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Sudan University of Science and Technology

College of Graduate Studies

Homotopy Perturbation Method for Solving Advection and Integral Equations

طريقة الإضطراب الهموتوبي لحل معادلات حركة الهواء الأفقية والمعادلات التكاملية

A Thesis submitted in partial fulfillment for the Degree of M.Sc. in Mathematics

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الآيـــــات

قال تعالى:

اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ ﴿ 1 ﴾ خَلَقَ الْإِنسَانَ مِنْ عَلَقٍ

﴿ 2 ﴾ اقْرَأْ وَرَبُّكَ الْأَكْرَمُ ﴿ 3 ﴾ الَّذِي عَلَّمَ بِالْقَلَمِ ﴿ 4 ﴾

عَلَّمَ الْإِنسَانَ مَا لَمْ يَعْلَمْ ﴿ 5 ﴾

صدق الله العظيم

سورة العلق الآيات (1 – 5)

Dedication

To my father's

Soul, to my mother, brothers, sisters and friends

Without their support and encouragement it would have been impossible for me to proceed.

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I am greatly indebted to my supervisor:

Dr. Mohamed Hassan Mohamed Khabir

For his valuable efforts with me throughout all the steps of this dissertation

Contains

Abstract

In this thesis we apply the homotopy perturbation method (HPM) to solve some partial differential and integral equations. Firstly we solve a time dependent homogenous and non-homogenous partial differential equations, subject to different kind of initial and boundary conditions. Then we apply the homotopy perturbation method (HPM) to solve non-linear advection differential equations. Further more, the homotopy perturbation method (HPM) is applied to solve some kind of integral equations.

الخـــلاصـة

في هذا البحـث قمنـا بتطبيـقطريقة الإضطراب الهموتوبي لحل بعض المعادلات التفاضلية والتكاملية. بداية قمنا بحل المعادلات التفاضلية المتجانسة وغير المتجانسة المرتبطة بالزمن وفق أنواع مختلفة من الشروط الإبتدائية والحدية، ثم بعد ذلك قمنا بتطبيق طريقة الإضطراب الهموتوبي لحل المعادلات التفاضلية غير الخطية، وأيضاً طبقنا هذه الطريقة لحل المعادلات التكاملية.

Chapter (1)

The homotopy perturbation method for linear and nonlinear operators

1.1 Introduction

The homotopy perturbation method (HPM), proposed first by He[1,2], for solving differential and integral equations. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, has a signification advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. The HPM is applied to Volterra's integro-differential equation [3], to nonlinear oscillators [4], bifurcation of nonlinear problems [5], bifurcation of delay-differential equations [6], nonlinear wave equations [7], boundary value problems [8], quadratic Riccati differential equation of fractional order [9], and to other fields [10-18]. The HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions.

1.2 Analysis of the method

The principles of the HPM and its applicability for various kinds of differential equations are given in [3-18]. We consider the nonlinear differential equation.

$$L(u) + N(u) = f(r)\ ,\ r \in \Omega \qquad (1.1)$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial x}) = 0,\ r \in \Gamma \qquad (1.2)$$

Where L is linear operator while N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is known analytic function.

The He's homotopy perturbation technique[1-13] defines the homotopy $u(r,p){:}\,\Omega\times[0,1]\rightarrow\Re$ which satisfies

$$H(u,p) = (1 - p)[L(u) - L(u_0)] + p[L(u) + N(u) - f(r)] = 0 \qquad (1.3)$$

or

$H(u,p) = L(u) - L(u_0) + PL(u_0) + p[N(u) - f(r)] = 0$ --------------- (1.4)

where$r \in \Omega$ and $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from Eqs,(1.3) and (1.4), we have

$H(u, 0) = L(u) - L(u_0) = 0$ --(1.5)

$H(u, 1) = L(u) - N(u) - f(r) = 0$ --(1.6)

The changing process of p from zero to unity is just that of $u(r, p)$ from $u_0(r)$to $u(r)$. In topology, this called deformation,$L(u) - L(u_0)$and$L(u) - N(u) - f(r)$ are homotopic. The basic assumption is that the solution of Eqs. (1.3) and (1.4) can be expressed as a power series in p:

$u = u_0 + pu_1 + p^2u_2 + \ldots$--(1.7)

The approximate solution of Eq, (1.1), therefore, can be readily obtained:

$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \ldots$---(1.8)

The convergence of the series (1.8) has been proved in [1,2].

1.3 Applications

To demonstrate the effectiveness of the proposed method, we have chosen several differential equations.

Example (1.1)

Consider the following homogeneous linear PDE[19]:

$u_t + u_x - u_{xx} = 0$

with the following conditions:

$u(x , 0) = e^x - x$, $u(0, t) = 1 + t$,$\frac{\partial u}{\partial x}(1 , t) = e - 1$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = L(u) - L(u_o) + pL(u_o) + p[N(u) - f(r)] = 0$

Then we have:

$[u_t - (u_o)_t] + p[(u_o)_t + p[u_x - u_{xx}] = 0$

Then, substituting the initial condition, we have

$[u_t - (e^x - x)_t] + p[(e^x - x)_t] + p[u_x - u_{xx}] = 0$

This gives

$u_t - 0 + p[(0)] + p[u_x - u_{xx}] = 0$

Then

$u_t + p[u_x - u_{xx}] = 0$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_t + p[(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_x$

$- (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{xx}] = 0$

To find u_o :

$p^0: (u_o)_t = 0$

$u_0 = C(x)$, $u(x, 0) = e^x - x$

$e^x - x = C(x) \rightarrow C(x) = e^x - x$

$u_0 = e^x - x$

To find u_1 :

$p^1: (u_1)_t + (u_0)_x - (u_0)_{xx} = 0$

$(u_1)_t + (e^x - x)_x - (e^x - x)_{xx} = 0$

$(u_1)_t + e^x - 1 - (e^x - 0) = 0$

$(u_1)_t - 1 = 0 \rightarrow (u_1)_t = 1$

$u_1 = t + C(x)$

$u(x, 0) = e^x - x$

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \ldots)(x, 0) = e^x - x$

Then:

$u_0(x\ ,0) = e^x - x$

$u_1\ (x\ ,0) = 0$, $u_2\ (x\ ,0) = 0$

$u_3\ (x\ ,0) = 0$ and so on

Then:

$u_1 = t + C(x), u_1(x\ ,0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_1 = t$

To find u_2 :

$p^2: (u_2)_t + (u_1)_x - (u_1)_{xx} = 0$

$(u_2)_t + (t)_x - (t)_{xx} = 0$

$(u_2)_t = 0$

$(u_2)_t = C(x)$ $, u_2\ (x\ ,0) = 0$

$0 = C(x) \rightarrow C(x) = 0$

$u_2 = 0$ $, u_3 = 0$ and $u_4 = 0$

The solution is a series form, given by:

$u(x\ ,t) = u_0 + u_1 + u_2 + \ldots$

$\quad\quad = e^x - x + t + 0 + 0 + \ldots$

The required solution is:

$u(x\ ,t) = e^x - x + t$

Example (1.2)

Consider the following homogeneous linear PDE(Klein-Gordon equation) [19]:

$u_{tt} + u - u_{xx} = 0$

With the following conditions:

$u(x, 0) = e^{-x} + x, \frac{\partial u}{\partial t}(x, 0) = 0$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = L(u) - L(u_o) + pL(u_o) + p[N(u) - f(r)] = 0$

Then we have:

$[u_{tt} - (u_o)_{tt}] + p[(u_o)_{tt} + p[u - u_{xx}] = 0$

Then, substituting the initial condition, we have

$[u_{tt} - (e^{-x} + x)_{tt}] + p[(e^{-x} + x)_{tt}] + p[u - u_{xx}] = 0$

This gives

$u_{tt} - 0 + p[(0)] + p[u - u_{xx}] = 0$

Then

$u_{tt} + p[u - u_{xx}] = 0$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{tt} + p[(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$

$- (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{xx}] = 0$

If:

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(x, 0) = e^{-x} + x$

Then:

$u_0(x, 0) = e^{-x} + x$

$u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_o :

p^0: $(u_o)_{tt} = 0$

$(u_o)_t = C_1(x), (u_o)_t(x, 0) = 0$

$0 = C_1(x) \rightarrow C_1(x) = 0$

$(u_o)_t = 0$

$u_0 = C_2(x), u_0(x, 0) = e^{-x} + x$

$e^{-x} + x = C_2(x), C_2(x) = e^{-x} + x$

$u_0 = e^{-x} + x$

To find u_1 :

$p^1: (u_1)_{tt} + u_0 - (u_0)_{xx} = 0$

$(u_1)_{tt} + (e^{-x} + x) - (e^{-x} + x)_{xx} = 0$

$(u_1)_{tt} + (e^{-x} + x) - (e^{-x}) = 0$

$(u_1)_{tt} + x = 0 \rightarrow (u_1)_{tt} = -x$

$(u_1)_t = -xt + C_1(x), (u_1)_t(x, 0) = 0$

$0 = 0 + C_1(x) \rightarrow C_1(x) = 0$

$(u_1)_t = -xt$

$u_1 = -\frac{xt^2}{2} + C_2(x), u_1(x, 0) = 0$

$0 = 0 + C_2(x) \rightarrow C_2(x) = 0$

$u_1 = -\frac{xt^2}{2!}$

To find u_2 :

$p^2: (u_2)_{tt} + u_1 - (u_1)_{xx} = 0$

$(u_2)_{tt} + (-\frac{xt^2}{2}) - (-\frac{xt^2}{2})_{xx} = 0$

$(u_2)_{tt} + (-\frac{xt^2}{2}) - (0) = 0$

$(u_2)_{tt} - \frac{xt^2}{2} = 0$

$(u_2)_t - \frac{xt^3}{3!} = C_1(x), \quad (u_2)_t(x, 0) = 0$

$0 + 0 = C_1(x) \rightarrow C_1(x) = 0$

$(u_2)_t - \frac{xt^3}{3!} = 0$

$u_2 - \frac{xt^4}{4!} = C_2(x), \quad u_2(x\ ,0)=0$

$0 + 0 - 0 = C_2(x) \rightarrow C_2(x) = 0$

$u_2 = \frac{xt^4}{4!}$

$u_3 = -\frac{xt^6}{6!}, u_4 = \frac{xt^8}{8!}$

The solution is a series form, given by:

