Chapter(1)

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Introduction:

Definition (1-1):

In this dissertation a survey of approximation theory and methods is given. Some algorithms are written and implemented. The theory mainly depends on approximation in normed linear spaces and inner product spaces.

In this chapter basic definitions existence and convergent the norms are given.

The remaining chapter cover some standard method for approximation.

Given a continuous function *f* defined on a closed interval [a, b] and a positive integer *n* , can we represent *f* by a polynomial :

$p(x) = \sum_{k=0}^{an} a_r \ x^k$, of degree at most *n* in such a way that the maximum error at any point x in [a, b] is controlled.

In particular, is it possible to construct *P* so that the error:

max $a \leq x \leq b \ |f(x) - p(x)|$ is minimized

Best Approximations in Normed spaces:

Chebysher's problem is perhaps best understood by rephrasing in modern terms. What we have here is a problem of best approximation in normed Linear space. Recall that a norm on a (real) vector space *x* is a nonnegative function on *x* satisfying:-

$||x|| \geq 0 \ and \ ||x|| = 0 \ \Leftrightarrow x = 0$

$||\alpha x|| = |\alpha|\ ||x||, for\ any\ x \in x, x \in IR$

$||k + y|| \leq ||x|| + \ ||y||, for\ any\ x, y \in X$

Any norm on X induces a in a metric or distance function by setting: $dis\ (x\ y) = ||x - y||$. The abstract version of vector problem(s) can now be restated:

Given a subset (or even subspace) Y of X and point $x\ \in X$, is there an element $x\ \in Y$ that is nearest to x ?? that is, can we find a vector $x\ \in Y$ such that $||x - y|| = \min_{z \in Y} ||x - Z||$.

It's not hard to see that a satisfactory answer to this question will require that we take Y to be a closed set in X, for otherwise point in $\bar{Y}\ |\ Y$ (sometimes called the boundary of the set Y) will not have nearest points indeed which point in the interval [0,1] is nearest to 1.

Less obvious existence (and certainly the uniqueness) of nearest points. For the time being we will consider.

The case where Y is a closed subspace of a normed Liner space X.

Finite, Dimensional Vector Spaces:

The key to the problem of polynomial approximation is the fact that each of the spaces *Pn*, is finite dimensional. To see how finite dimensional subspaces of arbitrary normed spaces.

Lema (1.2): Let *V* be a finite – dimensional vector space. Then all norms on *V* are equivalent. That is if ||.|| and |||.||| are norms on *V*, then there erist constant $0 < A < B < \infty$ such that

$A||x|| \leq \left|||x||\right| \leq B||x||$, it vectors $x \in V$

Proof: suppose that V is n-dimensional and that ||.|| is norm on V. fix a basis $e_1, \dots e_n$ for V and consider the norm.

$$\left\|\sum_{i=1}^{n} a_i e_i\right\| = \sum_{i=1}^{n} |a_i| = ||(a_i)_{c=1}^{n}||_1$$

For $x = \sum_{i=1}^{n} aiei \in V$. Because $e_1, \dots e_n$ is a basis for V, It's not hard to see that $||.||_1$ is indeed a norm on V.

The map $(ai)_{i=1}^{n} \to \sum_{i=1}^{n} aiei$ is obviously both one-to-one and onto. In fact this correspondence is an isometry between $\left(IR^n, ||.||_1\right)$ and $\left(V, ||.||_2\right)$ it now suffices to show that $||.||$ and $||.||_1$ are equivalent.

One inequality is easy to show, indeed notice that

$$\left\|\sum_{i=1}^{n} a_i e_i\right\| \leq \sum_{i=1}^{n} |a_i|\,||e_i|| \leq \left(\max_{1 \leq i \leq n} ||e_i||\right) \sum_{i=1}^{n} |a_i| = B \left\|\sum_{i=1}^{n} |a_i e_i|\right\|$$

The real work comes in establishing the other inequality now the inequality we've just established shows that the function $x \to ||x||$ is continuous on the space $\left(V, ||.||_1\right)$ indeed $||x|| - ||y|| \leq ||x - y|| \leq B||x - y||_1$, for any $x, y \in V$.

Thus $||.||$ assumes a minimum value on the compact set $S = \left\{x \in v: ||x||_1 = 1\right\}$, in particular, there is a some $A > 0$ such that $||x|| > A$ whenever

$||x||_1 = 1$, the inequality we need now follows from the homogeneity of the norm $\left\|\frac{x}{||x_1||}\right\| \geq A \Longrightarrow ||x|| \geq A||x||_1$

Corollary (1.3):

Every finite – dimensional normal space is complete [that is, every Cauchy sequence converges]. in particular if Y is a finite – dimensional subspace of a normed Linear space X, then Y is a closed subset of X.

Corollary (1.4):

Let Y be a finite – dimensional normed space, let $x \in y$ and Let $M > 0$. Then any closed ball:

$\{y \in Y: ||x - y|| \leq M\}$ is compact.

Proof:

Because translation is an isometry, it clearly suffices to show that the set $\{y \in Y: ||y|| \leq M\}$ (i.e. the ball about 0) is compact.

Suppose now that Y is n-dimensonal and that $e_1, \dots e_n$is a basis for Y. (from Lemma (1-2), we know that there is some $A > 0$ such that :

$$A\sum_{i=1}^{n}|a_i| \leq \left\|\sum_{i=1}^{n} a_i e_i\right\| \quad , \ for\ all$$

$$x = \sum_{i=1}^{n} a_i e_i \in Y.$$

In particular:

$$A|a_i| \leq \left\|\sum_{i=1}^{n} a_i e_i\right\| \leq M \Longrightarrow |a_i| \leq \frac{M}{A}, for\ i = 1, \dots, n$$

Thus, $\{y \in Y: ||y|| \leq M\}$ is a closed subset of the compact set:

$$\left\{x = \sum_{i=1}^{n} a_i e_i = |a_i| \leq M/A, i = 1, \dots, n\right\}$$

$$= [-M/A\ , M\ /A]^n$$

Theorem (1.5): Let Y be a finite – dimensional subspace of anormed Linear space X, and Let $x\ \in X,$ then there exist a (not necessarily unique) vector $y^* \in Y$:

$$||x - y^*|| = \min_{y \in Y} ||x - y||$$

for all $y \in Y$ that is there is best approximation to *x* by elements from Y.

Proof:

First notice that because $O \in Y$, we know that any nearest point y^* will satisfy:

$$||x - y^*|| \leq ||x|| = ||x - 0||$$

Thus it suffices to look for y^* in the compact:

set : $k = \{y \in Y: ||x - y|| \leq ||x||\}$

we need only note that the function $f(y) = ||x - y||$ is continuous :

$|f(y) - f(z)| = |\|x - y\| - \|x - z\| \leq \|y - z\||$, and hence attains a minimum value at some point $y^* \in K$.

Corollary (1.6): for each $f \in c[a,b]$ and each +ve integer n, there is a (not necessarily unique).

Polynomial $P_n^* \in P_n = \left||f - P_n^*|\right| = \min_{p \in P_n} \left||f - p|\right|$.

Lemma (1.7): Let Y be a finite dimensional subspace of anormed Linear space X, and suppose that each $x \in X$ has aunique nearest point $y_x \in Y$.

Then the nearest point map $x \to y_x$ is continuous.

Proof:

Let's write $p(x) = y_x$ for the nearest point map and Let's suppose that $x_n \to x$ in X. we want to show that $p(x_n) \longrightarrow P(x)$, and for this it's enough to show that there is a subsequence of $p(x_n)$ that converges to $p(x)$.

Because the sequence (x_n) is bounded in X.

Say $\left||x_n|\right| \leq M$ for all n, we have:

$$\left||P(x_n)|\right| \leq \left||P(x_n) - x_n|\right| + \left||x_n|\right| \leq 2\left||x_n|\right| \leq 2M.$$

Thus $P(x_n)$ is bounded sequence in Y, a finite, dimensional space. As such, by passing to subsequence we may suppose that $\left(P(x_n)\right)$ converges to some element $p_0 \in Y$.

