Chapter (2)

Duality transform for the coalgebra of ∂ x:B

Section 1:-

1. Introduction:

The main aim of this chapter is to construct a suitable framework for the dual GDQ structure in the case of a operator Y and a noncommutative algebra of scalars B. Approaching duality via a map of the dual E’ of the Banach algebra containing B and Y into matrices indexed by corepresentations, we need a certain GDQ structure on the matricidal functions. Since in case B = C the dual is a GDQ of analytic functions with respect to the difference quotient on the resolvent set of Y, dealing with general B requires a generalization of this. It turns out that we need to consider collections of metrical objects at all levels, very much like in K-theory or in the theory of operator spaces. Thus, for instance, instead of the scalar resolvent set, we will have an object combining all matricial B-resolvent sets, tied together by natural relations involving conjugation by matrices I GL (n;C) and direct sums. Quite generally, on such a matricially generalized open set Ω, he corresponding matricially generalized scalar analytic functions form a noncommutative algebra A(Ω) and there is a generalization ∂ of the difference quotient derivation comultiplication which yields a topological GDQ ring structure. In the c^*-context, if Ω = Ω* in a suitably defined sense, A(Ω) becomes a a *-algebra and there is also a notion of dual positivity.

The duality map appears as a transformation from E’ to an A(Ω), where Ω is he matricially generalized resovent set and the transformation intertwines GDQ ring structures and positivity on E’ with dual positivity on A(Ω) .

Besides section 1 to 6. Section 2 contains preliminaries on GDQ rings . section 3 is about the new corepresentations we found in section 4, we introduce multivariable GDQ ring and we give a reduction result to a one-variable GDQ ring in case$n = p^2$, with n the number of "variables." We also prove a result about how $\partial x.B$arises in general GDQ rings. Section 5 deals with full B-resolventss and resolvent sets, which are the metrical B-valued generalizations of usual resolvents and resolvent sets. Section 6 takes up the matricial generalization of functions and sets which go with the generalized resolvents. Section 7 gives the construction of the topological GDQ ring structure on the algebras A(Ω) of fully matricial functions. We have preferred to define the derivation –comultiplication as taking values in some "two-variable" A(Ω;Ω)instead of entering here the technical problems about tensor products and topologies on the A(Ω)'s. section 8 contains a discussion of dual positivity in A(Ω). Section 9 introduces the duality U-transform and discusses its intertwining properties of GDQ structure and positivity.

1. Preliminaries on GDQ rings:

Definition 2.1. A generalized difference quotient ring (a GDQ ring) is an object (A, μ, ∂) where A is an algebra over C and

(GDQ1) $\mu: A \otimes A, \rightarrow A$ is the multiplication map.

(GDQ2) $\partial: A \rightarrow A \otimes A$ is linear and coassoclative, that is

$(\partial \otimes id_A)^\circ \partial = (id_A \otimes \partial)^\circ \partial$

(GDQ3)∂ is a derivation, that is $\partial^\circ \mu = (di_A \otimes \mu)^\circ (\partial \otimes id_A) + (\mu \otimes id_A)^\circ \left(id_{A\otimes} \partial\right)$.

In general, A is not required to have a unit if 1∈A is a unit, then the GDQ ring will be called unital.

Remark 2.2. A GDQ ring can always be made uital by adjoining a unit and putting $\partial 1 = 0$

Definition 2.3 A quadruple $(A, \mu, \partial, L)$is a graded GDQ ring if (A, μ, ∂) is a GDQ ring and there is a linear map L:A→ A (the grading) so that.

(L1) $L - id_A$ is a derivation of the algebra (A, μ).

(L2) L is a coderivation of the coalgebra(A, μ), that is $\partial^\circ L = (L \otimes id_A + id_A \otimes l)^\circ \partial$.

Definition 2.4. An involution of a GDQ ring (A, μ, ∂) is a conjugate linear involution $A \ni\rightarrow \alpha^* \in A$ of the vector space A so that

(11) $(A, \mu, *)$is an algebra in involution.

(12) $\partial(\alpha^*) = \sigma_{12}\big((\partial_\alpha) *\big)$,where * on A⊗A is given by $(x \otimes y)^* = x^* \otimes y^*$and $\sigma_{12}(x \otimes y) = y \otimes x$.

If L is a grading, compatibility with the involution means that

(13)$L(\alpha^*) = \big((\alpha)\big)^*$.

If V is a vector space, we will denoted by V* its dual endowed with the topology of pointwise convergence. By $\widehat{\otimes}$ we denote the projective tensor product. The duality theorem can be restated in the following form.

Theorem 2.5.if$(A, \mu, \partial)$is a GDQ ring, then (A^*, ∂^*, μ^*) satisfies the GDQ rig conditions with ⊗ replaced by $\widehat{\otimes}$ if L is a grading and * is an involution for (A, ∂, μ), then $A = id_{A*} + l^* and\ \varepsilon^*(\alpha) = \overline{\varepsilon(\alpha^*)}$ satisfy, respectively, the grading and the involution conditions for $(A^*, \partial^*, \mu^*)$$\left(with\ \otimes\ replaced\ by\ \widehat{\otimes}\right)$

By $2\pi_p(A)$ we denote the Pxp matrices over A, an individual matrix written either in the form $\left(\alpha_{ij}\right)_{1\le i,j\le p}$or in the form $\sum_{1\le i,j\le p} \alpha_{ij} \otimes \varepsilon_{ij1}$ were $\alpha_{ij} \epsilon\ A\ and\ e_{ij}$are the matrix units. A corepresentation of (A, μ, ∂) is a matrix (A) so that.

$$\sum_{i.k} \partial_{\alpha_{ik}\otimes}\, e_{i.k} = \sum_{i.j.k} \partial_{\alpha_{ik}\otimes}\, e_{i.k} \qquad (2.1)$$

This can also be written

$$(\partial \otimes) id_{2\pi_p} \alpha \qquad (2.2)$$

The main result about corepresentation (see) [14] is the following the orem.

Theorem 2.6 Let(A, μ, ∂) be a unital GDQ ring and assume that $x \in A$ is so that $\partial x = 1 \otimes 1$. if invertible, the following are equivalent.

(I) $\alpha$is a corepresentation,

(II)$\alpha = \left(\left(n_{ij} - x\delta_{ij}\right)^{-1}\right)$where $n_{ij} \in N = \ker \delta$.

Since this is a functional analysis study, the algebraic facts will guide our functional analysis constructions, even if hey are not directly applicable. This is a familiar situation from the theory of kac algebras and C^*quantum groups, where finding the appropriate topological tensor products and topological duals are subtle analysis questions.

In particular, the vague idea that the dual object should be constructed by map ping $\varphi \epsilon A^*$ into the direct sum of

$\left(\varphi\left(\alpha_{ij}\right)\right) 1 \leq i,j \leq p \,\epsilon 2\pi_p$ where $\propto = \left(\propto_{ij}\right) i \leq j \leq p$ runs over a sufficiently large set of corepresentation of (A, μ, ∂) poses many analytical problems.

2. More corepresentation:
   Throughout this section (A, μ, ∂) will denote a unital GDQ ring and $x \epsilon A$ will be an element so that $\partial x = 1 \otimes 1$. we will exhibit corepresentations which enlarge the set provided by theorem 2.6.
   As in [14], it will be convenient to use d:$2\pi_p(A \otimes A), d = \partial \otimes id_{2\pi_p}$which is a derivation with respect to the bimodule structure given by the homomorphisms $\varphi_1, \varphi_{2 2\pi_p(A) \to 2\pi_P(A \otimes A)}$
   So that $\varphi_1\left(\alpha_{ij}\right) 1 \leq i.j \leq p$ =$\left(\alpha_{ij} \otimes 1\right) 1 \leq i.j \leq p, \varphi_2\left(\alpha_{i.j}\right) 1 \leq i.j \leq p = \left(1 \otimes \alpha_{ij}\right) 1 \leq i,j \leq p$.we will also denote $x \otimes 1_p$by $x \in 2\pi_p$(A) and write 1 for the unit $1 \otimes 1_p$ of $2\pi_p$ (A)

Proposition 3.1 if N ∈ $ker\partial$ and $\beta_1, \beta_{2,} \beta_3 \in 2\pi_p(N)$are such that $\beta_{2,} - \beta_{1,}\left(x \otimes 1_p\right)\beta_{3,}$ is invertible, then.

$$\propto = \beta_3\left(\beta_{2-}\beta_1\left(x \otimes 1_p\right)\beta_3\right)^{-1}\beta_1 \qquad (3.1)$$

Is acorepresentation

Proof. Let $y = \beta_2 - \beta_1\left(x \otimes 1_p\right)\beta_2 = \beta_1 - \beta_1 x \beta_{3.}$we have$d(y^{-1}) d(y) \varphi_2(y^{-1})$
(3.2)

Hence:

$$d(\propto) = d(\beta_3 y^{-1} \beta_1) = \beta_1(\beta_3) d(y^{-1}) \varphi 2(\beta_1)$$

$$= -\beta_1(\beta_3)\varphi_1(y^{-1})\left(-\varphi_1(\beta_1)\varphi_2(\beta_3)\right)\varphi_2(y^{-1})\varphi_2(\beta_1)$$

$$= \varphi_1(\alpha)\varphi_2(\alpha) \qquad (3.3)$$

Which is the desired result.

We also have he following general procedure for producing more corepresentations.

Lomma 3.2 Let$\varepsilon \epsilon \mathfrak{M}_p(A)$be a corepresentation and let $\beta \in \mathfrak{M}_p\,(ker\ \partial).if\ 1-\varepsilon\beta$ is invertible, then $\propto= (1-\varepsilon\beta)^{-1}\varepsilon$ is a corepresentation.