$u(x, t) = u_0 + u_1 + u_3 + \ldots$

$u(x, t) = e^{-x} + x - \frac{xt^2}{2!} + \frac{xt^4}{4!} - \frac{xt^6}{6!} + \ldots$

$= e^{-x} + x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots)$

The required solution is:

$u(x, t) = e^{-x} + xcos(t)$

Example (1.3)

Let us consider the problem:

$u_{tt} = u_{xx} + u$

With boundary conditions:

$u(0,t) = \cosh(t), \quad u_x(0,t) = 1$

and the initial conditions

$u(x,0) = \sin(x) + 1 \quad , \quad u_t(x,0) = 0$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u\ , p) = u_{tt} - (u_0)_{\boldsymbol{tt}} = p[\ u_{xx} + u]$

Then, substituting the initial condition, we have

$u_{tt} - (\sin(x) + 1)_{tt} = p[u_{xx} + u]$

Then we have:

$u_{tt} = p[u_{xx} + u]$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{tt} = p[(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{xx} + (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)]$$

If :

$u(x, 0) = \sin(x) + 1$

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(x, 0) = \sin(x) + 1$

Then:

$u_0(x, 0) = \sin(x) + 1, u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_o :

p^0: $(u_0)_{tt} = 0$

$(u_0)_t = C_1(x), (u_0)_t(x, 0) = 0$

$0 = C_1(x) \rightarrow C_1(x) = 0$

$(u_0)_t = 0$

$u_0 = C_2(x)$, $u_0(x, 0) = \sin(x) + 1$

$\sin(x) + 1 = C_2(x) \rightarrow$ $C_2(x) = \sin(x) + 1$

$u_0 = \sin(x) + 1$

To find u_1 :

p^1: $(u_1)_{tt} = (u_0)_{xx} + (u_0)$

$(u_1)_{tt} = (\sin(x) + 1)_{xx} + \sin(x) + 1$

$(u_1)_{tt} = -\sin(x) + \sin(x) + 1$

$(u_1)_t = t + C_1(x), (u_1)_t(x, 0) = 0$

$0 = 0 + C_1(x) \rightarrow C_1(x) = 0$

$(u_1)_t = t$

$u_1 = \frac{t^2}{2} + C_2(x), \quad u_1(x, 0) = 0$

$0 = 0 + C_2(x) \rightarrow C_2(x) = 0$

$u_1 = \frac{t^2}{2!}$

To find u_2 :

$p^2: (u_2)_{tt} = (u_1)_{xx} + (u_1)$

$(u_2)_{tt} = (\frac{t^2}{2!})_{xx} + \frac{t^2}{2!}$

$(u_2)_t = \frac{t^3}{6} + C_1, (u_2)_t(x, 0) = 0$

$0 = 0 + C_1(x) \rightarrow C_1(x) = 0$

$(u_2)_t = \frac{t^3}{6}$

$u_2 = \frac{t^4}{24} + C_2(x), \quad u_2(x, 0) = 0$

$0 = 0 + C_2(x) \rightarrow C_2(x) = 0$

$u_2 = \frac{t^4}{4!}$

To find u_3 :

$p^3: (u_3)_{tt} = (u_2)_{xx} + (u_2)$

$(u_3)_{tt} = (\frac{t^4}{24})_{xx} + \frac{t^4}{24}$

$(u_3)_t = \frac{t^5}{120} + C_1(x), (u_3)_t(x, 0) = 0$

$0 = 0 + C_1(x) \rightarrow C_1(x) = 0$

$(u_3)_t = \frac{t^5}{120}$

$u_3 = \frac{t^6}{720} + C_2(x), \quad u_3(x, 0) = 0$

$0 = 0 + C_2(x) \rightarrow C_2(x) = 0$

$u_3 = \frac{t^6}{6!}$

Then, The exact solution is:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \ldots$

$= \sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \ldots$

The required solution is:

$u(x, t) = \sin(x) + \cosh(t)$

Example (1.4)

Let us consider the one dimensional non-homogeneous problem

$u_t + u_{xx} + u - e^{-x}(1 + 2t) = 0$

Subject to boundary conditions:

$u(0,t) = t$, $u_x(0,t) = e^{-t} - t$

and the initial condition : $u(x,0)=x$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u , p) = u_t - (u_0)_{\boldsymbol{t}} + p[\, u_{xx} + u - e^{-x}(1 + 2t)] = 0$

Then, substituting the initial condition, we have

$u_t - (x)_{\boldsymbol{t}} + p[\, u_{xx} + u - e^{-x}(1 + 2t)] = 0$

Then we have:

$u_t + p[\, u_{xx} + u - e^{-x}(1 + 2t)] = 0$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_t + p[(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_{xx} + (p^0u_0 + p^1u_1 + p^2u_2 + \cdots) - e^{-x}(1 + 2t)] = 0$$

If :

$u(x, 0) = x$

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(x, 0) = x$

Then:

$u_0(x, 0) = x, u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_0 :

p^0: $(u_0)_t = 0$

$u_0 = C(x)$, $u_0(x, 0) = x$

$x = C(x) \rightarrow C(x) = x$

$u_0 = x$

To find u_1 :

p^1: $(u_1)_t + (u_0)_{xx} + u_0 - e^{-x}(1 + 2t) = 0$

$(u_1)_t + (x)_{xx} + x - e^{-x}(1 + 2t) = 0$

$(u_1)_t + x - e^{-x}(1 + 2t) = 0$

$u_1 + xt - e^{-x}(t + t^2) = C(x)$, $u_1(x, 0) = 0$

$0 + 0 - 0 = C(x) \rightarrow C(x) = 0$

$u_1 = e^{-x}(t + t^2) - xt$

To find u_2 :

p^2: $(u_2)_t + (u_1)_{xx} + u_1 = 0$

$(u_2)_t + [e^{-x}(t + t^2) - xt]_{xx} + [e^{-x}(t + t^2) - xt] = 0$

$(u_2)_t + [e^{-x}(t + t^2)] + [e^{-x}(t + t^2) - xt] = 0$

$(u_2)_t + 2e^{-x}(t + t^2) - xt = 0$

$u_2 + 2e^{-x}(\frac{t^2}{2} + \frac{t^3}{3}) - \frac{xt^2}{2} = C(x)$, $u_2(x, 0) = 0$

$0 + 0 - 0 = C \rightarrow C(x) = 0$

$u_2 = -2e^{-x}(\frac{t^2}{2} + \frac{t^3}{3}) + \frac{xt^2}{2}$

To find u_3 :

p^3: $(u_3)_t + (u_2)_{xx} + u_2 = 0$

$(u_3)_t + [-\ 2e^{-x}(\frac{t^2}{2} + \frac{t^3}{3}) + \frac{xt^2}{2}]_{xx} + [-\ 2e^{-x}\ (\frac{t^2}{2} + \frac{t^3}{3}) + \frac{xt^2}{2}] = 0$

$(u_3)_t + [-\ 2e^{-x}\left(\frac{t^2}{2} + \frac{t^3}{3}\right)] + [-\ 2e^{-x}\ (\frac{t^2}{2} + \frac{t^3}{3}) + \frac{xt^2}{2}] = 0$

$(u_3)_t - 4e^{-x}\left(\frac{t^2}{2} + \frac{t^3}{3}\right) + \frac{xt^2}{2} = 0$

$u_3 - 4e^{-x}(\frac{t^3}{6} + \frac{t^4}{12}) + \frac{xt^3}{6} = C(x)$, $u_3(x, 0) = 0$

$0\ -0 + 0 = C(x)$ $\rightarrow C(x)=0$

$u_3 = 4e^{-x}\ (\frac{t^3}{6} + \frac{t^4}{12}) - \frac{xt^3}{6}$

Then, The exact solution is:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \ldots$

$= x + e^{-x}(t + t^2) - xt - 2e^{-x}\ (\frac{t^2}{2} + \frac{t^3}{3}) + \frac{xt^2}{2}$

$+ 4e^{-x}\ (\frac{t^3}{6} + \frac{t^4}{12}) - \frac{xt^3}{6}$

$u(x, t) = x\ [\ 1 - t + \frac{t^2}{2} - \frac{t^3}{3} + \ldots] + e^{-x}\ [t + t^2 - t^2 - 2\frac{t^3}{3} + 4\ \frac{t^3}{6} - \ldots]$

The required solution is:

$u(x, t) = te^{-x} + xe^{-t}$

Example (1.5)

Consider the following nonhomogeneous PDE:

$u_{tt} + u_{xx} + (u_x)^2 = 2x + t^4$

with the following conditions:

$u(x\,,0) = 0\ \ , \frac{\partial u}{\partial t}\ (x\,,0) = a\ , u(o\,,t) = at \quad , \quad \frac{\partial u}{\partial x}(0\,,t) = t^2$

solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = L\,(u) - L(u_o) + pL\,(u_o) + p\,[N(u) - f(r)] = 0$

$[\,u_{tt} - (u_o)_{tt}] + p[(u_o)_{tt}] + \ p[u_{xx} + \ (u_x)^2 - 2x - t^4] = 0$

Then, substituting the initial condition, we have

$u_{tt} - (0)_{tt}\ ] + p[(0)_{tt}\ ] + p[u_{xx} + \ (u_x)^2 - 2x - t^4] = 0$

Then we have:

