# Chapter Two

# Laplace Transform

In this chapter, we will review only the basic concepts of the Laplace transform method. The details can be found in any text of ordinary differential equations. The Laplace transform method is a powerful tool used for solving differential and integral equations. The Laplace transform changes differential equations and integral equations to polynomial equations that can be easily solved, and hence by using the inverse Laplace transform gives the solution of the examined equation.

## Sec (2.1): Definition of Laplace Transform

**Definition (2.1.1)[13]:** Let $F(x)$ be defined for $x \geq 0$. The Laplace transform of $F(x)$, denoted by $F(s)$ or $L\{f(x)\}$, is an integral transform given by the Laplace integral

$$F(s) = L\{f(x)\} = \int_0^\infty e^{-sx} f(x)dx \qquad (1)$$

Where $s$ is real, and $L$ is called the Laplace transform operator.
The Laplace transform is an operation that transforms a function of t (i.e., a function of time domain), defined on $[0,\infty)$, to a function of s (i.e., of frequency domain) . $F(s)$, is the Laplace transform, or simply transform, of $f(x)$. Together the two functions $f(x)$ and $F(s)$ are called a Laplace transform pair. For functions of t continuous on $[0,\infty)$, the above transformation in the frequency domain is one-to-one. That is, different continuous functions will have different transforms.
**Theorem (2.1.2) [13]:** Suppose that:
1. $f$, is piecewise continuous on the interval $0 \leq x \leq A$ for any $A > 0$.
2. $|f(x)| \leq Ke^{ax}$, when $x \geq M$, for any real constant a, and some positive constants $K$ and $M$. (This means that f is “of exponential order”, i.e. its rate of growth is no faster than that of exponential functions.) Then the Laplace transform, $F(s) = L\{f(x)\}$, exists for $s > a$.

## The Convolution Theorem for Laplace Transform

This is an important theorem that will be used in solving integral equations.

The kernel $K(x,t)$ of the integral equation:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{2}$$

is termed difference kernel if it depends on the difference $x - t$ examples of the different kernels are $e^{x-t}, \sin(x-t)$ , and $\cosh(x-t)$. The integral equation (2) can be expressed as

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x-t)u(t)dt, \tag{3}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms of the functions $f_1(x)$ and $f_2(x)$ be given by

$$\begin{aligned} \ell\{f_1(x)\} &= F_1(s), \\ \ell\{f_2(x)\} &= F_2(s), \end{aligned} \tag{4}$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{5}$$

Or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt, \tag{6}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x) \tag{7}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$, is given by:

$$\ell\{(f_1 * f_2)(x)\} = \ell\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(\mathrm{s}). \tag{8}$$

**Example (2.1.3) [13]:**

Find the Laplace transform of

$$x^2 + \int_0^x e^{x-t} y(t) dt \tag{9}$$

**Solution:**

Notice that the kernel depends on the difference $x-t$ . The integral includes $f_1(x) = e^x$ and $f_2(x) = y(x)$. The integral is the convolution product. This means that if we take Laplace transform of each term we obtain

$$\ell\left[x^2\right] + \ell\left[\int_0^x e^{x-t} y(t) dt\right] = \ell\left[x^2\right] \ell\left[e^x\right] \ell\left[y(t)\right] \tag{10}$$

Using the table of Laplace transforms gives

$$\frac{2}{s^3} + \frac{1}{s-1} Y(s)$$

**Example (2.1.3) [13]:**

Find the Laplace transform of

$$xe^x = \int_0^x e^{x-t} y(t) dt \tag{11}$$

**Solution:**

Notice that $f_1(x) = e^x$ and $f_2(x) = y(x)$. The right hand side is the convolution product $(f_1 * f_2)(x)$. This means that if we take Laplace transforms of both sides we obtain

$$\ell\left[xe^x\right] = \ell\left[\int_0^x e^{x-t} y(t) dt\right] = \ell\left[e^x\right] \ell\left[y(t)\right] \tag{12}$$

Using the table of Laplace transforms gives

$$\frac{1}{(s-1)^2} + \frac{1}{s-1} Y(s) \tag{13}$$

That gives

$$Y(s) = \frac{1}{s-1} \tag{14}$$

From this we find the solution is

$$y(x) = \ell^{-1}\left\{\frac{1}{s-1}\right\} = e^{x}$$

## Sec (2.2): Laplace Transform Variational Iteration Method

Consider the following general nonlinear differential equation:

$$Ly(x) + Ny(x) = f(x) \tag{15}$$

where $L$ is a linear operator, $N$ is a nonlinear operator and $f(x)$ is a known analytical function. Before we begin the implementation, we shall present the variational iteration method scheme in constructing the correction functional.

