Sudan University of Science and Technology
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On the Solution of the Spectral Quasi-linearization Method of Selected Problems in Fluid Mechanics

دلالة حلول طريقة الطيف شبيهة الخطية لمسائل مختارة من ميكانيكا الموائع

A thesis Submitted in Partial Fulfillment for the Degree of M.Sc in Mathematics

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بسم الله الرحمن الرحيم

قال تعالى :

{أَلَمْ تَرَ أَنَّ اللَّهَ أَنزَلَ مِنَ السَّمَاء مَاء فَأَخْرَجْنَا بِهِ ثَمَرَاتٍ مُّخْتَلِفًا أَلْوَانُهَا وَمِنَ الْجِبَالِ جُدَدٌ بِيضٌ وَحُمْرٌ مُّخْتَلِفٌ أَلْوَانُهَا وَغَرَابِيبُ سُودٌ (٢٧) وَمِنَ النَّاسِ وَالدَّوَابِّ وَالأَنْعَامِ مُخْتَلِفٌ أَلْوَانُهُ كَذَلِكَ إِنَّمَا يَخْشَى اللَّهَ مِنْ عِبَادِهِ الْعُلَمَاء إِنَّ اللَّهَ عَزِيزٌ غَفُورٌ(٢٨)}

صدق الله العظيم

سورة فاطر

Abstract:-

In this thesis, we study the application of the spectral quasi-linearization method (SQLM) to different types of highly nonlinear ordinary differential equations which governed some fluid flow. Approximate numerical solutions are obtained. The residual error analysis is used to determine the speed of convergence, convergence rate and accuracy of the method. The main theme in this study is the application of recent semi-numerical method in the solution of nonlinear boundary value problems, particularly those arise in the study of fluid flow problems. First we gave a historical idea and details of the method of the solution. Second, we present a study of Oldroyd-B fluid flow over a moving surface. The highly nonlinear governing equations are solved using a novel spectral Quasi-linearization method (SQLM). This method combines a non-perturbation technique with the Chebyshev spectral collection method to produce an algorithm with accelerated and assured convergence. A parametric study addressing the effects of various flow parameters on the fluid properties, the skin friction coefficient is given. Also, an investigation of heat transfer in Oldroyd-B fluid flow over a moving surface is presented. The ordinary differential equations are solved numerically using the SQLM. Verification of the accuracy and correctness of the results is achieved by comparing our results with those in the literature review. The effects of the governing parameters on the flow characteristic are investigated. The result for the skin friction is presented in tabular form. We then, explore the use of a non-perturbation spectral quasi-linearization method to solve the coupled highly nonlinear system of differential equations due to a laminar flow with convective surface boundary conditions. The effects of Dufour, Soret and heat convective parameters are investigated. The velocity and temperature distributions as well as the skin-friction and heat transfer coefficients have been obtained and discussed for various physical parametric values.

Abstract (in Arabic):-

في هذا البحث، تطرقنا لاستخدام طريقةِ الطيف شبيهة الخطية (SQLM) على أنواعِ مختلفةِ مِنْ المعادلات التفاضليةِ العاديةِ اللاخطّيةِ التي توصف تدفقِ بَعْض الموائع. إن هذه الدراسةَ تهتم بتَطبيق الطرقِ العدديةِ على أنظمةِ المعادلات التفاضليةِ التى توصف تدفقَ المائع. تم استخدام تحليل خطأ المتبقى لتحديد سرعة التقارب ومعدله ودقة الطريقة. الموضوعِ الرئيسيِ فى هذه الدراسة هو تطبيقُ الطرقِ نصفِ العدديةِ في حَلِّ مشاكلِ القيم الحَدّيةِ اللاخطّيةِ، خصوصاً تلك التي تَظْهرُ في دراسةِ مشاكلِ تدفقِ الموائع. أولا أعطينَا فكرة تاريخية وتفاصيل طريقةِ الحَلِّ. ثانيا ندرس انسياب مائع أولدرويد-B على سطحِ متحرك. والمعادلاتَ اللاخطّيةَ التى توصف انسياب المائع تم حلها بإستخدام طريقة الSQLM. هذه الطريقةِ تَجْمعُ بين التقنية اللاضطربه وتقسيمات تشيبيشيف لاستخلاص خوارزميةَ متسارعه وضامنه للتقاربِ. والدراسة قد شملت تأثيرات بعض الوسائط الفيزيائية المتنوعة على خصائص الموائع ، مثل معامل احتكاك السطح. أيضا تناولنا انتقال الحرارةِ لانسياب المائع المسمى بالأولدرويد-B على سطحِ متحرك. المعادلات التفاضلية العادية حَلّتْ بشكل عددي بإستعمال طريقة SQLM. تحقّقنا من دقةِ وصحة نَتائِجِنا بالمُقَارَنَة مع نتائج دراسات سابقة. واكثر من ذلك قمنا بدراسة تأثيراتَ الوسائط الفيزيائية على الانسياب وكذلك قمنا بعرض نتائج معامل إحتكاكِ السطح في شكل جدول. ثم تناولنا استخدام طريقة SQLM اللاضطرابيه لحَلّ مجموعة معادلات لاخطّيةِ انية توصف انسياب طبقى مع سماحية السطح لانتقال الحرارة عبر حدوده. تناولنا تأثيرات وسائط دوفور وسوريت ووسيط سماحية انتقال الحرارة. تم ايجاد توزيعات السرعةَ ودرجةَ الحرارة بالإضافة إلى معاملا الإحتكاكِ وانتقال الحرارةِ ومناقشتها لمجموعة من قيم المؤثرات الفيزيائية.

Deduction

I would like to express my appreciation to my supervisor Dr. Mohammed Hassan Khabir. To my teacher Dr. Faiz G. Awad, who has cheerfully answered my queries, provided me with materials checked my examples, assisted me in a myriad ways with the writing and helpfully commented on earlier drafts of this work.

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Chapter 1

Introduction

Most engineering problems are governed by non-linear differential equations. When these equations are strongly non-linear, exact solutions are not easily obtained and we often resort to approximate numerical solutions. There are many well established numerical schemes such as the Runge-Kutta scheme, the Keller-box method, the shooting method, and finite element and volume methods. The main disadvantage of numerical solutions, however, is that they may not give any insights into the structure of the solution, particularly when the problem involves many embedded parameters. Numerical methods may also give discontinuous points on the solution curve, (see Paripour et al. [1]). Moreover, some numerical methods may not be stable or uniformly convergent. In such cases recourse is often made to either the classical series method or other perturbation methods to find approximate analytical solutions. Recent analytical techniques include the Lyapunov artificial small parameter method, the Adomain decomposition method, the homotopy perturbation method, (see He [2]) and the homotopy analysis method, (see Liao [3]). These methods may not always be convergent or valid. For example, the Adomain decomposition method has a small convergence region.

1.1 Quasi-Linearization Method:-

The quasi-linearization method (QLM) was originally developed by Bellman and Kalaba [4] as a generalization of the Newton-Raphson method to provide lower and upper bound solutions of nonlinear differential equations. The attraction of quasi-linearization is that the algorithm is easy to understand and the method generally converges rapidly if the initial guess is close to the true solution.
Bellman and Kalaba [4] established that the method converges quadratically. However; the original proof of quadratic convergence was subject to restrictive conditions of small step size and convexity or concavity of nonlinear functions, Maleknejad and Najafi [5]. These conditions were subsequently relaxed and the

method generalized to be applicable to a wider class of problems; see, for instance, papers by Mandelzweig and his coworkers [6-9] and Lakshmikantham [10, 11]. Parand et al. [12] used the quasi-linearization method to solve Volterra's model for population growth in a closed system. Other uses of the quasi-linearization method include application to reaction diffusion equations, Jiang and Vatsala [13], and to Volterra integro-differential equations, Ahmad [14], Pandit [15], and Ramos [16].

An often noted disadvantage of quasi-linearization is the instability of the method whenever a poor initial guess is chosen, Tuffuor and Labadie [17]. To improve the accuracy and convergence of the quasi-linearization method for all initial guesses, Motsa [18]) embed the QLM algorithm within the spectral method to obtain a sequence of integration schemes with arbitrary higher order convergence. This method gives excellent results in terms convergence and accuracy of solutions. The SQLM has been used successfully to solve nonlinear equations that govern the flow of fluids in bounded domains by (Motsa et al. [20]).

Chapter 2

Spectral Quasi Linearization Method (SQLM)

2.1 General governing equations system:-

Consider a system of m nonlinear ordinary differential equations in m unknowns functions $z_i(\eta)$, $i = 1, 2, \ldots, m$ where η is the dependent $\mathcal{L}$ variable. The system can be written as a sum of its linear $\mathcal{L}$ and nonlinear components $\mathcal{N}$ as

$$\mathcal{L}[z_1(\eta),\ z_2(\eta), \ldots,\ z_m(\eta)] + \mathcal{N}\ [z_1(\eta),\ z_2(\eta), \ldots,\ z_m(\eta)] = \mathcal{H}(\eta), \tag{2.1}$$

subject to the boundary conditions

$$A_i[z_1(a),\ z_2(a), \ldots,\ z_m(a)] = K_{a,i},\ B_i[z_1(b),\ z_2(b), \ldots,\ z_m(b)] = K_{b,i}, \tag{2.2}$$

where A_i and B_i are linear operators and $K_{a,i}$ and $K_{b,i}$ are constants for $\mathrm{i} = 1, 2, \ldots, \mathrm{m}$. Define the vector Z_i to be the vector of the derivatives of the variable z_i with respect to the dependent variable η, that is

$$Z_i = [z_i^{(0)}, z_i^{(1)}, \ldots, z_i^{n_i}], \tag{2.3}$$

where $z_i^{(0)} = z_i$, $z_i^{(p)}$ is the pth derivative of $\mathrm{z_i}$ with respect to η and n_i ($\mathrm{i} = 1, 2, \ldots, m$) is the highest derivative order of the variable $\mathrm{z_i}$ appearing in the system of equations. In addition, we define $\mathcal{L}_\mathrm{i}$ and $\mathcal{N}_\mathrm{i}$ to be the linear and nonlinear operators, respectively, that operate on the Z_i for $i = 1, 2, \ldots, m$ with these definitions, equation (2.1) and (2.2) can be written as

$$\mathcal{L}_\mathrm{i}[\mathrm{Z_1, Z_2, \ldots, Z_m}] + \mathcal{N}_\mathrm{i}[\mathrm{Z_1, Z_2, \ldots, Z_m}] = \sum_{\mathrm{j=1}}^{\mathrm{m}} \sum_{\mathrm{p=0}}^{\mathrm{n_j}} \alpha_{\mathrm{i,j}}^{[\mathrm{p}]} \mathrm{z_j^{(p)}} + \mathcal{N}_\mathrm{i}[\mathrm{Z_1, Z_2, \ldots, Z_m}], \tag{2.4}$$

where $\alpha_{i,j}^{[p]}$ are the constant coefficient of $z_j^{(p)}$, the derivative of z_j ($j = 1, 2, \ldots, m$) that appears in the ith equation for $i = 1, 2, \ldots, m$. Noting that, for each variable z_i, the derivatives in the boundary conditions can at most be one less than the highest derivative of z_i in the governing system (2.1), we define the

vector $\tilde{Z}_i$ to be the vector of the derivatives of the variable z_i with respect to the dependent variable η from 0 up to $(n_i - 1)$, that is

$$\tilde{Z}_i = \left[z_i^{(0)}, z_i^{(1)}, \dots, z_i^{(n_i-1)}\right]. \tag{2.5}$$