$u_{tt} + \ P[u_{xx} + \ (u_x)^2 - 2x - t^4] = 0$

$u_{tt} + \ P[u_{xx} + \ (u_x)^2 - 2x - t^4] = 0$

Substituting $\quad u = (p^0u_0\ +\ p^1u_1\ +\ p^2u_2\ + \cdots) \quad$ we have

$(p^0u_0\ +\ p^1u_1\ +\ p^2u_2\ + \cdots)_{tt} + p[(p^0u_0\ +\ p^1u_1\ +\ p^2u_2\ + \cdots)_{xx}$

$+ (p^0u_0\ +\ p^1u_1\ +\ p^2u_2\ + \cdots)_x{}^2 - 2x - t^4] = 0$

If:

$u(x, 0) = 0$

$(p^0u_0\ +\ p^1u_1\ +\ p^2u_2\ + \cdots)(x, 0) = \ 0$

Then:

$u_0(x, 0) = \ 0$

$u_1(x, 0) = \ 0$

$u_2(x, 0) = \ 0 \quad$ and so on

To find u_0 :

p^0: $(u_o)_{tt}\ = 0$

$(u_o)_t\ = C_1(x), \quad (u_o)_t(x\,,0) = a$

$a\ = C_1(x) \rightarrow C_1(x) = a$

$(u_o)_t\ = a$

$u_0 = at + C_2(x), u_0(x, 0) = 0$

$0 = 0 + C_2(x) \rightarrow C_2(x) = 0$

$u_0 = at$

To find u_1 :

$p^1: (u_1)_{tt} + (u_0)_{xx} + ((u_0)_x)^2 - 2x - t^4 = 0$

$(u_1)_{tt} + (at)_{xx} + ((at)_x)^2 - 2x - t^4 = 0$

$(u_1)_{tt} = 2x + t^4$

$(u_1)_t = 2xt + \frac{t^5}{5} + C_1(x) \quad , (u_1)_t(x, 0) = 0$

$0 = 0 + 0 + C_1(x) \rightarrow C_1(x) = 0$

$(u_1)_t = 2xt + \frac{t^5}{5}$

$u_1 = xt^2 + \frac{t^6}{30} + C_2(x), \quad u_1(x, 0) = 0$

$0 = 0 + 0 + 0 + C_2(x) \rightarrow C_2(x) = 0$

$u_1 = xt^2 + \frac{t^6}{30}$

To find u_2 :

$p^2: (u_2)_{tt} + (u_1)_{xx} + ((u_1)_x)^2 = 0$

$(u_2)_{tt} + (xt^2 + \frac{t^6}{30})_{xx} + [(xt^2 + \frac{t^6}{30})_x]^2 = 0$

$(u_2)_{tt} + 0 + [t^2]^2$

$(u_2)_{tt} = -t^4$

$(u_2)_t = -\frac{t^5}{5} + C_1(x) \quad , (u_2)_t(x, 0) = 0$

$0 = 0 + C_1(x) \rightarrow C_1(x) = 0$

$(u_2)_t = -\frac{t^5}{5}$

$(u_2)_t = -\frac{t^6}{30} + C_2(x), u_2(x, 0) = 0$

$0 = 0 + C_2(x) \to C_2(x) = 0$

$u_2 = -\frac{t^6}{30}$

To find u_3 :

p^3: $(u_3)_{tt} + (u_2)_{xx} + ((u_1)_x)^2 = 0$

$(u_3)_{tt} + (-\frac{t^6}{30})_{xx} + [(-\frac{t^6}{30})_x]^2 = 0$

$(u_3)_{tt} = 0$

$(u_3)_t = C_1(x)$ $\quad ,(u_3)_t(x , 0) = 0$

$0 = C_1(x) \to C_1(x) = 0$

$(u_3)_t = 0$

$u_3 = + C_2(x), u_3(x , 0) = 0$

$0 = 0 + C_2(x) \to C_2(x) = 0$

$u_3 = 0$

The solution is a series form, given by:

$u(x , t) = u_0 + u_1 + u_2 + u_3 + \dots$

$= at + xt^2 + \frac{t^6}{30} - \frac{t^6}{30} + 0 + \dots$

The required solution is:

$u(x , t) = at + xt^2$

1.4 Homptopy Perturbation Method for Solving Non-linear differential Equation

The proposed homptopy perturbation method solves effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions.

Example (1.6)

$u_t + \frac{1}{2}(u^2)_x = u(1-u) + g(x, t) \quad 0 \le x \le 1 \,, t > 0$

Case (1):

With $g(x, t) = 0$ then $u_t + \frac{1}{2}(u^2)_x = u(1-u)$

We start with an initial approximation $u_0(x, t) = e^{-x}$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u\,, p) = u_t - (u_0)_t + p[\frac{1}{2}(u^2)_x - u + u^2] = 0$

Then, substituting the initial condition, we have

$u_t - (e^{-x})_t + p[\frac{1}{2}(u^2)_x - u + u^2] = 0$

Then we have:

$u_t + p[\frac{1}{2}(u^2)_x - u + u^2] = 0$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_t$

$+ p[\frac{1}{2}[(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)^2]_x$

$-(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$

$+(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)^2] = 0$

If

$u(x, 0) = e^{-x}$

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(x, 0) = e^{-x}$

Then:

$u_0(x, 0) = e^{-x}, u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_0 :

p^0: $(u_0)_t = 0$

$u_0 = C(x), \qquad u_0(x, 0) = e^{-x}$

$e^{-x} = C(x) \rightarrow C(x) = e^{-x}$

$u_0 = e^{-x}$

To find u_1 :

$p^1: (u_1)_t \; + \frac{1}{2}[\,(u_0)^2]_x - u_0 + (u_0)^2 = 0$

$(u_1)_t \; + \frac{1}{2}[\,(e^{-x})^2]_x - e^{-x} + (e^{-x})^2 = 0$

$(u_1)_t \; + \frac{1}{2}[\,e^{-2x}]_x - e^{-x} + e^{-2x} = 0$

$(u_1)_t \; + \frac{1}{2}[-2e^{-2x}] - e^{-x} + e^{-2x} = 0$

$(u_1)_t - e^{-2x} - e^{-x} + e^{-2x} = 0$

$(u_1)_t = e^{-x}$

$u_1 = t\,e^{-x} + C(x), \qquad u_1(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_1 = t\,e^{-x}$

To find u_2 :

$p^2: (u_2)_t \; + \frac{1}{2}[\,2u_0u_1]_x - u_1 + 2u_0u_1) = 0$

$(u_2)_t \; + \frac{1}{2}[\,2(e^{-x})(t\,e^{-x})\,]_x - t\,e^{-x} + 2(e^{-x})(te^{-x}) = 0$

$(u_2)_t \; + \frac{1}{2}[2te^{-2x}]_x - te^{-x} + 2\,te^{-2x} = 0$

$(u_2)_t \; + \frac{1}{2}[-4te^{-2x}] - te^{-x} + 2te^{-2x} = 0$

$(u_2)_t - 2\,te^{-2x} - te^{-x} + 2te^{-2x} = 0$

$(u_2)_t = te^{-x}$

$u_2 = \frac{t^2}{2}e^{-x} + C(x), \qquad u_2(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_2 = \frac{t^2}{2!}\ e^{-x}$

To find u_3 :

p^3: $(u_3)_t\ \ + \frac{1}{2}[\ 2u_0 u_1 + (u_1)^2\ ]_x - u_2$ + $[2u_0 u_2 + (u_1)^2]$ = 0

$(u_3)_t\ \ + \frac{1}{2}[\ 2(e^{-x})\left(\frac{t^2}{2}\ e^{-x}\right) + (te^{-x})^2]_x - \frac{t^2}{2}\ e^{-x}$

$+[2(e^{-x})\left(\frac{t^2}{2}\ e^{-x}\right) + (te^{-x})^2]$ = 0

$(u_3)_t\ \ + \frac{1}{2}[t^2 e^{-2x}\ + t^2 e^{-2x}\ ]_x - \frac{t^2}{2}\ e^{-x}$ + $[t^2 e^{-2x}\ + t^2 e^{-2x}]$ = 0

$(u_3)_t\ \ + \frac{1}{2}[2t^2 e^{-2x}]_x - \frac{t^2}{2}\ e^{-x}$ + $[2t^2 e^{-2x}]$ = 0

$(u_3)_t\ \ + \frac{1}{2}[-4t^2 e^{-2x}] - \frac{t^2}{2}\ e^{-x}$ + $2t^2 e^{-2x}$ = 0

$(u_3)_t - \ 2t^2 e^{-2x} - \frac{t^2}{2}\ e^{-x}$ + $2t^2 e^{-2x}$ = 0

$(u_3)_t = \frac{t^2}{2}\ e^{-x}$

$u_3 = \frac{t^3}{6} e^{-x} + C(x)$, $u_3(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_3 = \frac{t^3}{3!}\ e^{-x}$

The exact solution:

$u(x, t) = \ u_0 + u_1 + u_2 + u_3 + \ldots$

$= e^{-x} + t\ e^{-x} + \frac{t^2}{2!}\ e^{-x} + \frac{t^3}{3!}\ e^{-x} + \ldots$

$= e^{-x}(\ 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots)$

The required solution is:

$u(x, t) = e^{-x}\ (e^t) = e^{t-x}$

Case (2):

With $g(x, t) = -e^{t-x}$ then $u_t + \frac{1}{2}[(u_1{}^2)]_x = u - u^2 - e^{t-x}$

with the initial approximation $u(x, 0) = 1 - e^{-x}$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = u_t - (u_0)_t + p[\frac{1}{2}[(u_1{}^2)]_x - u + u^2 + e^{t-x}] = 0$

Then, substituting the initial condition, we have

$u_t - (1 - e^{-x})_t + p[\frac{1}{2}[(u_1{}^2)]_x - u + u^2 + e^{t-x}] = 0$

Then we have

$u_t + p[\frac{1}{2}[(u_1{}^2)]_x - u + u^2 + e^{t-x}] = 0$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_t$

