4. Grassmannian resolvent sets:

4.1 Let E be a complex Branch algebra with 1 and 1 ∈B⊂E a Banach subalgebra,

Let further $\pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \overbrace{\lambda 1}\ \ Gr_1$(E), we define the n-th Grassmannian B-resolvent set of π to be the set $\check{p}$n

$\left\{(\pi; B) = \sigma \epsilon Gr_n(B) \lfloor \sigma \rfloor\right.$ is a complement of $\left\{ \quad \right\}$. If $\sigma =$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} / \bar{\lambda}_n$ where $\propto, \beta, \gamma, \delta, \in \mathfrak{M}_n(B)$is equivalent to requiring that

$$\begin{pmatrix} b & & 0 & \\ & \ddots & & \beta \\ 0 & & b & \\ d & & 0 & \\ & \ddots & & \delta \\ 0 & & d & \end{pmatrix} \in GL(2_n; E)$$

It is easily seen that $\bar{p}(\pi; B) = \left(\bar{p}_n(\pi; B)\right)_{n \in \mathbb{N}}$ is a fully matricial B-set of the Grassmannian. The direct sum property is obvious and the similarity property follows from the fact that S. **** if $s \in GL(n; \mathbb{C})$ and σ is a complement of **** if s.σ is "a complement" of s. $(\pi \oplus \ldots \oplus \pi)$. We shall call $\bar{p}(\pi; B)$ the full Grassmannian B-resolvent of π.

4.2 on $\bar{p}(\pi; B)$ we define the $\mathfrak{M}_n\ (E) - valued$analytic function $\mathcal{k}_n(\pi; B)$. be so that

$$\begin{pmatrix} b & & 0 & \\ & \ddots & & \beta \\ 0 & & b & \\ d & & 0 & \\ & \ddots & & \delta \\ 0 & & d & \end{pmatrix} = \begin{pmatrix} * & * \\ * & \xi \end{pmatrix},$$

Then we define $\mathcal{k}_n(\pi; B). = \beta\xi$ if $\tau \in GL_1\left(\mathfrak{M}_n(B)\right), \tau \in GL_1(E)$, then replacing β, δ, b, d by $\beta_\tau, \delta_\tau, b_1, d_1$ will lead to replacing ξ by $\tau^{-1\xi}$. since $\beta_\xi = (\beta_\tau)\left(\tau^{-1\xi}\right)$ we see that $\mathfrak{M}_n(\pi; B)(\sigma)$ is well-defined. We will call $\mathfrak{M}_n(\pi, B) the\ n-th$ grassmannian B-resolvent of π. It is easy to check that $\mathcal{k}(\pi; B)$ is a fully matricial E-valued analytic function on $\bar{p}\ (\pi; B)$.

4.3 as a first step toward fitting the "affine " resolvents into this framework, we shall see that happens if π is the graph of an element $\gamma \in E$, that is , if

$$\pi = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix} / \bar{\lambda} 1 \in G_{r1}(E)$$

5. The derivation $\bar{\partial}$ on fully matricial functions of the Grassinannian

5.1. Let Ω be a fully matricial open B-set of the Grassmannian. We shall denote by of (B) the algebra of $\mathbb{C}$-valued $(that\ is\ acalar)$ fully matricial analytic functions on Ω, under pointwise multiplication of the matricial values. More generally we get an algebra $\wp(\Omega)$ for a fully multimatricial $\left(B_1, \dots B_p\right)$-set and we shall denote the corresponding algebra, if Ω is a fully matricial open B-set of the Grassmannian, then ΩxΩ is a fully multimatricial(B,B)-set and we shall denote the corresponding algebra by $\wp$ $(\Omega \mathrm{x} \Omega)$, more generally we have algebra bras $\wp(\Omega; \dots; \Omega)$. This extends the construction in the affine case [16] . the aim of this section will be to extend the construction of the derivation ∂ from the affine case to a derivation $\bar{\partial}$ in the Grassmannian framework. Like in the affine case the construction rests on two technical lemmas.

5.2. Lemma. Let $\Omega = (\Omega_\mathrm{n})_{\mathrm{n} \in \mathrm{N}}$ be an open fully maricial B-set of the Grassmannian and let.

$$\pi = \frac{\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} /}{\bar{\lambda} n_j} \in \Omega_{nj} (j = 1,2).$$

Then for all N,y,z,1$\in \mathfrak{M}_{n_1 n_2}(B)$

$$\begin{pmatrix} a_j & x & b_1 & y \\ 0 & a_2 & 0 & b_2 \\ c_1 & & b & \\ d & z & d_1 & t \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \lambda_1 \overbrace{+ n_2} \in \Omega_{n_1 n_2}$$

Proof. Since Ω is open, for any given x,y,z,t there is $\varepsilon \neq 0$ so that the conclusion of the lemma holds with N,y,z,t replaced by $\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon t$. to obtain the result without ε, it suffices to use the GL $(n_1 + n_2)$ invariance with $s = \begin{pmatrix} 1_m & 0 \\ 0 & \varepsilon l_{n2} \end{pmatrix}$

5.3 Lemma. Let Ω be an open fully marticial B-set of the Grassmannian and $f \varepsilon \wp(\Omega)$ *and let* a_j, b_j, c_j, d_j, t *be* like in the preceding lemma. Then, there is k∈ $\mathfrak{M}_{n1n_2}$ (ℂ) so that,

$$f_{n1+n2}\left(\begin{pmatrix} a_j & 0 & b_1 & 0 \\ 0 & d_2 & 0 & b_2 \\ c_1 & 0 & d_1 & t \\ 0 & c_2 & 0 & d_2 \end{pmatrix} / \lambda_1 \overbrace{+ n_2}\right)$$

$$= \begin{pmatrix} f_{n1}\begin{pmatrix} a_j & 0 & b_1 \\ 0 & d_2 & 0 \\ c_1 & 0 & d_1 \end{pmatrix} / \lambda_1 \overbrace{+ n_2} & k \\ 0 & f_{n1}\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \lambda_{n2}\right) \end{pmatrix}$$

And k depends linearly on t. in fact we have

$$\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = \frac{d}{d_e} f_{n1+n2}\left(\begin{pmatrix} a_j & 0 & b_1 & 0 \\ 0 & d_2 & 0 & b_2 \\ c_1 & 0 & d_1 & t \\ 0 & c_2 & 0 & d_2 \end{pmatrix} / \lambda_{\overline{1+n}_2}\right)$$

Proof: assume the right hand side of the first equality is $\begin{pmatrix} u & k \\ h & v \end{pmatrix}$

GL$(n_1 + n_{2;}\ \mathbb{C})$ equivariance of $f_{n_1 n2}$ applied to the similarity $\begin{pmatrix} \varepsilon I_{n1} & 0 \\ c_2 & I_{n2} \end{pmatrix}$ we find that

$\begin{pmatrix} u & ek \\ \varepsilon I_h & v \end{pmatrix}$ converges as $\varepsilon \to 0$ to

$$= \begin{pmatrix} f_{n1}\left(\begin{pmatrix} a_j & b_1 \\ c_1 & d_1 \end{pmatrix} / \lambda_{n1}\right) & 0 \\ 0 & f_{n1}\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \lambda_{n2}\right) \end{pmatrix}$$

This, then , implies h=0 and that $\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ is given by the second formula in the statement of the lemma, since f as an analytic function is differentiable. In turn, this formula which identifies the map taking t to k with a partial differential of f_{n1+n2} shows that this map is a C-linear map.

5.4 to define $\bar{\partial}_{n1+n2}$ we shall use the isomorphism

$$\mathcal{Y}_{n1.n2:\mathfrak{M}_{n2}} \oplus \mathfrak{M}_{n2} \to \mathcal{Y}(\mathfrak{M}_{n1,n2})$$

Which takes A⊕B to the linear map $x \rightarrow AXB\ in\ \mathcal{Y}(\mathfrak{M}_{n1,n2})$

Which takes $A \oplus B$ *to the linear map* $X \rightarrow AXB$ *in* $\mathcal{Y}\left(\mathfrak{M}_{n1,n2}\right)$

Definition. Let $\Omega_1, f, a_j, b_j, c_j, d_j$ be like in 5.3 and let T∈ $\mathcal{S}\left(\mathfrak{M}_{n1,n2}\right)$ be the linear map, so that $T(t) = k$ when $\tau \in \mathfrak{M}_{n1,n2}$ (ℂ)⊂$\mathfrak{M}_{n1,n2}$ (B) and

$$\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = \frac{d}{d_e} f_{n1+n2}\left(\begin{pmatrix} a_j & 0 & b_1 & 0 \\ 0 & d_2 & 0 & b_2 \\ c_1 & 0 & d_1 & t \\ 0 & c_2 & 0 & d_2 \end{pmatrix} / \lambda_{\overbrace{n1 + n_2}}\right)$$

Then we define

$$\left(\bar{\partial}_{n1,n2} f_{n1,n2}\right)\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} / \check{\lambda}_{n1;} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \check{\lambda}_{n2;}\right) = a_{n1,n2}^{-1}(T) \in$$

$\mathfrak{M}_{n1} \oplus \mathfrak{M}_{n2}$

Note that if $z_{j, \in GL_1\left(\mathfrak{M}_{n2}(B)\right)}$ then $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} = GL_1\left(\mathfrak{M}_{n2}(B)\right)$ and

$$f_{n1+n2}\left(\begin{pmatrix} a_j & 0 & b_1 z_1 & 0 \\ 0 & d_2 & 0 & b_2 z_2 \\ c_1 & 0 & d_1 z_1 & \varepsilon\tau b_2 z_2 \\ 0 & c_2 & 0 & d_2 z_2 \end{pmatrix} / \lambda_{\overbrace{n1 + n_2}}\right)$$

$$= f_{n1+n2}\left(\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & \varepsilon\tau b_2 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} / \lambda_{\overbrace{n1 + n_2}}\right)$$

So that $\bar{\partial}_{n1,n2} f_{n1,n2}$ is well-defined.

It is also easy to see that $\bar{\partial}$ extends the definition of ∂ in the affine ease ([16]). Indeed if we take $a_j = I_{nj} \oplus 1, c_j = 0, b_j = I_{nj} \oplus I$ in the preceding formulae we get exactly he formulae in the affine case, corresponding to the embedding.

$$\mathfrak{M}_2(B) \ni \beta \to \begin{pmatrix} I_n \oplus I & I_n \oplus I \\ 0 & \beta \end{pmatrix} / \tilde{\lambda} \in Gr_n(B).$$

5.5 starting with this subsection and continuing in 5.6 and 5.7 we will check that ∂ turns $A(\Omega)$ into a "topological" infinitesimal bialgebra. Since section 5.5-5.7 are just a technical extension of the affine case.

The first step is to check that

$$\bar{\partial}\mathrm{f} = \left(\bar{\partial}_{m.n} f_{m+n}\right)_{(m,n)\in n^2} \mathfrak{F}(\Omega;\Omega),$$

Since analyticity of the $\bar{\partial}_{m.n} f_{m+n}$ is obvious, we are left with checking $GL(m)XGL(n)$ equivariance and the direct sum properties.

