Chapter Four

Variation Iteration Method for Solving Poruos Meduim Equation and Solving Fourth Order Parabolic PDEs with Variable Coefficients

The aim of this chapter, is to apply a new method called Variation Iteration perturbation method ((VIM)) to the porous medium equation. This method is a combination of the new integral “Variation Iteration” and the perturbation method.

The nonlinear term can be easily handled by perturbation method. The porous medium equations have importance in engineering and the sciences.

And we apply a new Modified Variational Iteration Method (MVIM) to solve one dimensional fourth order parabolic linear partial differential equations with variable coefficients. This method is a combination of the two initial conditions.

Sec (4.1): Variation Iteration Method for Solving Poruos Meduim Equation

Many of the physical phenomena and processes in various fields of engineering and science are governed by partial differential equations. The nonlinear heat equation describing various physical phenomena called the porous medium equation, is where m is a rational number. There are a number of physical applications where this simple model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion.

In this section, we apply a new method called Variation Iteration perturbation method ((VIM)) to solve porous medium equation. This method is a combination of the new integral “Variation Iteration” and the perturbation method.

The porous medium equation is:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{m}\frac{\partial u}{\partial x}\right) \tag{1}$$

Where m is a rational number.

The correction, functional for the porous medium equation is

$$u_{n+1}(x,t)=u_n(\mathrm{x},\mathrm{t})+\int_0^t \lambda(\xi)\left[\frac{\partial u(x,\xi)}{\partial \xi}-u^m\frac{\partial^2\tilde{u}(x,\xi)}{\partial x^2}-mu^{m-1}\left(\frac{\partial u(x,\xi)}{\partial x}\right)^2\right]d\xi \tag{2}$$

The Variation iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier λ that can be identified optimally via integration by parts and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Having determined the Lagrange multiplier $\lambda(\xi)$ the successive approximations $u_{n+1}, n \geq 0$ of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution:

$$u = \lim_{n\to\infty} u_n$$

Example (4.1.1) [43]:

Let us $m = -1$ in equation (1), we get,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left(u^{-1}\right)\frac{\partial u}{\partial x}\right) \tag{3}$$

With the initial condition as $u(x,0) = \frac{1}{x}$.

Solution:

The correction, functional for this equation is given by:

$$u_{n+1}(x,t) = u_n(\text{x,t}) + \int_0^t \lambda(\xi)\left[\frac{\partial u(x,\xi)}{\partial \xi} - u^{-1}\frac{\partial^2 \tilde{u}(x,\xi)}{\partial x^2} + u^{-2}\left(\frac{\partial u(x,\xi)}{\partial x}\right)^2\right] d\xi$$

Where we used $\lambda = -1$ for first order porous medium equation as shown (3) we can use the initial condition to select, $u_0(x,t) = u_0(x,0) = \frac{1}{x}$. Using this selection into the correction functional gives the following successive approximation,

$$u_0(x,0) = \frac{1}{x}.$$

$$u_1(x,t) = u_0(\text{x,t}) - \int_0^t \left[\frac{\partial u(x,\xi)}{\partial \xi} - u_0^{-1}\frac{\partial^2 \tilde{u}(x,\xi)}{\partial x^2} + u_0^{-2}\left(\frac{\partial u(x,\xi)}{\partial x}\right)^2\right] d\xi = \frac{t}{x^2}$$

$$u_2(x,t) = u_1(\text{x,t}) - \int_0^t \left[\frac{\partial u(x,\xi)}{\partial \xi} - u_1^{-1}\frac{\partial^2 \tilde{u}(x,\xi)}{\partial x^2} + u_1^{-2}\left(\frac{\partial u(x,\xi)}{\partial x}\right)^2\right] d\xi = \frac{t^2}{x^3}$$

Proceeding in a similar manner we can obtain further values, we get a solution in the form of a series,

$$u(x,t)=\frac{1}{x}+\frac{t}{x^2}+\frac{t^2}{x^3}+\ldots=\frac{1}{x-t}$$

This is the solution of (3) and which is exactly the exact solution given above,

Example (4.1.2) [43]:

Let us $m=1$ in equation (1), we get,

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)\quad,\qquad u(x,0)=x \tag{5}$$

Solution:

The correction, functional is:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t \lambda(\xi)\left[\frac{\partial u(x,\xi)}{\partial \xi}-u\frac{\partial^2\tilde{u}_n(x,\xi)}{\partial x^2}-\left(\frac{\partial u_n(x,\xi)}{\partial x}\right)^2\right]d\xi$$

Consider $\lambda=-1$, and $u_0(x,t)=u_0(x,0)=x$.

