Sudan University of Science and Technology

College of Graduate Studies

Solution of Linear and Nonlinear Partial Differential Equations by Mixing Adomian Decomposition Method and Sumudu Transform

حل المعادلات التفاضلية الجزئية الخطية و غير الخطية بخلط طريقتي تفكيك ادوميان و تحويل سمودو

A Thesis Submitted in Fulfillment Requirements for the Degree of Doctor of Philosophy in Mathematics

By

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Dedication

I would like to dedicate this achievement to all of my family.

My wife,

Who were my first math teachers, thank you for helping me becomes who I am.

Acknowledgement

All praise is to Almighty Allah for providing me with the blessing and the strength to complete this work. I would like to express my sincere thanks to Dr. Tarig Mohyeldin Elzaki for his continual support and encouragement. It has truly been a great honour to work with him. I would also like to thank Dr. Mohamed Hassan Mohamed, who cooperated in this study.

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Abstract

This study is fundamentally centering on the application of the Adomian decomposition method and Sumudu transform for solving the linear and nonlinear partial differential equations.

It has instituted some theorems, definitions, and properties of Adomian decomposition and Sumudu transform. This study is an elegant combination of the Adomian decomposition method and Sumudu transform. Consequently, it provides the solution in the form of convergent series. Then, it is applied to solve linear and nonlinear partial differential equations.

Finally, the solutions of linear and nonlinear partial differential equations by this method, and other methods are compared.

الخلاصة

تتمحور هذه الدراسة اساسا علي تطبيق طريقتي تفكيك ادوميان وتحويل سمودو لحل المعادلات التفاضلية الجزئية الخطية و غير الخطية.

ستقوم الدراسة بوضع بعض النظريات, التعريفات, والخصائص لطريقتي تفكيك ادوميان وتحويل سمودو.هذه الدراسة هي مزيج رائع من طريقتي تفكيك ادوميان وتحويل سمودو, وبناء علي ذلك,ايجاد الحل في شكل متسلسلة متقاربة. من ثم طبقت الدراسة لحل المعادلات التفاضلية الجزئية الخطية وغير الخطية.

واخيرا تمت مقارنة حلول المعادلات التفاضلية الجزئية الخطية وغير الخطية مع طرق مختلفة.

Introduction:-

Many of nonlinear phenomena are a necessary part in applied science and engineering fields. Nonlinear equations are noticed in a different type of physical problems such as fluid dynamics, plasma physics, solid mechanics, quantum field theory, propagation of shallow water waves, and many other models are controlled within its domain of validity by partial differential equations. The wide use of these equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically. In the past, active study attempts were given a large amount of attention to the study of getting exact or approximate solutions of this kind of equations.

Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving partial differential equations. For some examples of the traditional methods, such as, the separation of variables method, the method of characteristics, the σ - expansion method [60], the integral transforms and Hirota bilinear method [61].Moreover, the recently developed methods like, Adomian decomposition method (ADM), He’s semi – inverse method, the tanh method, the sinh – cosh method, the homotopy perturbation method (HPM) [62-73], the differential transform method (DTM) , the variational iteration method (VIM) [74-78], and the weighted finite difference.

Other techniques including the Laplace decomposition method (LDM) [79-85], the homotopy perturbation transform method (HPTM) [86-88], and variational iteration algorithms using the Laplace transform [89], have been also used.

In this research, our presentation will be based on applying the new method namely, the Adomian Decomposition Sumudu Transform Method (ADSTM) for solving linear and nonlinear differential equations, ordinary or partial and integral equations. This method is an elegant combination of the Sumudu transform method and decomposition method. The method has advantages of converting to the exact solution and can easily handle a wide class of both linear and nonlinear differential and integral equations.

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CHAPTER (1)
Linear Partial Differential Equations

1.1: Sumudu Transform

A long time ago, differential equations warred a necessary part in all aspects of applied science and engineering fields. Despite this, they are not easy to find an answer, either numerically or theoretically for these equations. In order to develop new techniques help in obtaining exact and approximate solutions of these equations is still a big problem need new methods.

Watugula [1] introduced a new integral transform and called it as Sumudu transform, which is defined as:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u} e^{\left(-\frac{t}{u}\right)} f(t)\, dt \ ; \tag{1}$$

Watugula [1] applied this transforms to the solution of ordinary differential equations. Because of its useful properties, the Sumudu transforms helps in solving complex problems in applied sciences and engineering mathematics. Henceforward, is the definition of the Sumudu transforms and properties describing the simplicity of the transform.

Definition (1.1.1): The Sumudu transform of the function $f(t)$ is defined by:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u} e^{\left(-\frac{t}{u}\right)} f(t)\, dt \tag{2}$$

Or

$$F(u) = S[f(t)] = \int_0^\infty f(u\,t)\, e^{-t}\, dt \tag{3}$$

For any function $f(t)$ and $-\tau_1 < u < \tau_2$

Theorem (1.1.2): If $S[f(t)] = F(u)$ and

$$g(t) = \begin{cases} f(t-\tau) , & t \geq \tau \\ 0 , & t \leq \tau \end{cases}$$

Then

$$S\left[g(t)\right] = e^{\left(-\frac{t}{u}\right)} G(u)$$

Theorem (1.1.3) [2]: If $c_1 \geq 0$, $c_2 \geq 0$ and $c \geq 0$ are any constant, $f_1(t)$, $f_2(t)$ and $f(t)$ any functions having the Sumudu transform $G_1(u)$, $G_2(u)$ and $G(u)$ respectively then:

i. $S\left[c_1\ f_1(t) + c_2\ f_2(t)\right] = c_1\ S\left[f_1(t)\right] + c_2\ S\left[f_2(t)\right]$

$= c_1 G_1(u) + c_2 G_2(u)$

ii. $S\left[f(c\,t)\right] = G(c\,u)$

iii. $\lim\limits_{t \to 0} f(t) = f(0) = \lim\limits_{u \to 0} G(u)$

Further are worded more, for several functions $f(t)$ defined for $t \geq 0$ in the neighborhood of infinity (i.e. as $t \to \infty$)

$$\lim_{t \to \infty} f(t) = \lim_{u \to \infty} G(u)$$

1.1.1: The Relation Between Sumudu and Laplace Transform

The Sumudu transform $F_s(u)$ of a function $f(t)$ defined for all real numbers $t \geq 0$. The Sumudu transform is essentially identical with the Laplace transform.
Given an initial $f(t)$ its Laplace transform $G(u)$ can be translated into the Sumudu transform $F_s(u)$ of f by means of the relation;

$$F(u) = \frac{G\left(\frac{1}{u}\right)}{u}$$

And it's inverse

$$G(s) = \frac{F_s\left(\frac{1}{s}\right)}{s}$$

Theorem (1.1.4): Let $f(t)$ with Laplace transform $G(s)$ then the Sumudu transform $F(u)$ of $f(t)$ is given by

$$F(u) = \frac{G\left(\frac{1}{u}\right)}{u} \; .$$

Proof:

Form definition (1.1.1) we get:

$$F(u) = \int_0^\infty e^{-t} f(u\,t)\, dt$$

If we set $w = u\,t$ and $dt = \frac{dw}{u}$ then

$$F(u) = \int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w)\frac{dw}{u} = \frac{1}{u}\int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w)dw$$

By definition of the Laplace transform we get:

$$F(u) = \frac{G\left(\frac{1}{u}\right)}{u}$$

Theorem (1.1.5): It deals with the effect of the differentiation of the function $f(t)$, k times on the Sumudu transform $F(u)$ if $S[f(t)] = F(u)$ then:

i. $S[f'(t)] = \frac{1}{u}[F(u) - f(0)]$

ii. $S[f''(t)] = \frac{1}{u^2}[F(u)] - \frac{1}{u^2} f(0) - \frac{1}{u} f'(u)$

iii. $S\left[f^{(n)}(t)\right] = \frac{1}{u^n}[F(u)] - \frac{1}{u^n}\sum_{k=0}^{n-1} u^k \; f^{(k)}(0) \; = \; u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k \; f^{(k)}(0)\right]$

Where $f^{(0)}(0) = f(0)$, $f^{(k)}(0)$, $k = 1, 2, 3, \cdots, n-1$ are the nth-order derivatives of the function $f(t)$ evaluated at $t = 0$.

Proof:

i. using integration by parts,

$$S\left[f'(t)\right] = \left[\frac{1}{u}\exp\left(-\frac{t}{u}\,f(t)\right)\right]_0^{\infty} + \frac{1}{u}\int_0^{\infty}\frac{1}{u}\exp\left(-\frac{t}{u}\right) f(t)\,dt$$

$$= -\frac{1}{u}f(0) + \frac{1}{u}F(u)$$

$$S\left[f'(t)\right] = \frac{1}{u}\left[F(u) - f(0)\right]$$

ii. Using integration by parts;

$$S\left[f''(t)\right] = \left[\frac{1}{u}e^{\left(-\frac{t}{u}\right)} f'(t)\right]_0^{\infty} + \frac{1}{u}\int_0^{\infty}\frac{1}{u}e^{\left(-\frac{t}{u}\right)} f'(t)\,dt$$

From (i)

$$= -\frac{1}{u}f'(0) + \frac{1}{u}S\left[f'(t)\right]$$

$$S\left[f''(t)\right] = \frac{1}{u^2}\left[F(u)\right] - \frac{1}{u^2}f(0) - \frac{1}{u}f'(0)$$

iii. By definition the Laplace transform for $f^{(n)}(t)$ is given by

$$G_n(s) = s^n G(s) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^{(k)}(0)$$

By using the relation between Sumudu and Laplace transform;

$$G_n\left(\frac{1}{u}\right) = \frac{G\left(\frac{1}{u}\right)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-(k+1)}}$$

Since $F_n(u) = \dfrac{G_n\left(\frac{1}{u}\right)}{u^n}$ we get:

$$u\,F_n(u) = \frac{u\,F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}\,u^{-1}}$$

$$F_n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}}$$

$$F_n(u) = u^{-n}F(u) - \sum_{k=0}^{n-1}u^{-n}u^{k}\,f^{(k)}(0)$$

$$S\left[f^{(n)}(t)\right] = F(u) = u^{-n}\left[F(u) - \sum_{k=0}^{n-1}u^{k}\,f^{(k)}(0)\right]$$

Theorem (1.1.6): Let $f(t)$ be a function with the Sumudu transform $F(u)$ then;

$$S\left[e^{at}\ f(t)\right]=\frac{1}{1-au}\ F\left(\frac{u}{1-au}\right)$$

Proof:

$$S\left[e^{at} f(t)\right]=\int_0^\infty f(ut)e^{aut}\ e^{-t}\ dt\ =\int_0^\infty f(ut)e^{(-(1-au)t)}\ dt$$

Let $w=(1-au)t \quad \Rightarrow \quad dt=\dfrac{dw}{1-au}$

$$S\left[e^{at}\ f(t)\right]=\frac{1}{1-au}\int_0^\infty f\left(\frac{uw}{1-au}\right)e^{-w}\ dw$$

$$S\left[e^{at}\ f(t)\right]=\frac{1}{1-au}\ F\left(\frac{u}{1-au}\right)$$

Theorem (1.1.7) [3]: This theorem deals with multiplication of the function $f(t)$ by a power series of t , if:

i. $S[t\ f(t)]=u^2\dfrac{d}{du}F(u)+u\,F(u)$

ii. $S\left[t^2\ f(t)\right]=u^4\dfrac{d^2}{du^2}F(u)+4u^3\dfrac{d}{du}F(u)+2u^2\,F(u)$

iii. $S\left[t^n\ f(t)\right]=u^n\sum_{k=0}^{n}a_k^n\,u^k\,F_k(u)$

iv. $S\left[t^{n+1}\ f(t)\right]=u^{n+1}\sum_{k=0}^{n+1}a_k^{n+1}\,u^k\,F_k(u)$

Theorem (1.1.8): Let $f(t)$ and $g(t)$ having Laplace transforms $F(s)$ and $G(s)$ respectively, and Sumudu transform $M(u)$ and $N(u)$, respectively.
Then the Sumudu transform of the convolution of f and g.

$$(f*g)(t)=\int_0^\infty f(t)g(t-\tau)d\tau$$

Is given by:

$$S[(f*g)(t)]=uM(u)N(u)$$

Proof:

First, recall that the Laplace transforms of $(f * g)$ is given by:

$$L[(f * g)(t)] = F(s)G(s)$$

By using the relation between Sumudu and Laplace transform;

$$S[(f * g)(t)] = \frac{1}{u} L[(f * g)(t)]$$

And since $$M(u) = \frac{F\left(\frac{1}{u}\right)}{u} \ , \ N(u) = \frac{G\left(\frac{1}{u}\right)}{u}$$

The Sumudu transform of $(f * g)$ is obtained as follows;

$$S[(f * g)(t)] = \frac{F\left(\frac{1}{u}\right)G\left(\frac{1}{u}\right)}{u} = u \frac{F\left(\frac{1}{u}\right)}{u} \frac{G\left(\frac{1}{u}\right)}{u} = u \ M(u)N(u)$$

$$S[(f * g)(t)] = u \ M(u)N(u)$$

Theorem (1.1.9): Let $G(u)$ denote the Sumudu transform of the function $f(t)$ let $f^{(n)}(t)$ denote the nth derivative of $f(t)$ with respect to t and let $F_n(u)$ denote the nth derivative of $F(u)$ with respect to, u, then the Sumudu transform of the function $t^n f^{(n)}(t)$ is given by:

$$S\left[t^n f^{(n)}(t)\right] = u^n F_n(u)$$

Proof:

Let the Sumudu transform of $f(t)$;

$$F(u) = \int_0^\infty f(u t)e^{-t} \ dt$$

Therefore, for $n = 0, 1, 2, \ldots$ we get:

$$F_n(u) = \int_0^\infty \frac{d^n}{du^n} f(u t)e^{-t} \ dt = \int_0^\infty t^n f(u t)e^{-t} \ dt$$

$$F_n(u) = \frac{1}{u^n}\int_0^\infty (u t)^n f^n(u t)e^{-t} \ dt = \frac{1}{u^n} S\left[t^n \ f^n(t)\right]$$

$$\Rightarrow \ S\left[t^n \ f^n(t)\right] = u^n F_n(u)$$

Corollary (1.1.10) [2]:

Let $F_n(u)$ denote the nth derivative of $F_n(u) = S[f(t)]$, then

i. $S[t\, f'(t)] = u \dfrac{d\,F(u)}{du} = u\,F_1(u)$

ii. $S\left[t^2\, f'(t)\right] = u^2\left[2F_1(u) + u\,F_2(u)\right]$

iii. $S\left[t^3\, f'(t)\right] = u^3\left[6F_1(u) + 6u\,F_2(u) + u^2\,F_3(u)\right]$

iv. $S\left[t^4\, f'(t)\right] = u^4\left[12F_2(u) + 8u\,F_3(u) + u^2\,F_4(u)\right]$

The Sumudu transform method will be illustrated by discussing the following examples.

Example (1.1.11): Consider the following inhomogeneous partial differential equation [24]:

$$U_x(x,y) + U_y(x,y) = x + y; \tag{4}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \quad U(0,y) = 0.$$

Taking the Sumudu transform of (4) we get:

$$S[U_x(x,y)] + S\left[U_y(x,y)\right] = S[x+y] \tag{5}$$

$$\frac{d}{dx}U(x,u) + \frac{1}{u}[U(x,u) - U(x,0)] = x + u$$

$$\frac{d}{dx}U(x,u) + \frac{1}{u}U(x,u) = x + u$$

Thus we have the ordinary differential equation:

$$\frac{d}{dx}U(x,u) + \frac{1}{u}U(x,u) = x + u \tag{6}$$

The integrating factor is;

$$F = e^{\int \frac{1}{u}dx} = e^{\frac{x}{u}} \tag{7}$$

Then

$$U(x,u) = e^{\frac{-x}{u}}\left[\int e^{\frac{x}{u}}(x+u)dx + c\right] = xu + c\,e^{\frac{-x}{u}} \tag{8}$$

Since $U(x,0) = 0$ then $c \to 0$

Then

$$U(x,u) = \quad xu \tag{9}$$

Taking the inverse Sumudu transform;

$$U(x,y) = \ S^{-1}[xu] \tag{10}$$

$$U(x,y) = x\,y \tag{11}$$

Example (1.1.12): Consider the following one – dimensional heat equation [25]:

$$U_{xx} = 4U_t(x,y)\ ; \tag{12}$$

With initial condition:

$$U(x,0) = 2\sin\frac{\pi}{2}x\ \ ; \tag{13}$$

And boundary conditions:

$$U(0,t) = 0 \quad , \quad U(2,t) = 0\ . \tag{14}$$

Taking the Sumudu transform of (12) and using initial condition (13) we get:

$$\frac{d^2}{dx^2}U(x,u) - \frac{4}{u}U(x,u) = -\frac{8}{u}\sin\frac{\pi}{2}x \tag{15}$$

This is the second order differential equation.
First we find the homogeneous solution:

$$U_c(x,u) = Ae^{\frac{2}{\sqrt{u}}x} + Be^{-\frac{2}{\sqrt{u}}x} \tag{16}$$

Using boundary conditions:

$$U(0,t) = 0 \quad \Rightarrow U(0,u) = 0$$

$$U(2,t) = 0 \quad \Rightarrow U(2,u) = 0$$

This gives

$$0 = A + B \ \rightarrow (i)$$

$$0 = Ae^{\frac{4}{\sqrt{u}}} + Be^{-\frac{4}{\sqrt{u}}} \rightarrow (ii)$$

From (i) and (ii) we have only a trivial solution $A = B = 0$.
Second we find particular solution:

$$U_p(x,u) = -\frac{8}{u}.\frac{\sin\frac{\pi}{2}x}{D^2 - \frac{4}{u}} = 32.\left[\frac{1}{\pi^2 u + 16}\right]\sin\frac{\pi}{2}x \tag{17}$$

The general solution is:

$$U(x,u) = U_c + U_p = 32.\left[\frac{1}{\pi^2 u + 16}\right]\sin\frac{\pi}{2}x \tag{18}$$

Taking the inverse Sumudu transform we get:

$$U(x,t) = 2\sin\frac{\pi}{2}x e^{-\frac{\pi^2}{16}t} . \tag{19}$$

Example (1.1.13): Consider the following Laplace equation [25]:

$$U_{xx} + U_{tt} = 0 ; \tag{20}$$

With initial conditions:

$$U(x,0) = 0 \ , U_t(x,0) = \cos x . \tag{21}$$

Taking the Sumudu transform of (20) and using initial condition (21) we get:

$$u^2 \frac{d^2}{dx^2}U(x,u) + U(x,u) = u\cos x ; \tag{22}$$

This is the second order differential equation which has the particular solution in the form:

$$U(x,u) = \frac{u\cos x}{u^2 D^2 + 1} = \frac{u\cos x}{-u^2 + 1} . \tag{23}$$

If we take the inverse Sumudu for Eq. (23), we obtain the solution of Eq. (20) in the form:

$$U(x,t) = \cos x \sinh t . \tag{24}$$

Example (1.1.13): Consider the following wave equation [25]:

$$U_{tt} - 4U_{xx} = 0 ; \tag{25}$$

With initial conditions:

$$U(x,0) = \sin \pi x \ , U_t(x,0) = 0 . \tag{26}$$

Taking the Sumudu transform of (25) and using initial condition (26) we get:

$$4u^2 \frac{d^2}{dx^2}U(x,u) - U(x,u) = -\sin \pi x ; \tag{27}$$

This is the second order differential equation which has the particular solution in the form:

$$U(x,u) = \frac{-\sin \pi x}{4u^2 D^2 - 1} = \frac{\sin \pi x}{4u^2 \pi^2 + 1} \ . \tag{28}$$

If we take the inverse Sumudu for Eq. (28), we obtain the solution of Eq. (25) in the form:

$$U(x,t) = \cos 2\pi t \sin \pi x \ . \tag{29}$$

1.2: Adomian Decomposition Method

Partial differential equations are a necessary part in applied science and engineering fields. The wide use of these equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically. However, most of the methods developed in mathematics are used in solving differential equations.

In this section, a semi – analytical method named, Adomian decomposition method (ADM) will be applied. The Adomian decomposition method (ADM) was developed between the 1970s and 1990s by George Adomian [4-9] have been attracting the attentions of many mathematicians’, physicists, engineers, and various graduate researchers. The method has the advantage of converging to the exact solution and can easily handle a wide class of both linear and nonlinear differential and the integral equations. The assumptions made by Adomian be modified in (1999) by Wazwaz [10-20].

The purpose of this method is to find the solution of complex systems without usual modeling. This method generates a solution in the form of a series whose terms are determined by a recursive relationship.
Into the Adomian decomposition method (ADM) showed that this method can be successfully used to solve intricate problems in engineering mathematics and applied science.

To give a clear view of Adomian decomposition method, we first consider the linear differential equation written in an operator form by:

$$Lu + Ru = g \;; \qquad (30)$$

Where L is a lower order derivative which is assumed to be invertible, R is another linear differential operator, and g is a source term.

Now apply the inverse operator L^{-1} to both sides of Eq. (30) and using the given condition to obtain:

$$u = f - L^{-1}\left(Ru\right) . \qquad (31)$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed.

The Adomian decomposition method consists of decomposition the unknown function u of any equation into a sum of an infinite number of components defined by the decomposition series:

$$u = \sum_{n=0}^{\infty} u_n \;; \qquad (32)$$

where the components $u_0, u_1, ..., u_n$ are usually recurrently determined.

Substituting Eq. (32) into both sides of (31) leads to:

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n \right) \right) . \qquad (33)$$

To construct the recursive relation needed for the determination the components $u_0, u_1, ..., u_n$ it is important to note the Adomian method suggests that the zeroth components u_0 is usually defined by the function f described above , i.e.

According, the formal recursive relation is defined by:

$$\begin{aligned} u_0 &= f \;; \\ u_{k+1} &= -L^{-1}\left(R(u_k)\right);\; k \geq 0 . \end{aligned} \qquad (34)$$

Having determined these components, we then substitute it into Eq. (32) to obtain the solution in a series form. As state above, the (ADM) produces a convergent series solution. The issue of convergence is addressed by several researchers [21-23].

The essential features of the decomposition method for linear and nonlinear equations; homogeneous and inhomogeneous; can be out lined as follows [24]:

1. Express the PDE, linear or nonlinear, in operator form.
2. Apply the inverse operator to both sides of equation written in an operator form.
3. Set the unknown function u into a decomposition series:

$$u = \sum_{n=0}^{\infty} u_n \tag{35}$$

   We next substitute the series (35) into both sides of the resulting equation.

4. Identify the zeroth components u_0 as the terms arising from the given conditions and from integrating the source term.
5. Determine the successive components of the series solution $u_k \; ; k \geq 0$ by applying the recursive scheme (34).
6. Substitute the determined components into (32) to obtain the solution in a series form.

The essential steps of the Adomian decomposition method will be illustrated by discussing the following examples.

Example (1.2.14): Consider the following inhomogeneous partial differential equation [24]:

$$u_x + u_y = x + y \quad ; \tag{36}$$

With the initial conditions:

$$u(x,0) = 0 \quad , \quad u(0,y) = 0 \ . \tag{37}$$

In an operator form, Eq. (36) becomes:

$$L_x u = x + y - L_y u \;; \tag{38}$$

Applying L_x^{-1} to both sides of (38) and using initial condition; hence we find:

$$u(x,y) = \frac{x^2}{2} + xy - L_x^{-1}\left(L_y u\right) \;. \tag{39}$$

As stated above, the decomposition method identifies the unknown function $u(x,y)$ as an infinite number of components $u_n(x,y)$, $n \geq 0$ given by:

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \;; \tag{40}$$

Substituting (40) into both sides of (39) we find:

$$\sum_{n=0}^{\infty} u_n(x,y) = \frac{x^2}{2} + xy - L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right) \;. \tag{41}$$

Consequently, the recursive scheme that will enable us to completely determine the successive components is thus constructed by:

$$\begin{aligned} u_0 &= \frac{x^2}{2} + xy, \\ u_{k+1} &= -L_x^{-1}\left(L_y\left(u_k\right)\right),\; k \geq 0 \,. \end{aligned} \tag{42}$$

This in turn gives;

$$\begin{aligned} u_0 &= \frac{x^2}{2} + xy, \\ u_1 &= -L_x^{-1}\left(L_y\left(u_0\right)\right) = -\frac{x^2}{2} \,, \\ u_2 &= -L_x^{-1}\left(L_y\left(u_1\right)\right) = 0 \,. \end{aligned} \tag{43}$$

Accordingly, $u_k(x,y) = 0$, $k \geq 2$.

Having determined the components of $u(x,y)$, we find:

$$u(x,y) = xy \;. \tag{44}$$

It is important to note here the exact solution given by (44) can also be obtained by determined the y- solution as discussed above.

Example (1.2.15): Consider the following one – dimensional heat equation [25]:

$$\frac{1}{4}u_{xx} = u_t \ ; \tag{45}$$

With the initial condition:

$$u(x,0) = 2\sin\frac{\pi}{2}x \ ; \tag{46}$$

And boundary conditions:

$$u(0,t) = 0 \quad , \quad u(2,t) = 0 \ . \tag{47}$$

In an operator form, Eq. (45) becomes:

$$L_t u = \frac{1}{4}L_{xx}u \ ; \tag{48}$$

Operating L_t^{-1} on both sides of (48) and using initial condition; hence we get:

$$u(x,t) = 2\sin\frac{\pi}{2}x + \frac{1}{4}L_t^{-1}\left(L_{xx}u\right) \ . \tag{49}$$

Using the decomposition (32) to both sides of (49) we obtain the recursive relation:

$$\begin{aligned} u_0 &= 2\sin\frac{\pi}{2}x, \\ u_{k+1} &= \frac{1}{4}L_t^{-1}\left(L_{xx}(u_k)\right), \ k \geq 0 . \end{aligned} \tag{50}$$

In view of (50) the components $u_n(x,t)$, $n \geq 0$ are determined by:

$$\begin{aligned} u_0 &= 2\sin\frac{\pi}{2}x, \\ u_1 &= \frac{1}{4}L_t^{-1}\left(L_{xx}(u_0)\right) = -\frac{\pi^2 t}{8}\sin\frac{\pi}{2}x, \\ u_2 &= \frac{1}{4}L_t^{-1}\left(L_{xx}(u_1)\right) = \frac{\pi^4 t^2}{256}\sin\frac{\pi}{2}x . \end{aligned} \tag{51}$$

And so on. The solution in a series given by:

$$u(x,t)=\sin\frac{\pi}{2}x\left(2-\frac{\pi^2 t}{8}+\frac{\pi^4 t^2}{256}-\ldots\right) \;; \tag{52}$$

In a closed form:

$$u(x,t) \;=\; 2\sin\frac{\pi}{2}x\,e^{-\frac{\pi^2}{16}t} \quad . \tag{53}$$

Example (1.2.16): Consider the following Laplace equation [25]:

$$u_{xx}+u_{tt}=0 \;; \tag{54}$$

With the initial conditions:

$$u(x,0)=0 \;\;, u_t(x,0)=\cos x \;. \tag{55}$$

In an operator form, Eq. (54) becomes:

$$L_{tt}u=-\;L_{xx}u \;; \tag{56}$$

Applying L_t^{-1} on both sides of (56) gives:

$$u(x,t)=t\cos x-L_t^{-1}(L_{xx}u) \;. \tag{57}$$

Substituting decomposition (32) into both sides of (57), and proceeding as before we obtain:

$$\begin{aligned} u_0 &= t\cos x, \\ u_1 &= -L_t^{-1}(L_{xx}(u_0))=\frac{t^3}{3!}\cos x, \\ u_2 &= -L_t^{-1}(L_{xx}(u_1))=\frac{t^5}{5!}\cos x\,. \end{aligned} \tag{58}$$

And so on. The solution in a series given by:

$$u(x,t)=\cos x\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\ldots\right) \;; \tag{59}$$

In a closed form:

$$u(x,t)=\cos x\sinh t \quad . \tag{60}$$

Example (1.2.17): Consider the following wave equation [25]:

$$u_{tt} - 4u_{xx} = 0 \; ; \tag{61}$$

With the initial conditions:

$$u(x,0) = \sin \pi x \;\; , \; u_t(x,0) = 0 \; . \tag{62}$$

In an operator form, Eq. (61) becomes:

$$L_{tt} u = \;\; 4 L_{xx} u \;\; ; \tag{63}$$

Applying L_t^{-1} on both sides of (63) gives:

$$u(x,t) = \sin \pi x + 4 L_t^{-1} (L_{xx} u) \; . \tag{64}$$

Substituting decomposition (32) into both sides of (64), and proceeding as before we obtain:

$$\begin{aligned} u_0 &= \sin \pi x, \\ u_1 &= 4 L_t^{-1} (L_{xx} (u_0)) = -2\pi^2 t^2 \sin \pi x, \\ u_2 &= -L_t^{-1} (L_{xx} (u_1)) = \frac{2\pi^4 t^4}{3} \sin \pi x \; . \end{aligned} \tag{65}$$

And so on. The solution in a series given by:

$$u(x,t) = \sin \pi x \left(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \ldots \right) ; \tag{66}$$

In a closed form:

$$u(x,t) = \sin \pi x \cos 2\pi t \quad . \tag{67}$$

Definition (1.2.18): The Noise Terms Phenomena

The idea of noise terms it means that opposite signs show in the first two components of the series solution that are happening only in inhomogeneous equations of any order.

The objective of this concept is demonstrating a fast convergence of the series solution.

In view of these remarks, we now outline the ideas of the noise terms [4, 10]:

1. The noise terms are defined as the identical terms with opposite signs that may appear in the components u_0 and u_1.
2. The noise terms appear only for specific of inhomogeneous equations whereas noise terms, do not appear for homogeneous equations.
3. The noise terms appear if the exact solution is part of zeroth component u_0 .
4. Verification that the remaining non-canceled terms satisfy the equation is necessary and essential.

The phenomenon of the useful noise terms will be explained by the following examples.

Example (1.2.19): Consider the inhomogeneous PDE [24]:

$$u_x + u_y = (1+x)e^y \quad ; \tag{68}$$

With the initial conditions:

$$u(0,y) = 0 \ . \tag{69}$$

The inhomogeneous PDE can be rewritten in an operator form by:

$$L_x u = (1+x)e^y - L_y u \ ; \tag{70}$$

Applying L_x^{-1} to both sides of (36) and using the given condition leads to:

$$u(x,y) = \left(x + \frac{x^2}{2!} \right) e^y - L_x^{-1}\left(L_y u \right) \ . \tag{71}$$

Substituting the decomposition (32) into both sides of (71), and proceeding as before, the components $u_0, u_1, ..., u_n$ are determined in a recursive manner by:

$$u_0 = \left(x + \frac{x^2}{2!} \right) e^y ,$$

$$u_1 = - L_x^{-1}\left(L_y\left(u_0\right)\right) = -\left(\frac{x^2}{2!} + \frac{x^3}{3!} \right) e^y . \qquad (72)$$

Considering the first two components u_0 and u_1 in (72), it is easily observed that the noise terms $\frac{x^2}{2!}e^y$ and $-\frac{x^2}{2!}e^y$ appears in u_0 and u_1 respectively. By canceling the noise terms in u_0, and by verifying that the remaining non-canceled terms of u_0 satisfying Eq. (68), we find that the exact solution is given by:

$$u(x, y) = x e^y \quad . \qquad (73)$$

Example (1.2.20): Consider the inhomogeneous PDE [24]:

$$u_x + u_y = (y + x) \quad ; \qquad (74)$$

With the initial conditions:

$$u(0, y) = 0 \quad . \qquad (75)$$

Proceeding as before, and applying the inverse operator L_x^{-1} to both sides of (74), and using the given condition we obtain;

$$u(x, y) = x\,y + \frac{x^2}{2!} - L_x^{-1}\left(L_y u\right) \quad . \qquad (76)$$

Preceding as before, the first two components u_0 and u_1 are given by:

$$u_0 = x\,y + \frac{x^2}{2!},$$

$$u_1 = - L_x^{-1}\left(L_y\left(u_0\right)\right) = -\frac{x^2}{2!}. \qquad (77)$$

Considering the first two components u_0 and u_1 in (77), it is easily observed that the noise terms $\frac{x^2}{2!}$ and $-\frac{x^2}{2!}$ appears in u_0 and u_1 respectively. By canceling the noise terms in u_0, and by verifying that the remaining non-canceled terms of u_0 satisfying Eq. (74), we find that the exact solution is given by:

$$u(x,y) = x\,y \quad . \tag{78}$$

1.2.1: The Modified Decomposition Method

In this section, we purpose to establish a new technique provides a rapid convergence of the series solution above the usualness of the decomposition method for linear and nonlinear differential equations called the modified decomposition method.

The modified decomposition method was developed by Wazwaz [11, 12]. Despite, the new technique is a slight variation in the Adomian recursive relation.

To give a clear description of the technique, we consider the PDE in an operator form:

$$Lu + Ru = g \;\; ; \tag{79}$$

Where L is the highest order derivative, R is a linear differential operator of less order or equal order to L, and g is the source term.

Now apply the inverse operator L^{-1} to both sides of Eq. (79) we obtain:

$$u = f - L^{-1}\left(Ru\right) . \tag{80}$$

where the function f represents the terms arising from integrating the source term g and using the given conditions that are assumed to be prescribed. We then proceed as discussed in section (1.2) and define the solution u as an infinite sum of components defined by:

$$u = \sum_{n=0}^{\infty} u_n \;; \tag{81}$$

To achieve this goal, the decomposition method admits the of the recursive relation:

$$\begin{aligned} &u_0 = f\,; \\ &u_{k+1} = -L^{-1}\left(R(u_k)\right);\; k \geq 0\,. \end{aligned} \tag{82}$$

The modified decomposition method introduces a slight variation to the recursive relation (82) that will lead to the determination of the components of u in a faster and easier way. For specific cases, the function f can be set as the sum of two partial functions, namely f_1 and f_2 . In other words, we can set:

$$f = f_1 + f_2 \;; \tag{83}$$

Using (83), we introduce a qualitative change in the formation of recursive relation (82). The modified recursive relation can be identified by:

$$\begin{aligned} &u_0 = f_1\,, \\ &u_1 = f_2 - L^{-1}\left(R(u_0)\right), \\ &u_{k+1} = -L^{-1}\left(R(u_k)\right);\; k \geq 1. \end{aligned} \tag{84}$$

It is worth mentioning that the modified decomposition method will be used for linear and nonlinear equations of any order. In the upcoming chapters, it will be used wherever it is appropriate [24].

