor on $T_x\,(R^n)$.

The fact that $\widetilde{\emptyset_\iota}$ and ψ_1 are of class C^∞ follows at once from the equations

$$\widetilde{\emptyset_\iota}\,(x)(\,x\,; v) = v_{\,i},$$
$$\widetilde{\psi_1}\,(x)\big((x; v_1), \dots, (x; v_k)\big) = \det X_1\,, \tag{3- 11}$$

where X is the matrix $X\ =\ [v_1 \dots v_k]$.

If ω is a $k-form$ defined on an open set A of R^n , then the $k-tensor\ \omega(x)$ can be written uniquely in the form

$$\omega(x) = \sum_{[I]} b_I(\,x)\,\psi_I(x), \tag{3- 12}$$

for some scalar functions $b_I(\,x)$. These functions are called the components of w relative to the standard elementary forms in R^n

Lemma (3-1-9):- $Let\ \omega\ be\ a\ k-form$ on the open set A of R^n . Then ω is of class C^r if and only if its component functions b_I are of class $C^r\ on\ A$.

Proof

Given ω, let us express it in terms of elementary forms by the equation

$$\omega = \sum_{[I]} b_I(x)\,\tilde{\psi}_I \qquad (3\text{- }13)$$

The functions ψ_I are of class C^∞ . Therefore, if the functions b_I are of class C^r , so is the function ω. Conversely, if ω is of class C^r as a function of $(x, v1, \dots, v_k)$, then in particular, given an ascending $k-tuple$
$J = \{j_1, \dots, j_k)$ from the set $\{ 1, \dots, n \}$, the function

$$\omega(x)((x;\ e_{j1}), \dots, (x;\ e_{jk})) \qquad (3\text{- }14)$$

is of class C^r as a function of x. But this function equals $b_J(x)$.

In the following, we shall need to deal not only with tensor fields in R^n , but with scalar fields as well. It is convenient to treat scalar fields as differential forms of order 0.

Definition. (3-1-10):- If A is open in R^n, and if $f : A \to R$ is a map of class C^r then f is called a s calar field in A . We also call f a differential form of order 0.

The sum of two such functions in another such, and so is the product by a scalar. We define the wedge product of two 0-forms f and g by the rule $f \wedge g = f \cdot g$, which is just the usual product of real-valued functions. More generally, we define the wedge product of the 0-form f and the $k-form\ \omega$ by the rule

$$(\omega \wedge f)(x) = (f \wedge \omega)(x) = f(x) \cdot \omega(x)\,; \qquad (3\text{- }15)$$

this is just the usual product of the tensor $\omega(x)$ and the scalar $f(x)$

Note that all the formal algebraic properties of the wedge product hold. Associativity, homogeneity, and distributivity are immediate; and anticommutativity holds because scalar fields are forms of order 0:

$$f \wedge g = (-1)^0\, g \wedge f \ \ and\ \ f \wedge \omega = (-1)^0 \omega \wedge f \qquad (3\text{- }16)$$

section(3 -2) :- scalar field and the action of differentiable maps

Now we will discuss the introduce a certain operator d on differential forms. In general, the operator d, when applied to a $k-form$, gives a $k+1$ form. We begin by defining d for 0-forms.

Definition(3-2-1):- Let A be open in R^n ; let $f : A \to R$ be a function of class C^r . We define a 1-form df on A by the formula

$$df(x)(x;\ v) = f'(x;\ v) = D\,f(x) \cdot v.$$

The 1-form df is called the differential of f. It is of class C^{r-1} as a function of x and v.

Theorem (3-2-2):- The operator d is linear on 0-forms.

Proof.

Let $f, g : A \to R$ be of class C^r. Let $h = a\,f + bg$. Then

$$Dh(x) = a\,Df(x) + b\,Dg\,(x),$$

so that

$$dh(x)(x;\ v) = a\,df(x)(x;\ v) + b\,dg(\,x)(x;\ v).$$

$Thus\ dh = a(df) + b(dg)\,, as\ desired.$

Using the- operator d, we can obtain a new way of expressing the elemen- tary 1-forms $\widetilde{\emptyset_\iota}\ in\ R^n$:

Lemma (3-2-3):- $Let\ \widetilde{\emptyset_1}, \ldots, \widetilde{\emptyset_n}$ be the elementary1 -forms in R^n $Let\ \pi_i : R^n \to . R$ be the i[th] projection function, defined by the equation

$$\pi_i(x_1, \ldots, x_n) = x_i. \qquad (3\text{- }17)$$

Then $d_{\pi i} = \widetilde{\emptyset_\iota}$

Proof.

Since π_i is a C^∞ function, $d_{\pi i}$ is a 1-form of class C^∞ . We Compute

$d_{\pi i}\,(x)(x\,;\ v) = D_{\pi i}\,(x).v$

$$= [0 \ldots .0\ 1\ 0 \ldots 0]\begin{bmatrix} v_1 \\ \cdot\cdot \\ v_n \end{bmatrix} = v_i \qquad (3\text{- }18)$$

$$Thus\ d_{\pi i} = \widetilde{\emptyset_\iota}\ . \qquad (3\text{- }19)$$

Now it is common in this subject to abuse notation slightly, denoting the i^{th} projection functionnot by π_i but by x_i . Then in this notation, $\emptyset_i$ is equal to dx_i. We shall use this notation henceforth:

Convention. $If\ x$ denotes the general point of R^n , we denote the i^{th} projection function mapping R^n to R by the symbol x_i . Then dx_i. equals the elementary $1-form\ \widetilde{\emptyset_\iota} in\ R^n\ .If\ I = (i_1, \ldots, i_k)$ is an ascending $k-\ tuple$ from the set $\{\,1, \ldots, n\}$, then we introduce the notation $dx_I = dx_{i1} \wedge \ldots\ldots \wedge dx_{ik}$ (3-20) for the elementary $k-form\ \ \psi_I in\ R^n$. The geneml $k-form$ can then be written uniquely in the form

$$\omega = \sum_{[I]} b_I\, dx_{I,}$$

(3- 21)

$for\ some\ scalar\ functions\ b_I$.

The forms $dx_i\ and\ dx_I$ are of course characterized by the equations

$$\left.\begin{aligned} &dx_i(\,x)\,(\,x;\ v) \,=\, v_i,\\ &dx_I\,(\,x)\,(\,(\,x;\ v_1\,),\dots,(\,x;\ v_k)) \,=\, det\ X_I \end{aligned}\right\} ,\qquad (2\text{-}22)$$

where X is the matrix $X\ =\ [v_1 \dots . v_k]$.

For convenience , we extend this notation to an arbitrary $k-tuple\ J$= $(j_1,\dots,j_k)$ from the set $\{\,1\,,\dots,n\,\}$, setting

$$dx_J = dx_{j1}\wedge\ \dots.\wedge dx_{jk} \qquad (3\text{-}23)$$

Theorem (3-2-4):-

$Let\ A\ be\ open\ in\ R^n\ ;\ let\ f\ \ A\ \rightarrow\ \ R\ be\ of\ class\ C^r$ Then

$$df\ =\ (D_1f)dx_1\ \ +\cdots+(D_n\,f)dx_n\,.$$

In particular, $df\ =\ 0$ if f is a constant function. In Leibnitz's notation, this equation takes the form

$$df\ =\frac{\partial f}{\partial x_1}\,dx_{\,1}\ +\cdots+\frac{\partial f}{\partial x_n}\,dx_n \qquad (3\text{-}24)$$

This formula sometimes appears in calculus books, but its meaning is not explained there.

Proof.

We evaluate both sides of the equation on the tangent vector $(x; v)$. We have

$$df(x)(x;\ v)\ =\ D\,f(x)\ \cdot\ v \qquad (3\text{-}25)$$

by definition, whereas

$$\sum_{i=1}^{n} D_i\,f\ (x)dx_i\ (x)(X;V) = \sum_{i=1}^{n} D_if(x)v_i$$

(3-26)

Convention. Henceforth, we restrict ourselves to manifolds, maps,vector fields, and forms that are of class C^∞.

now We discuss to define the differential operator d in general. It is in some sense a generalized directional derivative. A formula that makes this fact explicit appears in the exercises. Rather than using this formula to define d, we shall instead characterize d by its formal properties, as given in the theorem that follows.

Definition(3-2-5):- If A is an open set in R^n , let $\Omega^k\ (A)$ denote the set of all $k-forms$ on A (of class C^∞.). The sum of two such $k-forms$ is another k-form, and so is the product of a k-form by a scalar. It is easy to see that $\Omega^k\ (A)$ satisfies the axioms for a vector space; we call it the linear space of $k-forms\ on\ A$.

Theorem (3-2-6):- Let A be an open set in R^n . There exists a unique linear transformation $d: \Omega^k\ (A) \to \Omega^{k+1}\ (A), defined\ for\ k\ \geq\ 0, such\ that$:

(1) $If\ f\ is\ a\ 0-form, then\ df\ is\ the\ 1-form$

$$df(x)(x;\ v)\ =\ Df(x)\ \cdot\ v.$$

(2) $If\ w\ and\ \eta\ are\ forms\ of\ orders\ k\ and\ \ell$, respectively, then

$$d(\omega\wedge\eta\ ) = d\omega\ \wedge\eta + (-1)^k\omega\wedge d\eta$$

(3) $For\ every\ form\ \omega,$

$$d(d\omega)\ =\ 0.$$

We call d the differential operator, and we call $d\omega$ the differential of ω .

Proof.

Step 1. We verify uniqueness . First, we show that conditions (2) and (3) imply that for any forms $\omega_1 ,.., \omega_k$, we have $d(d\omega_1\wedge ... \wedge d\omega_k\ ) = 0$ (3- 27)

If $k\ =\ 1$, this equation is a consequence of (3). Supposing it true for $k\ -\ 1,$ we set $\eta\ =\ (d\omega_2\wedge ... \wedge d\omega_k)$ and use (2) to compute

$d(d\omega_1 \wedge\eta) = d\ (d\omega_1\ )\wedge\eta\ \neq\ d\omega_1\ \wedge d\eta$ (3- 28)

The first term vanishes by (3) and the second vanishes by the induction hypothesis.

Now we show that for any $k-form\ \omega,$ the form $d\omega$ is entirely determined by the value of d on 0-forms, which is specified by (1) . Since d is linear, it suffices to consider the case $\omega\ =\ f\ dx_1$ · We compute

$d\omega\ =\ d(f\ dx_1)\ =\ d(f\wedge dx_1) =\ df\ \wedge dx_1\ +\ f\ \wedge\ d(dx_1)\ by\ (2)\ \ =\ df\ \wedge dx_1\ ,$ (3- 29)

by the result just proved. Thus $d\omega$ is determined by the value of d on the 0-form f.