If $1-\beta\varepsilon$ is invertible, then $y = \ \varepsilon(1-\beta\varepsilon)^{-1}$ is a corepresentation

Proof because of symmetry , we will only prove he first assertion

We have

$$d((1-\varepsilon\beta)^{-1}\varepsilon)$$
$$= \varphi_1((1-\varepsilon\beta)^{-1}$$
$$-1)d\big((1-\varepsilon\beta)\big)\varphi_2((1-\varepsilon\beta)^{-1})\varphi_2(\varepsilon)$$
$$+\varphi_1((1-\varepsilon\beta)^{-1})\varphi_1(\varepsilon)\varphi_2(\varepsilon)$$

$$=\varphi_1((1-\varepsilon\beta)^{-1})\varphi_1(\varepsilon)\varphi_2(\varepsilon)\varphi_2(\beta)\varphi_2((1-\varepsilon\beta)^{-1})\varphi_2(\varepsilon)$$

$$+\varphi_1((1-\varepsilon\beta)^{-1})\varphi_1(\varepsilon)\varphi_2(\varepsilon) \qquad (3.4)$$

$$=\varphi_1((1-\varepsilon\beta)^{-1}\varepsilon)\varphi_2(\varepsilon+\varepsilon\beta)((1-\varepsilon\beta)^{-1}\varepsilon)$$

$$= \varphi_1(\alpha)\varphi_2(\varepsilon+((1-\varepsilon\beta)^{-1}-1)\varepsilon)$$

$$=\varphi_1(\propto)\varphi_2(\alpha)$$

3. Reduction of multivariable GDQ rings:
   Studying $\partial_{x\beta}$ dose not mean a limitation to one variable. In this section, we briefly explain how multivariable situations can easily be reduced to the $\partial_{x\beta}$ setting.

4.1

The typical multivariable situation deals with A=B $\big(X_{1,\dots.}X_n\big)$, the ring of noncommutative polynomials in the noncommutative variables $\left(X_{1,\dots\dots}X_n\right)$ and with noncommutative scalars B. This means that monomials are of the form $1x_1 = x_j 1 = \ x_1$. This n partial difference quotients $\partial_1 : A \rightarrow A \ \ \otimes A$ are the derivations such that $\partial_i x_j = \delta_\mu 1 \otimes 1 \; and \; \beta_1 \beta = 0$ thus each (A, μ, ∂) is a GDQ ring and he compatibility relations hold $\left(A, \mu, \sum_1 \ \lambda_\mu \partial_1 \dots \sum_1 \lambda_{in} \beta_1\right)$ is again a multivariable GDQ.

4.2

In case $n = p^2$, multivariable GDO ring $\left(A, \mu, \partial_1 \dots, \partial_{P_2}\right)$can be replaced by a one variable $\left(A, \dot{\bar{\mu}}, \partial\right)$ more precisely , we take $A = \mathfrak{M}_1(A) = \mathfrak{M}_p \otimes A$ where $\mathfrak{M}_p$ is short for $\mathfrak{M}_p(\mathbb{c})$. We may reindex $\partial_{1\dots,}\partial_p \left(\begin{matrix} possibly \; precended \; by \; a \; linear \; transformation \; if \\ we \; want \; to \; preserve \; some \; involution \end{matrix}\right) \; and \; replace \; them$by $\partial_{i.j} 1 \leq i, j \leq p$. Further let $\Delta = \sum_{i \leq k \leq p} e_{i,k} \in \mathfrak{M}_p \otimes \mathfrak{M}_p$

(4.2)

Note that

$$\top \otimes \mathsf{I}_P \Delta_{I,J} = \Delta_{I,J} \left(\mathsf{I}_P \otimes \top\right) \tag{4.3}$$

if $\top \in \mathfrak{M}_p$We then define

$$\partial : \bar{A}, \rightarrow \bar{A} \otimes \bar{A} \tag{4.4}$$

By

$$\partial(\top \otimes \alpha) = \sum_{1 \leq i,j \leq p} \left(\left(\top \otimes \mathsf{I}_P\right)_{\Delta_\mu}\right) \otimes \partial_\mu \alpha \in \left(\mathfrak{M}_p\right)^{\otimes_2} \otimes A^{\otimes_2} \approx \bar{A}^{\otimes_2} \tag{4.5}$$

Where the isomorphism takes $(\top_1 \otimes \top_2) \otimes (\alpha_1 \otimes \alpha_2) \; to \; (\top_1 \otimes \top_2) \otimes (\top_2 \otimes \alpha_2)$ note that we also have

$$\partial(\mathrm{T} \otimes \alpha) = \sum_{1 \leq i,j \leq p} \left(\Delta_{i,j} \left(1_p \otimes \mathrm{T}\right)\right) \otimes \partial_{ij} \alpha \tag{4.6}$$

That ∂ is a derivation is seen by the computation $\big(\partial(\top_1 \otimes \alpha_1)(\top_2 \otimes \alpha_2)\big) = \partial(\top_1 \top_2 \otimes \alpha) =$

$$\sum_{i,j} \Big((\top_1 \top_2 \otimes |_p) \Delta_\mu \Big) \otimes \partial_{i,j} \alpha_1 \alpha_2 \qquad (4.7)=$$

$$\sum_{i,j} \Big((\top_1 \otimes \top_2) \Delta_{i,j} (|_p \otimes \top_2) \Big) \otimes \Big((\alpha_1 \otimes |)\big(\partial_{i,j}\alpha_2\big) + \big(\partial_{i,j}\alpha_2\big)(| \otimes \alpha_2) \Big) =$$

$$\Big((\top_1 \otimes \alpha_k) \otimes (|_p \otimes |) \Big) \partial(\top_1 \otimes \alpha_{}) + \big(\partial(\top_1 \otimes \alpha_2)\big) \Big((|_p \otimes |) \otimes (\top_1 \otimes \alpha_2) \Big)$$

Before checking coassociativity remark that

$$\partial(e_{rs}) = \sum_{i,j}\left(e_{rj} \otimes \partial_{js}\right) \otimes \left(\partial_{ij}\alpha\right) \tag{4.8}$$

We have

$$\left((\partial \otimes id)^{\circ}\partial\right)(\partial_{rs} \otimes \alpha) = \sum_{i,j}(\partial \otimes id)\left(\left(\partial_{ri} \otimes \partial_{js}\right) \otimes \partial_{ij}\alpha\right)$$

$$= \sum_{k,j}\sum_{i,j}(\partial_{rk} \otimes \partial_{is}) \otimes \left(\left(\left(\partial_{kj} \otimes id\right)^{\circ}\partial_{ij}\right)\alpha\right)$$

While on the other hand,

$$\left((id \otimes \partial)^{\circ}\partial\right)(\partial_{rs} \otimes \alpha)$$
$$= \sum_{k,i}(id \otimes \partial)\left((id \otimes \partial)\left((e_{rk} \otimes e_{is})\right) \otimes \partial_{ki}\alpha\right)$$

$$\sum_{i,j}\sum_{k,i}(e_{rk} \otimes e_{li} \otimes e_{is}) \otimes \left(\left(\left(id \otimes \partial_{kj}\right)^{\circ}\partial_{ki}\right)\alpha\right)$$

and the coassociativity follows compatibility of $\partial_{ij}$and ∂_{kj}.

4.3

If here are elements $\in A$ so that $\delta_{ri}\delta_{sj}1 \otimes 1$, then it is easily seen that

$$\sum_{1 \le i.j \le p} e_{ij \otimes \Upsilon_{ij} \in \tilde{A}} \ (4.11)$$

Will have the property $\partial_{\Upsilon}\left(\ \mathsf{I}_p \otimes 1\right) \otimes \left(\ \mathsf{I}_p \otimes 1\right)$ also if

$$= z \ \sum_{1 \le \ ij \ \le \ p} e_{ij \otimes z_{ij,}} \tag{4.12}$$

Then $\partial z = 0$ is equivalent to $\partial_{rs} z_{ij} = 0$ for all $1 \leq r, s, i, j \leq p$ that

$$\text{Ker } \partial = \mathfrak{M}_p \otimes \left(\cap_{1 \leq i,j \leq p} \ker \partial_{ij} \right) \tag{4.13}$$

4.4

Returning to the multivariable GDQ ring A =B $\langle X_1, \dots, X_{P^2} \rangle$ and the partial free difference quotients $\partial_{1,\dots,\partial_{p^2}}$ with respect to $X_1, \dots, X_{P^2}$, he preceding construction combined with a linear transformation, gives the following. We consider $\tilde{A} = \mathfrak{M}_P \otimes B\langle X_1, \dots, X_{P^2} \rangle$ which is isomorphic to D $\langle X_1 \rangle$, where D = $\mathfrak{M}_P \otimes B$. The replacement for the multivariable GDQ rig is then D $\langle X_1 \rangle$ with comultiplication derivation $\partial_x : D$.

Note that in case B = $\mathbb{C}$ or B = $\mathfrak{M}_P$, this reduction has the pleasant feature that D, which is $\mathfrak{M}_P$ or $\mathfrak{M}_{Pq}$. Is finite dimensional.

Proposition: 4.1. Let (A, μ, ∂) be a GDQ ring with unit and assume that there is $X, \in, A$ such that $\partial X = 1 \otimes 1$. further let N=ker ∂.

Then he canonical homomorphism $\psi: N\langle X \rangle$ is endowed with the comultipliation $\partial_x : N$.

Proof. A derivation being completely determined by the way it acts on the generators of an algebra, the only assertion we really need to prove is the injectivity of ψ. Let ψ_k: $N^{\otimes(k+1)\rightarrow}$ A be the linear maps so that.

$$\psi_k(n_0 \otimes \ldots \otimes n_k) = n_\circ x_{n1} X \ldots n_k \quad (4.14)$$

We must prove that kerψ_k = 0 and the ranges of the ψ_k, $k \geq 0$ are linearly independent iterating ∂, we define $\partial^{(k)} = (\partial \otimes id_{k-1})^\circ \partial^{(k-1)}, \partial^{(1)} = \partial$. Then $\partial^{(k)} \psi_{k(N^{\otimes(k+1)}) \subset N^{\otimes(K-1)}} N^{\otimes(k+1)} \mid \partial^{(K)\circ} \psi_{K=} id_{N^{\otimes(K-1)}}$, and $\partial^{(K)\circ} \psi_1 = 0 \; if \; 1 < k$. The assertion follows from these facts.

4. The full B-resolvent

   Let E b a Banach algebra with unit, let $\in B \subset E$ be a closed subspace containing the unit, and let Y ∈ E be an element. The concepts we examine I this section will also serve as motivating examples in section 6 and it is good to note that the case when B is a banach subalgebra is of particular interest.