Now we need to show that $P_0 = P(x)$. But

$$\left||P(x_n) - x_n|\right| \leq \left||P(x) - x_n|\right| \text{, for any } n$$

Hence letting $n \longrightarrow \infty$ we get:

$$\left||P_0 - x|\right| \leq \left||P(x) - x|\right|$$

Because nearest point in Y are unique, we must have $P_0 = P(x)$.

Theorem (1.8): Let Y be a subspace of a normed Linear space X, and Let $x \in X$. The set Y_x, consisting of all best approximation to x out of y, is bounded convex set.

Proof:

As we' ve seen, the set Y_x is a subset of the ball $\{y \in x = \left||x-y|\right| \le \left||x|\right|\}$ and as such is bounded.

[more generally, the set Y_x is a sub set of the sphere $\left\{y \in x : \left||x-y|\right| = d\right\}$ where $d = dis(x, y) = \inf_{y \in Y} \left||x-y|\right|\Big\}$.

Next recall that a subset K of a vector space V is said to be convex if K contains the Line segment joining any pair of it's points.

Specifically, K is convex if:

$$x, y \in K \quad , \quad 0 \le \lambda \le 1 \Longrightarrow \lambda x + (1-\lambda)\, y \in k.$$

Thus given $y_1, y_2 \in Y_x$ and $0 \le \lambda \le 1$, we want to show that the vector $y^* = \lambda y_1 + (1-\lambda) y_2 \in Y_x$.

But $y_1, y_2 \in Y_x$ means that:

$$\left||x-y_1|\right| = \left||x-y_2|\right| = \min_{y \in Y} \left||x-y|\right|$$

Hence: $\left||x-y^*|\right| = \left||x-(\lambda y_1) + (1-\lambda) y_2|\right|$

$$= \left||\lambda(x-y_1) + (1-\lambda)(x-y_2)|\right|$$

$$\le \lambda \left||(x-y_1) + (1-\lambda)(x-y_2)|\right|$$

$$= \min_{y \in Y} ||x - y||$$

Consequently, $||x - y^*|| = \min_{y \in Y} ||x - y||$, that is $y^* \in Y_X$.

Corollary (1.9): if X has strictly convex norm, then for any subspace Y of X and any point $x \in$ X.

There can be at most one best approximation to x out of Y. that is Y_X is either empty or consist of a single point.

In order to arrive at condition that's a somewhat eiser to check it's translate our original definition into a statement about the triangle inquality in X.

Lemma (1.10): A normed space X has astrictly convex norm if and only if the triangle inequalify is strict on nonparallel vector, that is if:

$$x \neq xy, y \neq \alpha x, all\ \alpha \in IR \rightarrow ||x + y|| < ||x|| + ||y||$$

Proof:

First suppose that X is strictly convex, and Let x and y be nonparallel vactor in X, then in particular the vectors $x/||x||$ and $y/||y||$ must be different

Hence:

$$\left\| \left(\frac{||x||}{\left| ||x|| + ||y|| \right|} \right) \frac{x}{||x||} + \left(\frac{||y||}{||x|| + ||y||} \right) \frac{y}{||y||} \right\| < 1$$

That is $||x - y|| < ||x|| + ||y||$.

Next suppose that the triangle inequality is strict on nonparallel vectors, and let $x \neq y \in X$with $||x|| = r = ||y||$. if *x* and *y* are parallel, then we must have *y* = -*x*. in this case.

$$||\lambda x + (1 - \lambda)|| = |2\lambda - 1|||x|| < r$$

Because $-1 < 2\lambda - 1 < 1$ whenever $0 < \lambda < 1$. Otherwise *x* and *y* are nonparallel.

Thus for any $0 < \lambda < 1$, the vectors λx and $(1 - \lambda)y$ are likewise nonparallel and we have:

$$||\lambda x + (1 - \lambda)y|| < \lambda||x|| + (1 - \lambda)||y|| = r$$

Examples (1.11):

(1) The usual norm on *C[a,b]* is not strictly convex (and so the problem of uniqueness of best approximation is all the more interesting to tockle). For example if $f(x) = x$ and $g(x) = x^2$ in *C[0,1]* , then $f \neq g$ and $||f|| = 1 = ||g||$ while $||f + g|| = 2$.

(2) The usual norm on IR^n is strictly convex as is any one of the norms $||.||p$ for $1 < p < \infty$.

The norm $||.||1$ and $||.||\infty$, on the other hand are not strictly convex.

(2) Approximation by Algebraic Polynomials

(2-a) The weierstrass theorem:

Let's begin with some notation here we'll be concerned with the problem of the best (uniform) approximation of a given function $f \in C[a, b]$ by elements from P_n the subspace of algebraic polynomial of degree at most *n* in *c[a, b]*. we know that the problem has solution (possibly more than one), which we've chosen to write as P_n^* we set:

$$E_n(f) = \min_{P \in P_n} ||f - P|| = ||f - P_n^*||$$

Because $P_n < P_{n+1}$ for each n, it's clear that $E_n(f) \geq E_{n+1}(f)$ for each n. our goal here is to prove that $E_n(f) \to 0$. We'll complish this by proving.

Theorem (2.1): (the weierstrass theorem, 1885).

Let $f \in c[a,b]$. Then for every $\in > 0$, there is a polynomial p such that $||f - p|| < \in$.

[we have more than one proof for this theorem].

(1) It follows from weierstrass theorem that for some sequence of rolesnomials (q_k) we have $||f - q_k|| \to 0$, we may suppose that $q_k \in P_{n_x}$ where (n_k) is increasing.

Where it follows that $E_n(f) \to 0$; that is $P_n^* \to f$ this an important first step in determining the exact of $E_n(f)$as a function of f and n. We'll look for much more precise information by show the all proofs of weierstrass theorem.

Lemma (2.2): if the weierstrass theorem holds for c[0,1], then it also holds for c [a, b] and conversely. In fact c [0,1] and c [a, b] are for all practical purpose, identical they are linearly isometric as normed spaces, order isomorphic as lattices, and isomorphic as algebras (Rings).

Proof:

First notice that the function:

$$\sigma(x) = a + (b-a)x\,, \qquad\qquad o \le x \le 1$$

Defines a continuous, one-to-one map from [0,1] onto [a, b]. Given $f \in c[a,b]$, it follows that $g(x) = f(\sigma(x))$ defines an element of c[0,1], moreover

$$\max_{0\le x\le 1} |g(x)| = \max_{a\le t\le b} |f(t)|$$

Now given $\sigma > 0$, suppose that we can find apolyuomial p such that $||g-p|| <\in$, in other words suppose that:

$$\max_{0\le x\le 1} |f(a+(b-a)x)| <\in$$

Then: $$\max_{a\le t\le b} \left|f(t) - p\left(\frac{t-a}{b-a}\right)\right| <\in$$

But if $p(x)$ is a polynomial in x, then $q(t) = p\left(\frac{t-a}{b-a}\right) <\in$ is a polynomial in *t* satisfying $||f-q|| <\in$.

$\leftarrow$ if $g(x)$ is an element of c[0,1], then $f(t) = g\left(\frac{t-a}{b-a}\right)$ $a \le t \le b$, defines an element of c[a, b]. Moreover if $q(t)$ is a polynomial in t approximating $f(t)$, then $p(x) = q(a+(b-a)x)$ is a polynomial in *x* approximating $g(x)$.

Bernstein's Proof:

The proof of the weierstrass theorem we present there is due to the great Russian mathematician S.N Bernstein in 1912. Bernestein's proof is of interest of a variety of reasons, perhaps most important is that Besenstein actually displays a sequence of polynomials that approximate a given $f \in c[0,1]$.

Moreover, as well see later, bernestein's proof generalites bo yield a powerfull, unifying theorem called the (Bohmall-Korovkin theorem), if *f* is any bounded function on [0,1], we define the sequence of berestein polynomials for *f* by:-

$$\left(B_n(f)\right)_{(x)} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right).\binom{n}{k} x^k(1-x)^{n-k}, \qquad 0 \le x \le 1$$

Note that $B_n(f)$ is apolynomial of degree at most *n*.