The modified decomposition method will be illustrated by discussing the following examples.

Example (1.2.21): Consider the inhomogeneous PDE [24]:

$$u_x + u_y = 3x^2y^3 + 3x^3y^2 \;; \tag{85}$$

With the initial conditions:

$$u(0, y) = 0\;. \tag{86}$$

The inhomogeneous PDE can be rewritten in an operator form by:

$$L_x u = 3x^2y^3 + 3x^3y^2 - L_y u \;; \tag{87}$$

Applying L_x^{-1} to both sides of (87) and using the given condition leads to:

$$u(x,y)=x^3\,y^3+\frac{3}{4}x^4\,y^2-L_x^{-1}\left(L_y u\right)\ . \tag{88}$$

The function consists of two terms, hence we set:

$$f_1=x^3\,y^3\quad,\quad f_2=\frac{3}{4}x^4\,y^2\quad; \tag{89}$$

In view of (89) we introduce the modified recursive relation:

$$\begin{aligned}
&u_0=\ x^3\,y^3,\\
&u_1=\frac{3}{4}x^4\,y^2-L_x^{-1}\left(L_y\left(u_0\right)\right),\\
&u_{k+1}=-\,L^{-1}\left(L_y\left(u_k\right)\right);\ k\geq 1.
\end{aligned} \tag{90}$$

This gives:

$$\begin{aligned}
&u_0=\ x^3\,y^3,\\
&u_1=\frac{3}{4}x^4\,y^2-L_x^{-1}\left(L_y\left(u_0\right)\right)=0,\\
&u_{k+1}=0\ ,\ k\geq 1.
\end{aligned} \tag{91}$$

It then follows that the solution is:

$$u(x,y)=x^3\,y^3\quad . \tag{92}$$

Example (1.2.22): Consider the inhomogeneous PDE [24]:

$$u_x+u_y=\cosh x+\cosh y\quad; \tag{93}$$

With the initial conditions:

$$u(x,0)=\sinh x\ . \tag{94}$$

To effectively use the given condition, we rewrite (93) in an operator form by:

$$L_y u=\cosh x+\cosh y-L_x u\quad; \tag{95}$$

Applying L_y^{-1} to both sides of (95) and using the given condition gives:

$$u(x,y)=\sinh x+y\cosh x+\sinh y-L_y^{-1}\left(L_x u\right) \quad . \quad (96)$$

To determine the components of $u(x,y)$, we set the modified recursive relation:

$$\begin{aligned} &u_0=\sinh x+\sinh y, \\ &u_1=y\cosh x-L_y^{-1}\left(L_x\left(u_0\right)\right)=0, \\ &u_{k+1}=0\,;\,k\geq 1. \end{aligned} \quad (97)$$

The exact solution is:

$$u(x,y)=\sinh x+\sinh y \quad . \quad (98)$$

1.3: Adomian Decomposition Method and Sumudu Transform Method for Solving Linear Partial Differential Equations

The Adomian decomposition method proves to be powerful, effective and successfully used to handle most types of linear or nonlinear ordinary or partial differential equations, and linear or nonlinear integral equations. The method characteristics various advantages, which considerably from the usual methods. This method is a simple and directly without any restrictive assumption as usual is going in other methods.

In this section, we propose a new method, namely Adomian Decomposition Sumudu Transform Method (ADSTM) for solving linear partial differential equations. This method is a combination of Sumudu transform and decomposition method which was introduced by Devendra Kumar, Jagdev Singh and Sushila Rathore [26].

The objective of algorithm is to provide the solution in a rapid convergent series which can lead to the solution in a closed form.

To illustrate the basic idea of this method, we consider a general non-homogeneous partial differential equation with the initial conditions of the form:

$$\begin{aligned} &LU(x,t)+RU(x,t)=g(x,t), \\ &U(x,0)=h(x)\,,\,U_t(x,0)=f(x). \end{aligned} \quad (99)$$

Where L is the second order linear differential operator $L = \frac{\partial^2}{\partial t^2}$, R is other linear differential operator of less order than L, and g is a source term.

Taking the Sumudu transform of both sides of Eq. (99), we get:

$$S\left[LU(x,t)\right] + S\left[RU(x,t)\right] = S\left[g(x,t)\right]\ ; \qquad (100)$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have:

$$S\left[U(x,t)\right] = h(x) + u\,f(x) + u^2\,S\left[g(x,t)\right] - u^2\,S\left[RU(x,t)\right]\ . \qquad (101)$$

If we apply the inverse operator S^{-1} to both sides of the equation (101), we obtain:

$$U(x,t) = G(x,t) - S^{-1}\left[u^2\,S\left[RU(x,t)\right]\right]\ . \qquad (102)$$

Where, the function $G(x,t)$ represents the terms arising from integrating the source term g and the prescribed initial conditions.

Using the Adomian decomposition method which defines the solution by an infinite series of components given by:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t); \qquad (103)$$

where the components $U_0, U_1, U_2, \cdots$ are usually recurrently determined. Substituting (103) into both sides of (102) leads to:

$$\sum_{n=0}^{\infty} U_n(x,t) = G(x,t) - S^{-1}\left[u^2\,S\left[R\left(\sum_{n=0}^{\infty} U_n\right)\right]\right]\ . \qquad (104)$$

For simplicity, Equation (104) can be rewritten as;

$$U_0 + U_1 + U_2 + U_3 + \cdots = G - S^{-1}\left[u^2\,S\left[R\left(U_0 + U_1 + U_2 + U_3 + \cdots\right)\right]\right] \quad (105)$$

To construct the recursive relation needed for the determination of the components $U_0, U_1, U_2, \cdots$, it is important to note that the Adomian decomposition method suggests that the zeros component U_0 is usually defined by the function G described above, i.e. Accordingly, the formal recursive relation is defined by:

$$\begin{aligned} U_0 &= G(x,t), \\ U_1 &= -S^{-1}\left[u^2\,S\left[R(U_0)\right]\right], \\ U_2 &= -S^{-1}\left[u^2\,S\left[R(U_1)\right]\right] \\ U_3 &= -S^{-1}\left[u^2\,S\left[R(U_2)\right]\right] \\ &\ \ .\qquad\qquad . \qquad . \end{aligned} \qquad (106)$$

Substituting these components in the equation (103), we obtain the solution in a series form.

The Adomian decomposition Sumudu transform method will be illustrated by discussing the following examples.

Example (1.3.23): Consider the following inhomogeneous partial differential equation [24]:

$$U_x(x,y) + U_y(x,y) = x + y; \tag{107}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \quad U(0,y) = 0; \tag{108}$$

The $x-$solution:

Following discussion presented above, we obtain the recursive relation:

$$\begin{aligned} U_0 &= \frac{x^2}{2} + x\,y\,, \\ U_1 &= -\,S^{-1}\left[u^2\,S\left[(U_0)_y\right]\right] = -\frac{x^2}{2}, \\ U_2 &= -\,S^{-1}\left[u^2\,S\left[(U_1)_y\right]\right] = 0. \end{aligned} \tag{109}$$

Therefore the solution $U(x,t)$ in series form is given by;

$$\begin{aligned} U(x,t) &= U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots \\ &= \frac{x^2}{2} + x\,y - \frac{x^2}{2} \end{aligned} \tag{110}$$

And in closed form given as;

$$U(x,t) = x\,y \tag{111}$$

It is important to note here the exact solution given by (111) can also be obtained by determined the y- solution as discussed above.

Example (1.3.24): Consider the following one – dimensional heat equation [25]:

$$\frac{1}{4}U_{xx}(x,t) = U_t(x,t)\ ; \tag{112}$$

With the initial condition:

$$U(x,0)=2\sin\frac{\pi}{2}x \quad . \tag{113}$$

In a similar way above, we have the recursive relation:

$$\begin{aligned} U_0 &= 2\sin\frac{\pi}{2}x\,, \\ U_1 &= \frac{1}{4}S^{-1}\left[u\,S\left[(U_0)_{xx}\right]\right] = -\frac{\pi^2 t}{8}\sin\frac{\pi}{2}x, \\ U_2 &= \frac{1}{4}S^{-1}\left[u^2\,S\left[(U_1)_{xx}\right]\right] = \frac{\pi^4 t^2}{256}\sin\frac{\pi}{2}x. \end{aligned} \tag{114}$$

And so on. The solution in a series form given by:

$$U(x,t) = \sin\frac{\pi}{2}x\left(2 - \frac{\pi^2 t}{8} + \frac{\pi^4 t^2}{256} - \ldots\right) \tag{115}$$

And in a closed form of,

$$U(x,t) = 2\sin\frac{\pi}{2}x e^{-\frac{\pi^2}{16}t} \quad . \tag{116}$$

Example (1.3.25): Consider the following wave equation [25]:

$$U_{tt}(x,t) - 4U_{xx}(x,t) = 0 \;; \tag{117}$$

With the initial conditions:

$$U(x,0) = \sin\pi x \quad , U_t(x,0) = 0 \;. \tag{118}$$

Proceeding as before, we obtain:

$$\begin{aligned} U_0 &= \sin\pi x, \\ U_1 &= 4S^{-1}\left(L_{xx}(u_0)\right) = -2\pi^2 t^2 \sin\pi x, \\ U_2 &= 4S^{-1}\left(L_{xx}(u_1)\right) = \frac{2\pi^4 t^4}{3}\sin\pi x \;. \end{aligned} \tag{119}$$

And so on. The solution in a series given by:

$$U(x,t) = \sin\pi x\left(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \ldots\right); \tag{120}$$

In a closed form:

$$U(x,t) = \sin\pi x\cos 2\pi t \quad . \tag{121}$$

1.4: Adomian decomposition Method and Sumudu Transform Method for Solving Linear Systems Partial Differential Equations

In this section, we will present the combined Sumudu transform and Adomian decomposition method to solve some examples of linear system of partial differential equations; we first consider the system of partial differential equations written in an operator form;

$$\begin{aligned} U_t + V_x &= g_1 , \\ V_t + U_x &= g_2 , \end{aligned} ; \tag{122}$$

With the initial conditions;

$$\begin{aligned} U(x,0) &= f_1(x), \\ V(x,0) &= f_2(x). \end{aligned} \tag{123}$$

Using the differential operator property of the Sumudu transform and above initial conditions, we get;

$$\begin{aligned} S[U(x,t)] &= f_1(x) + u\,S[g_1 - V_x] \\ S[V(x,t)] &= f_2(x) + u\,S[g_2 - U_x] \end{aligned} \tag{124}$$

Now, applying the inverse Sumudu transform on both sides of (124), we get:

$$\begin{aligned} U(x,t) &= f_1(x) + S^{-1}\{u\,S[g_1 - V_x]\} \\ V(x,t) &= f_2(x) + S^{-1}\{u\,S[g_2 - U_x]\} \end{aligned} \tag{125}$$

where $g_1(x,t)$, $g_2(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We apply the Adomian decomposition method:

$$\begin{aligned} U(x,t) &= \sum_{n=0}^{\infty} U_n(x,t) \\ V(x,t) &= \sum_{n=0}^{\infty} V_n(x,t) \end{aligned} \tag{126}$$

Now, applying the Adomian decomposition method, we get:

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x,t) &= f_1(x) + S^{-1}\left[uS\left[g_1 - \left(\sum_{n=0}^{\infty} V_n(x,t) \right)_x \right]\right] \\ \sum_{n=0}^{\infty} V_n(x,t) &= f_2(x) + S^{-1}\left[uS\left[g_2 - \left(\sum_{n=0}^{\infty} U_n(x,t) \right)_x \right]\right] \end{aligned} \tag{127}$$

Following Adomian analysis, the system (127) is transformed into a set of recursive relation given by:

$$U_0(x,t) = f_1(x) + S^{-1}[u\,S[g_1]],$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S[(V_k)_x]],\ k \geq 0. \tag{128}$$

And

$$V_0(x,t) = f_2(x) + S^{-1}[u\,S[g_2]],$$
$$V_{k+1}(x,t) = -S^{-1}[u\,S[(U_k)_x]],\ k \geq 0. \tag{129}$$

To have a clear overview, forthwith are several examples to demonstrate the efficiency of the method.

Example (1.4.26): Consider the following system of partial differential equations [24]:

$$U_t + V_x = 0$$
$$V_t + U_x = 0 \tag{130}$$

With the initial conditions;

$$U(x,0) = e^x$$
$$V(x,0) = e^{-x} \tag{131}$$

To derive the solution by using the decomposition method, we follow the recursive relation (128) and (129) to obtain:

$$U_0(x,t) = e^x,$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S[(V_k)_x]],\ k \geq 0. \tag{132}$$

And

$$V_0(x,t) = e^{-x},$$
$$V_{k+1}(x,t) = -S^{-1}[u\,S[(U_k)_x]],\ k \geq 0. \tag{133}$$

The remaining components are thus determined by:

$$U_1(x,t) = t\,e^{-x}\ ,\ V_1(x,t) = -t\,e^{x}\ ,$$
$$U_2(x,t) = \frac{t^2}{2!}e^{x}\ ,\ V_2(x,t) = \frac{t^2}{2!}e^{-x},$$
$$U_3(x,t) = \frac{t^3}{3!}e^{-x}\ ,\ V_3(x,t) = -\frac{t^3}{3!}e^{x}, \tag{134}$$

And so on. Using (134) we obtain:

$$U(x,t)=e^{x}\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right)+e^{-x}\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right),$$
$$V(x,t)=e^{-x}\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right)-e^{x}\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right). \tag{135}$$

This has an exact analytical solution of the form;

$$(U,V)=\left(e^{x}\cosh t+e^{-x}\sinh t\ ,\ e^{-x}\cosh t-e^{x}\sinh t\right). \tag{136}$$

Example (1.4.27): Consider the following system of partial differential equations**:**

$$U_{tt}-V_{x}=2x^2-e^{t}$$
$$V_{t}+U_{xx}=2t^2+x\,e^{t} \tag{137}$$

With the initial conditions;

$$U(x,0)=0\quad,\quad U_{t}(x,0)=0\ ,\ V(x,0)=x \tag{138}$$

Taking Sumudu transform of equations (137) subject to the initial conditions, we get;

$$S[U(x,t)]=2x^2u^2-\frac{u^2}{1-u}+u^2\,S[V_x]$$
$$S[V(x,t)]=4u^3+\frac{x\,u}{1-u}+x-u\,S[U_{xx}] \tag{139}$$

The inverse Sumudu transform implies that:

$$U(x,t)=x^2t^2+t+1-e^{t}+S^{-1}\left[u^2\,S[V_x]\right]$$
$$V(x,t)=\frac{4}{3!}t^3+x\,e^{t}-S^{-1}\left[u\,S[U_{xx}]\right] \tag{140}$$

Now applying the Adomian decomposition method, we get;

$$\sum_{n=0}^{\infty}U_n(x,t)=x^2t^2+t+1-e^{t}+S^{-1}\left[u^2\,S\left[\sum_{n=0}^{\infty}V_n\right]_x\right]$$
$$\sum_{n=0}^{\infty}V_n(x,t)=\frac{4}{3!}t^3+x\,e^{t}-S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty}U_n\right]_{xx}\right] \tag{141}$$

The modified decomposition method defines the recursive relations in the form;

$$U_0(x,t)=x^2t^2,$$
$$U_1(x,t)=t+1-e^{t}+S^{-1}\left[u^2\,S[V_0]_x\right],$$
$$U_{k+1}(x,t)=S^{-1}\left[u^2\,S[V_k]_x\right],\ k\geq 1. \tag{142}$$

And

$$
\begin{aligned}
V_0(x,t) &= xe^t, \\
V_1(x,t) &= \frac{4}{3!}t^3 - S^{-1}\left[u\,S\left[U_0\right]_{xx}\right], \\
V_{k+1}(x,t) &= -\,S^{-1}\left[u\,S\left[U_k\right]_{xx}\right],\ k \geq 1.
\end{aligned} \tag{143}
$$

We obtain the following pairs of components;

$$
\begin{aligned}
(U_0, V_0) &= \left(x^2\,t^2, x\,e^t\right), \\
(U_1, V_1) &= (0,0), \\
(U_2, V_2) &= (0,0).
\end{aligned} \tag{144}
$$

This has an exact analytical solution of the form;

$$
(U, V) = \left(x^2 t^2\ ,\ x e^t\right)\ . \tag{145}
$$

1.5: Adomian Decomposition Method and Sumudu Transform for Solving Higher Dimensional Heat and Waves Equations

Heat and wave like models are the integral part of applied mathematics and engineering mathematics that arises from different physical phenomena. Several techniques such as characteristic, modified variation iteration, Adomian decomposition method, Hė s polynomials, and homotopy perturbation Sumudu transform method [27] have been used for solving these problems.

It is significant to note that the (ADM) is applied without any restrictive assumption or transformation. The main advantage of the (ADM) **is** that it can be applied straight to all types of differential equations both homogeneous and in-homogeneous boundary conditions.

In this section we introduce a new method called Adomian decomposition Sumudu transform method (ADSTM) for solving the heat and wave like equations in two and three dimensional spaces. It importance that the proposed method is an elegant combination of the Sumudu transforms method and the Adomian decomposition method which was introduced by D. Kumar, J. Singh and S. Rathore [26].

(ADSTM) provides the solution for nonlinear equations in the form of convergent series. These forms the motivation for us to apply (ADSTM) for solving nonlinear equations in understanding different physical phenomena.

1.5.1: Adomian Decomposition Method and Sumudu Transform Method (ADSTM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of form [27];

$$\begin{aligned} &DU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \\ &U(x,0) = h(x)\ ,U_t(x,0) = f(x), \end{aligned} \tag{146}$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D, N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (146), we get;

$$S[DU(x,t)] + S[RU(x,t)] + S[N(x,t)] = S[g(x,t)]. \quad (147)$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have;

$$S[U(x,t)] = u^2 S[g(x,t)] + h(x) + u f(x) - u^2 S[RU(x,t) + NU(x,t)]. \quad (148)$$

Now, applying the inverse Sumudu transform of both sides of (148), we get,

$$U(x,t) = G(x,t) - S^{-1}\left[u^2 S[RU(x,t) + NU(x,t)]\right]. \quad (149)$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, apply the Adomian decomposition method;

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \quad , \quad (150)$$

The nonlinear term can be decomposed as;

$$NU(x,t) = \sum_{n=0}^{\infty} A_n(U) \quad , \quad (151)$$

For some Adomian polynomials $A_n(U)$ that are given by;

$$A_n(U_0, U_1, U_2, ..., U_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n U_n\right)\right]_{\lambda=0}, \; n = 0,1,2,... \; , \quad (152)$$

Substituting Eq. (150) and Eq. (151) in Eq. (149), we get;

$$\sum_{n=0}^{\infty} U_n(x,t) = G(x,t) - S^{-1}\left[u^2 S\left[R\sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} A_n(U)\right]\right] \quad . \quad (153)$$

So that the recursive relation is given by;

$$\begin{aligned} &U_0(x,t) = G(x,t), \\ &U_{k+1}(x,t) = -S^{-1}\left[u^2 S[RU_k + A_k]\right], \; k \geq 0. \end{aligned} \quad (154)$$

1.5.2: Two Dimensional Heat Flow

The Adomian decomposition Sumudu transform method (ADSTM) can be used for solving the heat equation in a two dimensional space [24]:

$$U_t = k(U_{xx} + U_{yy}), \; 0 < x < a \,, \; 0 < y < b \,, t > 0 \; ; \quad (155)$$

With boundary conditions:

$$U(0,y,t)=0\ ,\ U(a,y,t)=0\ ,$$
$$U(x,0,t)=0\ ,\ U(x,b,t)=0\ , \tag{156}$$

And initial condition:

$$U(x,y,0)=f(x,y)\ . \tag{157}$$

Where $U=U(x,y,t)$ the temperature of any point is located at the position (x,y) of a rectangular plate at any time t, and k is the thermal diffusivity.

The distribution of heat flow in two dimensional spaces is governed by the following initial boundary value.

Example (1.5.28): Consider the following two-dimensional initial boundary value problem which describes the heat-like models [28];

$$U_t=\frac{1}{2}\left[x^2U_{xx}+yU_{yy}\right],\ 0<x,y<1\ ,\ t>0 \tag{158}$$

With boundary conditions as;

$$U(0,y,t)=0\ ,\ U(1,y,t)=2\sinh t$$
$$U(x,0,t)=0\ ,\ U(x,1,t)=2\cosh t\ , \tag{159}$$

And the initial condition as;

$$U(x,y,0)=y^2\ . \tag{160}$$

Taking Sumudu transform of both sides of (158) subject to the initial condition, we get;

$$S[U(x,y,t)]=y^2+\frac{1}{2}y^2uS[U_{xx}]+\frac{1}{2}x^2uS[U_{yy}]. \tag{161}$$

The inverse of Sumudu transform implies that;

$$U(x,y,t)=y^2+\frac{1}{2}y^2S^{-1}[uS[U_{xx}]]+\frac{1}{2}x^2S^{-1}[uS[U_{yy}]]. \tag{162}$$

The decomposition method defined the solution $U(x,y,t)$ as a series given by;

$$U(x,y,t)=\sum_{n=0}^{\infty}U_n(x,y,t)\ . \tag{163}$$

Now, applying the Adomian decomposition method, we get;

$$\sum_{n=0}^{\infty}U_n(x,y,t)=y^2+\frac{1}{2}y^2S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty}U_n(x,y,t)\right)_{xx}\right]\right]+$$
$$+\frac{1}{2}x^2S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty}U_n(x,y,t)\right)_{yy}\right]\right] \tag{164}$$

This leads to the recursive relation;

$$U_0(x,y,t) = y^2 ,$$
$$U_{k+1}(x,y,t) = \frac{1}{2} y^2 S^{-1}\left[u S\left[(U_k)_{xx}\right]\right] + \frac{1}{2} x^2 S^{-1}\left[u S\left[(U_k)_{yy}\right]\right] , \; k \geq 0. \qquad (165)$$

This gives;

$$U_0(x,y,t) = y^2,$$
$$U_1(x,y,t) = \frac{1}{2} y^2 S^{-1}\left[u S\left[(U_0)_{xx}\right]\right] + \frac{1}{2} x^2 S^{-1}\left[u S\left[(U_0)_{yy}\right]\right] = x^2 t,$$
$$U_2(x,y,t) = \frac{1}{2} y^2 S^{-1}\left[u S\left[(U_1)_{xx}\right]\right] + \frac{1}{2} x^2 S^{-1}\left[u S\left[(U_1)_{yy}\right]\right] = y^2 \frac{t^2}{2!}, \qquad (166)$$
$$U_3(x,y,t) = \frac{1}{2} y^2 S^{-1}\left[u S\left[(U_2)_{xx}\right]\right] + \frac{1}{2} x^2 S^{-1}\left[u S\left[(U_2)_{yy}\right]\right] = x^2 \frac{t^3}{3!}.$$

And so on. Therefore the solution $U(x,y,t)$ in series form is given by;

$$U(x,y,t) = x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) + y^2\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) \; ; \qquad (167)$$

And in closed form given as;

$$U(x,y,t) = x^2 \sinh t + y^2 \cosh t \; . \qquad (168)$$

Example (1.5.29): Consider the following two-dimensional initial boundary value problem which describes the heat-like models [24];

$$U_t = U_{xx} + U_{yy} - U , \qquad 0 < x, y < \pi \; , t > 0 \; ; \qquad (169)$$

With boundary conditions as;

$$U(0,y,t) = U(\pi,y,t) = 0$$
$$U(x,0,t) = U(x,\pi,t) = e^{-3t} \sin x \qquad (170)$$

And the initial condition as;

$$U(x,y,0) = \sin x \cos y \; . \qquad (171)$$

In a similar way as above, we have;

$$\sum_{n=0}^{\infty} U_n(x,y,t) = \sin x \cos y + S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} U_n(x,y,t)\right)_{xx}\right]\right] +$$
$$S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} U_n(x,y,t)\right)_{yy}\right]\right] + + S^{-1}\left[uS\left[\sum_{n=0}^{\infty} U_n(x,y,t)\right]\right] \; . \qquad (172)$$

This gives;

$$U_0(x, y, t) = \sin x \cos y$$

$$U_1(x, y, t) = S^{-1}[u\, S[(U_0)_{xx}]] + S^{-1}[u\, S[(U_0)_{yy}]] + S^{-1}[u\, S[U_0]]$$

$$= -3t \sin x \cos y \tag{173}$$

$$U_2(x, y, t) = S^{-1}[u\, S[(U_1)_{xx}]] + S^{-1}[u\, S[(U_1)_{yy}]] + S^{-1}[u\, S[U_1]] =$$

$$= \frac{(3t)^2}{2!} \sin x \cos y$$

Therefore the solution $U(x, y, t)$ in series form is given by;

$$U(x, y, t) = \left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!} - \cdots\right) \sin x \cos y\ ; \tag{174}$$

And in closed form given as;

$$U(x, y, t) = e^{-3t} \sin x \cos y\ . \tag{175}$$

1.5.3: Three Dimensional Heat Flow

The Adomian decomposition Sumudu transform method (ADSTM) can be used of solving the heat equation in a three dimensional space [24]:

$$U_t = k(U_{xx} + U_{yy} + U_{zz}),\ 0<x<a\ ,\ 0<y<b\ ,t>0\ ,\ 0<z<c\ ,t>0\ ; \tag{176}$$

With the boundary conditions:

$$\begin{aligned} U(0, y, z, t) &= 0\ ,\ U(a, y, z, t) = 0\ , \\ U(x, 0, z, t) &= 0\ ,\ U(x, b, z, t) = 0\ , \\ U(x, y, 0, t) &= 0\ ,\ U(x, y, c, t) = 0\ , \end{aligned} \tag{177}$$

And the initial condition:

$$U(x, y, z, 0) = f(x, y, z)\ . \tag{178}$$

Where $U = U(x, y, z, t)$ the temperature of any point is located at the position (x, y, z) of a rectangular plate at any time t , and k is the thermal diffusivity.

The distribution of heat flow in three dimensional spaces is governed by the following initial boundary value.

Example (1.5.30): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models as [28];

$$U_t = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right), \qquad 0 < x, y, z < 1, \ t > 0 \tag{179}$$

With the boundary conditions as;

$$\begin{aligned} U(0, y, z, t) &= 0, \quad U(\pi, y, z, t) = y^4 z^4 \left(e^t - 1\right) \\ U(x, 0, z, t) &= 0, \quad U(x, \pi, z, t) = x^4 z^4 \left(e^t - 1\right) \\ U(x, y, 0, t) &= 0, \quad U(x, y, \pi, t) = x^4 y^4 \left(e^t - 1\right) \end{aligned} \tag{180}$$

And the initial condition as;

$$U(x, y, z, 0) = 0 \tag{181}$$

Taking Sumudu transform of both sides of the equation (179) subject to the initial condition, we get;

$$S[U(x, y, z, t)] = x^4 y^4 z^4 t + \frac{1}{36} uS\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right] \tag{182}$$

The inverse of Sumudu transform implies that:

$$U(x, y, z, t) = x^4 y^4 z^4 t + \frac{1}{36} S^{-1}\left[uS\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right]\right] \tag{183}$$

Now, applying the Adomian decomposition method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, z, t) &= x^4 y^4 z^4 t + \frac{1}{36} x^2 S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t)\right)_{xx}\right]\right] \\ &+ \frac{1}{36} y^2 S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t)\right)_{yy}\right]\right] + \frac{1}{36} z^2 S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t)\right)_{zz}\right]\right]\Bigg) \end{aligned} \tag{184}$$

This gives;

$$U_0(x,y,z,t) = x^4y^4z^4t,$$

$$U_1(x,y,z,t) = \frac{1}{36}x^2S^{-1}\left[u\,S\left[(U_0)_{xx}\right]\right] + \frac{1}{36}y^2S^{-1}\left[u\,S\left[(U_0)_{yy}\right]\right] + \frac{1}{36}z^2S^{-1}\left[u\,S\left[(U_0)_{zz}\right]\right] = x^4y^4z^4\frac{t^2}{2!},$$

$$U_2(x,y,z,t) = \frac{1}{36}x^2S^{-1}\left[u\,S\left[(U_1)_{xx}\right]\right] + \frac{1}{36}y^2S^{-1}\left[u\,S\left[(U_1)_{yy}\right]\right] + \frac{1}{36}z^2S^{-1}\left[u\,S\left[(U_1)_{zz}\right]\right] = x^4y^4z^4\frac{t^3}{3!}, \tag{185}$$

$$U_3(x,y,z,t) = \frac{1}{36}x^2S^{-1}\left[u\,S\left[(U_2)_{xx}\right]\right] + \frac{1}{36}y^2S^{-1}\left[u\,S\left[(U_2)_{yy}\right]\right] + \frac{1}{36}z^2S^{-1}\left[u\,S\left[(U_2)_{zz}\right]\right] = x^4y^4z^4\frac{t^4}{4!}.$$

Therefore the solution $U(x,y,z,t)$ in series form is given by;

$$U(x,y\;,z,\;t) = x^4y^4z^4\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots\right); \tag{186}$$

And in closed form given as;

$$U(x,y,z,t) = x^4y^4z^4\left(e^{-t} - 1\right). \tag{187}$$

1.5.4: Two Dimensional Wave Equation

In this section, we will apply the newly developed Adomian decomposition method and Sumudu transform to handle the wave equation.

The propagation of waves in a two dimensional vibrating membrane of length a and width b is governed by the following initial-boundary value problem [24];

$$U_{tt} = c^2\left(U_{xx} + U_{yy}\right), \qquad 0 < x < a\;,\; 0 < x < b\;,\; t > 0 \tag{188}$$

Subject to the boundary conditions;

$$\begin{aligned} U(0,y,t) &= U(a,y,t) = 0 \\ U(x,0,t) &= U(x,b,t) = 0 \end{aligned} \tag{189}$$

And the initial condition;

$$U(x,y,0) = f(x,y)\;,\; U_t(x,y,0) = g(x,y) \tag{190}$$

As discussed before, the solution in the t- direction, in the $x-$space , or in the $y-$space will lead to identical results. However, the solution in the t-direction reduces the size of calculations compared with the other space solutions because it uses the initial conditions only. For this reason the solution in the $t-$direction will be discussed in this section.

Example (1.5.31): Consider the following two-dimensional initial boundary value problem which describes the heat-like models as [24];

$$U_{tt} = 2\left(U_{xx} + U_{yy}\right), \qquad 0 < x, y < \pi \ , \ t > 0 \tag{191}$$

With the boundary conditions as;

$$\begin{aligned} U(0, y, t) &= U(\pi, y, t) = 0 \\ U(x, 0, t) &= U(x, \pi, t) = 0 \end{aligned} \tag{192}$$

And the initial condition as;

$$U(x, y, 0) = \sin x \sin y , \quad U_t(x, y, 0) = 0 \tag{193}$$

Taking Sumudu transform of both sides of the equation (191) subject to the initial condition, we get;

$$S\left[U(x, y, t)\right] = \sin x \sin y + 2u^2 S\left[U_{xx} + U_{yy}\right] \tag{194}$$

The inverse of Sumudu transform implies that:

$$U(x, y, t) = \sin x \sin y + 2S^{-1}\left[u^2 S\left[U_{xx} + U_{yy}\right]\right] \tag{195}$$

Now, applying the Adomian decomposition method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, t) = \sin x \sin y + 2 S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} U_n(x, y, t)\right)_{xx}\right]\right] + \\ + 2 S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} U_n(x, y, t)\right)_{yy}\right]\right] \end{aligned} . \tag{196}$$

This gives;

$$\begin{aligned} U_0(x,y,t) &= (\sin x)(\sin y) \\ U_1(x,y,t) &= 2S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] + 2S^{-1}\left[u^2 S\left[(U_0)_{yy}\right]\right] = -\frac{(2t)^2}{2!}(\sin x)(\sin y) \\ U_2(x,y,t) &= 2S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right] + 2S^{-1}\left[u^2 S\left[(U_1)_{yy}\right]\right] = \frac{(2t)^4}{4!}(\sin x)(\sin y) \\ U_3(x,y,t) &= 2S^{-1}\left[u^2 S\left[(U_2)_{xx}\right]\right] + 2S^{-1}\left[u^2 S\left[(U_2)_{yy}\right]\right] = -\frac{(2t)^6}{6!}(\sin x)(\sin y) \end{aligned} \tag{197}$$

Therefore the solution $U(x, y, t)$ in series form is given by;

$$U(x, y, t) = \left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \frac{(2t)^6}{6!} - \cdots\right) \sin x \sin y \quad ; \qquad (198)$$

And in closed form given as;

$$U(x, y, t) = \sin x \sin y \cos(2t) \quad . \qquad (199)$$

Example (1.5.32): Consider the following two-dimensional initial boundary value problem which describes the heat-like models as [28];

$$U_{tt} = \frac{1}{12}\left(x^2 U_{xx} + y^2 U_{yy}\right), \qquad 0 < x, y < 1 \, , \, t > 0 \qquad (200)$$

Subject the Neumann boundary conditions as;

$$\begin{aligned} U_x(0, y, t) &= 0 \;\; , \;\; U_x(1, y, t) = 4 \cosh t \\ U_y(x, 0, t) &= U_y(x, 1, t) = 4 \sinh t \end{aligned} \qquad (201)$$

And the initial condition as;

$$U(x, y, 0) = x^4 \quad , \quad U_t(x, y, 0) = y^4 \qquad (202)$$

Taking Sumudu transform of both sides of the equation (194) subject to the initial condition, we get;

$$S[U(x, y, t)] = x^4 + u\, y^4 + \frac{1}{12} x^2 u^2 S[U_{xx}] + \frac{1}{12} y^2 u^2 S[U_{yy}] \qquad (203)$$

The inverse of Sumudu transform implies that:

$$U(x, y, t) = x^4 + t\, y^4 + \frac{1}{12} S^{-1}\left[u^2 S\left[x^2 U_{xx} + y^2 U_{yy}\right]\right] \qquad (204)$$

Now, applying the Adomian decomposition method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, t) = x^4 + ty^4 + \frac{1}{12} x^2 S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} U_n(x, y, t)\right)_{xx}\right]\right] + \\ + \frac{1}{12} y^2 S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} U_n(x, y, t)\right)_{yy}\right]\right] \end{aligned} \quad . \qquad (205)$$

This gives;

$$
\begin{aligned}
U_0(x,y,t) &= x^4 + t\,y^4 ,\\
U_1(x,y,t) &= \frac{1}{12}x^2 S^{-1}\left[u^2\, S\left[(U_0)_{xx}\right]\right] + \frac{1}{12}y^2 S^{-1}\left[u^2\, S\left[(U_0)_{yy}\right]\right]\\
&= x^4\frac{t^2}{2!} + y^4\frac{t^3}{3!},\\
U_2(x,y,t) &= \frac{1}{12}x^2 S^{-1}\left[u^2\, S\left[(U_1)_{xx}\right]\right] + \frac{1}{12}y^2 S^{-1}\left[u^2\, S\left[(U_1)_{yy}\right]\right]\\
&= x^4\frac{t^4}{4!} + y^4\frac{t^5}{5!}.
\end{aligned}
\tag{206}
$$

$$
U(x,y,t) = x^4\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + y^4\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)\ ; \tag{207}
$$

And in closed form given as;

$$
U(x,y,t) = x^4\cosh t + y^4\sinh t\ . \tag{208}
$$

1.5.5: Three Dimensional Wave Equation

The propagation of waves in a three dimensional volume of length a, width b, and height d is governed by the following initial boundary value problem [24];

$$
U_{tt} = c^2\left(U_{xx} + U_{yy} + U_{zz}\right),\ t>0\ ; \tag{209}
$$

With the boundary conditions:

$$
\begin{aligned}
&U(0,y,z,t) = 0\ ,\ U(a,y,z,t) = 0,\\
&U(x,0,z,t) = 0\ ,\ U(x,b,z,t) = 0,\\
&U(x,y,0,t) = 0\ ,\ U(x,y,c,t) = 0,
\end{aligned}
\tag{210}
$$

And the initial condition:

$$
U(x,y,z,0) = f(x,y,z)\ ,\ \ U_t(x,y,z,0) = g(x,y,z) \tag{211}
$$

Where $0 < x < a$, $0 < y < b$, $0 < z < d$, and $U = U(x,y,z,t)$ is the displacement of any point located at the position (x,y,z) of a rectangular volume at any time t, and c is the velocity of a propagating wave.