Definition 5.1 the set:

$$\rho(Y{:}\,B) = \left\{ \{b \in \mathfrak{M}_n(B)\} Y \otimes \mathrm{I}_n - b \; invertible \right\}$$

Will be called the nth B-resolvent of Y. The operator valued function $R_n(Y{:}\,B)(.)$:

$\rho_n(Y{:}\,B) \rightarrow \mathfrak{M}_n(\in)$defined by $R_n(Y{:}\,B)(b) = (Y \otimes \mathrm{I}_n - b)^{-1}$ will be called the nth B-resolvent of Y. The collection of functions $\left(R_n(Y;B)\right)_{n\geq 1}$will be called the full B-resolvent of Y.

Some basic about these concepts are summarized in the next proposition.

Proposition 5.2:

I. The set $\rho_n(Y:B) \otimes \rho_n(Y:B) \cap \left(\mathfrak{M}_m(B) \otimes \mathfrak{M}_m(B)\right)$

II. $\rho_n(Y:B) \otimes \rho_n(Y:B) = \rho_{m+n}\left((Y:B) \cap \mathfrak{M}_m(B) \otimes \mathfrak{M}_n(B)\right)$

III. $(5 \otimes)\rho_n(Y:B)(5 \otimes 1)^{-1} = \rho_n(Y:B)\ if\ S \in GL(n;\mathbb{c})$

(iv)if$b' \in \rho_m(Y:B), b" \in \rho_n(Y:B)$, and $\beta \in \mathfrak{M}_{m,n}(B)$ is an m x n matrix with entries in B then.

$$\begin{pmatrix} b' & \beta \\ \sigma & b" \end{pmatrix} \in \rho_{m+n}(Y:B)(5.2)$$

$(v)R_n(Y;B)$is a complex analytic function

(vi) if b' ∈ $\rho_n(Y:B)$ and b" ∈ $p_n(Y;B)R_{m+n}(Y:B)(b' \otimes b') \otimes R_n(Y;B)(b'')$.

(vii)if b∈ $\mathfrak{M}_n(B)$and S ∈ $GL(n;\mathbb{c})$, then

$$R_n(Y:B)(S \otimes 1)b(S \otimes 1)R_n(Y;B)(S \otimes 1)^{-1}(5.3)$$

Proof: most assertions are rather obvious and will be left to the reader we will only prove (iv). In view of (ii), b⊗.b' ∈ p_{m+n}(B), and in view of (i),

5. Fully matricia functions and sets
   Let G and H Banach spaces over $\mathbb{c}$ if S ∈ GL (n; $\mathbb{c}$) and ⊤ ∈ $\mathfrak{M}_n$, we denote by AdS the automorphism of $\mathfrak{M}_n$ so

that$(AdS)(\top) = STS^{-1}$. The corresponding automophism of $\mathfrak{M}_n \otimes H$ will be denoted by AdS⊗ $\mathsf{I}_{\aleph}$ or simply AdS ⊗ 1 and its action is $(AdS \otimes\ )(\top \otimes h) = STS^{-1} \otimes h$.

Definition 6.1 a fully matricial G-set is a sequence (Ω_n) is open or closed, respectively

Proposition 6.2: if $(\Omega_n)_{n\geq i}$ is a fully matricial open set and if g' $\in \Omega_n$, and $\gamma \in \mathfrak{M}_{m,n}(G), then \begin{pmatrix} g' & y \\ \sigma & g" \end{pmatrix} \in \Omega_{m+n}$

The proof is along the same lines as the proof of (iv) in proposition 5.2 in case G-ℂ using the Jordan form of a matrix, it is possible to describe the fully matricial ℂ $sets$

Proposition 6.3 (1)A fully matricialℂ-set $(\Omega_n)_{n\geq 1}$is described n a unique way by giving for each $\gamma \in$ ℂ an additive subsemigroup $L(\lambda) \subset N$. then $\top \in \Omega_n$ if and only if for each eigenvale$\lambda \in \sigma(\top)$,the length of the corresponding Jordan blocks I he Jordan form of $\top$are in $L(\lambda)$.

(ii) $(\Omega_n)_{n\geq 1}$is a closed $(res., open)$ fully matricial ℂ-set if and only if $\Omega_1$is closed $(respecvely\ open)\ and\{(\top \in \mathfrak{M}_n\ \ |\sigma(\top) \subset)\Omega_1\}$in particular, if the fully matricial ℂ-set is closed or open, the $L(\lambda)'s$ can only be $(\emptyset\ or\ N)$.

The proof of (1) is an exercise of combining the Jordan form with the similarity and direct sum properties of fully matricialses, which we leave to the reader we will only explain he different reasons in (ii), whenΩ_n is closed o open, why the $L(\lambda)'s$ can only be N or ∅, in both cases, using (FMS3), the discussion breaks down to showing that if $\top \in \Omega_n$ is an upper triangular matrix, then its (1,1),-entryλ_1 will be in Ω_n.

If he fully matricial set is closed, let $S(\lambda)$ be the diagonal matrix with entries $1, \lambda, \lambda$,then

$$\lim_{\lambda-\infty} s(\lambda)TS\,(\lambda)^{-1} = T', \tag{6.1}$$

Where $\top$ is the direct sum of the 1 x 1 matrix λ_j and an $(n-1)x(n-1)$ matrix Since Ω_n is closed,$\top' \in \Omega_n$, and by $\in \Omega_1$

If $\Omega_n$is open, we can find $\top' \in \Omega_n$ so that $\begin{pmatrix} \lambda, & * \\ \sigma & S \end{pmatrix}$, where $S \in \mathfrak{M}_{n-1}$ is so that $\sigma(S) \nexists \lambda_1$.thenusing $(FMS3)$and fully material G-sets, then $\left(\cap_{1\in 1} \Omega_1{}^{(1)}\right)_{n\geq 1}$is a fully matricial G-set.

In particular, the family of open fully matricial G-set is stable under such finite componentwise intersections. Similarly , the family of closed fully matricial G-set is stable under arbitrary componentwise intersection.

It seems natural to consider the topology $\left(view\ for\ instance\ as\ suvsets\ of\ \coprod_{n\geq 1}\mathfrak{M}_n(G)\right)$generaed by the open fully matricial G-sets.

Definition 6.5 A fully maricial H-valued function on a fully matricial G-set $(\Omega_n)$is a sequence $(R_n)_{n\geq 1}$so that

$(FMF1) R_n \colon \Omega_n \to \mathfrak{M}_n(H)$ Is a function,

$(FMF1)$if$g' \in \Omega_n\ and\ g" \in \Omega_n$ then $R_{m+n}(g' \otimes g") = R_n(g') \otimes R_n(g")$.

$(FMF1)$if$s \in GL(n;\mathbb{c}) and\ g \in \Omega_n$, then $R_n\big((AdS \otimes \mathsf{I}_G)(g)\big) = (AdS \otimes \mathrm{I}_H)\big(R_n(g)\big)$

As fully matricial function is continuous if each component. A fullymatricial function is analytic if the fully matricial G-set on which it is defined is open and the components R_n are analytic.

Remarks 6.6 A fully matricial function amounts to a sequence of functions whose graphs form a fully matricialGx H-set.

Lemma 6.7 Let $(R_n)_{n\geq 1}$be a continuous fully matricial H-valued function o the fully matricial G-set $(\Omega_n)_{n\geq 1}$. Assume that 9' $\in \Omega_{m1} g" \in \Omega_n, and\ \gamma \in \mathfrak{M}_{m,n}(G)$. then for some $h \in \mathfrak{M}_{m.n}(H)_1$

$$R_{m+n}\left(\begin{pmatrix} g' & \lambda y \\ \sigma & g" \end{pmatrix}\right) = \begin{pmatrix} R_m(g') & \lambda h \\ 0 & R_n(g") \end{pmatrix} (6.2)$$

For all $\lambda \in \mathbb{c}$

Proof. Let $R_{m+n}\left(\begin{pmatrix} g' & Y \\ \sigma & g" \end{pmatrix}\right) = \begin{pmatrix} h' h_{1.2} \\ h_{2.1} & h' \end{pmatrix}$ and let $\mathrm{S}(\in) = \varepsilon\ \mathsf{I}_m \otimes \mathsf{I}_n \in GL(m+n;\mathbb{c})$. then if $\lambda \neq 0$.

$$R_{m+n}\left(\begin{pmatrix} g' & \varepsilon\lambda y \\ \sigma & g" \end{pmatrix}\right) = R_{m+n}\left(AdS(\varepsilon\lambda) \otimes \mathsf{I}_G \begin{pmatrix} g' & y \\ \sigma & g" \end{pmatrix}\right)$$

$$= AdS(\varepsilon\lambda) \otimes \mathsf{I}_G \begin{pmatrix} h' h_{12} \\ \\ h_{21} & h" \end{pmatrix} (6.3)$$

$$= \Big(\lambda^{-1}\varepsilon - 1 h_{21}{}^{h'} \qquad {}_{h"}{}^{\varepsilon\lambda h_{12}}\Big)$$

Sincee R_{m+n}e is continuous and $lim_{\varepsilon\to 0}\begin{pmatrix} g' & \varepsilon\lambda \\ 0 & g" \end{pmatrix}\begin{pmatrix} g' & 0 \\ 0 & g" \end{pmatrix} \in \Omega_{m+n}$ we inter

$$\lim_{\varepsilon \to 0} \left(\lambda^{-1}\varepsilon - 1h_{21}{}^{h'} \quad {}_{h"}{}^{\varepsilon\lambda h_{12}} \right) = \begin{pmatrix} R_m(g') & 0 \\ 0 & R_n(g") \end{pmatrix} (6.4)$$

And hence $= h_{21} 0, h^{'} = R_n(g'),$ and h" = $R_n(g")$

Remark 6.8. The reader has probably recognized by now that the full B. resolvent set and that full B-resolvent are examples of a fully matricial B-set and of an analytic B-valued fully matricial function defined o an open fully matricial B-set respectively.

Definition 6.9A full matricial G-set $(\Omega_n)_{n\geq 1}$ will be called finite if it also satisfies.

$$\begin{pmatrix} g' & \lambda \\ 0 & g" \end{pmatrix} \in (\Omega_m)_{n\geq 1} \Rightarrow g' \in \Omega_m g' \in \Omega_m (6.5)$$

Remarks 6.10, full resolvent sets in a finite von Neumann provide examples of finite fully matricial sets. On the other hand, taking E=B =$\mathcal{B}(i^2(N)), Y = 0\ and\ m = n = i,$ the full B-resolvent of Y is not a finite fully matricial B-set since $\begin{pmatrix} S & K \\ 0 & S" \end{pmatrix}$, Where is the unilateral and K a rank-one operator making the matrix the bilateral shift, is in $p_1 + 1(0; B)$without S;S" being in $p_1 1(0; B)$.