Also it is easy to see that $\left(B_n(f)\right)(0) = f(0)$ and $(B_n(f)(1)) = f(1)$. in general $\left(B_n(f)\right)(x)$ is an average of the numbers $f\left(\frac{k}{w}, k = 0, \dots, n,\right)$

Berestien's theorem states $B_n(f) \rightrightarrows f$ for each $f \in c\{0,1\}$ surprisingly, the proof actually only requires that we check three easy cases:

$$f_0(x) = 1\,, f_1(x) = x, f_2(x) = x^2$$

Lemma (2.3):

(i) $B_n(f_0) = f_0$ and $B_n(f_1) = f_1$

(ii) $B_n(f_2) = \left(1 - \frac{1}{n}\right) f_2 + \frac{1}{n} f_1$, and hence $B_n(f_2) \rightrightarrows f_2$

(iii) $\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k(1-x)^{n-k} = \frac{x(1-x)}{n} \le \frac{1}{4n}, \quad if\ 0 \le x \le 1$

(iv) Given $\delta > 0$ and $0 \le x \le 1$, let F denots the k in $\{0, \dots, n\}$for which $\left|\frac{x}{n} - x\right| \ge \delta$. Then

$$\sum_{k \in F} \binom{n}{k} x^k(1-x)^{n-k} \le \frac{1}{4n\delta^2}$$

Proof: that $B_n(f_0) = f_0$ follows from the binomial formula

$$\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = [x + (1-x)]^n = 1$$

To x that $B_n(f_1) = f_1$, first notice that for $k \ge 1$we have:

$$\frac{k}{n}\binom{n}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} = \binom{n-1}{k-1}$$

Consequently:

$$\sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k} = x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1}(1-x)^{n-k}$$

$$= x \sum_{j=0}^{n-1} \binom{n-1}{j} (1-k)^{(n-1)-j} = x$$

Next, to compute $B_n(f_2)$, we rewrite twice

$$\left(\frac{k}{n}\right)^2 \binom{n}{k} = \frac{k}{n}\binom{n-1}{k-1} = \frac{n-1}{n}.\frac{k-1}{n-1}\binom{n-1}{k-1} + \frac{1}{n}\binom{n-1}{k-1},\ if\ k \geq 1$$

$$= \left(1 - \frac{1}{n}\right)\binom{n-2}{k-2} + \frac{1}{n}\binom{n-1}{k-1},\ if\ k \geq 2$$

Thus:

$$\sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \left(1 - \frac{1}{n}\right) \sum_{k=2}^{n} \binom{n-2}{k-2} x^k (1-x)^{n-k}$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} = \left(1 - \frac{1}{n}\right) x^2 + \frac{1}{n} x$$

Which establishes (ii) because.

$$\left|\left|B_n(f_2) - f_2\right|\right| = \frac{1}{n}\left|\left|f_1 - f_2\right|\right| \to 0\ as\ n \to \infty$$

To prove (iii) we combine the result in (i), (ii) and simplify. Because

$\left((k/n)-x\right)^2=$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}=\left(1-\frac{1}{n}\right)x^2+\frac{x}{n}-2x^2+x^2$$

$$=\frac{1}{n}x(1-x)\leq\frac{1}{4n},\ for\ 0\leq x\leq 1$$

Finally to prove (iv) note that $1\leq\left((k/n)-x\right)^2/\delta^2$

for $k\in F$, and hence:

$$\sum_{k\in F}\binom{n}{k}x^k(1-x)^{n-k}\leq\frac{1}{\delta^2}\sum_{k\in F}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leq\frac{1}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}$$

$$<\frac{1}{4n\delta^2}\text{ , (from (iii))}$$

Now we're ready for the proof of Bernstin's theorem:-

Proof:

Let $f\in c[0,1]$ and let $\in>0$. Then because *f* is uniformly continuous, there is a $\delta>0$ such that:

$|f(x)-f(y)|<\in/2$ whenever $|x-y|<\delta$. Now we use the previous lemma to estimate $||f-B_n(f)||$.

First notice that because the numbers $\binom{n}{k}x^k(1-x)^{n-k}$ are nonnegative and sum to 1, we have

$$|f(x) - B_n(f)(x)| = \left| f(x) - \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \right|$$

$$= \left| \sum_{k=0}^{n} (fx) - f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \right|$$

$$\leq \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k}$$

Now fix n (to be specified in a moment) and let F denote the set of k in $\{0, \ldots, n\}$ for which $|(k/n) - x| \geq \delta$. Then $|f(x) - f(k/n)| < \in/2$ for $k \notin F$.

While $|f(x) - f(k/n)| \leq 2\|f\|$ for $k \in F$. Thus

$$\left| f(x) - \big(B_n(f)\big)(x) \right| \leq \frac{\in}{2} \sum_{k \notin F} \binom{n}{k} x^k (1-x)^{n-k} + 2\|f\| \sum_{k \notin F} \binom{n}{k} x^k (1-x)^{n-k}$$

$$< \frac{\in}{2} . 1 + 2\|f\| . \frac{1}{4n\delta^2}$$, from (iv) of Lemma (2.3)

$$< \in, provided\ that\ n > \|f\|/\in \delta^2$$

Landau's proof:

Just because it's good for us, let's give a second proof of weierstrass's theorem. This one to landau in 1908. First given $f \in c[0,1]$, notice that it saffice's to approximate *f-p*, where *p* is any polynomial. In particular by subtracting the Linear function $f(0) + x\big(f(1) - f(0)\big)$,

may suppose that $f(0) = f(1) = 0$ and, that $f \equiv 0$ out side [0,1]. That is we may suppose that *f* is defined and uniformly continuous an all of IR.

Again we will display a sequence of polynomials that converge uniformly to *f*, this time we define:

$$L_n(x) = C_n \int_{-1}^{1} f(x+t)(1-t^2)\, dt$$

where C_n is chosen so that:

$$C_n \int_{-1}^{1} (1-t^2)^n\, dt = 1$$

Note that by our assumptions on *f* we may rewrite $L_n(x)$ as:

$$L_n(x) = C_n \int_{-x}^{1-x} f(x+t)(1-t^2)^n\, dt = C_n \int_{0}^{1} f(t)(1-(t-x)^2)^n\, dt$$

Written this way, it's clear that L_n is a polynomial in x of degree at most n.

We first need to estimate C_n. An easy induction argument will convince you that $(1-t^2)^n \geq 1 - nt^2$ and so we get:

$$\int_{-1}^{1} (1-t^2)^n\, dt \geq 2 \int_{0}^{1/\sqrt{n}} (1-nt^2)\, dt = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$$

from which it follows that $C_n < \sqrt{n}$.

In particular, for any $0 < \delta < 1$.

$$C_n \int_{\delta}^{1} (1-t^2)^n\, dt < \sqrt{n}(1-\delta^2)^n \to 0 \quad (n \to \infty)$$

Which is the inequality we'll need.

Next, let $\in > 0$ be given, and choose $0 < \delta < 1$ such that:

$|f(x) - f(y)| \leq \in/2$ when ever $|x - y| \leq \delta$.

Then because $C_n(1 - t^2)^n \geq 0$ and integrates to 1 we get

$$|L_n(x) - f(x)| = \left| C_n \int_{-1}^{1} [f(x+t) - f(x)](1 - t^2)^n \, dt \right|$$

$$\leq C_n \int_{-1}^{1} |f(x+t) - f(x)| \, (1 - t^2)^n \, dt$$

$$\leq \frac{\in}{2} C_n \int_{-\delta}^{\delta} (1 - t^2)^n \, dt + 4\|f\| C_n \int_{\delta}^{1} (1 - t^2)^n \, dt$$

$$\leq \frac{\in}{2} + 4\|f\|\sqrt{n}(1 - \delta^2)^n < \in$$

Provided that *n* is sufficiently large.

Improved Estimates:

To begin, we will need a bit more notation. The modulus of continuity of abounded function *f* on the interval [a,b] is defined by:

$$w_f(\delta) = w_f([a,b], \delta) = sup\,\{|f(x) - f(y)|: x, y \in [a,b], |x - y| \leq \delta\}$$

For any $\delta > 0$.

Note that $w_f(\delta)$ is a measure of the $(\in)$ that goes a long with , Literally we have written $\in = w_f(\delta)$ as a function of δ.

Lemma (2.5): Let *f* be abounded function on [a,b] and Let $\delta > 0$. Then $w_f(n\delta) \leq nw_f(\delta)$ for $n = 1,2, ...$

Consequently, $w_f(\lambda\delta) \leq (1 + \lambda)w_f(\delta)$ for any $\lambda > 0$.