The boundary conditions (2.2) can be written as

$$A_v[\tilde{Z}_1(a), \tilde{Z}_2(a), \dots, \tilde{Z}_m(a)] = \sum_{j=1}^{m} \sum_{p=0}^{n_j-1} \beta_{v,j}^{[p]} z_j^{(p)}(a) = K_{a,v}, \qquad v = 1,2, \dots, m_a, \tag{2.6}$$

$$B_\sigma[\tilde{Z}_1(b), \tilde{Z}_2(b), \dots, \tilde{Z}_m(b)] = \sum_{j=1}^{m} \sum_{p=0}^{n_j-1} \gamma_{\alpha,j}^{[p]} z_j^{(p)}(b) = K_{b,\sigma}, \qquad \sigma = 1,2, \dots, m_b, \tag{2.7}$$

where $\beta_{v,j}^{[p]}, \gamma_{\sigma,j}^{[p]}$ are the constant coefficients of $z_j^{(p)}$ in the boundary conditions, and m_a, m_b are the total number of prescribed boundary conditions at $x = a$ and $x = b$ respectively. We remark that the sum $m_a + m_b$ is equal to the sum of the highest orders of the derivatives corresponding to the dependent variables z_i, that is

$$m_a + m_b = \sum_{i=1}^{m} n_i, \tag{2.8}$$

2.2 Spectral Quasi Linearization Method (SQLM):-

Assume that the solution $z_i(\eta)$ of (2.4) at the $(r+1)^{th}$ iteration is $z_{i,r+1}$. If the solution at the previous iteration $z_{i,r}(\eta)$ is sufficiently close to $z_{i,r+1}$, the nonlinear component $\mathcal{N}_i$ of equation (2.4) can be linearized using one term Taylor series for multiple variables so that equation (2.4) can be approximated as,

$$\mathcal{L}_i\left[Z_{1,r+1}, \dots, Z_{m,r+1}\right] + \mathcal{N}_i[\dots] + \sum_{j=1}^{m} \sum_{p=0}^{n_j} \left(z_{j,r+1}^{(p)} - z_{j,r}^{(p)}\right) \frac{\partial \mathcal{N}_i}{\partial z_j^{(p)}}[\dots] = H_i(\eta), \tag{2.9}$$

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j} \beta_{v,j}^{[p]}\, z_{j,r+1}^{(p)}(a) = 0, \qquad v = 1,2,\dots,m_a, \tag{2.10}$$

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j} \gamma_{\sigma,j}^{[p]}\, z_{j,r+1}^{(p)}(b) = 0, \qquad \sigma = 1,2,\dots,m_b, \tag{2.11}$$

where

$$[\dots] = \left[Z_{1,r},\, Z_{2,r},\dots,Z_{m,r}\right]. \tag{2.12}$$

Equation (4.9) can be written in the following form

$$\mathcal{L}_i\left[Z_{1,r+1},\dots,Z_{m,r+1}\right] + \sum_{j=1}^{m}\sum_{p=0}^{n_j} z_{j,r+1}^{(p)} \frac{\partial \mathcal{N}_i}{\partial z_j^{(p)}}[\dots] = H_i(\eta) + \sum_{j=1}^{m}\sum_{p=0}^{n_j} z_{j,r}^{(p)} \frac{\partial \mathcal{N}_i}{\partial z_j^{(p)}}[\dots] - \mathcal{N}_i[\dots]. \tag{2.13}$$

In this work, the iterative scheme (2.13) is called the Spectral Quasi linearization Method (SQLM). The initial approximation, $z_{j,0}(\eta)$, required to start the iteration scheme (2.13) is chosen to be a function that satisfies the boundary conditions (2.2). As a guide, the initial guess can be obtained as a solution of the linear part of (2.1) subject to the boundary conditions (2.2), that is, we solve

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j} \alpha_{i,j}^{[p]}\, z_{j,0}^{(p)} = H_i(\eta), \tag{2.14}$$

subject to

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j} \beta_{v,j}^{[p]}\, z_{j,0}^{\{p)} = K_{a,v}, \qquad v = 1,2,\dots,m_a, \tag{2.15}$$

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j}\gamma_{\sigma,j}^{[p]}\, z_{j,0}^{(p)} = K_{b,v}, \qquad \sigma = 1,2,\dots,m_b. \tag{2.16}$$

2.3 Solution method:-

To solve the iteration scheme (2.13), it is convenient to use the Chebyshev spectral collocation method. For brevity, we omit the details of the spectral methods, and refer interested readers to ([21, 30]). Before applying the spectral method, it is convenient to transform the domain on which the governing equation is defined to the interval [-1, 1] on which the spectral method can be implemented. We use the transformation $x = (b - a)(\tau + 1)/2$ to map the interval $[\mathrm{a}, \mathrm{b}]$ to [-1,1]. The basic idea behind the spectral collocation method is the introduction of a differentiation matrix D which is used to approximate the derivatives of the unknown variables $z_i(x)$ at the collocation points as the matrix vector product

$$\frac{dz_i}{d\eta} = \sum_{k=0}^{\bar{N}} D_{lk}\, z_i(T_k) = DZ_i, \qquad l = 0,1,\dots,\bar{N}, \tag{2.17}$$

where $\bar{\mathrm{N}} + 1$ is the number of collocation points (grid points), $\mathrm{D} = 2\mathrm{D}/(\mathrm{b} - \mathrm{a})$, and $Z = [z(\tau_0), z(\tau_1), \dots, z(\tau_N)]^T$ is the vector function at the collocation points. Higher order derivatives are obtained as powers of D, that is

$$z_j^{(p)} = D^p Z_j, \tag{2.18}$$

The solution process is applied in two stages. First we determine the initial approximation $Z_{i,0} (i = 1, 2, \dots, m)$ then using the solution of the initial approximation, we solve the recursive iteration scheme (2.9). Applying the Chebyshev spectral method on the initial approximation equations (2.14) – (2.16) we obtain

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j}\alpha_{i,j}^{[p]} D^p Z_{j,0} = H_i(\eta), \tag{2.19}$$

subject to

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j}\beta_{v,j}^{[p]}\sum_{k=0}^{\bar{N}}D_{\bar{N}k}^{p}\,z_{j,0}(\tau_k) = K_{a,v}\,, \qquad v = 1,2,\dots,m_a, \tag{2.20}$$

$$\sum_{j=1}^{m}\sum_{p=0}^{n_j}\gamma_{\sigma,j}^{[p]}\sum_{k=0}^{\bar{N}}D_{\bar{N}k}^{p}\,z_{j,0}(\tau_k) = K_{b,v}\,, \qquad \sigma = 1,2,\dots,m_b. \tag{2.21}$$

Equations (2.19) can be written in matrix form as

$$\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & & \vdots \\ A_{1,m} & A_{2,m} & \cdots & A_{m,m} \end{bmatrix}\begin{bmatrix} Z_{1,0} \\ Z_{2,0} \\ \vdots \\ Z_{m,0} \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_m \end{bmatrix}, \tag{2.22}$$

where $Z_{i,0}$ are vectors of size $(\bar{\mathrm{N}}+1)\times 1$ and $\mathrm{A_{i,j}}$ are $(\bar{\mathrm{N}}+1)\times(\bar{\mathrm{N}}+1)$ matrices which are, respectively, defined as

$$Z_{i.0} = [z_{i,0}(\tau_0), z_{i,0}(\tau_1), \dots, z_{i,0}(\tau_N)]^T, A_{i,j} = \sum_{p=0}^{n_j}\alpha_{i,j}^{p}D^p\,, \quad i,j = 1,2,\dots,\mathrm{m}. \tag{2.23}$$

