Chapter Three

Combined Laplace Transform and Variational Iteration Method

It is well known that there are many nonlinear differential equations which are used in the study of several fields for example, physics, mechanics, etc. The solutions of these equations can give more understanding of the described process. But because of the complexity of the nonlinear differential equations and the limitations of mathematical methods, it is difficult to obtain the exact solutions for these problems. Thus, this complexity hinders further applications of nonlinear differential equations. A broad class of analytical and numerical methods were used to handle these problems such as Backlund transformation ,Hirota's bilinear method, Darboux transformation Symmetry method, the inverse scattering transformation, the tanh method, the A domain decomposition method ,the improved Adomian decomposition method, the exp-function method and other asymptotic methods for strongly nonlinear equations. In 1978, Inokutiet al. proposed a general use of Lagrange multiplier to solve nonlinear problems, which was intended to solve problems in quantum mechanics. Subsequently, in 1999, the variational iteration method(VIM) was first proposed by Ji-Huan.

The idea of the VIM is to construct an iteration method based on a correction functional that includes a generalized Lagrange multiplier. The value of the multiplier is chosen using variational theory so that each iteration improves the accuracy of the solution.

In this chapter, we have applied the modified variational iteration method (VIM) and Laplace transform to solve a new type of equations called convolution differential equations.

Sec(3.1):Combined Laplace Transform and Variational Iteration Method to Solve Convolution Differential Equations

In this section we combine Laplace transform and modified variational iteration method to solve new type of differential equation called convolution differential equations, it is possible to find the exact solutions or better approximate solutions of these equations. In this method, a correction functional is constructed by a general Lagrange multiplier, which can be identified via variational theory. This method is used for solving a convolution differential equation with given initial conditions. The solutions obtained by this method show the accuracy and efficiency of the method.

Definition(3.1.1) [26]:

Let $f(x), g(x)$ be integrable functions, then the convolution of $f(x), g(x)$ is definedas

$$f(x)*g(x)=\int_0^x f(x-t)\,g(t)\,dt$$

And the Laplace transform is defined as

$$L[f(x)]=F(s)=\int_0^\infty e^{-sx} f(x)\,dx$$

where $x > 0$ and is complex value.

And further the Laplace transform of first and second derivatives are given by

$$\text{(i)}\ L[f'(x)]=sL[f(x)]-f(0)$$
$$\text{(ii)}\ L[f''(x)]=s^2L[f(x)]-s\,f(0)-f'(0)$$

More generally

$$L[f^{(n)}(x)]=s^nL[f(x)]-s^{n-1}f(0)-s^{n-2}f'(0)-...-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Theorem (Convolution Theorem) (3.1.2)[26]:

If $L[f(x)]=F(s)$, $L[g(x)]=G(s)$, $L\left[u(x,t)\right]+N\left[u(x,t)\right]=g\left[u(x,t)\right]$ then:

$$L[f(x)*g(x)]=L[f(x)\,g(x)]=f(s)\,g(s)$$

or equivalently,

$$L^{-1}[F(s)\,G(s)]=f(x)*g(x)$$

Consider the differential equation

$$L[y(x)]+R[y(x)]+N[y(x)]+N^*[y(x)]=0 \qquad (1)$$

With the initial conditions

$$y(0)=h(x) \quad , \quad y'(0)=k(x) \qquad (2)$$

Where L is a linear second order operator, R is a linear operator less than L , N is the nonlinear operator, and $N^*[y(x)]$ is the nonlinear convolution term which is definite by

$u = u(x,t)$ $N^*[y(x)] = f(y, y', y'', ..., y^{(n)}) * g(y, y', y'', ..., y^{(n)})$

According to the variational iteration method, we can construct a correction functional as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)[Ly_n(\xi) + R\,\tilde{y}_n(\xi) + N\,\tilde{y}_n(\xi) + N^*\,\tilde{y}_n(\xi)]d\xi \qquad (3)$$

$Ry_n(\xi), N\,\tilde{y}_n(\xi)$ and $N^*\tilde{y}_n(\xi)$, are considered as restricted variations, i.e.

$\delta R\,\tilde{y}_n = 0, \delta\, N\,\tilde{y}_n = 0$ and $\delta\, N^*\,\tilde{y}_n = 0,\ \ \lambda$=-1

Then the variational iteration formula can be obtained as

$$y_{n+1}(x) = y_n(x) - \int_0^x [Ly_n(\xi) + R\,y_n(\xi) + Ny_n(\xi) + N^*\,\tilde{y}_n(\xi)]d\xi \qquad (4)$$

Eq. (4), can be solved iteratively using $y_0(x)$ as the initial approximation.

