Chapter Five

Comparison of Laplace Variational Iteration Method with Different Methods

In this chapter we study a comparative between Variations Iteration Method (VIM) and different powerful methods to solve ordinary and nonlinear partial differential equations, the Laplace Transform method (LTM), Adomian decomposition method (ADM), the Homotopy perturbation method (HPM) for nonlinear equations using He's polynomials, and the Laplace Transform Variations Iteration Method (LTVIM).

Sec (5.1): Comparing of Variations Iteration Method with the Combined Laplace Transform and Variations Iteration Method to Solve ODEs and PDEs

In this section, the main objective is to introduce a comparative study to solve ordinary and nonlinear partial differential equations using the variation iteration method and Laplace Transform.

5.1.1: Basic of Idea (VIM):

To illustrate the basic idea of this method, we consider the following differential equation

$$Lu + Nu = g(x) \qquad (1)$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. Then, we can construct a correct function as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left[Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi) \right] d\xi \,, \quad n = 0,1,2,\ldots \qquad (2)$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory. The second term on the right is called the correction and $\tilde{u}_n$ is considered as restricted variation, i.e. $\delta\tilde{u}_n = 0$.

5.1.2: Basic of Idea (LTVIM):

From eq. (1), we take Laplace Transform on both sides the correction, functional will be constructed in the following manner:

$$\ell[u_{n+1}(x)] = \ell[u_n(x)] + \ell\left[\int_0^x \lambda\left[Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right]d\xi\right], \quad n = 0,1,2,\ldots \qquad (3)$$

Example (5.1.3):

Solve the first nonlinear ordinary differential equation

$$y' - y^2 = 1,\ y(0) = 0. \qquad (4)$$

I: Using (VIM):

The correction, functional for equation (4) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(y_n'(\xi) - \tilde{y}_n^2(\xi) - 1\right)d\xi \qquad (5)$$

The stationary conditions

$$1 + \lambda|_{\xi=x} = 0,$$
$$\lambda'|_{\xi=x} = 0,$$

To follow immediately. This in turn gives:

$$\lambda = -1$$

Substituting this value of the Lagrange multiplier $\lambda = -1,$ into the functional (5) gives the iteration formula:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) - \tilde{y}_n^2(\xi) - 1\right)d\xi, n \geq 0 \qquad (6)$$

We can select $y_0(x) = y(0) = 0$ from the given condition. Using this selection into (6) we obtain the following successive approximations

$$y_0(x)=0,$$

$$y_1(x)=0-\int_0^x\left(y_0'(\xi)-\mathrm{y}_0^2(\xi)-1\right)d\xi=x,$$

$$y_2(x)=x-\int_0^x\left(y_1'(\xi)-\mathrm{y}_1^2(\xi)-1\right)d\xi=x+\frac{1}{3}\mathrm{x}^3,$$

$$y_3(x)=x+\frac{1}{3}\mathrm{x}^3-\int_0^x\left(y_2'(\xi)-\mathrm{y}_2^2(\xi)-1\right)d\xi=x+\frac{1}{3}\mathrm{x}^3+\frac{2}{15}\mathrm{x}^5+\frac{1}{63}\mathrm{x}^7,$$

$$y_4(x)=x+\frac{1}{3}\mathrm{x}^3+\frac{2}{15}\mathrm{x}^5+\frac{1}{63}\mathrm{x}^7-\int_0^x\left(y_3'(\xi)-\mathrm{y}_3^2(\xi)-1\right)d\xi$$

$$=x+\frac{1}{3}\mathrm{x}^3+\frac{2}{15}\mathrm{x}^5+\frac{17}{315}\mathrm{x}^7+\ldots,$$

:

$$y_n(x)==x+\frac{1}{3}\mathrm{x}^3+\frac{2}{15}\mathrm{x}^5+\frac{17}{315}\mathrm{x}^7+\frac{62}{2835}\mathrm{x}^9+\ldots,$$

The VIM admits the use of

$$y(x)=\lim_{n\to\infty}y_n(x),$$

That gives the exact solution by

$$y(\mathrm{x})=\tan\mathrm{x}. \tag{7}$$

II: Using (LTVIM)

Taking the Laplace Transform for two sides in eq.(4):

$$\begin{aligned}
&\ell[y']-\ell\left[y^2\right]=\ell[1],\\
&s\ell[y]-y(0)-\ell\left[y^2\right]=\frac{1}{s}\\
&s\ell[y]=\frac{1}{s}+\ell\left[y^2\right]\\
&\ell[y]=\frac{1}{s^2}+\frac{1}{s}\ell\left[y^2\right]
\end{aligned} \tag{8}$$