The (VIM) admits the use of the correction functional for Eq. (15) given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left[Ly_n(\xi) + \mathrm{N}\tilde{y}_n(\xi) - \mathrm{f}(\xi)\right]d\xi,\ n = 0,1,2,... \tag{16}$$

where $\lambda$ is a general Lagrange multiplier, which can be identified optimally via the variational theory. The subscript $n$ denotes, the $n$th approximation and $\tilde{y}_n$ is a restricted variation $(\delta\tilde{y}_n = 0)$. In a will of problems that appear in the literature, the general form of Lagrange multiplier is found to be of the form

$$\lambda = \bar{\lambda}(x - \xi)$$

In this section, we will make the assumption that $\lambda$ is expressed in this latter way. In such a case, the integration is, basically, the convolution; hence Laplace transform is appropriate to use. Operating with Laplace transform of both sides of (16) the correction, functional will be constructed in the following manner:

$$\ell\left[y_{n+1}(x)\right] = \ell\left[y_n(x)\right] + \ell\left[\int_0^x \bar{\lambda}(\mathrm{x}-\xi)\left[Ly_n(\xi) + \mathrm{N}\tilde{y}_n(\xi) - \mathrm{f}(\xi)\right]d\xi\right],\ n = 0,1,2,...$$

Therefore

$$\ell\left[y_{n+1}(x)\right] = \ell\left[y_n\right] + \ell\left[\bar{\lambda}(x) * \left(Ly_n(x) + N\tilde{y}_n(x) - f(x)\right)\right]$$

$$= \ell\left[y_n(x)\right] + \ell\left[\bar{\lambda}(x)\right]\ell\left[Ly_n(x) + N\tilde{y}_n(x) - f(x)\right]. \tag{17}$$

To find the optimal value of $\bar{\lambda}(x - \xi)$ we first take the variation with respect to $y_n(x)$. Thus

$$\frac{\delta}{\delta y_n}\ell\left[y_{n+1}(x)\right] = \frac{\delta}{\delta y_n}\ell\left[y_n(x)\right] + \frac{\delta}{\delta y_n}\ell\left[\bar{\lambda}(x)\right]\ell\left[Ly_n(x) + N\tilde{y}_n(x) - f(x)\right], \tag{18}$$

and hence upon applying the variation this simplifies to

$$\ell\left[\delta y_{n+1}\right] = \ell\left[\delta y_n\right] + \delta\ell\left[\bar{\lambda}\right]\ell\left[y_n\right]. \tag{19}$$

We assume that $L$ is a linear differential operator with constant coefficients contain given by

$$L(y) \equiv a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_2 y'' + a_1 y' + a_0 y, \tag{20}$$

where $a_j$'s are constants. It is important to note that if the coefficients contain only non-constant terms of the form $\chi^k$, then the Laplace variation approach is still valid.

The Laplace transform of the operator $L$ is given by

$$\ell\left[a_n y^{(n)}\right] = a_n s^n \ell[y] - a_n \sum_{k=1}^{n} s^{k-1} y^{(n-k)}(0), \tag{21}$$

so the variation with respect to $y$ is

$$(22)\, \delta\ell\left[a_n y^{(n)}\right] = a_n s^n \ell[\delta y].$$

The other term in the operator $L$, namely $a_{n-1} y^{(n-1)}, \dots, a_1 y' + a_0 y$ yields similar results. Hence, using (22), Eq. (19) reduces to

$$(23)\, \ell[\delta y_{n+1}] = \ell[\delta y_n] + \ell\left[\bar{\lambda}\right]\left(\sum_{k=0}^{n} a_k s^k\right)\ell[\delta y_n] = \left[1 + \ell\left[\bar{\lambda}\right]\left(\sum_{k=0}^{n} a_k s^k\right)\right]\ell[\delta y_n]$$

The extremum condition of $y_{n+1}$ requires that $\delta y_{n+1} = 0$. This means that the right-hand side of Eq. (23) should be set to zero. Hence, we have the stationary condition

$$\ell\left[\bar{\lambda}\right] = -\frac{1}{\sum_{k=0}^{n} a_k s^k} \tag{24}$$

Taking the Laplace inverse of the last equation gives the optimal value of $\bar{\lambda}$. For this value , we have the following iteration formulation:

$$\ell[y_{n+1}(x)] = \ell[y_n(x)] + \ell\left[\int_0^x \bar{\lambda}(x-\xi)[Ly_n(\xi) + Ny_n(\xi) - f(\xi)]d\xi\right],$$

where $n \geq 0$.