Example (1.5.33): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the wave-like models [24];

$$U_{tt} = 3\left(U_{xx} + U_{yy} + U_{zz}\right), \qquad 0 < x, y, z < \pi \;, \; t > 0 \quad (212)$$

Subject to the following boundary conditions;

$$\begin{aligned} U(0, y, z, t) &= U(\pi, y, z, t) = 0 \\ U(x, 0, z, t) &= U(x, \pi, z, t) = 0 \\ U(x, y, 0, t) &= U(x, y, \pi, t) = 0 \end{aligned} \quad (213)$$

And the initial condition as;

$$U(x, y, z, 0) = 0 \quad , \quad U_t(x, y, z, 0) = 3\sin x \sin y \sin z \quad (214)$$

Taking Sumudu transform of both sides of the equation (206) subject to the initial condition, we get;

$$S\left[U(x, y, z, t)\right] = 3u \sin x \sin y \sin z + 3u^2 S\left[U_{xx} + U_{yy} + U_{zz}\right] \quad (215)$$

The inverse of Sumudu transform implies that:

$$U(x, y, z, t) = 3t \sin x \sin y \sin z + 3S^{-1}\left[u^2 S\left[U_{xx} + U_{yy} + U_{zz}\right]\right] \quad (216)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, z, t) &= 3t \sin x \sin y \sin z + 3S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t)\right)_{xx}\right]\right] \\ &+ 3S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t)\right)_{yy}\right]\right] + 3S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t)\right)_{zz}\right]\right] \end{aligned} \quad (217)$$

This gives;

$$\begin{aligned} U_0(x, y, z, t) &= 3t \sin x \sin y \sin z \\ U_1(x, y, z, t) &= 3S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] + 3S^{-1}\left[u^2 S\left[(U_0)_{yy}\right]\right] \\ &+ 3S^{-1}\left[u^2 S\left[(U_0)_{zz}\right]\right] = -\frac{(3t)^3}{3!} \sin x \sin y \sin z \\ U_2(x, y, z, t) &= 3S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right] + 3S^{-1}\left[u^2 S\left[(U_1)_{yy}\right]\right] \\ &+ 3S^{-1}\left[u^2 S\left[(U_1)_{zz}\right]\right] = \frac{(3t)^5}{5!} \sin x \sin y \sin z \end{aligned} \quad (218)$$

Therefore, the solution $U(x, y, z, t)$ in series form is given by;

$$U(x, y, z, t) = 3\sin x \sin y \sin z \left(3t - \frac{(3t)^3}{3!} + \frac{(3t)^5}{5!} - \frac{(3t)^7}{7!} - \cdots \right) \quad ; \quad (219)$$

And in closed form given as;

$$U(x, y, z, t) = \sin x \sin y \sin z \sin(3t) \quad . \quad (220)$$

Example (1.5.34): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models [28];

$$U_{tt} = \left(x^2+y^2+z^2\right) + \frac{1}{2}\left(x^2U_{xx} + y^2U_{yy} + z^2U_{zz}\right), \quad 0 < x, y, z < 1, \; t > 0 \tag{221}$$

Subject to the following boundary conditions;

$$\begin{aligned} U(0,y,z,t) &= y^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right), \quad U(1,y,z,t) = \left(1+y^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right) \\ U(x,0,z,t) &= x^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right), \quad U(x,\pi,z,t) = \left(1+x^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right) \\ U(x,y,0,t) &= \left(x^2+z^2\right)\left(e^t-1\right) \quad , \quad U(x,y,\pi,t) = \left(x^2+y^2\right)\left(e^t-1\right)+\left(e^{-t}-1\right) \end{aligned} \tag{222}$$

And the initial condition as;

$$U(x,y,z,0) = 0 \quad , \quad U(x,y,z,0)_t = x^2+y^2-z^2 \tag{223}$$

Taking Sumudu transform of both sides of the equation (221) subject to the initial condition, we get;

$$\begin{aligned} S\left[U(x,y,z,t)\right] = &\left(x^2+y^2+z^2\right)u^2 + \left(x^2+y^2-z^2\right)u \\ &+ \frac{1}{2}u^2 S\left[x^2U_{xx} + y^2U_{yy} + z^2U_{zz}\right] \end{aligned} \tag{224}$$

The inverse of Sumudu transform implies that:

$$\begin{aligned} U(x,y,z,t) = &\left(x^2+y^2+z^2\right)\frac{t^2}{2} + \left(x^2+y^2-z^2\right)t \\ &+ \frac{1}{2} S^{-1}\left[u^2 S\left[x^2U_{xx} + y^2U_{yy} + z^2U_{zz}\right]\right] \end{aligned} \tag{225}$$

Now, applying the Adomian decomposition method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x,y,z,t) = &\left(x^2+y^2+z^2\right)\frac{t^2}{2} + \left(x^2+y^2-z^2\right)t \\ &+ \frac{1}{2}x^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty}U_n(x,y,z,t)\right)_{xx}\right]\right] \\ &+\frac{1}{2}y^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty}U_n(x,y,z,t)\right)_{yy}\right]\right]+\frac{1}{2}z^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty}U_n(x,y,z,t)\right)_{zz}\right]\right]\Bigg) \end{aligned} \tag{226}$$

This gives;

$$U_0(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^2}{2} + \left(x^2+y^2-z^2\right)t$$

$$U_1(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^4}{4!} + \left(x^2+y^2-z^2\right)\frac{t^3}{3!}$$

$$U_2(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^6}{6!} + \left(x^2+y^2-z^2\right)\frac{t^5}{5!}$$

$$U_3(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^8}{8!} + \left(x^2+y^2-z^2\right)\frac{t^7}{7!} \qquad (227)$$

$$\vdots$$

Therefore the solution $U(x, y, z, t)$ in series form is given by;

$$U(x, y, z, t) = \left(x^2+y^2\right)\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots\right) + z^2\left(-t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots\right) \quad ; \quad (228)$$

And in closed form given as;

$$U(x, y, z, t) = \left(x^2+y^2\right)e^t + z^2 e^{-t} - \left(x^2+y^2+z^2\right) \quad . \quad (229)$$

CHAPTER (2)
Nonlinear Partial Differential Equations

2.1: Adomian Decomposition Method

Many of nonlinear phenomena are a necessary part in applied science and engineering fields. Nonlinear equations are noticed in a different type of physical problems such as fluid dynamics, plasma physics, solid mechanics, and quantum field theory.

The wide use of these equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically.
In the past, active study attempts were given a large amount of attention to the study of getting exact or approximate solutions of this kind of equations. It is significant to note that several powerful methods have been advanced for this purpose.

The Adomian decomposition method will be used in this chapter and in other chapters to deal with nonlinear equations. The Adomian decomposition method proves to be powerful, effective and successfully used to handle most types of linear or nonlinear ordinary or partial differential equations, and linear or nonlinear integral equations. This method is a simple and directly without any restrictive assumption as usual is going in other methods.

In the following, the Adomian scheme for calculating a wide variety of forms of nonlinearity.

2.1.1: Calculation of Adomian Polynomials

It is well known that the Adomian decomposition method suggests the unknown linear function u may be represented by the decomposition series;

$$u = \sum_{n=0}^{\infty} u_n \ , \tag{1}$$

where the components $u_n, n \geq 0$ can be elegantly computed in a recursive way. However, the nonlinear term $F(u)$, such as $u^2, u^3, u^4, \sin u, e^u, u u_x, u_x{}^2$, etc. can be expressed by an infinite series of the so- called Adomian polynomials A_n given in the form;

$$F(u)=\sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, ..., u_n). \qquad (2)$$

The Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression;

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,... \qquad (3)$$

Assuming that the nonlinear function is $F(u)$, therefore, by using (3), Adomian polynomials are given by;

$$\begin{aligned} A_0 &= F(u_0), \\ A_1 &= u_1 F'(u_0), \\ A_2 &= u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0), \\ A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0). \end{aligned} \qquad (4)$$

Other polynomials can be generated in a similar manner.
Substituting (4) into (2) gives;

$$\begin{aligned} F(u) &= A_0 + A_1 + A_2 + A_3 + ... \\ &= F(u_0) + (u_1 + u_2 + u_3 + ...)F'(u_0) + \frac{1}{2!}\left(u_1^2 + 2u_1 u_2 + u_2^2 + ...\right)F''(u_0) + \\ &\quad + \frac{1}{3!}\left(u_1^3 + 3u_1^2 u_2 + 3u_1^2 u_3 + ...\right)F'''(u_0) + ... \\ &= F(u_0) + (u-u_0)F'(u_0) + \frac{1}{2!}(u-u_0)^2 F''(u_0) + ... \end{aligned}$$

The last expansion confirms a fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used.
In the following, we will calculate Adomian polynomials for several forms of nonlinearity.

I. Nonlinear Polynomials

If $F(u) = u^2$

The polynomials can be found as follows:

$$A_0 = F(u_0) = u_0^2,$$
$$A_1 = u_1 F'(u_0) = 2u_0 u_1,$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = 2u_0 u_2 + u_1^2,$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = 2u_0 u_3 + 2u_1 u_2 .$$

And so on. Proceeding as before, we find $u^3, u^4, u^5, \ldots$, etc.

II. Nonlinear Derivatives

Case1. $F(u) = (u_x)^2$

The Adomian polynomials for this nonlinearity given by;

$$A_0 = u_{0x}^2,$$
$$A_1 = 2u_{0x} u_{1_x},$$
$$A_2 = 2u_{0x} u_{2x} + u_{1x}^2,$$
$$A_3 = 2u_{0x} u_{3x} + 2u_{1x} u_{2x}.$$

And so on. In a similar, we get $u_x^3, u_x^4, u_x^5, \ldots$, etc.

Case2. $F(u) = u u_x = \frac{1}{2} L_x(u^2)$

The A_n polynomials in this case given by;

$$A_0 = F(u_0) = u_0 u_{0_x},$$
$$A_1 = \frac{1}{2} L_x(2u_0 u_1) = u_{0x} u_1 + u_0 u_{1_x},$$
$$A_2 = \frac{1}{2} L_x(2u_0 u_2 + u_1^2) = u_{0_x} u_2 + u_0 u_{2_x} u_0 + u_1 u_{1_x},$$
$$A_3 = \frac{1}{2} L_x(2u_0 u_3 + 2u_1 u_2) = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0 .$$

And so on.

III. Trigonometric Nonlinearity

If $F(u) = \sin u$

The Adomian polynomials for this form nonlinearity are given by;

$$A_0 = \sin u_0 ,$$
$$A_1 = u_1 \cos u_0 ,$$
$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^{\,2} \sin u_0 ,$$
$$A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^{\,3} \cos u_0 .$$

And so on. In a similar way, we find $F(u) = \cos u$.

IV. Hyperbolic Nonlinearity

If $F(u) = \sinh u$

The A_n polynomials for this case are given by;

$$A_0 = \sinh u_0 ,$$
$$A_1 = u_1 \cosh u_0 ,$$
$$A_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1^{\,2} \sinh u_0 ,$$
$$A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!} u_1^{\,3} \cosh u_0 .$$

And so on. In a parallel manner, Adomian polynomials can be calculated for $F(u) = \cosh u$.

V. Exponential Nonlinearity

If $F(u) = e^{u}$

The Adomian polynomials in this form of nonlinearity are given by;

$$A_0 = e^{u_0} ,$$
$$A_1 = u_1 e^{u_0} ,$$
$$A_2 = \left(u_2 + \frac{1}{2!} u_1^{\,2} \right) e^{u_0} ,$$
$$A_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1^{\,3} \right) e^{u_0} .$$

And so on. Proceeding as a before, we find $F(u) = e^{-u}$.

VI. Logarithmic Nonlinearity

If $F(u) = \ln u$, $u > 0$

The A_n polynomials for logarithmic nonlinearity are given by;

$$A_0 = \ln u_0 ,$$

$$A_1 = \frac{u_1}{u_0} ,$$

$$A_2 = \frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^{\,2}}{u_0^{\,2}} ,$$

$$A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^{\,2}} + \frac{1}{3}\frac{u_1^{\,3}}{u_0^{\,3}}.$$

And so on. In a similar way, we find $F(u) = \ln(1+u)$,$-1 < u \le 1$.

2.1.2: A new Algorithm for Calculating Adomian Polynomials (The Alternative Algorithm for Calculating Adomian Polynomials)

It is well known about the main disadvantage of the calculating Adomian polynomials A_n , that it is a difficult method to perform calculation so called nonlinear terms. There is an alternative algorithm to reduce the demerits of formula introduced before, which depends mainly on algebraic, trigonometric identities and on Taylor expansions.

In the alternative algorithm which is a simple and reliable may be employed to calculate Adomian Polynomials A_n.

The new algorithm will be clarified by discussing the following suitable forms of nonlinearity [13].

I. Nonlinear Polynomials

If $F(u) = u^2$

We first set

$$u = \sum_{n=0}^{\infty} u_n . \tag{5}$$

Substituting (5) into $F(u) = u^2$ gives;

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + ...)^2 . \tag{6}$$

Expanding the expression at the right- hand side gives;

$$F(u) = u_0^{\,2} + 2u_0 u_1 + 2u_0 u_2 + u_1^{\,2} + 2u_0 u_3 + 2u_1 u_2 + \ldots \quad (7)$$

The expansion in (7) can be rearranged by grouping all terms with the sum of subscripts of the components is the same. This means that we can rewrite (7) as;

$$F(u) = \underbrace{u_0^{\,2}}_{A_0} + \underbrace{2u_0 u_1}_{A_1} + \underbrace{2u_0 u_2 + u_1^{\,2}}_{A_2} + \underbrace{2u_0 u_3 + 2u_1 u_2}_{A_3} + \ldots \quad . \quad (8)$$

This gives Adomian polynomials for $F(u) = u^2$ by;

$$A_0 = u_0^{\,2},$$
$$A_1 = 2u_0 u_1,$$
$$A_2 = 2u_0 u_2 + u_1^{\,2},$$
$$A_3 = 2u_0 u_3 + 2u_1 u_2 .$$

And so on. Proceeding as before, we get $u^3, u^4, u^5, \ldots$, etc.

II. Nonlinear Derivatives

Case1. If $F(u) = u_x^{\,2}$.

We first set;

$$u_x = \sum_{n=0}^{\infty} u_{n\,x} \quad . \quad (9)$$

Substituting (9) into $F(u) = u_x^{\,2}$ giving;

$$F(u) = \left(u_{0x} + u_{1x} + u_{2x} + u_{3x} + u_{4x} + \ldots\right)^2 . \quad (10)$$

Squaring the right – hand side gives;

$$F(u) = u_{0x}^{\,2} + 2u_{0x} u_{1x} + 2u_{0x} u_{2x} + u_{1x}^{\,2} + 2u_{0x} u_{3x} + 2u_{1x} u_{2x} + \ldots \; . \quad (11)$$

Grouping the terms as discussed above, we find;

$$F(u) = \underbrace{u_{0x}^{\,2}}_{A_0} + \underbrace{2u_{0x} u_{1x}}_{A_1} + \underbrace{2u_{0x} u_{2x} + u_{1x}^{\,2}}_{A_2} + \underbrace{2u_{0x} u_{3x} + 2u_{1x} u_{2x}}_{A_3} + \ldots \; . \quad (12)$$

Adomian polynomials are given by;

$$A_0 = u_{0x}^{\,2},$$
$$A_1 = 2u_{0x} u_{1_x},$$
$$A_2 = 2u_{0x} u_{2x} + u_{1x}^{\,2},$$
$$A_3 = 2u_{0x} u_{3x} + 2u_{1x} u_{2x} .$$

Case2. $F(u) = u\,u_x$

Note that this form of nonlinearity appears in advection problems and in nonlinear Burgers equations. We first set;

$$u = \sum_{n=0}^{\infty} u_n \quad , \quad u_x = \sum_{n=0}^{\infty} u_{n x} \quad . \tag{13}$$

Substituting (13) into $F(u) = u u_x$ yields;

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + \ldots) \times (u_{0x} + u_{1x} + u_{2x} + u_{3x} + u_{4x} + \ldots); \tag{14}$$

Multiplying the two factors gives;

$$\begin{aligned} F(u) = {} & u_0 u_{0x} + u_{0x} u_1 + u_0 u_{1x} + u_{0x} u_2 + u_{1x} u_1 + u_{2x} u_0 + u_{0x} u_3 + u_{1x} u_2 + \\ & + u_{2x} u_1 + u_{3x} u_0 + u_{0x} u_4 + u_0 u_{4x} + u_{1x} u_3 + u_1 u_{3x} + u_2 u_{2x} + \ldots \end{aligned} \quad . \tag{15}$$

Proceeding with grouping the terms are obtain;

$$\begin{aligned} F(u) = {} & \underbrace{u_0 u_{0x}}_{A_0} + \underbrace{u_{0x} u_1 + u_0 u_{1x}}_{A_1} + \underbrace{u_{0x} u_2 + u_{1x} u_1 + u_{2x} u_0}_{A_2} + \\ & + \underbrace{u_{0x} u_3 + u_{1x} u_2 + u_{2x} u_1 + u_{3x} u_0}_{A_3} \ldots \end{aligned} \tag{16}$$

Consequently, the Adomian polynomials are given by;

$$\begin{aligned} A_0 &= u_0 u_{0_x} , \\ A_1 &= u_{0x} u_1 + u_0 u_{1_x} , \\ A_2 &= u_{0_x} u_2 + u_0 u_{2_x} u_0 + u_1 u_{1_x} , \\ A_3 &= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0 . \end{aligned}$$

Proceeding as before, we find $F(u) = u^2 u_x$.

III. Trigonometric Nonlinearity

If $F(u) = \sin u$

First, we should be separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute;

$$u = \sum_{n=0}^{\infty} u_n \; ; \tag{17}$$

Into $F(u) = \sin u$ to obtain;

$$F(u) = \sin\left[u_0 + (u_1 + u_2 + u_3 + u_4 + \ldots)\right] . \tag{18}$$

To separate A_0 , recall the trigonometric identity;

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi \; . \tag{19}$$

This means that;

$$F(u) = \sin u_0 \cos(u_1 + u_2 + u_3 + u_4 + ...) + \cos u_0 \sin(u_1 + u_2 + u_3 + u_4 + ...) \ . \quad (20)$$

Separating $F(u_0) = \sin u_0$ from other factors and using Taylor expansion for $\cos(u_1 + u_2 + u_3 + u_4 + ...)$ and $\sin(u_1 + u_2 + u_3 + u_4 + ...)$ gives;

$$\begin{aligned} F(u) &= \sin u_0 \left(1 - \frac{1}{2!}(u_1 + u_2 + ...)^2 + \frac{1}{4!}(u_1 + u_2 + ...)^4 - ... \right) + \\ &+ \cos u_0 \left((u_1 + u_2 + ...) - \frac{1}{3!}(u_1 + u_2 + ...)^3 + ... \right) \end{aligned} \ , \quad (21)$$

So that;

$$F(u) = \sin u_0 \left(1 - \frac{1}{2!}\left(u_1^{\ 2} + 2u_1 u_2 + ...\right) + ... \right) + \cos u_0 \left((u_1 + u_2 + ...) - \frac{1}{3!} u_1^{\ 3} + ... \right). \quad (22)$$

The last expansion can be rearranged by grouping all terms with the same sum of subscripts. This leads to;

$$\begin{aligned} F(u) &= \underbrace{\sin u_0}_{A_0} + \underbrace{u_1 \cos u_0}_{A_1} + \underbrace{u_2 \cos u_0 - \frac{1}{2!} u_1^{\ 2} \sin u_0}_{A_2} + \\ &\quad + \underbrace{u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^{\ 3} \cos u_0}_{A_3} + ... \end{aligned} \quad (23)$$

This completes the calculation of the Adomian polynomials for nonlinear operator $F(u) = \sin u$, therefore we write;

$$\begin{aligned} A_0 &= \sin u_0 \ , \\ A_1 &= u_1 \cos u_0 \ , \\ A_2 &= u_2 \cos u_0 - \frac{1}{2!} u_1^{\ 2} \sin u_0 \ , \\ A_3 &= u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^{\ 3} \cos u_0 \ . \end{aligned}$$

And so on. In a similar way, we find $F(u) = \cos u$.

IV. Hyperbolic Nonlinearity

If $F(u) = \sinh u$

We first substitute

$$u = \sum_{n=0}^{\infty} u_n \ ; \quad (24)$$

Into $F(u) = \sinh u$ to obtain;

$$F(u) = \sinh\left[u_0 + (u_1 + u_2 + u_3 + u_4 + ...)\right] . \tag{25}$$

To calculate A_0 , recall the hyperbolic identity;

$$\sinh(\theta+\phi) = \sinh\theta\cosh\phi + \cosh\theta\sinh\phi . \tag{26}$$

Accordingly, Eq. (26) becomes;

$$F(u) = \sinh u_0 \cosh(u_1 + u_2 + u_3 + u_4 + ...) + \cosh u_0 \sinh(u_1 + u_2 + u_3 + u_4 + ...). \tag{27}$$

Separating $F(u_0) = \sinh u_0$ from other factors and using Taylor expansion for $\cosh(u_1 + u_2 + u_3 + u_4 + ...)$ and $\sinh(u_1 + u_2 + u_3 + u_4 + ...)$ gives;

$$F(u) = \sinh u_0\left(1 + \frac{1}{2!}(u_1 + u_2 + ...)^2 + \frac{1}{4!}(u_1 + u_2 + ...)^4 + ...\right) +$$

$$+ \cosh u_0\left((u_1 + u_2 + ...) + \frac{1}{3!}(u_1 + u_2 + ...)^3 + ...\right)$$

$$= \sinh u_0\left(1 + \frac{1}{2!}\left(u_1^{\ 2} + 2u_1 u_2 + ...\right) + ...\right) + \cosh u_0\left((u_1 + u_2 + ...) + \frac{1}{3!}u_1^{\ 3} + ...\right)$$

By grouping all terms with the same sum of subscripts we find

$$F(u) = \underbrace{\sinh u_0}_{A_0} + \underbrace{u_1 \cosh u_0}_{A_1} + \underbrace{u_2 \cosh u_0 + \frac{1}{2!}u_1^{\ 2} \sinh u_0}_{A_2} +$$

$$\underbrace{+ u_3 \cosh u_0 + u_1 u_2 \sinh u_0 - \frac{1}{3!}u_1^{\ 3} \cosh u_0}_{A_3} + ...$$

Consequently, the Adomian polynomials for $F(u) = \sinh u$ are given by;

$$A_0 = \sinh u_0 ,$$

$$A_1 = u_1 \cosh u_0 ,$$

$$A_2 = u_2 \cosh u_0 + \frac{1}{2!}u_1^{\ 2} \sinh u_0 ,$$

$$A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!}u_1^{\ 3} \cosh u_0 .$$

Similarly as before, we find $F(u) = \cosh u$.

V. Exponential Nonlinearity

If $F(u) = e^u$.

Substituting

$$u = \sum_{n=0}^{\infty} u_n ; \tag{28}$$

Into $F(u) = e^{u}$ gives;

$$F(u) = e^{(u_0 + u_1 + u_2 + u_3 + u_4 + ...)} \quad . \tag{29}$$

Or equivalently;

$$F(u) = e^{u_0} \times e^{(u_1 + u_2 + u_3 + u_4 + ...)} \quad . \tag{30}$$

Keeping the term $F(u_0) = e^{u_0}$ and using Taylor expansion for the other factors we obtain;

$$F(u) = e^{u_0} \times \left(1 + (u_1 + u_2 + u_3 + ...) + \frac{1}{2!}(u_1 + u_2 + u_3 + ...)^2 + ... \right) \quad . \tag{31}$$

By grouping all terms with an identical sum of subscripts we find

$$F(u) = \underbrace{e^{u_0}}_{A_0} + \underbrace{u_1 e^{u_0}}_{A_1} + \underbrace{\left(u_2 + \frac{1}{2!} u_1{}^2 \right) e^{u_0}}_{A_2} + \underbrace{\left(u_3 + u_1 u_2 + \frac{1}{3!} u_1{}^3 \right) e^{u_0}}_{A_3} + ... \; . \tag{32}$$

It then follows that;

$$A_0 = e^{u_0} ,$$

$$A_1 = u_1 e^{u_0} ,$$

$$A_2 = \left(u_2 + \frac{1}{2!} u_1{}^2 \right) e^{u_0} ,$$

$$A_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1{}^3 \right) e^{u_0} .$$

And so on. Proceeding as a before, we find $F(u) = e^{-u}$.

VI. Logarithmic Nonlinearity

If $F(u) = \ln u$, $u > 0$

Substituting

$$u = \sum_{n=0}^{\infty} u_n \; ; \tag{33}$$

Into $F(u) = \ln u$ gives;

$$F(u) = \ln(u_0 + u_1 + u_2 + u_3 + u_4 + ...) \; . \tag{34}$$

Eq. (34) can be written as;

$$F(u) = \ln\left(u_0 \left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + ... \right) \right) \; . \tag{35}$$

Using the identity $\ln(\alpha \beta) = \ln \alpha + \ln \beta$, Eq. (35) becomes;

$$F(u) = \ln(u_0) + \ln\left(1+\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right). \qquad (36)$$

Separating $F(u_0) = \ln(u_0)$ and using Taylor expansion of the remaining term, we obtain;

$$F(u) = \ln(u_0) + \left\{ \begin{array}{l} \left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right) - \frac{1}{2}\left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right)^2 + \frac{1}{3}\left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right)^3 \\ -\frac{1}{4}\left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right)^4 + \ldots \end{array} \right\} \qquad (37)$$

Proceeding as before, Eq. (37) can be rewritten as;

$$F(u) = \underbrace{\ln(u_0)}_{A_0} + \underbrace{\frac{u_1}{u_0}}_{A_1} + \underbrace{\frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^{\,2}}{u_0^{\,2}}}_{A_2} + \underbrace{\frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^{\,2}} + \frac{1}{3}\frac{u_1^{\,3}}{u_0^{\,3}}}_{A_3} + \ldots \; . \qquad (38)$$

Based on this, the Adomian polynomials are given by;

$$A_0 = \ln(u_0),$$

$$A_1 = \frac{u_1}{u_0},$$

$$A_2 = \frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^{\,2}}{u_0^{\,2}},$$

$$A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^{\,2}} + \frac{1}{3}\frac{u_1^{\,3}}{u_0^{\,3}}.$$

And so on. In a like manner, we obtain $F(u) = \ln(1+u), -1 < u \le 1$.

2.2: Adomian Decomposition Method and Sumudu Transform Method for Solving Nonlinear Partial Differential Equations

In this section, we will concentrate our study on the nonlinear PDEs. There are many nonlinear partial differential equations which are quite useful and applicable in engineering and physics.

The nonlinear phenomena that appear in the many scientific fields' such as solid state physics, plasma physics, fluid mechanics and quantum field theory can be modeled by nonlinear differential equations. The significance of obtaining exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is yet an important problem that needs new methods to develop new techniques for obtaining analytical solutions. Several powerful mathematical methods are used for this purpose.

In this section, we propose a new method, namely Adomian Decomposition Sumudu Transform Method (ADSTM) for solving nonlinear equations. This method is a combination of Sumudu transform and decomposition method which was introduced by D. Kumar, J. Singh and S. Rathore [26].

(ADSTM) provides the solution for nonlinear equations in the form of convergent series. This forms the motivation for us to apply (ADSTM) for solving nonlinear equations in understanding different physical phenomena.

To illustrate the basic idea of this method, we consider a general non-homogeneous partial differential equation with the initial conditions of the form:

$$\begin{aligned} &DU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \\ &U(x,0) = h(x) \ ,U_t(x,0) = f(x). \end{aligned} \quad ; \qquad (39)$$

where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D, N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (39), we get:

$$S[DU(x,t)] + S[RU(x,t)] + S[N(x,t)] = S[g(x,t)] \quad ; \qquad (40)$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have:

$$S[U(x,t)] = u^2 S[g(x,t)] + h(x) + u\, f(x) - u^2 S[RU(x,t) + NU(x,t)] \quad . \qquad (41)$$

If we apply the inverse operator S^{-1} to both sides of equation (41), we obtain:

$$U(x,t) = G(x,t) - S^{-1}\left[u^2 S[RU(x,t) + NU(x,t)]\right] \quad . \qquad (42)$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, apply the Adomian decomposition method:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \qquad ; \qquad (43)$$

The nonlinear term can be decomposed as:

$$NU(x,t) = \sum_{n=0}^{\infty} A_n(U) \qquad ; \qquad (44)$$

For some Adomian polynomials $A_n(U)$ that are given by:

$$A_n(U_0, U_1, U_2, ..., U_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} \lambda^n U_n\right)\right]_{\lambda=0}, \; n = 0,1,2,....$$

Substituting Eq. (43) and Eq. (44) in Eq. (42), we get:

$$\sum_{n=0}^{\infty} U_n(x,t) = G(x,t) - S^{-1}\left[u^2 S\left[R\sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} A_n(U)\right]\right] \quad . \qquad (45)$$

Accordingly, the formal recursive relation is defined by:

$$\begin{aligned} U_0(x,t) &= G(x,t), \\ U_{k+1}(x,t) &= -S^{-1}\left[u^2 S[RU_k + A_k]\right]. \; k \geq 0. \end{aligned} \qquad (46)$$

The Adomian decomposition Sumudu transform method will be illustrated by discussing the following examples.