Now, take the inverse Laplace Transform:

$$y(x)=x+\ell^{-1}\left[\frac{1}{s}\ell\left(y^2\right)\right] \tag{9}$$

The correction functions:

$$y_{n+1}(x) = x + \ell^{-1}\left[\frac{1}{s}\ell\left(y_n^2(x)\right)\right] \qquad (10)$$

Choose, $y_0(x) = 0$

Then,

$$\begin{aligned}
y_1(x) &= x + \ell^{-1}\left[\frac{1}{s}\ell\left(y_0^2(x)\right)\right] = x,\\
y_2(x) &= x + \ell^{-1}\left[\frac{1}{s}\ell\left(y_1^2(x)\right)\right] = x + \frac{x^3}{3},\\
y_3(x) &= x + \ell^{-1}\left[\frac{1}{s}\ell\left(y_2^2(x)\right)\right] = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{x^7}{63},\\
y_4(x) &= x + \ell^{-1}\left[\frac{1}{s}\ell\left(y_3^2(x)\right)\right] = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17x^7}{315} + \frac{38}{2835}x^9 + \ldots
\end{aligned} \qquad (11)$$

$\vdots$

$$y_n(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17x^7}{315} + \frac{38}{2835}x^9 + \ldots$$

Then:

$$y_n(x) \approx \tan x. \qquad (12)$$

Example (5.1.4) :

Consider the nonlinear partial differential equation,

$$u_t + u^2 u_x = 0,\ u(x,0) = 2x,\ t > 0 \qquad (13)$$

where $u = u(x,t)$.

I: Using (VIM):

Proceeding as in the previous example, we find:

$$\lambda = -1. \qquad (14)$$

This gives the iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n^2(x,\xi)\frac{\partial u_n(x,\xi)}{\partial \xi} \right) d\xi,\ n \ge 0. \qquad (15)$$

Choose $u_0(x,t)=2x$ from the given initial condition yields the successive approximations,

$$
\begin{aligned}
&u_0(x,t)=2x,\\
&u_1(x,t)=2x-8x^2t,\\
&u_2(x,t)=2x-8x^2t+64x^3t^2-\frac{640}{3}x^4t^3+\ldots\\
&u_3(x,t)=2x-8x^2t+64x^3t^2-640x^4t^3+\ldots\\
&:\\
&u_n(x,t)=2x-8x^2t+64x^3t^2-640x^4t^3+7168x^5t^4+\ldots
\end{aligned}
\tag{16}
$$

As concluded before, we can easily observe that:

$$u(x,\mathrm{t})=2\,\mathrm{x}\ , \qquad t=0, \tag{17}$$

And for $t>0$, the series solution (17) can be formally expressed in a closed form by,

$$u(x,t)=\frac{1}{4}\left(\sqrt{1+16xt}-1\right). \tag{18}$$

Combining (17) and (18) gives the solution in the form:

$$u(x,t)=\begin{cases}2x,\mathrm{t}=0,\\ \frac{1}{4}\left(\sqrt{1+16xt}-1\right),t>0.\end{cases} \tag{19}$$

II: Using (LTVIM)

Taking the Laplace Transform for two sides in eq.(13):

$$
\begin{aligned}
&\ell\left[u_t\right]+\ell\left[u^2u_x\right]=0\\
&s\ell\left[u(x,t)\right]-u(x,0)+\ell\left[u^2u_x\right]=0\\
&s\ell\left[u(x,t)\right]=2x-\ell\left[u^2u_x\right]\\
&\ell\left[u(x,t)\right]=\frac{2x}{s}-\frac{1}{s}\ell\left[u^2u_x\right]
\end{aligned}
\tag{20}
$$

Now, take the inverse Laplace Transform:

$$u(x,t)=2x-\ell^{-1}\left[\frac{1}{s}\ell\left(u^2u_x\right)\right] \tag{21}$$

The correction functions:

$$u_{n+1}(x,t) = 2x - \ell^{-1}\left[\frac{1}{s}\ell\left(u_n{}^2 u_{nx}\right)\right] \tag{22}$$

