

**Sudan University of Science and Technology
College of Graduate Studies**

**Application of Homotopy Perturbation Method to
Linear and Nonlinear Partial Differential Equations**

**تطبيق طريقة إرتجاج الهوموتوبيا على المعادلات التفاضلية الجزئية الخطية
وغير الخطية**

**A Thesis submitted in Fulfillment Requirements for the Degree of
Doctor of Philosophy in Mathematics**

By

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Dedication

It is our genuine gratefulness and warmest regard that we dedicate this work

To my family and my friends

Acknowledgment

I would like to pay special thankfulness, warmth and appreciation to the persons below who made my research successful and assisted me at every point to cherish my goal:

My Supervisor, Dr. Tarig Elzaki for his vital support and assistance. His encouragement made it possible to achieve the goal.

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My Mother and Father, family members and friends, without whom I was nothing; they not only assisted me financially but also extended their support morally and emotionally.

Abstract

During the recent years, the studies and researches concentrated on the topic of Homotopy perturbation method, for being one the modern and effective methods for solving miscellaneous types of differential equations, ordinary or partial, linear or nonlinear. Homotopy perturbation was paid much attention by many searchers, it has become a fruitful field for study and research, and for this reason the main goal of this thesis is to study a class of partial differential equations by using Homotopy perturbation method. This method was introduced by Ji-Huan He (1999) and has gone through many modification and development which allowed researchers to apply it on various problems.

The necessary papers have been collected for the topic of the study and summarizing the results of the study and present the chapters of the thesis that included a general introduction and five chapters through which we sought present the basic concepts necessary for understanding the content of the thesis.

الخلاصة

فى السنوات الأخيرة تنامت الدراسات والأبحاث حول موضوع طريقة الأضطراب الهموتوبى، لكونها من الطرق الحديثة والفعالة لحل أنماط متنوعة من المعادلات التفاضلية العادية والجزئية، الخطية وغير الخطية على حد سواء، ونالت إهتمام كثير من الباحثين فأصبحت مجالاً خصباً للدراسة والبحث ولهذا السبب فإن هذه الرسالة تهدف بشكل رئيسى الى دراسة تطبيق طريقة الأضطراب الهموتوبى فى مجال المعادلات التفاضلية الجزئية.

ومما هو جدير بالذكر، انه منذ بدايات 1999 م ظهرت طريقة الأضطراب الهموتوبى والتي تطوراً مذهباً بعدما ارسى مفاهيمها العالم (Ji-Huan He) ، ولقد لعبت هذه الطريقة دوراً مهماً ورئيسياً فى إيجاد حلول كثير من المسائل فى المجالات المختلفة.

تم جمع الأبحاث اللازمة لموضوع الدراسة، وتلخيص ماورد فيها من نتائج وعرض نتائجها من خلال أبواب الرسالة التي تضمن مقدمة عامة وخمسة أبواب حرصنا من خلالها على تقديم المفاهيم الأساسية اللازمة لفهم محتوى الرسالة .

Introduction

The differential equations have been a branch of modern mathematics since the nineteenth and the twentieth centuries and ever since they have been of interest to many of the leading mathematicians as they are regarded as a vitally important tool in the mathematical library by which solutions to the problems of interest can be found elegantly and then translated into the real world as a written material that contributes to solving the problems in question in a way that achieves the desired interest. Thus, they play a key role in the translation of many natural and physical phenomena into mathematical models that can be studied from a purely mathematical perspective so that suitable solutions can be obtained mathematically and then translated back into practical, real-life applications. Consequently, a clear picture about the possible solutions and Available options is obtained

Recent years have seen a new approach with various ways to the natural and physical phenomena that translated into initial and boundary value problems.

It is well-known that perturbation and asymptotic approximations of nonlinear problems often break down as nonlinearity becomes strong. Therefore, they are only valid for weakly nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) in general.

The homotopy perturbation method (HPM) is an analytic approximation method for highly nonlinear problems, proposed by Ji-Huan He [1] in 1999. It is coupling method of a homotopy technique and a perturbation technique. In contrast to the traditional perturbation methods, the (HPM) method does not require a small parameter in the equation. In this method, according to the homotopy technique, a homotopy with an embedding parameter $p \in [0,1]$ is constructed and the embedding parameter is considered as a “small parameter”. Thus, the (HPM) can take full advantage of the traditional perturbation methods.

Secondly, different from all of other analytic techniques, the HPM provides us a convenient way to guarantee the convergence of solution series so that it is valid even if the nonlinearity becomes rather strong. Besides being based on the homotopy in the topology, it provides us with extremely large freedom to choose the equation type of linear sub-problems, base function of solution, initial guess and, As a result, complicated nonlinear ODEs and PDEs can often be solved in a simple way. In short, the HPM provides us a useful tool for solving highly nonlinear problems in science, finance and engineering.

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CHAPTER ONE

BASIC CONCEPTS OF HOMOTOPY PERTURBATION METHOD

1.1: Concept and Definitions of Homotopy

The Homotopy perturbation method (HPM), He's [1,2,3,4,5,6,7,8,9] proposed by Ji-Huan He [1] is based on the concept of the homotopy, a fundamental concept in topology and differential geometry (Armstrong, 1983). The concept of the homotopy can be traced back to rules Henri Poincare (1854 – 1982), a French mathematician, shortly speaking, a homotopy describes a kind of continuous variation or deformation in mathematics. For example, a circle can be continuously deformed into a Square or an Ellipse, the shape of a coffee cup can deform continuously into the shape of a doughnut. However the shape of a coffee cup cannot be distorted continuously into the shape of a football, essentially a homotopy defines a connection between different things in mathematics, which contain some characteristics in some aspects.

Definition (1.1.1)

Let X and Y be a topological space, if $f, g : X \rightarrow Y$ are continuous maps, is said that f is homotopic to g if there exists a continuous map $H : X \times [0,1] \rightarrow Y$ $H(x, y) \in Y, x \in X, 0 \leq p \leq 1, H(x,0) = f(x), H(x,1) = g(x)$, Such that the map is called a homotopy between f and g , $f \sim g$, denotes that f and g are homotopic.

We think a homotopy as a continuous one-parameter family of maps from X to Y imagine the parameter p as representing time, then the homotopy represents a continuous deforming of the map f to the map g as p goes from 0 to 1.

Definition (1.1.2)

Let $C[a,b]$ denote the a set of all continuous real functions in the interval $a \leq x \leq b$, in general, if a continuous function $f \in C[a,b]$ can be deformed continuously into another continuous function $g \in C[a,b]$, one constant a homotopy

$$H : f(x) \sim g(x)$$

In the way

$$H(x; p) = (1 - p)f(x) + pg(x) \tag{1}$$

However a continuous real function cannot be deformed continuously into a discontinuous function, for example $\sin x$ cannot be deformed continuously into step function

$$S(x) = \begin{cases} 1, & x < 0 \\ 0, & x = 0 \\ -1, & x > 0 \end{cases}$$

Definition (1.1.3)

The embedding parameter $p \in [0,1]$ in a homotopy of functions or equations is called homotopy parameter.

The concept of homotopy defined above for functions can be easily expanded to the equations.

Example (1.1.4)

The two different real functions $\sin(fx)$ and $8x(x-1)$ in interval $x \in [0,1]$ can be connected by constructing such a family of function

$$H(x; p) = (1 - p)\sin(fx) + p[8x(x-1)] \tag{2}$$

where the embedding parameter $p \in [0,1]$

We note that $H(x; p)$ depends on not only the independent variable $x \in [0,1]$ but also the embedding parameter $p \in [0,1]$. Especially, $p = 0$ we have

$$H(x;0) = \sin(fx), \quad x \in [0,1]$$

And when $p = 1$, it holds

$$H(x;1) = 8x(x-1), \quad x \in [0,1]$$

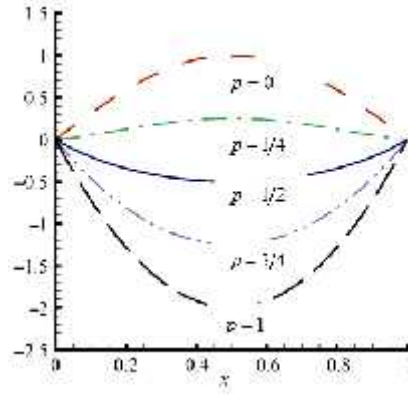
Respectively, so as the embedding parameter $p \in [0,1]$ increase from 0 to 1, the real function $H(x; p)$ varies continuously from a trigonometric function $\sin(fx)$ to a polynomial $8x(x-1)$, as shown in Fig. (1.1.5) then $H(x; p)$ homotopy, $\sin(fx)$ and $8x(x-1)$ are homotopic denoted by

$$H : \sin(fx) \sim 8x(x-1)$$

Fig (1.1.5) continuous deformation of the homotopy

$$H : \sin(fx) \sim 8x(x-1)$$

Dashed Line; $p = 0$ Dash dotted Line $p = \frac{1}{4}$, Solid line; $p = \frac{1}{2}$ Dash-double dotted. Line; $p = \frac{3}{4}$ long dashed; $p = 1$



Definition (1.1.6)

Given an equation denoted by v_1 which has at least one solution u , Let v_0 denote a proper, simpler equation, called the initial equation, whose solution u_0 is known, if one can construct a homotopy of equation $\tilde{v}(p): v_0 \sim v_1$, such that, as the homotopy-parameter $p \in [0,1]$ increases from 0 to 1, $\tilde{v}(p)$ deforms (or varies) continuously from the initial equation v_0 to the original equation v_1 , while its solution varies continuously from the known solution u_0 of v_0 to the unknown solution u of v_1 , then this kind of homotopy of equations is called the zeroth-order deformation equation.

Example (1.1.7)

Let us consider such a family of algebraic equations

$$v(p) : (1+3p)x^2 + \frac{y^2}{(1+3p)} = 1, \quad p \in [0,1] \tag{3}$$

Where $p \in [0,1]$ is the embedding parameter, when $p=0$, we have a circle equation

$$v_0 : x^2 + y^2 = 1 \tag{4}$$

Whose solution is a circle $y = \pm\sqrt{1-x^2}$, when $p=1$ we have the ellipse equation

$$v_1 : 4x^2 + \frac{y^2}{4} = 1 \tag{5}$$

Whose solution is an ellipse $y = \pm 2\sqrt{1-4x^2}$.

Thus, as the embedding parameter p increases from 0 to 1, Eq. (3) varies continuously from a circle equation v_0 into the ellipse equation v_1 , while its solution y deforms continuously from a circle $y = \pm\sqrt{1-x^2}$ to the ellipse $y = \pm 2\sqrt{1-4x^2}$, as shown in Fig (1.1.8).

So, more precisely speaking, the solution y of (3) is dependent not only on x but also on $p \in [0,1]$, and thus (3) should be expressed more precisely in the form

$$v(p): (1+3p)x^2 + \frac{y^2(x,p)}{(1+3p)} = 1, \quad p \in [0,1] \quad (6)$$

Which defines two homotopies: one is homotopy of the equation

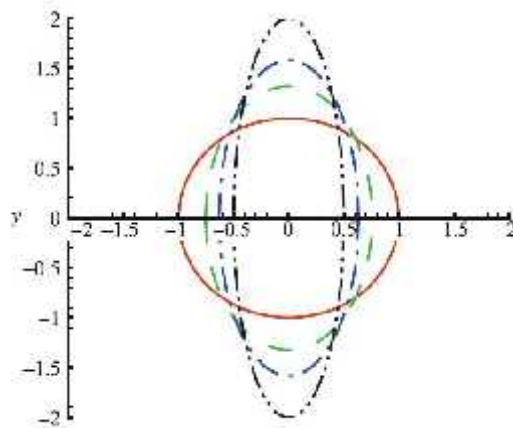
$$v(p): v_0 \sim v_1$$

Where v_0 and v_1 denote (4) and (5), respectively, the other is homotopy of function,

$$y(x,p) : \pm\sqrt{1-x^2} \sim \pm 2\sqrt{1-4x^2}$$

In other words, the solution $y(x,p)$ is also homotopy notice that such kind of continuous deformation is completely defined by (6), we call (6) the zeroth-order deformation equation, the same idea can be easily extended to other types of equations, such as differential equations, integral equations and so on.

Fig (1.1.8) Consider deformation of equation of the solution $y(x,p)$ of the homotopy (3) solid line: $p = 0$ Dashed Line: $p = \frac{1}{4}$ Dash-dotted Line $p = \frac{1}{2}$ Dash-double-dotted Line: $p = 1$.



Note that we can construct many different homotopies which connect the circle equation (4) and the ellipse equation (5) for example the following zeroth order deformation equation

$$v(x, \sim) : (1 + 3p^\sim)x^2 + \frac{y^2(x; p)}{(1 + 3p^\sim)}, \quad p \in [0,1] \quad (7)$$

Where v_0 and v_1 denote (4) and (5) respectively. For different values of \sim , it define a different homotopy since $\sim \in (0, +\infty)$, there exists an infinite number of different homotopies of equations, which connect the circle of equation (4) and the ellipse of equation (5), and correspondingly, an infinite number of homotopies of functions which connect the circle

$$y = \pm\sqrt{1-x^2} \quad \text{And the ellipse} \quad y = \pm 2\sqrt{1-4x^2}$$

This illustrates the great flexibility of constructing a homotopy for given two homotopic functions or equation. All of these belong to the basic concepts in topology a differential geometry (Armstrong [10]). Some new concepts can be derived not that the homotopy

$$H(x; p) = (1 - p)\sin(fx) + p[8x(x - 1)]$$

Can be rewritten in the form

$$H(x; p) = \sin(fx) + p[8x(x - 1) - \sin(fx)]$$

And we have

$$\frac{\partial H(x, p)}{\partial p} = 8x(x - 1) - \sin x, \quad p \in [0,1] \quad (8)$$

Which describes the ratio (or the speed) of the continuous deformation from $\sin(fx)$ to $8x(x - 1)$, called the first order homotopy-derivative,

Definition (1.1.9)

The homotopy:

$$H(x; p) = (1 - p)f(x) + pg(x) \quad x \in [a, b]$$

Completely defines the corresponding first order homotopy-derivative

$$\frac{\partial H(x; p)}{\partial p} = g(x) - f(x), \quad p \in [0,1] \quad (9)$$

Unfortunately the (HPM) is based on the simple fundamental concept of homotopy, and other knowledge in topology is almost unnecessary.

1.2: Perturbation Theory

Many of the functions that arise from everyday problems cannot easily be evaluated exactly, particularly those defined in terms of integrals or differential equation, in these situations we usually have two options. We can use computer to seek complicated numerical solutions or we can look to construct an analytical approximation to the solution using asymptotic expansions, Asymptotic method has particular importance in many areas of applied mathematics, with the physical problems studied in fluid dynamics providing the main motivation for much of the important development in the subject history.

Henri Poincare' who introduced the term asymptotic expansion during 1886 [11] studying irregular integrals of linear equations. In this section, we will focus on the methods applicable to problems presented as differential equations, particularly the area of regular and singular perturbation theory.

In the classical asymptotic analysis the asymptotic variable is taken as the independent variable of the differential equation, in the perturbation theory, the asymptotic behavior is studied with respect to the small physical parameter.

Perturbation theory deals with problems that contain a small parameter conventionally denoted by ν , solutions are sought as ν , approaches 0.

1.2.1: Regular Perturbation

The general method with perturbation problems is to seek an expansion with respect to asymptotic sequence $\{1, \nu, \nu^2, \dots\}$ as $\nu \rightarrow 0$, the regular or (Poincare') expansion is then

$$U(\nu, x) = U_0(x) + \nu U_1(x) + \nu^2 U_2(x) + \dots \quad \text{as } \nu \rightarrow 0$$

For gauge function U_0, U_1 , which we will determine

Example (1.2.10) Consider the initial value problem

$$\begin{aligned} \frac{d^2 y}{dt^2} &= -\nu \frac{dy}{dt} - 1 & (10) \\ y(0) &= 0, \quad \frac{dy}{dt}(0) = 1 \end{aligned}$$

The equation here represents projectile motion where air friction taken into account $\nu = \frac{kv_0}{mg}$. Assuming a solution expanded in terms of ν by Taylor

$$y(t) = y_0(t) + \nu y_1(t) + \nu^2 y_2(t) + \dots \quad (11)$$

This is now substituted in the differential equation and initial conditions (10) to determine function y_0, y_1 and y_2 give a 3-term expansion of course this can be carried out to finite as many terms of the expansion as necessary but in

practiced situations only a small number of terms are usually needed substituting gives, after rearranging

$$\begin{aligned} \frac{d^2 y_0}{dt^2} + 1 + v \left(\frac{d^2 y_1}{dt^2} + \frac{dy_0}{dt} \right) + v^2 \left(\frac{d^2 y_2}{dt^2} + \frac{dy_1}{dt} \right) + O(v^3) &= 0, \\ y_0(0) + v y_1(0) + v^2 y_2(0) + O(v^3) &= 0, \\ \frac{dy_0(0)}{dt} - 1 + v \frac{dy_1(0)}{dt} + v^2 \frac{dy_2(0)}{dt} + O(v^3) &= 0, \end{aligned}$$

The next step is then to equate to zero all the terms of each order of v

$$\begin{aligned} v^0 : \frac{d^2 y_0}{dt^2} + 1 &= 0, \quad y_0(0) = 0, \quad \frac{dy_0(0)}{dt} = 1, \\ v^1 : \frac{d^2 y_1}{dt^2} + \frac{dy_0}{dt} &= 0, \quad y_1(0) = 0, \quad \frac{dy_1(0)}{dt} = 0, \\ v^2 : \frac{d^2 y_2}{dt^2} + \frac{dy_1}{dt} &= 0, \quad y_2(0) = 0, \quad \frac{dy_2(0)}{dt} = 0, \end{aligned}$$

Solving these equations gives,

$$\begin{aligned} y_0(t) &= t - \frac{t^2}{2}, \\ y_1(t) &= -\frac{t^2}{2} + \frac{t^3}{6}, \\ y_2(t) &= \frac{t^3}{6} - \frac{t^4}{24}, \end{aligned}$$

Now putting these into Eq. (10), gives third approximation

$$y(t) \sim \left(t - \frac{t^2}{2!} \right) + v \left(-\frac{t^2}{2!} + \frac{t^3}{3!} \right) + v^2 \left(\frac{t^3}{3!} - \frac{t^4}{4!} \right) \quad (12)$$

The exact of Eq. (10), is

$$y(t) = \frac{(1+v)}{v^2} (1 - e^{-vt}) - \frac{t}{v} \quad (13)$$

We can expand this as a Taylor series gives,

$$y(t) = \left(t - \frac{t^2}{2!} \right) + v \left(-\frac{t^2}{2!} + \frac{t^3}{3!} \right) + v^2 \left(\frac{t^3}{3!} - \frac{t^4}{4!} \right) + O(v^3) \quad (14)$$

Noticing this identical the solution obtain in (11) using Perturbation method above.

1.2.2: Singular Perturbation

A Perturbation problem is said to be singular when the regular methods produce an expansion that fails at some point, to be valid over the complete domain. To introduce a singular perturbation type problem we look to the previous example.

Example (1.2.11)

Consider the problem

$$vx^2 + 2x - 1 = 0 \quad (15)$$

We note that v here is the coefficient of the leading order term x^2 . Following the regular expansion,

$$x \sim x_0 + vx_1 + v^2x_2 + \dots \quad (16)$$

And equating the coefficients gives as the solution,

$$x \sim \frac{1}{2} - \frac{v}{8} + \dots \quad (17)$$

Clearly the regular method has failed. The problem is quadratic which has two solutions, but only one have produced. In many cases, this situation is easy to spot by setting $v = 0$ to give the unperturbed equation when v is the leading order term's sole coefficient the equation is reduced in the unperturbed equation, in this example to a linear equation with only one solution.

There are several types of singular perturbation problem that all require a different method to lackey them, two of the most common and widely applicable method, Matched asymptotic expansions and the method of multiple scales.

When v is the multiplier of the highest derivatives or leading term of a polynomial equation it is known as a boundary layer problem or occasionally a matching problem.

1.2.3: Matched Asymptotic Expansions

In the method of matched asymptotic expansions can be useful for differential equations with an v coefficient multiplying the highest order derivative usually these contain a boundary layer preventing the complete set of boundary conditions being satisfied by regular perturbation solution where the regular solution fails we introduce new coordinates to describes the solution inside the boundary layer and produce two separate approximation valid over different sections of the domain, these solutions must be matched together and combined to single expansion valid universally [11].

1.2.4: Method of Multiple Scales

A second type of singular perturbation problem fails not due the loss of the leading order term, but instead these problems fail to be valid when the independent variable becomes large in the unbounded domain. Problems like this are common in system dependent on time, thus an approximation found may be valid initially but will deviate of multiple scales are to introduce two time scales, a fast one t_2 and a slow on t_1 , expand a regular perturbation solution in term of this new coordinates, the secular terms found in each stage can be suppressed by equation the arbitrary functions from one term in the expansion with next. Thus we have single solution valid over the complete domain that can easily be expanded with lower order terms where desired [11].

1.3: Homotopy Perturbation method

After the appearance of supercomputers, it is not difficult for us to find the solution of linear problems. It is however still difficult to solve nonlinear problems, especially by means of analytical methods. Although the nonlinear analytical techniques are fast developing, they still do not completely satisfy mathematicians and engineers.

Until recently, nonlinear analytical techniques for solving nonlinear problems have been dominated by perturbation methods, which have found wide applications in engineering. But like other nonlinear techniques, perturbation methods have their own limitations. Firstly, almost all perturbation methods are based on small parameters so that the approximate solution can be expressed in a series of small parameters. This so-called small parameter assumption greatly strictest application of perturbation techniques, as is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to ideal results, however, an unsuitable choice of small parameters result in bad effects, sometimes seriously. Thirdly, even if there exist suitable parameters, the approximate solutions solved by the perturbation methods are valid, in most cases, only for the small values of the parameters.

It is obvious that all these limitation come from the small parameter assumption.

So it is very necessary to develop a kind of new non-linear analytical method which does not require small parameters at all.

To eliminate the small parameter assumption in 1997, Liu [11] proposes a new perturbation technique, where an artificial parameter is embedded in an equation at its appropriate place, and the embedding parameter is used as a

“small parameter”. Unfortunately, there is an uncertainty about an appropriate artificial parameter and often enough the approximation obtained by such method will not be uniform, so that its applicability range is severely limited. Just recently in order to be freed from the limitation of “small parameter” assumption, Liu [11,12] proposes a new technique which base on homotopy in topology, does not require small parameter in equations, using the interesting property of homotopy, he transforms a nonlinear problem into an initial number of linear problem without using the perturbation techniques. To illustrate Lu’s basic idea of artificial parameter consider the following example.

Example (1.3.12)

Consider the following differential equation [11]:

$$\frac{du(t)}{dt} + u^2(t) = 1$$

With initial condition:

$$u(0) = 0 \tag{18}$$

Embedding an artificial parameter B in Eq. (18) resulting,

$$\frac{du(t)}{dt} = (1-u)(1+Bu) \tag{19}$$

In Liu’s method the embedding parameters are considered as small parameter.

Assume the solution in the following

$$u(t, B) = u_0(t) + Bu_1(t) + \dots \tag{20}$$

Substituting Eq. (20) in Eq. (19) equating the term of like power B , as resulting, Liu obtained the following first-order approximation

$$u(t, B) = u_0(t) + Bu_1(t) = (1 - e^{-t}) + Be^{-t}(e^{-t} + t - 1) \tag{21}$$

The substituting $B = 1$ resulting a good approximate solution of the original Eq. (18).

In Liu’s method, however, the artificial parameters are embedded much artificially or technically in most cases, the method will fail to obtain a uniformly valid approximation, for example, if we embed the artificial parameter as follows:

$$\frac{du(t)}{dt} = (1 - Bu)(1 + u) \tag{22}$$

Or

$$\frac{du}{dt} + Bu^2(t) = 1 \tag{23}$$

The approximate solutions obtained from Eq. (22) or Eq. (23) will not be uniformly valid. The problem lies on the fact that the artificial parameters can in no way be considered as a small parameter!

It thus becomes desirable to adjust the perturbation approach in such a manner that the embedding parameters are always small.

To this end, we will give a heuristical method based on the homotopy in topology [12,13]. The homotopy technique or the continuous mapping technique embeds a parameter p that typically ranges from zero to one, when the embedding parameter is zero, the equation is one of a linear system, when it is one; the equation is the same as the original one. So the embedded parameter $p \in [0,1]$ can be considered as a small parameter. That homotopy constructs universal perturbation equation with an appropriate artificial parameter. The coupling method of the homotopy technique and the perturbation technique is called the homotopy perturbation method, noted (HPM) was proposed by Ji-Huan He in [1] and [2-9]. More details will be discussed below.

1.3.1: Basic Idea of Homotopy Perturbation Method

To illustrate the basic ideas of the (HPM), we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (24)$$

With the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \quad (25)$$

Where A is a general differential operation, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω .

The operator A can generally speak, be divided in two parts L and N , where L is linear, while N is non linear, Eq. (24) therefore, can be rewritten as follows

$$L(u) + N(u) - f(r) = 0 \quad (26)$$

By homotopy technique [13,14], we construct a homotopy $v(r, p) : \Omega \times [0,1] \rightarrow R$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega \quad (27a)$$

Or

$$H(v, p) = L(v) - L(u_0) + p[N(v) - f(r)] = 0 \quad (27b)$$

Where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq.(24) which satisfies the boundary conditions, Eq.(27a) or Eq.(27b) is called perturbation equation with embedding parameter obviously from Eq.(27) we have

$$H(v,0) = L(v) - L(u_0) = 0, \quad (28)$$

$$H(v,1) = A(v) - f(r) = 0. \quad (29)$$

The changing process of p from zero to unity is just that of $v(r, p)$ from trivial solution $u_0(r)$ to original solution $u(r)$, in topology this is called deformation, and $L(v) - L(u_0)$, $A(v) - f(r)$, are called homotopic.

Here the imbedding parameter p can be considered as “small parameter”

Assume that the solution of Eq. (26) can be written as a power series in p

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (30)$$

Setting $p = 1$ result in the approximate solution of Eq. (24)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (31)$$

The coupling of the perturbation method and the homotopy method is called homotopy perturbation method, which has elimination limitations of the traditional perturbation methods. On the other hand the (HPM) can take full advantage of the traditional perturbation techniques.

Example (1.3.13)

Let us first consider a nonlinear algebraic equation [2]

$$f(x) = 0, \quad x \in R. \quad (32)$$

To solve Eq. (32) by (HPM) we construct a homotopy $R \times [0,1] \rightarrow R$ which satisfies

$$H(\langle, p) = (1 - p)[f(\langle) - f(x_0)] + pf(\langle) = 0, \quad x \in R, \quad p \in [0,1] \quad (33a)$$

$$H(\langle, p) = f(\langle) - f(x_0) + pf(x_0) = 0, \quad x \in R, \quad p \in [0,1] \quad (33b)$$

Where x_0 is initial approximation of Eq. (32) it is obvious that

$$H(\langle, 0) = f(x) - f(x_0) = 0$$

$$H(\langle, 1) = f(\langle) = 0$$

The changing process of p from zero to unity is just that of $H(\langle, p)$, from $f(\langle) - f(x_0)$ to $f(\langle)$ and $f(\langle) - f(x_0)$, $f(\langle)$ are homotopic.

Applying the perturbation technique, we can assume that the solution of Eq. (33a) and (33b) can be expressed as a series in p

$$\langle = \langle_0 + p\langle_1 + p^2\langle_2 + \dots \quad (34)$$

To obtain it is the approximate solution of Eq. (33), we first expand $f(\zeta)$ into a Taylor series

$$f(\zeta) = f(\zeta_0) + f'(\zeta_0)(p\zeta_1 + p^2\zeta_2 + \dots) + \frac{1}{2!}f''(\zeta_0)(p\zeta_1 + p^2\zeta_2 + \dots)^2 + \dots \quad (35)$$

Substituting Eq. (35) into Eq. (33) and equating the coefficients of like powers of p , we obtain

$$p^0 : f(\zeta_0) - f(x_0) = 0, \quad (36)$$

$$p^1 : f'(\zeta_0)\zeta_1 + f(x_0) = 0, \quad (37)$$

$$p^2 : f'(\zeta_0)\zeta_2 + \frac{1}{2!}f''(\zeta_0)\zeta_1^2 = 0. \quad (38)$$

From Eq. (37) ζ_1 can be solved

$$\zeta_1 = -\frac{f(x_0)}{f'(\zeta_0)}. \quad (39)$$

If, for example, its first-order approximation is sufficient, then we have

$$\zeta = \zeta_0 - p\frac{f(\zeta_0)}{f'(\zeta_0)} \quad (40)$$

Then substitution $p = 1$ in Eq. (28) yields the first order approximate solution of Eq. (26)

$$x = \zeta_0 - \frac{f(\zeta_0)}{f'(\zeta_0)} \quad (41)$$

Using Eq. (41) as an initial approximation in Eq. (32) repeatedly, we have the following iteration formula:

$$x_{n+1} = \zeta_n - \frac{f(\zeta_n)}{f'(\zeta_n)} \quad (42)$$

From Eq. (36) we can obtain one of its solutions $\zeta_0 = x_0$, under this condition Eq. (42) can be re-written down as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (43)$$

Which is the well known Newton iteration formula.

By the same manipulation, from Eq. (38), ζ_2 can be solved, and the following formula can be obtained

$$x_{n+1} = \zeta_n - \frac{f(\zeta_n)}{f'(\zeta_n)} - \frac{f''(\zeta_n)}{2f'(\zeta_n)} \left\{ \frac{f(\zeta_n)}{f'(\zeta_n)} \right\}^2 \quad (44)$$

The iteration formula (44) is called Newton-Like iteration formula with second-order approximation.

The approximate solution obtained by the above iteration formula (44) converges to its exact solution faster than the Newton iteration formula (43) for example,

$$f(x) = x^2 + x - 2 = 0 \quad (45)$$

Supposing $x_0 = 0$ be one of its initial approximate solution, from Eq. (36) we have $\kappa_0^{(1)} = 0$ and $\kappa_0^{(2)} = -1$. By Newton-Like iteration formula (44) we can immediately obtain its exact solutions $x_1^{(1)} = -2$ and $x_1^{(2)} = 1$ by only one iteration step.

Example (1.3.14) Consider the simple ordering differential equation

$$y' + y^2 = 0, \quad x > 0, \quad x \in \Omega, \quad y(0) = 1 \quad (46)$$

We construct the following homotopy

$$(1-p)(Y' - y_0') + p(Y' + Y^2) = 0 \quad (47)$$

Suppose the solution of equation Eq. (47) has the form

$$y = Y_0 + pY_1 + p^2Y_2 + \dots \quad (48)$$

Substituting (48) into (47) and equating the form with identical powers of p

$$\begin{aligned} p^0 : Y_0' &= y_0' \\ p^1 : Y_1' + y_0' + Y_0^2 &= 0, \quad Y_1(0) = 0 \\ p^2 : Y_2' + 2y_0y_1 &= 0, \quad Y_2(0) = 0 \\ &\vdots \end{aligned} \quad (49)$$

For simplicity we start with initial approximation $Y_0 = y_0 = 1$ and solving above system, we get,

$$Y_1 = -x, \quad Y_2 = x^2$$

Then we have the second order approximation of Eq. (46)

$$Y = 1 - px + p^2x^2 + \dots \quad (50)$$

And the exact solution given by;

$$y = \lim_{p \rightarrow 1} Y = 1 - x + x^2 + \dots = \frac{1}{1+x} \quad (51)$$

Example (1.3.15) Consider the partial differential equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad (52)$$

with initial condition

$$u(x,0) = 2x \quad (53)$$

and boundary condition

$$u(0,t) = 0, \quad u_x(0,t) = \frac{2}{1+2t} \quad (54)$$

To solve Eq. (52) with initial condition (53), i.e.; (t -solution) we construct the following homotopy;

$$(1-p) \left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right) + p \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} \right) = 0$$

Or

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} + \frac{\partial u_0}{\partial t} \right) = 0 \quad (55)$$

Assume the solution of Eq. (55) has the form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (56)$$

Substituting Eq. (56) and (53) into Eq. (55) and equation the terms of Like Power of p ,

$$\begin{aligned} p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} \\ p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_0}{\partial x^2} = 0, \quad v_1(x,0) = 0 \\ p^2 : \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} = 0, \quad v_2(x,0) = 0 \\ \vdots \end{aligned} \quad (57)$$

Start with $v_0(x,t) = u_0(x,t) = u_0(x,0) = 2x$, so we derive the following

$$\begin{aligned} v_1 &= \int_0^t \left(-\frac{\partial u_0}{\partial t} - v_0 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_0}{\partial x^2} \right) dt = -4xt, \\ v_2 &= \int_0^t \left(-v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} \right) dt = 8xt \\ &\vdots \end{aligned}$$

The approximation solution of Eq. (52)

$$u = \lim_{p \rightarrow 1} v = 2x - 4xt + 8xt + \dots \quad (58)$$

And in closed form

$$u(x,t) = \frac{2x}{1+2t} \quad (59)$$

Which is an exact solution

Similarly, to solve Eq. (52) in the (x -direction) with boundary conditions (54) we construct the following homotopy

$$(1-p) \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} \right) + p \left(\frac{\partial^2 v}{\partial x^2} + v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} \right) = 0$$

Or

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} + p \left(\frac{\partial v}{\partial t} - v \frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial t} \right) = 0 \quad (60)$$

Substituting Eq.(56) into Eq. (60) and Eq.(54) equation the terms of Like power p ,

$$\begin{aligned}
p^0 : \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial t^2} &= 0 \\
p^1 : \frac{\partial^2 v_1}{\partial x^2} - \frac{\partial v_0}{\partial t} - v_0 \frac{\partial v_0}{\partial x} + \frac{\partial^2 u_0}{\partial t^2} &= 0, \quad v_1(0, t) = v_{1x}(0, t) \\
p^2 : \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial v_1}{\partial t} - v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} &= 0, \quad v_2(0, t) = v_{2x}(0, t) = 0 \\
&\vdots
\end{aligned} \tag{61}$$

Start with $v_0(x, t) = u_0(x, t) = \frac{2x}{1+2t}$

And we derive the following

$$\begin{aligned}
v_1 &= \int_0^x \int_0^x \left(\frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} - \frac{\partial^2 u_0}{\partial t^2} \right) dx dx = 0 \\
v_k &= 0, \quad k \geq 0
\end{aligned}$$

Then the exact solution

$$u(x, t) = v_0 = \frac{2x}{1+2t}$$

is the same solution given by t -direction.