Example (2.2.1): Consider the nonlinear ordinary differential equation [24]**:**

$$y' - y^2 = 1 \;,\; y(0) = 0 \quad ; \qquad (47)$$

Taking the Sumudu transform to both sides (47) and using the initial condition gives:

$$S[y(x)] = u + uS[y^2] \quad . \qquad (48)$$

Applying S^{-1} to both sides of (48) gives:

$$y(x) = x + S^{-1}\left[uS[y^2]\right] \quad . \qquad (49)$$

The decomposition method suggests that the solution $y(x)$ can be expressed by the decomposition series:

$$y(x)=\sum_{n=0}^{\infty} y_n \; ; \tag{50}$$

The components of $y(x)$ can be elegantly determined by using the recursive relation:

$$\begin{aligned} y_0(x) &= x, \\ y_{k+1}(x) &= S^{-1}\left[u\,S\left[A_k\right]\right],\; k\geq 0. \end{aligned} \tag{51}$$

Note that the Adomian polynomials A_n for the nonlinear term y^2 were determined using Eq. (8), and we found:

$$\begin{aligned} A_0 &= y_0^{\,2}, \\ A_1 &= 2y_0\,y_1, \\ A_2 &= 2y_0\,y_2 + y_1^{\,2}, \\ A_3 &= 2y_0\,y_3 + 2y_1\,y_2. \end{aligned}$$

And so on. Using these polynomials into (51), the first few components can be determined recursively by:

$$\begin{aligned} y_0(x) &= x, \\ y_1(x) &= S^{-1}\left[u\,S(A_0)\right] = \frac{x^3}{3}, \\ y_2(x) &= S^{-1}\left[u\,S(A_1)\right] = \frac{2}{15}x^5, \\ y_3(x) &= S^{-1}\left[u\,S(A_2)\right] = \frac{17}{315}x^7. \end{aligned} \tag{52}$$

Consequently, the solution in a series form is given by;

$$y(x)=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\dots \; ; \tag{53}$$

And in a closed form of:

$$y(x)=\tan x \; . \tag{54}$$

Example (2.2.2): Consider the nonlinear ordinary differential equation [24]**:**

$$y' = 1 - x^2 + y^2 \ , \quad y(0) = 0 \ ; \tag{55}$$

In a similar way as above, we have:

$$\sum_{n=0}^{\infty} y_n(x) = x - \frac{1}{3}x^3 + S^{-1}\left[u\,S\left(\sum_{n=0}^{\infty} A_n \right) \right] \ . \tag{56}$$

The modified recursive relation is defined by:

$$\begin{aligned} y_0 &= x, \\ y_1 &= -\frac{1}{3}x^3 + S^{-1}[u\,S(A_0)], \\ y_{k+2}(x) &= S^{-1}[u\,S(A_K)], k \ge 0. \end{aligned} \tag{57}$$

Consequently, the first few components are given by:

$$\begin{aligned} y_0 &= x, \\ y_1 &= -\frac{1}{3}x^3 + S^{-1}[u\,S(A_0)] = 0, \\ y_{k+2}(x) &= 0, k \ge 0. \end{aligned} \tag{58}$$

The exact solution is given by:

$$y(x) = x \quad . \tag{59}$$

Example (2.2.3): Consider the nonlinear ordinary differential equation [24]**:**

$$y'' + (y')^2 + y^2 = 1 - \sin x \ , \ y(0) = 0 \ , \ y'(0) = 1 \ . \tag{60}$$

In a similar way above, we have:

$$\sum_{n=0}^{\infty} y_n(x) = \frac{x^2}{2} + \sin x - S^{-1}\left[u^2\,S\left(\sum_{n=0}^{\infty} A_n \right) \right] \ . \tag{61}$$

This leads to the recursive relation;

$$\begin{aligned} y_0(x) &= \sin x + \frac{x^2}{2}, \\ y_{k+1}(x) &= -S^{-1}\left[u^2\,S(A_k)\right], \ k \ge 0. \end{aligned} \tag{62}$$

This relation leads to the identification:

$$y_0(x) = \sin x + \frac{x^2}{2} ,$$
$$y_1(x) = -S^{-1}\left[u^2 S\left[\left(y_0'\right)^2 + y_0^2\right]\right] = -\frac{x^2}{2} + \ldots, \tag{63}$$

The zeroth component contains the trigonometric function $\sin x$, therefore it is recommended that the noise term phenomenon be used here. By canceling the noise terms $\frac{x^2}{2}$ and $-\frac{x^2}{2}$ between $y_0(x)$ and $y_1(x)$, the exact solution given by:

$$y(x) = \sin x \quad . \tag{64}$$

Example (2.2.4): Consider the following nonlinear partial differential equation [24]**:**

$$U_t + UU_x = 0 \quad ; \tag{65}$$

With the initial condition:

$$U(x,0) = x \; . \tag{66}$$

Taking the Sumudu transform of both sides of Eq. (65) and using the initial condition, we have:

$$S[U(x,t)] = x - u\,S[UU_x] \; . \tag{67}$$

Applying S^{-1} to both sides of Eq. (67) implies that:

$$U(x,t) = x - S^{-1}[u\,S[UU_x]] \;\; ; \tag{68}$$

Following the technique, if we assume an infinite series of the form (68), we obtain:

$$\sum_{n=0}^{\infty} U_n(x,t) = x - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n(U)\right]\right] \; . \tag{69}$$

Where $A_n(U)$ are Adomian polynomials that represent the nonlinear terms.
The first few components of the Adomian polynomials are given by;

$$A_0(U) = U_0 U_{0_x} ,$$
$$A_1(U) = U_0 U_{1_x} + U_1 U_{0_x} , \tag{70}$$
$$. \quad . \quad . \quad .$$

This gives the recursive relation:

$$U_0(x,t) = x ,$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S[A_k]] , \; k \geq 0. \tag{71}$$

The first few components are given by:

$$\begin{aligned} U_0(x,t) &= x\ , \\ U_1(x,t) &= -S^{-1}[u\,S[A_0]] = -xt\,, \\ U_2(x,t) &= -S^{-1}[u\,S[A_1]] = xt^2\,, \\ U_3(x,t) &= -S^{-1}[u\,S[A_2]] = -xt^3. \end{aligned} \tag{72}$$

And so on. The solution in a series form is given by:

$$U(x,t) = x(1 - t + t^2 - t^3 + \ldots)\ ; \tag{73}$$

And in a closed form of:

$$U(x,t) = \frac{x}{1+t}\ . \tag{74}$$

Example (2.2.5): Consider the following nonlinear partial differential equation [29, 11]:

$$U_t + UU_x = x + xt^2; \tag{75}$$

With the initial condition:

$$U(x,0) = 0\ . \tag{76}$$

Proceeding as in **Example (2.2.4)**, Eq. (75) becomes:

$$\sum_{n=0}^{\infty} U_n(x,t) = xt + \frac{xt^3}{3} - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n(U)\right]\right]\ . \tag{77}$$

The modified decomposition method admits the of a modified recursive relation given by:

$$\begin{aligned} U_0(x,t) &= xt\ , \\ U_1(x,t) &= \frac{xt^3}{3} - S^{-1}[u\,S[A_0]] \\ U_{k+1}(x,t) &= -S^{-1}[u\,S[A_k]],\ k \geq 1. \end{aligned} \tag{78}$$

Consequently, we obtain:

$$U_0(x,t) = xt\ ,$$
$$U_1(x,t) = \frac{xt^3}{3} - S^{-1}\left[u\,S\left[xt^2\right]\right] = 0 \tag{79}$$
$$U_{k+1}(x,t) = 0,\ k \geq 1.$$

In few of Eq. (79), the exact solution is given by:

$$U(x,t) = xt\ . \tag{80}$$

Example (2.2.6): Consider the nonlinear partial differential equation [11]:

$$U_{tt} + U_x^{\ 2} + U - U^2 = t\,e^{-x}\quad ; \tag{81}$$

With the initial condition

$$U(x,0) = 0,\ U_t(x,0) = e^{-x}\quad . \tag{82}$$

By taking Sumudu transform for (81) and using (82) we obtain:

$$S\left[U(x,t)\right] = u^3 e^{-x} + u\,e^{-x} - u^2\,S\left[U_x^{\ 2} - U^2 + U\right]\ . \tag{83}$$

Applying S^{-1} to both sides of (83) we obtain;

$$U(x,t) = te^{-x} + \frac{1}{6}t^3 e^{-x} - S^{-1}\left[u^2\,S\left[U_x^{\ 2} - U^2 + U\right]\right]. \tag{84}$$

Substituting;

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t)\ ; \tag{85}$$

And the nonlinear terms of;

$$U_x^{\ 2} = \sum_{n=0}^{\infty} A_n\ ,\ U^2 = \sum_{n=0}^{\infty} B_n\ . \tag{86}$$

Into (84) gives;

$$\sum_{n=0}^{\infty} U_n(x,t) = te^{-x} + \frac{1}{6}t^3 e^{-x} - S^{-1}\left[u^2\,S\left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} U_n(x,t) - \sum_{n=0}^{\infty} B_n\right)\right] \tag{87}$$

This gives the modified recursive relation;

$$U_0(x,t) = t\,e^{-x}\ ,$$
$$U_1(x,t) = \frac{1}{6}t^3\,e^{-x} - L_t^{-1}\left(A_0 + U_0 - B_0\right) \tag{88}$$
$$U_{k+1}(x,t) = -\ L_t^{-1}\left(A_k + U_k - B_k\right),\ k \geq 1.$$

The first few of the components are given by;

$$U_0(x,t) = t\,e^{-x} \;,$$
$$U_1(x,t) = \frac{1}{6}t^3\,e^{-x} - L_t^{-1}\left(A_0 + U_0 - B_0\right) = 0, \qquad (89)$$
$$U_{k+1}(x,t) = 0,\; k \geq 1.$$

The solution in a closed form is given by;

$$U(x,t) = t\,e^{-x} \;. \qquad (90)$$

Example (2.2.7): Consider the following nonlinear partial differential equation [24]:

$$U_{tt} + U_t U_{xx} = -t + U \quad ; \qquad (91)$$

With the initial conditions

$$U(x,0) = \sin x \;,\; U_t(x,0) = 1 \;. \qquad (92)$$

By applying the aforesaid method subject to the initial condition, we have:

$$S\left[U(x,t)\right] = u + \sin x - u^3 + u^2\,S\left[U + U_t U_{xx}\right] \;. \qquad (93)$$

The inverse of Sumudu transform implies that:

$$U(x,t) = t + \sin x - \frac{t^3}{6} + S^{-1}\left[u^2\,S\left[U + U_t\,U_{xx}\right]\right] \;. \qquad (94)$$

Now, applying the same procedure as in the previous **Example (2.2.6)**, we arrive in recursive relation given below:

$$U_0(x,t) = t + \sin x - \frac{t^3}{6} \;,$$
$$U_{k+1}(x,t) = S^{-1}\left[u^2\,S\left[U_k + A_k\right]\right],\; k \geq 0. \qquad (95)$$

This relation leads to the identification:

$$U_0(x,t) = t + \sin x - \frac{t^3}{6} \;,$$
$$U_1(x,t) = \frac{t^3}{6} + \frac{t^4}{4!}\sin x - \frac{t^5}{5!}, \qquad (96)$$

The zeroth component contains the trigonometric function $\sin x$, therefore it is recommended that the noise term phenomenon be used here. By canceling the noise terms $\frac{x^3}{6}$ and $-\frac{x^3}{6}$ between $U_0(x)$ and $U_1(x)$, the exact solution given by:

$$U(x,t) = t + \sin x \;. \qquad (97)$$

Example (2.2.8): Consider the following nonlinear partial differential equation [24]**:**

$$U_{tt} + U^2 - U_x{}^2 = 0 \quad ; \tag{98}$$

With the initial conditions

$$U(x,0) = 0 \ , U_t(x,0) = e^x \ . \tag{99}$$

By taking Sumudu transform for (98) and using (99) we obtain:

$$S[U(x,t)] = u\,e^x + u^2\,S\left[U_x{}^2 - U^2\right] . \tag{100}$$

By applying the inverse Sumudu transform of (100), we get:

$$U(x,t) = t\,e^x + S^{-1}\left[u^2\,S\left[U_x{}^2 - U^2\right]\right] ; \tag{101}$$

This assumes a series solution of the function $U(x,t)$ is given by:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \ ; \tag{102}$$

Using (102) into (101), we get:

$$\sum_{n=0}^{\infty} U_n(x,t) = t\,e^x + S^{-1}\left[u^2\,S\left[\sum_{n=0}^{\infty} A_n(U) - \sum_{n=0}^{\infty} B_n(U)\right]\right] . \tag{103}$$

Where $A_n(U)$ and $B_n(U)$ are Adomian polynomials that represents nonlinear terms. The few components of the Adomian polynomials are given as follows:

$$\begin{aligned} &A_0(U) = U_{0_x}{}^2 \ , \ A_1(U) = 2U_{0_x}\,U_{1_x} \ , \\ &B_0(U) = U_0{}^2 \quad , \quad B_1(U) = 2U_0\,U_1 \ , \end{aligned} \tag{104}$$

And so on. From the above equations we obtain:

$$\begin{aligned} &U_0(x,t) = t\,e^x \ , \\ &U_{k+1}(x,t) = S^{-1}\left[u^2\,S\left[A_k - B_k\right]\right], \ k \geq 0. \end{aligned} \tag{105}$$

The first few terms of $U_n(x,t)$ follows immediately upon setting:

$$\begin{aligned} &U_1(x,t) = S^{-1}\left[u^2\,S\left[A_0 - B_0\right]\right] = S^{-1}\left[u^2\,S\left[U_{0_x}{}^2 - U^2{}_0\right]\right] = 0 \\ &U_{k+1}(x,t) = 0 \ , \ k \geq 1. \end{aligned} \tag{106}$$

Therefore the solution obtained by ADSTM is given as follows:

$$U(x,t) = t\,e^x . \tag{107}$$

2.3: Adomian Decomposition Method and Sumudu Transform Method for Solving Systems of Nonlinear Partial Differential Equations

In this section, the system of nonlinear PDEs will be examined by using (ADSTM). Systems of nonlinear PDEs arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of chemical reaction – diffusion model. To achieve our goal in handling systems of nonlinear PDEs, we write a system as:

$$\begin{aligned} U_t(x,t) + V_x(x,t) + N_1(U,V) &= g_1, \\ V_t(x,t) + U_x(x,t) + N_2(U,V) &= g_2, \end{aligned} \tag{108}$$

With initial data:

$$U(x,0) = f_1(x)\ ,\ V(x,0) = f_2(x) \tag{109}$$

Where N_1 and N_2 are nonlinear terms, and g_1, g_2 are source terms. Applying the Sumudu transforms to the system (108) and using (109) yields:

$$\begin{aligned} S[U(x,t)] &= f_1(x) + u\,S[g_1] - u\,S[V_x(x,t)] - u\,S[N_1(U,V)], \\ S[V(x,t)] &= f_2(x) + u\,S[g_2] - u\,S[U_x(x,t)] - u\,S[N_2(U,V)]. \end{aligned} \tag{110}$$

Using inverse Sumudu transform of (110) gives:

$$\begin{aligned} U(x,t) &= f_1(x) + S^{-1}[u\,S[g_1]] - S^{-1}[u\,S[V_x(x,t)]] - S^{-1}[u\,S[N_1(U,V)]], \\ V(x,t) &= f_2(x) + S^{-1}[u\,S[g_2]] - S^{-1}[u\,S[U_x(x,t)]] - S^{-1}[u\,S[N_2(U,V)]]. \end{aligned} \tag{111}$$

The linear unknown function $U(x,t)$ and $V(x,t)$ can be decomposed by infinite series of components:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t)\ ,\ V(x,t) = \sum_{n=0}^{\infty} V_n(x,t) \quad ; \tag{112}$$

However, the nonlinear terms N_1 and N_2 should be represented by Adomian polynomials $A_n(U)$ and $B_n(U)$ as follows:

$$N_1(U,V) = \sum_{n=0}^{\infty} A_n\ ,\ N_2(U,V) = \sum_{n=0}^{\infty} B_n \quad ; \tag{113}$$

Substituting (112) and (113) into (111) gives:

$$\sum_{n=0}^{\infty} U_n(x,t) =$$

$$f_1(x) + S^{-1}[u\,S[g_1]] - S^{-1}[u\,S[V_x(x,t)]] - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n\right]\right],$$

$$\sum_{n=0}^{\infty} V_n(x,t) = \tag{114}$$

$$f_2(x) + S^{-1}[u\,S[g_2]] - S^{-1}[u\,S[U_x(x,t)]] - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} B_n\right]\right].$$

The recursive relations can be constructed from (114) given by:

$$\begin{aligned} U_0(x,t) &= f_1(x) + S^{-1}[u\,S[g_1]], \\ U_{k+1}(x,t) &= -S^{-1}\left[u\,S[V_k]_x\right] - S^{-1}[u\,S[A_k]],\ k \geq 0. \end{aligned} \tag{115}$$

And

$$\begin{aligned} V_0(x,t) &= f_2(x) + S^{-1}[u\,S[g_2]], \\ V_{k+1}(x,t) &= -S^{-1}\left[u\,S[U_k]_x\right] - S^{-1}[u\,S[B_k]],\ k \geq 0. \end{aligned} \tag{116}$$

To have a clear overview, forthwith are several examples to demonstrate the efficiency of the method.

Example (2.3.9): Consider the following nonlinear system of partial differential equations [30]:

$$\begin{aligned} U_t(x,t) + VU_x + U &= 1, \\ V_t(x,t) - UV_x - V &= 1, \end{aligned} \quad ; \tag{117}$$

With the initial conditions:

$$U(x,0) = e^x,\ V(x,0) = e^{-x}\quad . \tag{118}$$

Taking the Sumudu transform of the system (117) and using initial conditions (118) we obtain:

$$\begin{aligned} S[U(x,t)] &= e^x + u - u\,S[U] - u\,S[VU_x], \\ S[V(x,t)] &= e^{-x} + u + u\,S[V] + u\,S[UV_x]. \end{aligned} \tag{119}$$

Using inverse Sumudu transform from (119) gives:

$$\begin{aligned} U(x,t) &= e^x + t - S^{-1}[u\,S[U]] - S^{-1}[u\,S[VU_x]], \\ V(x,t) &= e^{-x} + t + S^{-1}[u\,S[V]] + S^{-1}[u\,S[UV_x]]. \end{aligned} \tag{120}$$

The modified decomposition method defines the recursive relations in the form:

$$\begin{aligned} U_0(x,t) &= e^x \\ U_1(x,t) &= t - S^{-1}\left[u\,S\left[U_0 + A_0\right]\right], \\ U_{k+1}(x,t) &= -\,S^{-1}\left[u\,S\left[U_k + A_k\right]\right],\ k \geq 0. \end{aligned} \tag{121}$$

And

$$\begin{aligned} V_0(x,t) &= e^{-x} \\ V_1(x,t) &= t + S^{-1}\left[u\,S\left[V_0 + B_0\right]\right], \\ V_{k+1}(x,t) &= S^{-1}\left[u\,S\left[V_k + B_k\right]\right],\ k \geq 0. \end{aligned} \tag{122}$$

The Adomian polynomials for the nonlinear term VU_x are given by:

$$A_0 = V_0 U_{0_x} \quad , \quad A_1 = V_1 U_{0_x} + V_0 U_{1_x} \ ,$$
$$A_2 = V_2 U_{0_x} + V_1 U_{1_x} + V_0 U_{2_x}$$

And so on. And for the nonlinear term UV_x by:

$$B_0 = U_0 V_{0_x} \quad , \quad B_1 = U_1 V_{0_x} + U_0 V_{1_x} \ ,$$
$$B_2 = U_2 V_{0_x} + U_1 V_{1_x} + U_0 V_{2_x}$$

And so on. Using the derived Adomian polynomials into (121) and (122), we obtain the following pairs of components:

$$\begin{aligned} (U_0, V_0) &= \left(e^x, e^{-x}\right), \\ (U_1, V_1) &= \left(-\,t e^x, t e^{-x}\right), \\ (U_2, V_2) &= \left(\frac{t^2}{2!} e^x, \frac{t^2}{2!} e^{-x}\right), \\ (U_3, V_3) &= \left(-\frac{t^3}{3!} e^x, \frac{t^3}{3!} e^{-x}\right). \end{aligned} \tag{123}$$

And so on. Accordingly, the solution in a series form is given by:

$$(U, V) = \left(e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right),\ e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)\right) \ ; \tag{124}$$

And in a closed form of:

$$(U, V) = \left(e^{x-t},\ e^{-x+t}\right) \ . \tag{125}$$

Example (2.3.10): Consider the following nonlinear coupled of partial differential equations [24]**:**

$$\begin{aligned} U_t - V_x W_y &= 1, \\ V_t - W_x U_y &= 5, \\ W_t - U_x V_y &= 5, \end{aligned} \quad ; \qquad (126)$$

With the initial conditions:

$$U(x,y,0) = x+2y \ , \ V(x,y,0) = x-2y \ , W(x,y,0) = -x+2y \ . \qquad (127)$$

Following the analysis presented above, we obtain:

$$\begin{aligned} U(x,y,t) &= x+2y+t+S^{-1}\left[u\,S\left[V_x W_y\right]\right], \\ V(x,y,t) &= x-2y+5t+S^{-1}\left[u\,S\left[W_x U_y\right]\right], \\ W(x,yt) &= -x+2y+5t+S^{-1}\left[u\,S\left[U_x V_y\right]\right]. \end{aligned} \qquad (128)$$

Substituting the decomposition representations for linear and nonlinear terms into (128) yields:

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x,y,t) &= x+2y+t+S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n\right]\right], \\ \sum_{n=0}^{\infty} V_n(x,y,t) &= x-2y+5t+S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} B_n\right]\right], \\ \sum_{n=0}^{\infty} W_n(x,y,t) &= -x+2y+5t+S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} C_n\right]\right]. \end{aligned} \qquad (129)$$

For brevity, we list the first three Adomian polynomials for A_n, B_n and C_n as follows:

For $V_x W_y$, we find:

$$A_0 = V_{0_x} W_{0_y} \quad , \quad A_1 = V_{1_x} W_{0_y} + V_{0_x} W_{1_y} \ ,$$
$$A_2 = V_{2_x} W_{0_y} + V_{1_x} W_{1_y} + V_{0_x} W_{2_y}$$

For $W_x U_y$, we find:

$$B_0 = W_{0_x} U_{0_y} \quad , \quad B_1 = W_{1_x} U_{0_y} + W_{0_x} U_{1_y} \ ,$$
$$B_2 = W_{2_x} U_{0_y} + W_{1_x} U_{1_y} + W_{0_x} U_{2_y}$$

For $U_x V_y$, we find:

$$C_0 = U_{0_x} V_{0_y} \quad , \quad C_1 = U_{1_x} V_{0_y} + U_{0_x} V_{1_y} \ ,$$
$$C_2 = U_{2_x} V_{0_y} + U_{1_x} V_{1_y} + U_{0_x} V_{2_y}$$

Substituting these polynomials into the appropriate recursive relations we find:

$$\begin{aligned}
&(U_0, V_0, W_0) = (x+2y+t, x-2y+5t, -x+2y+5t),\\
&(U_1, V_1, W_1) = (2t, -2t, -2t),\\
&(U_2, V_2, W_2) = (0,0,0),\\
&(U_k, V_k, W_k) = (0,0,0), k \geq 3.
\end{aligned} \tag{130}$$

The exact solution of the system of nonlinear PDE is given by:

$$(U, V, W) = (x+2y+3t,\ x-2y+3t, -x+2y+3t) \quad . \tag{131}$$

CHAPTER (3)
Linear and Nonlinear Physical Models

In this chapter, we will concentrate our study on the linear and nonlinear particular applications that appear in applied science. The wide use of these physical significant problems is the most important reason why they have drawn mathematician's attention in recent years.

Nonlinear partial differential equations have witnessed remarkable improvement. Nonlinear problems appear in the many scientific fields' such as gravitation, chemical reaction, fluid dynamics, dispersion, nonlinear optics, plasma physics, acoustics, and others. Several approaches have been used such as the Adomian decomposition method, the variational iteration method, and the characteristics method and perturbation techniques to examine these problems.

(ADSTM) gives the solution of nonlinear equations in the form of convergent series. The main advantage of this method is its potentiality of combining two powerful methods for deriving exact and approximate solution of nonlinear equations. This forms the motivation for us to apply (ADSTM) for solving nonlinear equations in understanding different physical phenomena.

The following section offers the effectiveness of the Adomian decomposition Sumudu transform method (ADSTM) in solving linear and nonlinear physical models.

3.1: The Nonlinear Advection Problem

The nonlinear partial differential equation of the advection problem is of the form;

$$U_t + UU_x = f(x,t) \; ; \tag{1}$$

With the initial condition

$$U(x,0) = g(x) \; . \tag{2}$$

In this section, we approach the advection problem by utilizing the Adomian decomposition Sumudu transform method to find a rapidly convergent power series solution.

Operating Sumudu transform from Eq. (1) and using initial condition yields;

$$S[U(x,t)] = g(x) + uS[f(x,t)] - uS[UU_x] \tag{3}$$

Taking the inverse Sumudu transform of (3) gives;

$$U(x,t) = g(x) + S^{-1}[u\,S[f(x,t)]] + S^{-1}[u\,S[UU_x]] \; . \tag{4}$$

Substituting the linear term $U(x,t)$, by the series;

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \; ; \tag{5}$$

And the nonlinear term UU_x the a series of the Adomian polynomials;

$$UU_x = \sum_{n=0}^{\infty} A_n \; ; \tag{6}$$

Into Eq. (4), gives;

$$\sum_{n=0}^{\infty} U_n(x,t) = g(x) + S^{-1}[u\,S[f(x,t)]] + S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n\right]\right] . \tag{7}$$

Following Adomian approach, we obtain the recursive relation;

$$\begin{aligned} &U_0(x,t) = g(x) + S^{-1}[u\,S[f(x,t)]], \\ &U_{k+1}(x,t) = -\,S^{-1}[u\,S[A_k]], k \geq 0. \end{aligned} \tag{8}$$

The following examples will be used to illustrate the algorithm discussed above.

Example (3.1.1): Consider the homogeneous nonlinear partial differential equation [31]**:**

$$U_t + UU_x = 0 \quad ; \tag{9}$$

With the initial condition:

$$U(x,0) = x \; . \tag{10}$$

Taking the Sumudu transform of both sides of Eq. (9), and using the initial condition, we have:

$$S[U(x,t)] = x - u\,S[UU_x] \tag{11}$$

Applying S^{-1} to both sides of Eq. (11); implies that:

$$U(x,t) = x - S^{-1}[u\,S[UU_x]] \; ; \tag{12}$$

Following the technique, if we assume an infinite series of the form (12); we obtain:

$$\sum_{n=0}^{\infty} U_n(x,t) = x - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n(U)\right]\right] \; . \tag{13}$$

Where $A_n(U)$ are the Adomian polynomials that represent the nonlinear term.

The first few the components of the Adomian polynomials are given by;

$$A_0 = U_0 U_{0_x},$$
$$A_1 = U_0 U_{1_x} + U_1 U_{0_x}, \quad (14)$$
$$A_2 = U_0 U_{2_x} + U_1 U_{1_x} + U_2 U_{0_x}.$$

This gives the recursive relation:

$$U_0(x,t) = x,$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S[A_k]],\ k \geq 0. \quad (15)$$

The first few the components are given by:

$$U_0(x,t) = x,$$
$$U_1(x,t) = -S^{-1}[u\,S[A_0]] = -xt,$$
$$U_2(x,t) = -S^{-1}[u\,S[A_1]] = xt^2, \quad (16)$$
$$U_3(x,t) = -S^{-1}[u\,S[A_2]] = -xt^3.$$

And so on. The solution in a series form is given by:

$$U(x,t) = x(1 - t + t^2 - t^3 + \ldots)\ ; \quad (17)$$

And in a closed form of:

$$U(x,t) = \frac{x}{1+t}\ . \quad (18)$$

Example (3.1.2): Consider the following inhomogeneous advection problem [32]:

$$U_t + UU_x = 2t + x + t^3 + xt^2\ ; \quad (19)$$

With the initial condition:

$$U(x,0) = 0\ . \quad (20)$$

Following discussion presented above, we obtain the recursive relation;

$$U_0(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt}{3},$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S[A_k]], k \geq 0. \quad (21)$$

This gives;

$$U_0(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3},$$
$$U_1(x,t) = -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8. \quad (22)$$

It is easily observed that two noise term appears in the components $U_0(x,t)$ and $U_1(x,t)$. By canceling these terms from $U_0(x,t)$, the remaining non-canceled term of $U_0(x,t)$ may provide the exact solution.

The exact solution is given by;

$$U(x,t) = t^2 + xt \ . \tag{23}$$

3.2: The Goursat Problem

The Goursat partial differential equation arises in a variety of physical phenomenon and applied sciences.

The Goursat problem arises in partial differential equation with mixed derivatives, and its standard form given by;

$$\begin{aligned} &u_{xy} = f(x,y,u,u_x,u_y), \\ &u(x,0)=g(x),\ u(0,t)=h(y), \\ &g(0)=h(0)=u(0,0),\ 0\le x\le a\ ,\ 0\le y\le b. \end{aligned} \tag{24}$$

In this section, we outline a reliable strategy of (ADSTM) of solving the Goursat problem. To mention the basic idea of this method, we consider a general nonlinear no homogeneous Goursat problem of the form;

$$\begin{aligned} &DU(x,t)+RU(x,t)+NU(x,t) = g(x,t) \\ &U(x,0)=h(x),\ U(0,t)=f(t). \end{aligned} \tag{25}$$

Where D is the second order linear mixed differential operator, N represent the general nonlinear operator, and $g(x,t)$ the source term.

Taking the Sumudu transform of both sides of Eq. (25) with respect to t and using initial condition, we get;

$$S[U_x(x,t)]=h(x)+u\,S[g(x,t)]-u\,S[RU(x,t)+NU(x,t)] \tag{26}$$

Now, applying the inverse Sumudu transform of both sides of (26) gives;

$$U_x(x,t)=h(x)+G(x,t)-S^{-1}[u\,S[RU(x,t)+NU(x,t)]] \tag{27}$$

Again, taking the Sumudu transform of both sides of (27) with respect to x, we get;

$$S[U(x,t)]= f(t)+uS[h(x)]+u\,S[G(x,t)]-u\,S[S^{-1}[u\,S[RU(x,t)+NU(x,t)]]] \tag{28}$$

Now, again applying the inverse Sumudu transform of both sides of (28) gives;

$$U(x,t)=H(x,t)-S^{-1}[u\,S[S^{-1}[u\,S[RU(x,t)+NU(x,t)]]]] \tag{29}$$

Where $H(x,t)$ represent the term arising from the source term and the prescribed initial conditions. We represent solution as an infinite series;

$$U(x,t)=\sum_{n=0}^{\infty}U_n(x,t)\ , \tag{30}$$

And the nonlinear term can be decomposed as;

$$NU(x,t)=\sum_{n=0}^{\infty}A_n\ \ ; \tag{31}$$

Then Eq. (29) becomes;

$$\sum_{n=0}^{\infty}U_n(x,t)=H(x,t)-S^{-1}\left[u\,S\left[\ S^{-1}\left[u\,S\left[\ R\left(\sum_{n=0}^{\infty}U_n(x,t)\right)+\sum_{n=0}^{\infty}A_n\ \right]\right]\right]\right] \tag{32}$$

The recursive relation is given by;

$$\begin{aligned}&U_0(x,t)=H(x,t),\\&U_{k+1}(x,t)=-\ S^{-1}\left[u\,S\left[S^{-1}\left[u\,S\left[R(U_k)+A_k\right]\right]\right]\right],\ k\geq 0.\end{aligned} \tag{33}$$

The following examples will be used to illustrate the algorithm discussed above.

Example (3.2.3): Consider the following homogeneous Goursat problem [33]:

$$\begin{aligned}&U_{xt}=U\ ,\\&U(x,0)=e^{x}\ \ ,U(0,t)=e^{t}\ ,U(0,0)=1.\end{aligned} \tag{34}$$

Following discussion presented above, we obtain the recursive relation is given by;

$$\begin{aligned}&U_0(x,t)=e^{x}-1+e^{t}\ ,\\&U_{k+1}(x,t)=\ S^{-1}\left[u\,S\left[S^{-1}\left[u\,S\left[(U_k)\right]\right]\right]\right],\ k\geq 0.\end{aligned} \tag{35}$$

The first few of the components are;

$$\begin{aligned}&U_0(x,t)=e^{x}-1+e^{t}\ ,\\&U_1(x,t)=-x+xe^{t}-xt-t+te^{x}\ ,\\&U_2(x,t)=-\frac{x^2}{2!}t-\frac{x^2}{2!}+\frac{x^2}{2!}e^{t}-\frac{x^2}{2!}\frac{t^2}{2!}-x\frac{t^2}{2!}-\frac{t^2}{2!}+\frac{t^2}{2!}e^{x}\,.\end{aligned} \tag{36}$$

And so on. Therefore the solution in a closed form is;

$$U(x,t)=e^{x+t}\quad . \tag{37}$$

3.3: The Klein- Gordon Equation

The linear and nonlinear Klein-Gordon equations are considered one of the most important partial differential equations in quantum field theory. Those equations arise in the study of relativistic physics and are used to describe dispersive wave phenomena in general. In addition, it also appears in nonlinear optics and plasma physics.

3.3.1: The Linear Klein- Gordon Equation

The linear Klein-Gordon equation is very important in quantum mechanics. It is derived from the relativistic energy formula, and it is standard form given by;

$$U_{tt}(x,t) - U_{xx}(x,t) + aU(x,t) = h(x,t) \quad ; \tag{38}$$

Subject to the initial conditions:

$$U(x,0) = f(x) \ , \ U_t(x,t) = g(x) \ . \tag{39}$$

Where a is a constant, $h(x,t)$ is a source term. It is interesting to note that if $a = 0$ Eq. (38) becomes inhomogeneous wave equation.

In this section, the (ADSTM) will be applied to handle the linear Klein-Gordon equations. To achieve this goal, we apply the Sumudu transform of both sides of Eq. (38) and using the initial condition, we obtain;

$$S[U(x,t)] = f(x) + u\,g(x) + u^2 h(x,t) + u^2 S[U_{xx}(x,t)] - u^2\,S[aU(x,t)] \tag{40}$$

Now, applying inverse Sumudu transform and using the decomposition series for the linear term, $U(x,t)$ and proceeding as before we obtain the a recursive relation;

$$\begin{aligned} U_0(x,t) &= f(x) + t\,g(x) + S^{-1}\left[u^2\,S[h(x,t)]\right] \\ U_{k+1}(x,t) &= S^{-1}\left[u^2\,S[(U_k)_{xx}]\right] - S^{-1}\left[u^2\,S[a(U_k)]\right],\ k \geq 0. \end{aligned} \tag{41}$$

The following examples will be used to illustrate the algorithm discussed above.