In view of the equivariance property of $\propto_{m.n}$ $(see[16], 7.7)$ is suffices to remark that if S' $\in GL(m)$ and S"$\in GL(n)$ then assuming t,k$\in \mathfrak{M}_{m.n}$ and

$$\begin{pmatrix}0 & k\\ 0 & 0\end{pmatrix} = \frac{d}{d_e} f_{m+n}\left(\begin{pmatrix} a_j & 0 & b_1 & 0\\ 0 & d_2 & 0 & b_2\\ c_1 & 0 & d_1 & \varepsilon t b_2\\ 0 & c_2 & 0 & d_2\end{pmatrix} / \lambda_{\widetilde{m+n}}\right)$$

We also have

$$\begin{pmatrix}0 & S^1 k s^{"-1}\\ 0 & 0\end{pmatrix}$$

$$= \frac{d}{d_e} f_{m+n}\left(\begin{pmatrix} S' a_j S'^{-1} & 0 & S'^1 b_1 S'^{-1} & 0\\ 0 & S^{"} a_2 S^{"-1} & 0 & S^{"} b_2 S^{"-1}\\ S' c_1 S'^{-1} & 0 & S' d_1 S'^{-1} & \varepsilon\left(S' t S^{"-1} S" b_2 S^{"-1}\right)\\ 0 & S^{"} c_2 S^{"-1} & 0 & d_2\end{pmatrix} / \lambda_{\widetilde{m+n}}\right)$$

The last equality is a consequence of the GL(m+n) equivariance of f_{m+n} applied to $\begin{pmatrix}S' & 0\\ 0 & S"\end{pmatrix}$. we thus have proved that $\bar{\partial}_{m.n} f_{m+n}$ satisfies GL(m+n) equivariance.

The direct sum properties to be checked are if $\pi \in \Omega_{m,} \sigma \in \Omega_n$ and

$$m = m' + m'',\ \pi \in \Omega_{m,} \pi" \in \Omega_{m"}$$
$$n = n' + n'',\ \sigma' \pi \in \Omega_{n',} \sigma" \in \Omega_{n"}$$

then

$$\left(\bar{\partial}_{m.n} f_{m+n}\right)(\pi' \oplus \pi", \sigma) = \bar{\partial}_{m'.n} f_{m'+n}(\pi', \sigma) \oplus \bar{\partial}_{m".n} f_{m"+n}\left(\pi^{"}, \sigma\right)$$

$$\left(\bar{\partial}_{m.n} f_{m+n}\right)(\pi, \sigma' \oplus \sigma") = \bar{\partial}_{m'.n'} f_{m'+n}\left(\pi\ \ , \sigma'\right) \oplus \bar{\partial}_{m.n} f_{m+n}\left(\pi\ \ , \sigma"\right)$$

We will only discuss the first equality to be checked , the second being obtainable along similar lines

Since the isomorphism $\propto$ had the property;

$$\propto_{m"+n}^{-1}(T_1 \oplus T_2) = f_{m+n}(\pi, \sigma' \oplus \sigma") = \propto_{m',n}^{-1}(T_1) \oplus \propto_{m",n}(T_2)$$

$if T_1 \in \mathcal{Y}(\mathfrak{M}_{m',n}), T_2 \in \mathcal{Y}(\mathfrak{M}_{m',n})$ it is easily seen that what we must prove boils down to the following.

We have

$$= f_{m\prime+m"+n}\left(\begin{pmatrix} a'_1 & 0 & 0 & \beta'_1 & 0 & 0 \\ 0 & a_1^" & 0 & 0 & b_1^" & 0 \\ 0 & 0 & a_2 & 0 & 0 & b^2 \\ c'_1 & 0 & 0 & d_1^" & 0 & t^1 \\ 0 & c_1^" & 0 & 0 & d_1^" & t^" \\ 0 & 0 & c_2 & 0 & 0 & d_2 \end{pmatrix} / \lambda_{m\prime} \widetilde{+ m"} + n\right)$$

$$= \begin{pmatrix} fm\left(\begin{pmatrix} a'_1 & b'_1 \\ c'_1 & d'_1 \end{pmatrix} / \widetilde{\lambda m'}\right) & 0 & k^1 \\ 0 & fm"\left(\begin{pmatrix} a_1^" & b_1^" \\ c_1^" & d_1^" \end{pmatrix} / \widetilde{\lambda m'}\right) & k^" \\ d_1^" & 0 & fn\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \widetilde{\lambda n}\right) \end{pmatrix}$$

Where

$$f_{m'+n}\left(\begin{pmatrix} a'_1 & 0 & b'_1 & 0 \\ 0 & a_2 & 0 & b^2 \\ c'_1 & 0 & d'_1 & t^1 \\ 0 & c_2 & 0 & d^2 \end{pmatrix} / \lambda_{\overline{m' + n}}\right)$$

$$= \begin{pmatrix} fm\left(\begin{pmatrix} a'_1 & b'_1 \\ c'_1 & d'_1 \end{pmatrix} / \widetilde{\lambda m'}\right) & k^1 \\ 0 & fn\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \widetilde{\lambda n}\right) \end{pmatrix}$$

and

$$f_{m''+n}\left(\begin{pmatrix} a''_1 & 0 & b''_1 & 0 \\ 0 & a_2 & 0 & b^2 \\ c''_1 & 0 & d''_1 & t'' \\ 0 & c_2 & 0 & d_2 \end{pmatrix} / \lambda_{\overline{m'' + n}}\right)$$

And

$$= \begin{pmatrix} fm''\left(\begin{pmatrix} a''_1 & b''_1 \\ c''_1 & d''_1 \end{pmatrix} / \widetilde{\lambda m''}\right) & k'' \\ 0 & fn\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \widetilde{\lambda n}\right) \end{pmatrix}$$

If we define k'' by the last two equalities (with lemma 5.3 in mind)
we get

$$=f_{m'+m''+n}\left(\begin{pmatrix} a'_1 & 0 & 0 & b'_1 & 0 & 0 \\ 0 & a''_1 & 0 & 0 & b''_1 & 0 \\ 0 & 0 & a_2 & 0 & 0 & b_2 \\ c'_1 & 0 & 0 & d'_1 & 0 & t^1 \\ 0 & c''_1 & 0 & 0 & d''_1 & t'' \\ 0 & 0 & c_2 & 0 & 0 & d_2 \end{pmatrix} / \lambda_{m'}\,\widetilde{\overline{+m''}} + n\right)$$

=

$$\begin{pmatrix} fm\left(\begin{pmatrix} a'_1 & b'_1 \\ c'_1 & d'_1 \end{pmatrix} / \widetilde{\lambda m'}\right) & * & * \\ 0 & fm''\left(\begin{pmatrix} a''_1 & b''_1 \\ c''_1 & d''_1 \end{pmatrix} / \widetilde{\lambda m'}\right) & k'' \\ 0 & 0 & fn\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \widetilde{\lambda n}\right) \end{pmatrix}$$

and

$$f_{m''+m'+n}\left(\begin{pmatrix} a''_1 & 0 & 0 & b''_1 & 0 & 0 \\ 0 & a'_1 & 0 & 0 & b'_1 & 0 \\ 0 & 0 & a_2 & 0 & 0 & b_2 \\ c'_1 & 0 & 0 & d''_1 & 0 & t'' \\ 0 & c'_1 & 0 & 0 & d'_1 & t' \\ 0 & 0 & c_2 & 0 & 0 & d_2 \end{pmatrix} / \lambda_{m''}\,\widetilde{\overline{+m'}} + n\right)$$

=

$$\begin{pmatrix} fm\left(\begin{pmatrix} a''_1 & b''_1 \\ c''_1 & d''_1 \end{pmatrix} / \widetilde{\overline{\partial}\lambda m''}\right) & * & * \\ 0 & fm''\left(\begin{pmatrix} a'_1 & b'_1 \\ c'_1 & d'_1 \end{pmatrix} / \widetilde{\lambda m'}\right) & k' \\ 0 & 0 & fn\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \widetilde{\lambda n}\right) \end{pmatrix}$$

Using a similarity which permutes the first two summands in $\mathbb{C}^{m'} \oplus \mathbb{C}^{m"} \oplus \mathbb{C}^{"}$, we get that the 13-block in the formula for $f_{m'+m"+n}(\dots)$ is K'. that all we must still do is to show that the 12-block in that formula is zero. This in turn is immediate from Lemma 5.3 applied to $f(m' + m") + n \; and \; f_{m'+m"}$. thus we concluded checking that

$$\left(\bar{\partial}_{m.n} f_{m+n}\right)_{(m,n)\in N^2 \in \mathcal{S}(\Omega;\Omega)}.$$

5.6. Our next task is to show that $\bar{\partial}: \mathcal{S}(\Omega) \to \mathcal{f}(\Omega:\Omega)$is a derivation.

Lemma. Let $f, g \in \mathcal{S}(\Omega)$ and let

$$\pi' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} / \widetilde{\lambda_m} \in \Omega_m \pi" = \begin{pmatrix} a" & b" \\ c" & d" \end{pmatrix} / \widetilde{\lambda_n} \in \Omega_m \text{ and } t \in \mathfrak{M}_{m,n.}$$

Then we have

$$\alpha_{m.n}\left(\bar{\partial}_{m.n}(fg)_{m+n}\right)\left(\pi' : \pi"\right)\Big)(t) = f_m(\pi')_{m.n}\left(\bar{\partial}_{m.n} g_{m+n}\right)(\pi', \pi")(t)$$

$$= \alpha_{m.n}\left(\bar{\partial}_{m.n} f_{m,n}\right)\left(\pi' : \pi"\right)\Big)(t) g_n(\pi")$$

Proof. To simplify notations put

$\xi = \alpha_{m.n}\left(\bar{\partial}_{m.n} f_{m,n}\right)\left(\pi' : \pi"\right)\Big)(t) \in \mathfrak{M}_{m,n.}$

$\eta = \alpha_{m.n}\left(\bar{\partial}_{m.n} f_{m,n}\right)\left(\pi' : \pi"\right)\Big)(t) \in \mathfrak{M}_{m,n.}$

$\xi = \alpha_{m.n}\left(\bar{\partial}_{m.n}(fg)_{m+n}\right)\left(\pi' : \pi"\right)\Big)(t) \in \mathfrak{M}_{m,n.}$

and

$$\pi\left(\begin{pmatrix} a' & 0 & b' & 0 \\ 0 & a'' & 0 & b'' \\ c' & 0 & d' & tb'' \\ 0 & c' & 0 & d'' \end{pmatrix} / \lambda_{m''} + \overbrace{n \ n} \in \Omega_{m+n}\right)$$

Then, by Lemma 5.3 and Definition 5.4 we have

$$(fg)_{m+n}(\pi) = \begin{pmatrix} fm(\pi')_{gm}(\pi') & \xi \\ 0 & f_m(\pi'')_{gn}(\pi'') \\ fm(\pi) = \begin{pmatrix} f_m(\pi') & \xi \\ 0 & f_n(\pi'') \end{pmatrix} & \end{pmatrix}$$

and

$$g_{m+n}(\pi) = \begin{pmatrix} g_m(\pi') & \eta \\ 0 & g_n(\pi'') \end{pmatrix}$$

The Lemma then follows from the equality of matrices derived from

$$(fg)_{m+n}(\pi) = f_{m+n}(\pi) g_{m+n}(\pi).$$

Corollary. $\rightarrow \mathfrak{F}(\Omega;\Omega)$is a derivation.

Proof; take into account that if $f, g \in \mathfrak{F}(\Omega;\Omega)$then the $\mathfrak{F}(\Omega) -$ bimodule structure $\rightarrow \mathfrak{F}(\Omega;\Omega)$ is given by the homomorphisms $f \rightarrow \oplus 1$ and $g \rightarrow 1 \oplus g$ where $(f \oplus 1)_{m,n}(\pi', \pi'') = 1_m \oplus g_n(\pi'')$, and that if $A \in \mathfrak{M}_{m,} B \in \mathfrak{M}_{m,}$ T$\in \mathcal{Y}(\mathfrak{M}_{m,n})$ then .

$$\alpha_{m,n}^{-1}(AT(.)B) = (A \oplus I_n)\alpha_{m,n}^{-1}\left(T(.)\right)(I_m \oplus B)$$

The corollary is immediately inferred from the lemma.