Example (1.3.16) Fredholm integral equation

Now we consider the Fredholm integral equation of the second kind in the general case

$$u(x) = f(x) + \int_a^b K(x, t)u(t)dt \tag{62}$$

To solve Eq. (62), we construct the following homotopy

$$(1-p)[u(x) - f(x)] + p \left[u(x) - f(x) - \int_a^b K(x, t)u(t)dt \right] = 0$$

Or

$$u(x) = f(x) + p \int_a^b K(x, t)u(t)dt \tag{63}$$

Assume the solution of Eq. (63) has the form

$$u = u_0 + pu_1 + p^2u_2 + \dots \tag{64}$$

Substituting Eq. (64) into Eq. (63) and equate the terms of the Like power p , we have

$$\begin{aligned}
p^0 : u_0 &= f(x) \\
p^1 : u_1 &= \int_a^b K(x, t)(u_0)dt
\end{aligned}$$

$$\begin{aligned}
p^2 : u_2 &= \int_a^b K(x,t)(u_1)dt \\
&\vdots \\
p^j : u_j &= \int_a^b K(x,t)(u_{j-1})dt
\end{aligned} \tag{65}$$

The approximation solution given by setting $p = 1$ in Eq. (64)

$$u = u_0 + u_1 + u_2 + \dots \tag{66}$$

Example (1.3.17) Consider the integral equation

$$u(x) = \sqrt{x} + \int_0^1 xt u(t)dt \tag{67}$$

In view of Eq. (63), we obtain

$$u(x) = \sqrt{x} + p \int_0^1 xt u(t)dt \tag{68}$$

Substituting Eq. (64) into Eq. (68), we have the following result

$$\begin{aligned}
p^0 : u_0(x) &= \sqrt{x} \\
p^1 : u_1(x) &= \int_0^1 xt \sqrt{t} dt = \frac{2}{5} x \\
p^2 : u_2(x) &= \int_0^1 xt \cdot \frac{2}{5} t dt = \frac{2}{15} x^2 \\
p^3 : u_3(x) &= \int_0^1 xt \cdot \frac{2}{15} t^2 dt = \frac{2}{45} x^3 \\
&\vdots
\end{aligned} \tag{69}$$

Then the solution obtains by setting $p = 1$ in Eq. (64)

$$\begin{aligned}
u &= u_0 + u_1 + u_2 + \dots \\
&= \sqrt{x} + \left[\frac{2}{5} + \frac{2}{5.3} \right] x^2 + \frac{2}{5.3^2} x^3 + \dots \\
&= \sqrt{x} + \left[\frac{5}{6} \sum_{i=1}^n \left(\frac{1}{3} \right)^i \right] x^2
\end{aligned} \tag{70}$$

1.3.2: The Advantages of the Homotopy Perturbation Method

The homotopy perturbation method has been receiving much attention in recent years in applied mathematics, in general and particular in the area of series solution. The method to be powerful effective and easy to use, It was formally shown by many researchers that the advantage of the HPM.

The perturbation equation can be easily constructed by homotopy in topology and the embedding parameter $p \in [0,1]$ is considered as “perturbation parameter” the novel method can take full advantage of the traditional perturbation techniques. The initial approximation can be freely selected which can be identified via various methods. The approximation obtained by this method are valid not for small parameter but also for the very large parameter. Also, the homotopy perturbation method can easily handle a wide class of algebraic equation, ordinary differential equations, partial differential equation, integral equations, integral differential equation and fractional equation homogeneous or inhomogeneous and linear or nonlinear in a straightforward manner without any need for restrictive assumptions, such as linearization or discretion. There is no need in using this method to convert inhomogeneous conditions to homogenous conditions are required by other techniques. The HPM requires less computational work if compared with other methods, and demonstrates a fast convergence of the solution.

A disadvantage of the HPM is to need an initial value

1.4: The Noise Terms Phenomenon

The noise terms phenomenon [15,16] gives useful tool in that, if it appears, it gives a fast convergence of the solution by using two iterations only it is significant to note that the noise terms may appear only for the inhomogeneous problems.

The noise terms defined as an identical terms, with opposite signs that may appear in various components $u_k, k \geq 1$, it is important to note that these terms may appear for inhomogeneous problem whereas homogenous problems do not generate noise terms. It was formally shown that by canceling the noise terms that appears in u_0 and u_1 from u_0 , even though u_1 contains further terms, the remaining non-cancelled terms of u_0 may give the exact solution of an inhomogeneous problem. This can be justified through substitution. Therefore, it is necessary to verify that the non-cancelled terms of u_0 satisfying PDE under discussion. A necessary condition for the generation of the noise terms of inhomogeneous problems is that the zeroth component u_0 must contain the exact solution u among other terms.

On the another hand, if the non-cancelled terms of u_0 did not satisfy the given problem or the noise term did not appear between u_0 and u_1 , then it is necessary to determine more components of u to determine the solution in a series form.

Example (1.4.18) Consider the following inhomogeneous PDE [18]

$$u_x + u_y = (1+x)e^y, u(0, y) = 0, u(x, 0) = x \quad (71)$$

Clearly, x -direction is invertible and therefore to solve Eq. (71) we construct the following homotopy:

$$\frac{\partial v}{\partial x} - \frac{\partial u_0}{\partial x} = p \left[(1+x)e^y - \frac{\partial v}{\partial y} - \frac{\partial u_0}{\partial x} \right] \quad (72)$$

Assume the solution of Eq. (72) has the following form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (73)$$

Substituting Eq. (73) into Eq. (72) and equating the terms with Like power p ,

$$\begin{aligned} p^0 : \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial x} &= 0 \\ p^1 : \frac{\partial v_1}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial u_0}{\partial x} &= (1+x)e^y, v_1(0, y) = 0 \\ p^2 : \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} &= 0, v_2(0, y) = 0 \\ p^3 : \frac{\partial v_3}{\partial x} + \frac{\partial v_2}{\partial y} &= 0, v_3(0, y) = 0 \\ &\vdots \end{aligned} \quad (74)$$

Start with $v_0(x, y) = u_0(x, y) = u_0(0, y) = 0$ and integrating above system with

$\int_0^x (\circ) dx$ we get,

$$\begin{aligned} v_1 &= \int_0^x \left(-\frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial x} + (1+x)e^y \right) dx = \left(x + \frac{x^2}{2!} \right) e^y, \\ v_2 &= \int_0^x \left(-\frac{\partial v_1}{\partial y} \right) dx = - \left(\frac{x^2}{2!} + \frac{x^3}{3!} \right) e^y, \\ v_3 &= \int_0^x \left(-\frac{\partial v_2}{\partial y} \right) dx = \left(\frac{x^3}{3!} + \frac{x^4}{4!} \right) e^y, \\ &\vdots \end{aligned}$$

It is easily observed that the noise terms $\frac{x^2}{2!}e^y$ and $-\frac{x^2}{2!}e^y$ appear in the first two component respectively, by canceling the noise term $\frac{x^2}{2!}e^y$ in v_1 and verifying

that the remaining non-cancelled term of v_0 satisfy Eq.(71), we find that the exact solution is given by;

$$u(x,t) = xe^y \quad (75)$$

Notice that the exact solution is verified through substituting in Eq. (74) and not upon the appearance of the noise term, in addition, the other noise terms that appear between other components will vanish in the limit.

Example (1.4.19) Consider the following partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \cos x, \quad 0 < x < f \quad t > 0 \quad (76)$$

With initial condition

$$u(0, x) \quad (77)$$

And boundary condition

$$u(t,0) = 1 - e^{-t}, \quad u(t,f) = e^{-t}, \quad t \geq 0 \quad (78)$$

To solve Eq. (76) with initial condition (77) we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} - p \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial u_0}{\partial t} + \cos x \right) = 0 \quad (79)$$

Assume the solution of Eq. (79) has the form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (80)$$

Substituting Eq. (80) into Eq. (79) and (77) and equating the terms of the Like power p ,

$$\begin{aligned} p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} &= 0 \\ p^1 : \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial u_0}{\partial t} &= \cos x, \quad v_1(0, x) = 0 \\ p^2 : \frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} &= 0, \quad v_2(0, x) = 0 \\ p^3 : \frac{\partial v_3}{\partial t} - \frac{\partial^2 v_2}{\partial x^2} &= 0, \quad v_3(0, x) = 0 \\ &\vdots \end{aligned} \quad (81)$$

Start with $v_0(t, x) = u_0(t, x) = u_0(0, x) = 0$ and integrating above system with

$\int_0^t (\circ) dx$ we get,

$$\begin{aligned} v_1 &= \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t} - \cos x \right) = t \cos x, \\ v_2 &= \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} \right) = -\frac{t^2}{2!} \cos x, \end{aligned}$$

$$v_3 = \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} \right) = \frac{t^3}{3!} \cos x ,$$

$$\vdots$$

We can easily observed that the components does not contain noise terms this confirm our benefit that the PDE is an inhomogeneous equation but the noise terms between the first two components did not exist in this problem, then the series solution obtain by

$$u = \lim_{p \rightarrow 1} v = t \cos x - \frac{t^2}{2!} \cos x + \frac{t^3}{3!} \cos x + \dots \quad (82)$$

In closed form

$$u(x, t) = (1 - e^t) \cos x \quad (83)$$

Which is an exact solution

1.5: The Modified of the Homotopy Perturbation Method (MHPM)

In this section, we will present the modification of the homotopy perturbation method (MHPM) [17].

The (MHPM) demonstrate a rapid convergence of the series solution compared with standard HPM in addition the modified algorithm may give the exact solution for the problem by using two iterations only.

Now the standard HPM in Eq. (27) given by

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0$$

The modified form of the HPM can be established based on the assumption that the function $f(r)$ can be divided in two parts, namely $f_0(r)$ and $f_1(r)$

$$f(r) = f_0(r) + f_1(r) \quad (84)$$

Or on the assumption that the function $f(r)$ can be replaced by a series of infinite components under this assumption that $f(r)$ be expressed in Taylor series

$$f(r) = \sum_{n=0}^{\infty} f_n(r) \quad (85)$$

According to the first assumption $f(r) = f_0(r) + f_1(r)$ we can construct the homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow R$ which satisfies

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f_1(r)] = f_0(r) \quad (86)$$

Here, a slight variation was proposed only on the components u_0 and u_1 . The suggestion was that only the part f_0 be assigned to the zeroth component u_0 , whereas the remaining part f_1 be combined with the component u_1 if we set $f_1(r) = f(r)$ and $f_0(r) = 0$, then the homotopy (86) reduce to the homotopy (27)

Note (1.5.20)

The important point that the success of the method depends on the proper selection of the function f_0 and f_1 .

Now, according to the second assumption $f(r) = \sum_{n=0}^{\infty} f_n(r)$ we can construct the homotopy $v(r, p) : \Omega \times [0,1] \rightarrow R$ which satisfies

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + pN(v) = \sum_{n=0}^{\infty} p^n f_n(r) \quad (87)$$

If $f(r)$ consists of two terms only then the homotopy (87) reduce to the homotopy (86).

In this case the term f_0 is combined with the component u_0 , f_1 is combined with component u_1 , f_2 is combined with component u_2 and so on, this suggestion will facilitate the calculations of the terms u_0, u_1, u_2, \dots and hence accelerate the rapid convergence of the series solution.

Note (1.5.21)

It is easy to observe that the algorithm of the (MHPM) based on the homotopy given in Eq. (86) and Eq. (87) reduces the number of terms involved in each component and hence the size of the calculation is minimized compared to the standard HPM.

Moreover, this reduction of terms in each component facilitates the construction of the homotopy perturbation solution.

It is to be also noted that the (MHPM) will be applied, wherever it is appropriate to all differential equations of any order. To demonstrate the effectiveness of the (MHPM) we compare the (MHPM) with standard (HMP) in the following examples.

Example (1.5.22) Consider the nonlinear differential equation [17]

$$u'' + \frac{2}{t}u' + u^3 = 6 + t^6 \quad (88)$$

Subject to the initial conditions

$$u(0) = 0, \quad u'(0) = 0. \quad (89)$$

The standard HPM: To solve Eq. (88) by HPM we construct the following homotopy:

$$u'' + \frac{2}{t}u' + p[u^3 - t^6 - 6] = 0 \quad (90)$$

Suppose the solution of Eq. (90) has the form

$$u = u_0 + pu_1 + p^2u_2 + \dots \quad (91)$$

Substituting (91) and the initial conditions (89).into the homotopy (90) and equating the term with identical powers of p , we obtain the following set of linear differential equations

$$\begin{aligned}
p^0 : u_0'' + \frac{2}{t}u_0' &= 0, \quad u_0(0) = 0, \quad u_0'(0) = 0 \\
p^1 : u_1'' + \frac{2}{t}u_1' &= -u_0^3 + t^6 + 6, \quad u_1(0) = 0, \quad u_1'(0) = 0 \\
p^2 : u_2'' + \frac{2}{t}u_2' &= 3u_0^2u_1, \quad u_2(0) = 0, \quad u_2'(0) = 0 \\
p^3 : u_3'' + \frac{2}{t}u_3' &= -3u_0u_1^2 - 3u_0^2u_2, \quad u_3(0) = 0, \quad u_3'(0) = 0 \\
&\vdots
\end{aligned} \tag{92}$$

Consequently, solving the above equation, we obtain

$$\begin{aligned}
u_0 &= 0, \\
u_1 &= t^2 + \frac{t^8}{72}, \\
u_2 &= 0, \\
u_3 &= 0, \\
u_4 &= -\frac{t^8}{72} - \frac{3t^{14}}{14.15.72} - \frac{3t^{20}}{20.21.(72)^2} - \frac{t^{26}}{26.27.(72)^3}.
\end{aligned}$$

And so, in this manner the rest of HPM can be obtained. The solution for Eq. (88) given by setting $p = 1$ in Eq. (91)

$$u = u_0 + u_1 + u_2 + u_3 + \dots = t^2 \tag{93}$$

The modified HPM: in view of the homotopy (86), we construct the following homotopy:

$$u'' + \frac{2}{t}u' + p[u^3 - t^6] = 6 \tag{94}$$

Substituting (91) into (94) and equation term with identical powers of p , we obtain the following set of linear differential equations

$$\begin{aligned}
p^0 : u_0'' + \frac{2}{t}u_0' &= 6, \quad u_0(0) = 0, \quad u_0'(0) = 0 \\
p^1 : u_1'' + \frac{2}{t}u_1' &= -u_0^3 + t^6, \quad u_1(0) = 0, \quad u_1'(0) = 0 \\
p^2 : u_2'' + \frac{2}{t}u_2' &= 3u_0^2u_1, \quad u_2(0) = 0, \quad u_2'(0) = 0 \\
p^3 : u_3'' + \frac{2}{t}u_3' &= -3u_0u_1^2 - 3u_0^2u_2, \quad u_3(0) = 0, \quad u_3'(0) = 0
\end{aligned} \tag{95}$$

Consequently, solving the above equation the first few components of the homotopy perturbation solution of Eq. (88) are derived as follows

$$\begin{aligned} u_0 &= t^2, \\ u_k &= 0, \quad k \geq 1 \end{aligned}$$

The exact solution

$$u(t) = t^2 \quad (96)$$

Follows immediately the success of obtaining the exact solution by using two iterations is the result of the proper selection of $f_0(r)$ and $f_1(r)$.

Example (1.5.23) Consider the partial differential equation [17]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x \cos t + x^2 \cos^2 t \quad (97)$$

Subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = 0. \quad (98)$$

The modified HPM: in the view of the homotopy (86), we construct the following homotopy:

$$\frac{\partial^2 u}{\partial t^2} + p \left[-\frac{\partial^2 u}{\partial x^2} + u^2 - x^2 \cos^2 t \right] = -x \cos t \quad (99)$$

Assume the solution of Eq. (99) in the form

$$u = u_0 + pu_1 + p^2u_2 + \dots \quad (100)$$

Substituting Eq. (100) and the initial conditions into the homotopy (99) and equating the terms identical power of p , we obtain the following set of linear differential equation

$$\begin{aligned} p^0 : \frac{\partial^2 u_0}{\partial t^2} &= -x \cos t, \quad u_0(x,0) = x, \quad u_{0t}(x,0) = 0 \\ p^1 : \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} + u_0^2 &= x^2 \cos^2 x, \quad u_1(x,0) = 0, \quad u_{1t}(x,0) = 0 \\ p^2 : \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} + 2u_0u_1 &= 0, \quad u_2(x,0) = 0, \quad u_{2t}(x,0) = 0 \\ &\vdots \end{aligned} \quad (101)$$

Solving the above equation, we obtain

$$\begin{aligned} u_0 &= x \cos t \\ u_k &= 0, \quad k \geq 1 \end{aligned}$$

The exact solution

$$u(x,t) = x \cos t \quad (102)$$

Follows immediately, it's clear that we used two iteration only to obtain exact solution

Example (1.5.24) Consider the linear differential equation [17]

$$u' + 2tu = 2te^{-t^2} \quad (103)$$

Subject to the initial condition:

$$u(0) = 0 \quad (104)$$

The standard HPM: to solve Eq. (103) with initial condition (104) by HPM we construct the following homotopy:

$$u' + p[2tu - 2e^{-t^2}] = 0 \quad (105)$$

Assume the solution of Eq. (105) has the form,

$$u = u_0 + pu_1 + p^2u_2 + \dots \quad (106)$$

Substituting Eq. (106) and initial condition (104) into the homotopy (105) and equating the terms with identical power of p , we obtain

$$\begin{aligned} p^0 : u'_0 &= 0, \quad u_0(0) = 0 \\ p^1 : u'_1 + 2tu_0 &= 2te^{-t^2}, \quad u_1(0) = 0 \\ p^2 : u'_2 + 2tu_1 &= 0, \quad u_2(0) = 0 \\ p^3 : u'_3 + 2tu_2 &= 0, \quad u_3(0) = 0 \end{aligned} \quad (107)$$

Consequently, solving the above equation, we get

$$\begin{aligned} u_0 &= 0, \\ u_1 &= 1 - e^{-t^2}, \\ u_2 &= 1 - t^2 - e^{-t^2}, \\ u_3 &= 1 - t^2 + \frac{t^4}{2} - e^{-t^2}, \\ u_4 &= 1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} - e^{-t^2}. \end{aligned}$$

And so on. In this manner the rest of components of the homotopy perturbation solution can be obtained, if we compute more terms we can show that the solution converges to

$$u(t) = t^2 e^{-t^2} \quad (108)$$

The modified HPM: in the view of the homotopy (87) and using the Taylor expansion

$$te^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{n!} \quad (109)$$

We construct the following homotopy

$$u' + p[2tu] = 2 \sum_{n=0}^{\infty} p^n \frac{(-1)^n t^{2n+1}}{n!} \quad (110)$$

Substituting (110) and the initial condition (104) into the homotopy (110) and equating the terms with identical powers of p , we obtain

$$\begin{aligned}
 p^0 : u'_0 &= 2t, \quad u_0(0) = 0 \\
 p^1 : u'_1 + 2tu_0 &= 2t^3, \quad u_1(0) = 0 \\
 p^2 : u'_2 + 2tu_1 &= t^5, \quad u_2(0) = 0 \\
 p^3 : u'_3 + 2tu_2 &= \frac{-t^7}{3}, \quad u_3(0) = 0 \\
 &\vdots
 \end{aligned} \tag{111}$$

Consequently, solving the above equation, the first few components of the homotopy perturbation solution for Eq. (103) are derived as follows

$$\begin{aligned}
 u_0 &= t^2, \\
 u_1 &= -t^4, \\
 u_2 &= \frac{t^6}{2!}, \\
 u_3 &= \frac{-t^8}{3!}.
 \end{aligned}$$

The solution in a series form is given by setting $p = 1$ in Eq. (106)

$$u = t^2 - t^4 - \frac{t^6}{2!} - \frac{t^8}{3!} + \dots \tag{112}$$

And in closed form

$$u(t) = t^2 e^{-t^2} \tag{113}$$

CHAPTER TWO

APPLICATION OF HOMOTOPY PERTURBATION METHOD TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS

2.1: Introduction

It is well known that most of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). In physics, for example, the heat flow and the wave propagation phenomena are well described by partial differential equations.

A partial differential equation is an equation that contains an unknown function of several variables, and one or more of its partial derivatives. There are two types of partial differential equation: linear and nonlinear partial differential equations. The linear partial differential equations are very important in mathematics as well as in applied sciences; In particular, the wave equation, heat equation and Laplace's equation are known as three fundamental linear partial differential equations and occur in many branches of physics, in applied mathematics and in engineering. It is to be noted that several methods are usually used in solving linear partial differential equation. Including, spectral method, characteristic method, variation iteration method and Adomian's decomposition method. In this chapter, we applied the homotopy perturbation method and the related improvements of the modified technique and noise terms phenomena will be effectively used .the homotopy perturbation method has been used extensively to solve nonlinear boundary and initial value problems. The method attacks the problem in a direct way and in a straightforward fashion without using linearization, or any other restrictive assumption that may change the physical behavior of the model under discussion. Therefore, homotopy perturbation method is of great interest to many researchers and scientists

2.2: First-Order Linear Partial Differential Equation

Partial differential equations of the first order are used to model traffic flow on a crowded road, blood flow through an elastic-walled tube, shock waves and as special cases of the general theories of gas dynamics and hydraulics. In this section, we will apply the homotopy perturbation and the related phenomenon of the noise terms and the modified homotopy perturbation method to the first-order linear partial differential equation homogeneous and inhomogeneous.

Example (2.2.1) Consider the following homogeneous partial differential equation [18]

$$x \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - 3u = 0, \quad u(x,0) = x^2. \quad (1)$$

To solve Eq. (1) by (HMP), we construct the following homotopy:

$$\frac{\partial v}{\partial y} - \frac{\partial u_0}{\partial y} + p \left(x \frac{\partial v}{\partial x} - 3v + \frac{\partial u_0}{\partial y} \right) = 0 \quad (2)$$

Assume the solution of Eq. (2) has the following form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (3)$$

Substituting Eq. (3) into Eq. (2) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial y} + x \frac{\partial v_0}{\partial x} - 3v_0 + \frac{\partial u_0}{\partial y} = 0, \quad v_1(x,0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial y} + x \frac{\partial v_1}{\partial x} - 3v_1 = 0, \quad v_2(x,0) = 0 \\ p^3 : \quad & \frac{\partial v_3}{\partial y} + x \frac{\partial v_2}{\partial x} - 3v_2 = 0, \quad v_3(x,0) = 0 \\ & \vdots \end{aligned} \quad (4)$$

Starting with $v_0(x, y) = u_0(x, y) = x^2$, and Applying the inverse operator $\int_0^y (\circ) dy$ to

above system, we obtain

$$\begin{aligned} v_1(x, y) &= \int_0^y \left(-x \frac{\partial v_0}{\partial x} + 3v_0 - \frac{\partial u_0}{\partial y} \right) dy = x^2 y, \\ v_2(x, y) &= \int_0^y \left(-x \frac{\partial v_1}{\partial x} + 3v_1 \right) dy = \frac{x^2 y^2}{2!}, \end{aligned}$$

$$v_3(x, y) = \int_0^y \left(-x \frac{\partial v_2}{\partial x} + 3v_2 \right) dy = \frac{x^2 y^3}{3!},$$

$$\vdots$$

Then the approximate solution of Eq. (1) obtain by setting $p = 1$ in Eq. (3)

$$u(x, y) = x^2 + x^2 y + \frac{x^2 y^2}{2!} + \frac{x^2 y^3}{3!} + \dots = x^2 e^y \quad (5)$$

Which is an exact solution.

Example (2.2.2) Consider the following inhomogeneous partial differential equation [19]

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - xu = ye^{xy} + x, \quad u(0, y) = 0. \quad (6)$$

Standard HPM: To solve Eq. (6) by (HMP), we construct the following homotopy:

$$\frac{\partial v}{\partial x} - \frac{\partial u_0}{\partial x} + p \left(\frac{\partial v}{\partial y} - xv + \frac{\partial u_0}{\partial y} - ye^{xy} - x \right) = 0 \quad (7)$$

Assume the solution of Eq. (6) has the following form

$$v = v_0 + pv_1 + p^2 v_2 + \dots \quad (8)$$

Substituting Eq. (8) in to Eq. (7) and equating the terms of like power p ,

$$p^0 : \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial x} = 0,$$

$$p^1 : \frac{\partial v_1}{\partial x} + \frac{\partial v_0}{\partial y} - xv_0 + \frac{\partial u_0}{\partial y} = ye^{xy} + x, \quad v_1(0, y) = 0 \quad (9)$$

$$p^2 : \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} - xv_1 = 0, \quad v_2(0, y) = 0$$

$$p^3 : \frac{\partial v_3}{\partial x} + \frac{\partial v_2}{\partial y} - xv_2 = 0, \quad v_3(0, y) = 0$$

$$\vdots$$

Starting with $v_0(x, y) = u_0(x, y) = 0$, and Applying the inverse operator $\int_0^x (\circ) dx$ to

the above system, we obtain:

$$v_1(x, y) = \int_0^x \left(-\frac{\partial v_0}{\partial y} + xv_0 - \frac{\partial u_0}{\partial y} + ye^{xy} + x \right) dx = e^{xy} - 1 + \frac{x^2}{2!},$$

$$v_2(x, y) = \int_0^x \left(-\frac{\partial v_1}{\partial y} + xv_1 \right) dx = -\frac{x^2}{2} + \frac{x^4}{8},$$

$$v_3(x, y) = \int_0^x \left(-\frac{\partial v_2}{\partial y} + xv_2 \right) dx = -\frac{x^4}{8} + \frac{x^5}{40},$$

$$\vdots$$

It is easily observed the noise terms $\frac{x^2}{2!}$ and $-\frac{x^2}{2!}$ appears in v_1 and v_2 respectively. By canceling the noise term $\frac{x^2}{2!}$ in v_1 , and by verifying that the remaining non-canceled terms of v_1 satisfy Eq. (7) we find the exact solution given by

$$u(x, y) = v_1(x, y) = e^{xy} - 1 \quad (10)$$

Modified HPM: To solve Eq. (6) by (MHMP), we construct the Following homotopy:

$$\frac{\partial v}{\partial x} + p \left(\frac{\partial v}{\partial y} - xv - x \right) = ye^{xy} \quad (11)$$

Assume the solution of Eq. (11) has the form Eq. (8) substituting Eq. (8), into Eq. (11) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad \frac{\partial v_0}{\partial x} &= ye^{xy}, & v_0(0, y) &= 0 \\ p^1 : \quad \frac{\partial v_1}{\partial x} + \frac{\partial v_0}{\partial y} - xv_0 &= x, & v_1(0, y) &= 0 \\ p^2 : \quad \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} - xv_1 &= 0, & v_2(0, y) &= 0 \\ & \vdots & & \end{aligned} \quad (12)$$

Applying the inverse operator $\int_0^x (\circ) dx$ to above system, we obtain

$$v_0 = \int_0^x (ye^{xy}) dx = e^{xy} - 1,$$

$$v_k = 0, \quad k \geq 1.$$

It then follows that the solution is

$$u(x, y) = v_0(x, y) = e^{xy} - 1 \quad (13)$$

This example clearly shows that the solution can be obtained by using two iterations, and hence the volume of calculation is reduced.

2.3: Second -Order Linear Partial Differential Equation

In this section, we consider the second-order quasi-linear partial differential equation,

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + d = 0 \quad (14)$$

With initial conditions:

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial y} = g(x), \quad (15)$$

Or

$$u(0, y) = f(y), \quad \frac{\partial u(0, y)}{\partial x} = g(y). \quad (16)$$

Where a, b, c and d may be functions of x, y, z but not of, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y^2}$

i.e., the second-order derivative occurs only to the first degree. An equation (14) is said to be hyperbolic, parabolic or elliptic accordingly as $b^2 - 4ac$ is positive, zero, or negative. Numerical methods of solving Eq. (14), which are common, used as a characteristics method [20], needed large size of computation work and usually the round-off error causes the loss of accuracy. Her homotopy perturbation method needs less computation and leads higher accuracy. We have applied homotopy perturbation method for special cases in which the coefficient in the Eq. (14) do not depend on partial derivatives and u . To solve Eq. (14) with the initial conditions (15), according to the homotopy perturbation, we construct the following homotopy:

$$(1-p) \left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} \right) + p \left(\frac{a}{c} \frac{\partial^2 v}{\partial x^2} + \frac{b}{c} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{d}{c} \right) = 0$$

Or,

$$\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = p \left(-\frac{a}{c} \frac{\partial^2 v}{\partial x^2} - \frac{b}{c} \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u_0}{\partial y^2} - \frac{d}{c} \right) \quad (17)$$

Assume the solution of Eq. (16) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (18)$$

Putting (18) in to (17) and comparing the coefficient of identical degrees of p ,

$$p^0 : \frac{\partial^2 v_0}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = 0,$$

$$p^1 : \frac{\partial^2 v_1}{\partial y^2} = -\frac{a}{c} \frac{\partial^2 v_0}{\partial x^2} - \frac{b}{c} \frac{\partial^2 v_0}{\partial x \partial y} - \frac{d}{c} - \frac{\partial^2 u_0}{\partial y^2}, v_1(x,0) = 0, \frac{\partial v_1}{\partial y}(x,0) = 0 \quad (19)$$

$$p^2 : \frac{\partial^2 v_2}{\partial y^2} = -\frac{a}{c} \frac{\partial^2 v_1}{\partial x^2} - \frac{b}{c} \frac{\partial^2 v_1}{\partial x \partial y}, v_2(x,0) = 0, \frac{\partial v_2}{\partial y}(x,0) = 0$$

For simplicity, we take $v_0 = u_0 = f(x) + g(x)y$. Accordingly, we have:

$$\begin{aligned} v_1 &= \int_0^y \int_0^y \left(-\frac{a}{c} \frac{\partial^2 v_0}{\partial x^2} - \frac{b}{c} \frac{\partial^2 v_0}{\partial x \partial \kappa} - \frac{d}{c} \right) d\kappa dy, \\ v_2 &= \int_0^y \int_0^y \left(-\frac{a}{c} \frac{\partial^2 v_1}{\partial x^2} - \frac{b}{c} \frac{\partial^2 v_1}{\partial x \partial \kappa} \right) d\kappa dy, \\ &\vdots \end{aligned} \quad (20)$$

The approximate solution of Eq. (14) can be obtained by setting $p = 1$.

$$u = v_0 + v_1 + v_2 + \dots$$

Similarly, to solve Eq. (14) with initial condition (15) we construct the following homotopy:

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = p \left(-\frac{c}{a} \frac{\partial^2 v}{\partial y^2} - \frac{b}{a} \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u_0}{\partial x^2} - \frac{d}{a} \right) \quad (21)$$

With initial approximation $v_0 = u_0 = f(y) + g(y)x$. Suppose the solution of Eq.