Then the solution is $y(x) = \lim_{n\to\infty} y_n(x)$

Consider the nonlinear convolution ordinary differential equation

$$L[y(x)] + R[y(x)] + N[y(x)] + N^*[y(x)] = 0 \qquad (5)$$

With the initial conditions

$$y(0) = h(x) \quad , \qquad y'(0)=k(x) \qquad (6)$$

Where $L$is a linear operator, R is a linear operator less than $L$and $N$is the nonlnonlinear operatord $N^*[y(x)] = f(y, y', y'', ..., y^{(n)}) * g(y, y', y'', ..., y^{(n)})$ is the nonlnonlinear convolutionm.

In this section we assume that $L = \frac{d^2}{dx^2}$

Take Laplace transform of both sides of eq (5), to find

$$\ell[Ly(x)] + \ell[Ry(x)] + \ell[Ny(x)] + \ell[N^*y(x)] = 0 \qquad (7)$$

$$s^2\ell y - sy(0) - y'(0) = -\ell\{Ry(x) + Ny(x) + N^*y(x)\} = 0 \qquad (8)$$

By using the initial conditions and taking the inverse Laplace transform we have

$$y(x) = p(x) - \ell^{-1}\left[\frac{1}{s^2}Ry(x) + Ny(x) + N^{*}y(x)\right] = 0 \qquad (9)$$

Where $p(x)$ represents the terms arising from the source term and the prescribed initial conditions.

Now the first derivative of eq (9) is given by

$$\frac{dy(x)}{dx} = \frac{dp(x)}{dx} - \frac{d}{dx}\ell^{-1}\left[\frac{1}{s^2}\ell\{Ry(x) + Ny(x) + N^{*}y(x)\}\right] = 0 \qquad (10)$$

By the correction function of the irrational method we have

$$y_{n+1}(x) = y_n(x) - \int_0^x \left\{ (y_n(\xi))_\xi - \frac{d}{d\xi}p(\xi) - \frac{d}{d\xi}\ell^{-1}\left[\frac{1}{s^2}\ell\{Ry(\xi) + Ny(\xi) + N^{*}y(\xi)\}\right] \right\} d\xi$$

Then the new correction function (new modified VIM) is given by

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left[\frac{1}{s^2}\ell\{Ry_n(x) + Ny_n(x) + N^{*}y_n(x)\}\right], \; n \ge 0 \qquad (11)$$

In the last we find the solution in the form $y(x) = \lim_{n\to\infty} y_n(x)$, if inverse Laplace transform exist.

In particular, consider the nonlinear ordinary differential equations with convolution terms

$$1 - y''(x)-2+2y' * y''-y' *(y'')^2 = 0 \text{ , } y(0)=y'(0)=0 \qquad (12)$$

Take Laplace transform of eq (12), and making use of initial conditions, we have

$$s^2\ell y(x) - \frac{2}{s} = \ell\left[y' *\left(y''\right)^2 \text{ -}2y' *y''\right]$$

The inverse Laplace transform implies that

$$y(x) = x^2 + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[y' *\left(y''\right)^2 \text{ -}2y' *y''\right]\right\}$$

By using the new modified (eq (11)), we have the new correction function

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[y' * \left(y''\right)^2 \text{ -2} y' * y'' \right]\right\}$$

or

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s^2}\left[\ell(y') * \ell\left(y''\right)^2 \text{ -2}\ell(y') * \ell(y'') \right]\right\} \tag{13}$$

Then we have

$$y_0(x) = x^2$$

$$y_1(x) = x^2 + \ell^{-1}\left\{\frac{1}{s^2}[\ell(4)\ell(2x) - 2\ell(2x)\ell(2)\right\} = x^2$$

$$y_2(x) = x^2 \quad , \quad y_3(x) = x^2, \quad ...., \ y_n(x) = x^2$$

This means that

$$y_0(x) = y_1(x) = y_2(x) = ... = y_n(x) = x^2$$

Then the exact solution of Eq.(12) is $y(x) = x^2$

$$y'\text{-}(y')^2\text{-}2x + y' * \left(y''\right)^2 = 0 \ , \ y(0)=1 \tag{14}$$

Take Laplace transform of Eq. (14), and use the initial condition, we obtain

$$s\,\ell y - 1 - \frac{2}{s^2} = \ell\left[\left(y'\right)^2 \text{ -} y' * \left(y''\right)^2\right]$$

Take the inverse Laplace transform to obtain

$$y(x) = 1 + x^2 + \ell^{-1}\left\{\frac{1}{s}\ell\left[\left(y'\right)^2 \text{ -} y' * \left(y''\right)^2\right]\right\}$$