Example (3.3.4): Consider the following linear Klein – Gordon equation [34]:

$$U_{tt} - U_{xx} + U = 0 \ ; \tag{42}$$

Subject to the initial conditions

$$U(x,0) = 0 \ , \ U_t(x,t) = x \ . \tag{43}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} U_0(x,t) &= xt \\ U_{k+1}(x,t) &= S^{-1}\left[u^2\,S[(U_k)_{xx}]\right] - S^{-1}\left[u^2\,S[(U_k)]\right],\ k \geq 0. \end{aligned} \tag{44}$$

That in turn gives;

$$
\begin{aligned}
U_0(x,t) &= xt \\
U_1(x,t) &= S^{-1}\left[u^2\, S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2\, S\left[(U_0)\right]\right] = -\frac{xt^3}{3!}, \\
U_2(x,t) &= S^{-1}\left[u^2\, S\left[(U_1)_{xx}\right]\right] - S^{-1}\left[u^2\, S\left[(U_1)\right]\right] = \frac{xt^5}{5!}, \\
U_3(x,t) &= S^{-1}\left[u^2\, S\left[(U_2)_{xx}\right]\right] - S^{-1}\left[u^2\, S\left[(U_2)\right]\right] = -\frac{xt^7}{7!}.
\end{aligned}
\tag{45}
$$

And so on. In view of (45) the series solution is given by;

$$U(x,t) = x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right); \tag{46}$$

And the exact solution is given by;

$$U(x,t) = x\sin t\ . \tag{47}$$

Example (3.3.5): Consider the following linear Klein – Gordon equation [34]:

$$U_{tt} - U_{xx} + U = 2\sin x\ ; \tag{48}$$

Subject to the initial conditions:

$$U(x,0) = \sin x\ ,\ U_t(x,t) = 1\ . \tag{49}$$

Proceeding as in Example **(3.3.4)**, we set the relation;

$$
\begin{aligned}
U_0(x,t) &= \sin x + t + t^2 \sin x, \\
U_{k+1}(x,t) &= S^{-1}\left[u^2\, S\left[(U_k)_{xx}\right]\right] - S^{-1}\left[u^2\, S\left[(U_k)\right]\right],\ k \geq 0.
\end{aligned}
\tag{50}
$$

That in turn gives;

$$
\begin{aligned}
U_0(x,t) &= \sin x + t + t^2 \sin x, \\
U_1(x,t) &= -t^2 \sin x - \frac{t^3}{3!} - \frac{t^4}{4!}\sin x\ , \\
U_2(x,t) &= \frac{t^6}{90}\sin x + \frac{t^5}{5!} + \frac{t^4}{4!}\sin x\ , \\
U_3(x,t) &= -\frac{t^6}{90}\sin x - \frac{t^7}{7!} - \frac{2t^8}{7!}\sin x\ .
\end{aligned}
\tag{51}
$$

Therefore the solution in series form is given by;

$$U(x,t) = \sin x + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right); \tag{52}$$

And the exact solution is given by;

$$U(x,t) = \sin x + x\sin t\ . \tag{53}$$

3.3.2: The Nonlinear Klein- Gordon Equation

The nonlinear Klein-Gordon equation comes from quantum field theory and describes the nonlinear wave interaction and it is standard form given by;

$$U_{tt}(x,t) - U_{xx}(x,t) + aU(x,t) + F(U(x,t)) = h(x,t) \; ; \qquad (54)$$

Subject to the initial conditions

$$U(x,0) = f(x) \; , \; U_t(x,t) = g(x) \; . \qquad (55)$$

Where a is a constant and $h(x,t)$ is a source term and $F(U(x,t))$ is a nonlinear function of $U(x,t)$.

In this section, the (ADSTM) will be applied to handle the nonlinear Klein-Gordon equations. To achieve this goal, we apply the Sumudu transform of both sides of Eq. (54) and using the initial condition, we obtain;

$$S[U(x,t)] = f(x) + u\,g(x) + u^2 h(x,t) + u^2 S[U_{xx}(x,t)] - u^2\, S[aU(x,t) + F(U(x,t))] \quad (56)$$

Now, applying inverse Sumudu transform and using the decomposition series for the linear term, $U(x,t)$, the infinite series of the Adomian polynomials for the nonlinear term, $F(U(x,t))$, and proceeding as before we obtain the a recursive relation;

$$\begin{aligned} U_0(x,t) &= f(x) + t\,g(x) + S^{-1}\left[u^2\, S[h(x,t)]\right] \\ U_{k+1}(x,t) &= S^{-1}\left[u^2\, S[(U_k)_{xx}]\right] - S^{-1}\left[u^2\, S[a(U_k) + A_k]\right], \; k \ge 0. \end{aligned} \qquad (57)$$

The following examples will be used to illustrate the algorithm discussed above.

Example (3.3.6): Consider the following nonlinear Klein – Gordon equation [24]:

$$U_{tt} - U_{xx} + U^2 = x^2 t^2 \; ; \qquad (58)$$

Subject to the initial conditions:

$$U(x,0) = 0 \; , \; U_t(x,t) = x \; . \qquad (59)$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} U_0(x,t) &= xt + \frac{1}{12} x^2 t^4 , \\ U_{k+1}(x,t) &= S^{-1}\left[u^2\, S[(U_k)_{xx}]\right] - S^{-1}\left[u^2\, S[A_k]\right], \; k \ge 0. \end{aligned} \qquad (60)$$

So the Adomian polynomials A_n are given as follows;

$$\begin{aligned} A_0 &= U_0^{\,2} , \\ A_1 &= 2U_0 U_1 , \\ A_2 &= 2U_0 U_2 + U_1^{\,2} . \end{aligned}$$

And so on. Using modified recursive relation Eq. (60) can be rewritten in the scheme;

$$U_0(x,t) = xt,$$
$$U_1(x,t) = \frac{1}{12}x^2 t^4 + S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_0]\right], \quad (61)$$
$$U_{k+1}(x,t) = S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_k]\right],\ k \geq 1.$$

This lead to;

$$U_0(x,t) = xt,$$
$$U_1(x,t) = \frac{1}{12}x^2 t^4 + S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_0]\right] = 0, \quad (62)$$
$$U_{k+1}(x,t) = 0,\ k \geq 1.$$

Therefore, the exact solution is given by;

$$U(x,t) = xt. \quad (63)$$

Example (3.3.7): Consider the following nonlinear Klein – Gordon equation [24]:

$$U_{tt} - U_{xx} + U^2 = 2x^2 - 2t^2 + x^4 t^4; \quad (64)$$

Subject to the initial conditions:

$$U(x,0) = 0,\ U_t(x,t) = 0. \quad (65)$$

Proceeding as in Example **(3.3.6)**, we set the relation;

$$U_0(x,t) = x^2 t^2,$$
$$U_1(x,t) = -\frac{1}{6}t^4 + \frac{1}{30}x^4 t^6 + S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_0]\right] = 0, \quad (66)$$
$$U_{k+1}(x,t) = 0,\ k \geq 1.$$

This formally gives the exact solution;

$$U(x,t) = x^2 t^2. \quad (67)$$

3.3.3: The Sine - Gordon Equation

The sine – Gordon equation appeared first in differential geometry. This equation becomes the focus of a lot of research work because it appears in many physical phenomena such as the propagation of magnetic flux and the stability of fluid motion.

The standard form of the sine – Gordon equation is given by;

$$U_{tt}(x,t) - c^2 U_{xx}(x,t) + \alpha \sin U = 0; \quad (68)$$

Subject to the initial conditions

$$U(x,0)= f(x)\ ,\ U_t(x,t)= g(x)\ . \tag{69}$$

Where c and α are constants.

In this section, the sine – Gordon equation will be handled by using (ADSTM). Taking the Sumudu transforms of both sides of (68) and using the initial conditions we have;

$$S[U(x,t)]= f(x)+u\,g(x)+u^2c^2\,S[U_{xx}(x,t)]-u^2\,S[\alpha \sin U] \tag{70}$$

Applying inverse Sumudu transform to (70) gives;

$$U(x,t)= f(x)+t\,g(x)+S^{-1}\left[u^2c^2\,S[U_{xx}(x,t)]\right]-S^{-1}\left[u^2\,S[\alpha \sin U]\right] \tag{71}$$

Using the decomposition series for the linear term $U(x,t)$, the infinite series of the Adomian polynomials for the nonlinear term $\sin U$, and proceeding as before we obtain the recursive relation;

$$\begin{aligned} &U_0(x,t)= f(x)+t\,g(x), \\ &U_{k+1}(x,t)= S^{-1}\left[u^2c^2\ S[(U_k)_{xx}]\right]-S^{-1}\left[u^2\alpha\,S[A_k]\right],\ k\geq 0. \end{aligned} \tag{72}$$

This will lead to the determination of the solution in a series form. This can be illustrated as follows.

Example (3.3.8): Consider the following Sine-Gordon equation with the given initial conditions [24]:

$$U_{tt}(x,t)-U_{xx}(x,t)= \sin U\ ; \tag{73}$$

Subject to the initial conditions;

$$U(x,0)=\frac{\pi}{2}\ ,\ U_t(x,t)=0\ . \tag{74}$$

Using the recursive scheme (72) yields;

$$\begin{aligned} &U_0(x,t)=\frac{\pi}{2}, \\ &U_{k+1}(x,t)= S^{-1}\left[u^2S[(U_k)_{xx}]\right]+S^{-1}\left[u^2S[A_k]\right],\ k\geq 0. \end{aligned} \tag{75}$$

The first few the Adomian polynomials for $\sin U$ are given as;

$$\begin{aligned} A_0 &= \sin U_0\,, \\ A_1 &= U_1\cos U_0\,, \\ A_2 &= U_2\cos U_0-\frac{1}{2!}U_1^{\,2}\sin U_0\,, \\ A_3 &= U_3\cos U_0-U_1U_2\sin U_0-\frac{1}{3!}U_1^{\,3}\cos U_0\,. \end{aligned} \tag{76}$$

Combining (75) and (76) leads to;

$$U_0(x,t) = \frac{\pi}{2},$$

$$U_1(x,t) = S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_0]\right] = \frac{t^2}{2!},$$

$$U_2(x,t) = S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_1]\right] = 0 \quad , \tag{77}$$

$$U_3(x,t) = S^{-1}\left[u^2 S\left[(U_2)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_2]\right] = -\frac{1}{240} t^6 .$$

And so on. The series solution is;

$$U(x,t) = \frac{\pi}{2} + \frac{t^2}{2!} - \frac{1}{240} t^6 + \ldots \quad . \tag{78}$$

3.4: The Burgers Equation

The Burgers equation is considered one of the fundamental model equations in fluid mechanics. This equation demonstrates the coupling between diffusion and convection processes.

The equation appears in various areas of applied mathematics and physics, such as modeling of gas dynamic and is used to describe the structure of shock waves. In addition, it also appears in traffic flow and acoustic transmission.

The standard form of Burgers equation is given by;

$$U_t(x,t) + UU_x = V\, U_{xx}(x,t) \, , \, t > 0 \quad ; \tag{79}$$

Subject to the initial conditions:

$$U(x,0) = f(x). \tag{80}$$

Where V is a constant that defines the kinematic viscosity.

In this section, we apply the (ADSTM) to solve nonlinear Burgers equation. Taking the Sumudu transforms of both sides of (79) and using the initial conditions we have;

$$S[U(x,t)] = f(x) + Vu\, S[\, U_{xx}(x,t)] - u\; S[UU_x] \; . \tag{81}$$

Applying the inverse operator S^{-1} of (81) leads to;

$$U(x,t) = f(x) + S^{-1}\left[aVu\, S[U_{xx}(x,t)]\right] - S^{-1}\left[u\, S[UU_x]\right] \tag{82}$$

Using the decomposition series for the linear term $U(x,t)$ and the series of the Adomian polynomials for the nonlinear term UU_x gives;

$$\sum_{n=0}^{\infty} U_n(x,t) = f(x) + S^{-1}\left[Vu\, S\left[\left(\sum_{n=0}^{\infty} U_n(x,t) \right)_{xx} \right] \right] - S^{-1}\left[u\, S\left[\sum_{n=0}^{\infty} A_n \right] \right] \tag{83}$$

Identifying the zeroth component $U_0(x,t)$ by the terms that arise from the initial condition, and following the decomposition method, we obtain the recursive relation;

$$\begin{aligned} &U_0(x,t) = f(x), \\ &U_{k+1}(x,t) = S^{-1}\left[V\, uS\left[(U_k)_{xx}\right]\right] - S^{-1}\left[uS\left[A_k\right]\right],\ k \geq 0. \end{aligned} \tag{84}$$

The Adomian polynomials for the nonlinear term UU_x have been derived in the form;

$$\begin{aligned} A_0 &= U_{0_x} U_0 , \\ A_1 &= U_{0_x} U_1 + U_{1_x} U_0 , \\ A_2 &= U_{0_x} U_2 + U_{1_x} U_1 + U_{2_x} U_0 , \\ A_3 &= U_{0_x} U_3 + .U_{1_x} U_2 + U_{2_x} U_1 + U_{3_x} U_0 . \end{aligned} \tag{85}$$

In view of (84) and (85), the first few the components can be identified by;

$$\begin{aligned} U_0(x,t) &= f(x), \\ U_1(x,t) &= S^{-1}\left[VuS\left[(U_0)_{xx}\right]\right] - S^{-1}\left[uS\left[A_0\right]\right], \\ U_2(x,t) &= S^{-1}\left[V\, uS\left[(U_1)_{xx}\right]\right] - S^{-1}\left[uS\left[A_1\right]\right], \\ U_3(x,t) &= S^{-1}\left[V\, uS\left[(U_2)_{xx}\right]\right] - S^{-1}\left[uS\left[A_2\right]\right]. \end{aligned} \tag{86}$$

The following examples will be used to illustrate the discussion carried out above by using Sumudu decomposition method.

Example (3.4.9): Consider the following one – dimensional Burgers equation [24]:

$$U_t = U_{xx} - UU_x \quad ; \tag{87}$$

Subject to the initial conditions:

$$U(x,0) = x. \tag{88}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} &U_0(x,t) = x, \\ &U_{k+1}(x,t) = S^{-1}\left[uS\left[(U_k)_{xx}\right]\right] - S^{-1}\left[uS\left[A_k\right]\right],\ k \geq 0. \end{aligned} \tag{89}$$

Using the Adomian polynomials we obtain;

$$\begin{aligned} U_0(x,t) &= x, \\ U_1(x,t) &= S^{-1}\left[a\, uS\left[(U_0)_{xx}\right]\right] - S^{-1}\left[uS\left[A_0\right]\right] = -xt , \\ U_2(x,t) &= S^{-1}\left[a\, uS\left[(U_1)_{xx}\right]\right] - S^{-1}\left[uS\left[A_1\right]\right] = xt^2 , \\ U_3(x,t) &= S^{-1}\left[a\, uS\left[(U_2)_{xx}\right]\right] - S^{-1}\left[uS\left[A_2\right]\right] = -xt^3 . \end{aligned} \tag{90}$$

Summing these iterates gives the series solution;

$$U(x,t) = x\left(1 - t + t^2 - t^3 + \ldots\right) ; \tag{91}$$

Consequently, the exact solution is given by;

$$U(x,t)=\frac{x}{1+t} \quad . \tag{92}$$

Example (3.4.10): Consider the following one – dimensional Burgers equation [24]:

$$U_t = U_{xx} - UU_x \quad ; \tag{93}$$

Subject to the initial conditions:

$$U(x,0)=1-\frac{2}{x}, \; x>0. \tag{94}$$

Proceeding as in Example **(3.4.9)**, we set the relation;

$$\begin{aligned} &U_0(x,t)=1-\frac{2}{x}, \\ &U_{k+1}(x,t)=S^{-1}\left[uS\left[(U_k)_{xx}\right]\right]-S^{-1}\left[uS\left[A_k\right]\right], \; k\geq 0. \end{aligned} \tag{95}$$

That gives;

$$\begin{aligned} &U_0(x,t)=1-\frac{2}{x}, \\ &U_1(x,t)=-\frac{2}{x^2}\,t \;, \\ &U_2(x,t)= -\frac{2}{x^3}t^2, \\ &U_3(x,t)= -\frac{2}{x^4}t^3. \end{aligned} \tag{96}$$

The Eq. (96) can be rewritten as series form;

$$U(x,t)=1-\frac{2}{x}\left(1+\frac{t}{x}+\frac{t^2}{x^2}+\frac{t^3}{x^3}+\ldots\right)\; ; \tag{97}$$

Thus, the exact solution is given by;

$$U(x,t)=1-\frac{2}{x-t} \quad . \tag{98}$$

3.4.1: System of (1+2)-Dimensional Burgers Equations

Systems of partial differential equations have attracted much attention in studying evolution equations describing wave propagation, in investigating the shallow water waves, and in examining the chemical reaction–diffusion model of Brusselator. Several numerical methods used for solving this system [35-41].

In this section, we present (ADSTM) to obtain a closed form solution of the system of (1+2)-dimensional Burgers equations.

Consider the following system of two – dimensional Burgers equation [35]:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right),$$
$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right). \tag{99}$$

With the initial conditions;

$$U(x,y,0)=f(x,y),\ x,y\in D\ ,V(x,y,0)=g(x,y)\ ,\ x,y\in D. \tag{100}$$

And the boundary conditions;

$$U(x,y,t)=f_1(x,y,t),x,y\in\partial D\ ,V(x,y,t)=f_2(x,y,t),x,y\in\partial D \tag{101}$$

where $D=\{(x,y)|a\le x\le b\ ,\ a\le y\le b\}$ and ∂D is its boundary, $U(x,y,t)$ and $V(x,y,t)$ are the velocity components to be determined, f , g , f_1 and f_2 are known functions, and R is the Reynolds number.

Taking the Sumudu transform of the system (99) and using initial conditions (100), we obtain;

$$S[U(x,y,u)]=f(x,y)-u\,S\left[U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}\right]+u\,S\left[\frac{1}{R}\left(\nabla^2 U\right)\right],$$
$$S[V(x,y,u)]=g(x,y)-u\,S\left[U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}\right]+u\,S\left[\frac{1}{R}\left(\nabla^2 V\right)\right]. \tag{102}$$

Using inverse Sumudu transform from (102) gives;

$$U(x,y,t)=f(x,y)-S^{-1}\left[u\,S\left[U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}\right]\right]+S^{-1}\left[u\,S\left[\frac{1}{R}\left(\nabla^2 U\right)\right]\right],$$
$$V(x,y,t)=g(x,y)-S^{-1}\left[u\,S\left[U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}\right]\right]+S^{-1}\left[u\,S\left[\frac{1}{R}\left(\nabla^2 V\right)\right]\right]. \tag{103}$$

Using the decomposition series for the linear terms $U(x,y,t)$, $V(x,y,t)$, the infinite series of the Adomian polynomials for the nonlinear terms UU_x, VU_y, UV_x, VV_y ; and proceeding as before, we obtain the recursive relations;

$$U_0(x,y,t) = f(x,y)$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S[A_k + B_k]] + S^{-1}\left[u\,S\left[\frac{1}{R}\left(\nabla^2(U_k)\right)\right]\right],\ k \geq 0. \quad (104)$$

And

$$V_0(x,y,t) = g(x,y)$$
$$V_{k+1}(x,t) = -S^{-1}[u\,S[C_k + D_k]] + S^{-1}\left[u\,S\left[\frac{1}{R}\left(\nabla^2(V_k)\right)\right]\right],\ k \geq 0. \quad (105)$$

The Adomian polynomials for the nonlinear term UU_x are given by;

$$A_0 = U_0 U_{0_x} \quad , \quad A_1 = U_1 U_{0_x} + U_0 U_{1_x} \ ,$$
$$A_2 = U_2 U_{0_x} + U_1 U_{1_x} + U_0 U_{2_x}$$

And so on.

And the nonlinear term VU_y is given by;

$$B_0 = V_0 U_{0_y} \quad , \quad B_1 = V_1 U_{0_y} + V_0 U_{1_y} \ ,$$
$$B_2 = V_2 U_{0_y} + V_1 U_{1_y} + V_0 U_{2_y}$$

And so on. And for the nonlinear term UV_x;

$$C_0 = U_0 V_{0_x} \quad , \quad C_1 = U_1 V_{0_x} + U_0 V_{1_x} \ ,$$
$$C_2 = U_2 V_{0_x} + U_1 V_{1_x} + U_0 V_{2_x}$$

And so on. And for the nonlinear term VV_y;

$$D_0 = V_0 V_{0_y} \quad , \quad D_1 = V_1 V_{0_y} + V_0 V_{1_y} \ ,$$
$$D_2 = V_2 V_{0_y} + V_1 V_{1_y} + V_0 V_{2_y}$$

And so on. The following example will be used to illustrate the discussion carried out above by using (ADSTM).

Example (3.4.11): Consider the following system of two – dimensional Burgers equation (99), with the following the initial conditions [35]:

$$U(x,y,0) = x + y,\ x,y \in D\ ,\ V(x,y,0) = x - y,\ x,y \in D \quad (106)$$

Following the discussion presented above, we find recursive relations;

$$U_0(x,y,t) = x + y\,,$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S[A_k + B_k]] + S^{-1}\left[u\,S\left[\frac{1}{R}\left(\nabla^2(U_k)\right)\right]\right],\ k \geq 0. \quad (107)$$

And

$$V_0(x,y,t) = x - y\,,$$
$$V_{k+1}(x,t) = -S^{-1}[u\,S[C_k + D_k]] + S^{-1}\left[u\,S\left[\frac{1}{R}\left(\nabla^2(V_k)\right)\right]\right],\ k \geq 0. \quad (108)$$

Using the derived the Adomian polynomials above into (107) and (108), we obtain the following pairs of components, upon setting $R=1$, we have;

$$\begin{aligned}
(U_0, V_0) &= (x + y, x - y),\\
(U_1, V_1) &= (-2xt, -2yt),\\
(U_2, V_2) &= \left(2xt^2 + 2yt^2, 2xt^2 - 2yt^2\right),\\
(U_3, V_3) &= \left(-4xt^3, -4yt^3\right).\\
(U_4, V_4) &= \left(4xt^4 + 4yt^4\,,\ 4xt^4 - 4yt^4\right)
\end{aligned} \quad (109)$$

And so on. Accordingly, the solution in a series form is given by;

$$(U, V) = \begin{pmatrix} x\left(1+2t^2+4t^4+\ldots\right) - 2xt\left(1+2t^2+\ldots\right) + y\left(1+2t^2+4t^4+\ldots\right), \\ x\left(1+2t^2+4t^4+\ldots\right) - 2yt\left(1+2t^2+\ldots\right) - y\left(1+2t^2+4t^4+\ldots\right) \end{pmatrix}\ ; \quad (110)$$

And in a closed form of;

$$(U(x,y,t), V(x,y,t)) = \left(\frac{x + y - 2xt}{1 - 2t^2}, \frac{x - y - 2yt}{1 - 2t^2}\right)\ . \quad (111)$$

3.5: The Telegraph Equation

Telegraph equations appear in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems.
The general linear telegraph equation is;

$$a\,U_{tt}(x,t) + b\,U_t(x,t) + c\,U(x,t) = U_{xx}(x,t)\ ; \quad (112)$$

Subject to the initial conditions:

$$U(x,0) = \alpha\ ,\ U_t(x,t) = \beta\ . \quad (113)$$

Where a, b and, c are constants related to the inductance, capacitance, and conductance of the cable respectively, and, α, β are functions of , x .

Assuming $a=0$ and $c=0$, because of electrical properties of the cable, then we obtain:

$$U_{xx}(x,t)= b\,U_t(x,t) \quad ; \tag{114}$$

Which is the standard linear heat equation mentioned before in chapter one.
On the other hand, the electrical properties may lead to $b=0$ and $c=0$, hence we obtain:

$$U_{xx}(x,t)= a\,U_{tt}(x,t) \quad ; \tag{115}$$

Which is the standard linear wave equation presented before in chapter one.

In this section, we apply the Sumudu transform method to solve general linear telegraph equation.
Applying Sumudu transform of the equation (112) and making use the initial conditions to find;

$$a\left[\frac{G(x,u)}{u^2}-\frac{\alpha}{u^2}-\frac{\beta}{u}\right]+b\left[\frac{G(x,u)}{u}-\frac{\alpha}{u}\right]+c\,G(x,u)=\frac{d^2}{d\,x^2}G(x,u) \tag{116}$$

Or equivalently;

$$u^2\frac{d^2}{d\,x^2}G(x,u)-\left(a+bu+cu^2\right)G(x,u)=-\left(a\alpha+au\beta+b\alpha u\right) \tag{117}$$

This is the second order linear differential equation. The particular solution of this equation is obtained as:

$$G(x,u)=\frac{-\left(a\,\alpha+au\,\beta+b\alpha\,u\right)}{u^2D^2-\left(a+bu+cu^2\right)}=F(u)G(x)\ ,\ D=\frac{d}{d\,x} \tag{118}$$

Now apply the inverse Sumudu transform to find the solution of the general telegraph equation (109) in the Form;

$$U(x,t)=G(x)S^{-1}\left[F(u)\right]=G(x)\,f(t)\quad . \tag{119}$$

The following examples will be used to illustrate the algorithm discussed above.

Example (3.5.12): Consider the following telegraph equation [42]:

$$U_{tt}+2U_t+U=U_{xx}\ ; \tag{120}$$

Subject to the initial conditions;

$$U(x,0)=e^x\ ,\ U_t(x,t)=-2e^x\quad . \tag{121}$$

Following the discussion presented above, we find;

$$G(x,u)=\frac{-e^x}{u^2D^2-(1+u)^2}=\frac{e^x}{1+2u} \tag{122}$$

Now apply the inverse Sumudu transform to find the solution;

$$U(x,t) = e^{x}\ S^{-1}\left[\frac{1}{1+2u}\right] = e^{x-2t} \quad . \tag{123}$$

Example (3.5.13): Consider the following telegraph equation [42]:

$$U_{tt} + 4U_{t} + 4U = U_{xx} \ ; \tag{124}$$

Subject to the initial conditions;

$$U(x,0) = 1 + e^{2x} \ , \ U_{t}(x,t) = -2 \quad . \tag{125}$$

Proceeding as in Example **(3.6.12)**, we find;

$$G(x,u) = \frac{-(1+2u)}{u^{2}D^{2} - (1+2u)^{2}} - \frac{(1+4u)e^{2x}}{u^{2}D^{2} - (1+2u)^{2}} = \frac{1}{1+2u} + e^{2x} \tag{126}$$

Now apply the inverse Sumudu transform to find the solution;

$$U(x,t) = e^{-2t} + e^{2x} \quad . \tag{127}$$

3.6: The Schrödinger Equation

Linear and nonlinear Schrödinger equations are one of the most important partial differential equations in quantum mechanics.
Those equations arise in the study of time evolution of the wave function.

3.6.1: The Linear Schrödinger Equation

The linear Schrödinger equation describes the time evolution of a free particle with mass m, and it is standard form given by;

$$U_{t} = i\,U_{xx}\,,\ i^{2} = -1\ ,\ t > 0 \quad ; \tag{128}$$

Subject to the initial conditions:

$$U(x,0) = f(x). \tag{129}$$

Where $f(x)$ is continuous function and square integrable.

In this section, the (ADSTM) will be applied to handle the linear Schrodinger equations. To achieve this goal, we apply the Sumudu transform of both sides of Eq. (128) and using the initial condition, we obtain;

$$S[U(x,t)] = f(x) + iu\ S[U_{xx}] \tag{130}$$

Applying inverse Sumudu transform of (130) gives;

$$U(x,t) = f(x) + S^{-1}\left[iuS[U_{xx}]\right] \tag{131}$$

Using the decomposition series for the linear term $U(x,t)$ and proceeding as before, we obtain the recursive relation;

$$\begin{aligned} &U_0(x,t) = f(x), \\ &U_{k+1}(x,t) = S^{-1}\left[iuS[(U_k)_{xx}]\right],\ k \geq 0. \end{aligned} \tag{132}$$

We can easily determine the first few the components by;

$$\begin{aligned} &U_0(x,t) = f(x), \\ &U_1(x,t) = S^{-1}\left[iuS[(U_0)_{xx}]\right], \\ &U_2(x,t) = S^{-1}\left[iuS[(U_1)_{xx}]\right], \\ &U_3(x,t) = S^{-1}\left[iuS[(U_2)_{xx}]\right]. \end{aligned} \tag{133}$$

Other components can be determined as well. This completes the determination of the series solution.

The analysis introduced above will be illustrated by discussing the following examples.

Example (3.6.14): Consider the following linear Schrödinger equation [43]:

$$U_t = i\,U_{xx}\ ; \tag{134}$$

Subject to the initial conditions:

$$U(x,0) = e^{ix}. \tag{135}$$

Following the discussion presented above, we find;

$$\begin{aligned} &U_0(x,t) = e^{ix}, \\ &U_1(x,t) = S^{-1}\left[iuS[(U_0)_{xx}]\right] = -it\,e^{ix}, \\ &U_2(x,t) = S^{-1}\left[iuS[(U_1)_{xx}]\right] = -\frac{1}{2!}t^2 e^{ix}, \\ &U_3(x,t) = S^{-1}\left[iuS[(U_2)_{xx}]\right] = \frac{1}{3!}it^3 e^{ix}. \end{aligned} \tag{136}$$

Accordingly, the series solution is given by;

$$U(x,t) = e^{ix}\left(1 - it + \frac{1}{2!}(it)^2 - \frac{1}{3!}(it)^3 + \ \ldots\right); \tag{137}$$

That gives the exact solution by;

$$U(x,t) = e^{i(x-t)}\ . \tag{138}$$

Example (3.6.15): Consider the following linear Schrödinger equation [24]:

$$U_t = i\,U_{xx} \quad ; \tag{139}$$

Subject to the initial conditions:

$$U(x,0) = \sinh x . \tag{140}$$

Proceeding as in Example **(3.6.14)**, we find;

$$\begin{aligned} U_0(x,t) &= \sinh x ,\\ U_1(x,t) &= S^{-1}\left[i\,uS\left[(U_0)_{xx}\right]\right] = i\,t\,e^{ix} ,\\ U_2(x,t) &= S^{-1}\left[i\,uS\left[(U_1)_{xx}\right]\right] = -\frac{1}{2!}t^2 \sinh x ,\\ U_3(x,t) &= S^{-1}\left[i\,uS\left[(U_2)_{xx}\right]\right] = -\frac{1}{3!}i\,t^3 \sinh x . \end{aligned} \tag{141}$$

Accordingly, the series solution is given by;

$$U(x,t) = \sinh x \left(1 + i\,t + \frac{1}{2!}(i\,t)^2 + \frac{1}{3!}(i\,t)^3 + \ \ldots\right) ; \tag{142}$$

That gives the exact solution by;

$$U(x,t) = e^{i\,t} \sinh x \quad . \tag{143}$$

3.6.2: The Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation is a solitary wave equation, where the speed of propagation is independent of the amplitude of the wave function, and it is standard form given by;

$$iU_t + U_{xx} + \alpha |U|^2 U = 0 \quad ; \tag{144}$$

Subject to the initial conditions:

$$U(x,0) = g(x). \tag{145}$$

where α is constant term and $U(x,t)$ is complex.

In this section, the (ADSTM) will be applied to handle the nonlinear Schrödinger equations. To achieve this goal, we apply the Sumudu transform of both sides of Eq. (144) and using the initial condition, we obtain;

$$S[U(x,t)] = g(x) + i\,u\ S[U_{xx}] + i\,u\alpha\, S\left[|U|^2 U\right] \tag{146}$$

Applying inverse Sumudu transform of (146) gives;

$$U(x,t) = g(x) + S^{-1}\left[i\,uS[U_{xx}]\right] + S^{-1}\left[iu\alpha\, S\left[|U|^2 U\right]\right] \tag{147}$$

Using the decomposition series for the linear term $U(x,t)$, the infinite series of Adomian polynomials for the nonlinear term $F(U(x,t)) = |U|^2 U = U^2 \overline{U}$, and proceeding as before, we obtain the recursive relation;

$$U_0(x,t) = g(x),$$
$$U_{k+1}(x,t) = S^{-1}\left[i\,uS\left[(U_k)_{xx}\right]\right] + S^{-1}\left[iu\alpha S[\,A_k\,]\right],\ k \ge 0. \tag{148}$$

Besides, some the components of A_n are computed below:

$$A_0 = U_0^{\,2}\overline{U_0},$$
$$A_1 = 2U_0U_1\overline{U_0} + U_0^{\,2}\overline{U_1}, \tag{149}$$
$$A_2 = 2U_0U_2\overline{U_0} + U_1^{\,2}\overline{U_0} + 2U_0U_1\overline{U_1} + U_0^{\,2}\overline{U_2}.$$

And so on. In conjunction with (148) and (149), we can easily determine the first few the components by;

$$U_0(x,t) = g(x),$$
$$U_1(x,t) = S^{-1}\left[i\,uS\left[(U_0)_{xx}\right]\right] + S^{-1}\left[iu\alpha S[\,A_0\,]\right],$$
$$U_2(x,t) = S^{-1}\left[i\,uS\left[(U_1)_{xx}\right]\right] + S^{-1}\left[iu\alpha S[\,A_1\,]\right], \tag{150}$$
$$U_3(x,t) = S^{-1}\left[i\,uS\left[(U_2)_{xx}\right]\right] + S^{-1}\left[iu\alpha S[\,A_2\,]\right].$$

Other components can be determined as well. This completes the determination of the series solution. The analysis introduced above will be illustrated by discussing the following examples.

Example (3.6.16): Consider the following nonlinear Schrödinger equation [44]:

$$iU_t + U_{xx} - 2|U|^2 U = 0 \quad ; \tag{151}$$

Subject to the initial conditions:

$$U(x,0) = e^{ix}. \tag{152}$$

Following the discussion presented above, we find;

$$U_0(x,t) = e^{ix},$$
$$U_1(x,t) = S^{-1}\left[i\,uS\left[(U_0)_{xx}\right]\right] - S^{-1}\left[2iuS[\,A_0\,]\right] = -3ite^{ix},$$
$$U_2(x,t) = S^{-1}\left[i\,uS\left[(U_1)_{xx}\right]\right] - S^{-1}\left[2iuS[\,A_1\,]\right] = \frac{1}{2!}(3it)^2 e^{ix}, \tag{153}$$
$$U_3(x,t) = S^{-1}\left[i\,uS\left[(U_2)_{xx}\right]\right] - S^{-1}\left[2iuS[\,A_2\,]\right] = -\frac{1}{3!}(3it)^3 e^{ix}.$$

In a few of (153), the series solution is given by;

$$U(x,t) = e^{ix}\left(1 - (3it) + \frac{1}{2!}(3it)^2 - \frac{1}{3!}(3it)^3 + \ \ldots\right); \tag{154}$$

The exact solution is;

$$U(x,t) = e^{i(x-3t)} \quad . \tag{155}$$

Example (3.6.17): Consider the following nonlinear Schrödinger equation [44]:

$$iU_t + U_{xx} + 2|U|^2 U = 0 \quad ; \tag{156}$$

Subject to the initial conditions:

$$U(x,0) = e^{ix} . \tag{157}$$

Proceeding as in **Example (3.6.16),** we find;

$$\begin{aligned} U_0(x,t) &= e^{ix} , \\ U_1(x,t) &= ite^{ix} , \\ U_2(x,t) &= -\frac{1}{2!}t^2 e^{ix} , \\ U_3(x,t) &= -\frac{1}{3!}it^3 e^{ix} . \end{aligned} \tag{158}$$

In a few of (158), the series solution is given by;

$$U(x,t) = e^{ix}\left(1 + (it) + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \ ...\right) ; \tag{159}$$

The exact solution is;

$$U(x,t) = e^{i(x+t)} \quad . \tag{160}$$

3.7: The Korteweg – de Vries Equation (KdV)

The nonlinear KdV equation is an important mathematical model with wide applications in quantum mechanics and nonlinear optics.