5.7 we pass now to the proof of the co-associativity property of $\bar{\partial}$. Like in the affine case $([16], 7.10) since\ \mathcal{S}(\Omega;\Omega)\ and\ \mathcal{S}\ (\Omega;\Omega;\Omega)$ have not been identified with some topological tensor products of two and respectively three copies of $\mathcal{S}(\Omega)$, we will have to define the maps id$\oplus\ \bar{\partial}$; $\mathcal{S}(\Omega;\Omega) \to \mathcal{S}\ (\Omega;\Omega;\Omega)$ and $\bar{\partial}$id$\oplus$: $\mathcal{S}(\Omega;\Omega) \to \mathcal{S}\ (\Omega;\Omega;\Omega.)$

Let $\acute{k} \in \mathcal{S}(\Omega;\Omega)$ and output K=$\acute{k}_{m,n+p}$ which is an analytic function on $\Omega_m \times \Omega_{n+p}$ with values in $\mathfrak{M}_m \oplus \mathfrak{M}_{n+p}$ let further $\pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \widetilde{\lambda_m} \in \Omega_n$

$$\pi^n = \frac{\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}}{\widetilde{\lambda_m}} \in \Omega_p.$$ we define:

$$\left((id \oplus \tilde{\partial})_{m.n} k \right) (\pi, \pi', \pi'')$$

=

$$\sum_{\substack{I \le a.b \le m \\ I \le c,d \le n \\ I \le c.f \le p}} \left(\frac{d}{d\varepsilon} k \left(\pi; \begin{pmatrix} a & 0 & b' & 0 \\ 0 & a'' & 0 & b'' \\ c' & 0 & d' & \varepsilon(e_{de} \oplus 1) b'' \\ 0 & c'' & 0 & d'' \end{pmatrix} / \widetilde{\lambda_n + p} \middle| z = 0 \right) \right)_{(a,b)(c,n)+f e_{ah}^{(m)} \oplus e_{ed}^{(n)} \oplus e_{ed}^{(p)}}$$

Where e_{ij}^{r} are the matrix-units in $\mathfrak{M}_r$ and he index $(a,b)(c,n+f)$ indicates the co-efficient of $e\frac{(m)}{ah} \oplus e\frac{(n+p)}{c.n+f}$ of an element of $\mathfrak{M}_m \oplus \mathfrak{M}_{n+p}$. it is easy to see that if $k' = f \oplus g$, where $f, g \in . \mathcal{S}(\Omega)$ then $(id \oplus \bar{\partial})(f \oplus g) = f \oplus \bar{\partial}$. we also leave it to the reader to check that $id \oplus \bar{\partial}$ takes values in $\mathcal{S}(\Omega;\Omega;\Omega)$.This involves arguments of the type used in showing that $\bar{\partial}$ takes values in $\mathcal{S}(\Omega;\Omega)$

Similarly, we define

$\left((\bar{\partial}\oplus id)_{\pi,\pi',\pi''}\right)$

$$\sum_{\substack{I\le u,h\le m\\ I\le c,d\le n\\ I\le c.f\le p}}\left(\left(\left(\frac{d}{d\varepsilon}k\begin{pmatrix} a & 0 & b' & 0\\ 0 & a'' & 0 & b''\\ c' & 0 & d' & \varepsilon(e_{de}\oplus 1)b''\\ 0 & c'' & 0 & d''\end{pmatrix}/\lambda_n\widetilde{+n},\pi''\right)\right|c=0\right)_{(am+d),(e,f)}$$

$$e\frac{(n)}{ch}\oplus e\frac{(p)}{ef}$$

Checking that $(id\oplus\bar{\partial})o\bar{\partial}$, after all these questions are put aside, boils down, like in the affine case to permuting the order in which we take two derivatives.

Lemma. $if\, h' \in A(\Omega)\; and\; h = h'_{m+n+p,}$ then

$$(id\oplus\bar{\partial})_{m,n,p}\bar{\partial}_{m,n+p}h = (\bar{\bar{\partial}}\ \oplus id)_{m,n,p}\bar{\partial}_{m,n+p}h$$

Proof. Using the notations already introduced in this subsection, we have

$$\left((id\oplus\bar{\partial})_{m,n,p}\bar{\partial}_{m,n+p}h\right) = (\pi,\pi',\pi'')_{(a,b)(e,d)(e,f)}$$

$$=$$

$$\frac{d}{d_{\varepsilon_2}}\left(\frac{d}{d_{\varepsilon_2}}\left(\left(\left(h\begin{pmatrix} a & 0 & 0 & b & 0 & 0\\ 0 & a' & 0 & 0 & b' & 0\\ 0 & 0 & a'' & 0 & 0 & b''\\ c & 0 & 0 & d & \varepsilon_1(e_{b,c}\oplus 1)b' & 0\\ 0 & c' & 0 & 0 & d_1'' & t''\\ 0 & 0 & c'' & 0 & 0 & \varepsilon_1(e_{b,c}\oplus 1)b''\end{pmatrix}/\lambda_{m+n}\widetilde{+p}\right)\right)_{a,m+n+f}\right|\ \right)\right|\varepsilon_2=0$$

Similarly we have

$$\left((\bar{\partial}\oplus id)_{m,n,p}o\bar{\partial}_{m,n+p}h\right)(\pi,\pi',\pi'')_{(a,b)(e,d)(e,f)}$$

$$\frac{d}{d_{\varepsilon_2}}\left(\frac{d}{d_{\varepsilon_2}}\left(\left(\left(h\begin{pmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & a' & 0 & 0 & b' & 0 \\ 0 & 0 & a'' & 0 & 0 & b'' \\ c & 0 & 0 & d & \varepsilon_2(e_{b,c}\oplus 1)b' & 0 \\ 0 & c' & 0 & 0 & d' & \varepsilon_2(e_{b,c}\oplus 1)b'' \\ 0 & 0 & c'' & 0 & 0 & d'' \end{pmatrix} / \widetilde{\lambda_{m+n}} + p\right)\right)_{a,m+n+f}\right|_{\varepsilon_0}\right)\Bigg|_{\varepsilon_2 = 0}$$

Clearly the two quantities are equal [the only difference is that inside the 6x6 matrix we have replaced ε_1 by ε_2 and ε_2 by $\varepsilon_1$so that the equality is just a permutability of partial derivative].

6. The resolvent equation and he duality transform.

6.1. we shall use the same framework as in section 4 and 5, to carry out the computations which yield the functional equation for the Grassmannian resolvent $(\mathfrak{R}_n(\pi, B)(.)_{n\in\mathbb{N}})$ where

$$\pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \widetilde{\lambda 1} = G_{r_1}(E)$$

Let

$$\sigma' = \begin{pmatrix} \alpha'' & \beta' \\ y' & \delta' \end{pmatrix} / \widetilde{\lambda m} \in \check{p}_n(\pi; B)$$

and let

$$\sigma'' = \begin{pmatrix} \alpha'' & \beta'' \\ y'' & \delta'' \end{pmatrix} / \widetilde{\lambda m} \in \check{p}_n(\pi; B)$$

We then consider

$$\sigma = \begin{pmatrix} \alpha' & 0 & \beta' & 0 \\ 0 & \alpha'' & 0 & \beta'' \\ y' & 0 & \delta' & t\beta'' \\ 0 & y'' & 0 & \delta'' \end{pmatrix} / \lambda(\widetilde{m+n}) \in \check{p}_{m+n}(\pi; B)$$

Where $t \in \mathfrak{M}_{m.n}(\mathbb{C}) \subset \mathfrak{M}_{m.n}(B)$. to compute $\Re_{m+n}(\pi; B)(\sigma)$ we must examine the matrix.

$$\equiv = \begin{pmatrix} I_{m+n} \oplus b & \beta' & 0 \\ & 0 & \beta'' \\ & \delta' & t\beta'' \\ I_{m+n} \oplus b & 0 & \delta'' \end{pmatrix}$$

Permuting indices 2 and 3 in the above matrix, viewed as a 4x4 block-matrix, we get

$$\odot = \begin{pmatrix} I_m \oplus b & \beta' & 0 & 0 \\ I_m \oplus b & \delta' & 0 & t\beta'' \\ 0 & 0 & I_n \oplus b & \beta'' \\ 0 & 0 & I_n \oplus d & \delta'' \end{pmatrix}$$

$$= \begin{pmatrix} * & * & & y \\ * & \xi' & & \\ 0 & 0 & * & * \\ 0 & 0 & * & \xi'' \end{pmatrix}$$

where

$$y = -\begin{pmatrix} * & * \\ * & \xi' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & t\beta'' \end{pmatrix} \begin{pmatrix} * & * \\ * & \xi'' \end{pmatrix}$$

$$= \begin{pmatrix} * & * \\ * & -\xi + \beta''\xi'' \end{pmatrix}$$

and $\beta'\xi' = \Re_m(\pi; B)(\delta')$, $\beta''\xi'' = \Re_m(\pi; B)(\delta')$ that gives that

$$\odot = \begin{pmatrix} * & * & * & 0 \\ * & \xi' & * & \xi' + \beta''\xi'' \\ 0 & 0 & * & * \\ 0 & 0 & * & \xi'' \end{pmatrix}$$

so that switching indices 2 and 3 we get

$$\equiv=\begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & \xi' & \xi' + \beta''\xi'' \\ 0 & 0 & \xi'' \end{pmatrix}$$

He last formula implies

$$\mathfrak{R}_{m+n(\pi;\boldsymbol{B})(\sigma)} = \begin{pmatrix} \beta' & 0 \\ 0 & \beta'' \end{pmatrix} \begin{pmatrix} \xi' & \xi' + \beta''\xi'' \\ 0 & \xi'' \end{pmatrix}$$

$$= \begin{pmatrix} \beta' & 0 \\ 0 & \beta'' \end{pmatrix} \begin{pmatrix} \xi' & \xi' + \beta''\xi'' \\ 0 & \xi'' \end{pmatrix}$$

$$=$$

$$\left(\mathfrak{R}_m \begin{pmatrix} (\pi;\boldsymbol{B})(\sigma') & \mathfrak{R}_m(\pi;\boldsymbol{B})(\sigma')\mathfrak{R}_n t(\pi;\boldsymbol{B})(\sigma'') \\ 0 & \widetilde{\mathfrak{R}_m}(\pi;\boldsymbol{B})(\sigma'') \end{pmatrix} \right)$$

Comparing this with the definition of $\overline{\partial}_{m,n+p}\ \widetilde{\mathfrak{R}_m}(\pi;\boldsymbol{B})$we find that we have proved the following result.