$+ p[\frac{1}{2}[(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)^2]_x$

$-(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$

$+(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)^2 + e^{t-x}] = 0$

If

$u(x, 0) = 1 - e^{-x}$

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(x, 0) = 1 - e^{-x}$

Then:

$u_0(x, 0) = 1 - e^{-x}, u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_0 :

p^0: $(u_0)_t = 0$

$u_0 = C(x)$, $u_0(x, 0) = 1 - e^{-x}$

$1 - e^{-x} = C(x) \rightarrow C(x) = 1 - e^{-x}$

$u_0 = 1 - e^{-x}$

To find u_1 :

p^1: $(u_1)_t \; + \frac{1}{2}[\, (u_0)^2]_x - u_0 + (u_0)^2 + e^{t-x} = 0$

$(u_1)_t \; + \frac{1}{2}[\, (1 - e^{-x})^2]_x - (1 - e^{-x}) + (1 - e^{-x})^2 + e^{t-x} = 0$

$(u_1)_t \; + \frac{1}{2}\, [\, 1 - 2e^{-x} + e^{-2x}\,]_x - (1 - e^{-x}) + (1 - 2e^{-x} + e^{-2x}]$

$+ e^{t-x} = 0$

$(u_1)_t \; + \frac{1}{2}\, [2e^{-x} - 2e^{-2x}] - (1 - e^{-x}) + (1 - 2e^{-x} + e^{-2x}\,)$

$+ e^{t-x} = 0$

$(u_1)_t + e^{-x} - e^{-2x} - 1 + e^{-x} + 1 - 2e^{-x} + e^{-2x} + e^{t-x} = 0$

$(u_1)_t + 2e^{-x} - 2e^{-x} + e^{t-x} = 0$

$(u_1)_t = -e^{t-x}$

$u_1 = - e^{t-x} + C(x), \qquad u_1(x, 0) = 0$

$0 = - e^{-x} + C(x) \rightarrow C(x) = e^{-x}$

$u_1 = - e^{t-x} + e^{-x}$

To find u_2 :

p^2: $(u_2)_t \; + \frac{1}{2}[\, (u_0 u_1)_x - u_1 + (2u_0 u_1) = 0$

$(u_2)_t \; + \frac{1}{2}\, [\, 2(1 - e^{-x})(- e^{t-x} + e^{-x})]_x$

$- (- e^{t-x} - e^{-x}) + [2(1 - e^{-x})(- e^{t-x} + e^{-x}) = 0$

$(u_2)_t \; + \frac{1}{2}\, [\, -2e^{t-x} + 2e^{-x} + 2e^{t-2x} - 2e^{-2x}\,]_x$

$- (- e^{t-x} - e^{-x}) + (-2e^{t-x} + 2e^{-x} + 2e^{t-2x} - 2e^{-2x}\,)] = 0$

$(u_2)_t \; + \frac{1}{2}[2e^{t-x} - 2e^{-x} - 4e^{t-2x} + 4e^{-2x}]$

$- \left(- e^{t-x} - e^{-x}\right) + (-2e^{t-x} + 2e^{-x} + 2e^{t-2x} - 2e^{-2x}\,)] = 0$

$(u_2)_t + e^{t-x} - e^{-x} - 2e^{t-2x} + 2e^{-2x} + e^{t-x} + e^{-x} - 2e^{t-x}$

$+ 2e^{-x} + 2e^{t-2x} - 2e^{-2x} = 0$

$(u_2)_t = 0$

$u_2 = C(x)$, $u_2(x, 0) = 0$

$0 = C(x) \rightarrow C(x) = 0$

$u_2 = 0$andthen $u_3 = 0$, $u_4 = 0$

Then,The exact solution is:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots$

$= 1 - e^{-x} - e^{t-x} + e^{-x} + 0 + \dots$

$= e^{-x}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots)$

The required solution is:

$u(x, t) = 1 - e^{t-x}$

Chapter (2)

Homomtopy perturbation method for solving advection equations

2.1 Introduction

Nonlinear phenomena have important effects on applied mathematics, physics and issues related to engineering: many such physical phenomena are modeled in terms of nonlinear partial differential equations. For example, the advection problems which are of the form.

$u_t(x\ ,t) + uu_x = h(x\ ,t),\ u(x\ ,0) = g(x)$

First we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form.

$Du(x, t) + Ru(x\ ,t) + Nu(x, t) = g(x, t)$

$u(x, 0) = h(x),\ u_t(x\ ,0) = f(x)$

2.2 Application:

In order to illustrate the solution procedure of the homotopy perturbation method, we first consider the nonlinear advection equations.

Example (2.1)

Consider the following homogeneous advection problem [20, 21]:

$u_t + uu_x = 0$

$u(x, 0) = -x$

solution:

By homotopy technique we construct a homotopy which is solutions

$H(u\ ,p) = u_t - (u_0)_t + p[u\,u_x] = 0$

Then, substituting the initial condition, we have

$u_t - (-x)_t + p[u\,u_x] = 0$

Then we have :

$u_t + p[u\, u_t] = 0$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_t + p[\,(u_0 + u_1 + u_2 + \ldots)\,(u_0 + u_1 + u_2 + \cdots)_x] = 0$

$$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_t + p[\underbrace{u_0 u_{0x}}_{p^0} + \underbrace{u_0 u_{1x} + u_1 u_{0x}}_{p^1} + \underbrace{u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}}_{p^2} + \ldots]$$

If

$u(x, 0) = -x$

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(x, 0) = -x$

Then:

$u_0(x, 0) = -x, u_1(x, 0) = 0 \quad , u_2(x, 0) = 0$

To find u_0 :

p^0: $(u_0)_t = 0$

$u_0 = C(x), u_0(x, 0) = -x$

$-x = C(x) \rightarrow C(x) = 0$

$u_0 = -x$

To find u_1 :

p^1: $(u_1)_t + u_0 u_{0x} = 0$

$(u_1)_t + (-x)(-x)_x = 0$

$(u_1)_t + (-x)(-1) = 0$

$(u_1)_t + x = 0$

$(u_1)_t = -x$

$u_1 = -xt + C(x), u_1(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_1 = -xt$

To find u_2 :

p^2: $(u_2)_t + u_0 u_{1x} + u_1 u_{0x} = 0$

$(u_2)_t + (-x)(-xt)_{\boldsymbol{x}} + (-xt)(-x)_{\boldsymbol{x}} = 0$

$(u_2)_t + (-x)(-t) + (-xt)(-1) = 0$

$(u_2)_t + xt + xt = 0$

$(u_2)_t = -2xt$

$u_2 = -xt^2 + C(x), u_2(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_2 = -xt^2$

In a similar, we have:

$u_3 = -xt^3$

$u_4 = -xt^4$

Then:

The solution $u(x, t)$ is given by:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots$

$= -x - xt - xt^2 - xt^3 - \dots$

$u(x, t) = -x\,(1 + t + t^2 + t^3 + \dots) = \frac{-x}{(1-t)}$

The required solution is:

$u(x, t) = \frac{x}{t-1}$

Example (2.2)

We now consider the nonhomogeneous advection problem [20, 21]:

$u_t + uu_x = 2t + x + t^3 + xt^2$

$u(x, 0) = 0$

solution:

By homotopy technique we construct a homotopy which is solutions

$H(u , p) = u_t - (u_0)_t + p[u\, u_x - 2t - x - t^3 - xt^2] = 0$

Then, substituting the initial condition, we have

$u_t - (0)_t + p[u\, u_x - 2t - x - t^3 - xt^2] = 0$

Then we have:

$u_t + p[u\, u_x - 2t - x - t^3 - xt^2] = 0$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$u_t + p[\,(u_0 + u_1 + u_2 + \ldots)\,(u_0 + u_1 + u_2 + \cdots)_x$

$- 2t - x - t^3 - xt^2\,] = 0$

$$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_t + p[\underbrace{u_0u_{0x}}_{p^0} + \underbrace{u_0u_{1x} + u_1u_{0x}}_{p^1} +$$

$$\underbrace{u_0u_{2x} + u_1u_{1x} + u_2u_{0x}}_{p^2} + \ldots - 2t - x - t^3 - xt^2] = 0$$

If :

$u(x, 0) = 0$

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(x, 0) = 0$

Then:

$u_0(x, 0) = 0, u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_0 :

p^0: $(u_0)_t = 0$

$u_0 = C(x), u_0(x, 0) = 0$

$0 = C(x) \rightarrow C(x) = 0$

$u_0 = 0$

To find u_1 :

p^1: $(u_1)_t + u_0 u_{0x} - 2t - x - t^3 - xt^2 = 0$

$(u_1)_t + (0)(0)_{\boldsymbol{x}} - 2t - x - t^3 - xt^2 = 0$

$(u_1)_t = 2t + x + t^3 + xt^2$

$u_1 = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3} + C(x), u_1(x, 0) = 0$

$0 = 0 + 0 + 0 + 0 + C \rightarrow C(x) = 0$

$u_1 = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3}$

To find u_2 :

p^2: $(u_2)_t + u_0 u_{1x} + u_1 u_{0x} = 0$

$(u_2)_t + (0)u_{1x} + u_1 (0)_x = 0$

$u_2 = C(x), u_2(x, 0) = 0$

$0 = C(x) \rightarrow C(x) = 0$

$u_2 = 0$

To find u_3 :

p^3: $(u_3)_t + u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} = 0$

$(u_3)_t + (0)u_{2x} + (t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3})[(t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3})_x$

$+ 0(0)_x = 0$

$(u_3)_t + (t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3})[t + \frac{t^3}{3}] = 0$

$(u_3)_t + t^3 + xt^2 + \frac{t^5}{4} + x\frac{t^4}{3} + \frac{t^5}{3} + x\frac{t^4}{3} + \frac{t^7}{12} + x\frac{t^6}{9} = 0$