(21) has the form (18), according to the mentioned procedure we have:

$$\begin{aligned} p^0 : \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} &= 0, \\ p^1 : \frac{\partial^2 v_1}{\partial x^2} &= -\frac{c}{a} \frac{\partial^2 v_0}{\partial x^2} - \frac{b}{a} \frac{\partial^2 v_0}{\partial x \partial y} - \frac{d}{a} - \frac{\partial^2 u_0}{\partial x^2}, v_1(0, y) = 0, \frac{\partial v_1}{\partial x}(x,0) = 0 \\ p^2 : \frac{\partial^2 v_2}{\partial x^2} &= -\frac{c}{a} \frac{\partial^2 v_1}{\partial y^2} - \frac{b}{a} \frac{\partial^2 v_1}{\partial x \partial y}, v_2(x,0) = 0, \frac{\partial v_2}{\partial x}(x,0) = 0 \\ &\vdots \end{aligned} \quad (22)$$

So we have

$$\begin{aligned} v_1 &= \int_0^x \int_0^x \left(-\frac{c}{a} \frac{\partial^2 v_0}{\partial x^2} - \frac{b}{a} \frac{\partial^2 v_0}{\partial y \partial \kappa} - \frac{d}{a} \right) d\kappa dx, \\ v_2 &= \int_0^x \int_0^x \left(-\frac{c}{a} \frac{\partial^2 v_1}{\partial x^2} - \frac{b}{a} \frac{\partial^2 v_1}{\partial y \partial \kappa} \right) d\kappa dx, \\ &\vdots \end{aligned} \quad (23)$$

Setting $p = 1$, result in the approximation solution of Eq. (14)

$$u = v_0 + v_1 + v_2 + \dots$$

Example (2.3.3) Consider the following equation with initial conditions [21]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2 \frac{\partial^2 u}{\partial y^2} + 1 = 0, \quad (24)$$

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial y} = x.$$

According to the homotopy (16), we have;

$$\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = p \left(\frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{2} \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} - \frac{\partial^2 u_0}{\partial y^2} \right) \quad (25)$$

Beginning with $v_0 = u_0 = x + xy$ and from (20) we have:

$$v_1 = \int_0^y \int_0^y \left(\frac{1}{2} \frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2} \frac{\partial^2 v_0}{\partial x \partial y} + \frac{1}{2} - \frac{\partial^2 u_0}{\partial y^2} \right) dx dy = \frac{1}{2} y^2$$

$$v_2 = \int_0^y \int_0^y \left(\frac{1}{2} \frac{\partial^2 v_1}{\partial x^2} + \frac{1}{2} \frac{\partial^2 v_1}{\partial x \partial y} \right) dx dy = 0$$

$$v_k = 0, \quad k \geq 2$$

So the first-order approximate obtain by setting $p = 1$ in Eq. (18)

$$u = v_0(x, y) + v_1(x, y) = x + xy + \frac{1}{2} y^2 \quad (26)$$

Which is an exact solution.

The results are compared with characteristics in table (2.3.4)

Table (2.3.4):

The solution of $u(x, y)$ for different values of x and y

x	y	$u(x, y)$ (HPM)	$u(x, y)$ (characteristics method)
0.139	0.074	0.213	0.212
0.448	0.077	0.525	0.526
0.758	0.075	0.833	0.834
0.819	0.152	0.971	0.971

Example (2.3.5) Consider the following equation with initial conditions [21]

$$\frac{\partial^2 u}{\partial x^2} - 4x^2 \frac{\partial^2 u}{\partial y^2} = 0, \quad (27)$$

$$u(x,0) = x^2, \quad \frac{\partial u(x,0)}{\partial y} = 0.$$

According to homotopy (16), we have:

$$\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = p \left(\frac{1}{4x^2} \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial y^2} \right) \quad (28)$$

Beginning with $v_0 = u_0 = x^2$ and from (20) we have;

$$v_1 = \int_0^y \int_0^y \left(\frac{1}{4x^2} \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial y^2} \right) d\kappa dy = \frac{1}{4} \frac{y^2}{x^2},$$

$$v_2 = \int_0^y \int_0^y \left(\frac{1}{4x^2} \frac{\partial^2 v_1}{\partial x^2} \right) d\kappa dy = \frac{1}{32} \frac{y^4}{x^6},$$

$$v_3 = \int_0^y \int_0^y \left(\frac{1}{4x^2} \frac{\partial^2 v_2}{\partial x^2} \right) d\kappa dy = \frac{7}{640} \frac{y^6}{x^{10}},$$

$$v_4 = \int_0^y \int_0^y \left(\frac{1}{4x^2} \frac{\partial^2 v_3}{\partial x^2} \right) d\kappa dy = \frac{11}{2048} \frac{y^8}{x^{14}}.$$

So the fourth-order approximate obtain by setting $p = 1$ in Eq. (18)

$$u \approx x^2 + \frac{1}{4} \frac{y^2}{x^2} + \frac{1}{32} \frac{y^4}{x^6} + \frac{7}{640} \frac{y^6}{x^{10}} + \frac{11}{2048} \frac{y^8}{x^{14}} \quad (29)$$

The results are compared with characteristics in table (2.3.6)

Table (2.3.6)

The solution of $u(x, y)$ for different values of x and y

x	y	$u(x, y)$ (HPM)	$u(x, y)$ (characteristics method)
0.133	0.067	0.1442	0.1444
0.833	0.067	0.8911	0.8911
0.067	0.133	0.0848	0.0844
0.767	0.133	0.8779	0.8778

2.4: The Heat Equation

In this section, we will study the physical problem of Heat conduction in a rod of length L . The temperature distribution of a rod is governed by an initial-boundary value problem [18] that is often defined in the general form by:

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} - u = f(x), \quad 0 < x < L, \quad t > 0 \quad (30)$$

With initial condition:

$$u(x,0) = g(x), \quad (31)$$

And boundary conditions:

$$u(0,t) = f_0(t), \quad u(L,t) = f_1(t). \quad (32)$$

where $u(x,t)$ represents the temperature of the rod at the position x at time t and k is the thermal diffusivity of the material that measures the rod ability to heat conduction. It is interesting to note that Eq. (30) arise in two different types, namely,

- homogeneous heat equation : this type of equation is often given by

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0 \quad (33)$$

Further, heat equation with a literal loss is formally derived as a homogeneous PDE in the form,

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} + u = 0 \quad (34)$$

- Inhomogeneous Heat Equation : this type of equations is often given by

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = f(x) \quad (35)$$

Where $f(x)$ is called the heat source which is independent of time.

Many researchers have applied the HPM to the problem, homogeneous or inhomogeneous, and it was formally proven by [22, 23, 24] that the method attacks the problem, homogeneous or inhomogeneous, in a straightforward manner without any need for transformation formulas. Further, there is no need to change the inhomogeneous boundary conditions to homogeneous conditions as required by the method of separation of variables [18], and finite difference method [20], and Pdé approximate [18], and other methods.

In order to solve Eq. (30) with the initial condition (31), (i.e., t – solution) by the HPM, we choose the initial approximation $v_0 = g(x)$ and construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p \left(k \frac{\partial^2 v}{\partial x^2} - v - \frac{\partial u_0}{\partial t} + f(x, t) \right) \quad (36)$$

Assume the solution of Eq. (16) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (37)$$

Putting (37) into (36) and comparing the coefficients of identical degrees of p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} - k \frac{\partial^2 v_0}{\partial x^2} + v_0 + \frac{\partial u_0}{\partial t} = f(x, t), \quad v_1(x, 0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} - k \frac{\partial^2 v_1}{\partial x^2} + v_1 = 0, \quad v_2(x, 0) = 0, \\ & \vdots \end{aligned} \quad (38)$$

We can start with $v_0 = u_0 = g(x)$. And Applying the inverse operator $\int_0^t (\circ) dt$ to

above system we obtain the following recreation formula

$$\begin{aligned} v_1 &= \int_0^t \left(k \frac{\partial^2 v_0}{\partial x^2} - v_0 - \frac{\partial u_0}{\partial t} - f(x, t) \right) dt, \\ v_j &= \int_0^t \left(k \frac{\partial^2 v_{j-1}}{\partial x^2} - v_{j-1} \right) dt, \quad j \geq 2. \end{aligned} \quad (39)$$

The approximate solution of Eq. (30) can be obtained by setting $p = 1$.

$$u = v_0 + v_1 + v_2 + \dots$$

An important conclusion can be made here; the (t -solution) is obtained by using the initial condition only without using the boundary conditions. The obtained solution can be used to show that it satisfies the given boundary conditions. However, we can also obtain the (x -solution) In fact; the solution obtained in this way requires the use of boundary conditions and initial condition as well. This leads to an important conclusion that solving the PDE in the t direction reduces the size of computational work. This important observation will be confirmed through examples that will be discussed later. To give a clear overview of the content of the HPM method, we have chosen several examples, homogeneous and inhomogeneous, to illustrate the discussion given above.

Example (2.4.7) Consider the homogeneous one dimension diffusion equation [18]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < f, \quad t > 0. \quad (40)$$

With initial condition

$$u(x,0) = g(x), \quad (41)$$

And boundary conditions

$$u(0,t) = 0, \quad u(f,t) = 0. \quad (42)$$

According to the homotopy (36), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial u_0}{\partial t} \right) \quad (43)$$

Beginning with $v_0 = u_0 = g(x)$ and from the recreation formula (39) we have

$$v_1 = \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t} \right) dt = g''(x)t,$$

$$v_2 = \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} \right) dt = g^{(4)}(x) \frac{t^2}{2!},$$

$$v_3 = \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} \right) dt = g^{(6)}(x) \frac{t^2}{3!},$$

⋮

Other components can be determined in a like manner as far as we like. The accuracy level can be effectively improved by increasing the number of components determined. Then the series solution of Eq. (40) obtain by setting $p = 1$ in Eq. (37)

$$u(x,t) = \sum_{n=0}^{\infty} g^{(2n)}(x) \frac{t^n}{n!} \quad (44)$$

Example (2.4.8) Consider the homogeneous one dimension diffusion equation [18]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \quad 0 < x < f, \quad t > 0 \quad (45)$$

With initial condition

$$u(x,0) = \sin x, \quad (46)$$

And boundary conditions

$$u(0,t) = 0, \quad u(f,t) = 0. \quad (47)$$

According to homotopy (36), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p \left(\frac{\partial^2 v}{\partial x^2} - u - \frac{\partial u_0}{\partial t} \right) \quad (48)$$

Beginning with $v_0 = u_0 = \sin x$ and from the recreation formula (39) we have;

$$v_1 = \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - v_0 - \frac{\partial u_0}{\partial t} \right) dt = -2t \sin x,$$

$$v_2 = \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} - v_1 \right) dt = -\frac{(2t)^2}{2!} \sin x,$$

$$v_3 = \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} - v_2 \right) dt = -\frac{(2t)^3}{3!},$$

⋮

Then the approximate solution of Eq. (42) obtain by setting $p = 1$ in Eq. (37)

$$u(x,t) = \sin x \left(1 - (2t) + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \dots \right) = e^{-2t} \sin x \quad (48)$$

Which is an exact solution.

Example (2.4.9) Consider the one-dimensional initial boundary value problem which describes the heat-like models [22]

$$\frac{\partial u}{\partial t} = \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \quad (49)$$

With initial condition

$$u(x,0) = x^2, \quad (50)$$

And boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^t. \quad (51)$$

According to homotopy (36), we have:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p \left(\frac{1}{2} x^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial u_0}{\partial t} \right) \quad (52)$$

Beginning with $v_0 = u_0 = \sin x$ and from recreation formula (39) we have

$$v_1 = \int_0^t \left(\frac{1}{2} x^2 \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t} \right) dt = x^2 t,$$

$$v_2 = \int_0^t \left(\frac{1}{2} x^2 \frac{\partial^2 v_1}{\partial x^2} \right) dt = x^2 \frac{t^2}{2!},$$

$$v_3 = \int_0^t \left(\frac{1}{2} x^2 \frac{\partial^2 v_2}{\partial x^2} \right) dt = x^2 \frac{t^3}{3!},$$

$$\vdots$$

Then the approximate solution of Eq. (42) obtain by setting $p = 1$ in Eq. (37)

$$u(x,t) = x^2 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = x^2 e^t \quad (53)$$

Which is an exact solution.

Example (2.4.10) Consider the inhomogeneous one dimension diffusion equation [23]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \cos x, \quad 0 < x < f, \quad t > 0 \quad (54)$$

With initial condition:

$$u(x,0) = 0, \quad (55)$$

And boundary conditions:

$$u(0,t) = 1 - e^{-t}, \quad u(f,t) = e^{-t} - 1. \quad (56)$$

According to homotopy (36), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial u_0}{\partial t} + \cos x \right) \quad (57)$$

Beginning with $v_0 = u_0 = 0$ and from the recreation formula (36) we have;

$$v_1 = \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t} + \cos t \right) dt = t \cos x,$$

$$v_2 = \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} \right) dt = -\frac{t^2}{2!} \cos x,$$

$$v_3 = \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} \right) dt = \frac{t^3}{3!} \cos x,$$

\vdots

Then the approximate solution of Eq. (54) obtain by setting $p = 1$ in Eq. (37)

$$u(x,t) = \cos x \left(t - \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = (1 - e^{-t}) \cos x \quad (58)$$

Which is an exact solution.

2.5: The Wave Equations

Since the governing equations on many experiments in engineering as well as science leads to the wave equation. The wave equation usually describes water waves, the vibrations of a string or a membrane, the propagation of electromagnetic and sound waves, or the transmission of electric signals in a cable. Analytical methods commonly used for solving the wave equation are Very restricted and can be used in very special cases so they can not be used to solve equations of numerous realistic scenarios. Numerical techniques, which are commonly used, encounter difficulties in terms of the size of computational works needed and usually the round-off error causes the loss of accuracy. The homotopy perturbation method has been widely used with promising results in linear and nonlinear partial differential equations that describe wave propagations [22,25,26].in this section; we will apply the HPM to handle the wave equation. This method has proven to be very effective and results in considerable saving in computation time.

Now consider the following homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0 \quad (59)$$

With initial conditions:

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \quad (60)$$

And boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0. \quad (61)$$

In order to solve Eq. (39) with the initial conditions (31), (i.e., t – solution) by the HPM, we construct the following homotopy:

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p \left(c^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial t^2} \right) \quad (62)$$

Assume the solution of Eq. (59) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (63)$$

Putting (63) into (62) and comparing the coefficients of identical degrees of p ,

$$\begin{aligned}
p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} &= 0, \\
p^1 : \frac{\partial^2 v_1}{\partial t^2} - c^2 \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial t^2} &= 0, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \\
p^2 : \frac{\partial^2 v_2}{\partial t^2} - c^2 \frac{\partial^2 v_1}{\partial x^2} &= 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \\
p^3 : \frac{\partial^2 v_3}{\partial t^2} - c^2 \frac{\partial^2 v_2}{\partial x^2} &= 0, \quad v_3(x,0) = 0, \quad \frac{\partial v_3}{\partial t}(x,0) = 0 \\
&\vdots
\end{aligned} \tag{64}$$

We always start with $v_0 = f(x) + t g(x)$ as initial approximate. Applying the

inverse operator $\int_0^t \int_0^t (\circ) dt dt$ to above system we obtain;

$$\begin{aligned}
v_1 &= \int_0^t \int_0^t \left(c^2 \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial t^2} \right) dt dt = c^2 \left(\frac{t^2}{2!} f''(x) + \frac{t^3}{3!} g''(x) \right), \\
v_2 &= \int_0^t \int_0^t \left(c^2 \frac{\partial^2 v_1}{\partial x^2} \right) dt dt = c^{(4)} \left(\frac{t^4}{4!} f^{(4)}(x) + \frac{t^5}{5!} g^{(4)}(x) \right), \\
v_3 &= \int_0^t \int_0^t \left(c^2 \frac{\partial^2 v_2}{\partial x^2} \right) dt dt = c^{(6)} \left(\frac{t^4}{6!} f^{(4)}(x) + \frac{t^7}{7!} g^{(6)}(x) \right) \\
&\vdots
\end{aligned}$$

By continuing the calculation, we thus have the solution given by

$$\begin{aligned}
u &= v_0 + v_1 + v_2 + \dots \\
&\left[f(x) + c^2 f''(x) \frac{t^2}{2!} + c^4 f^{(4)}(x) \frac{t^4}{4!} + c^6 f^{(6)}(x) \frac{t^6}{6!} + \dots \right] \\
&+ \left[g''(x)t + c^2 g(x) \frac{t^3}{3!} + c^4 g^{(4)}(x) \frac{t^5}{5!} + c^6 g^{(6)}(x) \frac{t^7}{7!} + \dots \right]. \tag{65}
\end{aligned}$$

It is important to note that we can also obtained the (x -solution) by using the boundary conditions in this way requires more work because the boundary condition $v_x(0,t)$ is not always available. To give a clear overview of the HPM method, we have selected homogeneous and inhomogeneous equations to illustrate the procedure discussed above.

Example (2.5.11) Consider the wave equation in the infinite domain [26]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty \quad t > 0 \quad (66)$$

With initial conditions:

$$u(x,0) = \sin x, \quad \frac{\partial u}{\partial t}(x,0) = \cos x. \quad (67)$$

With $f(x) = \sin x$ and $g(x) = \cos x$, we find

$$f^{(2n)}(x) = (-1)^n \sin x, \quad n = 0,1,2,\dots \quad (68)$$

And

$$g^{(2n)}(x) = (-1)^n \cos x, \quad n = 0,1,2,\dots \quad (69)$$

Substituting Eq. (68) and (69) into (65) produces

$$u(x,t) = \sin x \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \right) + \cos x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \right)$$

And in a closed form by

$$u(x,t) = \sin(x+t) \quad (70)$$

This is the same as D'Alembert solution [18]

Example (2.5.12) Consider the following inhomogeneous wave equation [18]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 6t + 2x, \quad 0 < x < f, \quad t > 0 \quad (71)$$

With initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \sin x. \quad (72)$$

And boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = t^2 + \sin t, \quad \frac{\partial u}{\partial x}(f,t) = t^2 - \sin t. \quad (73)$$

To solve Eq. (71) with the initial condition (72), by the HPM, we construct the following homotopy:

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial t^2} + 6t + 2x \right) \quad (74)$$

Assume the solution of Eq. (74) has the form Eq. (63) substituting (63) in (74) and comparing the coefficients of identical degrees of p ,

$$\begin{aligned}
p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} &= 0, \\
p^1 : \frac{\partial^2 v_1}{\partial t^2} - \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial t^2} &= 6t + 2x, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \\
p^2 : \frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} &= 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \\
p^3 : \frac{\partial v_3}{\partial t} - \frac{\partial^2 v_2}{\partial x^2} &= 0, \quad v_3(x,0) = 0, \quad \frac{\partial v_3}{\partial t}(x,0) = 0 \\
&\vdots
\end{aligned} \tag{75}$$

Start with $v_0 = t \sin x$ as initial approximate. Applying the inverse operator

$\int_0^t \int_0^t (\circ) dt dt$ to above system we obtain;

$$\begin{aligned}
v_1 &= \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial t^2} + 6t + 2x \right) dt dt = t^3 + t^2 x - \frac{t^3}{3!} \sin x, \\
v_2 &= \int_0^t \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} \right) dt dt = \frac{t^5}{5!} \sin x, \\
&\vdots
\end{aligned}$$

Then the approximate solution of Eq. (71) obtain by setting $p = 1$ in Eq. (63)

$$u(x,t) = t^3 + t^2 x + \left(t - \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \sin x = t^3 + t^2 x + \sin x \sin t \tag{76}$$

Which is an exact solution.

Example (2.5.13) Consider two-dimension initial boundary value problem which describes the wave-like models [26]

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{12} \left(x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} \right), \quad 0 < x, y < 1, \quad t > 0 \tag{77}$$

Subject to the initial conditions:

$$u(x, y, 0) = x^4, \quad \frac{\partial u}{\partial t}(x, y, 0) = y^4. \tag{78}$$

And the Neumann boundary conditions:

$$\begin{aligned}
\frac{\partial u}{\partial x}(0, y, t) &= 0, \quad \frac{\partial u}{\partial x}(1, y, t) = 4 \cosh t, \\
\frac{\partial u}{\partial y}(x, 0, t) &= 0, \quad \frac{\partial u}{\partial y}(x, 1, t) = 4 \sinh t.
\end{aligned} \tag{79}$$

To solve Eq. (77) with the initial condition (78), by the HPM, we construct the following homotopy:

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p \left(\frac{1}{12} x^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{12} y^2 \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} \right) \quad (80)$$

Assume the solution of Eq. (80) has the form Eq. (63) substituting (63) in (80) and comparing the coefficient of identical degrees of p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial t^2} - \frac{1}{12} x^2 \frac{\partial^2 v_0}{\partial x^2} - \frac{1}{12} y^2 \frac{\partial^2 v_0}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = 0, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \quad (81) \\ p^2 : \quad & \frac{\partial v_2}{\partial t} - \frac{1}{12} x^2 \frac{\partial^2 v_1}{\partial x^2} + \frac{1}{12} y^2 \frac{\partial^2 v_1}{\partial y^2} = 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \\ & \vdots \end{aligned}$$

Start with $v_0 = x^2 + y^4 t$ as initial approximate. Applying the inverse operator

$\int_0^t \int_0^t (\circ) dt dt$ to above system we obtain;

$$\begin{aligned} v_1 &= \int_0^t \int_0^t \left(\frac{1}{12} x^2 \frac{\partial^2 v_0}{\partial x^2} + \frac{1}{12} y^2 \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 u_0}{\partial t^2} \right) dt dt = x^4 \frac{t^2}{2!} + y^4 \frac{t^3}{3!}, \\ v_2 &= \int_0^t \int_0^t \left(\frac{1}{12} x^2 \frac{\partial^2 v_1}{\partial x^2} + \frac{1}{12} y^2 \frac{\partial^2 v_1}{\partial y^2} \right) dt dt = x^4 \frac{t^4}{4!} + y^2 \frac{t^5}{5!}, \\ & \vdots \end{aligned}$$

Then the approximate solution of Eq. (77) obtained by setting $p = 1$ in Eq. (63)

$$u(x,t) = x^4 \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \right) + x^4 \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right) = x^4 \cosh t + y^4 \sinh t \quad (82)$$

Which is an exact solution.

2.6: The Laplace Equation

The Laplace equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. The two-dimensional Laplace equation has the following form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (83)$$

Or

$$\nabla^2 = 0 \quad (84)$$

Where ∇^2 is laplacian.

The Dirichlet boundary conditions for Laplace's equation consist in finding a solution of u on domain D such that on the boundary of D is equal to some given function [15,18]. One physical interpretation of this problem which arises in heat equations is as follows: fix the temperature on the boundary of the domain and wait until the temperature in the interior does not change anymore; the temperature distribution in the interior will then be given by the solution to the corresponding Dirichlet problem. The Neumann boundary conditions for Laplace's equation specify not the function itself on the boundary of D , but its Normal derivative [15,18]. Physically, this is similar to the construction of a potential for a vector field whose effect is known at the boundary of D alone. In this section, we will apply the (HPM) to Laplace's equation with specified boundary conditions [27,28].

Now consider the two dimension Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 < x < a, 0 < y < b \quad (85)$$

With boundary conditions

$$\begin{aligned} u(0, y) &= 0, & u(a, y) &= f(y), \\ u(x, 0) &= 0, & u(x, b) &= 0. \end{aligned} \quad (86)$$

In order to solve Eq. (85), by the HPM, with boundary conditions (86) (i.e., y - solution), we construct the following homotopy:

$$\frac{\partial v^2}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = p \left(-\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial y^2} \right) \quad (87)$$

Assume the solution of Eq. (86) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (88)$$

Putting (88) into (86) and comparing the coefficient of identical degrees of p ,

$$p^0 : \frac{\partial^2 v_0}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = 0,$$

$$p^1 : \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} = 0, \quad v_1(0, y) = 0, \quad \frac{\partial v_1}{\partial y}(0, y) = 0 \quad (89)$$

$$p^2 : \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_1}{\partial x^2} = 0, \quad v_2(0, y) = 0, \quad \frac{\partial v_2}{\partial y}(0, y) = 0$$

⋮

We note that $u_y(x, 0) = g(x)$, boundary condition that is not given but will be determined now start with $v_0 = y g(x)$ as initial approximate. Applying the

inverse operator $\int_0^y \int_0^y (\circ) dy dy$ to above system we obtain;

$$v_1 = \int_0^y \int_0^y \left(-\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} \right) dy dy = -\frac{y^3}{3!} g''(x),$$

$$v_2 = \int_0^y \int_0^y \left(-\frac{\partial^2 v_1}{\partial x^2} \right) dy dy = \frac{y^5}{5!} g^{(4)}(x), \quad (90)$$

⋮

By continuing the calculation, we thus have the solution given by

$$u = v_0 + v_1 + v_2 + \dots$$

$$= y g(x) - \frac{y^3}{3!} g''(x) + \frac{y^5}{5!} g^{(4)}(x) + \dots \quad (91)$$

To complete the determination of the solution of $u(x, y)$, we should determine $g(x)$ This can be easily done by using the inhomogeneous boundary condition $u(a, y) = f(y)$. Substituting $x = a$ into (89), using the Taylor expansion for $f(y)$ and equating the coefficients of like terms in both sides' leads to the complete determination of $g(x)$.

Example (2.6.14) Consider the two dimension Laplace equation [27]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 < x, y < f, \quad (92)$$

With Dirichlet boundary conditions:

$$u(0, y) = 0, \quad u(f, y) = \sinh f \sin y,$$

$$u(x, 0) = 0, \quad u(x, f) = 0. \quad (93)$$

According to homotopy (86), we have;

$$\frac{\partial v^2}{\partial y^2} - \frac{\partial^2 u_0}{\partial y^2} = p \left(-\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial y^2} \right)$$

Beginning with $v_0 = y g(x)$ and according to Eq. (90) the solution of (92) reads;

$$u(x, y) = yg(x) - \frac{y^3}{3!} g''(x) + \frac{y^5}{5!} g^{(4)}(x) + \dots$$

To determine the function $g(x)$, we use the inhomogeneous boundary condition $u(f, y) = \sinh f \sin y$, and by using the Taylor expansion of $\sin y$ we obtain

$$yg(f) - \frac{y^3}{3!} g''(f) + \frac{y^5}{5!} g^{(4)}(f) + \dots = \sinh \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots \right) \quad (94)$$

Equating the coefficient of like terms on both sides gives

$$g(f) = g''(f) = g^{(4)}(f) = \dots = \sinh f \quad (95)$$

This means that

$$g(x) = \sinh x \quad (96)$$

The only function that when substituted in (90) will also satisfy the remaining Boundary conditions, consequently, the solution is given by

$$u(x, y) = \sinh x \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots \right) \quad (97)$$

And in a closed form

$$u(x, y) = \sinh x \sin y \quad (98)$$

Example (2.6.15) Consider the two dimension Laplace equation [27]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 < x, y < f, \quad (99)$$

With Neumann boundary conditions:

$$\begin{aligned} u_x(0, y) &= 0, & u_x(f, y) &= 0, \\ u_y(x, 0) &= \cos x, & u_y(x, f) &= \cosh f \cos x. \end{aligned} \quad (100)$$

In order to solve Eq. (99), by the HPM, with boundary conditions (100) (i.e., x -solution), we construct the following homotopy:

$$\frac{\partial v^2}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = p \left(-\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u_0}{\partial x^2} \right) \quad (101)$$

Assume the solution of Eq. (101) has the form (88) substituting (88) into (101)

And comparing the coefficient of identical degrees of p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 u_0}{\partial x^2} = 0, \quad v_1(x, 0) = 0, \quad \frac{\partial v_1}{\partial x}(x, 0) = 0 \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} = 0, \quad v_2(x, 0) = 0, \quad \frac{\partial v_2}{\partial x}(x, 0) = 0 \\ & \vdots \end{aligned} \quad (102)$$

We note that $u(0, y) = h(y)$, boundary condition that is not given but will be determined now start with $v_0 = h(y)$ as initial approximate. Applying the inverse

operator $\int_0^x \int_0^x (\circ) dx dx$ to above system we obtain;

$$\begin{aligned} v_1 &= \int_0^x \int_0^x \left(-\frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 u_0}{\partial x^2} \right) dx dx = -\frac{x^2}{2!} h''(y), \\ v_2 &= \int_0^x \int_0^x \left(-\frac{\partial^2 v_1}{\partial y^2} \right) dx dx = \frac{x^4}{4!} h^{(4)}(y), \\ &\vdots \end{aligned} \tag{103}$$

By continuing the calculation, we thus have the solution given by

$$\begin{aligned} u &= v_0 + v_1 + v_2 + \dots \\ &= h(y) - \frac{x^2}{2!} h''(y) + \frac{x^4}{4!} h^{(4)}(y) + \dots \end{aligned} \tag{104}$$

To determine $h(y)$, we use the boundary condition $u_y(x, f) = \cosh f \cos x$ to obtain

$$u_y(x, f) = h'(f) - \frac{x^2}{2!} h'''(f) + \frac{x^4}{4!} h^{(5)}(f) + \dots = \cosh f \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \tag{105}$$

Equating the coefficients of like terms on both sides we get;

$$h'(f) = h'''(f) = h^{(5)}(f) + \dots = \cosh f \tag{106}$$

Then

$$h(y) = \sinh y$$

Consequently, the solution is given by;

$$u(x, y) = \sinh y \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \tag{107}$$

It is worth pointing out that the Neumann problem has a property that the solution is determined up to an arbitrary additive constant which cannot be defined by this method or even separation of variables [18]. So, the solution is given in the closed form as

$$u(x, y) = C + \sinh y \cos x$$

The important things which we want to mentioned here, the results of this section were published as scientific paper in [80,81].

CHAPTER THREE

APPLICATION OF HOMOTOPY PERTURBATION METHOD TO LINEAR AND NONLINEAR PHYSICAL MODELS

3.1: Introduction

This chapter is devoted to treatments of linear and nonlinear particular applications that appear in applied sciences. A wide variety of physically significant problems modeled by linear and nonlinear partial differential equations has been the focus of extensive studies for the last decades. A huge size of research and investigation has been invested in these scientific applications. Nonlinear PDEs have undergone remarkable developments. Nonlinear problems arise in different areas including gravitation, chemical reaction, fluid dynamics, dispersion, nonlinear optics, plasma physics, acoustics, inviscid fluids and others.

The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact or approximate solutions. Most new nonlinear equations do not have a precise analytic solution; so, numerical methods have largely been used to handle these equations. There are also analytic techniques for nonlinear equations. Some of the classic analytic methods are Lyapunov's artificial small parameter method, perturbation techniques, ϵ -expansion method, and Hirota bilinear method. In recent years, many authors have paid attention to studying the solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM), He's semi-inverse method, the tanh method, the sinh-cosh method, the differential transform method and the variational iteration method (VIM).

In this chapter, the homotopy perturbation method, the modified homotopy perturbation method, and the self-canceling noise-terms phenomenon will be employed in the treatments of these physical models.

3.2: He's Polynomials

The homotopy perturbation method has been outlined before in previous chapters and has been applied to a wide class of linear partial differential equation; the method has been applied directly and straightforward manner to homogeneous and inhomogeneous problems without any restrictive assumption or linearization. The method considered the solution as a summation of an infinite series usually converging to the solution. The homotopy perturbation method will be applied in this chapter to handle nonlinear partial differential equations. An important remark should be made here concerning the representation of the nonlinear terms that appear in the equation, although the linear term is expressed as an infinite series of components. The homotopy perturbation method requires special representation for nonlinear terms such as nonlinear polynomials, trigonometric nonlinearity, hyperbolic nonlinearity, exponential nonlinearity and logarithmic nonlinearity that arise in nonlinear equations. The method introduces a formula logarithm to establish a proper representation for all nonlinear terms, the representation of nonlinear term is necessary to handle the nonlinear equations in an effective and successful way, in the [29] Asghar Ghorbani introduced the a logarithm for calculating the polynomials that expressed the nonlinear terms as polynomials, and this polynomial are called He's polynomials.