5.1 Returning to the context of proposition 6.4 we associate with an open fully matricial G-set Ω=$(\Omega_n)_{n\geq 1}$ the set $\acute{\Omega} = \coprod_{n\geq 1}\Omega_n \subset \coprod_{n\geq 1}\mathfrak{M}_n(G)$. it is then also natural to associate with Ωthe analytic or continuous fully matricial H-valued functions on Ω and the sheaves on $\coprod_{n\geq 1}\mathfrak{M}_n(G)$ which they generate.

5.2 Abbreviations:

From now on, we will also use the abbreviations FMGS for full matricial G-set and FMF for full matricial G-function. Also FMS will abbreviate fully matricial set and FMAF will abbreviate fully matricial analytic function.

Section (2):-

2. The GDQ ring of scalar fully matricial analytic function.

2.1 let $(\Omega_n)_{n\geq 1}$be an open FMG.S.to avoid amending our assumptions on G to introduce more structure, we will assume that G is an operator system, that is, it is isomorphic to a space of operators on Hillbert space which is selfedjointand unital, is correspondingly endowed with involution and unit, and is matrix, normed. (the reader could simplify and assume that G is a unitalC^*-algebra). We should also clarify from the beginning that the term GOQ ring in the title of this section has been used rather loosely: the tensor product required for hecomultiplication would be a topological one, and we will actually circumvent this question interpreting the tensor product as “some.

Two-variable functions “our aim here is to clarify the function theory aspect of the comultiplication and to return to the precise topological GDQ ring structure later.

2.2

Let A Ω denote the ℂ-valued FMAF on Ω if $r = (r_n)_{n\geq 1}\ and\ s = (s_n)_{n\geq 1}\ are$ A Ω, then $r + 1 = (r_n s_n)_{n\geq 1}\ and\ rs = (r_n s_n)_{n\geq 1} are\ in$ A Ω which is thus naturally a non-commutative ring. Moreover $1 = (1_n)_{n\geq 1}.\,where\ 1_n$ denotes the constant function on $\Omega_n$with value the identity $\alpha x n$ matrix is the unit in A Ω.

Let $\Omega^* = (\Omega_m^n)_{n\geq 1}$where $\Omega_m^n = (\mathsf{T}^* \mid \mathsf{T} \in \Omega_n) if\ r \in$ A Ω, we define $r^* = (r_n^*)_{n\geq 1}$ ∈A(Ω)by $r_n^*(g) = \left(r_n(9^*)\right)^*$ thus $r \to r^*$ is a conjugae linear antisomorphism of A Ω and A(Ω^*) in case $\Omega = \Omega^*$. This makes A Ω a unital algebra with involution.

7.3 Let k = $(k_n)_{n\geq 1}$be a sequence of subsets $k_m \subset \Omega_m$ satisfying FMS1 and FMS2. we will say that K is properly included in Ω if

$$\sup_{n\in N} \sup_{n\in N}$$

and if there is $\varepsilon > 0$ such that $k_n + \varepsilon\left(\mathfrak{M}_n(G)\right)_1 \subset \Omega_n$ for all $n \in N.(here,\ \|.\|_n)$is the norm and $\left(\mathfrak{M}_n(G)\right)_i$ the unit hall in$\mathfrak{M}_n(G)$ clearly, this definition uses the fact that G is matrix normed if $if\ r \in A(\Omega)$. We define.

$$\| r.\|_k \ \sup_{n\in N} \sup_{kn\in k_n} \| r_n.(k_n)\|_k$$

Where $\| r.\|_n$ is the norm on $\mathfrak{M}_n(C)$. *unless* $\| r.\|_{k<\infty}$ *for all properly included k, it may be natural to add this consider the corresponding subalgebra* $A_{pr}(\Omega)$ *of* $A(\Omega)$.

2.4

The comultiplication derivation will be defined plecewise. That is for fixed matrix sizes. We will use algebras of matrix-valued analytic functions $n_{1,\dots\dots,n_2(\Omega_{n1,\dots\dots}\Omega_{n1,}),}$ where $n = n_1 + \cdots + n_{p1}$consisting of analytic maps $if \rightarrow \Omega_n$ x …x$\Omega_n \rightarrow \mathfrak{M}_n, \otimes \dots \otimes \mathfrak{M}_n$

Which are GL$(n_p) - equivariant$:

$$f\left(AdS,\otimes\, l_G\left(g^{(1)}\right)\right),\dots,\left(AdS_p,\otimes\, l_G\left(g^{(p)}\right)\right)$$

$$=\left(AdS,\otimes \dots \otimes \left(AdS_p\right)\right) f\left(g^{(1)} \dots .\, g^{(p)}\right)$$

Where S,∈ $GL(n_1), g^{(1)} \in \Omega_n$

A result similar to Lemma 6.7 holds for function in $A_{m+n}(\Omega_{m+n})$.

Lemma 7.1 Let $f_{m+n} \in A_{m+n(\Omega_{m+n})}$, then,

(i) If $g' \in \Omega_n$ and $9''' \in \Omega_n$ there are $a' \in \mathfrak{M}_n$ and $\alpha'' \in \mathfrak{M}_n$ so that

$$f_{m+n}(9' \otimes 9'') = \alpha'' \otimes \alpha''; \tag{2.3}$$

Where $S_j \in GL(n_j) 9^{(1)} \in \Omega_n$

A result similar to Lemma 6.7 holds for function in $A_{m+n}(\Omega_{m+n})$

Lemma.2.1 $Let\ f_{m+n} \in A_{m+n}(\Omega_{m+n})$, then

(i) $if\, 9' \in \Omega_n$and 9" $\in \Omega_{n,}$ there are $\alpha' \in \mathfrak{M}_n$ so that $f_{m+n}(9' \otimes 9'') = \alpha' \otimes \alpha''$; (2.4)

(ii) $if\, 9', 9'', \alpha', \alpha''$are as in(i) and Y ∈ $\mathfrak{M}_{m+n}$(G). there is $h \in \mathfrak{M}_{m+n}$so that

$$f_{m+n}\left(\begin{pmatrix} 9' & \lambda\gamma \\ 0 & 9'' \end{pmatrix}\right) = \begin{pmatrix} \alpha' & \lambda h \\ 0 & 9'' \end{pmatrix} \forall\lambda\in \tag{2.5}$$

Proof.(i) if$f_{m+n}(9' \otimes 9'')$= $\begin{pmatrix} h' & \lambda 12 \\ h21 & h'' \end{pmatrix}$, then in view of the equivariance applied to S= $\varepsilon l_m \otimes l_n$, we get $h_{12} = h_{12}, h_{21} = \varepsilon^{-1} h_{21}$ so that $h_{21}\ 0, h_{21} = 0$

$$f_{m+n}\left(\begin{pmatrix} 9' & \gamma \\ 0 & 9'' \end{pmatrix}\right) = \begin{pmatrix} h' h_{12} & \\ h_{21} & h'' \end{pmatrix} \tag{2.6}$$

Conjugation with$\varepsilon l_m \otimes l_n$ yields

$$f_{m+n}\left(\begin{pmatrix} 9' & \varepsilon\gamma \\ 0 & 9" \end{pmatrix}\right) = \begin{pmatrix} h'h_{12} & \\ \varepsilon^{-1}h_{21} & h" \end{pmatrix} \qquad (2.7)$$

And since

$$\lim_{\varepsilon \to 0} = \begin{pmatrix} h'\varepsilon h_{12} & \\ \varepsilon^{-1}h_{21} & h" \end{pmatrix} \begin{pmatrix} h' & 0 \\ 0 & \alpha" \end{pmatrix} \qquad (2.8)$$

we infer that $h' = \alpha', h" = \alpha"$, and $h_{21} = 0$ hence

$$f_{m+n}\left(\begin{pmatrix} 9' & \varepsilon\gamma \\ 0 & 9" \end{pmatrix}\right) = \begin{pmatrix} \alpha' & \varepsilon h_{12} \\ 0 & 9" \end{pmatrix} \qquad [2.9]$$

2.5

There is a canonical identification $\propto of\ \mathfrak{M}_n$ with the linear operators $(\mathcal{L}\mathfrak{M}_{m,n})$on $(\mathcal{L}\mathfrak{M}_{m,n})$if $\alpha \in \mathfrak{M}_n b \in \mathfrak{M}_n$,and $c \in \mathfrak{M}_{m,n}$,

$\big(\alpha(\alpha \otimes b)\big)(c) = acb \in \mathfrak{M}_{m,n}$(2.10)

If m and n need to be specified, we will write $\alpha_{m,n,}$

2.6

We define

$$\partial_{m,n}: A_{m+n}(\Omega_{m+n}) \to A_{m+n}(\Omega_m: \Omega_n) \qquad (2.11)$$

As follows.Let $f \in A_{m+n}(\Omega_{m+n}), h \in \mathfrak{M}_{m,n}, 1 \in G, 9' \in \Omega_n$, and let $\gamma_{m,n}$: $\mathfrak{M}_{m,n}$ be the map which puts $\mathfrak{M}_{m,n}$ into the right m x n corner of $\mathfrak{M}_{m,n}$ that is , $\gamma_{m,n}(\varepsilon_{jk}) = e_{j,m,k}$ and $\gamma_{m,n}$ is linear. By Lemma 7.1

$$\frac{d}{d\varepsilon} f\left(9' \otimes 9" + \varepsilon Y_{m,n}(h) \otimes l\right) \mid_{\varepsilon=0} = Y_{m,n}(h) \qquad (2.12)$$

for some $h' \in \mathfrak{M}_{m,n}$. Hence, for each $(9', 9") \in \Omega_m \otimes \Omega_n$, *we get a map* $\mathfrak{M}_{m,n} \ni h \to h' \in \mathfrak{M}_{m,n}$applying α^{-1} to this map gives an element in $\mathfrak{M}_m \otimes \mathfrak{M}_m$which is our definition of $(\partial_{m+n} f)(9', 9") \in \mathfrak{M}_m \otimes \mathfrak{M}_m$. This can also be written as a formula. Since the differential of f at $9' \otimes 9"$ is linear map, we have

$$(\partial_{m+n} f)(9', 9") = \sum_{\substack{1 \le ij \le m \\ 1 \le k \le n}} \left(\frac{d}{dc} f\left(9' \otimes 9" + \varepsilon e_{j.k+m} \otimes 1\right) \right)_{i,m+1} e_{ij}{}^{(m)}$$