Using eq (11) to find the new correction function in the form

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s}\ell\left[\left(y'\right)^2 \text{ -} y'_n * \left(y''_n\right)^2\right]\right\}$$

or

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s}\left[\ell[\left(y'\right)^2] \text{ -}\ell[y'_n]\ell[\left(y''_n\right)^2]\right]\right\} \tag{15}$$

Then we have

$$y_0(x) = 1 + x^2$$

$$y_1(x) = 1 + x^2 + \ell^{-1}\frac{1}{s}\left\{\ell(4x^2) - \ell(2x)\ell(4)\right\} = 1 + x^2 + \ell^{-1}\frac{1}{s}\left\{\frac{8}{s^3} - \left(\frac{2}{s^2}\right)\left(\frac{4}{s}\right)\right\} = 1 + x^2$$

………………………………………..

$$y_0(x) = y_1(x) = y_2(x) = ... = y_n(x) = 1 + x^2$$

Then the exact solution of Eq. (14) is:

$$y(x) = 1 + x^2$$

Sec(3.2): Solution of Nonlinear Partial Differential Equations by the Combined Laplace Transform and the New Modified Variational Iteration Method

In this section, we present a reliable combined Laplace transform and the new modified variational iteration method to solve some nonlinear partial differential equations. The analytical results of these equations have been obtained in terms of convergent series with easily compute able components. The nonlinear terms in these equations can be handled by using the new modified variational iteration method. This method is more efficient and easy to handle such nonlinear partial differential equations.

The basic definition of the Laplace transform is given as follows:
Laplace transform of the function $f(t)$ is

$$L[f(\mathrm{x})] = \mathrm{F(s)} = \int_0^\infty e^{-sx} f(\mathrm{x})\,\mathrm{dx}, \qquad \mathrm{Re\ s} > 0 \tag{16}$$

And the inverse Laplace transform is given by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} F(s)\,\mathrm{d}s, \qquad \alpha > 0$$

Obviously $L$and $L^{-1}$are linear integral operators.
In this section, we combined Laplace transform and variational iteration method to solve nonlinear partial differential equations.
To obtain Laplace transform of partial derivative we use integration by parts, and then we have:

$$L\left(\frac{\partial f(\mathrm{x,t})}{\partial t}\right) = sF(\mathrm{x,s}) - \mathrm{f(x,0)}\ ,$$

$$L\left(\frac{\partial^2 f(\mathrm{x,t})}{\partial t^2}\right) = s^2 F(\mathrm{x,s}) - s\,\mathrm{f(x,0)} - \frac{\partial f(\mathrm{x,0})}{\partial t}$$

$$L\left(\frac{\partial f(\mathrm{x,t})}{\partial t}\right) = \frac{d}{dx}\left[F(\mathrm{x,s})\right],$$

$$L\left(\frac{\partial^2 f(\mathrm{x,t})}{\partial t^2}\right) = \frac{d^2}{dx^2}\left[F(\mathrm{x,s})\right].$$

where $f(x,s)$ is the Laplace transform of (x,t).
We can easily extend this result to the nth partial derivative by using mathematical induction.

To illustrate the basic concept of the He's VIM, we consider the following general differential equations

$$L[u(x,t)]+N[u(x,t)]=g(x,t) \tag{17}$$

with the initial condition

$$u(x,0)=h(x) \tag{18}$$

where $L$is a linear operator of the first order, $N$is nonlinear operator and $g(x,t)$is inhomogeneous term. According to variational iteration method we can construct a correction functional as follows.

$$u_{n+1}=u_n+\int_0^t \lambda\left[Lu_n(x,s)+N\tilde{u}_n(\mathrm{x,s})-\mathrm{g(x,s)}\right]ds \tag{19}$$

where λ is a Lagrange multiplier $(\lambda=-1)$, the subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n=0$.
Equation (19) is called a correction functional.
The successive approximation u_{n+1}, of the solution u will be readily obtained by using thedetermined Lagrange multiplier and any selective function u_0, consequently, the solution is given by

$$u=\lim_{u\to\infty} u_n$$

In this section we assume that L is an operator of the first order $\frac{\partial}{\partial t}$ in equation (17).
Taking Laplace transform on both sides of equation (17), to get

$$L\left[Lu(\mathrm{x,t})\right]+L\left[Nu(\mathrm{x,t})\right]=L\left[g(x,t)\right] \tag{20}$$

Using the differentiation property of Laplace transform and initial condition (18), we have:

$$sL\left[u(\mathrm{x,t})\right]-h(\mathrm{x})=\mathrm{L}\left[g(\mathrm{x,t})\right]-L\left[Nu(\mathrm{x,t})\right] \tag{21}$$