Thus, the size of the coefficient matrix in (2.12) is $\mathrm{m}(\bar{\mathrm{N}}+1)\times\mathrm{m}(\bar{\mathrm{N}}+1)$ and the column vector on the right hand side has dimension $\mathrm{m}(\bar{\mathrm{N}}+1)\times 1$. Applying the Chebyshev spectral collocation on the recursive iteration scheme (2.13) gives

$$\sum_{j=1}^{m}[A_{i,j} + \Pi_{i,j}]\,Z_{j,r+1=}\,\Phi_{i,r}, \qquad j = 1,2,\dots,m, \tag{2.24}$$

where $Z_{i.r+1}$=$[z_{i,r+1}(\tau_0),\ z_{i,r+1}(\tau_1),\dots,\ z_{i,r+1}(\tau_N)]^{\mathrm{T}}$, $\mathrm{A_{i,j}}$ is as defined in (2.13) and $\Pi_{\mathrm{i,j}}$ and Φ_{i} are given by

$$\Pi_{i,j} = \sum_{p=0}^{n_j}\left[\frac{\partial N_i}{\partial z_j^{(p)}}\right]_d \mathrm{D}^p, \tag{2.25}$$

and

$$\Phi_{i,r} = H_i(\eta) + \sum_{j=1}^{m} \sum_{p=0}^{n_j} z_{j,r}^{(p)} \frac{\partial N_i}{\partial z_j^{(p)}} [\dots] - N_i[Z_{1,r}, Z_{2,r}, \dots, Z_{m,r}], \tag{2.26}$$

$$\begin{bmatrix} \Delta_{1,1} & \Delta_{1,2} & \cdots & \Delta_{1,m} \\ \Delta_{2,1} & \Delta_{2,2} & \cdots & \Delta_{2,m} \\ \vdots & \vdots & & \vdots \\ \Delta_{1,m} & \Delta_{2,m} & \cdots & \Delta_{m,m} \end{bmatrix} \begin{bmatrix} Z_{1,r+1} \\ Z_{2,r+1} \\ \vdots \\ Z_{m,r+1} \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_m \end{bmatrix}, \tag{2.27}$$

where $Z_{i,r}, \Phi_{\mathrm{i,r}}$ are vectors of size $(\overline{\mathrm{N}} + 1) \times 1$ and $\Delta_{\mathrm{i,j}}$ are $(\overline{\mathrm{N}} + 1) \times (\overline{\mathrm{N}} + 1)$ matrices Starting from, the recursive sequence (2.27) is solved iteratively for $r = 0, 1, 2, 3, \dots$

Chapter 3

On the solution of the SQLM of the Oldroyd-B fluid flow over moving surface

3.1 Mathematical Formulation:-

Consider the three dimensional flow of an incompressible Oldroyd-B fluid flow bounded by a moving surface at $z = 0$. The fluid occupies the space $z = 0$, and the motion in fluid is caused by a non-conducting moving surface. The governing equations are given (see Hayat et al. [31]) in the similarity form

$$f''' - f'^2 + ff'' + \beta_1 (2ff'f'' - f^2 f''') + \beta_2 \left(f''^2 - ff''''\right) = 0, \qquad (3.1)$$

where the prime symbol represent the derivative with respect to η. Subject to the boundary conditions are

$$f(0) = 0, \quad f'(0) = 1, \qquad \text{at} \quad \eta = 0,$$

$$f'(\infty) \longrightarrow 0, \qquad f''(\infty) \longrightarrow 0, \ \text{when}\, \eta \longrightarrow \infty, \qquad (3.2)$$

The primary interest parameters are $\beta_1$and β_2 are the Deborah numbers in terms of relaxation and retardation times respectively (see Hayat et al. [31]), these are given by

$\beta_1 = a\lambda_1$ and $\beta_2 = b\lambda_2$ where λ_1 and λ_1 are the relaxation and retardation times, respectively.

3.2 Numerical solutions:-

Before applying the spectral quasi linearization method, it is convenient to transform the domain on which the governing equation is defined to the interval [-1,1] on which the spectral method can be implemented. We use the transformation $x = (b - a)(\tau + 1)/2$ to map the interval $[a, b]$to [-1,1].To apply the multi-variables on (3.1), firstly we have to choose an initial guess in which satisfy the boundary conditions $f = 1 - e^{-\eta}$.

The iteration scheme can be given in the following form:

$$(-\beta_2 f_r)f''''_{r+1} + (1-\beta_1 f_r^2)f'''_{r+1} + (f_r + 2\beta_1 f_r f_r' + 2\beta_2 f_r'')f''_{r+1} + (2\beta_1 f_r f_r'' - 2f_r')f'_{r+1} + (f_r'' + 2\beta_1 f_r' f_r'' - 2\beta_1 f_r f_r''' - \beta_2 f_r'''')f_{r+1} = f_r f_r'' - f_r'^2 + 2\beta_1 f_r f_r' f_r'' - \beta_1 f_r^2 f_r''' + \beta_2 f_r''^2 - \beta_2 f_r f_r'''', \quad (3.3)$$

To solve the above iteration scheme, it is convenient to use the Chebyshev spectral collocation method. The iteration scheme (3.3) can be written in the framework of the SQLM in the form

$$\mathbf{A}\, f_{r+1} = \Phi_r, \quad (3.4)$$

where f_{r+1} and Φ_r are vectors of size $(\mathrm{N}+1)\times 1$ in current iteration and previous iteration respectively and $\mathbf{A}$ is $(\mathrm{N}+1)\times(\mathrm{N}+1)$ matrix defined as

$$\mathbf{A} = \mathbf{diag}[-\beta_2 f_r]\mathbf{D}^4 + \mathbf{diag}[1-\beta_1 f_r^2]\mathbf{D}^3 + \mathbf{diag}\,[f_r + 2\beta_1 f_r f_r' + 2\beta_2 f_r'']\,\mathbf{D}^2 + \mathbf{diag}\,[-2f_r' + 2\beta_1 f_r f_r'']\mathbf{D} + \mathbf{diag}[f_r'' + 2\beta_1 f_r' f_r'' - 2\beta_1 f_r f_r''' - \beta_2 f_r'''']\,\mathbf{I}, \quad (3.5)$$

$$\Phi_r = f_r f_r'' - f_r'^2 + 2\beta_1 f_r f_r' f_r'' - \beta_1 f_r^2 f_r''' + \beta_2 f_r''^2 - \beta_2 f_r f_r'''' \quad (3.6)$$

with $\mathbf{I}$ being an $(\mathrm{N}+1)\times(\mathrm{N}+1)$ identity matrix.

The boundary conditions become

$$f_{r+1}(\tau_0) = 0,\ f'_{r+1}(\tau_0) = 1,\ f'_{r+1}(\tau_N) = 0, f''_{r+1}(\tau_N) = 0. \quad (3.7)$$

3.3 Results and discussions:-

We remark that the proposed SQLM method for solving the ordinary differential equations presented in this study strongly depends on the length of the governing domain $(b-a)$ and the number N of collocation points (grid points).
In particular, since equation is defined on the semi-infinite interval $[0,\infty)$, a crucial factor of the SQLM iteration scheme is to find the appropriate finite value

η_∞ which must be selected to be large enough to numerically approximate infinity and the behavior of the governing flow parameters at infinity. In order to select the appropriate value of η_∞, we start with an initial guess $f = 1 - e^{-\eta}$ satisfy the boundary conditions (3.2) we solve the governing SQLM scheme equations over $[0,\ \eta_\infty]$ to obtain the solutions of flow parameters $f(\eta)$.
It can be observed from the data in Table 1 that the SQLM converges after 12 iterations. Clearly, what this means is that an increase in the number of iterations used improves the accuracy until convergence is attained.

Table 1: The skin friction coefficient $f''(0)$ for $\beta_1 = 0.3,\ \beta_2 = 0.4$.

Order of approximation	SQLM
1	1.020945
2	0.977974
5	0.965586
10	0.965234
12	0.965232
13	0.965232

That point is said to be a saturation point where an increase in number of iterations will only increase computational time and not make any positive significance in the accuracy of the solution. When convergence is attained for each method, it can be seen that the results are in excellent agreement with each other.

Fig. 1 and 2 are plotted to represent the influence of the Deborah numbers both $\beta_1$and β_2 on the velocity profiles. It is clear that from Fig. 1 a higher Deborah numbers are indicative that the Oldroyd-B fluid is stretched. The fluid velocity $f(\eta)$ and the momentum boundary layer thickness decrease with increasing β_1 which is not an unexpected result since it is well known that the viscoelastic fluid resists the motion of the fluid.

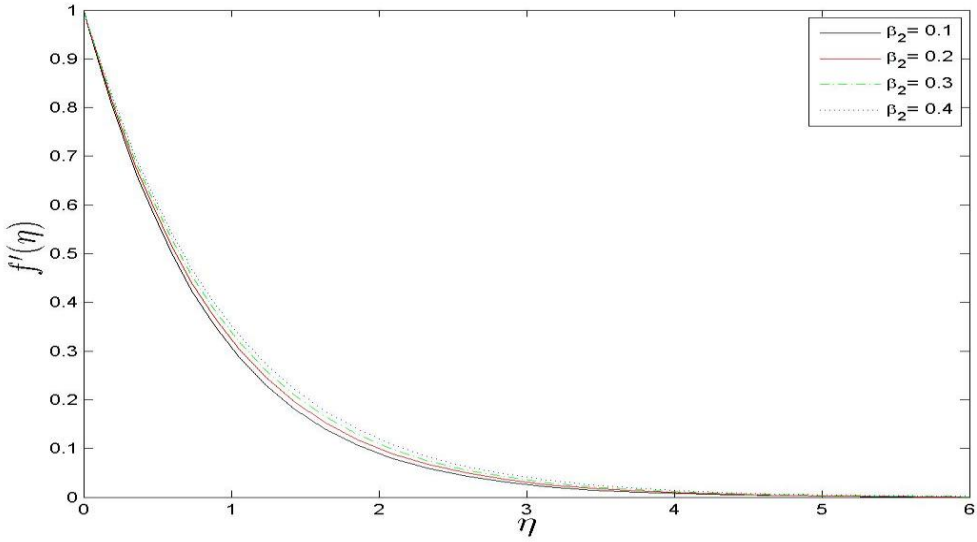

Fig. 1: The variation of the velocity $f'(\eta)$ with β_2 for $\beta_1 = 0.2$.