Where $(.)_{i,m+1}$ denotes the $i, m+1$ entry of the (m+n)x(m+n) matrix,

It is clear that $\partial_{m.n} f$ defined in this way is an analytic function $\Omega_n x \Omega_n \to \mathfrak{M}_m \otimes \mathfrak{M}_n$

2.7

To check that $\partial_{m,n} f$ is a GL(m) x $GL(n)$ equivariant map, remark first that if $S' \in GL(m)$ *and* $S" \in GL(n)$, *then*

$$\frac{d}{d\varepsilon} f(AdS' \otimes l_G \otimes l)9" \otimes (AdS" \otimes l_G \otimes l)g" + \epsilon\gamma_{m,n}(S'hS"^{-1}) \mid_{\varepsilon=0}$$

$$= Ad(S' \otimes S")\left(\frac{d}{d} f\left(9' \otimes 9" + \varepsilon\gamma_{m,n(h)}\right) \otimes 1\right) \mid_{\varepsilon=0}$$

$$= S'\gamma_{m,n}(h')S^{-1}$$

Thus, we must check that if $\top \epsilon \mathcal{L}(\mathfrak{M}_{m,n})$ is given by $\tilde{\top}(S'hS"^{-1}) =$

$S'(\top(h))S"^{-1}$then$\alpha^{-1}(\acute{\top}) = ((AdS') \otimes (AdS"))\alpha^{-1}(\top)$. this is the same as he following equivariance for α: if $\top = \alpha(\mathcal{L})$ and $\acute{\top}\alpha((AdS' \otimes AdS")\mathcal{L})$, then $\acute{\top}(S'hS"^{-1}) = S'\top(h)S"^{-1}$

It suffices to see this for$\mathcal{L} = a \otimes c$, then $\top(h) = ahc$ and$\top(h) = (S'\alpha S'^{-1})h(S"cS"^{-1})$ so that $\acute{\top}(ShS"^{-1}) = S'ahcS"^{-1} = S'\top(h)S"^{-1}$, that is the equivariance we wanted to check.

Hence $\partial_{m,n} f_{m,n} \epsilon A_{m,n}(\Omega_m : \Omega_n)$

The derivation property will be obtained from the following lemma.

Lemma 7.2 if $f, \acute{f} \in A_{m,n}(\Omega_{m,n}), 9' \in \Omega_n 9" \in \Omega_n a', b' \in \mathfrak{M}_m$, α,b $\in \mathfrak{M}_n$, $(9' \otimes g") = a' \otimes a'$, and $\tilde{f}(9', \otimes 9") = b' \otimes b"$, then.

$$\left(\partial_{m,n}(f\tilde{f})\right)(9', 9") = (a' \otimes \mathrm{I}_n)\left(\partial_{m,n}\tilde{f}\right)(9', 9") + \left(\partial_{m,n} f\right)(9', 9")(\mathrm{I}_m \otimes b")(2.15)$$

Proof, I view of Lemma 7.1 we have

$$f\left(9' \otimes 9" + \lambda\gamma_{m,n}(h) \otimes 1\right) = f(9' \otimes 9") + \lambda \frac{d}{d\varepsilon} f\left(9' \otimes 9" + \varepsilon\gamma_{m,n(h)\otimes 1}\right) \mid_{\varepsilon=0}$$

And the same holds with f replaced by $\tilde{f}$ multiplying, we get

$$\left(f\tilde{f}\right)(9' \otimes 9" + \lambda\gamma_{mn}(h) \otimes 1) = f(g' \otimes 9")\tilde{f}(g' \otimes g")$$

$$+\lambda f(g' \otimes 9") \frac{d}{d\varepsilon} \tilde{f}\left(9' \otimes 9" + \varepsilon\gamma_{m,n}(h) \otimes 1\right) \mid_{\varepsilon=0}$$

$$+\lambda\frac{d}{d\varepsilon}f\big(9'\otimes 9" + \varepsilon\gamma_{m,n}(h)\otimes 1\big)\mid_{\varepsilon=0}\tilde{f}(g'\otimes 9")$$

$$+\,O(\lambda^2) \qquad (2.17)$$

This gives

$$\frac{d}{d\varepsilon}\big(f\tilde{f}\big)(9'\otimes 9" + c\gamma_{mn}(h)\otimes 1)\mid_{\varepsilon=0}$$

$$=(a'\otimes a"+)\frac{d}{d\varepsilon}\tilde{f}\big(g'\otimes 9" + \varepsilon\gamma_{m,n}(h)\otimes 1\big)\mid_{\varepsilon=0}$$

$$+\frac{d}{d\varepsilon}f\big(9'\otimes 9" + \varepsilon\gamma_{m,n}(h)\otimes 1\big)\mid_{\varepsilon=0}(b'\otimes b",). \qquad (2.18)$$

Taking result follows from

$\alpha'\alpha(\mathcal{L})(h) = \alpha\big((\alpha'\otimes\ \mathsf{I}_n)\mathcal{L}\big)(h).$

$\alpha(\mathcal{L})b"h = \alpha\big(\mathcal{L}(\ \mathsf{I}_m\otimes b")\big)(h)$

If $\mathcal{L}\epsilon\mathfrak{M}_m\otimes\mathfrak{M}_m$.

Corollary 2.3 $if\ r = (r_n)_{n\geq 1}\epsilon A(\Omega), s = (s_n)_{n\geq 1}\epsilon A(\Omega), 9'\epsilon\Omega_m$ and 9” $\epsilon\Omega_n$

Then.

$$\big(\partial_{m,n}(r_s)_{m+n}\big)(9',9") = (r_m(9')\otimes\ \mathsf{I}_n)\big(\partial_{m,n}S_{m+n}\big)(9',9")$$

$$+\big(\partial_{m,n}\mathsf{T}_{m+n}\big)(9',9")\big(\ \mathsf{I}_m\otimes S_n(9")\big) \qquad (2.21)$$

This is immediate from Lemma 7.2 when we take into account that $\mathsf{T}_{m+n}(9'\otimes 9") = \mathsf{T}_m(9')\ and\ s_{m+n}(9'\otimes 9") = s_m(9')\ \otimes S_n(9")$.

2.8

To combine the maps $\partial_{m,n}$ into a derivation for A(Ω), we will need to define "several variables FMAFS,"

Let $\Omega^{(1)}, j = 1, \dots p\ be\ FMG.S$

We define the scalar p-variables FMAPS on $\Omega^{(1)}$ x…x$\Omega^{(1)}$to be families of analytic functions$\otimes \dots \otimes \mathfrak{M}_n$

Are GL$(n_1)x \dots .xGL(n_1) - equivariant\ and\ so\ that$

$$f_{n1\dots n1-1,n1+n1-n1+1\dots\dots n1}(9_{1\dots\dots}9'_j \otimes 9''_j, \dots .9_p)$$

$$= f_{n1\dots n1-1,n1+n1-\cdots\dots n1}(9_{1\dots\dots}9'_j, \dots .9_p)$$

(2.22)

$$\otimes f_{n1\dots n1-n1-n1+1\dots\dots n1}(9_{1\dots\dots}9''_j, \dots .9_p)$$

The scalar p-variables FMAFS on $\Omega^{(1)}\ x \dots x\Omega^{(p)} will\ be\ denoted\ by\ A(\Omega^{(1)}; \dots .: \Omega^{(p)}) clearly, A\Omega^{(p)}, \dots ; \Omega^{(p)}$ is an algebra with unit.

$if\ f \in A\Omega^{(1)}, \dots \Omega^{(p)}and\acute{f} \in A\Omega^{(1)}, \dots ; \Omega^{(4)}$, then we define $f \otimes f \in A\Omega^{(1)}, \dots ; \Omega^{(p)} : \acute{\Omega}^{(1)}; \dots ; \Omega^{(p)}; \Omega^{(1)}; \dots . \acute{\Omega^{(4)}}\ by$

$$(f \otimes \acute{f})_{n;\dots\dots np,..n,\dots .no}(9_1, \dots, 9_p, 9_i, \dots ., 9_4) \quad (2.23)$$

$f_{n,\dots,n_p}(9_1,\dots 9_p) \otimes \acute{f}_{i,\dots\dots n_a}(9_{1,\dots\dots}9_d)$ (2.24)

Lomma7.4 if $r \in A(\Omega)$, then $(\partial_{m,n} \top_{m+n})_{m \geq 1} \in A(\Omega : \Omega)$

Proof: Analytical and equivariance have already been checked and we are left with $(\partial_{m,n} \top_{m+n})(9' \otimes 9", 9) =$ $(\partial_{m,n} \top_{m2n})(9', 9) \otimes (\partial_{m2,n} \top_{m2+n})(9', 9)$, where $g' \in \Omega_{\mathfrak{M}2} g" \in \Omega_{\mathfrak{M}2}, g \in \Omega_{\mathfrak{M}2} m = m_2 + m_2$and $(\partial_{m,n+m2} \top_{m+n})(g.9' \otimes 9")$= $(\partial_{m,n+m1} \top_{m+n2})(9 \otimes 9') \otimes$

$\left(\partial_{m,n2+m2}\top_{m+n2}\right)(9'9")$ where now $g \in \Omega, g' \in \Omega_{m1,} g"g \in \Omega_{n2}, and\ n = n_{1+}n_2$ he direct sums are in the sense of

$(\mathfrak{M}_m \otimes \mathfrak{M}_n) \otimes (\mathfrak{M}_{m2} \otimes \mathfrak{M}_n)$=$(\mathfrak{M}_{m1} \otimes \mathfrak{M}_{n2}) \otimes \mathfrak{M}_n \subset \mathfrak{M}_{n1+m2} \otimes \mathfrak{M}_n,$

and in the second case

$(\mathfrak{M}_m \otimes \mathfrak{M}_{n1}) \otimes (\mathfrak{M}_m \otimes \mathfrak{M}_{n2})$=$\mathfrak{M}_{m\otimes}(\mathfrak{M}_{m1}, \mathfrak{M}_{n2}) \subset \mathfrak{M}_m \otimes \mathfrak{M}_{n+n},$

We will only sketch how one checks the first of the two equalities for $\partial_{m,n}$the second one being similar.