3.2.1: Homotopy Perturbation Method

Now consider the functional equation

$$u - N(u) = f \quad (1)$$

Where N is nonlinear operator from Hilbert space H to H , u is unknown function, and f is known function in H .

Consider Eq. (1), in the form,

$$L(v) = v - f(x) - N(v) = 0 \quad (2)$$

With solution $u(x)$. As a possible remedy, we can define homotopy $H(v, p)$ as follows:

$$H(v,0) = F(v), \quad H(v,1) = L(v)$$

Where $F(p)$ is an integral operator with known solution v_0 which can be obtained easily, typically we may choose a convex homotopy in the form

$$H(v, p) = (1 - p)F(v) + pL(v) = 0 \quad (3)$$

And continuously trace implicitly defined curve from starting point $H(v_0,0) = F(v)$, to the solution function $H(u,1) = L(v)$, the embedding parameter p monotonically increase from zero to unit as the trivial problem $F(v) = 0$ is

continuously deformed form to original problem $L(v)=0$, the embedding parameter $p \in [0,1]$ can be considered as an expanding parameter

$$v = v_0 + pv_1 + pv_2 + \dots \quad (4)$$

When $p \rightarrow 1$, Eq. (3), corresponds to Eqs. (2) and (4) becomes the approximate of Eq. (2) i.e.

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (5)$$

Theorem (3.2.1)

Suppose $N(v)$ is nonlinear function, $v = \sum_{k=0}^{\infty} p^k v_k$ then we have

$$\frac{\partial^n}{\partial p^n} N(v)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0} \quad (6)$$

Proof: Since

$$v = \sum_{k=0}^{\infty} p^k v_k = \sum_{k=0}^n p^k v_k + \sum_{k=n+1}^{\infty} p^k v_k \quad (7)$$

We have such result as following

$$\begin{aligned} \frac{\partial^n}{\partial p^n} N(v)_{p=0} &= \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k + \sum_{k=n+1}^{\infty} p^k v_k\right)_{p=0} \\ &= \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0} \end{aligned} \quad (8)$$

Therefore we obtain

$$\frac{\partial^n}{\partial p^n} N(v)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0}$$

Theorem (3.2.2)

The He's polynomial can be calculated from the formula

$$H_n = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0}, \quad n = 0,1,2,\dots \quad (9)$$

Proof:

Taking $F(v) = v(x) - f(x)$ and substituting (2) in to (3) we have

$$H(v, p) = v(x) - f(x) - pN(v) = 0 \quad (10)$$

According to Maclaurine expansion of $N(v)$ with respect to p we have;

$$\begin{aligned} N(v) &= N(v)_{p=0} + \left(\frac{\partial}{\partial p} N(v)_{p=0}\right)p + \left(\frac{1}{2!} \frac{\partial^2}{\partial p^2} N(v)_{p=0}\right)p^2 + \dots \\ &+ \left(\frac{1}{n!} \frac{\partial^n}{\partial p^n} N(v)_{p=0}\right)p^n + \dots \end{aligned} \quad (11)$$

Substituting (4) in to (11), we have;

$$\begin{aligned} N(v) = & N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0} + \left(\frac{\partial}{\partial p} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0}\right) p + \left(\frac{1}{2!} \frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0}\right) p^2 + \dots \\ & + \left(\frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0}\right) p^n + \dots \end{aligned} \quad (12)$$

According to theorem (3.2.1)

$$\begin{aligned} N(v) = & N(v_0) + \left(\frac{\partial}{\partial p} N\left(\sum_{k=0}^1 p^k v_k\right)_{p=0}\right) p + \left(\frac{1}{2!} \frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^2 p^k v_k\right)_{p=0}\right) p^2 + \dots \\ & + \left(\frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0}\right) p^n + \dots \end{aligned} \quad (13)$$

Substituting (4) and (13) in to (10) and equating the indicated powers p , we have

$$\begin{aligned} p^0 : & v_0 - f(x) = 0 \Rightarrow v_0 = f(x), \\ p^1 : & v_1 - N(v_0) = 0 \Rightarrow v_1 = N(v_0), \\ p^2 : & v_2(x) - \frac{\partial}{\partial p} N\left(\sum_{k=0}^1 p^k v_k\right)_{p=0} = 0 \Rightarrow v_2(x) = \frac{\partial}{\partial p} N\left(\sum_{k=0}^1 p^k v_k\right)_{p=0}, \\ p^3 : & v_3(x) - \frac{1}{2!} \frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^2 p^k v_k\right)_{p=0} = 0 \Rightarrow v_3(x) = \frac{1}{2!} \frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^2 p^k v_k\right)_{p=0}, \\ & \vdots \\ p^{n+1} : & v_{n+1}(x) - \frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0} = 0 \Rightarrow v_{n+1}(x) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0}, \end{aligned}$$

Then the He polynomials is defined as follows

$$H_n(v_0, v_1, v_2, \dots) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0}, \quad n = 0, 1, 2, \dots$$

Therefore, the approximate solution obtained by the homotopy perturbation method can be expressed in He polynomials

$$\begin{aligned} u(x) = & f(x) + \underbrace{N(v_0)}_{H_0} + \underbrace{\frac{\partial}{\partial p} N\left(\sum_{k=0}^1 p^k v_k\right)_{p=0}}_{H_1} + \underbrace{\frac{1}{2!} \frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^2 p^k v_k\right)_{p=0}}_{H_2} + \dots \\ & + \underbrace{\frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0}}_{H_n} + \dots \end{aligned} \quad (14)$$

The nonlinear term $N(u)$ can be also expressed in He polynomials

$$N(u) = \sum_{n=0}^{\infty} H_n(v_0, \dots, v_n) = H_0(v_0) + H_1(v_0, v_1) + \dots + H_n(v_0, \dots, v_n) \quad (15)$$

Where

$$H_n(v_0, \dots, v_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N \left(\sum_{k=0}^n p^k v_k \right)_{p=0}, \quad n = 0, 1, 2, \dots$$

Alternatively, the approximate solution can be expressed as follows

$$u(x) = f(x) + \sum_{n=0}^{\infty} H_n(v_0, \dots, v_n) \quad (16)$$

It is interesting to point out that we can obtain He polynomials and the solution simultaneously making the solution procedure much more attractive and fascinating. Now the general formula of He polynomial

$$H_n = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N \left(\sum_{k=0}^n p^k v_k \right)_{p=0}, \quad n = 0, 1, 2, \dots$$

Can be simplified as follows

$$\begin{aligned} H_0 &= N(v_0) \\ H_1 &= v_1 N'(v_0) \\ H_2 &= v_2 N'(v_0) + \frac{1}{2!} v_1^2 N''(v_0) \\ H_3 &= v_3 N'(v_0) + v_1 v_2 N''(v_0) + \frac{1}{3!} v_1^3 N'''(v_0) \\ H_4 &= v_4 N'(v_0) + \left(\frac{1}{2!} v_2^2 + v_1 v_3 \right) N''(v_0) + \frac{1}{2!} v_1^2 v_2 N'''(v_0) + \frac{1}{4!} v_1^4 N^{(4)}(v_0) \end{aligned} \quad (17)$$

Other polynomials can be generated in a similar manner.

Notes (3.2.3): Two important observations can be made here,

First: H_0 depends only on v_0 , H_1 depends only on v_0 and v_1 , H_2 depends only on v_0 , v_1 and v_2 and so on.

Second: the series $\sum_{n=0}^{\infty} H_n$ is general Taylor series about a function v_0 and not about a point as usually used

Proof:

$$\begin{aligned} N(v) &= \sum_{n=0}^{\infty} H_n = H_0 + H_1 + H_2 + \dots \\ &= N(v_0) + v_1 N'(v_0) + \frac{1}{2!} v_1^2 N''(v_0) + v_2 N'(v_0) + v_3 N'(v_0) + v_1 v_2 N''(v_0) + \dots \\ &= N(v_0) + (v_0 + v_1 + v_2 + \dots) N'(v_0) + \frac{1}{2!} (v_1^2 + 2v_1 v_2 + 2v_1 v_3 + v_2^2 + \dots) N''(v_0) \\ &\quad + \frac{1}{3!} (v_1^3 + 3v_1^2 v_2 + 3v_1^2 v_3 + 6v_1 v_2 v_3 + \dots) N'''(v_0) + \dots \\ &= N(v_0) + (v - v_0) N' + \frac{1}{2!} (v - v_0)^2 N''(v_0) + \dots \end{aligned}$$

Then we have

$$N(v) = \sum_{n=0}^{\infty} H_n = \sum_{n=0}^{\infty} \frac{1}{n!} (v - v_0)^n N^{(n)}(v_0). \quad (18)$$

A definition which is called it as He's polynomials has presented by Ghorbani in [29]. The Adomian decomposition method [30,31,32] is a method to solve functional equations [33,34,35,36]. The crucial part of this method is calculating Adomian polynomials. The homotopy perturbation method is used to calculate Adomian Polynomials [37], making the solution procedure in Adomian method remarkable simple and straightforward. It is well-known that the main disadvantage of the Adomian method is the complex and difficult procedure for calculation the so- Adomian polynomials, in [38] Ozis, and Yildirim compared Adomian's method and He's homotopy perturbation method for solving certain nonlinear problems. Li also has shown that the ADM and HPM for solving nonlinear equations are equivalent [39]. Hossein Jafari [40], proved that He's polynomials are only Adomian's polynomials with different name. We will also show that the standard Adomian decomposition method and the standard HPM are equivalent when applied for solving nonlinear functional equations.

In the following an attempt is made to calculate homotopy polynomials for different forms of nonlinearity that may arise in nonlinear ordinary or partial differential equations.

3.2.2: Calculation of Homotopy Polynomials H_n

I. Nonlinear Polynomials

Case 1: $N(u) = u^2$

The polynomials can be obtained as follows:

$$\begin{aligned} H_0 &= u_0^2, \\ H_1 &= 2u_0u_1, \\ H_2 &= 2u_0u_2 + u_1^2, \\ H_3 &= 2u_0u_3 + 2u_1u_2. \end{aligned}$$

Case 2: $N(u) = u^3$

The polynomials are given by

$$\begin{aligned} H_0 &= u_0^3, \\ H_1 &= 3u_0^2u_1, \\ H_2 &= 3u_0^2u_2 + 3u_0u_1^2, \\ H_3 &= 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3. \end{aligned}$$

Case 3: $N(u) = u^4$

Proceeding as before we find

$$\begin{aligned} H_0 &= u_0^4, \\ H_1 &= 4u_0^4 u_1, \\ H_2 &= 4u_0^3 u_2 + 6u_0^2 u_1^2, \\ H_3 &= 4u_0^3 u_3 + 4u_0^3 u_0 + 12u_0^2 u_1 u_2. \end{aligned}$$

In a parallel manner, homotopy polynomials can be calculated for nonlinear polynomials of higher degrees.

II. Nonlinear Derivatives

Case 1: $N(u) = (u_x)^2$

$$\begin{aligned} H_0 &= u_{0_x}^2, \\ H_1 &= 2u_{0_x} u_{1_x}, \\ H_2 &= 2u_{0_x} u_{2_x} + u_{1_x}^2, \\ H_3 &= 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x}. \end{aligned}$$

Case 2: $N(u) = u_x^3$

The homotopy polynomials are given by

$$\begin{aligned} H_0 &= u_{0_x}^3, \\ H_1 &= 3u_{0_x}^2 u_{2_x} + 3u_{0_x} u_{1_x}^2, \\ H_2 &= 3u_{0_x}^2 u_{2_x} + 3u_{0_x} u_{1_x}^2, \\ H_3 &= 3u_{0_x}^2 u_{3_x} + 6u_{0_x} u_{1_x} u_{2_x} + u_{1_x}^3. \end{aligned}$$

Case 3: $N(u) = u u_x$

The homotopy polynomials for this nonlinearity are given by

$$\begin{aligned} H_0 &= u_0 u_{0_x}, \\ H_1 &= u_{0_x} u_1 + u_0 u_{1_x}, \\ H_2 &= u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \\ H_3 &= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0. \end{aligned}$$

III. Trigonometric Nonlinearity

Case 1: $N(u) = \sin u$

The homotopy polynomials of this form of nonlinearity are given by

$$\begin{aligned} H_0 &= \sin u_0, \\ H_1 &= u_1 \cos u_0, \\ H_2 &= u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0, \end{aligned}$$

$$H_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0 .$$

Case 2: $N(u) = \cos u$

Proceeding as before gives

$$H_0 = \cos u_0 ,$$

$$H_1 = -u_1 \sin u_0 ,$$

$$H_2 = -u_2 \sin u_0 - \frac{1}{2!} u_1^2 \cos u_0 ,$$

$$H_3 = -u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{3!} u_1^3 \sin u_0 .$$

IV. Hyperbolic Nonlinearity

Case 1: $N(u) = \sinh u$

The H_n polynomials of this form of nonlinearity are given by

$$H_0 = \sinh u_0 ,$$

$$H_1 = u_1 \cosh u_0 ,$$

$$H_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1^2 \sinh u_0 ,$$

$$H_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 - \frac{1}{3!} u_1^3 \cosh u_0 .$$

Case 2: $N(u) = \cosh u$

The homotopy polynomials are given by

$$H_0 = \cosh u_0 ,$$

$$H_1 = u_1 \sinh u_0 ,$$

$$H_2 = u_2 \sinh u_0 + \frac{1}{2!} u_1^2 \cosh u_0 ,$$

$$H_3 = u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!} u_1^3 \sinh u_0 .$$

V. Exponential Nonlinearity

Case 1: $N(u) = e^u$

The homotopy polynomials for this form of nonlinearity are given by

$$H_0 = e^{u_0} ,$$

$$H_1 = u_1 e^{u_0} ,$$

$$H_2 = \left(u_2 + \frac{1}{2!} u_1^2 \right) e^{u_0} ,$$

$$H_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1^3 \right) e^{u_0} .$$

Case 2: $N(u) = e^{-u}$

Proceeding as before gives

$$H_0 = e^{-u_0},$$

$$H_1 = -u_1 e^{-u_0},$$

$$H_2 = \left(-u_2 + \frac{1}{2!} u_1^2 \right) e^{-u_0},$$

$$H_3 = \left(-u_3 + u_1 u_2 - \frac{1}{3!} u_1^3 \right) e^{-u_0}.$$

VI. Logarithmic Nonlinearity

Case 1: $N(u) = \ln u$, $u > 0$

The H_n polynomials for logarithmic nonlinearity are given by

$$H_0 = \ln u_0,$$

$$H_1 = \frac{u_1}{u_0},$$

$$H_2 = \frac{u_2}{u_0} - \frac{u_1^2}{2 u_0^2},$$

$$H_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{u_1^3}{3 u_0^3}.$$

Case 2: $N(u) = \ln(1+u)$, $-1 < u \leq 1$

The H_n polynomials are given by

$$H_0 = \ln(1 + u_0),$$

$$H_1 = \frac{u_1}{1 + u_0},$$

$$H_2 = \frac{u_2}{u_0} - \frac{u_1^2}{2u_0^2}$$

$$H_3 = \frac{u_3}{1 + u_0} - \frac{u_1 u_2}{(1 + u_0)^2} + \frac{u_1^3}{3 (1 + u_0)^3}$$

3.3: The Nonlinear Advection problem

Nonlinear phenomena have important effects on applied mathematics, physics and issues related to engineering; many such physical phenomena are modeled in terms of nonlinear partial differential equations. For example, the advection Problem which are of the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x, t) \quad u(x, 0) = g(x) \quad (19)$$

Arise in various branches of physics, engineering and applied sciences. The problem has been handled by using the characteristic method and numerical methods such as Fourier series and Runge-Kutta method. In this section, we approach the advection problem by utilizing the homotopy perturbation method to obtain the exact solution [41]. The modified Homotopy perturbation method and the phenomenon of self-canceling noise term will be used where appropriate.

To solve Eq. (19) by (HMP), we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial t} - f(x, t) \right) = 0 \quad (20)$$

Assume the solution of Eq. (20) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (21)$$

Substituting Eq. (21) into Eq. (20) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} + v_0 \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial t} = f(x, t), \quad v_1(x, 0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} = 0, \quad v_2(x, 0) = 0 \\ & \vdots \\ p^j : \quad & \frac{\partial v_j}{\partial t} + \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} = 0, \quad v_j(x, 0) = 0, \quad j \geq 2 \end{aligned} \quad (22)$$

Starting with $v_0(x, y) = u_0(x, y) = g(x)$, as initial approximate and applying the

inverse operator $\int_0^t (\circ) dt$ to above system, we obtain the following recreation

formula,

$$v_1 = \int_0^t \left(-v_0 \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial t} + f(x, t) \right) dt,$$

$$v_j = -\int_0^t \left(\sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} \right) dt, \quad j \geq 2 \quad (23)$$

The approximate solution of (20) can be obtained by setting $p = 1$ in Eq. (21)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (24)$$

Example (3.3.4) Consider the homogeneous Advection problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad u(x,0) = x^2. \quad (25)$$

According to homotopy (36), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial t} \right) = 0 \quad (26)$$

Beginning with $v_0 = u_0 = x^2$ and from recreation formula (23) we have;

$$\begin{aligned} v_1 &= \int_0^t \left(-v_0 \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial t} \right) dt = -2x^3 t \\ v_2 &= \int_0^t \left(-v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} \right) dt = 5x^4 t^2 \\ v_3 &= \int_0^t \left(-v_2 \frac{\partial v_0}{\partial x} - v_1 \frac{\partial v_1}{\partial x} - v_0 \frac{\partial v_2}{\partial x} \right) dt = \frac{42}{3} x^3 t^3 \\ &\vdots \end{aligned}$$

Then the approximate solution of Eq. (25) obtain by setting $p = 1$ in Eq. (21)

$$\begin{aligned} u &= \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \\ u &= x^2 - 2x^3 t + 5x^4 t^2 - \frac{42}{3} x^3 t^3 + \dots \end{aligned} \quad (27)$$

Example (3.3.5) Consider the inhomogeneous Advection problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 2t + x + t^3 + xt^2, \quad u(x,0) = 0. \quad (28)$$

Standard HPM: According to homotopy (20), we have

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial t} - 2t - x - t^3 - xt^2 \right) = 0 \quad (29)$$

Beginning with $v_0 = u_0 = 0$ and from recreation formula (23), we have;

$$\begin{aligned} v_1 &= \int_0^t \left(-v_0 \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial t} + 2t + x + t^3 + xt^2 \right) dt = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\ v_2 &= \int_0^t \left(-v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} \right) dt = 0, \end{aligned}$$

$$v_3 = \int_0^t \left(-v_2 \frac{\partial v_0}{\partial x} - v_1 \frac{\partial v_1}{\partial x} - v_0 \frac{\partial v_2}{\partial x} \right) dt = -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2xt^5}{15} - \frac{7t^6}{72} - \frac{xt^7}{63} - \frac{t^8}{98},$$

⋮

It is important to recall here that the noise terms appear between the components v_1 and v_3 , where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm \frac{t^4}{4}, \pm \frac{xt^3}{3}$ Between the components v_1 and v_3 can be cancelled and the remaining terms of v_1 still satisfy the equation. The exact solution is therefore

$$u = t^2 + xt \quad (30)$$

Modified HPM: To solve Eq. (28) by (MHMP), we construct the following homotopy:

$$\frac{\partial v}{\partial t} + p \left(v \frac{\partial v}{\partial x} - t^3 - xt^2 \right) = 2t + x \quad (31)$$

Assume the solution of Eq. (28) has the form Eq. (21) substituting Eq. (21) in to Eq. (28) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad \frac{\partial v_0}{\partial t} &= 2t + x, & v_0(x,0) &= 0 \\ p^1 : \quad \frac{\partial v_1}{\partial t} + v_0 \frac{\partial v_0}{\partial x} &= t^3 + xt^2, & v_1(x,0) &= 0 \\ p^2 : \quad \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} &= 0, & v_2(x,0) &= 0 \\ & \vdots & & \end{aligned} \quad (32)$$

Applying the inverse operator $\int_0^t (\circ) dt$ to above system, we obtain

$$\begin{aligned} v_0 &= \int_0^t (2t + x) dx = t^2 + xt, \\ v_1 &= \int_0^t \left(-v_0 \frac{\partial v_0}{\partial x} - t^3 - xt^2 \right) dt = 0, \\ v_k &= 0, \quad k \geq 1. \end{aligned}$$

It then follows that the solution is

$$u(x, y) = v_0(x, y) = t^2 + xt \quad (33)$$

This example clearly shows that the solution can be obtained by using two iterations, and hence the volume of calculation is reduce

Example (3.3.6) Consider the inhomogeneous Advection problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\sin(x+t) - \frac{1}{2} \sin 2(x+t), \quad u(x,0) = \cos x. \quad (34)$$

Standard HPM: According to homotopy (20), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial t} + \sin(x+t) + \frac{1}{2} \sin 2(x+t) \right) = 0 \quad (35)$$

Beginning with $v_0 = u_0 = \cos x$ and from recreation formula (23) we have;

$$\begin{aligned} v_1 &= \int_0^t \left(-v_0 \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial t} - \sin(x+t) - \frac{1}{2} \sin 2(x+t) \right) dt \\ &= \frac{1}{2} t \sin 2x + \cos(x+t) - \cos x + \frac{1}{4} \cos 2(x+t) - \frac{1}{4} \cos 2x, \\ v_2 &= \int_0^t \left(-v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} \right) dt \\ &\quad - \frac{1}{4} t^2 \sin x \sin 2x + \frac{1}{2} t^2 \cos x \cos 2x - \sin x \sin(x+t) + \sin^2 x \\ &\quad + \cos x \cos(x+t) + \cos^2 x + t \sin 2x - \frac{1}{8} \sin x \sin 2(x+t) + \frac{1}{8} \sin x \sin 2x \\ &\quad + \frac{1}{4} \cos x \cos 2(x+t) - \frac{1}{4} \cos x \cos 2x + \frac{1}{4} t \sin x \cos 2x + \frac{1}{2} t \cos x \sin 2x, \end{aligned}$$

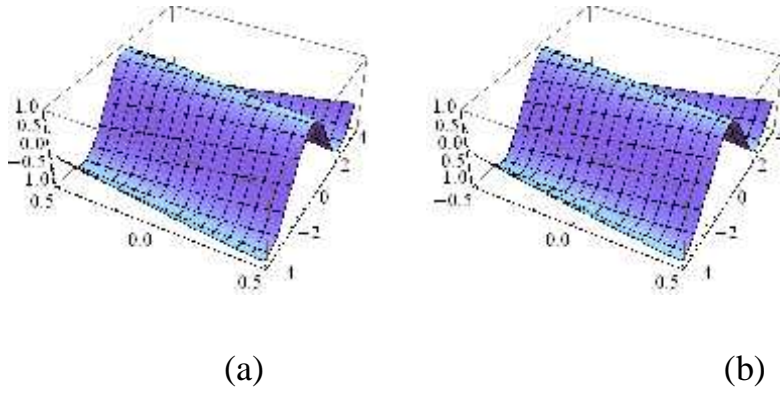
Then the approximate solution of Eq. (34) obtain by setting $p = 1$ in Eq. (21)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

$$\begin{aligned} u &= \frac{1}{16} \left(\cos x - 2t^2 \cos x + 12 \cos 2x + 3 \cos 3x - 6t^2 \cos 3x + 16 \cos(x+t) - \cos(2t+x) \right. \\ &\quad \left. 16 \cos(2x+4) + 4 \cos(2x+2t) - 3 \cos(3x+2t) - 2t \sin x - 8t \sin 2x - 6t \sin 3x \right) + \dots \quad (36) \end{aligned}$$

The behavior of the solution(36) obtained by HPM and the exact solution (39) is shown in Fig (3.3.7) we achieve a good agreement with the actual solution by using two terms only in HPM derived about.

Fig (3.3.7) the surfaces show the exact solution in (a) and the approximate solution given by HPM in (b).



Modified HPM: To solve Eq. (34) by (MHPM), we construct the Following homotopy:

$$\frac{\partial v}{\partial t} + p \left(v \frac{\partial v}{\partial x} - \frac{1}{2} \sin 2(x+t) \right) = -\sin(x+t) \quad (37)$$

Assume the solution of Eq. (37) has the form Eq. (21) substituting Eq. (21) in to Eq. (37) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} = -\sin(x+t), \quad v_0(x,0) = \cos x \\ p^1 : \quad & \frac{\partial v_1}{\partial t} + v_0 \frac{\partial v_0}{\partial x} = -\frac{1}{2} \sin 2(x+t), \quad v_1(x,0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} = 0, \quad v_2(x,0) = 0 \\ & \vdots \end{aligned} \quad (38)$$

Applying the inverse operator $\int_0^t (\circ) dt$ to above system, we obtain

$$\begin{aligned} v_0 &= \cos x + \int_0^t (-\sin(x+t)) dx = \cos(x+t), \\ v_1 &= \int_0^t \left(-v_0 \frac{\partial v_0}{\partial x} + \frac{1}{2} \sin 2(x+t) \right) dt = 0, \\ v_k &= 0, \quad k \geq 1. \end{aligned}$$

It then follows that the solution is

$$u(x, y) = v_0(x, y) = \cos(x+t) \quad (39)$$

Which is an exact solution.

3.4: The Klein-Gordon Equation

The Klein–Gordon and sine-Gordon equations model many problems in classical and quantum mechanics, solitons and condensed matter physics. The Klein–Gordon equation arise in physics in linear and nonlinear form and it is has been extensively studied by using traditional method such as finite difference method, finite element method, Backlund transformations. The parametric finite-difference method, discrete singular convolution algorithm, predictor-corrector scheme, variable separated ODE method tanh method, mapping method, Jacobi elliptic function expansion method and inverse spectral transform, are presented handling the Sin-Gordon equation. Approximate analytical solutions of Klein–Gordon equation such a domain decomposition method, variation iteration method were present to solve the Sine-Gordon equation. In this section, the HPM will be applied to obtain exact solution if exist and approximate to the solution for concrete problems, the modified homotopy perturbation method used where appropriate. [42,43,44,45]

3.4.1: Linear Klein-Gordon Equation

The linear Klein-Gordon equation in its standard form is given by

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + au = h(x, t), \quad (40)$$

Subject to the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \quad (41)$$

In order to solve Eq. (40) with the initial condition (41), (i.e., t – solution) by the HPM, we construct the following homotopy:

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} + av - h(x, t) \right) = 0 \quad (42)$$

Assume the solution of Eq. (42) has the following form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (43)$$

Putting (43) and (41) into (42) and comparing the coefficients of identical degrees of p ,

$$\begin{aligned} p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} &= 0, \\ p^1 : \frac{\partial^2 v_1}{\partial t^2} - \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial t^2} + av_0 &= h(x, t), \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \end{aligned} \quad (44)$$

$$p^2 : \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial^2 v_1}{\partial x^2} + av_1 = 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0$$

$$\vdots$$

We always start with $v_0 = f(x) + t g(x)$ as initial approximate. Applying the inverse operator $\int_0^t \int_0^t (\circ) dt dt$ to above system we obtain the following recreation formula;

$$v_1 = \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - av_0 - \frac{\partial^2 u_0}{\partial t^2} + h(x,t) \right) dt dt,$$

$$v_j = \int_0^t \int_0^t \left(\frac{\partial^2 v_{j-1}}{\partial x^2} - av_{j-1} \right) dt dt, \quad (45)$$

The approximate solution of (41) can be obtained by setting $p = 1$ in Eq. (43)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (46)$$

Example (3.4.8) Consider the homogeneous linear Klein Gordon equation [42]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u, \quad (47)$$

Subject to the initial conditions

$$u(x,0) = 1 + \sin x, \quad \frac{\partial u}{\partial t}(x,0) = 0. \quad (48)$$

According to homotopy (42), we have;

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} - v \right) = 0 \quad (49)$$

Beginning with $v_0 = 1 + \sin x$ and from the recreation formula (39) we have;

$$v_1 = \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} + v_0 - \frac{\partial^2 u_0}{\partial t^2} \right) dt dt = \frac{t^2}{2},$$

$$v_2 = \int_0^t \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} + v_1 \right) dt dt = \frac{t^4}{24},$$

$$v_3 = \int_0^t \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} + v_2 \right) dt dt = \frac{t^6}{720},$$

$$\vdots$$

Hence, the approximate series solution is,

$$u(x, t) = 1 + \sin x + \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720} + \dots \quad (50)$$

And this will, in the limit of infinitely many terms, yield the closed-form solution

$$u(x, t) = \sin x + \cosh t \quad (51)$$

Example (3.4.9) Consider the inhomogeneous linear Klein Gordon equation [42]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 2u = -2 \sin x \sin t \quad (52)$$

Subject to the initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = \sin x. \quad (53)$$

According to homotopy (42), we have;

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} + 2v + 2 \sin x \sin t \right) = 0 \quad (54)$$

Beginning with $v_0 = t \sin x$ and from the recreation formula (39) we have;

$$v_1 = \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - 2v_0 - \frac{\partial^2 u_0}{\partial t^2} - 2 \sin x \sin t \right) dt dt = -\sin x \left(-\frac{t^3}{6} - 2 \sin t \right) - 2t \sin x,$$

$$v_2 = \int_0^t \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} - 2v_1 \right) dt dt = \frac{1}{6} \sin x \left(\frac{t^5}{20} - 12 \sin t - 2t^3 \right) + 2t \sin x,$$

$$v_3 = \int_0^t \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} - 2v_2 \right) dt dt = \frac{1}{120} \sin x \left(\frac{t^7}{42} - +240 \sin t - 2t^5 + 40t^3 \right) - 2t \sin x,$$

⋮

Hence, the approximate series solution is

$$u(x, t) = \sin x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} + \dots \right) \quad (55)$$

And this will, in the limit of infinitely many terms, yield the closed-form solution

$$u(x, t) = \sin x \sin t \quad (56)$$

3.4.2: Nonlinear Klein-Gordon Equation

The nonlinear Klein-Gordon equation in its standard form is given by:

$$\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^2 u}{\partial x^2} + \varsigma u + \chi u^k = h(x, t), \quad (57)$$

Subject to the initial conditions

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x). \quad (58)$$

Where γ , ς and χ are known constants, when $k = 2$ we have quadratic nonlinearity and when $k = 3$ we have cubic nonlinearity

Now to solve Eq. (57) with the initial condition (58), (i.e., t -solution) by the HPM, we construct the following homotopy:

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} + \gamma \frac{\partial^2 v}{\partial x^2} + \varsigma v + \chi v^k - h(x, t) \right) = 0 \quad (59)$$

Assume the solution of Eq. (59) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (60)$$

Putting (58) and (60) into (59) and comparing the coefficient of identical degrees of p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial t^2} + \frac{\partial^2 u_0}{\partial t^2} + \gamma \frac{\partial^2 v_0}{\partial x^2} + \varsigma v_0 + \chi v_0^k = h(x, t), \quad v_1(x, 0) = 0, \quad \frac{\partial v_1}{\partial t}(x, 0) = 0 \quad (61) \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial t^2} + \gamma \frac{\partial^2 v_0}{\partial x^2} + \varsigma v_1 + \chi k v_1 v_0^{k-1} = 0, \quad v_2(x, 0) = 0, \quad \frac{\partial v_2}{\partial t}(x, 0) = 0 \\ & \vdots \end{aligned}$$

We always start with $v_0 = f(x) + t g(x)$ as initial approximate. Applying the

inverse operator $\int_0^t \int_0^t (\circ) dt dt$ to above system we obtain:

$$\begin{aligned} v_1 &= - \int_0^t \int_0^t \left(\gamma \frac{\partial^2 v_0}{\partial x^2} + \varsigma v_0 + \chi v_0^k + \frac{\partial^2 u_0}{\partial x^2} - h(x, t) \right) dt dt, \\ v_2 &= - \int_0^t \int_0^t \left(\gamma \frac{\partial^2 v_1}{\partial x^2} + \varsigma v_1 + \chi k v_1 v_0^{k-1} \right) dt dt, \\ & \vdots \end{aligned} \quad (62)$$

The approximate solution of (57) can be obtained by setting $p = 1$ in Eq. (43)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (63)$$

Example (3.4.10) Consider the inhomogeneous nonlinear Klein Gordon equation [42]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x \cos t + x^2 \cos^2 t, \quad (64)$$

Subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = 0. \quad (65)$$

Standard HPM: According to the homotopy (59), we have;

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} + v^2 + x \cos t - x^2 \cos^2 t \right) = 0 \quad (66)$$

Beginning with $v_0 = x$ and from (62) we have;

$$\begin{aligned} v_1 &= \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - v_0^2 - \frac{\partial^2 u_0}{\partial t^2} - x \cos t + x^2 \cos^2 t \right) dt dt \\ &= -x + \frac{1}{8}x^2 - \frac{3}{4}x^2 t^2 + x \cos t - \frac{1}{8}x^2 \cos 2x, \\ v_2 &= \int_0^t \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} - 2v_0 v_1 \right) dt dt = -\frac{1}{16}x^2 + \frac{1}{8}t^2 - \frac{t^4}{24} - 2x^2 + t^2 x^2 \\ &\quad + \frac{x^3}{16} - \frac{t^2 x^3}{8} + \frac{t^4 x^3}{24} + 2x^2 \cos t - \frac{1}{16} \cos 2x - \frac{1}{16} x^3 \cos 2x, \\ &\quad \vdots \end{aligned}$$

The approximate solution of (57) can be obtained by setting $p = 1$ in Eq. (43)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (66)$$

Modified HPM: To solve Eq. (64) by (MHMP), we construct the Following homotopy;

$$\frac{\partial v^2}{\partial t^2} + p \left(-\frac{\partial^2 v}{\partial x^2} + v^2 - x^2 \cos^2 t \right) = -x \cos t \quad (67)$$

Assume the solution of Eq. (64) has the form Eq. (43) substituting Eq. (43) into Eq. (64) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial t^2} = -x \cos t, \quad v_0(x,0) = x, \quad \frac{\partial v_0}{\partial t}(x,0) = 0 \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial t^2} = \frac{\partial^2 v_0}{\partial x^2} + v_0^2 - x^2 \cos^2 t, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial t^2} = \frac{\partial^2 v_1}{\partial x^2} - 2v_0 v_1 = 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \\ & \quad \vdots \end{aligned} \quad (68)$$

Applying the inverse operator $\int_0^t \int_0^t (\circ) dt dt$ to above system, we obtain:

$$\begin{aligned} v_0 &= x + \int_0^t \int_0^t (x \cos t) dt dt = x \cos t, \\ v_1 &= \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} + v_0^2 - x^2 \cos^2 t \right) dt dt = 0, \\ v_k &= 0, \quad k \geq 1. \end{aligned}$$

It then follows that the solution is

$$u(x, y) = v_0(x, y) = x \cos t \quad (69)$$

Which is an exact solution.