Then:

$$u_1(x,t)=u_0(x,t)-\int_0^t \left[\frac{\partial u_0(x,\xi)}{\partial \xi}-u_0\frac{\partial^2\tilde{u}_0(x,\xi)}{\partial x^2}-\left(\frac{\partial u_0(x,\xi)}{\partial x}\right)^2\right]d\xi=x+t$$

$$u_2(x,t)=u_1(x,t)-\int_0^t \left[\frac{\partial u_1(x,\xi)}{\partial \xi}-u_1\frac{\partial^2\tilde{u}_1(x,\xi)}{\partial x^2}-\left(\frac{\partial u_1(x,\xi)}{\partial x}\right)^2\right]d\xi=x+t$$

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$$u(x,t)=x+t+0+0=x+t$$

Example (4.1.3) [43]:

Let us $m = -4/3$ in equation (1), we get:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right) \tag{6}$$

With initial condition as $u(x,0) = (2x)^{-3/4}$.

Solution:

The correction, functional is:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \left[\frac{\partial u(x,\xi)}{\partial \xi} - u^{-4/3}\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \frac{4}{3}u^{-7/3}\left(\frac{\partial u_n(x,\xi)}{\partial x}\right)^2\right]d\xi$$

Consider $\lambda = -1$, and $u(x,0) = (2x)^{-3/4}$.

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\xi)}{\partial \xi} - u_0^{-4/3}\frac{\partial^2 \tilde{u}_0(x,\xi)}{\partial x^2} + \frac{4}{3}u_0^{-7/3}\left(\frac{\partial u_0(x,\xi)}{\partial x}\right)^2\right]d\xi$$

$$= (2x)^{-3/4} - \int_0^t \left[\tfrac{-21}{4}(2x)^{-\frac{7}{4}} + 3(2x)^{-\frac{7}{4}}\right]d\zeta = (2x)^{-3/4} + 9\times 2^{-15/4}\times x^{-7/4}\times t$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\xi)}{\partial \xi} - u_1^{-4/3}\frac{\partial^2 \tilde{u}_1(x,\xi)}{\partial x^2} + \frac{4}{3}u_1^{-7/3}\left(\frac{\partial u_1(x,\xi)}{\partial x}\right)^2\right]d\xi = 189\times 2^{-31/4}x^{-11/4}\times t^2$$

Then:

$$u(x,t) = (2x)^{-3/4} + 9\times 2^{-15/4}\times x^{-7/4}\times t + 189\times 2^{-31/4}x^{-11/4}\times t^2$$

This result can be verified through substitution.

Sec (4.2): Modified Variational Iteration Method for Solving Fourth Order Parabolic PDEs With Variable Coefficients

Numerous issues of physical hobby are portrayed by direct halfway differential comparisons with beginning and limit conditions. One of them is fourth request illustrative fractional differential mathematical statements with variable coefficients; these comparisons arise on the transverse vibration issue [51]. As of late, numerous exploration specialists have paid consideration on to discover the arrangement of these mathematical statements by utilizing different strategies. Among these is the variation cycle technique [Bazaar and Ghazvini (2007)], Adomian deterioration strategy [Wazwaz (2001) and Biazaretal (2007)], homotopy irritation strategy [Mehdi Dehghan and JalilManafian (2008)], homotopy examination technique [NajeebAlam Khan (2010)] and Laplace disintegration calculation [Majid Khan, Muhammad AsifGondal and Yasir Khan (2011)]. In this section, we utilize the Modified Variational Iteration Method. This strategy is a valuable procedure for illuminating straight and nonlinear differential mathematical statements. The guideline purpose is to incorporate beginning conditions for fathoming higher request straight fractional differential mathematical statements with variable coefficients. This technique gives the arrangement as joined arrangement prompts the accurate arrangement.