First, remark that α^{-1} behaves well with respect to direct sums, that is , if $\top_1 \in £\left(\mathfrak{M}_{m,n}\right), \top_2 \in £\left(\mathfrak{M}_{m,n}\right)$ and$\top_{1,} \otimes \top_2\left(\in £\mathfrak{M}_{m1,n} \otimes \mathfrak{M}_{m2,n}\right) =\in £\left(\mathfrak{M}_{m1+m2,n}\right)$ then $\alpha^{-1}_{m1+m2,n}\left(\top_{1,} \otimes \top_2\right) = \alpha^{-1}_{m1+m2,n}\left(\top_{1,}\right)\alpha^{-1}_{m1+m2,n}\left(\top_{2,}\right)$ thus , it will suffice to check that.$\left(\alpha_{m1+m2,n}\partial_{\alpha_{m+m,n}\tau_{m+n}}\right)(g' \otimes g") \otimes \left(\alpha_{m1+m2,n}\partial_{\alpha_{m+m,n}\tau_{m+n}}\right)(g' \otimes g")$

$$=\left(\alpha_{m2+m2,n}\partial_{\alpha_{m2+m,n}\tau_{m2+n2}}\right)(g'g') \qquad (2.26)$$

In view of Lemma7.1 and of the direct sum property of an FMAF, what we must prove amounts to the following Let $h'_{1\in\mathfrak{M}_{m+n}}$ and $h_{2,}h'_2 \in \mathfrak{M}_{m2+n}$ be such that.

Where α'= (9') ,$\alpha" = \top m_2(9")$, and $\alpha = \top_n(9)$, then we will have

$$\tau_{m+n}\left(\begin{pmatrix} 9' & h_1 \otimes \mid \\ 0 & 9 \end{pmatrix}\right) = \begin{pmatrix} \alpha' & h_j \\ 0 & \alpha \end{pmatrix} (2.27)$$

$$\tau_{m+n}\left(\begin{pmatrix} 9" & h_2 \otimes \mid \\ 0 & 9 \end{pmatrix}\right) = \begin{pmatrix} \alpha" & h'_2 \\ 0 & \alpha \end{pmatrix}$$

$$\mathsf{T}_{m1+m2+n}\left(\begin{pmatrix} 9' & 0 & h_1 \otimes \mathsf{I} \\ 0 & 9" & h_2 \otimes \mathsf{I} \\ 0 & 0 & 9 \end{pmatrix}\right) = \begin{pmatrix} \alpha' & 0 & h_j \\ 0 & \alpha" & h_2 \\ 0 & 0 & \alpha \end{pmatrix} (2.28)$$

Since

$$\mathsf{T}_{m1+m2+n}\left(\begin{pmatrix} 9' & 0 & h_1 \otimes \mathsf{I} \\ 0 & 9" & h_2 \otimes \mathsf{I} \\ 0 & 0 & 9 \end{pmatrix}\right) = \begin{pmatrix} \mathsf{T}_{m1+m2}\left(\begin{pmatrix} 9' & 0 \\ 0 & 9" \end{pmatrix}\right) * \\ 0 \\ r_n(9) \end{pmatrix} 2.29)$$

By Lemma.2.1 and $\mathsf{T}_{m1+m2}(9' \otimes 9") = \alpha' \otimes \alpha"$, all we need to check is that the (1,3) and (2,3) blook entries of the result are h_j and h_j. This can be done b several application of direct sum and GL equivariance properties.

$$\mathsf{T}_{m1+m2+n}\left(\begin{pmatrix} 9' & h_1 \otimes \mathsf{I} & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9" h_2 \otimes \mathsf{I} \\ 0 & 0 & 0 & 9 \end{pmatrix}\right) = \begin{pmatrix} \alpha & h_j' & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha" & h_j' \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

$$\mathsf{T}_{m1+m2+n}\left(\begin{pmatrix} 9' & 0 & h_1 \otimes \mathsf{I} & 0 \\ 0 & 9" & 0 & h_2 \otimes \mathsf{I} \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}\right) = \begin{pmatrix} \alpha' & 0 & h_j' & 0 \\ 0 & \alpha" & 0 & h_j' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

$$\mathsf{T}_{m1+m2+n}\left(\begin{pmatrix} 9' & 0 & h_1 \otimes \mathsf{I} & 0 \\ 0 & 9" & h_2 \otimes \mathsf{I} & h_2 \otimes \mathsf{I} \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}\right) = \begin{pmatrix} \alpha' & 0 & h_j' & 0 \\ 0 & \alpha" & h_j' & h_j' \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

On the one hand, and on the other hand, he last equality can be continued with

$$\begin{pmatrix} \top_{m1+m2+n}\left(\begin{pmatrix} 9' & 0 & h_1 \otimes \mathsf{I} \\ 0 & 9" & h_1 \otimes \mathsf{I} \\ 0 & 0 & 9 \end{pmatrix}\right) & * \\ 0 & 0 \end{pmatrix}$$

$if\ r \in A\Omega_1$, we will denote by ∂_r the element $\left(\partial_{m,n}\top_{m+n}\right)_{m\geq 1} \in A(\Omega\colon\Omega)$before going further, we also record he following fact which appeared in the preceding roofs.

Lemma 7.5 Let $9' \in \Omega_m, 9" \in \Omega_n,$ and $h_1 h' \in \mathfrak{M}_{m,n}$ be such that

$$\top_{m+n}\left(\begin{pmatrix} 9' & h_{\otimes\ \mathsf{I}} \\ 0 & 9" \end{pmatrix}\right) = \begin{pmatrix} r_m(9') & h' \\ 0 & \top_n(9") \end{pmatrix} (2.32)$$

Then $h' = \left(\alpha_{m.n}\left(\partial_{m,n}\top_{m+n}\right)(9', 9")\right)(h)$

In particular, he map taking h is linear and takes sht to sh’t$if\ s \in GL(m)$and $t \in GL(n)$.

2.10

We pass to the coassociativityproperty of∂ since we have not identified $A(\Omega;\Omega)$with a tensor product $A(\Omega) \otimes A(\Omega)$, we will define maps $(id \otimes \partial)\ A(\Omega;\Omega) \to A(\Omega;\Omega,\Omega)$and respectively $(\partial \otimes id)\ A(\Omega;\Omega) \to A(\Omega;\Omega,\Omega)$, he most convenient seems to use the formula for matrix entries given at the end of section 2.6 thus, for $k \in A_{m,n+p}(\Omega_{m;};,\Omega)$, $9 \in \Omega_{m;}$, $9' \in \Omega_n$ and 9"$\in \Omega_{p;}$, we define

$$\left(id \otimes \partial_{m.n.p}k\right)(9,9',9,9")$$
$$\sum_{\substack{1\le a,b \le m \\ 1\le c,d\le n \\ 1\le e\le f\le p}} \left(\frac{d}{d\varepsilon}k(9;9' \otimes 9" + \varepsilon ed_{n+e}) \mid_{\varepsilon=0}\right)_{(a,b)(c,n+f)} e_{ab}^{(m)} \otimes e_{ef}^{(\top)}$$

(2.33)

Where the index $(a,b)(c,n+f)$ sands for the coefficient $e_{ab}^{(m)} \otimes e_{c,n+f}^{(n+p)}$.

In particular , if $\in A_m(\otimes_m)$ and $\acute{f} \in A_{n+p}(\Omega_{n+p})$, then $(id \otimes \partial)_{m,np}(f \otimes \acute{f}) = f \otimes (\partial_{n,p}\acute{f})$

We leave it to the reader to check that $(id \otimes \partial)_{m,n,p}(\Omega_{m,}\Omega_{n,}, \Omega_{p,})$ part of the verification can e done using $9 \in \Omega_m, 9" \in \Omega_m$, functional $\varphi \in (\mathfrak{M})'_m$, the functions $(\varphi \otimes id)k(9;.) \in A(\Omega_{n+p})$, he fact that $\left((\varphi \otimes id_{\mathfrak{M}_n \otimes} id_{\mathfrak{M}_n})(id \otimes \partial)_{m,n.p}k\right)(9;9';9") = \partial_{n,p}\left(\left(\varphi \otimes id_{\mathfrak{M}_{n+p}}\right)k(g;)\right)(9'9")$, and he results we already have for $\partial_{n,p}$ using this type of argument, one then checks that k=$(k_{n1,n2})_{n1\ge1,n2\ge1} \in A(\Omega,\Omega)$, then $((id \otimes \partial)_{n1,n2,n3}k_{n1,n2,n3})_{n1\ge1,n2\ge1,n3\ge\ge1} \in A(\Omega,\Omega,\Omega)$and a similar result is obtained for $\partial \otimes id$

Checking that $(\partial \otimes id)^\circ\partial = (\partial \otimes id)^\circ\partial$, after we have put aside all these questions, boils down to the following result..

Lemma 7.6 *if* $k \in A_{m+np}(\Omega_{m+n+p},)$ then $(id \otimes \partial)_{m,n,p}\partial_{m,n,p}k = (\partial \otimes id)_{m,n,p}\partial_{m+n+p}k$.