**Example (2.2.1) [14]:**

Consider the boundary-value problem

$$y'' + (y')^2 + y^2 = 1 - \sin x, \qquad (25)$$
$$y(0) = 0, y'(0) = 1$$

**Solution:**

The Laplace variational iteration functional will be constructed in the following manner:

$$\ell[y_{n+1}(x)] = \ell[y_n(x)] + \ell\left[\int_0^x \bar{\lambda}(x-\xi)\left(y_n''(\xi) + (y_n')^2 + y_n^2 - 1 + \sin x\right)d\xi\right], \qquad (26)$$

or equivalently, upon applying the properties of Laplace transform, we have

$$\ell[y_{n+1}(x)] = \ell[y_n(x)] + \ell\left[\bar{\lambda}(x) * \left(\left[y_n''(x) + (y_n'(\mathrm{x}))^2 + y_n^2(x) - 1 + \mathrm{sinx}\right]\right)\right]$$

$$= \ell[y_n(x)] + \ell\left[\bar{\lambda}(x)\right]\ell\left[y_n''(x) + (y_n'(\mathrm{x}))^2 + y_n^2(x) - 1 + \mathrm{sinx}\right]$$
$$= \ell[y_n(x)] + \ell\left[\bar{\lambda}(x)\right]\left(s^2\ell(y_n(x)) - sy_n(0) - y_n'(0) + \ell(y_n'(\mathrm{x}))^2 + \ell(y_n^2(x)) - \frac{1}{s} + \frac{1}{s^2+1}\right)$$

Taking the variation with respect to $y_n(x)$ on both sides of the latter equation, leads to

$$\frac{\delta}{\delta y_n}\ell[y_{n+1}(x)] = \frac{\delta}{\delta y_n}\ell[y_n(x)] + \frac{\delta}{\delta y_n}\ell\left[\bar{\lambda}(x)\right]\left(s^2\ell(y_n(x)) - 1 + \ell(y_n'(\mathrm{x}))^2 + \ell(y_n^2(x)) - \frac{1}{s} + \frac{1}{s^2+1}\right),$$

and upon simplification we get

$$\ell[\delta y_{n+1}] = \ell[\delta y_n] + \ell\left[\bar{\lambda}\right]\left(s^2\ell[\delta y_n]\right) = \ell[\delta y_n]\left(1 + s^2\ell\left[\bar{\lambda}\right]\right) \qquad (27)$$

The extremum condition of $y_{n+1}$ requires that $\delta y_{n+1} = 0$. This means that the right-hand side of Eq. (27) should be set to zero. Hence, we have $1 + \ell\left[\bar{\lambda}\right]s^2 = 0$, that is, $\ell\left[\bar{\lambda}\right] = -\frac{1}{s^2}$ . Therefore:

$$\bar{\lambda}(x) = -x \qquad (28)$$

Substituting Eq. (28) into Eq. (26) result in the following iterative scheme

$$\ell[y_{n+1}(x)] = \ell[y_n(x)] - \ell\left[\int_0^x (x-\xi)\left(y_n''(\xi) + (y_n')^2(\xi) + y_n^2(\xi) - 1 + \sin\xi\right)d\xi\right] \quad (29)$$
$$= \ell[y_n(x)] - \ell[x]\,\ell\left[y_n''(x) + (y_n')^2(x) + y_n^2(x) - 1 + \sin x\right]$$

Let $y_0 = \sin x$

$$\begin{aligned} \ell[y_1] &= \ell[y_0] - \ell[x]\,\ell\left[y_0'' + y_0'^2 + y_0^2 - 1 + \sin x\right] \\ &= \ell[\sin x] - \ell[x]\,\ell\left[-\sin x + \cos x^2 + \sin x^2 - 1 + \sin x\right] \\ &= \ell[\sin x] \\ &= \frac{1}{s^2+1} \end{aligned} \quad (30)$$