Lemma.

$$\left(id_E \oplus \overline{\partial}_{m,n}(\pi;\boldsymbol{B})\right)(\sigma';\sigma'')$$
$$= -\widetilde{\mathfrak{R}_m}(\pi;\boldsymbol{B})(\sigma') \oplus_E \widetilde{\mathfrak{R}_m}(\pi;\boldsymbol{B})(\sigma'')$$

In the statement of the lemma $id_E \oplus \overline{\partial}_{m,n}$ refers to applying $\overline{\partial}$ to an E-valued fully matricial analytic function. The $\oplus_E$ among two matrices with entries in E amounts to

$$\left(\sum_{1\leq 1, j\leq m} c'_{ij} \oplus e_g^{(m)}\right) \oplus_E \left(\sum_{1\leq k, 1\leq n} c''_{kj} \oplus e_{kj}^{(n)}\right) = \sum_{1,j,k,1} c'_{ij} c''_{k1} \oplus e_g^{(m)} \oplus e_{l1}^{(m)}$$

We can write the resolvent equation also in a more compact

Proposition:

$$\left(id_E \oplus \overline{\partial}\,\right) \mathfrak{R}(\pi\colon B) = -\widetilde{\mathfrak{R}}(\pi\colon B) \oplus_E \widetilde{\mathfrak{R}}(\pi\colon B)$$

6.2 Martix entries of resolvents. An extension of the duality transform of [16], from the case of $y \in E$ to the case of $\pi \in G_{r_1}(E)$, includes in particular also the possibility of working with "unbounded operators Y" represented by their graph and therefore the definition of the albebra $\mathfrak{R}d(Y, B)$in [16] , 9,1 which includes Y, does not appear. By $\mathscr{y}\mathscr{k}(\pi, B)$ we shall denote the set of matrix coefficient of $\{-\mathfrak{R}_n(\pi; B)(\sigma)\} n\epsilon\mathbb{N}, \sigma\epsilon\bar{p}_n(\pi, B)$, By $\mathscr{y}\mathfrak{R}(\pi, B)$we shall denoe the linear span of $\mathscr{y}\mathfrak{R}(\pi, B)$.

Lemma: $\mathscr{y}\mathfrak{R}(\pi, B)$ is closed under multiplication in particular $\mathscr{y}\mathfrak{R}(\pi, B)$ is a sub-algebra of E.

Proof. The lemma is a consequence of the computations in 6.1 indeed let a,b be the (i, j) and respectively the (k, I) matrix coefficient of $\mathfrak{R}_n(\pi; B)(\alpha')$ and $-\mathfrak{R}_n(\pi; B)(\sigma'')$ and let α be defined like in 6.1 with $t = e_{jk}$ then the computation of $-\mathfrak{R}_n(\pi; B)(\sigma)$ we did shows that its $(i, m + I)$- entry is exactly the (f, I)- entry of

$$\left(-\mathfrak{R}_m(\pi; B)(\alpha')\right)_{e_{jk}} - \mathfrak{R}_n(\pi; B)(\sigma'')$$

Which is ah,

6.3 he duality transform. Let E_I be the closure in E of $\mathcal{Y}\mathfrak{R}(\pi, B)$ we will define the duality transform associated with π and B on he topological dual E_I^d of E_I in general, he bialgebra structure is only " partially" defined on E_I^d for analysis reasons. Which cannot be dealt in this generality, we will therefore often look for formulations which avoid such problems or we will introduce extra assumptions (as we did in [16]). Some important instances when these assumptions are satisfied will be shown in §12.

If $\varphi \epsilon E_I^d$, we define $\mathcal{Y}(\varphi) \epsilon / d\big(\bar{p}(\pi: B)\big)$ by $\mathcal{U}(\varphi) = (\mathcal{U}(\varphi)_n)_{n\epsilon\mathbb{N}}$

$$\mathcal{U}(\varphi)_n(\sigma) = \big(id_{\mathfrak{M}_n} \oplus \varphi\big)\big(\mathfrak{R}_n(\pi; B)(\sigma)\big)$$

for $\sigma \epsilon \bar{p}_n(\pi: B)$ Since $\oplus\varphi$ is $\mathfrak{M}_n$ linear on $\mathfrak{M}_n\ (B)$ we infer that $\mathcal{U}(\varphi)$ is fully matricial since $\overline{\mathfrak{R}}(\pi: B)$ is fully matricial. He continuity assumption on σ is necessary to obtain the analyticity of $\mathcal{U}(\varphi)$

we also remark that $\mathcal{U}(\varphi) = 0$ implies $\varphi = 0$ that is $\mathcal{U}$ is injective.
$\mathcal{U}(\varphi) = 0$ implies $\pi: B = 0$ and E_I is the closure of $\mathcal{Y}\mathfrak{R}(\pi: B)$

up to now $\mathcal{Y}\mathfrak{R}(\pi: B) = 0$ is only an algebra so we have only a coalgebra structure on the dual (modulo technical problems). He behavior of $\mathcal{Y}$ with respect to this comultiplication is recorded in he next proposition.

Proposition: if $\varphi \epsilon E_I^d$, $\sigma' \epsilon \bar{p}_n(\pi: B)$, then we have

$$\big(id_{\mathfrak{M}_n} \oplus id_{\mathfrak{M}_n} \oplus \varphi\big)\left(\overline{\mathfrak{R}}_m(\pi: B)(\sigma') \oplus_E \overline{\mathfrak{R}}_n(\pi: B)(\rho')\right)$$

$$= \bar{\partial}_{m,n}(\mathcal{Y}(\sigma)_{m+n})(\sigma' : \sigma'')$$

Proof. The proposition is exactly what we obtain from Lemma 6.1 when we apply $id_{\mathfrak{M}_n} \oplus id_{\mathfrak{M}_n} \oplus \varphi$ to the equality thee.

To justify out assertion that the above proposition shows the behavior of $\mathcal{U}$ with respect to the comultiplication, note that the right hand side is the (m,n) component of $\bar{\partial}\, \mathcal{U}(\varphi)$, while he left hand side corresponds to $(\mathscr{Y} \oplus \mathscr{Y})(\varphi\ o\ \mu)$ with μ denoting he multiplication on $\mathscr{Y}\Re(\pi: B)$(see also the proof of Lemma 6.2)

6.4 further properties of the duality transform arise when there is an appropriate derivation-comultiplication, on $\mathscr{Y}\Re(\pi: B)$. To avoid questions such as the action of the derivation on elements of the Grassmannian, we will resort to a somewhat tautological (from the point of view of the duality transform) characterization of he derivation.

We will assure there is a derivation.

$$\partial_{\pi:B}: \mathscr{Y}\Re(\pi: B) \rightarrow \mathscr{Y}\Re(\pi: B) \oplus \mathscr{Y}\Re(\pi: B)$$

Such that

$$\left(id_{\mathfrak{M}_n} \oplus \partial_{\pi:B}\right)\overline{\Re}_n(\pi: B)(\sigma) = \overline{\Re}_n(\pi: B)(\sigma) \oplus_{\mathfrak{M}_n} \overline{\Re}_n(\pi: B)(\sigma)$$

For all $n \epsilon \mathbb{N}$ and $\sigma \epsilon \bar{p}_n(\pi: B)$

For the universal unitary and hermitian Grassmannian elements this will be proved in §12

Remark that in view of lemma 6.2 the linear map $\partial_{\pi:B}$ is completely determined by he relation we assume. Thus he assumption means that this unique linear map exist and that it is a derivation. Note also that Lemma 6.2 similarly implies that $\partial_{\pi:B}$, if it exist is co-associative .

Proposition. If $\varphi_{1,}\varphi_2, \varphi_3\ \epsilon E_1^d$ are such that

$$\varphi_{1,} = (\alpha) = (\varphi_2 \oplus \varphi_3) o \partial_{\pi:B}\ (\sigma) \text{ if } \alpha\epsilon\ \mathcal{Y}\Re(\pi: B)$$

Then we have

$$\mathcal{U}(\varphi_{1,}) = \mathcal{U}(\varphi_{2,})\mathcal{U}(\varphi_{3,})$$

Proof. The proposition is almost obvious in view of the way we defined $\partial_{\pi:B}$

Of course, as the reader probably already observed, the condition characterizing $\partial_{\pi:B}$ replaces in the Grassmannian context the condition $\partial B = 0, \partial Y = 1 \oplus 1$ we required in the affine case $(see[16], 9.2)$, which corresponds to $\pi = \begin{pmatrix} 0 & 1 \\ 1 & y \end{pmatrix} / \overline{\boldsymbol{\lambda}1}$.

6.5 he duality transform of traces. In this section we return to the context of 6.3 , that is we will not use the derivation – comultiplication of $\mathcal{Y}\Re(\pi: B)$. We will record here that [16], proposition 9.5 on transforms of traces in the affine case extends immediately to the Grassmannian setting.

Proposition. An element $\varphi\epsilon E_1^d$ satisfies he trace-condition $\varphi([E_1, E_1]) = 0$ if and only if.

$$\bar{\partial}_{m,n}(\mathcal{U}(\varphi)_{m+n})(\sigma_1 : \sigma_2) = \varepsilon o \bar{\partial}_{m,n}(\mathcal{U}(\varphi)_{m+n})(\sigma_1 : \sigma_1)$$

for all $\sigma_1 \epsilon \bar{p}_m(\pi; B), \sigma_2 \epsilon \bar{p}_n(\pi; B)$ and indices I,j,k,I. the last equality is then equivalent, by proposition 6.3 to

$$\bar{\partial}_{m,n}\left(\mathcal{U}(\varphi)_{m+n}\right)(\sigma_1{:}\,\sigma_2) = \varepsilon o\bar{\partial}_{m,n}\left(\mathcal{U}(\varphi)_{m+n}\right)(\sigma_2{:}\,\sigma_1)$$

7. More on the fully matricial affine space

Roughly, a large part of this section is about the analogue of polynomials in he context of fully matricial analytic function on the fully matricial affine space. Besides providing a way to construct fully matricial analytic function, this material will also underline the series expansions in §13.

7.1 The polynomial sub-bialgebra $\wp(B^d)$ of $\mathrm{d}\left(\mathfrak{M}(B)\right)$. throughout 7.1 it will suffice to assume that B is a complex Banach space and $1\epsilon B$ is a non-zero vector(used in the definition of ∂), there is no need for a multiplication on B here.

The full matricial affine space over B that is the largest fully matricial B-set will be denoted $\mathfrak{M}(B)_{n\epsilon\mathbb{N}}$.

By $\mathbb{1}\epsilon.\,d\left(\mathfrak{M}(B)\right)$ we denote the unit element $\mathbb{1} = (I_n \oplus 1)_{n\epsilon\mathbb{N}}$ (constant functions). If $\varphi\epsilon B^d$ (the topological dual of B) we define $z(\varphi) = (z(\varphi)_n)_{n\epsilon\mathbb{N}}\epsilon d\left(\mathfrak{M}(B)\right)$ by

$$z(\varphi)_n\left(\begin{pmatrix} b_{11} & \cdots & b_{In} \\ \vdots & & \vdots \\ b_{nI} & \cdots & b_{nn}\end{pmatrix}\right) = \begin{pmatrix} \varphi(b_{11}) & \cdots & \varphi(b_{In}) \\ \vdots & & \vdots \\ \varphi(b_{nI}) & \cdots & \varphi(b_{nn})\end{pmatrix}\epsilon\mathfrak{M}_n(\mathbb{C})$$

Since $z(\varphi)_n$ is linear the definition of ∂ immediately gives

$$\partial z(\varphi) = \varphi(1)\mathbb{1}\oplus\mathbb{1}$$

We shall denote by $\mathrm{y}(B^d)$ the subalgebra of $\mathrm{d}\left(\mathfrak{M}(B)\right)$ generated by $\mathbb{1}$ and $\left\{z(\varphi)\ \middle|\varphi\epsilon B^d\right\}$, it is easy to see that $\mathrm{y}(B^d)$ is isomorphic to the tensor – algebra $\mathrm{I}(B^d)$ over the vector space B^d. Indeed if $\varphi_{1,\dots}\varphi_n$ are linearly independent in B^d we can find $b_{1,\dots}b_n\epsilon B$ so that $\varphi_i\left(b_j\right) = \delta_{ij}\ if\ P\epsilon\mathbb{C} <$

