CHAPTER ONE

Algebra of Differential Forms

Section (1.1): Differential Forms in $\mathbb{R}^n$

The goal of this section is to define in $\mathbb{R}^n$ "field of alternate forms" that will be used later to obtain geometric results.

In order to fix the ideas, we will work initially with the 3-dimensinonal space $\mathbb{R}^3$.

Let p be a point of $\mathbb{R}^3$. The set of vectors $q - p, q \in \mathbb{R}^3$ (that have origin at p) will be called the *tangent space of* $\mathbb{R}^3$ at p and will be denoted by $\mathbb{R}^3_p$. The vectors $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ of the canonical basis of $\mathbb{R}^3_0$ will be indentified with their translates $(e_1)_p, (e_2)_p, (e_3)_p$ at the point p.

A *vector field* in $\mathbb{R}^3$ is a map v that associates to each point $p \in \mathbb{R}^3$ a vector $v(p) \in \mathbb{R}^3_p$. We can write v as

$$v(p) = a_1(p)e_1 + a_2(p)e_2 + a_3(p)e_3, \tag{1.1}$$

thereby defining three functions $a_i: \mathbb{R}^3 \longrightarrow \mathbb{R}, i = 1, 2, 3,$ that characterize the vector field v. We say that v is *differentiable* if the functions a_i are differentiable. To each tangent space $\mathbb{R}^3_p$ we can associate the *dual space* $\left(\mathbb{R}^3_p\right)^*$ which is the set of linear maps $\varphi: \mathbb{R}^3_p \longrightarrow \mathbb{R}$. A basis for $\left(\mathbb{R}^3_p\right)^*$ is obtained by taking $(dx_i)_p, i = 1, 2, 3,$ where $x_i: \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the map which assigns to each point its i^{th}-coordinate. The set

$$\left\{(dx_i)_p; i = 1, 2, 3\right\},$$

is in fact the dual basis of $\left\{(e_i)_p\right\}$ since

$$(dx_i)_p\left(e_j\right) = \frac{\partial x_i}{\partial x_j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases} \tag{1.2}$$

Definition (1.1.1):

A *field of linear forms* (or an *exterior form of degree 1*) in $\mathbb{R}^3$ is a map ω that an associates to each $p \in \mathbb{R}^3$ an element $\omega(p) \in \mathbb{R}_p^3$; ω can be written as

$$\omega(p) = a_1(p)(dx_1)_p + a_2(p)(dx_2)_p + a_3(p)(dx_3)_p,$$
$$\omega = \sum_{i=1}^{3} a_i \, dx_i \,, \tag{1.3}$$

where a_i is real functions in $\mathbb{R}^3$. If the functions a_i are differentiable, ω is called a *differential form of degree 1*.

Now let $\Lambda^2(\mathbb{R}_p^3)^*$ be the set of maps $\varphi: \mathbb{R}_p^3 \times \mathbb{R}_p^3 \longrightarrow \mathbb{R}$ that are bilinear (i.e., φ is linear in each variable) and alternate (i.e., $\varphi(v_1, v_2) = -\varphi(v_2, v_1)$). With the usual operations of functions, the set $\Lambda^2(\mathbb{R}_p^3)^*$ becomes a vector space.

When φ_1 and φ_2 belong to $(\mathbb{R}_p^3)^*$, we can obtain an element $\varphi_1 \wedge \varphi_2 \in \Lambda^2(\mathbb{R}_p^3)^*$ by setting

$$(\varphi_1 \wedge \varphi_2)(v_1, v_2) = \det(\varphi_i(v_j)), \tag{1.4}$$

then, the element $(dx_i)_p \wedge (dx_j)_p \in \Lambda^2(\mathbb{R}_p^3)^*$ will be denoted by $(dx_i \wedge dx_j)_p$. It is easy to see that the set $\{(dx_i \wedge dx_j)_p, i < j\}$ is a basis for $\Lambda^2(\mathbb{R}_p^3)^*$. Furthermore,

$$(dx_i \wedge dx_j)_p = -(dx_j \wedge dx_i)_p, \quad i \neq j, \tag{1.5}$$

and

$$(dx_i \wedge dx_i)_p = 0. \tag{1.6}$$

where the symbol " $\wedge$ " is called wedge product.

Definition (1.1.2):

A *field of bilinear alternating forms* or an *exterior form of degree* 2 in $\mathbb{R}^3$ is a correspondence ω that associates to each $p \in \mathbb{R}^3$ an element $\omega(p) \in \Lambda^2(\mathbb{R}_p^3)^*$; ω can be written in the form

$$\omega(p) = a_{12}(p)(dx_1 \wedge dx_2)_p + a_{13}(p)(dx_1 \wedge dx_3)_p + a_{23}(p)(dx_2 \wedge dx_3)_p$$

or

$$\omega = \sum_{i<j} a_{ij}\, dx_i \wedge dx_j\,, \qquad i,j = 1,2,3, \tag{1.7}$$

Where a_{ij} are real functions in $\mathbb{R}^3$. When the functions a_{ij} are differentiable ω is a *differential form of degree 2.*

We will now generalize the notion of differential form to $\mathbb{R}^n$, let $p \in \mathbb{R}^n$, $\mathbb{R}^n_p$ the tangent space of $\mathbb{R}^n$ at p and $\left(\mathbb{R}^n_p\right)^*$ its dual space. Let $\Lambda^k(\mathbb{R}^n_p)^*$ be the set of all k-linear alternating maps

$$\varphi\colon \underbrace{\mathbb{R}^n_p \times \ldots \times \mathbb{R}^n_p}_{k \text{ times}} \longrightarrow \mathbb{R}.$$

(Alternating means that φ changes sign with the interchange of two consecutive arguments). With the usual operations, $\Lambda^k(\mathbb{R}^n_p)^*$ is a vector space. Given $\varphi_1, \ldots, \varphi_k \in (\mathbb{R}^n_p)^*$, we can obtain an element $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k$ of $\Lambda^k(\mathbb{R}^n_p)^*$ by setting

$$(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k)(v_1, v_2, \ldots, v_k) = \det\left(\varphi_i(v_j)\right), \qquad i,j = 1, \ldots, k. \tag{1.8}$$

it follows from the properties of determinants that $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k$ is in fact k-linear and alternate. In particular, $\left(dx_{i_1}\right)_p \wedge \left(dx_{i_2}\right)_p \wedge \ldots \wedge (dx_k)_p \in \Lambda^k(\mathbb{R}^n_p)^*$, $i_1, i_2, \ldots, i_k = 1, \ldots, n$. We will denote this element by $\left(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}\right)_p$.

Note (1.1.3): The set

$$\left\{\left(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}\right)_p, i_1 < i_2 < \cdots < i_k, i_j \in \{1, \ldots, n\}\right\}, \tag{1.9}$$

is a basis for $\Lambda^k(\mathbb{R}^n_p)^*$.

Definition (1.1.4):

An *exterior* k*-form in* $\mathbb{R}^n$ is a map ω that associates to each $p \in \mathbb{R}^n$ an element $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)^*$; by Note (1.1.3), ω can be written as

$$\omega(p) = \sum_{i_1<\cdots<i_k} a_{i_1\ldots i_k}(p)\left(dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right)_p, \qquad i_j \in \{1, \ldots, n\} \qquad (1.10)$$

where $a_{i_1\ldots i_k}$ are real functions in $\mathbb{R}^n$. When the $a_{i_1\ldots i_k}$ are differentiable functions, ω is called a *differential* k*-form*.

For notational convenience, we will denote by I the k-tuple $(i_1, \ldots, i_k)$, $i_1 < \cdots < i_k, i_j \in \{1, \ldots, n\}$, and will use the following notation for ω:

$$\omega = \sum_I a_I \, dx_I. \qquad (1.11)$$

We also set the convention that a differential 0-form is a differentiable function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$.

Example (1.1.5): In $\mathbb{R}^4$ we have the following types of exterior forms (where a_i, a_{ij}, etc., are real functions in $\mathbb{R}^4$):

0-forms, functions in $\mathbb{R}^4$,

1-form:

$a_1dx_1 + a_2dx_2 + a_3dx_3 + a_4dx_4$,

2-forms:

$$a_{12}dx_1 \wedge dx_2 + a_{13}dx_1 \wedge dx_3 + a_{14}dx_1 \wedge dx_4 + a_{23}dx_2 \wedge dx_3 + a_{24}dx_2 \wedge dx_4 + a_{34}dx_3 \wedge dx_4 ,$$

3-forms:

$$a_{123}dx_1 \wedge dx_2 \wedge dx_3 + a_{124}dx_1 \wedge dx_2 \wedge dx_4 + a_{134}dx_1 \wedge dx_3 \wedge dx_4 + a_{234}dx_2 \wedge dx_3 \wedge dx_4,$$

4-forms,

$a_{1234}dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

From now on, we will restrict ourselves to differential k-forms and we will call them simply k-forms.

We are going to define some operations on k-forms in $\mathbb{R}^n$.

First, if ω and φ are two k-forms:

$$\omega = \sum_I a_I \, dx_I \, , \qquad \varphi = \sum_I b_I \, dx_I,$$

We can define their *sum*

$$\omega + \varphi = \sum_I (a_I + b_I) dx_I.$$

Next, if ω is a k-form and φ is an s-form, we can define their *exterior product* $\omega \wedge \varphi$, which is an $(s+k)$-form, as follows

Definition (1.1.6):

$$\begin{aligned} \omega &= \sum a_I \, dx_I \, , \qquad I = (i_1, \dots, i_k), i_1 < \cdots < i_k, \\ \varphi &= \sum b_J \, dx_J \, , \qquad J = (j_1, \dots, j_s), j_1 < \cdots < j_s. \end{aligned} \tag{1.12}$$

By definition,

$$\omega \wedge \varphi = \sum_{IJ} a_I b_J \, dx_I \wedge dx_J. \tag{1.13}$$

Example (1.1.7): Let

$$\omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3,$$

be a 1-form in $\mathbb{R}^3$ and

$$\varphi = x_1 dx_1 \wedge dx_2 + dx_1 \wedge dx_3,$$

be a 2-form in $\mathbb{R}^3$. Then, since

$$dx_i \wedge dx_i = 0,$$

and

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \qquad i \neq j,$$

we obtain

$$\begin{aligned}\omega \wedge \varphi &= x_2 dx_2 \wedge dx_1 \wedge dx_3 + x_3 x_1 dx_3 \wedge dx_1 \wedge dx_2 \\ &= (x_1 x_3 - x_2) dx_1 \wedge dx_2 \wedge dx_3.\end{aligned}$$

Remark (1.1.8):

The definition of exterior product is made in such a way that if $\varphi_1, \dots, \varphi_k$ are 1-forms, then the exterior product $\varphi_1 \wedge \dots \wedge \varphi_k$ agrees with the k-form previously defined by

$$\varphi_1 \wedge \dots \wedge \varphi_k(v_1, \dots, v_k) = \det\left(\varphi_i(v_j)\right). \tag{1.14}$$

This follows immediately from the definition.

The exterior product of forms in $\mathbb{R}^n$ has the following properties.

Proposition (1.1.9):

Let ω be a k-form, φ be an s-form and θ be an r-form.

Then:

a) $(\omega \wedge \varphi) \wedge \theta = \omega \wedge (\varphi \wedge \theta),$
b) $(\omega \wedge \varphi) = (-1)^{ks}(\varphi \wedge \omega),$
c) $\omega \wedge (\varphi + \theta) = \omega \wedge \varphi + \omega \wedge \theta,$ if $r = s$.

Proof:

(a) and (c) are straightforward. To prove (b), we write

$$\omega = \sum a_I \, dx_I\,, \qquad I = (i_1, \dots, i_k), \qquad i_1 < \cdots < i_k,$$
$$\varphi = \sum b_J \, dx_J\,, \qquad J = (j_1, \dots, j_s), \qquad j_1 < \cdots < j_s\,.$$

then

$$\begin{aligned}\omega \wedge \varphi &= \sum_{IJ} a_I b_J dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_s} \\ &= \sum_{IJ} b_J a_I (-1) dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} \wedge dx_{j_1} \wedge dx_{i_k} \wedge \dots \wedge dx_{j_s} \\ &= \sum_{IJ} b_J a_I (-1)^k dx_{j_1} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_s}.\end{aligned}$$

since J has s elements, we obtain, by repeating the above argument for each $dx_{ji}, ji \in J$,

$$\begin{aligned}\omega \wedge \varphi &= \sum_{IJ} b_J a_I (-1)^{ks} dx_{j_1} \wedge \dots \wedge dx_{j_s} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}. \\ &= (-1)^{ks} \varphi \wedge \omega.\end{aligned}$$

Remark (1.1.10):

Although $dx_i \wedge dx_i = 0$, it is not true that for any form $\omega \wedge \omega = 0$.

For instance, if

$$\omega = x_1 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_4,$$

then

$$\omega \wedge \omega = 2x_1 x_2 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

One of the most important features of differential forms is the way they behave under differentiable maps. Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable map. Then f induces a map f^* that takes k-forms in $\mathbb{R}^m$ into k-forms in $\mathbb{R}^n$ and is defined as follows. Let ω be a k-form in $\mathbb{R}^m$. By definition, $f^*\omega$ is the k-form in $\mathbb{R}^n$ given by

$$(f^*\omega)(p)(v_1, \dots, v_k) = \omega\big(f(p)\big)\left(df_p(v_1), \dots, df_p(v_k)\right). \tag{1.15}$$

Here $p \in \mathbb{R}^n, v_1, \dots, v_k \in \mathbb{R}_p^n$, and $df_p: \mathbb{R}_p^n \longrightarrow \mathbb{R}_{f(p)}^m$ is the differential of the map f at p. We set the convention that if g is the 0-form,

$$f^* g = g \circ f. \tag{1.16}$$

We are going to show that the operation f^* on forms is equivalent to "substitution of variables". Before that, we need some properties of f^*.

Proposition (1.1.11):

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable map, ω and φ be a k-forms on $\mathbb{R}^m$ and $g: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a 0-form on $\mathbb{R}^m$. Then:

a) $f^*(\omega + \varphi) = f^*\omega + f^*\varphi$,
b) $f^*(g\omega) = f^*(g) f^*(\omega)$,
c) If $\varphi_1, \dots, \varphi_k$ are 1-forms in $\mathbb{R}^m$, $f^*(\varphi_1 \wedge \dots \wedge \varphi_k) = f^*(\varphi_1) \wedge \dots \wedge f^*(\varphi_k)$.

Proof:

The proofs are very simple. Let $p \in \mathbb{R}^n$ and let $v_1, \dots, v_k \in \mathbb{R}_p^n$.

Then

a)
$$\begin{aligned} f^*(\omega + \varphi)(p)(v_1, \dots, v_k) &= (\omega + \varphi)\big(f(p)\big)\left(df_p(v_1), \dots, df_p(v_k)\right) \\ &= (f^*\omega)(p)(v_1, \dots, v_k) + (f^*\varphi)(p)(v_1, \dots, v_k) \\ &= (f^*\omega + f^*\varphi)(p)(v_1, \dots, v_k). \end{aligned}$$

b)
$$\begin{aligned} &f^*(g\omega)(p)(v_1, \dots, v_k) = (g\omega)\big(f(p)\big)\left(df_p(v_1), \dots, df_p(v_k)\right) = (g \circ f)(p) \\ &\quad f^*\omega(p)(v_1, \dots, v_k) = f^*g(p) \cdot f^*\omega(p)(v_1, \dots, v_k). \end{aligned}$$

c) By omitting the indication of the point p, we obtain

$$\begin{aligned} f^*(\varphi_1 \wedge \dots \wedge \varphi_k)(v_1, \dots, v_k) &= (\varphi_1 \wedge \dots \wedge \varphi_k)\left(df_p(v_1), \dots, df_p(v_k)\right) \\ &= \det\left(\varphi_i\left(df(v_j)\right)\right) = \det\left(f^*\varphi_i(v_j)\right) \\ &= (f^*\varphi_1 \wedge \dots \wedge f^*\varphi_k)(v_1, \dots, v_k). \end{aligned}$$

Example (1.1.12): (Polar coordinates). Let ω be the 1-form in $\mathbb{R}^n - \{0,0\}$ by

$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{y^2 + x^2}dy.$$

Let U be the set in the plane (r, θ) given by

$$U = \{r > 0; 0 < \theta < 2\pi\}$$

and let $f: U \longrightarrow \mathbb{R}^2$ be the map

$$f(r, \theta) = \begin{cases} x = r \ \cos\theta \\ y = r \ \sin\theta \end{cases}$$

Let us compute $f^*\omega$. Since

$$dx = \cos\theta dr - r\sin\theta d\theta, \qquad dy = \sin\theta \, dr + r\cos\theta \, d\theta,$$

we obtain

$$f^*\omega = -\frac{y}{x^2 + y^2}(\cos\theta dr - r\sin\theta d\theta) + \frac{x}{y^2 + x^2}(\sin\theta \, dr + r\cos\theta \, d\theta)$$
$$= d\theta.$$

Notice that (a) of Proposition (1.1.11) states that the addition of differential forms commutes with the “substitution of variables”. We will now show that the same holds for the exterior product.

Proposition (1.1.13):

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a differentiable map. Then

a) $f^*(\omega \wedge \varphi) = (f^*\omega) \wedge (f^*\varphi)$, where ω and φ any two forms in $\mathbb{R}^m$.
b) $(f \circ g)^*\omega = g^*(f^*\omega)$, where $g: \mathbb{R}^p \longrightarrow \mathbb{R}^n$ is a differentiable map.

We are now going to define an operation on differential form that generalizes the differentiation of functions. Let $g: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a 0-form (i.e., a differentiable function). Then the differential

$$dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i \tag{1.17}$$

is a 1-form. We want to generalize this process by defining an operation that takes k-forms into $(k + 1)$-forms.

Definition (1.1.14):

Let $\omega = \sum a_I dx_I$ be a k-form in $\mathbb{R}^n$. The *exterior differential* $d\omega$ of ω is defined by

$$d\omega = \sum_I da_I \wedge dx_I. \tag{1.18}$$

Example (1.1.15): Let

$$\omega = xyzdx + yzdy + (x + z)dz$$

and let us compute $d\omega$:

$$\begin{aligned} d\omega &= d(xyz) \wedge dx + d(yz) \wedge dy + d(x + z) \wedge dz \\ &= (yzdx + xzdy + xydz) \wedge dx + (zdy + ydz) \wedge dy + (dx + dz) \wedge dz \\ &= -xzdx \wedge dy + (1 - xy)dx \wedge dz - ydy \wedge dz. \end{aligned}$$

Example (1.1.16): Let ω the 1-form on $\mathbb{R}^2$

$$\omega = fdx + gdy, \tag{1.19}$$

where f and g are functions on $\mathbb{R}^2$. to simplify notation, write

$$f_x = \partial f/\partial x, \qquad f_y = \partial f/\partial y.$$

then

$$d\omega = df \wedge dx + fd(dx) + dg \wedge dy + gd(dy),$$

by linearity and the product rule. Since $d^2 = 0$,

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= \left(f_x dx + f_y dy\right) \wedge dx + \left(g_x dx + g_y dy\right) \wedge dy. \end{aligned}$$

Since

$$dx \wedge dx = 0,$$
$$dy \wedge dy = 0, \qquad (1.20)$$

this becomes

$$\begin{aligned} d\omega &= f_y dy \wedge dx + g_x dx \wedge dy \\ &= (g_x - f_y) dx \wedge dy. \end{aligned} \qquad (1.21)$$

Proposition (1.1.17):

a) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$, where $\omega_1$and ω_2 are k-forms

b) $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$, where ω is a k-form and φ is an s-form.

c) $d(d\omega) = d^2\omega = 0$.

d) $d(f^*\omega) = f^*(d\omega)$, where ω is a k-form in $\mathbb{R}^m$ and $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a differentiable map.

Proof:

a) Is straightforward.

b) Let $\omega = \sum_I a_I dx_I$, $\varphi = \sum_J b_J dx_J$. Then

$$\begin{aligned} &= \sum_{IJ} d(a_I b_J) \wedge dx_I \wedge dx_J \\ &= \sum_{IJ} b_J da_I \wedge dx_I \wedge dx_J + \sum_{IJ} a_I db_J \wedge dx_I \wedge dx_J \\ &= d\omega \wedge \varphi + (-1)^k \sum_{IJ} a_I dx_I \wedge db_J \wedge dx_J \\ &= d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi. \end{aligned}$$

c) Let us first assume that ω is a 0-form, i.e., ω is a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ that associates to each $(x_1, \dots, x_n) \in \mathbb{R}^n$ that value $f(x_1, \dots, x_n) \in \mathbb{R}$. Then

$$d(df) = d\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\right) = \sum_{j=1}^{n} d\left(\frac{\partial f}{\partial x_j}\right) \wedge dx_j$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \right).$$

Since

$$\partial^2 f / \partial x_i \partial x_j = \partial^2 f / \partial x_j \partial x_i,$$

and

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \qquad i \neq j,$$

we obtain that

$$d(df) = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) dx_i \wedge dx_j = 0.$$

Now let $\omega = \sum a_I dx_I$. By (a), we can restrict ourselves to the case $\omega = a_I dx_I$ with $a_I \neq 0$. By (b), we have that

$$d\omega = da_I \wedge dx_I + a_I d(dx_I).$$

But $d(dx_I) = d(1) \wedge dx_I = 0$. Therefore,

$$d(d\omega) = d(da_I \wedge dx_I) = d(da_I) \wedge dx_I + da_I \wedge d(dx_I) = 0,$$

since $d(da_I) = 0$ and $d(dx_I) = 0$, which proves (c).

d) We will first prove the result for a 0-form. Let $g: \mathbb{R}^m \longrightarrow \mathbb{R}$ be a differentiable function that associates to each $(y_1, \dots, y_m) \in \mathbb{R}^m$ the value $g(y_1, \dots, y_m)$. Then

$$f^*(dg) = f^* \left(\sum_i \frac{\partial g}{\partial y_i} dy_i \right) = \sum_{ij} \frac{\partial g}{\partial y_i} \frac{\partial f_i}{\partial x_j} dx_j$$

$$= \sum_j \frac{\partial (g \circ f)}{\partial x_j} dx_j = d(g \circ f) = d(f^* g).$$

Now, let $\varphi = \sum_I a_I dx_I$ be a k-form. By using the above, and the fact that f^* commutes with the exterior product, we obtain

$$\begin{aligned} d(f^*\varphi) &= d\left(\sum_I f^*(a_I) f^*(dx_I)\right) \\ &= \sum_I d(f^*(a_I)) \wedge f^*(dx_I)) \\ &= \sum_I f^*(da_I) \wedge f^*(dx_I) \\ &= f^*\left(\sum_I da_I \wedge dx_I\right) = f^*(d\varphi) \end{aligned}$$

Which prove (d).

(1.2): Differential Forms on Manifolds

We now extend the notion of a differential form in $\mathbb{R}^n$ to differentiable manifolds (See Section 1.1). Given a vector space V, we will denote by $\Lambda^k(V)$ the set of all alternating, k-linear maps $\omega: V \times \dots \times V \longrightarrow \mathbb{R}$, where $V \times \dots \times V$ contains k factors.

Definition (1.2.1):

Let M be a differentiable manifold. *An exterior k-form ω in M* is the choice, for every $p \in M$, of an element $\omega(p)$ of the space $\Lambda^k(T_pM)^*$ of alternating k-linear forms of the tangent space T_pM.

Given an exterior k-form ω and a parameterization $f_i: U_i \longrightarrow M$, around $p \in f_i(U_i)$, we defined the *representation* of ω in this parameterization as the exterior k-form ω_i in $U_i \subset \mathbb{R}^n$ given by

$$\omega_i(v_1, \dots, v_k) = \omega\big(df_i(v_1), \dots, df_i(v_k)\big), \qquad v_1, \dots, v_k \in \mathbb{R}^n. \qquad (1.22)$$

Definition (1.2.2):

A *differential form of order* k (or a differential k-form) in a differentiable manifold M is an exterior k-form such that, in some coordinates system (hence, in all), its differentiable.

The important fact is, that all the operations defined for differential forms in $\mathbb{R}^n$ can be extend to differential forms in M through their local representations. For instance, if ω is differential form in M, $d\omega$ is the differential form in M whose local representation is $d\omega_i$, then $d\omega$ is a well defined differential form on M.

(1.2.1): Exterior Derivative on a Coordinate Chart

We showed in Section (1.1) the exterior differentiation on $\mathbb{R}^n$.

More precisely, suppose (U, φ) is coordinate chart on a manifold M. Then any k-form ω on U is uniquely a linear combination

$$\omega = \sum a_I dx_I, \tag{1.23}$$

If d is an exterior differentiation on U, then by using Proposition (1.1.17)

$$d\omega = \sum (da_I) \wedge dx_I + \sum a_I ddx_I \tag{1.24a}$$

$$= \sum da_I \wedge dx_I \tag{1.24b}$$

$$= \sum \frac{\partial a_I}{\partial x_J} dx_J \wedge dx_I. \tag{1.24c}$$

Hence, if an exterior differentiation d exists on U, then its uniquely defined by Equation (1.24c).

To show existence, we define d by formula (1.24c). The proof that d satisfies Proposition (1.1.17), like the derivative of function on $\mathbb{R}^n$, an antiderivation D on $\Lambda^*(M)$ has the property that for a k-form ω, the value of $D\omega$ at a point p depends only on the values of ω in neighborhood of p. To explain this, we make a digression on local operators.

(1.2.2): Local Operators

An endomorphism of a vector space W is often called an *operator* on W. For example, if $W = C^{\infty}(\mathbb{R})$ is a vector space of C^{∞} functions on $\mathbb{R}$, then the derivative d/dx is an operator on W

$$\frac{d}{dx} f(x) = f^{'}(x). \tag{1.25}$$

The derivative has the property that the value of $f'(x)$ at a point p depends only on the values of f in a small neighborhood of p. More precisely, if $f = g$ on open set U in $\mathbb{R}$, then $f'{=}g'$ on U. We say that the derivative is *local operator* on $C^{\infty}(\mathbb{R})$.

Definition (1.2.3):

An operator $D: \Lambda^{*}(M) \longrightarrow \Lambda^{*}(M)$ is said to be *local* if for all $k \geq 0$, where a k-form $\omega \in \Lambda^{k}(M)$ restricts to 0 on an open set U, then $D\omega \equiv 0$ on U.

Here the restricting to 0 on U, we mean that $\omega_p = 0$ at every point p in U, and the symbol " $\equiv 0$" means “identically zero”: $(D\omega)_p = 0$ at every point p in U. An equivalent definition of local operator is that for all $k \geq 0$, whenever two k-forms $\omega, \varphi \in \Lambda^{*}(M)$ agree on an open set U, then $D\omega = D\varphi$ on U.

(1.2.3): Extension of a Local Form to a Global Form

Sometimes we are given a differential form φ that is defined only on open subset U of a manifold M. We can use a bump function to extend φ to a global form $\tilde{\varphi}$ on M that agrees with φ near some point. (By a global form, we mean a differential form defined at every point of M).

Proposition (1.2.4):

Suppose φ is a C^{∞} differential form on an open subset U of M. For any $p \in U$, there is a C^{∞} global form $\tilde{\varphi}$ on M that agrees with φ on a neighborhood of p in U.

(1.3): The Exterior Derivatives, Interior Product, and Lie Derivative

(1.3.1): Exterior Derivatives

The *exterior derivative* [1,2,3] of a k-form ω is a $(k+1)$-form which we denote by $d\omega$. We will define $d\omega$ in the case $k=0,\ df(X)=Xf$ (for every vector field X). There are several approaches to its definition, each of which gives important information about the operator d.

a) In term of coordinates d merely operates on the component function:

$$d\omega = \left(d\omega_{(i_1 \quad i_k)}\right) \wedge dx^{i_1} \cdots dx^{i_k}. \tag{1.26}$$

It is not immediately clear that this defines anything at all, since the right side might depend on the choice of coordinates x^i. However, it is easily verified that this formula satisfies the axioms for d given below. Since the axioms are coordinate free and determine d, it is a consequence that Equation (1.26) is invariant under change of coordinates.

In the case of $M=\mathbb{R}^3$ and Cartesian coordinates x, y, z the formula bears a strong, nonaccidental resemblance to grad, curl, and div:

$$\begin{aligned}
df &= f_x\,dx + f_y\,dy + f_z\,dz,\\
d(f\,dx + g\,dy + h\,dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz\\
&= \left(f_x\,dx + f_y\,dy + f_z\,dz\right) \wedge dx\\
&\quad + \left(g_x\,dx + g_y\,dy + g_z\,dz\right) \wedge dy + \left(h_x\,dx + h_y\,dy + h_z\,dz\right) \wedge dz\\
&= \left(h_y - g_z\right)dy\,dz + (f_z - h_x)dz\,dx\\
&\quad + \left(g_x - f_y\right)dx\,dy\\
d(f\,dy\,dz + g\,dz\,dx + h\,dx\,dy) &= df \wedge dy\,dz + dg \wedge dz\,dx + dh \wedge dx\,dy\\
&= \left(f_x + g_y + h_z\right)dx\,dy\,dz.
\end{aligned} \tag{1.27}$$

(We have indicated partial derivatives by subscripts.) The discrepancies from the usual formulas for grad, curl, and div can be erased by introducing the Euclidean inner product on $\mathbb{R}^3$, for which dx, dy, dz is an orthonormal basis at each point. This gives us an isomorphism between contravariant and covariant vectors, ai+bj+ck $= a\partial_x + b\partial_y + c\partial_z \leftrightarrow a\,dx + b\,dy + c\,dz$; we shall ignore this isomorphism and deal with only the covariant vectors.

b) There are a few important properties of d which are also sufficient to determine d completely, that is axioms for d:

1) If f is a 0-form, then df coincides with the previous definition; that is, $df(X) = Xf$ for every vector field X.
2) There is a wedge-product rule which d satisfies; as a memory device, we think of d as having degree 1, so a factor of $(-1)^k$ is product when d commutes with a k-form: If ω is a k-form and τ a q-form, then

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau, \tag{1.28}$$

that is, d derivation.

3) When d is applied twice the result is 0, written $d^2 = 0$: $d(d\omega) = 0$ for every k-form ω. [As an axiom for the determination of d it would suffice to assume $d(df) = 0$ only for 0-forms f, but the more general result (3) is a theorem which we need.]
4) The operator d is linear. Only the additivity need be assumed, because commutation with constant scalar multiplication is consequence of (1) and (2): if ω and τ are k-forms, then

$$d(\omega + \tau) = d\omega + d\tau. \tag{1.29}$$

The coordinate definition by Equation (1.26) is an easy consequence of these axioms, because by (2) and (3),

$$\begin{aligned} d\left(dx^{i_1} \cdots dx^{i_k}\right) &\\ &= \left(d^2 x^{i_1}\right) \wedge dx^{i_2} \cdots dx^{i_k} - dx^{i_1} \wedge \left(d^2 x^{i_2}\right) \wedge dx^{i_3} \cdots dx^{i_k} + \cdots \\ &+ (-1)^{k-1} dx^{i_1} \cdots dx^{i_{k-1}} \wedge d^2 x^{i_k} = 0. \end{aligned} \tag{1.30}$$

Thus we have

$$\begin{aligned} d\left(f\, dx^{i_1} \cdots dx^{i_k}\right) &= df \wedge dx^{i_1} \cdots dx^{i_k} + f d\left(dx^{i_1} \cdots dx^{i_k}\right) \\ &= df \wedge dx^{i_1} \cdots dx^{i_k}, \end{aligned} \tag{1.31}$$

This, with additivity (4), gives Equation (1.26).

The converse, that formula Equation (1.26) satisfies the axioms, is a little harder. Of course, (1) and (4) are trivial. To prove (2) we need to product rule for functions: $d(fg) = (df)g + f\, dg$. The components of $\omega \wedge \tau$ are sums of products of the components of ω and τ. Applying the product rule for functions gives two indexed sums, which we want to factor to get (2), and this is done by shifting the components of τ and their differentials over the coordinate differentials corresponding to ω, which in the second case requires a sign $(-1)^k$:

$$\begin{aligned} d(\omega \wedge \tau) &= \frac{1}{k!\, q!} d\left(\omega_{i_1 \cdots i_k} \tau_{j_1 \cdots j_q}\right) \wedge dx^{i_1} \cdots dx^{i_k} dx^{j_1} \cdots dx^{j_q} \\ &= \frac{1}{k!\, q!} \Big[d\omega_{i_1 \cdots i_k} \wedge dx^{i_1} \cdots dx^{i_k}\, \tau_{j_1 \cdots j_q} dx^{j_1} \cdots dx^{j_q} \\ &\quad + \omega_{i_1 \cdots i_k} (-1)^k dx^{i_1} \cdots dx^{i_k} \wedge d\tau_{j_1 \cdots j_q} \wedge dx^{j_1} \cdots dx^{j_q} \Big]. \end{aligned} \tag{1.32}$$

(The factor $1/k!\,q!$ is inserted because we are unable to keep $i_1 \cdots i_k j_1 \cdots j_q$ in increasing order when we are only given $i_1 \cdots i_k$ and $j_1 \cdots j_q$ in increasing order, so we have switched to the full sum and consequent duplication of terms, $k!$ for ω and $q!$ for τ.)

Axiom (3) is known as the *Poincare' lemma*, although there is some confusion historically, so that in some places the converse, "if $d\omega = 0$, then there is some τ such that $\omega = d\tau$," is referred to as the Poincare' lemma. The proof that Equation (1.26) satisfies (3), $d^2 = 0$, uses the equality of mixed derivative on functions in either order, a symmetric property, which combines with the skew-symmetry of wedge product to give 0.

c) There is an intrinsic formula for d in terms of values of forms on arbitrary vector fields. This formula involves bracket and shows that the ability to form an intrinsic derivative of k-forms is related to the ability to form an intrinsic bracket of two vector fields. We only give the formula in the low-degree cases for which it has the greatest use.

f "a 0-form":

$$df(X) = Xf, \tag{1.33}$$

ω "a 1-form":

$$\begin{aligned} d\omega(X,Y) &= \frac{1}{2}\{X\omega(Y) - Y\omega(X) - \omega[X,Y]\} \\ &= \frac{1}{2}(X\langle Y,\omega\rangle - Y\langle X,\omega\rangle - \langle [X,Y],\omega\rangle) \end{aligned} \tag{1.34}$$

ω "a 2-form":

$$d\omega(X,Y,Z) = \frac{1}{3}\{X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y)\}. \tag{1.35}$$

[The annoying factors $\frac{1}{2},\frac{1}{3},\cdots$ can be eliminated by using another definition of wedge products. This alternative definition, which does not alter the essential properties of wedge product, is obtained by magnifying our present wedge product of a k-form and a q-form by the factor $(k+q)!/k!\,q!$. Both products are in common use and we shall continue with our original definition.]