Take $u_0(x,t) = 2x$,

Then

$$\begin{aligned}
&u_0(x,t) = 2x \\
&u_1(x,t) = 2x - \ell^{-1}\left[\frac{1}{s}\ell\left(u_0^2 u_{0x}\right)\right] = 2x - 8x^2 t \\
&u_2(x,t) = 2x - \ell^{-1}\left[\frac{1}{s}\ell\left(u_1^2 u_{1x}\right)\right] = 2x - 8x^2 t + 64x^3 t^2 - \frac{640}{3}x^4 t^3 + 256x^5 t^4 \\
&u_3(x,t) = 2x - \ell^{-1}\left[\frac{1}{s}\ell\left(u_2^2 u_{2x}\right)\right] = 2x - 8x^2 t + 64x^3 t^2 - 640x^4 t^3 + 4608x^5 t^4 \\
&: \\
&u_{n+1}(x,t) = 2x - 8x^2 t + 64x^3 t^2 - 640x^4 t^3 + 7168x^5 t^4 + \ldots
\end{aligned} \tag{23}$$

If $t = 0,$ then $u(x,t) = 2x$ .And if $t > 0,$ then $u(x,t) = \frac{1}{4}\left(\sqrt{1+16xt} - 1\right).$

Then: $u(x,t) = \begin{cases} 2x, \mathrm{t} = 0, \\ \frac{1}{4}\left(\sqrt{1+16xt} - 1\right), t > 0. \end{cases}$

Notes on (VIM) and (LTVIM):

From the past examination, we can watch that:

The two methods are effective and productive. Laplace Transform and Variation Iteration method gives the exact solution, where these components given in (3). However, application of the Laplace Transform to the solution of linear partial differential equations has been illustrated.

Sec (5.2): Comparison Between Laplace Transform and Variations Iteration Method and Homotopy Perturbation Transform Method for Nonlinear Equations Using He's Polynomials

In this section, a combined form of the Laplace transform method with the homotopy perturbation method is proposed to solve nonlinear equations. This method is called the homotopy perturbation transform method (HPTM). The nonlinear terms can be easily handled by the use of He's polynomials. The proposed scheme finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method. Nonlinear phenomena have important effects on applied mathematics, physics and issues related to engineering; manysuch physical phenomena are modeled in terms of nonlinear partial differential equations. For example, the advection problems which are of the form

$$u_t(x,\mathrm{t}) + \mathrm{uu}_x = h(x,t),\ u(x,0) = g(x) \qquad (24)$$

A rise in various branches of physics, engineering and applied sciences. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact or approximate solutions.

5.2.1:Basic Idea of HPTM

To illustrate the basic idea of this method, we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \qquad (25)$$

$$u(x,0) = h(x),\ u_t(x,0) = f(x)$$

where D is the second order linear differential operator $D = \partial^2/\partial t^2$, R is the linear differential operator of less order than D, N represents the general non-linear differential operator and $g(x,t)$ is the source term.

Taking the Laplace transform (denoted throughout this section by L) on both sides of Eq. (25):

$$L\left[Du(x,t)\right]+L\left[Ru(x,t)\right]+L\left[Nu(x,t)\right]=L\left[g(x,t)\right]. \tag{26}$$

Using the differentiation property of the Laplace transform, we have:

$$L\left[u(x,t)\right]=\frac{h(x)}{s}+\frac{f(x)}{s^2}-\frac{1}{s^2}L\left[Ru(x,t)\right]+\frac{1}{s^2}L\left[gu(x,t)\right]-\frac{1}{s^2}L\left[Nu(x,t)\right]. \tag{27}$$

Operating with the Laplace inverse on both sides of Eq. (27) gives:

$$u(x,t)=G(x,t)-L^{-1}\left[\frac{1}{s^2}L\left[Ru(x,t)+Nu(\mathrm{x},\mathrm{t})\right]\right], \tag{28}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method,

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t) \tag{29}$$

and the nonlinear term can be decomposed as,

$$Nu(x,t)=\sum_{n=0}^{\infty}p^n H_n(u) \tag{30}$$

For some He's polynomials H_n that are given by:

$$H_n(u_0,...,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}\left(p^i u_i\right)\right)\right]_{p=0}, n=0,1,2,3,... \tag{31}$$

Substituting Eqs. (29) and (30) in Eq. (28) we get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=G(x,t)-p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)+\sum_{n=0}^{\infty}p^n H_n(\mathrm{u})\right]\right]\right), \tag{32}$$

Which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$
\begin{aligned}
&p^0 : u_0(x,t) = G(x,t),\\
&p^1 : u_1(x,t) = -\frac{1}{s^2} L\left[Ru_0(x,t) + H_0(u)\right],\\
&p^2 : u_2(x,t) = -\frac{1}{s^2} L\left[Ru_1(x,t) + H_1(u)\right],\\
&p^3 : u_3(x,t) = -\frac{1}{s^2} L\left[Ru_2(x,t) + H_2(u)\right],\\
&\vdots
\end{aligned}
\tag{33}
$$

5.2.2: Application

In order to elucidate the solution procedure of the homotopy perturbation transform method, we first consider the nonlinear advection equations.