$X_1, \dots, X_n >$ is a polynomial in the noncommulting indeterminates $X_1, \dots, X_n$, so that P$\neq 0$, then there is $N \epsilon \mathbb{N}$ so that we can find $N \times N$ matrices $A_k \epsilon \mathfrak{M}_n(\mathbb{c})$, $1 \leq k \leq n$ so that $p(A_{1,\dots,}A_n) \epsilon \mathfrak{M}_n(\mathbb{C})$. thus $z(\varphi_1), \dots., z(\varphi_n)$ are algebraically free. His suffices to guarantee that the natural nuital homomorphism $T(B^d) \rightarrow d(\mathfrak{M}(B))$ defined by the linear map $B^d \ni \varphi \rightarrow z(\varphi) \epsilon\ d(\mathfrak{M}(B))$ is injunctive

The fact that ∂z$(\varphi) = \varphi(1)\mathbb{1}\oplus\mathbb{1}$ implies that $y(B^d)$ is a subcoalgebra of $d(\mathfrak{M}(B))$ that is

$$\partial z(B^d) \supset y(B^d) \oplus y(B^d)$$

Also the structure of ∂ on $y(B^d)$ is easy to identify. Let $1^{\perp} = \left\{\varphi \epsilon B^d \middle| \varphi(1) = 0\right\}$and choose some element $O \epsilon B^d$ so that $O(1) = 1$. Let then $z1^{\perp} \subset y(1^{\perp})$. then clearly $y(B^d)$ identifies with $(y(1^{\perp})) < z(O) >$ and $z(1^{\perp})$ is in ker ∂ while $\partial z(O) = 1 \oplus 1$. This means that he bialgebra $y(B^d)$ with he structure induced from d$(\mathfrak{M}(B))$ is isomorphic to $(y(1^{\perp})) < X >, \partial_{X:g(1^{\perp})})$ noted in particular that.

$$\text{Ker}\ \partial \cap \quad y(B^d) = y(1^{\perp})$$

Moreover, if B is a Banach space with a continuous conjugate-linear involution $(z(\varphi))^* = z(\varphi^*)$ where$\varphi^*(b) = \overline{\varphi(b^*)}$.

Also, at the end of 8.2 we will point out in a remark an additional feature of $y(B^d)$.

7.2 Decomposable and reducible points in $\mathfrak{M}(B)$. Like in 7.1 we will only require that B be a Banach space.

In view of the similarity and direct sum requirements for "fully matricial" objects, we are led to look at properties of points connected with these requirements .

Definition: An element $\beta\epsilon\mathfrak{M}_n(B)$ is decomposable if there are $\beta'\epsilon\mathfrak{M}_p(B), \beta^{"}\epsilon\mathfrak{M}_g(B)$ and S$\epsilon GL(n\colon\mathbb{C})$,so that $n = p, p > 0, q > 0$ and $S^{-1} = \beta^1 \oplus \beta^{"}$. an element $\beta\epsilon\mathfrak{M}_n(B)$ is reducible if there are $\beta'\epsilon\mathfrak{M}_p(B), \beta^{"}\epsilon\mathfrak{M}_p(B)$, $y\epsilon\mathfrak{M}_y(B)$ and $S\epsilon GL(n\colon\mathbb{C})$so that.

$$S\beta S^{-1}=\begin{pmatrix}\beta' & y \\ 0 & \beta^{"}\end{pmatrix}$$

and $p > 0, q > 0$. an element $\beta\epsilon\mathfrak{M}_n(B)$ is approximately decomposable (resp, reducible) if it is in the closure of the decomposable (resp, reducible) elements. Elements which are not decomposable (reducible , approximately decomposable, approximately educible) will be called indecomosable (resp, irreducible, strongly indecompsable, strongly irreducible).

7.3 to conclude this section of remarks about the fully matricial affine space, we should point out that there is a fully matricial action of the additive group B on $\mathfrak{M}(B)$. for each $h\epsilon B$ there are fully matricial maps $T(b) = (T(b)_n)_{n\epsilon\mathbb{N}}$: $\mathfrak{M}(B)$ where $T(b)_n(\beta) = \beta + b\oplus I_n$which give an action of B on $\mathfrak{M}(B)$.

In case B is a Banach algebra, there is also a multiplication action given by fully matricial maps $(L(b)_n)_{n\epsilon\mathbb{N}}$ $R(b) = (R(b)_n)_{n\epsilon\mathbb{N}}$ so that $L(b)\beta = (b\oplus I_n)\beta$ and $R(b)\beta(b\oplus I_n)$.

Note also that even if B is only a Banach space there is a multiplicative action of $\mathbb{C}$ on $\mathfrak{M}(B)$.

8. More on the fully matricial B-Grassmannian and on $\bar{\partial}$

In this section we present further properties of the fully matricial B-Grassmannian $G_r(B) = \left(R_{r_n}(B)\right)_{n\epsilon\mathbb{N}}$. This includes the action by fully matricial automorphisms of $GL(2;B)\ on\ G_r\ (B)$ and the existence of a coderivation A such that A-id plays the role of a grading of the bialgebras $d(\Omega)$. We also discuss the properties of A in connection with the duality transform.

8.1 The GL(2;B) action on Gr (B). we recall that in 3.2 we defined $g\pi\ if$

$$h = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \epsilon GL(2;B)$$

Gives rise to elements $h_n \epsilon GL_2(\mathfrak{M}_n(B))$ where $h_n = \begin{pmatrix} L_n \oplus b_{11} & L_n \oplus b_{12} \\ L_n \oplus b_{12} & L_n \oplus b_{22} \end{pmatrix}$ we define $C(h); G_r(B)$ by mapping $\pi_n G_r$ to $h_n \pi_n$ it is easy to check that $h_{m+n}(\pi_m \oplus \pi_n) = (h_m \pi_n) \oplus (h_n \pi_n)$ and that $h_n(\delta, \pi_n) = \delta\ (h_n \pi_n)\ if\ \delta \epsilon GL(n;\mathbb{C})$. this establishes that $\mathbb{C}(.)$is an action of $GL(2;B)$ by fully marticial automorphisms of $G_r(B)$.

It is easily seen that $C(h)$ preserves transversality in each $G_r(B)$.

Clearly, when B= $\mathbb{C}$ the $GL(2;\mathbb{C})$ - action on $Gr_1(\mathbb{C})$ is the usual action on the remann sphere by fractional linear transformation.

2.8 the education A Let $f = (f_n)_{n\epsilon\mathbb{N}} \epsilon d(\Omega)$, where $\Omega = (\Omega_n)_{n\epsilon\mathbb{N}}$ is a fully matricial open B-set of he Grassmannin. We define Af=$\left(A_{nf_n}\right)_{n\epsilon\mathbb{N}} \epsilon d(\Omega)$. By

$$Af = \frac{d}{d_{\varepsilon_2}}\left(e' f o C\left(\begin{pmatrix}1 & 0\\ 0 & e'\end{pmatrix}_n \pi_n\right)\right) \Big| f = 0$$

Which, componentwise amounts:

$$(A_n f_n)(\pi_n)\frac{d}{d_{\varepsilon_2}}\left(e' f\left(\begin{pmatrix}1 & 0\\ 0 & e'\end{pmatrix}_n \pi_n\right)\right) \Big| f = 0$$

Since

$$Af = f\frac{d}{d_{\varepsilon_2}}\left(f o C\left(\begin{pmatrix}1 & 0\\ 0 & e'\end{pmatrix}_n \pi_n\right)\right) \Big| f = 0$$

It follows that A-id is it derivation of $d(\Omega)$

To prove that A is a coderivation amounts to proving that

$$\bar{\partial} o A = (A \oplus id + id \oplus A) o \bar{\partial}$$

This will be a consequence of the following lemma

Lemma. We have

$$\bar{\partial}\left(f o C\left(\begin{pmatrix}1 & 0\\ 0 & e'\end{pmatrix}\right)\right) = e'(\bar{\partial} f) o \left(C\left(\begin{pmatrix}1 & 0\\ 0 & e'\end{pmatrix}\right) \times C\left(\begin{pmatrix}1 & 0\\ 0 & e'\end{pmatrix}\right)\right)$$

Proof. Let $\pi_n = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} / \overline{\boldsymbol{\lambda} m}\,, \pi_n = \pi_n = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \overline{\boldsymbol{\lambda} n}$ and let T and T' be define

$$\mathrm{T}{=}\bar{\partial}_{m.n}\left(\bar{\partial}_{m.n}\left(f o C\left(\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}\right)\right)(\pi_{m,}\pi_n)\right)$$

$$\mathrm{T}{=}\bar{\partial}_{m.n}\left(\bar{\partial}_{m.n}(f)\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_m \pi_{m,}\right)\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}\pi_n\right)\right)$$

Since $\bar{\partial}_{m.n}$ is an isomorphism, it will suffice to prove that $\mathrm{T}(\pi)$ $=T'(e's)$ for all $\delta \epsilon \mathfrak{M}_{m,n}(\mathbb{C})$. indeed, we have.

$$f_{m,n}\left(\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_{m+n}\begin{pmatrix} d_1 & 0 & b_1 & 0 \\ 0 & a_2'' & 0 & b_2 \\ c_1 & 0 & d_1 & sb_2 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} / \lambda_{\widetilde{m'' + n}}\right)$$

$=$

$$f_{m,n}\left(\begin{pmatrix} d_1 & 0 & b_1 & 0 \\ 0 & a_2'' & 0 & b_2 \\ e'c_1 & 0 & e'd_1 & e'sb_2 \\ 0 & e'c_2 & 0 & d_2 \end{pmatrix} / \lambda_{\widetilde{m'' + n}}\right)$$

$=$

$$\begin{pmatrix} f_m\left(\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_m \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} / \overline{\boldsymbol{\lambda} m}\right) & T(s) \\ 0 & f_m\left(\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_m \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} / \overline{\boldsymbol{\lambda} m}\right) \end{pmatrix}$$

$$\begin{pmatrix} f_m\left(\begin{pmatrix} a_1 & b_1 \\ e'c_1 & e'd_2 \end{pmatrix} / \overline{\boldsymbol{\lambda} m}\right) & T(e's) \\ 0 & f_n\left(\begin{pmatrix} a_2 & b_2 \\ e'c_2 & e'd_2 \end{pmatrix} / \overline{\boldsymbol{\lambda} m}\right) \end{pmatrix}$$

Which implies $T(X) = T'(e^{'}s)$

To conclude the proof of the fact that A is a conderivation it will suffice to remark that taking the derivative $\frac{d}{di}$ at t=0 of the equality in the preceding lemma gives

$$\bar{\partial}(Af - f) = \bar{\partial}f + \big((A - id)\oplus id\oplus(A - id)\big)\bar{\partial}f$$

Which immediately implies.

$$\overline{\partial A}f=(A\oplus id + id\oplus A)\bar{\partial}f$$

Proposition. A-id is a derivation of $s(\Omega)$ and A is also a coderivation , that is $\bar{\partial}oA = (A\oplus id + id\oplus A)o\bar{\partial}$

8.3 he derivation D of $yR(\pi; B)$. in the next section we will show that the coderivation A discussed in the previous section is natural form the point of view of the duality transform. This will involve describing what the natural coderivation l on yR $(\pi; B)$ should be so that for the duality described in [1.5], theorem 5.3, he dual coderivation corresponds under the duality transform to A. since in 6.4 we assumed the existence of a derivation-comultiplication $\partial_{\pi;B} on\ yR(\pi; B)$, we will handle L similarly based on an additional assumption.