Appling the inverse Laplace transform on both sides of equation (21) to find:

$$u(\mathrm{x,t})=G(\mathrm{x,t})-\mathrm{L}^{-1}\left\{\frac{1}{s}Nu[\mathrm{x,t}]\right\}, \tag{22}$$

where $G(x,t)$ represents the terms arising from the source term and the prescribed initialcondition.
Take the first partial derivative with respect to t of equation (22), to obtain:

$$\frac{\partial}{\partial t}u(\mathrm{x},\mathrm{t})-\frac{\partial}{\partial t}G(\mathrm{x},\mathrm{t})+\frac{\partial}{\partial t}\mathrm{L}^{-1}\left\{\frac{1}{s}L\left[Nu(\mathrm{x},\mathrm{t})\right]\right\} \tag{23}$$

By the correction function of the variational iteration method

$$u_{n+1}=u_n-\int_0^t\left\{(u_n)_\xi(\mathrm{x},\xi)-\frac{\partial}{\partial\xi}G(\mathrm{x},\xi)+\frac{\partial}{\partial\xi}\mathrm{L}^{-1}\left\{\frac{1}{\xi}L\left[Nu(\xi,\mathrm{t})\right]\right\}\right\}d\xi$$

or

$$u_{n+1}=G(\mathrm{x},\mathrm{t})-\mathrm{L}^{-1}\left\{\frac{1}{s}L\left[Nu_n(\mathrm{x},\mathrm{t})\right]\right\} \tag{24}$$

Equation (24) is the new modified correction functional of Laplace transform and the variational iteration method, and the solution $u$is given by

$$u(\mathrm{x},\mathrm{t})=\lim_{u\to\infty}u_n(\mathrm{x},\mathrm{t})$$

In this section, we solve some nonlinear partial differential equations by using the new modified
variational iteration Laplace transform method, therefore we have:

Example (3.2.1) [32]:

Consider the following nonlinear partial differential equation

$$u_t+uu_x=0 \quad , \qquad \mathrm{u(x,0)=-x} \tag{25}$$

Taking Laplace transform of equation (25), subject to the initial condition, we have:

$$L\left[u(\mathrm{x},\mathrm{t})\right]=-\frac{x}{s}-\frac{1}{s}L\left[uu_x\right]$$

The inverse Laplace transform implies that:

$$u(\mathrm{x},\mathrm{t})=-\mathrm{x}-\mathrm{L}^{-1}\left\{\frac{1}{s}L\left[uu_x\right]\right\}$$

by the new correction function we find:

$$u_{n+1}(\mathrm{x},\mathrm{t})=-\mathrm{x}-\mathrm{L}^{-1}\left\{\frac{1}{s}L\left[u_n(u_n)_x\right]\right\}$$

Now we apply the new modified variational iteration Laplace transform method,

$u_0(\mathrm{x,t}) = -\mathrm{x}$

$u_1(\mathrm{x,t}) = -\mathrm{x} - \mathrm{L}^{-1}\left\{\frac{1}{s}L\left[x\right]\right\} = -x - \mathrm{L}^{-1}\left[\frac{x}{s^2}\right] = -x - xt$

$u_2(\mathrm{x,t}) = -\mathrm{x} - \mathrm{L}^{-1}\left[x\left(\frac{1}{s^2}+\frac{2}{s^3}+\frac{2}{s^4}\right)\right] = -x - xt - xt^2 - \frac{1}{3}xt^3$

. . .

. . .

. . .

So we deduce the series solution to be

$$u(\mathrm{x,t}) = -\mathrm{x}(1+\mathrm{t}+\mathrm{t}^2+\mathrm{t}^3+...) = \frac{x}{t-1},$$

which is the exact solution.

Example (3.2.2) [32]:

Consider the following nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\,\frac{\partial^2 u}{\partial x^2} \quad , \quad \mathrm{u(x.0)=x^2} \tag{26}$$

Taking Laplace transform of equation (26), subject to the initial condition, we have:

$$L\left[u(\mathrm{x,t})\right] = \frac{x^2}{s} + \frac{1}{s}L\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\,\frac{\partial^2 u}{\partial x^2}\right]$$

Take the inverse Laplace transform to find that:

$$u(\mathrm{x,t}) = \mathrm{x}^2 + L^{-1}\left\{\frac{1}{s}L\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\,\frac{\partial^2 u}{\partial x^2}\right]\right\}$$

The new correction functional is given as

$$u_{n+1}(\mathrm{x,t}) = \mathrm{x}^2 + L^{-1}\left\{\frac{1}{s}L\left[\left(\frac{\partial u_n}{\partial x}\right)^2 + u_n\,\frac{\partial^2 u_n}{\partial x^2}\right]\right\}$$

This is the new modified variational iteration Laplace transform method. The solution in series form is given by:

$$u_0(x,t) = x^2$$

$$u_1(x,t) = x^2 + L^{-1}\left\{\frac{6x^2}{s^2}\right\} = x^2 + 6x^2 t$$

$$u_2(x,t) = x^2(1 + 6t + 36t^2 + 72t^3)$$

. . .