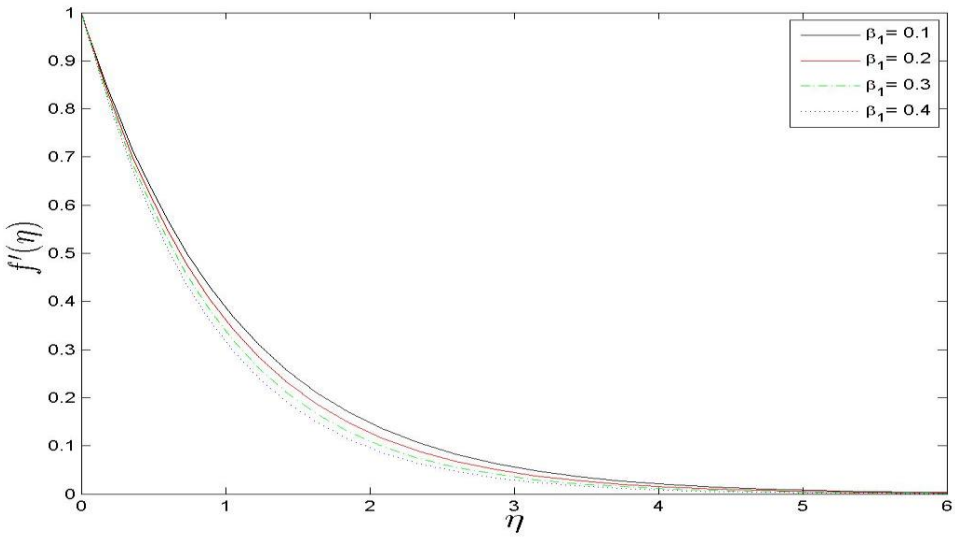

Fig. 2: The variation of the velocity $f'(\eta)$ with β_1 for $\beta_2 = 0.3$.

Fig. 3, one can observe that an increase in the number of collocation points reduces the residual error very quickly to a point where the optimal residual is achieved (that is the minimum residual error attainable). It can be observed that the optimal residual of the SQLM is about 10^{-21}.

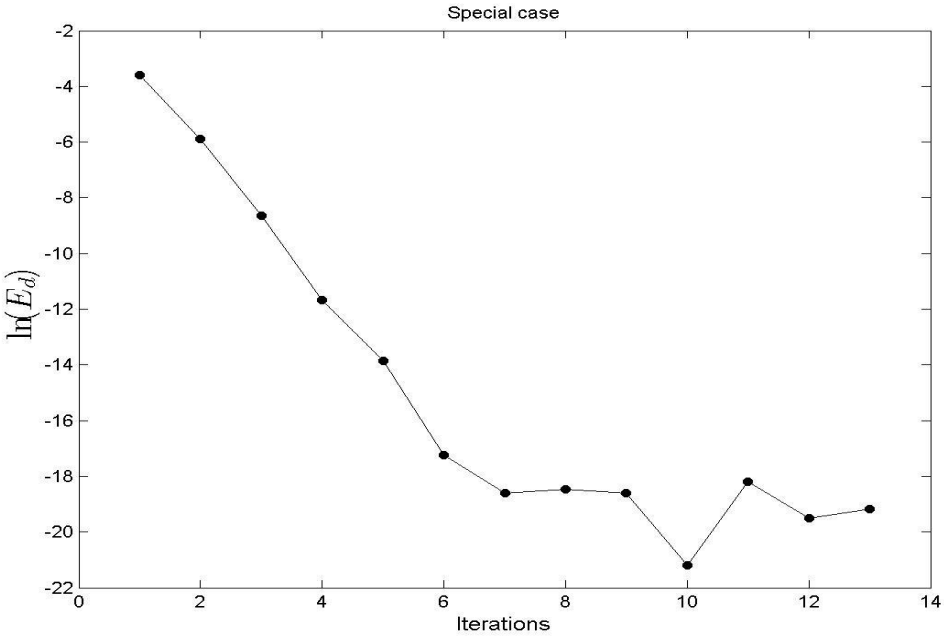

Fig. 3: The error E_d through difference order for $\beta_1 = 0.3,\ \beta_2 = 0.4$.

Chapter 4

Heat transfer in Oldroyd-B fluid flow bounded by a stretching surface

4.1 Mathematical Formulation:-

In this chapter we will extend the problem in chapter 3 to investigate the heat transfer in the three-dimensional flow of an incompressible Oldroyd-B fluid bounded by a stretching surface at $z = 0$. Following Hayat et al. [32]. The non-dimensional equations governing the flow can be written as:

$$f''' + (f+g)f'' - f'^2 + \beta_1 (2(f+g)f'f'' - (f+g)^2 f''') + \beta_2\big((f'' + g''f'' - (f+g)f''''= 0, \qquad (4.1)$$

$$g''' + (f+g)g'' - g'^2 + \beta_1 (2(f+g)g'g'' - (f+g)^2 g''') + \beta_2\big((f'' + g''g'' - (f+g)g''''= 0, \qquad (4.2)$$

$$\theta'' + \Pr(f+g)\theta' = 0, \qquad (4.3)$$

subject η to the boundary conditions are

$$f = 0,\ g = 0,\ f' = 1,\ g' = \beta,\ \theta' = -\gamma\big(1 - \theta(0)\big), \qquad \text{at } \eta = 0,$$

$$f' \to 0, \qquad f'' \to 0, \qquad g' \to 0, \qquad \theta \to 0, \quad \text{when } \eta = \infty \qquad (4.4)$$

The parameters of primary interest are the Deborah numbers β_1 and β_2, the Prandtl number $\Pr$ and the Biot number γ.

4.2 Numerical solution:-

The main idea behind this approach is identifying multiple nonlinear terms from the system (4.1) – (4.3), linearizing the terms and applying Chebychev pseudo-spectral collocation method to integrate the coupled system obtained.

Therefore we have to choose initial approximations which satisfy the boundary conditions (4.4) as follows

$$f(\eta) = 1 - e^{-\eta}, g(\eta) = \beta - \beta e^{-\eta}, \theta(\eta) = \frac{\gamma}{\gamma+1} e^{-\eta}.$$

To apply the SQLM on equations (4.1) – (4.3) can start from the following iteration scheme

$$-\beta_2(f_r + g_r)\, f''''_{r+1} + (1 - \beta_1(f_r + g_r)^2) f'''_{r+1} + (f_r + g_r + 2\beta_1(f_r + g_r) f'_r + \beta_2(2f''_r + g''_r)) f''_{r+1} + (-2f'_r + 2\beta_1(f_r + g_r) f''_r) f'_{r+1} + (f''_r + 2\beta_1 f'_r f''_r - 2\beta_1(f_r + g_r) f'''_r - \beta_2 f''''_r) f_{r+1} + (\beta_2 f''_r) g''_{r+1} + (f''_r + 2\beta_1 f'_r f''_r - 2\beta_1(f_r + grfr''' - \beta 2fr'''')gr+1 = fr''(fr+gr) - fr'2 + 4\beta 1(fr+gr)fr'fr'' - 2\beta_1(f_r + g_r)^2 f'''_r + \beta_2(f''_r + g''_r) f''_r - \beta_2(f_r + g_r) f''''_r,$$

(4.5)

$$-\beta_2(f_r + g_r) g''''_{r+1} + (1 - \beta_1(f_r + g_r)^2) g'''_{r+1} + (f_r + g_r + 2\beta_1(f_r + g_r) g'_r + \beta_2(2f''_r + g''_r)) g''_{r+1} + (-2g'_r + 2\beta_1(f_r + g_r) f''_r) g'_{r+1} + (g''_r + 2\beta_1 g'_r g''_r - 2\beta_1(f_r + g_r) g'''_r - \beta_2 g''''_r) g_{r+1} + (\beta_2 g''_r) f''_{r+1} + (g''_r + 2\beta_1 g'_r g''_r - 2\beta_1(f_r + g_r) g''''_r - \beta_2 g''''_r))\, f_{r+1} = g''_r(f_r + g_r) - g'^2_r + 4\beta_1(f_r + g_r) g'_r g''_r - 2\beta_1(f_r + g_r)^2 g'''_r + \beta_2(f''_r + g''_r) g''_r - \beta_2(f_r + g_r) g''''_r, \quad (4.6)$$

$$\theta''_{r+1} + \Pr\,(f_r + g_r)\theta'_{r+1} = 0. \quad (4.7)$$

Applying the chebyshev pseudo spectral method on (4.5) – (4.7) we obtained

$$\mathbf{A_{11}} f_{r+1} + \mathbf{A_{12}} g_{r+1} = \mathrm{R}_1, \quad (4.8)$$

$$\mathbf{A_{21}} f_{r+1} + \mathbf{A_{22}} g_{r+1} = \mathrm{R}_2, \quad (4.9)$$

$$\mathbf{B}\, \theta_{r+1} = \mathrm{R}_3, \quad (4.10)$$

where

$$\mathbf{A_{11}} = \mathbf{diag}[-\beta_2(f_r + g_r)]\mathbf{D}^4 + \mathbf{diag}\,[1 - \beta_1(f_r + g_r)^2]\mathbf{D}^3 + \mathbf{diag}[f_r + g_r + 2\beta_1(f_r + g_r) f'_r + \beta_2(2f''_r + g''_r)]\mathbf{D}^2 + \mathrm{diag}\,[-2f'_r + 2\beta_1(f_r + g_r]\mathbf{D} + \mathbf{diag}\,[f''_r + 2\beta_1 f'_r f''_r - 2\beta_1(f_r + g_r f''_r - \beta_2 f''''_r]\,\mathbf{I}, \quad (4.11)$$

$$\mathbf{A_{12}} = \mathbf{diag}[\beta_2 f_r'']\mathbf{D}^2 + \mathbf{diag}[f_r'' + 2\beta_1 f_r' f_r'' - 2\beta_1(f_r + g_r)f_r''' - \beta_2 f_r''''] \mathbf{I}, \quad (4.12)$$

$$\mathbf{A_{21}} = \mathbf{diag}[\beta_2 g_r'']\mathrm{D}^2 + \mathbf{diag}[g_r'' + 2\beta_1 g_r' g_r'' - 2\beta_1(f_r + g_r)g_r'' - \beta_2 g_r''''] \mathbf{I}, \quad (4.13)$$

$$\mathbf{A_{22}} = \mathbf{diag}[-\beta_2(f_r + g_r)]\mathbf{D}^4 + \mathbf{diag}[1 - \beta_1(f_r + g_r)^2]\mathbf{D}^3 + \mathrm{diag}[f_r + g_r + 2\beta_1(f_r + g_r)g_r' + \beta_2(2f_r'' + g_r'')]\mathbf{D}^2 + \mathbf{diag}[-2g_r' + 2\beta_1(f_r + g_r)]\mathbf{D} + \mathbf{diag}[g_r'' + 2\beta_1 g_r' g_r'' - 2\beta_1(f_r + g_r)g_r'' - \beta_2 g_r''''] \mathbf{I}, \quad (4.14)$$