The KdV equation has several applications to physical problems. It approximately describes the evolution of long water waves. In addition, it used in various fields such as, shallow water waves, acoustic waves in a plasma, and long internal waves in a density.

In this section, we consider the nonlinear KdV in the following form:

$$U_t + aUU_x + bU_{xxx} = 0 \quad ; \tag{161}$$

Subject to the initial condition:

$$U(x,0) = f(x) \ . \tag{162}$$

Where a and b are constants.

The solutions of (161) are called Solitons or Solitary waves.

In this section, we will use (ADSTM) to study the nonlinear KdV equation.

Applying Sumudu transform of both sides of (161) and using initial condition yields;

$$S[U(x,t)] = f(x) - bu\, S[U_{xxx}] - au\, S[UU_x] \tag{163}$$

Applying inverse Sumudu transform to (163) gives;

$$U(x,t) = f(x) - S^{-1}[buS[U_{xxx}]] - S^{-1}[au\, S[UU_x]] \tag{164}$$

Using the decomposition series for the linear term, $U(x,t)$, the infinite series of the Adomian polynomials for the nonlinear term, $F(U(x,t)) = UU_x$, and proceeding as before, we obtain the recursive relation;

$$\begin{aligned} &U_0(x,t) = f(x), \\ &U_{k+1}(x,t) = -\, S^{-1}[buS[(U_k)_{xxx}]] - S^{-1}[auS[A_k]],\ k \ge 0. \end{aligned} \tag{165}$$

The components; $U_n, n \ge 0$ can be elegantly calculated by:

$$\begin{aligned} &U_0(x,t) = f(x), \\ &U_1(x,t) = -\, S^{-1}[buS[(U_0)_{xxx}]] - S^{-1}[auS[A_0]], \\ &U_2(x,t) = -\, S^{-1}[buS[(U_1)_{xxx}]] - S^{-1}[auS[A_1]], \\ &U_3(x,t) = -\, S^{-1}[buS[(U_2)_{xxx}]] - S^{-1}[auS[A_2]]. \end{aligned} \tag{166}$$

Where the Adomian polynomials A_n for the nonlinearity UU_x were derived before and used in advection and Burgers problems.

The discussion presented above will be illustrated as follows:

Example (3.7.18): Consider the following homogeneous nonlinear KdV equation [24]:

$$U_t - 6UU_x + U_{xxx} = 0 \ ; \tag{167}$$

Subject to the initial condition;

$$U(x,0) = 6x \ . \tag{168}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} &U_0(x,t) = 6x, \\ &U_{k+1}(x,t) = -\, S^{-1}[uS[(U_k)_{xxx}]] + S^{-1}[6uS[A_k]],\ k \ge 0. \end{aligned} \tag{169}$$

That gives the first few the components by;

$$\begin{aligned} &U_0(x,t) = 6x, \\ &U_1(x,t) = -\, S^{-1}[uS[(U_0)_{xxx}]] + S^{-1}[6uS[A_0]] = 6^3\, xt, \\ &U_2(x,t) = -\, S^{-1}[buS[(U_1)_{xxx}]] + S^{-1}[auS[A_1]] = 6^5\, xt^2, \\ &U_3(x,t) = -\, S^{-1}[buS[(U_2)_{xxx}]] + S^{-1}[6uS[A_2]] = 6^7\, xt^3. \end{aligned} \tag{170}$$

In a few of (170), the series solution is given by;

$$U(x,t) = 6x\left(1 + (36t) + (36t)^2 + (36t)^3 + \ldots\right); \qquad (171)$$

The exact solution is;

$$U(x,t) = \frac{6x}{1-36t}, \ |36t| < 1 \ . \qquad (172)$$

Example (3.7.19): Consider the following homogeneous nonlinear KdV equation [24]:

$$U_t - 6UU_x + U_{xxx} = 0 \ ; \qquad (173)$$

Subject to the initial condition;

$$U(x,0) = \frac{1}{6}(x-1) \ . \qquad (174)$$

Proceeding as in Example **(3.7.18)**, we find recursive relation;

$$\begin{aligned} &U_0(x,t) = \frac{1}{6}(x-1), \\ &U_{k+1}(x,t) = -S^{-1}\left[uS[(U_k)_{xxx}]\right] + S^{-1}[6uS[A_k]], \ k \geq 0. \end{aligned} \qquad (175)$$

That gives the first few the components by;

$$\begin{aligned} &U_0(x,t) = \frac{1}{6}(x-1), \\ &U_1(x,t) = -S^{-1}\left[uS[(U_0)_{xxx}]\right] + S^{-1}[6uS[A_0]] = \frac{1}{6}(x-1)t, \\ &U_2(x,t) = -S^{-1}[b\,uS[(U_1)_{xxx}]] + S^{-1}[auS[A_1]] = \frac{1}{6}(x-1)t^2, \\ &U_3(x,t) = -S^{-1}[buS[(U_2)_{xxx}]] + S^{-1}[6uS[A_2]] = \frac{1}{6}(x-1)t^3. \end{aligned} \qquad (176)$$

In a few of (176), the series solution is given by;

$$U(x,t) = \frac{1}{6}(x-1)\left(1 + t + t^2 + t^3 + \ldots\right); \qquad (177)$$

The exact solution is;

$$U(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right), \ |t| < 1 \ . \qquad (178)$$

Example (3.7.20): Consider the following inhomogeneous nonlinear mKdV equation [45]:

$$U_t - U^2U_x + U_{xxx} = x\left(1 - t^3x\right) \ ; \qquad (179)$$

Subject to the initial condition:

$$U(x,0) = 0 \ . \tag{180}$$

Following the analysis presented before, we obtain:

$$U(x,t) = xt - \frac{t^4}{4}x^2 \ - S^{-1}\left[uS[U_{xxx}]\right] + S^{-1}\left[u\,S\left[U^2 U_x\right]\right] \tag{181}$$

This gives the modified recursive relation;

$$\begin{aligned} &U_0(x,t) = xt, \\ &U_1(x,t) = -\ \frac{t^4}{4}x^2 \ - \ S^{-1}\left[uS\left[(U_0)_{xxx}\right]\right] + S^{-1}\left[uS[\,A_k\,]\right], \\ &U_{k+1}(x,t) = -\ S^{-1}\left[uS\left[(U_k)_{xxx}\right]\right] + S^{-1}\left[uS[\,A_k\,]\right],\ k \geq 1. \end{aligned} \tag{182}$$

The first few components of the solution are given by;

$$\begin{aligned} &U_0(x,t) = xt, \\ &U_1(x,t) = -\ \frac{t^4}{4}x^2 \ - \ S^{-1}\left[uS\left[(U_0)_{xxx}\right]\right] + S^{-1}\left[uS[\,A_k\,]\right] = 0\ , \\ &U_{k+1}(x,t) = 0\ ,\ k \geq 1. \end{aligned} \tag{183}$$

The exact solution is;

$$U(x,t) = xt \quad . \tag{184}$$

Example (3.7.21): Consider the following homogeneous nonlinear FKdV equation [46]:

$$U_t \ + \ U^2 U_{xx} \ + U_x U_{xxx} \ - 20U^2 U_{xxx} \ + U_{xxxxx} = 0 \tag{185}$$

Subject to the initial condition:

$$U(x,0) = \frac{1}{x} \ . \tag{186}$$

Taking Sumudu transform of both sides of Eq. (185) subject to the initial condition, we get;

$$S[U(x,t)] = \frac{1}{x} + u\,S\left[20U^2 U_{xxx} - U_{xxxxx} - U_x - U^2 U_{xx} - U_x U_{xx}\right] \tag{187}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = \frac{1}{x} + S^{-1}\left[u\,S\left[20U^2 U_{xxx} - U_{xxxxx} - U_x - U^2 U_{xx} - U_x U_{xx}\right]\right] \tag{188}$$

Using the decomposition series for the linear term, $U(x,t)$, the infinite series of the Adomian polynomials for the nonlinear terms, and proceeding as before, we obtain the recursive relation;

$$\begin{aligned} &U_0(x,t) = \frac{1}{x}, \\ &U_{k+1}(x,t) = \ S^{-1}\left[uS\left[\,20A_k \ - B_k \ - C_k \ - (U_k)_{xxxxx} - (U_k)_x\right]\right]\ ,\ k \geq 0. \end{aligned} \tag{189}$$

The first few components of adomian polynomials are given by;

$$A_0 = U_0^2 U_{0\,xxx} \quad , \quad A_1 = 2U_0 U_1 U_{0\,xxx} + U_0^2 U_{1\,xxx} \;,$$
$$A_2 = 2U_0 U_2 U_{0\,xxx} + U_1^2 U_{0\,xxx} + 2U_0 U_1 U_{1\,xxx} + U_0^2 U_{2\,xxx} \;;$$
$$B_0 = U_0^2 U_{0\,xx} \quad , \quad B_1 = 2U_0 U_1 U_{0\,xx} + U_0^2 U_{1\,xx} \;,$$
$$B_2 = 2U_0 U_2 U_{0\,xx} + U_1^2 U_{0\,xx} + 2U_0 U_1 U_{1\,xx} + U_0^2 U_{2\,xx} \;; \qquad (190)$$
$$C_0 = U_{0\,x} U_{0\,xx} \quad , \quad C_1 = U_{0\,x} U_{1\,xx} + U_{0\,xx} U_{1\,x} \;,$$
$$C_2 = U_{2\,x} U_{0\,xx} + U_{1\,x} U_{1\,xx} + U_{0\,x} U_{2\,xx} \;.$$

And so on. In conjunction with (189) and (190), we can easily determine the first few the components by;

$$U_0(x,t) = \frac{1}{x},$$
$$U_1(x,t) = S^{-1}\left[uS\left[20A_0 - B_0 - C_0 - (U_0)_{xxxxx} - (U_0)_x\right]\right] = \frac{t}{x^2} \;,$$
$$U_2(x,t) = S^{-1}\left[uS\left[20A_1 - B_1 - C_1 - (U_1)_{xxxxx} - (U_1)_x\right]\right] = \frac{t^2}{x^3} \;, \qquad (191)$$
$$U_3(x,t) = S^{-1}\left[uS\left[20A_2 - B_2 - C_2 - (U_2)_{xxxxx} - (U_2)_x\right]\right] = \frac{t^3}{x^4} \;.$$

In a few of (191), the series solution is given by;

$$U(x,t) = \frac{1}{x}\left(1 + \frac{t}{x} + \frac{t^2}{x^2} + \frac{t^3}{x^3} + \ldots\right) \;; \qquad (192)$$

The exact solution is;

$$U(x,t) = \frac{1}{x-t} \quad . \qquad (193)$$

Example (3.7.22): Consider the following homogeneous nonlinear FKdV equation [46]:

$$U_t + UU_x - UU_{xxx} + U_{xxxxx} = 0 \qquad (194)$$

Subject to the initial condition:

$$U(x,0) = e^x \;. \qquad (195)$$

Proceeding as in Example **(3.7.21)**, we find recursive relation;

$$U_0(x,t) = e^x \;,$$
$$U_{k+1}(x,t) = S^{-1}\left[uS\left[A_k - B_k - (U_k)_{xxxxx}\right]\right] \;,\; k \geq 0. \qquad (196)$$

That gives the first few the components by;

$$
\begin{aligned}
&U_0(x,t) = e^x , \\
&U_1(x,t) = S^{-1}\left[uS\left[\, A_0 - B_0 - (U_0)_{xxxxx}\right]\right] = -t\, e^x , \\
&U_2(x,t) = S^{-1}\left[uS\left[\, A_1 - B_1 - (U_1)_{xxxxx}\right]\right] = \frac{t^2}{2!} e^x , \\
&U_3(x,t) = S^{-1}\left[uS\left[A_2 - B_2 - (U_2)_{xxxxx}\right]\right] = -\frac{t^3}{3!} e^x .
\end{aligned}
\tag{197}
$$

Summing these iterations yields the series solution;

$$
U(x,t) = e^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right) ; \tag{198}
$$

That leads to the exact solution;

$$
U(x,t) = e^{x-t} . \tag{199}
$$

3.8: The Fourth Order Parabolic Equation

The fourth order parabolic equation with variable equation arises in the transverse vibration, and it is standard form given by;

$$
\frac{\partial^2 U}{\partial t^2} + \psi(x)\frac{\partial^4 U}{\partial x^4} = f(x,t) \; , \psi(x) > 0 \; , a < x < b \; , t > 0 \; ; \tag{142}
$$

Subject to the initial conditions:

$$
U(x,0) = g(x) \; , U_t(x,0) = h(x) \; ; \tag{143}
$$

And the boundary conditions:

$$
\begin{aligned}
&U(a,t) = p(t) \; , U(b,t) = r(t) , \\
&U_{xx}(a,t) = s(t) , U_{xx}(b,t) = q(t).
\end{aligned}
\tag{144}
$$

Where the functions $g(x), h(x), p(t)$, $r(t)$, $s(t)$ and $q(t)$ are continuous functions.

In this section, we use coupling of new integral transform Sumudu transform and Adomian decomposition method to solve one dimensional fourth order parabolic linear partial differential equation with variable coefficients.

Applying Sumudu transform of both sides of Eq. (142) and using initial conditions, we have;

$$
S[U(x,t)] = g(x) + u\, h(x) + u^2 S[f(x,t)] - u^2 S\left[\psi(x)\frac{\partial^4 U}{\partial x^2}\right] \tag{145}
$$

Operating inverse Sumudu transform of both sides of (145) gives;

$$U(x,t)= G(x,t) - S^{-1}\left[u^2 S\left[\psi(x)\frac{\partial^4 U}{\partial x^2}\right]\right] \tag{146}$$

Where $G(x,t)$ represents the term arising from the source term and prescribed initial conditions.

Using the decomposition series for the linear term $U(x,t)$ and proceeding as before, we obtain the recursive relation;

$$\begin{aligned} &U_0(x,t) = G(x,t), \\ &U_{k+1}(x,t) = -\, S^{-1}\left[u^2 S\left[\psi(x)(U_k)_{xxxx}\right]\right],\ k \geq 0. \end{aligned} \tag{147}$$

The discussion presented above will be illustrated as follows:

Example (3.8.23): Let us consider fourth order homogeneous parabolic PDE [47]:

$$\frac{\partial^2 U}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U}{\partial x^4} = 0\ ,\ \ \frac{1}{2}<x<1,\ t>0\ \ ; \tag{148}$$

Subject to the initial conditions:

$$U(x,0) = 0\ \ ,\ U_t(x,0) = 1 + \frac{x^5}{120}\ \ ; \tag{149}$$

And the boundary conditions:

$$\begin{aligned} &U\left(\frac{1}{2},t\right) = \left(1+\frac{(1/2)^5}{120}\right)\sin t\ \ ,\ U(1,t) = \frac{121}{120}\sin t, \\ &U_{xx}\left(\frac{1}{2},t\right) = \frac{1}{6}\left(\frac{1}{2}\right)^3 \sin t\ \ \ \ ,\ U_{xx}(1,t) = \frac{1}{6}\sin t. \end{aligned} \tag{150}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} &U_0(x,t) = \left(1 + \frac{x^5}{120}\right)t, \\ &U_{k+1}(x,t) = -\, S^{-1}\left[u^2 S\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)(U_k)_{xxxx}\right]\right],\ k \geq 0. \end{aligned} \tag{151}$$

That gives;

$$U_0(x,t) = \left(1 + \frac{x^5}{120}\right)t,$$
$$U_1(x,t) = -S^{-1}\left[u^2 S\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)(U_0)_{xxxx}\right]\right] = -\left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!},$$
$$U_2(x,t) = -S^{-1}\left[u^2 S\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)(U_1)_{xxxx}\right]\right] = \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!}, \qquad (152)$$
$$U_3(x,t) = -S^{-1}\left[u^2 S\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)(U_2)_{xxxx}\right]\right] = -\left(1 + \frac{x^5}{120}\right)\frac{t^7}{7!}.$$

And so on. The solution in a series form is;

$$U(x,t) = \left(1 + \frac{x^5}{120}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right); \qquad (153)$$

And in a closed form solution of;

$$U(x,t) = \left(1 + \frac{x^5}{120}\right)\sin t \quad . \qquad (154)$$

3.9: The Pade/ Approximants

In this section, we purpose to establish a new technique gives better approximation of the function than truncating its Taylor series, and it may well still work where the Taylor series does not converge. It is significant to note that several powerful methods [19, 38, 48-50] have been advanced for this purpose by using this new technique. The new technique was developed around (1890) by Henri Pade/ and called Pade/ approximant. A Pade/ approximant are the fraction of two polynomials constructs from the coefficients of the Taylor expansion of a function.

The Pade/ approximation of a function symbolized by [m / n] and defined by:

$$[m/n] = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n} \qquad (155)$$

Where we considered $b_0 = 1$, and numerator, denominator have no common factors. If we selected $m = n$, then the approximants [n / n] are called diagonal approximants.

In the following, we will introduce the simple and the straightforward method to construct Pade/ approximants. We denote the m, n Pade/ approximants to $f(x)$.

Suppose that $f(x)$ has a Taylor series given by;

$$f(x)=\sum_{k=0}^{\infty} c_k\, x^k \quad ; \tag{156}$$

Assuming that $f(x)$ can be manipulated by the diagonal Pade/ approximant defined in (155), where $m = n$. This admits the use of;

$$\frac{a_0+a_1\,x+a_2\,x^2+...+a_n\,x^n}{1+b_1\,x+b_2\,x^2+...+b_n\,x^n}=c_0+c_1\,x+c_2\,x^2+...+c_{2n}\,x^{2n} \tag{157}$$

By using cross multiplication in (157), we find;

$$\begin{aligned} a_0+a_1\,x+a_2\,x^2+...+a_m\,x^n = c_0+(c_1+b_1\,c_0)x+(c_2+b_1\,c_1+b_2\,c_2)\,x^2 + \\ +(c_3+b_1\,c_2+b_2\,c_1+b_3\,c_0)x^3+... \end{aligned} \tag{158}$$

Equating powers of x leads to;

$$\begin{aligned} a_0 &= c_0\,, \\ a_1 &= c_1+b_1c_0\,, \\ a_2 &= c_2+b_1c_1+b_2c_0\,, \\ a_3 &= c_3+b_1c_2+b_2c_1+b_3c_0 \\ . \quad & . \quad . \quad . \\ a_n &= c_n+\sum_{k=1}^{\infty} b_k c_{n-k} \end{aligned}$$

Notice that $x^{n+1}, x^{n+2}, ..., x^{2n}$ should be equated to zero.

The simple procedure outlined above will be illustrated by discussing the following examples.

Example (3.9.24): Find the Pade/ approximants [2/2] for the function [24]:

$$f(x)=\sqrt{\frac{1+3x}{1+x}} \quad ; \tag{159}$$

The Taylor series for $f(x)$ of (149) is given by;

$$f(x)=1+x-\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4+\frac{75}{8}x^5-\frac{327}{16}x^6+\frac{753}{16}x^7+o(x^8)\,. \tag{160}$$

The [2/2] approximant is defined by;

$$[2/2]=\frac{a_0+a_1\,x+a_2\,x^2}{1+b_1\,x+b_2\,x^2}\quad . \tag{161}$$

To determine the five coefficients of the two polynomials, the [2/2] approximant must fit the Taylor series of $f(x)$ in (160) through the orders of $1\,,\,x\,,\,...\,,\,x^4$, hence we set;

$$\frac{a_0+a_1\,x+a_2\,x^2}{1+b_1\,x+b_2\,x^2}=1+x-\frac{3}{2}x^2+\frac{5}{2}x^3-\frac{37}{8}x^4+...\quad . \tag{162}$$

Cross multiplying yields;

$$a_0 + a_1 x + a_2 x^2 = 1 + (1 + b_1) x + \left(b_1 + b_2 - \frac{3}{2} \right) x^2 + + \left(\frac{5}{2} - \frac{3}{2} b_1 + b_2 \right) x^3 + \left(\frac{5}{2} b_1 - \frac{3}{2} b_2 - \frac{37}{8} \right) . x^4 \quad . \qquad (163)$$

Equating powers of x leads to;

$$a_0 = 1 ,$$
$$a_1 = 1 + b_1 ,$$
$$a_2 = b_1 + b_2 - \frac{3}{2} ,$$
$$0 = \frac{5}{2} - \frac{3}{2} b_1 + b_2 ,$$
$$0 = \frac{5}{2} b_1 - \frac{3}{2} b_2 - \frac{37}{8} .$$

The solution of this system of equations is;

$$a_0 = 1 \ , \ a_1 = \frac{9}{2} \ , \ a_2 = \frac{19}{4} \ , \ b_1 = \frac{7}{2} \ , \ b_2 = \frac{11}{4} \quad .$$

Consequently, the [2/2] Pade/ approximant is;

$$[2/2] = \frac{1 + \frac{9}{2} x + \frac{19}{4} x^2}{1 + \frac{7}{2} x + \frac{11}{4} x^2} \quad . \qquad (164)$$

However, the limit of Pade/ approximant (164) as $x \to \infty$ is $\frac{a_2}{b_2}$. In other words, as $x \to \infty$ we obtain;

$$\lim_{x \to \infty} f(x) = \sqrt{3} \approx 1.73205 \ ; \qquad (165)$$

And

$$\lim_{x \to \infty} [2/2] = \frac{19}{11} \approx 1.72727 \ . \qquad (166)$$

Example (3.9.25): Consider the coupled Burgers system of equation [48]:

$$\begin{aligned} U_t - 2UU_x - U_{xx} + (UV)_x &= 0 , \\ V_t - 2VV_x - V_{xx} + (UV)_x &= 0 . \end{aligned} \qquad (167)$$

With the initial data;

$$U(x,0) = \sin x \ , \ V(x,0) = \sin x . \qquad (168)$$

Applying L_t^{-1} of the system (167) and using the initial data (168) yields;

$$\begin{aligned} U(x,t) &= \sin x + L_t^{-1}[U_{xx}] + 2L_t^{-1}[UU_x] - L_t^{-1}[UV_x + VU_x], \\ V(x,t) &= \sin x + L_t^{-1}[V_{xx}] + 2L_t^{-1}[VV_x] - L_t^{-1}[UV_x + VU_x]. \end{aligned} \tag{169}$$

The Adomian decomposition method suggests that the linear terms $U(x,t)$ and $V(x,t)$ is decomposed by an infinite series of components:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t), \quad V(x,t) = \sum_{n=0}^{\infty} V_n(x,t). \tag{170}$$

However, the nonlinear terms UU_x, UV_x, VU_x and VV_x should be represented by the Adomian polynomials A_n and B_n, C_n and D_n respectively as follows;

$$UU_x = \sum_{n=0}^{\infty} A_n, \quad UV_x = \sum_{n=0}^{\infty} B_n, \quad VU_x = \sum_{n=0}^{\infty} C_n, \quad VV_x = \sum_{n=0}^{\infty} D_n \tag{171}$$

Substituting (170) and (171) into (169), gives the recursive relations by;

$$\begin{aligned} U_0(x,t) &= \sin x, \\ U_{k+1}(x,t) &= L_t^{-1}[(U_k)_{xx}] + 2L_t^{-1}[A_k] - L_t^{-1}[B_k + C_k], \; k \geq 0. \end{aligned} \tag{172}$$

And

$$\begin{aligned} V_0(x,t) &= \sin x, \\ V_{k+1}(x,t) &= L_t^{-1}[(V_k)_{xx}] + 2L_t^{-1}[D_k] - L_t^{-1}[B_k + C_k], \; k \geq 0. \end{aligned} \tag{173}$$

Using the derived Adomian polynomials in Burgers equation **Example (3.5.8)**, we obtain the following pairs of components;

$$\begin{aligned} (U_0, V_0) &= (\sin x, \sin x), \\ (U_1, V_1) &= (-t\sin x, -t\sin x), \\ (U_2, V_2) &= \left(\frac{t^2}{2!}\sin x, \frac{t^2}{2!}\sin x\right), \\ (U_3, V_3) &= \left(-\frac{t^3}{3!}\sin x, -\frac{t^3}{3!}\sin x\right). \end{aligned} \tag{174}$$

And so on. Accordingly, the solution in a series form is given by;

$$U(x,t) = \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right),$$
$$V(x,t) = \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right). \tag{175}$$

Applying the Sumudu transform to $U_n(x,t)$ and $V_n(x,t)$ yields;

$$S[U(x,t)] = \sin x\left(1 - u + u^2 - u^3 + \ldots\right),$$
$$S[V(x,t)] = \sin x\left(1 - u + u^2 - u^3 + \ldots\right). \tag{176}$$

The [m / n] Pade/ approximant of each one of Eq. (155) with $m \geq 1$ and $n \geq 1$ yields;

$$\left[\frac{m}{n}\right] = \frac{1}{1+u}\sin x \tag{177}$$

Using the inverse Sumudu transform to [m / n], the exact solution is obtained;

$$U(x,t) = e^{-t}\sin x,$$
$$V(x,t) = e^{-t}\sin x. \tag{178}$$

CHAPTER (4)
Volterra Integro-Differential Equations

It is well known that linear and nonlinear Volterra integro – differential equation arise in many scientific fields such as the population dynamic, neutron diffusion and semiconductor devices.

The Volterra integro – differential equation appears in the form:

$$U^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)U(t)\,dt \quad , \qquad (1)$$

Where $U^{n}(x) = \dfrac{d^{n}U}{d\,x^{n}}$, and the initial conditions $U(0), U'(0), \ldots\ldots.., U^{n-1}(0)$ are prescribed. The kernel $K(x,t)$ and the function $f(x)$ are given real- valued functions.

It is our goal in this chapter to study the nonlinear Volterra integro – differential equations of the second and first kind given by:

$$U^{(n)}(x) = f(x) + \int_0^x K(x,t)F(U(t))\,dt \quad , \qquad (2)$$

However, the standard form of the nonlinear Volterra integro – differential equation (1) of the first kind is the form:

$$\int_0^x K_1(x,t)F(U(t))\,dt + \int_0^x K_2(x,t)U^{n}(t)\,dt = f(x) \qquad (3)$$

Several techniques such that , homotopy perturbation method [51-54], modified Laplace Adomian decomposition method [55], variation iteration method, the series solution method, the differential transform method [56] and combined Laplace transform – Adomian decomposition method [14] has been used for solving these problems.

It is the aim of this chapter to develop a combined form of the Sumudu transform method with the Adomian decomposition method to establish the exact solution or approximations of high degree of accuracy for the nonlinear Volterra of a second and first kind (2) and (3) respectively.

The advantage of these methods is its capability of combining the two powerful methods for obtaining exact solutions for nonlinear equations.

4.1: Nonlinear Volterra Integro – Differential Equations of The Second Kind

To illustrate the basic idea of this section, we consider the kernel $K(x,t)$ of the equation (2) as difference kernel that depends on the difference $(x-t)$.
The nonlinear Volterra integro- differential equation (2) can be expressed as;

$$U^{(n)}(x) = f(x) + \int_0^x K(x-t)\, F(U(t))dt \ , \tag{4}$$

Consider two functions $f_1(x)$ *and* $f_2(x)$ that possess the conditions.

Let Sumudu transform of the functions $f_1(x)$ *and* $f_2(x)$ given by:

$$S[f_1(x)] = F_1(u) \quad , \quad S[f_2(x)] = F_2(u) \tag{5}$$

The Sumudu convolution product of these two functions is defined by;

$$S[(f_1 * f_2)(x)] = S\left[\int_0^x f_1(x-t)\, f_2(x)dt\right] = u\, F_1(u)F_2(u) \tag{6}$$

To solve the nonlinear Volterra integro- differential equation by using Sumudu transform, it is essential to use the Sumudu transform of the derivatives of $U(x)$ are defined by;

$$S\left[U^{(n)}(x)\right] = \frac{S[U(x)]}{u^n} - \frac{U(0)}{u^n} - \frac{U'(0)}{u^{n-1}} - \dots - \frac{U^{n-1}(0)}{u} \tag{7}$$

This simply gives;

$$\begin{aligned} S[U'(x)] &= \frac{S[U(x)]}{u} - \frac{U(0)}{u} = u^{-1}U(u) - u^{-1}U(0), \\ S[U''(x)] &= u^{-2}U(u) - u^{-2}U(0) - u^{-1}U'(0), \\ S[U'''(x)] &= u^{-3}U(u) - u^{-3}U(0) - u^{-2}U'(0) - u^{-1}U''(0). \end{aligned} \tag{8}$$

And so on for derivatives of higher order, where $U(u) = S[U(x)]$.
Applying Sumudu transform of both sides of Eq. (2), to get;

$$\begin{aligned} u^{-n}\, S[U(x)] - u^{-n}U(0) - u^{1-n}\, U'(0) - \dots - u^{-1}\, u^{n-1}(0) = \\ S[f(x)] + u\, S[K(x-t)]S[F(U(x))] \end{aligned} \tag{9}$$

Or equivalently;

$$\begin{aligned} S[u(x)] = U(0) + u\,U'(0) + \ \dots + \ u^{n-1}\, U^{n-1}(0) + \\ u^n\, S[f(x)] + u^{n+1}\, S[K(x-t)]S[F(U(x))] \end{aligned} \tag{10}$$

Taking the inverse Sumudu transform of both sides of Eq. (10), to get;

$$\begin{aligned} u(x) = U(0) + x\,U'(0) + \ \dots + \frac{x^{n-1}}{(n-1)!}U^{n-1}(0) + \\ S^{-1}\left[u^n\, S[f(x)]\right] + S^{-1}\left[u^{n+1}\, S[K(x-t)]S[F(U(x))]\right] \end{aligned} \tag{11}$$

Now, we apply the Adomian decomposition method;

$$U(x)=\sum_{n=0}^{\infty}U_n(x) \quad ; \qquad (12)$$

And the nonlinear terms can be decomposed as;

$$F(U(x))=\sum_{n=0}^{\infty}A_n(U) \quad . \qquad (13)$$

For some the Adomian polynomials $A_n(U)$ that are given by;

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty}\lambda^n U_n\right)\right]_{\lambda=0}, \; n=0,1,2,\ldots \qquad (14)$$

Substituting Eq. (12) and Eq. (13) into Eq. (11) leads to;

$$\begin{aligned}\sum_{n=0}^{\infty}U_n(x)=\; & U(0)+xU'(0)+\ldots\ldots+\frac{x^{n-1}}{(n-1)!}U^{n-1}(0)+S^{-1}[u^n S[f(x)]] \\ & +S^{-1}\left[u^{n+1}S[K(x-t)]S\left[\sum_{n=0}^{\infty}A_n(U)\right]\right]\end{aligned} \qquad (15)$$

So that the recursive relation is given by;

$$\begin{aligned}&U_0(x)=U(0)+xU'(0)+\ldots\ldots.+\frac{x^{n-1}}{(n-1)!}U^{n-1}(0)+S^{-1}[u^n S[f(x)]],\\ &U_{k+1}(x)=S^{-1}[u^{n+1}S[K(x-t)]S[A_k]], \; k\geq 0 .\end{aligned} \qquad (16)$$

The combined Sumudu transform – Adomian decomposition method for solving nonlinear Volterra integro- differential equations of the second kind will be illustrated by studding the following examples.