Example (5.2.3) [66]:

Consider the following homogeneous advection problem:

$$
\begin{aligned}
&u_t + uu_x = 0,\\
&u(x,0) = -x.
\end{aligned}
\tag{34}
$$

I: Using (HPTM)

By applying the aforesaid method subject to the initial condition, we have,

$$u(x,s) = -\frac{x}{s} - \frac{1}{s} L\left[uu_x\right]. \tag{35}$$

The inverse of the Laplace transform implies that:

$$u(x,t) = -x - L^{-1}\left[\frac{1}{s} L\left[uu_x\right]\right]. \tag{36}$$

Now, we apply the homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = -x - p\left(L^{-1}\left[\frac{1}{s} L\left[\sum_{n=0}^{\infty} p^n H_n(\mathrm{u})\right]\right]\right), \tag{37}$$

where $H_n(u)$ are He’s polynomials [48,49] that represent the nonlinear terms.The first few components of He’s polynomials, for example, are given by:

$$
\begin{aligned}
&H_0(u) = u_0 u_{0x} , \\
&H_1(u) = u_0 u_{1x} + u_1 u_{0x} , \\
&H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} , \\
&\vdots
\end{aligned} \tag{38}
$$

Comparing the coefficient of like powers of p, we have:

$$
\begin{aligned}
&p^0 : u_0(x,t) = -x , \\
&p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s} L\left[H_0(u)\right]\right] = -xt \\
&p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s} L\left[H_1(u)\right]\right] - xt^2
\end{aligned} \tag{37}
$$

Proceeding in a similar manner, we have:

$$
\begin{aligned}
&p^3 : u_3(x,t) = -xt^3 , \\
&p^4 : u_4(x,t) = -xt^4 , \\
&\vdots
\end{aligned} \tag{38}
$$

so that the solution $u(x,t)$ is given by:

$$
u(x,t) = -x\left(1 + t + t^2 + t^3 + t^4 + \ldots\right), \tag{39}
$$

in series form, and

$$
u(x,t) = \frac{x}{t-1}, \tag{40}
$$

in closed form.

II: Using (LTVIM)

Taking the Laplace transform of the equation (32), subject to the initial condition, we have:

$$
\ell\left[u(\mathrm{x},\mathrm{t})\right] = -\frac{x}{s} - \frac{1}{s}\ell\left[uu_x\right] \tag{41}
$$

The inverse Laplace transform implies that:

$$
u(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \ell^{-1}\left\{\frac{1}{s}\ell\left[uu_x\right]\right\} \tag{42}
$$

By the correction function, we find:

$$u_{n+1}(\mathrm{x,t}) = -\mathrm{x} - \ell^{-1}\left\{\frac{1}{s}\ell\left[u_n\left(u_n\right)_x\right]\right\} \tag{43}$$

Now we apply the variational iteration Laplace transform method,

$$\begin{aligned}
&u_0(\mathrm{x,t}) = -\mathrm{x} \\
&u_1(\mathrm{x,t}) = -\mathrm{x} - \ell^{-1}\left\{\frac{1}{s}\ell\left[x\right]\right\} = -x - \ell^{-1}\left[\frac{x}{s^2}\right] = -x - xt \\
&u_2(\mathrm{x,t}) = -\mathrm{x} - \ell^{-1}\left[x\left(\frac{1}{s^2}+\frac{2}{s^3}+\frac{2}{s^4}\right)\right] = -x - xt - xt^2 - \frac{1}{3}xt^3 \\
&\quad \vdots \qquad\qquad \vdots
\end{aligned} \tag{44}$$

So we deduce the series solution to be,

$$u(\mathrm{x,t}) = -\mathrm{x}(1+\mathrm{t}+\mathrm{t}^2+\mathrm{t}^3+\ldots) = \frac{x}{t-1} \tag{45}$$

Which is the exact solution.