. . .

. . .

The series solution is given by

$$u(x,t) = x^2(1 + 6t + 36t^2 + 72t^3 + \ldots) = \frac{x^2}{1-6t}$$

Example (3.2.3) [32]:

Consider the following nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = 2u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} \quad , \quad u(x.0)=\frac{x+1}{2} \tag{27}$$

Using the same method in the above examples to find the new correction functional in the form:

$$u_{n+1}(x,t) = \frac{x+1}{2} + L^{-1}\left\{\frac{1}{s}L\left[2u_n\left(\frac{\partial u_n}{\partial x}\right)^2 + u^2{}_n\frac{\partial^2 u_n}{\partial x^2}\right]\right\}$$

Then we have:

$$u_0(x,t) = \frac{x+1}{2}$$

$$u_1(x,t) = \frac{x+1}{2} + L^{-1}\left\{\frac{x+1}{4}\frac{1}{s^2}\right\} = \frac{x+1}{2}\left[1 + \frac{t}{2}\right]$$

$$u_2(x,t) = \frac{x+1}{2}(1 + \frac{t}{2} + \frac{3}{8}t^2 + \frac{1}{8}t^3 + \frac{1}{64}t^4)$$

. . .

. . .

. . .

The series solution is given by

$$u(x,t) = \frac{x+1}{2}(1 + \frac{t}{2} + \frac{3}{8}t^2 + \ldots) = \frac{x+1}{2}(1-t)^{-\frac{1}{2}},$$

which is the exact solution of equation (27).

Example (3.2.4) [32]:
Consider the following nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{\partial u}{\partial x}\right)^2+u-u^2=\mathrm{t}e^{-x} \quad , \quad \mathrm{u(x.0)=0}, \frac{\partial u}{\partial t}=e^{-x} \tag{28}$$

Taking the Laplace transform of the equation (28), subject to the initial conditions, we have:

$$s^2 L\left[u(\mathrm{x,t})\right]-e^{-x}=L\left[te^{-x}+u^2-\left(\frac{\partial u}{\partial x}\right)^2-u\right]$$

Take the inverse Laplace transform to find that:

$$u(\mathrm{x,t})=te^{-x}+L^{-1}\left\{\frac{1}{s^2}L\left[te^{-x}+u^2-\left(\frac{\partial u}{\partial x}\right)^2-u\right]\right\}$$

The new correct functional is given as

$$u_{n+1}(\mathrm{x,t})=te^{-x}+L^{-1}\left\{\frac{1}{s^2}L\left[te^{-x}+u_n{}^2-\left(\frac{\partial u_n}{\partial x}\right)^2-u_n\right]\right\}$$

This is the new modified variational iteration Laplace transform method.
The solution in series form is given by:

$$\begin{aligned} u_0(\mathrm{x,t})&=te^{-x}\\ u_1(\mathrm{x,t})&=te^{-x}\\ u_2(\mathrm{x,t})&=te^{-x}\\ &\vdots \end{aligned} \tag{29}$$

The series solution is given by

$$u(\mathrm{x,t})=te^{-x} \tag{30}$$

Sec(3.3): A Comparative Study of Variational Iteration Method and He- Laplace Method

Consider the following nonlinear differential equation:

$$y'' + p_1 y' + p_2 y + p_3 f\,(\mathrm{y}) = \mathrm{f(x)} \tag{31}$$

$$y\,(0) = \alpha, \qquad \mathrm{y}'(0) = \beta \tag{32}$$

where $p_1, p_2, p_3, \alpha, \beta$ are constants. $f\,(y)$ is a nonlinear function and $f\,(x)$ is the source term. Taking Laplace transformation (denoted throughout this section by L) on both side of Equation (31), we have

$$L\left[y''\right] + L\left[p_1 y'\right] + L\left[p_2 y\right] + L\left[p_3 f\,(\mathrm{y})\right] = L\left[f\,(\mathrm{x})\right] \tag{33}$$

By using linearity of Laplace transformation, the result is

$$L\left[y''\right] + p_1 L\left[y'\right] + p_2 L\left[y\right] + p_3 L\left[f\,(\mathrm{y})\right] = L\left[f\,(\mathrm{x})\right] \tag{34}$$