Example (4.1.1): Consider the initial value problem [57]:

$$U'(x)=\frac{9}{4}-\frac{5}{2}x-\frac{1}{2}x^2-3e^{-x}-\frac{1}{4}e^{-2x}+\int_0^x(x-t)U^2(t)dt, \; U(0)=2 \quad . \qquad (17)$$

Notice that the kernel $K(x-t)=(x-t)$. Taking Sumudu transform of both sides of Eq. (17) gives;

$$S[U'(x)]=S\left[\frac{9}{4}-\frac{5}{2}x-\frac{1}{2}x^2-3e^{-x}-\frac{1}{4}e^{-2x}\right]+S[(x-t)*U^2(x)]. \qquad (18)$$

So that:

$$u^{-1}U(u)-u^{-1}U(0)=\frac{9}{4}-\frac{5}{2}u-u^2-\frac{3}{1+u}-\frac{1}{4(1+2u)}+u^2S[U^2(x)], \qquad (19)$$

Or equivalently;

$$U(u)=2+\frac{9}{4}u-\frac{5}{2}u^2-u^3-\frac{3u}{1+u}-\frac{u}{4(1+2u)}+u^3S[U^2(x)], \qquad (20)$$

Applying the inverse Sumudu transform of both sides of Eq. (21) gives;

$$U(x)= 2+\frac{9}{4}x-\frac{5}{4}x^2-\frac{x^3}{3!}-3+3e^{-x}-\frac{1}{8}+\frac{1}{8}e^{-2x}+S^{-1}\left[u^3\,S\left[U^2(x)\right]\right], \quad (21)$$

Or equivalently;

$$U(x)= 2-x+\frac{1}{2}x^2-\frac{5}{3!}x^3+\frac{5}{4!}x^4-\frac{7}{5!}x^5+.....+S^{-1}\left[u^3\,S\left[U^2(x)\right]\right] \quad (22)$$

Substituting the series assumption for $U(x)$ and the Adomian polynomials for $U^2(x)$ as given above in Eq. (12) and Eq. (13) respectively, and using the recursive relation (16) to obtain;

$$\begin{aligned} U_0(x)&= 2-x+\frac{1}{2}x^2-\frac{5}{3!}x^3+\frac{5}{4!}x^4-\frac{7}{5!}x^5+.....\ , \\ U_{k+1}(x)&= S^{-1}\left[u^3\,S\left[A_K\right]\right],\ k\geq 0\ . \end{aligned} \quad (23)$$

Recall that the Adomian polynomials for $F(U(x))=U^2(x)$ are given by;

$$\begin{aligned} A_0 &= {U_0}^2\ , \\ A_1 &= 2U_0\,U_1\ , \\ A_2 &= 2U_0\,U_2+{U_1}^2\ , \\ A_3 &= 2U_0\,U_3+2U_1\,U_2\ . \end{aligned}$$

Substituting these polynomials into the recursive relation (23) to find;

$$\begin{aligned} U_0(x)&= 2-x+\frac{1}{2}x^2-\frac{5}{3!}x^3+\frac{5}{4!}x^4-\frac{7}{5!}x^5+...\ , \\ U_1(x)&= \frac{2}{3}x^3-\frac{1}{6}x^4+\frac{1}{20}x^5+..., \end{aligned} \quad (24)$$

Using (12), to find the series solution of eq. (17), in the form;

$$U(x)= 2-x+\frac{1}{2}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4-\frac{1}{5!}x^5+... \quad ; \quad (25)$$

This is converging to the exact solution;

$$U(x)= 1+e^{-x}\ . \quad (26)$$

Example (4.1.2): Consider the following integro-differential equation [57]:

$$U''(x)=-1-\frac{1}{3}(\sin x+\sin 2x)+2\cos x+\int_0^x \sin(x-t)U^2(t)dt,\ U(0)=-1\ ,\ U'(0)=1 \quad (27)$$

Taking Sumudu transform of (27), to find;

$$S[U''(x)]=S\left[-1-\frac{1}{3}(\sin x+\sin 2x)+2\cos x\right]+S\left[\sin(x-t)*U^2(x)\right], \quad (28)$$

So that;

$$u^{-2}U(u)-u^{-2}U(0)-u^{-1}U'(0)=-1-\frac{u}{3(1+u^2)}-\frac{2u}{3(1+4u^2)}+\frac{2}{(1+u^2)}+ +\frac{u^2}{(1+u^2)}S\left[U^2(x)\right], \tag{29}$$

Or equivalently;

$$U(u)=-1+u-u^2-\frac{u^3}{3(1+u^2)}-\frac{2u^3}{3(1+4u^2)}+\frac{2u^2}{1+u^2}+\frac{u^4}{1+u^2}S\left[U^2(x)\right], \tag{30}$$

Applying inverse Sumudu of both sides of Eq. (30) gives;

$$U(x)=-1+x+\frac{x^2}{2!}-\frac{1}{3!}x^3-\frac{1}{12}x^4+\frac{1}{40}x^5+ +\frac{1}{360}x^6-\frac{11}{5040}x^7+\ldots+S^{-1}\left[\frac{u^4}{1+u^2}S\left[U^2(x)\right]\right] \tag{31}$$

Proceeding as before, we find;

$$U_0(x)=-1+x+\frac{x^2}{2!}-\frac{1}{3!}x^3-\frac{1}{12}x^4+\frac{1}{40}x^5+\frac{1}{360}x^6-\frac{11}{5040}x^7+\ldots,$$
$$U_1(x)=\frac{1}{4!}x^4-\frac{1}{60}x^5-\frac{1}{720}x^6+\frac{1}{504}x^7+\ldots, \tag{32}$$

Using (12), to find the series solution of eq. (27), in the form;

$$U(x)=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)\ ; \tag{33}$$

Which is converges to the exact solution;

$$U(x)=\sin x-\cos x\ . \tag{34}$$

4.2: Nonlinear Volterra Integro – Differential Equations of the First Kind

To illustrate the basic idea of this section, we consider the kernels $K_1(x,t)$ and $K_2(x,t)$ of equation (3) as difference kernel that depends on the difference $(x-t)$.

The nonlinear Volterra integro- differential equation (3) can be expressed as;

$$\int_0^x K_1(x-t)F(U(t))dt+\int_0^x K_2(x-t)U^n(t)\,dt=f(x)\ ; \tag{35}$$

Recall that;

$$S\left[U^{(n)}(x)\right]=\frac{S[U(x)]}{u^n}-\frac{U(0)}{u^n}-\frac{U'(0)}{u^{n-1}}-\ldots-\frac{U^{n-1}(0)}{u}\ . \tag{36}$$

Applying Sumudu transform of both sides of Eq. (35) to get;

$$S[K_1(x-t)*F(U(x))]+S\left[K_2(x-t)*U^n(x)\right]=S[f(x)] \tag{37}$$

Or equivalently;

$$u\,K_1(u)\;S[F(U(x))]+u\,K_2(u)S\left[U^n(x)\right]=F(u) \qquad (38)$$

Using (36) and solving for $U(x)$, we find;

$$S\left[U(x)\right]=u^{n-1}\left(\frac{F(u)+K_2(u)\psi(u)-u\,K_1(u)\;S[F(U(x))]}{K_2(u)}\right)\;; \qquad (39)$$

where $$\psi(u)=\frac{U(0)}{u^{n-1}}+\frac{U'(0)}{u^{n-1}}+\ldots+U^{(n-1)}(0)\quad . \qquad (40)$$

Now we use the Adomian decomposition method to handle (39), substituting (12) and (13) into (39), we get;

$$S\left[\sum_{n=0}^{\infty}U_n(x)\right]=u^{n-1}\frac{F(u)}{K_2(u)}+U(0)+u\,U'(0)+\ldots+u^{n-1}U^{(n-1)}(0)-$$
$$-\frac{u^nK_1(u)\;S\left[\sum_{n=0}^{\infty}A_n\right]}{K_2(u)} \qquad (41)$$

The Adomian decomposition method admits the use of the following recursive relation;

$$U_0(u)=u^{n-1}\frac{F(u)}{K_2(u)}+U(0)+u\,U'(0)+\ldots+u^{n-1}U^{(n-1)}(0),$$
$$S[U_{k+1}(x)]=-\frac{u^n\,K_1(u)}{K_2(u)}\;S[A_k],\;k\geq 0\,. \qquad (42)$$

Applying the inverse Sumudu transform to the first part of (42) gives $U_0(x)$, that will define $A_0(U)$. This in turn will lead to the complete determination of the components of $U_k(x), k\geq 0$.

The proposed scheme will be illustrated by using the following examples.

Example (4.2.3): Consider the following first kind of nonlinear Volterra integro – differential equation [58]:

$$\int_0^x (x-t)U^2(t)dt+\int_0^x e^{x-t}\,U'(t)d\,t=-\frac{1}{4}-\frac{1}{2}x+x\,e^x+\frac{1}{4}e^{2x}\;,\;U(0)=1\quad . \qquad (43)$$

Proceeding as before, we find the recursive relation;

$$U_0(u)=1-\frac{(1-u)}{4}-\frac{1}{2}u(1-u)+\frac{u}{1-u}+\frac{1-u}{4(1-2u)}\,,$$
$$S[U_{k+1}(x)]=-u^2(1-u)\;S[U^2(x)],\;k\geq 0\,. \qquad (44)$$

Taking the inverse Sumudu transform of Eq. (44) gives;

$$U_0(x) = 1 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{24}x^5 + \dots ,$$

$$U_1(x) = -\frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \frac{1}{12}x^5 + \dots , \qquad (45)$$

$$U_2(x) = \frac{1}{12}x^4 + \frac{1}{20}x^5 + \dots ,$$

The series solution is given by;

$$U(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots , \qquad (46)$$

In a closed form given of;

$$U(x) = e^x \; . \qquad (47)$$

Example (4.2.4): Consider the following first kind of nonlinear Volterra integro – differential equation [59]:

$$\int_0^x (x-t)U^2(t)dt + \int_0^x (x-t)U''(t)d\,t = -\frac{15}{32} - \frac{3}{4}x^2 + \frac{1}{2}\cos 2x - \frac{1}{32}\cos 4x , \qquad (48)$$

$$U(0) = 2 , U'(0) = 0$$

Proceeding as before, the recursive relation is;

$$U_0(u) = \frac{49}{32} + \frac{3}{2}u^2 + \frac{1}{1+4u^2} - \frac{1}{8(1+16u^2)} ,$$

$$S[U_{k+1}(x)] = -u^2 \; S[U^2(x)], \; k \geq 0 . \qquad (49)$$

Taking the inverse Sumudu transform of Eq. (49) gives;

$$U_0(x) = 2 + \frac{2}{15}x^6 + \dots ,$$

$$U_1(x) = -2x^2 + \dots ,$$

$$U_2(x) = \frac{2}{3}x^4 + \dots , \qquad (50)$$

$$U_3(x) = -\frac{2}{9}x^6 + \dots ,$$

The series solution is therefore given by;

$$U(x) = 1 + \left(1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \dots \right) , \qquad (51)$$

That converges to the exact solution;

$$U(x) = 1 + \cos 2x . \qquad (52)$$

4.3: Systems of Nonlinear Volterra Integro – Differential Equations of The Second Kind

In this section, we will study systems of nonlinear Volterra integro – differential equations of the second kind by combining Sumudu transform – Adomian decomposition method.

Consider systems of nonlinear Volterra integro – differential equations of the second kind as follows:

$$\begin{aligned} U^{(n)}(x) &= f_1(x) + \int_0^x [K_1(x-t)\, F_1(u(t)) + R_1(x-t) G_1(V(t))]dt \;, \\ V^{(n)}(x) &= f_2(x) + \int_0^x [K_2(x-t)\, F_2(u(t)) + R_2(x-t) G_2(V(t))]dt . \end{aligned} \tag{53}$$

Where F_i , G_i , $i = 1,2$ nonlinear functions of $U(x), V(x)$, K_i , R_i , $i = 1,2$ are the kernels and $f_i(x)$, $i = 1,2$ are real – valued functions.

Applying Sumudu transform of both sides of (54), we have;

$$\begin{aligned} & u^{-n}\, S[U(x)] - u^{-n} U(0) - u^{1-n}\, U'(0) - ... - u^{-1}\, U^{n-1}(0) = \\ & \qquad S[f_1(x)] + S[K_1(x) * F_1(U(x)) + R_1(x) * G_1(V(x))], \\ & u^{-n}\, S[V(x)] - u^{-n} V(0) - u^{1-n} V'(0) - ... - u^{-1}\, V^{n-1}(0) = \\ & \qquad S[f_2(x)] + S[K_2(x) * F_2(U(x)) + R_2(x) * G_2(V(x))]. \end{aligned} \tag{54}$$

Or equivalent;

$$\begin{aligned} & S[U(x)] = U(0) + u\,U'(0) + ... + u^{n-1}\, U^{n-1}(0) + u^n\, S[f_1(x)] + \\ & \qquad + u^n\, S[K_1(x) * F_1(U(x))] + u^n\, S[R_1(x) * G_1(V(x))] \\ & S[V(x)] = V(0) + u\,V'(0) + ... + u^{n-1}\, V^{n-1}(0) + u^n\, S[f_2(x)] + \\ & \qquad + u^n\, S[K_2(x) * F_2(U(x))] + u^n\, S[R_2(x) * G_2(V(x))]. \end{aligned} \tag{55}$$

Now, we apply the Adomian decomposition method;

$$U(x) = \sum_{n=0}^{\infty} U_n(x) \;\; , \;\; V(x) = \sum_{n=0}^{\infty} V_n(x) \tag{56}$$

And the nonlinear terms can be decomposed as;

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x) \tag{57}$$

Substituting (56) and (57) into (55) gives;

$$S\left[\sum_{n=0}^{\infty}U_n(x)\right]=U(0)+uU'(0)+\ldots+u^{n-1}U^{n-1}(0)+u^n S[f_1(x)]+$$
$$+u^n S\left[K_1(x)*\sum_{n=0}^{\infty}A_n\right]+u^n S\left[R_1(x)*\sum_{n=0}^{\infty}B_n\right], \tag{58}$$
$$S\left[\sum_{n=0}^{\infty}V_n(x)\right]=V(0)+uV'(0)+\ldots+u^{n-1}V^{n-1}(0)+u^n S[f_2(x)]+$$
$$+u^n S\left[K_2(x)*\sum_{n=0}^{\infty}C_n\right]+u^n S\left[R_2(x)*\sum_{n=0}^{\infty}D_n\right].$$

The Adomian decomposition method admits the use of the following recursive relations;

$$\begin{aligned} S[U_0(u)]&=U(0)+uU'(0)+\ldots+u^{n-1}U^{n-1}(0)+u^n S[f_1(x)] \\ S[U_{k+1}(u)]&=u^n S[K_1(x)*A_k]+u^n S[R_1(x)*B_k],\ k\geq 0. \end{aligned} \tag{59}$$

And

$$\begin{aligned} S[V_0(u)]&=V(0)+uV'(0)+\ldots+u^{n-1}V^{n-1}(0)+u^n S[f_2(x)] \\ S[V_{k+1}(u)]&=u^n S[K_2(x)*C_k]+u^n S[R_2(x)*D_k],\ k\geq 0. \end{aligned} \tag{60}$$

Applying the inverse Sumudu transform to the first part of (59) and (60) gives $U_0(x), V_0(x)$, that will define A_0, B_0, C_0, D_0. This in turn will lead to the complete determination of the components $U_k(x), V_k\ k\geq 0$.

The combined Laplace transform Adomian-decomposition method for solving systems of nonlinear Volterra integro-differential equations of the second kind will be illustrated by studying the following examples.

Example (4.3.5): Consider the system of nonlinear Volterra integro – differential equation [58];

$$\begin{aligned} U''(x)&=\frac{7}{3}e^x-e^{2x}-\frac{1}{3}e^{4x}+\int_0^x e^{x-t}\left(U^2(t)+V^2(t)\right)dt\ , \\ V''(x)&=\frac{2}{3}e^x+3e^{2x}+\frac{1}{3}e^{4x}+\int_0^x e^{x-t}\left(U^2(t)-V^2(t)\right)dt\ . \end{aligned} \tag{61}$$

With the initial conditions:

$$U(0)=1\,,\ U'(0)=1\,,\ V(0)=1\,,\ V'(0)=2. \tag{62}$$

Taking Sumudu transforms of both sides of (61) and using initial conditions, we obtain;

$$U(u)=1+u+\frac{7u^2}{3(1-u)}-\frac{u^2}{(1-2u)}-\frac{u^2}{3(1-4u)}+\left(\frac{u^3}{1-u}\right)S\left[U^2(x)+V^2(x)\right],$$
$$V(u)=1+2u+\frac{2u^2}{3(1-u)}+\frac{3u^2}{(1-2u)}+\frac{u^2}{3(1-4u)}+\left(\frac{u^3}{1-u}\right)S\left[U^2(x)-V^2(x)\right]. \quad (63)$$

By using (59) and (60), we have;

$$U_0(u)=1+u+\frac{7u^2}{3(1-u)}-\frac{u^2}{(1-2u)}-\frac{u^2}{3(1-4u)},$$
$$U_{k+1}(u)=\left(\frac{u^3}{1-u}\right)S[\,A_k+B_k\,],\; k\geq 0. \quad (64)$$

And

$$V_0(u)=1+2u+\frac{2u^2}{3(1-u)}+\frac{3u^2}{(1-2u)}+\frac{u^2}{3(1-4u)},$$
$$V_{k+1}(u)=\left(\frac{u^3}{1-u}\right)S[\,A_k-B_k\,],\; k\geq 0. \quad (65)$$

Taking the inverse Sumudu transform of both sides of (64) and (65), we obtain the solution as follows:

$$U(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots,$$
$$V(x)=1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}+\dots. \quad (66)$$

Then the solution for the above system is given by;

$$(U(x),V(x))=\left(e^x,\ e^{2x}\right). \quad (67)$$

CHAPTER (5)

Comparison of Adomian Decomposition Method and Sumudu Transform Method with Other Methods

Our aim of this chapter is to introduce a comparative study between Adomian Decomposition Sumudu Transform Method (ADSTM) and different powerful methods to solve several linear and nonlinear partial differential equations, and nonlinear integral equations, namely, The Sumudu transform method (STM), The Adomian Decomposition Sumudu Transform Method (ADSTM), The Adomian Decomposition Sumudu Transform Method with a Pade$^{/}$ approximant (ADSTM-PA method), The Homotopy Perturbation Method (HPM), and The Variational Iteration Method (VIM).

The Adomian decomposition Sumudu transform method is a combination of Sumudu transform and Adomian decomposition method. This method is a simple and directly without any restrictive assumption as usual is going in other methods for obtaining exact or approximant solutions for nonlinear problems.

5.1: Comparison of Adomian Decomposition Method and Sumudu Transform Method with Sumudu Transform for Solving Linear Partial Differential Equations

Partial differential equations are a necessary part in applied science and engineering fields. The wide use of these equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically.

In this section, the main objective is to introduce a comparative study to solve linear partial differential equations using Adomian decomposition method and Sumudu transform method with Sumudu transform method.

5.1.1: Basic Idea of (ADSTM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of form;

$$DU(x,t)+RU(x,t)+NU(x,t)=g(x,t)$$
$$U(x,0)=h(x)\ ,U_t(x,0)=f(x), \tag{1}$$

Where D is the second order linear differential operator $D=\frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D , N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (1), we get;

$$S[DU(x,t)]+S[RU(x,t)]+S[N(x,t)]=S[g(x,t)]. \tag{2}$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have;

$$S[U(x,t)]=u^2\,S[g(x,t)]+h(x)+u\,f(x)-u^2\,S[RU(x,t)+NU(x,t)]. \tag{3}$$

Now, applying the inverse Sumudu transform of both sides of (3), we get,

$$U(x,t)=G(x,t)-S^{-1}\left[u^2\,S[RU(x,t)+NU(x,t)]\right]$$

Where $G(x,t)$represents the term arising from the source term and the prescribed initial conditions. Now, apply the Adomian decomposition method;

$$U(x,t)=\sum_{n=0}^{\infty}U_n(x,t). \tag{5}$$

The nonlinear term can be decomposed as;

$$NU(x,t)=\sum_{n=0}^{\infty}A_n(U) \tag{6}$$

For some Adomian polynomials $A_n(U)$ that are given by;

$$A_n(U_0,U_1,U_2,......,U_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n U_n\right)\right]_{\lambda=0}\quad ,\ n=0,1,2,.... \tag{7}$$

Substituting Eq. (5), and Eq. (6), in Eq. (4), we get;

$$\sum_{n=0}^{\infty}U_n(x,t)=G(x,t)-S^{-1}\left[u^2\,S\left[R\sum_{n=0}^{\infty}U_n(x,t)+\sum_{n=0}^{\infty}A_n(U)\right]\right] \tag{8}$$

So that the recursive relation is given by;

$$U_0(x,t)=G(x,t),$$
$$U_{k+1}(x,t)=-S^{-1}\left[u^2\,S[RU_k+A_k]\right],\ k\geq 0. \tag{9}$$

5.1.2: Basic Idea of (STM)

The Sumudu transform is an integral transform similar to the Laplace transform, introduced in the early 1990s by Watugala [1] to solve linear differential equations and control engineering problems.

Note that these definitions will use in this section**.**

Definition (5.1.1): The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(u)$, defined by:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u} \exp\left[-\frac{t}{u}\right] f(t)\, dt \tag{10}$$

Or

$$F(u) = S[f(t)] = \int_0^\infty \exp\,[-t] f(u\,t)\, dt \tag{11}$$

Definition (5.1.2): The double Sumudu transform of a function $f(x,t)$, defined for all real numbers $(t \geq 0, x \geq 0)$, is defined by:

$$S[f(x,t)] = \int_0^\infty \frac{1}{u} \exp\left[-\frac{t}{u}\right] f(x,t)\, dt \tag{12}$$

In the same line of ideas, the Sumudu transform of the second partial derivative with respect to *t* is of the form [3],

$$S\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{u} F(x,u) - \frac{1}{u} F(x,0)$$

$$S\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{u^2} F(x,u) - \frac{1}{u^2} F(x,0) - \frac{1}{u} \frac{\partial F(x,0)}{\partial t} \tag{13}$$

Similarly, the Sumudu transform of the second partial derivative with respect to *x* is of the form [3],

$$S\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx} F(x,u)$$

$$S\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2} F(x,u) \tag{14}$$

5.1.3: Application

In this section, we demonstrate the analysis of two methods by applying two methods to the following of two partial differential equations.

Example (5.1.3): Consider the following one – dimensional heat equation:

$$\frac{1}{4}u_{xx} = u_t \ ; \tag{15}$$

With the initial condition:

$$u(x,0) = 2\sin\frac{\pi}{2}x \ ; \tag{16}$$

I: Using (ADSTM)

Following discussion presented above, we obtain the recursive relation:

$$\begin{aligned} U_0 &= 2\sin\frac{\pi}{2}x, \\ U_1 &= \frac{1}{4}S^{-1}\left[u\,S\left[(U_0)_{xx}\right]\right] = -\frac{\pi^2 t}{8}\sin\frac{\pi}{2}x, \\ U_2 &= \frac{1}{4}S^{-1}\left[u^2\,S\left[(U_1)_{xx}\right]\right] = \frac{\pi^4 t^2}{256}\sin\frac{\pi}{2}x. \end{aligned} \tag{17}$$

And so on. The solution in a series form given by:

$$U(x,t) = \sin\frac{\pi}{2}x\left(2 - \frac{\pi^2 t}{8} + \frac{\pi^4 t^2}{256} - \dots\right) \tag{18}$$

And in a closed form of,

$$U(x,t) = 2\sin\frac{\pi}{2}x\, e^{-\frac{\pi^2}{16}t} \ . \tag{19}$$

II: Using (STM)

Taking the Sumudu transform of (15) and using initial condition (16) we get:

$$\frac{d^2}{dx^2}U(x,u) - \frac{4}{u}U(x,u) = -\frac{8}{u}\sin\frac{\pi}{2}x \tag{20}$$

This is the second order differential equation. First, we find the homogeneous solution;

$$U_c(x,u) = Ae^{\frac{2}{\sqrt{u}}x} + Be^{-\frac{2}{\sqrt{u}}x} \tag{21}$$

Using boundary conditions:

$$U(0,t)=0 \quad \Rightarrow U(0,u)=0$$
$$U(2,t)=0 \quad \Rightarrow U(2,u)=0$$

This gives:

$$0 = A+B \rightarrow (i)$$
$$0 = Ae^{\frac{4}{\sqrt{u}}} + Be^{-\frac{4}{\sqrt{u}}} \rightarrow (ii)$$

From (i) and (ii), we have only a trivial solution $A = B = 0$.

Second, we find particular solution;

$$U_p(x,u) = -\frac{8}{u}.\frac{\sin\frac{\pi}{2}x}{D^2 - \frac{4}{u}} = 32.\left[\frac{1}{\pi^2 u + 16}\right]\sin\frac{\pi}{2}x \tag{22}$$

The general solution is:

$$U(x,u) = U_c + U_p = 32.\left[\frac{1}{\pi^2 u + 16}\right]\sin\frac{\pi}{2}x \tag{23}$$

Taking the inverse Sumudu transform we get:

$$U(x,t) = 2\sin\frac{\pi}{2}xe^{-\frac{\pi^2}{16}t} \quad . \tag{24}$$

Example (5.1.4): Consider the following wave equation:

$$u_{tt} - 4u_{xx} = 0 \; ; \tag{25}$$

With the initial conditions:

$$u(x,0) = \sin\pi x \;\; , \; u_t(x,0) = 0 \;\; . \tag{26}$$

I: Using (ADSTM)

Proceeding as before, we obtain:

$$u_0 = \sin \pi x,$$
$$u_1 = 4S^{-1}\left(L_{xx}(u_0)\right) = -2\pi^2 t^2 \sin \pi x, \tag{27}$$
$$u_2 = 4S^{-1}\left(L_{xx}(u_1)\right) = \frac{2\pi^4 t^4}{3} \sin \pi x \ .$$

And so on. The solution in a series given by:

$$u(x,t) = \sin \pi x \left(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \ldots \right); \tag{28}$$

In a closed form:

$$u(x,t) = \sin \pi x \cos 2\pi t \ . \tag{29}$$

II: Using (STM)

Taking the Sumudu transform of (25) and using initial condition (26) we get:

$$u^2 \frac{d^2}{dx^2} U(x,u) + U(x,u) = u \cos x \ ; \tag{30}$$

This is the second order differential equation which has the particular solution in the form:

$$U(x,u) = \frac{u \cos x}{u^2 D^2 + 1} = \frac{u \cos x}{-u^2 + 1} \ . \tag{31}$$

If we take the inverse Sumudu for Eq. (31), we obtain the solution of Eq. (25) in the form:

$$U(x,t) = \cos x \sinh t \ . \tag{32}$$

Notes on (STM) and (ADSTM):

From the previous analysis, we can observe that:

The two methods are powerful and efficient. Adomian decomposition Sumudu transform method provides the components of the exact solution, where these components must follow the summation given in (5). However, application of the Sumudu transform to the solution of linear partial differential equations has been demonstrated.

5.2: Comparison of Adomian Decomposition Method and Sumudu Transform Method with Homotopy Perturbation Method for Solving Nonlinear Partial Differential Equations

In this section, the main objective is to introduce a comparative study to solve nonlinear partial differential equations using Adomian decomposition Sumudu transform method and homotopy perturbation method.

5.2.1: Basic Idea of HPM

Consider the following general nonlinear differential equation,

$$u - N(u) = f \,, \tag{33}$$

Where N is nonlinear operator from Hilbert space H to H , u is an unknown function, and f is a known function in H .

The homotopy perturbation method u as a series with components u_n, and $N(u)$ as a series with components H_n, homotopy polynomials, which can be calculated using the formula:

$$H_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} u_i \lambda^i\right)\Bigg|_{\lambda=0} \tag{34}$$

To illustrate the homotopy perturbation method (HPM), we consider (33) as;

$$L(v) = v(x) - f(x) - N(v) = 0 \tag{35}$$

with solution $u(x)$. As a possible remedy, we can define homotopy $H(v, p)$ as follows:

$$H(v,0) = F(v) \quad , \quad H(v,1) = L(v)$$

Where $F(v)$is an integral operator with known solutions, which can be obtained easily. Typically, we may choose a convex homotopy in the form;

$$H(v,p) = (1-p)F(v) + pL(v) = 0 \tag{36}$$

And continuously trace an implicitly defined curve from a starting point $H(v_0,0)$to a solution function $H(u,0)$. The embedding parameter p monotonically increase from zero in the unit as the trivial problem $F(v)=0$ is continuously deformed to the original problem.

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots \qquad (37)$$

When $p \to 1$, Eq. (36) corresponds to Eq. (35) and (37) becomes the approximate solution of Eq. (35), i.e.

$$v = \lim_{p \to 1} v_0 + v_1 + v_2 + v_3 + \cdots \qquad (38)$$

5.2.2: Application

In this section, we demonstrate the analysis of two methods by applying two methods to the following two Kortewege – deVries (KdV) partial differential equations.

Example (5.2.5): Consider the following inhomogeneous nonlinear KdV equation [45]:

$$u_t + uu_x + u_{xxx} = \sin x + t\cos x(t\sin x - 1) \; ; \qquad (39)$$

With the initial condition:

$$u(x,0) = 0 \; . \qquad (40)$$

I: Using (HPM)

To solve equation (39)-(40) by homotopy perturbation method, we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p\left(-u\,\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} + \sin x + t\cos x(t\sin x - 1) - \frac{\partial u_0}{\partial t} \right) \qquad (41)$$

Assume the solution of Eq. (41) to be in the form:

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots \qquad (42)$$

Substituting (42) into (41) and comparing coefficients of terms with identical powers of p, leads to:

$$p^0 : \frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y} = 0$$

$$p^1 : \frac{\partial v_1}{\partial t} = -u_0\,\frac{\partial u_0}{\partial x} - \frac{\partial^3 u_0}{\partial x^3} + \sin x + t^2 \cos x \sin x - t\cos x - \frac{\partial u_0}{\partial t}, \qquad (43)$$

.

.

The given initial value admits the use of:

$$u_0(x,0) = 0 \tag{44}$$

The solution reads:

$$u_0(x\,,\,t) = 0$$

$$u_1(x\,,\,t) = t\sin x + \frac{t^3}{3}\sin x\cos x - \frac{t^2}{2}\cos x\,, \tag{45}$$

$$u_2(x\,,\,y) = -\frac{t^3}{3}\sin x\cos x + \frac{t^2}{2}\cos x - \frac{t^4}{3}\sin^2 x - \frac{t^4}{3}\sin^3 x + \ldots.$$

Examining the components u_1 and u_2 in Eq. (45), we can easily observe that the last two terms in u_1 and the first two terms in u_2 are the self-canceling (noise terms). Hence, the non-noise terms in u_1 yields the exact solution of equations (39)-(40), given by:

$$u(x,t) = t\sin x \quad . \tag{46}$$

Notes on (HPM):

From the previous analysis, we can observe that:

- **HPM** can be applied it to various nonlinear problems. The main disadvantage is that we should suitably choose an initial guess.
- **HPM** needs some modification to the rapid convergence of the series solution.

To overcome these disadvantages of **HPM**, the following **ADSTM** method is suggested.

II: Using (ADSTM)

By taking Sumudu transform for (39) and using (40) we obtain:

$$S[U(x,t)] = u\sin x + 2u^3\cos x\sin x - u^2\cos x - u\,S[UU_x + U_{xxx}]\,. \tag{47}$$

Applying S^{-1} to both sides of (47) we obtain;

$$U(x,t)=t\sin x+\frac{t^3}{3}\cos x\sin x-\frac{t^2}{2}\cos x-S^{-1}\left[u\,S\left[UU_x+U_{xxx}\right]\right] \;. \quad (48)$$

Substituting;

$$U(x,t)=\sum_{n=0}^{\infty}U_n(x,t)\;; \quad (49)$$

And the nonlinear terms of;

$$UU_x=\sum_{n=0}^{\infty}A_n\;,\;\;. \quad (50)$$

Into (48) gives;

$$\sum_{n=0}^{\infty}U_n(x,t)=t\sin x+\frac{t^3}{3}\cos x\sin x-\frac{t^2}{2}\cos x-S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty}A_n+\left(\sum_{n=0}^{\infty}U_n(x,t)\right)_{xxx}\right]\right] \quad (51)$$

This gives the modified recursive relation;

$$\begin{aligned}&U_0(x,t)=t\sin x\;,\\&U_1(x,t)=\frac{t^3}{3}\cos x\sin x-\frac{t^2}{2}\cos x-S^{-1}\left[u\,S\left[A_0+(U_0)_{xxx}\right]\right]\\&U_{k+1}(x,t)=-\;S^{-1}\left(A_k+U_k\right),\;k\geq 1.\end{aligned} \quad (52)$$

The first few of the components are given by;

$$\begin{aligned}&U_0(x,t)=t\sin x\;,\\&U_1(x,t)=\frac{t^3}{3}\cos x\sin x-\frac{t^2}{2}\cos x-S^{-1}\left[u\,S\left[A_0+(U_0)_{xxx}\right]\right]=0\;,\\&U_{k+1}(x,t)=0\;,\quad k\geq 1.\end{aligned} \quad (53)$$

The solution in a closed form is given by;

$$U(x,t)=t\sin x\;\;. \quad (54)$$

Example (5.2.6): Consider the following inhomogeneous nonlinear KdV equation [45]:

$$u_t+uu_x+u_{xxx}=(xt+\sin x)(t+\cos x)+x+\cos x\;; \quad (55)$$

With the initial condition:

$$u(x,0)=\sin x\;. \quad (56)$$

I: Using (HPM)

Using a homotopy perturbation method like in **Example (5.2.5),** we obtain the following components:

$$u_0(x\,,t)=\sin x$$
$$u_1(x\,,t)=xt+\frac{1}{3}xt^3+\frac{1}{2}xt^2\cos x+t\cos x+t\sin x\cos x+\frac{1}{2}t^2\sin x\ , \tag{57}$$
$$u_2(x\,,t)=-t\cos x\ -t\sin x\cos x-\ \frac{1}{2}t^2\sin x-\ldots\,.$$

It is obvious that the last three terms in u_1 and the first three terms in u_2 are the self-canceling (noise terms). Keeping the remaining non-noise terms in u_1 leads to the exact solution of equations (55)-(56), given by:

$$u(x,t)=\ xt\ +\ \sin x \qquad . \tag{58}$$

II: Using (ADSTM)

Proceeding as in **Example (5.2.5)**, Eq. (55) becomes:

$$\sum_{n=0}^{\infty}U_n(x,t)=\ xt\ +\sin x\ +\ \frac{1}{3}xt^3\ +\frac{1}{2}xt^2\cos x\ +\ t\cos x\ +\ t\sin x\cos x\ +\ \frac{1}{2}t^2\sin x-$$
$$-\ S^{-1}\left[u\,S\left[\ \sum_{n=0}^{\infty}A_n\ +\left(\sum_{n=0}^{\infty}U_n(x,t)\right)_{xxx}\ \right]\right] \tag{59}$$

The modified decomposition method admits the of a modified recursive relation given by:

$$U_0(x,t)=\ xt\ +\ \sin x\ ,$$
$$U_1(x,t)=\ \frac{1}{3}xt\ +\ \frac{1}{2}xt^2\cos x\ +\ t\cos x\ +\ t\sin x\cos x\ +\ \frac{1}{2}t^2\sin x-$$
$$-\ S^{-1}\left[u\,S\left[\ A_0\ +\ \left(U_0(x,t)\right)_{xxx}\ \right]\right]\ , \tag{60}$$
$$U_{k+1}(x,t)=-\ \ S^{-1}\left[u\,S\left[\ A_k\ +\ \left(U_k(x,t)\right)_{xxx}\ \right]\right]k\geq 1.$$

Consequently, we obtain:

$$U_0(x,t)\ \ =xt+\sin x\ ,$$
$$U_1(x,t)\ \ =\ 0\ , \tag{61}$$
$$U_{k+1}(x,t)=\ \ 0,\ k\geq 1.$$

In a few of Eq. (61), the exact solution is given by:

$$U(x,t) = xt + \sin x \quad . \tag{62}$$

Remarks:

In this section, we accurately employed the modified decomposition method that accelerates to the rapid convergence of the series solution. The comparison with the homotopy perturbation method (HPM), the decomposition Sumudu transforms method (ADSTM) gives better performance in many cases, and this implies the decomposition Sumudu transforms method an advantage over the homotopy perturbation method.

5.3: A Comparative Study Numerical Methods for Solving Integro – Differential Equations

Our aim of this section, is to introduce a comparative study to solve integro-differential equations by using different numerical methods, namely; the Adomian decomposition Sumudu transform method (ADSTM), the homotopy perturbation method (HPM), the Adomian decomposition Sumudu transform Sumudu method with the Pade approximant (ADST -PA method), and the variational iteration method (VIM).