Inverse Laplace transform yields

$$y_1 = \sin x$$

And so on, then the exact solution

$$y = \sin x$$

**Example (2.2.2) [14]:**

Consider the boundary-value problem

$$y' = -1 + xy + y^2, y(0) = 0$$

**Solution:**

The Laplace variational iteration  correction, functional is expressed as:

$$\ell[y_{n+1}(x)] = \ell[y_n(x)] + \ell\left[\int_0^x \bar{\lambda}(x-\xi)\left(y_n'(\xi) + 1 - \xi\, y(\xi) - y^2(\xi)\right)d\xi\right], \quad (31)$$

Appling the Laplace transform, Eq. (31) becomes

$$\ell[y_{n+1}(x)] = \ell[y_n(x)] + \ell\left[\bar{\lambda}(x) * \left(y' + 1 - xy - y^2\right)\right]$$

$$= \ell\left[\bar{\lambda}(x)\right]\left(s\,\ell[y_n(x)] - y_n(0) - \frac{1}{s} - \ell[xy_n] - \ell\left[y_n^2\right]\right) \quad (32)$$

Taking the variation with respect to $y_n(x)$ and making the above correction, functional stationary, noting that $\delta\tilde{y}_n = 0,$ we have:

$$\ell[\delta y_{n+1}]=\ell[\delta y_n]\left(1+s\,\ell\left[\bar{\lambda}\right]\right)=0. \tag{33}$$

This implies that $\ell\left[\bar{\lambda}\right]=-\frac{1}{s}$ ; hence

$$\bar{\lambda}(x)=-1 \tag{34}$$

Substituting Eq. (34) into Eq. (31), we get:

$$\ell[y_{n+1}(x)]=\ell[y_n(x)]-\ell\left[\int_0^x\left(y_n'(\xi)+1-\xi y_n(\xi)-y_n^2\right)d\xi\right]$$

$$\ell[y_{n+1}(x)]=\ell[y_n(x)]-\ell[1]\ell\left[y_n'(x)+1-\mathrm{xy}_n(x)-y_n^2(x)\right]. \tag{35}$$

Let $y_0=y(0)=0.$ Then form the scheme (35), the first iteration is

$$\ell[y_1]=\ell[0]-\ell[1]\ell[1]=-\frac{1}{s}\left(\frac{1}{s}\right)=-\frac{1}{s^2}$$

Take inverse Laplace

$$y_1=-x$$

$$\ell[y_2]=\ell[-x]-\ell[1]\ell\left[-1+1-x^2+x^2\right]=-\frac{1}{s^2}$$

$$y_2=-x$$

And so on , then the exact solution is:

$$y=-x$$

## Sec (2.3): Solution of Differential Equations of Lane-Emden Type by Combining Integral Transform and Variational Iteration Method

In this section, we introduce a dependable joined the modified Laplace Transform and the new modified variational iteration method to solve some nonlinear differential equations for Lane-Emden type. This method may be efficient and not difficult.

The basic definition of the modification of Laplace Transform is given as below:

the transform of the function $f(t)$ is

$$\ell[f(t)] = \int_0^\infty e^{-st} f(t) dt \tag{36}$$

let us consider the following general differential equation

$$L[u(x,t)] + N[u(x,t)] = g(x,t), u(x,0) = h(x), \tag{37}$$

where $L$ is a linear operator of the first order, $N$ is a nonlinear operator and $g(x,t)$ is inhomogeneous term. According to the variational iteration method, we can construct a correction functional as

$$u_{n+1} = u_n + \int_0^t \lambda [Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s)] ds, \tag{38}$$

where $\lambda$ is a Lagrange multiplier $(\lambda = -1)$, the subscript n denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

The successive approximation $u_{n+1}$ of the solution $u$ will be readily obtained upon using the determined Lagrange multiplier and any selective function $u_0$.

Consequently, the solution is given by $u = \lim_{n\to\infty} u_n$. In this section, we assume that L is an operator of the first order $\frac{\partial}{\partial t}$ in the equation (37).

Let us take the modified Laplace Transform of both sides and apply the differentiation property of new transform. Then we get

$$\ell\left[Lu(x,t)\right]+\ell\left[Nu(x,t)\right]=\ell[g(x,t)] \tag{38}$$

and

$$\ell\left[u(x,t)\right]=\frac{1}{s}\ell[g(x,t)]+\frac{1}{s}h(x)-\frac{1}{s}\ell\left[Nu(x,t)\right], \tag{39}$$