Applying the formula on Laplace transform, we obtain

$$s^2 L\left[y\right] - sy\,(0) - \mathrm{y}'(0) + p_1\left\{sL\left[y\right] - y\,(0)\right\} + p_2 L\left[y\right] + p_3 L\left[f\,(\mathrm{y})\right] = L\left[f\,(\mathrm{x})\right] \tag{35}$$

Using initial conditions in Equation (5), we have

$$(\mathrm{s}^2 + p_1\,\mathrm{s})\,\mathrm{L}\left[y\right] = \alpha s + \beta + \alpha p_1 - p_2 L\left[y\right] - p_3 L\left[\mathrm{f}(y)\right] + L\left[f\,(\mathrm{x})\right] \tag{36}$$

or

$$L\left[y\right] = \frac{(\alpha\,\mathrm{s} + \beta + \alpha\,\mathrm{p}_1)}{(\mathrm{s}^2 + \mathrm{p}_1\mathrm{s})} - \frac{p_2}{(\mathrm{s}^2 + \mathrm{p}_1\mathrm{s})} L\left[y\right] - \frac{p_3}{(\mathrm{s}^2 + \mathrm{p}_1\mathrm{s})} L\left[f\,(\mathrm{y})\right] + L\left[f\,(\mathrm{x})\right] \tag{37}$$

Taking the inverse Laplace transform, we have

$$y(x) = F(x) - L^{-1}\left(\frac{p_2}{(s^2+p_1 s)} L\left[y\right]\right) - L^{-1}\left(\frac{p_3}{(s^2+p_1 s)} L\left[f\ (y)\right]\right) \tag{38}$$

where $F(x)$ represents the term arising from the source term and the prescribed initial conditions.

Example (3.3.1) [38]:

Consider the following first order nonlinear differential equation:

$$y' + y^2 = 0, \quad y \geq 0 \tag{39}$$

$$y(0) = 1 \tag{40}$$

If y_0 is an initial approximation or trial-function then we can write down following expression for correction:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda_n \left\{ y'_n(\tau) + y_n^2(\tau) \right\} d\tau \tag{41}$$

where the last term of right is called "correction", λ_n is a general Lagrange multiplier. The above functional is called correction functional, the Lagrange multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation (trial- function) and is the best within the flexibility of the trial- function, accordingly we can identify the multiplier by variational theory. Making the above correction, functional stationary with $y(0) = 1$ so that, we can obtain following stationary conditions:

$$-\lambda'_n(\tau) + 2\,y_n(\tau)\lambda(\tau) = 0 \tag{42}$$

$$1 + \lambda_n(\tau)\,|_{\tau=0} = 0 \tag{43}$$

The Lagrange multiplier, therefore, can be identified as follows:

$$\lambda_n = -\exp\left\{2\int_t^\tau y_n(\xi)\,\mathrm{d}\,\xi\right\} \tag{44}$$

To simplify the multiplier, we approximate Equation (44) as follows:

$$\lambda_n(\tau) = -\exp\left\{2\int_t^\tau y_0(\xi)\,\mathrm{d}\,\xi\right\} \tag{45}$$

Substituting Equation (45) in Equation (41) yields following variational iteration formula

$$y_{n+1}(\mathrm{t}) = \mathrm{y}_n(\mathrm{t}) - \lambda_n(\tau) = -\exp\left(2\int_t^\tau y_0(\xi)\,\mathrm{d}\,\xi\left\{y'_n(\tau) + \mathrm{y}_n{}^2(\tau)\right\}\right)d\,\tau \tag{46}$$

We start with by above iteration formula, we can obtain the following results,

$$y_1(\mathrm{t}) = 1 - \int_0^t e^{2(\tau-\mathrm{t})}d\,\tau = 1 - \tfrac{1}{2}(1-\mathrm{e}^{-2t}) = \tfrac{1}{2}(1+\mathrm{e}^{-2t}) \tag{47}$$

$$y_2(\mathrm{t}) = \tfrac{1}{2}(1+\mathrm{e}^{-2t}) - \int_0^t e^{2(\tau-\mathrm{t})}\left\{-e^{-2\tau} + \tfrac{1}{4}(1+\mathrm{e}^{-2t})^2\right\}d\,\tau$$

$$= \tfrac{1}{2}(1+\mathrm{e}^{-2t}) - \int_0^t e^{2(\tau-\mathrm{t})}\left\{\tfrac{1}{4} - \tfrac{1}{2}\mathrm{e}^{-2\tau} + \tfrac{1}{4}\mathrm{e}^{-4\tau}\right\}d\,\tau \tag{48}$$

$$= \tfrac{1}{2}(1+\mathrm{e}^{-2t}) - \tfrac{1}{8}(1-\mathrm{e}^{-2t}) + \tfrac{1}{2}t\,\mathrm{e}^{-2t} - \tfrac{1}{8}(\mathrm{e}^{-2t} - \mathrm{e} - 4t)$$