$u_3 + x\frac{t^3}{3} + \frac{t^4}{4} + \frac{2}{15}xt^5 + \frac{7}{72}t^6 + x\frac{t^7}{63} + \frac{t^8}{96} = C(x), u_3(x, 0) = 0$

$0 + 0 + 0 + \ldots + 0 = C(x) \rightarrow C(x) = 0$

$u_3 = -(x\frac{t^3}{3} + \frac{t^4}{4} + \frac{2}{15}xt^5 + \frac{7}{72}t^6 + x\frac{t^7}{63} + \frac{t^8}{96})$

Then, The exact solution is:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots$

$= 0 + t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3} + 0 - x\frac{t^3}{3} - \frac{t^4}{4} - \frac{2}{15}xt^5 + \dots$

The required solution is:

$u(x, t) = t^2 + xt$

Example (2.3)

Consider a nonlinear partial differential equation [22]

$u_t + uu_x = x + xt^2$

With initial condition$u(x, 0) = 0$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = u_t - (u_0)_t + p[u\, u_x - x - xt^2] = 0$

Then, substituting the initial condition, we have

$u_t - (0)_t + p[u\, u_x - x - xt^2] = 0$

Then we have:

$u_t + p[u\, u_x - x - xt^2] = 0$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_t + p[(u_0 + u_1 + u_2 + \dots)(u_0 + u_1 + u_2 + \dots)_x - x - xt^2] = 0$

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_t + p[\underbrace{u_0u_{0x}}_{p^0} + \underbrace{u_0u_{1x} + u_1u_{0x}}_{p^1} + \underbrace{u_0u_{2x} + u_1u_{1x} + u_2u_{0x}}_{p^2} + \dots - x - xt^2] = 0$

If :

$u(x, 0) = 0$

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(x, 0) = 0$

Then:

$u_0(x, 0)= 0, u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_0 :

p^0: $(u_0)_t = 0$

$u_0 = C(x), u_0(x, 0) = 0$

$0 = C(x) \rightarrow C(x) = 0$

$u_0 = 0$

To find u_1 :

p^1: $(u_1)_t + u_0 u_{0x} - x - xt^2 = 0$

$(u_1)_t + (0)(0)_x - x - xt^2 = 0$

$(u_1)_t = x + xt^2$

$u_1 = xt + x\frac{t^3}{3} + C(x), u_1(x, 0) = 0$

$0 = 0 + 0 + C(x) \rightarrow C(x) = 0$

$u_1 = xt + x\frac{t^3}{3}$

To find u_2 :

p^2: $(u_2)_t + u_0 u_{1x} + u_1 u_{0x} = 0$

$(u_2)_t + (0)u_{1x} + u_1(0)_x = 0$

$(u_2)_t = 0$

$u_2 = C(x), u_2(x, 0) = 0$

$0 = C(x) \rightarrow C(x) = 0$

$u_2 = 0$

To find u_3 :

p^3: $(u_3)_t + u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} = 0$

$(u_3)_t + (0)u_{2x} + (xt + x\frac{t^3}{3})(xt + x\frac{t^3}{3})_x + u_2(0)_x = 0$

$(u_3)_t + (xt + x\frac{t^3}{3})(t + \frac{t^3}{3}) + = 0$

$(u_3)_t + xt^2 + x\frac{t^4}{3} + x\frac{t^4}{3} + x\frac{t^6}{9} = 0$

$(u_3)_t + xt^2 + \frac{2}{3}xt^4 + x\frac{t^6}{9} = 0$

$u_3 + x\frac{t^3}{3} + \frac{2}{15}xt^5 + x\frac{t^7}{63} = C(x)$, $\quad u_3(x, 0) = 0$

$0 + 0 + 0 + 0 = C(x) \rightarrow C(x) = 0$

$u_3 = -x\frac{t^3}{3} - \frac{2}{15}xt^5 - x\frac{t^7}{63}$

Then,The exact solution is:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots$

$= 0 + xt + x\frac{t^3}{3} + 0 - x\frac{t^3}{3} - \dots$

The solution is:

$u(x, t) = xt$

Example (2.4)

Consider another nonlinear partial differential equation [22]

$u_{xx} - u_x u_{yy} = - x + u$

With initial condition

$u(0,y) = \sin y$, $u_x(x, 0) = 1$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = u_{xx} - (u_0)_{xx} = p[u_x u_{yy} - x + u]$

Then, substituting the initial condition, we have

$u_{xx} - (\sin y)_{xx} = p[\, u_x u_{yy} - x + u]$

Then we have:

$u_{xx} = p[\, u_x u_{yy} - x + u]$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{xx} =$

$p[(u_0 + u_1 + u_2 + \ldots)_x (u_0 + u_1 + u_2 + \ldots)_{yy} - x$

$+ (u_0 + u_1 + u_2 + \ldots)]$

$$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{xx} = p[\underbrace{(u_0)_x (u_0)_{yy}}_{p^0}$$

$$+ \underbrace{(u_0)_x (u_1)_{yy} + (u_1)_x (u_0)_{yy}}_{p^1} + \ldots - x + p^0 u_0 + p^1 u_1 + p^2 u_2 + \ldots)]$$

If:

$u(0, y) = \sin y$

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(x, 0) = \sin y$

Then:

$u_0 = \sin y$, $u_1 = 0$, $u_2 = 0$ and so on

To find u_0 :

p^0: $(u_0)_{xx} = 0$

$(u_0)_x = C_1(x), (u_0)_x(x, 0) = 1$

$1 = C_1(x) \rightarrow C_1(x) = 1$

$(u_0)_x = 1$

$u_0 = x + C_2(x), u_0(0, y) = \sin y$

$\sin y = 0 + C_2(x) \rightarrow C_2(x) = \sin y$

$u_0 = x + \sin y$

To find u_1 :

p^1: $(u_1)_{xx} = (u_0)_x (u_0)_{yy} - x + u_0$

$(u_1)_{xx} = (x + \sin y)_x (x + \sin y)_{yy} - x + (x + \sin y)$

$(u_1)_{xx} = (1 + 0)(0 - \sin y) - x + x + \sin y$

$(u_1)_{xx} = -\sin y - x + x + \sin y = 0$

$(u_1)_x = C_1(x)$, $(u_1)_x(x, 0) = 0$

$0 = C_1(x) \rightarrow C_1(x) = 0$

$(u_1)_x = 0$

$u_1 = C_2(x)$,$u_1(0,y) = 0$

$0 = C_2(x) \rightarrow C_2(x) = 0$

$u_1 = 0$

To find u_2 :

p^2: $(u_2)_{xx} = (u_0)_x (u_1)_{yy} + (u_1)_x (u_0)_{yy} + u_1$

$(u_2)_{xx} = (u_0)_x (0) u_{yy} + (0)_x (u_0)_{yy} + 0$

$(u_2)_{xx} = 0$

$(u_2)_x = C_1(x)$,$(u_2)_x(x, 0) = 0$

$0 = C_1(x) \rightarrow C_1(x) = 0$

$(u_2)_x = 0$

$u_2 = C_2(x)$,$u_2(0, y) = 0$

$0 = C_2(x) \rightarrow C_2(x) = 0$

$u_2 = 0$, $u_3 = 0$, $u_4 = 0$

Then, The exact solution is:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \ldots$

$= x + \sin y + 0 + 0 + \ldots$

The required solution is:

$u(x, t) = x + \sin y$

Example (2.5)

Consider the following homogeneous nonlinear PDE(Burger equation) [19]:

$u_t - u\,(u_x) - u_{xx} = 0$

With the following conditions:

$u(x, 0) = 1 - x \quad , \quad u(0, x) = \frac{1}{1+t}\,,\ u(1, t) = 0$

solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = L(u) - L(u_o) + pL(u_o) + P\,[N(u) - f(r)] = 0$

$[\,u_t - (u_o)_t] + p[(u_o)_t] + p[-u\,(u_x) - u_{xx}\,] = 0$

Then, substituting the initial condition, we have

$[\,u_t - (1 - x)_t\,] + p[(1 - x)_t\,] + p[-u\,(u_x) - u_{xx}\,] = 0$

Then we have:

$u_t - 0 + p[0] + p[-u\,(u_x) - u_{xx}\,] = 0$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_t + p[-(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots) - (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_{xx}] = 0$

If :

$u(x, 0) = 1 - x$

$(p^0u_0 + p^1u_1 + p^2u_2 + \ldots)(x, 0) = 1 - x$

Then:

$u_1(x, 0) = 0$

$u_2(x\ , 0)$=0 and so on

To find u_0 :

p^0: $(u_o)_t$ = 0

u_0 = $C(x)$,$u(x, 0) = 1 – x$

$1 – x= C(x)\ ,C(x) =1 – x$

u_0 =1 – x

To find u_1 :

p^1: $(u_1)_t - u_0(u_0)_x - (u_0)_{xx}$ = 0

$(u_1)_t - (1 – x)(1 - x)_x - (1 - x)_{xx}$ = 0

$(u_1)_t - (1 - x)(– 1) – (0) = 0$

$(u_1)_t + (1 - x) = 0$

$u_1 + (1 - x)t = C, u_1(x\ , 0)=0$

$0 + (1 - x)0 = C(x) \rightarrow C(x) = 0$

$u_1 = -(1 - x)t$

$u_2 = (1 - x)t^2, \quad u_3 = -(1 - x)t^3$

Then, The exact solution is:

$u(x, t) = u_0 + u_1 + u_2 + \ldots$

$= (1 – x\ )t^0 - (1 - x)t^1 + (1 - x)t^2 - (1 - x)t^3$

$= (1 – x\ )[\ 1 – t\ + t^2 - t^3 + \ldots]$

The required solution is:

$u(x, t) = (1 – x\ )(1 + t)^{-1} = \frac{(1-x)}{(1+t)}$

Example (2.6)

Consider the following homogeneous nonlinear PDE [19]:

$u_t - u - u(u_{xx}) - u_x^2 = 0$

With the following conditions:

$u(x, 0) = \sqrt{x}, \quad u(0, t) = 0 \quad , \; u(1, t) = e^t$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = L(u) - L(u_o) + pL(u_o) + p[N(u) - f(r)] = 0$

$[u_t - (u_o)_t] = p[(u_o)_t] + p[u + u(u_{xx}) + (u_x)^2]$

Then, substituting the initial condition, we have

$[u_t - (\sqrt{x})_t] = p[(\sqrt{x})_t] + p[u + u(u_{xx}) + {u_x}^2]$

Then we have:

$u_t = p[u + u(u_{xx}) + {u_x}^2]$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_t + p[(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$

$+ (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_{xx}$

$+ [(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_x{}^2] = 0$

If:

$u(x, 0) = \sqrt{x}$

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(x, 0) = \sqrt{x}$

Then:

$u_0(x, 0) = \sqrt{x}$

$u_1(x, 0) = 0$

$u_2(x, 0) = 0$ and so on

To find u_0 :

$p^0: (u_o)_t = 0$

$u_0 = C(x), u_0(x, 0) = \sqrt{x}$

$\sqrt{x} = C(x), C(x) = \sqrt{x}$

$u_0 = \sqrt{x}$

To find u_1 :

p^1: $(u_1)_t = u_0 + u_0(u_0)_{xx} + [(u_0)_x]^2$

$(u_1)_t = \sqrt{x} + \sqrt{x}(\sqrt{x})_{xx} + [(\sqrt{x})_x]^2$

$(u_1)_t = \sqrt{x} + \sqrt{x}(-\frac{1}{4}x^{\frac{-3}{2}}) + \frac{1}{4}x^{-1}$

$(u_1)_t = \sqrt{x} - \frac{1}{4}x^{-1} + \frac{1}{4}x^{-1}$

$(u_1)_t = \sqrt{x}$

$u_1 = (\sqrt{x})t + C(x)$, $u_1(x, 0) = 0$

$0 = 0 + C(x) \Longrightarrow C(x) = 0$

$u_1 = \sqrt{x}\, t$

To find u_2 :

p^2: $(u_2)_t = u_1 + u_0(u_1)_{xx} + u_1(u_0)_{xx} + 2(u_0)_x(u_1)_x$

$(u_2)_t = \sqrt{x}\, t + \sqrt{x}\,[\sqrt{x}\, t]_{xx} + \sqrt{x}\, t[\sqrt{x}\,]_{xx} + 2(\sqrt{x}\,)_x(\sqrt{x}\ \mathrm{t})_x$

$(u_2)_t = \sqrt{x}\, t + \sqrt{x}\,(-\frac{1}{4}x^{\frac{-3}{2}}t) + \sqrt{x}\, t(-\frac{1}{4}x^{\frac{-3}{2}}t) + 2\,(\frac{1}{2}x^{\frac{-1}{2}})(\frac{1}{2}x^{\frac{-1}{2}}t\,)$

$(u_2)_t = \sqrt{x}\, t - \frac{1}{4}x^{-1}t - \frac{1}{4}x^{-1}t + \frac{1}{2}x^{-1}t$

$(u_2)_t = \sqrt{x}\, t - \frac{1}{2}x^{-1}t + \frac{1}{2}x^{-1}t$

$(u_2)_t = \sqrt{x}\, t$

$u_2 = \frac{\sqrt{x}\, t^2}{2} + C(x)$, $u_2(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_2 = \frac{\sqrt{x}\, t^2}{2!}$

The exact solution:

$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots$

$u(x, t) = \sqrt{x} + \sqrt{x}\, t + \frac{\sqrt{x}\, t^2}{2!} + \frac{\sqrt{x}\, t^3}{3!} + \dots$

$= \sqrt{x}(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots)$

The required solution is:

$u(x, t) = \sqrt{x}e^t$

2.3 Homotopy Perturbation Method for System of advection Problems

To illustrate this method for system of advection equations, we take the following example.

Example (2.7)

We consider system of coupled nonlinear partial differentials [18]

$u_t + vu_x + u = 1 \qquad t > 0$

$v_t - uv_x - v = 1$

With initial condition

$u(x,0) = e^x \quad , v(x,0) = e^{-x}$

Solution:

$u_t + vu_x + u = 1$

By homotopy technique we construct a homotopy which is solutions

$H(u , p) = u_t - (u_0)_{\boldsymbol{t}} + p[vu_x + u - 1] = 0$

Then, substituting the initial condition, we have

$u_t - (e^x)_{\boldsymbol{t}} + p[vu_x + u - 1] = 0$

Then we have:

$u_t + p[vu_x + u - 1] = 0$

Substituting $\quad u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots) \quad$ we have

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)_t + p[(v_0 + v_1 + v_2 + \ldots)(u_0 + u_1 + u_2 + \ldots)_x + (p^0u_0 + p^1u_1 + p^2u_2 + \ldots) - 1] = 0$

If :

$u(x, 0) = e^x$

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(x, 0) = e^x$

Then:

$u_0(x, 0) = e^x, u_1(x, 0) = 0$, $u_2(x, 0) = 0$ and so on

To find u_0 :

$p^0: (u_0)_t = 0$

$u_0 = C(x), u_0(x, 0) = e^x$

$e^x = C(x) \rightarrow C(x) = e^x$

$u_0 = e^x$

By homotopy technique we construct a homotopy which is solutions

$H(v, p) = v_t - (v_0)_{\boldsymbol{t}} = p[uv_x + v + 1] = 0$

Then we substitute the initial condition we have

$v_t - (e^{-x})_{\boldsymbol{t}} = p[uv_x + v + 1] = 0$

Then we have:

$v_t = p[uv_x + v + 1] = 0$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$(p^0v_0 + p^1v_1 + p^2v + \cdots)_t + p[(u_0 + u_1 + u_2 + \ldots)(v_0 + v_1 + v_2 + \ldots)_x + (v_0 + v_1 + v_2 + \ldots) + 1] = 0$

If :

$v(x, 0) = e^{-x}$

$(p^0v_0 + p^1v_1 + p^2v_2 + \cdots)(x, 0) = e^{-x}$

Then:

$v_0(x,0) = e^{-x}, v_1(x,0) = 0$, $v_2(x,0) = 0$ and so on

To find v_0 :

$p^0: (v_0)_t = 0$

$v_0 = C(x), v_0(x,0) = e^{-x}$

$e^{-x} = C(x) \rightarrow C(x) = e^{-x}$

$v_0 = e^{-x}$

To find u_1 :

$p^1: (u_1)_t + v_0(u_0)_x + u_0 - 1 = 0$

$(u_1)_t + e^{-x}(e^x)_x + (e^x) - 1 = 0$

$(u_1)_t + e^{-x}(e^x) + (e^x) - 1 = 0$

$(u_1)_t + 1 + e^x - 1 = 0$

$(u_1)_t = -e^x$

$u_1 = -te^x + C(x)$, $u_1(x,0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_1 = -te^x$

To find v_1 :

$p^1: (v_1)_t = u_0(v_0)_x + v_0 + 1$

$(v_1)_t = e^x(e^{-x})_x + (e^{-x}) + 1$

$(v_1)_t = e^x(-e^{-x}) + (e^{-x}) + 1$

$(v_1)_t = -1 + e^{-x} + 1$

$(v_1)_t = e^{-x}$

$v_1 = te^{-x} + C(x)$, $v_1(x,0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$v_1 = te^{-x}$

To find u_2 :

p^2: $(u_2)_t + v_0(u_1)_x + v_1(u_0)_x + u_1 = 0$

$(u_2)_t + e^{-x}(-\ te^x)_x \ +te^{-x}(e^x)_x + (-\ te^x) = 0$

$(u_2)_t \ + e^{-x}\ (-\ te^x) + te^{-x}(e^x) - \ te^x = 0$

$(u_2)_t - \ te^0 + te^0 - \ te^x \ = 0$

$(u_2)_t \ = te^x$

$u_2 = \frac{t^2}{2!}e^x + C(x), \qquad u_2(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$u_2 = \frac{t^2}{2!}e^x$

To find v_2 :

p^2: $(v_2)_t = u_0(v_1)_x + u_1(v_0)_x + v_1$

$(v_2)_t = e^x(te^{-x})_x - te^x(e^{-x}) + te^{-x}$

$(v_2)_t = e^x\ (-\ te^{-x}) - te^x\ (-e^{-x}) + te^{-x}$

$(v_2)_t = \ -te^0 \ + te^0 + \ te^{-x}$

$(u_2)_t \ = te^{-x}$

$u_2 = \frac{t^2}{2!}e^{-x} + C(x) \qquad , \qquad u_2(x, 0) = 0$

$0 = 0 + C(x) \rightarrow C(x) = 0$

$v_2 = \frac{t^2}{2!}e^{-x}$

The solution is a series form is given by:

$u(x, t) = \ u_0 + u_1 + u_2 + u_3 + \ldots$

$= e^x - te^x + \frac{t^2}{2!}e^x + \frac{t^3}{3!}e^x$

$= e^x\ [1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \ + \ldots]$

The required solution is:

$u(x, t) = e^{x}.e^{-t} = e^{x-t}$

The solution is a series form, given by:

$v(x, t) = v_0 + v_1 + v_2 + v_3 + \ldots$

$= e^{-x} + te^{-x} + \frac{t^2}{2!}e^{-x} + \ldots$

$= e^{-x}[\ 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots]$

The required solution is:

$v(x, t) = e^{-x}.e^{t} = e^{t-x}$

Chapter (3)

Homotopy PerturbationMethod for solving integral equations

3.1 Introduction

Various kinds of analytical methods and numerical methods [1, 2] were used to solve integral equations. In this chapter, we apply homotopy perturbation method to solve integral equations. To illustrate the basic idea of the method, we consider the following general nonlinear system:

$L[\, u(t)\,] + N\, [\, u(t)\,] = g(t)$

Where L is a linear operator, N is a nonlinear operator and $g(t)$ is a given continuous function. The basic character of the method is to construct a correction functional for the system, which reads

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\, [\, L\, u_n(s) + N\tilde{u}_n(s) - g(s)\,]ds$$

Where λ is a language multiplier which can be identified optimally via variational theory, u_n is the n-th approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta\tilde{u}_n = 0$

3.2 HomotopyPerturbation Method integral equations of the second kind

First, we consider the Volterra integral equations of the second kind, which read:

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt$$

where$k(x,t)$is the kernel of the integral equation.