Proof:

Given $x < y$ with $|x - y| \leq n\delta$, split the interval [*x, y*]into *n* pieces, each of Length at most δ.

Specifically, if we set $Z_k = x + k(y - x)/n$, for $k = 0,1, \dots, n$

Then $|Z_k - Z_{k-1}| \leq \delta$ for any ≥ 1 , and so

$$|f(z) - f(y)\ | = |\textstyle\sum_{k=1}^{n} f(z_k) - f(z_{k-1} - 1)|$$

$$\leq \textstyle\sum_{k=1}^{n} |f(zx) - f(z_{k-1} - 1)|$$

$$\leq nw_f(\delta)$$

Thus $w_f(\delta) \leq nw_f(\delta)$.

Theorem (2.6): for any bounded function *f* on [0, 1] we have:

$$\left||f - B_n(f)|\right| \leq \frac{3}{2} w_f\left(\frac{1}{\sqrt{n}}\right)$$

In particular, if $f \in C[0,1]$ then $E_n(f) \leq \frac{3}{2} w_f\left(\frac{1}{\sqrt{n}}\right) \to 0$ as $\to \infty$.

Proof: we first do some term juggling:

$$|f(x) - B_n(f)(x)| = \left|\sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}\right)\right|$$

$$\leq \sum_{k=0}^{n} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq \sum_{k=0}^{n} w_f\left(\left|x - \frac{k}{n}\right|\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq w_f\left(\frac{1}{\sqrt{n}}\right) \sum_{k=0}^{n} \left[1 + \sqrt{n}\right] \left|x - \frac{k}{n}\right|\Big] \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq w_f\left(\frac{1}{\sqrt{n}}\right) \left[1 + \sqrt{n} \sum_{k=0}^{n} \left|x - \frac{k}{n}\right| x^k (1-x)^{n-k}\right]$$

Where the third inequality follows from lemma (2.5) (by taking

$\left(\lambda = \sqrt{n}\left|x - \frac{k}{n}\right| \textit{and} \;\; \delta = \frac{1}{\sqrt{n}}\right)$.

All that remains is to estimate the some, and for this we'll use coachy-schwart, (and our earlier observations about Bernstein Polynomials).

Because each of the terms $\binom{n}{k} x^k (1-x)^{n-k}$ is nonnegative we have:

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right| \binom{n}{k} x^k (1-x)^{n-k} =$$

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right| \left[\binom{n}{k} x^k (1-x)^{n-k}\right]^{\frac{1}{2}} . \left[\binom{n}{k} x^k (1-x)^{n-k}\right]^{\frac{1}{2}}$$

$$\leq \left[\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^2 \binom{n}{k} x^k (1-x)^{n-k}\right]^{\frac{1}{2}} . \left[\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}\right]^{\frac{1}{2}}$$

$$\leq \left[\frac{1}{4n}\right]^{\frac{1}{2}} = \frac{1}{2\sqrt{n}}$$

Finally $|f(x) - B_n(f)(x)| \leq w_f\left(\frac{1}{\sqrt{n}}\right)\left[1 + \sqrt{n}.\frac{1}{2\sqrt{n}}\right] = \frac{3}{2} w_f\left(\frac{1}{\sqrt{n}}\right)$

The Bohman – Korovkin theorem:

The real value to us in Bernstein's approach is that the map $f \to B_n(f)$, while providing a simple formula for an approximating polynomial, is also Linear and positive. In other words:

$$B_n(f+g) = B_n(f) + B_n(g)$$

$$B_n(\alpha f) = \alpha B_n(f), \alpha \in R \qquad and$$

$$B_n(f) \geq 0 \quad whenever \quad f \geq 0$$

Lemma (7.8): if T : c[a, b] → c[a,b] is both positive and Linear, then T is continuous.

Proof:

First note that a positive, Linear map is also monotone.

That is, T satisfies $T(f) \leq T(g)$ whenever $f \leq g$.

Thus for any $f \in c[a, b]$ we have:

$$-f, f \leq |f| \Rightarrow -T(f), T(f) \leq T(|f|) .$$

That is $|T(f)| \leq T(|f|)$. But now $|f| \leq ||f||.1$

Where 1 denotes the constant 1 function, and so we get:

$$|T(f)| \leq T(|f|) \leq ||f||T(1)$$

Thus: $$||T(f)|| \leq ||f||||T(1)||$$

For any $f \in c[a, b]$. Finally, because T is Linear it follows that T is Lipchitz with constant $||T(1)||$:

$$||T(f) - T(g)|| = ||T(f - g)|| \leq ||T(1)||||f - g||$$

Consequently T is continuous.

Now positive, Linear maps abound in analysis, so this is fortunate turn of events. What's more, Bernstein's theorem generalizes very nicely when placed in this new setting.

The following elegant theorem was proved (independently) by Bohman-Korovkin in roughly 1952.

Theorem (2.9): Let $T_n = c[0,1] \to c[0,1]$ be a sequence of positive, Linear maps, and suppose that $T_n(f) \to f$uniformaly in each of the three cases $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$.

Then $T_n(f) \to f$uniformaly for every $f \in c[0,1]$.

Chapter (2)

Least–squares Approximation of a function

(2.1.1) Least–squares Approximation of a function

We have describe least-squares approximation to fit a set of discrete data. Here we describe continuous least-squares approximation of a function $f(x)$ by using polynomials.

- First – consider approximation with monomial basis:

$$\{1, x, x^2, \dots, x^n\}$$

Least-squares approximation of a function using monomial polynomial:-

Given a function $f(x)$, continuous on [*a*,*b*], find a polynomial $P_n(x)$ of degree at most n:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Such that the integral of the square of the error is minimized. That is:

$$E = \int_a^b [f(x) - P_n(x)]^2 \, dx \qquad is\ minimized.$$

The polynomial $P_n(x)$ is called least-squares polynomial sinse E is a function of $a_0, a_1, \dots, a_n$, we denote this by $E\ (a_0, a_1, \dots, a_n)$.

For minimization we must have:

$$\frac{\partial E}{\partial a_i} = 0, \qquad i = 0,1, \dots, n$$

As before, these condition, will give rise to a system of (n+1) normal equations in (n+1) unknowns: $a_0, a_1, \dots, a_n$ solution of these equations will yield the unknowns $a_0, a_1, \dots, a_n$.

(2.1.2) Setting up the Normal equations:-

Since:

$$E = \int_a^b [f(x) - a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n]^2 \, dx$$

$$\frac{\partial E}{\partial a_0} = -2 \int_a^b [f(x) - a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n]^2 \, dx$$

$$\frac{\partial E}{\partial a_1} = -2 \int_a^b x[f(x) - a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n]^2 \, dx$$

$$\vdots$$

$$\frac{\partial E}{\partial a_n} = -2 \int_a^b x^n[f(x) - a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n]^2 \, dx$$

$$\frac{\partial E}{\partial a_0} = 0 \Rightarrow a_0 \int_a^b 1dx + a_1 \int_a^b xdx + a_2 \int_a^b x^2 dx + \cdots + a_n \int_a^b x^n dx = \int_a^b f(x)dx$$