$$= \tfrac{1}{2}(1+\mathrm{e}^{-2t}) + \tfrac{1}{2}t\,\mathrm{e}^{-2t} - \tfrac{1}{8}(1-\mathrm{e}^{-4t})$$

if, suppose, $y_2(t)$ is sufficient, the approximation at $x = 0.4$ is $\mathrm{y}_2(0.4) = 0.6678$, while its exact one is $\mathrm{y}_2(0.4) = 0.6667$,, the 0.17% accuracy is remarkably good in view of the crudeness of its initial approximation. The process can, in principle, be continued as far as desired, however, the resulting integrals quickly become very cumbersome, so some simplification in the process of identification of Lagrange multiplier will be discussed at below:

We re-consider the correction, functional Equation (41) as follows:

$$y_{n+1}(\mathrm{t}) = \mathrm{y}_n(\mathrm{t}) + \int_0^t \lambda \left\{ y_n'(\tau) + y_n^2(\tau) \right\} d\tau \tag{49}$$

Where the nonlinear term y_n^2 is considered as non-variational variation or constrained variation $i.e. \delta y_n^2 = 0$. The Lagrange multiplier, therefore, can be readily identified and the following variation iteration formula can be obtained:

$$y_{n+1}(\mathrm{t}) = \mathrm{y}_n(\mathrm{t}) + \int_0^t \left\{ y_n'(\tau) + y_n^2(\tau) \right\} d\tau \tag{50}$$

Putting $n = 0,1,$ in Equation (50), we can obtain the following

$$y_1(\mathrm{t}) = 1 - \int_0^t (0+1) d\tau = 1 - t$$

$$y_2(\mathrm{t}) = 1 - t - \int_0^t \left\{ -1 + (1-\tau)^2 \right\} d\tau = 1 - t + t^2 - \tfrac{1}{3} t^3$$

Similarly, putting $n = 2, 3, \ldots, n-1$ the nth approximation can be obtained, which converges to its exact solution, a little more slowly due to the approximate identification of the Lagrange multiplier.

Remark

The variational iteration technique mentioned above can be readily extended to partial differential equations (PDEs). We will illustrate its process.

Example (3.3.2) [38]:

Consider the following equation

$$\nabla^2 u + \left(\frac{\partial u}{\partial y}\right)^2 = 2y + x^4$$

$$\begin{aligned} &u(0,y)=0, \qquad u(1,y)= y + a \\ &u(x,0)=ax, \qquad u(x,1)=x^2+ax \end{aligned} \tag{51}$$

which has the exact solution $u = x\,(xy + a)$.

Supposing the initial approximation equation (51) is u_0, its correction variational functional in x-direction and y-direction can be expressed respectively as follows:

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^t \lambda_1\left[\frac{\partial^2 u_n(\xi,y)}{\partial \xi^2}+\frac{\partial^2 \hat{u}_n(\xi,y)}{\partial y^2}+\left(\frac{\partial \hat{u}_n(\xi,y)}{\partial y}\right)^2-2y-\xi^4\right]d\xi$$

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^t \lambda_2\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\frac{\partial^2 \hat{u}_n(x,\xi)}{\partial x^2}+\left(\frac{\partial \hat{u}_n(x,\xi)}{\partial \varsigma}\right)^2-2\xi-x^4\right]d\xi \tag{52}$$

where $\hat{u}_n$ is a non variational. Their stationary conditions are written down respectively as follows

$$\frac{\partial^2 \lambda_1(\xi)}{\partial \xi^2} = 0, \quad \lambda_1(\xi)|_{\xi=x} = 0, \quad 1\text{-}\frac{\partial \lambda_1(\xi)}{\partial \xi}|_{\xi=x} = 0 \tag{53}$$

and

$$\frac{\partial^2 \lambda_2(\xi)}{\partial \xi^2} = 0, \quad (\xi)|_{\xi=y} = 0, \quad 1\text{-}\frac{\partial \lambda_2(\xi)}{\partial \xi}|_{\xi=y} = 0 \tag{54}$$

The Lagrange multipliers can be easily identified

$$\lambda_1 = \xi - x, \qquad \lambda_2 = \xi - y \tag{55}$$

The iteration formulae in x-direction and y-directions can be, therefore, expressed respectively as follows

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x (\xi - \mathrm{x})\left[\frac{\partial^2 u_n(\xi,\mathrm{y})}{\partial \xi^2} + \frac{\partial^2 u_n(\xi,\mathrm{y})}{\partial y^2} + \left(\frac{\partial u_n(\xi,\mathrm{y})}{\partial y}\right)^2 - 2y - \xi^4\right] d\xi$$