Step 2. We now define d. Its value for 0-forms is specified by (1) . The computation just made tells us how to define it for forms of positive order: If A is an open set in R^n and if w is a k-form on A, we write w uniquely in the form

$$\omega = \sum_{[I]} f_I\ dx_I,$$

and define

$$d\omega\ =\ \sum_{[I]}\ df_I\wedge dx_I\ \cdot$$

(3-29)

We check that dw is of class C^r . For this purpose, we first compute

$$d\omega = \sum_{[I]} \left[\sum_{j=1}^{n} (D\, f) dx_j) \wedge dx_I \right]$$

(3-30)

To express $d\omega$ as a linear combination of elementary $k + 1$ forms, one proceeds as follows: First, delete all terms for which j is the same as one of the indices in the $k - tuple$ I. Second , take the remaining terms and rearrange the dx; so the indices are in ascending order. Third, collect like terms. One sees in this way that each component of $d\omega$ is a linear combination of the functions D i f,so that it is of class C^r . Thus dw is of class C^r (Note that if w were only of class C^r then dw would be of class C^{r-1}We show d is linear on $k - forms$ with $k > 0$. Let

$$\omega = \sum_{[I]} f_1\, dx_I \quad and \;\; \eta = \sum_{[I]} g_I\, dx_I$$

be $k - forms$. Then

$$d(a\omega + b\eta) = d \sum_{[I]} (af_I + bg_I) dx_I$$

$$= \sum_{[I]} d(af_I + bg_I) \wedge dx_I \qquad by\; definition,$$

$$= \sum_{[I]} (adf_I + bdg_I) \wedge dx_I \qquad \text{since d is linear on } 0 - \text{forms},$$

$$= a\, d\omega + b\, d\eta. \tag{3-31}$$

Step 3. We now show that if J is an arbitrary $k - tuple$ of integers from the set $\{ 1 , \dots , n\}$, then

$$d(f \wedge d_j) = df \wedge dx_J. \tag{3-32}$$

Certainly this formula holds if two of the indices in J are the same, since $dx_J = 0$ in this case. So suppose the indices in J are distinct. Let I be the $k - tupl$e obtained by rearranging the indices in J in ascending order; let π be the permutation involved. Anticommutativity of the wedge product implies that $dx = (sgn\, \pi) dx_J$. Because d is linear and the wedge product is homogeneous, the formula $d(f \wedge dx_I) = df \wedge dx_I$, which holds by definition, implies that

$$(sgn\, \pi)\, d(f \wedge dx_J) = (sgn\, \pi)\, df \wedge dx_J\, . \tag{3-33}$$

Our desired result follows.

Step 4. We verify property (2), in the case $k = 0$ and $\ell = 0$ we cmpute

$$d(f\; A\; g) = \sum_{j=1}^{n} D_j(f \cdot g) dx_j$$

$$= \sum_{j=1}^{n} (D_j f).g dx_j + \sum_{j=1}^{n} f \ .(D_j g) dx_j$$

$$= (df) \wedge g + f \wedge (dg) \tag{3-34}$$

Step 5. We verify property (2) in general First, we consider the case where both forms have positive order. Since both sides of our desired equation are linear in w and in η, it suffices to consider the case

$$\omega = f dx_I \quad and \quad \eta = g dx_J$$

We compute

$$\begin{aligned} d(\omega \wedge \eta) &= d(f\ g\ dx_I \wedge dx_J) \\ &= d(fg) \wedge dx_I \wedge dx_J \quad by\ Step\ 3, \\ &= (df \wedge g + f \wedge dg) A\ dx_I \wedge dx_J \quad by\ Step\ 4, \\ &= (df \wedge dx_I)\ A\ (g \wedge dx_J) + (-1)^k (f\ A\ dx_I) \wedge (dg \wedge dx_J) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned} \tag{3- 35}$$

The sign $(-1)^k$ comes from the fact that dx_I is a k −form and dg is a 1-form. Finally, the proof in the case where one of k or ℓ is zero proceeds as in the argument just given. If $k = 0$, the term dx_I is missing from the equations, while if $\ell = 0$, the term dx_J is missing. We leave the details to you.

Step 6. We show that if f is a 0-form, then $d(df) = 0$. We have

$$\begin{aligned} d(df) &= d \sum_{j=1}^{n} D_j f\ dx_j , \\ &= \textstyle\sum_{j=1}^{n} d(D_j f) \wedge dx_j \quad by\ definition, \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} D_i D_j f dx_i \wedge dx_j \end{aligned} \tag{3-36}$$

To write this expression in standard form, we delete all terms for which $i = j$, and collect the remaining terms as follows:

$$d(df) = \sum_{i<j} (D_i D_j f - D_j D_i f) dx_i \wedge dx_j . \tag{3-37}$$

The equality of the mixed partial derivatives implies that $d(df) = 0$.

Step 7. We show that if ω is a k − form with k > 0, then d(dω) = 0. Since d is linear, it suffices to consider the case $\omega = f dx_I$ · Then

$d(d\omega) = d(df \wedge dx_I) = d(df) \wedge dx_I - df \wedge d(dx_I)$,

by property (2). Now $d(df) = 0$ $by\ Step\ 6$, and

$$d(dx_I) = d(1) \wedge dx_I = 0 \tag{3- 38}$$

by definition. Hence $d(d\omega) = 0$.

Definition(3-2-7):-Let A be an open set in R^n . A 0-form f on A is said to be exact on A if it is constant on A; a k-form w on A with $k > 0$ is said to be exact on A if there is a k - 1 form θ on A such that w = $d\theta$. A k-form w on A with k $\geq$ 0 is said to be closed if $dw = 0$.

Every exact form is closed; for if f is constant, then $df = 0$, while if w = $d\theta$ then $dw = d(d\theta) = 0$. Conversely, every closed form on A is exact on A if A equals all of R^n, or more generally, if A is a "star-convex" subset of R^n . But the converse does not holds in general, as we shall see. If every closed k-form on A is exact on A then we say that A is homologically trivial in dimension k. We shall explore this notion further in Chapter 4

EXAMPLE (3-2-8):- Let A be the open set in R^2 consisting of all points (x, y) for which x≠0. Set $f(x, y) = x/|x|$ for $(x, y) \epsilon A$. Then f is of class C^∞ on A,and $df = 0$ on A. But f is not exact on A because f is not constant on A.

Finally, it is time to show that what we have been doing with tensor fields and forms and the differential operator is a true generalization to R^n of familiar facts about vector analysis in R^3. We know that if A is an open set in R^n, then the set $\Omega^k(A)$ of k-forms on A is a linear space. It is also easy to check that the set of all C^∞ vector fields on A is a linear space. We define here a sequence of linear transformations from scalar fields and vector fields to forms. These transformations act as operators that "translate" theorems written in the language of scalar and vector fields to theorems written in the language of forms, and conversely.

Definition(3-2-9):- Let A be open in R^n . Let f $A \rightarrow$R be a scalar field in A .We define a corresponding vector field in A, called the gradient of f, by the equation

$$grad\, f)(x) = x; D_1 f(x)e_1 + \ldots + D_n f(x)e_n \qquad (3\text{-} 39)$$

If $G(x) = (x;\ g(x))$ is a vector field in A, where g : $A \rightarrow R^n$ is given by the equation

$$g(x) = g_1(x)e_1 + \ldots + g_n(x)e_n \qquad (3\text{-}40$$

then we define a corresponding scalar field in A called the divergence of G, by the equation

$$divG)(x) = Dg_1(x)e_1 + \ldots + Dg_n(x)e_n \qquad (3\text{-}41)$$

These operators are of course familiar from calculus in the case $n = 3$. The following theorem shows how these operators correspond to the operator d:

Theorem (3-2-10) :- Let A be an open set in R^n. There exist vector space isomorphisms α_i; and β_j as in the following diagram:

$$\begin{array}{ccc}
\text{Scalar fields in } A & \xrightarrow{\alpha_0} & \Omega^0(A) \\
\downarrow grad & & \downarrow d \\
\text{Vector fields in } A & \xrightarrow{\alpha_1} & \Omega^1(A) \\
 & & \\
\text{Vector fields in} & \xrightarrow{\beta n-1} & A\ \Omega^{n-1}(A) \\
\downarrow div & & \downarrow d \\
\text{Scalar fields in } A & \xrightarrow{\beta n} & \Omega^n(A)
\end{array}$$

fig(3-5)

such that

$$d \circ \alpha_0 = \alpha_1 \circ grad \quad \text{and} \quad d \circ \beta_{n-1} = \beta_1 \circ div$$

Proo f.

Let f and h be scalar fields in A; let

$$F(x) = (x; \sum f_i(x)e_i) \quad \text{and} \quad G(x) = (x; \sum g_i(x)e_i)$$

be vector fields in A. We define the transformations α_i and β_j by the equations

$$\alpha_0 f = f \quad , \quad \alpha_1 F = \sum_{i=1}^{n} f_i dx_i$$

$$\beta_{n-1}G = \sum_{n=1}^{n}(-1)^{i-1} g_i dx_1 \cap .... \cap dx_i \cap ... dx_n$$

$$\beta_n h = h dx_1 \cap ... \cap dx_n \qquad (3\text{- } 42 \text{ })$$

The fact that each α_i and β_i is a linear isomorphism, and that the two equations hold , is left as an exercise.

This theorem is all that can be said about applications to vector fields in general . However, in the case of R^3, we have a "curl" operator, and something more can be said .

Definition :(3-2-11) :- Let A be open in R^3; let

$$F = (x; \sum f_i dx_i e_i) \qquad (3\text{-}43)$$

be a vector field in A . We define another vector field in A , called the curl of F, by the equation

$$(curlF)(x) = (x; (D_2 f_3 - D_3 f_2)e_1 + (D_3 f_1 - D_1 f_3)e_2 + (D_1 f_2 - D_2 f_1)e_3 \qquad (3\text{-}44)$$

A convenient trick for remembering the definition of the curl operator is to think of it as obtained by evaluation of the symbolic determinant

$$\det\begin{bmatrix} e_1 & e_1 & e_1 \\ D_1 & D & D \\ f_1 & f_1 & f_1 \end{bmatrix}$$

Theorem (3-2-12):- Let A be an open set in R^3 • There exist vectorspace isomorphisms α_i; and β_i as in the following diagram:

$$\begin{array}{ccc}
\text{Scalar fields in } A & \xrightarrow{\alpha_0} & \Omega^0(A) \\
\downarrow grad & & \downarrow d \\
\text{Vector fields in } A & \xrightarrow{\alpha_1} & \Omega^2(A) \\
\downarrow curl & & \downarrow d \\
\text{Vector fields in } A & \xrightarrow{\beta 2} & A\ \Omega^0(A) \\
\downarrow div & & \downarrow d
\end{array}$$

Scalar fields in A $\xrightarrow{\beta 3}$ $\Omega^3(A)$

fig(3-6)

such that

$d \circ \alpha_0 = \alpha_1 \circ grad$ and $d \circ \alpha_0 = \beta_2 \circ curl$ and $d \circ \beta_3 = \alpha_1 \circ div$

Proof.

The maps α_i a; and β_i are those defined in the proof of the preceding theorem. Only the second equation needs checking; we leave it to you.