Example (3.1)

$$u(x) = x + \int_0^x (t-x)\, u(t)dt$$

with an initial condition

$u_0(0) = x$

Solution:

$u(x) = f(x) + \int_0^x k(x,t)u(t)dt$

By homotopy technique we construct a homotopy which is solutions

$H(u, p) = u(x) - u_0(x) = p[\, x + \int_0^x (t-x)\, u(t)dt]$

Then, substituting the initial condition, we have

$u(x) - x = p[\, x + \int_0^x (t-x)\, u(t)dt]$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(x) - x =$

$p[\, x + \int_0^x (t-x)\,(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(t)\, dt\,]$

To find u_0 :

p^0: $u_0(x) - x = 0$

$$u_0(x) = x$$

To find u_1 :

p^1: $u_1(x) = x + \int_0^x (t-x)\, u_0(t)dt$

$= x + \int_0^x (t-x)\,(t)dt$

$= x + \int_0^x (t^2 - xt)\, dt$

$u_1(x) = x + \left[\frac{t^3}{3} - x\frac{t^2}{2}\right]_0^x = x + \frac{x^3}{3} - \frac{x^3}{2} = x - \frac{1}{6}x^3$

$$u_1(x) = x - \frac{1}{3!}x^3$$

To find u_2 :

p^2: $u_2(x) = x + \int_0^x (t-x)\, u_1(t)dt$

$= x + \int_0^x (t-x)\left(t - \frac{1}{6}t^3\right)dt$

$= x + \int_0^x \left(t^2 - \frac{1}{6}t^4 - xt + \frac{1}{6}xt^3\right)dt$

$$u_2(x) = x + \left(\frac{t^3}{3} - \frac{1}{30}t^5 - x\frac{t^2}{2} + \frac{1}{24}xt^4 \right]_0^x$$

$$u_2(x) = x + \left(\frac{x^3}{3} - \frac{1}{30}x^5 - \frac{x^2}{2} + \frac{1}{24}x^5 \right]$$

$$u_2(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$u_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

The solution $u(x)$ in a series:

$$u(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Which is the exact solution

Example (3.2)

$u(x)$= cos x + $\frac{1}{2}\int_0^{\pi/2} \sin x\, u(t)dt$

We can assume an initial approximation

$u_0(x)$ = cos x

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u\ , p) = u(x) - u_0(x) = p[\cos x\ + \frac{1}{2}\int_0^{\pi/2} \sin x\, u(t)dt]$

Then, substituting the initial condition, we have

$u(x)$– cos x= $p[\cos x\ + \frac{1}{2}\int_0^{\pi/2} \sin x\, u(t)dt]$

Substituting $u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ we have

$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$ –cos x =

$p[\cos x + \frac{1}{2}\int_0^{\pi/2} \sin x\, (p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(t)\, dt\ ]$

To find u_0 :

$p^0: u_0(x) - \cos x = 0$

$u_0(x) = \cos x$

To find u_1 :

$p^1: u_1(x) = \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x \, u_0(t)dt$

$= \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x \, (\cos t)dt$

$u_1(x) = \cos x + \frac{1}{2}\sin x \, (\sin t)\Big|_0^{\pi/2}$

$u_1(x) = \cos x + \frac{1}{2}\left[\sin x \sin \frac{\pi}{2}\right]$

$u_1(x) = \cos x + \frac{1}{2}\sin x$

$u_1(x) = \cos x + \frac{2^1 - 1}{2^1}\sin x$

To find u_2 :

$p^2: u_2(x) = \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x \, u_1(t)dt$

$= \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x\left(\cos t + \frac{1}{2}\sin t\right)dt$

$u_2(x) = \cos x + \frac{1}{2}\sin x(\sin t - \frac{1}{2}\cos t)\Big|_0^{\pi/2}$

$u_2(x) = \cos x + \frac{1}{2}\left[\sin x[(1-0) - (0 - \frac{1}{2})]\right]$

$u_2(x) = \cos x + \frac{1}{2}\frac{3}{2}\left[\sin x\right]$

$u_2(x) = \cos x + \frac{3}{4}\sin x$

$u_2(x) = \cos x + \frac{2^2 - 1}{2^2}\sin x$

To find u_3 :

$p^3: u_3(x) = \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x \, u_2(t)dt$

$= \cos x + \frac{1}{2}\int_0^{\pi/2} \sin x \left(\cos t + \frac{3}{4} \sin t\right) dt$

$u_3(x) = \cos x + \frac{1}{2} \left[\sin x(\sin t - \frac{3}{4} \cos t)\right]_0^{\pi/2}$

$u_3(x) = \cos x + \frac{1}{2} \left[\sin x \, [(1 - 0) - (0 - \frac{3}{4})]\right]$

$u_3(x) = \cos x + \frac{1}{2}\frac{7}{4}\left[\sin x\right]$

$u_2(x) = \cos x + \frac{7}{8} \sin x$

$u_3(x) = \cos x + \frac{2^3-1}{2^3} \sin x$

Then, The exact solution is :

$u(x) = \cos x + \lim_{n\to\infty} \left(\frac{2^n-1}{2^n} \right) \sin x = \cos x + \sin x$

Example (3.3)

$u(x) = x + \int_0^1 (x\,t^2 + x^2 t) \, u(t) \, dt$

With $u_0(x) = x$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u , p) = u(x) - u_0(x) = p[x + \int_0^1 (x\,t^2 + x^2 t) \, u(t) \, dt]$

Then, substituting the initial condition, we have

$u(x) - x = p[x + \int_0^1 (x\,t^2 + x^2 t) \, u(t) \, dt]$

Substituting $u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)$ we have

$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(x) - x =$

$p[x + \int_0^1 (x\,t^2 + x^2 t)(p^0 u_0 \; + \; p^1 u_1 \; + \; p^2 u_2 \; + \cdots)(t)\; dt \;]$

To find u_0 :

$p^0 : u_0(x) - \; x = 0$

$$u_0(x) = x$$

To find u_1 :

$p^1 : u_1(x) = \; x + \int_0^1 (x\,t^2 + x^2 t)\; u_0(t)\; dt$

$= x + \; \int_0^1 (x\,t^2 + x^2 t)(t)\; dt$

$u_1(x) = x + \; \int_0^1 \; x\,t^3 + x^2 t^2 dt$

$u_1(x) = x + \left(x\frac{t^4}{4} + x^2 \; \frac{t^3}{3} \; \right]_0^1$

$u_1(x) = \; x + \frac{x}{4} + \frac{x^2}{3}$

$u_1(x) = \frac{5}{4}\; x + \frac{1}{3}\; x^2$

To find u_2 :

$p^2 : u_2(x) = \; x + \int_0^1 (x\,t^2 + x^2 t)\; u_1(t)\; dt$

$= x + \; \int_0^1 (x\,t^2 + x^2 t)(\frac{5}{4}\; t \; + \frac{1}{3}\; t^2 \;)\; dt$

$u_2(x) = x + \; \int_0^1 \frac{5}{4}\; x\,t^3 + \frac{1}{3} x t^4 + \frac{5}{4} x^2\; t^2 \; + \frac{1}{3} x^2\; t^3\; dt$

$u_2(x) = x + \left(\frac{5}{16} x t^4 + \frac{1}{15} x\; t^5 \; + \frac{5}{12} x^2\; t^3 \; + \frac{1}{12} x^2\; t^4 \qquad \right]_0^1$

$u_2(x) = \; x + \frac{5}{16} x + \frac{1}{15} x + \frac{5}{12} x^2 + \frac{1}{12} x^2$

$u_2(x) = \frac{331}{240}\; x + \frac{1}{2}\; x^2$

To find u_3 :

$p^3 : u_3(x) = \; x + \int_0^1 (x\,t^2 + x^2 t)\; u_2(t)\; dt$

$= x + \int_0^1 (x\,t^2 + x^2 t)(\frac{331}{240}\ t + \frac{1}{2}\ t^2)\ dt$

$u_3(x) = x + \int_0^1 \frac{331}{240}\ x\,t^3 + \frac{1}{2} x t^4 + \frac{331}{240} x^2\ t^2\ + \frac{1}{2} x^2\ t^3\ dt$

$u_3(x) = x + \left[\frac{331}{960} x t^4 + \frac{1}{10} x\ t^5\ + \frac{331}{720} x^2\ t^3\ + \frac{1}{8} x^2\ t^4\ \right]_0^1$

$u_3(x) = x + \frac{331}{960} x + \frac{1}{10} x + \frac{331}{720} x^2 + \frac{1}{8} x^2$

$u_3(x) = \frac{1387}{960}\ x + \frac{421}{720}\ x^2$

The required solution is:

$u(x) = \frac{20}{119}\ (9x + 4\ x^2)$

Example (3.4)

$u(x) = x + \lambda \int_0^1 xt\,u^2(t)\ dt \qquad 0 \leq \lambda \leq 1$

We begin with $u_0(x) = x$

Solution:

By homotopy technique we construct a homotopy which is solutions

$H(u\ , p) = u(x) – u_0(x) = p[x\ + \lambda \int_0^1 xt\,u^2(t)\ dt]$

Then, substituting the initial condition, we have

$u(x) – x = p[x\ + \lambda \int_0^1 xt\,u^2(t)\ dt]$

Substituting $\quad u = (p^0 u_0\ +\ p^1 u_1\ +\ p^2 u_2\ + \cdots) \quad$ we have