(1.3.2): Lie Derivatives

If X is a C^∞ vector field, then X operates on C^∞ scalar fields to give C^∞ scalar fields. The Lie derivation with respect to X is an extension of this operation to an operator L_X on all C^∞ tensor fields which preserve type of tensor fields.

The tensor field derived from T in the above way by differentiating with respect to the parameters of the integral curves of X is called the *Lie derivative of T with respect to X* and is denoted $L_X T$.

In the following proposition we list some of the elementary properties of the Lie derivatives.

Proposition (1.3.1):

Let M be a smooth manifold. Suppose X, Y are smooth vector fields on M, σ, τ are smooth covariant tensor fields, ω, η are differential forms, and f is a smooth function (thought of as a 0-tensor field).

a) $L_X f = Xf$.
b) $L_X(f\sigma) = (L_X f)\sigma + f L_X \sigma$.
c) $L_X(\sigma \otimes \tau) = (L_X \sigma) \otimes \tau + \sigma \otimes L_X \tau$.
d) $L_X(\omega \wedge \eta) = L_X \omega \wedge \eta + \omega \wedge L_X \eta$.
e) $L_X(Y \rfloor \omega) = (L_X Y) \rfloor \omega + Y \rfloor L_X \omega$.
f) For any smooth vector fields $Y_1, \dots, Y_k$,

$$L_X\big(\sigma(Y_1, \dots, Y_k)\big) = (L_X \sigma)(Y_1, \dots, Y_k) + \sigma(L_X Y_1, \dots, Y_k) + \cdots + \sigma(Y_1, \dots, L_X Y_k) \quad (1.36)$$

Corollary (1.3.2):

If X is a smooth vector field and σ is a smooth covariant tensor field, then

$$(L_X \sigma)(Y_1, \dots, Y_k) = X(\sigma(Y_1, \dots, Y_k) - \sigma([X, Y_1], Y_2, \dots, Y_k) - \cdots \\ \dots - \sigma(Y_1, \dots, Y_{k-1}, [X, Y_k])). \quad (1.37)$$

It follows that $L_X \sigma$ is smooth.

Corollary (1.3.3):

If $f \in C^{\infty}(M)$, then

$$L_X(df) = d(L_X f). \qquad (1.38)$$

Proof:

Using Corollary (1.3.2), we compute

$$\begin{aligned}(L_X df)(Y) &= X\big(df(Y)\big) - df[X,Y] \\ &= XYf - [X,Y]f \\ &= XYf - (XYf - YXf) \\ &= YXf \\ &= d(Xf)(Y) \\ &= d(L_X f)(Y) \ \blacksquare\end{aligned}$$

Closely associated with differential forms is the notion of vector field.

Since the vector field X on M is an operation on the space D of differentiable functions on M, we can take the iterates of this operation. For instance, if X and Y are differentiable vector fields and $\varphi: M \longrightarrow \mathbb{R}$ is a differentiable function, we can consider the functions $Y(X\varphi)$ and $X(Y\varphi)$. In general, such iterated operations do not lead to vector fields, since they involve derivatives of order higher than the first. However, the following holds.

Lemma (1.3.4):

Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exist a unique vector field Z on M such that, for each

$$Z\varphi = (XY - YX)\varphi, \qquad \varphi \in D \qquad (1.39)$$

Proof:

We first prove that if such a Z exists, then it is unique. For that, let $f: U \longrightarrow M$ be a parameterization, and let

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \qquad Y = \sum_j b_j \frac{\partial}{\partial x_j} \tag{1.40}$$

be the expressions of X and Y, respectively, in the parameterization f. Then

$$\begin{aligned} XY\varphi &= X\left(\sum_i b_j \frac{\partial \varphi}{\partial x_j}\right) = \sum_{ij} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + \sum_{ij} a_i b_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \\ YX\varphi &= Y\left(\sum_i a_i \frac{\partial \varphi}{\partial x_i}\right) = \sum_{ij} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + \sum_{ij} a_i b_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \end{aligned} \tag{1.41}$$

Hence

$$(XY - YX)\varphi = \sum_j \left(\sum_i \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial}{\partial x_j}\right)\varphi. \tag{1.42}$$

It follows that if Z exists with the required property, it must be expressed as above in many coordinate system, hence it is unique.

To prove existence, just define Z_α in each coordinate neighborhood $f_\alpha(U_\alpha) \subset M$ by the above expression. By uniqueness, $Z_\alpha = Z_\beta$ in $f_\alpha(U_\alpha) \cap f_\beta(U_\beta)$, hence Z is well defined on M.

Definition (1.3.5):

The vector field determined by the above lemma is called the *bracket*

$$[X, Y] = XY - YX \text{ of } X \text{ and } Y, \tag{1.43}$$

and it is clearly differentiable.

The bracket operation has the following properties:

Proposition (1.3.6):

Let X, Y and Z be differentiable vector fields, a and b be real numbers, and φ, θ be a differentiable functions. Then

a) $[X,Y] = -[Y,X]$,
b) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$,
c) $\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big] = 0$, (Jacobi's identity),
d) $[\theta X, \varphi Y] = \theta\varphi[X,Y] + \theta \cdot X(\varphi)Y - \varphi \cdot Y(\theta)X$.

Proof:

(a) and (b) are immediate. To prove (c), we observe that

$$\begin{aligned}\big[[X,Y],Z\big] &= [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX \\ &= \big[X,[Y,Z]\big] + \big[Y,[Z,X]\big]\end{aligned} \tag{1.44}$$

and use (a) to obtain (c). The proof of (d) is direct with simple computation.

There exists an interesting relation between exterior differentiation of differential forms and the bracket operation. For the case of 1-forms, this relation is as follows:

Proposition (1.3.7):

Let ω be a differentiable 1-form on a differentiable manifold M and let X and Y be a differentiable vector fields on M. Then

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]). \tag{1.45}$$

Proof:

It is enough to prove that it is true locally, say in a coordinate neighborhood of each point. In any such neighborhood with coordinates $x^1, \dots, x^n$, $\omega = \sum_{i=1}^{n} a_i dx^i$ and it is easy to see that the equation of the proposition holds for all ω if it holds for every ω of the form $f\,dg$, where f, g are C^∞ functions on the neighborhood.

Suppose, then, that $\omega = f\, dg$ and let X, Y be C^∞-vector fields. Then, evaluating both sides of the equation of the lemma separately, we obtain

$$\begin{aligned} d\omega(X,Y) &= df \wedge dg(X,Y) = df(X)dg(Y) - dg(X)df(Y) \\ &= (Xf)(Yg) - (Xg)(Yf), \end{aligned} \tag{1.46}$$

and

$$\begin{aligned} X\omega(Y) - Y\omega(X) - \omega([X,Y]) &= X\big(fdg(Y)\big) - Y\big(fdg(X)\big) - fdg([X,Y]) \\ &= X\big(f(Yg)\big) - Y\big(f(Xg)\big) - f(XYg - YXg) \\ &= (Xf)(Yg) - (Xg)(Yf) \end{aligned} \tag{1.47}$$

(1.3.3). Interior Products

The *interior product* by X is an operator i_X on k-forms for every vector field X, which maps a k-form into a $(k-1)$-form; essentially this is done by fixing the first variable of the k-form ω at X, leaving the remaining $k-1$ variables free to be the variable of $i_X\omega$ (except for normalizing factor k). In formulas, for vector fields $X_1, \ldots, X_{k-1}$,

$$[i_X\omega](X_1, \ldots, X_{k-1}) = k\omega(X, X_1, \ldots, X_{k-1}). \tag{1.48}$$

For 0-forms we define

$$i_X f = 0. \tag{1.49}$$

An important notion that comes up in studying differential forms is the notion of contracting an k-form. Given an k-form $\omega \in \Lambda^k(M)$ and a vector field X, then

$$i_X\omega = \omega(X, \cdot, \cdots, \cdot), \tag{1.50}$$

is called the *contraction* with X, and is differential $(k-1)$-form on M. Another notation for this is $i_X\omega = X \rfloor \omega$. Contraction is a linear mapping

$$i_X : \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M).$$

Contraction is also linear in X, i.e. for vector field X, Y it holds that

$$\begin{aligned} i_{X+Y}\omega &= i_X\omega + i_Y\omega, \\ i_{\lambda X}\omega &= \lambda \cdot i_X\omega. \end{aligned} \tag{1.51}$$

Proposition (1.3.8):

The operator i_X is a derivation of forms, that is, for a k-form ω and a q-form τ it satisfies the product rule:

$$i_X(\omega \wedge \tau) = i_X\omega \wedge \tau + (-1)^k \omega \wedge i_X\tau. \tag{1.52}$$

[As with d, if we think of i_X as having degree -1, then in passing over the k-form ω we get a factor of $(-1)^k$.]

Multiplication of these interior product operators is skew-symmetric, that is, $i_X i_Y = -i_Y i_X$:

$$\begin{aligned} i_X i_Y \omega(\cdots) &= k i_Y \omega(X, \dots) \\ &= k(k-1)\omega(Y, X, \dots) \\ &= -k(k-1)\omega(X, Y, \dots) \\ &= -i_Y i_X \omega(\cdots). \end{aligned} \tag{1.53}$$

It follows that the operation $i_X i_Y$ depends only on $X \wedge Y$, so we define

$$i_{(X\wedge Y)} = i_X i_Y. \tag{1.54}$$

Then we extend linearly to obtain i_A for every skew-symmetric contravariant tensor A of degree 2. The operator i_A maps $(k-1)$-forms into $(k-2)$-forms. Similarly, we can define i_B, for any skew-symmetric contravariant tensor B of degree r, mapping k-forms into $(k-r)$-forms.

Since i_X is derivation it is determined by it action on 0-forms and 1-forms. One needs only express an arbitrary k-form in terms of 0-forms and 1-forms and apply the product rule repeatedly.

(1.3.4): Lie Derivatives of Differential Forms

In the case of differential forms, the exterior derivative yields a much more powerful formula for computing Lie derivatives. Although Corollary (1.3.2) gives a general formula for computing the Lie derivative of any tensor field, this formula has a serious drawback: In order to calculate what $L_X\sigma$ does to vectors $Y_1, \dots, Y_k$ at a point $p \in M$, it is necessary first to extend the vectors to vector *fields* in a neighborhood of p. The formula in the next proposition overcomes this disadvantage.

As promised, Proposition (1.3.9) gives a formula for the Lie derivative of a differential form that can be computed easily in local coordinates, without having to go to the trouble of letting the form act on vector fields. In fact, this leads to an easy algorithm for computing Lie derivatives of *arbitrary* tensor fields, since any tensor field can be written locally as a linear combination of tensor products of 1-forms.

Since Lie derivatives are brackets in one case, and the exterior derivative operator d given terms of brackets and evaluations of forms on vector fields by (c) in Section (1.3), it is not too surprising that there is a relation between the operators $L_X, i_X,$ and d, operating on forms.

Proposition (1.3.9):

For any vector field X and any differential k-form ω on a smooth manifold M,

$$L_X\omega = X\rfloor(d\omega) + d(X\rfloor\,\omega). \tag{1.55}$$

Proposition (1.3.10):

If $x \in V,$ then $x\rfloor$ is an anti-derivative.

Notice that $e(x)$ is neither a derivative nor an anti-derivation.

Definition (1.3.11):

A subring $I \subset \Lambda(V^*)$ is called an *ideal*, if:

i. $\phi \in I$ implies $\phi \wedge \beta \in I,\ \forall\, \beta \in \Lambda(V^*)$,

ii. $\phi \in I$ implies that all its components in $\Lambda(V^*)$ are contained in I.

A subring satisfying the second condition is called *homogeneous*. As a consequence of i. and ii. we conclude that $\phi \in I$ implies $\beta \wedge \phi \in I, \forall\, \beta \in \Lambda(V^*)$. Thus all our ideal are homogeneous and two-sided.

Given an ideal $I \subset \Lambda(V^*)$, we wish to determine the smallest subspace $W^* \subset V^*$ such that I is generated, as an ideal, by a set S of elements of $\Lambda(W^*)$. An element of I is then a sum of elements of the form $\sigma \wedge \beta, \sigma \in S,\ \ \beta \in \Lambda(V^*)$. If $x \in W = (W^*)^{\perp}$, we have, since the interior product $x\llcorner$ is an anti-derivation,

$$\begin{gathered} x \rfloor \sigma = 0 \\ x \rfloor (\sigma \wedge \beta) = \pm\, \sigma\, (x \rfloor \beta) \in I. \end{gathered} \tag{1.56}$$

Therefore, we define

$$A(I) = \{x \in V : x \rfloor I \subset I\}, \tag{1.57}$$

where the last condition means that $x \rfloor \phi \in I,\ \forall\, \phi \in I. A(I)$ is clearly a subspace of V. It will play later an important role in differential systems, for which reason we will call it the *Cauchy characteristic space* of I. Its annihilator

$$C(I) = A(I)^{\perp} \subset V^* \tag{1.58}$$

will be called the *retracting subspace* of I.

Theorem (1.3.12): (Retracting theorem)

Let I be an ideal of $\Lambda(V^*)$. Its retracting subspace $C(I)$ is the smallest subspace of V^* such that $\Lambda\big(C(I)\big)$ contains a set S of elements generating I as an ideal. The set S also generates an ideal J in $\Lambda\big(C(I)\big)$, to be called a retracting ideal of I. Then there exists a mapping

$$\Delta: \Lambda(V^*) \longrightarrow \Lambda\big(C(I)\big),$$

of graded algebras such that $\Delta(I) = J$.

Proposition (1.3.13):

The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.

(1.3.5): Cartan's Magic Formula

A very important formula for the Lie derivative is given by the following.

Theorem (1.3.14):

For X a vector field and ω k-form on a manifold M, on differential forms, Lie derivatives are given by the operator equation

$$L_X\omega = i_X d\omega + d i_X \omega, \tag{1.59}$$

or, in the "hook" notation,

$$L_X\omega = (X \rfloor d\omega) + d(X \rfloor \omega). \tag{1.60}$$

(We also remember this as $L = id + di$.)

"the formula (1.60) is known as Cartan's magic formula"

Proof:

We have seen that L_X is derivation of degree 0 of skew-symmetric tensors; that is, it preserves degree and satisfies the product rule. We shall show that $i_X d + d i_X$ also is derivation:

$$
\begin{aligned}
[i_X d + d i_X](\omega \wedge \tau) \\
&= i_X(d\omega \wedge \tau + (-1)^k \omega \wedge d\tau) \\
&+ d(i_X\omega \wedge \tau + (-1)^k \omega \wedge i_X\tau) \\
&= i_X d\omega \wedge \tau + (-1)^{k+1} d\omega \wedge i_X\tau + (-1)^k i_X\omega \wedge d\tau + (-1)^{2k}\omega \\
&\wedge i_X d\tau + d i_X\omega \wedge \tau + (-1)^{k-1} i_X\omega \wedge d\tau + (-1)^k d\omega \wedge i_X\tau \\
&+ (-1)^{2k}\omega \wedge d i_X\tau
\end{aligned}
$$

$$= (i_X d + d i_X)\omega \wedge \tau + \omega \wedge (i_X d + d i_X)\tau. \tag{1.61}$$

Thus if L_X and $i_X d + d i_X$ agree on 0-forms and 1-forms, then they agree on all k-forms.

On 0-forms we have $L_X f = Xf$, where as

$$i_X df + d i_X f = i_X f + d0 = df(X) = Xf, \tag{1.62}$$

on a 1-form df we have $L_X df = d(Xf)$, since

$$L_X\langle Y, df\rangle = X\langle Y, df\rangle = X(Yf), \tag{1.63}$$

on the other hand,

$$
\begin{aligned}
L_X\langle Y, df\rangle &= \langle L_X\, Y, df\rangle + \langle Y, L_X\, df\rangle \\
&= \langle [X, Y], df\rangle + \langle Y, L_X\, df\rangle \\
&= XYf - YXf + \langle Y, L_X\, df\rangle,
\end{aligned}
\tag{1.64}
$$

so

$$\langle Y, L_X\, df\rangle = YXf = \langle Y, d(Xf)\rangle, \tag{1.65}$$

but

$$
\begin{aligned}
[i_X d + d i_X] df &= i_X d^2 f + d i_X df \\
&= 0 + d(Xf).
\end{aligned}
\tag{1.66}
$$

We do not need to check values on the more general 1-forms $g\, df$ because of the product rule being satisfied by each operator.

Corollary (1.3.5):

If X is a vector field and ω is a differential form, then

$$L_X(d\omega) = d(L_X\omega). \tag{1.67}$$

Proof:

This follows from the preceding proposition and the fact that $d^2 = 0$:

$$
\begin{aligned}
L_X d\omega &= X \rfloor d(d\omega) + d(X \rfloor d\omega) \\
&= d(X \rfloor d\omega) \\
d(L_X\omega) &= d\big(X \rfloor d\omega + d(X \rfloor \omega)\big) \\
&= d(X \rfloor d\omega)
\end{aligned}
\tag{1.68}
$$

The operators d and L_X commute on forms; that is, for every k-form ω, then

$$dL_X\omega = L_X d\omega, \tag{1.69}$$

when written in the form

$$(k+1) d\omega(X, \dots) = [L_X\omega - d(i_X\omega)](\dots), \tag{1.70}$$

the relation gives a means of determining d on k-forms from Lie derivatives and d on $(k-1)$-forms. This suggests that when we wish to develop some property of d and we have some corresponding property of Lie derivatives, we should try an induction on the degree of the forms involved.

CHAPTER TWO

Exterior Differential System and Basic Theorems

Section (2.1): The Concept of an Exterior Differential Systems

An ***exterior differential system*** [4,5,6] is a system of equations on a manifold defined by equating to zero a number of exterior differential forms. It is called a Pfaffian system when all the forms are linear.

Consider a differentiable manifold M of dimension n. Its cotangent bundle, whose fibers are the cotangent spaces, $T_x^*(M)$, $x \in M$we will denote by T^*M. From T^*M we construct the bundle ΛT^*M, whose fibers are

$$\Lambda T_x^* = \sum_{0 \le p \le n} \Lambda^p T_x^*, \tag{2.1}$$

which have the structure of the graded algebra. The bundle ΛT^*M has the sub-bundles $\Lambda^p T^*M$. Similar definitions are valid for the tangent bundle TM.

A section of the bundle

$$\Lambda^p T^*M = \bigcup_{x \in M} \Lambda^p T_x^* \longrightarrow M, \tag{2.2}$$

is called an exterior differential form of degree p, or a form of degree p or simply a p-form. As we mention in chapter one. By abuse of language we will call a differential form a section of the bundle ΛT^*M its p-th component is a p-form. All sections are supposed to be sufficiently smooth, refer to equation (1.11).

Let $\Omega^p(M) = C^\infty$-sections of $\Lambda^p T^*M$ and let $\Omega^*(M) = \oplus \Omega^p(M)$.

Definition (2.1.1):

An exterior differential system (EDS) is a pair (M, I) where M is a smooth manifold and $I \subset \Omega^*(M)$ is a graded ideal in the ring $\Omega^*(M)$ of differential forms on M that closed under exterior differentiation $d: \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$, i.e., for any ϕ in I, its exterior derivative $d\phi$ also lies in I.

The exterior differential systems considered in this section will always be finitely generated.

The main interest in an EDS (M, I) centers on the problem of describing the submanifolds $f: M \longrightarrow N$ for which all the elements of I vanish when pulled back to N, i.e., for which $f^*\phi = 0,\ \forall\, \phi \in I$. Such submanifolds are said to be *integral manifold* of I.

The most common way of specifying an EDS (M, I) is to give a list of generators of I. For $\phi_1, \dots, \phi_s \in \Omega^*(M)$, the 'algebraic' ideal consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots + \gamma^s \wedge \phi_s, \tag{2.3}$$

will be denoted $\langle \phi_1, \dots, \phi_s \rangle_{\text{alg}}$ while the differential ideal I consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots + \gamma^s \wedge \phi_s + \beta^1 \wedge d\phi_1 + \cdots + \beta^s \wedge d\phi_s, \tag{2.4}$$

will be denoted $\langle \varphi_1, \dots, \varphi_s \rangle$.

By our conventions $I = \bigoplus I^q$ is a direct sum of its homogeneous pieces $I^q = I \cap \Omega^q(M)$, and by differentiation; and by differential closure we have $d\phi \in I$ whenever $\phi \in I$. We sometime refer to an ideal $I \subset \Omega^*(M)$ satisfying $dI \subseteq I$ as differential ideal.

In practice, I will almost always generated as a differential ideal by a finite collection $\{\phi_A\}, 1 \leq A \leq N$ of a differential form; forms of degree zero, i.e., functions, are not excluded. An integral manifold of I is given by an immersion

$$f : N \to M,$$

satisfying $f^*\varphi = 0$ for $1 \leq A \leq N$. Then

$$\begin{aligned} &f^*(\beta \wedge \varphi_A) = 0, \\ &f^*(d\varphi_A) = 0, \end{aligned} \tag{2.5}$$

and so $f^*\varphi = 0$ for all φ in the differential ideal generated by the $\{\varphi_A\}$.

The fundamental problem in exterior differential systems is to study the integral manifolds. We may think of these as solutions to the system

$$\varphi_A = 0, \tag{2.6}$$

of exterior equations. When is written out in local coordinates, this is the system of P.D.E.'s.

The notion is of such generality that includes all the ordinary and partial differential equations, as the following:

Notion (2.1.2):

Ordinary Differential Equations Formulated as Exterior Differential Systems:

Consider the system of ordinary differential equations:

$$\begin{aligned} &\frac{dy}{dx} = F(x, y, z), \\ &\frac{dz}{dx} = G(x, y, z), \end{aligned} \tag{2.7}$$

where F and G are smooth functions on some domain $M \subset \mathbb{R}^3$. This can be modeled by the EDS (M, I) where

$$I = \langle dy - F(x, y, z)dx, dz - G(x, y, z)dx \rangle. \tag{2.8}$$

It's clear that the 1-dimensional integral manifold of I are just the integral curves of the vector field

$$X = \frac{\partial}{\partial x} + F(x,y,z)\frac{\partial}{\partial y} + G(x,y,z)\frac{\partial}{\partial z}. \tag{2.9}$$

as clarification example:

let: $y' = e^{xyz}$ and $z' = \sinh(x+y-z)$,

we can model this system by an EDS with $M \subset \mathbb{R}^3$

$$I = \langle dy - e^{xyz}dx, dz - \sinh(x+y-z)\,dx\rangle.$$

Notion (2.1.3):

Consider the second-order differential equation in $M \subset \mathbb{R}^3$,

$$y'' = F(x,y,y'), \tag{2.10}$$

can be drive an exterior differential system:

$$\begin{aligned} dy &= y'dx, \\ \frac{d}{dx}(y) &= \frac{d}{dx}(y'), \\ dy' &= \frac{d}{dx}(y')\cdot dx, \\ dy' - y''\cdot dx &= 0, \\ dy' - F(x,y,y')dx &= 0. \end{aligned} \tag{2.11}$$

Notion (2.1.4):

Partial Differential Equations Formulated as Exterior Differential Systems:

A useful application of exterior differential systems is the analysis of systems of partial differential equations PDEs. Before the theory to be described shortly can be applied, the partial differential equations must be re-expressed as an exterior differential system. It should be stressed that while the recipe given here is straightforward, it is by no means the only possible one, and often not the most efficient one since it tends to lead to manifold M of higher dimensions than necessary.

Consider any system of partial differential equations can be described by an exterior differential system.

$$F\left(x^i, z, \frac{\partial z}{\partial x^i}\right) = 0, \qquad 1 \leq i \leq n. \tag{2.12}$$

By introducing the partial derivatives as new variables, it can be written as an exterior differential system

$$\begin{aligned} &F(x^i, z, p_i) = 0, \\ &dz - \sum p_i \, dx^i = 0\,, \end{aligned} \tag{2.13}$$

in the $2n + 1$-dimensional space (x^i, z, p_i).

Obviously, one can 'encode' higher order partial differential equation as well, by simply in regard to the intermediate partial derivatives as dependent variables in their own right, constrained by the obvious PDE needed to make them second order scalar PDE.

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{2.14}$$

written in the classical notation

$$\begin{aligned} &F(x, y, u, p, q, r, s, t) = 0 \\ &du - pdx - qdy = 0 \\ &dp - rdx - sdy = 0 \\ &dq - sdx - tdy = 0 \end{aligned} \tag{2.15}$$

We would explain this to mean that the equation $F = 0$ define a smooth hypersurface $M^7 \subset \mathbb{R}^8$ and the differential equation is then can be modeled by the differential ideal $I \subset \Omega^*(M)$ giving by

$$I = \langle du - pdx - qdy, dp - rdx - sdy, dq - sdx - tdy \rangle. \tag{2.16}$$

The assumption that the PDE be reasonable is then that not all of the partial (F_r, F_s, F_t) vanish along the locus $F = 0$, so that $x, y, u, p, q$and two of r, s and t can be taken as a local coordinate on M.

From the Notion (2.1.4) we can illustration that any system of differential equations can be written as an exterior differential system. However, not all exterior differential systems arise in this way. The following notion marks the birth of differential systems:

Notion (2.1.5):

The equation

$$a_1(x)dx^1 + \cdots + a_n(x)dx^n = 0, x = (x^1, \ldots, x^n), \qquad (2.17)$$

is called a *Pfaffian equation*. Pfaff's to determine its integral manifolds of maximal dimension.

From the (2.17) we notice two important concepts. One is an exterior differential system with independence condition (I, Ω) which is given by the closed differential ideal I together with a decomposable p-form

$$\Omega = \omega^1 \wedge \ldots \wedge \omega^p. \qquad (2.18)$$

An integral manifold of (I, Ω) is an integral manifold of I satisfying the additional condition $f^*\Omega \neq 0$. This is the case when we wish to keep some variables independent, as in the case when the system arises from a system of partial differential equations. For instance, in notion, we take

$$\Omega = dx^1 \wedge \ldots \wedge dx^n. \qquad (2.19)$$

The partial differential equation (2.12) is equivalent to the system to independence condition, where I is generated by the left-hand members of (2.13) and Ω is given by (2.19). Whether an independence condition should be imposed depends on the particular problem.

Definition (2.1.6):

Let M and N be smooth manifolds. The *jet bundle* of k-jets of functions of M two N will be denoted by. The k-jet of a smooth function $\phi: M \longrightarrow N$ at a point x will be denoted by $j_x^k \phi$. If we have coordinates $x^1, \dots, x^m$ for M and $y^1, \dots, y^n$for N, then we can introduce coordinates x^i, y, p_i for $J^1(M, N)$. The 1-jet of the function $\phi: M \longrightarrow N$ at x given by

$$(x^i, y, p_i) = \left(x^i, \phi(x), (\partial\phi/\partial x^i)(x)\right). \tag{2.20}$$

On the second order jet bundle $J^2(M, N)$ we have coordinates x^i, y, p_i, p_{ij}, etc.

Let M, N are differentiable manifolds and $f, g: M \longrightarrow N$ be two maps. Then $f$and g are said to be k-jet at $p \in M$ if:

i. $f(p) = g(p) = q$.
ii. For all maps $u: \mathbb{R} \longrightarrow N$ and $v: N \longrightarrow \mathbb{R}$ with $u(0) = p$, the differentiable maps $v \circ f \circ u$ and $v \circ g \circ u$ have the same k-jet at 0.

On every jet bundle there is a natural ideal of the contact form. In the local coordinates this ideal is generated by contact forms of the form

$$\theta_I = dp_I - p_I dx, \tag{2.21}$$

where I a multi-index $I = (i_1, \dots, i_k)$.

Every transformation of the base manifold $M \times N$ can be prolonged to the unique transformation on the jet bundle $J^k(M, N)$ that preserves contact ideal.

(2.2): Basic Theorems

It is common to go between geometric formulations of a problem, which are coordinate free, and choosing coordinates to get at particular features of the problem. The theorems of Frobenius and Pfaff are useful for finding coordinate systems appropriate to a given problem.

Conceivably the simplest exterior differential systems are those whose differential ideal I is generated algebraically by the form of degree one. Let the generators be

$$\phi^1, \dots, \phi^{n-r}, \tag{2.22}$$

which it supposed to be linearly independent. The condition that I is closed gives

$$d\phi^i \equiv 0, \qquad mod\ \ \phi^1, \dots, \phi^{n-r}, \qquad 1 \le i \le n-r. \tag{2.23}$$

The above condition (2.23) is called the Frobenius condition it's equivalent to requiring that any 1-form $\phi \in I$ satisfies $d\phi = a_i \omega_i$ for some 1-forms $\omega_i \in I$. The exterior derivative of any 1-form is zero modulo the 1-forms of I. A differential system

$$\phi^1 = \cdots = \ \phi^{n-r} = 0, \tag{2.24}$$

satisfying (2.23) is called completely integrable.

Geometrically the ϕ span at every point $x \in M$ a subspace W_x of the dimension $n - r$ in the cotangent space $T_x^* M$ or, what is the same, a subspace $W_x^{\perp}$ of dimension r in the tangent space T_x. The usual presentation of the Frobenius theorem is in terms of distributions on vector fields on C^{∞}. A smooth distribution D is called *involutive* “we will be discussing it in chapter 3.3” if and only if $[X, Y] \in D$ for any smooth vector fields $X, Y \in D$. Notice that the condition (2.23) is intrinsic, i.e., independent of local coordinates, and is also invariant under a linear change of the ϕ with C^{∞}-coefficients.

Theorem (2.2.1): Frobenius Theorem

Let I be a differential ideal having as generators the linearly independent forms $\phi^1, \dots, \phi^{n-r}$ of degree one, so the above condition (2.23) is satisfied. In a sufficiently small neighborhood there is a coordinate system $y^1, \dots, y^n$ such that I generated by $dy^{r+1}, \dots, dy^n$.

The Frobenius theorem gives a normal form of completely integrable system, i.e., the system can be written locally as

$$dy^{r+1} = \cdots = dy^n = 0, \tag{2.25}$$

in a suitable coordinate system. The maximal integral manifolds are

$$y^{r+1} = \text{const}, \dots, y^n = \text{const}, \tag{2.26}$$

and are therefore of dimension r. We say the system defines a *foliation*, of dimension r and codimension $n-r$ of the manifold M. The individual integral manifolds are called the *leaves*.

The simplest non-trivial case of the Frobenius theorem is the system generated by single one form in three spaces. Thus

$$I = \{R\,dx + S\,dy + T\,dz\}, \tag{2.27}$$

and the condition (2.23) are the necessary and sufficient condition that there exists an integrating factor for the form for the one form $\omega = R\,dx + S\,dy + T\,dz$. That is there exists is a function μ such that $\mu\omega$ is true.

The condition (2.23) has a formulation in terms of vector fields, which is also useful. We add to $\phi^1, \dots, \phi^{n-r}$ the r forms $\phi^{n-r+1}, \dots, \phi^n$, so that $\phi^i,\ 1 \le i \le n$, are linearly independent. Then we have

$$\begin{aligned} d\phi^i &= \frac{1}{2}\sum_{j,k} e^i_{jk}\,\phi^j \wedge \phi^k, \qquad 1 \le i, j, k \le n \\ e^i_{jk} &+ e^i_{kj} = 0. \end{aligned} \tag{2.28}$$

The condition (2.23) can be expressed as

$$e_{pq}^{a} = 0, \qquad 1 \leq a \leq n - r, \qquad n - r + 1 \leq p, q \leq n. \quad (2.29)$$

Let f be a smooth function. The equation

$$df = \sum (X_i f)\phi^i, \tag{2.30}$$

define n operators or vector fields X_i, which form a dual base to ϕ^i. Exterior differential of (2.30) gives

$$\frac{1}{2}\sum_{i,j}\left(X_i\left(X_j(f)\right) - X_j\big(X_i(f)\big)\right)\phi^i \wedge \phi^j + \sum_j X_i(f)d\phi^i = 0. \tag{2.31}$$

By substituting (2.28) into (2.31), we get

$$[X_i, X_j]f = (X_i X_j - X_j X_i)f = -\sum e_{ij}^{k}\, X_k f \tag{2.32}$$

Then it follows the condition (2.29) can be written in the form

$$[X_p, X_q]f = -\sum e_{pq}^{s}\, X_s f\,, \qquad n - r \leq p, q, s \leq n \tag{2.33}$$

Equation (2.32) is the dual version of (2.28). The vectors $X_{n-r+1}, \dots, X_n$ span at each point $x \in M$ the subspace $W_x^{\perp}$ of the distribution. Hence the condition (2.23) or (2.29) or (2.33) can be expressed as follows:

Proposition (2.2.2):

Let a distribution M be defined by the subspace $W_x^{\perp} \subset T_x, \dim W_x^{\perp} = r$. The condition (2.23) says that, for any two vector fields X, Y such that $X_x, Y_x \in W_x^{\perp}$ their bracket $[X, Y]_x \in W_x^{\perp}$.