Remark. In the affine case of (B), we have $Ar(B^d) \subset r\big(B^d\big)$ and

$$\mathrm{A}\big(z(\varphi_1)\dots z(\varphi_n)\big) = (n+1)_z(\varphi_1)\dots z(\varphi_n)$$

Like in 6.2 we let $\pi\epsilon Gr_1(E)$ and we consider $yR(\pi, B)$. the assumption about L is roughly that on $yR(\pi, B)$. there is a linear map D corresponding to the infinitesimal deformation of π, into $\begin{pmatrix}1 & 0\\0 & e'\end{pmatrix}_1 \pi\ with\ t \to 0$. we will show that D must then be a derivation of $yR(\pi, B)$ with values in itself. [in case $\pi\ is\ he\ graph\ of\ an\ element\ y\epsilon E$, the deformation is $y \to e^t y$ with $t \to 0$.

More precisely our assumption can be formulated as follows: we assume there is a linear map D: $yR(\pi, B) \to E$ so that

$$(id\mathfrak{M}_n \oplus D)\bar{R}_n(\pi, B)(\sigma) = \frac{d}{dt}\bar{R}_n\left(\begin{pmatrix}1 & 0\\0 & e'\end{pmatrix}_n \pi, B\right)\Big| t = 0$$

For all $\sigma\epsilon\bar{p}_n(\pi; B), n\epsilon\mathbb{N}$. there is a simple identity which we will use to show that D takes values in $yR(\pi, B)$.

Lemma. We have

$\sigma\epsilon\bar{p}_n\left(\left(\begin{pmatrix}1 & 0\\0 & e'\end{pmatrix}_1 \pi, B\right)\right) iff \begin{pmatrix}1 & 0\\0 & e'\end{pmatrix}_n\Big) \sigma\epsilon\bar{p}_n(\pi, B)$ moreover

$$\bar{R}_n \begin{pmatrix}1 & 0\\0 & e'\end{pmatrix}_1 \pi, B\Big)(\sigma) = e^{-1}\bar{R}_n(\pi; B)\begin{pmatrix}1 & 0\\0 & e^{-1}\end{pmatrix}_n (\sigma)\Big)$$

Proof. Let

$\pi\begin{pmatrix}1 & 0\\0 & e'\end{pmatrix}/\bar{\lambda}1, \sigma = \begin{pmatrix}\alpha & \beta\\\gamma & \delta\end{pmatrix}/\bar{\lambda}_n\ and\ let\ also$

$$\begin{pmatrix}I_n \oplus a & I_n \oplus b\\I_n \oplus c & I_n \oplus d\end{pmatrix}/\bar{\lambda}_n$$

and $\alpha_{n=I_n} \oplus a, b' = I_n \oplus \mathrm{b}, c, d' = I_n \oplus d.$ then

$\sigma \epsilon \bar{p}_n \left(\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_1 \pi, B \right)$ means $\begin{pmatrix} \alpha & \beta \\ e'd' & \delta \end{pmatrix}$ is invertible and this is obviously equivalent to $\begin{pmatrix} b' & \beta \\ d' & e^{-1}\delta \end{pmatrix}$ being invertible, which is that $\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_1 \sigma \epsilon \bar{p}_n(\pi; B)$

moreover $\begin{pmatrix} b' & \beta \\ e'd' & \delta \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ * & \xi \end{pmatrix}$ and $\begin{pmatrix} \alpha & \beta \\ d' & e'\delta \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ * & \xi \end{pmatrix}$ then $\xi = e'\xi$.

He last part of the lemma follows from the two Grassmannian resolvents being equal to$\beta\xi$and$\beta\xi$ respectively.

with the notations used in the proof of the lemma , to show that

$$\mathrm{D}\big(yR(\pi, B)\big) \subset yR(\pi, B)$$

We must prove in view of the definition of D hat the entries of $\frac{d}{dt}\left(e^{-1}\beta\xi(t)\right)\Big|_{1-0}$ are in $yR(\pi, B)$ or equivalently he entries of $\frac{d}{dt}\left(e^{-1}\beta\xi(t)\right)\Big|_{1-0}$ since $\beta\xi(0) = \beta\xi(0)$ is a resolvent, its entries are in $yR(\pi, B)$, so we are left with showing $\beta\left(\frac{d}{dt}\xi(t)\right)\Big|_{1-0}$ has entries in $yR(\pi, B)$ we have

$$\begin{pmatrix} * & * \\ * & \frac{d}{dt}\xi \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} b' & \beta \\ d' & e^{-1}\delta \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ * & \xi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & e^{-1}\delta \end{pmatrix}\begin{pmatrix} * & * \\ * & \xi \end{pmatrix}$$

hence we infer that.

$$\left(\frac{d}{dt}\xi(t)\right)\Big|_{1-0} = \xi(0)\delta\xi(0)$$

and we must show that $\beta\xi(0)\delta\xi(0)$ has entries in $yR(\pi,B)$. It is easily seen that the (2,4) block entry of the 4x4 block matrix

$$\Gamma^{-1}\begin{pmatrix} b' & \beta & 0 & 0 \\ d' & \delta & 0 & \delta \\ 0 & 0 & b' & \beta \\ 0 & 0 & d' & \delta \end{pmatrix}$$

Is precisely $\xi(0)\delta\xi(0)$on the other hand if S is the permutation matrix.

$$S = \begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & 0 & I_1 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}$$

We see that

$$S\Gamma^{-1}S^{-1} = \begin{pmatrix} b' & 0 & \beta & 0 \\ 0 & b' & 0 & \beta \\ d' & 0 & \delta & \delta \\ 0 & d' & 0 & \delta \end{pmatrix} = \begin{pmatrix} * & * \\ * & Z \end{pmatrix}$$

Where $\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} Z$ is an $R_n(\pi,B)(\mu)$for some $\mu = \bar{p}_{2n}(\pi,B)$.

Hence he entries of $\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} Z$ are in $yR(\pi,B)$. returning to Γ^{-1} we see that the $(2,4)$ block entry of Γ^{-1} coincides with the$(3,4)$ block entry of $S\Gamma^{-1}S^{-1}$ which is the $(1,2)$ – block entry of Z (the blocks are n x n). this concludes the proof that D maps $yR(\pi,B)$into itself.

To prove that D is a derivation we return to the proof of lemma 6.2 where we showed $yR(\pi, B)$ is closed under multiplication. With the notation of Lemma 6.2 we have

- $R_{m+n}(\pi, B)(\sigma)_{i.m+1} = R_m\big((\pi, B)(\sigma')\big)_{ij} R_n\big((\pi, B)(\sigma'')\big)_{kl}$

Where $\sigma', \sigma'', (ij), (kj)$ were given. then the definition of D applied to the above equality shows that D is a derivation. Concluding we have proved.

Proposition: Under our assumption D is a derivation of $yR(\pi, B)$ into itself.

8.4 The coderivation L of $yR(\pi, B)$. In this section we assume the existence of $\partial_{\pi B}$ with the properties outlined in 6.4 and we also assume the existence of the linear map D like in 8.3 and which implies that D is a derivation $yR(\pi, B)$. In addition, we will assume that $\partial_{\pi B}$ is closed as an operator on $yR(\pi, B)$ endowed with the norm from E.

We define

$$\text{L:D+id: } yR(\pi, B) \rightarrow yR(\pi, B)$$

Clearly L-id is a derivation of $yR(\pi, B)$.

Lemma. The map L is a coderivation of $yR(\pi, B)$, $(yR(\pi, B), \partial_{\pi B})$ that is

$$,\partial_{\pi B}\ o\ L = (id \oplus L + L \oplus id) o, \partial_{\pi B}$$

Proof. since $yR(\pi, B)$ is the linear span of $yR(\pi, B)$ it suffices to check that the equality to be proved holds for the entries of $yR(\pi, B)$. In view of the definitions of $, \partial_{\pi B}$ and L this boils down to showing that

$$(id_{\mathfrak{M}_n} \oplus, \partial_{\pi B}) \left(\frac{d}{dt} \bar{R}_n\ (\pi, B) \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix}_n \sigma \right) \right) \Bigg|_{t=0}$$

It is immediate that the right hand side equals.

$$\frac{d}{dt} \left(\bar{R}_n\ (\pi, B) \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix}_n \sigma \right) \oplus_{\mathfrak{M}_n} \bar{R}_n\ (\pi, B) \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix}_n \sigma \right) \right) \Bigg|_{t=0}$$

$$= \frac{d}{dt} (id_{\mathfrak{M}_n} \oplus, \partial_{\pi B}) \bar{R}_n\ (\pi, B) \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix}_n \sigma \right) \Bigg|_{t=0}$$

Thus the equality to be proved reduces to showing that.

$$\frac{d}{dt} (id_{\mathfrak{M}_n} \oplus , \partial_{\pi B}) \bar{R}_n (\pi, B) \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix}_n \sigma \right) \Bigg|_{t=0}$$

$$= (id_{\mathfrak{M}_n} \oplus \partial_{\pi B}) \left(\frac{d}{dt}\ \bar{R}_n (\pi, B) \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix}_n \sigma \right) \Bigg|_{t=0} \right).$$

Clearly, the last equality is a consequence of the assumption that $\partial_{\pi B}$ is closed.

8.5 the coderivation L and A and the duality transform. In this section the same assumptions as in 8.4 will hold throughout.

Let E_1 be the closure of $yR(\pi, B)$ in E and let $\varphi \epsilon \partial_j^d$ so that φ is in the domain of L^d, that is φ o L defined on $yR(\pi, B)$ is bounded (extends to an element of E_1^d).

Recall that the n-th component of the duality transform is defined by

$$\mathscr{Y}=(\varphi)_n(\sigma) = \left(id_{\mathfrak{M}_n} \oplus \varphi\right)\left(\bar{R}_n(\pi, B)(\sigma)\right).$$

We have

$$\mathscr{Y}=(L^d\varphi)_n(\sigma) = \left(id_{\mathfrak{M}_n} \oplus \varphi\right)\left(id_{\mathfrak{M}_n} \oplus L\right)\left(\bar{R}_n(\pi, B)(\sigma)\right).$$

$$= \left(id_{\mathfrak{M}_n} \oplus \varphi\right) \frac{d}{dt}\ \bar{R}_n(\pi, B)\left(\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}_n (\sigma)\right)\Bigg|_{t=0}.$$

$$= \frac{d}{dt}\left(id_{\mathfrak{M}_n} \oplus \varphi\right) \left(\bar{R}_n(\pi, B)\left(\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}_n (\sigma)\right)\right)\Bigg|_{t=0}$$

$$= -\frac{d}{dt}\Upsilon(\varphi)_n \left(\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}_n (\sigma)\right)\Bigg|_{t=0}$$

$$= -\left((A - id)y(\varphi)_n\right)(\sigma) = (id - A)y(\varphi)_n(\sigma)$$

Thus we have proved the following proposition.

Proposition we have $y(L^d\varphi) = (id - A)y(\varphi)$.

Note that the way the coderivation should be transformed under duality given in [15]. Them. 5.3. is in agreement with the above proposition.

9. The Grassmannian involution:

Throughout this section B will be a unital Banach algebra with involution. We will discuss the corresponding involutions on Gr (B) and bialgebras $\mathscr{A}(\Omega)$, and the properties of the duality transform related to the involutions.

9.1 the involution on Gr (B). On the affine fully matricial space the involution amounts simply to the conjugate-linear antiautomorpism $T \rightarrow T^*$on $\mathfrak{M}_n(B)$, $n\epsilon\mathbb{N}$. he extension to an antiholomorphic automorphism of he fully matricial B-Grassmannian has some additional technical points.