Example (5.2.4) [66]:

We now consider the nonhomogeneous advection problem,

$$\begin{aligned}
&u_t + uu_x = 2t + x + t^3 + xt^2, \\
&u(x,0) = 0.
\end{aligned} \tag{46}$$

I: Using (HPTM)

In a similar way as above, we have:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 + xt + \frac{t^4}{4} + x\,\frac{t^3}{3} - p\left(L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right). \tag{47}$$

Comparing the coefficient of like powers of p, we have:

$$p^0 : u_0(x,t) = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3},$$
$$p^1 : u_1(x,t) = -\frac{1}{4}t^4 - \frac{1}{3}t^3 - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8,$$
$$p^2 : u_2(x,t) = \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} + \frac{38}{2835}xt^9 + \frac{143}{2880}t^8 + \frac{22}{315}xt^7 + \frac{7}{12}t^6 + \frac{2}{15}xt^5,$$
:

It is imperative to review here that the commotion terms show up between the parts $u_0(x,t)$ and $u_1(x,t)$ where the clamor terms are those sets of terms that are indistinguishable yet conveying inverse signs. All the more exactly, the clamor terms between the segments $u_0(x,t)$ and $u_0(x,t)$, can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation.The exact solution is therefore,

$$u(x,t) = t^2 + xt. \tag{48}$$

II: Using (LTVIM)

Taking the Laplace Transform for two sides in eq. (46) :

$$\begin{aligned} &\ell[u_t] + \ell[uu_x] = \ell\left[2t + x + t^3 + xt^2\right], \\ &s\ell[u(x,t)] + \ell[uu_x] = \frac{2}{s^2} + \frac{x}{s} + \frac{3!}{s^4} + \frac{2!x}{s^3}, \\ &\ell[u(x,t)] = \frac{2}{s^3} + \frac{x}{s^2} + \frac{3!}{s^5} + \frac{2!x}{s^4} - \frac{1}{s}\ell[uu_x] \end{aligned} \tag{49}$$

Then we have:

$$u(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - \ell^{-1}\left[\frac{1}{s}\ell[uu_x]\right] \tag{50}$$

by the correction function, we find:

$$u_{n+1}(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - \ell^{-1}\left[\frac{1}{s}\ell[u_n u_{nx}]\right] \tag{51}$$

Choose, $u_0(x,t) = 0$

then:

$$u_0(x,t)=0,$$

$$u_1(x,t)=t^2+xt+\frac{t^4}{4}+\frac{xt^3}{3}-\ell^{-1}\left[\frac{1}{s}\ell\left[u_0u_{0x}\right]\right]=t^2+xt+\frac{t^4}{4}+\frac{xt^3}{3},$$

$$u_2(x,t)=t^2+xt+\frac{t^4}{4}+\frac{xt^3}{3}-\ell^{-1}\left[\frac{1}{s}\ell\left[u_1u_{1x}\right]\right]=t^2+xt-\frac{xt^4}{24}-\frac{t^5}{5}-\frac{xt^5}{90}-\frac{t^6}{24}-\frac{xt^7}{63}-\frac{t^8}{96},$$

:

$$\mathrm{u}_n(x,t)=t^2+xt+\frac{t^4}{4}+\frac{xt^3}{3}-t^3-xt^2+\frac{19t^6}{120}-\frac{xt^6}{45}-\frac{66t^7}{2160}+\ldots,$$

$$\mathrm{u}_n(x,t)=t^2+xt+\frac{t^4}{4}+\frac{xt^3}{3}+noise\ \ term$$

The exact solution is

$$\mathrm{u}(x,t)=t^2+xt+\frac{t^4}{4}+\frac{xt^3}{3} \qquad (52)$$

Sec (5.3): A comparison Between the Variational Iteration Method and Adomain Decomposition Method

In this section, we present a comparative study between the variational iteration method and Adomian decomposition method. The study outlines the significant features of the two methods. The analysis will be illustrated by investigating the homogeneous and the nonhomogeneous advection problems.
This section outlines a reliable comparison between two powerful methods that were recently developed. The first is the variational iteration method (VIM) developed by He and used by many others. Thesecond is the decomposition method (ADM) developed by Adomian in [72,73], and used heavily in the literaturein [76,77] and the references therein. The two methods give rapidly convergent series with specific significant features for each scheme. The homogeneous and the nonhomogeneous advection problem