Example (2.2.3):

Consider the overdetermined first order system of partial differential equations for the function u of the variables x and y given by

$$u_x = -Fu, \qquad u_y = -Gu. \tag{2.34}$$

Here F and G are arbitrary functions of x, y. The first order jet bundle $J^1(\mathbb{R}^2, \mathbb{R})$ has coordinates x, y, u, u_x, u_y and contact form

$$\theta = du - u_x dx - u_y dy. \tag{2.35}$$

Let M be the submanifold of the first order jet bundle defined by the two equations (2.34) and use x, u and y as coordinates on M. The solutions of the system (2.34) are locally in one-to-one correspondence the single 1-form

$$\theta = du + Fu\, dx + Gu\, dy, \tag{2.36}$$

with independence condition $\Omega = dx \wedge dy$.

$$\begin{aligned} d\theta &= \left(F_y u\, dy + F\, du\right) \wedge dx + (G_x u\, dx + G\, du) \wedge dy, \\ &= \left(F_y - G_x\right) u\, dx \wedge dy \quad \bmod \theta. \end{aligned} \tag{2.37}$$

The distribution is integrable at points where the compatibility condition $\left(F_y - G_x\right)u$ is satisfied. Since the system is linear, $u(x, y) = 0$ is always solution. Near points where $\left(F_y - G_x\right)u \neq 0$ there are no other solutions to the system. At each point (x_0, y_0) where $F_y - G_x = 0$ on small neighborhood, the distribution is integrable. It follows from the Frobenius theorem that there is a unique integral manifold of the system through the point (x_0, y_0, u_0).

(2.2.1): Cauchy Characteristics

Let I be differential ideal. A vector field ξ such that $\xi \rfloor I \subset I$ is called a *Cauchy characteristic vector field* of I. At a point $x \in M$ we define

$$A(I)_x = \{\xi_x \in T_x M \mid \xi_x \rfloor I_x \subset I_x\}, \tag{2.38}$$

and $C(I)_x = A(I)_x^{\perp} \subset T_x^* M$. These concepts reduce to the ones treated in the last section. In particular, we will call $C(I)_x$ the retracting space at x and call $\dim C(I)_x$ *the class of* I at x. We have now a family of ideals I_x depending on the parameter $x \in M$. When restricting to a point x we have a purely algebraic situation. See Proposition (1.3.9).

Theorem (2.2.4)

Let I be finitely generated differential ideal whose retracting space $C(I)$ has constant dimension $s = n - r$. Then there is a neighborhood in which in which there are coordinates $(x^1, \dots, x^r;\ y^1, \dots, y^s)$ such that I has a set of generators are forms in $y^1, \dots, y^s$ and their differentials.

Proof.

Using proposition (1.3.10) the differential system defined by $C(I)$ (or what is the same, the distribution defined by $A(I)$) is completely integrable. We may choose coordinates $(x^1, \dots, x^r;\ y^1, \dots, y^s)$ so that the foliation so defined is given by

$$y^{\sigma} = \text{const}, \qquad 1 \leq \sigma \leq s. \tag{2.39}$$

By the retraction theorem, I has a set of generators which are forms in dy^{σ}, $1 \leq \sigma \leq s$. But their coefficients may involve $x^{\rho}, 1 \leq \rho \leq r$. The theorem follows when we show that we can choose a new set of generators for I which are forms in the y^{σ} coordinates in which the x^{ρ} don't enter. To exclude the trivial case, we suppose the I is a proper ideal, so that it contains no none-zero functions.

Let I_q be the set of $q-$forms in $I, q = 1,2, \dots$. Let $\varphi^1, \dots, \varphi^p$ be linearly independent 1-forms in I_1 such that any form in I_1 is their linear combination. Since I is closed, $d\varphi^i \in I, 1 \leq i \leq p$. For a fixed ρ we have $\partial/\partial x^{\rho} \in A(I)$, which implies

$$\frac{\partial}{\partial x^{\rho}} \rfloor d\varphi^i = L_{\partial/\partial x^{\rho}} \varphi^i \in I_1, \tag{2.40}$$

since the left-hand side is of degree 1. It follows

$$\frac{\partial \varphi^i}{\partial x^\rho} = L_{\partial/\partial x^\rho}\varphi^i = \sum_j a^i_j \varphi^j, \quad 1 \le i, j \le p, \tag{2.41}$$

where the left-hand side stand for the form obtained from φ^i by taking the partial derivatives of the coefficients with respect to x^ρ.

For this fixed ρ we regard x^ρ as the variable and $x^1, \dots, x^{\rho-1}, x^{\rho+1}, \dots, x^r, y^1, \dots, y^s$ as parameters.

Consider the system of ordinary differential equations

$$\frac{dz^i}{dx^\rho} = \sum_j a^i_j\, z^j, \qquad 1 \le i, j \le p. \tag{2.42}$$

Let $z^i_{(k)},\ 1 \le k \le p$, be a fundamental system of solutions, so that

$$\det\left(z^i_{(k)}\right) \ne 0. \tag{2.43}$$

Then we shall replace φ^i by $\tilde{\varphi}^k$ defined by

$$\varphi^i = \sum z^i_{(k)} \tilde{\varphi}^k. \tag{2.44}$$

By differentiating (2.44) with respect to x^ρ and using (2.42), (2.41) we get

$$\frac{\partial \tilde{\varphi}^k}{\partial x^\rho} = 0, \tag{2.45}$$

so that $\tilde{\varphi}^k$ doesn't involve x^ρ. Applying the process to the other x's, we arrive at a set of generators of I_1 which are forms in y^σ.

Suppose this process carried out for $I_1, \dots, I_{q-1}$, so that they consist of forms in y^σ. Now let J_{q-1} be the ideal generated by $I_1, \dots, I_{q-1}$. Let $\psi^\alpha \in I_q,\ 1 \le \alpha \le r$, be linearly independent $\operatorname{mod} J_{q-1}$, such that any q-form of I_q is congruent mod J_{q-1} to a linear combination of them. By the above argument such forms include

$$\frac{\partial}{\partial x^\rho} \rfloor d\psi^\alpha = L_{\partial/\partial x^\rho}\psi^\alpha. \tag{2.46}$$

Hence we have

$$\frac{\partial \psi^\alpha}{\partial x^\rho} = \sum b^\alpha_\beta \psi^\beta, \quad \text{mod}\, J_{q-1}, \;\; 1 \leq \alpha, \beta \leq r, \tag{2.47}$$

using the above argument, we can replace the ψ^α by $\tilde{\psi}^\beta$ such that $\frac{\partial \tilde{\psi}^\alpha}{\partial x^\rho} \in J_{q-1}$, this means that we can write

$$\frac{\partial \tilde{\psi}^\alpha}{\partial x^\rho} = \sum_h \tau^\alpha_h \wedge \omega^\alpha_h, \tag{2.48}$$

where $\tau^\alpha_h \in I_1 \cup \cdots \cup I_{q-1}$ and are therefore forms in y^σ. Let θ^α_h be defined by

$$\frac{\partial \theta^\alpha_h}{\partial x^\rho} = \omega^\alpha_h, \tag{2.49}$$

then the forms

$$\tilde{\tilde{\psi}}^\alpha = \tilde{\psi}^\alpha - \sum_h \tau^\alpha_h \wedge \theta^\alpha_h, \tag{2.50}$$

don't involve x^ρ and can be used to replaced ψ^α. Applying this process to all x^ρ, $1 \leq \rho \leq r$, we find a set of generators for I_q, which are forms in y^σonly. ■

Definition (2.2.5):

The leaves defined by the distribution $A(I)$ are called the *Cauchy characteristics*.

Notice that generally r is zero, so that a differential system generally doesn't have Cauchy characteristics (i.e., they are points). The Theorem (2.2.4) allows us to locally reduce a differential ideal to a system in which there are no extraneous variables in the sense that all coordinates are needed to express I in any coordinate system. Thus the class of I equal the minimal number of variables needed to describe the system.

Now we will introduce a useful corollary of Theorem (2.2.4) which illustrates its geometric content is the following:

Corollary (2.2.6):

Let $f: M \to M'$ be a fibration with vertical distribution $V \subset T(M)$ with connected fibers over $x \in M'$ given by $(\ker f_*)_x$. Then a form α on M is the pull-back $f^*\alpha'$ of a form α' on M if and only if

$$v \rfloor \alpha = 0 \quad \text{and}\; v \rfloor d\alpha = 0, \;\; \forall\, v \in V. \tag{2.51}$$

We will apply this theorem to the first order partial differential equation

$$F\left(x', z, \frac{\partial z}{\partial x^i}\right) = 0, \qquad 1 \le i \le n. \tag{2.52}$$

Following the Notion (2.1.4) that we mentioned in the first section in this chapter, and we get equation (2.12); this equation can be formulated as the exterior differential systems in equation (2.13). To these equations we add their exterior derivatives to obtain

$$\begin{aligned} &F\left(x^i, z, p_i\right) = 0,\\ &dz - \sum p_i \, dx^i = 0\\ &\sum \left(F_{x^i} + F_z p_i\right) dx^i + \sum F_{p_i} dp_i = 0,\\ &\sum dx^i \wedge dp_i = 0. \end{aligned} \tag{2.53}$$

These equations are in the $(2n+1)$-dimensional space $\left(x^i, z, p_i\right)$. The corresponding differential ideal is generated by the left-hand members of (2.53).

To determine the space $A(I)$ consider the vector

$$\xi = \sum u^i \, \partial/\partial x^i + u \, \partial/\partial z + \sum v_i \, \partial/\partial p_i, \qquad (2.54)$$

and express the condition that the interior product $\xi \rfloor$ keeps the ideal stable. This gives

$$\begin{aligned} &u - \sum p_i \, u^i = 0, \\ &\sum \left(F_{x^i} + F_z \, p_i\right) u^i + F_{p_i} v_i = 0, \\ &\sum \left(u^i dp_i - v_i dx^i\right) = 0. \end{aligned} \qquad (2.55)$$

By comparing the third equation in (2.55) by third equation in (2.53) we have get

$$u^i = \lambda F_{p_i}, \qquad v_i = -\lambda\left(F_{x^i} + F_z p_i\right), \qquad (2.56)$$

then the first equation in (2.55) gives

$$u = \lambda \sum p_i F_{p_i}. \qquad (2.57)$$

The parameter λ being arbitrary, equations (2.56), (2.57) show that $\dim A(I) = 1$, i.e., the characteristic vectors at each point from a one-dimensional space. This fundamental fact is the key of the theory of partial differential equations of the first order. The *characteristic curves* in the space (x^i, z, p_i), or *characteristic strips* in the classical terminology, are the integral curves of the differential system

$$\frac{dx^i}{F_{p_i}} = -\frac{dp_i}{F_{x^i} + F_z p_i} = \frac{dz}{\sum p_i F_{p_i}}. \qquad (2.58)$$

These are the *equations of Charpit and Lagrange*. To construct an integral manifold of dimension n it suffices to take an $(n-1)$-dimensional integral manifold transverse to the Cauchy characteristic vector field and draw the characteristic strips through its points. Putting it in another way, an n-dimensional integral manifold is generated by characteristic strips.

Now we'll explain the meaning of the terminology "strips". We remark that points in (x^i, p_i)-space may be thought of as hyper-planes $\sum p_i dx^i = 0$ in the tangent spaces $T_x(\mathbb{R}^n)$. A curve in (x^i, z, p_i)-space projects to a curve in (x^i, p_i)-space, which is geometrically a 1-parameter family of the tangent hyper-planes.

Example (2.2.7):

Consider the initial value problem for the partial differential equation

$$z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1, \tag{2.59}$$

with initial data given along $y = 0$ by $z(x, 0) = \sqrt{x}$

Let us introduce coordinates $J^1(2, 1)$ by (x, y, z, p, q). This initial data $D: \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ where $D(x) = (x, 0, \sqrt{x})$ is extended to a map $\delta: \mathbb{R} \to J^1(2, 1)$ where the image satisfies the equation and the strip condition

$$0 = \delta^*(dz - pdx - qdy) = \frac{dx}{2\sqrt{x}} - pdx, \tag{2.60}$$

$p = \frac{1}{2\sqrt{x}},\ q = \frac{1}{2}$ and δ is unique. In general, there are several choices of δ due to non-linearity of the equation. The extend data becomes

$$\delta(x) = \left(x, 0, \frac{1}{2\sqrt{x}}, \frac{1}{2}\right). \tag{2.61}$$

If we parameterize the equation by $i: \Omega \to J^1(2,1)$ where $i(x, y, z, p) = (x, y, z, p, 1 - zp)$, then the data can be pulled back to a map $\Delta: \mathbb{R} \to \Omega$, where $\Delta(s) = \left(s, 0, \sqrt{s}, \frac{1}{2\sqrt{s}}\right)$.

The Cauchy characteristic vector field is

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - p^2\frac{\partial}{\partial p} + \frac{\partial}{\partial z} \tag{2.62}$$

and the corresponding flow is given by

$$\frac{dx}{dt} = z, \qquad \frac{dy}{dt} = 1, \qquad \frac{dz}{dt} = 1, \qquad \frac{dp}{dt} = -p^2. \tag{2.63}$$

The solution for the given data representing the union of characteristic curves along the data is

$$x = \frac{t^2}{2} + t\sqrt{s} + s, \qquad y = t, \qquad z = t + \sqrt{s}. \tag{2.64}$$

And eliminating s and t gives an implicit equation $z(x, y)$ by

$$z^2 + zy = x - (y^2/2). \tag{2.65}$$

Theorem (2.2.8):

Consider the Eikonal differential equation

$$\sum \left(\partial z/\partial x^i\right)^2 = 1, \qquad 1 \leq i \leq n. \tag{2.66}$$

If $z = z(x^1, \dots, x^n)$ is a solution valid for all $(x^1, \dots, x^n) \in E^n$, where ($E^n = n$-dimensional Euclidean space), then z is a linear function in x^i, i.e.,

$$z = \sum a_i x^i + b, \tag{2.67}$$

where $a^i, b$are constants satisfying $\sum a_i^2 = 1$.

Proof:

We'll denote by E^{n+1} the space of $(x^1, \dots, x^n, z)$, and identify E^n with the hyper-plane $z = 0$. The solution can be interpreted as a graph Γ in E^{n+1} having a one-one projection to E^n. For the equation (2.66) the denominators in the middle term of (2.58) are zero, so that the Cauchy characteristic satisfy

$$p_i = \text{const.}$$

The equation (2.58) can be integrated and the Cauchy characteristic curves, when projected to E^{n+1}, are the straight lines

$$x^i = x_0^i + p_i t, \qquad z = z_0 + t \tag{2.68}$$

where x_0^i, p_i, z_0 are constant. Hence the graph Γ must have the property that it's generated by "Cauchy lines" (2.68), whose projections in E^n form a foliation of E^n.

By developing the notation in the first equation of (2.68), we write it as

$$x^{*i} = x^i + \frac{\partial z}{\partial x^i} t, \tag{2.69}$$

where $z = z(x^1, \dots, x^n)$ is a solution of equation (2.66). For a given $t \in R$ this can be interpreted as a diffeomorphism $f_t: E^n \to E^n$ defined by

$$f_t(x) = x^* = \left(x^{*1}, \dots, x^{*n}\right), \qquad x, x^* \in E^n. \tag{2.70}$$

Geometrically it maps $x \in E^n$ to the point x^* at a distance t along the Cauchy line through x; this makes sense, because the Cauchy lines are oriented. Its Jacobean determinant is

$$J(t) = \det\left(\delta_j^i + \frac{\partial^2 z}{\partial x^i \partial x^j} t\right) \neq 0, \tag{2.71}$$

but this implies

$$\frac{\partial^2 z}{\partial x^i \partial x^j} = 0, \tag{2.72}$$

and hence that z is linear. For if (2.72) isn't true, then the symmetric matrix $\partial^2 z / \partial x^i \partial x^j$ has a real non-zero eigenvalue, say λ, and $J(-1/\lambda) = 0$, which is contradiction. ■

Remark (2.2.9):

The function

$$z = \left(\sum_i (x^i)^2\right)^{1/2} \tag{2.73}$$

satisfies (2.66), except at $x^i = 0$. Hence Theorem (2.2.8) needs the hypothesis that (2.66) is valid for all $x \in E^n$.

(2.2.2): Theorems of Pfaff and Darboux

There is another case in which there is a simple exterior differential system is one which consists of a single equation

$$\phi = 0 \tag{2.74}$$

where ϕ is a form of degree one. The above equation was studied by Pfaff in the early 19th century. The differential ideal $I(\Lambda)$ is generated by ϕ and $d\phi$. The integer $r = r(x)$ define by

$$\phi \wedge (d\phi)^r \neq 0, \qquad \phi \wedge (d\phi)^{r+1} = 0, \tag{2.75}$$

is called *the rank* of the equation (2.74). It depends on the point $x \in M$. Clearly, the rank remains unchanged under the transformation $\phi \to a\phi, \quad a \neq 0$. It is also easy to see that the rank is locally constant.

Putting it in a different way, the 2-form $d\phi \bmod \phi$, has an even rank $2r$ in the sense of linear algebra.

The following proposition is an immediate consequence of the Frobenius Theorem:

Proposition (2.2.10):

The rank of the system $\Lambda = \phi = 0$ is identically zero if and only if the system is completely integrable.

When the rank of Λ is not zero the closure of the system,

$$\Lambda \subset (\Lambda \cup d\Lambda),$$

imposes extra conditions; the problem of finding integral manifolds of (2.74) is clarified by the normal form, given by the theorem:

Theorem (2.2.11): (The Pfaff Problem)

Suppose the equation $\Lambda = \{\phi = 0\}$ has the constant rank r. Then there are local coordinates $\omega^1, \dots, \omega^n$ such that

$$\phi = a(d\omega^1 + \omega^2 d\omega^3 + \cdots + \omega^{2r} d\omega^{2r+1}), \qquad (2.76)$$

where $a = a(\omega^1, \dots, \omega^n)$ is never zero. In particular, the system may be replaced by the equation

$$d\omega^1 + \omega^2 d\omega^3 + \cdots + \omega^{2r} d\omega^{2r+1} = 0 \qquad (2.77)$$

Proof:

For $r = 0$ the result follows from Proposition (2.2.10). We will do an induction on r. It is routinely verified that the dual associated space $A\big(I(\Lambda)\big)^{\perp}$ has dimension $2r + 1$. So there are local coordinates $w^1, \dots, w^n$ such that $I(\Lambda)$ can be generated by some forms in $w^1, \dots, w^{2r+1}$. In particular, the form θ can be written as

$$\theta = a\theta', \qquad a = a(w^1, \dots, w^n) \neq 0, \qquad (2.78)$$

where θ' is a form in $w^1, \dots, w^{2r+1}$ only. We need to normalize θ'. For this we work inside the $w^1, \dots, w^{2r+1}$-space, which we denote by $U^{2r+1} \subset M$. Consider the differential ideal generated by $d\theta', I(d\theta') \subset \Lambda^*(U^{2r+1})$. Since $(d\theta')^r \neq 0$, the dual associated space has dimension $2r$, hence the associated space $A\big(I(d\theta')\big)$ is one-dimensional. The ideal $I(d\theta')$ is then generated by a 2-form α in $2r$ variables $y^1, \dots, y^{2r}; d\theta'$ differs from α by a factor. Write

$$d\theta' = b\alpha, \qquad b = b(w^1, \dots, w^{2r+1}) \neq 0. \qquad (2.79)$$

Now $(d\theta')^r = b^r \alpha^r \neq 0$, hence

$$\alpha^r = c \ dy^1 \wedge \cdots \wedge dy^{2r}, \tag{2.80}$$

for some $c = c(y^1, \dots, y^{2r}) \neq 0$. Using the fact that $(d\theta')^r$ is a closed form,

$$db \ \wedge dy^1 \wedge \cdots \wedge dy^{2r} = 0. \tag{2.81}$$

This means that b is a function of the $y^{i,}$s. So $d\theta'$ is a form in the $y^{i,}$s. Since $d\theta'$ is closed there is nonzero 1-form γ in the $y^{i,}$s with $d\gamma = d\theta'$. Being in a $2r$-dimensional space the equation $\gamma = 0$ can't have rank r. So it must have rank $r - 1$. By induction hypothesis we can then write

$$\gamma = \lambda \, (dz^1 + z^2 dz^3 + \cdots + z^{2r-2} dz^{2r-1}), \tag{2.82}$$

where $\lambda = \lambda \, (y^1, \dots, y^{2r}) \neq 0$. Since $d\theta' = d\gamma$ we can find a function f such that $\theta' = df + \gamma$. Put

$$\omega^1 = f, \omega^2 = \lambda, \omega^3 = z^1, \omega^4 = \lambda z^2 \ , \dots, \ \ \omega^{2r} = \lambda z^{2r-2}, \omega^{2r+1} = z^{2r-1}$$

it follows that

$$\theta' = d\omega^1 + \omega^2 d\omega^3 + \cdots + \omega^{2r} d\omega^{2r+1}. \tag{2.83}$$

Let M be a $(2r+1)$-dimensional manifold, and consider the equation (2.74) of maximal rank r. The above theorem tells us that the general maximal integral manifold of dimension r and is given by

$$\begin{aligned} &\omega^1 = f \, (\omega^3, \omega^5, \dots, \omega^{2s+1}), \qquad s < r \\ &\omega^2 = \frac{\partial f}{\partial \omega^3} \ , \dots, \omega^{2r} = \frac{\partial f}{\partial \omega^{2r+1}}, \end{aligned} \tag{2.84}$$

where f is arbitrary function.

Corollary (2.2.12): (Symmetric normal form)

In a neighborhood suppose the equation (2.74) has a constant rank r. Then there exist independent functions $z, y^1, \dots, y^r, \ x^1, \dots, x^r$ such that the equation becomes

$$dz + \frac{1}{2}\sum_{i=1}^{r}\left(y^i dx^i - x^i dy^i\right) = 0 \tag{2.85}$$

Proof

It suffices to apply the change of coordinates

$$\begin{aligned} &\omega^1 = z - 1/2\sum x^i y^i\,, \\ &\omega^{2i} = y^i, \qquad \omega^{2i+1} = x^i, \qquad 1 \le i \le r. \end{aligned} \tag{2.86}$$

Related to the Pfaffian problem are the normal forms for the forms themselves and not the ideals generated by them. For 1-forms and closed 2-forms we have the following theorems.

Theorem (2.2.13): (Darboux Theorem)

If Ψ is a closed 2-form satisfying

$$\Psi^r \neq 0, \qquad \Psi^{r+1} = 0, \qquad r = \text{const}. \tag{2.87}$$

Locally there exist coordinates $\omega^1, \dots, \omega^n$ such that

$$\Psi = d\omega^1 \wedge d\omega^2 + \cdots + d\omega^{2r-1} \wedge d\omega^{2r}. \tag{2.88}$$

We consider the case of 1-form ϕ. The rank r is defined by the condition

$$\phi \wedge (d\phi)^r \neq 0, \qquad \phi \wedge (d\phi)^{r+1} = 0. \tag{2.89}$$

There is a second integer s defined by

$$(d\phi)^s \neq 0, \qquad (d\phi)^{s+1} = 0. \tag{2.90}$$

Elementary arguments illustrate there are two cases:

a) $s = r$
b) $s = r + 1$.

Example (2.2.14):

Consider a single first order PDE of the form

$$F(x^1, \dots, x^n, u, \partial u/\partial x^1, \dots, \partial u/\partial x^n) = 0. \tag{2.91}$$

Assuming F its cut a hyper-surface

$$M = \{(x^1, \dots, x^n, u, p_1, \dots, p_n) \mid F(x^1, \dots, x^n, u, p_1, \dots, p_n) = 0\} \subset J^1(\mathbb{R}^n, \mathbb{R}).$$

For

$$\phi = du - p_1 dx^1 - \cdots - p_n dx^n, \tag{2.92}$$

the PDE is encoded by the EDS

$$I = \langle \phi \rangle.$$

We have $\phi \wedge (d\phi)^n = 0$, while $\phi \wedge (d\phi)^{n-1}$ is nowhere vanishing, and therefore by the Pfaff theorem we can find local coordinates $(z, y^1, \dots, y^{n-1}, v, q_1, \dots, q_{n-1})$ on M on which I is given by

$$\langle dv - q_1 dy^1 - \cdots - q_{n-1} dy^{n-1} \rangle. \tag{2.93}$$

Now we observe that an n-dimensional integral manifold of I is locally of the form

$$v = g(y^1, \dots, y^{n-1}), \qquad q_i = \frac{\partial g}{\partial y_i}(y^1, \dots, y^{n-1}), \tag{2.94}$$

for some function $g: \mathbb{R}^{n-1} \to \mathbb{R}$. In particular, its tangent to the vector field $z = \partial/\partial z$.

Theorem (2.2.15):

Let ϕ be a 1-form. In a neighborhood suppose r and s be constant. Then ϕ has the normal form

$$\begin{aligned} \phi &= y^0 dy^1 + \cdots + y^{2r} dy^{2r+1}, \quad if\ r + 1 = s \\ \phi &= dy^1 + y^2 dy^3 + \cdots + y^{2r} dy^{2r+1}, \ if\ r = s \end{aligned} \tag{2.95}$$

Proof:

let I be a differential ideal generated by ϕ and $d\phi$. By Theorem (2.2.11) there're coordinates $y^1, \dots, y^n$ in a neighborhood such that

$$\phi = u(dy^1 + y^2 dy^3 + \cdots + y^{2r} dy^{2r+1}). \tag{2.96}$$

By changing the notation that allows us to write

$$\phi = z^0 dy^1 + z^2 dy^3 + \cdots + z^{2r} dy^{2r+1}, \tag{2.97}$$

then

$$\begin{gathered}(d\phi)^{r+1} = c dz^0 \wedge dy^1 \wedge dz^2 \wedge dy^3 \wedge \cdots \wedge dz^{2r} \wedge dy^{2r+1}, \\ c = \text{const}, \qquad \text{c} \neq 0.\end{gathered} \tag{2.98}$$

If $s = r + 1, s \neq 0$, and the functions $z^0, z^2, \dots, z^{2r}, y^1, y^3, \dots, y^{2r+1}$ are independent. This proves the normal form of first equation (2.95).

Consider next the case $r = s$. Then $d\phi$ is a 2-form of rank $2r$. By Darboux Theorem (2.2.13) we can write

$$\begin{aligned} d\phi &= d\omega^1 \wedge d\omega^2 + \cdots + d\omega^{2r-1} \wedge d\omega^{2r} \\ &= d(\omega^1 d\omega^2 + \cdots + \omega^{2r-1} d\omega^{2r}). \end{aligned} \tag{2.99}$$

Hence the form

$$\phi = (\omega^1 d\omega^2 + \cdots + \omega^{2r-1} d\omega^{2r}), \tag{2.100}$$

is closed, and is equal to dv. By changing the notation is gives us the second equation (2.95).

Remark (2.2.16):

- A manifold of dimension $2r + 1$ provided with a 1-form ϕ, defined up to a factor, such that

$$\phi \wedge (d\phi)^r \neq 0, \tag{2.101}$$

is called a *contact manifold*. An example is the projectiveized cotangent bundle of a manifold, whose points are the non-zero 1-forms on the base manifold defined up to a factor.

- A manifold of dimension $2r$ provided with a closed 2-form of maximum rank $2r$ is called a *symplectic manifold*.

Both contact manifolds and symplectic manifolds are play a fundamental role in theoretical mechanics and partial differential equations.

Theorem (2.2.17): (Caratheodory Theorem)

Suppose the rank of the Pfaffian equation $\phi = 0$ be constant. It has local accessibility property if and only if

$$\phi \wedge d\phi \neq 0. \qquad (2.102)$$

(2.3): Pfaffian Systems

A pfaffian system is an exterior differential system

$$\phi^1 = 0 = \cdots = \phi^s = 0, \qquad (2.103)$$

where ϕ^i are *Pfaffian forms* and linearly independent, and $s = \text{const}$, that is exterior differential forms of degree one. We will denote the Pfaffian system by I and s its dimension. Therefore, the two form is described by

$$d\phi^i \ \text{mod}\ (\phi^1, \dots, \phi^s) \ \ 1 \leq i \leq s. \qquad (2.104)$$

The Frobenius condition is equivalent to saying that they are zero. Geometrically the ϕ^i span at every point $x \in M$ a subspace W_x^* of dimension s in the cotangent space T_x^*.

We can view $I \in \Omega^1(M)$ as the sub-module over $C^\infty(M)$ of 1-forms

$$\phi = \sum_i f_i \phi^i. \qquad (2.105)$$

We denote by $\{I\} \subset \Omega^*(M)$ the algebraic ideal generated by I. Therefore, $\beta \in \{I\}$ is of the form

$$\beta = \sum_i \vartheta_i \wedge \phi^i, \tag{2.106}$$

where ϑ_i are differential forms. The exterior derivative induces a mapping $\delta: I \longrightarrow \Omega^2(M)|\{I\}$ that is linear over $C^\infty(M)$. We set $I^{(1)} = \ker \delta$, and call $I^{(1)}$ the first derived system. Thus, we have

$$0 \longrightarrow I^{(1)} \longrightarrow I \xrightarrow{\delta} dI|\{I\} \longrightarrow 0, \tag{2.107}$$

and we get from Frobenius case $I^{(1)} = I$. Now I is the space of C^∞ sections of sub-bundle $W \subset T^*M$ with fibers $W_x = \operatorname{span}\left(\phi^1(x), \dots, \phi^s(x)\right)$. The images of $W \otimes \Lambda^q T^*M \longrightarrow \Lambda^{q+1}T^*M$ are sub-bundles $W^{q+1} \subset \Lambda^{q+1}T^*M$, and the mapping δ is induced form a bundle mapping

$$W \xrightarrow{\bar{\delta}} \Lambda^2 T^*M|W^2. \tag{2.108}$$

Suppose that $\bar{\delta}$ has constant rank, so $I^{(1)}$ is the sections of a sub-bundle $W_1 \subset W \subset T^*M$.

Continuing with this construction we arrive at a filtration

$$I^{(k)} \subset \cdots \subset I^{(1)} \subset I^{(0)} = I, \tag{2.109}$$

defined inductively by

$$I^{(k+1)} = \left(I^{(k)}\right)^{(1)}, \tag{2.110}$$

where the above filtration corresponds to a flag of bundles $W_k \subset \cdots \subset W_1 \subset W$. So when $k = N$, such N is smallest integer, $W_{N+1} = W_N$, $I^{(N+1)} = I^{(N)}$. When (2.109) is represented the derived flag of I_0 and N the derived length. Note that $I^{(N)}$ is the largest integrable subsystem contained in I. Then, we can also define the integers

$$
\begin{aligned}
p_0 &= \dim I^{(N)}\,, \\
p_{N-i} &= \dim I^{(i)} \big| I^{(i+1)}\,, \qquad 0 \leq i \leq N-1, \\
p_{N+1} &= \dim C(I) | I\,.
\end{aligned} \tag{2.111}
$$

An integral manifold of I annihilates all the elements of its derived flag, and in particular those of $I^{(N)}$. A function g with differential $dg \in I^{(N)}$ is called a first integral of I, since its constant on all integral manifolds of I. There are two other integers, which can be defined for a Pfaffian system I. The wedge length or the Engel half-rank of I is the smallest integer ρ such that

$$
(d\phi)^{\rho+1} \equiv \ \mod \{I\}, \qquad \forall \phi \in I. \tag{2.112}
$$

The Cartan rank of I is the smallest integer v such that there exist $\pi^1, \dots, \pi^v$ in $\Omega^1(M)|I$ with $\pi^1 \wedge \dots \wedge \pi^v \neq 0$, and

$$
d\phi \wedge \pi^1 \wedge \dots \wedge \pi^v \equiv 0 \mod \{I\}, \qquad \forall \phi \in I. \tag{2.113}
$$

Then, we will introduce a simple properties concerning the wedge length and the Cartan rank.

Definition (2.3.1):

The Cartan system of I is defined as the Pfaffian system generated as a differential ideal by the 1-forms that annihilate all Cauchy characteristic vector fields. The class of an exterior differential system is by definition the rank of its Cartan system. Then we have the following proposition:

Proposition (2.3.2):

The Cartan system $C(I)$ of any exterior differential system I is a completely integrable Pfaffian system. The following retraction theorem shows that the first integrals of the Cartan system $C(I)$ provide a minimal set of local coordinates with which one can express the generators of I.

Proposition (2.3.3):

Let I be a Pfaffian system and ρ its wedge length. Then all $(\rho + 1)$-fold products of the elements in $dI \mod \{I\}$ are zero.

Proof

If I is given by the equation (2.103), an element of the module I is

$$\phi = \varepsilon_1 \phi^1 + \cdots + \varepsilon_s \phi^s \tag{2.114}$$

where ε_i are arbitrary smooth functions and can be considered as indeterminate. Then, the assumption implies

$$(\varepsilon_1 d\phi^1 + \cdots + \varepsilon_s d\phi^s)^{\rho+1} \equiv 0 \mod \{I\}, \tag{2.115}$$

then, by expanding the left-hand side of the above equation and equating to zero the coefficients of the resulting polynomial in the ε_i, that which proof the proposition ■

Proposition (2.3.4):

Between the wedge length ρ and the Cartan rank v the following inequalities hold

$$\rho \leq v \leq 2\rho. \tag{2.116}$$

Proof

The condition that $d\phi \wedge \pi^1 \wedge \cdots \wedge \pi^v \equiv 0 \mod \{I\},\ \forall \phi \in I$, can be written in the form

$$d\phi \equiv 0 \mod \{I, \pi^1, \dots, \pi^v\}. \tag{2.117}$$

Hence

$$(d\phi)^{v+1} \equiv 0 \mod \{I\}, \qquad \rho \leq v. \tag{2.118}$$

Bu using the definition of ρ there exists $\eta \in I$, such that

$$(d\phi)^{\rho} \not\equiv 0, \qquad (d\eta)^{\rho+1} \equiv 0 \mod \{I\}. \tag{2.119}$$

Then, by Darboux Theorem (2.2.13) and by Proposition (2.2.3), we get

$$d\phi \wedge (d\eta)^{\rho} \equiv 0 \mod \{I\}, \qquad \forall \phi \in I. \tag{2.120}$$

It follows that $v \leq 2\rho$.

Remark (2.3.5):

The bounds for v in (2.116) are sharp. The lower bound is achieved by a system consisting of a single equation. To achieve the upper bound, consider in $\mathbb{R}^{3\rho+3}$ with the coordinates $(x_{1k}, x_{2k}, x_{3k}, y^1, y^2, y^3), 1 \leq k \leq \rho$, the Pfaffian system

$$\begin{aligned} \phi^1 &= dy^1 + \sum_k x_{2k} dx_{3k}\,, \\ \phi^2 &= dy^2 + \sum_k x_{3k} dx_{1k}\,, \\ \phi^3 &= dy^3 + \sum_k x_{1k} dx_{2k}\,. \end{aligned} \tag{2.121}$$

This system has $v = 2\rho$.