Proof.Let $\in \Omega_{m,} g" \in \Omega_{m,} 9' \in \Omega_n$, and 9"$\in \Omega_p$. We have

$$\left((id \otimes \partial)_{m,n,p} \circ \partial_{m,n+p} k\right)(g, g', g")_{(a,b)(c,d)(e,f)}$$

$$= \frac{d}{d_{\varepsilon 2}}\left(\frac{d}{d_{\varepsilon 1}}\left(k\left(9 \otimes g' \otimes g" + \varepsilon_1 e_{m+d,m+n+e} + \varepsilon_2 e_{b,m+c}\right)\right)_{a,m+n+f} \mid_{\varepsilon_{2=0}}\right)$$

The equality of the two quantities is thus quite obvious

Lemma 7.7 *Let* $\Omega_1 \subset \mathbb{C}$ be an open set, G= $\mathbb{C}$, and $\Omega_2 = \Omega_n = (a \in \mathfrak{M}_n \mid \sigma(a) \subset \Omega_1)$. Further let $f = (f_n)_{n\geq 1} \in A(\Omega)$, where $\Omega = (\Omega_n)_{n\geq 1}$ so that $f_n(\alpha) = f_1(a)$, where the right hand side has he meaning of functional calculus. Then if $z_{1,} z_2 \in (\Omega_1, z_1 \neq z_2)$

$$\left(\partial_{1,1} f_2\right)\left(z_{1,} z_2\right) = \frac{f_{1(z_1)-f_1(z_2)}}{z_1 - z_2} \tag{2.36}$$

Proof Let $\left(\partial_{1,1} f_2\right)\left(z_{1,1} z_2\right) = \lambda \in \mathfrak{M}_1^{\otimes 2}\mathbb{C}$. Then

$$\begin{pmatrix} f_1(z_1) & \lambda \\ 0 & f_1(z_2) \end{pmatrix} = z_1\left(\begin{pmatrix} z_1 & 1 \\ 0 & z_2 \end{pmatrix}\right) \tag{2.37}$$

Since $\left(\begin{pmatrix} z_1 & 1 \\ 0 & z_2 \end{pmatrix}\right) = Ad\begin{pmatrix} 1 & \begin{pmatrix} 1^{(z_1 - z_2)^{-1}} & \\ 0 & 1 \end{pmatrix} \\ 0 & \end{pmatrix}\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$, It follows that

$$\begin{pmatrix} f_1(z_1) & \lambda \\ 0 & f_1(z_2) \end{pmatrix} = \mathrm{Ad}\begin{pmatrix} 1 & (z_1 - z_2) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_1(z_1) & \lambda \\ 0 & f_2(z_2) \end{pmatrix} \tag{2.38}$$

Which gives the desired result.

8. Dual positivity in A Ω

Let Ω be an openFMG-S over the operator space G and assume Ω =Ω*. We will see the map $\alpha\sigma = \nabla$. I particular, if $f9' \in \Omega_{n,}$ then $\left(\nabla_{m,n} f_{m+n}\right)(g', g")$is an element in $\mathcal{L}\left(\mathfrak{M}_{m,n}\right)$

Definition 8.1 an element $f \in A(\Omega)$is dual positive if $f = f^*$ and for any $g \in \Omega_{n,}$, n∈ N,

$$\left(\nabla_{n,n} f\right)(g, g^*): \mathfrak{M}_n \to \mathfrak{M}_n \tag{8.1}$$

Is a positive map $(i.e, \textit{transforms positive operaors into positive operators})$

Proposition 3.2 $if\ f \in A(\Omega)$the following are equivalent

(i) f is dual positive.

(ii) f=f^* and for any $g^{(1)} \in \Omega_{n(1),}\ 1 \le j \le p, \otimes_{1\le i,j\le p} \left(\nabla_{n(1),n(1)} f\right)\left(g^{(1)}, g^{(1)^*}\right)$ *is a positive linear map of* $\otimes_{i,j,}\ \mathfrak{M}_{n(1),n(1),}$, identified with $\mathfrak{M}_{n(1)+\cdots+n(p),}$into itself.

(iii) $f = f^*$, and for any 9 $\in \Omega_{n,}$, the map $\left(\nabla_{n,n} f\right)(9', 9^*): \mathfrak{M}_n \to \mathfrak{M}_n$ is completely positive.

Proof. Clearly $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$

$(1) \Longrightarrow (ii). II.$ it suffices to show that the map$\otimes_{1\le i,j\le p}\left(\nabla_{n(1),n(j)} f_{n(1)+n(1)}\right)\left(9^{(1)}, g^{(i)}, g^{(j)*}\right)$ coincides with the map.

$$\left(\nabla_{n,n} f_{n,n}\right)\left(g^{(1)\otimes\dots\otimes} g^{p}, g^{(1)*\otimes\dots\otimes} g^{(p)*}\right), \tag{3.2}$$

Where $n = n(1) + \cdots + n(p)$. indeed, in view of the definition ofα, this is the same as establishing hat $\otimes_{1\le i,j\le p}\, \partial_{n(i),n(1)} f_{n(1)+n(j)}\left(9^{(1)}, 9^{(1)*}\right)$ as an element of $\otimes_{1\le i,j\le p}\, \mathfrak{M}_{n(1)} \otimes \mathfrak{M}_{n(1)} \subset \mathfrak{M}_{n\otimes}\mathfrak{M}_n$ coincides with $\partial_{n,n} f_{n+n}\left(9^{(1)} \otimes \dots \otimes 9^{(p),} 9^{(10)*,} \otimes \dots \otimes 9^{(p)*,}\right) \in (\mathfrak{M}_n, \otimes \mathfrak{M}_n)$. This in turn is an immediate consequence of the fact that $\partial f \in A(\Omega, \Omega)$.

$(ii) \Longrightarrow (iii). if i^{ij} \in \mathfrak{M}_n, 1 \le i, j \le p$.form a p x p matrix with n x n block, which is positive in $\mathfrak{M}_{np}$, we must show that the np x np matrix formed from the blocks $\left(\nabla_{n,n} f_{n+n}(g, g^*)\right)\left(t^{i,j}\right)$ is also positive. This is precisely he statement in (ii) in case $n(1) = \cdots = n(p)$=n and $9^{(1)} = \cdots = g^{(p)} = g$.

9. The full resolventtransform

4.1

The dual GDQ ring corresponds to a map of the dual of the GDQ ring into a GDQ ring of the $A(\Omega)$ type. As long as we do not use an involution we will use the context of Section 5 thus. E will be a Banach algebra with unit, $1 \in B \subset E$ a Banach subalgebra, and $Y \in E$ an element Let $p(Y{:}B) = \left(p_n(Y{:}B)\right)_{n\ge1}$be the full B-resolvent set of Y and R(Y;B) = $\left(R_n(Y{:}B)\right)_{n\ge1}$ the full B-Resolvent.

By $\mathcal{R}$A$(Y:B)$; we will denote the subalgebra of E generated by B, $\langle Y \rangle$, and the matrix coefficients of $\{R_n(Y;B)(b) \mid n \in N. b \in p_n(Y;B)\}$

4.2

We will assume that there is a derivation comultiplication

$$\partial: \mathcal{R}A(Y;B) \to \mathcal{R}A(Y;B) \otimes \mathcal{R}A(Y;B)$$

So that $\mathcal{R}A(Y;B)$ is a GDQ ring , $\partial B = 0$, and $\partial y = 1 \otimes 1$. if such a ∂ exists, then it is unique, that is, completely determined by he condition $\partial B = 0$ and $\partial y = 1 \otimes 1$ indeed, the $R_n(Y;B)(b)$ will then be corepresentation, and the corresponding equation determines ∂on the matrix coefficients. Thus, ∂ is completely determined on the generators of $\mathcal{R}A(Y;B)$;hence, being a derivation, it is completely determined on $\mathcal{R}A(Y;B)$.

4.3

We will also assume that $\mathcal{R}A(Y;B)$ is dense in E

Let $(Y;B)$ dente he matrix coefficients of $R_n(Y;B)(Y;B)(b)(b \in p_n(Y;B), n \in, N)$

Lemma 9.1 $\mathcal{C}R(Y;B)$is closed under multiplication. The assumptions in section 4.3 imply that the linear span of R$(Y;B)$ is dense in E.

Proof.Remark first that if $\alpha \in \mathfrak{M}_m(E)$, $x \in \mathfrak{M}_{m.n}(E)$, and α^{-1}exist, then

$$\begin{pmatrix} \alpha & -x \\ 0 & \acute{\alpha} \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} & \alpha^{-1} x \breve{\alpha}^{-1} \\ 0 & \acute{\alpha}^{-1} \end{pmatrix} \quad (4.2)$$

In particular, the $(i, l+m)$, entry of this 2x2 block matrix is the (i, l)entry of $a^{-1}x\grave{a}^{-1}$ choosing x to be the (j, k) matrix unit, we find that for this choice of x, one of the matrix coefficients of $\begin{pmatrix} \alpha & -x \\ 0 & \acute{\alpha} \end{pmatrix}^{-1}$ is he product of the (k, l) entry of d^{-1} taking a and a' to be $\beta - Y \otimes I_n$,respectively, we get that $\mathcal{C}\mathrm{R}(Y; B)$is closed under multiplication.

Thus, the linear span of $\mathcal{C}\mathrm{R}(Y; B)$is an algebra, and to prove the second assertion, it suffices to prove that its closure contain and B. since the linear span of invertible elements in B is B it will suffice to prove that the invertible elements in B and Y are in the closure of the linear span of $\mathcal{C}\mathrm{R}(Y; B)$if $b \in B$ is invertible, then so is $(\lambda b - Y)$for λ large enough. $(\lambda b - Y)^{-1} \in \mathcal{C}\mathrm{R}(Y; B)$, and $\lim_{\lambda\to\infty} \lambda^{-1} (\lambda b - Y)^{-1}$=$b^{-1}$. The assertion about Y follows from.

$y = \lim_{\varepsilon\to 0} e^{-1} (\varepsilon^{-1}(\varepsilon^{-1} - Y)^{-1} - \varepsilon^{-2}(\varepsilon^{-2} - Y)^{-1})$(4.3)

4.4

Let E' denote the dual of the banach space E the full resolvent transform is defined to be the map.

$$\sqcup: E' \to Ap\big((Y; B)\big) \qquad (4.4)$$

So that $\sqcup(\varphi) = \big(\sqcup_n(\varphi)\big)_{n\geq 1,}$ where

$\sqcup_n(\varphi)(.) = (\varphi \otimes id_{\mathfrak{M}n})\Big(R_n\big(Y{:}B(.)\big) \in A_n(p_n(Y{:}B)\Big)$(4.5)

$\big($*Remark that* $\sqcup_n(\varphi)(.)\big)$is fully matricial analytic because $R_n(Y{:}B)(.)$ is fully matricial an-alytic

Proposition 8.2 *if* $\varphi_{1,}\varphi_2\varphi_3 = \in E'$ are such that

$\varphi_1(\alpha) = (\varphi_2 \otimes \varphi_3)(\partial\alpha)$(4.6)

For all $a \in \mathcal{R}A(Y;B)$, then

$\sqcup(\Omega_1) = \sqcup(\Omega_2) \sqcup (\Omega_3)$(4.7)

Proof: it is actually sufficient that the assumption holds for a $\in \mathcal{R}A(Y;B)$in order to get the conclusion indeed, applying the assumption to each matrix coefficient of $R_n(Y;B)(b) = \alpha$, we have that.