We will first define he orthogonal $\pi^{\perp}$ of $n\epsilon Gr_n(B)$and then we shall define, $\pi^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_n \pi^{\perp}$

$if\pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} /\bar{\lambda}n$ we define $\pi^{\perp} = \begin{pmatrix} z^* & x^* \\ t^* & y^* \end{pmatrix} /\bar{\lambda}n$ where

$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$

To check that $\pi^{\perp}$ is well defined we begin with a simple algebraic lemma.

Lemma. Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$=$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ and $\begin{pmatrix} a' & b \\ c' & d \end{pmatrix}^{-1} =$ $\begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$ then$\begin{pmatrix} x & y \\ z' & t' \end{pmatrix}$ is invertible and there is w invertible so that wx $= x'$,wy=y'

Proof. Since $\begin{pmatrix} x & y \\ z' & t' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$

Is invertible, we inter $\begin{pmatrix} x & y \\ z' & t' \end{pmatrix}$ is invertible.

On the other hand $\begin{pmatrix} x & y \\ z' & t' \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} w & 0 \\ * & 1 \end{pmatrix}$

Is invertible, so that w is invertible and we have

$$\begin{pmatrix} x & y \\ z' & t' \end{pmatrix} = \begin{pmatrix} w & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$$

Which gives $x = wx^{-1}, y = wy^{-1}$.

Corollary. The map $\pi \to \pi^{\perp}$ is well-define.

Proof. We have two things to check.

First, using the same notation as in the lemma, since

$$\pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \bar{\lambda} n = \begin{pmatrix} a' & b \\ c' & d \end{pmatrix} / \bar{\lambda} n$$

We must show that $\begin{pmatrix} z^* & x^* \\ t^* & y^* \end{pmatrix} / \bar{\lambda} n = \begin{pmatrix} z^{1*} & x^{1*} \\ t^{1*} & y^{1*} \end{pmatrix} / \bar{\lambda} n$

this is indeed so, since w^* is invertible and $x^*, w^* =$ $x^*, y^*, w^* = y^*$.

Secondly if u is invertible and $\begin{pmatrix} x & y \\ z' & t' \end{pmatrix} = \begin{pmatrix} a & bu \\ c & du \end{pmatrix}^{-1}$

Then it is easily seen that $x = x^n$,y $= y''$ and hence clearly.

$$\begin{pmatrix} z^{n*} & x^{n*} \\ t^{n*} & y^{n*} \end{pmatrix} /\bar{\lambda}n = \begin{pmatrix} z^* & x^* \\ t^* & y^* \end{pmatrix} /\bar{\lambda}n$$

Remark also that the definition of $\pi^{\perp}$ can also be written.

$$\pi^{\perp} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) /\bar{\lambda}n = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{*-1} \right) /\bar{\lambda}n$$

Proposition . we have $\pi^{**} = \pi$ the maps $\pi = \pi^*$ and $\pi \to \pi^{\perp}$ are antiholomorphic automorphisms of $Gr_n(B)$.

The antiholomorphicity needs only to be checked in charts.

$$\left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \right) /\bar{\lambda}n | f \epsilon \mathfrak{M}_n(B) \right\}.$$

If $\pi = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) /\bar{\lambda}n \ then \ \pi^{\perp} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) /\bar{\lambda}n$

which clearly is antiholomorphic as a function of $\epsilon \mathfrak{M}_n(B)$

that the definition of π^* extends the definition of the involution on the affine space is easily seen. Indeed , then $\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} /\bar{\lambda}n, \pi^{\perp} = \begin{pmatrix} 1 & -d^* \\ 0 & 1 \end{pmatrix} /\bar{\lambda}n$ and $\pi^* = \begin{pmatrix} 0 & 1 \\ -1 & d^* \end{pmatrix} /\bar{\lambda}n$

we conclude this subsection remarking that in the formula for $\pi^{\perp}$ the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ can be replaced by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, since this does not affect the second column in the result. Hence the formula for π^* can be written also in the form

$$\pi^* = \left(-\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*-1}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}\Big)/\bar{\lambda}n$$

note also that this gives $(C(g)\pi)^* = C\left({W_g}^{*-1}W^{-1}\right)\pi^*$ where W= $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $g\epsilon GL(2.B)$

9.2 the involution and he bialgebras $d(\Omega)$. it is easy to see that

$$(\pi\oplus\sigma)^* = \pi^*\oplus\sigma^*$$

and that $(\delta.\pi)^* = S^{*-1}.\pi^*$ where $\sigma\epsilon Gr_m(B), \delta\epsilon GL(n;\mathbb{C})$.

It follows that if $\Omega = (\Omega_n)_{n\epsilon\mathbb{N}}$ is a fully matricial set of the B-Grassmannian hen the same holds for $\Omega^* = (\Omega_n^*)_{n\epsilon\mathbb{N}}$ where Ω_n^* =$\Omega_n^* = \{\pi^*\Omega_n^* | \pi\epsilon\Omega_n\}$. clearly Ω is open if Ω^* is open.

If $f_n: \Omega_n \to \mathfrak{M}_n$ we define $f_n^*: \Omega_n^* \to \mathfrak{M}_n$ by $\left(f_n(\pi)\right)^*$ =$f_n^*(\pi^*)$, where $\pi\epsilon\Omega_n$ if f_n is analytic then so it f_n^* $and\ if\ f = (f_n)_{n\epsilon\mathbb{N}}\epsilon\ d(\Omega)$ then $f^* = (f_n^*)_{n\epsilon\mathbb{N}}\ d(\Omega)$ and the map $f = f^*$ is a conjugate linear antisomorphism. More generally there is a conjugate linear antiisomorphism $f = f^*$of $d\left(\Omega_1 : \Omega_2; \dots; \Omega_p\right)$ and $d\left(\Omega_1^* : \Omega_2^* \ ; \dots; \Omega_p^*\right)$ where

$$f^*_{n_1 \dots n_p}\left(\omega_1^*; \dots; \omega_p^*\right) = \left(f_{n_1 \dots n_p}\left(\omega_1; \dots; \omega_p\right)\right)^*$$

If $\Omega = \Omega^*$ then $d(\Omega)$ is an algebra with involution. More generally $\Omega_j = \Omega_j^*, 1 \leq j \leq p$.

The automorphism permuting the variables, that is

$$\left(\sigma_{1,2} f\right)_{m,n}(\sigma, \pi) = E_{m,n} o f_{n,m}(\sigma, \pi)$$

With $E_{m,n}: \mathfrak{M}_m \oplus \mathfrak{M}_n \ \rightarrow: \mathfrak{M}_n \oplus \mathfrak{M}_m$ the tensorial permutation isomorphism.

Proposition: $if\ f \in d(\Omega)$, then

$$\bar{\partial} f^* = \sigma_{1,2}\left(\bar{\partial} f\right)^*$$

In particular if $\Omega = \Omega^*$, this is the compatibility of the involution and comultiplication of $d(\Omega)$.

Proof. If $L \epsilon\ y\left(\mathfrak{M}_{m,n}\right)$ and $L^* \epsilon\ y\left(\mathfrak{M}_{m,n}\right)$ is defined by $L^*(y) = (L^*(y^*)^*$ then $\epsilon_{m,n}\left(\left(\alpha_{m,n}(L)\right)^*\right)$

In view of this it is easily seen that the proposition will follow if we prove that.

$$f_{m,n}^*\left(\left(S\Gamma\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & tb' \\ 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{pmatrix}\Gamma^{-1}\right)/\overline{\bar{\lambda} n + n}\right)$$

$$S\left(f_{m,n}\left(\left(S\Gamma\begin{pmatrix}\alpha' & \beta' & 0 & 0\\ y' & \delta' & 0 & t^*\beta\\ 0 & 0 & \alpha & \beta\\ 0 & 0 & y & \delta\end{pmatrix}\Gamma^{-1}\right)/\widetilde{\bar{\lambda}n+n}\right)\right)^* S^{-1}$$

WhereΓ, Γ', S are permutation matrices, the first two having the effect by conjugation of permuting second and third rows and columns in 4x4 block matrices and S permuting first and second rows and columns in a 2x2 block matrix (the sizes of blocks corresponding to m+m+n+n, n+n+m+m and n+m respectively). The other notations used are $t\epsilon\mathfrak{M}_{m,n}(\mathbb{C})$and

$$\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}/\bar{\lambda}n,\right)^* = \begin{pmatrix}\alpha & \beta\\ y & \delta\end{pmatrix}/\bar{\lambda}n$$

$$\left(\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}/\bar{\lambda}n,\right)^* = \begin{pmatrix}\alpha' & \beta'\\ y' & \delta'\end{pmatrix}/\bar{\lambda}n$$

Remarks that the right hand side of the equality to be proved is equal to.

$$\left(f_{m,n}\left(S\left(S\Gamma\begin{pmatrix}\alpha' & \beta' & 0 & 0\\ y' & \delta' & 0 & t^*\beta\\ 0 & 0 & \alpha & \beta\\ 0 & 0 & y & \delta\end{pmatrix}\Gamma'^{-1}\right)/\widetilde{\bar{\lambda}n+m}\right)\right)^*$$

Hence by the definition of $f_{m,n}$ it will suffice to show that

$$= S\left(\Gamma'\begin{pmatrix}\alpha' & \beta' & 0 & 0\\ y' & \delta' & 0 & t^*\beta\\ 0 & 0 & \alpha & \beta\\ 0 & 0 & y & \delta\end{pmatrix}\Gamma'^{-1}\right)/\widetilde{\bar{\lambda}n+n}$$

Writing the equality in the form $A//\widetilde{\bar{\lambda}n+n} = B//\widetilde{\bar{\lambda}n+n}$ the problem amounts to showing that $A^{-1}B$ is a lower triangular 2x2 block matrix with invertible diagonal blocks. Denoting by $\odot$ and Ξ the 4x4 explicitly written matrices in A and B and by Σ and W the matrices

$$\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \; and \; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \text{ we have}$$

$$A = W\Gamma \odot^{*-1} \Gamma^{-1}W^{-1}$$

$$\mathrm{B} = \Sigma\Gamma'\Xi\Gamma^{-1}\Sigma^{-1}$$

Hence

$$A^{-1}B = -W\Gamma \odot^{*} \Gamma^{-1}W^{-1}\Sigma\Gamma'\Xi\Gamma'^{-1}\Sigma^{-1}$$

$$= - W\Gamma \odot^{*} U\Xi U^{-1}\Gamma^{-1} \, W\Gamma \odot^{*} \Gamma^{-1}W^{-1}$$

Where U=$\Gamma^{-1}W^{-1}\,\Sigma\Gamma'$ it is easily seen that

$$U\begin{pmatrix} 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \\ 0 & -I_m & 0 & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix}$$

And hence that

$$U\Xi U^{-1}\begin{pmatrix} \delta & -y & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ t^{*}\beta & 0 & \delta' & -y' \\ 0 & 0 & -\beta' & \alpha' \end{pmatrix}$$

to compute $\Xi^{*}U\Xi^{-1}$ remark first that in view of the formula for π^{*}, we may assume

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} -\delta^* & \beta^* \\ y^* & -\alpha^* \end{pmatrix}$$

And its primed analogue. We get.

$$\bigodot^* U\Xi U^{-1} \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & b'^* t^* \end{matrix} & \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^* \end{pmatrix} \begin{pmatrix} \delta & -y & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ t^*\beta & 0 & \delta' & y' \\ 0 & 0 & \beta' & \alpha' \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ a'^* t^* \beta & 0 & I & 0 \\ 0 & b'^* t^* \alpha & 0 & I \end{pmatrix}$$

this in turn gives

$$-W\Gamma(\bigodot^* U\Xi U^{-1})\Gamma'^{-1}W^{-1} = -W = \begin{pmatrix} I & 0 & 0 & 0 \\ a'^* t^* \beta & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & b'^* t^* \alpha & I \end{pmatrix}$$

$$= -\begin{pmatrix} I & 0 & 0 & 0 \\ b'^* t^* \alpha & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & a'^* t^* \beta & I \end{pmatrix}$$

Which is a matrix of the desired kind.