Similarly:

$$\frac{\partial E}{\partial a_i} = 0 \Longrightarrow a_0 \int_a^b x^i dx + a_1 \int_a^b x^{i+1} dx + a_2 \int_a^b x^{i+2} dx + \cdots + a_n \int_a^b x^{i+n} dx$$

$$= \int_a^b x^i f(x) dx$$

$$i = 1,2,3, \dots, n$$

So the $(n+1)$ normal equations in this case are:

$$i = 0: a_0 \int_a^b 1 dx + a_1 \int_a^b x dx + a_2 \int_a^b x^2 dx + \cdots + a_n \int_a^b x^n dx = \int_a^b f(x) dx$$

$$i = 1: a_0 \int_a^b x dx + a_1 \int_a^b x^2 dx + a_2 \int_a^b x^3 dx + \cdots + a_n \int_a^b x^{n+1} dx = \int_a^b x f(x) dx$$

$$\vdots$$

$$i = n: a_0 \int_a^b x^n dx + a_1 \int_a^b x^{n+1} dx + a_2 \int_a^b x^{n+2} dx + \cdots + a_n \int_a^b x^{2n} dx$$

$$= \int_a^b x^n f(x) dx$$

Denote:

$$\int_a^b x^i \, dx = \delta_i \ , \quad i = 0,1,2,3, \dots 2n \quad and$$

$$b_i = \int_a^b x^i \, dx \quad , \quad i = 0\,,1\,,2\,,3\,,\dots,n$$

Then the above (*n*+*1*) equations can be written as:

$$a_0\delta_0 + a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n = b_0$$

$$a_0\delta_1 + a_1\delta_2 + a_2\delta_3 + \cdots + a_n\delta_{n+1} = b_1$$

$$a_0\delta_n + a_1\delta_{n+1} + a_2\delta_{n+2} + \cdots + a_n\delta_{2n} = b_n$$

or in matrix notation:

$$\begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \cdots & \delta_n \\ \delta_1 & \delta_2 & \delta_3 & \cdots & \delta_{n+1} \\ \vdots & & & & \\ \delta_n & \delta_{n+1} & \delta_{n+2} & \cdots & \delta_{2n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Denote:

$$\delta = (\delta_i) \;,\; a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix},\; b = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Then we have the system of normal equations:

$$Sa = b$$

The solution of these equations will yield the coefficients $a_0, a_1, \dots, a_n$ of the least-squares polynomial $P_n(x)$.

A special case: let the interval be [*0,1*], then

$$\delta_i = \int_0^1 x^i \, dx = \frac{1}{i+1}, \quad i = 0, 1, \dots, 2n$$

Thus in this case the matrix of the normal equations:

$$\delta = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & & \cdots & \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n} \end{pmatrix}$$

Which is Hilbert Matrix. It is well-known to be ill conditioned.

Algorithm (2.2): (Least-squares approximation using monomial polynomials).

Inputs: (i) $f(x)$ - A continuous function on [*a, b*]

(ii) n – The degree of the desired Least-squares polynomial.

Output: the coefficients $a_0, a_1, \dots, a_n$ of desired least-squares polynomial:

$$P_n(x) = a_0 + a_1 x, \dots, a_n x^n$$

Step 1: compute, $\delta_0, \delta_1, \dots, \delta_{2n}$:

for *i = 0 , 1 , … , 2n* do

$$\delta_i = \int_0^1 x^i f(x) dx$$

End:

Step 2 : compute $b_0, b_1, \dots, b_n$:

for $i = 0, 1, \dots, n$ do

$$b_i = \int_a^b x^i f(x) dx$$

End

Step 3. Form the matrix δ from the numbers $\delta_0, \delta_1, \dots, \delta_{2n}$ and the vector b from the numbers $b_0, b_1, \dots, b_n$

$$\delta = \begin{pmatrix} \delta_0 & \delta_1 & \cdots & \delta_n \\ \delta_1 & \delta_2 & \cdots & \delta_{n+1} \\ \vdots & & & \\ \delta_n & \delta_{n+1} & \cdots & \delta_{2n} \end{pmatrix}, \quad b = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Step 4:

Solve the $(n+1) \times (n+1)$ system of equations for $a_0, a_1, \dots, a_n$

$$Sa = b \quad , \quad where \quad a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Example (2.3):

Find Linear and quadratic Least-squares approximation

to $f(x) = e^x \quad on \quad [-1, 1]$

Solution:

Linear Approximation: n = 1 , $P_1(x) = a_0 + a_1 x$

$$\delta_0 = \int_{-1}^{1} dx = 2 \quad , \quad \delta_1 = \int_{-1}^{1} x dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\delta_2 = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

Thus: $\delta = \begin{pmatrix} \delta_0 & \delta_1 \\ \delta_1 & \delta_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$

$$b_0 = \int_{-1}^{1} e^x\,dx = e - \frac{1}{e} = 2.3504$$

$$b_1 = \int_{-1}^{1} x\,e^x\,dx = \frac{2}{e} = 0.7358$$

The normal system is: $\begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$

This gives $a_0 = 1.1752$, $a_1 = 1.1037$

The Linear Least-squares polynomial is

$$P_1(x) = 1.1752 + 1.1037\,x$$

Accuracy check: $P_1(0.5) = 1.7270$, $e^{0.5} = 1.6487$

Relative Error:

$$\frac{|e^{0.5} - P_1(0.5)|}{e^{0.5}} = \frac{|1.6487 - 1.7270|}{|1.6487|} = 0.0475$$

Quadratic fitting: n = 2 , $P_2(x) = a_0 + a_1 x + a_2 x^2$

$$\delta_0 = 2 \;,\quad \delta_1 = 0 \;,\quad \delta_2 = \frac{2}{3}$$

$$\delta_3 = \int_{-1}^{1} x^3\, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0\,, \qquad \delta_4 = \int_{-1}^{1} x^4\, dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$$

$$b_0 = \int_{-1}^{1} e^x\, dx = e - \frac{1}{e} = 2.3504$$

$$b_1 = \int_{-1}^{1} x\, e^x\, dx = \frac{2}{e} = 0.7358$$

$$b_2 = \int_{-1}^{1} x^2 e^x\, dx = e - \frac{5}{e} = 0.8789$$

The system of normal equations is

$$\begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2.3504 \\ 0.7358 \\ 0.8789 \end{pmatrix}$$

The solution of this system is

$$a_0 = 0.9963 \quad , \qquad a_1 = 1.1037 \quad , \quad a_2 = 0.5368$$

The quadratic least-squares polynomial is

$$P_2(x) = 0.9963 + 1.1037x + 0.5368\, x^2$$

Accuracy check: $P_2(0.5) = 1.6889$, $e^{0.5} = 1.6487$

Relative error =

$$\frac{|P_2(0.5) - e^{0.5}|}{|e^{0.5}|} = \frac{|1.6824 - 1.6487|}{|1.6487|} = 0.0204$$

Example (2.4): Find Linear and Quadratic Least-squares polynomial approximation to $f(x) = x^2 + 5x + 6 \ \ in$ [0,1].

Solution:

Linear fit: $P_1(x) = a_0 + \ a_1 x$

$$\delta_0 = \int_0^1 dx = 1 \ , \delta_1 = \int_0^1 x \ \ dx = \frac{1}{2} \quad , \qquad \delta_2 = \int_0^1 x^2 dx = \frac{1}{3}$$

$$b_0 = \int_0^1 (x^2 + 5x + 6)\, dx \ \ = \frac{1}{3} + \frac{5}{2} + 6 \ = \frac{53}{6}$$

$$b_1 = \int_0^1 x\ (x^2 + 5x + 6)\, dx = \frac{1}{4} + \frac{5}{3} + \frac{6}{2} \ = \frac{59}{12}$$

The normal equation are:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 53/6 \\ 59/12 \end{pmatrix} \Rightarrow \quad \begin{matrix} a_0 = 5.8333 \\ a_1 = 6 \end{matrix}$$

The linear least-squares polynomial $P_1(x) = 5.8333 + 6x$

Accuracy check

Exat value $f(0.5) = 8.75 \ , P_1(0.5) = 8.833$

Relative error :

$$\frac{|8.833 - 8.75|}{|8.75|} = 0.0095$$

Quadratic least squares approximation.

$$P_2(x) = a_0 + a_1 x + a_2\, x^2$$

$$\delta = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}, \qquad b_0 = \frac{53}{6} \quad , \qquad b_1 = \frac{59}{12}$$

$$b_2 = \int_0^1 x^2(x^2 + 5x + 6)\, dx \;= \int_0^1 (x^4 + 5x^3 + 6x^2)\, dx \;= \frac{1}{5} + \frac{5}{4} + \frac{6}{3} = \frac{69}{20}$$

The solution of the Linear system is:

$$a_0 = 6 \;\;,\;\; a_1 = 5 \;\;, a_2 = 1$$

$P_2(x) = 6 + 5\, x + \; x^2$

(2.1.3) Use the orthogonal polynomial in least-squares Approximation:

The least-squares approximation using monomial polynomials as described above is not numerically effective, since the system matrix δ of normal equations is very often ill-conditioned, for example, when the interval [*0,1*], we have seen that δ is Hilbert matrix, which is notoriously ill-conditioned for even modest values of *n*.