$(p^0 u_0\ +\ p^1 u_1\ +\ p^2 u_2\ + \cdots)(x) – x =$

$p[x + \lambda \int_0^1\ x\,t(p^0 u_0\ +\ p^1 u_1\ +\ p^2 u_2\ + \cdots)^2(t)\ dt]$

To find u_0 :

$p^0 : u_0(x) -\ x = 0$

$$u_0(x) = x$$

To find u_1 :

$p^1: u_1(x) = x + \lambda \int_0^1 xt\, u_0{}^2(t)dt$

$= x + \lambda \int_0^1 xt\,(t^2)dt$

$= x + \lambda \int_0^1 x\, t^3\, dt$

$u_1(x) = x + \lambda \left[x \frac{t^4}{4}\right]_0^1$

$u_1(x) = x + \lambda \left[\frac{1}{4} x - 0\right]$

$$u_1(x) = x + \frac{1}{4}\lambda\, x$$

$$u_1(x) = (1 + \frac{1}{4}\lambda)\, x$$

To find u_2 :

$p^2: u_2(x) = x + \lambda \int_0^1 xt\, u_1{}^2(t)dt$

$= x + \lambda \int_0^1 xt\,(t + \frac{1}{4}\lambda t)^2\, dt$

$= x + \lambda \int_0^1 xt\,(t^2 + \frac{1}{2}\lambda\, t^2 + \frac{1}{16}\lambda^2\, t^2)dt$

$u_2(x) = x + \lambda \int_0^1 (xt^3 + \frac{1}{2}\lambda\, xt^3 + \frac{1}{16}\lambda^2\, xt^3)dt$

$u_2(x) = x + \lambda x\left[\frac{t^4}{4} + \frac{1}{8}\lambda\, xt^4 + \frac{1}{64}\lambda^2\, xt^4 \right]_0^1$

$u_2(x) = x + \lambda \left(\frac{1}{4}x + \frac{1}{8}\lambda x + \frac{1}{64}\lambda^2 x \right)$

$$u_2(x) = (1 + \frac{1}{4}\lambda + \frac{1}{8}\lambda^2 + \frac{1}{64}\lambda^3)\, x$$

The exact solution is:

$u(x) = \frac{2}{\lambda}\,(1 - \sqrt{1-\lambda}\,)\, x$

Example (3.5)

$u'' = 1 + xe^{-x} - \int_0^x e^{x-t}\, u(t)dt$

Where

$u(0) = 0$, $u'(0) = 1$

Solution:

Suppose the initial approximation in the form:

$u_0\,(x) = a + b\, e^x$

$u_1\,(x) = u_0\,(x) - \int_0^x e^{x-t}\, u_0(t)dt$

$u_1\,(x) = a + b\, e^x - \int_0^x e^{x-t}(a + b\, e^x)\, dt$

$= a + b\, e^x - \int_0^x (ae^{x-t} + b\, e^x)\, dt$

$u_1\,(x) = a + b\, e^x - \left[- ae^{x-t} + bt\, e^x\right]_0^x$

$u_1\,(x) = a + b\, e^x - [- ae^0 + bx\, e^x - (- ae^x + 0)]$

$= a + b\, e^x + a - bx\, e^x - ae^x$

$u_1\,(x) = 2a + (b - a)e^x - bx\, e^x$

$u_1\,(0) = 2a + (b - a)e^0 - 0$

$0 \quad = 2a + b - a$

$a + b = 0$ -- (1)

$u_1{}'(x) = (b - a)e^x - b\,[xe^x + e^x]$

$u_1{}'(0) = (b - a)e^0 - b\,[0 + e^0]$

$u_1{}'(0) = b - a - b$

$1 = - a$, $a = - 1$

From (1):

$a + b = 0$ ------------------------ (1)

$-1 + b = 0$, $b = 0$

$u_0\ (x) = a + be^x$

Then

$u_0\ (x) = -1\ + e^x$

$u''(x) = 1 + xe^x - \int_0^x e^{x-t}\, u(t)dt$

By homotopy technique we construct a homotopy which is solutions

$H(u\ , p) = u''(x) - u_0{}''(x) = p[1 + xe^x\ - \int_0^x e^{x-t}\, u(t)dt]$

Then, substituting the initial condition, we have

$u''(x) - (e^x - 1)''(x) = p[1 + xe^x\ - \int_0^x e^{x-t}\, u(t)d]$

Then we have:

$u''(x) - e^x = p[1 + xe^x\ - \int_0^x e^{x-t}\, u(t)d]$

Substituting $u = (p^0u_0\ +\ p^1u_1\ +\ p^2u_2\ + \cdots)$ we have

$(p^0u_0\ +\ p^1u_1\ +\ p^2u_2\ + \cdots)''(x) - e^x =$

$p[1 + xe^x\ - \int_0^x e^{x-t}\, u(t)dt]$

To find u_0 :

p^0: $u_0{}''(x) - e^x = 0$

$$u_0{}''(x) = e^x$$

$u_0{}'(x) = e^x + C_1(x), u_0{}'(0) = 1$

$1 = 1 + C_1(x) \rightarrow C_1(x) = 0$

$u_0{}'(x) = e^x$

$u_0(x) = e^x + C_2(x), u_0(x) = 0$

$0 = 1 + C_2(x) \rightarrow C_2(x) = -1$

$$u_0 = e^x - 1$$

To find u_1 :

$p^1: u_1''(x) = 1 + xe^x - \int_0^x e^{x-t} u_0(t)dt$

$= 1 + xe^x - \int_0^x (e^{x-t}(e^t - 1)dt$

$= 1 + xe^x - \int_0^x (e^x - e^{x-t}) dt$

$u_1''(x) = 1 + xe^x - \left[te^x + e^{x-t}\right]_0^x$

$u_1''(x) = 1 + xe^x - [(xe^x + e^0) - (0 + e^x)]$

$u_1''(x) = 1 + xe^x - xe^x - 1 + e^x$

$$u_1''(x) = e^x$$

$u_1''(x) = e^x + C_1(x), u_1'(0) = 1$

$1 = 1 + C_1(x) \rightarrow C_1(x) = 0$

$u_1'(x) = e^x$

$u_1(x) = e^x + C_2(x), u_1(0) = 0$

$0 = 1 + C_2(x) \rightarrow C_2(x) = -1$

$u_1(x) = e^x - 1$

The exact solution:

$u(x) = e^x - 1$

Conclusion:

In this thesis, we have applied the homotopy perturbation method (HPM) to find the analytic solution for some linear and nonlinear differential equations, advection equations and some integral equations. The proposed method is applied without using linearization, discretization or restrictive assumptions. It may be concluded that the (HPM) is very powerful and efficient in finding the analytic solution for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in the problems that studied in this thesis.

References:

[1] J. H. He, Homotopy Perturbation technique, Comput. Methods Appl. Mech. Eng. 178(1999)257 – 262.

[2] J. H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems, Int J. Non-Linear Mech. 35 (1) (2000) 37-43

[3] M. El-Shahed, Application of He's homotopy perturbation method to Volterra's integro-differential equation, Int. J. Nonlinear Sci. Numer. Simulat. 6(2) (2005) 163-168.

[4] J. H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Appl. Math. Comput. 151(2004)287-292.

[5] J. H. He, Homotopy perturbation method for bifurcation of nonlinear problems, Int. J. Nonlinear Sci. Numer. Sumilat. 6(2) (2005) 207- 208.

[6] J. H. He, Periodic solutions and bifurcation of delay-differential equations, Phys. Lett. A 374(4-6) (2005) 228-230.

[7] J. H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons Fractals 26(3) (2005) 695 – 700.

[8] J. H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350(1-2) (2006) 87-88.

[9] Z. Odibat, S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, Chaos Solitons Fractals, in press.

[10] J. H. He, Homotopy perturbation method: a new nonlinear analytic technique, Appl. Math. Comput. 135(2003)73-79.

[11] J. H. He, Comparison of homotopy perturbation method and homotopy analysis method, Appl. Math. Comput. 156 (2004) 527-539.

[12] J. H. He, Asymptotology by homotopy perturbation method, Appl. Math. Comput. 156 (2004) 591-596.

[13] J. H. He, Limit cycle and bifurcation of nonlinear problems, Chaos Solitons Fractals 26(3) (2005) 827 – 833.

[14] A. Siddiqui, R. Mahmood, Q. Ghori, Thin film flow of a third grade fluid on moving a belt by He's homotopy perturbation method, Int. J. Nonlinear Sci. Numer. Simulat. 7(1) (2006) 7 – 14.

[15] A. Siddiqui, M. Ahmed, Q. Ghori, Couette and Poiseuille flows for non-Newtonian fluids, Int. J. Nonlinear Sci. Numer. Simulat. 7(1) (2006) 15 – 26.

[16] J. H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Mod. Phys. B 20(10) (2006) 1141 – 1199.

[17] S. Abbasbandy, Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method,Appl. Math. Comput. 172 (2006) 485-490.

[18] S. Abbasbandy, Numerical solutions of the integral equations: Homotopy perturbation method and Adomian's decomposition method, Appl. Math. Comput. 173 (2006) 493 – 500.

[19] M. A. Jafari and A. Aminataei, "Improved homotopy perturbation method, " *International mathematical Forum*, vol. 5, no. 29 – 32, pp. 1567-1579, 2010.

[20] A. M. Wazwaz. A comparison between the variational iteration method and adomian decomposition method. Journal of Computational and Applied Mathematics 207(2007) 129 – 136.

[21] Y. Khan, F. Austin, Application of the Lapiace decomposition method to nonlinear homogeneous and non-homogeneous advection equations, Zeitschrifuer Naturforschung 65a (2010)1 – 5.

[22] A. M Wazwaz, Partial differential equations methods and application, Netherland Balkema Publisher. 2002.