Applying the inverse of modified Laplace Transform of both sides of the equation, we have

$$u(x,t)=G(x,t)-\ell^{-1}\left[\frac{1}{s}\ell\left(N\,(u(x,t))\right)\right], \tag{40}$$

where $G(x,\mathrm{t})$ represents the terms arising from the source term and the prescribed initial condition. Taking the first partial derivative with respect to $t$, we have

$$u_t(x,t)-\frac{\partial}{\partial t}G(x,t)+\frac{\partial}{\partial t}\ell^{-1}\left[\frac{1}{s}\ell\left(N\,(u(x,t))\right)\right]=0. \tag{41}$$

Or, alternatively

$$u_{n+1}(x,t)=G(x,t)-\ell^{-1}\left[\frac{1}{s}\ell\left(N\,(u_n(x,t))\right)\right], \tag{42}$$

Thus, we can obtain the solution u by

$$u(x,t)=\lim_{n\to\infty}u_n(x,t) \tag{43}$$

## Illustrative examples

In this section, we solve some examples of nonlinear differential equations of Lane-Emden type by using the modified Laplace Transform variational iteration method.

**Example (2.3.1) [20]:** (The isothermal gas sphere equation) The isothermal gas sphere equation is

$$y''+\frac{2}{x}y'+e^{y}=0 \tag{44}$$

subject to the boundary conditions, $y(0)=0, y'(0)=0$ . This model can be used to view the isothermal gas spheres, where the temperature remains constant.

**Solution:**

Now, taking the transform on the given equation, we have:

$$\ell[y'']+\ell\left[\frac{2}{x}y'+e^{y}\right]=0$$

$$s^{2}\ell[y(x)]=-\ell\left[\frac{2}{x}y'+e^{y}\right] \tag{45}$$

$$\ell[y(x)]=\frac{-1}{s^{2}}\ell\left[\frac{2}{x}y'+e^{y}\right]$$

taking the inverse to obtain,

$$y(x)=\ell^{-1}\left[\frac{-1}{s^{2}}\ell\left[\frac{2}{x}y'+e^{y}\right]\right] \tag{46}$$

According to the equation (42), the correction function is given by

$$y_{n+1}(x)=\ell^{-1}\left[\frac{-1}{s^{2}}\ell\left[\frac{2}{x}y'_{n}+e^{y_{n}}\right]\right] \tag{47}$$

Now let us apply the modified Sumudu transform variational iteration method, the solution in the series is given by

$$y_0=0,$$

$$y_1=\ell^{-1}\left[-\frac{1}{s^2}\ell\left(\frac{2}{x}y_0'+e^{y_0}\right)\right]=\ell^{-1}\left[-\frac{1}{s^2}\ell(1)\right]=-\frac{1}{2}x^2,$$

$$y_2=\ell^{-1}\left[-\frac{1}{s^2}\ell\left(\frac{2}{x}y_1'+e^{y_1}\right)\right]=\ell^{-1}\left[-\frac{1}{s^2}\ell\left(-2+e^{-\frac{1}{2}x^2}\right)\right]$$

$$=\ell^{-1}\left[-\frac{1}{s^2}\ell\left(-1-\frac{1}{2!}x^2+\frac{3}{4!}x^4-\frac{15}{6!}\mathrm{x}^6+\ldots\right)\right]$$

$$=\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{240}x^6+\frac{1}{2688}x^8+\ldots$$

Continue this process, and we can obtain the solution in the form:

$$y(x)\cong-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{240}x^{6}+\frac{1}{2688}x^{8}+\ldots \tag{48}$$

**Example (2.3.2) [20]:** The Emden-Fowler type equations

$$y'' + \frac{2}{x} y' + \sin(y) = 0, (x > 0) \tag{49}$$

subject to the boundary conditions, $y(0) = 1, y'(0) = 0$ .

**Solution:**

Now, taking the transform on the given equation, we have

$$\ell[y''] + \ell\left[\frac{2}{x} y' + \sin y\right] = 0$$

$$s^2 \ell[y(x)] - s = -\ell\left[\frac{2}{x} y' + \sin y\right] \tag{50}$$

$$\ell[y(x)] = \frac{1}{s} - \frac{1}{s^2} \ell\left[\frac{2}{x} y' + \sin y\right]$$

taking the inverse to obtain

$$y(x) = 1 - \ell^{-1}\left[\frac{1}{s^2} \ell\left[\frac{2}{x} y' + \sin y\right]\right] \tag{51}$$

According to the equation (51), the correction function is given by

$$y_{n+1}(x) = 1 - \ell^{-1}\left[\frac{1}{s^2} \ell\left[\frac{2}{x} y'_n + \sin y_n\right]\right] \tag{52}$$