Example (3.4.11) Consider the nonlinear Klein Gordon equation with cubic nonlinearity [42]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + u + u^3 = 2x + xt^2 + x^3 t^6, \quad (70)$$

Subject to the initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0. \quad (71)$$

According to the homotopy (59), we have;

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial t^2} + \frac{\partial^2 v}{\partial x^2} + v + v^3 - 2x - xt^2 - x^3 t^6 \right) = 0 \quad (72)$$

Beginning with $v_0 = 0$ and from (62) we have;

$$\begin{aligned} v_1 &= \int_0^t \int_0^t \left(-\frac{\partial^2 v_0}{\partial x^2} - v_0 - v_0^3 - \frac{\partial^2 u_0}{\partial t^2} + 2x + xt^2 + x^3 t^6 \right) dt dt \\ &= xt^2 + \frac{1}{56} x^3 t^8 + \frac{1}{12} xt^4 \\ v_2 &= \int_0^t \int_0^t \left(-\frac{\partial^2 v_1}{\partial x^2} - v_1 - v_1^3 - 3v_0 v_1 \right) dt dt = -\frac{1}{56} x^3 t^8 - \frac{1}{12} xt^4 - \frac{1}{360} xt^6 - \frac{1}{840} xt^{10} \\ &\quad - \frac{1}{336} x^3 t^{10} - \frac{1}{6336} x^3 t^{12} - \frac{1}{31496} x^3 t^{14} + \frac{3}{10192} x^5 t^{14} - \frac{1}{26880} x^5 t^{16} \\ &\quad - \frac{1}{822528} x^5 t^{18} + \frac{3}{1191680} x^7 t^{20} - \frac{1}{5795328} x^7 t^{22} - \frac{1}{114150400} x^9 t^{26} \\ &\quad \vdots \end{aligned}$$

It is obvious that the self-canceling ‘noise’ terms appear between various components, looking into the last terms v_1 and the first term v_2 is the self-

canceling ‘noise’ terms. Hence, the non-noise term in v_1 yields the exact solution of Equation (70), given by

$$u(x,t) = xt^2 \quad (73)$$

3.4.2: Sine-Gordon Equation

The standard form of sin-Gordon equation is given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + r \sin u = 0 \quad (74)$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \quad (75)$$

Where c and r are constants.

Now to solve Eq. (74) with the initial conditions (75), (i.e., t – solution) by the HPM, the homotopy taking $\sin u = u - \frac{u^3}{6} + \frac{u^5}{120}$, and we construct the following homotopy:

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} + r \left(v - \frac{v^3}{6} + \frac{v^5}{120} \right) \right) = 0 \quad (76)$$

Assume the solution of Eq. (76) has the following form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (77)$$

Putting (77) into (76) and comparing the coefficients of identical degrees of p ,

$$\begin{aligned} p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} &= 0, \\ p^1 : \frac{\partial^2 v_1}{\partial t^2} + \frac{\partial^2 u_0}{\partial t^2} - c^2 \frac{\partial^2 v_0}{\partial x^2} + r v_0 - \frac{r}{6} v_0^3 + \frac{r}{120} v_0^5 &= 0, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \\ p^2 : \frac{\partial^2 v_2}{\partial t^2} - c^2 \frac{\partial^2 v_1}{\partial x^2} + r v_1 - \frac{r}{2} v_0^2 v_1 - \frac{r}{24} v_0^4 v_1 &= 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \\ &\vdots \end{aligned} \quad (78)$$

We always start with $v_0 = f(x) + t g(x)$ as initial approximate. And solving above

system with $\int_0^t \int_0^t (\circ) dt dt$ give an approximate solution

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (79)$$

Example (3.4.12) Consider the sine-Gordon equation [42,43]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0 \quad (80)$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 4 \sec hx. \quad (81)$$

The exact solution is

$$u(x,t) = 4 \tan^{-1}(t \sec hx) \quad (82)$$

According to homotopy (76), we have;

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v - \frac{v^3}{6} + \frac{v^5}{120} \right) = 0 \quad (83)$$

Assume the solution of Eq. (83) has the form (77) substituting (77) into (83)

And comparing the coefficients of identical degrees of p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 v_0}{\partial x^2} + v_0 - \frac{1}{6} v_0^3 + \frac{1}{120} v_0^5 = 0, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial^2 v_1}{\partial x^2} + v_1 - \frac{1}{2} v_0^2 v_1 - \frac{1}{24} v_0^4 v_1 = 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \\ & \vdots \end{aligned} \quad (84)$$

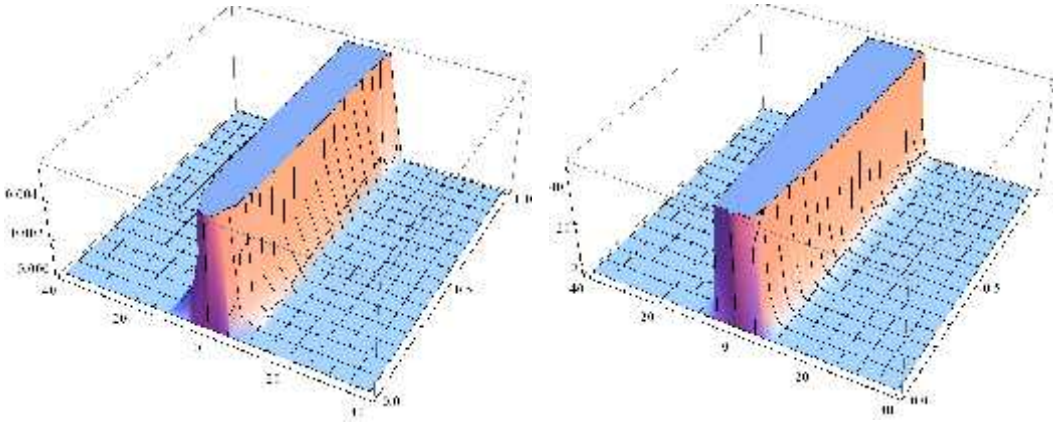
Start with $v_0 = 4t \sec hx$ and integrating above system with $\int_0^t \int_0^t (\circ) dt dt$, we get;

$$\begin{aligned} v_1 &= \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - v_0 + \frac{1}{6} v_0^3 - \frac{1}{120} v_0^5 - \frac{\partial^2 u_0}{\partial t^2} \right) dt dt \\ &= \frac{4 \sec h^5 x}{315} (-16t^7 + 42t^5 \cosh^3 x - 150t^3 \cosh^2 x), \\ v_2 &= \int_0^t \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} - v_0 + \frac{1}{2} v_0 v_1 - \frac{1}{24} v_0^4 v_1 \right) dt dt = \frac{4t^5 \sec h^5 x}{2027025} (7040t^8 - 3369t^6 \cosh^2 x \\ &\quad - 4290t^4 \cosh^4 x + 14300t^4 \cosh^2 x - 2059t^2 \cosh^4 x + 51480t^2 \cosh^6 x \\ &\quad - 270270 \cosh^4 x + 405405 \cosh^4 x), \end{aligned}$$

Hence, the 3-term HPM solution is

$$\begin{aligned} u(x,t) &= \frac{4t \sec h^9 x}{2027025} (7040t^8 - 33696t^{10} \cosh^2 x - 4290t^8 \cosh^4 x + 143000t^8 \cosh 2x \\ &\quad - 308880t^6 \cosh^4 x + 51480t^8 \cosh^4 x + 143000t^6 \cosh^6 x + 405405t^4 \cosh^4 x \\ &\quad - 675675 \cosh^6 x + 2027025 \cosh^8 x) \end{aligned} \quad (85)$$

Fig (3.4.13) shows a very good agreement between the 3-term HPM (85) and the exact solution (82)



(a)

(b)

(a) The exact solution, (b) the approximate solution given by HPM

3.4.3: The Modified of HPM to Sine-Gordon Equation

The sine –Gordon equation it is inevitable that we have to solve equation that involves $\sin u$, this makes it very complicate to solve sine-Gordon equation, to avoid this disadvantage, we apply the modified HP M [43]

Now to solve Eq. (74) by the modified of HPM, we construct the following homotopy;

$$\frac{\partial v^2}{\partial t^2} - pc^2 \frac{\partial^2 v}{\partial x^2} + r \sin(pv) = 0 \quad (86)$$

To obtain the approximate solution of Eq. (86) we consider the Taylor expansion of $\sin v$ in the following

$$\sin v = v - \frac{v^3}{3!} + \frac{v^5}{5!} - \dots \frac{(-1)^n v^{2n-1}}{(2n-1)!} \quad (87)$$

Assume the solution of Eq. (86) has the form (77) substituting (77) and (87) into (86) and comparing the coefficients of identical degrees of p ,

$$\begin{aligned} p^0 : \frac{\partial^2 v_0}{\partial t^2} &= 0, \quad v_1(x,0) = f(x), \quad \frac{\partial v_1}{\partial t}(x,0) = g(x) \\ p^1 : \frac{\partial^2 v_1}{\partial t^2} - c^2 \frac{\partial^2 v_0}{\partial x^2} + r v_0 &= 0, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0 \\ p^2 : \frac{\partial^2 v_2}{\partial t^2} - c^2 \frac{\partial^2 v_1}{\partial x^2} + r v_1 &= 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \end{aligned} \quad (88)$$

$$p^3 : \frac{\partial^2 v_3}{\partial t^2} - c^2 \frac{\partial^2 v_2}{\partial x^2} + r v_2 - r \frac{v_0^3}{3!} = 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0$$

$$\vdots$$

Solving these equations by applying $\int_0^t \int_0^t (\circ) dt dt$, give an approximate solution

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (89)$$

Obviously it is easy calculate more components to improve that accuracy of approximate solution.

Example (3.4.14) Consider the sine –Gordon (80) with initial conditions (81) To solve Eq. (80) by MHPM according to homotopy (86), we have;

$$\frac{\partial v^2}{\partial t^2} - p \frac{\partial^2 v}{\partial x^2} + \sin(pv) = 0 \quad (89)$$

Assume the solution of Eq. (89) has the form (77) substituting (77) and (87) in to (89) and comparing the coefficient of identical degrees of p ,

$$p^0 : \frac{\partial^2 v_0}{\partial t^2} = 0, \quad v_0(x,0) = 0, \quad \frac{\partial v_0}{\partial t}(x,0) = 4 \sec hx$$

$$p^1 : \frac{\partial^2 v_1}{\partial t^2} - \frac{\partial^2 v_0}{\partial x^2} + v_0 = 0, \quad v_1(x,0) = 0, \quad \frac{\partial v_1}{\partial t}(x,0) = 0$$

$$p^2 : \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial^2 v_1}{\partial x^2} + v_1 = 0, \quad v_2(x,0) = 0, \quad \frac{\partial v_2}{\partial t}(x,0) = 0 \quad (90)$$

$$p^3 : \frac{\partial^2 v_3}{\partial t^2} - \frac{\partial^2 v_2}{\partial x^2} + v_2 - \frac{v_0^3}{3!} = 0, \quad v_3(x,0) = 0, \quad \frac{\partial v_3}{\partial t}(x,0) = 0$$

$$\vdots$$

By applying the inverse operator $\int_0^t \int_0^t (\circ) dt dt$, to Eqs. (90) We obtain

$$v_0 = 4t \sec hx,$$

$$v_1 = \int_0^t \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - v_0 \right) dt dt = \frac{-4t^3}{3} \sec h^3 x,$$

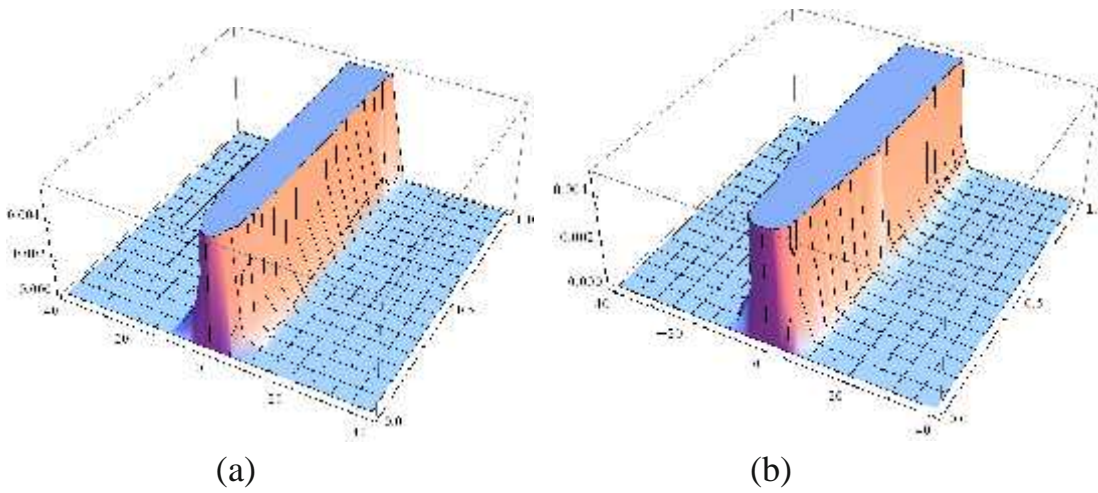
$$v_2 = \int_0^t \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} - v_1 \right) dt dt = \frac{4t^3}{5} (2 - \cosh x) \sec h^3 x,$$

$$v_3 = \int_0^t \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} - v_2 + \frac{v_0^3}{3!} \right) dt dt = -\frac{8t^3}{15} \sec h^3 x - \frac{2t^7}{315} \sec h^4 x (1 - 2 \sec hx),$$

Then this approach leads to the third-order approximation

$$u(x,t) = 4t \sec hx - \frac{4t^3}{3} \sec h^3 x + \frac{4t^5}{15} (2 - \cosh x) \sec h^5 x - \frac{8t^5}{15} \sec h^3 x - \frac{2t^7}{315} \sec h^4 x (1 - 2 \sec hx) \quad (91)$$

Fig (3.4.15) we plot the results for the analytical solution (82) and the approximate solutions (91) obtained with MHPM



(a) The exact solution, (b) the approximate solution given by MHPM

3.5: The Burgers' Equation

Burgers' equation is used to describe various kinds of phenomena such as turbulence and the approximate theory of flow through a shock wave traveling in a viscous fluid; one and two dimensional Burgers' equations are quite famous in wave theory, which has applications in gas dynamics and in plasma physics. Great potential of research work has been invested on Burgers equation. Several exact solutions have been derived by using distinct approaches. In this section, we have employed HPM, to Solve one and two dimensional Burgers' equation [46, 47, 48, 49, 50, 51].

3.5.1: On-Dimensional Burger's Equation

Consider the following one- dimensional Burgers' Equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \quad \epsilon > 0 \text{ is parameter} \quad (92)$$

Subject to the conditions

$$u(x,0) = f_0(x), \quad 0 \leq x \leq l \quad (93)$$

And

$$u(0,t) = f_1(t), \quad u_x(0,t) = f_2(t), \quad t > 0 \quad (94)$$

To solve Eq. (94) by (HMP), we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} - \epsilon \frac{\partial^2 v}{\partial x^2} + \frac{\partial u_0}{\partial t} \right) = 0 \quad (95)$$

Assume the solution of Eq. (95) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (96)$$

Substituting Eq. (96) into Eq. (95) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} + v_0 \frac{\partial v_0}{\partial x} - \epsilon \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial u_0}{\partial t} = 0, \quad v_1(x,0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} - \epsilon \frac{\partial^2 v_1}{\partial x^2} = 0, \quad v_2(x,0) = 0 \\ & \vdots \\ p^j : \quad & \frac{\partial v_j}{\partial t} + \sum_{k=0}^{j-1} \left(v_k \frac{\partial v_{j-k-1}}{\partial x} \right) - \epsilon \frac{\partial^2 v_{j-1}}{\partial x^2} = 0, \quad v_j(x,0) = 0, \end{aligned} \quad (97)$$

Starting with $v_0(x, y) = u_0(x, y) = f_0(x)$, so we derive the following recurrent relation

$$v_j = \int_0^t \left(\epsilon \frac{\partial^2 v_{j-1}}{\partial x^2} - \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} \right) dt, \quad j = 1, 2, 3, \dots \quad (98)$$

An approximate to the solution of (92) can be obtained by setting $p = 1$

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (99)$$

Similarly, to solve Eq. (92) with boundary conditions (94) we construct the following homotopy:

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{\epsilon} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) \right) = 0 \quad (100)$$

With initial approximate $v_0(x, y) = u_0(x, y) = f_1(t) + x f_2(t)$

Suppose the solution of Eq. (100) has the form (96), then

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial x^2} - \frac{1}{\epsilon} \left(\frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} \right) + \frac{\partial^2 u_0}{\partial x^2} = 0, \quad v_1(0, t) = 0, \quad \frac{\partial v_1}{\partial x}(0, t) = 0 \quad (101) \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial x^2} - \frac{1}{\epsilon} \left(\frac{\partial v_1}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} \right) = 0, \quad v_2(0, t) = 0, \quad \frac{\partial v_2}{\partial x}(0, t) = 0 \\ & \vdots \\ p^j : \quad & \frac{\partial^2 v_j}{\partial x^2} - \frac{1}{\epsilon} \left(\frac{\partial v_{j-1}}{\partial t} + \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} \right) = 0, \quad v_j(x, 0) = 0, \quad \frac{\partial v_j}{\partial x}(0, t) = 0 \end{aligned}$$

Which yields

$$v_j = \frac{1}{\epsilon} \int_0^x \int_0^x \left(\frac{\partial v_{j-1}}{\partial t} + \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} \right) dx dx, \quad j = 1, 2, 3, \dots \quad (102)$$

An approximate to the solution of (92) can be obtained by setting $p = 1$

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Example (3.5.16) Consider the following one-dimensional Burgers' equation [46]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad (103)$$

Subject to the initial condition

$$u(x, 0) = 2x \quad (104)$$

According to homotopy (95), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} + \frac{\partial u_0}{\partial t} \right) = 0 \quad (105)$$

Beginning with $v_0 = u_0 = 2x$ and from the recreation formula (98) we have;

$$\begin{aligned} v_1 &= \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} - v_0 \frac{\partial v_0}{\partial x} \right) dt = -4xt, \\ v_2 &= \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} - v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} \right) dt = 8xt^2, \\ v_3 &= \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} - v_2 \frac{\partial v_0}{\partial x} - v_1 \frac{\partial v_1}{\partial x} - v_0 \frac{\partial v_2}{\partial x} \right) dt = -16xt^3, \\ &\vdots \end{aligned}$$

Then the approximate solution of Eq. (103) obtain by setting $p = 1$ in Eq. (96)

$$u(x, t) = 2x - 4xt + 8xt^2 - 16xt^3 + \dots \quad (106)$$

In closed form

$$u(x, t) = \frac{2x}{1 + 2t} \quad (107)$$

Example (3.5.17) Consider the following one-dimensional Burgers' equation [46]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} \quad (108)$$

Subject to the boundary conditions

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(0, t) = \frac{1}{t} - \frac{f^2}{2\epsilon t^2}. \quad (109)$$

According to homotopy (100), we have;

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{\epsilon} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) \right) = 0 \quad (110)$$

Beginning with $v_0 = u_0 = \left(\frac{1}{t} - \frac{f^2}{2\epsilon t^2} \right) x$ and from the recreation formula (102) we

have;

$$\begin{aligned} v_1 &= \frac{1}{\epsilon} \int_0^x \int_0^x \left(\frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} \right) dx dx = \frac{x^3 f^4}{24v^3 t^4}, \\ v_2 &= \frac{1}{\epsilon} \int_0^x \int_0^x \left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_1}{\partial x} \right) dx dx = -\frac{x^5 f^6}{240v^5 t^6}, \\ v_3 &= \frac{1}{\epsilon} \int_0^x \int_0^x \left(\frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_2}{\partial x} \right) dx dx = \frac{17x^7 f^8}{40320v^7 t^8}, \\ &\vdots \end{aligned}$$

Then the approximate solution of Eq. (108) obtain by setting $p = 1$ in Eq. (96)

$$u(x,t) = \frac{x}{t} - \frac{f}{t} \left(\frac{fx}{2vt} - \frac{f^3 x^3}{3(2)^3 v^3 t^3} + \frac{2f^5 x^5}{15(2)^5 v^5 t^5} - \frac{17f^7 x^7}{315(2)^7 v^7 t^7} + \dots \right) \quad (111)$$

In closed form

$$u(x,t) = \frac{x}{t} - \frac{f}{t} \tanh\left(\frac{fx}{2vt}\right) \quad (112)$$

Which is an exact solution.

3.5.2: Two-Dimensional Burgers' Equations

Consider the following system of two-dimensional Burgers' Equations [46]

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{1}{R} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= \frac{1}{R} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \end{aligned} \quad (113)$$

Subject to the conditions

$$\begin{aligned} u(x, y, 0) &= f(x, y), \quad (x, y) \in D, \\ v(x, y, 0) &= g(x, y), \quad (x, y) \in D. \end{aligned} \quad (114)$$

And

$$\begin{aligned} u(x, y, t) &= f_1(x, y, t), \quad (x, y) \in \partial D, \quad t > 0, \\ V(x, y, 0) &= f_2(x, y, t), \quad (x, y) \in \partial D \quad t > 0. \end{aligned} \quad (115)$$

Here, $D = \{(x, y), a \leq x \leq b, a \leq y \leq b\}$, ∂D denotes the boundary of D , $u(x, y, t)$ and $v(x, y, t)$ are the velocity components to be determined, R is the Reynolds number. In order to solve Eq. (113) with the initial conditions (114), (i.e., t -solution) by the HPM, we construct the following homotopy:

$$\begin{aligned} \frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - \frac{1}{R} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) + \frac{\partial u_0}{\partial t} \right) &= 0 \\ \frac{\partial V}{\partial t} - \frac{\partial v_0}{\partial t} + p \left(U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} - \frac{1}{R} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) + \frac{\partial v_0}{\partial t} \right) &= 0 \end{aligned} \quad (116)$$

Suppose the solution of Eq. (116) has the form

$$\begin{aligned} U &= U_0 + pU_1 + p^2U_2 + \dots \\ V &= V_0 + pV_1 + p^2V_2 + \dots \end{aligned} \quad (117)$$

Substituting (17) into (16), and equating the terms with the identical powers of p ,

$$\begin{aligned}
p^0 : & \begin{cases} \frac{\partial U_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ \frac{\partial V_0}{\partial t} - \frac{\partial v_0}{\partial t} = 0, \end{cases} \\
p^1 : & \begin{cases} \frac{\partial U_1}{\partial t} + \frac{\partial u_0}{\partial t} + U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial y} - \frac{1}{R} \left(\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} \right) = 0, U_1(x,0) = 0, \\ \frac{\partial V_1}{\partial t} + \frac{\partial v_0}{\partial t} + U_0 \frac{\partial V_0}{\partial x} + V_0 \frac{\partial V_0}{\partial y} - \frac{1}{R} \left(\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} \right) = 0, V_1(x,0) = 0, \end{cases} \\
& \vdots \\
p^j : & \begin{cases} \frac{\partial U_j}{\partial t} + \sum_{k=0}^{j-1} \left(U_k \frac{\partial U_{j-k-1}}{\partial x} + V_k \frac{\partial U_{j-k-1}}{\partial y} \right) - \frac{1}{R} \left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \frac{\partial^2 U_{j-1}}{\partial y^2} \right) = 0, U_j(x,0) = 0, \\ \frac{\partial V_j}{\partial t} + \sum_{k=0}^{j-1} \left(U_k \frac{\partial V_{j-k-1}}{\partial x} + V_k \frac{\partial V_{j-k-1}}{\partial y} \right) - \frac{1}{R} \left(\frac{\partial^2 V_{j-1}}{\partial x^2} + \frac{\partial^2 V_{j-1}}{\partial y^2} \right) = 0, V_j(x,0) = 0 \end{cases}
\end{aligned} \tag{118}$$

With initial approximate

$$u_0 = U_0 = f(x, y), \quad v_0 = V_0 = g(x, y).$$

And we have the following recurrent equations;

$$\begin{aligned}
U_j &= \frac{1}{R} \int_0^t \left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \frac{\partial^2 U_{j-1}}{\partial y^2} \right) dt - \int_0^t \sum_{k=0}^{j-1} \left(U_k \frac{\partial U_{j-k-1}}{\partial x} + V_k \frac{\partial U_{j-k-1}}{\partial y} \right) dt, \quad j = 1, 2, 3, \dots \\
V_j &= \frac{1}{R} \int_0^t \left(\frac{\partial^2 V_{j-1}}{\partial x^2} + \frac{\partial^2 V_{j-1}}{\partial y^2} \right) dt - \int_0^t \sum_{k=0}^{j-1} \left(U_k \frac{\partial V_{j-k-1}}{\partial x} + V_k \frac{\partial V_{j-k-1}}{\partial y} \right) dt, \quad j = 1, 2, 3, \dots
\end{aligned} \tag{119}$$

An approximation to the solution of (113) can be obtained by setting $p=1$,

$$u = \lim_{p=1} U = U_0 + U_1 + U_2 + \dots$$

$$v = \lim_{p=1} V = V_0 + V_1 + V_2 + \dots$$

Example (3.5.18) Consider the following two-dimensional Burgers' equation (113) ($R=1$) [46] with following initial conditions

$$u(x, y, 0) = x + y$$

$$v(x, y, 0) = x - y \tag{120}$$

According to homotopy (116), we have;

$$\begin{aligned} \frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) + \frac{\partial u_0}{\partial t} \right) &= 0 \\ \frac{\partial V}{\partial t} - \frac{\partial v_0}{\partial t} + p \left(U \frac{\partial V}{\partial x} + V \frac{\partial U}{\partial y} - \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) + \frac{\partial v_0}{\partial t} \right) &= 0 \end{aligned} \quad (121)$$

Start with

$$u_0 = U_0 = x + y$$

$$v_0 = V_0 = x - y$$

And from recreation the formula (119) we have;

$$U_1 = \int_0^t \left(\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} - U_0 \frac{\partial U_0}{\partial x} - V_0 \frac{\partial U_0}{\partial x} \right) dt = -2xt,$$

$$V_1 = \int_0^t \left(\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} - U_0 \frac{\partial V_0}{\partial x} - V_0 \frac{\partial V_0}{\partial x} \right) dt = -2yt,$$

$$U_2 = \int_0^t \left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2} - U_1 \frac{\partial U_0}{\partial x} - U_0 \frac{\partial U_1}{\partial x} - V_1 \frac{\partial U_0}{\partial x} - V_0 \frac{\partial U_1}{\partial x} \right) dt = 2xt^2 + 2yt^2,$$

$$V_2 = \int_0^t \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} - U_1 \frac{\partial V_0}{\partial x} - U_0 \frac{\partial V_1}{\partial x} - V_1 \frac{\partial V_0}{\partial x} - V_0 \frac{\partial V_1}{\partial x} \right) dt = 2xt^2 - 2yt^2,$$

$$U_3 = \int_0^t \left(\frac{\partial^2 U_2}{\partial x^2} + \frac{\partial^2 U_2}{\partial y^2} - U_2 \frac{\partial U_0}{\partial x} - U_1 \frac{\partial U_1}{\partial x} - U_0 \frac{\partial U_2}{\partial x} - V_2 \frac{\partial U_0}{\partial x} - V_1 \frac{\partial U_1}{\partial x} - V_0 \frac{\partial U_2}{\partial x} \right) dt = -4xt^3$$

$$V_3 = \int_0^t \left(\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} - U_2 \frac{\partial V_0}{\partial x} - U_1 \frac{\partial V_1}{\partial x} - U_0 \frac{\partial V_2}{\partial x} - V_2 \frac{\partial V_0}{\partial x} - V_1 \frac{\partial V_1}{\partial x} - V_0 \frac{\partial V_2}{\partial x} \right) dt = -4yt^3,$$

⋮

Then the approximate solution of Eq. (120) obtained by setting $p = 1$ in Eq.

(117)

$$u(x, y, t) = x + y - 2xt + 2xt^2 + 2yt^2 - 4xt^3 + 4xt^4 + 4yt^4 + \dots$$

$$= x(1 + 2t^2 + 4t^4 + \dots) + y(1 + 2t^2 + 4t^4 + \dots) - 2xt(1 + 2t^2 + 4t^4 + \dots) = \frac{x + y - 2xt}{1 - 2t^2} \quad (122)$$

$$v(x, y, t) = x - y - 2yt + 2xt^2 - 2yt^2 - 4yt^3 + 4xt^4 + 4yt^4 + \dots$$

$$= x(1 + 2t^2 + 4t^4 + \dots) - y(1 + 2t^2 + 4t^4 + \dots) - 2yt(1 + 2t^2 + 4t^4 + \dots) = \frac{x - y - 2yt}{1 - 2t^2} \quad (123)$$

Which are exact solution.

3.5.3: (1+2) Burgers' Equations

Example (3.5.19) Consider the following (1+2) Burgers' Equation [47]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u \frac{\partial u}{\partial x} \quad (124)$$

Subject to the condition

$$u(x, y, 0) = x + y \quad (125)$$

To solve Eq. (124) by (HMP), we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\frac{\partial u_0}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - v \frac{\partial v}{\partial x} \right) = 0 \quad (126)$$

Assume the solution of Eq. (124) has the form (96) Substituting Eq. (96) into Eq. (126) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 v_0}{\partial y^2} - v_0 \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial t} = 0, \quad v_1(x, 0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial y^2} - v_0 \frac{\partial v_1}{\partial x} - v_1 \frac{\partial v_0}{\partial x} = 0, \quad v_2(x, 0) = 0 \\ p^3 : \quad & \frac{\partial v_3}{\partial t} - \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial^2 v_2}{\partial y^2} - v_2 \frac{\partial v_0}{\partial x} - v_1 \frac{\partial v_1}{\partial x} - v_0 \frac{\partial v_2}{\partial x} = 0, \quad v_3(x, 0) = 0 \\ & \vdots \end{aligned} \quad (127)$$

Starting with $v_0(x, y) = u_0(x, y) = x + y$, and integrating above system with $\int_0^t (\circ) dt$,

we get;

$$\begin{aligned} v_1 &= \int_0^t \left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} + v_0 \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial t} \right) dt = (x + y)t, \\ v_2 &= \int_0^t \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} - v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} \right) dt = (x + y)^2 t, \\ v_3 &= \int_0^t \left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + v_2 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_2}{\partial x} \right) dt = (x + y)t^3, \end{aligned}$$

An approximate to the solution of (124) can be obtained by setting $p = 1$

$$u = \lim_{p \rightarrow 1} v = (x + y) + (x + y)t + (x + y)t^2 + (x + y)t^3 + \dots \quad (128)$$

The exact solution is expressed as

$$u(x, y, t) = \frac{x + y}{1 - t} \quad (129)$$

3.6: The Nonlinear Schrödinger Equation

The nonlinear Schrödinger equations occur in various areas of physics, including nonlinear optics, plasma physics, superconductivity and quantum mechanics. The Schrödinger equation generally exhibits solitary type solutions. A soliton, or solitary wave, is a wave where the speed of propagation is independent of the amplitude of the wave. Solitons usually occur in fluid mechanics. The inverse scattering method is usually used to handle the nonlinear Schrödinger equation where solitary type solutions were derived.