$$\mathrm{R}_1 = f_r''(f_r + g_r) - f_r'^2 + 2\beta_1(f_r + g_r)f_r' f_r'' - \beta_1(f_r + g_r)^2 f_r''' + \beta_2(f_r'' + g_r'')f_r'' - \beta_2(f_r + g_r)f_r'''', \quad (4.15)$$

$$\mathrm{R}_2 = g_r''(f_r + g_r) - g_r'^2 + 2\beta_1(f_r + g_r)g_r' g_r'' - \beta_1(f_r + g_r)^2 g_r''' + \beta_2(f_r'' + gr''gr'' - \beta 2fr + grgr'''', \quad (4.16)$$

$$\mathbf{B} = \mathbf{D}^2 + \mathbf{diag}[(\mathrm{Pr}(f_r + g_r))]\mathbf{D}, \quad (4.17)$$

$$\mathrm{R}_3 = \mathbf{0}, \quad (4.18)$$

with $\mathbf{I}$ being an $(\mathrm{N}+1) \times (\mathrm{N}+1)$ identity matrix , $\mathbf{0}$ being $(\mathrm{N}+1) \times 1$ vector of zero.
The boundary conditions become

$$f_{r+1}(\tau_0) = 0,\ f_{r+1}'(\tau_0) = 1,\ g_{r+1}(\tau_0) = 0,\ g_{r+1}'(\tau_0) = \beta,\ \theta'_{r+1}(\tau_0) = -\gamma(1 - \theta(\tau_0)),$$
$$f_{r+1}'(\tau_N) = 0,\ f_{r+1}''(\tau_N) = 0,\ g_{r+1}'(\tau_N) = 0,\ \theta_{r+1}'(\tau_N) = 0. \quad (4.19)$$

Thus, starting from the initial approximations, the approximate solutions f_{r+1}, can be obtained iteratively by solving the equations (4.5) - (4.7).

4.3 Results and discussions:-

In this section we present the results of the implementation of the spectral Quasi linearization method (SQLM) to the governing equations (4.1) – (4.3) along with boundary conditions (4.4). All the results presented for the SQLM simulations were obtained using $\mathrm{N} = 20$ collocation points and$\eta_\infty = 25$. In order to assess the accuracy and performance of the proposed SQLM approach, the present results are compared with those obtained by Hayat et al.[32] using the homotopy analysis method.

It can be observed from the data in Table 1, Table 2 and Table 3 that the SQLM converges after 8 iterations while it takes 20 iterations for the HAM to converge to a consistent solution. Clearly, what this means is that an increase in the number of iterations used improves the accuracy until convergence is attained. That point is said to be saturation point where an increase in number of iterations will only increase computational time and not make any positive significance in the accuracy of the solution. When convergence is attained for method, it can be seen that the results are in excellent agreement with each other.

Table 1: Computations showing comparison with HAM and SQLM for skin fraction $-f''(0)$ when $\beta_1 = 0.3,\ \beta_2 = 0.4,\ Pr = 1,\ \gamma = 0.8,\ \beta = 0.5.$

Order of approximation	HAM	SQLM
1	0.94875	1.01501
2	-	0.97476
3	-	0.96698
7	-	0.96453
8	-	0.96451
10	0.96460	0.96451
15	0.96449	0.96451
20	0.96450	0.96451

Table 2: Computations showing comparison with HAM and SQLM for skin fraction $-g''(0)$ when $\beta_1 = 0.3,\ \beta_2 = 0.4,\ Pr = 1,\ \gamma = 0.8,\ \beta = 0.5.$

Order of approximation	HAM	SQLM
1	0.41313	0.44200
2	-	0.41436
3	-	0.40830
7	-	0.40623
8	-	0.40622
10	0.40614	0.40622
15	0.40619	0.40622
20	0.40622	0.40622

Table 3: Computations showing comparison with HAM and SQLM for - $\theta'(0)$ for β_1= 0.3, β_2 = 0.4, Pr = 1, γ = 0.8, β = 0.5.

Order of approximation	HAM	SQLM
1	0.41481	0.40645
2	-	0.39176
3	-	0.38871
7	-	0.38790
8	-	0.38790
10	0.38771	0.38790
15	0.38791	0.38790
20	0.38790	0.38790

Tables 1, 2 and 3 it is observed that the method are computationally efficient (in terms of speed) as the method take less than a second to generate solutions . To further verify the results obtained using this method, the solutions obtained using the SQLM is validated against solutions obtained by other known numerical methods. The influences of the parameters on the dimensionless system are studied in this study as well. Hayat et al. [31] studied the same problem using the homotopy analysis method. Table 1 and 2 show solution of the Skin frictions

coefficients $-f''(0)$ and $-g''(0)$ and in Table 3 show solution of heat transfer coefficients -$\theta'(0)$ obtained for fixed Prandtl number $\mathrm{Pr} = 1$,Biot number $\gamma = 0.8$.

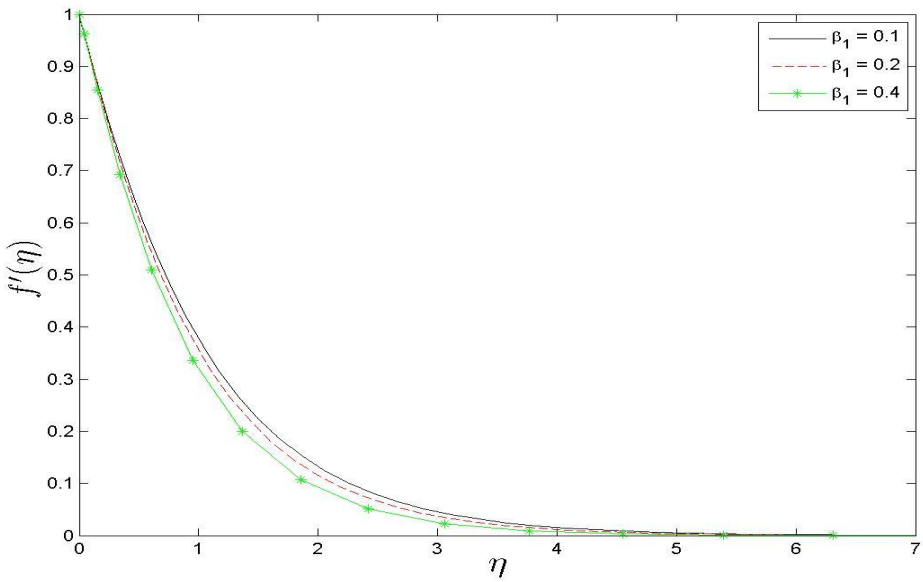

Fig. 1: The variation of the velocity component f' with β_1 for β_2 = 0.4, Pr = 1, γ = 0.8, β = 0.5.

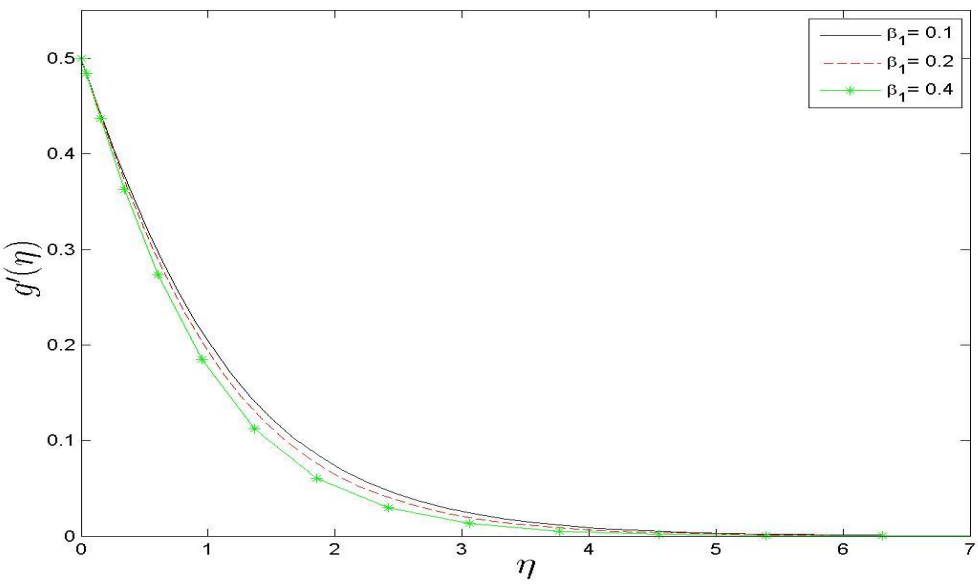

Fig. 2: The variation of $g'(\eta)$ for $\beta_2 = 0.4, Pr = 1, \gamma = 0.8, \beta = 0.5$.

Fig. 1 - 7 documented the effect of the governing parameters and the number of collocation points on the accuracy of the solutions obtained by the SQLM. Fig. 1 and 2 illustrate the variation of the velocities $f'(\eta)$and $g'(\eta)$ with β_1. It is clear that the velocities $f'(\eta)$ and $g'(\eta)$ decrease with increasing values of β_1.

Fig. 3 and 4 illustrate the variation of the velocities $f'(\eta)$and $g'(\eta)$ with β_2. It is clear that the velocities $f'(\eta)$ and $g'(\eta)$ increase with increasing values of β_2 .