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y (\xi - \mathrm{y})\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \left(\frac{\partial u_n(x,\xi)}{\partial \xi}\right)^2 - 2\xi - x^4\right] d\xi$$

(To ensure the approximations satisfy the boundary conditions at x=0 and y=0, we modify the varittional iteration formulae in x-direction and y-direction as follows

$$u_{n+1}(x,y) = u_n(x,y) + \int_1^x (\xi - \mathrm{x})\left[\frac{\partial^2 u_n(\xi,\mathrm{y})}{\partial \xi^2} + \frac{\partial^2 u_n(\xi,\mathrm{y})}{\partial y^2} + \left(\frac{\partial u_n(\xi,\mathrm{y})}{\partial y}\right)^2 - 2y - \xi^4\right] d\xi$$

(56)

$$u_{n+1}(x,y) = u_n(x,y) + \int_1^y (\xi - \mathrm{y})\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \left(\frac{\partial u_n(x,\xi)}{\partial \xi}\right)^2 - 2\xi - x^4\right] d\xi$$

Now we start with an arbitrary initial approximation:

$u_0 = A + Bx$, where A and B are constant to be determined, by the variational iteration formula in x-direction, we have

$$u_1(x, y) = A + Bx + \int_0^x (\xi - x)\left[0 + 0 - 2y - \xi^4\right] d\xi \tag{57}$$

$$= A + Bx + x^2 y + \frac{1}{30} x$$

By imposing the boundary conditions at $x = 0$ and $y = 0$ and $B = a - 1/30$, thus we have

$$u_1(x, y) = x\,(xy + a) + \frac{1}{30} x\,(x^5 - 1) \tag{58}$$

By (56) we have:

$$u_2(x, y) = x\,(xy + a) + \tfrac{1}{30} x\,(x^5 - 1) + \int_0^x (\xi - x)\left[2y + \xi^4 + 0 + \xi^4 - 2y - \xi^4\right] d\xi \tag{59}$$

$$= x\,(xy + a)$$

which is an exact solution. The approximation can also be obtained by

y-direction.

Example (3.3.3) [38]:

Consider the following nonlinear PDE:

$$\frac{\partial u}{\partial t} = x^2 - \tfrac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2 \qquad u(x, 0) = 0 \tag{60}$$

Its t-direction correction functional can be constructed as

$$u_{n+1}(x,t) = u_n(x,\mathrm{t}) + \int_0^t \lambda \left[\frac{\partial u_n(\mathrm{x},\tau)}{\partial \tau} - x^2 + \tfrac{1}{4} \left(\frac{\partial \hat{u}_n(\mathrm{x},\tau)}{\partial x} \right)^2 \right] d\tau \qquad (61)$$

In which $\hat{u}_n$ is a non variational variation.The multiplier can be identified and its variational iteration formula *t*-direction can be obtained

$$u_{n+1}(x,t) = u_n(x,\mathrm{t}) + \int_0^t \lambda \left[\frac{\partial u_n(\mathrm{x},\tau)}{\partial \tau} - x^2 + \tfrac{1}{4} \left(\frac{\partial u_n(\mathrm{x},\tau)}{\partial x} \right)^2 \right] d\tau \qquad (62)$$

We start with an initial approximation $u_0 = 0$, by above iteration, we can obtain successively its approximation:

$$u_1(x,t) = 0 - \int_0^t (-\mathrm{x}^2) d\tau,$$

$$u_2(x,t) = x^2 t - \int_0^t \left[\mathrm{x}^2 - \mathrm{x}^2 + \tfrac{1}{4}(2x\tau)^2 \right] d\tau,$$
$$= \mathrm{x}^2 t - \tfrac{1}{3}\mathrm{x}^2 t^3,$$

$$u_3(x,t) = x^2 t - \tfrac{1}{3} x^2 t^3 - \int_0^t \left[\mathrm{x}^2 - \mathrm{x}^2 t^2 - x^2 + \tfrac{1}{4}(2x\tau - \tfrac{2}{3}\mathrm{x}\tau^3)^2 \right] d\tau,$$
$$= \mathrm{x}^2 t - \tfrac{1}{3}\mathrm{x}^2 t^3 - \int_0^t \left[-\tfrac{2}{3} x^2 \tau^4 - \tfrac{1}{9}\mathrm{x}^2 \tau^6 \right] d\tau$$
$$= \mathrm{x}^2 t - \tfrac{1}{3}\mathrm{x}^2 t^3 + \tfrac{2}{15}\mathrm{x}^2 t^3 - \tfrac{1}{63}\mathrm{x}^2 t^7$$