Theorem (3-2-13):-Let A be open in R^k let $\alpha : A \to R^m$ be a C^∞ map. Let B be open in R^m and contain $\alpha(A)$; let $\beta : B \to R^n$ be a C^∞ map. Let C^∞ map ω η θ be forms defined in an open set C of R^n containing assume w and 'f/ have the same order. The transformations α^* and β^* have the following properties:

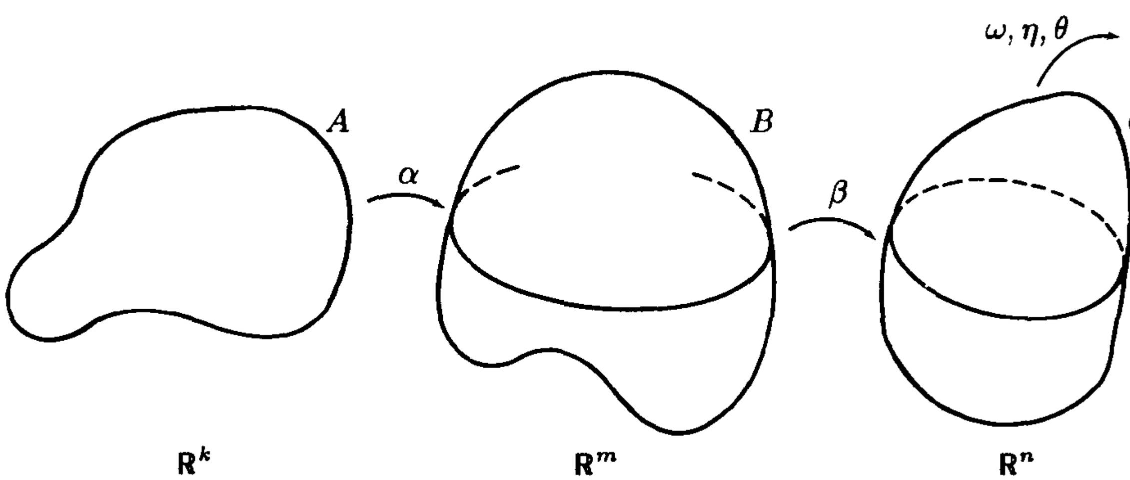
(1) β^* (aw + η) = $a(\beta^*$ w) + $b(\beta^*$ η).

(2) β^* $(w \cap \theta)$ = β^* w A β^* θ)

(3) $(\beta \circ \alpha)w = \alpha^*(\beta * w)$

Proof.

See Figure (3-5) . In the case of forms of positive order, properties (1) and (3) are merely restatements, in the language of forms, of Theorem (2-1-9) and (2) is a restatement of (6) of Theorem (2-3-1).



(3-5)

This theorem shows that α^* preserves the vector space structure and the wedge product. We now show it preserves the operator d. For this purpose(and later purposes as well) , we obtain a formula for computing $\alpha^*\omega$. If A is open in R^k and $\alpha : A \to R^n$, we derive this formula in two cases-when w is a $1 - form$ and when w is a $k - form$. This is all we shall need .

Since α^* is linear and preserves wedge products, and since $\alpha^* f$ equals $f \circ \alpha$, it remains only to compute α^* for elementary $1 - forms$ and elementary$k - forms$. Here is the required formula

Chapter (4)
Application forms and manifolds:-

Section(4-1) Closed Forms and Exact Forms:-

in the following we will discuss what additional conditions, either on Aor on both A and w, are needed in order to ensure that w is exact

Theorem (4-1-1) :- (Leibnitz's rule) Let Qbe a rectangle in R^n; Let $f : Q \times [a,b] \to R$ be a continuous function. Denote f by $f(x,t)$ for $x \in Q$ and $t \in [a,b]$. Then the function : $F(x) = \int_{t=a}^{t=b} f(x,t)$

(4 – 1)

is continuous on Q. Furthermore, if $\partial f / \partial x_j$ is continuous on $Q \times [a,b]$

then $$\frac{\partial F}{\partial x_j}(x) = \int_{t=a}^{t=b} \frac{\partial f}{\partial x_j}(x,t) \qquad (4-2)$$

This formula is called Leibnitz 's rule for differentiating under the integral sign.

Proof.

Step 1. We show that F is continuous. The rectangle $Q \times [a,b]$is compact; therefore f is uniformly continuous on $Q \times [a,b]$. That is, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x_1,t_1) - f(x_0,t_0)| < \varepsilon$ whenever $|(x_1,t_1) - (x_0,t_0)| < \delta$.It follows that when $|x_1, x_0| < \delta$,

$$|F(x_1) - F(x_0)| \le \int_{t=a}^{t=b} |f(x_1,t) - f(x_0,t) \le \varepsilon\,(b-a)|. \qquad (4\text{-}3)$$

Continuity of F follows.

Step 2. In calculating the integral and derivatives involved in Leibnitz's rule, only the variables x_j and t are involved; all others are held constant .Therefore it suffices to prove the theorem in the case where $n = 1$ and Q is an interval $[c,d]$ in R .

Let us set, for $x \in [c,d]$, $$G(x) = \int_{t=a}^{t=b} D_1 f(x,t) \qquad (4\text{-}4)$$

We wish to show that $F'(x)$ exists and equals $G(x)$. For this purpose, we apply (of all things) the Fubini theorem. We are given that $D_1 f$ is continuous on $[c,d]\times[a,b]$.Then

$$\left.\begin{aligned} \int_{x=c}^{x=x_0} G(x) &= \int_{x=c}^{x=x_0}\int_{t=a}^{t=b} D_1 f(x,t) \\ &= \int_{t=a}^{t=b}\int_{x=c}^{x=x_0} D_1 f(x,t) \\ &= \int_{t=a}^{t=b}\left[f(x_0,t)-f(c,t)\right] \\ &= F(x_0)-F(c) \end{aligned}\right\} \tag{4-5}$$

the second equation follows from the Fubini theorem, and the third from the fundamental theorem of calculus. Then for $x\in[c,d]$, we have

$$\int_c^x G = F(x)-F(c) \tag{4-6}$$

Since G is continuous by Step 1, we may apply the fundamental theorem of calculus once more to conclude that

$$G(x)=F'(x)\,. \tag{4-7}$$

We now obtain a criterion for determining when two closed forms differ by an exact form. This criterion involves the notion of a differentiable homotopy.

Definition(4-1-2):- Let A and B be open sets in R^n and R^m , respectively; Let $g,h: A\to B$ be C^{oo} maps. We say that g and h are differentiable homotopic if there is a C^{oo} map $H : A\times I\to B$ such that

$$H(x,0)=g(x) \text{ and } H(x,1)=h(x)$$

for $x\in A$. The map H is called a differentiable homotopybetween g and h

For each t, the map $x\to H(x,t)$ is a C^{oo} map of A into B; if we think of t as "time," then H gives us a way of "deforming" the map g into the map h, as t goes from 0 to 1.

Theorem (4-1-3) :- Let A and B be open sets in R^n and R^m, respectively. Let $g,h:A\to B$ be C^{oo} maps that are differentiably homotopic .Then there is a linear transformation

$$p:\Omega^{k+1}(B)\to\Omega^k(A),$$

defined for $k\geq 0$, such that for any form η of order $k>0$,

$$dp\eta+pd\eta=h^*\eta-g^*\eta, \qquad (4\text{-}8)$$

while for a form f of order 0,

$$pdf=h^*f-g^*f. \qquad (4\text{-}9)$$

This theorem implies that if η is a closed form of positive order, then $h^*\eta$ and $g^*\eta$ differ by an exact form, since $h^*\eta-g^*\eta=dp\eta$ if η is closed. On the other hand, if f is a closed 0-form, then $h^*f-g^*f=0$.

Note that d raises the order of a form by 1, and p lowers it by 1. Thus if η has order $k>0$, all the forms in the first equation have order k; and all the forms in the second equation have order 0. Of course, pf is not defined if f is a 0-form.

Proof.

Step 1. We consider first a very special case. Given an open set A in R^n, let U be a neighborhood of $A\times I$ in R^{n+1}, and let $\alpha,\beta:A\to U$, be the maps given by the equations

$$\alpha(x)=(x,0)\text{ . and }\beta(x)=(x,1)$$

(Then α and β are differentiable). We define, for any $k+1$ form η defined in U, a k- form $p\eta$ defined in A, such that

$$\left.\begin{array}{lll}(*) & dp\eta+pd\eta=\beta^*\eta-\alpha^*\eta & \text{if order } \eta>0,\\ & pdf=\beta^*f\alpha^*f & \text{if order } f=0\end{array}\right] \qquad (4\text{-}10)$$

To begin, let x denote the general point of R^n, and let t denote the general point of R. Then $dx_1,....,dx_n$, dt are the elementary 1-forms in R^{n+1}. If g is any continuous scalar function in $A \times I$, we define a scalar function Tg on A by the formula

$$(Ig)(x) = \int_{t=0}^{t=1} g(x,t) \tag{4-11}$$

Then we define P as follows: If $k \geq 0$, the general $k+1$ form η in R^{n+1} can be written uniquely as

$$\eta = \sum_{[1]} f_1 dx_1 + \sum_{[J]} g_J dx_J \wedge dt\ . \tag{4-12}$$

Here I denotes an ascending $k+1$ tuple, and J denotes an ascending k-tuple, from the set $1,\ldots,n$. We define P by the equation

$$p\eta = \sum_{[1]} p(f_1 dx_1) + \sum_{[J]} p(g_J dx_J \wedge dt) \tag{4-13}$$

where

$p(f_1 dx_1) = 0$ and $\qquad p(g_J dx_J \wedge dt) = (-1)^k (Tg_J) dx_J$

Then $p\eta$ is a k-form defined on the subset A of R^n.

Linearity of p follows at once from the uniqueness of the repressEntation of η and linearity of the integral operator I.

To show that $p\eta$ is of class C^{oo}, we need only show that the function I g is of class C^{oo}; and this result follows at once from Leibnitz's rule, since g is of class C^{oo}.

Note that in the special case $k=0$, the form η is a 1-form and is written As

$$\eta = \sum_{i=1}^{n} f_i dx_i + gdt \tag{4-14}$$

in this case, the tuple J is empty, and we have

$$p\eta = 0 + p(gdt) = Ig \tag{4-15}$$

Although the operator P may seem rather artificial, it is in fact a rather natural one. Just as d is in some sense a "differentiation operator," theoperator P is in some sense an "integration operator," one that "integrates η in the direction of the last coordinate. An alternate of P .

Step 2. We show that the formulas

$$p(fdx_1)=0 \text{ and } p(gdx_J \wedge dt)=(-1)^k (Tg)dx_J \tag{4-16}$$

hold even when I is an arbitrary $k+1$ tuple, and J is an arbitrary k-tuple,from the set $\{1,\ldots,n\}$. The proof is easy. If the indices are not distinct, then these formulas hold trivially, since $dx_I=0$ and $dx_J=0$ in this case. If the indices are distinct and in ascending order, these formulas hold by definition. Then they hold for any sets of distinct indices, since rearranging the indices changes the values of dx_I and dx_J; only by a sign.

Step 3 We verify formula (4-10) of Step 1 in the case $k=0$. We have

$$\left.\begin{aligned} P(df) &= P\left(\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}dx_j\right)+P\left(\frac{\partial f}{\partial t}dt\right) \\ &= 0+(-1)^0 I\left(\frac{\partial f}{\partial t}\right) \\ &= fo\beta - fo\alpha \\ &= \beta^* f - \alpha^* f \end{aligned}\right\} \tag{4-17}$$

where the third equation follows from the fundamental theorem of calculus .

Step 4 We verify formula (4-10) in the case $k>0$. Note that because α is the map $\alpha(x)=(x,0)$, then

$$\alpha^*(dx_i)=d\alpha_i=dx_i \qquad \text{for} \quad i=1,\ldots,n,$$
$$\alpha^*(dt)=d\alpha_{n+1}=0. \tag{4-18}$$

A similar remark holds for β^* .