$$u_t + uu_x = f(x,t) \tag{53}$$

where$u = u(x,t)$, will be utilized as a vehicle for this study. For $f(x,t) = 0$, Eq.(53) lessens to the homogeneous shift in the weather conditions model. The nonlinear shift in the weather conditions equation (53) emerges in the portrayal of different physical procedures. The presence of nontrivial careful arrangements is the subject of physical interest. Such correct arrangements are critical on the grounds that numerical arrangements may not distinguish the exploratory marvel under scrutiny .
A considerable measure of exploration work has been coordinated for the investigation of the nonlinear issues, and on the shift in weather condition issue specifically. In this section, our work stems for the most part of two of the most as of late created strategies, the VIM and ADM. The two techniques, which precisely process the arrangements in an arrangement structure or in a definite structure, are of awesome enthusiasm for connecting sciences .
The principle point of interest of the two techniques is that it can be connected specifically for a wide range of different and essential conditions, homogeneous or inhomogeneous. Another critical favorable position is that the strategies are prepared to do extraordinarily decreasing the measure of computational work while as yet keeping up the high precision of the numerical arrangement. The viability what's more, the value of both strategies is exhibited by finding accurate answers for the models that will be researched. In any case, every strategy has its own particular trademark and note worthiness that will be inspected.

5.3.1: Basic Idea of (ADM):

Adomian decomposition method defines the unknown function $u(x)$ by an infinite series,

$$u(\mathrm{x}) = \sum_{n=0}^{\infty} u_n(\mathrm{x}), \tag{54}$$

where the components $u_n(\mathrm{x})$ are usually determined recurrently. The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by:

$$F(\mathrm{u}) = \sum_{n=0}^{\infty} \mathrm{A}_n, \tag{55}$$

where A_n are the so-called Adomian polynomials of $u_n, u_1, \ldots, \mathrm{u}_n$ defined by:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \ \text{n=0, 1, 2,...}, \tag{56}$$

or equivalently:

$$\begin{aligned}
A_0 &= F(\mathrm{u}_0), \\
A_1 &= u_1 F'(\mathrm{u}_0), \\
A_2 &= u_2 F'(\mathrm{u}_0) + \tfrac{1}{2}u_1^2 F''(\mathrm{u}_0) \\
A_3 &= u_3 F'(\mathrm{u}_0) + u_1 u_2 F''(\mathrm{u}_0) + \tfrac{1}{3}u_1^3 F'''(\mathrm{u}_0), \\
A_4 &= u_4 F'(\mathrm{u}_0) + (u_1 u_3 + \tfrac{1}{2}\mathrm{u}_2^2)F''(\mathrm{u}_0) + \tfrac{1}{2}u_1^2 u_2 F'''(\mathrm{u}_0) + \tfrac{1}{24}\mathrm{u}_1^4 \mathrm{F}^{(iv)}(\mathrm{u}_0).
\end{aligned} \tag{57}$$

It is presently surely understood that these polynomials can be created for all classes of nonlinearity as indicated by particular calculations characterized by (56). As of late, an option calculation for building Adomian polynomials. The variational cycle strategy gives a few progressive approximations through utilizing the emphasis of the rectification useful .

In what takes over, a homogeneous and a nonhomogeneous shift in weather condition issue will be analyzed by utilizing the two plans exhibited previously. The two physical models will be utilized for illustrative purposes with respect to the examination objective.

Example (5.3.1) [72]:

We first consider the homogeneous advection problem,

$$u_t + uu_x = 0,$$
$$u(\mathrm{x},0) = -\mathrm{x}. \qquad (58)$$

I: Using (VIM)

The correction, functional for (58) reads as:

$$u_{n+1}(\mathrm{x},\mathrm{t}) = \mathrm{u}_n(\mathrm{x},\mathrm{t}) + \int_0^t \lambda(\xi)\left(\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial \xi} + \tilde{u}_n(\mathrm{x},\xi)\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial x}\right) d\xi, \qquad (59)$$

This yields the stationary conditions

$$\lambda'(\xi) = 0,$$
$$1 + \lambda(\xi) = 0. \qquad (60)$$

This in turn gives

$$\lambda = -1. \qquad (61)$$

Substituting this value of the Lagrangian multiplier into functional (59) gives the iteration formula

$$u_{n+1}(\mathrm{x},\mathrm{t}) = \mathrm{u}_n(\mathrm{x},\mathrm{t}) - \int_0^t \left(\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial \xi} + \tilde{u}_n(\mathrm{x},\xi)\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial x}\right) d\xi, \mathrm{n} \geq 0. \qquad (62)$$