Now let us apply the modified Laplace transform variational iteration method, the solution in the series is given by

$$y_0 = 1,$$

$$y_1 = 1 - \ell^{-1}\left[\frac{1}{s^2} \ell\left(\frac{2}{x} y'_0 + \sin y_0\right)\right] = 1 - \ell^{-1}\left[\frac{1}{s^2} \ell\left(\sin(1)\right)\right] = 1 - \frac{k_1}{2} \mathrm{x}^2, \mathrm{k}_1 = \sin(1).$$

$$y_2 = 1 - \ell^{-1}\left[-\frac{1}{s^2} \ell\left(\frac{2}{x} y'_1 + \sin y_1\right)\right] = 1 - \ell^{-1}\left[\frac{1}{s^2} \ell\left(-2k_1 + \sin(1 - \frac{k_1}{2} \mathrm{x}^2)\right)\right] \tag{53}$$

$$= 1 - \ell^{-1}\left[\frac{1}{s^2} \ell\left(-2k_1 + k_1 - \frac{k_1 k_2}{2!} x^2 + \frac{k_1^2}{3!} x^3 - \frac{k_1^3}{4!} \mathrm{x}^4 + \dots\right)\right], k_2 = \cos(1)$$

$$= 1 + \frac{k_1}{2} x^2 + \frac{k_1 k_2}{24} x^4 - \frac{k_1^2}{120} x^5 + \frac{k_1^3}{720} x^6 + \dots$$

Continue this process, we obtain the solution in the form:

$$y(x) \cong 1 + \frac{k_1}{2} x^2 + \frac{k_1 k_2}{24} x^4 - \frac{k_1^2}{120} x^5 + \frac{k_1^3}{720} x^6 + \dots \tag{54}$$

**Example (2.3.3) [20]:** The Emden-Fowler type equations

$$y''+\frac{2}{x}y'+\sinh(y)=0,(x>0) \tag{55}$$

subject to the boundary conditions $y(0)=1, y'(0)=0$ .

**Solution:**

Now, taking the transform on the given equation, we have:

$$\ell\left[y''\right]+\ell\left[\frac{2}{x}y'+\sinh y\right]=0$$

$$s^2\ell\left[y(x)\right]-s=-\ell\left[\frac{2}{x}y'+\sinh y\right] \tag{56}$$

$$\ell\left[y(x)\right]=\frac{1}{s}-\frac{1}{s^2}\ell\left[\frac{2}{x}y'+\sinh y\right]$$

then we follow the same procedure in the previous examples to get

$$y_{n+1}(x)=1-\ell^{-1}\left[\frac{1}{s^2}\ell\left[\frac{2}{x}y_n'+\sinh y_n\right]\right] \tag{57}$$

and apply the modified Laplace Transform variational iteration method, to get

$$y_0=1,$$

$$y_1=1-\ell^{-1}\left[\frac{1}{s^2}\ell\left(\frac{2}{x}y_0'+\sinh y_0\right)\right]=1-\ell^{-1}\left[\frac{1}{s^2}\ell\left(\sinh(1)\right)\right]=1-\frac{k_1}{2}\mathrm{x}^2=1-\frac{e^2-1}{4e}\mathrm{x}^2,\mathrm{k}_1=\sinh(1).$$

$$y_2=1-\ell^{-1}\left[\frac{1}{s^2}\ell\left(\frac{2}{x}y_1'+\sinh y_1\right)\right]=1-\ell^{-1}\left[\frac{1}{s^2}\ell\left(-2k_1+\sinh(1-\frac{e^2-1}{4e}\mathrm{x}^2)\right)\right]$$

$$=1-\ell^{-1}\left[\frac{1}{s^2}\ell\left(-2k_1+k_1-\frac{2k_1k_2}{2!}x^2+\frac{10k_1^3}{4!}\mathrm{x}^4+...\right)\right],k_2=\cosh(1)$$

$$=1+\frac{k_1}{2!}x^2+\frac{2k_1k_2}{24}x^4-\frac{10k_1^3}{720}x^6+...$$

$$=1+\frac{e^2-1}{4e}x^2+\frac{e^4-1}{48e^2}x^4-\frac{e^6-3e^4+3e^2-1}{576e^3}x^6+...$$

Continue this process, we obtain the solution in the form:

$$y(x)\cong 1+\frac{e^2-1}{4e}x^2+\frac{e^4-1}{48e^2}x^4-\frac{e^6-3e^4+3e^2-1}{576e^3}x^6+...$$