In the present study, we consider the nonlinear integro-differential equation of the following type [51]:

$$u'(x) = f(x) + \int_0^x K(t, u(t), u'(t))dt , \tag{63}$$

With the initial condition;

$$u(0) = \alpha \ , \ \ 0 \le x \le 1 . \tag{64}$$

Where $f(x)$ is known as the source term and $K(t, u(t), u'(t))$ is a linear or nonlinear function depending on the problem discussed.

5.3.1: Basic Idea of (ADSTM)

In this section, Adomian decomposition Sumudu transform method is applied to the following classes of nonlinear integro-differential equation (63).

The method consists of first applying the Sumudu transformation of both sides of Eq. (63);

$$S[U'(x)] = S[f(x)] + S\left[\int_0^x K(t,u(t),u'(t))dt\right]. \tag{65}$$

Using the formulas of the Sumudu transform, we get;

$$u^{-1}\ S[U(x)] - u^{-1}U(0) = S[f(x)] + S\left[\int_0^x K(t,u(t),u'(t))dt\right]. \tag{66}$$

Using the initial condition (64), we have;

$$S[U(x)] = \alpha + u\,S[f(x)] + u\,S\left[\int_0^x K(t,u(t),u'(t))dt\right]. \tag{67}$$

In the Sumudu decomposition method we assume the solution as an infinite series, given as follows;

$$U = \sum_{n=0}^{\infty} U_n\ , \tag{68}$$

where the terms u_n are to be recursively computed. Also the nonlinear term $K(t,u(t),u'(t))$ is decomposed as an infinite series of Adomian polynomials:

$$K(t,u(t),u'(t)) = \sum_{n=0}^{\infty} A_n\ , \tag{69}$$

Where $A_n = A_n(u_1, u_2, u_3, ..., u_n)$ are determined by the following recursive relation:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[\sum_{i=0}^{\infty}(\lambda^i y_i)\right]\right]_{\lambda=0}. \tag{70}$$

Using (68) and (69), we rewrite (67) as;

$$S\left[\sum_{n=0}^{\infty} U_n\right] = \alpha + u\,S[f(x)] + u\,S\left[\int_0^x \left[\sum_{n=0}^{\infty} A_n\right]dt\right]. \tag{71}$$

Applying the linearity of the Sumudu transform, we have;

$$S\left[\sum_{n=0}^{\infty} U_n\right] = \alpha + u\,S[f(x)] + u\left[\int_0^x \sum_{n=0}^{\infty} S[A_n]dt\right]. \tag{72}$$

Now, we define the following iterative algorithm:

$$\begin{aligned} S[U_0] &= \alpha + u\,S[f(x)], \\ S[U_{k+1}] &= u\,S\left[\int_0^x A_k\ dt\right],\ k \geq 0. \end{aligned} \tag{73}$$

As the result, the components $U_0, U_1, U_2, U_3, ..., U_n$ are identified and the series solution is thus entirely determined. However, in many cases the exact solution in the closed form may also be obtained.
From a numerical point of view, the approximation;

$$U(x) = \lim_{n \to \infty} [\phi_n] \quad , \tag{74}$$

Where

$$\phi_n = \sum_{k=0}^{n-1} U_k(x) \quad , \tag{75}$$

Used in the Sumudu decomposition scheme for computing the approximate solution. It is also clear that a better approximation can be evaluated more components of the series solution (68) of $U(x)$.

5.3.2: Basic Idea of The Pade/ Approximant

Here we will investigate the construction of the Pade/ approximates for the functions studied. The main advantage of the Pade/ approximation gives a better approximation of the function than truncating its Taylor series.
The Pade/ approximation of a function, symbolized by [m / n], is a rational function defined by;

$$[m/n] = \frac{a_0 + a_1 x + a_2 x^2 + ... + a_m x^m}{1 + b_1 x + b_2 x^2 + ... + b_n x^n} \tag{76}$$

Where we considered $b_0 = 1$, and numerator, denominator have no common factors.
In The (ADSTM-PA method) we use the method of the Pade/ approximation as an after – treatment method to the solution obtained by the Adomian decomposition Sumudu transform method. This after – treatment method improves the accuracy of the proposed method.

5.3.3: Basic Idea of The Homotopy Perturbation Method (HPM)

To explain (HPM), we consider (63) as;

$$L(u) = u'(x) - f(x) - \int_0^x K(t, u(t), u'(t))dt = 0 , \tag{77}$$

With solution $f(x)$.Now, we can define homotopy $H(u,p)$ by;

$$H(u,0)=F(u) \quad , \quad H(u,1)=L(u) \ , \tag{78}$$

Where $F(u)$ is a functional operator with a solution v_0, obtained easily. Now, we choose a convex homotopy by;

$$H(u,p)=(1-p)F(u)+p\,L(u)=0 \quad . \tag{79}$$

And continuously trace an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. Here the parameter p is monotonically increasing from zero to unit along – with the trivial problem $F(u)=0$ is continuously deformed to the original problem $L(u)=0$.

The (HPM) uses the homotopy parameter p as an expending parameter to obtain;

$$u=v_0+p\,v_1+p^2 v_2+p^3 v_3+\dots \quad , \tag{80}$$

When $p\to 1$, Eq. (80) becomes the approximate solution of (13), i.e.

$$f=\lim_{p\to 1} v_0+v_1+v_2+\dots \quad . \tag{81}$$

Series (79) is convergent for most cases, and the rate of convergence depends on $L(u)$.

5.3.4: The Variational Iteration Method (VIM)

To clarify the basic ideas of (VIM), we consider Eq. (63) as correction functional as follows;

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\zeta)\left(\left(u_n'(\zeta)-f(\zeta)-\int_0^\zeta K(r,u(r),u'(r))dr\right)\right)d\zeta \ . \tag{82}$$

Where λ is general Lagrange multiplier which can be identified optimally via integrated by parts. The successive approximations $u_{n+1}(x)$, $n\geq 0$ for the solution $u(x)$ will be readily obtained upon using the Lagrange multiplier and by using the selected function. Consequently, the exact solution may be obtained by using:

$$u(x)=\lim_{n\to\infty} u_n \quad .$$

5.3.5: Application

In this section, we demonstrate the analysis of all the numerical methods by applying the methods to the following two integro- differential equations. A comparison is also given in the forms of graphs and tables, presented here.

Example (5.3.7): Consider the following integro- differential equation [51]:

$$U'(x) = -1 + \int_0^x U^2(t)\,dt\,, \tag{83}$$

With the initial condition;

$$U(0) = 0\ ,\ \ 0 \le x \le 1\ . \tag{84}$$

I: Use (ADSTM)

Taking the Sumudu transform of both sides of (83) gives;

$$u^{-1}\ S[U(x)] - u^{-1}U(0) = -1 + S\left[\int_0^x U^2(t)\,dt\right]. \tag{85}$$

Using the initial condition (84), we have;

$$S[U(x)] = -u + u\,S\left[\int_0^x U^2(t)\,dt\right]. \tag{86}$$

By the assumption (68) and (69), we rewrite (86) as;

$$S\left[\sum_{n=0}^{\infty} U_n\right] = -u + u\,S\left[\int_0^x \left[\sum_{n=0}^{\infty} A_n\right] dt\right]\ . \tag{87}$$

where the nonlinear term $K(t, u(t), u'(t)) = U^2$ is decomposed in terms of the Adomian polynomials as suggested in (69). Few terms of the Adomian polynomials for U^2 are given as follows:

$$A_0 = U_0^{\,2}\,,$$
$$A_1 = 2U_0U_1\,,$$
$$A_2 = 2U_0U_2 + U_1^{\,2}\,,$$
$$A_3 = 2U_0U_3 + 2U_1U_2\,.$$

And so on. Following the Adomian decomposition Sumudu transform method, we define an iterative scheme;

$$S[U_0] = -u\ ,$$
$$S[U_{k+1}] = u\, S\left[\int_0^x A_k\, dt\right]\ ,\ k \geq 0. \tag{88}$$

Applying the inverse Sumudu transform, finally we get the value of $U_0, U_1, U_2, \ldots$.

$$\begin{aligned}
U_0 &= S^{-1}[-u] = -x, \\
U_1 &= S^{-1}\left[u\, S\left[\int_0^x A_0\, dt\right]\right] = \frac{x^4}{12}, \\
U_2 &= S^{-1}\left[u\, S\left[\int_0^x A_1\, dt\right]\right] = -\frac{x^7}{252}, \\
U_3 &= S^{-1}\left[u\, S\left[\int_0^x A_2\, dt\right]\right] = \frac{x^{10}}{6048}, \\
U_4 &= S^{-1}\left[u\, S\left[\int_0^x A_3\, dt\right]\right] = -\frac{x^{13}}{157248}.
\end{aligned} \tag{89}$$

Similarly, we can also find other components. Finally, the solution takes the following form;

$$U(x) = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{x^{13}}{157248} + \ldots\ . \tag{90}$$

Notes on (ADSTM):

From the previous analysis, we can observe that:

1. **ADSTM** can obtain a series solution, not converge, which must be truncated. The truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method.

2. **ADSTM** needs some modification to overcome the Taylor series does not converge.

To overcome these disadvantages of **ADSTM**, the following **ADSTM -PA** method is suggested.

II: The Proposed (ADST -PA Method)

Here, we purpose to establish Pade approximant to give a better approximation of function truncating its Taylor series see **(section 3.9)** in chapter three.

The [m / n] Pade/ approximant of the infinite series (28), with $m \geq 4$ and $n \geq 4$, which gives the following fraction approximation to the solution:

$$U(x) = \frac{-x + \frac{x^2}{28}}{1 + \frac{x^3}{21}} \quad . \tag{91}$$

III: Use Homotopy Perturbation Method (HPM)

The homotopy of (83) can be readily written in the form;

$$H(v, p) = v'(x) + 1 - p \int_0^x u^2(t) dt = 0 \quad , \tag{92}$$

This homotopy can continuously trace an implicitly defined curve from a starting point $H(v,0)$ to a solution function $H(v,1)$. Collecting the coefficients of like power of p and setting to be equal zero, we have;

$$\begin{aligned}
&p^0 : v_0{}'(x) + 1 = 0 \;\Rightarrow\; v_0(x) = -x \,, \\
&p^1 : v_1{}'(x) - \int_0^x v_0{}^2 \, dt = 0 \;\Rightarrow\; v_1(x) = \frac{x^4}{12} \,, \\
&p^2 : v_2{}'(x) - \int_0^x (2 v_0 v_1) dt = 0 \;\Rightarrow\; v_2(x) = -\frac{x^7}{252} \,, \\
&p^3 : v_3{}'(x) - \int_0^x (2 v_0 v_2 + v_1{}^2) dt = 0 \;\Rightarrow\; v_3(x) = \frac{x^{10}}{6048} \,, \\
&p^4 : v_4{}'(x) - \int_0^x (2 v_0 v_3 + 2 v_1 v_2) dt = 0 \;\Rightarrow\; v_4(x) = -\frac{x^{13}}{157248} \,.
\end{aligned} \tag{93}$$

Therefore, we obtain;

$$u(x) = \lim_{p \to 1} v_0 + v_1 + v_2 + \ldots \quad , \tag{94}$$

Or

$$u(x) = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{x^{13}}{157248} + \dots \quad . \tag{95}$$

Step Size	ADSTM	ADST-Pade	HPM	VIM
0	0	0	0	0
0.1250	- 0.1250	-0.1244	-0.1250	- 0.1250
0.2500	- 0.2497	-0.2476	-0.2497	- 0.2497
0.3750	- 0.3734	-0.3691	-0.3734	- 0.3734
0.5000	- 0.4948	-0.4882	-0.4948	- 0.4948
0.6250	- 0.6124	-0.6040	-0.6124	- 0.6124
0.7500	- 0.7242	-0.7155	-0.7242	- 0.7242
0.8750	- 0.8277	-0.8215	-0.8277	- 0.8277
1.000	- 0.9205	-0.9205	-0.9205	- 0.9205
1.1250	- 1.0000	-1.0112	-1.0000	- 1.0001
1.2500	- 1.0638	-1.0926	-1.0638	- 1.0640
1.3750	- 1.1097	-1.1635	-1.1097	- 1.1104
1.5000	- 1.1353	-1.2231	-1.1353	- 1.1376

Table 1: Comparison of (ADSTM), (HPM), (VIM) and (ADST -PA) for Example 1

The numerical results shown in **Table 1**: imply the effectiveness of numerical methods discussed here. These methods give highly accurate in the very little iteration.

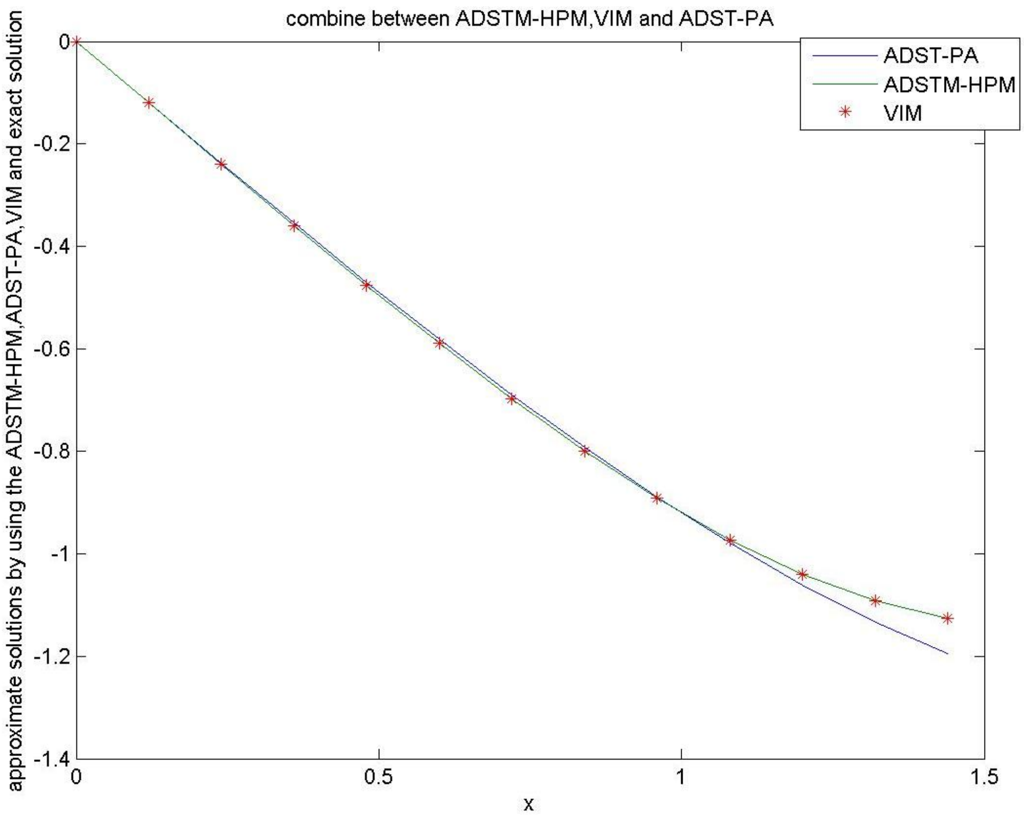

Fig 1: Combine between (ADSTM), (HPM), (VIM) and (ADST-PA), for Example 1.

IV: Use Variational Iteration Method (VIM)

The correction, functional for the equation (83) is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) + 1 - \int_0^t u_n^2(r)\,dr \right) dt, \tag{96}$$

We used here $\lambda = -1$ for first order integro- differential equation. We can use the initial condition to select $u_0(x) = u(0) = 0$. Using this selection into the correction functional gives the following successive approximations:

$$u_0 = 0 ,$$
$$u_1 = -x ,$$
$$u_2 = -x + \frac{x^4}{12} , \tag{97}$$
$$u_3 = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{12960} ,$$
$$u_4 = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{37\,x^{13}}{7076160} + .\frac{109x^{16}}{914457600} + .. \; .$$

And so on for other approximations. The (VIM) admits the use of:

$$u(x) = \lim_{n\to\infty} u_n \quad , \tag{98}$$

This gives the following approximation solution:

$$u(x) = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{37\,x^{13}}{7076160} + .\frac{109x^{16}}{914457600} + ... \; . \tag{99}$$

Notes on (VIM):

From the previous analysis, we can observe that:

.VIM can obtain a series solution, not exactly like Adomian decomposition method. The VIM may not lead to faster convergence (repeated calculations) in each step.

Example (5.3.8): Consider the following integro- differential equation [51]:

$$U'(x) = 1 + \int_0^x U(t)U'(t)dt , \tag{100}$$

Given the initial condition;

$$U(0) = 0 \ , \ \ 0 \le x \le 1 . \tag{101}$$

With the exact solution:

$$U(x) = \sqrt{2}\ \tan\left(\frac{x}{\sqrt{2}}\right) . \tag{102}$$

I: Use (ADSTM)

Taking the Sumudu transform of both sides of (100) gives;

$$u^{-1}\ S[U(x)] - u^{-1}U(0) = 1 + S\left[\int_0^x U(t)U'(t)\,dt\right]. \tag{103}$$

Using the initial condition (101), we have;

$$S[U(x)] = u\ + u\,S\left[\int_0^x U(t)U'(t)\,dt\right]. \tag{104}$$

By the assumption (68) and (69), we rewrite (104) as;

$$S\left[\sum_{n=0}^{\infty} U_n\right] = u\ + u\,S\left[\int_0^x \left[\sum_{n=0}^{\infty} B_n\right] dt\right]. \tag{105}$$

where the nonlinear term $K(t, u(t), u'(t)) = U(t)U'(t)$ is decomposed in terms of the Adomian polynomials as suggested in (69). We have a few terms of the Adomian polynomials of $u(t)u'(t)$ which are given by:

$$\begin{aligned}
B_0 &= U_0 U_{0t}\,,\\
B_1 &= U_0 U_{1t} + U_1 U_{0t}\,,\\
B_2 &= U_0 U_{2t} + U_1 U_{1t} + U_2 U_{0t}\,,\\
B_3 &= U_0 U_{3t} + U_3 U_{0t} + U_1 U_{2t} + U_2 U_{1t}\,.
\end{aligned}$$

And so on. Following the Sumudu transform decomposition method, we define an iterative scheme;

$$\begin{aligned}
S[U_0] &= u\,,\\
S[U_{k+1}] &= u\,S\left[\int_0^x B_k\,dt\right],\ k \geq 0.
\end{aligned} \tag{106}$$

Applying the inverse Sumudu transform, we can evaluate $u_0, u_1, u_2, \ldots$ as:

$$\begin{aligned}
U_0 &= S^{-1}[u] = x,\\
U_1 &= S^{-1}\left[u\,S\left[\int_0^x B_0\,dt\right]\right] = \frac{x^3}{6}\,,\\
U_2 &= S^{-1}\left[u\,S\left[\int_0^x B_1\,dt\right]\right] = \frac{x^5}{30}\,,\\
U_3 &= S^{-1}\left[u\,S\left[\int_0^x B_2\,dt\right]\right] = \frac{17x^7}{2520}\,.
\end{aligned} \tag{107}$$

Similarly, we can also find other components. Finally, the solution takes the following form;

$$U(x) = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{17x^7}{2520} + \dots \quad . \tag{108}$$

II: The Proposed (ADST -PA Method)

The [m / n] Pade/ approximant of the infinite series (108), with $m \geq 4$ and $n \geq 4$, which gives the following fraction approximation to the solution:

$$U(x) = \frac{x - \frac{x^3}{21}}{1 - \frac{3x^2}{14} + \frac{x^4}{420}} \quad . \tag{109}$$

III: Use Homotopy Perturbation Method (HPM)

The homotopy of (100) can be readily written in the form;

$$H(v,p) = v'(x) - 1 - p\int_0^x u(t)u'(t)dt = 0 \quad , \tag{110}$$

Proceeding as before in **Example (5.4.7)**, we obtain;

$$\begin{aligned}
&p^0 : v_0'(x) - 1 = 0 \Rightarrow v_0(x) = x ,\\
&p^1 : v_1'(x) - \int_0^x v_0 v_{0t}\, dt = 0 \Rightarrow v_1(x) = \frac{x^3}{6} ,\\
&p^2 : v_2'(x) - \int_0^x (v_0 v_{1t} + v_1 v_{0t})dt = 0 \Rightarrow v_2(x) = \frac{x^5}{30} ,\\
&p^3 : v_3'(x) - \int_0^x (v_0 v_{2t} + v_1 v_{1t} + v_2 v_{0t})dt = 0 \Rightarrow v_3(x) = \frac{17x^7}{2520} .
\end{aligned} \tag{111}$$

Therefore, we obtain;

$$u(x) = \lim_{p \to 1} v_0 + v_1 + v_2 + \dots \quad , \tag{112}$$

Or

$$u(x) = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{17x^7}{2520} + \dots \quad . \tag{113}$$

Step Size	Exact Sol.	Error (ADSTM)	Error (ADST-PA)	Error (HPM)
0.0000	0.0000	0.0000	0.0000	0.0000
0.1250	0.1253	0.0000	0.0000	0.0000
0.2500	0.2526	0.0000	0.0000	0.0000
0.3750	0.3840	0.0000	0.0000	0.0000
0.5000	0.5219	0.0001	0.0000	0.0001
0.6250	0.6691	0.0003	0.0000	0.0003
0.7500	0.8292	0.0010	0.0000	0.0010
0.8750	1.0069	0.0030	0.0000	0.0030
1.0000	1.2085	0.0081	0.0000	0.0081
1.1250	1.4431	0.0198	0.0000	0.0198
1.2500	1.7243	0.0452	0.0000	0.0452
1.3750	2.0737	0.0979	0.0000	0.0979
1.5000	2.5275	0.2051	0.0001	0.2051

Table 2: Comparison of (ADSTM), (HPM) and (ADST -PA method), for Example 2.

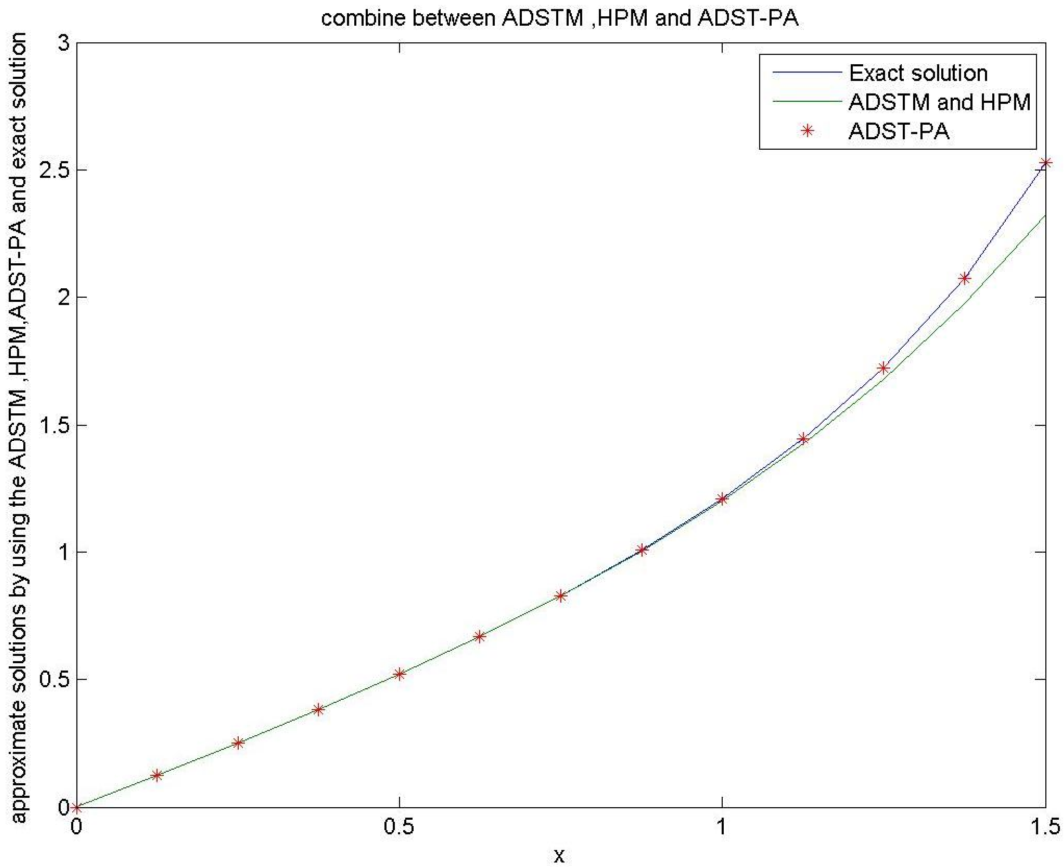

Fig 2: Combine between (ADSTM), (HPM) and (ADST-PA), for Example 2.

IV: Use Variational Iteration Method (VIM)

The correction, functional for the equation (100) is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 1 - \int_0^t u_n(r) u_n'(r)\, dr \right) dt, \tag{114}$$

Proceeding as before in **Example (5.3.7)**, we obtain;

$$u_0 = 0 ,$$
$$u_1 = x ,$$
$$u_2 = x + \frac{x^3}{6} , \tag{115}$$
$$u_3 = x + \frac{x^3}{12} + \frac{x^5}{30} + \frac{x^7}{504} ,$$
$$u_4 = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{17x^7}{2520} + \frac{19x^9}{22680} + \ldots .$$

And so on for other approximations. The (VIM) admits the use of:

$$u(x) = \lim_{n\to\infty} u_n , \tag{116}$$

This gives the following approximation solution:

$$u(x) = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{17x^7}{2520} + \frac{19x^9}{22680} + \ldots . \tag{117}$$

Step Size	Exact Sol.	Error (ADST-PA)	Error (VIM)
0.0000	0.0000	0.0000	0.0000
0.1250	0.1253	0.0000	0.0000
0.2500	0.2526	0.0000	0.0000
0.3750	0.3840	0.0000	0.0000
0.5000	0.5219	0.0000	0.0000
0.6250	0.6691	0.0000	0.0000
0.7500	0.8292	0.0000	0.0001
0.8750	1.0069	0.0000	0.0002
1.0000	1.2085	0.0000	0.0009
1.1250	1.4431	0.0000	0.0029
1.2500	1.7243	0.0000	0.0087
1.3750	2.0737	0.0000	0.0242
1.5000	2.5275	0.0001	0.0644

Table 3: Comparison of (ADST -PA method) and (VIM), for Example 2.

Numerical results shown in **Table 2, 3:** illustrate the importance of (ADST -PA method) over other numerical methods. In (ADST -PA method), we have used only 4 iterations and [4 /4] Pade/ approximation of the solution obtained by (ADSTM), (HPM) and (VIM).

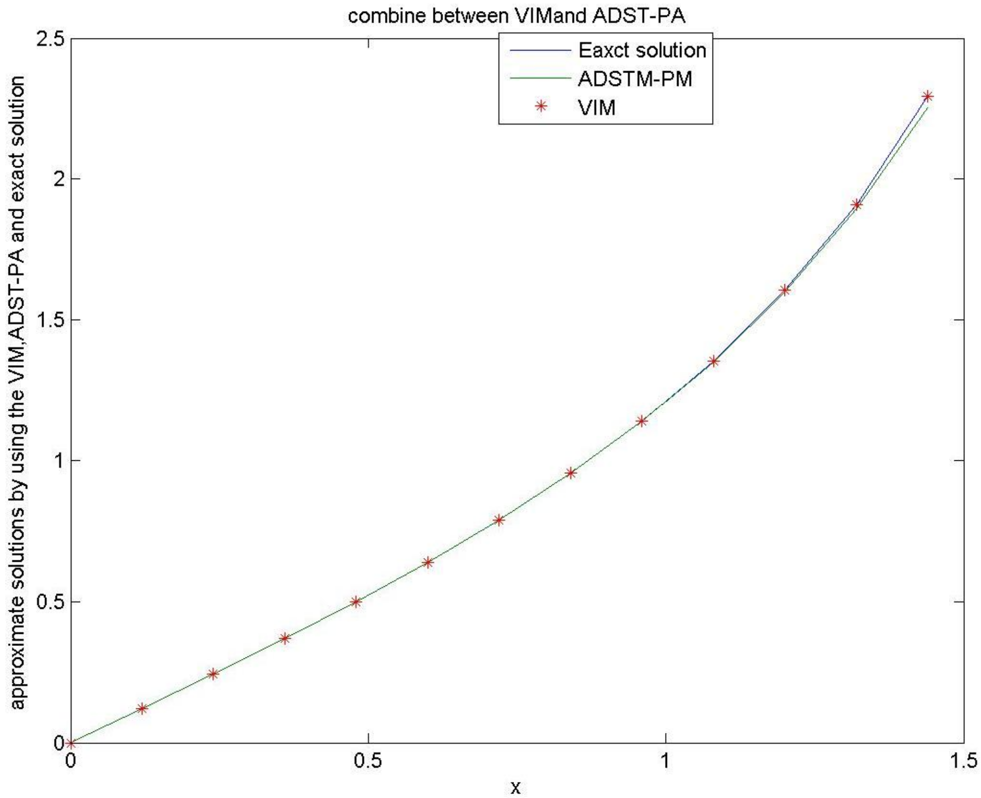

Fig 3: Combine between (ADST-PA) and (VIM), for Example 2.

Concluding Remarks:

In this section, we have studied a few recent familiar numerical methods for solving integro-differential equations. The numerical studies in this section showed that all the method gives highly accurate results for given equations. The (ADSTM), the (HPM) and the (VIM) are simple and easy. Despite this, they are not converging to a closed form. Since the method of the (ADSTM) is based on an approximation of the solution function in this study by the truncating of approximation the solution, this kind of approximation is an inaccurate solution, which will greatly restrict the application area of the method. To overcome these demerits, we use the Pade approximations. This fact is also verified by the second example given in the study.

Table: Laplace and Sumudu transform of some function

$f(t)$	$F(s)=L[f(t)]$	$F(u)=S[f(t)]$
1	$\frac{1}{s}$	1
t	$\frac{1}{s^2}$	u
$\frac{t^{n-1}}{(n-1)!} \quad n=1,2,\cdots$	$\frac{1}{s^n}$	u^{n-1}
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{u}}$
$2\sqrt{\frac{t}{\pi}}$	$1/s^{\frac{3}{2}}$	$\sqrt{u}$
$\frac{t^{a-1}}{\Gamma(a)}, a>0$	$\frac{1}{s^a}$	u^{a-1}
e^{at}	$\frac{1}{s-a}$	$\frac{1}{1-au}$
$t\,e^{at}$	$\frac{1}{(s-a)^2}$	$\frac{u}{(1-au)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{at}$, $n=1,2,\ldots$	$\frac{1}{(s-a)^n}$	$\frac{u^{n-1}}{(1-au)^n}$
$\frac{1}{\Gamma(k)}t^{k-1}e^{at}$, $k>0$	$\frac{1}{(s-a)^k}$	$\frac{u^{k-1}}{(1-au)^k}$
$\frac{1}{(a-b)}\left(e^{at}-e^{bt}\right)$, $a \neq b$	$\frac{1}{(s-a)(s-b)}$	$\frac{u}{(1-au)(1-bu)}$
$\frac{1}{(a-b)}\left(a\,e^{at}-b\,e^{bt}\right)$, $a \neq b$	$\frac{s}{(s-a)(s-b)}$	$\frac{1}{(1-au)(1-bu)}$
$\frac{1}{w}\sin wt$	$\frac{1}{s^2+w^2}$	$\frac{u}{1+u^2w^2}$
$\cos wt$	$\frac{s}{s^2+w^2}$	$\frac{1}{1+w^2u^2}$
$\frac{1}{a}\sinh at$	$\frac{1}{s^2-a^2}$	$\frac{u}{1-a^2u^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$	$\frac{1}{1-a^2u^2}$
$\frac{1}{w}e^{at}\sin wt$	$\frac{1}{(s-a)^2+w^2}$	$\frac{u}{(1-au)^2+w^2u^2}$

$e^{at}\cos wt$	$\dfrac{s-a}{(s-a)^2+w^2}$	$\dfrac{1-au}{(1-au)^2+w^2u^2}$
$\dfrac{1}{w^2}(1-\cos wt)$	$\dfrac{1}{s(s^2+w^2)}$	$\dfrac{u^2}{1+w^2u^2}$
$\dfrac{1}{w^3}(wt-\sin wt)$	$\dfrac{1}{s^2(s^2+w^2)}$	$\dfrac{u^3}{1+w^2u^2}$
$\dfrac{1}{2w^3}(\sin wt - wt\ \cos wt)$	$\dfrac{1}{(s^2+w^2)^2}$	$\dfrac{u^3}{(1+w^2u^2)^2}$
$\dfrac{t}{2w}\sin wt$	$\dfrac{s}{(s^2+w^2)^2}$	$\dfrac{u^2}{(1+w^2u^2)^2}$
$\dfrac{1}{2w}(\sin wt + wt\cos wt)$	$\dfrac{s^2}{(s^2+w^2)^2}$	$\dfrac{u}{(1+w^2u^2)^2}$
$\dfrac{1}{2k^2}\sin kt \sinh kt$	$\dfrac{s}{s^4+4k^4}$	$\dfrac{u^2}{1+4k^4u^4}$
$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$	$\dfrac{1}{s^4-k^4}$	$\dfrac{u^3}{1-k^4u^4}$
$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$	$\dfrac{s}{s^4-k^4}$	$\dfrac{u^2}{1-k^4u^4}$
$j_0(at)$	$\dfrac{1}{\sqrt{s^2+a^2}}$	$\dfrac{1}{\sqrt{1+a^2u^2}}$
$H(t-a)$	$\dfrac{1}{s}e^{-as}$	$e^{-\frac{a}{u}}$
$\delta(t-a)$	e^{-as}	$\dfrac{1}{u}e^{-\frac{a}{u}}$
$\dfrac{2}{t}(1-\cos wt)$	$\ln\left(\dfrac{s^2+w^2}{s^2}\right)$	$\dfrac{1}{u}\ln(1+w^2u^2)$
$\dfrac{2}{t}(1-\cosh at)$	$\ln\dfrac{s^2-a^2}{s^2}$	$\dfrac{1}{u}\ln(1-a^2u^2)$
$\dfrac{1}{t}\sin wt$	$\tan^{-1}\dfrac{w}{s}$	$\dfrac{1}{u}\tan^{-1} wu$

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