As stated before, we can use any selective function for u_0; preferably we use the initial condition $u_0 = -x$. Consequently, using (62) yields the following successive approximations:

$$u_0(\mathrm{x},\mathrm{t}) = -\mathrm{x},$$
$$\mathrm{u}_1(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \mathrm{xt},$$
$$\mathrm{u}_2(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \mathrm{xt} - \mathrm{xt}^2 - \tfrac{1}{3}xt^3,$$
$$u_3(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \mathrm{xt} - \mathrm{xt}^2 - xt^3 - \tfrac{2}{3}xt^4 - small\ \ terms,$$
$$u_4(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \mathrm{xt} - \mathrm{xt}^2 - xt^3 - xt^4 - \tfrac{14}{15}xt^5 - small\ \ terms,$$
$$u_5(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \mathrm{xt} - \mathrm{xt}^2 - xt^3 - xt^4 - xt^5 - small\ \ terms,$$
$$\vdots$$
$$\mathrm{u}_n(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \mathrm{xt} - \mathrm{xt}^2 - xt^3 - xt^4 - xt^5 - small\ \ terms.$$

Recall that

$$u = \lim_{n\to\infty} u_n \qquad (63)$$

which gives

$$u(\mathrm{x},\mathrm{t}) = -\mathrm{x}(1 + \mathrm{t} + \mathrm{t}^2 + t^3 + t^4 + ...), \qquad (64)$$

which leads to the closed form solution

$$u(x,t)=\frac{x}{t-1}. \tag{65}$$

II: Using (ADM)

We first rewrite Eq. (58) in an operational form

$$Lu=-uu_x,$$
$$u(x,0)=-x, \tag{66}$$

where the differential operator L is,

$$L=\frac{\partial}{\partial t}. \tag{67}$$

The inverse L^{-1} is assumed as an integral operator given by:

$$L^{-1}(.)=\int_0^t (.)\,dt. \tag{68}$$

Applying the inverse operator L^{-1} on both sides of (66) and using the initial condition we find,

$$u(x,t)=-x-L^{-1}(uu_x). \tag{69}$$

Substituting (54) and (55) into the functional equation (66) gives:

$$\sum_{n=0}^{\infty}u_n(x,t)=-x-L^{-1}\left(\sum_{n=0}^{\infty}A_n\right), \tag{70}$$

where A_n are the so-called Adomian polynomials. Identifying the zeroth component $u_0(x,t)$ by $-x$, the remaining components $u_n(x,t),\ n\geq 1$, can be determined by using the recurrence relation

$$u_0(x,t)=-x,$$
$$u_{k+1}(x)=-L^{-1}(A_k),\ k\geq 0\ , \tag{71}$$

where A_k are Adomian polynomials that represent the nonlinear term, uu_x and given by:

$$A_0=u_0u_{0_x},$$
$$A_1=u_0u_{1_x}+u_1u_{0_x},$$
$$A_2=u_0u_{2_x}+u_1u_{1_x}+u_2u_{0_x},$$
$$\vdots \tag{72}$$

Other polynomials can be generated in a similar way to enhance the accuracy of the approximation.
Combining (71) and (72) yields:

$$u_0(\mathrm{x,t}) = -x\,,$$
$$u_1(\mathrm{x,t}) = -\mathrm{xt},$$
$$\mathrm{u}_2(\mathrm{x,t}) = -\mathrm{xt}^2,$$
$$u_3(\mathrm{x,t}) = -\mathrm{xt}^3,$$
$$u_4(\mathrm{x,t}) = -\mathrm{xt}^4,$$
$$\vdots \tag{73}$$

In view of (73), the solution $u(x,t)$ is readily obtained in a series form by:

$$u(\mathrm{x,t}) = -\mathrm{x}(1+\mathrm{t}+\mathrm{t}^2+t^3+t^4+.....)\,, \tag{74}$$

or in a closed form by:

$$u(\mathrm{x,t}) = \frac{x}{t-1}. \tag{75}$$

Example (5.3.2) [72]:

We first consider the nonhomogeneous advection problem,

$$u_t + uu_x = 2t + x + t^3 + xt\;. \tag{76}$$

I: Using (VIM)
The correction, functional for (76) reads as

$$u_{n+1}(\mathrm{x,t}) = \mathrm{u}_n(\mathrm{x,t}) - \int_0^t \lambda(\xi)\left(\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial \xi} + \tilde{u}_n(\mathrm{x},\xi)\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial x} - (2\xi + \mathrm{x} + \xi^3 + \mathrm{x}\,\xi^2)\right)d\xi$$