9.3 The involution and he coderivation A. in this subsection we check the compatibility of A with the involution.

Proposition. $if\ f\ \epsilon d(\Omega)$ then we have

$$Af^* = (Af)^*$$

$\left(the\ same\ A\ denotes\ the\ coderivations\ in\ d(\Omega)\ and\ in\ d(\Omega^*)\right)$

Proof. $if\ \pi=\begin{pmatrix} a & b \\ c & d \end{pmatrix}/\bar{\lambda}n\epsilon\Omega_{n,}$ then

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_n \pi\right)^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_n \begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_n^{*-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \pi$$

$$=\begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \pi^* = \begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}_n^{*-1} \pi^*$$

From which the proposition follows immediately using he formula for a.

9.4. the involution and Grassmannian resolvents. In this subsection we check the behavior of resolvents with respect to the involution.

We will need an algebraic lemma which provides explicit formulae for resolvents.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} r & s \\ u & v \end{pmatrix}$$

Then the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is invertible if x $\beta+\gamma\delta$ is invertible, which is also *iff* $rb+sd$ is invertible. Moreover we then have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} (rb+sd)^{-1}r & (rb+sd)^{-1}s \\ (x\beta+y\delta)^{-1}x & (x\beta+\gamma\delta)^{-1}y \end{pmatrix}$$

Proof. Since

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & x\beta + y\delta \\ 1 & z\beta + t\delta \end{pmatrix}$$

We get the "iffx β+ $y\delta$ is invertible" part of the statement

And

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} 0 & x\beta + y\delta \\ 1 & z\beta + t\delta \end{pmatrix}^{-1}\begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$=\begin{pmatrix} * & * \\ (x\beta + y\delta)^{-1} & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$=\begin{pmatrix} * & * \\ (x\beta + y\delta)^{-1} & (x\beta + y\delta)^{-1} \end{pmatrix}$$

Similarly since:

$$\begin{pmatrix} y & \delta \\ u & v \end{pmatrix}\begin{pmatrix} b & \beta \\ d & \delta \end{pmatrix} = \begin{pmatrix} x\beta + y\delta \\ z\beta + t\delta \end{pmatrix}$$

We get he "iff rb+sd is invertible" part of the statement and

$$\begin{pmatrix} b & \beta \\ d & \delta \end{pmatrix}^{-1} = \begin{pmatrix} rb + sd & 0 \\ ub + ud & 1 \end{pmatrix}^{-1}\begin{pmatrix} r & s \\ u & v \end{pmatrix}$$

$$= \begin{pmatrix} rb + sd & 0 \\ * & * \end{pmatrix}\begin{pmatrix} r & s \\ u & v \end{pmatrix}$$

$$= \begin{pmatrix} (rb + sd)^{-1}r & (rb + sd)^{-1}s \\ * & * \end{pmatrix}^{-1}$$

The framework for resolvent will be a unitasl Banach algebra with involution E and abanach subalgebra with the same involution $I\epsilon\beta \subset E$

Proposition:

Let $\pi\epsilon Gr_1(E)$ and $\sigma\epsilon Gr_n$ be such that $\sigma\epsilon\bar{p}_n(\pi;B)$then $\sigma^*\epsilon\bar{p}_n(\pi^*;B)$and $\left(\bar{R}_n(\pi;B)(\sigma)\right)^* = \bar{R}_n(\pi^*;B)(\sigma^*)$.

Proof. Remark that it suffices to prove the proposition when B=E and n=1 indeed, replacing E by $\mathfrak{M}_n(E)$ we get the reduction to the case m=1.

Let $\pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} /\bar{\lambda}n, \sigma = \begin{pmatrix} \alpha & \beta \\ y & \delta \end{pmatrix} /\bar{\lambda}n$ and use the notation for the inverses of the two matrices which we used in the Lemma . then we have . $\bar{R}_1(\pi;E)(\sigma) = \beta(x\beta + \gamma\delta)^{-1}y$ and on the other hand we have .

$$\pi^* \begin{pmatrix} -t^* & -y^* \\ -z^* & -x^* \end{pmatrix} /\bar{\lambda}1$$

$$\sigma^* \begin{pmatrix} -p^* & -^{\sigma *} \\ -u^* & -\gamma^* \end{pmatrix} /\bar{\lambda}1$$

and

$$\begin{pmatrix} -t^* & y^* \\ z^* & x^* \end{pmatrix}^{-1} = \begin{pmatrix} -d^* & b^* \\ c^* & a^* \end{pmatrix}$$

Applying again the lemma, to these new matrices, we get that $\sigma^* \epsilon \bar{p}_1(\pi^*; E)$ is equivalent to the invertibility of $-d^*s^* - b^*r^*$ and $\bar{R}_1(\pi^*; E)(\sigma^*) = -s^*(d^*s^* - b^*r^*)^{-1}b^*$

Since $-d^*s^* - b^*r^*$ is invertible iff rb+sd is invertible, the equivalence of $\sigma^* \epsilon \bar{p}_1(\pi^*; E)$ with $\sigma \epsilon \bar{p}_1(\pi; E)$is precisely the equivalence of he invertibility of rb+sd and of x $\beta + yd$.

To conclude the proof of he proposition we must show that

$$(\beta(x\beta + y\delta)^{-1}y)^* + s^*(d^*s^* + b^*r^*)^{-1}b^* = 0$$

Or equivalently, that

$$\beta(x\beta + y\delta)^{-1}y + b(rb + sd)^{-1}s = 0$$

This is a consequence of the last assertion of the lemma, which gives that

$$\begin{pmatrix} b & \beta \\ d & \delta \end{pmatrix} = \begin{pmatrix} (rb + sd)^{-1}r & (rb + sd)^{-1}s \\ (rb + sd)^{-1}x & (rb + sd)^{-1}y \end{pmatrix} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

9.5. the involution and he duality transform. Like in the previous section $I \epsilon B \subset E$ will be Banach algebras with involution. Since we will consider the duality transforms with respect to π, B and with respect to π^*, B will use the notations $y_z(.)$ and respectively $y_z(.)$ to distinguish the two.

Proposition:

We have $\left(yR(\pi;B)\right)^{*}yR(\pi^{*};B)$ and $\left(y_{z}(\varphi^{*})\right)$

He proof is a straightforward consequence of proposition 9.4 and of the definition of $yR(\pi^{*};B)$ and of the duality transform and will therefore be omitted.

10. Dual Positively

10.1. The definition: the grassmannian extension of the notion of dual positivity is quite straightforward. Here B will be a unital Banach algebra with involution.

Definition: if $\Omega = \Omega^{*}$ an element $f \epsilon d(\Omega)$ is dual-positive if $f = f^{*}$ and $\Delta_{n.nf}(\sigma, \sigma^{*})$ is a positive map of $\mathfrak{M}_n$ in to $\mathfrak{M}_n$ for all $\sigma \epsilon \Omega_n$ and $n \epsilon \mathbb{N}(\nabla_{m.n} f)(\sigma', \sigma'')$ denotes the map $\alpha_{m.n} \bar{\partial}_{m.n} f\left(\sigma', \sigma''\right)\big)$.

Like in the affine case we have a few equivalent conditions.

Proposition. If $\Omega = \Omega^{*}$ and $f \epsilon d(\Omega)$ the following are equivalent:

i. f is dual positive.
ii. $f = f^{*}$ and for any $\sigma(j) \epsilon \Omega_{n(j),}$ the map $1 \leqq j \leqq p, \bigoplus_{1 \leqq i,j \leqq p} \left(\nabla_{n(j),n(f)} f\right)\left(\sigma^{(1)}, \sigma^{(j)*}\right)$ is a positive linear map of $\bigoplus_{i,j} \mathfrak{M}_n(1) + \cdots + n(p)$ into itself.

iii. $f = f^*$ and for any $\sigma\epsilon\Omega_{n,}$ the map

$(\nabla_{n,nf})(\sigma,\sigma^*): \mathfrak{M}_n \to \mathfrak{M}_n$ is completely positive.

The proof from the affine case [16] prop. 8.2, immediately carries over to this more general case and will not be repeated.

10.2 he duality transforms of the positive functional . here $1\epsilon p \supset E$ will be an inclusion of unital Banach algebras with involution. By E_1 we shall denote the closure of $y. R(\pi; B)$ where $\pi = \pi^*\epsilon G_{r1}(E)$. A functional $\varphi\epsilon E_1^D$ is positive , denoted $\varphi \geqq 0$, if $\varphi = \varphi^*$ and $\varphi(y^*y) \geqq 0$ for all $y\epsilon E_1$ (the hermiticity follows actually from the second requirement)

Proposition. $if\,\delta\epsilon E_1^d$ then $\varphi \geqq 0$ iff-$u(\varphi)$ is dual positive in $d(\tilde{p}(\pi, B))$.

Proof. (a) $\varphi \geqq 0 \Rightarrow -u(\varphi)$dual positive. We shall use proposition 6.3 which implies that

$$\left(id_{\mathfrak{M}_n} \oplus id_{\mathfrak{M}_n}\varphi\right)\left(\tilde{R}(\pi; B)(\sigma)\sigma_E\check{R}(\pi; B)(\sigma^*)\right) = -\tilde{\partial}_{n,n}y(\varphi)(\sigma,\sigma^*)$$

Since $\pi = \pi^*$ we have

$$\tilde{R}(\pi; B)(\sigma) = \left(R(\pi^*; B)(\sigma^*)\right)^*$$

Hence, if $\chi_{ij}\epsilon E_1$ are such that $\tilde{R}(\pi; B)(\sigma) = \sum_{i,j} e_{ij} \oplus \chi_{ij}$ then

$$-\tilde{\partial}_{n,n}y(\varphi)(\sigma,\sigma^*) = \sum\nolimits_{1\leq i.j,k,i\leq n} \varphi\left(X_{ij}X_{jk}^*\right)e_y \oplus e_{kl}$$

We must check that:

$$u_{n.n}\left(-\tilde{\partial}_{n.n}y(\varphi)(\sigma,\sigma^*)\right)\left(\sum_{p,y}c_p c_y e_{py}\right)\geq 0$$

In view of the definition of α, this is equivalent to

$$\sum_{1\leq i,j,k1\leq n}\varphi\left(\alpha_j\alpha^*\right)e_{il}\geq 0$$

Or equivalently, for all $\lambda_j \dots \lambda_n \epsilon C$,

$$\sum_{1\leq i,j,k1\leq n}\varphi\left(\alpha_j\alpha_i^*\right)\tilde{\lambda}_i\widetilde{\lambda_j}\geq 0$$

Putting

$$y=\sum_{1\leq 1\leq n}\tilde{\lambda}_i\alpha_{i,}\,we\ get\ \varphi(\gamma\gamma^*)\geq 0$$

$(b)-y(\varphi)$ dual positive $\Rightarrow \varphi\geq 0$ we have

$$\left(\left(-\nabla_{\mathrm{n,n}}\mathrm{y}(\varphi)(\sigma,\sigma^*)\right)e_j k\right)_{ij}\varphi\left(\left(\tilde{R}_n(\pi;B)(\sigma)_y\right)\left(\tilde{R}_n(\pi;B)(\sigma)\right)^*_{jk}\right)_*$$