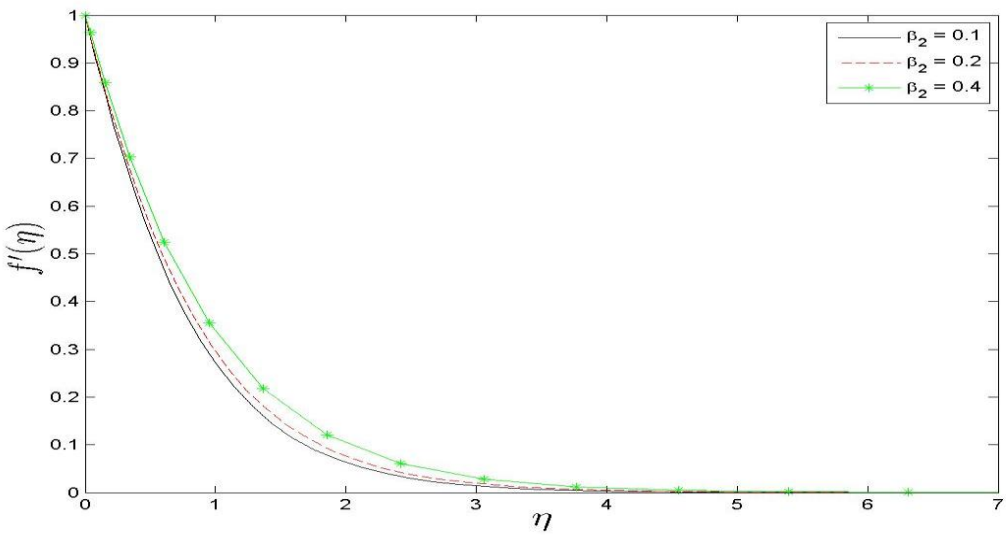

Fig. 3 : The variation of the velocity component f' with β_2 for β_1 = 0.4, Pr = 1, γ = 0.8, β = 0.5.

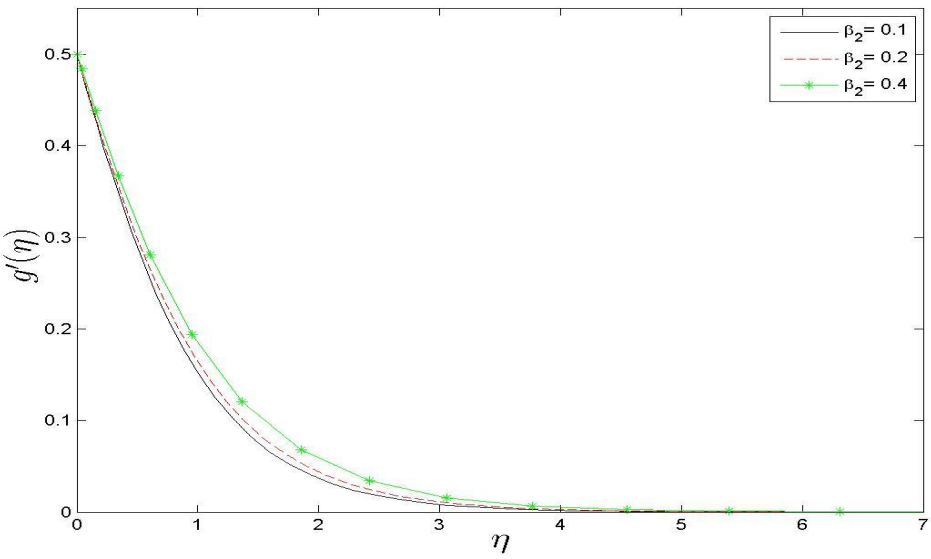

Fig. 4: The variation of the velocity component g' with β_2 for β_1 = 0.4, Pr = 1, γ = 0.8, β = 0.5.

Fig. 5 and 6 show the effects of the Deborah numbers β_1 and β_2 on the dimensionless temperature respectively. It can be seen from the Fig. 5 that the thermal boundary layer thickness increases with an increasing in β_1, thus leading to temperature profile increasing in the boundary. On the other hand, we observe that dimensionless temperature decreases with an increase in β_2 that is due to increase in the thermal boundary layer thinness.

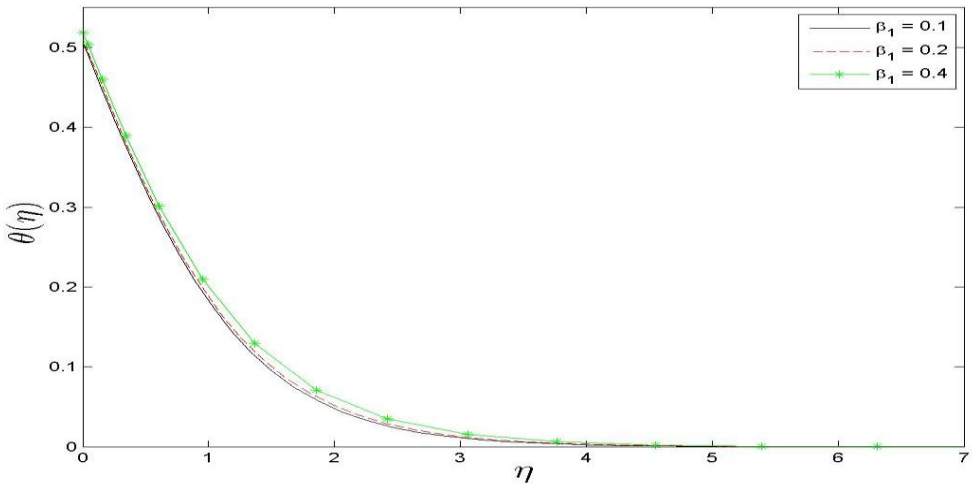

Fig. 5: The variation of the temperature $\theta(\eta)$ with β_1 for β_2 = 0.4, Pr = 1, γ = 0.8, β = 0.5.

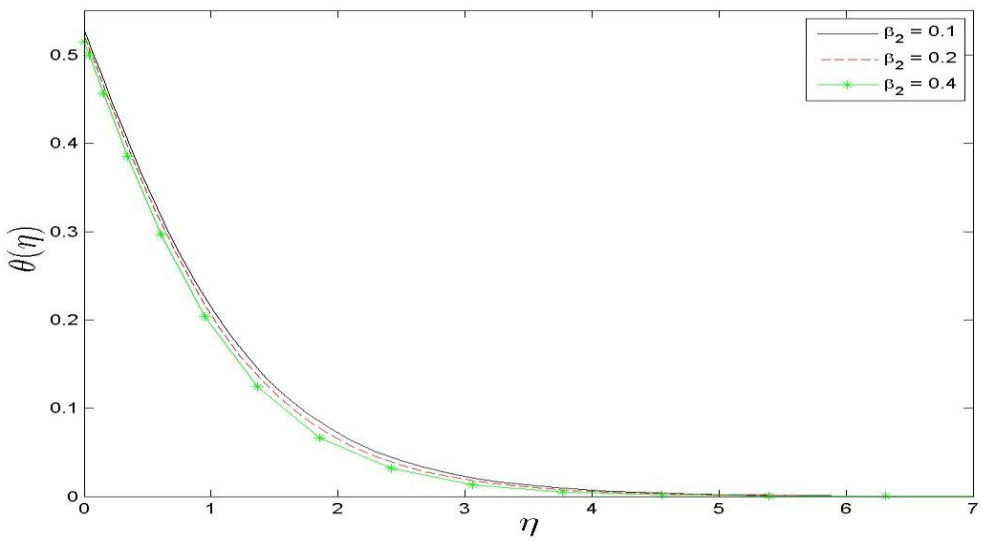

Fig. 6: The variation of the temperature $\theta(\eta)$ with β_2 for β_1 = 0.4, Pr = 1, γ = 0.8, β = 0.5.

From Fig. 7, we observed that an increase in the number of collocation points reduces the residual error very quickly to a point where the optimal residual is achieved (that is the minimum residual error attainable). It can be observed that the optimal residual of the SQLM is about10^{-18} . When the behavior of the SQLM is studied closely, observation can be made that accuracy is best when the number of collocation points used range between 7 and 8 and then the accuracy slowly declines. This implies that very large amount of collocation points used will cause lesser accuracy to be attained and this observation is supported by Motsa [18].

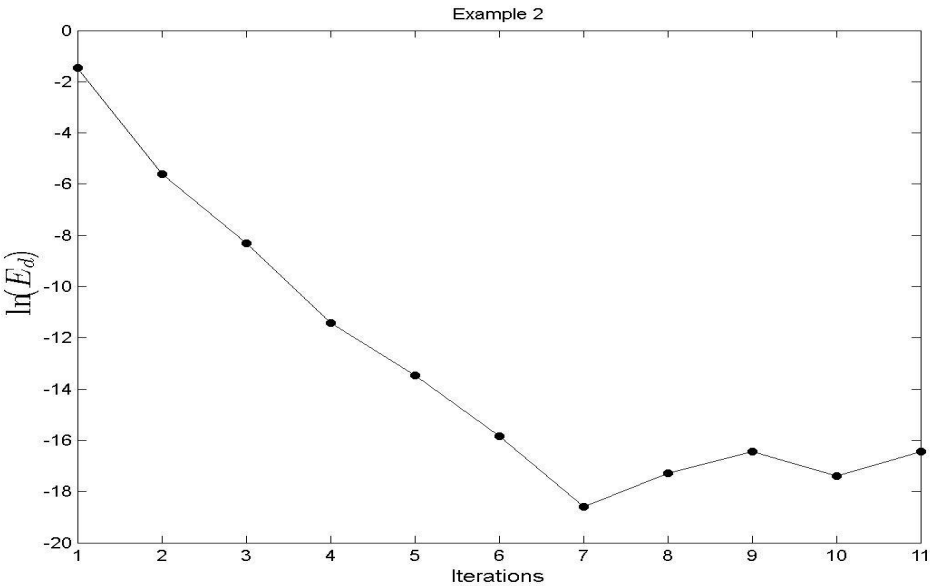

Fig. 7: the error E_d between velocity and temperature.