When n=5 , the condition number of this matrix cond(s) = O(105). Such computations can, however be made computationally effective by using aspecial type of polynomials, called Orthogonal polynomials.

Definition (2.5): the set of functions $\{\phi_0, \phi_1, \dots, \phi_n\}$is called a set of orthogonal functions, with respect to a weight function $w(x)$if:-

$$\int_a^b w(x)\phi_j(x)\phi_i(x)dx = \begin{cases} 0\ , & if \quad i \neq j \\ c_j, & if \quad i = j \end{cases}$$

Where C_j is a real positive number.

Furthermore, if $C_j = 1\ , j = 0, 1, \dots, n$,then the orthogonal set is called an orthonormal set.

Using this interesting property, least-squares compatations can be more numerically effective as shown below.

Without any loss of generality, let's assume that $w(x) = 1$.

Idea: the idea is to find a least-squares approximation of $f(x)$ on [a,b] by means of a polynomial of the form: $P_n(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x)$, where $\{\phi_n\}_{k=0}^{n}$ is a set of orthogonal polynomials. That is the basis for generating $P_n(x)$ in this case is a set of orthonormal polynomials.

Given the set of orthogonal polynomials $\{\phi_i(x)\}_{i=0}^{n}$, a polynomial $P_n(x)$of degree $\leq n$, can be written as:

$$P_n(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x)$$

For some $a_0, a_1, \dots, a_n$.

Finding a least-squares approximation of $f(x)$ on [a,b].

Using orthogonal polynomials, then can be stated as follows:

Least-squares approximation of a function using orthogonal polynomials:

Given $f(x)$, continuous on [a,b], find $a_0, a_1, \dots, a_n$ using a polynomial of the form:

$P_n(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x)$, where $\{\phi_k(x)\}_{k=0}^{n}$is agiven set of orthogonal polynomials on [a, b], such that the error function:

$$E(a_0, a_1, \dots, a_n) = \int_a^b \left[f(x) - \left(a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x)\right)\right]^2 dx$$

Is minimized.

As before we set

$\frac{\partial E}{\partial a_i} = 0, i = 0,1, \dots, n$ now:

$$\frac{\partial E}{\partial a_0} = -2\int_a^b \phi_0(x)\left[f(x) - a_0\phi_0(x) - a_1\phi_1(x), \dots, a_n\phi_n(x)\right]dx$$

Setting this equal to zero, we get:

$$\int_a^b \phi_0(x)\, f(x)dx = \int_a^b \left(a_0\phi_0(x) + \cdots + a_n\phi_n(x)\right)\phi_0(x)dx$$

Since:

$\{\phi_k(x)\}_{k=0}^{n}$ is an orthogonal set, we have:

$$\int_a^b \phi_0^2(x)dx = C_0 \quad and \quad \int_a^b \phi_0(x)\phi_i(x)\, dx = 0,\ \ i \neq 0$$

Applying the above orthogonal property, we see from above that

$$\int_a^b \phi_0(x) f(x)\, dx = C_0 a_0$$

that is

$$a_0 = \frac{1}{C_0} \int_a^b \phi_0(x) f(x)\, dx$$

Similarly

$$\frac{\partial E}{\partial a_1} = -2 \int_a^b \phi_1(x) \left[f(x) - a_0 \phi_0(x) - a_1 \phi_1(x), \dots, a_n \phi_n(x) \right] dx$$

Again from the orthogonal property of $\{\phi_j(x)\}_{j=0}^{n}$ we have :

$$\int_a^b \phi_1^2(x) dx = C_1 \quad and \quad \int_a^b \phi_1(x) \phi_i(x)\, dx = 0, \;\; i \neq 1$$

So, sitting $\frac{\partial E}{\partial a_1} = 0$, we get : $a_1 \frac{1}{C_1} \int_a^b \phi_1(x) f(x)\, dx$ in general , we have :

$$a_k \frac{1}{C_k} \int_a^b \phi_k(x) f(x)\, dx \;\;, \quad k = 0,1, \dots, n$$

Where

$$C_k \int_a^b \phi_k^2(x)\,dx$$

(2.1.4) Expressions for a_k with weight function $w(k)$.

If the weight function $w(x)$ is included, then $a_k$is modified to:

$$a_k = \frac{1}{C_k}\int_a^b w(x)f(x)\phi_k(x)\,dx \quad , \quad k = 0,1,\dots,n$$

Where

$$C_k \int_a^b = \int_a^b w(x)\phi_k^2(x)\,dx$$

Code(2.6): least-squats approximation using orthogonal polynomials

```
clc
syms x
f= @(x) exp(x);
n = 2;
n= 2*n;
a = -1;
b = 1;
s = [];
B= [];
for
i=1:(n-1)
r = (i-1);
```

```
s(i) = int(x.^r,a,b);
end
S= [s(:,1:n-2); s(:,2:n-1)];
for
i=1:(n-2)
r = (i-1);
B(i) = int((f* x.^r),a,b);
end
B=B';
A = inv(S)*B;
%the solution is
%  the linear leas-squares
A =     1.1752
1.1036
S =    2.0000         0
0    0.6667
B =    2.3504
  0.7358
```

Chapter (3)

Lagrang approximation

The most straightforward method of computing the interpolation polynomial is to form the system $Ax = b$, where $bi = yi,\ i = 0, \dots, n$ and the entries of are defined by $a_{i_j} = P_j(xi), i, j = 0, \dots, n$, where $x_0, x_1, \dots, x_n$ are the points at which the data $y_0, y_1, \dots, y_n$ are obtained, and $P_j(x) = x^j,\ j = 0, \dots, n.$ The basis $\{1, x, \dots, x^n\}$ of the space of polynomials of degree $n + 1$ is called the monomial basis, and the corresponding matrix A is called the Vandermonde matrix for the points $x_0, x_1, \dots, x_n$.

Unfortunately, this matrix can be ill-conditioned, especially when interpolation points are close together. In Lagrange interpolation, the matrix A is simply the identity matrix, by virtue of the fact that the interpolating polynomial is written in the form:

$$P_n(x) = \sum_{j=0}^{n} y_i \, \mathcal{L}_{n_j}(x),$$

where the polynomials$\left\{\mathcal{L}_{n_j}\right\}_{j=0}^{n}$ have the property that :

$$\mathcal{L}_{n_j}(x_i) = \begin{cases} 1 & , \quad if \quad i = j \\ 0 & , \quad if \quad i \neq j \end{cases}$$

The polynomials $\left\{\mathcal{L}_{n_j}\right\}_{j=0}^{n}$ are called the Lagrange polynomials for the interpolation points $x_0, x_1, \dots, x_n$. They are defined by

$$\mathcal{L}_{n_j}(x) = \prod_{k=0\ ,k\neq j}^{n} \frac{x - x_k}{x_j - x_k}$$

As the following result indicates, the problem of polynomial interpolation can be solved using Lagrange polynomials.

Theorem: Let $x_0, x_1, \dots, x_n$ be *n* + 1 distinct numbers, and let $f(x)$ be a function defined on a domain containing these numbers. Then the polynomial defined by:

$$P_n(x) = \sum_{j=0}^{n} f(x_j)\, \mathcal{L}_{n_j}$$

is the unique polynomial of degree *n* that satisfies

$$P_n(x_j) = f(x_j), \qquad j = 0,1, \dots, n$$

The polynomial $P_n(x)$ is called the interpolating polynomial of $f(x)$. We say that $P_n(x)$interpolates $f(x)$ at the points $x_0, x_1, \dots, x_n$.