Now because d and P and α^* and β^* are linear, it suffices to verify our formula for the forms fd_I and $gdx_J \wedge dt$. We first consider the case $\eta = fdx_I$. Let us compute both sides of the equation. The left side is

$$dP\eta + Pd\eta = d(0)+P(d\eta)$$
$$=\left[\sum_{j=1}^{n}P\left(\frac{\partial f}{\partial x_j}dx_j \wedge dx_I\right)\right]+P\left(\frac{\partial f}{\partial t}dt\wedge dx_I\right)$$

$$= 0 + (-1)^{k+1} P(\frac{\partial f}{\partial t} dx_I \wedge dt)$$

By step 2

$$= I(\frac{\partial f}{\partial t})dx_I = \quad [fo\beta - fo\alpha]dx_I \tag{4-19}$$

The right side of our equation is

$$\beta^*\eta - \alpha^*\eta = (fo\beta)\beta^*(dx_I) - (fo\alpha)\alpha^*(dx_I) = [fo\beta - fo\alpha]dx_I \,. \tag{4-20}$$

Thus our result holds in this case.

We now consider the case when $\eta = gdx_J \wedge dt$. Again, we compute both sides of the equation . We have

(1) $$d(P\eta) = d[(-1)^k (Ig)dx_J)]$$

$$= (-1)^k \sum_{j=1}^{n} D_j (Ig)dx_j \wedge dx_J \,. \tag{4-21}$$

On the other hand,

$$d\eta = \sum_{j=1}^{n} (D_j g)dx_j \wedge dx_J \wedge dt + (D_n +_1 g)dt \wedge dx_J \wedge dt \tag{4-22}$$

so that by Step 2,

(2) $$P(d\eta) = (-1)^{k+1} \sum I(D_j g)dx_j \wedge dx_J \tag{4-23}$$

Adding (1) and (2) and applying Leibnitz's rule, we see that

$$d(P\eta) + P(d\eta) = 0 \tag{4-24}$$

On the other hand, the right side of the equation is

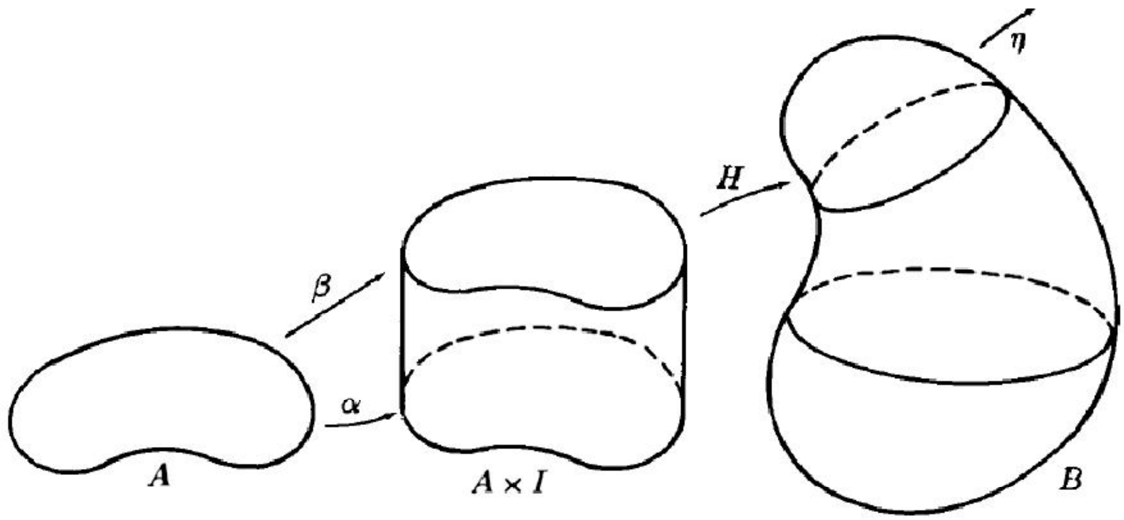
$$\beta^*(gdx_J \wedge dt) - \alpha^*(gdx_J \wedge dt) = 0 \,, \tag{4-25}$$

Step 5. We now prove the theorem in general. We are given C^{oo} maps $g, h : A \to B$, and a differentiable homotopy $H : A \times I \to B$ between them. Let $\alpha, \beta : A \to A \times I$ be the maps of Step 1 , and let P be the Linear transformation of forms whose properties are stated in Step 1. We then define our desired linear transformation $p : \Omega^{k+1}(B) \to \Omega^K (A)$ by the equation

$$p\eta = P(H^*\eta) \qquad (4\text{-}26)$$

See Figure (4-1) Since $H^*\eta$ is a $k+1$ form defined in a neighborhood of $A \times I$, then $P(H^*\eta)$ is a k -form defined in A.

Note that since H is a differentiable homotopy between g and h, $Ho\alpha = g$ and $Ho\beta = h$.



Figure(4-1)

Then if $k > 0$, we compute

$$\begin{aligned} dp\eta + pd\eta &= dP(H^*\eta) + P(H^*d\eta) \\ &= dP(H^*\eta) + P(dH^*d\eta) \\ &= \beta^*(H^*\eta) - \alpha^*(H^*\eta) \quad \text{by step 1,} \\ &= h^*\eta - g^*\eta \qquad (4\text{-}27) \end{aligned}$$

as desired. An entirely similar computation applies if $k = 0$

Definition (4-1-4):- Let A be an open set in R^n. We say that A is star-convex with respect to the point p of A if for each $x \in A$, the line segment joining x and p lies in A.

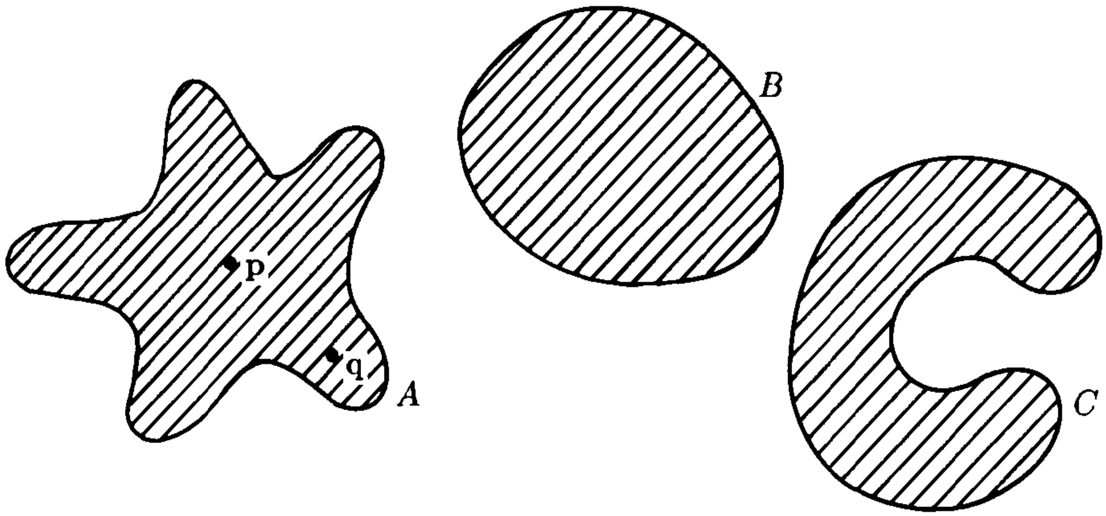


Figure (4-2)

EXAMPLE (4-1-5) :- In Figure (4-2), the set A is star-convex with respect to the point p, but not with respect to the point q. The set B is star-convex with respect to each of its points; that is, it is convex. The set C is not star-convex with respect to any of its points.

Theorem (4-1-6) (The Poincare lemma) :- Let A be a star-convex open set in R^n. If w is a closed k-form on A, then w is exact on A.

Proof.

We apply the preceding theorem. Let p be a point with respect to which A is star-convex. Let $h: A \to A$ be the identity map and let $g: A \to A$ be the constant map carrying each point to the point P. Then g and h are differentiably homotopic; indeed, the map

$$H(x,t) = th(x) + (1-t)g(x) \tag{4-28}$$

carries $A \times I$ into A and is the desired differentiable homotopy. (For each t, the point $H(x,t)$ lies on the line segment between $h(x) = x$ and $g(x) = p$, so that it lies in A.) We call H the straight-line homotopy between g and h.

Let p be the transformation given by the preceding theorem. If f is a 0-form on A, we have

$$p(df) = h^* f - g^* f = foh - fog \,. \tag{4-29}$$

Then if $df = 0$, we have for all $x \in A$,

$$0 = f(h(x)) - f(g(x)) = f(x) - f(p) \tag{4-30}$$

so that f is constant on A.
If w is a k-form with $k > 0$, we have

$$dpw + pdw = h^* w - g^* w. \tag{4-31}$$

Now $h^* w = w$ because h is the identity map, and $g^* w = 0$ because g is a constant map. Then if $dw = 0$, we have

$$dpw = w \tag{4-32}$$

so that w is exact on A.

Definition. (4-1-7):- If V is a vector space, and if W is a linear subspace of V, we denote by V/W the set whose elements are the subsets of V of the form

$$v + W = \{v + w \mid w \in W\}. \tag{4-33}$$

Each such set is called a coset of V, determined by W. One shows readily that if $v_1 - v_2 \in W$, then the cosets $v_1 + W$ and $v_2 + W$ are equal, while if $v_1 - v_2 \notin W$, then they are disjoint. Thus V/W is a collection of disjo- int subsets of V whose union is V. (Such a collection is called a partition of V.) We define vector space operations in V/W by the equations

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$
$$c(v + W) = (cv) + W \tag{4-34}$$

With these operations, V/W becomes a vector space. It is called the quotient space of V by W.

We must show these operations are well-defined. Suppose $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. Then $v_1 - v_1'$ and $v_2 - v_2'$ are in W, so that their sum, which equals $(v_1 + v_2) - (v_1' + v_2')$, is in W. Then

$$(v_1 + v_2) + W = (v_1' + v_2') + W \tag{4-35}$$

Thus vector addition is well-defined. A similar proof shows that multipli- cation by a scalar is well-defined. The vector space properties are easy to check; we leave the details to you.

Now if V is finite-dimensional, then so is V/W; we shall not however need this result. On the other hand, V/W may be finite-dimensional even in cases where V and W are not.

Definition(4-1-8):- Suppose V and V' are vector spaces, and suppose W and W' are linear subspaces of V and V', respectively. If $T: V \to V'$ is a linear transformation that carries W into W', then there is a linear transformation

$$\widetilde{T}: V/W \to V'/W'$$

defined by the equation $\widetilde{T}(v+W) = T(v) + W'$; it is said to be induced by T. One checks readily that $\widetilde{T}$ is well-defined and linear.

Definition(4-1-9):- Let A be an open set in R^n. The set $\Omega^k(A)$ of all k-forms on A is a vector space. The set $C^k(A)$ of closed k-forms on A and the set $E^k(A)$ of exact k-forms on A are linear subspaces of $\Omega^k(A)$. Since every exact form is closed, $E^k(A)$ is contained in $C^k(A)$. We define the deRham group of A in dimension k to be the quotient vector space

$$H^k(A) = C^k(A)/E^k(A) \qquad (4\text{-}36)$$

If w is a closed k-form on A (i.e., an element of $C^k(A)$), we often denote its coset $w + E^k(A)$ simply by $\{w\}$.

It is immediate that $H^k(A)$ is the trivial vector space, consisting of the zero vector alone, if and only if A is homologically trivial in dimension k.