Proposition (2.3.6):

With our notations the following inequalities hold

$$s + 2\rho \leq \dim C(I) \leq s + \rho + p_N\rho. \tag{2.122}$$

Proof

We remark that $C(I)$ is the retracting subspace of I. Then using the definition of ρ the left-hand side inequality is obvious.

Then we will prove the inequality at the right-hand side we recall that by (2.111)

$$p_N = \dim I|I_1.$$

We choose a basis of I such that

$$(d\phi^1)^\rho \not\equiv 0 \mod \{I\}. \tag{2.123}$$

Darboux theorem shows the left-hand side is a monomial, which can be written

$$(d\phi^1)^\rho = \beta^1 \wedge \cdots \wedge \beta^{2\rho} \neq 0 \mod \{I\}, \tag{2.124}$$

when β^i are1-forms. By proposition (2.3.3) we have

$$(d\phi^1)^\rho \wedge d\phi^j \equiv 0 \mod \{I\}, \qquad 2 \leq j \leq p_N, \tag{2.125}$$

then it is follows.

Remark (2.3.7):

The lower bound for $\dim C(I)$ is achieved by a system consisting of a single equation. To reach the upper bound consider the contact system

$$I = \left\{dz^\lambda - \sum p_i^\lambda dx^i\right\}, \qquad 1 \leq i \leq m, 1 \leq j \leq n, \tag{2.126}$$

in the space $\left(x^i, z^\lambda, p_i^\lambda\right)$. For this system we have

$$I_1 = 0, \qquad s = p_N = n, \qquad \rho = v = m,$$

and

$$\dim C(I) = mn + m + n.$$

Theorem (2.3.8): (Bryant normal form)

Let $I = \{\phi^1, \dots, \phi^s\}$ be a differential system with $I_1 = 0$. If

$$\dim C(I) = s + vs + v, \qquad s \geq 3, \tag{2.127}$$

there is local coordinate system containing the coordinates $x^i, z^\lambda, p_i^\lambda, 1 \leq i \leq v$, $1 \leq \lambda \leq s$, such that

$$I = \left\{dz^\lambda - \sum p_i^\lambda dx^i\right\}. \tag{2.128}$$

Proof

Using the definition of v there exist $\pi^1, \dots, \pi^v$, such that

$$\begin{aligned} &\pi^1 \wedge \cdots \wedge \pi^v \neq 0 \mod I, \\ &d\phi^\lambda \wedge \pi^1 \wedge \cdots \wedge \pi^v \equiv 0 \mod I. \end{aligned} \tag{2.129}$$

The second relation can have written as

$$d\phi^\lambda \equiv \sum \eta_i^\lambda \wedge \pi^i \mod I. \tag{2.130}$$

The relation (2.127) implies that the forms $\phi^\lambda, \pi^i, \eta_i^\lambda$ are linearly independent. By exterior differentiation of the last relation, we get

$$\sum \eta_i^\lambda \wedge \pi^i \equiv 0 \mod \{I, \pi^1, \dots, \pi^v\}, \tag{2.131}$$

which implies

$$d\pi^i \equiv 0 \mod \left\{I, \pi^1, \dots, \pi^v, \eta_i^\lambda\right\}. \tag{2.132}$$

Since $s \geq 3$, this possible only when

$$d\pi^i \equiv 0 \mod I, \pi^1, \dots, \pi^v. \tag{2.133}$$

It follows that the system

$$J = \{\phi^1, \dots, \phi^s, \pi^1, \dots, \pi^v\}, \tag{2.134}$$

is completely integrable, so we can write

$$J = \{d\xi^1, \dots, d\xi^{s+v}\}, \tag{2.135}$$

where ξ^i are the first integrals. Then we have

$$\phi^\lambda = \sum b_A^\lambda d\xi^A, \qquad 1 \leq A \leq s + v, \tag{2.136}$$

in which we can assume that the $(s \times s)$-minor at the left-hand side of the matrix $\left(b_A^\lambda\right) \neq 0$. We write

$$\xi^{\lambda} = z^{\lambda}, \qquad \xi^{s+i} = x^{i}, \qquad 1 \leq \lambda \leq s, 1 \leq i \leq v, \tag{2.137}$$

therefore, as we have $x^i, z^\lambda, p_i^\lambda$ are linearly independent, we can suppose

$$I = \left\{dz^{\lambda} - \sum p_i^{\lambda} dx^i\right\}. \tag{2.138}$$

Remark (2.3.9):

Theorem (2.3.8) is true for $s = 1$, in which case it reduce to the Pfaffian problem. It is not true for $s = 2$. An important counter-example is the following

Consider in $\mathbb{R}^5$ a Pfaffian system

$$I = \{\phi^1, \phi^2\}, \tag{2.139}$$

satisfying

$$d\phi^1 \equiv \phi^3 \wedge \phi^4, \qquad d\phi^2 \equiv \phi^3 \wedge \phi^5, \ \bmod I, \tag{2.140}$$

where $\phi^i,\ i = 1, 2, \dots, 5$ are linearly independent 1-forms. We have $I_1 = 0$ and

$$s = 2, \qquad v = 1, \dim C(I) = 5, \tag{2.141}$$

then, the hypotheses of Theorem (2.3.8) are satisfied.

Remark (2.3.10):

The conclusion of theorem (2.3.8) remains valid, if the condition (2.127) is replaced by

$$\dim C(I) = s + \rho s + \rho. \tag{2.142}$$

The proof is depending on an algebraic argument to show that $\rho = v$.

Example (2.3.11):

Let $\mathbb{R}^4$ be endowed with coordinates (x, y, z, u), and let M be the open subset $\mathbb{R}^4$ defined by $u > 0,\ y > 0,\ x + z > 0$. On M consider the Pfaffian system I defined by

$$I = \{\theta^1, \theta^2, d\theta^1, d\theta^2\}, \tag{2.143}$$

where

$$\begin{aligned} \theta^1 &= u^2(x+z)(dx + 2\,dz), \\ \theta^2 &= y^4(dy + u\,du). \end{aligned} \tag{2.144}$$

The surfaces given the intersection of the parabolic cylinder $c_1 = 2y + u^2$ with the hyperplanes $c_2 = x + 2z$ in M, where c_1, c_2 are real constant, are the integral manifolds of maximal dimension of I.

Example (2.3.12):

We work on $\mathbb{R}^{2n+1}$ with coordinates $(x, u^1, \dots, u^n, p_1, \dots, p_n)$, and consider the Pfaffian system

$$I = \{du^i - p^i dx, dp^i \wedge dx, \qquad 1 \le i \le n\}. \tag{2.145}$$

This Pfaffian system admits all the curves $(x(t), u^1(t), \dots, u^n(t), p_1(t), \dots, p_n(t))$ such that

$$u_t^i - p^i x_t = 0, \tag{2.146}$$

as 1-dimensional integral manifolds. The integral manifolds of I thus depend on n arbitrary C^∞ function of one variable.

Example (2.3.13):

On consider the non-Pfaffian system

$$I = \left\{\sum_{i=1}^{n} dp_i \wedge dx^i\right\}, \tag{2.147}$$

on $\mathbb{R}^{2n}$, with coordinates $(u^1, \dots, u^n, p_1, \dots, p_n)$. The sub-manifolds $\left(u^1, \dots, u^n, p_1 = \frac{\partial f}{\partial x^1}, \dots, p_n = \frac{\partial f}{\partial x^n}\right)$ are integral manifolds of I of dimension n for any choice of a C^∞ function f of n variables.

Example (2.3.14):

On $\mathbb{R}^8$ with coordinates (x, y, u, p, q, r, s, t), we consider the Pfaffian system

$$I = \{\theta^1, \theta^2, \theta^3, d\theta^1, d\theta^2, d\theta^3, dF\}, \tag{2.148}$$

where

$$\begin{aligned} \theta^1 &= du - pdx - qdy, \\ \theta^2 &= dp - rdx - sdy, \\ \theta^3 &= dq - sdx - tdy, \end{aligned} \tag{2.149}$$

and $F: \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is a smooth function such that $(F_r, F_s, F_t) \neq (0, 0, 0)$. The surfaces

$$\big(x(w,z), y(w,z), u(w,z), p(w,z), q(w,z), r(w,z), s(w,z), t(w,z)\big),$$

such that

$$F\big(x(w,z), y(w,z), u(w,z), p(w,z), q(w,z), r(w,z), s(w,z), t(w,z)\big) = 0,$$

Note that if

$$\left|\frac{\partial(x,y)}{\partial(w,z)}\right| \neq 0, \tag{2.150}$$

then the integral surface of I can be locally parametrized as graphs of the form

$$\big(x, y, u(x,y), p(x,y), q(x,y), r(x,y), s(x,y), t(x,y)\big), \tag{2.151}$$

with

$$p = u_x, \qquad q = u_y, \qquad r = u_{xx}, \qquad s = u_{xy}, \qquad t = u_{yy}, \tag{2.152}$$

and $u(x, y)$ will be a solution of the second-order partial differential equation

$$F\big(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\big) = 0. \tag{2.153}$$

We remark that one could have equivalently considered on the hypersurface M_7 of $\mathbb{R}^8$ the Pfaffian system

$$I_F = \{i^*\theta^1, i^*\theta^2, i^*\theta^3, i^*d\theta^1, i^*d\theta^2, i^*d\theta^3\}, \tag{2.154}$$

obtained by pulling-back under the inclusion map $i: M_7 \longrightarrow \mathbb{R}^8$ the generators of I to M_7. The integral manifolds of the form (2.151) will also correspond to solutions of the second-order partial differential equation (2.153).

Therefore, by adding the forms ϕ^{n-1}, ϕ^n to the left-hand side of the form that we introduced in (2.103), so that $\phi^1, \dots, \phi^n$ are linearly independent. Then we have

$$d\phi^i \equiv T^i \phi^{n-1} \wedge \phi^n \mod I, \qquad 1 \leq i \leq s. \tag{2.155}$$

If $T^i = 0$, I is completely integrable, and $I^1 = I$. We discard this case and suppose $(T^1, \dots, T^s) \neq 0$. The ϕ^i are defined up to the non-singular linear transformation

$$\begin{pmatrix} \phi^1 \\ \vdots \\ \phi^n \end{pmatrix} \longrightarrow \begin{pmatrix} u_1^1 & u_s^1 & 0 & 0 \\ \cdots & & & \\ u_1^s & u_s^s & 0 & 0 \\ u_1^{n-1} & u_s^{n-1} & u_{n-1}^{n-1} & u_n^{n-1} \\ u_1^n & u_s^n & u_{n-1}^n & u_n^n \end{pmatrix} \begin{pmatrix} \phi^1 \\ \vdots \\ \phi^n \end{pmatrix}. \tag{2.156}$$

By choosing the above matrix u properly, we can suppose

$$T^1 = \cdots = T^{s-1} = 0, \qquad T^s = 1, \tag{2.157}$$

then

$$d\phi^1 \equiv \cdots \equiv d\phi^{s-1} \equiv 0, \qquad d\phi^s \equiv \phi^{s-1} \wedge \phi^n, \mod I. \tag{2.158}$$

Under the choice $I^{(1)}$ is generated by $\phi^1, \dots, \phi^{s-1}$, and we have $\dim I^{(1)} = s - 1$.

In the case $s = 2$, $n = 4$ we have the theorem:

Theorem (2.3.15): (Engel's normal form)

Let I be a Pfaffian system of two equations in four variables with derived flag satisfying

$$\dim I^{(1)} = 1, \qquad I^{(2)} = 0. \tag{2.159}$$

Then locally there are coordinates x, y, y', y'' such that

$$I = \{dy - y'dx, dy' - y''dx\}. \tag{2.160}$$

If the system is put into Engel normal form, then the "general solution" is visible given by

$$y = f(x), \qquad y' = f'(x), \qquad y'' = f''(x), \tag{2.161}$$

where $f(x)$ is an arbitrary function, then the general solution represented the Pfaffian system with independence condition: (I, dx) so that $dx \neq 0$.

The following theorem, known as the Pfaff normal form, provides a description of the integral manifolds of maximal dimension of a smooth Pfaffian system which is generated as a differential ideal by a single one-form ($s = 1$), and which is not completely integrable.

Theorem (2.3.16): (Pfaff normal form)

Let M_n be a C^∞ manifold and let

$$I = \{\theta, d\theta\}, \tag{2.162}$$

be a C^∞ Pfaffian system on M_n. Suppose that there exists an integer $r \geq 0$ such that we have $(d\theta)^r \wedge \theta \neq 0$, $(d\theta)^{r+1} \wedge \theta = 0$, on M_n. Then there exist local coordinates $(x^1, \dots, x^r, z, p^1, \dots, p^r, u^{2r+2}, \dots, u^n)$ such that

$$I = \left\{dz - \sum_{i=1}^{r} p_i dx^i, \sum_{i=1}^{r} dp_i \wedge dx^i\right\}. \tag{2.163}$$

The Goursat normal form theorem, which we now present, applies to a class of Pfaffian systems which are generated as a differential ideal by more than a single one-form, and which are not completely integrable.

Theorem (2.3.19): (The Goursat normal form)

Let M_n be a C^∞ manifold and let

$$I = \{\theta^1, \dots, \theta^r, d\theta^1, \dots, d\theta^r\}, \tag{2.164}$$

be a Pfaffian system of class C^∞ on M_n. Suppose that there exist 1-forms α and π, where α and π are not congruent to zero modulo I, such that,

$$\begin{aligned} &d\theta^1 \equiv \theta^2 \wedge \pi \ \bmod \{\theta^1\}, \\ &d\theta^2 \equiv \theta^3 \wedge \pi \ \bmod \{\theta^1, \theta^2\}, \\ &\vdots \\ &\theta^{r-1} \equiv \theta^r \wedge \pi \ \bmod \{\theta^1, \dots, \theta^{r-1}\}, \\ &\theta^r \equiv \alpha \wedge \pi \ \bmod \{\theta^1, \dots, \theta^r\}. \end{aligned} \tag{2.165}$$

Then there exist local coordinates $(x, y, y^1, \dots, y^r)$ such that

$$I = \{dy - y^1 dx, \dots, dy^{r-1} - y^r dx, dy^1 \wedge dx, \dots, dy^r \wedge dx\}. \tag{2.166}$$

It is easy to see that the integral manifolds of maximal dimension of a system satisfying the Goursat normal form are locally parametrized by one arbitrary C^∞ function of one variable.

CHAPTER THREE

Cartan-Kähler Theory

Section (3.1): Introduction to Caratn-Kähler Theory

In the previous chapter, we have discussed how problems in differential geometry and partial differential equations can often be reformed as problems about integral manifolds of appropriate exterior differential systems. Moreover, in differential geometry, particularly in the theory and applications of the moving frame and Cartan's methods of the equivalence [8,9,10], the problems to be studied often appear naturally in the form of exterior differential system anyway.

This motivates the problem of finding a general method of constructing integral manifold. When the exterior differential system I has a particularly simple form, standard differential calculus and the techniques of ordinary differential equations allow a complete (local) description of the integral manifolds of I. We illustrate in the previous chapter some examples are supplied by the theorems of Foebenius, Pfaaff-Darboux.

However, the differential systems arising in practice are usually more complicated that the ones dealt with in the previous chapter. Certainly, one can't expect to construct the general integral manifold of a differential system I using ordinary differential equation techniques alone. However, at least locally, this problem can be expressed as a problem in partial differential equations. It's instructive to see how this could be possible.

Suppose that we are interested in finding the n-dimensional manifolds of the set S, where $S \subset \Omega^*(M)$ be an arbitrary set of differential forms on M. To simplify our notation, we will agree on the index ranges $1 \leq i, j, k \leq n$ and $1 \leq a, b, c \leq m - n$ and use of the summation convention. We choose local coordinates $x^1, \ldots, x^n, y^1, \ldots, y^{m-n}$ centered at z on a z-neighborhood $U \subset M$. Let $\Omega = dx^1 \wedge \ldots \wedge dx^n$ and $G_n(TU, \Omega)$ denote the dense open subset of $G_n(TU)$

consisting of the n-planes $P \subset T_w U$ on which Ω restricts to be non-zero. Then there are well defined functions p_i^a on $G_n(TU, \Omega)$ so that, for each $p \in G_n(TU, \Omega)$, the vectors

$$X_i(P) = \left(\frac{\partial}{\partial x^i} + p_i^a(P)\frac{\partial}{\partial y^a}\right)\Big|_w, \tag{3.1}$$

from a basis of P. In the fact the functions x^i, y^a, p_i^a form a coordinate system on $G_n(TU, \Omega)$.

Now for each q-form φ on U with $q \leq n$ and every multi-index $J = (j_1, \dots, j_q)$ with $1 \leq j_1 \leq j_2 \leq \cdots \leq j_q \leq n$ we may define a function $F_{\varphi,J}$ on $G_n(TU, \Omega)$ by sitting

$$F_{\varphi,J}(P) = \varphi\left(X_{j1}(P), \dots, X_{jq}(P)\right). \tag{3.2}$$

(Note that, when $F_{\varphi,J}$ is expressed in the coordinates x^i, y^a, p_i^a, it's linear in the $(k \times k)$-minors of the matrix $p = (p_i^\alpha)$, where $k \leq q$.)

Any submanifold $V \subset U$ of dimension n which passes through $z \in U$ and satisfies $\Omega|_V \neq 0$ can be described in a neighborhood of z as a "graph" $y^a = u^a(x) = u^a(x^1, \dots, x^n)$ of the set $m - n$ functions u^a of the n variables x^i. For each $w = (x, u(x))$ in V, the p-coordinates of $T_w V \in G_n(TU, \Omega)$ are simply the partials $p_i^a = \partial u^a / \partial x^i$ evaluated at x. It follows that V is an integral manifold of S if and only if the function u satisfies the system of the first order partial differential equations

$$F_{\varphi,J}(x, u, \partial u / \partial x) = 0, \tag{3.3}$$

for all $\varphi \in S$ and all J with $\deg(\varphi) = |J| \leq n$.

Thus, constructing integral manifolds of S is locally equivalent to solving a system of first order partial differential equations of the form (3.3). Conversely, any first order system of partial differential equations for the functions

$u^1, \ldots, u^{m-n}$ as functions of $x^1, \ldots, x^n$ which is linear in the minors of the Jacobean matrix $\partial u/\partial x$ can be expressed as the condition that the graph $\big(x, u(x)\big)$ in $\mathbb{R}^m$ be an integral manifold of an appropriate set S of differential forms on $\mathbb{R}^m$.

It is natural to ask about methods of solving systems of P.D.E. of the form (3.3). It is rare that the system (3.3) can be placed in a form to which the classical existence theorems in P.D.E. can be applied directly. In general, even for simple systems S, the corresponding system of the equations (3.3) is over-determined, meaning that there are independent equations in (3.3) that unknowns u. For example, if $m = 2n$ and S consists of the single differential form $\varphi = dy^1 \wedge dx^1 + \cdots + dy^n \wedge dx^n$, then (3.3) becomes the system of the equations $\partial u^i/\partial x^j = \partial u^j/\partial x^i$, which is over-determined when $n > 3$. Even when (3.3) isn't over-determined, it can't generally be placed in one of the classical forms (e.g., Cauchy-Kowalevski).

Nevertheless, we can generalize the system (3.3) by initial value problem.

Example (3.1.1):

Consider the following system of the first order P.D.E.s for one function $u(x, y)$:

$$u_x = F(x, y, u), \qquad u_y = G(x, y, u), \tag{3.4}$$

if we found a solution (3.4) which satisfies $u(0,0) = c$, then we may try to construct such as a solution by first solving the initial value problem

$$v_x = F(x, 0, v), \qquad \text{where} \qquad v(0) = c, \tag{3.5}$$

for v as a function of x

$$u_y = G(x, y, u), \qquad \text{where} \qquad u(x, 0) = v(x). \tag{3.6}$$

Assuming that F and G are smooth in a neighborhood of $(x, y, u) = (0,0, c)$, standard ordinary differential equation theory tells us that this process will yield a smooth function $u(x, y)$ defined on a neighborhood of $(x, y) = (0,0)$. However, the function u may not satisfy the equation $u_x = F(x, y, u)$ except along the line $y = 0$. In fact, if we set $H(x, y) = u_x(x, y) - u_x$, then $H(x, 0) = 0$, and we may compute that

$$H_y(x, y) = \big(G(x, y, u)\big)_x - F_y(x, y, u) - F_u(x, y, u)\, G(x, y, u) \tag{3.7}$$

$$= G_u(x, y, u)\, H(x, y) + R(x, y, u), \tag{3.8}$$

where

$$R(x, y, u) = FG_u - GF_u + G_x - F_y. \tag{3.9}$$

Suppose that F and G satisfy that identity $R \equiv 0$. Then H satisfies the differential equation with initial condition

$$H_y = G_u(x, y, u)H \quad \text{and} \quad H(x, 0) = 0.$$

By the usual uniqueness theorem in O.D.E., it follows that $H(x, y) \equiv 0$, so u satisfies the system of the equations (3.4). It follows that the condition $R \equiv 0$ is sufficient condition for the existence of the local solutions of (3.4) where $u(0,0)$ is allowed to be an arbitrary constant as long as $\big(0,0, u(0,0)\big)$ is in the common domain of F and G.

Note that if we consider the differential system I which is generated by the 1-form $\vartheta = du - Fdx - Gdy$, then $d\vartheta \equiv -Rdx \wedge dy \,.\mathrm{mod}\, \vartheta$, so the condition $R \equiv 0$ is equivalent to the condition that I be generated algebraically by ϑ.

Let us pursue the case of the first order equations with two independent variables a little further. Given a system of P.D.E. $R\big(x, y, u, u_x, u_y\big) = 0$, where u is regarded as a vector-valued function of the independent variables x and y, then, under certain mild constant rank assumptions, it will be possible to place the equations in the following (local) normal form

$$u_x^0 = F(x, y, u) \tag{3.10}$$

$$u_y^0 = G(x, y, u, u_x)$$
$$u_y^1 = H(x, y, u, u_x) \tag{3.11}$$

by making suitable changes of coordinates in (x, y) and decomposing u into $u = (u^0, u^1, u^2)$ where each of the u^α is a (vector-valued) unknown function of x and y. Note that the original system may thus be (roughly) regarded as being composed of an "over-determined" part (for u^0), a "determined" part (for u^1), and "underdetermined" part (for u^2). (This "normal form" generalizes in a straightforward way to the case of n independent variables, in which case the unknown functions u are split into $(n + 1)$ vector-valued components.)

The "Cauchy-Kowalevski approach" to solving the system in the real analytic case can then be described as follows: suppose that the collection u^α consist of $s_\alpha \geq 0$ unknown functions. For simplicity's sake, we assume that F, G, D are real analytic and well-defined on the entire $\mathbb{R}^k$ where k has the appropriate dimension. Then we choose s_0 constants, which we write as f^0, s_1analytic functions of x, which we write as $f^1(x)$, and s_2 is analytic functions of x and y, which we write as $f^2(x, y)$. We then first solve the following system of O.D.E. with initial conditions for s_0 functions v^0 of x:

$$v_x^0 = F\big(x, 0, v^0, f^1(x), f^2(x, 0)\big)$$
$$v^0(0) = f^0, \tag{3.10a}$$

and then second solve the following system of P.D.E. with initial conditions for $s_0 + s_1$ functions (u^0, u^1) of x and y:

$$u_y^0 = G\big(x, y, u^0, u^1, f^2(x, y), u_x^0, u_x^1, f_x^2(x, y)\big),$$
$$u_y^1 = D\big(x, y, u^0, u^1, f^2(x, y), u_x^0, u_x^1, f_x^2(x, y)\big),$$
$$u^0(x, 0) = v^0(x), \tag{3.11a}$$
$$u^1(x, 0) = f^1(x).$$

This process yields a function $u(x, y) = \big(u^0(x, y), u^1(x, y), u^2(x, y)\big)$, where $u^2(x, y)$ is defined to be $f^2(x, y)$ which is uniquely determined by the collection

$f=\{f^0, f^1(x), f^2(x,y)\}$. While it is clear that the $u(x,y)$ thus constructed satisfies (3.11), it isn't at all clear that u satisfies (3.10). In fact, if we set

$$H(x,y)=u_x^0(x,y)-f\big(x,y,u(x,y)\big), \tag{3.12}$$

then $H(x,y)\equiv 0$ since $u(x,0)$ satisfies (3.10.1), but in general $H(x,y)\not\equiv 0$ for the generic choice of initial data f.

Example (3.1.2):

Consider the following system of three equations for three unknown functions u^1, u^2, u^3 of three independent variables x^1, x^2, x^3. Here we write $\partial_j$for $\partial/\partial x^j$ and v^1, v^2, v^3 are some given functions of x^1, x^2, x^3

$$\begin{aligned}\partial_2 u^3-\partial_3 u^2&=u^1+v^1,\\ \partial_3 u^1-\partial_1 u^3&=u^2+v^2,\\ \partial_1 u^2-\partial_2 u^1&=u^3+v^3.\end{aligned} \tag{3.13}$$

The approach to treating (3.13) as a sequence of Cauchy problems (with $(s_0, s_1, s_2, s_3)=(0,1,1,1)$ is as follows:

1) Choose three functions $\omega^1(x^1), \omega^2(x^1,x^2), \omega^3(x^1,x^2,x^3)$.
2) Solve the equation in $\mathbb{R}^2$, $\partial_2 w=-\partial_1\omega^2-\bar{\omega}^3-\bar{v}^3$ with the initial condition $w(x^1,0)=\omega^1(x^1)$ where

$$\begin{aligned}\bar{\omega}^3(x^1,x^2)&=\omega^3(x^1,x^2,0)\\ \bar{v}^3(x^1,x^2)&=v^3(x^1,x^2,0)\end{aligned} \tag{3.14}$$

3) Solve the pair of the equations $\partial_3 u^1=\partial_1\omega^3+u^2+v^2$ and $\partial_3 u^2=\partial_2\omega^3-u^1-v^1$ with the initial conditions $u^1(x^1,x^2,0)=w(x^1,x^2)$ and $u^2(x^1,x^2,0)=\omega^2(x^1,x^2)$.
4) Set u^3 equal to ω^3.

However, the resulting set of functions u^a will not generally be a solution to third equation (3.13). If we set $E=\partial_1 u^2-\partial_2 u^1-u^3-v^3$, then, of course $E(x^1,x^2,0)=0$, but if we compute $\partial_3 E=-\{\partial_1(u^1+v^1)+\partial_2(u^2+v^2)+$

$\partial_3(u^3 + v^3)\}$, we see that E vanishes identically if and only if the functions u^a satisfy the additional equation

$$0 = \partial_1(u^1 + v^1) + \partial_2(u^2 + v^2) + \partial_3(u^3 + v^3). \qquad (3.15)$$

This suggests modifying our Cauchy sequence by adjoining (3.15), thus getting a new system with $(s_0, s_1, s_2, s_3) = (0, 1, 2, 0)$ and then proceeding as follows:

1)* Choose three functions $\omega^1(x^1), \omega^2(x^1, x^2), \omega^3(x^1, x^2)$.

2)* Solve the equation in $\mathbb{R}^2$, $\partial_2 w = -\partial_1\omega^2 - \omega^3 - \bar{v}^3$ with the initial condition $w(x^1, 0) = \omega^1(x^1)$ where

$$\bar{v}^3(x^1, x^2) = v^3(x^1, x^2, 0). \qquad (3.16)$$

3)* Solve the triple of equations with initial conditions

$$\begin{aligned}
\partial_3 u^1 &= \partial_1 u^3 + u^2 + v^2, & u^1(x^1, x^2, 0) &= w(x^1, x^2) \\
\partial_3 u^2 &= \partial_2 u^3 - u^1 - v^1, & u^2(x^1, x^2, 0) &= \omega^2(x^1, x^2). \\
\partial_3 u^3 &= -\partial_1(u^1 + v^1) - \partial_2(u^2 + v^2) - \partial_3 v^3, & u^3(x^1, x^2, 0) &= \omega^3(x^1, x^2)
\end{aligned}$$

It then follows easily that the resulting u^a also satisfy third equation (3.13). In the example just given, the "compatibility condition" took the form of extra equation which must be adjoined to the given equations so that the Cauchy sequence approach would work.

(3.2): Integral Elements and Polar Space

All through this section, M will be a smooth manifold of dimension m and I will be a differential ideal on M.

Definition (3.2.1):

Let $z \in M$ and $E \in T_zM$a linear subspace of T_zM. E Is an *integral element* of I if $\varphi_E = 0, \forall\varphi \in I$. We denote by $\mathcal{V}_p(I)$ the set of p-dimensional integral elements of I.

Example (3.2.2)

Let $M = \mathbb{R}^5$ and let I be generated by the two 1-forms

$$\begin{aligned} \vartheta^1 &= dx^1 + (x^3 - x^4 x^5) dx^4, \\ \vartheta^2 &= dx^2 + (x^3 + x^4 x^5) dx^5, \end{aligned} \tag{3.17}$$

so that,

$$I = \{\vartheta^1, \vartheta^2, d\vartheta^1 = \vartheta^3 \wedge dx^4, d\vartheta^2 = \vartheta^3 \wedge dx^5\}, \tag{3.18}$$

is generated algebraically, then, we have written

$$\vartheta^3 = dx^3 + x^5 dx^4 - x^4 dx^5 \tag{3.19}$$

for each $p \in M$, let

$$H_p = \left\{ v \in T_p\mathbb{R}^5 \mid \vartheta^1(v) = \vartheta^2(v) = 0 \right\} \subset T_p\mathbb{R}^5, \tag{3.20}$$

then $H \subset T\mathbb{R}^5$ is a rank 3 distribution. A 1-dimensional subspace $E \subset T_p\mathbb{R}^5$ is an integral element of I if and only if $E \subset H_p$. Thus, $V_1(I) \cong \mathbb{p}H$ and it is a smooth manifold of dimension 7. Now let

$$K_p = \left\{ v \in T_p\mathbb{R}^5 \mid \vartheta^1(v) = \vartheta^2(v) = \vartheta^3(v) = 0 \right\}. \tag{3.21}$$

then $K \subset H$ is a rank 2 distribution on $\mathbb{R}^5$. It is easy to see that, for each $p \in \mathbb{R}^5$, K_p is the unique 2-dimensional integral element of I based at p. Thus, $V_2(I) \cong \mathbb{R}^5$. Moreover, I have no integral elements of dimension greater than two.

The example (3.2.2) it illustrate several concepts in this section, and also shows, the relationship between the integral elements of the differential system and its integral manifolds can be subtle. In general, even the problem of describing the spaces $V_n(I)$ can be complicated. Whatever remains of this section will be dedicated to devoted fundamental properties of integral elements of I and of the subsets $V_n(I)$.

Definition (3.2.3):

N is an integral manifold of I if and only if each tangent space of N is an integral element of I.

Proposition (3.2.4):

If E is a p-dimensional integral elements of I, then every subspace of E are also integral elements of I.

We denote by $I_p = I \cap \mathcal{A}^p(M)$ the set of differential p-forms of I.

Proposition (3.2.5):

$$\mathcal{V}_p(I) = \{E \in G_p(TM) | \varphi_E = 0, \forall \varphi \in I_p\}. \qquad (3.22)$$

Definition (3.2.6):

An integral element of dimension n on which $\Omega \neq 0$ is called *admissible*. A n-dimensional integral element on which $\Omega \neq 0$ is called admissible.

Definition (3.2.7):

Let E an integral element of I. Let $\{e_1, \dots, e_p\}$ a basis of $E \subset T_z M$. The *polar space* of E, denoted by $H(E)$, is the vector space defined as follow

$$H(E) = \{v \in T_z M | \varphi(v, e_1, \dots, e_p) = 0, \forall \varphi \in I_{p+1}\}. \qquad (3.23)$$

Note that $E \subset H(E)$. This implies that a differential form is alternate. The polar space is most valuable role in exterior differential system theory as we shall illustrate by the following proposition.

Proposition (3.2.8)

Let $E \subset \mathcal{V}_p(I)$ be an p-dimensional integral elements of I. A $(p+1)$-dimensional vector space $E^+ \subset T_z M$ which contaians E is an integral element of I if and only if $E^+ \subset H(E)$.

In order to check if a given p-dimensional integral element of an exterior differential ideal I is contained in a $(p+1)$-dimensional integral element of I, we introduce the following function $r: \mathcal{V}_p(I) \longrightarrow \mathbb{Z}$, $r(E) = \dim H(E) - (p+1)$ is a relative integer, $\forall E \in \mathcal{V}_p(I)$.

Notice that $r(E) \geq 1$. If $r(E) = -1$, then E is contained in any $(p+1)$-dimensional integral element of I.

While determining the structure of $\mathcal{V}_p(I)$ can be difficult, one sees that the problem of understanding the $(p+1)$-dimensional extensions that are integral elements of a given p-dimensional integral element is essentially a linear one.

(3.2.1): Ordinary and Regular Integral Elements, and Integral Flags

Let Ω a differential n-form on a m-dimensional manifold M. Let $G_n(TM, \Omega) = \{E \in G_n(TM) | \Omega_{\mathrm{E}} \neq 0\}$, where $G_n(TM)$ is the Grassmanian of TM, i.e., the set of n-dimensional subspace of TM. We denote the set of integral elements of I by $\mathcal{V}_n(I, \Omega) = \mathcal{V}_n(I) \cap G_n(TM, \Omega)$ which $\Omega_E \neq 0$.

Definition (3.2.9):

An integral element $E \in \mathcal{V}_n(I)$ is called *Kähler-ordinary* if there exists a differential n-form Ω such that $\Omega_E \neq 0$. Moreover, if the function r is locally constant in some neighborhood of E, then is said *Kähler-regular*.