The nonlinear Schrödinger equation will be handled differently in this section, by using the homotopy perturbation method [52,53,54,55]. Consider the following Schrödinger equation with the following initial condition [52]

$$\begin{aligned} i \frac{\partial u(X,t)}{\partial t} &= -\frac{1}{2} \nabla^2 u + \epsilon_d(X)u + s_d |u|^2 u, \quad X \in \mathfrak{R}^d, t \geq 0 \\ u(X,0) &= u^0(X), \quad X \in \mathfrak{R}^d \end{aligned} \quad (130)$$

Where $\epsilon_d(X)$ is the trapping potential and s_d is a real constant

$$|u|^2 = u\bar{u} \text{ And } \bar{u} \text{ is conjugate of } u$$

To solve Eq. (130) by homotopy perturbation method, we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(i \left(-\frac{1}{2} \nabla^2 v + \epsilon_d(X)v + s_d v^2 \bar{v} \right) + \frac{\partial u_0}{\partial t} \right) = 0 \quad (131)$$

Assume the solution of Eq. (130) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (132)$$

Substituting Eq. (132) into Eq. (131), and equating the terms of like power p ,

$$\begin{aligned} p^0: \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1: \quad & \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} - i \left(\frac{1}{2} \nabla^2 v_0 - \epsilon_d(X)v_0 - s_d v_0^2 \bar{v}_0 \right) = 0, \quad v_1(x,0) = 0, \\ p^2: \quad & \frac{\partial v_2}{\partial t} - i \left(\frac{1}{2} \nabla^2 v_1 - \epsilon_d(X)v_1 - s_d \sum_{i=0}^1 \sum_{k=0}^{1-i} v_i v_k \bar{v}_{1-k-i} \right) = 0, \quad v_2(x,0) = 0, \\ p^3: \quad & \frac{\partial v_3}{\partial t} - i \left(\frac{1}{2} \nabla^2 v_2 - \epsilon_d(X)v_2 - s_d \sum_{i=0}^2 \sum_{k=0}^{2-i} v_i v_k \bar{v}_{2-k-i} \right) = 0, \quad v_3(x,0) = 0, \\ & \vdots \\ p^j: \quad & \frac{\partial v_j}{\partial t} - i \left(\frac{1}{2} \nabla^2 v_{j-1} - \epsilon_d(X)v_{j-1} - s_d \sum_{i=0}^{j-1} \sum_{k=0}^{j-1-i} v_i v_k \bar{v}_{j-k-i-1} \right) = 0, \quad v_j(x,0) = 0. \end{aligned} \quad (133)$$

Starting with $v_0(X, t) = u_0(X, t) = u^0(X)$, having this assumption we get the following recurrent relation recreation

$$v_j = i \int_0^t \left(\frac{1}{2} \nabla^2 v_{j-1} - \epsilon_d(X) v_{j-1} - S_d \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \bar{v}_{j-k-i-1} \right) dt, \quad j = 1, 2, 3, \dots \quad (134)$$

An approximate to the solution of (130) can be obtained by setting $p = 1$,

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Example (3.6.20) Consider the following one-dimensional Schrödinger equation with the following initial condition [52]

$$\begin{aligned} i \frac{\partial u(x, t)}{\partial t} &= -\frac{1}{2} \frac{\partial^2 u}{\partial x^2} - |u|^2 u, \quad t \geq 0 \\ u(x, 0) &= e^{ix}. \end{aligned} \quad (135)$$

According to homotopy (131), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(i \left(-\frac{1}{2} \frac{\partial^2 v}{\partial x^2} - v^2 \bar{v} \right) + \frac{\partial u_0}{\partial t} \right) = 0 \quad (136)$$

Starting with $v_0 = u_0 = e^{ix}$ and by using (134) we obtain the recurrence relation

$$v_j = i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_{j-1}}{\partial x^2} + \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \bar{v}_{j-k-i-1} \right) dt, \quad j = 1, 2, 3, \dots \quad (137)$$

The solution reads

$$\begin{aligned} v_1 &= i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_0}{\partial x^2} + v_0^2 \bar{v}_0 \right) dt = \frac{1}{2} ite^{ix}, \\ v_2 &= i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_1}{\partial x^2} + 2v_0 v_1 \bar{v}_0 + v_0^2 \bar{v}_1 \right) dt = -\frac{1}{8} t^2 e^{ix}, \\ v_3 &= i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_2}{\partial x^2} + 2v_0 v_2 \bar{v}_0 + v_1^2 \bar{v}_0 + 2v_0 v_1 \bar{v}_1 + v_0^2 \bar{v}_2 \right) dt = -\frac{1}{48} it^3 e^{ix}, \\ &\vdots \end{aligned}$$

An approximate to the solution of (135) can be obtained by setting $p = 1$

$$\begin{aligned} u(x, t) &= e^{ix} + \frac{1}{2} ite^{ix} - \frac{1}{8} it^2 e^{ix} - \frac{1}{48} it^3 e^{ix} + \dots \\ &= e^{ix} + \frac{1}{1!} \left(\frac{1}{2} it \right) e^{ix} + \frac{1}{2!} \left(\frac{1}{2} it \right)^2 e^{ix} + \frac{1}{3!} \left(\frac{1}{2} it \right)^3 e^{ix} + \dots \end{aligned} \quad (138)$$

In closed form

$$u(x, t) = e^{i \left(x + \frac{1}{2} t \right)} \quad (139)$$

Example (3.6.21) Consider the following one-dimensional Schrödinger equation with the following initial condition [52]

$$i \frac{\partial u(x,t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u \cos^2 x + |u|^2 u, \quad t \geq 0$$

$$u(x,0) = \sin x. \quad (140)$$

According to homotopy (131), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(i \left(-\frac{1}{2} \frac{\partial^2 v}{\partial x^2} + v \cos^2 x + v^2 \bar{v} \right) + \frac{\partial u_0}{\partial t} \right) = 0 \quad (141)$$

Starting with $v_0 = u_0 = \sin x$ and by using (134) we obtain the recurrence relation

$$v_j = i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_{j-1}}{\partial x^2} - v_{j-1} \cos^2 x - \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \bar{v}_{j-k-i-1} \right) dt, \quad j = 1, 2, 3, \dots \quad (142)$$

The solution reads

$$v_1 = i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_0}{\partial x^2} - v_0 \cos^2 x - v_0^2 \bar{v}_0 \right) dt = -\frac{3}{2} it \sin x,$$

$$v_2 = i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_1}{\partial x^2} - v_1 \cos^2 x - 2v_0 v_1 \bar{v}_0 - v_0^2 \bar{v}_1 \right) dt = -\frac{9}{2} t^2 \sin x,$$

$$v_3 = i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_2}{\partial x^2} - v_2 \cos^2 x - 2v_0 v_2 \bar{v}_0 - v_1^2 \bar{v}_0 - 2v_0 v_1 \bar{v}_1 - v_0^2 \bar{v}_2 \right) dt = \frac{9}{16} it^3 \sin x,$$

$$\vdots$$

An approximate to the solution of (135) can be obtained by setting $p = 1$

$$u(x,t) = \sin x - \frac{3}{2} it \sin x - \frac{9}{2} t^2 \sin x + \frac{9}{16} it^3 \sin x + \dots$$

$$= \sin x + \frac{1}{1!} \left(-\frac{3}{2} it \right) \sin x + \frac{1}{2!} \left(-\frac{3}{2} it \right)^2 \sin x + \frac{1}{3!} \left(-\frac{3}{2} it \right)^3 \sin x + \dots \quad (143)$$

In closed form

$$u(x,t) = e^{-\frac{3}{2} it} \sin x \quad (144)$$

Which is an exact solution.

Example (3.6.22) Consider the following two-dimensional Schrödinger equation [52]

$$i \frac{\partial u(x,t)}{\partial t} = -\frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \epsilon(x,y)u + |u|^2 u, \quad (x,y) \in [0,2f] \times [0,2f]$$

$$u(x,y,0) = \sin x \sin y, \quad (145)$$

Where $\epsilon(x,y) = 1 - \sin^2 x \sin^2 y$

According to homotopy (131), we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(i \left(-\frac{1}{2} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \epsilon(x,y)v + v^2 \bar{v} \right) + \frac{\partial u_0}{\partial t} \right) = 0 \quad (146)$$

Starting with $v_0 = u_0 = \sin x \sin y$ and by using (134) we obtain the recurrence relation

$$v_j = i \int_0^t \left(\frac{1}{2} \left(\frac{\partial^2 v_{j-1}}{\partial x^2} + \frac{\partial^2 v_{j-1}}{\partial y^2} \right) - \epsilon(x,y)v_{j-1} - \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \bar{v}_{j-k-i-1} \right) dt, \quad j = 1, 2, 3, \dots \quad (147)$$

The solution reads

$$v_1 = i \int_0^t \left(\frac{1}{2} \left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} \right) - \epsilon(x,y)v_0 - v_0^2 \bar{v}_0 \right) dt = -2it \sin x \sin y,$$

$$v_2 = i \int_0^t \left(\frac{1}{2} \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} \right) - \epsilon(x,y)v_1 - 2v_0 v_1 \bar{v}_0 - v_0^2 \bar{v}_1 \right) dt = -2t^2 \sin x \sin y,$$

$$v_3 = i \int_0^t \left(\frac{1}{2} \left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} \right) - \epsilon(x,y)v_2 - 2v_0 v_2 \bar{v}_0 - v_1^2 \bar{v}_0 - 2v_0 v_1 \bar{v}_1 - v_0^2 \bar{v}_2 \right) dt = -\frac{4}{3} it^3 \sin x \sin y$$

$$\vdots$$

An approximate to the solution of (135) can be obtained by setting $p = 1$

$$u(x,t) = \sin x \sin y - 2it \sin x \sin y - 2t^2 \sin x \sin y - \frac{4}{3} it^3 \sin x \sin y + \dots$$

$$= \sin x \sin y + \frac{1}{1!} (-2it) \sin x \sin y + \frac{1}{2!} (-2it)^2 \sin x \sin y + \frac{1}{3!} (-2it)^3 \sin x \sin y + \dots \quad (148)$$

In closed form

$$u(x,t) = e^{-2it} \sin x \sin y \quad (149)$$

Which is an exact solution.

3.7: The Goursat problem

The Goursat problem arises in linear and nonlinear partial differential equations with mixed derivatives. The standard form of the Goursat problem is given by:

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial t} &= f(x, t, u, u_x, u_y), & a \leq x \leq b, a \leq t \leq b \\ u(x, 0) &= g(x), \quad u(0, t) = h(t), \quad g(0) = h(0) = u(0, 0). \end{aligned} \quad (150)$$

This equation has been examined by several methods, such as Runge–Kutta method, Adomian decomposition method, variational iteration method and geometric mean averaging, for the functional values of $f(x, t, u, u_x, u_y)$. It is worth to note that the major advantage of He's HPM is that the perturbation equation can be freely constructed in many ways (therefore it is dependent to the problems that are interested) by homotopy in topology and the initial approximation can also be freely selected. In this section we employ the HPM [56,57], to solve linear and nonlinear Goursat problem with different initial conditions

3.7.1: The Homogeneous Linear Goursat Problem

We consider the homogenous linear Goursat problem [56]

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial t} &= f(u), \\ u(x, 0) &= g(x), \quad u(0, t) = h(t), \quad g(0) = h(0) = u(0, 0). \end{aligned} \quad (151)$$

In order to solve Equation (151) by HPM, we construct the following homotopy:

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = p \left(v - \frac{\partial^2 u_0}{\partial x \partial t} \right) = 0 \quad (152)$$

Assume the solution of Eq. (152) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (153)$$

Substituting Eq. (153) into Eq. (152) and equating the terms of like power p ,

$$\begin{aligned}
p^0 &: \frac{\partial^2 v_0}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = 0, \\
p^1 &: \frac{\partial^2 v_1}{\partial x \partial t} = v_0 - \frac{\partial^2 u_0}{\partial x \partial t} = 0, v_1(x,0) = 0, v_1(0,t) = 0, v_1(0,0) = 0 \\
p^2 &: \frac{\partial^2 v_2}{\partial x \partial t} = v_1, v_2(x,0) = 0, v_2(0,t) = 0, v_2(0,0) \\
&\vdots \\
p^j &: \frac{\partial^2 v_j}{\partial x \partial t} = v_{j-1}, v_{j-1}(x,0) = 0, v_{j-1}(0,t) = 0, v_{j-1}(0,0)
\end{aligned} \tag{154}$$

Now start with two different initial approximate, in the first way, we start with

$$v_0 = u_0 = u(x,0) + u(0,t) - u(0,0), \tag{155}$$

And in a second way, we start with

$$v_0 = u_0 = g(x) \tag{156}$$

Integrating (154) with $\int_0^x \int_0^t (\circ) dx dt$, we have the following recurrent equations

$$\begin{aligned}
v_1 &= \int_0^x \int_0^t \left(v_0(x,t) - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt, \\
v_j &= \int_0^x \int_0^t (v_j(x,t)) dx dt, \quad j \geq 2
\end{aligned} \tag{157}$$

An approximate to the solution of (151) can be obtained by setting $p = 1$

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Example (3.7.23) Consider the homogeneous Goursart problem [56]

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial t} &= u, \\
u(x,0) &= e^x, u(0,t) = e^t, u(0,0) = 1.
\end{aligned} \tag{158}$$

According to homotopy (152), we have;

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = p \left(v - \frac{\partial^2 u_0}{\partial x \partial t} \right) = 0 \tag{159}$$

First adaptation of HPM: start with $v_0 = u_0 = e^x + e^t - 1$, as initial approximate, and from Eq. (157) we have;

$$v_1 = \int_0^x \int_0^t \left(v_0(x,t) - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt = te^x + xe^t - x - t - xt,$$

$$v_2 = \int_0^x \int_0^t v_1(x,t) dx dt = \frac{t^2}{2} e^x + \frac{x^2}{2} e^t - \frac{x^2}{2} - \frac{t^2}{2} - \frac{x^2 t}{2} - \frac{xt^2}{2} - \frac{x^2 t^2}{4},$$

$$v_3 = \int_0^x \int_0^t v_2(x,t) dx dt = \frac{t^3}{6} e^x + \frac{x^3}{6} e^t - \frac{x^3}{6} - \frac{t^3}{6} - \frac{x^3 t}{6} - \frac{xt^3}{6} - \frac{x^2 t^3}{6} - \frac{x^3 t^2}{6} - \frac{x^3 t^3}{36},$$

Then the series solutions expression by HPM can be written in the form

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Then, the approximate solution in a series form is

$$u(x,t) = e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) + e^t \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = e^{x+t} \quad (160)$$

Second adaptation of HPM: start with $v_0 = u_0 = g(x) = e^x$, as initial approximate, and from Eq. (157), we have;

$$v_1 = \int_0^x \int_0^t \left(v_0(x,t) - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt = te^x,$$

$$v_2 = \int_0^x \int_0^t v_1(x,t) dx dt = \frac{t^2}{2} e^x,$$

$$v_3 = \int_0^x \int_0^t v_2(x,t) dx dt = \frac{t^3}{6} e^x,$$

An approximate to the solution of (158) can be obtained by setting $p = 1$

$$u(x,t) = e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = e^{x+t} \quad (161)$$

We obtained the solution by choosing a suitable homotopy with different initial conditions. We showed two adaptations of homotopy: firstly we obtained the approximate analytical solution of the equation in the form of a convergent power series with easily computable components and secondly we obtained the exact solution with less computational work compared with first method.

3.7.2: The Inhomogeneous Linear Goursat Problem

We consider the inhomogeneous linear Goursat problem [56]

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial t} &= f(u) + w(x, t), \\ u(x, 0) &= g(x), u(0, t) = h(t), \quad g(0) = h(0) = u(0, 0). \end{aligned} \quad (162)$$

In order to solve Eq. (162) by HPM, we construct the following homotopy:

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = p \left(v + w(x, t) - \frac{\partial^2 u_0}{\partial x \partial t} \right) = 0 \quad (163)$$

Assume the solution of Eq. (163) has the form (153)

Substituting Eq. (153) into Eq. (162) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial x \partial t} = v_0 + w(x, t) - \frac{\partial^2 u_0}{\partial x \partial t} = 0, \quad v_1(x, 0) = 0, v_1(0, t) = 0, v_1(0, 0) = 0 \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial x \partial t} = v_1, \quad v_2(x, 0) = 0, v_2(0, t) = 0, v_2(0, 0) \\ & \vdots \\ p^j : \quad & \frac{\partial^2 v_j}{\partial x \partial t} = v_{j-1}, \quad v_{j-1}(x, 0) = 0, v_{j-1}(0, t) = 0, v_{j-1}(0, 0) \end{aligned} \quad (164)$$

The power of HPM is that we can select the proper zeroth approximation

Now start with two different initial approximate, in the first way we start with

$$v_0 = u_0 = u(x, 0) + u(0, t) - u(0, 0), \quad (165)$$

And preferably by using the boundary conditions in the case

Integrating (154) with $\int_0^x \int_0^t (\circ) dx dt$, we have the following recurrent equations

$$\begin{aligned} v_1 &= \int_0^x \int_0^t \left(v_0(x, t) + w(x, t) - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt, \\ v_j &= \int_0^x \int_0^t (v_{j-1}(x, t)) dx dt, \quad j \geq 2. \end{aligned} \quad (166)$$

Example (3.7.24) Consider the homogeneous Goursart problem [56]

$$\frac{\partial^2 u}{\partial x \partial t} = u - t,$$

$$u(x,0) = e^x, u(0,t) = t + e^t, u(0,0) = 1. \quad (167)$$

According to homotopy (163), we have;

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = p \left(v - t - \frac{\partial^2 u_0}{\partial x \partial t} \right) = 0 \quad (168)$$

First adaptation of HPM: start with $v_0 = u_0 = e^x + e^t + t - 1$, as initial approximate, and from Eq. (166) we have;

$$v_1 = \int_0^x \int_0^t \left(v_0(x,t) - t - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt = te^x + xe^t - x - t - xt,$$

$$v_2 = \int_0^x \int_0^t v_1(x,t) dx dt = \frac{t^2}{2} e^x + \frac{x^2}{2} e^t - \frac{x^2}{2} - \frac{t^2}{2} - \frac{x^2 t}{2} - \frac{xt^2}{2} - \frac{x^2 t^2}{4},$$

$$v_3 = \int_0^x \int_0^t v_2(x,t) dx dt = \frac{t^3}{6} e^x + \frac{x^3}{6} e^t - \frac{x^3}{6} - \frac{t^3}{6} - \frac{x^3 t}{6} - \frac{xt^3}{6} - \frac{x^2 t^3}{6} - \frac{x^3 t^2}{6} - \frac{x^3 t^3}{36},$$

Then, the approximate solution in a series form is

$$\begin{aligned} u(x,t) &= t + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) + e^t \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \\ &\quad - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = t + e^{x+t} \end{aligned} \quad (169)$$

Second adaptation of HPM: start with $v_0 = u_0 = t + e^x$, as initial approximate, and from Eq. (166) we have;

$$v_1 = \int_0^x \int_0^t \left(v_0(x,t) - t - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt = te^x,$$

$$v_2 = \int_0^x \int_0^t v_1(x,t) dx dt = \frac{t^2}{2} e^x,$$

$$v_3 = \int_0^x \int_0^t v_2(x,t) dx dt = \frac{t^3}{6} e^x,$$

An approximate to the solution of (167) can be obtained by setting $p = 1$

$$u(x,t) = t + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = t + e^{x+t} \quad (170)$$

Which is an exact solution of Equation (167)

Example (3.7.25) Consider the homogeneous Goursart problem [56]

$$\frac{\partial^2 u}{\partial x \partial t} = u + 4xt - x^2 t^2, \\ u(x,0) = e^x, u(0,t) = e^t, u(0,0) = 1. \quad (171)$$

According to homotopy (163), we have;

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = p \left(v + 4xt - x^2 t^2 - \frac{\partial^2 u_0}{\partial x \partial t} \right) = 0 \quad (172)$$

First adaptation of HPM: start with $v_0 = u_0 = e^x + e^t - 1$, as initial approximate, and from Eq. (166) we have;

$$v_1 = \int_0^x \int_0^t \left(v_0(x,t) + 4xt - x^2 t^2 - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt = te^x + xe^t - x - t - xt + x^2 t^2 - \frac{x^3 t^3}{9}, \\ v_2 = \int_0^x \int_0^t v_1(x,t) dx dt = \frac{t^2}{2} e^x + \frac{x^2}{2} e^t - \frac{x^2}{2} - \frac{t^2}{2} - \frac{x^2 t}{2} - \frac{xt^2}{2} - \frac{x^2 t^2}{4} + \frac{x^3 t^3}{9} - \frac{x^4 t^4}{144},$$

Then, the approximate solution in a series form is

$$u(x,t) = x^2 t^2 + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) + e^t \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \\ - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = x^2 t^2 + e^{x+t} \quad (172)$$

Second adaptation of HPM: start with $v_0 = u_0 = x^2 t^2 + e^x$, as initial approximate, and from Eq. (166) we have

$$v_1 = \int_0^x \int_0^t \left(v_0(x,t) + 4xt - x^2 t^2 - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt = te^x, \\ v_2 = \int_0^x \int_0^t v_1(x,t) dx dt = \frac{t^2}{2} e^x, \\ v_3 = \int_0^x \int_0^t v_2(x,t) dx dt = \frac{t^3}{6} e^x,$$

An approximate to the solution of (171) can be obtained by setting $p = 1$

$$u(x,t) = x^2 t^2 + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = x^2 t^2 + e^{x+t} \quad (173)$$

Which is an exact solution of Equation (171)

Also we observe that, the second adaptation in the inhomogeneous problems is less computational work compared with first method

3.7.2: The Nonlinear Goursat Problem

We consider the non-linear Goursat problem

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial t} &= N(u), \\ u(x,0) &= g(x), u(0,t) = h(t), \quad g(0) = h(0) = u(0,0). \end{aligned} \quad (174)$$

Where $N(u)$ is nonlinear function

In order to solve Eq. (174) by HPM, we construct the following homotopy:

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = p \left(N(v) - \frac{\partial^2 u_0}{\partial x \partial t} \right) = 0 \quad (175)$$

Assume the solution of Eq. (175) has the form (153)

Substituting Eq. (153) in to Eq. (175) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial x \partial t} = H(v_0) - \frac{\partial^2 u_0}{\partial x \partial t} = 0, \quad v_1(x,0) = 0, v_1(0,t) = 0, v_1(0,0) = 0 \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial x \partial t} = H(v_0, v_1), \quad v_2(x,0) = 0, v_2(0,t) = 0, v_2(0,0) = 0 \\ & \vdots \end{aligned} \quad (176)$$

$$\text{Where } H(v_0, v_1, \dots, v_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N \left(\sum_{i=0}^n v_i p^i \right)_{p=0}$$

Start with $v_0 = u_0 = u(x,0) + u(0,t) - u(0,0)$ and integrating (176) with $\int_0^x \int_0^t (\circ) dx dt$,

then an approximate to the solution of (174), can be obtained by setting $p = 1$,

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Example (3.7.26): Consider the non-linear Goursart problem [56]

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial t} &= e^{x+t} e^u, \\ u(x,0) &= \ln 2 - 2 \ln(1 + e^x), u(0,t) = \ln 2 - 2 \ln(1 + e^t), \quad u(0,0) = -\ln 2. \end{aligned} \quad (177)$$

In order to solve Equation (177) by HPM, we construct the following homotopy:

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = p \left(e^{x+t} e^v - \frac{\partial^2 u_0}{\partial x \partial t} \right) = 0 \quad (178)$$

Assume the solution of Eq. (163) has the form (153)

Substituting Eq. (153) into Eq. (162) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial^2 v_0}{\partial x \partial t} - \frac{\partial^2 u_0}{\partial x \partial t} = 0, \\ p^1 : \quad & \frac{\partial^2 v_1}{\partial x \partial t} = e^{x+t} e^{v_0} - \frac{\partial^2 u_0}{\partial x \partial t} = 0, \quad v_1(x,0) = 0, v_1(0,t) = 0, v_1(0,0) = 0 \\ p^2 : \quad & \frac{\partial^2 v_2}{\partial x \partial t} = e^{x+t} v_1 e^{v_0}, \quad v_2(x,0) = 0, v_2(0,t) = 0, v_2(0,0) = 0 \\ p^3 : \quad & \frac{\partial^2 v_3}{\partial x \partial t} = e^{x+t} \left(\frac{1}{2} v_1^2 + v_2 \right) e^{v_0}, \quad v_3(x,0) = 0, v_3(0,t) = 0, v_3(0,0) = 0 \\ & \vdots \end{aligned} \quad (179)$$

We can start with $v_0 = u_0 = 3 \ln 2 - 2 \ln(1 + e^x) - 2 \ln(1 + e^t)$ and integrating (179)

with $\int_0^x \int_0^t (\circ) dx dt$, we get;

$$\begin{aligned} v_1 &= \int_0^x \int_0^t \left(e^{x+t} e^{v_0} - \frac{\partial^2 u_0}{\partial x \partial t} \right) dx dt = 2 \left[\frac{(e^x - 1)(e^t - 1)}{(e^x + 1)(e^t + 1)} \right], \\ v_2 &= \int_0^x \int_0^t e^{x+t} v_1 e^{v_0} dx dt = \left[\frac{(e^x - 1)(e^t - 1)}{(e^x + 1)(e^t + 1)} \right]^2, \\ & \vdots \end{aligned}$$

And so on to note that the integrals involved above can be obtained by substituting $z = 1 + e^y, dz = e^y dy$ in view

Then the approximate solution in a series form obtained by setting $p = 1$

$$u(x,t) = 3 \ln 2 - 2 \ln(1 + e^x) - 2 \ln(1 + e^t) + 2 \left[\frac{(e^x - 1)(e^t - 1)}{(e^x + 1)(e^t + 1)} \right] + \left[\frac{(e^x - 1)(e^t - 1)}{(e^x + 1)(e^t + 1)} \right]^2 + \dots \quad (180)$$

$$u(x,t) = 3 \ln 2 - 2 \ln(1 + e^x) - 2 \ln(1 + e^t) + 2 \left(\sum_{n=1}^{\infty} \frac{K^n(x,t)}{n} \right) \quad (181)$$

$$\text{Where } K(x,t) = \frac{(e^x - 1)(e^t - 1)}{(e^x + 1)(e^t + 1)}$$

Recall that the Taylor expansion for $\ln(1 - t)$ is given by

$$\ln(1 - y) = - \left(y + \frac{y^2}{2} + \frac{y^3}{3} + \dots \right) = - \sum_{n=1}^{\infty} \frac{y^n}{n} \quad (182)$$

This means that Eq. (181)

$$u(x,t) = 3\ln 2 - 2\ln(1 + e^x) - 2\ln(1 + e^t) + 2\ln[1 - K(x, y)] \quad (183)$$

$$u(x,t) = 3\ln 2 - 2\ln(1 + e^x) - 2\ln(1 + e^t) + 2\ln\left[1 - \frac{(e^x - 1)(e^t - 1)}{(e^x + 1)(e^t + 1)}\right] \quad (184)$$

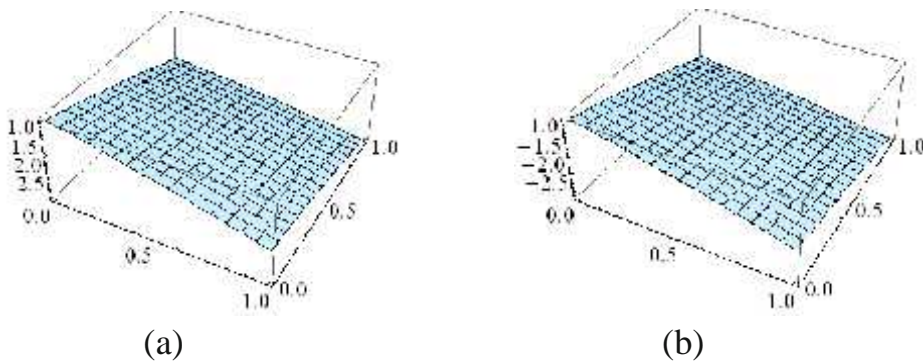
$$u(x,t) = \ln 2 - 2\ln(e^x + e^y) \quad (185)$$

The results for the exact solution (185) and the approximate solution (180) obtained using the HPM are shown in Table (3.7.27) and Fig (3.7.28) it can be seen from Table (3.7.4) that the solution obtained by the HPM is nearly identical with the exact solution. It is to be noted that only the second-order approximate solution was used. To increase the accuracy of the results or to decrease the error, we increase the number of components.