Example: We will use Lagrange interpolation to find the unique polynomial $P_3(x)$, of degree 3 or less, that agrees with the following data:

i	x_i	y_i
0	-1	3
1	0	-4
2	1	5
3	22	-6

In other words, we must have $P_3(-1) = 3, P_3(0) = -4, P_3(1) = 5, P_3(2) = -6$

First, we construct the Lagrange polynomials $\left\{\mathcal{L}_{3_j}\right\}_{j=0}^{3}$ using the formula:

$\mathcal{L}_{n_j}(x) = \prod_{i=0\ ,\ i\neq j}^{3} \frac{x-x_i}{x_j-x_i}$ this yields:

$$\mathcal{L}_{3,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$= \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} = \frac{x(x^2-3x+2)}{(-1)(-2)(-3)}$$

$$= -\frac{1}{6}(x^3-3x^2+2x)$$

$$\mathcal{L}_{3,1}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$= \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}$$

$$= \frac{(x^2-1)(x-2)}{(1)(-1)(-2)}$$

$$= \frac{1}{2}(x^3-2x^2-x+2)$$

$$\mathcal{L}_{3,2}(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$= \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}$$

$$= \frac{X(x^2-x-2)}{2(1)(-1)} \qquad = -\frac{1}{2}(x^3-x^2-2x)$$

$$\mathcal{L}_{3,3}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}$$

$$= \frac{x(x^2-1)}{(3)(2)(1)}$$

$$= \frac{1}{6}(x^3 - x)$$

By substituting x_i for x in each Lagrange polynomial $\mathcal{L}_{3_j}(x)$, for *j = 0, 1, 2, 3* it can be verified that:

$$\mathcal{L}_{3_j}(x_i) = \begin{cases} 1 & , \quad if \quad i = j \\ 0 & , \quad if \quad i \neq j \end{cases}$$

It follows that the Lagrange interpolating polynomial $P_3(x)$ given by:

$$P_3(x) = \sum_{j=0}^{3} y_i \, \mathcal{L}_{3_j}(x)$$

$$= y_0\mathcal{L}_{3,0}(x) + y_1\mathcal{L}_{3,1}(x) + y_2\mathcal{L}_{3,2}(x) + y_3\mathcal{L}_{3,3}(x)$$

$$= (3)\left(-\frac{1}{6}\right)(x^3 - 3x^2 + 2x) + (-4)\left(\frac{1}{2}\right)(x^3 - 2x^2 - x + 2) + (5)\left(-\frac{1}{2}\right)(x^3 - x^2 + 2x) + (-6)(\tfrac{1}{6})(x^3 + x)$$

$$= -\frac{1}{2}(x^3 - 3x^2 + 2x) + (-2)(x^3 - 2x^2 - x + 2) - \frac{5}{2}(x^3 - x^2 - 2x) - (x^3 - x)$$

$$= \left(-\frac{1}{2} - 2 - \frac{5}{2} - 1\right)x^3 + \left(\frac{3}{2} + 4 + \frac{5}{2}\right)x^2 + (-1 + 2 + 5 + 1)x - 4$$

$$= -6x^3 + 8x^2 + 7x - 4$$

Substituting each x_i , $for \quad i = 0,1,2,3$ into $P_3(x)$, we can verify that we obtain $P_3(xi) = yi$ in each case.

While the Lagrange polynomials are easy to compute, they are difficult to work with. Furthermore, if new interpolation points are added, all of the Lagrange polynomials must be recomputed. Unfortunately, it is not uncommon, in practice, to add to an existing set of interpolation points. It may be determined after computing the k'th-degree interpolating polynomial $P_k(x)$ of a function $f(x)$ that $P_k(x)$ is not a sufficiently accurate approximation of $f(x)$ on some domain. Therefore, an interpolating polynomial of higher degree must be computed, which requires additional interpolation points.

To address these issues, we consider the problem of computing the interpolating polynomial recursively. More precisely, let $k > 0$, and let $P_k(x)$ be the polynomial of degree k that interpolates the function $f(x)$ at the points $x_0, x_1, \dots, x_k$. Ideally, we would like to be able to obtain $P_k(x)$ from polynomials of degree *k−1* that interpolate $f(x)$ at points chosen from among $x_0, x_1, \dots, x_k$. The following result shows that this is possible.

Theorem: Let *n* be a positive integer, and let $f(x)$ be a function defined on a domain containing the *n+1* distinct points $x_0, x_1, \dots, x_n$, and let $P_n(x)$ be the polynomial of degree *n* that interpolates $f(x)$ at the points $x_0, x_1, \dots, x_n$.

For each $i = 0,1, \dots, n$, we define $P_{n-1,i}(x)$ to be the polynomial 4 of degree *n− 1* that interpolates $f(x)$ at the points $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$. If *i* and *j* are distinct nonnegative integers not exceeding *n*, then

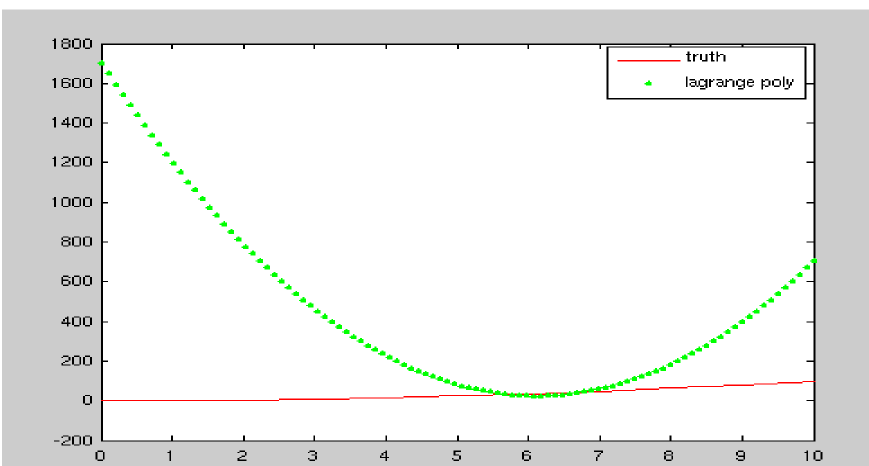
$$P_n(x) = \frac{(x - x_j)P_{n-1,j}(x) - (x - x_i)P_{n-1,i}(x)}{x_i - x_j}$$

This result leads to an algorithm called Neville's Method that computes the value of $P_n(x)$ at a given point using the values of lower-degree interpolating polynomials at k.

Algorithm in detail: Let $x_0, x_1, \dots, x_n$ be distinct numbers, and let $f(x)$ be a function defined on a domain containing these numbers. Given a number x^* , the following algorithm computes $y^* = P_n(x^*)$, where $P_n(x)$ is the nth interpolating polynomial of $f(x)$ that interpolates $f(x)$ at the points $x_0, x_1, \dots, x_n$.

```
function [ y ] =lagrange(x,pointx,pointy)
%x=0:10;
%y=x.^2-1;
%xx = linspace(0,10);
%yy = lagrange(xx,x,y);
%plot(x,y,'r-',xx,yy,'g.')
%legend ('truth','lagrange poly')
n = size(pointx,2);
L = ones(n,size(x,2));
%Error cheching
if (size(pointx,2)~=size(pointy,2))
fprintf(1,'\n ERROR! \n Pointx and Pointy must have the same number of elements\n');
        %initialize your sum
y=NaN;
else
          %Initialize Li
For
```

```
      i=1:n
for
              j=1:n
if (i~=j)
            L(i,:)=L(i,:).*(x -pointx(i)) / (pointx(i)-pointx(j));
end
end
end
y=0;
for
i=1:n
y= y + pointy(i) * L(i,:);
end
end
end
```



Theorem [interpolation error]

If f is $n+1$ times continuously differentiable on [a,b] and $P_n(x)$ is the unique polynomial of degree n that interpolates $f(x)$ at the $n+1$ distinct points $x_0, x_1, \dots, x_n$ in [a,b] then for each $x \in$ [a, b]:

$$f(x) - P_n(x) = \prod_{j=0}^{n} \left(x - x_j\right) \frac{f^{(n+1)}\left(\ \}(x)\right)}{(n+1)!}$$

where $\}(x) \in [a, b]$

Chapter (4)

Chebyshev Polynomials

(4.1)

Here we try to reduce the error in the polynomial approximation by minimizing the term:

$$\prod_{k=0}^{n}(x - x_k)$$

Note that for any $x \in [-1,1]$ there exist a $\in$ such that $x = \cos\theta$.