Example (3.2.10):

We will show that all of the 2-dimensional integral elements of I are Kähler-regular. Let $\Omega = dx^4 \wedge dx^5$. Then every element $E \in G_2(T\mathbb{R}^5, \Omega)$ has a basis $\{X_4, X_5\}$ of the form

$$\begin{aligned} X_4(E) &= \partial/\partial x^4 + p_4^1(E)\,\partial/\partial x^1 + p_4^2(E)\,\partial/\partial x^2 + p_4^3(E)\,\partial/\partial x^3 \\ X_5(E) &= \partial/\partial x^5 + p_5^1(E)\,\partial/\partial x^1 + p_5^2(E)\,\partial/\partial x^2 + p_5^3(E)\,\partial/\partial x^3 \end{aligned}. \quad (3.24)$$

the functions $x^1, \dots, x^5, p_5^1, \dots, p_5^3$ for a coordinate system on $G_2(T\mathbb{R}^5, \Omega)$. Computation gives

$$\begin{aligned}
(\vartheta^1 \wedge dx^4)\Omega &= -p_5^1 \\
(\vartheta^1 \wedge dx^5)\Omega &= p_4^1 + (x^3 - x^4x^5) \\
(\vartheta^2 \wedge dx^4)\Omega &= -p_5^2 - (x^3 + x^4x^5) \\
(\vartheta^2 \wedge dx^5)\Omega &= p_4^2 \\
(\vartheta^3 \wedge dx^4)\Omega &= -p_5^3 + x^4 \\
(\vartheta^3 \wedge dx^5)\Omega &= -p_4^3 - x^5
\end{aligned} \tag{3.25}$$

every point of $\mathcal{V}_2(I)$ is kähler-ordinary. Since none of these elements has any extension to a 3-dimensional integral element, it follows that $r(E) = -1$, $\forall\, E \in \mathcal{V}_2(I)$. Thus, every element of $\mathcal{V}_2(I)$ is also kähler-regular.

Definition (3.2.11):

An integral flag of I on $z \in M$ of length n is a sequence of integral elements E_k of I: $(0)_z \subset E_1 \subset \cdots \subset E_n \subset T_zM$.

Proposition (3.2.12): (Cartan's Bound)

Given $E \in \mathcal{V}_n(I)$ and a flag $F = (E_i)$ in E, then there is an open E-neighborhood $U \subset \mathrm{Gr}_n(TM)$ such that $\mathcal{V}_n(I) \cap U$ is contained in smooth submanifold of U of codimension $c(F) = c(E_0) + \cdots + c(E_{n-1})$.

Theorem (3.2.13): (Cartan's Test)

Let $I \subset A^*(M)$ be an exterior ideal which doesn't contain 0-forms (functions on M). Let $(0)_z \subset E_1 \subset \cdots \subset E_n \subset T_zM$ be an integral flag of I. For any $k < n$, we denote by c_k the codimension of the polar space $H(E_k)$ in T_zM. Then $\mathcal{V}_n(I) \subset G_n(TM)$ is at least of $c_0, c_1, \dots, c_{n-1}$ codimension at E_n. Moreover, E_n is an ordinary integral flag if and only if E_n has a neighborhood U in $G_n(TM)$ such that $\mathcal{V}_n(I) \cap U$ is a manifold of $c_0 + c_1 + \cdots + c_{n-1}$ codimension in U.

Example (3.2.14):

By using Theorem (3.2.13), we can give a quick proof the none-elements in $\mathcal{V}_2(I)$ are ordinary. For any integral flag $(0)_z \subset E_1 \subset E_2 \subset T_z\mathbb{R}^5$, we know that $c_0 \leq 2$ since there are two independent 1-forms in I. Also, since $E_2 \subset H(E_1)$, it follows that $c_1 \leq 3$. Since there is unique 2-dimensional integral element at each point of $\mathbb{R}^5$ it follows that $\mathcal{V}_2(I)$ has codimension six in $G_2(T\mathbb{R}^5)$. Since $c_0 + c_1 < 6$, it follows, by theorem (3.2.13), that none of the integral flags of length two can be ordinary. Hence there are no ordinary integral elements of dimension two.

Example (3.2.15):

Let $M = \mathbb{R}^6$ with coordinates $x^1, x^2, x^3, u^1, u^2, u^3$. Let I be the differential system generated by the 2-form

$$\begin{aligned}\vartheta = d(u_1 dx^1 + u_2 dx^2 + u_3 dx^3)\\ - (u_1 dx^2 \wedge dx^3 + u_2 dx^3 \wedge dx^1 + u_3 dx^1 \wedge dx^2).\end{aligned} \tag{3.26}$$

Of course, I generated algebraically by the forms $\{\vartheta, d\vartheta\}$. Then, we have

$$d\vartheta = -(du_1 \wedge dx^2 \wedge dx^3 + du_2 \wedge dx^3 \wedge dx^1 + du_3 \wedge dx^1 \wedge dx^2). \tag{3.27}$$

We can use the theorem (3.2.13) to show that all of 3-dimensional integral elements of I on which $\Omega = dx^1 \wedge dx^2 \wedge dx^3$ does not vanish are ordinary. Let $E \in \mathcal{V}_3(I, \Omega)$ be fixed with base point $z \in \mathbb{R}^6$. Let (e_1, e_2, e_3) be basis of E which is dual of basis (dx^1, dx^2, dx^3) of E^*. Let E_1 be the line spanned by e_1, let E_2 be the 2-plane spanned by the pair $\{e_1, e_2\}$, and E_3 be E. Then $(0)_z \subset E_1 \subset E_2 \subset E_3$ is an integral flag. Since I is generated by $\{\vartheta, d\vartheta\}$, it follows that $c_0 = 0$. Moreover, since $\vartheta(v, e_1) = \tau_1(v)$ where $\tau_1 \equiv du_1 \text{mod}\ (dx^1, dx^2, dx^3)$, it follows that $c_1 = 1$. Note that, since $E_3 \subset H(E_2)$, it follows that $c_2 \leq 3$. On the hand, we have the formula

$$\begin{aligned}\vartheta(v, e_1) &= \tau_1(v),\\ \vartheta(v, e_2) &= \tau_2(v),\\ d\vartheta(v, e_1, e_2) &= -\tau_3(v),\end{aligned} \tag{3.28}$$

where in each case, $\tau_k \equiv du_k \mod (dx^1, dx^2, dx^3)$. Since the 1-form τ_k are clearly independent and annihilate $H(E_2)$, it follows that $c_2 \geq 3$. Combined with the pervious argument, we have $c_2 = 3$. It follows by theorem (3.2.13) that the codimension of $\mathcal{V}_3(I)$ in $G_3(T\mathbb{R}^6)$ at E is at least $c_0 + c_1 + c_3 = 4$. Now, we shall illustrate that $\mathcal{V}_3(I, \Omega)$ is smooth submanifold of $G_3(T\mathbb{R}^6)$ of codimension four, and thence, by theorem (3.2.13), conclude that E is ordinary. To do this, we introduce functions p_{ij} on $G_3(T\mathbb{R}^6, \Omega)$ with the property that, for each $E \in G_3(T\mathbb{R}^6, \Omega)$ based at $z \in \mathbb{R}^6$, the forms $\tau_i = du_i - p_{ij}(E)\, dx^j \in T_z^*(\mathbb{R}^6)$ are a basis for the 1-forms which annihilate E. Then the functions (x, u, p) form a coordinate system on $G_3(T\mathbb{R}^6, \Omega)$. It is easy to compute that

$$\begin{aligned}\vartheta_E = (p_{23} - p_{32} - u_1) dx^2 \wedge dx^3 + (p_{31} - p_{13} - u_2) dx^3 \wedge dx^1 \\ + (p_{12} - p_{21} - u_3) dx^1 \wedge dx^2 \end{aligned} \qquad (3.28)$$

$$d\vartheta_E = -(p_{11} + p_{22} + p_{33}) dx^1 \wedge dx^2 \wedge dx^3. \qquad (3.29)$$

It follows that the condition that $E \in G_3(T\mathbb{R}^6, \Omega)$ be an integral element of I is equivalent to the vanishing of four functions on $G_3(T\mathbb{R}^6, \Omega)$ whose differentials are independent. Thus, $\mathcal{V}_3(I, \Omega)$ is smooth manifold of codimension 4 in $G_3(T\mathbb{R}^6, \Omega)$.

Proposition (3.2.16):

Let $I \cap A^*(M)$ an exterior ideal which don't contains 0-forms. Let $E \subset \mathcal{V}_n(I)$ be an integral element of I at the point $z \in M$. Let $\omega_1, \omega_2, \dots, \omega_n, \tau_1, \tau_2, \dots, \tau_s$ (where $s = \dim M - n$) be a coframe in an open neighborhood of $z \in M$ such that $E = \{v \in T_z M | \tau_a(v) = 0, \ \forall a = 1, \dots, s\}$. For all $p \leq n$, we define $E_p = \{v \in E | \omega_k(v) = 0, \ \forall\, k > p\}$. Let $\{\varphi_1, \dots, \varphi_r\}$ be the set differential forms which generate the exterior ideal I, where φ_ρ is of $(d_\rho + 1)$ degree. For all ρ, there exists an expansion

$$\varphi_\rho = \sum_{|J| = d_\rho} \tau_\rho^J \wedge \omega_J + \tilde{\varphi}_\rho \qquad (3.29)$$

where the 1-forms τ_ρ^J are linear combinations of the τ's and the terms $\tilde{\varphi}_\rho$ are, either of degree 2 or more in the τ's or else vanish at z.

Moreover, we have the formula

$$H(E_p) = \{v \in T_zM | \tau_a(v) = 0, \forall\, \rho \text{ and} \sup J \leq p\} \qquad (3.30)$$

In particular, for the integral flag $(0)_z \subset E_1 \subset \cdots \subset E_n \cap T_zM$ of I, c_p is the number of the linear independent forms $\left\{\tau_\rho^J\big|_z \text{such that} \sup J \leq p\right\}$.

(3.3): Cartan-Kähler Theory

Cartan's theory [8-12] was developed to deal in a coordinate-free, geometric way with questions of existence and uniqueness of local, real-analytic solutions of the systems of partial differential equations arising in differential geometry. It may be regarded as a synthesis and summary of the nineteenth century work on the geometric theory of partial differential equations, associated with such names Pfaff, Jacobi, Frobenius, Lie, and Darboux. Many of the intricate and fascinating details of this work are unknown to mathematicians today because of the intervening revision in mathematical thought and concept.

The Cartan-kähler theorem depends on the fundamental existence theorem of Cauchy and Kowalevski dealing with differential equations, and Cauchy-Kowalevski theorem uses the power series method. Consequently, Cartan-kähler theory is a real-analytic and local theory. This theorem is a coordinate-free, geometric generalization of the classical Cauchy-Kowalevski theorem, which we state presently.

We shall use the index ranges $1 \leq i, j \leq n$ and $1 \leq a, b \leq s$.

Theorem (3.3.1): (Cauchy-kowalevski)

Let y be a coordinate on $\mathbb{R}$, let $x = (x^i)$ be coordinates on $\mathbb{R}^n$, let $z = (z^a)$ be coordinates on $\mathbb{R}^s$, and let $p = (p_i^a)$ be coordinates on $\mathbb{R}^{ns}$. Let $D \subset \mathbb{R}^n \times \mathbb{R} \times$

$\mathbb{R}^s \times \mathbb{R}^{ns}$ be an open domain, and let $G: D \longrightarrow \mathbb{R}^s$ be a real analytic mapping. $D_0 \subset \mathbb{R}^n$ be an open domain, and let $f: D_0 \longrightarrow \mathbb{R}^s$ be real analytic mapping so that the "1-graph"

$$\Psi_f = \{(x, y_0, f(x), Df(x)) | x \in D_0\} \tag{3.31}$$

lies in D for some constant y_0. (Here, $Df(x) \in \mathbb{R}^{ns}$, the Jacobian of f, described by the condition that $p_i^a(Df(x)) = \partial f^a(x)/\partial x_i$.)

Then there exists an open neighborhood $D_1 \subset D_0 \times \mathbb{R}$ of $D_0 \times \{y_0\}$ and a real analytic mapping $F: D_1 \longrightarrow \mathbb{R}^s$ which satisfies the P.D.E. with initial condition

$$\begin{aligned} \partial F/\partial y &= G(x, y, F, \partial F/\partial x) \\ F(x, y_0) &= f(x), \ \forall\, x \in D_0. \end{aligned} \tag{3.32}$$

Moreover, F is unique in the sense that any other real-analytic solution of (3.32) agrees with F on some neighborhood of $D_0 \times \{y_0\}$.

We now turn to the statement of the Cartan-kähler theorem. If $I \subset \Lambda^*(M)$ is a differential ideal, we shall say that an integral manifold of $I, V \subset M$, is a kähler-regular integral manifold if the tangent space $T_v V$ is a kähler-regular integral element of $I, \forall\, v \in V$. If V is connected, kähler-regular integral manifold of I, then we define $r(V)$ to be $r(T_v V)$ where v is any element of V. The following theorem is generalization of the well-known Frobenius's theorem.

Theorem (3.3.2): (Cartan-Kähler)

Let $I \subset \Lambda^*(M)$ be a real analytic exterior differential ideal. Let $P \subset M$ a p-dimensional connected real analytic Kähler-Regular integral manifold of I. Suppose that $r = r(P) \geq 0$. Let $R \subset M$ be a real analytic submanifold of M of codimension r which contains P and such that $T_x R$ and $H(T_x P)$ are transversals in $T_x M, \forall\, x \in P \subset M$.

Then, there exists a $(p+1)$-dimensional connected real analytic integral manifold X of I, such that $P \subset X \subset R$. X is unique in the sense that another integral manifold of I having the stated properties, coincides with X on an open neighborhood of P.

Proof. See [1]

The anal-city condition of exterior differential ideal is crucial because of the requirements in the Cauchy-Kowalevski theorem used in the proof of the Cartan-Kähler theorem.

Cartan-Kähler's theorem has important corollary. Actually, this corollary is often more used than the theorem and it is sometimes called the Cartan-Kähler theorem.

Corollary (3.3.3): (Cartan-Kähler)

Let I be an analytic exterior differential on a manifold M. If $E \subset T_z M$ is an ordinary integral element of I, there exists an integral manifold of I passing through z and having E as a tangent space at the point z.

Example (3.3.4):

As a simple illustration of the Cartan-kähler theorem we consider the partial differential equation in the unknown function $u(x,y)$ given by

$$\partial^2 u/\partial y^2 = \partial u/\partial x, \tag{3.33}$$

on $\mathbb{R}^6 = \{(x,y,u,p,q,r)\}$ we put

$$\begin{aligned} \vartheta_1 &= du - p\,dx - q\,dy, \\ \vartheta_2 &= du - p\,dy - r\,dx, \end{aligned} \tag{3.34}$$

then the above partial differential equation translated to the closed exterior differential system on $M = \mathbb{R}^6$ given by

$$\Lambda = \{\vartheta_1, \vartheta_2, \omega_1 = dp \wedge dx + dq \wedge dy, \omega_2 = dp \wedge dy + dr \wedge dx\}. \tag{3.35}$$

Since $\vartheta_1 \wedge \vartheta_2 \neq 0$ then, every point is a regular integral point, and $s_0 = 2$. Fix an origin, $0 = (0, 0, u_0, p_0, q_0, r_0) \in M$. To find an integral submanifold

$$g(x) = \left(x, 0, \Phi^1(x), \dots, \Phi^4(x)\right) \tag{3.36}$$

we must solve

$$\begin{aligned} \partial u/\partial x = p, \qquad d\Phi^1/dx = \Phi^2, \\ \partial q/\partial x = r, \qquad d\Phi^3/dx = \Phi^4 \end{aligned} \tag{3.37}$$

with $u = u_0, q = q_0$ at $x = 0$. Let Φ^1 and Φ^3 be arbitrary real-analytic functions in x such that $\Phi^1(0) = u_0$ and $\Phi^3(0) = q_0$. (We can specify u and $\partial u/\partial u$ along $y = 0$). Now, we have $g(x)$. By taking a tangent vector

$$v = \left(1, 0, v_u, v_p, v_q, v_r\right) \in M_0. \tag{3.38}$$

Then the equations $\vartheta_1(v) = \vartheta_2(v) = 0$ implies $v_u = p_0, v_q = r_0$. The dual polar space of $E_0^1 = [v]$ is spanned by ϑ_1, ϑ_2 and the two 1-forms

$$\begin{aligned} i_v\omega_1 = v_p\, dx - dp + r_0\, dy \\ i_v\omega_2 = v_r\, dx - dr + r_p\, dy \end{aligned}. \tag{3.39}$$

the polar matrix of these four 1-forms with respect to (dx, dy, du, dp, dq, dr) is

$$\begin{bmatrix} p_0 & q_0 & 1 & 0 & 0 & 0 \\ r_0 & p_0 & 0 & 0 & -1 & 0 \\ v_p & r_0 & 0 & -1 & 0 & 0 \\ v_r & v_p & 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.40}$$

this matrix has rank four for any quadruple $\left(p_0, r_0, v_p, v_r\right)$. In particular,

$$s_0 + s_1 = 4, \qquad s_1 = 2,$$

and $\sigma_2 = 0$. Therefore, there exists a unique solution

$$f(x, y) = \left(x, y, F^1(x, y), \dots, F^4(x, y)\right), \qquad \text{with } f(x, 0) = g(x). \tag{3.41}$$

(3.4): Involutive Differential Systems

The calculation of the reduced Cartan characters can lead to two outcomes. The first is that the system is involutive [4, 5, 6, 52, 62]. This means that a solution to our problem exists in the current setting, and the last non vanishing (pseudo-)character tells us how many free functions we can specify in the solution of the problem. In particular, since any particular solution provides an equivalence of our original problem, the free functions can be interpreted as parameterizing the self-equivalence, or symmetry, of our system. The symmetry in this case is an infinite dimensional Lie pseudo-group parameterized by the arbitrary functions obtained. The other outcome is that the system is not involutive. This simply means that the space we are working with is too small: we need to prolong the system we are considering. The prolonged space is the most natural setting for discussing any physical problems: here there is no “hidden constraints”.

(3.4.1): Independence Conditions. Involution

The solutions approved by the Cartan-Kähler theorem did not regard our requirement of independence variables. To move on, let us define differential system with independence condition as a differential system for which we require that solutions must keep certain 1-form $\omega_1, \dots, \omega_n$ independent. This mean to

$$\omega_i \wedge \omega_j \wedge \dots \wedge \omega_k \neq 0, \qquad i, j, \dots, k \text{ all distinct}$$

on solution for any decision of any quantities of $\omega_i, \omega_j, \dots$. If we require the coordinates $x_1, \dots, x_n$ to be independently, we simply take the independent forms to be $dx_1, \dots, dx_n$, and the vectors spanning the integral elements must be in the form

$$\partial/\partial x^a + \sum B_a^i \, \partial/\partial z_i. \tag{3.42}$$

Immediately, we see that we must not end up with equations of the form

$$0 = \Omega^k = C_{ij\dots k} \, \omega_i \wedge \omega_j \wedge \dots \wedge \omega_k. \tag{3.43}$$

Unless $C_{ij...k} = 0$, no solution will satisfy the independence conditions. Such terms are called *essential torsion*. So we can summarize that an exterior differential system with independence condition (I, Ω) is an *involution* at $z \in M$ if there exists an ordinary integral element $E \subset T_z M$ for (I, Ω). To carry on in such cases, we need to add to our differential system the algebraic equations

$$C_{ij...k} = 0, \tag{3.44}$$

where $C_{ij...k}$ are non-zero constants, we called that such a differential system is *incompatible*. A discussion of the interpretation of the remaining characters within the context of formal theory can be found in [3-5].

Example (3.4.1):

Any P.D.E. system

$$F^{\lambda}\left(x^i, z^a, \partial z^a / \partial x^i, \ldots, \partial^k z^a / \partial x^I\right) = 0, \qquad \partial x^I = \partial x^{i_1} \ldots \partial x^{i_k}, \tag{3.45}$$

may be written as a differential system with independence condition. For e.g., in the second order case $(k = 2)$ we introduce variables p_i^a, $p_{ij}^a = p_{ji}^a$ and the system is defined on the space with coordinates $\left(x^i, z^a, p_i^a, p_{ij}^a\right)$ and is generated by the equations

$$\begin{aligned} F^{\lambda}\left(x^i, z^a, p_i^a, p_{ij}^a\right) &= 0 \\ dz^a - p_i^a dx^i &= 0, \\ dp_i^a - p_{ij}^a dx^j &= 0 \end{aligned} \tag{3.46}$$

with the independence condition $\omega = dx^1 \wedge \ldots \wedge dx^n \neq 0$. An admissible integral manifold of Ω may locally be written as $f: x \to \left(x, f^a(x), f_i^a(x), f_{ij}^a(x)\right)$, and it corresponds to a solution to the P.D.E. system in the usual sense.

Theorem (3.4.2):

If (I, Ω) is an involution and E is an admissible integral element, then there exist an admissible integral manifold $W \subset M$ through z, such that $T_z W = E$.

The statement of this theorem is powerful in its simplicity, but it is difficult to apply in practice if one does not have a manageable criterion for determining if an exterior differential system with independence condition is in involution. There is such a criterion, known as E. Cartan's involutivity test. In what follows we shall present this test in the case of quasi-linear Pfaffian systems, and refer the reader to [6-10] for the detailed treatment of the general case. Quasi-linear systems are more manageable, and then it will define as follows. Consider a Pfaffian system with independence condition (I, Ω), whrer

$$I = \{\vartheta^1, \dots, \vartheta^s, d\vartheta^1, \dots, d\vartheta^s\}, \qquad \Omega = \omega^1 \wedge \cdots \wedge \omega^p, \tag{3.47}$$

and let $\pi^1, \dots, \pi^l,\ l = n - s - p$, be 1-form such that

$$\vartheta^1 \wedge \cdots \wedge \vartheta^s \wedge \omega^1 \wedge \cdots \wedge \omega^p \wedge \cdots \wedge \pi^1 \wedge \cdots \wedge \pi^l \neq 0. \tag{3.48}$$

we have, for $1 \leq i \leq s$,

$$\begin{aligned} d\vartheta^i \equiv & \sum_{\alpha=1}^{l} \sum_{b=1}^{p} A^i_{\alpha b} \pi^\alpha \wedge \omega^b + \frac{1}{2} \sum_{a,b=1}^{p} B^i_{ab} \omega^a \wedge \omega^b \\ & + \frac{1}{2} \sum_{\alpha,\beta=1}^{l} C^i_{\alpha\beta} \pi^\alpha \wedge \pi^\beta \mod \vartheta^1, \dots, \vartheta^s . \end{aligned} \tag{3.49}$$

Definition (3.4.3):

A Pfaffian system with independence condition (I, Ω) is said to be *quasi-linear* if

$$C^i_{\alpha\beta} = 0, \qquad 1 \leq i \leq s, \qquad 1 \leq \alpha, \beta \leq l. \tag{3.50}$$

It is important to notice that if (I, Ω) is quasi-linear, then $\mathcal{V}_p(I, \Omega)$ is an affine bundle over M. The admissible integral elements E can be taken in the form

$$\left(\pi^{\alpha} - \sum_{b=1}^{p} t_b^{\alpha} \omega^b\right)\Bigg|_E = 0, \qquad 1 \le \alpha \le l, \tag{3.51}$$

and the polar equations of E are given by

$$\sum_{\alpha=1}^{l} \left(A_{\alpha b}^{i} t_c^{\alpha} - A_{\alpha c}^{i} t_b^{\alpha}\right) + B_{bc}^{i} = 0,\ \ 1 \le b, c \le p \text{ and } 1 \le i \le s. \tag{3.52}$$

These equations are indeed linear. We denote the dimension of their solution space by d. We now define the *reduce characters* $\sigma_1', \dots \sigma_r',\ r \le p$ of (I, Ω) by

$$\sigma_1' + \cdots + \sigma_r' = \max_{v_1,\dots,v_r \in \mathbb{R}^l} \operatorname{rk} \begin{pmatrix} \sum_{\alpha=1}^{l} v_1^{\alpha}\, A_{\alpha b}^{i} \\ \vdots \\ \sum_{\alpha=1}^{l} v_r^{\alpha}\, A_{\alpha b}^{i} \end{pmatrix}. \tag{3.53}$$

Necessary and sufficient conditions for involutivity are given by the following theorem, known as *Cartan's involutivity test*:

Theorem (3.4.5):

We have

$$d \le \sum_{i=1}^{p} i \sigma_i', \tag{3.54}$$

with equality if and only if the system (I, Ω) is involution. If $\sigma_q' = k \neq 0$ with q maximal, then the admissible local integral manifolds are parameterized by kC^{ω} functions of q variables.

The reminder of this section is detected to illustration of the Cartan's test on series of examples.

Example (3.4.6):

Consider on $\mathbb{R}^3$ with coordinates (x, u, p) the Pfaffian system with independence condition (I, Ω) where

$$I = \{\vartheta, d\vartheta\}, \qquad \Omega = dx, \tag{3.55}$$

and

$$\vartheta = du - pdx. \tag{3.56}$$

We have

$$d\vartheta = dp \wedge dx = \pi \wedge \Omega. \tag{3.57}$$

The admissible integral elements are of the form

$$\pi - t\Omega, \qquad d = 1, \qquad \sigma_1' = 1. \tag{3.58}$$

The involutivity test is therefore satisfied and the admissible integral manifolds are one-dimensional, as expected. They are parameterized by one arbitrary function of one variable.

Example (3.4.7):

We consider the scalar partial differential equation

$$F\left(x^i, u, \partial u/\partial x^i, \partial^2 u/\partial x^i \partial x^j\right) = 0, \qquad 1 \le i, j \le p, \tag{3.59}$$

where F is assumed to be C^ω in all its arguments. We apply the Cartan-Kähler theorem that the solutions of the P.D.E. (3.59) are parameterized by two analytic functions of $p - 1$ variables.

To the P.D.E. (3.59), we associate on $\mathbb{R}^{(p(p+5)/2)+1}$, with local coordinates $\left(x^i, u, u_i, u_{ij}\right)$, $1 \le i, j \le p$, an exterior differential system with independence condition (I, Ω), by letting I be the differential ideal generated by

$$F(x^i, u, u_i, u_{ij}) = 0, \qquad \vartheta_0 = du - \sum_{i=1}^{p} dx^i, \tag{3.60}$$

$$\vartheta_i = du_i - \sum_{j=1}^{n} u_{ij} dx^j, \qquad 1 \le i \le p, \tag{3.61}$$

where we assume that

$$\det(\partial F/\partial u_{ij})\big|_{F=0} \neq 0, \tag{3.62}$$

and the independence condition Ω be defined by

$$\Omega = dx^1 \wedge \cdots \wedge dx^p. \tag{3.63}$$

The structure equations of (I, Ω) are given by

$$dF = 0, \qquad d\vartheta_0 \equiv 0, \qquad d\vartheta_i \equiv \sum_{j=1}^{n} \pi_{ij} \wedge dx^j, \qquad \mathrm{mod}\, \vartheta_0, \dots, \vartheta_p, \tag{3.64}$$

where $\pi_{ij} = -du_{ij}$. In order to compute the reduce characters of (I, Ω), it is convenient to exploit the non-degeneracy condition (3.62) to put the above structure equations in normal form. Under a change of coframe of the form

$$\bar{\omega}^i = \sum_{j=1}^{p} a^i{}_j dx^j, \qquad \bar{\vartheta}_i = \sum_{j=1}^{p} (a^{-1})^j{}_i\, \vartheta_j, \tag{3.65}$$

we obtain the following transformation law for the 1-form π_{ij},

$$\bar{\pi}_{ij} = -\sum_{k,l=1}^{p} du_{kl}\, a^k{}_i\, a^l{}_j\,. \tag{3.66}$$

Using the rank condition (3.62), we can locally choose $(a^i{}_j)$ so as to have

$$\sum_{i,j=1}^{p} (\partial F/\partial u_{ij})\, (a^{-1})_i{}^k\, (a^{-1})_j{}^l = \delta_{ij}\varepsilon_i, \tag{3.67}$$

where $\varepsilon_i^2 = 1$. The structure equation $dF = 0$ then takes the form

$$\sum_{i=1}^{p} \varepsilon_i \bar{\pi}_{ii} + \sum_{k=1}^{p} b_k \bar{\omega}^k \equiv 0, \qquad \mathrm{mod}\ \bar{\vartheta}_0, \dots, \bar{\vartheta}_p, \tag{3.68}$$

for some functions $b_k,\ 1 \le k \le p$. Now we define

$$\bar{\bar{\pi}}_{ii} = \bar{\pi}_{ii} + \varepsilon_i b_i \bar{\omega}_i, \qquad \bar{\bar{\pi}}_{ij} = \bar{\pi}_{ij}, \qquad 1 \le i \ne j \le p. \tag{3.69}$$

Dropping bars, the structure equations (3.64) become

$$d\vartheta_0 \equiv 0, \qquad d\vartheta_i \equiv \sum_{j=1}^{p} \pi_{ij} \wedge \omega^j, \qquad \mathrm{mod}\ \vartheta_0, \dots, \vartheta_p, \tag{3.70}$$

where

$$\sum_{i=1}^{p} \varepsilon_i \pi_{ii} \equiv 0, \qquad \pi_{ij} \equiv \pi_{ji}, \qquad \mathrm{mod}\ \vartheta_0, \dots, \vartheta_p, \tag{3.71}$$

where $1 \le i, j \le p$. We want now to compute the reduced characters of (I, Ω) using (3.59). We have

$$s_1' = p, s_2' = p - 1, \dots, s_{p-1}' = 2, s_p' = 0, \tag{3.72}$$

where the final drop from 2 to 0 is due to trace condition (3.71). Thus we have

$$\sum_{i=1}^{p} i s_i' = \frac{p(p+1)(p+2)}{6} - p. \tag{3.73}$$

In order to apply the involutivity test, we consider the admissible integral elements E of (I, Ω), which are given by

$$\vartheta_0|_E = 0, \qquad \vartheta^i\big|_E = 0, \qquad \left(\pi_{ij} - \sum_{k=1}^{p} L_{ijk}\omega^k\right)\Bigg|_E = 0, \tag{3.74}$$

where $1 \le i, j, k \le p$, and where we have

$$L_{ijk} = L_{jik} = L_{ikj}, \qquad \sum_{i=1}^{p} \varepsilon_i L_{ijk} = 0. \tag{3.75}$$

The dimension of the solution space of the polar equations of E is given by

$$d = \binom{p+2}{p-1} - p = \frac{p(p+1)(p+2)}{6} - p, \tag{3.76}$$

and the system is in involution, with top character $\sigma'_{p-1} = 2$. The local C^{ω} solutions are thus parameterized by two arbitrary functions of $p-1$ variables, as claimed.

(3.5): Exterior Differential Systems and Prolongations

The prolongations of a differential system are the differential system obtained by adjoining to the original differential system its differential consequences. The concept of *prolongation tower*, which will be defined below, gives an abstract formulation of the operation of the prolongation. A general conjecture of Elie Cartan's, proved by Kuranishi, [15], for a wide class of differential systems, state that an analytic differential system with independence condition it's takes a finite number of prolongations for it to be either involutive or incompatible, or has no solutions. This result is known as Cartan-Kuranishi Theorem. Our purpose is to review some of the basic aspects of the prolongation theorem [16]. We assume that all manifolds and the differential systems under consideration are of class C^{ω}.

The prolongation tower of an exterior differential system with independence condition (I, Ω) on an n-dimensional manifold M is defined as a follows. Let $f: W_p \to M$ be an immersion and let $f_*: W_p \to G_p(M)$ denote the map into the Grassmann bundle of p-planes in TM determined by f. The Grassmann bundle $G_p(M)$ is endowed with a canonical exterior differential system $C^{(1)}$ defined the property that ${f_*}^* C^{(1)} = 0$ for any immersion $f: W_p \to M$. Using affine fiber coordinates $(x^i, u^\alpha, u_i^\alpha)$, $1 \le i \le p$, $1 \le \alpha \le n$, on Grassmann bundle $G_p(M)$, the system $C^{(1)}$ is defined as the differential ideal generated by the 1-form

$$\vartheta^\alpha = du^\alpha - \sum_{i=1}^{p} u_i^\alpha dx^i. \tag{3.77}$$

We choose component $V_p(I)$ of the sub-variety of $G_p(M)$ defined by the p-dimensional admissible integral elements of I and assume $V_p(I)$ to be C^ω manifold.

Definition (3.5.1):

The first prolongation of I is the exterior differential system $I^{(1)}$ defined by

$$I^{(1)} = C^{(1)}\big|_{V_p(I)}. \tag{3.78}$$

For notational simplicity, we use the notation M^1 to denote the $V_p(I)$. We also assume that the map $\pi^{1,0}: (M^{(1)}, I^{(1)}) \to (M, I)$ is a C^ω submersion. The prolongation tower of I is then defined by induction,

$$\cdots \xrightarrow{\pi^{k+1,k}} \left(M^{(k)}, I^{(k)}\right) \xrightarrow{\pi^{k,k-1}} \cdots \xrightarrow{\pi^{2,1}} \left(M^{(1)}, I^{(1)}\right) \xrightarrow{\pi^{1,0}} (M, I). \tag{3.79}$$

The infinite prolongation $(M^{(\infty)}, I^{(\infty)})$ of (M, I) is then defined as the inverse limit of this tower

$$M^{(\infty)} := \lim_{\leftarrow} M^{(k)}, \qquad I^{(\infty)} = \bigcup_{k \ge 0} I^{(k)}. \tag{3.80}$$

(3.5.1): A Version of Wahquist-Estabrook Prolongation

In recent years, we have seen almost an explosion in the number of nonlinear partial differential equations (NLPDE) which can be solved by various exact techniques. What I shall try to do here is to outline briefly the history of these developments, put them into perspective, point out where foresight was important in the development, and finally end up with the bath made by Frank Estabrook-Hugo Wahlquist [17-19]. Wahlquist and Estabrook recently introduce an algebraic structure which has the form of an incomplete Lie algebra and which could be associated with a nonlinear partial differential equation. They called this the prolongation structure and indicated how it could be used, not only as solution technique, but as a means of understanding the underlying algebraic structure equation concerned. This is particularly applicable to the class of equations which exhibits soliton properties.