Table (3.7.27) Numerical results of the exact solution (185) and the approximate solution (180)

(x, t)	Exact solution	Approximate solution
(0.0, 0.0)	-0.6931471806	-0.6931471806
(0.2, 0.2)	-1.093147180	-1.093147839
(0.4, 0.4)	-1.493147181	-1.493187785
(0.6, 0.6)	-1.893147179	-1.893582447
(0.8, 0.8)	-2.293147179	-2.295398534
(1.0, 1.0)	-2.693147179	-2.700896101

Fig. (3.7.28): The surfaces show the approximate solutions obtained by HPM and the exact solution respectively. (a) HPM plot (Eq. (180)); (b) Exact plot Eq. (185)



3.8: The Korteweg-de Vries (KdV) Equation

The Korteweg-de Vries (KdV) in the general form given by [58]

$$\frac{\partial u}{\partial t} + \chi u^m \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = f(x, t, u^m, u_x) \quad (186)$$

Where $m = 1, 2$

The KdV equation arises in a number of different physical applications Problems, in the study of shallow water waves, in particular, the KdV equation is used to describe long waves traveling in canals, and the KdV equation has solitary waves as solution hence it can have number of solitons, several numerical and analytical techniques were employed to the KdV equation such as inverse scattering method, Backlund transform method, Adomian decomposition method, and variational iteration method. In this section, we will use HPM to study the nonlinear KdV equation [58,59,60,61,62] the phenomenon of self-canceling “noise terms” will be used where appropriate, now we discuss two special cases for the Eq. (186)

I- for $m = 1$, $\chi = \pm 6$ or ± 1 we obtain one of the standard KdV equations

$$\frac{\partial u}{\partial t} + \chi u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = f(x, t) \quad (187)$$

II- for $m = 2$, $\chi = \pm 6$ or ± 1 equation (186) called modified KdV, (MKdV)

Equation given by

$$\frac{\partial u}{\partial t} + \chi u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = f(x, t) \quad (188)$$

3.8.1: The KdV Equation

Consider the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \chi u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= f(x, t), \\ u(x, 0) &= g(x). \end{aligned} \quad (189)$$

To solve Eq. (189) by (HMP), we construct the following homotopy;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\chi v \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} + \frac{\partial u_0}{\partial t} - f(x, t) \right) = 0 \quad (190)$$

Assume the solution of Eq. (189) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (191)$$

Substituting Eq. (191) into Eq. (189) and equating the terms of like power p ,

$$\begin{aligned}
p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\
p^1 : \quad & \frac{\partial v_1}{\partial t} + \kappa v_0 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_0}{\partial x^3} + \frac{\partial u_0}{\partial t} = f(x,t), \quad v_1(x,0) = 0 \\
p^2 : \quad & \frac{\partial v_2}{\partial t} + \kappa \left(v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} \right) + \frac{\partial^3 v_1}{\partial x^3} = 0, \quad v_2(x,0) = 0 \\
& \vdots \\
p^j : \quad & \frac{\partial v_j}{\partial t} + \frac{\partial^3 v_{j-1}}{\partial x^3} + \sum_{k=0}^{j-1} \left(\kappa v_k \frac{\partial v_{j-k-1}}{\partial x} \right) = 0, \quad v_j(x,0) = 0, \quad j \geq 2
\end{aligned} \tag{192}$$

Starting with $v_0(x, y) = u_0(x, y) = g(x)$, having this assumption we get the following iterative equations;

$$\begin{aligned}
v_1 &= -\int_0^t \left(\kappa v_0 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_0}{\partial x^3} + \frac{\partial u_0}{\partial t} - f(x,t) \right) dt, \\
v_j &= -\int_0^t \left(\frac{\partial^3 v_{j-1}}{\partial x^3} + \sum_{k=0}^{j-1} \kappa v_k \frac{\partial v_{j-k-1}}{\partial x} \right) dt, \quad j \geq 2
\end{aligned} \tag{193}$$

An approximate to the solution of (189) can be obtained by setting $p = 1$

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Example (3.8.29) Consider the special case of homogeneous nonlinear KdV equation [58]

$$\begin{aligned}
\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= 0, \\
u(x,0) &= \frac{x}{6}.
\end{aligned} \tag{194}$$

According to the homotopy (192) we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\frac{\partial u_0}{\partial t} - 6v \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} \right) = 0 \tag{195}$$

We start with $v_0 = u_0 = \frac{x}{6}$, as initial approximate, and from Eq. (193) we have;

$$\begin{aligned}
v_1 &= \int_0^t \left(6v_0 \frac{\partial v_0}{\partial x} - \frac{\partial^3 v_0}{\partial x^3} - \frac{\partial u_0}{\partial t} \right) dt = \frac{xt}{6}, \\
v_2 &= \int_0^t \left(6 \left(v_1 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_1}{\partial x} \right) - \frac{\partial^3 v_1}{\partial x^3} \right) dt = \frac{xt^2}{6}, \\
v_3 &= \int_0^t \left(6 \left(v_2 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_2}{\partial x} \right) - \frac{\partial^3 v_2}{\partial x^3} \right) dt = \frac{xt^3}{6}, \\
&\vdots
\end{aligned}$$

The approximate solution can be obtained by setting $p = 1$ in Eq. (191)

$$u = \lim_{p \rightarrow 1} v = \frac{x}{6} + \frac{xt}{6} + \frac{xt^2}{6} + \frac{xt^3}{6} + \dots \quad (196)$$

This series has closed form

$$u(x,t) = \frac{x}{6(1-t)} \quad (197)$$

Which is the exact solution of the problem

Example (3.8.30) Consider the special case of homogeneous nonlinear KdV equation [58]

$$\begin{aligned}
\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= 0, \\
u(x,0) &= \frac{-k^2}{2} \operatorname{sech}^2 \left[\frac{k}{2} x \right].
\end{aligned} \quad (198)$$

According to homotopy (192) we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\frac{\partial u_0}{\partial t} - 6v \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} \right) = 0 \quad (199)$$

We start with $v_0 = u_0 = \frac{-k^2}{2} \operatorname{sech}^2 \left[\frac{k}{2} x \right]$, as initial approximate, and from Eq.

(193) we have;

$$\begin{aligned}
v_1 &= \int_0^t \left(6v_0 \frac{\partial v_0}{\partial x} - \frac{\partial^3 v_0}{\partial x^3} - \frac{\partial u_0}{\partial t} \right) dt = \frac{-k^5}{2} \operatorname{sech}^2 \left[\frac{k}{2} x \right] \tanh \left[\frac{k}{2} x \right] t, \\
v_2 &= \int_0^t \left(6 \left(v_1 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_1}{\partial x} \right) - \frac{\partial^3 v_1}{\partial x^3} \right) dt = \frac{k^8}{8} \operatorname{sech}^4 \left[\frac{k}{2} x \right] (2 - \cosh[kx]) t^2, \\
v_3 &= \int_0^t \left(6 \left(v_2 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_2}{\partial x} \right) - \frac{\partial^3 v_2}{\partial x^3} \right) dt = \frac{k^{11}}{48} \operatorname{sech} \left[\frac{k}{2} x \right] \left(11 \sinh \left[\frac{k}{2} x \right] - \sinh \left[\frac{3k}{2} x \right] \right) t^3 \\
&\vdots
\end{aligned}$$

The approximate solution can be obtained by setting $p = 1$ in Eq. (191)

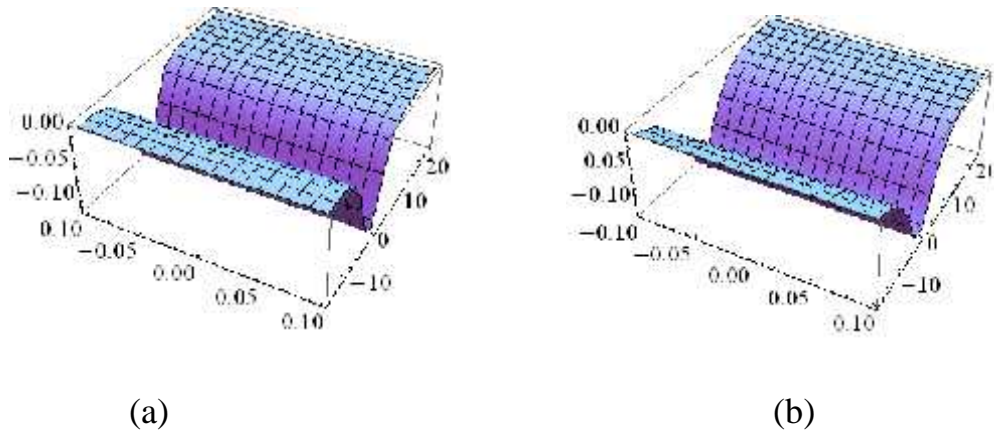
$$u = \frac{-k^2}{2} \operatorname{sech}^2\left[\frac{k}{2}x\right] - \frac{k^5}{2} \operatorname{sech}^2\left[\frac{k}{2}x\right] \tanh\left[\frac{k}{2}x\right] t + \frac{k^8}{8} \operatorname{sech}^4\left[\frac{k}{2}x\right] (2 - \cosh[kx]) t^2 + \frac{k^{11}}{48} \operatorname{sech}^4\left[\frac{k}{2}x\right] \left(11 \sinh\left[\frac{k}{2}x\right] - \sinh\left[\frac{3k}{2}x\right]\right) t^3 + \dots \quad (200)$$

This solution is convergent to the exact solution

$$u(x,t) = \frac{-k^2}{2} \operatorname{sech}^2\left[\frac{k}{2}(x - k^2 t)\right] \quad (201)$$

The behavior of the solution(200) obtained by HPM and the exact solution (201) is shown in Fig (3.8.31) we achieve a good agreement with the actual solution by using four terms only in HPM derived about.

Fig (3.8.31) The surfaces show the approximate solutions obtained by HPM and the exact solution respectively. (a) HPM plot (Eq. (200)); (b) Exact plot Eq. (201)



Example (3.8.32) Consider the special case of homogeneous nonlinear KdV equation [58]

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= \sin x + t \cos x(t \sin x - 1), \\ u(x,0) &= 0.\end{aligned}\tag{202}$$

According to the homotopy (192) we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\frac{\partial u_0}{\partial t} + v \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - \sin x - t \cos x(t \sin x - 1) \right) = 0\tag{203}$$

We start with $v_0 = u_0 = 0$, as initial approximate, and from Eq. (193) we have;

$$\begin{aligned}v_1 &= \int_0^t \left(-v_0 \frac{\partial v_0}{\partial x} - \frac{\partial^3 v_0}{\partial x^3} - \frac{\partial u_0}{\partial t} + \sin x + t \cos x(t \sin x - 1) \right) dt \\ &= t \sin x - \frac{1}{2} t^2 \cos x + \frac{1}{3} t^3 \sin x \cos x, \\ v_2 &= \int_0^t \left(-v_1 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_1}{\partial x} - \frac{\partial^3 v_1}{\partial x^3} \right) dt \\ &= \frac{1}{2} t^2 \cos x - \frac{1}{3} t^3 \sin x \cos x + \frac{1}{3} t^4 \cos 2x + \frac{1}{6} t^3 \sin x, \\ &\vdots\end{aligned}$$

We can easily observe that the last two terms in v_1 and the first two terms in v_2 are the self-canceling ‘noise’ terms. Hence, the non-noise term in v_1 yields the exact solution of Equations (202), given by

$$u(x,t) = t \sin x\tag{204}$$

This can be justified through substitution .It is worth mentioning that the remaining ‘noise’ terms of v_2 will be canceled by other noise terms of the other components v_j , $j \geq 2$.

3.8.1: The Modified KdV Equation (MKdV)

Consider the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \kappa u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= f(x, t), \\ u(x, 0) &= g(x). \end{aligned} \quad (205)$$

To solve Eq. (189) by (HMP), we construct the following homotopy

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\kappa v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} + \frac{\partial u_0}{\partial t} - f(x, t) \right) = 0 \quad (206)$$

Assume the solution of Eq. (206) has form Eq. (191) Substituting Eq. (191) Into Eq. (206) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} + \kappa v_0^2 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_0}{\partial x^3} + \frac{\partial u_0}{\partial t} = f(x, t), \quad v_1(x, 0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} + \kappa \left(2v_0 v_1 \frac{\partial v_0}{\partial x} + v_0^2 \frac{\partial v_1}{\partial x} \right) + \frac{\partial^3 v_1}{\partial x^3} = 0, \quad v_2(x, 0) = 0 \\ & \vdots \\ p^j : \quad & \frac{\partial v_j}{\partial t} + \frac{\partial^3 v_{j-1}}{\partial x^3} + \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} \left(\kappa v_i v_k \frac{\partial v_{j-k-i-1}}{\partial x} \right) = 0, \quad v_j(x, 0) = 0, \quad j \geq 2 \end{aligned} \quad (207)$$

Starting with $v_0(x, y) = u_0(x, y) = g(x)$, having this assumption we get the following iterative equations:

$$\begin{aligned} v_1 &= -\int_0^t \left(\kappa v_0^2 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_0}{\partial x^3} + \frac{\partial u_0}{\partial t} - f(x, t) \right) dt, \\ v_j &= -\int_0^t \left(\frac{\partial^3 v_{j-1}}{\partial x^3} + \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} \left(\kappa v_i v_k \frac{\partial v_{j-k-i-1}}{\partial x} \right) \right) dt, \quad j \geq 2 \end{aligned} \quad (208)$$

An approximate to the solution of (205) can be obtained by setting $p = 1$

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Example (3.8.33) Consider the inhomogeneous MKdV equation [58]

$$\begin{aligned} \frac{\partial u}{\partial t} - u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} &= x(1 - xt^3), \\ u(x,0) = g(x) &= 0. \end{aligned} \quad (209)$$

According to homotopy (206) we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\frac{\partial u_0}{\partial t} - v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} + \frac{\partial u_0}{\partial t} - x(1 - xt^3) \right) = 0 \quad (210)$$

We start with $v_0 = u_0 = 0$, as initial approximate, and from Eqs. (206) we have;

$$\begin{aligned} v_1 &= \int_0^t \left(v_0^2 \frac{\partial v_0}{\partial x} - \frac{\partial^3 v_0}{\partial x^3} - \frac{\partial u_0}{\partial t} + x(1 - xt^3) \right) dt = xt - \frac{x^2 t^4}{4}, \\ v_2 &= \int_0^t \left(2v_0 v_1 \frac{\partial v_0}{\partial x} + v_0^2 \frac{\partial v_1}{\partial x} - \frac{\partial^3 v_1}{\partial x^3} \right) dt = \frac{x^2 t^4}{4} - \frac{x^3 t^7}{7} + \frac{x^4 t^{10}}{32} - \frac{x^5 t^{13}}{416}, \\ &\vdots \end{aligned}$$

We can easily observe that the last term in v_1 and the first term in v_2 are the self-canceling ‘noise’ terms. Hence, the non-noise term in v_1 yields the exact solution of Equations (209), given by

$$u(x,t) = xt \quad (211)$$

Which is an exact solution of the problem

3.9: The K (n, n) Equation

The genuinely nonlinear dispersive equation K(n,n) which generalizes of KdV given by:

$$u_t + (u^n)_x + (u^n)_{xxx} = 0, \quad n, a > 1 \quad (212)$$

The K(n,n) equation is characterized by the genuinely nonlinear term $(u^n)_x$, and genuinely nonlinear dispersion term, $(u^n)_{xxx}$, the balance between them gives rise to the so-called compacton, solitary wave with compact support and without tails or wings. In this section, we will use HPM [63,64,65] to derive the numerical and exact compacton solution of the nonlinear dispersive K(n,n) equation of the following initial conditions:

$$\begin{aligned} u_t + (u^n)_x + (u^n)_{xxx} &= 0, \\ u(x,0) &= f(x). \end{aligned} \quad (213)$$

To solve Eq. (213) by (HMP), we construct the following homotopy

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\frac{\partial}{\partial x} (v^n) + \frac{\partial^3}{\partial x^3} (v^n) + \frac{\partial u_0}{\partial t} \right) = 0 \quad (214)$$

Assume the solution of Eq. (214) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (215)$$

Substituting Eq. (191) into Eq. (189) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} + \frac{\partial}{\partial x} (v_0^n) + \frac{\partial^3}{\partial x^3} (v_0^n) + \frac{\partial u_0}{\partial t} = 0, \quad v_1(x,0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} + \frac{\partial}{\partial x} (n v_1 v_0^{n-1}) + \frac{\partial^3}{\partial x^3} (n v_1 v_0^{n-1}) = 0, \quad v_2(x,0) = 0 \\ & \vdots \end{aligned} \quad (216)$$

Starting with $v_0(x, y) = u_0(x, y) = f(x)$ and Applying the inverse operator $\int_0^t (\circ) dt$ to

Above system, we obtain:

$$\begin{aligned}
v_1 &= -\int_0^t \left(\frac{\partial}{\partial x} (v_0^n) + \frac{\partial^3}{\partial x^3} (v_0^n) + \frac{\partial u_0}{\partial t} \right) dt, \\
v_2 &= -\int_0^t \left(\frac{\partial}{\partial x} (nv_1 v_0^{n-1}) + \frac{\partial^3}{\partial x^3} (nv_1 v_0^{n-1}) \right) dt, \\
&\vdots
\end{aligned} \tag{217}$$

An approximate to the solution of (213) can be obtained by setting $p = 1$

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

Example (3.9.34) Consider the K(2,2) equation [63]

$$\begin{aligned}
u_t + (u^2)_x + (u^2)_{xxx} &= 0, \\
u(x,0) &= \frac{4}{3} c \cos^2\left(\frac{1}{4}x\right).
\end{aligned} \tag{218}$$

According to homotopy (214) we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(\frac{\partial}{\partial x} (v^2) + \frac{\partial^3}{\partial x^3} (v^2) + \frac{\partial u_0}{\partial t} \right) = 0 \tag{219}$$

We start with $v_0 = u_0 = \frac{4}{3} c \cos^2\left(\frac{1}{4}x\right)$, as initial approximate, and from Eqs. (217)

we have;

$$\begin{aligned}
v_1 &= -\int_0^t \left(\frac{\partial}{\partial x} (v_0^2) + \frac{\partial^3}{\partial x^3} (v_0^2) + \frac{\partial u_0}{\partial t} \right) dt = \frac{1}{2} c^2 t \sin\left(\frac{1}{2}x\right), \\
v_2 &= -\int_0^t \left(\frac{\partial}{\partial x} (2v_0 v_1) + \frac{\partial^3}{\partial x^3} (2v_0 v_1) \right) dt = -\frac{1}{12} c^3 t^2 \cos\left(\frac{1}{2}x\right), \\
v_3 &= -\int_0^t \left(\frac{\partial}{\partial x} (v_1^2 + 2v_0 v_2) + \frac{\partial^3}{\partial x^3} (v_1^2 + 2v_0 v_2) \right) dt = -\frac{1}{72} c^4 t^3 \sin\left(\frac{1}{2}x\right), \\
&\vdots
\end{aligned}$$

The approximate solution will be as follows:

$$u(x,t) = \frac{4}{3} c \cos^2\left(\frac{1}{4}x\right) + \frac{1}{2} c^2 t \sin\left(\frac{1}{2}x\right) - \frac{1}{12} c^3 t^2 \cos\left(\frac{1}{2}x\right) - \frac{1}{72} c^4 t^3 \sin\left(\frac{1}{2}x\right) + \dots \tag{220}$$

This gives the solution in a close form

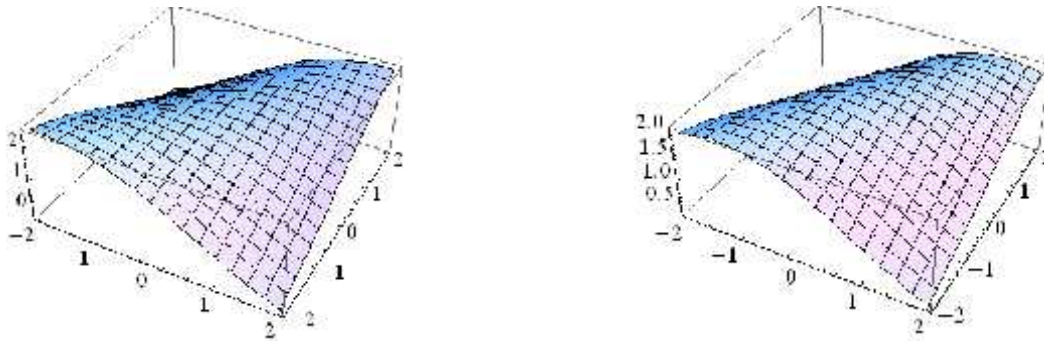
$$u(x,t) = \begin{cases} \frac{4}{3} c \cos^2\left(\frac{1}{4}(x-ct)\right) & , |x-ct| \leq 2f \\ 0 & , \text{other wise} \end{cases} \tag{221}$$

Which is an exact solution

The behavior of the solution (220) obtained by the HPM and the exact solution (221) are shown in Figs. (3.9.35) and (3.9.36) we have plotted these equations with some different values of c, t versus distance x .

Fig (3.9.35)

The surfaces show the approximate solutions obtained by HPM and the exact solution, respectively. (a) HPM plot (Eq. (220)); (b) Exact plot (Eq. (221)).

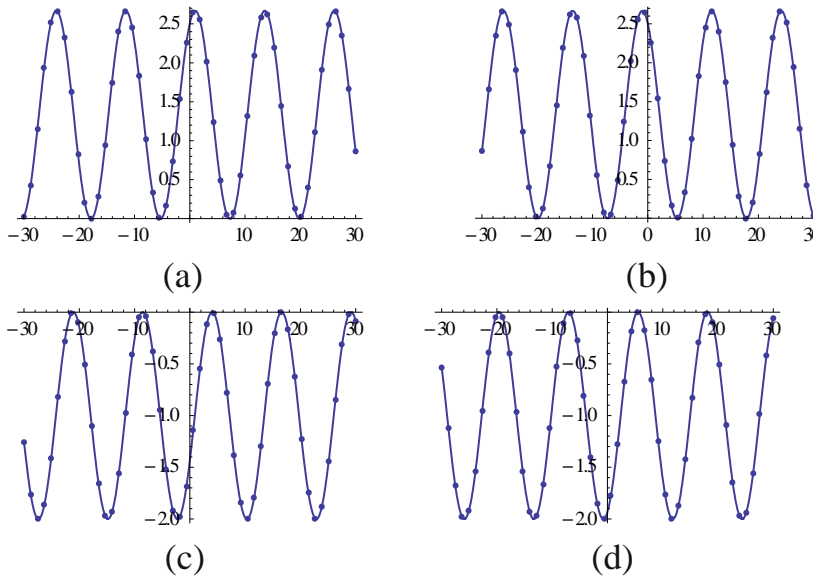


(a)

(b)

Fig (3.9.36)

The comparison of the results by HPM and the exact solutions for different values of c and t , versus distance x (a) $c = 2, t = (1/2)$; (b) $c = 2, t = (-1/2)$; (c) $c = (3/2), t = (3/2)$; (d) $c = (-3/2), t = (-1/2)$.



(a)

(b)

(c)

(d)

Example (3.9.37) Consider the K(3,3) equation [63]

$$u_t + (u^3)_x + (u^3)_{xxx} = 0,$$

$$u(x,0) = \sqrt{\frac{3c}{2}}c \cos\left(\frac{1}{3}x\right). \quad (222)$$

According to homotopy (214) we have;

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(\frac{\partial}{\partial x}(v^3) + \frac{\partial^3}{\partial x^3}(v^3) + \frac{\partial u_0}{\partial t}\right) = 0 \quad (223)$$

We start with $v_0 = u_0 = \sqrt{\frac{3c}{2}}c \cos\left(\frac{1}{3}x\right)$, as initial approximate, and from Eqs. (217) we have;

$$v_1 = -\int_0^t \left(\frac{\partial}{\partial x}(v_0^3) + \frac{\partial^3}{\partial x^3}(v_0^3) + \frac{\partial u_0}{\partial t} \right) dt = \frac{1}{6}c\sqrt{6ct} \sin\left(\frac{1}{3}x\right),$$

$$v_2 = -\int_0^t \left(\frac{\partial}{\partial x}(3v_0^2v_1) + \frac{\partial^3}{\partial x^3}(3v_0^2v_1) \right) dt = -\frac{1}{36}c^2\sqrt{6ct^2} \cos\left(\frac{1}{3}x\right),$$

$$v_3 = -\int_0^t \left(\frac{\partial}{\partial x}(3v_0^2v_2 + 3v_0v_1^2) + \frac{\partial^3}{\partial x^3}(3v_0^2v_2 + 3v_0v_1^2) \right) dt = -\frac{1}{324}c^3\sqrt{6ct^3} \sin\left(\frac{1}{3}x\right)$$

$$\vdots$$

The approximate solution will be as follows;

$$u(x,t) = \sqrt{\frac{3c}{2}}c \cos\left(\frac{1}{3}x\right) + \frac{1}{6}c\sqrt{6ct} \sin\left(\frac{1}{3}x\right) - \frac{1}{36}c^2\sqrt{6ct^2} \cos\left(\frac{1}{3}x\right) - \frac{1}{324}c^3\sqrt{6ct^3} \sin\left(\frac{1}{3}x\right) + \dots \quad (224)$$

This gives the solution in a close form

$$u(x,t) = \begin{cases} \sqrt{\frac{3c}{2}}c \cos\left(\frac{1}{3}(x-ct)\right) & , |x-ct| \leq \frac{3f}{2} \\ 0 & , \text{other wise} \end{cases} \quad (225)$$

Which is exact solution

The behavior of the solution (224) obtained by HPM and the exact solution (225) are shown in Figs (3.9.38) and (3.9.39); with different values of c and t , versus distance x . We achieve a good agreement with the actual solution by using four terms only in homotopy perturbation method derived about.

Fig (3.9.38)

The surfaces show the approximate solutions obtained by HPM and the exact solution, respectively. (a) HPM plot (Eq. (224)); (b) Exact plot (Eq. (225)).

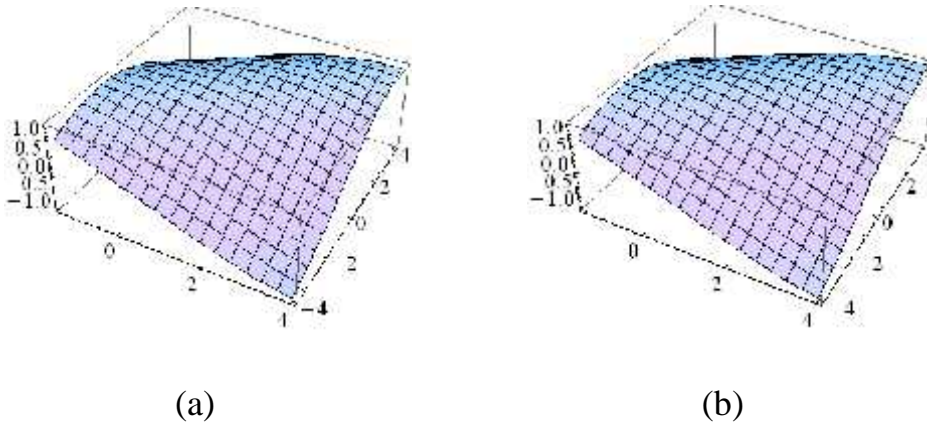
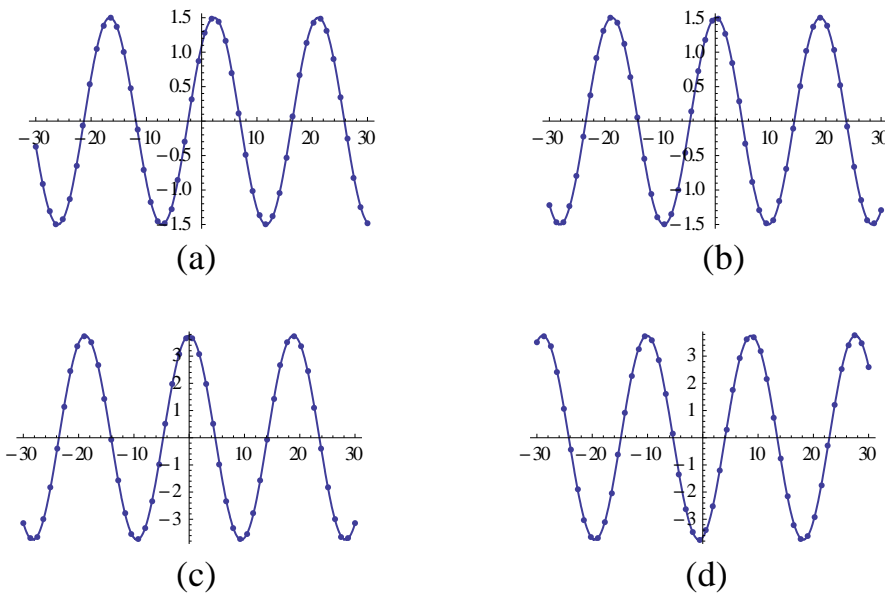


Fig (3.9.39)

The comparison of the results by HPM and the exact solutions for different values of c and t , versus distance x (a) $c = 3/2, t = 3/2$; (b) $c = 3/2, t = (1/10)$ (c) $c = (5/2), t = 0$; (d) $c = (5/2), t = (7/2)$.



CHAPTER FOURE

CONVERGENCE OF THE HOMOTOPY PERTUTBATION METHOD

4.1: Introduction

In this chapter, we will study the convergence of (HPM); the convergence concept of the (HPM) was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. Ji-He examined the convergence of (HPM) in [1]. In addition, J. Biazar presented the sufficient Condition of convergence [68,69]. However, this theorem requires knowledge of the exact solution in prior, in [66,67,70] the Authors have shown that the (HPM) converges to the exact desired solution, without a priori knowledge of the exact solution.

4.2: Theorems of convergence of HPM

To investigate the theorem of the (HPM), we consider the functional equation;

$$A(u) - f(r) = 0, \quad r \in \Omega, \quad (1)$$

With boundary conditions

$$B\left(u, \frac{\partial u}{\partial y}\right) = 0, \quad r \in \Gamma,$$

Where A is differential operator, B is boundary operator, $f(r)$ is a known analytic function, and Γ is boundary of the domain Ω . Generally speaking the operator A can be divided in two parts L and N , where L a linear is, and N is a non linear operator Eq. (1), therefore, can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \quad (2)$$

We construct a homotopy which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1]$$

Or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad (3)$$

Where u_0 is an initial approximation of Eq. (1), assume the solution of Eq. (3)

Has the following form

$$v = v_0 + pv_1 + pv_2 + \dots = \sum_{i=0}^{\infty} p^i v_i \quad (4)$$

When $p \rightarrow 1$, Eq. (3) corresponds to Eqs. (2) and (4) becomes the approximate of Eq. (2) i.e.

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (5)$$

Let's rewrite the Eq. (3) as the following:

$$L(v) - L(u_0) = p[f(r) - L(u_0) - N(v)], \quad (6)$$

Substituting (4) in (6) leads to:

$$L\left(\sum_{i=0}^{\infty} p^i v_i\right) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right], \quad (7)$$

So

$$\sum_{i=0}^{\infty} L(p^i v_i) - L(u_0) = p\left[f(r) - L(u_0) - N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right], \quad (8)$$

Now we set

$$N\left(\sum_{i=0}^{\infty} v_i p^i\right) = \sum_{i=0}^{\infty} H_i p^i \quad (9)$$

Where

$$H_n(v_0, v_1, \dots, v_n) = \left(\frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^n v_i p^i\right)\right)_{p=0}, \quad n = 0, 1, 2, \dots \quad (10)$$

Is He's polynomials, substituting (9) in (8), we drive;

$$\sum_{i=0}^{\infty} L(p^i v_i) - L(u_0) = p\left[f(r) - L(u_0) - \sum_{i=0}^{\infty} H_i p^i\right], \quad (11)$$

By equating the terms with identical powers in p :

$$\begin{cases} p^0 : L(v_0) - L(u_0) = 0, \\ p^1 : L(v_1) = f(r) - L(u_0) - H_0, \\ p^2 : L(v_2) = -H_1, \\ \vdots \\ p^{n+1} : L(v_{n+1}) = -H_n, \\ \vdots \end{cases} \quad (12)$$

So we derive

$$\begin{cases} v_0 = u_0, \\ v_1 = L^{-1}[f(r)] - u_0 - L^{-1}(H_0), \\ v_2 = -L^{-1}(H_1), \\ \vdots \\ v_{n+1} = -L^{-1}(H_n), \\ \vdots \end{cases} \quad (13)$$

Theorem (4.2.1) Homotopy perturbation method used the solution of Eq. (1) is equivalent to determining the following sequence;[66]

$$\begin{aligned} s_n &= v_1 + \dots + v_n, \\ s_0 &= 0, \end{aligned} \quad (14)$$

By using the iterative scheme:

$$s_{n+1} = -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}[f(r)], \quad (15)$$

Where

$$N_n \left(\sum_{i=0}^n v_i \right) = \sum_{i=0}^n H_i, \quad n = 0, 1, 2, \dots \quad (16)$$

Proof: For $n = 0$, from Eq. (15), we have;

$$\begin{aligned} s_1 &= -L^{-1}N_0(s_0 + v_0) - u_0 + L^{-1}[f(r)], \\ &= -L^{-1}(H_0) - u_0 + L^{-1}[f(r)] \end{aligned} \quad (17)$$

Then

$$v_1 = -L^{-1}(H_0) - u_0 + L^{-1}[f(r)] \quad (18)$$

For $n = 1$:

$$\begin{aligned} s_2 &= -L^{-1}N_1(s_1 + v_0) - u_0 + L^{-1}[f(r)], \\ &= -L^{-1}(H_0 + H_1) - u_0 + L^{-1}[f(r)], \end{aligned} \quad (19)$$

Substituting (18) in (19) we get;

$$= -L^{-1}(H_1) + v_1$$

According to $s_2 = v_1 + v_2$, we get;

$$v_2 = -L^{-1}(H_1) \quad (20)$$

This theorem will be proved by strong induction let's assume that

$$v_{k+1} = -L^{-1}(H_k) \quad \text{For } k = 1, 2, \dots, n-1,$$

So

$$\begin{aligned} s_{n+1} &= -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}[f(r)], \\ &= -L^{-1} \left(\sum_{i=0}^n H_i \right) - u_0 + L^{-1}[f(r)] \\ &= -\sum_{i=0}^n L^{-1}(H_i) - u_0 + L^{-1}[f(r)] = v_1 + v_2 + \dots + v_n - L^{-1}(H_n) \end{aligned} \quad (21)$$

Then, from (14), it can drive;

$$v_{n+1} = -L^{-1}(H_n) \quad (22)$$

Which is the same as the result of (13) from HPM, and the theorem is proved.