Consider a one dimensional, linear, non-homogeneous fourth order parabolic partial differential equation with variable coefficients of the form,

$$\frac{\partial^2 u}{\partial t^2} + \psi(x)\frac{\partial^4 u}{\partial x^4} = \phi(x,t), \tag{7}$$

Where, $\psi(x)$ is a variable coefficient, with the following initial conditions,

$$u(x,0) = f(x), \text{ and } \frac{\partial u}{\partial t}(x,0) = h(x), \tag{8}$$

and the boundary conditions are,

$$u(a,t) = \beta_1(t), u(b,t) = \beta_2(t),$$

$$\frac{\partial^4 u}{\partial x^4}(a,t)=\beta_3(t),\frac{\partial^4 u}{\partial x^4}(b,t)=\beta_4(t), \tag{9}$$

Apply modified variational iteration method of Eq. (7),

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\psi(x)\frac{\partial^4 \tilde{u}_n}{\partial x^4}(x,\xi)-\phi(x,\xi)\right]d\xi, \tag{10}$$

where λ is a Lagrange multiplier $(\lambda=\xi-t)$, the subscripts n denote the n th approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n=0.$ Equation (10) is called a correction functional.

The successive approximation u_{n+1} of the solution u will be readily obtained by using the determined Lagrange multiplier and any selective function u_0 , consequently, the solution is given by,

$$u=\lim_{n\to\infty} u_n$$

To show that the method is effected, we have solved homogeneous and non-homogeneous one dimensional fourth order linear partial differential equations with the initial and boundary conditions.

Example (4.2.1) [51]:

Consider the fourth order homogenous partial differential equations,

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u}{\partial t^4}=0, \quad \frac{1}{2}<x<1, t>0 \tag{11}$$

With the following initial conditions,

$$u(x,0)=0,\frac{\partial u}{\partial t}(x,0)=1+\frac{x^5}{120}, \tag{12}$$

and the boundary conditions,

$$\begin{aligned}&u(0.5,t)=\left(1+\frac{(0.5)^5}{120}\right)\sin t, u(1,t)=\frac{121}{120}\sin t,\\&\frac{\partial^2 u}{\partial t^2}(0.5,t)=0.02084\sin t,\frac{\partial^2 u}{\partial t^2}(1,t)=\frac{1}{6}\sin t.\end{aligned} \tag{13}$$

Appling Modified Variational Iteration Method to Eq.(11), we get:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 \tilde{u}_n}{\partial x^4}(x,\xi)\right]d\xi, \tag{14}$$

take $\lambda=\xi-t$, then:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u_n}{\partial x^4}(x,\xi)\right]d\xi,$$

take, $u_0(x,t)=u(x,0)=\left(1+\frac{x^5}{120}\right)t$, then:

$$u_1(x,t)=\left(1+\frac{x^5}{120}\right)t+\int_0^t(\xi-t)\left[\left(\frac{1}{x}-\frac{x^4}{120}\right)x\xi\right]d\xi=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!}$$

$$u_2(x,t)=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!}+\int_0^t(\xi-t)\left[-\left(1+\frac{x^5}{120}\right)t+\left(\frac{1}{x}-\frac{x^4}{120}\right)\left(x\xi-x\frac{\xi^3}{3!}\right)\right]d\xi$$

$$=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!}+\left(1+\frac{x^5}{120}\right)\frac{t^5}{5!}$$

$$u_3(x,t)=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!}+\left(1+\frac{x^5}{120}\right)\frac{t^5}{5!}+$$

$$\int_0^t(\xi-t)\left[\left(1+\frac{x^5}{120}\right)\left(-t+\frac{t^3}{6}\right)+\left(\frac{1}{x}-\frac{x^4}{120}\right)\left(x\xi-x\frac{\xi^3}{3!}+x\frac{\xi^5}{5!}\right)\right]d\xi$$

$$=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!}+\left(1+\frac{x^5}{120}\right)\frac{t^5}{5!}-\left(1+\frac{x^5}{120}\right)\frac{t^7}{7!}$$

:

$$u_n(x,t)=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!}+\left(1+\frac{x^5}{120}\right)\frac{t^5}{5!}-\left(1+\frac{x^5}{120}\right)\frac{t^7}{7!}+\dots$$

$$=\left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right)$$

The exact solution is:

$$u(x,t)=\left(1+\frac{x^5}{120}\right)\sin t \tag{15}$$

Example (4.2.2) [51]:

Consider fourth order homogenous partial differential equation,

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 u}{\partial x^4}=0,\ 0<x<1, t>0 \tag{16}$$

with the following initial conditions,

$$u(x,0)=x-\sin x,\ \frac{\partial u}{\partial t}(x,0)=-x+\sin x \tag{17}$$

And the boundary conditions,

$$\begin{aligned} &u(0,t)=0, u(1,t)=e^{-t}(1-\sin 1),\\ &\frac{\partial^2 u}{\partial t^2}(0,t)=0, \frac{\partial^2 u}{\partial t^2}(1,t)=e^{-t}\sin 1. \end{aligned} \tag{18}$$

Solution:

Appling Modified Variational Iteration Method to Eq.(16), we get:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 \tilde{u}_n(x,\xi)}{\partial x^4}\right]d\xi$$

take , $\lambda=\xi-t$, then:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right]d\xi$$

Using initial conditions from Eq. (17), we get:

$$u_0(x,t)=(x-\sin x)+(-x+\sin x)t$$

$$u_1(x,t)=u_0(x,t)+\int_0^t (\xi-t)\left[\frac{\partial^2 u_0(x,\xi)}{\partial \xi^2}+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 u_0(x,\xi)}{\partial x^4}\right]d\xi$$

$$=(x-\sin x)+(-x+\sin x)t+(x-\sin x)\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right)$$

$$u_2(x,t)=u_1(x,t)+\int_0^t (\xi-t)\left[\frac{\partial^2 u_1(x,\xi)}{\partial \xi^2}+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 u_1(x,\xi)}{\partial x^4}\right]d\xi$$

$$=(x-\sin x)+(-x+\sin x)t+(x-\sin x)\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right)+(x-\sin x)\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right)$$

:

$$u_n(x,t)=(x-\sin x)+(-x+\sin x)t+(x-\sin x)\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right)+(x-\sin x)\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right)+...$$
$$=(x-\sin x)\left[1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+...\right]$$

The exact solution:

$$u(x,t)=(x-\sin x)e^{-t}$$

Example (4.2.3) [**51**]:

Consider fourth order homogenous partial differential equation,

$$\frac{\partial^2 u}{\partial t^2}+(1+x)\frac{\partial^4 u}{\partial x^4}=\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos t,\ 0<x<1, t>0 \tag{19}$$

with the following initial conditions,

$$u(x,0)=\frac{6}{7!}x^7,\frac{\partial u}{\partial t}(x,0)=0. \tag{20}$$

And the boundary conditions,

$$\begin{aligned} &u(0,t)=0, u(1,t)=\frac{6}{7!}\cos t,\\ &\frac{\partial^2 u}{\partial t^2}(0,t)=0,\frac{\partial^2 u}{\partial t^2}(1,t)=\frac{1}{20}\cos t. \end{aligned} \tag{21}$$

Solution:

Appling Modified Variational Iteration Method to Eq. (19), to find:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+(1+x)\frac{\partial^4 \tilde{u}_n(x,\xi)}{\partial x^4}-\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

take, $\lambda=\xi-t$, then:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+(1+x)\frac{\partial^4 u_n(x,\xi)}{\partial x^4}-\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

Using initial conditions from Eq. (20), to get:

$$u_0(x,t)=\frac{6}{7!}x^7,$$

$$u_1(x,t)=u_0(x,t)+\int_0^t(\xi-t)\left[\frac{\partial^2 u_0(x,\xi)}{\partial\xi^2}+(1+x)\frac{\partial^4 u_0(x,\xi)}{\partial x^4}-\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

$$=\frac{6}{7!}x^7\cos t-(x^4+x^3)\frac{t^2}{2!}+\left(x^4+x^3\right)(1-\cos t)$$

$$=\frac{6}{7!}x^7\cos t+\left(x^4+x^3\right)(1-\cos t)+noiseterms$$

$$u_2(x,t)=u_1(x,t)+\int_0^t(\xi-t)\left[\frac{\partial^2 u_1(x,\xi)}{\partial\xi^2}+(1+x)\frac{\partial^4 u_1(x,\xi)}{\partial x^4}-\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos\xi\right]d\xi$$

$$=\frac{6}{7!}x^7\cos t-(1+x)\frac{t^2}{2!}$$

$$=\frac{6}{7!}x^7\cos t+noiseterms$$

$\vdots$

$$u_n(x,t)=\frac{6}{7!}x^7\cos t$$

This is the approximate solution.