Firstly, we give a basic outline of how to write down the prolongation structure in order to understand the basic principles involved. The procedure is somewhat different from their original approach. It relies on finding an underlying Pfaffian system which constitutes a closed differential ideal and reduces to the equation on the transversal integral manifold. This can be regarded as a generalization of the Frobenius Theorem to establish complete integrability [49, 57]. The method of prolongation introduces over the base manifold a type of fiber bundle which is endowed with a Cartan-Ehresmann connection. The vanishing of the connection form is the necessary and sufficient condition for the existence of this type of prolongation. The theorem developed here for this operation is very suitable for applications and some of these will be mentioned further on Chapter four and Five. Thus, in this second approach, a differential system which gives the equation on the transversal integral manifold is found, and this differential system is used to solve for the quantities which appear in the connection forms. These two approaches are quite complementary to each other and it might be of interest to put them together here.

The aim of prolongation, of course, is to reduce the study of the integral manifolds of an arbitrary differential system to the case of an involutive differential system, the case to which the vast majority of the theory applies.

Following Wahlquist-Estabrook (EW), consider the space $M = \mathbb{R}^n(x, t, u, p, q, \dots)$ in which there is defined a closed exterior differential system

$$\alpha_1 = 0, \dots, \alpha_l = 0, \tag{3.81}$$

and let I be the ideal generated by the system (3.81), then we got

$$I = \left\{ \omega = \sum_{i=1}^{l} \sigma_i \wedge \alpha_i : \sigma_i \in \Lambda(M) \right\}. \tag{3.82}$$

If ideal (3.82) is closed, we have $dI \subset I$ and so (3.81) is integrable by Frobenius Theorem (2.2.1). It is important to stress that system (3.81) is chosen such that the solutions $u = u(x,t)$ of an equation

$$u_t = F(x, t, u, u_x, u_{xx}, \dots), \tag{3.83}$$

correspond with the two-dimensional integral manifolds of (3.81). These are the integral manifolds given by sections S of the projection

$$\pi: M \longrightarrow \mathbb{R}^2, \qquad \pi(x, t, u, p, q, \dots) = (x, t). \tag{3.84}$$

These sections S are given by mapping

$$S: \mathbb{R}^2 \longrightarrow M, \qquad S(x,t) = (x, t, u(x,t), p(x,t), q(x,t), \dots). \tag{3.85}$$

that mean $dx \wedge dt|_S = \pi^*(dx \wedge dt) \neq 0,$ where $\pi^*: \Lambda(\mathbb{R}^2) \longrightarrow \Lambda(S)$. Now, we have the fiber bundle $(\widetilde{M}, \tilde{p}, M)$ over M with $M \subset \widetilde{M}$ and $\tilde{p}$ a projection of $\widetilde{M}$ onto M, so points of $\widetilde{M}$ and M are denote respectively by $\widetilde{m}, m$, then $\tilde{p}(\widetilde{m}) = m$. A Cartan-Ehresmann connection in the fiber bundle $(\widetilde{M}, \tilde{p}, M)$ is a system of the 1-form $\widetilde{\omega}_i = 1, \dots, k$ in $T^*(\widetilde{M})$ with the property that the mapping $\tilde{p}_*$ from the vector space $H_{\widetilde{m}} = \{\tilde{X} \in T_{\widetilde{m}} |\, \widetilde{\omega}_i(\tilde{X}) = 0, i = 1, \dots k\}$, to the tangent space T_m is a bijection. We consider in $\widetilde{M}$ the exterior differential system

$$\begin{aligned} \tilde{\alpha}_i &= \tilde{p}^* \alpha_i = 0, \quad && i = 1, \dots, l, \\ \widetilde{\omega}_j &= 0, && j = 1, \dots, k \end{aligned} \tag{3.86}$$

where $\widetilde{\omega}_j$ a Cartan-Ehresmann connection in $(\widetilde{M}, \tilde{p}, M)$.

The system (3.86) is called a Cartan prolongation of (3.81) if (3.86) is closed and whenever S is a transversal solution of (3.81), there should also exist a transversal local solution $\tilde{S}$ of (3.86) with $\tilde{p}(\tilde{S}) = S$. There is then a theorem which states that (3.86) is a Cartan prolongation of (3.81) if and only if (3.86) is closed. A necessary and sufficient condition for the existence of this is given by

$$d\tilde{\omega}_i = \tilde{\beta}_i^j \wedge \tilde{\omega}_j, \qquad \text{mod } p^*(I) \tag{3.87}$$

For the considerations here, the fiber bundle will be the trivial fiber bundle given by the form $\widetilde{M} = M \times \mathbb{R}^k$ with $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ and the connections will be

$$\tilde{\omega}_i = dy_i - \eta_i, \qquad \eta_i = A_i dx + B_i dt, i = 1, \ldots, k, \tag{3.88}$$

where, A_i, B_i are defined as C^∞ functions on $\widetilde{M}$. Substituting into the prolongation condition (3.87), we gets

$$-d\eta_i = \tilde{\beta}_i^j \wedge \left(dy_j - \eta_j\right), \qquad \text{mod } \tilde{p}^*(I). \tag{3.89}$$

From this, it follows that $\tilde{\beta}_i^j$ may be chosen such that they do not depend on $dy_m - \eta_m$ $m = 1, \ldots, k,\ m \neq j$ .So

$$\tilde{\beta}_i^j = a_i^j dx + \beta_i^j dt + c_i^j du + d_i^j dp + \cdots \ \text{ mod } \tau_i^j\left(dy_j - \eta_j\right).$$

Comparing forms on both sides of (3.89), we get

$$a_i^j = \partial A_i / \partial y_j\,, \qquad b_i^j = \partial B_i / \partial y_i,$$

with $c_i^j = d_i^j = \cdots = 0$ because $du \wedge dy_j, dp \wedge dy_j, \ldots$ don't occur on the left of (3.89). the prolongation condition reduces to

$$-d\eta_i = \frac{\partial \eta_i}{\partial y_j} \wedge \left(dy_j - \eta_j\right), \qquad \text{mod } \tilde{p}^*(I). \tag{3.90}$$

Finally, introduce the vertical valued one-form $\eta = \eta_i \frac{\partial}{\partial y_i}$ along with s definitions

$$\begin{aligned} d\eta &= (d_M \eta_i) \frac{\partial}{\partial y_i}, \\ [\eta, \omega] &= \left(\eta_j \wedge \frac{\partial \omega_i}{\partial y_j} + \omega_j \wedge \frac{\partial \eta_i}{\partial y_j}\right) \frac{\partial}{\partial y_i}, \end{aligned} \tag{3.91}$$

d_M means differentiation with respect to just the variables of the base manifold.

Moreover, we can show that the prolongation condition boils down to the following condition

$$d\eta + \frac{1}{2}[\eta, \eta] = 0, \qquad \text{mod } \tilde{p}^*(I) \tag{3.92}$$

Then, a necessary and sufficient condition for the connection forms (3.88) to be a Cartan-prolongation is the vanishing of its curvature form.

We now present a statement of prolongation theorem as given in [7]

Theorem (3.5.2):

There exists an integer k such that for all $l \geq k$, each of these systems $\left(I^{(l)}, \Omega^{(k)}\right)$ is involutive. Furthermore, if $M^{(k)}$ is empty for some $k \geq 1$, then (I, Ω) has no n-dimensional integral manifolds.

Theorem (3.5.3): (Maurer-Cartan)

If N is connected and γ is smooth g-valued 1-form on N that satisfies $d\gamma = \frac{1}{2}[\gamma, \gamma]$, then there exists a smooth map $g: N \to G$, unique up to composition with a constant left translation, so $g^*\eta = \gamma$.

Example (3.5.4):

Consider the system of partial differential equations of a single variable $u = u(x, y, z)$ defined on $\mathbb{R}^3$

$$u_{xx} = u_{yy} = u_{zz}. \tag{3.93}$$

Following the general procedure for transformation P.D.E. into exterior differential equations, the space we are working with is hence formed by the following variables

- u (1 variable);
- x, y, z (3 variables);
- u_x, u_y, u_z (3 variables);
- $u_{xx} = u_{yy} = u_{zz},\ u_{xy} = u_{yx},\ u_{xz} = u_{zx},\ u_{yz} = u_{zy}$ (4 variables).

So the differential equation is defined on an 11-dimensional space formed by the variables above. The solution we seek for is a 3-dimensional integral manifold for which $dx \wedge dy \wedge dz \neq 0$, i.e., there are 8 variables that need to become dependent on x, y, z. The contact forms are

$$\begin{cases} \omega_u = du - u_x dx - u_y dy - u_z dz \\ \omega_x = du_x - u_{xx} dx - u_{xy} dy - u_{xz} dz \\ \omega_y = du_y - u_{xy} dx - u_{yy} dy - u_{yz} dz \\ \omega_z = du_z - u_{xz} dx - u_{yz} dy - u_{zz} dz \end{cases} \tag{3.94}$$

which are set to zero. Recall that the zeros Caratn character s_0 is the number of the equations for which any linear integral elements of the system must be satisfied while ignoring the $dx, dy, dy,$ i.e., it is the rank of the matrix

$$\left(\begin{array}{cccccccc} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \tag{3.95}$$

derived from the contact 1-forms, where the first row not part of the matrix. Obviously, this number is always equal to the number of independent 1-form equations above (since we are not allowed to have linear independence among the independence variables), and here we have $s_0 = 4$, even without forming the matrix explicitly.

Under exterior differentiation we have

$$\begin{cases} d\omega_u = -du_x \wedge dx - du_y \wedge dy - du_z \wedge dz \\ d\omega_x = -du_{xx} \wedge dx - du_{xy} \wedge dy - du_{xz} \wedge dz \\ d\omega_y = -du_{xy} \wedge dx - du_{yy} \wedge dy - du_{yz} \wedge dz \\ d\omega_z = -du_{xz} \wedge dx - du_{yz} \wedge dy - du_{zz} \wedge dz \end{cases} \tag{3.96}$$

Then, we will use the equations $\omega_u = \omega_x = \omega_y = \omega_z = 0$, which give expressions for du, du_x, du_y, du_z to simplify these equations. Then, we get

$$\begin{cases} d\omega_u = 0 \\ d\omega_x = -du_{xx} \wedge dx - du_{xy} \wedge dy - du_{xz} \wedge dz \\ d\omega_y = -du_{xy} \wedge dx - du_{yy} \wedge dy - du_{yz} \wedge dz \\ d\omega_z = -du_{xz} \wedge dx - du_{yz} \wedge dy - du_{zz} \wedge dz \end{cases} \tag{3.97}$$

Therefore s_1, we give dx, dy, dz the values $\varepsilon_1 x, \varepsilon_1 y,$ and $\varepsilon_1 z$ respectively, and $s_0 + s_1$ the rank of the matrix

$$\left(\begin{array}{cccccccc} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 x & \varepsilon_1 y & 0 & \varepsilon_1 z \\ 0 & 0 & 0 & 0 & \varepsilon_1 y & \varepsilon_1 x & \varepsilon_1 z & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 z & 0 & \varepsilon_1 y & \varepsilon_1 x \end{array}\right) \tag{3.98}$$

and we have $s_0 + s_1 = 7$, hence $s_1 = 3$. Then, for s_2, we form the matrix

$$\begin{pmatrix} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 x & \varepsilon_1 y & 0 & \varepsilon_1 z \\ 0 & 0 & 0 & 0 & \varepsilon_1 y & \varepsilon_1 x & \varepsilon_1 z & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 z & 0 & \varepsilon_1 y & \varepsilon_1 x \\ 0 & 0 & 0 & 0 & \varepsilon_2 x & \varepsilon_2 y & 0 & \varepsilon_2 z \\ 0 & 0 & 0 & 0 & \varepsilon_2 y & \varepsilon_2 x & \varepsilon_2 z & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2 z & 0 & \varepsilon_2 y & \varepsilon_2 x \end{pmatrix} \quad (3.99)$$

The rank is 8, so $s_2 = 1$. Note that the matrix has already attained its maximal rank, so $s_3 = 0$. Obviously if we remove the first four columns, which can be non-zero only for the first four rows since we must enforce the 1-form equations, then we can calculate more easily the numbers $s_1, s_1 + s_2$, etc., which correspond to the ranks of the series of the matrices stacked together.

Then, we can apply Cartan's test. This corresponds to sitting $\varepsilon_1 y = \varepsilon_1 z = \varepsilon_2 x = \varepsilon_2 z = 0$ above, and then the characters can be read of directly from the 2-form equations as

$$s_1 = 3\big(du_{xx}, du_{xy}, du_{xz}\big), \qquad s_2 = 1\big(du_{yz}\big), \qquad s_3 = 0. \quad (3.100)$$

s_1 corresponding to the independent forms which multiply dx, shown in parentheses, etc. Note that this shortcut works also when we have higher order forms in our equations as long as the differentials of dependent variables enter only linearly, i.e., we don’t have terms such as

$$du_{xx} \wedge du, \qquad dx \wedge dy \wedge du_x \wedge du_y.$$

If such forms are present, we cannot use this shortcut and the calculation of even the reduced characters become very difficult, since first we need to find the general 3-dimensional linear elements, which is already more difficult since now there would be quadratic or higher order relations among the parameters, and then calculate using the elements, the steps of calculation required being roughly quadratic in the total number of variables. Actually in such a case, unless the number of variables is exceedingly small, a better way to proceed is to immediately effect a prolongation so as to get rid of all of the original higher order form equations, and the new systems is guaranteed to include only linear forms in the dependent variables.

We want to see the free parameters in an integral element

$$\begin{cases} du_{xx} = u_{xxx}dx + u_{xxy}dy + u_{xxz}dz \\ du_{xy} = u_{xyx}dx + u_{xyy}dy + u_{xyz}dz \\ du_{xz} = u_{xzx}dx + u_{xzy}dy + u_{xzz}dz \\ du_{yz} = u_{yzx}dx + u_{yzy}dy + u_{yzz}dz \end{cases} \tag{3.101}$$

The above expression holds 12 parameters. Substituting this back to the two form equations, we see that the free parameters are

$$\begin{cases} u_{xxx} = u_{xyy} = u_{xzz} \\ u_{xxy} = u_{xyx} = u_{yzz} \\ u_{xxz} = u_{xzx} = u_{yzy} \\ u_{xyz} = u_{xzy} = u_{yzx} \end{cases} \tag{3.102}$$

so here the numbers of free parameters ($N = 4$), actually, even this substitution is unnecessary, since it is obvious that the free parameters are just the independent third order partial derivatives of u. We have

$$N = 4 < s_1 + 2s_2 + 3s_3 = 5, \tag{3.103}$$

so Cartans test fails, the system is not involutive and prolongation is necessary. We can also see where things could go wrong as we laboriously wrote down the matrices: the real characters would correspond to matrices whose top-row labels include also dx, dy, dz. Now already at s_2, the rank of the reduced polar matrix is already constrained by the number of columns, and if we have more columns the rank could grow further, and consequently imply linear dependence among dx, dy, dz, which at the same time will imply the existence of further constraints on the free parameters in the integral element than implied by the counting of reduced Cartan characters. If this happens, which is the present case, it shows that the reduced character and real characters are not equal, and we are not in the involutive case. Prolongation corresponds, on the other hand, adding to the labels $du_{xx}, du_{xy}, du_{xz}, du_{yz}$, so we will not be constrained by the number of columns so soon.

Now, for prolongation we take $u_{xxx}, u_{xxy}, u_{xxz}, u_{xyz}$ to be the new variables, adjoining (3.101) to the list of one-form equations (hence for the prolonged system, $s_0 = 4 + 4 = 8$), and we need to differentiate (3.101) to get some new two form equations (the original one are now all identities). We now have $11 + 4 = 15$ variables, and the number of variables that we want to get rid of is 12. We have

$$\begin{cases} d^2 u_{xx} = du_{xxx} \wedge dx + du_{xxy} \wedge dy + du_{xxz} \wedge dz \\ d^2 u_{xy} = du_{xxy} \wedge dx + du_{xxx} \wedge dy + du_{xyz} \wedge dz \\ d^2 u_{xz} = du_{xxz} \wedge dx + du_{xyz} \wedge dy + du_{xxx} \wedge dz \\ d^2 u_{yz} = du_{xyz} \wedge dx + du_{xxz} \wedge dy + du_{xxy} \wedge dz \end{cases} \tag{3.104}$$

For this new system, the reduced characters are

$$s_1 = 4 \left(du_{xxx}, du_{xxy}, du_{xxz}, du_{xyz}\right), \qquad s_2 = 0, \qquad s_3 = 0. \tag{3.105}$$

So $s_0 + s_1 + s_2 + s_3 = 12$, the number of dependent variables, as it should be.

For completeness, we give the polar matrix for calculating s_2 of which we have removed the columns corresponding to $du, du_x, du_y, du_z, du_{xx}, du_{xy}, du_{xz}, du_{yz}$

$$\begin{pmatrix} du_{xxx} & du_{xxy} & du_{xxz} & du_{xyz} \\ \hline \varepsilon_1 x & \varepsilon_1 y & \varepsilon_1 z & 0 \\ \varepsilon_1 y & \varepsilon_1 x & 0 & \varepsilon_1 z \\ \varepsilon_1 z & 0 & \varepsilon_1 x & \varepsilon_1 y \\ 0 & \varepsilon_1 z & \varepsilon_1 y & \varepsilon_1 x \\ \varepsilon_2 x & \varepsilon_2 y & \varepsilon_2 z & 0 \\ \varepsilon_2 y & \varepsilon_2 x & 0 & \varepsilon_3 z \\ \varepsilon_2 z & 0 & \varepsilon_2 x & \varepsilon_3 y \\ 0 & \varepsilon_2 z & \varepsilon_2 y & \varepsilon_3 x \end{pmatrix} \tag{3.106}$$

For the parameters,

$$\begin{cases} du_{xxx} = u_{xxxx}dx + u_{xxxy}dy + u_{xxxz}dz \\ du_{xxy} = u_{xxyx}dx + u_{xxyy}dy + u_{xxyz}dz \\ du_{xxz} = u_{xxzx}dx + u_{xxzy}dy + u_{xxzz}dz \\ du_{xyz} = u_{xyzx}dx + u_{xyzy}dy + u_{xyzz}dz \end{cases} \tag{3.107}$$

again there are 12 of them. The free ones can be algorithmically obtained by substituting these expressions into the two form equations, and we have

$$\begin{cases} u_{xxxx} = u_{xxyy} = u_{xxzz} \\ u_{xxxy} = u_{xxyx} = u_{xyzz}, \\ u_{xxxz} = u_{xxzx} = u_{xyzy}, \\ u_{xxyz} = u_{xxzy} = u_{xyzx}, \end{cases} \tag{3.108}$$

so the number of free parameters is $N = 4$. Again, this substitution can be avoided by noting that the free parameters are just the independent fourth order partial derivatives of u. Now

$$N = 4 = s_1 + 2s_2 + 3s_3, \tag{3.109}$$

so Cartans test is satisfied, the system is involutive, and the general solution of the differential equation depends on four functions of one variable, by the Cartan-Kähler theorem.

CHAPTER FOUR

The Extended Estabrook-Wahlquist Method

Section (4.1): Introduction

Wahlquist and Estabrook [17-20] made a significant advance when they found that prolongations can be constructed for nonlinear equations as we showed in Chapter Three Section Five. A kind of prolongation over a fiber bundle was found which corresponds to the Pfaffian system which gives the equation upon projecting to the transversal integral manifold. This approach yields results which can be exploited to develop Lax pairs and to study the Bäcklund properties [21,22, 29] of the system, as will be seen here. The objective here is to find prolongation structures that can be obtained for a large class of equations given by a two-by-two problem based on an $SU(2)$ Lie algebra and expressed in terms of differential forms. This results in a geometric approach which does not assume the form of any specific equation at the outset. The integrability condition for the Pfaffian system can be expressed as the vanishing of a traceless two-by-two matrix of two forms. This gives by construction the nonlinear equation to be studied. A prolongation structure for a nonlinear equation consists of a system of Pfaffian equations for a set of pseudopotentials, that is functions, which serve as potentials for conservation laws in a generalized sense. It will be shown how these prolongations for the two-by-two system can be derived recursively at first. In the first type of prolongation discussed here, forms are used which satisfy an integrability condition and define a type of connection in terms of pseudopotentials.

Sec (4.1.1): Prolongation Structure for Two-By-Two Problem

Consider the Pfaffian system which given by

$$\xi_i = 0, \qquad \xi_i = dy_i - \Omega_{ij} y_j, \qquad i, j = 1, 2. \tag{4.1}$$

In (4.1), Ω is a trackless two-by-two matrix which consists of a set of one forms. They can be thought of as quite general, but may be taken to constitute a one-parameter family of forms which, projected onto the solution manifold, depend on the independent variables, the dependent variables and their derivatives. The form of the matrix of one-forms Ω is given explicitly as

$$\Omega = \Omega_{ij} = \omega_l \sigma_l = \begin{bmatrix} \omega_3 & \omega_1 - i\omega_2 \\ \omega_1 + i\omega_2 & \omega_3 \end{bmatrix}, \tag{4.2}$$

where σ_l, $l = 1, 2, 3$ are the Pauli matrices. Using (4.2) for Ω, the exterior differential system in (4.1) takes the form,

$$\begin{aligned} \xi_1 &= dy_1 - y_1\omega_3 - y_2(\omega_1 - i\omega_2), \\ \xi_2 &= dy_2 - y_1(\omega_1 + i\omega_2) - y_2\omega_3. \end{aligned} \tag{4.3}$$

The integrability conditions for (4.1) are expressed as the vanishing of a traceless two-by-two matrix of two-forms Ψ,

$$\Psi = 0, \qquad \Psi = d\Omega - \Omega \wedge \Omega \tag{4.4}$$

This gives by construction the nonlinear equation which is of interest. The components of Ψ can be expressed in the form

$$\Psi = \Psi_{ij} = \vartheta_l \sigma_l, \qquad \vartheta_l = d\omega_l - i\varepsilon_{lmn}\omega_m \wedge \omega_n. \tag{4.5}$$

where ε_{lmn} represents in (4.5) the totally antisymmetric constants of an $SU(2)$ Lie algebra, which is the case considered now. The nonlinear system to be considered is specified then by

$$\Psi = 0, \qquad \vartheta_l = 0, \qquad l = 1, 2, 3. \tag{4.6}$$

By exterior differential of Ψ in (4.4), it is found that

$$d\Psi = \Omega \wedge \Psi - \Psi \wedge \Omega. \tag{4.7}$$

This establishes that the exterior derivatives of the two-forms $\{\vartheta_l\}$ are contained in the ring generated by the set $\{\vartheta_l\}$.

It is important to realize that (4.1) and integrability condition (4.4) are both invariant under the following type of gauge transformation

$$y \longrightarrow y' = Qy, \Omega \longrightarrow \Omega' = Q\Omega Q^{-1} + dQQ^{-1}, \Psi \longrightarrow \Psi' = Q\Psi Q^{-1} \tag{4.8}$$

In (4.8), Q is an arbitrary space-time dependent two-by-two matrix with determinant one. In other words, the gauge transformation of Ω does not change the solution manifold of the nonlinear equation. The matrix of one-forms Ω has the interpretation of being a connection on a gauge field, the two-form is Ψ, a curvature or gauge field strength, and the closure property (4.7), a Bianchi identity.

Theorem (4.1.1):

The exterior derivatives of the forms ξ_1 and ξ_2 have the form,

$$\begin{aligned} d\xi_1 &= -y_2(\vartheta_1 - i\vartheta_2) - y_1\vartheta_3 + \omega_3 \wedge \xi_1 + (\omega_1 - i\omega_2) \wedge \xi_2, \\ d\xi_2 &= -y_1(\vartheta_1 + i\vartheta_2) + y_2\vartheta_3 + (\omega_1 + i\omega_2) \wedge \xi_1 - \omega_3 \wedge \xi_2. \end{aligned} \tag{4.9}$$

Consequently, the derivatives of ξ_1 and ξ_2 are contained in the ring of forms spanned by $\{\vartheta_i\}$ and forms $\{\xi_i\}$.

Proof:

Expressions for the dy_i follow from (4.3), and from (4.5) $d\omega_i$ can be obtained by writing

$$d\omega_1 = \vartheta_1 + 2i\omega_2 \wedge \omega_3, \qquad d\omega_2 = \vartheta_2 - 2i\omega_1 \wedge \omega_3, \qquad d\omega_3 = \vartheta_3 + 2i\omega_1 \wedge \omega_2.$$

The exterior derivative of ξ_1 from (4.3) is given by

$$
\begin{aligned}
d\xi_1 = -y_1\vartheta_3 - 2iy_1\omega_1 \wedge \omega_2 + \omega_3 \wedge \big(\xi_1 + y_1\omega_3 + y_2(\omega_1 - i\omega_2)\big) \\
- y_2(\vartheta_1 + 2i\omega_2 \wedge \omega_3 - i\vartheta_2 - 2\omega_1 \wedge \omega_3) + (\omega_1 - i\omega_2) \\
\wedge (\xi_2 + y_1(\omega_1 + i\omega_2) - y_2\omega_3) \\
= -y_2\vartheta_1 + iy_2\vartheta_2 - y_1\vartheta_3 + (\omega_1 - i\omega_2) \wedge \xi_2 - 2iy_1\omega_1 \wedge \omega_2 + y_2\omega_3 \\
\wedge (\omega_1 - i\omega_2) - 2iy_2\omega_2 \wedge \omega_3 + 2y_2\omega_1 \wedge \omega_3 + 2iy_1\omega_1 \wedge \omega_2 \\
- y_2(\omega_1 - i\omega_2) \wedge \omega_3
\end{aligned}
$$

The second line in the final result vanishes and we are left with the first equation in (4.9). The proof of second equation in (4.9) is proceeds in the same way.

Corollary (4.1.2):

The exterior derivatives (4.9) can be expressed in terms of the matrix elements of Ω and Ψ in (4.2) and (4.5) for $i = 1, 2$ as follows,

$$
d\xi_i = -\Psi_{ij}y_j - \Omega_{ij} \wedge \xi_j. \tag{4.10}
$$

The one-forms ξ_1 and ξ_2 can be used to generate an ideal which assumes a standard Riccati form by taking particular linear combinations of them. The new one-forms which result are called ξ_3 and ξ_4, and are defined by calculating in the following way; first

$$
y_1^2\xi_3 = y_1\xi_2 - y_2\xi_1 = y_1dy_2 - y_2dy_1 - y_1^2(\omega_1 + i\omega_2) + 2y_1y_2\omega_3 + y_2^2(\omega_1 - i\omega_2).
$$

Therefore, ξ_3 is given by

$$
\xi_3 = d(y_2/y_1) - (\omega_1 + i\omega_2) + 2(y_2/y_1)\omega_3 + (y_2/y_1)^2(\omega_1 - i\omega_2). \tag{4.11}
$$

In a similar fashion,

$$
y_2^2\xi_4 = y_2\xi_1 - y_1\xi_2 = y_2dy_1 - y_1dy_2 - y_2^2(\omega_1 - i\omega_2) - 2y_1y_2\omega_3 + y_1^2(\omega_1 + i\omega_2),
$$

and so ξ_4 is given by

$$
\xi_4 = d(y_1/y_2) - (\omega_1 - i\omega_2) - 2(y_1/y_2)\omega_3 + (y_1/y_2)^2(\omega_1 + i\omega_2). \tag{4.12}
$$

Introducing the new projective variables ξ_3 and ξ_3 defined to be

$$y_3 = y_2/y_1\,, \qquad y_4 = y_1/y_2 \tag{4.13}$$

into the expression for ξ_3 and ξ_4, Pfaffian system (4.1) takes the Riccati form,

Theorem (4.1.3):

The exterior derivatives of the forms ξ_3 and ξ_4 in (4.12), (4.13) are given by

$$\begin{aligned} d\xi_3 &= -(\vartheta_1 + i\vartheta_2) + y_3^2(\vartheta_1 - i\vartheta_2) + 2y_3\vartheta_3 - 2\big(\omega_3 + y_3(\omega_1 - i\omega_2)\big) \wedge \xi_3, \\ d\xi_4 &= -(\vartheta_1 - i\vartheta_2) + y_4^2(\vartheta_1 + i\vartheta_2) - 2y_4\vartheta_4 + 2\big(\omega_3 - y_4(\omega_1 + i\omega_2)\big) \wedge \xi_4. \end{aligned} \tag{4.14}$$

Consequently, the derivatives of ξ_3 and ξ_4 are contained in the ring of forms spanned by $\{\vartheta_i\}$ and forms $\{\xi_i\}$.

Thus, theorem (4.1.3) is proved along the same lines as the previous theorem where dy_3 and dy_4 are obtained from (4.12) and (4.13) respectively.
The results of theorem (4.1.3) can be cast in the general form,

$$\begin{aligned} d\xi_3 &= \beta_1 \wedge \xi_3 + \sum_{j=1}^{3} c_{3j}\vartheta_j\,, \\ d\xi_4 &= \beta_2 \wedge \xi_4 + \sum_{j=1}^{3} c_{4j}\vartheta_j\,, \end{aligned} \tag{4.15}$$

where the coefficients c_{ij} are function valued quantities and the β_i are 1-forms defined to be

$$\begin{aligned} \beta_1 &= -2\omega_3 - 2y_3(\omega_1 - i\omega_2), \\ \beta_2 &= 2\omega_3 - 2y_4(\omega_1 + i\omega_2). \end{aligned} \tag{4.16}$$

It is interesting to note that the exterior derivatives of the forms β_i in (4.16) have the same generic form as that expressed on the right side of (4.15). Based on these results, there are a series of prolongation results which can be stated and proved along lines similar to the ones given. These results will be collected together in Theorem (4.1.4):

Theorem (4.1.4):

a) Define 1-forms ξ_5 and ξ_6 to have the form,

$$\begin{aligned} \xi_5 &= dy_5 + 2\omega_3 + 2y_3(\omega_1 - i\omega_2), \\ \xi_6 &= dy_6 - 2\omega_3 + 2y_4(\omega_1 + i\omega_2). \end{aligned} \tag{4.17}$$

The exterior derivatives of ξ_5 and ξ_6 can be expressed in the form,

$$\begin{aligned} d\xi_5 &= 2y_3(\vartheta_1 - i\vartheta_2) + 2\vartheta_3 + 2\xi_3 \wedge (\omega_1 - i\omega_2), \\ d\xi_6 &= 2y_4(\vartheta_1 + i\vartheta_2) - 2\vartheta_3 + 2\xi_4 \wedge (\omega_1 + i\omega_2). \end{aligned} \tag{4.18}$$

b) Define the pair of the 1-form

$$\begin{aligned} \xi_7 &= dy_7 - e^{y_5}(\omega_1 - i\omega_2), \\ \xi_8 &= dy_8 - e^{y_6}(\omega_1 + i\omega_2). \end{aligned} \tag{4.19}$$

The exterior derivatives of ξ_7 and ξ_8 can be expressed in the form,

$$\begin{aligned} d\xi_7 &= -e^{y_5}\big(\vartheta_1 - i\vartheta_2 + \xi_5 \wedge (\omega_1 - i\omega_2)\big), \\ d\xi_8 &= -e^{y_6}\big(\vartheta_1 + i\vartheta_2 + \xi_6 \wedge (\omega_1 + i\omega_2)\big). \end{aligned} \tag{4.20}$$

Theorem (4.1.4) is shown along lines identical to that used for Theorem (4.1.1) by evaluating exterior derivatives of the relevant forms, substituting known derivatives and then simplifying the resulting expression.

Sec (4.2): Estabrook-Wahlquist EW Procedure to Lax-Integrable System

Some NLEEs have appeared as the compatibility conditions for systems of linear partial differential equations of the first order, and such NLEEs have been referred to as Lax integrable. Using Lax pairs, people can construct the gauge transformations (GTs) [24,26], Darboux transformations (DTs) [27,28], of such NLEEs. For a given NLEE, it is difficult to determine whether it can be associated with a Lax pair. One of the methods to test the Lax integrability is the prolongation structure (PS) method proposed via exterior differentials.

We embark in this section on an attempt to systematize the derivation of Lax-integrable systems with variable coefficients. Of the many techniques which have been employed for constant coefficient integrable systems. The method directly proceeds to attempt construction of the Lax Pair or linear spectral problem; whose compatibility condition is the integrable system under discussion. While not at all guaranteed to work, any successful implementation of the technique means that Lax-integrability has already been verified during the study, and in addition the Lax Pair is algorithmically obtained. If the technique fails, that does not necessarily imply non-integrability of the equation contained in the compatibility condition of the assumed Lax Pair.

(4.2.1): The Exterior Differential Expression of Lax Integrability

In the standard Estabrook-Wahlquist method one begins with a constant coefficient NLPDE and assumes an implicit dependence on $u(x,t)$ and its partial derivatives of the spatial and time evolution matrices (M,N) involved in the linear scattering problem, or its Lax representation can be written in the following matrix form (4.21)

$$\begin{aligned} \psi_x &= M(\lambda)\psi, \\ \psi_t &= N(\lambda)\psi, \end{aligned} \tag{4.21}$$

where x and t are the independent variables, the subscripts denote the partial differentials, λ is the spectral parameter, ψ is the eigen-function associated with λ, while M and N are (evolution matrices connected via a zero-curvature condition) whose elements are dependent on λ. The corresponding NLEE can be obtained from the compatibility condition of system (4.21), which can be written as

$$M_t - N_x + [M,N] = 0, \tag{4.22}$$

Define the system of 1-forms

$$\omega = M\,dx + N\,dt. \tag{4.23}$$

The system (4.21) can be equivalent to the following Pfaff system

$$\sigma = d\psi - \omega\psi = 0. \tag{4.24}$$

By using the exterior differentiation on (4.24), we get

$$d\sigma = -d\omega\,\psi + \omega \wedge d\psi, \tag{4.25}$$

Combining (4.24) and (4.25), we get

$$\begin{aligned} d\sigma &= -d\omega\,\psi + \omega \wedge (\sigma + \omega\psi) \\ &= \omega \wedge \sigma - (d\omega - \omega \wedge \omega)\psi. \end{aligned} \tag{4.26}$$

Through the Frobenius theorem, Pfaff system (4.24) is completely integrable if and only if the following system is satisfied

$$\Omega = d\omega - \omega \wedge \omega = 0, \tag{4.27}$$

where Ω is a square matrix whose elements are 2-forms. It can be verified that system (4.27) is equivalent to compatibility condition (4.22). Thus, system (4.27) also corresponds to the NLEE whose Lax representation is given by (4.21). On the relation between complete integrability and Lax integrability, we have the following proposition:

Proposition (4.2.1)

The (1+1)-dimensional NLEE whose Lax representation is given by (4.21) must be completely integrable.