Now if A and B are open sets in R^n and R^m, respectively, and if $g: A \to B$ is a C^{oo} map, then g induces a linear transformation $g^*: \Omega^k(B) \to \Omega^k(A)$ of forms, for all k. Because g^* commutes with d, it carries closed forms to closed forms and exact forms to exact forms; thus g^* induces a linear transformation

$$g^*: H^k(B) \to H^k(A)$$

of deRham groups. (For convenience, we denote this induced transformation also by g^*, rather than by $\widetilde{g}^*$.)

Studying closed forms and exact forms on a given set A now reduces to calculating the deRham groups of A. There are several tools that are used in computing these groups. We consider two of then here. One involves the notion of homotopy equivalence. The other is a special case of a general theorem called the Mayer- Vietoris theorem. Both are standard tools in algebraic topology.

Theorem (4-1-10) (Homotopy equivalence theorem):- Let A and B be open sets in R^n and R^m, respectively. Let $g: A \to B$ and $h: B \to A$ be C^{oo} maps. If $goh: B \to B$ is differe ntiably homotopic to the identity map i_B of B, and if $hog: A \to A$ is differentiably homotopic to the identity map i_A of A, then g^* and h^* are linear isomorphisms of the deRham groups.

If goh equals i_B and hog equals i_A, then of course g and h are diffeomorphisms. If g and h satisfy the hypotheses of this theorem , then they are called (differentiable) homotopy equivalences.

Proof

If η is a closed k-form on A, for $k \geq 0$, then Theorem (4-1-3) implies that

$(hog)^* \eta - (i_A)^* \eta$

is exact. Then the induced maps of the deRham groups satisfy the equation

$$g^*(h^*(\{\eta\}) = \{\eta\} \qquad (4\text{-}37)$$

so that $g^* o h^*$ is the identity map of $H^k(A)$ with itself. A similar argu -ment shows that $h^* o g^*$ is the identity map of $H^k(B)$. The first fact implies that g^* maps $H^k(B)$ onto $H^k(A)$, since given η in $H^k(A)$, it equals $g^*(h^*\{\eta\})$.

The second fact implies that g^* is one-to-one, since the equation $g^*\{w\} = 0$ implies that $h^*(g^*\{w\}) = 0$, whence $\{w\} = 0$.

By symmetry, h^* is also a linear isomorphism.

Lemma (4-1-11):- Let U and V be open sets in R^n ; let $U \cup V$; and suppose $A = U \cap V$ is non-empty. Then there exists a C^{oo} function $\phi: X \to [0,1]$ such that ϕ is identically 0 in a neighborhood of $U - A$ and ϕ is identically 1 in a neighborhood of $V - A$.

Proof

See Figure (4-3) Let $\{\phi_i\}$ be a partition of unity on X dominated by the open covering $\{U,V\}$. Let S_i = Support ϕ_i; for each i. Divide the index set of the collection $\{\phi_i\}$ into two disjoint subsets J and K, so that for every $i \in J$, the set S_i is contained in U, and for every $i \in K$, the set S_i; is contained in V. (For example, one could let J consist of all i such that $S_i \subset U$, and let K (consist of the remaining i .) Then let

$$\phi(x) = \sum_{i \in K} \phi_i(x) \tag{4-38}$$

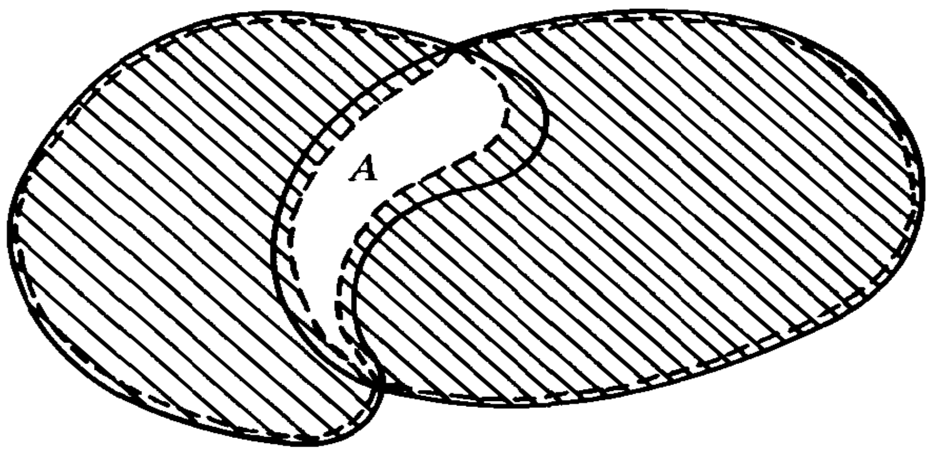


Figure (4-3)

The local finiteness condition guarantees that ϕ is of class C^{oo} on X, since each $x \in X$ has a neighborhood on which ϕ equals a finite sum of C^{oo} functions.

Let $a \in U - A$; we show ϕ is identically 0 in a neighborhood of a. First, we choose a neighborhood W of a that intersects only finitely many sets S_i; From among these sets S_i, take those whose indices belong to K , and let D be their union. Then D is closed, and D does not contain the point a. The set $W - D$ is thus a neighborhood of a, and for each $i \in K$, the function ϕ_1 vanishes on $W - D$. It follows that $\phi(x) = 0$ for $x \in W - D$. Since

$$1 - \phi(x) = \sum_{i = j} \phi_i(x), \tag{4-39}$$

symmetry implies that the function $1 - \phi$ is identically 0 in a neighborhood of $V - A$.

Theorem (4-1-12) (Mayer-Vietoris-special case) :-Let U and V be open sets in R^n with U and V homologically trivial in all dimensions.Let $X = U \cup V$; suppose $A = U \cap V$ is non-empty. Then $H^0(X)$ is trivial, and for $k \geq 0$, the space $H^{k+1}(X)$ is linearly isomorphic to the space $H^k(A)$.

Proof.

We introduce some notation that will be convenient. If B, C are open sets of R^n with $B \subset C$, and if η is a k-form on C, we denote by $\eta \setminus B$ the restriction of η to B. That is, $\eta \setminus B = j^* \eta$ /, where j is the inclusion map $j : B \to C$. Since j^* commutes with d, it follows that the restriction of a closed or exact form is closed or exact, respectively. It also follows that if $A \subset B \subset C$, then $(\eta \setminus B) \setminus A = \eta \setminus A$.

Step 1. We first show that $H^0(X)$ is trivial. Let f be a closed 0-form on X. Then $f \setminus U$ and $f \setminus V$ are closed forms on U and V, respectively.
Because U and V are homologically trivial in dimension 0, there are constant functions c_1 an d c_2 such that $f \setminus U = c_1$ and $f \setminus V = c_2$. Since $U \cap V$ is non-empty, $c_1 = c_2$; thus f is constant on X.

Step 2. Let $\phi : X \to [0,1]$ be a C^{oo} function such that ϕ vanishes in a neighborhood U' of $U - A$ and $1 - \phi$ vanishes in a neighborhood V' of $V - A$. For $k \geq 0$, we define

$$\delta : \Omega^k(A) \to \Omega^{k+1}(X)$$

by the equation

$$\delta(w) = \begin{cases} d\phi \wedge w & on\ A \\ 0 & on\ \ U' \cup V' \end{cases} \qquad (4\text{-}40)$$

Since $d\phi = 0$ on the set $U' \cup V'$, the form $\delta(w)$ is well-defined; since A and $U' \cup V'$ are open and their union is X, it is of class C^{oo} on X. The map δ is clearly linear. It commutes with the differential operator d, up to sign, since

$$d(\delta(w)) = \begin{cases} (-1)d\phi \wedge w & on\ A \\ 0 & on\ \ U' \cup V' \end{cases} = -\delta(dw) . \quad (4\text{-}41)$$

Then δ carries closed forms to closed forms, and exact forms to exact forms, so it induces a linear transformation

$$\tilde{\delta} H^k(A) \to H^{k+1}(X)$$

We show that $\tilde{\delta}$ is an isomorphism.

Step 3. We first show that $\tilde{\delta}$ is one-to-one. For this purpose, it suffices to show that if w is a closed k-form in A such that $\delta(w)$ is exact, then w is itself exact. So suppose $\delta(w) = d\theta$ for some k-form θ on X. We define k-forms w_1 and w_2 on U and V, respectively, by the equations

$$w_1 = \begin{cases} dw & on\ A \\ 0 & on\ \ U' \end{cases} and w_2 = \begin{cases} (1-\phi)w on & A \\ 0 On & V' \end{cases} \quad (4\text{-}42)$$

Then w_1 and w_2 are well-defined and of class C^{oo} . See Figure (4-4)
We compute on ,

$$dw_1 = \begin{cases} d\phi \wedge w + 0 & on\ A \\ 0 & on\ \ U' \end{cases} \quad (4\text{-}43)$$

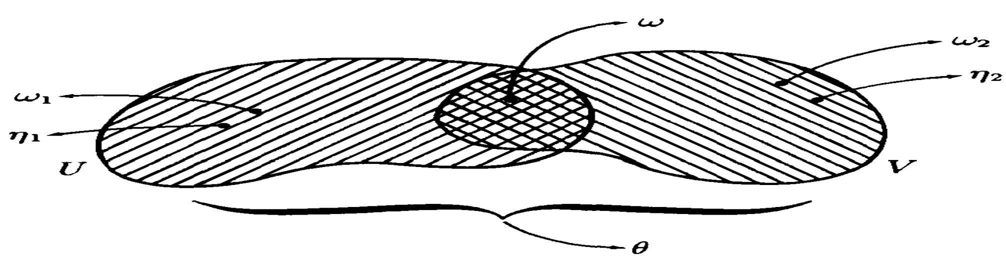


Figure (4-4)

the first equation follows from the fact that $dw_1 = 0$. Then

$$dw_1 = \delta(w) \setminus U = d\theta \setminus U \quad (4\text{-}44)$$

It follows that $w_1 - \theta \setminus U$ is a closed k-form on U . An entirely similar proof shows that $dw_2 = -d\theta \setminus V$ (4-45) so that $w_2 + \theta \setminus V$, is a closed k-form on V .

Now U and V are homologically trivial in all dimensions. If $k > 0$, this implies that there are $k-1$ forms η_1 and η_2 on U and V, respectively, such that $w_1 - \theta \backslash U = d\eta_1$ and $w_2 + \theta \backslash V = d\eta_2$. Restricting to A and adding, we have

$$w_1|A + w_2|A = d\eta_1 \backslash A + d\eta_2 \backslash A \tag{4-46}$$

which implies that

$$\phi w + (1-\phi)w = d(\eta_1|A + \eta_2|A) \tag{4-47}$$

Thus w is exact on A.

If $k=0$, then there are constants c_1 and c_2 such that

$$w_1 - \theta \backslash U = c_1 \text{ and } w_2 + \theta \backslash V = c_2$$

Then $\phi w + (1-\phi)w = w_1|A + w_2|A = c_1 + c_2$ (4-48)

Step 4. We show $\bar{\delta}$ maps $H^K(A)$ onto $H^{K+1}(X)$. For this purpose, it suffices to show that if η is a closed $k+1$ form in X, then there is a closed k-form w in A such that $\eta - \delta(w)$ is exact.