Let us define the set of polynomial $T_n(x) = \cos n\,\theta$.

where $\cos\theta = x \;\; for \; -1 \le x \le 1.$ These polynomials called chebyshev polynomials we recall the trigonometric formula:

$$\cos(n+1)\theta + \cos(n-1)\cos\theta = 2\cos n\,\theta\cos\theta \;\ldots\ldots\ldots\ldots (4.1)$$

Note that we can rewrite the trigonometric identity given in (4.1) as:

$$T_{n+1}(x) + T_{n-1}(x) = 2T_n(x)x,$$

Which can be Rearrange to give the recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad \ldots\ldots\ldots (4.2)$$

The chebyshev polynomials can be generated by:

$T_0(x) = \cos\theta = 1$

$T_1(x) = \cos 1\theta = x$

$T_2(x) = \cos 2\theta = 2x^2 - 1$

$T_3(x) = \cos 3\theta = 2x(2x^2 - 1) - x = 4x^3 - 3x$

$T_4(x) = \cos 4\theta = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$

$T_5(x) = \cos 5\theta = 2x(4x^3 - 3x) - (2x^2 - 1) = 16x^5 - 20x^3 + 5x$

$\ldots etc.$.

Observe that if *n* is even $T_n(x)$ contains even powers of *n* and if *n* odd then $T_n(x)$ contains odd powers of x.

Properties of the Chebyshev Polynomials:

1) $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$ $, \forall n$
2) $|T_n|$ attains its maximum value of 1 on $x \in [-1,1]$ at n+1 points, including both endpoints and takes the values $\pm$ 1 alternately on these points.
3) T_n has *n* distinct zeros in the interior of [-1,1].

Proof:

(1) $-1 \leq \cos n\theta \leq 1$, so by definition $-1 \leq T_n(x) \leq 1$ for $x \in [-1,1]$

(2) The maximum of $|T_n(x)| = 1$ since the maximum of $|cosn\theta| = 1$, now $cosn\theta = \pm 1$

when $n\theta = k\pi \quad , \quad k = 0, 1, \ldots, n$

$$\theta = \frac{k\pi}{n} \quad , \quad k = 0, 1, \ldots, n \qquad \ldots\ldots (4.3)$$

So, $x = cos\theta = \cos\left(\frac{k\pi}{n}\right), k = 0,1, \ldots, n$

There are (n+1) values of $\theta \left(0, \frac{\pi}{n}, \frac{2\pi}{2}, \ldots, \pi\right)$ for which $|T_n(x)| = |cosn\theta| = 1$. Hence there are (n+1) values of $x \in [-1,1](k = 1), \cos\frac{\pi}{2}, \cos\frac{2\pi}{2}, \ldots, \cos\frac{(k-1)\pi}{n}$, at which $|T_n(x)| = 1$. We call this set of (n+1) points.

The extreme points of $T_n(x)$.

Furthermore, observe that $cosn\theta = \cos(k\pi) = (-1)^k, k = 0,1, \dots, n.$ So $T_n(x)$ oscillates between +1 and -1 at these (n+1) points.

Proof of property (3):

Obviously, between each maximum and minimum of +1 and -1 is a zero and hence, there are n zeros in [-1,1] given by:

$$cosn\,\theta = 0 \quad , \quad n\theta = (2k+1)\frac{\pi}{2}\,, \;\; k = 0, 1, 2, \dots, (n-1)$$

$$\Longrightarrow \theta = (2k+1)\frac{\pi}{2n}\,, \;\; k = 0, 1, 2, \dots, (n-1)$$

In terms of *x* let use denote the k[th] zero as

$$x_k = cos\,\theta = cos\left((2k+1)\frac{\pi}{2n}\right), \;\; k = 0, 1, 2, \dots, (n-1)$$

Observe that: $x_0 = cos\,\theta = \cos\left(\frac{\pi}{2n}\right) and\; x_{n-1} = cos\left(\frac{(2n-2+1)\pi}{2n}\right)$

$= cos\left(\pi - \frac{\pi}{2n}\right)$, which implies

$$x_0 = -\,x_{n-1}$$

So in general, we have:

$$x_k - x_{n-1-k}$$

i.e. the zeros are placed symmetrically about *x = 0* in [-1 , 1].

(4.2) Minimizing the Error Bound:

The objective here is to minimize:

$$\max\left\{\left|\prod_{k=0}^{n}(x-x_k)\right|\right\}$$

Over $x_0 \leq x \leq x_n$ by selecting sceitable nodes $x_0, x_1, \ldots, x_n$ for interpolation.

Consider $\prod(x-x_k)$ for $-1 \leq x \leq 1$

$$\prod_{k=0}^{n}(x-x_k) = (x-x_0)(x-x_1) \ldots (x-x_n), x_k \in [-1,1]$$

$$= x^{n+1} - x^n \sum_{k=0}^{n} x_k + \cdots + x_0 x_1 \ldots . x_n$$

i.e. a polynomial of degree (n+1) with leading coefficient 1. This polynomial has n+1 roots (zeros) namely the nodes $x_0, x_1, \ldots, x_k, \ldots, x_n$.

Now we know that $T_{n+1}(x)$ is a polynomial of degree (n+1) with (n+1) zeros in [-1,1] and:

$$|T_{n+1}(x)| \leq 1 \quad , \quad \forall\, x \in [-1\,,1]$$

From the recurrence relation $T_{n+1} = 2xT_n - T_{n-1}$ we can see that:

$$T_{n+1}(x) = 2^n x^{n+1} + \cdots.$$

And so has leading coefficient 2^n.

If we choose the nodes of the interpolation $(x_0, x_1, \ldots, x_k, \ldots, x_n)$to be equal to zeros of $T_{n+1}(x)$.

Then this is equivalent to saying:

$$\prod_{k=0}^{n}(x - x_k) = \frac{1}{2^n}T_{n+1}. \qquad Now$$

$$|T_{n+1}(x)| \leq 1 \Rightarrow \left|\prod_{k=0}^{n}(x - x_k)\right| \leq \frac{1}{2^n} \Rightarrow \max_{-1\leq x\leq 1}\prod_{k=0}^{n}(x - x_k) = \frac{1}{2^n}$$

Thus: $\left||\prod_{k=0}^{n}(x)|\right|_\infty = \frac{1}{2n}$

Is this the minimum value for $\left||\prod_{k=0}^{n}(x)|\right|_\infty$ we can find:

Theorem:

$$\left|\left|\frac{1}{2^n}T_{n+1}\right|\right|_\infty \leq ||q(x)||_\infty \quad , \qquad x \in [-1,1]$$

for all $q(x) \in P_{n+1}$ [set of polynomial of degree n+1].

With leading coefficients of 1.

Proof:

Suppose the theorem is false, that is assume there exist a polynomial $r(x)$ of degree n+1, with leading coefficients 1, such that:

$$||r(x)||_\infty < \frac{1}{2^n} = \left|\left|\frac{1}{2^n}T_{n+1}\right|\right|_\infty$$

Consider $\left[r(x) - \frac{1}{2^n}T_{n+1}\right]$. This is a polynomial of degree *n* since the leading terms cancel (both have same coefficient of 1).

From the 2nd property of T_{n+1}we know it has n+2 extreme points which oscillate in sign.

Also from the definition of $r(x)$, we know these extremes are larger than the extremes of $r(x)$.

Hence at the extreme points of T_{n+1} $\left[r(x) - \frac{1}{2^n} T_{n+1}\right]$ will oscillate in sign this means $\left[r(x) - \frac{1}{2^n} T_{n+1}\right]$ has (n+1) zeros (at least).

But a polynomial of degree n has at most n zeros.

Thus there can be no such $r(x)$, which implies that:

$$\min_{q(x) \in P_{n+1}} ||q(x)||_\infty = \left\| \frac{1}{2^n} T_{n+1} \right\|_\infty$$

Where $q(x) \in P_{n+1}$ is of the form $q(x) = x^{n+1} + \cdots$

since $\prod_{k=0}^{n}(x - x_k) = (x - x_0)(x - x_1) \dots (x - x_n)$in a polynomial belonging to P_{n+1} leading coefficient 1.

We have:-

$$\min \left\{ \max_{-1 \le x \le 1} \left| \prod_{k=0}^{n} (x - x_k) \right| \right\} = \min \left\| \prod_{k=0}^{n} (x - x_k) \right\|_\infty = \frac{1}{2^n}$$

Where $\prod_{k=0}^{n}(x - x_k) = \frac{1}{2^n} T_{n+1}$. in other words, $\prod_{k=0}^{n}(x - x_k)$ is minimized (with minimum value $\frac{1}{2^n}$), by choosing $x_0, x_1, \dots, x_n$ as the zero of the chebyshev polynomial $T_{n+1}(x)$.

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