Proof

Differentiating system (4.27), we have

$$\begin{aligned} \Omega &= -d\omega \wedge \omega + \omega \wedge d\omega \\ &= -(\Omega + \omega \wedge \omega) \wedge \omega + \omega \wedge (\Omega + \omega \wedge \omega) \\ &= -\Omega \wedge \omega + \omega \wedge \Omega. \end{aligned} \tag{4.28}$$

Thus, system (4.27) is completely integrable according to the Frobenius theorem. Because system (4.27) corresponds to the (1+1)-dimensional NLEE whose Lax representation is given by (4.21). ■

(4.2.1): An Equivalent Definition of Lax Integrability

We replace the (1+1)-dimensional NLEE whose Lax representation is given by (4.21), by a system of 2-forms. Hereby, we take the nonlinear Schrödinger equation (NLSE)

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{4.29}$$

as an example to illustrate this procedure, where u is a complex function of x and t. Equation (4.29) can describe the propagation of light in the nonlinear optical fiber or Bose–Einstein condensate confined to highly anisotropic cigar-shaped trap. Equation (4.29) has also appeared in the studies of small-amplitude gravity wave on the surface of deep inviscid water. Let $U \subset \mathbb{R}^2$ be coordinated by $x,\ t$ and, $V = U \times \mathbb{R}^4 \subset \mathbb{R}^6$ be a fiber bundle with U as the base manifold and V is coordinated by $x, t, u, \bar{u}, p, \bar{p}$, where $\bar{u}$ and $\bar{p}$ are complex conjugates of u and p respectively. Then, equation (4.29) can be represented by the following system of 2-forms which are defined on the manifold V

$$\begin{aligned} \alpha_1 &= p dx \wedge dt - du \wedge dt, \\ \alpha_2 &= \bar{p} dx \wedge dt - d\bar{u} \wedge dt, \\ \alpha_3 &= -i du \wedge dx + dp \wedge dt + 2u^2 \bar{u} dx \wedge dt, \\ \alpha_4 &= i d\bar{u} \wedge dx + d\bar{p} \wedge dt + 2\bar{u}^2 u dx \wedge dt. \end{aligned} \tag{4.30}$$

If map $s: U \longrightarrow V$ is a cross section of V with the property

$$s^* \alpha_i = 0, \qquad i = 1, 2, 3, 4 \tag{4.31}$$

where s^* denotes the pull back of the map s, it can be verified that $u(x,t)$ is a solution of equation (4.29). Conversely, for any given solution $u(x,t)$ of equation (4.29), the map $s: U \longrightarrow V$ by $(x,t) \longrightarrow (x, t, u(x,t), \bar{u}(x,t), u_x(x,t), \bar{u}_x(x,t))$ is a cross section of V which satisfies property (4.31). Without causing any confusion, we write property (4.31) just as $\{\alpha_i = 0\}$ for simplicity. For any (1+1)-dimensional NLEE, the corresponding system of 2-forms $\{\alpha_i = 0\}$ can be constructed as above.

Two systems of 2-forms $\{\alpha_i = 0\}$ and $\{\beta_i = 0\}$ are said to be equivalent if $\beta_i = f_i^{\ j}\alpha_j$, and the rank of matrix $\left(f_i^{\ j}\right)$ is full, where β_i's are 2-forms. It should be noted that the Einstein summation convention is used here and below. With those preparations, we give the following equivalent definition of Lax integrability:

Proposition (4.2.2):

A (1+1)-dimensional NLEE is Lax integrable if and only if there exist a square matrix ω of 1-forms about dx and dt, such that the system $\Omega = d\omega - \omega \wedge \omega = 0$ is equivalent to $\{\alpha_i = 0\}$.

Proof

a) Assume that the Lax representation for the (1+1)-dimensional NLEE is given by (4.21). if ω is set to be $M\,dx + N\,dt$, then ω is exactly what we wanted according to the chain of the equivalent relations:

$$\begin{aligned} \Omega = d\omega - \omega \wedge \omega = 0 & \\ \Leftrightarrow M_t - N_x + [M, N] = 0 & \\ \Leftrightarrow (1+1) - \text{dimensional NLEE whose} & \\ \text{Lax representation is given by (4.21)} & \\ \Leftrightarrow \{\alpha_i = 0\}. & \end{aligned} \tag{4.32}$$

b) Assume that the square matrix of 1-forms about dx and dt is ω. Then, ω can be decomposed into the dx part and dt part, i.e., ω can be written in the form $\omega = Mdx + Ndt$. It can be verified that M and N are just the Lax pair we are searching for. ■

Proposition (4.32) defines the (1+1)-dimensional NLEE whose Lax representation is given by (4.21) in terms of the differential form and exterior differentials.

(4.2.2): Some Remarks on Prolongation Structure Method

The prolongation structure method has been used to test the Lax integrability. Just like what we have done above, we hereby replace the (1+1)-dimensional NLEE whose Lax representation is given by (4.21) by the system of 2-forms $\{\alpha_i = 0\}$ which are defined on the manifold V. We introduce the Pfaff system on the vector bundle $E = V \times \mathbb{R}^n$,

$$\sigma_i = dy_i - F_i dx - G_i dt, \qquad i = 1, 2, \dots, n \tag{4.33}$$

where y_i are called the pseudo-potentials and they are the coordinates of $\mathbb{R}^n$, $n \in \mathbb{N}$, σ_i is 1-forms, F_i and G_i are the functions on the vector bundle E. The prolongation structure method requires that the ideal generated by the system $\{\alpha_i = 0,\ \sigma_i = 0\}$ is a closed ideal. According to Frobenius theorem and proposition (4.21), we know that $\{\alpha_i = 0\}$ must generate a closed ideal. Thus, the prolongation condition can be written as

$$d\sigma_i = m_i^j \alpha_j + n_i^j \wedge \sigma_j, \qquad i = 1, 2, \dots, n \tag{4.34}$$

where m_i^j are functions to be determined and n_i^j are 1-forms.

Via the comparison between equations (4.24) and (4.33), it can be seen that the pseudo-potentials y_i correspond to the eigenfunction ψ, while F_i and G_i correspond to ω. What's more, there is a correspondence between prolongation condition (4.34) and equation (4.26). In fact, from that correspondence, we can see that prolongation condition (4.34) is to guarantee that Pfaff system (4.33) is completely integrable on the solution manifold $\{\alpha_i = 0\}$ of the (1+1)-dimensional NLEE whose Lax representation is given by (4.21). Then, Lax equation (4.21) for the (1+1)-dimensional NLEE can be obtained with the complete integrability condition although the fact that the Lax integrability is a stronger integrability property than complete integrability.

(4.2.3): Geometric Interpretation of Lax Equation

Assume that there is a connection D defined on the vector bundle $E = V \times \mathbb{R}^n$, and the sections $s_1, \dots, s_n$ form a frame of sections of E. Using the frame of sections S, where S is the transpose of the row vector $(s_1, \dots, s_n)$, and the connection D, we can obtain an $n \times n$ connection matrix ω which is defined by the formula $DS = \omega S$. We will give the geometric interpretation of ω in this part. Elements of the connection matrix ω are functions on the manifold V. If the section $s = \eta^i s_i$ is a parallel section of E with η^i as functions on the manifold V, i.e., $Ds = 0$, then η^i should satisfy the following equations

$$d\eta^i + \eta^j \omega_j^i = 0, \qquad i = 1, \dots, n. \tag{4.35}$$

The equation (4.35) can be written in the matrix form

$$d\eta + \eta\omega = 0. \tag{4.36}$$

The connection D on the vector bundle E can induce a connection D' on the dual vector bundle $E^* = V \times (\mathbb{R}^n)^*$, where $(\mathbb{R}^n)^*$ denotes the dual space of $\mathbb{R}^n$. If we choose the dual frame of the sections $S^* = (s^{1*}, \dots, s^{n*})^T$ of E^*, i.e., $\langle s_i, s^{j*} \rangle = \delta_i^j$, where $\langle\ ,\ \rangle$ represent the inner product in the vector bundles E and E^*, then we have the equation $D'S^* = -\omega S^*$. That is to say, the induced connection matrix on the dual vector bundle E^* is $-\omega$. If the section $s^* = \theta_i s^{i*}$ is a parallel section of E^* with θ_i as functions on the manifold V, i.e., $D's^* = 0$, θ_i should satisfy the following equation

$$d\theta_i - \theta_j \omega_i^j = 0. \tag{4.37}$$

Also the equation (4.37) can be written in the matrix form

$$d\theta - \omega\theta = 0. \tag{4.38}$$

It can be seen that equation (4.38) is exactly Lax equation (4.24). then, Lax equation (4.24) can be interpreted as the parallel section equation on the dual vector bundle E^* with the connection matrix equal to $-\omega = -(Mdx + Ndt)$, while the eigenfunction ψ corresponds to the vector formed by the coordinates of parallel section of E^* under the dual frame of sections S^*.

In the differential geometry, formula $\Omega = d\omega - \omega \wedge \omega$ represents the curvature matrix with ω as the connection matrix. Then, Pfaff system (4.24) is completely integrable if and only if the curvature matrix Ω vanishes. If zero-curvature condition (4.27) is satisfied, then there exists n linearly-independent parallel sections, i.e., the Lax equation $d\psi = \omega\psi$ has n linearly independent solutions. With those solutions, we can construct the multi-soliton solutions using the Darboux transformation.

Sec (4.3): Exterior Differential Expression of the Gauge Transformation and Darboux Transformation

Assume that we have the following two Lax equations

$$\psi_x = M\psi, \qquad \psi_t = N\psi, \tag{4.39}$$

$$\psi'_x = M'\psi', \qquad \psi'_t = N'\psi', \tag{4.40}$$

where ψ' is the eigenfunction associated with spectral parameter λ, while M' and N' are matrices. If there exists a gauge transformation (GT) $\psi' = T\psi$ with T as a matrix, which converts Lax equation (4.39) into (4.40), the gauge transformation T should satisfy the following system

$$\begin{aligned} T_x &= M'T - TM, \\ T_t &= N'T - TN. \end{aligned} \tag{4.41}$$

It can be verified that system (4.41) can be written in the form

$$\omega' = T\omega T^{-1} + dT \cdot T^{-1}, \tag{4.42}$$

where ω is defined by equation (4.23), while ω' is $M'dx + N'dt$. Equation (4.42) is the transformation formula of the connection matrix under the transformation of the basis of sections $S' = TS$. The transformation formula for the curvature matrix Ω is

$$\Omega' = T\Omega T^{-1} \tag{4.43}$$

where $\Omega = d\omega - \omega \wedge \omega$, while $\Omega' = d\omega' - \omega' \wedge \omega'$. System (4.41) is also equivalent to the following Pfaff system

$$dT - \omega' T + T\omega = 0. \tag{4.44}$$

Because the curvature matrices Ω and Ω' both vanish from zero-curvature condition (4.47), it can be proved that Pfaff system (4.44) is completely integrable according to the Frobenius theorem. That is to say, every (1+1)-dimensional NLEE with Lax integrability is gauge equivalent to one another. We can choose the Korteweg–de Vries equation *(*KdV) as an example

$$u_t + u_{xxx} + 6uu_x = 0, \tag{4.45}$$

where $u = u(x,t)$, the equation (4.45) is encountered in many physical areas such as the shallow water waves in the ocean, internal gravity waves in the lake of changing cross section and ion-acoustic waves in the plasma.

Assume that the n-dimensional Lax representation for equation (4.45) is $d\psi = \omega\psi$ is equivalent to the system $\Omega = d\omega - \omega \wedge \omega = 0$. Thus, we have the following proposition:

Proposition (4.3.1):

The (1+1)-dimensional NLEE with Lax integrability is equivalent to the system $T\Omega T^{-1} = 0$ for some $n \times n$ invertible matrix T whose elements are functions on the manifold V.

Proof:

If a (1+1)-dimensional NLEE with Lax integrability is equivalent to the system $T\Omega T^{-1} = 0$, then the Lax equation corresponding to that NLEE is $d\psi' = \omega'\psi'$, where $\omega' = T\omega T^{-1} + dT \cdot T^{-1}$. Then, that NLEE is Lax integrable. Conversely, we have known that every (1+1)-dimensional NLEE with Lax integrability is gauge equivalent to equation (4.45) via the above discussion. Then, there exists an invertible matrix T, such that the given (1+1)-dimensional NLEE with Lax integrability is equivalent to the system $T\Omega T^{-1} = 0$. ■

Gauge transformations (GTs) of the same (1+1)-dimensional NLEE with Lax integrability are useful in the analysis of this kind of the NLEEs. We say that T is a GT if the two systems $T\Omega T^{-1} = 0$ and $\Omega = 0$ are equivalent, i.e., they correspond to the same (1+1)-dimensional NLEE with Lax integrability. The set of all GTs in fact forms a group, which we call the gauge group. Because GTs preserve solution manifold, gauge group can also be called the symmetry group. The Darboux transformation is a particular gauge transformation. The set of all DTs also forms a group which is a subgroup of the gauge group.

Proposition (4.3.2):

For the (1+1)-dimensional NLEE whose Lax representation is given by (4.21), if there exists a Darboux transformation of the form $T = \lambda I - S$, then the $n \times n$ matrix S, which is independent on the spectral parameter λ, must satisfy the following equation

$$dS + [S, \omega(S)] = 0. \tag{4.46}$$

Note (4.3.3):

if we expand $\omega(\lambda) = M(\lambda)dx + N(\lambda)dt$ into the power series of the spectral parameter λ as $\omega(\lambda) = \sum_{i=0}^{m} \omega_i \lambda^i$, then $\omega(S)$ which appears in equation (4.46) represents $\sum_{i=0}^{m} \omega_i S^i$, where m is positive integer.

Proof:

Substituting $\omega(\lambda) = \sum_{i=0}^{m} \omega_i \lambda^i$, $\omega'(\lambda) = \sum_{i=0}^{m} \omega_i' \lambda^i$, $T = \lambda I - S$ into equation (4.44), and collecting the coefficients of the same power of the spectral parameter λ, we get

$$\lambda^{m+1}: \omega_m' = \omega_m \,, \tag{4.47}$$

$$\lambda^j: \omega_{j-1}' = \omega_{j-1} - S\omega_j + \omega_j' S, \qquad j = 1, \dots, m \tag{4.48}$$

$$\lambda^0: dS + S\omega_0 - \omega_0' S = 0 \,. \tag{4.49}$$

The relation between ω_0' and ω_0 can be derived from equations (4.47) and (4.48) as

$$\omega_0' = \omega_0 + \sum_{k=1}^{m} [\omega_k, S]\, S^{k-1} \,. \tag{4.50}$$

Substituting equation (4.50) into (4.49), we obtain equation (4.46).

Equation (4.46) can be decomposed into the following system

$$\begin{aligned} S_x + [S, M(S)] &= 0 \,, \\ S_t + [S, N(S)] &= 0 \,, \end{aligned} \tag{4.51}$$

where $M(S)dx + N(S)dt = \omega(S)$.

Proposition (4.3.4):

Equation (4.46) is completely integrable if and only if the following condition is satisfied

$$d\omega(S) - \omega(S) \wedge \omega(S) = 0 \,. \tag{4.52}$$

Equivalently, system (4.51) is completely integrable if and only if

$$M_t(S) - N_x(S) + [M(S), N(S)] = 0 \,. \tag{4.53}$$

Proof:

Consider the Pfaff system

$$\tau = dS + [S, \omega(S)] = dS + [S, M(S)]dx + [S, N(S)]dt = 0, \qquad (4.54)$$

then, by differentiating the (4.54), we get

$$\begin{aligned} d\tau &= [dS, M(S)] \wedge dx + [S, dM(S)] \wedge dx + [dS, N(S)] \wedge dt + [S, dN(S)] \wedge dt \\ &= [dS, M(S)] \wedge dx + [S, M_t(S) - N_x(S)]dt \wedge dx + [dS, N(S)] \wedge dt\,. \end{aligned}$$

Substituting dS by $\tau - [S, M(S)]dx - [S, N(S)]dt$ into above system, we have

$$\begin{aligned} d\tau &= [\tau, M(S)] \wedge dx + [\tau, N(S)] \wedge dt + [S, M_t(S) - N_x(S)]dt \wedge dx \\ &\quad - \big[[S, N(S)], M(S)\big]dt \wedge dx - \big[[S, M(S)], N(S)\big]dx \wedge dt. \end{aligned} \qquad (4.55)$$

By impact of the Jacobi identity

$$\big[M(S), [N(S), S]\big] + \big[S, [M(S), N(S)]\big] + \big[N(S), [S, M(S)]\big] = 0,$$

the equation (4.55) is written as

$$d\tau = [\tau, M(S)] \wedge dx + [\tau, N(S)] \wedge dt + \big[S, M_t - N_x + [M, N]\big]dt \wedge dx. \quad (4.56)$$

Thus, the Pfaff system $\tau = 0$ is completely integrable if and only if the condition $M_t(S) - N_x(S) + [M(S), N(S)] = 0$ is satisfied according to the Frobenius theorem. ■

The matrix S can be constructed as follows: Let Λ be a $n \times n$ diagonal matrix whose diagonal elements are $\lambda_1, \dots, \lambda_n$, where $\lambda_1, \dots, \lambda_n$ are complex spectral parameter. H represents the $n \times n$ invertible matrix $(\psi_1, \dots, \psi_n)$, where ψ_i, $i = 1, \dots, n$ satisfy the equations $d\psi_i = \omega(\lambda_i)\psi_i$. Define S as $H\Lambda H^{-1}$, and we will prove that $S = H\Lambda H^{-1}$ satisfies condition (4.50). for that, two lemmas will be given.

Lemma (4.3.5):

$S = H\Lambda H^{-1}$ satisfies the equation

$$dS^i = \omega(S)S^i - S^i\omega(S). \tag{4.57}$$

Proof:

Differentiating $H = (\psi_1, \dots, \psi_n)$, we have

$$\begin{aligned} dH &= (d\psi_1, \dots, d\psi_n) = (\omega(\lambda_1)\psi_1, \dots, \omega(\lambda_n)\psi_n) \\ &= \left(\sum_{i=0}^{m} \omega_i \lambda_1^i \psi_1, \dots, \sum_{i=0}^{m} \omega_i \lambda_n^i \psi_n\right) = \sum_{i=0}^{m} \omega_i H\Lambda^i . \end{aligned} \tag{4.58}$$

Using the equation (4.58) then, we can calculate the differential of S

$$\begin{aligned} dS &= d(H\Lambda H^{-1}) = dH\Lambda H^{-1} + H\Lambda dH^{-1} \\ &= dH\Lambda H^{-1} - H\Lambda H^{-1} dH H^{-1} \\ &= \left(\sum_{i=0}^{m} \omega_i H\Lambda^i\right)\Lambda H^{-1} - H\Lambda H^{-1}\left(\sum_{i=0}^{m} \omega_i H\Lambda^i\right)H^{-1} \\ &= \left(\sum_{i=0}^{m} \omega_i H\Lambda^i H^{-1}\right)\mathrm{H}\Lambda H^{-1} - H\Lambda H^{-1}\left(\sum_{i=0}^{m} \omega_i H\Lambda^i H^{-1}\right) \\ &= \left(\sum_{i=0}^{m} \omega_i S^i\right)S - S\left(\sum_{i=0}^{m} \omega_i S^i\right) = \omega(S)S - S\omega(S). \end{aligned} \tag{4.59}$$

Then we get

$$dS^i = dSS^{i-1} + SdSS^{i-2} + \cdots + S^{i-1}dS, \tag{4.60}$$

where

$$dSS^{i-1} = \big(\omega(S)S - S\omega(S)\big)S^{i-1} = \omega(S)S^i - S\omega(S)S^{i-1}, \tag{4.61a}$$

$$SdSS^{i-2} = S\big(\omega(S)S - S\omega(S)\big)S^{i-2} = S\omega(S)S^{i-1} - S^2\omega(S)S^{i-2}, \tag{4.61b}$$

$$\vdots$$

$$S^{i-1}dS = S^{i-1}\big(\omega(S)S - S\omega(S)\big) = S^{i-1}\omega(S)S - S^i\omega(S). \tag{4.61c}$$

From equations (4.60), (4.61a), (4.61b), and (4.61c), we obtain equation (4.57) ■

Lemma (4.3.6):

$S = H\Lambda H^{-1}$ satisfies the equation

$$\sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge \omega(S) S^i = 0. \qquad (4.62)$$

Proof:

The compatibility condition of equation $d\psi = \omega(\lambda)\psi$ is

$$\Omega(\lambda) = d\omega(\lambda) - \omega(\lambda) \wedge \omega(\lambda) = 0.$$

$\Omega(\lambda)$ can be expanded into the power series of the spectral parameter λ as

$$\Omega(\lambda) = \sum_{i=0}^{m} d\omega_i \lambda^i - \sum_{i=0}^{m} \omega_i \lambda^i \wedge \omega(\lambda) = 0. \qquad (4.63)$$

Moreover, (4.63) can be written in the following identities

$$\sum_{i=0}^{m} d\omega_i \lambda_1^i = \sum_{i=0}^{m} \omega_i \wedge \omega(\lambda_1) \lambda_1^i = \sum_{i,j=0}^{m} \omega_i \wedge \omega_j \lambda_1^{i+j}, \qquad (5.44a)$$

$$\vdots$$

$$\sum_{i=0}^{m} d\omega_i \lambda_n^i = \sum_{i=0}^{m} \omega_i \wedge \omega(\lambda_n) \lambda_n^i = \sum_{i,j=0}^{m} \omega_i \wedge \omega_j \lambda_n^{i+j}. \qquad (5.44b)$$

From equations (4.64a) and (4.64b), we can derive

$$\sum_{i=0}^{m} d\omega_i \lambda_1^i \psi_1 = \sum_{i,j=0}^{m} \omega_i \wedge \omega_j \lambda_1^{i+j} \psi_1, \qquad (4.65a)$$

$$\vdots$$

$$\sum_{i=0}^{m} d\omega_i \lambda_n^i \psi_n = \sum_{i,j=0}^{m} \omega_i \wedge \omega_j \lambda_n^{i+j} \psi_n. \qquad (4.65b)$$

Therefore, equations (4.65a) and (4.65b) can be written in the matrix form

$$\sum_{i=0}^{m} d\omega_i H\Lambda^i = \sum_{i,j=0}^{m} \omega_i \wedge \omega_j H\Lambda^{i+j}\,. \tag{4.66}$$

Then, multiplying both sides of equation (4.66) by H^{-1}, we get equation (4.62). ■

Proposition (4.3.7):

$S = H\Lambda H^{-1}$ satisfies the equation (4.52)

Proof:

Using Lemma (4.3.5) and Lemma (4.3.6)

Via the direct calculation

$$\begin{aligned} d\omega(S) &= d\left(\sum_{i=0}^{m} \omega_i S^i\right) = \sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge dS^i \\ &= \sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge \big(\omega(S)S^i - S^i\omega(S)\big) \\ &= \sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge \omega(S)S^i + \sum_{i=0}^{m} \omega_i \wedge S^i\omega(S) \\ &= \sum_{i=0}^{m} \omega_i \wedge S^i\omega(S) = \omega(S) \wedge \omega(S)\,, \end{aligned} \tag{4.67}$$

the proof is complete, where the second step is due to Lemma (4.3.5), while the last step is due to Lemma (4.3.6). ■

CHAPTER FIVE

Applications on Exterior Differential Systems

Section (5.1): Prolongation Structures of Nonlinear Evolution Equations

This study about "Prolongation structures of nonlinear evolution equations," Wahlquist and Estabrook introduced for "prolonging" a partial differential equation and applied it to the generalization Korteweg-de Vries (KdV) equation. They found that the prolongation determined a structure which "comes close to defining a Lie algebra" and that, by considering a special case, they could associate to the KdV equation a 5-dimensional. The algebra [or rather a subalgebra isomorphic to $Sl(2,\mathbb{R})$] was then used to obtain the inverse scattering problem and Bäcklund transformation appropriate to the KdV equation. The purpose of the present chapter is to investigate the algebraic structure of the WE prolongation of generalization KdV. Once nonlinear terms are included in linear dispersive equations, solitary waves can result which can be stable enough to persist indefinitely. It is well known that many important nonlinear evolution equations which have numerous applications in mathematical physics appear as sufficient conditions for the integrability of systems of linear partial differential equations of first order, and such systems are referred to as integrable. This is not just an oddity, since algebraic structures such as those which appear in AKNS systems; which the AKNS refer to "method of Ablowitz, Kaup, Newell and Segur (1974) A.K.N.S", can arise very naturally from nonlinear evolution equations. This is very well exemplified by applying the prolongation technique that we introduced in previous chapters. These prolongations have a very useful application since Bäcklund transformations can be calculated based on them as well [29]. A Bäcklund transformation has important practical consequences, since such transformations can be used to calculate solutions to an associated equation, usually referred to as the potential equation, based on solutions of the initial equation. Sometimes these transformations can be used to obtain new solutions to the same initial equation, in which case they are referred to as auto Bäcklund transformations.

Recently an exterior differential system which defines a generalized KdV equation on the transverse manifold was obtained [30]. A particular case of this equation has appeared in [31] recently. The symmetries of this equation were determined and some solutions were found as well [32]. This permitted the determination of a certain form of integrability. Also, a particular type of prolongation over a fiber bundle was found corresponding to this differential system, as well as a specific form for a Bäcklund transformation with its associated potential equation. Here, the same differential system is studied, but a fully general calculation of the prolongation over the same bundle is carried out in detail for this generalized KdV equation. This allows the prolongation structure for any case of the given parameters in the equation. For completeness, the general theory for obtaining such prolongations based on the given exterior system of differential forms that defines the equation upon sectioning to a transversal integral manifold will be outlined first. Transversal integral manifolds give solutions of the equation. Finally, this work is extended to a study of a differential system of one-forms which define an equation that includes the Camassa-Holm equation and Degasperis-Procesi equations as specific cases [33-35]. The Camassa-Holm equation has been of interest because it has been shown to have peaked soliton solutions. The Camassa-Holm equation has a lot in common with the KdV equation, but there are significant differences as well. The KdV equation is globally well-posed when considered on a suitable Sobolev space, while Camassa-Holm is in general not. The first derivative of a solution of the latter can become infinite in finite time. The associated prolongation equations are developed and found to be much more restrictive than the previous case. However, it is shown that at least one solution to the prolongation system can be found. Finally, for each system a brief discussion concerning how conservation laws arise and can be expressed in this context will be discussed based on the defining exterior differential system.

(5.1.1): Differential System and Associated Differential Equation

As the best known equation exhibiting all these phenomena, the KdV equation provides an excellent prototype upon which to exercise and illustrate any new development. Accordingly, this section concerned with obtaining the prolongation structure of the KdV equation and illustrating its relation to the many known techniques for treating this equation. Since the analysis is performed in the perhaps unfamiliar language of Cartan's exterior differential forms. First we will give a brief introduction, defining the notation and setting up the KdV equation in terms of differential forms. While we do not emphasize the geometrical interpretation of our analysis (which is so well expressed by the differential form language), even analytically this notation is unquestionably superior for any treatment of conservation laws and integrability conditions.

These ideas are applied to a class of equation that includes the nonlinear Kortewege-de Vries (KdV) equation. We may have written in the form

$$u_t + (u^n)_{xxx} + \gamma \frac{n}{n+s} (u^{n+s})_x = 0. \tag{5.1}$$

where, γ is a real constant, nonzero. A more compact form is obtained if we set $m = n + s \neq 0$ and define a new constant $\beta = n\gamma/(n+s)$, then the (5.1) takes the form

$$u_t + (u^n)_{xxx} + \beta (u^m)_x = 0. \tag{5.2}$$

To begin the investigation, an exterior differential system which is relevant to the partial differential equation must be introduced. An exterior differential system is given which is defined over base manifold $M = \mathbb{R}^5$, which supports the differential forms. Consider the system of the 2-forms given by

$$\begin{aligned} \alpha_1 &= nu^{n-1} du \wedge dt - p\, dx \wedge dt = 0, \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt = 0, \\ \alpha_3 &= du \wedge dt - dq \wedge dt - \gamma p u^s dx \wedge dt = 0, \end{aligned} \tag{5.3}$$

then, take the differentiating forms in (5.3), we get

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2, \\ d\alpha_2 &= -dq \wedge dx \wedge dt = dx \wedge \alpha_3, \\ d\alpha_3 &= -\gamma spu^{s-1} du \wedge dx \wedge dt - \gamma u^s dp \wedge dx \wedge dt \\ &= dx \wedge \left(\gamma \frac{s}{n} pu^{s-n} \alpha_1 + \gamma pu^s \alpha_2\right). \end{aligned} \tag{5.4}$$

Therefore, it can be seen that of all these exterior derivatives vanish modulo $\{\alpha_j\}_{j=1}^{3}$. Any regular 2-dimensional solution manifold in the 5-dimensional space $S_2 = \{u(x,t),\ u_x = p(x,t),\ p_x(x,t) = q(x,t)\}$ satisfying a specific partial differential equation of the form (4.3) will annul this set of forms. The system that mentioned in (5.3) is integrable. The exact form of this equation which corresponds to (5.3) can be found explicitly by sectioning the forms into the solution manifold. It follows that

$$\begin{aligned} 0 &= \alpha_1 |\, S = ((u^n)_x - p) dx \wedge dt, \\ 0 &= \alpha_2 |\, S = (p_x - q) dx \wedge dt, \\ 0 &= \alpha_3 |\, S = (u_t + q_x + \gamma pu^s) dt \wedge dx. \end{aligned} \tag{5.5}$$

thus, the result that give us the equation (5.2).

(5.1.2): Determining Prolongation Algebra

To generate a prolongation algebra, system (5.3) is substituted into the

Based on the forms in system (5.3), the prolongation method outlined in previous chapter can be carried out, and the resulting system of equations can be solved quite generally. A very general prolongation corresponding to (5.1) can be calculated in terms of an algebra of vector fields. Then, to generate a prolongation algebra, the system (5.3) is substituted into prolongation condition (3.92) which lead us to

$$\begin{aligned} &A_t dt \wedge dx + A_u du \wedge dx + A_p dp \wedge dx + A_q dq \wedge dx + B_x dx \wedge dt \\ &B_u du \wedge dt + B_p dp \wedge dt + B_q dq \wedge dt + [A, B] dx \wedge dt \\ &= \lambda_1 (nu^{n-1} du \wedge dt - p dx \wedge dt) + \lambda_2 (dp \wedge dt - q dx \wedge dt) \\ &\qquad + \lambda_3 (du \wedge dx - dq \wedge dt - \gamma pu^s) dx \wedge dt \end{aligned} \tag{5.6}$$

Comparing the coefficients on the both side of two forms of (5.6) then, we get

$$\begin{aligned} &A_u = \lambda_3, \qquad A_p = 0, \qquad A_q = 0, \\ &B_u = n\lambda_1 u^{n-1}, \qquad B_p = \lambda_2, \qquad B_q = -\lambda_3, \\ &-A_t + B_x + [A, B] = -p\lambda_1 - q\lambda_2 - \gamma p u^s \lambda_3. \end{aligned} \tag{5.7}$$

Subscripts indicate partial differentiation with respect to the variable indicated. Translations in x and t constitute symmetries of equation (5.1), and so a simplifying assumption would be to suppose that A, B are independent on (x, t). So that, $A_x = A_t = 0, B_x = B_t = 0$, means it must be that A, B are also invariant under translations in these variables. This introduces a considerable simplification into (5.7) reducing it to

$$\begin{aligned} &A_p = 0, \qquad A_q = 0, \qquad A_u = -B_q, \\ &-[A, B] = \frac{1}{n} u^{1-n} p B_u + q B_p - \gamma p u^s B_q. \end{aligned} \tag{5.8}$$

Theorem (5.1.1):

The system (5.8) can be reduced to a single expression which specifies the algebra of brackets of a set of basis vector field X_i. The structure of these algebra is dependent on the relative values of m and n.