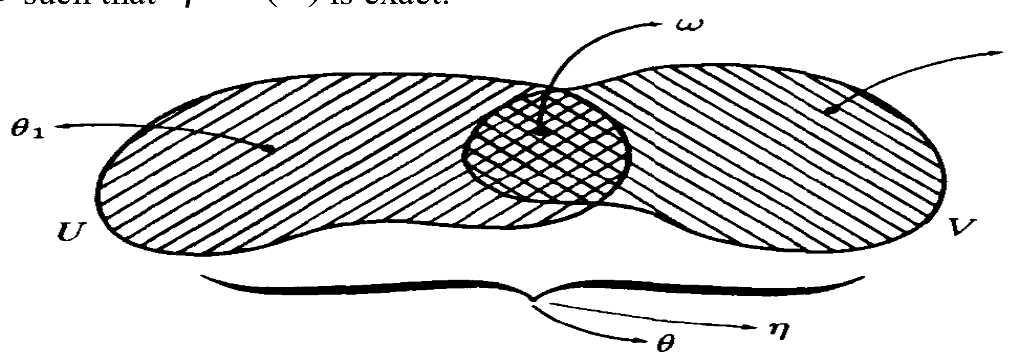


Figure (4-5)

Given η, the forms $\eta \backslash U$ and $\eta \backslash V$ are closed; hence there are k-forms θ_1 and θ_2 on U and V respectively, such that

$$d\theta_1 = \eta \backslash U \qquad \text{and} \qquad d\theta_2 = \eta \backslash V$$

Let w be the k-form on A defined by the equation

$$w = \theta_1|A - \theta_2|A; \tag{4-49}$$

then w is closed because $dw = d\theta_1|A - d\theta_2|A = \eta|A - \eta|A = 0$. We define a k-form θ on X by the equation

$$\theta = \begin{cases} (1-\phi)\theta_1 + \phi\theta_2 & on\ A \\ \theta_1 & on\ \ U' \\ \theta_2 & onV' \end{cases} \quad (4\text{-}50)$$

Then θ is well-defined and of class C^{oo}. See Figure (4-5) We show that

$$\eta - \delta(w) = d\theta; \quad (4\text{-}51)$$

We compute $d\theta$ on A and U' and V' separately. Restricting to A, we have

$$\left.\begin{aligned} d\theta \backslash A &= [-d\phi \wedge (\theta_1|A)] + [(1-\phi)[+[d\phi \wedge (\theta_2|A) + \phi(d\theta_2 \mid A)] \\ &= \phi\eta \mid A + (1-\phi)\eta \mid A + d\phi \wedge [\theta_2 \mid A - \theta_1 \mid A] \\ &= \eta \mid A + d\phi \wedge (-w) \\ &= \eta \mid A - \delta(w) \mid A. \end{aligned}\right\} \quad (4\text{-}52)$$

Restricting to U' and to V', we compute

$$\left.\begin{aligned} d\theta \backslash U' &= d\theta_1 \backslash U' = \eta \backslash U' = \eta \backslash U' - \delta(w) \backslash U', \\ d\theta \backslash V' &= d\theta_2 \backslash V' = \eta \backslash V' = \eta \backslash V' - \delta(w) \backslash V' \end{aligned}\right\} \quad (4\text{-}53)$$

since $\delta(w)U' = 0$ and $\delta(w) \backslash V' = 0$ by definition. It follows that

$$d\theta = \eta - \delta(w), \quad (4\text{-}54)$$

Now we can calculate the deRham groups of punctured euclidean space.

Theorem (4-1-13):- Let $n \geq 1$. Then

$$\dim H^k(R^n - 0) = \begin{cases} 0 & for\ k \neq n-1 \\ 1 & for\ \ k = n-1 \end{cases} \quad (4\text{-}55)$$

Proof.

Step 1. We prove the theorem for $n = 1$. Let $A = R^1 - 0$;write $A = A_0 \cup A_1$, where A_0 consists of the negative reals and A_1 consists of the positive reals. If w is a closed k-form in A, with k > 0, then $w \backslash A_0$ and $w \backslash A_1$ are closed. Since A_0 and A_1 are star-convex, there are $k-1$ forms η_0 and η_1 on A_0 and A_1 , respectively, such that $d\eta_i = w \setminus A_i$ for $i = 0,1$. Define $\eta = \eta_0$ on A_0 and $\eta = \eta_1$ on A_1. Then η_1 is well-defined and of class C^{oo}, and $d\eta = w$.

Now let f_0 be the 0-form in A defined by setting $f_0(x) = 0$ for $x \in A_0$ and $f_0(x) = 1$ for $x \in A_1$. Then f_0 is a closed form, and f_0 is not exact.

We show the coset $\{f_0\}$ forms a basis for $H^0(A)$. Given a closed 0-form f on A, the forms $f \mid A_0$ and $f \mid A_1$ are closed and thus exact. Then there are constants c_0 and c_1 such that $f \mid A_0 = c_0$ and $f \mid A_1 = c_1$. It follows that

$$f(x) = c_1 f_0(x) + c_0 \qquad (4\text{-}56)$$

for $x \in A$. Then $\{f\} = c_1\{f_0\}$,

Step 2 If B is open in R^n, then $B \times R$ is open in R^{n+1} We show that for all k,

$$\dim H^k(B) = \dim H^k(B \times R) \qquad (4\text{-}57)$$

We use the homotopy equivalence theorem. Define $g : B \to B \times R$ by the equation $g(x) = (x,0)$, and define $h : B \times R \to B$ by the equation $h(x,s) = x$. Then $ho\, g$ equals the identity map of B with itself. On the other hand, $go\, h$ is differentiably homotopic to the identity map of $B \times R$ with itself; the straight-line homotopy will suffice. It is given by the equation

$$H((x,s),t) = t(x,s) + (1-t)(x,0) = (x,st) \qquad (4\text{-}58)$$

Step 3. Let $n \geq 1$. We assume the theorem true for n and prove it for $n+1$.

Let U and V be the open sets in R^{n+1} defined by the equations

$$\left.\begin{aligned} U &= R^{n+1} - \{(0,\ldots,0,t) \mid t \geq 0\}, \\ V &= R^{n+1} - \{(0,\ldots,0,t) \mid t \leq 0\} \end{aligned}\right\} \qquad (4\text{-}59)$$

Thus U consists of all of R^{n+1} except for points on the half-line $0 \times H^1$, and V consists of all of R^{n+1} except for points on the half-line $0 \times L^1$. Figure (4-6) illustrates the case $n = 3$. The set $A = U \cap V$ is non-empty; indeed, A consists of all points of $R^{n+1} = R^n \times R$ not on the line $0 \times R$; that is,

$$A = (R^n - 0) \times R \qquad (4\text{-}60)$$

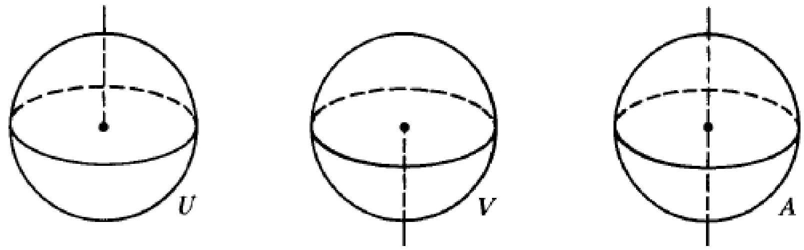


Figure (4-6)

If we set $X = U \cup V$, then

$$X = R^{n+1} - 0 \quad (4\text{-}61)$$

The set U is star-convex relative to the point $p = (0,\dots,0,-1)$ of R^{n+1}, and the set V is star-convex relative to the point $q = (0,\dots,0,1)$, as you can readily check. It follows from the preceding theorem that $H^0(X)$ is trivial, and that

$$\dim H^{k+1}(X) = \dim H^k(A) \text{ .for } k \geq 0 \quad (4\text{-}62)$$

Now Step 2 tells us that $H^k(A)$ has the same dimension as $H^k(R^n - 0)$, and the induction hypothesis implies that the latter has dimension 0 if $k \neq n-1$, and dimension 1 if $k = n-1$. The theorem follows.

Theorem (4-1-10):- Let $A = R^n - 0$, with $n \geq 1$.
(a) If $k \neq n-1$, then every closed k-form on A is exact on A.(b) There is a closed $n-1$ form η_0 on A that is not exact. If η is any closed $n-1$ form on A, then there is a unique scalar c such that $\eta - c\eta_0$ is exact.

This theorem guarantees the existence of a closed $n-1$ form in $R^n - 0$ that is not exact, but it does not give us a formula for such a form. In the exercises of the last chapter, however, we obtained such a formula. If η_0 is the $n-1$ form in $R^n - 0$ given by the equation

$$\eta_0 = \sum (-1)^{i-1} f_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n \quad (4\text{-}63)$$

where $f_i() = x_i / \|X\|^n$, then it is easy to show by direct computation that η_0 is closed, and only somewhat more difficult to show that the integral of η_0 over S^{n-1} is non-zero, so that by Stokes' theorem it cannot be exact. Using this result, we now derive the following criterion for a closed $n-1$ form in $R^n - 0$ to be exact:

Theorem (4-1-11):- . Let $A = R^n - 0$, with $n > 1$. If η_0 is a closed $n-1$ form in A, then η_0 is exact in A if and only if $\int_{S^{n-1}} \eta = 0$. (4-63)

Proof

If η is exact, then its integral over S^{n-1} is 0, by Stokes' theorem.On the other hand, suppose this integral is zero. Let η_0 be the form just defined. The preceding theorem tells us that there is a unique scalar c such that $\eta - \eta_0$ is exact. Then

$$\int_{S^{n-1}} \eta = c \int_{S^{n-1}} \eta_0 \qquad (3\text{-}64)$$

by Stokes' theorem. Since the integral of η_0 over S^{n-1} is not 0, we must have $c = 0$. Thus η is exact.

Section(4-2) Differentiable and Riemnnian manifolds

In the following we will describe briefly how this can be accomplished, and indicate how mathematicians really look at manifolds and forms.

Dfinition.(4-2-1) :- Let M be a metric space. Suppose there is a collection of homeomorphisms $\alpha_i : U_i \to V_i$, where U_i; is open in H^k or R^k , and V_i; is open in M , such that the sets V_i; cover M. (To say that a; is a homeomorphism is to say that a; carries U_i; onto V_i; in a one-to-one fashion , and that both α_i and α_i^{-1} are continuous.) Suppose that the maps α_i overlap with class C^∞ This means that the transition function α_i^{-1} o α_j is of class C^{oo} whenever $V_i \cap V_j$ is nonempty. The maps α_i are called coordinate patches on M , and so is any other homeomorphism $\alpha : U \to V$, where U is open in H^k or R^k , and V is open in M , that overlaps the α_i with class C^{oo} . The metric space M , together with this collection of coordinate patches on M , is called a differentiable k -manifold (of class C^{oo}) .

In the case $k = 1$, we make the special convention that the domains of the coordinate patches may be open sets in L^1 as well as R^1 or H^1 , just as we did before .

If there is a coordinate patch $\alpha : U \to V$ about the point p of M such that U is open in R^k , then p is called an interior point of M . Otherwise, p is called a boundary point of M . The set of boundary points of M is denoted ∂M .If $\alpha : U \to V$ is a coordinate patch on M about p , then p belongs to ∂M if and only if U is open in H^k and $p = \alpha(x)$ for some $x \in R^{k-1} x_0$.

In the following will denote a differentiable k manifold.

Definition(4-2-2) :- Given coordinate patches α_0 , α_1 on M , we say they overlap positively if det $D(\alpha_1^{-1} o \alpha_0) >$ 0. If M can be covered by coordinate patches that overlap positively, then M is said to be orientable. An orientation of M consists of such a covering of M , along with all other coordinate patches that overlap these positively. An oriented manifold consists of a manifold M together with an orientation of M .