$$\sqcup_n (\varphi_1)(b) = (\varphi_2 \otimes id_{\mathfrak{M}_n})(\partial \otimes id_{\mathfrak{M}_n}\alpha)$$

$= (\varphi_2 \otimes id_{\mathfrak{M}_n})(\partial \otimes id_{\mathfrak{M}_n}\alpha)$(4.8)

$$= \sqcup_n (\varphi_2)(b) \sqcup_n (\varphi_3)(b)$$

4.5

Before stating the duality property involving he co-multiplication of $A(P(Y;B))$, we need to clarify a notation we will use. If $b = \sum_{i,j} b_{i,j} \otimes e_{i,j}^{(m)} \in \mathfrak{M}_n(B) \sum_{k,I} b'_{k,I} \otimes e_{i,j}^{(m)} \in \mathfrak{M}_n(B)$and $b' = \sum_{k,I} b'_{k,I} \otimes e_{i,j}^{(n)} \in \mathfrak{M}_n(B)$, We denote by $b \otimes_B$ b' $\in \mathfrak{M}_{mn}(B)$ the mn x mn matrix, or equivalently, the element in $\mathfrak{M}_m \otimes \mathfrak{M}_n$given by $\sum_{i,j,kl} e_{i,j}^m \otimes e_{ki}^n \otimes (b_{i,j}b'_{kl})$. Equivalently, if $\alpha \otimes \beta \in \mathfrak{M}_n \otimes B$ and $\alpha' \otimes \beta' \in \mathfrak{M}_n \otimes B$, then $(\alpha' \otimes \beta' \in \mathfrak{M}_n \otimes B)$ then$(\alpha \otimes \beta) \otimes_B (\alpha' \otimes \beta') = \alpha \otimes \alpha' \otimes \beta\beta'$.

Proposition 4.3 *if* $\varphi \in \otimes E'$, $b_1 \in p_m(Y;B)$,and $b_2 \in p_n(Y;B)$, then

$$(\varphi \otimes id_{\mathfrak{M}_n})(R_m(Y;B)(b_1) \otimes_E R_n(Y;B)(b_2))$$

$$= -\partial_{m,n}(\sqcup_{m+n}(\varphi))(b_{1i:}b_2)$$

Proof. Returning to the computations on which the proof of Lemma 9.1 relies, Let $\alpha = b_1 Y \otimes \mathsf{I}_m$ and let $x = 1 \otimes e_{i,j}^{(m)}$ so that

$$(R_m)(Y;B)(b_1) \otimes_E R_n\big(Y;B(b_2)\big)_{(i,j)(k,i)} = \left(\begin{pmatrix} \alpha & -x \\ 0 & \acute{\alpha} \end{pmatrix}^{-1}\right)_{i,l+m} \qquad (4.10)$$

On the other hand, section 7.6 and Lemma 7.1 give that

$$\Big(\partial_{m,n}\big(\sqcup_{m+n}(\varphi)\big)(b_1 b_2)\Big)_{(i,j)(k,i)} = \left(-\sqcup_{m+n}(\varphi)\begin{pmatrix} b_1 & -x \\ 0 & b_2 \end{pmatrix}^{-1}\right)_{i,m+1}$$

$$= -\left((\varphi) \otimes id_{\mathfrak{M}_{n+n}}\begin{pmatrix} \alpha & -x \\ 0 & \acute{\alpha} \end{pmatrix}^{-1}\right)_{i,m+1}$$

Which implies the desired result.

Remark 4.4 proposition 4.2 and 4.3 express the fact that the ⊔-transform relates “ dual GDQ structure” on E’ with the “ topological GDQ structure” of $A\big(p(Y;B)\big)$ endowed with he comultiplication – ∂ these duality statements take this indirect form because of he rather algebraic setting of our discussion $i.e. without analytic assumptions on the$

$comultiplication of \mathcal{R}A(Y;B) and a closer examination$

$of\ he\ topological$tensor product in the GDQ structure of $A\big(p(Y;B)\big)$

Proposition.4.5 (i) the map ⊔is injective.

(ii)An element $\varphi \in E$’ satisfies the trace-condition $\varphi([E,E]) = 0$ if and only if

$$\partial_{m,n}\big(\sqcup_{m+n}(\varphi)\big)(b_1b_2) = \varepsilon^\circ\partial_{n,m}\big(\sqcup_{m+n}(\varphi)\big)(b_2b_1)(4.13)$$

Which follows from the trace condition. The converse, that is, all these equalities taken together imply that $\varphi$is a trace, follows from Lemma 9.1

4.6

We will now consider dual-positivity. We assume for the rest of section 4 that E and B are C^*-algebras and that $Y = Y^*$. Note that $\big(p_n(Y;B)\big)^* = p_n(Y;B)^*$and $R_n(Y;B)(b) = \big(R_n(Y;B)(b^*)\big)^* = R(Y;B)$ under these assumptions

Propositions 4.6 Let $\phi \in E'$ then

(i) $\left(\sqcup(\varphi^*)\right)^*$
(ii) $\sqcup(\varphi)^* = \sqcup(\varphi)$ if ad only if $\varphi\varphi^*$
(iii) $\varphi \otimes \geq 0$ if and only if $\sqcup(\varphi) \geq 0$ in the sense of dual positivity in $A\left(p(Y;B)\right)$

Proof.(i)if$b \in p_n(Y;B)$ and t denotes transpose of a matrix, then

$$\frac{\left(\sqcup_n(\varphi^*)\right)(b) = \left(\varphi^* \otimes id_{\mathfrak{M}_n}\right)\left((b - Y \otimes \mathsf{I}_n)\right)^{-1}}{}$$
$$= \frac{(\varphi \otimes id_{\mathfrak{M}_n})(((b-Y\otimes \mathsf{I}_n)^{-1})^{*t})}{=(\varphi \otimes id_{\mathfrak{M}_n})(((b-Y\otimes \mathsf{I}_n)^{-1})^{*t})}$$

(iii) It follows from (i) and the injectivity of $\sqcup$.

(iv) We first prove the only if part. Assume $\varphi \geq 0$ and let $h \in \mathfrak{M}_n, h \geq 0$. By Lemma 7.1 and he definition of dual positivity , we must check that in the 2n x 2n matrix $\varphi \otimes id_{\mathfrak{M}_2}\left(\left(b \otimes b^* - Y \otimes \mathsf{I}_{2n} - 1 \otimes Y_{n,n}(h)\right)^{-1}\right)$, The right n x n corner block is positive since this block is precisely $\left(\varphi \otimes id_{\mathfrak{M}_2}\right)((b - \otimes \mathsf{I}_n)^{-1})$, the assertion follows from the assumptions $\varphi \geq 0$ *and* $h \geq 0$.

To prove the converse, note that from the proof of the only if part, the dual positivity of $-\sqcup(\varphi)$ implies $\left((\varphi \otimes id_{\mathfrak{M}_2})((b - Y \otimes \mathsf{I}_n)^{-1})(1 \otimes h)(b^* - Y \otimes \mathsf{I}_n)^{-1}\right) \geq 0 h \in \mathfrak{M}_{n.}$ this is turn implies $\varphi(££^*) \geq 0$for any £in the linear span of $\mathcal{CR}(Y;B)$. indeed, if £ $=c_1\eta_1 + \cdots + c_1\eta_1$

Where $c_j \in and\ \eta_1$ is some matrix coefficient of$(b_1 - Y \otimes \mathrm{I}_n)^{-1}$

Then it is easily seen that

$$££^{*}=(1 \otimes k)(b - Y \otimes \mathrm{I}_n)^{-1}(1 \otimes h)\left(b^{*} - Y \otimes \mathrm{I}_n\right)^{-1}(1 \otimes k)$$

For some $h \geq 0, h \in \mathfrak{M}_{n.}\ n = n_1 + \cdots + n_p$, and $p = b_1 \otimes \ldots \otimes b_p$ hence $\varphi(££^{*})=$ k$\varphi \otimes id_{\mathfrak{M}_2}\left((b - Y \otimes \mathrm{I}_n)^{-1}(1 \otimes h)(b^{*} - Y \otimes \mathrm{I}_n)^{-1}k^{*} \geq 0\right.$

Remark 4.7 the dual positivity of $\sqcup$ $(\varphi)$is equivalent to the dual positivity of $\bar{\sqcup}$ $(\varphi)$with respect to $-\partial$ which is then in agreement with ∂ intertwining the GDQ structures of £′ and $\left(A\left(p(Y;B)\right),-\partial\right)$

Remarks 4.8 to characterize states in E via their $\sqcup$-transform, in addition to dual.Positivity of$\sqcup$ (φ) one requires $\varphi(1) = 1$, which is equivalent to ln $lim_{n\infty} n \sqcup_1 (\varphi)(n1) = 1$

$(n1 \in p_1(p(Y;B)$for$(n \geq \|Y\|)$

Remark 4.9 one situation in free probability, where the dual multiplication appears, is the definition of the conjugate variable $\ni (X;B)$ see [13,15,16] in the corresponding W^{*}probability context, $(M,t), B(X)$⊂M, with $1 \in B$ a von Neumann subalgera t a trace state, and assuming $B(X)$ to be weakly dense in M, let $\varphi(.) = \tau\big(3(X:B)\big) be\ the$ functional defined by $\partial(X:B)$ then , if $\alpha \in B(X)$, we have $\varphi(\alpha) = (\tau \otimes \tau)(\partial_{x:B\alpha})$or, denoting by $\neq$, he dual multiplication $\tau \neq \tau\varphi$ identifying $l_2(M,\tau)$with a part of the predual M_{*} of M and hence τ with 1, the same relation would be written in the form $1 \neq 1 = 3(X;B.)$

Similarly, the higher conjugates see 13 amount to$(p+1)$ fold dual products $\tau \neq \cdots \neq \tau$ or in the other notation $\tau \neq \cdots \neq \tau$

Note added in proof. We have recently learned more about work in combinatorics on bialgebras with derivation comultiplication. Around the same time with our paper .the selfduality of the structure was also found independently by Aguiar in [1]. Besides “infinitesimal bialgebras” and “GDQ rings,” other names for related structures have been used: “eHopfalbebras,” “ infinitesimalHopfalbebras,” and “Newtonian bialgebras.”

Concerning the compatibility relations [4] satisfied by the partial difference quotient derivations, we have learned that certain structures with several comultiplications satisfying such compatibility relations have been considered by Leroux in [8]