Proceeding as before, we find the stationary conditions,
$\lambda'(\xi) = 0,$

$$1 + \lambda(\xi) = 0\,. \tag{77}$$

This in turn gives

$$\lambda = -1\,. \tag{78}$$

Substituting this value of the Lagrangian multiplier into functional, gives the iteration formula,

$$u_{n+1}(\mathrm{x,t}) = t^2 + \mathrm{xt} + \tfrac{1}{4}\mathrm{t}^4 + \tfrac{1}{3}\mathrm{xt}^3\mathrm{u}_n(\mathrm{x,t}) - \int_0^t \lambda(\xi)\left(\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial \xi} + \tilde{u}_n(\mathrm{x},\xi)\frac{\partial \mathrm{u}_n(\mathrm{x},\xi)}{\partial x}\right)d\xi\,,$$

Acquired after incorporating the source nonhomogeneous term, $n \geq 0$. As expressed some time recently, we can utilize any specific function for u_0; ideally we utilize the initial condition $u(x,0)=0$.

$$
\begin{aligned}
&u_0(\mathrm{x,t}) = t^2 + \mathrm{xt} + \tfrac{1}{4}\mathrm{t}^4 + \tfrac{1}{3}\mathrm{xt}^3,\\
&\mathrm{u}_1(\mathrm{x,t}) = t^2 + \mathrm{xt} + \tfrac{1}{4}\mathrm{t}^4 + \tfrac{1}{3}\mathrm{xt}^3,\\
&\mathrm{u}_2(\mathrm{x,t}) = t^2 + \mathrm{xt} - \tfrac{2}{15}\mathrm{xt}^5 - \tfrac{7}{72}\mathrm{t}^6,\\
&u_3(\mathrm{x,t}) = \mathrm{t}^2 - xt - small\ \ terms,\\
&u_4(\mathrm{x,t}) = \mathrm{t}^2 - xt - small\ \ terms,\\
&u_5(\mathrm{x,t}) = \mathrm{t}^2 - xt - small\ \ terms,\\
&:\\
&\mathrm{u}_n(\mathrm{x,t}) = \mathrm{t}^2 - xt - small\ \ terms.
\end{aligned} \tag{79}
$$

Recall that

$$u = \lim_{n\to\infty} u_n , \tag{80}$$

which gives the exact solution,

$$\mathrm{u(x,t)} = \mathrm{t}^2 - xt \ . \tag{81}$$

II: Using (ADM)

We first rewrite Eq. (75) in an operational form,

$$
\begin{aligned}
Lu &= 2t + x + t^3 + xt^2 - uu_x ,\\
u(\mathrm{x},0) &= 0
\end{aligned} \tag{82}
$$

where the differential operator L is,

$$L = \frac{\partial}{\partial t} . \tag{83}$$

The inverse L^{-1} is assumed as an integral operator given by:

$$L^{-1}(.) = \int_0^t (.)\,\mathrm{dt} . \tag{84}$$

Applying the inverse operator L^{-1} on both sides of (82) and using the initial condition we find:

$$u(\mathrm{x,t}) = t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3 - L^{-1}(\mathrm{uu}_x) . \tag{85}$$

Substituting (54) and (55) into the functional equation (82) gives:

$$\sum_{n=0}^{\infty} u_n(\mathrm{x,t}) = t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3 - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{86}$$

where A_n are the so-called Adomian polynomials. Identifying the zeroth component $u_0(\mathrm{x,t})$ by $t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3$, the remaining components $u_n(\mathrm{x,t}),\ \mathrm{n} \geq 1$, can be determined by using the recurrence relation,

$$u_0(\mathrm{x},\mathrm{t}) = t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3,$$
$$u_{k+1}(\mathrm{x}) = -\mathrm{L}^{-1}(\mathrm{A}_k),\ \mathrm{k} \geq 0, \tag{87}$$

where A_k are Adomian polynomials that were evaluated before in the homogeneous case. This in turn gives the components

$$u_0(\mathrm{x},\mathrm{t}) = t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3,$$
$$u_1(\mathrm{x},\mathrm{t}) = -\tfrac{1}{4}t^4 - \tfrac{1}{3}xt^3 - \tfrac{2}{15}\mathrm{xt}^5 - \tfrac{7}{72}\mathrm{t}^6 - \tfrac{1}{63}\mathrm{xt}^7 - \tfrac{1}{96}\mathrm{t}^8, \tag{88}$$
$$\vdots$$

It is essential to review here that the commotion terms show up between the two parts u_0 and u_1. The noise terms are distinguished as the indistinguishable terms with inverse signs. We then drop the noise terms between the parts $\pm\frac{1}{4}t^4$ and $\pm\frac{1}{4}xt^3$, and legitimize that the remaining terms of fulfill the condition. Thus, the exact solution is

$$u(\mathrm{x},\mathrm{t}) = t^2 + xt\ . \tag{89}$$