Given an orientation { α_i } of M , the collection {$\alpha_i \; o \; r$}, where $r : R^k \to R^k$ is the reftec tion map , gives a different orientation of M ; it is called the orientation opposite to the given one.

Suppose M is a differentiable k-manifold with non-empty boundary. Then ∂M is a differentiable $k-1$ manifold without boundary. The maps $\alpha o b$, where α is a coordinate patch on M about $p \in \partial M$ and $b : R^{k-1} \to R^k$ is the map

$$b(x_1,...,x_{k-1}) = (x_1,...,x_{k-1},0), \qquad (4-1)$$

If the patches α_0 and α_1 on M overlap positively, so do the coordinate patches α_0 o b and α_1 o b on ∂M; the proof is that of prceeding Theorem Thus if M is oriented and ∂M is nonempty, then ∂M can be oriented simply by taking coordinate patches on M belonging to the orientation of M about points of ∂M , and composing them with the map b. If k is even, the orientation of ∂M obtained in this way is called the induced orientation of ∂M; if k is odd, the opposite of this orientation is so called.

Definition(4-2-3):- Let M and N be differentiable manifolds of dimensions k and n, respectively. Suppose A is a subset of M ; and suppose $f : A \to N$. We say that f is of class C^{oo} if for each $x \in A$, there is a coordinate patch $\alpha : U \to V$ on M about x, and a coordinate patch $\beta : W \to Y$ on N about $y = f(x)$, such that the composite $\beta^{-1} o \, f \, o \, \alpha$ is of class C^{oo} , as a map of a subset of R^k into R^n. Because the transition functions are of class C^{oo},this condition is independent of the choice of the coordinate patches. See Figure (4-7)

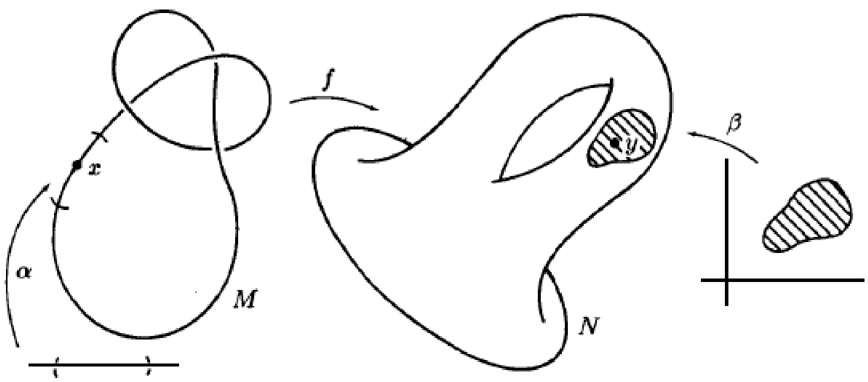


Figure (4-7)

Of course, if M or N equals euclidean space, this definition simplifies, since one can take one of the coordinate patches to be the identity map of that euclidean space.

A one-to-one map $f:M\to N$ carrying M onto N is called a diffeomorphism if both f and f^{-1} are of class C^{oo}.

Now we define what we mean by a tangent vector to M. Since we have here no surrounding euclidean space to work with, it is not obvious what a tangent vector should be.

Our usual picture of a tangent vector to a manifold M in R^n at p point p of M is that it is the velocity vector of a C^{oo} curve $\gamma:[a,b]\to M$ that passes through p. This vector is just the pair $(p;D\gamma(t_0))$ where $p=\gamma(t_0)$ and $D\gamma$ is the derivative of γ ·

Let us try to generalize this notion. If M is an arbitrary differentiable manifold, and γ is a C^{oo} curve in M, what does one mean by the "derivative" of the function γ ? Certainly one cannot speak of derivatives in the ordinary sense, since M does not lie in euclidean space. However, if $\alpha:U\to V$ is a coordinate patch in M about the point p, then the composite function $\alpha^{-1}o\gamma$ is a map from a subset of R^1 into R^k, so we can speak of its derivative. We

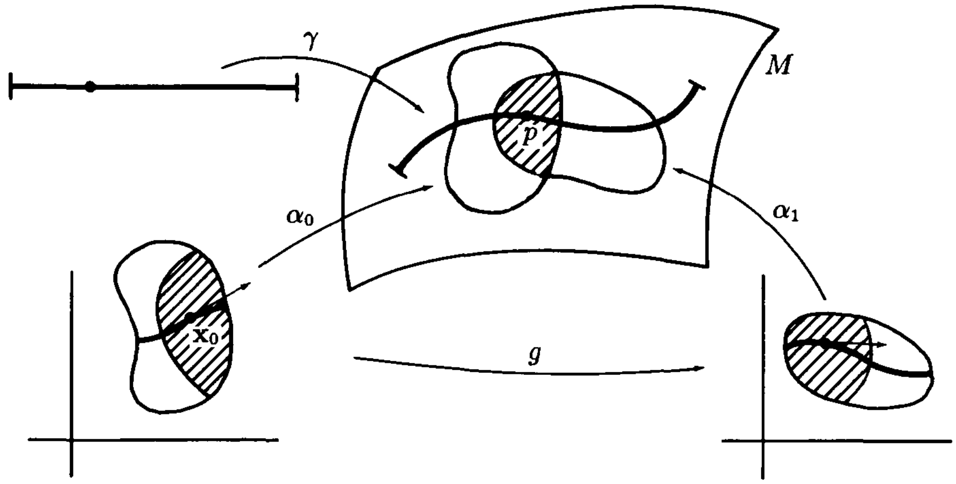


Figure (4-8)

can thus think of the "derivative" of γ at to as the function v that assigns, to each coordinate patch a about the point p , the matrix

$$v(\alpha) = D(\alpha^{-1} \circ \gamma)(t_0) \qquad (4-2)$$

where $p = \alpha(t_0)$.

Of course, the matrix $D(\alpha^{-1} \circ \gamma)$ depends on the particular coordinate patch chosen; if α_0 and α_1 are two coordinate patches about p , the chain rule implies that these matrices are related by the equation

$$v(\alpha_1) = \mathrm{D}g(x_0) \cdot v(\alpha_0), \qquad (4-3)$$

where $g = \alpha_1^{-1} o\ \alpha_0$ is the transition function $g = \alpha_1^{-1} \circ \alpha_0$, and $x_0 = \alpha_0^{-1}(p)$. See Figure (4-8)

. The pattern of this example suggests to us how to define a tangent vector to M in general.

Definition(4-2-4) :- Given $p \in M$, a tangent vector to M at p is a function v that assigns, to each coordinate patch $\alpha : U \to V$ in M about p , a column matrix of size k by 1 which we denote $v(\alpha)$. If α_0 and α_1 are two coordinate patches about p , we require that

$$v(\alpha_1) = \mathrm{D}g(x_0) \cdot v(\alpha_0) \qquad (4-4)$$

where $g = \alpha^{-1}$ o α_0 is the transition function and $x_0 = \alpha_0^{-1}(p)$. The entries of the matrix $v(\alpha)$ are called the components of v with respect to the coordinate patch α.

It follows from (4-4) that a tangent vector v to M at p is entirely determined once its components are given with respect to a single coordinate system. It also follows from (*) that if v and w are tangent vectors to M at p, then we can define $av + bw$ unambiguously by setting

$$(av + bw)(\alpha) = av(\alpha) + bw(\alpha) \qquad (4\ \text{-}\ 5)$$

for each α. That is, we add tangent vectors by adding their components in the usual way in each coordinate patch. And we multiply a vector v by a scalar similarly.

The set of tangent vectors M at p is denoted $T_p(M)$; it is called the tangent space to M at p. It is easy to see that it is a k-dimensional space; indeed, if α is a coordinate patch about p with $\alpha(x) = p$, one checks readily that the map $v \to (x; v(\alpha))$, which carries $T_p(M)$ onto $T_x(R^k)$, is a linear isomorphism. The inverse of this map is denoted by

$\alpha_* : T_x(R^k) \to T_p(M)$ It satisfies the equation $\alpha_*(x; v(\alpha)) = v$.

Given a C^{oo} curve $\gamma : [a,b] \to M$ in M, with $\gamma(t_0) = p$, we define the velocity vector v of this curve corresponding to the parameter value t_0 by the equation

$$v(\alpha) = D(\alpha^{-1} o\, \gamma)(t_0); \qquad (4\ \text{-}\ 6)$$

then v is a tangent vector to M at p. One readily shows that every tangent vector to M at p is the velocity vector of some such curve.

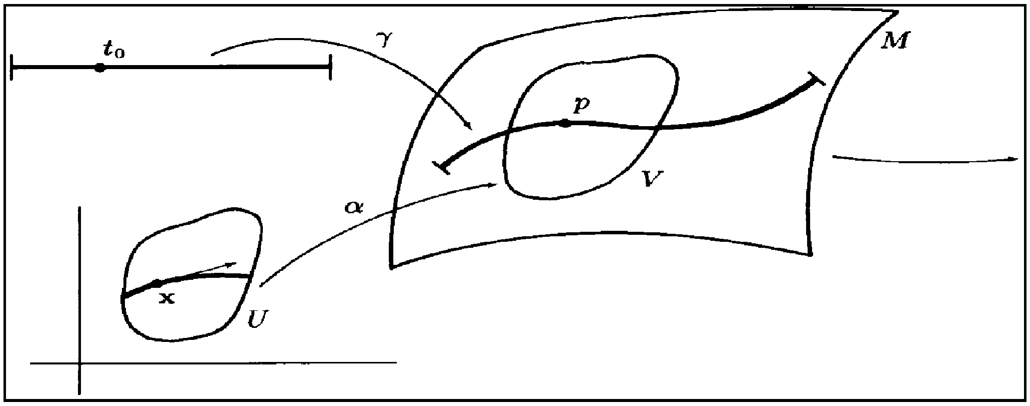


Figure (4-9)

See Figure (4-9) Note that this derivative depends only on f and the velocity vector v, not on the particular curve γ ·

This formula leads us to define the operator X_v as follows:

If v is a tangent vector to M at p, and if f is a C^{oo} real-valued function defined near p, choose a coordinate patch $\alpha : U \to V$ about p with $\alpha(x) = p$, and define the derivative of f with respect to v by the equation

$$X_v(f) = D(fo\alpha)(x).v(\alpha). \qquad (4\ -7)$$

One checks readily that this number is independent of the choice of a. One checks also that $X_{v+w}(f) = X_v + X_w$ and $X_{cv} = cX_v$. Thus the sum of vectors corresponds to the sum of the corresponding operations, and similarly for a scalar multiple of a vector.

Note that if $M = R^k$, then the operator X_v is just the directional derivative of f with respect to the vector v.

The operator X_v satisfies the following properties, which are easy to check:

(1) (Locality). If f and g agree in a neighborhood ofp, then $X_v(f) = X_v(g)$.

(2) (Linearity). $X_v(af + bg) = aX_v(f) + bX_v(g)$.

(3) (Product rule). $X_v(f.g) = X_v(f)g(p) + f(p)X_v(g)$.

These properties in fact characterize the operator X_v . One has the following Theorem: Let X be an operator that assigns to each C^{oo} real-valued function f

defined near p a number denoted $X(f)$, such that X satisfies conditions (1)-(3). Then there is a unique tangent vector v to M at p such that $X = X_v$.