Proof

The differential equations in (5.8i) imply the following results

$$A = A(u, y), \qquad B = B(u, p, q, y), \qquad B = -qA_u(u, y) + \hat{B}(u, p, y). \tag{5.9}$$

Substituting B from (5.8) into (5.9) and collecting terms in q gives

$$q\left(-\frac{1}{n} u^{1-n} p A_{uu} + \hat{B}_p - [A, A_u]\right) + \frac{1}{n} p u^{1-n} \hat{B}_u + \gamma p u^s A_u + [A, \hat{B}] = 0. \tag{5.10}$$

Since $A, \hat{B}$ do not depend on q, then, it follows from (5.10)

$$\hat{B}_p = \frac{1}{n} u^{1-n} p A_{uu} + [A, A_u]. \tag{5.11}$$

As A does not depend on p, this can be integrated to give $\hat{B}$

$$\hat{B}(u,p,y) = \frac{1}{2n}u^{1-n}p^2A_{uu} + [A, A_u]p + B''(u,y). \qquad (5.12)$$

Substituting (5.12) into (5.10) as well as $\hat{B}_u$, there results

$$\frac{1}{2n}u^{1-2n}\big((1-n)A_{uu} + uA_{uuu}\big)p^3 + u^{1-n}[A, A_{uu}]p^2 + u^{1-n}B''_u p + n\gamma p u^s A_u$$
$$+ n\left[A, \frac{1}{2n}u^{1-n}p^2A_{uu} + [A, A_u]p + B''\right] = 0 \qquad (5.13)$$

Since A, B''do not depend on p, the coefficient of p^3 must vanish giving the equation

$$(1-n)A_{uu} + uA_{uuu} = 0. \qquad (4.14)$$

Then, (5.14) can be solved for A to give

$$A(u,y) = X_1(y) + X_2(y)u + X_3(y)u^{n+1}, \qquad (5.15)$$

where the $X_i(y)$ are vertical vector fields. Consequently, (5.13) simplifies to

$$u^{1-n}\left([A, A_{uu}] + \frac{1}{2}[A, A_{uu}]\right)p^2 + \big(n\gamma u^s A_u + u^{1-n}B''_u + n\big[A, [A, A_u]\big]\big)p$$
$$+ n[A, B''] = 0. \qquad (5.16)$$

The coefficient of p^2 implies $[A, A_{uu}] = 0$, which using (5.15) immediately establishes two basic commutators of the vector fields X_1, X_2, and X_3

$$[X_1, X_3] = 0, \qquad [X_2, X_3] = 0. \qquad (5.17)$$

The coefficient of p implies the condition

$$n\gamma u^s A_u + u^{1-n}B''_u + n\big[A, [A, A_u]\big] = 0. \qquad (5.18)$$

Solving for B''_u and let $s = m - n$, we get

$$B''_u = n\gamma u^{m-1}A_u - nu^{n-1}\big[A, [A,\ A_u]\big] \qquad (5.19)$$

By differentiation (5.15) we get $A_u = X_2 + (n+1)u^n X_3$ and substituting to (5.19)

$$B_u'' = n\gamma u^{m-1}(X_2 + (n+1)u^n X_3) - nu^{n-1}\big[X_1 + X_2 u + X_3 u^{n+1}, [X_1, X_2]\big] \quad (5.20)$$

Suppose at this point that X_1, X_2 do not commute with each other, then a new vector field can be defined as

$$X_7 = [X_1, X_2]. \quad (5.21)$$

Sitting $X = X_3, Y = X_1$ and $Z = X_2$ in the Jacobi identity $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$ gives

$$\big[X_3, [X_1, X_2]\big] + \big[X_1, [X_2, X_3]\big] + \big[X_2, [X_3, X_1]\big] = 0 \quad (5.22)$$

furthermore, by restitution (5.17) on (5.22) we get

$$[X_3, X_7] = 0, \quad (5.23)$$

thus, B_u'' reduces to the form

$$B_u'' = n\gamma u^{m-1}(X_2 + (n+1)u^n X_3) - nu^{n-1}(X_5 + uX_6). \quad (5.24)$$

Two new commutators have been introduced to write (5.24) defined as

$$[X_1, X_7] = X_5, \qquad [X_2, X_7] = X_6. \quad (5.25)$$

Using (5.25) in the Jacobi identity, the following brackets result

$$[X_2, X_5] = [X_1, X_6], \qquad [X_3, X_5] = 0. \quad (5.26)$$

Finally, integrating B_u'' with respect to u yields an expression for B''

$$B'' = \frac{n}{m}\gamma u^m X_2 + \frac{n(n+1)}{n+m}\gamma u^{m+n} X_3 - u^n X_5 - \frac{n}{n+1} X_6 + X_4. \quad (5.27)$$

Only one term in (5.16) remains to be satisfied, namely $[A, B''] = 0$. Thus substituting A, B'' into this bracket and using linearity to expand out, we have

$$\left[X_1 + uX_2 + u^{n+1}X_3, \frac{n}{m}\gamma u^m X_2 + \frac{n(n+1)}{m+n}\gamma u^{m+n}X_3 - u^n X_5 - \frac{n}{n+1}u^{n+1}X_6 + X_4\right]$$
$$= \frac{n}{m}\gamma u^m [X_1, X_2] - u^n [X_1, X_5] - \frac{n}{n+1}u^{n+1}[X_2, X_5] + [X_1, X_4] - u^{n+1}[X_2, X_5]$$
$$- \frac{n}{n+1}u^{n+2}[X_2, X_6] + u[X_2, X_4] - \frac{n}{n+1}u^{2n+2}[X_3, X_6] + u^{n+1}[X_3, X_4]$$

Therefore, the vector fields must be interrelated in such a way that the following holds among the coefficients of each power of u

$$\begin{gathered}[X_1, X_4] + \frac{n}{m}\gamma u^m [X_1, X_2] + u^{n+1}\left(-\frac{2n+1}{n+1}[X_2, X_5] + [X_3, X_4]\right) \\ + u[X_2, X_4] - u^n [X_1, X_5] - \frac{n}{n+1}u^{n+2}[X_2, X_6] - \frac{n}{n+1}u^{2n+2}[X_3, X_6] = 0\end{gathered} \tag{5.28}$$

Theorem (5.1.2):

There exist nontrivial algebras for the X_i specified by (5.17), (5.21), (5.23), (5.26) and the coefficients of powers of u in (5.28), which depend on the relative values of m and n.

Proof

It is required to equate the independent powers of u equal to zero. This has to be done on a case by case basis by putting individual restrictions on m and n, and not all cases are given.

a) Suppose none of the powers of u in (5.28) are equal, hence $n \neq m \neq 1,0$. Equating each power of u to zero gives the following algebra

$$[X_3, X_6] = 0, \qquad [X_2, X_4] = 0, \qquad [X_1, X_4] = 0.$$

At this point, X_1 and X_2 have be required to commute, since $X_7 = 0$ must hold. However, from (5.25), it follows that $X_5 = X_6 = 0$. Moreover, $[X_1, X_3] = 0$ implies that X_1 and X_3 differ by a constant, hence X_2 and X_3 also differ by a constant. Finally, $[X_1, X_4] = 0$ implies that X_1 and X_4 differ by a constant. Therefore, we can put

$$X_1 = \varepsilon X, \qquad X_2 = \sigma X, \qquad X_3 = X, \qquad X_4 = \alpha X. \tag{5.29}$$

Substituting these results into A and B, they take the form

$$A = (\varepsilon + \sigma u + u^{n+1})X, \tag{5.30}$$
$$B = -(\sigma + (n+1)u^n)qX + \frac{1}{2}(n+1)p^2 X + \frac{n}{m}\gamma\sigma u^m X + \frac{n(n+1)}{m+n}\gamma u^{m+n} X + \alpha X$$

b) Suppose $n \neq 1$ and $m \neq 1, 2, 3, 4$. Then the same algebra as (5.29) results and $A,\ B$ are given by (5.30) with n set equal to one.

c) Suppose now that $n \neq m \neq 0,1$, then prolongation equation (5.28) reduce to

$$\begin{gathered}[X_1, X_4] + u[X_2, X_4] + u^{n+1}\left(-[X_2, X_5] - \frac{n}{n+1}[X_1, X_6] + [X_3, X_4]\right) \\ +u^n(\gamma X_7 - [X_1, X_5]) - \frac{n}{n+1}u^{n+2}[X_2, X_6] - \frac{n}{n+1}u^{2n+2}[X_3, X_6] = 0\end{gathered} \tag{5.31}$$

This equation is satisfied provided that the following brackets hold

$$\begin{gathered}[X_3, X_6] = 0, \qquad [X_2, X_6] = 0, \qquad \frac{2n+1}{n+1}[X_2, X_5] = [X_3, X_4] \\ \gamma X_7 = [X_1, X_5], \qquad [X_2, X_4] = 0, \qquad [X_1, X_4] = 0\end{gathered} \tag{5.32}$$

in addition to the brackets given in (5.23), (5.25), and (5.26). This algebra has a simpler three elements realization which satisfies all the commutation relations provided that

$$X_3 = 0, \qquad X_4 = 0, \qquad X_5 = \gamma X_2, \qquad X_6 = X_2. \tag{5.33}$$

The nonzero commutation relations are given by

$$[X_1, X_2] = X_7, \qquad [X_2, X_7] = X_2, \qquad [X_1, X_7] = -\gamma X_2 \tag{5.34}$$

The algebra closes and a finite three-elements algebra results.

d) Suppose that $m = n + 1 \neq 0,1$, then prolongation equation (5.28) implies the algebra

$$[X_1,X_4]=0, \qquad \gamma\frac{n}{n+1}[X_1,X_2]-\frac{2n+1}{n+1}[X_2,X_5]+[X_3,X_4]=0,$$
$$[X_1,X_5]=0, \qquad [X_2,X_4]=0, \qquad [X_2,X_6]=0, \qquad [X_3,X_6]=0 \tag{5.35}$$

Recalling that (5.26) must be satisfied, a three element algebra results if we take

$$X_2=X_3, \qquad X_4=0, \qquad X_5=-\frac{\gamma n}{2n+1}X_1, \qquad X_6=\frac{\gamma n}{2n+1}X_2 \tag{5.36}$$

There is a closed algebra in this case with three nontrivial brackets

$$[X_1,X_2]=X_7, \qquad [X_1,X_7]=-\frac{\gamma n}{2n+1}X_1, \qquad [X_2,X_7]=\frac{\gamma n}{2n+1}X_2 \tag{5.37}$$

e) The linear case $m=n=1$ generates the following bracket relation

$$\begin{aligned}&[X_1,X_4]=[X_2,X_6]=[X_3,X_6]=0,\\ &\gamma X_7+[X_2,X_4]-[X_1,X_5]=0,\\ &2[X_3,X_4]-2[X_2,X_5]-[X_1,X_6]=0,\end{aligned} \tag{5.38}$$

f) The case $m=2, n=1$ corresponds to the classical KdV equation and the brackets must satisfy

$$[X_2,X_6]=0, \qquad \frac{1}{2}\gamma X_7[X_2,X_7]=\frac{n3}{2}[X_3,X_4]$$
$$[X_2,X_4]-[X_1,X_5]=0, \qquad [X_3,X_6]=0, \qquad [X_1,X_4]=0. \tag{5.39}$$

Since (5.26) must be satisfied, this system is satisfied if we put

$$X_3=X_4=0, \qquad X_5=-\frac{\gamma}{3}X_1, \qquad X_6=\frac{\gamma}{3}X_2 \tag{5.40}$$

There are three nontrivial commutators which take the form

$$[X_1,X_2]=X_7, \qquad [X_1,X_7]=-\frac{\gamma}{3}X_1, \qquad [X_2,X_7]=\frac{\gamma}{3}X_2 \tag{5.41}$$

Now, we want to achieve a class of prolongation for the system (5.8), these condition imply $A=A(u,y)$, $B=B(u,p,q,y)$. Let us take the following form for the vector fields A

$$A = A(u, y) = X_1 + uX_2, \qquad X_i = X_i(y), \qquad i = 1, 2. \tag{5.42}$$

Using $A_u = X_2$ and (4.8), A in (4.42) is sufficient to determine B in the form

$$B = -qX_2 + C(u, p, y). \tag{5.43}$$

Thence, the second equation in (5.8) takes the form

$$[X_1 + uX_2, -qX_2 + C] = -\frac{p}{n}u^{1-n}C_u - qC_p - \gamma pu^s X_2.$$

Simplifying the above formula, it follows

$$\frac{p}{n}C_u + qu^{n-1}C_p = -\gamma pu^{s+n-1}X_2 + qu^{n-1}[X_1, X_2] - u^{n-1}[X_1, C] - u^n[X_2, C] \tag{5.44}$$

Now, by defining the vector field $X_3 = [X_1, X_2]$, then whenever C is independent of q, we obtain form (5.44) that

$$C(u, p, y) = pX_3 + D(u, y). \tag{5.45}$$

Substituting C in (5.45) into (5.44), we get

$$\begin{aligned}\frac{p}{n}D_u &= p\{-\gamma u^{s+n-1}X_2 - u^{n-1}[X_1, X_3] - u^n[X_2, X_3]\} \\ &\quad - u^{n-1}\{[X_1, D] - u[X_2, D]\}.\end{aligned} \tag{5.46}$$

Furthermore, the last term on (5.46) must vanish because D does not depend on p, then we have two condition on D

$$\begin{aligned}&[X_1, D] - u[X_2, D] = 0, \\ &\frac{1}{n}D_u = -\gamma u^{m-1}X_2 - u^{n-1}[X_1, X_3] - u^n[X_2, X_3]\end{aligned} \tag{5.47}$$

where $m = s + n$. By integrating in (5.47) with respect to u the second equation for D

$$D(u, y) = -\gamma\frac{n}{m}u^m X_2 - u^n[X_1, X_3] - \frac{n}{n+1}u^{n+1}[X_2, X_3] + X_4 \tag{5.48}$$

Substituting D from (5.48) into the first equation with commutator in (5.47), it can simplify to the following

$$-\gamma\frac{n}{m}u^m[X_1,X_2]+u^n\big[X_1,[X_1,X_3]\big]+[X_1,X_4]$$
$$-u^{n+1}\left\{\frac{n}{n+1}\big[X_1,[X_2,X_3]\big]+\big[X_2,[X_1,X_3]\big]\right\}$$
$$+\frac{n}{n+1}u^{n+2}\big[X_2,[X_2,X_3]\big]-u[X_2,X_4]=0. \qquad (5.49)$$

Some of the brackets in the form (5.49) will vanish, if that m and n not be equal to one,

$$[X_2,X_4]=0, \qquad [X_1,X_4]=0$$

To satisfy these brackets, one way in which this can be done is to take $X_4=\mu X_2$ and $X_4=\varepsilon X_1$, from which it follows that $X_1=\lambda X_2$, where μ,ε and λ are real constants. Moreover, substituting these results into the definition of X_3, it follows that $X_3=0$. Using all of these results in (5.49), it follows that the remaining terms in (5.49) vanish, hence (5.49) is satisfied identically and we have one solution. To summarize these results for the vector field, we have

$$X_1=\lambda X_2, \qquad X_2=X, \qquad X_3=0, \qquad X_4=\mu X_2. \qquad (5.50)$$

Since there is only one independent vector field left, we have set $X=X_2$ in (5.50) in this case, the prolongation structure reduces to the following set of the vector fields

$$\begin{aligned}
&A=(\lambda+u)X,\\
&B=-qX+C,\\
&C=D=-\gamma\frac{n}{m}u^mX+\mu X=\left(-\gamma\frac{n}{m}u^m+\mu\right)X,\\
&\qquad X=X(y), \qquad \lambda,\mu\in\mathbb{R}.
\end{aligned} \qquad (5.51)$$

(5.1.3): Conservation Laws

Conservation laws [36] describe quantities that remain invariant during the evolution of the PDE. This provides simple and efficient methods for the study of many qualitative properties of solutions, including stability, evolution of solutions, and decomposition into solutions, as well as the theoretical description of the solution manifolds. A solution equation is a PDE with a wave like solution known as a solitary wave. A solitary wave is localized, traveling wave and several nonlinear partial differential equations have a solution of this type. A soliton is specific of stable solitary wave which is described in terms of its interaction with solitary waves [37,38].

We will take under consideration the conservation laws which associated with the KdV equation correspond to the existence of the exact 2-forms contained in the ring of the α_j Let us suppose that we can find a set of functions $f_i(x,t,u,p,q)$ such that the two-form

$$\sigma = f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 \tag{5.52}$$

satisfies $d\sigma = 0$, the condition for exactness. This the integrability condition for the existence of a 1-form ω such that

$$\sigma = d\omega \tag{5.53}$$

which conversely implies, $d\sigma = 0$.

Differentiation of (5.52) and substituting (5.4) we get

$$d\sigma = \left(df_1 + f_3\gamma\frac{s}{n}pu^{s-n}dx\right)\wedge\alpha_1 + (df_2 + (f_1 + f_3\gamma pu^s)dx)\wedge\alpha_2 + (df_3 - f_2dx)\wedge\alpha_3$$

Therefore $d\sigma \in I$, and this clearly vanishes $mod\ \hat{p}^*(I)$.

Remark (5.1.3):

A form σ with the structure (5.52) corresponding to equation (5.1), take into consideration the 1-form σ which is given in the terms of the α_i in (5.3) with $f_1 = -\gamma u^s,\ f_2 = 0,$ and $f_3 = 1$ as

$$\sigma = \alpha_3 - \gamma u^s \alpha_1 \tag{5.54}$$

calculate the exterior derivative of (5.54), we get that $d\sigma = 0$ then, verifies that this (5.54) vanishes identically. Substituting $\alpha_1,\ \alpha_2$ into (5.54), we get

$$\sigma = -\gamma \frac{n}{m} d(u^m) \wedge dt + du \wedge dx - dq \wedge dt. \tag{5.55}$$

Then, we find in accordance with (5.53) that σ can be derived from the 1-form

$$\omega = -\left(\gamma \frac{n}{m} u^m + q\right) dt + udx \tag{5.56}$$

This is exactly the 2-form σ that was given in (5.54). The associated conservation law results from an application of Stokes theorem.

$$\oint_{M_1} \omega = \int_{M_2} d\omega \tag{5.57}$$

This has been written for any simply-connected, 2-dimensional manifold M_2 with closed 1-dimensional boundary M_1. The equations imply that ω and $d\omega$ are to be evaluated on their respective manifolds.

Returning to ω again, we can add to ω any exact 1-form dy, where y is an arbitrary scalar function. Then, ω can also hold

$$d\omega = dy - \left(\gamma \frac{n}{m} u^m + q\right) dt + udx \tag{5.58}$$

such that $\sigma = d\omega$. Now y may be regarded simply as a coordinate in an extended 6-dimensional space of variables $\{x, t, u, p, q, y\}$ and the 1-form ω may be included with the original set of forms. Since $d\omega$ is known to be in the ring of the original set, the new set of forms remains a closed ideal.

(5.2): Prolongation of a Differential System Related to the Camassa-Holm Equation and the Degasperis-Procesi equations.

It is the intention here to review some of the mathematical background which will let us study some interrelated equations which have been of interest recently. First we will give a brief introduction, defining the notation and setting up the Camassa-Holm equation in terms of differential forms. While we do not emphasize the geometrical interpretation of our analysis (which is so well expressed by the differential form language), even analytically this notation is unquestionably superior for any treatment of conservation laws and integrability conditions.

These ideas are applied to a class of equations that includes the Camassa-Holm and Degasperis-Procesi equations. These equations are of the form:

$$(u - u_{xx})_t + u(u - u_{xx})_x + \beta(u - u_{xx})u_x = 0, \qquad (5.59)$$

where, $\beta = \text{constant}$, nonzero.

An exterior differential system which reproduces the given equation on the transverse manifold is developed for each case. The derivatives of the forms in this set are shown to be expressible in terms of the same forms, so the integrability of each equation is established. Finally, conservation laws for the two equations will be written down developed from the original set of one-forms.

Let us begin by introducing the system of exterior differential which is related to several equations which are of interest in mathematical physics at the moment. In particular, the Camassa-Holm and Degasperis-Procesi equations are to be included in this group. Define the following system of two forms

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt, \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge d\,, \\ \alpha_3 &= du \wedge dx - dq \wedge dx + dq \wedge dt + (u - q)dx \wedge dt, \end{aligned} \qquad (5.60)$$

then, by differentiating the forms in (5.60), we get

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = (1/q)\alpha_2 \wedge dp, \\ d\alpha_2 &= -dq \wedge dx \wedge dt = \alpha_3 \wedge dx, \\ d\alpha_3 &= du \wedge dx \wedge dt - dq \wedge dx \wedge dt = \alpha_3 \wedge dt. \end{aligned} \tag{5.61}$$

Therefore, it can be seen that of all these exterior derivatives vanish modulo $\{\alpha_j\}_{j=1}^{3}$. Any regular 2-dimensional solution manifold in the 5-dimensional space $S_2 = \{u(x,t),\ u_x = p(x,t),\ p_x(x,t) = q(x,t)\}$ satisfying a specific partial differential equation will annul this set of forms. The exact form can be found explicitly by sectioning the forms into the solution manifold. It follows that

$$\begin{aligned} 0 &= \alpha_1 |\, S = (u_x - p)dx \wedge dt \\ 0 &= \alpha_2 |\, S = (p_x - q)dx \wedge dt \\ 0 &= \alpha_3 |\, S = ((u-q)_t - (u-q) - q_x)dt \wedge dx, \end{aligned} \tag{5.62}$$

thus, the result that give us the equation

$$(u - u_{xx})_t - (u - u_{xx}) - u_{xxx} = 0, \tag{5.63}$$

this is the specific equation whose integrability is implied by system (5.60).

Consider the differential system:

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt \\ \alpha_3 &= du \wedge dx - dq \wedge dx + du \wedge dt - dq \wedge dt + (u-q)dx \wedge dt. \end{aligned} \tag{5.64}$$

Then, by differentiating the forms in (5.64), we get

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2 \\ d\alpha_2 &= -dq \wedge dx \wedge dt = dx \wedge (-\alpha_3 + \alpha_1) \\ d\alpha_3 &= -dx \wedge \alpha_3 \end{aligned} \tag{5.65}$$

Upon sectioning these forms, and the equation which belong to (5.64) arises from the section $\alpha_3|_S = 0$ is given by

$$(u - u_{xx})_t - (u - u_{xx})_x - (u - u_{xx}) = 0. \tag{5.66}$$

The final two cases which will be introduced include equations which are being actively studied at the moment

Define the following system of two forms, let β be a real, nonzero constant

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt \\ \alpha_3 &= -du \wedge dx + dq \wedge dx - \beta u\, dq \wedge dt + \beta(2u - q) du \wedge dt \end{aligned} \tag{5.67}$$

then, by differentiating the forms in (5.67), we get

$$\begin{aligned} d\alpha_1 &= dx \wedge dp \wedge dt = dx \wedge \alpha_2 \\ d\alpha_2 &= dx \wedge dq \wedge dt = (1/\beta u) dx \wedge (-\alpha_3 + \beta u \alpha_1 + \beta(u - q)\alpha_1) \\ d\alpha_3 &= 0 \end{aligned} \tag{5.68}$$

Obviously all of the (5.68) vanish modulo the set of the α_j in (5.67). Upon sectioning these forms, and the equation obtained from the restriction α_3

$$\alpha_3|_S = ((u - q)_t + \beta u u_x - \beta u q_x + \beta(u - q) u_x) dx \wedge dt, \tag{5.69}$$

from sectioning α_1 and α_2, we have get

$$(u - u_{xx})_t - \beta\big(u(u - u_{xx})\big)_x = 0. \tag{5.70}$$

The following system leads to an important class of partial differential equations which are of much current interest. The Camassa-Holm and Degasperis-Procesi equations are to be included in this group. Define the system of forms:

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt \\ \alpha_3 &= -du \wedge dx + dq \wedge dx - u\, dq \wedge dt + u du \wedge dt + \beta(u - q) du \wedge dt \end{aligned} \tag{5.71}$$

Differentiating (5.71), we have:

$$
\begin{aligned}
d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2 \\
d\alpha_2 &= dx \wedge \big(-\alpha_3 + u\big((1+\beta)u - q\big)\alpha_1\big) \\
d\alpha_3 &= (1-\beta)dq \wedge du \wedge dt \\
&= (1-\beta)dq \wedge \alpha_1 + (1-\beta)p\, dq \wedge dx \wedge dt \\
&= (1-\beta)dq \wedge \alpha_1 + (1-\beta)p\, (\alpha_3 + du \wedge dx) \wedge dt \\
&= (1-\beta)dq \wedge \alpha_1 + (1-\beta)p\, dt \wedge \alpha_3 - (1-\beta)p\, dx \wedge du \wedge dt \\
&= (1-\beta)[dq \wedge \alpha_1 + p\, dt \wedge \alpha_3 - p\, dx \wedge \alpha_1]
\end{aligned} \tag{5.72}
$$

All of the details for calculating $d\alpha_3$ have been shown here. Obviously all of the $d\alpha_j$vanish modulo the set of α_j. from sectioning α_1 and α_2, we have get other cases and the equation results from evaluating the section as follows:

$$
\begin{aligned}
0 &= \alpha_1|_S = (u_x - p)dx \wedge dt \\
0 &= \alpha_2|_S = (p_x - q)dx \wedge dt \\
0 &= \alpha_3|_S = ((u-q)_t + u(u-q)_x + \beta(u-q)u_x)dx \wedge dt
\end{aligned} \tag{5.73}
$$

These results imply the partial differential equation:

$$(u-q)_t + u(u-q)_x + \beta(u-q)u_x = 0, \tag{5.74}$$

then, by putting $\beta = 3\ and\ \rho = u - q$ the equation (5.74) becomes the Degasperis-Procesi equation

$$\rho_t + u\,\rho_x + 3\rho\, u_x = 0 \tag{5.75}$$

again by putting $\beta = 2\ and\ \rho = u - q$ the equation (5.74) becomes the Camassa-Holm equation

$$\rho_t + u\,\rho_x + 2\rho\, u_x = 0. \tag{5.76}$$

(5.3.1): Prolongation Equations

A very general prolongation corresponding to (5.59) can be calculated in terms of an algebra of vector fields which are defined on fibers above the base manifold that supports the forms (5.71). Then, to generate a prolongation algebra, the system (5.71) is substituted into prolongation condition (3.92) which lead us to

$$
\begin{aligned}
&A_t dt \wedge dx + A_u du \wedge dx + A_p dp \wedge dx + A_q dq \wedge dx + B_x dx \wedge dt \\
&B_u du \wedge dt + B_p dp \wedge dt + B_q dq \wedge dt + [A, B] dx \wedge dt \\
&= \lambda_1 (du \wedge dt - p dx \wedge dt) + \lambda_2 (dp \wedge dt - q dx \wedge dt) \\
&+ \lambda_3 (-du \wedge dx + dq \wedge dx + u du \wedge dt - u dq \wedge dt + \beta (u - q) du \wedge dt)
\end{aligned} \tag{5.77}
$$

Comparing the coefficients on the both side of two forms of (5.77) then, we get

$$
\begin{aligned}
&A_u = -\lambda_3, \qquad A_p = 0, \qquad A_q = \lambda_3, \\
&B_u = \lambda_1 + u\lambda_3 + \beta(u - q)\lambda_3, \qquad B_p = \lambda_2, \qquad B_q = -u\lambda_3\,, \\
&-A_t + B_x + [A, B] = -p\lambda_1 - q\lambda_2.
\end{aligned} \tag{5.78}
$$

Subscripts indicate partial differentiation with respect to the variable indicated. Translations in x and t constitute symmetries of equation (5.59), and so a simplifying assumption would be to suppose that A, B are independent on (x, t). So that, $A_x = A_t = 0, B_x = B_t = 0$, means it must be that A, B are also invariant under translations in these variables. This introduces a considerable simplification into (5.64) reducing it to

$$
\begin{aligned}
&A_p = 0, \qquad A_q = -\frac{1}{u} B_q, \qquad A_u = -\frac{1}{u} B_q, \\
&[A, B] = pB_u + p\left(1 + \left(1 - \frac{1}{u}\right)\beta\right) B_q + qB_p.
\end{aligned} \tag{5.79}
$$

(5.3.2): Conservation laws

Let us suppose that we can find a set of functions $f_i(x, t, u, p, q)$ such that the 2-form, which look like the form (5.52) satisfies $d\sigma = 0$, the condition for exactness. This the integrability condition $\sigma = d\omega$ for the existence of a 1-form ω, which conversely implies that $d\sigma = 0$.

Take into consideration the 1-form σ given by

$$
\sigma = \alpha_3 \tag{5.80}
$$

Actually, σ can be derived from a single 1-form. Let ω be defined to be:

$$
\omega = (q - u)dx + \beta\, u(u - q)dt \tag{5.81}
$$

by taking the exterior derivative of ω, we have get

$$d\omega = -du \wedge dx + dq \wedge dx + \beta u\, du \wedge dt - \beta u\, dq \wedge dt + \beta(u-q)du \wedge dt$$

This is precisely the form in (5.80).

Consideration the form (5.71), we can similarly determine a form σ.

Take into consideration the 1-form σ which is given by

$$\sigma = \alpha_3 - (1-\beta)\, q\alpha_1 + (1-\beta)\, p\alpha_2 \tag{5.82}$$

by taking the exterior derivative of σ, we have get $d\sigma = 0$. Now it can be shown that σ can be derived from a 1-form, namely ω defined by

$$\omega = (q-u)dx + \frac{1}{2}(u^2 - 2uq + \beta u^2 + p^2)dt, \tag{5.83}$$

by taking the exterior derivative of ω, we have get

$$\begin{gathered} d\omega = -du \wedge dx + dq \wedge dx + u\, du \wedge dt - u\, dq \wedge dt + \\ \beta u\, du \wedge dt - q\, du \wedge dt + (1-\beta)p\, dp \wedge dt \end{gathered} \tag{5.84}$$

This is exactly the 2-form σ that was given in (5.82). The associated conservation law results from an application of Stokes theorem (5.57).

This has been written for any simply-connected, 2-dimensional manifold M_2 with closed 1-dimensional boundary M_1. The equations imply that ω and $d\omega$ are to be evaluated on their respective manifolds.

$$\omega = dy + (q-u)dx + \frac{1}{2}(u^2 - 2uq + \beta u^2 + p^2)dt. \tag{5.85}$$

such that $\sigma = d\omega$. Now v may be regarded simply as a coordinate in an extended 6-dimensional space of variables $\{x, t, u, p, q, y\}$ and the 1-form ω may be included with the original set of forms. Since $d\omega$ is known to be in the ring of the original set, the new set of forms remains a closed ideal.

Sec (5.3): Conclusion

It has been seen that exterior differential systems have been constructed for some very important classes of partial differential equation. As well as giving some information about the associated integrability of these equations, it has been shown that the prolongation structure of these systems can be studied. This is more than just of theoretical interest, since Bäcklund transformations can be constructed based on these results. The relationship of differential systems to Bäcklund transformations has been discussed by Estabrook and Wahlquist.

Let us show how to use the results of remark (5.1.3) to obtain such a result:

Consider X to be one of the generators of $sl(2,\mathbb{R})$, so the solution of (5.86) is based on a sub-algebra.

It can be represented in matrix form as

$$X_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad X_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5.86}$$

To find the Maurer-Cartan algebra of $GL(n,\mathbb{R})$, we consider the left invariant forms ω_i^j as elements of a matrix

$$\omega - \left(\omega_i^j\right) - y^{-1}dy \tag{5.87}$$

where, y is the natural embedding of the group into $\mathbb{R}^{2n}$. Then $y^{-1}dy$ is the Maurer-Cartan form. The Maurer-Cartan algebra can be written as

$$d\omega + \omega \wedge \omega = 0. \tag{5.88}$$

In this case we take $SL(2,\mathbb{R}) = \{X \in GL(2,\mathbb{R}) \det(X) = 1\}$. Exponential map can be used to obtain the Maurer-Caratn algebra. To obtain a form (5.88) that is more convenient, we introduce the ω^i by

$$\omega = \begin{bmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} \omega^1 & \omega^3 - \omega^2 \\ -\omega^3 - \omega^2 & -\omega^1 \end{bmatrix} \tag{5.89}$$

Substituting (5.89) into (5.88), it follows that the ω^i satisfying the Maurer-Cartan relations

$$d\omega^1 = \omega^3 \wedge \omega^2, \qquad d\omega^2 = \omega^1 \wedge \omega^3, \qquad d\omega^3 = \omega^1 \wedge \omega^2 \tag{5.90}$$

Calculating (5.86) and substituting into (5.51), we calculate A and B to be

$$A = \begin{bmatrix} 0 & \lambda + u \\ -\lambda - u & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & -q - \gamma \frac{n}{m} u^m + \mu \\ q + \gamma \frac{n}{m} u^m - \mu & 0 \end{bmatrix}, \tag{5.91}$$

and the cocycle is given by

$$\sigma = \begin{bmatrix} 0 & \lambda + u \\ -\lambda - u & 0 \end{bmatrix} dx + \begin{bmatrix} 0 & -q - \gamma \frac{n}{m} u^m + \mu \\ q + \gamma \frac{n}{m} u^m - \mu & 0 \end{bmatrix} dt \tag{5.92}$$

If we let Maurer-Cartan form have the structure (5.89), then the σ^i and found to be

$$\begin{aligned} \sigma^1 &= 0, \\ \sigma^2 &= 0, \\ \sigma^3 &= (\lambda + u)dx - \left(q + \gamma \frac{n}{m} u^m - \mu\right) dt. \end{aligned} \tag{5.93}$$

Using (3.88), we can choose the connection $\widetilde{\omega}$ from $\mathbb{R}$ with coordinates y and $X = \partial/\partial y$. By using the results that we get from (5.51) and (5.91)

$$\widetilde{\omega} = dy - (\lambda + u)dx - \left(q + \gamma \frac{n}{m} u^m - \mu\right) dt, \tag{5.94}$$

solutions of the system (5.2) determine transversal sections of the fiber bundle such that, upon substituting $p = (u^n)_x$ and $q = (u^n)_{xx}$, we have get

$$y_x = (\lambda + u), \qquad y_t = -\left((u^n)_{xx} + \gamma \frac{n}{m} u^m - \mu\right). \tag{5.95}$$

Since (5.95) implies that $u = y_x - \lambda$, it can be eliminated in the second equation of (5.9) to yield an equation for $y = y(x,t)$

$$y_t + ((y_x - \lambda)^n)_{xx} + \gamma \frac{n}{m}(y_x - \lambda)^m - \mu = 0. \tag{5.96}$$

to make the monograph more concise we put $\lambda = 0$ as well giving

$$y_x - \lambda = u^n, \qquad u = (y_x)^{\frac{1}{n}}, \tag{5.97}$$

where n is exponent in (5.97) representing positive root $n = 2r, r \in \mathbb{N}$. Eliminating u from the second equation in (5.96), we have an equation for y

$$y_t + ((y_x)^n)_{xx} + \gamma \frac{n}{m}(y_x)^m - \mu = 0. \tag{5.97}$$

It follows that $\lambda = \mu = 0$, a potential equation in terms of y results

$$y_t + ((y_x)^n)_{xx} + \gamma \frac{n}{m}(y_x)^m = 0. \tag{5.98}$$

Although the prolongation or the solution of the vector fields (5.51) is not extremely complicated, in effect a Bäcklund transformation has been determined in the form of the equations presented in (5.95). This set of equations transforms the original equation into the form of its potential equation. Given a solution u of (5.2) then integrating (5.95) gives a corresponding solution y to (5.98).

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