Chapter 5

Dufour and Soret effects on a laminar fluid flow with convective surface boundary conditions

5.1 Mathematical Formulation:-

Let us consider a steady, laminar, hydromagnetic coupled heat and mass transfer by mixed convection flow over a vertical plate. The fluid is assumed to be Newtonian, electrically conducting and its property variations due to temperature and chemical species concentration are limited to fluid density. The density variation and the effects of the buoyancy are taken into account in the momentum equation (Boussinesq approximation). In addition, there is no applied electric field and all of the Hall effects and Joule heating are neglected. Since the magnetic Reynolds number is very small for most fluid used in industrial applications, we assume that the induced magnetic field is negligible. Let the x-axis be taken along the direction of plate and y-axis normal to it. If u, v, T and C are the fluid x-component of velocity, y component of velocity, temperature and concentration respectively. Under the Boussinesq and boundary-layer approximations, following Makinde [34], the dimensionless governing equations for this problem can be written as:

$$f''' + \frac{1}{2}ff'' - \mathrm{Ha}_x(f' - 1) + Gr_x\theta + Gc_x\phi = 0, \tag{5.1}$$

$$\theta'' + \frac{1}{2}\mathrm{Pr}\, f\theta' + \mathrm{D_f}\phi'' = 0, \tag{5.2}$$

$$\phi'' + \frac{1}{2}\mathrm{Sc} f\phi' + \mathrm{Sr}\theta'' = 0, \tag{5.3}$$

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \quad \theta'(0) = Bi_x\left[\theta(0) - 1\right], \tag{5.4}$$

$$f'(0) = 1,\ \phi(0) = 1,\ \theta(\infty) = 0,\ \phi(\infty) = 0, \tag{5.5}$$

where Sr is soret numbers, D_f is Dufour number, Sc is Schmidt number and Pr is Pranditl number define as follows

$$Sr = \frac{Dk}{C_s C_p}\frac{T_w - T_\infty}{C_w - C_\infty}, \qquad D_f = \frac{Dk}{C_s C_p}\frac{C_w - C_\infty}{T_w - T_\infty}, \qquad Sc = \frac{v}{D}, \qquad Pr = \frac{v}{\alpha},$$

$$Bi_x = \frac{h_f}{k}\sqrt{\frac{vx}{U_\infty}}.$$

5.2 Numerical solution:-

To demonstrate the applicability of the proposed spectral quasi-linearization method (SQLM) algorithm as an appropriate tool for solving nonlinear governing equations. We start by linearizing the nonlinear terms in equations (5.1) – (5.3) by using one term Taylor series for multiple variables, so we will obtained the following iteration scheme

$$f'''_{r+1} + \frac{1}{2}f_r f''_{r+1} + \frac{1}{2}f''_r f_{r+1} - \frac{1}{2}f_r f''_r - Ha_x f'_{r+1} + Ha_x + Gr_x\theta_{r+1} + Gc_x\phi_{r+1} = 0, \tag{5.6}$$

$$\theta''_{r+1} + \frac{1}{2}\mathrm{Pr} f_r \theta'_{r+1} + \frac{1}{2}\mathrm{Pr}\theta'_r f_{r+1} - \frac{1}{2}\mathrm{Pr}\theta'_r f_r + \mathrm{D_f}\phi''_{r+1} = 0, \tag{5.7}$$

$$\phi''_{r+1} + \frac{1}{2}Sc f_r \phi'_{r+1} + \frac{1}{2}Sc\phi''_r f_{r+1} - \frac{1}{2}Sc\phi'_r f_r + Sr\theta''_{r+1} = 0, \tag{5.8}$$

$$f'''_{r+1} + \frac{1}{2}f_r f''_{r+1} - Ha_x f'_{r+1} + \frac{1}{2}f''_r f_{r+1} + Gr_x\theta_{r+1} + Gc_x\phi_{r+1} = \frac{1}{2}f_r f''_r - Ha_x, \tag{5.9}$$

$$\frac{1}{2}Pr\theta'_r f_{r+1} + \theta''_{r+1} + \frac{1}{2}Pr f_r\theta'_{r+1} + D_f\phi''_{r+1} = \frac{1}{2}Pr\theta'_r f_r, \tag{5.10}$$

$$\frac{1}{2}Sc\phi''_r f_{r+1} + Sr\theta''_{r+1} + \phi''_{r+1} + \frac{1}{2}Sc f_r\phi'_{r+1} = \frac{1}{2}Sc\phi'_r f_r, \tag{5.11}$$

Applying the chebyshev pseudo spectral method on (5.9) – (5.11) we obtain

$$\mathbf{A_{11}} f_{r+1} + \mathbf{A_{12}}\theta_{r+1} + \mathbf{A_{13}}\phi_{r+1} = \mathrm{R_{1,r}}, \tag{5.12}$$

$$\mathbf{A_{21}} f_{r+1} + \mathbf{A_{22}} \theta_{r+1} + \mathbf{A_{23}} \phi_{r+1} = \mathrm{R}_{2,\mathrm{r}}, \tag{5.13}$$

$$\mathbf{A_{31}} f_{r+1} + \mathbf{A_{32}} \theta_{r+1} + \mathbf{A_{33}} \phi_{r+1} = \mathrm{R}_{3,\mathrm{r}}, \tag{5.14}$$

where

$$\mathbf{A_{11}} = \mathbf{D}^3 + \mathbf{diag}[\tfrac{1}{2} f_r]\mathbf{D}^2 - \mathrm{Ha}_x \mathbf{D} + \mathbf{diag}[\tfrac{1}{2} f_r'']\mathbf{I}, \tag{5.15}$$

$$\mathbf{A_{12}} = \mathrm{Gr}_x \mathbf{I}, \tag{5.16}$$

$$\mathbf{A_{13}} = Gc_x \mathrm{I}, \tag{5.17}$$

$$\mathbf{A_{21}} = \tfrac{1}{2} \mathbf{diag}[\mathrm{Pr}\theta_r']\mathbf{I}, \tag{5.18}$$

$$\mathbf{A_{22}} = \mathbf{D}^2 + \tfrac{1}{2} \mathbf{diag}[\mathrm{Pr} f_r]\mathbf{D}, \tag{5.19}$$

$$\mathbf{A_{23}} = \mathrm{D_f} \mathbf{D}^2, \tag{5.20}$$

$$\mathbf{A_{31}} = \mathbf{diag}[\tfrac{1}{2} \mathrm{Sc}\phi_r'']\mathbf{I}, \tag{5.21}$$

$$\mathbf{A_{32}} = \mathrm{Sr}\mathbf{D}^2, \tag{5.22}$$

$$\mathrm{A}_{33} = \mathbf{D}^2 + \mathbf{diag}[\tfrac{1}{2} \mathrm{Sc} f_r]\mathbf{D}, \tag{5.23}$$

$$\mathrm{R}_{1,\mathrm{r}} = \tfrac{1}{2} f_r f_r'' - \mathrm{Ha}_x, \tag{5.24}$$

$$\mathrm{R}_{2,\mathrm{r}} = \tfrac{1}{2} \mathrm{Pr} f_r \theta_r', \tag{5.25}$$

$$R_{3,r} = \tfrac{1}{2} \mathrm{Sc} f_r \phi_r', \tag{5.26}$$

with $\mathbf{I}$ being an (N+1) × (N+1) identity matrix .The boundary condition become

$f_{r+1}(\tau_0) = 0$, $f_{r+1}'(\tau_0) = 0$, $\theta_{r+1}'(\tau_0) = Bi_x[\theta(\tau_0) - 1]$, $\phi_{r+1}(\tau_0) = 1$,

$f_{r+1}'(\tau_N) = 1$, $\theta_{r+1}(\tau_N) = \phi_{r+1}(\tau_N) = 0$.

Before applying the spectral method, it is convenient to transform the domain on which the governing equation is defined to the interval $[-1,1]$ where the spectral method can be implemented. For the convenience of the numerical computations, the semi-infinite domain in the space direction is approximated by the truncated domain $[0,\eta_\infty]$, where η_∞ is a finite number selected to be large enough to represent the behavior of the flow properties when η is very large.

5.3 Results and discussions:-

The governing systems of ODEs (5.1) - (5.3) have been solved numerically using the spectral quasi-linearization method (SQLM) as described in chapter 2 starting from the initial guesses $f(\eta) = e^{-\eta} + \eta - 1,\ \theta(\eta) = \frac{Bi_x}{Bi_x+1} e^{-\eta},\ \phi(\eta) = e^{-\eta}$.

Table 1: Computations showing comparison with Aziz [33] and Makindie [34] results for Ha_x=0, Gr_x=Gc_x=0, Pr=0.72, Sc=0.63, D_f=Sr=0.

Bi_x	$\theta(0)$			-$\theta'(0)$		
	Aziz[33]	Makinde[34]	SQLM	[33]	[34]	SQLM
0.05	0.1447	0.14466	0.144661	0.0428	0.04276	0.042767
0.1	0.2528	0.25275	0.252758	0.0747	0.07472	0.074724
0.2	0.4035	0.40352	0.403523	0.1193	0.11929	0.119295
0.4	0.5750	0.57501	0.575014	0.1700	0.16999	0.169994
0.6	0.6699	0.66991	0.669916	0.1981	0.19805	0.198051
0.8	0.7302	0.73016	0.730170	0.2159	0.21586	0.281853
1	0.7718	0.77182	0.771822	0.2282	0.22817	0.228178
5	0.9441	0.94417	0.944174	0.2791	0.27913	0.279131
10	0.9713	0.97128	0.971285	0.2871	0.28714	0.287146
20	0.9854	0.98543	0.985434	0.2913	0.29132	0.291329

Our results have been presented graphically and tablet for the velocity, temperature, and concentration profiles for various input parameters. Values of other flow properties such as skin friction $f''(0)$, surface heat transfer rate at the

$\theta'(0)$, and mass transfer rate at the wall $\phi'(0)$ are also discussed in order to illustrate some special features of the solution and to compare the accuracy, convergence, and efficiency of the SQLM algorithms proposed in this study.

The accuracy of the present results was verified by comparing them with other results from literature which have been reported to be accurate to within a certain number of decimal digits.

Table 2: $Ha_x = 1, Gr_x = Gc_x = 0.1, Pr = 0.72, Sc = 0.63, D_f = Sr = 0.5.$

$\boldsymbol{Bi_x}$	$\boldsymbol{\theta(0)}$	$\boldsymbol{-\theta'(0)}$	$\boldsymbol{-\phi'(0)}$
0.1	0.358016	0.064198	0.346640
1	0.769099	0.230901	0.311131
20	0.984051	0.318976	0.292277
100	0.996758	0.324201	0.291157

Fig. 1 shows the variation of velocity component $f'(\eta)$ with $\mathrm{B}i_x$. It is clear that the velocity profile increases with increasing values of Hartmann number Bi_x.