This theorem suggests an alternative approach to defining tangent vectors. One could define a tangent vector to M at p to be simply an operator X satisfying conditions (1)-(3). The set of these operators is a linear space if we add operators in the usual way and multiply by scalars in thusual way, and thus it can be identified with the tangent space to M at p

Many authors prefer to use this definition of tangent vector. It has the appeal that it is "intrinsic"; that is, it does not involve coordinate patches explicitly.

Now we will define the forms on M .

Definition(4-2-5) :- An ℓ-form on M is a function w assigning to each $p \in M$,an alternating ℓ-tensor on the vector space $T_p(M)$. That is,

$$w(p) \in A^{\ell}(T_p(M)) \qquad \text{for each } p \in M .$$

We require w to be of class C^{oo} in the following sense: If $\alpha : U \to V$ is a coordinate patch on M about p, with $\alpha(x) = p$, one has the linear transformation

$$T = \alpha_* : T_x(R^k) \to T_p(M) \qquad (4 - 8)$$

and the dual transformation

$$T^* : A^{\ell}(T_p(M) \to A^{\ell}(T_x(R^k))$$

If w is an ℓ-form on M , then the ℓ-form $\alpha^* w$ is defined as usual by setting

$$(\alpha^* w)(x) = T^*(w(p)) \qquad (4 - 9)$$

We say that w is of class C^{oo} near p if $\alpha^* w$ is of class C^{oo} near x in the usual sense . This condition is independent of the choice of coordinate patch.

Thus w is of class C^{oo} if for every coordinate patch α on M , the form $\alpha^* w$ is of class C^{oo} in the sense defined earlier.

Henceforth, we assume all our our forms are of class C^{oo} .

Let $\Omega^{\ell}(M)$ denote the space of ℓ-forms on M . Note that there are no elementary forms on M that would enable us to write w in canonical form, as there were in R^n . However, one can write $\alpha^* w$ in canonical form as

$$\alpha^{*}w=\sum_{\lceil i\rceil} f_1\, dx_1 \qquad (4\ \text{-}\ 10\ )$$

where the $d\,x_1$ are the elementary forms in R^k. We call the functions f_1 the components of w with respect to the coordinate patch α. They are of course of class C^{oo}.

Definition(4-2-6):- If w is an ℓ -form on $_M$, we define the differential of w as follows: Given $p\in M$, and given tangent vectors $v_1,\ldots\ldots,v_{t+1}$ to M at p ,choose a coordinate patch $\alpha:U\to V$ on M about p with $\alpha(x)=p$. Then define

$dw(p)(v_1,\ldots,v_{t+1})=d(\alpha^{*}w)(x)((x;v_1(\alpha)),\ldots,(x;v_{t+1}(\alpha)))$ (4-11) That is, we define dw by choosing a coordinate patch α , pulling w back to a form $\alpha^{*}w$ in R^k , pulling $v_1,\ldots,v_{\ell+1}$ back to tangent vectors in R^k, and then applying the operator d in R^k . One checks that this definition is independent of the choice of the patch α. Then dw is of class C^{oo} .

We can rewrite this equation as follows: Let $a_i=v_i(\alpha)$. The preceding equation can be written in the form

$$dw(p)(\alpha_{*}(x;a_1),\ldots,\alpha_{*}(x;a_{t+1}))=d(\alpha^{*}w)(x)((x;a_1),\ldots,(x;a_{t+1})) \qquad (4\ \text{-}\ 12\ )$$

This equation says simply that $\alpha^{*}(dw)=d(\alpha^{*}w)$. Thus one has an alternate version of the preceding definition:

Definition(4-2-7) :- If w is an ℓ -form on M , then dw is defined to be the unique $\ell+1$ form on M such that for every coordinate patch a on M ,

$$\alpha^{*}(dw)=d(\alpha^{*}w) \qquad (4\ \text{-}\ 13\ )$$

Here the " d " on the right side of the equation is the usual differential operator d in R^k, and the " d " on the left is our new differential operator in M .

Now we define the integral of a-form over M . We need first to discuss partitions of unity. Because we assume M is compact, matters are especially simple.

Theorem (4-2-8);- Let M be a compact differentiable manifold. Given a covering of M by coordinate patches, there exist functions $\phi_i : M \rightarrow R$ of class C^{oo}, for $i = 1, \ldots\ldots, \ell$, such that:

(1) $\phi_i(p) \geq 0$ for each $p \in M$.

(2) For each i, the set Support ϕ_i; is covered by one of the given coordinate patches.

(3) $\sum \phi_i(p) = 1$ for each $p \in M$.

Proof.

Given $p \in M$, choose a coord inate patch $\alpha : U \rightarrow V$ about p .Let $\alpha(x) = p$; choose a non-negative C^{oo} function $f : U \rightarrow V$ whose support. is compact and is contained in U, such that f is positive at the point x. Define $\psi_p : M \rightarrow R$ by setting

$$\psi_p(y) = \begin{Bmatrix} f(\alpha^{-1}(y)) & if \;\; y \in V \\ 0 & otherwise. \end{Bmatrix} \qquad (4 - 14)$$

Because $f(\alpha^{-1}((y))$ vanishes outside a compact subset of V, the function ψ_p is of class C^{oo} on M.

Now ψ_p is positive on an open set U_p about p. Cover M by finitely many of the open sets U_p, say for $p = p_1, \ldots\ldots, p_\ell \cdot$ Then set

$$\lambda = \sum_{j=1}^{t} \psi_{pj} \text{ and } \phi_i = (1/\lambda)\psi_{pi} \qquad (4\text{-}15)$$

Definition(4-2-9) :-Let M be a compact, oriented differentiable k-manifold.Let w be a k -form on M. If the support of w lies in a single coordinate patch $\alpha : U \rightarrow V$ belonging to the orientation of M, define

$$\int_M w = \int_{\ln t u} \alpha^* w \qquad (4 - 16)$$

In general, choose $\phi_1, \ldots\ldots, \phi_\ell$ in the prece ding theorem and define

$$\int_M w = \sum_{i=1}^{t}\left[\int_M \phi_i w\right]. \qquad (4-17)$$

Theorem (4-2-10):- (Stokes' theorem):- Let M be a compact, oriented differentiable k-manifold. Let w be $a\ k-1$ form on M. If ∂M is nonempty, give ∂M the induced orientation; then

$$\int_M dw = \int_{\partial M} w. \qquad (4-18)$$

If ∂M is empty, then $\int_M dw = 0$.

Proof.

The proof given earlier goes through verbatim. Since all the computations were carried out by working within coordinate patches, no changes are necessary. The special conventions involved when $k=1$ and ∂M is a 0-manifold are handled exactly as before.

Not only does Stokes' theorem generalize to abstract differentiable manifolds, but the results in Chapter 8 concerning closed forms and exact forms generalize as well. Given M, one defines the deRham group $H^k(M)$ of M in dimension k to be the quotient of the space of closed k-forms on M by the space of exact k-forms. One has various methods for computing the dimensions of these spaces, including a general Mayer- Vietoris theorem. If M is written as the union of the two open sets U and V in M, it gives relations between the deRham groups of M and U and V and $U \cap V$. These topics are explored in [$B-T$] .

The vector space $H^k(M)$ is obviously a diffeomorphism invariant of M. It is an unexpected and striking fact that it is also a topological invariant of M. This means that if there is a homeomorphism of M with N, then the vector spaces $H^k(M)$ and $H^k(N)$ are linearly isomorphic. This fact is a consequence of a celebrated theorem called deRham 's theorem, which states that the algebra of closed forms on M modulo exact forms is isomorphic to a certain algebra, defined in algebraic topology for an arbitrary topological space, called the "cohomology algebra of M with real coefficients.

In the following We will have indicated how Stokes' theorem and the deRham groups generalize to abstract differentiable manifolds. Now we consider some of the other topics we have treated. Surprisingly, many of these do not generalize as readily.

Consider for instance the notions of the volume of a manifold M , and of the integral $\int_M f\, dV$ of a scalar function over M with respect to volume.These notions do not generalize to abstract differentiable manifolds.
Why should this be so ? One way of answering this question is to note that, one can define the volume of a compact oriented k-manifold M in R^n by the formula

$$v(M) = \int_M w_v \ , \qquad (4 - 19)$$

where w_v is a "volume form" for M , that is, w_v is a k-form whose value is 1 on any orthonormal basis for $T_p(M)$ belonging to the natural orientation of this tangent space. In this case, $T_p(M)$ is a linear subspace of $T_p(R^n) = p \times R^n$, so $T_p(M)$ has a natural inner product derived from the dot product in R^n. This notion of a volume form cannot be generalized to an arbitrary differentiable manifold M because we have no inner product on $T_p(M)$ in general, so we do not know what it means for a set of vectors to be orthonormal.

In order to generalize our definition of volume to a differentiable manifold M , we need to have an inner product on each tangent space $T_p(M)$

Definition(4-2-11) :- Let M be a differentiable k-manifold . A Riemannian metric on M is an inner product (v, w) defined on each tangent space $T_p(M)$;it is required to be of class C^{oo} as a 2-tensor field on M . A Riemannian manifold consists of a differentiable manifold M along with a Riemannian metric on M .

Now it is true that for any differentiable manifold M , there exists a Rie-mannian metric on M . The proof is not particularly difficult; one uses a partition of unity. But the Riemannian metric is certainly not unique.
Given a Riemannian metric on M , one has a corresponding volume function $V(v_1, \ldots, v_k)$ defined for k-tuples of vectors of $T_p(M)$. Then one can define the integral of a scalar function just as before.

Definition(4-2-12):- Let M be acompact Riemannian manifold of dimension k .Let $f : M \to R$ be a continuous function. If the support of

f is covered by a single coordinate patch $\alpha : U \to V$, we define the integral of f over M by the equation

$$\int_M f\,dV = \int_{\ln t} (f o \alpha)\, V(\alpha_*(x;e_1),....,\alpha_*(x;e_k)) \qquad (4\ \text{-}\ 20)$$

The integral of f over M is defined in general by using a partition of unity, as in

The volume of M is defined by the equation $v(M) = \int_M dV$

(4 - 21)

If M is a compact oriented Riemannian manifold , one can interpret the integral $\int_M w$ of a k-form over M as the integral $\int_M \lambda\, dV$ of a certain scalar function , just as we did before, where $\lambda(p)$ is the value of $w(p)$ on an orthonormal k-tuple of tangent vectors to M at p that belongs to the natural orientation of $T_p(M)$ (derived from the orientation of M). If $\lambda(p)$ is identically 1 , then w is called the volume form of the Riemannian manifold M , and is denoted by w_v. Then

$$v(M) = \int_M w_v . \qquad (4\ \text{-}\ 22)$$

For a Riemannian manifold M , a host of interesting questions arise .For instance, one can define what one means by the length of a smooth parametrized curve $\gamma : [a,b] \to M$; it is just the integral

$$\int_{t=a}^{t=b} \| \gamma_*(t;e_1) \| . \qquad (4\ \text{-}\ 23)$$

The integrand is the norm of the velocity vector of the curve γ, defined of course by using the inner product on $T_p(M)$,. Then one can discuss "geo-desics," which are "curves of minimal length" joining two points of M . One goes on to discuss such matters as "curvature." All this is dealt with in a subject called Riemannian geometry, which I hope you are tempted to investigate!