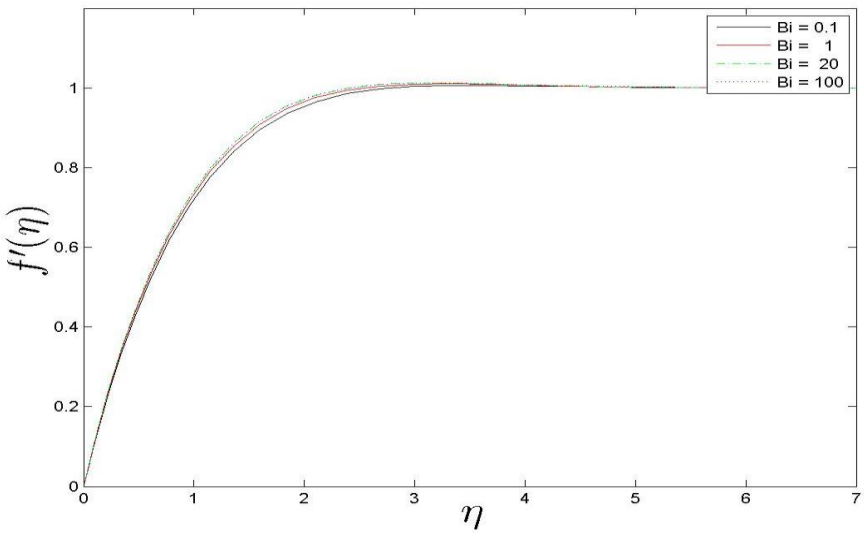

Fig.1: Effect of Bi_x on the velocity profile for $Ha_x = 1, Gr_x = Gc_x = 0.1, Pr = 0.72, Sc = 0.63,\ Sr = 0.5.$

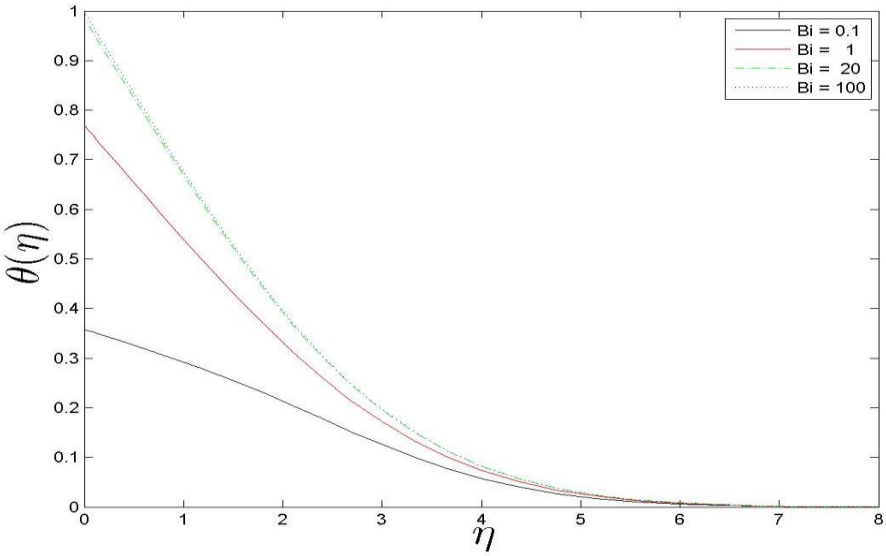

Fig. 2: temperature profile for Ha_x=1,Gr_x=Gc_x=0.1, Pr=0.72, Sc=0.63, $\mathrm{D_f}$= Sr =0.5,

Fig. 2: effects of the convective heat transfer parameter on the temperature profile. We observe that an increase in Bi_x leads to an increase of the thermal boundary layer thickness, hence, the Bi_x number enhances the temperature of the fluid along the surface.

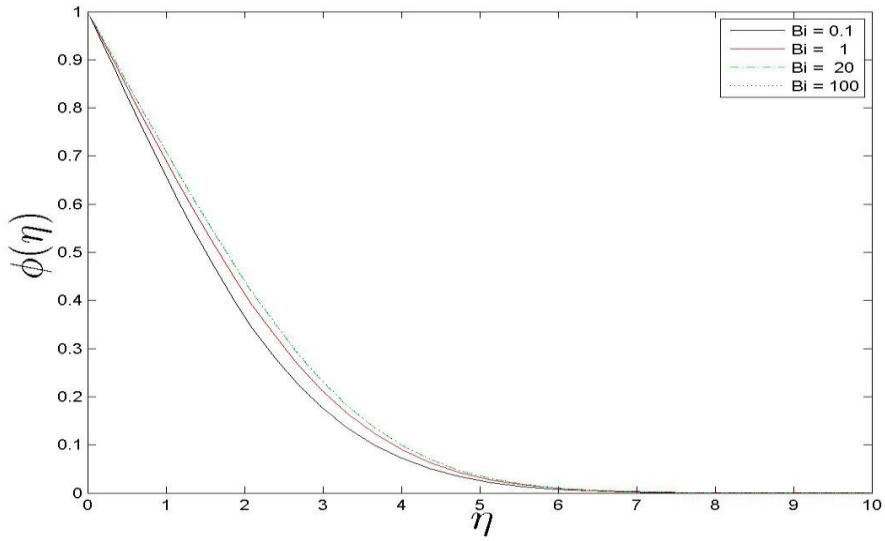

Fig 3: Concentration profile for Ha_x=1, Gr_x=Gc_x=0.1, Pr=0.72, Sc=0.63, $\mathrm{D_f}$= Sr =0.5.

Fig. 3 displays the effects of the convective heat transfer parameter the concentration profile. The concentration profile increases with increasing values

of the convective heat transfer parameter Bi_x. This is because increasing Bi_x enhances the thinness of thermal boundary layer, consequently, the concentration boundary layer thickness increases.

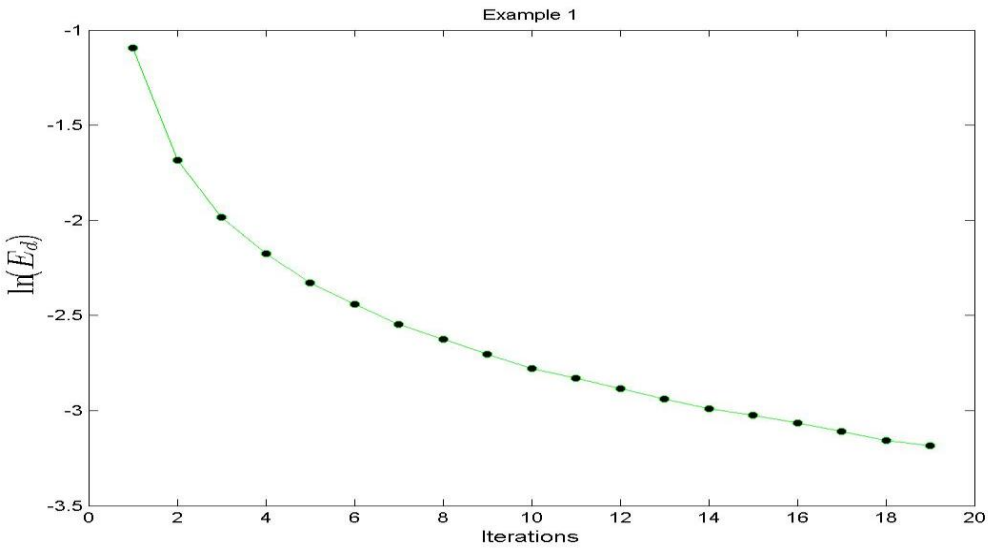

Fig 4: SQLM variation of solution errors against iterations when $Ha_x = 1, Gr_x = Gc_x = 0.1, Pr = 0.72, Sc = 0.63, D_f = Sr = 0.5$ and $Bi_x = 0.1$.

Fig. 4: depicts the SQLM variation of solution errors against iterations. It is that increases in the number of collocation points together with the scheme iteration number reduce the residual error very quickly to a point where the optimal residual is achieved (that is the minimum residual error attainable). It can be observed that the optimal residual of the SQLM is about10^{-19}.

Chapter 6

Conclusion

The main objective of this thesis was to apply the SQLM to solve highly nonlinear coupled differential equations. The accuracy and validity of the numerical solutions obtained using spectral quasi-linearization approach were dertemined by comparing with results from literature for limiting cases. The effects of various physical parameters on the fluid properties as well as flow characteristics including skin friction and heat transfer coefficients were presented in tabular. In this chapter, we highlight some of the conclusions that have been drawn from this study.

Chapter 3. Oldroyd-B fluid flow due to a moving surface was investigated. The nonlinear ordinary differential equation produced by Hayat et al. [21] was solved numerically using the SQLM. Validation was made on the approximate solution obtained using this numerical method using residual error analysis. We found that the SQLM is so accurate and the solution converges at most after 12 iterations.

Chapter 4. Heat transfer in the three-dimensional flow of an incompressible Oldroyd-B fluid bounded by a stretching surface was investigated using SQLM. The model equations were described by a nonlinear system of ordinary differential equations. The approximate solutions obtained were validated against solutions from published literature. From the tabulated results we observed That the SQLM was efficient in terms of accuracy and it converges so fast after 8 iterations while it takes 20 iterations for the HAM to converge to a consistent solution.

Chapter 5. The effects of Dufour and Soret on a laminar fluid flow with convective surface boundary conditions were investigated. The governing equations were solved using the SQLM technique. The effects of the governing parameters on the skin friction as well as the heat transfer rate were presented in both tabulated.

In conclusion, one common phenomenon observed was that the SQLM is very efficient and accurate numerical method. The steps taken to develop their

respective algorithms were found to be very straight forward and easy to come up with when applying the numerical methods on ODEs. Further research can be carried out under other conditions to observe if the behavior of the methods remains the same as what was seen in this study.

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