Theorem (4.2.2) let B be a Banach space [66]

(a) The series solution $\sum_{i=0}^{\infty} v_i$ obtained by (13), convergence to $s \in B$, if

$$\exists(0 < \gamma < 1), \text{ s.t } (\forall n \in N \Rightarrow \|v_n\| \leq \gamma \|v_{n-1}\|) \quad (23)$$

(b) $s = \sum_{i=1}^{\infty} v_i$, satisfies in

$$s = -L^{-1}N(s + v_0) - u_0 + L^{-1}[f(r)] \quad (24)$$

Proof: (a) we have

$$\|s_{n+1} - s_n\| = \|v_{n+1}\| \leq \gamma \|v_n\| \leq \gamma^2 \|v_{n-1}\| \leq \dots \leq \gamma^{n+1} \|v_0\| \quad (25)$$

For any $n, m \in N, n \geq m$, we

$$\begin{aligned} \|s_n - s_m\| &= \|(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \dots + (s_{m+1} - s_m)\| \\ &\leq \|s_n - s_{n-1}\| + \|s_{n-1} - s_{n-2}\| + \dots + \|s_{m+1} - s_m\| \\ &\leq \gamma^n \|v_0\| + \gamma^{n-1} \|v_0\| + \dots + \gamma^{m+1} \|v_0\| \\ &\leq (\gamma^n + \gamma^{n-1} + \dots + \gamma^{m+1}) \|v_0\| \\ &\leq \gamma^{m+1} (\gamma^{n-m-1} + \gamma^{n-m-2} + \dots + \gamma + 1) \|v_0\| \\ &\leq \gamma^{m+1} \left(\frac{1 - \gamma^{n-m}}{1 - \gamma} \right) \|v_0\| \end{aligned} \quad (26)$$

Since $0 < \gamma < 1$, we have $1 - \gamma^{n-m} < 1$; then,

$$\lim_{n, m \rightarrow \infty} \|s_n - s_m\| = 0 \quad (27)$$

Then $\{s_n\}$, is Cauchy sequence in Banach space and it is convergent, i.e.,

$$\exists s \in B, \text{ s.t } = \lim_{n \rightarrow \infty} s_n = \sum_{i=1}^{\infty} v_i = s \quad (28)$$

(b) From Eq. (15), we have;

$$\begin{aligned} \lim_{n \rightarrow \infty} s_{n+1} &= -L^{-1} \lim_{n \rightarrow \infty} N_n(s_n + v_0) - u_0 + L^{-1}[f(r)], \\ &= -L^{-1} \lim_{n \rightarrow \infty} N_n \left(\sum_{i=0}^n v_i \right) - u_0 + L^{-1}[f(r)], \end{aligned} \quad (29)$$

$$s = -L^{-1} \lim_{n \rightarrow \infty} \sum_{i=0}^n H_i - u_0 + L^{-1}[f(r)],$$

$$= -L^{-1} \sum_{i=0}^{\infty} H_i - u_0 + L^{-1}[f(r)],$$

But by Eqs. (9) and (16) for $p = 1$, we drive;

$$\sum_{i=0}^{\infty} H_i = N \left(\sum_{i=0}^{\infty} v_i \right) \quad (30)$$

So

$$s = -L^{-1} N \left(\sum_{i=0}^{\infty} v_i \right) - u_0 + L^{-1}[f(r)],$$

$$s = -L^{-1} N(s + v_0) - u_0 + L^{-1}[f(r)]$$

Lemma (4.2.3) Eq. (24) is equivalent to; [66]

$$L(u) + N(u) - f(r) = 0 \quad (31)$$

Proof: we rewrite Eq. (24) as fallows;

$$s + u_0 = -L^{-1} N(s + v_0) + L^{-1}[f(r)] \quad (32)$$

By applying the operator L to Eq. (32) we derive;

$$L(s + u_0) = -N(s + v_0) + f(r) \quad (33)$$

But $u_0 = v_0$, then;

$$L(s + v_0) = -N(s + v_0) + f(r) \quad (34)$$

By considering $u = s + v_0 = \sum_{i=0}^{\infty} v_i$, Eq. (31), has been derived which is the original Equation. Then solution of Eq. (24) is the same solution of $A(u) - f(r) = 0$.

It is worth mention in other wards we proved in theorem (4.1.2) that the series $\sum_{i=0}^{\infty} p^i v_i$ defined in (4) converges absolutely at $p = 1$ to the solution $s \in B$, over the domain of definition of t , also we proved that if the series solution defined in (5) is convergent, then it converges to the exact solution of the nonlinear Problem (1)

Theorem (4.2.4) Error Estimate [67]

The maximum absolute truncation error of the series solution $\sum_{i=0}^{\infty} v_i$ of the problem (1) is estimated to be

$$E_M \leq \frac{\}^{M+1}}{1-\} \|v_0\| \quad (35)$$

Proof: Making use of inequality (26) of Theorem (4.1.2), we immediately obtain

$$\|u - s_M\| \leq \}^{M+1} \left(\frac{1-\}^{n-M}}{1-\} \right) \|v_0\| \quad (36)$$

And taking into account $1-\}^{n-m} < 1$, Eq. (36) directly leads to the desired formula (35). This completes the proof.

Theorem (4.2.5) (Sufficient Condition of Convergence).[68]

Suppose that X and Y be Banach space and $N : X \rightarrow Y$ is a contraction mapping that is

$$\forall v, \tilde{v} \in X; \|N(v) - N(\tilde{v})\| \leq \} \|v - \tilde{v}\|, \quad 0 < \} < 1 \quad (37)$$

Which according to Banach's fixed point theorem, having the fixed point u , that is $N(u) = u$.

The sequence generated by the HPM will be regarded as,

$$s_n = N(s_{n-1}), \quad s_{n-1} = \sum_{i=0}^{n-1} v_i, \quad n = 1, 2, 3, \dots$$

And suppose that $s_0 = v_0 = u_0 \in B_r(u)$ where $B_r(u) = \{u^* \in X \mid \|u^* - u\| < r\}$, and then we have the following statements:

- (i) $\|s_n - u\| \leq \}^n \|u_0 - u\|$,
- (ii) $s_n \in B_r(u)$,
- (iii) $\lim_{n \rightarrow \infty} s_n = u$.

Proof: (i) By induction method on, for $n = 1$ we have

$$\|s_1 - u\| \leq \|N(s_0) - N(u)\| \leq \} \|v_0 - u\|$$

Assume that $\|s_{n-1} - u\| \leq \}^{n-1} \|v_0 - u\|$ as an induction hypothesis, then

$$\|s_n - u\| = \|N(s_{n-1}) - N(u)\| \leq \} \|s_{n-1} - u\| \leq \} \}^{n-1} \|v_0 - u\| \leq \}^n \|v_0 - u\|$$

(ii) Using (i), we have

$$\|s_n - u\| \leq \}^n \|v_0 - u\| \leq \}^n r < r \Rightarrow s_n \in B_r(u).$$

(iii) Because of $\|s_n - u\| \leq \}^n \|v_0 - u\|$, and $\lim_{n \rightarrow \infty} \}^n = 0$, we drive $\lim_{n \rightarrow \infty} \|s_n - u\| = 0$

$$\lim_{n \rightarrow \infty} s_n = u$$

Example (4.2.6) Consider the following Burgers' equation [46]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad (x, y) \in R \times [0,1), \quad (38)$$

Subject to the initial conditions

$$u(x,0) = 2x. \quad (39)$$

With the exact solution

$$u(x,t) = \frac{2x}{1+2t} \quad (40)$$

To solve Eq. (38) with initial condition (39) by (HMP), we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} + \frac{\partial u_0}{\partial t} \right) = 0 \quad (41)$$

Assume the solution of Eq. (40) has the following form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (42)$$

Substituting Eq. (42) in to Eq. (41) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} + v_0 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial u_0}{\partial t} = 0, \quad v_1(x,0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} = 0, \quad v_2(x,0) = 0 \\ & \vdots \\ p^j : \quad & \frac{\partial v_j}{\partial t} + \sum_{k=0}^{j-1} \left(v_k \frac{\partial v_{j-k-1}}{\partial x} \right) - \frac{\partial^2 v_{j-1}}{\partial x^2} = 0, \quad v_j(x,0) = 0, \end{aligned} \quad (43)$$

Starting with $v_0(x, y) = u_0(x, y) = 2x$, so we derive the following recurrent relation

$$v_j = \int_0^t \left(\frac{\partial^2 v_{j-1}}{\partial x^2} - \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} \right) dt, \quad j = 1,2,3,\dots \quad (44)$$

The solution reads

$$\begin{aligned} v_1(x,t) &= -4xt \\ v_2(x,t) &= 8xt^2 \\ v_3(x,t) &= -16xt^3 \\ & \vdots \\ v_n(x,t) &= (-1)^n 2^{n+1} xt^n \end{aligned}$$

Suppose that $N : R \times \left[0, \frac{1}{2}\right) \rightarrow R^2, s_n = N(s_{n-1})$, then:

$$s_0 = v_0 = u_0, v_n = \sum_{j=0}^n \int_0^t \left(\frac{\partial^2 v_{j-1}}{\partial x^2} - \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} \right) dt, \quad n = 1, 2, 3, \dots \quad (45)$$

And $t \leq \frac{\lambda}{2}, 0 < \lambda < 1$

According to the theorem for nonlinear mapping N , a sufficient condition for convergence of the HPM is strictly contraction N . Therefore, we have:

$$\begin{aligned} \|v_0 - u\| &= \left\| 2x - \frac{2x}{1+2t} \right\| = 4 \left\| \frac{xt}{1+2t} \right\|, \\ \|s_1 - u\| &= \|v_0 + v_1 - u\| = 8 \left\| \frac{xt^2}{1+2t} \right\| \leq 8 \left(\frac{\lambda}{2} \right) \left\| \frac{xt}{1+2t} \right\| = \lambda \|v_0 - u\|, \\ \|s_2 - u\| &= \|v_0 + v_1 + v_2 - u\| = 16 \left\| \frac{xt^3}{1+2t} \right\| \leq 16 \left(\frac{\lambda}{2} \right)^2 \left\| \frac{xt}{1+2t} \right\| = \lambda^2 \|v_0 - u\|, \\ &\vdots \\ \|s_n - u\| &= \left\| \sum_{j=0}^n v_j - u \right\| = 2^{n+2} \left\| \frac{xt^{n+1}}{1+2t} \right\| \leq 2^{n+2} \left(\frac{\lambda}{2} \right)^n \left\| \frac{xt}{1+2t} \right\| = \lambda^n \|v_0 - u\|. \end{aligned} \quad (46)$$

Therefore,

$$\lim_{n \rightarrow \infty} \|s_n - u\| \leq \lim_{n \rightarrow \infty} \lambda^n \|v_0 - u\| = 0, \quad (47)$$

That is

$$u(x, t) = \lim_{n \rightarrow \infty} s_n = \frac{2x}{1+2t} \quad (48)$$

Which is an exact solution

Example (4.2.7) Consider the following Schrödinger equation [52]

$$i \frac{\partial u(x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 u}{\partial x^2} - |u|^2 u, \quad (x, t) \in R \times [0, 2] \quad (49)$$

Subject to initial condition

$$u(x, 0) = e^{ix}. \quad (50)$$

With the exact solution

$$u(x, t) = e^{i\left(x + \frac{1}{2}t\right)} \quad (51)$$

To solve Eq. (49) with initial condition (50) by (HMP), we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(i \left(-\frac{1}{2} \frac{\partial^2 v}{\partial x^2} - v^2 \bar{v} \right) + \frac{\partial u_0}{\partial t} \right) = 0 \quad (52)$$

Assume the solution of Eq. (52) has the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (53)$$

Substituting Eq. (53) into Eq. (52) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \quad & \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \quad & \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} - i \left(\frac{1}{2} \frac{\partial^2 v_0}{\partial x^2} + v_0^2 \bar{v}_0 \right) = 0, \quad v_1(x,0) = 0 \\ p^2 : \quad & \frac{\partial v_2}{\partial t} - i \left(\frac{1}{2} \frac{\partial^2 v_1}{\partial x^2} + 2v_0 v_1 \bar{v}_0 + v_0^2 \bar{v} \right) = 0, \quad v_2(x,0) = 0 \\ p^3 : \quad & \frac{\partial v_3}{\partial t} - i \left(\frac{1}{2} \frac{\partial^2 v_2}{\partial x^2} + 2v_0 v_2 \bar{v}_0 + v_1^2 \bar{v}_0 + 2v_0 v_1 \bar{v}_1 + v_0^2 \bar{v}_2 \right) = 0, \quad v_3(x,0) = 0 \\ & \vdots \\ p^j : \quad & \frac{\partial v_j}{\partial t} - i \left(\frac{1}{2} \frac{\partial^2 v_{j-1}}{\partial x^2} + \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \bar{v}_{j-k-i-1} \right) = 0, \quad v_j(x,0) = 0, \end{aligned} \quad (54)$$

Starting with initial condition $v_0 = u_0 = e^{ix}$ Eq. (54) gives

$$v_j = i \int_0^t \left(\frac{1}{2} \frac{\partial^2 v_{j-1}}{\partial x^2} + \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \bar{v}_{j-k-i-1} \right) dt, \quad j = 1, 2, 3, \dots \quad (55)$$

Which has solutions

$$\begin{aligned} v_1 &= \frac{1}{2} i t e^{ix} = \frac{1}{1!} \left(\frac{1}{2} i t \right) e^{ix} \\ v_2 &= -\frac{1}{8} t^2 e^{ix} = \frac{1}{2!} \left(\frac{1}{2} i t \right)^2 e^{ix} \\ v_3 &= -\frac{1}{48} i t^3 e^{ix} = \frac{1}{3!} \left(\frac{1}{2} i t \right)^3 e^{ix}, \\ & \vdots \\ v_n &= \frac{1}{n!} \left(\frac{1}{2} i t \right)^n e^{ix}. \end{aligned}$$

Suppose that $N : R \times [0,2] \rightarrow C \times C$, $s_n = N(s_{n-1})$, $s_0 = v_0 = u_0$ then:

$$v_n = \sum_{j=0}^n \int_0^t i \left(\frac{1}{2} \frac{\partial^2 v_{j-1}}{\partial x^2} + \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \bar{v}_{j-k-i-1} \right) dt, \quad n = 1, 2, 3, \dots \quad (56)$$

Thus,

$$\|v_0 - u\| = \left\| e^{ix} - e^{i\left(\frac{x+t}{2}\right)} \right\| = \left\| 1 - e^{\frac{1}{2}it} \right\|,$$

$$\|s_1 - u\| = \|v_0 + v_1 - u\| = \left\| 1 + \frac{1}{2}it - e^{\frac{1}{2}it} \right\| \leq \left\| 1 - e^{\frac{1}{2}it} \right\| \left\| \frac{1 + \frac{1}{2}it - e^{\frac{1}{2}it}}{1 - e^{\frac{1}{2}it}} \right\|,$$

Since, for all $t \in [0,2]$ we have $\left\| \frac{1 + \frac{1}{2}it - e^{\frac{1}{2}it}}{1 - e^{\frac{1}{2}it}} \right\| \leq \} = 0.507 < 1$, therefore, (57)

$$\|s_1 - u\| \leq \} \left\| 1 - e^{\frac{1}{2}it} \right\| = \} \|v_0 - u\|$$

$$\|s_2 - u\| = \left\| 1 + \frac{1}{2}it - \frac{t^2}{8} - e^{\frac{1}{2}it} \right\| \leq \left\| 1 + \frac{1}{2}it - e^{\frac{1}{2}it} \right\| \left\| \frac{1 + \frac{1}{2}it - \frac{t^2}{8} - e^{\frac{1}{2}it}}{1 + \frac{1}{2}it - e^{\frac{1}{2}it}} \right\|,$$

But, for all $t \in [0,2]$, $\left\| \frac{1 + \frac{1}{2}it - \frac{t^2}{8} - e^{\frac{1}{2}it}}{1 + \frac{1}{2}it - e^{\frac{1}{2}it}} \right\| \leq 0.336 < \}$, thus

$$\|s_2 - u\| \leq \}^2 \|v_0 - u\|$$

$$\|s_3 - u\| = \left\| 1 + \frac{1}{2}it - \frac{1}{48}it^3 - \frac{t^2}{8} - e^{\frac{1}{2}it} \right\| \leq \left\| 1 + \frac{1}{2}it - \frac{t^2}{8} - e^{\frac{1}{2}it} \right\| \left\| \frac{1 + \frac{1}{2}it - \frac{1}{48}it^3 - \frac{t^2}{8} - e^{\frac{1}{2}it}}{1 + \frac{1}{2}it - \frac{t^2}{8} - e^{\frac{1}{2}it}} \right\|,$$

But, for all $t \in [0,2]$, $\left\| \frac{1 + \frac{1}{2}it - \frac{1}{48}it^3 - \frac{t^2}{8} - e^{\frac{1}{2}it}}{1 + \frac{1}{2}it - \frac{t^2}{8} - e^{\frac{1}{2}it}} \right\| \leq 0.251 < \}$, thus

$$\|s_3 - u\| \leq \}^3 \|v_0 - u\|,$$

Therefore,

$$\lim_{n \rightarrow \infty} \|s_n - u\| \leq \lim_{n \rightarrow \infty} \|s_n - v_0\| = 0$$

That is

$$u(x, t) = \lim_{n \rightarrow \infty} s_n = e^{i(x + \frac{1}{2}t)} \quad (58)$$

Which is an exact solution

Example (4.2.8) Consider the fourth-order parabolic equation [52]

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{y+z}{2 \cos x} - 1 \right) \frac{\partial^4 u}{\partial x^4} + \left(\frac{z+x}{2 \cos y} - 1 \right) \frac{\partial^4 u}{\partial y^4} + \left(\frac{x+y}{2 \cos z} - 1 \right) \frac{\partial^4 u}{\partial z^4} = 0$$

$$0 < x, y, z < \frac{f}{3}, 0 \leq t \leq 1 \quad (59)$$

Subject to initial condition

$$u(x, y, z, 0) = -\frac{\partial u}{\partial t}(x, y, z, 0) = x + y + z - (\cos x + \cos y + \cos z), \quad (60)$$

And the boundary conditions

$$\begin{aligned} u(0, y, z, t) &= e^{-t}(-1 + y + z - \cos y - \cos z), \\ u\left(\frac{f}{3}, y, z, t\right) &= e^{-t}\left(\frac{2f-3}{6} + y + z - \cos y - \cos z\right), \\ u(x, 0, z, t) &= e^{-t}(-1 + x + z - \cos x - \cos z), \\ u\left(x, \frac{f}{3}, z, t\right) &= e^{-t}\left(\frac{2f-3}{6} + x + z - \cos x - \cos z\right), \\ u(x, y, 0, t) &= e^{-t}(-1 + x + y - \cos x - \cos y), \\ u\left(x, y, \frac{f}{3}, t\right) &= e^{-t}\left(\frac{2f-3}{6} + x + z - \cos x - \cos y\right), \\ \frac{\partial u}{\partial x}(0, y, z, t) &= \frac{\partial u}{\partial y}(x, 0, z, t) = \frac{\partial u}{\partial z}(x, y, 0, t) = e^{-t}, \\ \frac{\partial u}{\partial x}\left(\frac{f}{3}, y, z, t\right) &= \frac{\partial u}{\partial y}\left(x, \frac{f}{3}, z, t\right) = \frac{\partial u}{\partial z}\left(x, y, \frac{f}{3}, t\right) = \frac{\sqrt{3}+2}{2}e^{-t}. \end{aligned} \quad (61)$$

The exact solution is

$$u(x, y, z, t) = e^{-t}(x + y + z - \cos x - \cos y - \cos z), \quad (62)$$

For solving Eq. (60) with the initial condition (61), we construct a homotopy

$v(r, p) : \Omega \times [0, 1] \rightarrow R^4$ which satisfies

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\left(\frac{y+z}{2 \cos x} - 1 \right) \frac{\partial^4 v}{\partial x^4} + \left(\frac{z+x}{2 \cos y} - 1 \right) \frac{\partial^4 v}{\partial y^4} + \left(\frac{x+y}{2 \cos z} - 1 \right) \frac{\partial^4 v}{\partial z^4} + \frac{\partial^2 u_0}{\partial t^2} \right) = 0 \quad (63)$$

Assume the solution of Eq. (63) has the following form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (64)$$

Substituting Eq. (64) in to Eq. (63) and equating the terms of like power p ,

$$\begin{aligned} p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial u_0^2}{\partial t} &= 0, \\ p^1 : \frac{\partial^2 v_1}{\partial t^2} + \left(\frac{y+z}{2\cos x} - 1 \right) \frac{\partial^4 v_0}{\partial x^4} + \left(\frac{z+x}{2\cos y} - 1 \right) \frac{\partial^4 v_0}{\partial y^4} + \left(\frac{x+y}{2\cos z} - 1 \right) \frac{\partial^4 v_0}{\partial z^4} + \frac{\partial^2 u_0}{\partial t^2} &= 0, \\ p^2 : \frac{\partial^2 v_2}{\partial t^2} + \left(\frac{y+z}{2\cos x} - 1 \right) \frac{\partial^4 v_1}{\partial x^4} + \left(\frac{z+x}{2\cos y} - 1 \right) \frac{\partial^4 v_1}{\partial y^4} + \left(\frac{x+y}{2\cos z} - 1 \right) \frac{\partial^4 v_1}{\partial z^4} &= 0, \\ &\vdots \\ p^j : \frac{\partial v_j}{\partial t} + \left(\frac{y+z}{2\cos x} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial x^4} + \left(\frac{z+x}{2\cos y} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial y^4} + \left(\frac{x+y}{2\cos z} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial z^4} &= 0, \end{aligned} \quad (65)$$

For simplicity we take $v_0 = u_0 = (x + y + z - \cos x - \cos y - \cos z)(1 - t)$.

So we have

$$\begin{aligned} v_j &= -\int_0^t \int_0^t \left(\left(\frac{y+z}{2\cos x} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial x^4} + \left(\frac{z+x}{2\cos y} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial y^4} + \left(\frac{x+y}{2\cos z} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial z^4} \right) dt dt, \\ &j = 1, 2, \dots \end{aligned} \quad (66)$$

Which has solutions

$$\begin{aligned} v_1 &= (x + y + z - \cos x - \cos y - \cos z) \left(\frac{t^2}{2!} - \frac{t^3}{3!} \right), \\ v_2 &= (x + y + z - \cos x - \cos y - \cos z) \left(\frac{t^4}{4!} - \frac{t^5}{5!} \right), \\ v_3 &= (x + y + z - \cos x - \cos y - \cos z) \left(\frac{t^6}{6!} - \frac{t^7}{7!} \right), \\ &\vdots \\ v_n &= (x + y + z - \cos x - \cos y - \cos z) \left(\frac{t^{2n}}{(2n)!} - \frac{t^{2n+1}}{(2n+1)!} \right), \end{aligned}$$

Suppose that $N : R^3 \times [0,1] \rightarrow R^4$ $s_n = N(s_{n-1})$, $s_0 = v_0 = u_0$ then:

$$\begin{aligned} v_j &= -\sum_{j=0}^n \int_0^t \int_0^t \left(\left(\frac{y+z}{2\cos x} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial x^4} + \left(\frac{z+x}{2\cos y} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial y^4} + \left(\frac{x+y}{2\cos z} - 1 \right) \frac{\partial^4 v_{j-1}}{\partial z^4} \right) dt dt, \\ &j = 1, 2, \dots \end{aligned} \quad (67)$$

Thus

$$\|v_0 - u\| = \|1 - t - e^{-t}\|,$$

$$\|s_1 - u\| = \left\| 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} - e^{-t} \right\| \leq \|1 - t - e^{-t}\| \left\| 1 - \frac{\frac{t^2}{2!} - \frac{t^3}{3!}}{e^{-t} + t - 1} \right\|,$$

Since, for all $t \in [0,1]$ we have $\left\| 1 - \frac{\frac{t^2}{2!} - \frac{t^3}{3!}}{e^{-t} + t - 1} \right\| \leq \} = 0.94 < 1$, therefore, (68)

$$\|s_1 - u\| \leq \} \|1 - t - e^{-t}\| = \} \|v_0 - u\|,$$

$$\|s_2 - u\| = \left\| 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} - e^{-t} \right\|$$

$$\leq \left\| 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} - e^{-t} \right\| \left\| 1 - \frac{\frac{t^4}{4!} - \frac{t^5}{5!}}{e^{-t} + t - \frac{t^2}{2!} - \frac{t^3}{3!} - 1} \right\|$$

But, for all $t \in [0,1]$, $\left\| 1 - \frac{\frac{t^4}{4!} - \frac{t^5}{5!}}{e^{-t} + t - \frac{t^2}{2!} - \frac{t^3}{3!} - 1} \right\| \leq 0.35 < \}$, thus

$$\|s_2 - u\| \leq \}^2 \|v_0 - u\|$$

$$\|s_3 - u\| = \left\| 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} - e^{-t} \right\|$$

$$\leq \left\| 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} - e^{-t} \right\| \left\| 1 - \frac{\frac{t^6}{6!} - \frac{t^7}{7!}}{e^{-t} + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} - 1} \right\|$$

But, for all $t \in [0,2]$, $\left\| 1 - \frac{\frac{t^6}{6!} - \frac{t^7}{7!}}{e^{-t} + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} - 1} \right\| \leq 0.018 < \}$, thus

$$\|s_3 - u\| \leq \}^3 \|v_0 - u\|,$$

⋮

$$\|s_n - u\| \leq \}^n \|v_0 - u\|$$

Therefore,

$$\lim_{n \rightarrow \infty} \|s_n - u\| \leq \lim_{n \rightarrow \infty} \}^n \|v_0 - u\| = 0 \quad (69)$$

That is

$$\begin{aligned} u(x, y, z, t) &= \lim_{n \rightarrow \infty} \sum_{j=0}^n (x + y + z - \cos x - \cos y - \cos z) \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \\ &= (x + y + z - \cos x - \cos y - \cos z) e^{-t} \end{aligned} \quad (70)$$

Which is an exact solution

4.3: Convergence rate of HPM

Definition (4.3.9) for every $i \in N$, we define;

$$\} _i = \begin{cases} \frac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \neq 0, \\ 0 & \|v_i\| = 0. \end{cases} \quad (71)$$

In theorem (4.2.2), $\sum_{i=0}^{\infty} v_i$ converges to exact solution, when $0 \leq \} _i < 1$.

If v_i and v'_i are obtained by two different homotopy, and $\} _i < \} _i'$ for each $i \in N$, the rate of convergence of $\sum_{i=0}^{\infty} v_i$ is higher than $\sum_{i=0}^{\infty} v'_i$.

Example (4.3.10) Consider the Lan-Emden equation in the following form

$$\begin{aligned} u'' + \frac{2}{x} u' + u &= x^5 + 30x^3 \\ u(0) = 0, \quad u'(0) &= 0 \end{aligned} \quad (72)$$

With the exact solution

$$u(x) = x^5 \quad (73)$$

To solve Eq. (72) by (HMP), we consider the linear part as follows

$$Lu = u'' \quad (74)$$

And construct the following homotopy;

$$u'' - U_0'' = p \left(x^5 + 30x^3 - \frac{2}{x} u' - u - u_0'' \right) \quad (75)$$

Let's consider the solution u as the summation series;

$$u = \sum_{i=0}^{\infty} u_i \quad (76)$$

Substituting (76) into (75) leads to;

$$\sum_{i=0}^{\infty} u'' - U_0'' = p \left(x^5 + 30x^3 - \frac{2}{x} \sum_{i=0}^{\infty} u'_i - \sum_{i=0}^{\infty} u_i - u_0'' \right) \quad (77)$$

Beginning with the $u_0 = 0$, we get;

$$u_1 = \frac{1}{42} x^7 + \frac{3}{2} x^5,$$

$$u_2 = -\frac{11}{252} x^7 - \frac{3}{4} x^5 - \frac{1}{4024} x^9,$$

$$u_3 = \frac{25}{36288} x^9 - \frac{7}{216} x^7 + \frac{3}{8} x^5 + \frac{1}{332640} x^{11},$$

$$u_4 = -\frac{137}{19958400} x^{11} - \frac{271}{435456} x^9 - \frac{179}{9072} x^7 + \frac{3}{16} x^5 + \frac{1}{51891840} x^{13},$$

$$u_5 = \frac{7}{148262400} x^{13} + \frac{8419}{119850400} x^{11} + \frac{2245}{5225472} x^9 + \frac{601}{54432} x^7 + \frac{3}{32} x^5 + \frac{1}{10897286400} x^{15}$$

By considered $\|f(x)\| = \max_{0 \leq x \leq 1} |f(x)|$, we have

$$\}_1 = 0.5210503471$$

$$\}_2 = 0.5139910140,$$

$$\}_3 = 0.5093374003,$$

$$\}_4 = 0.5062439696,$$

$$\}_5 = 0.5041785188,$$

$$\}_6 = 0.5027965117.$$

If the linear part of equation is consider as follows;

$$Lu = u'' + \frac{2}{x} u' = x^{-2} \frac{d}{dx} \left(x^2 \frac{du}{dx} \right). \quad (78)$$

Then we construct the following homotopy

$$\left(v'' + \frac{2}{x} v' \right) - \left(u_0'' + \frac{2}{x} u_0' \right) = p \left(x^5 + 30x^3 - v - u_0'' - \frac{2}{x} u_0' \right) \quad (79)$$

Where

$$L^{-1}(u) = \int_0^x x^{-2} \int_0^x \frac{d}{dx} (x^2 u(x)) dx dx \quad (80)$$

Suppose the solution of Eq. (74) has the following form;

$$v = \sum_{i=0}^{\infty} p^i v_i \quad (81)$$

Substituting (76) into (74) and equating the term of like powers, we get;

$$p^0 : \left(v_0'' + \frac{2}{x} v_0' \right) - \left(u_0'' + \frac{2}{x} u_0' \right) = 0,$$

$$p^1 : v_1'' + \frac{2}{x} v_1' + v_0 + u_0'' + \frac{2}{x} u_0' = x^5 + 30x^3, \quad u_1(0) = 0, \quad u_1'(0) = 0$$

$$p^2 : v_2'' + \frac{2}{x} v_2' + v_1 = 0, \quad u_2(0) = 0, \quad u_2'(0) = 0$$

⋮

Starting with $v_0 = u_0 = 0$, then the solution reads

$$v_1 = \frac{1}{56} x^7 + x^5,$$

$$v_2 = -\frac{1}{5040} x^9 - \frac{1}{56} x^7,$$

$$v_3 = \frac{1}{665280} x^{11} + \frac{1}{5040} x^9,$$

$$v_4 = -\frac{1}{121080960} x^{13} - \frac{1}{665280} x^{11},$$

$$u_5 = \frac{1}{29059430400} x^{15} + \frac{1}{121080960} x^{13},$$

Then;

$$\} _1 = 0.0177387914$$

$$\} _2 = 0.0110722610,$$

$$\} _3 = 0.00756010906,$$

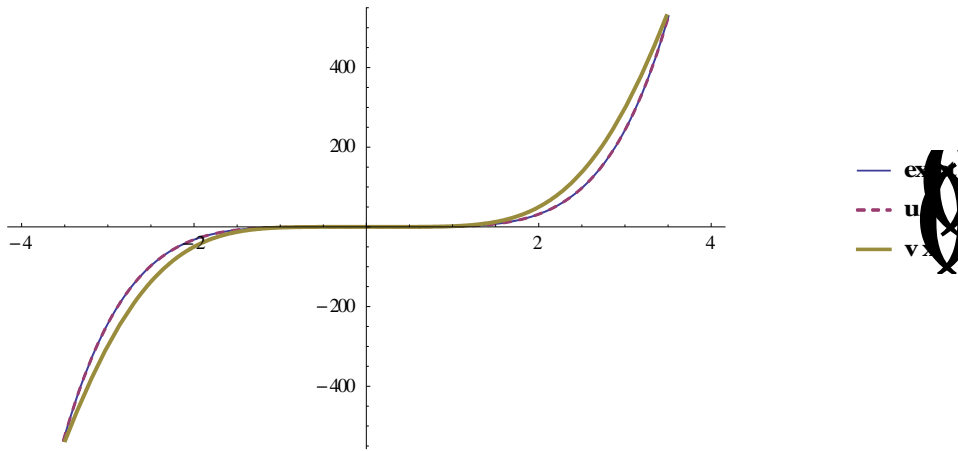
$$\} _4 = 0.00548724954,$$

$$\} _5 = 0.00416293765,$$

$$\} _6 = 0.00326590091.$$

By comparison between the obtained results in the above Example, it can be concluded that the rate of convergence of homotopy (79) is higher than homotopy (77) (see Fig. 4.3.11).

Fig (4.3.11): Plots of solution of HPM and exact solution for Ex. (4.3.10).



The important things which we want to mentioned here, the results of this section were published as scientific paper in [79].

CHAPTER FIVE

HOMOTOPY PERTUTBATIONT TRANSFORM METHOD

(5.1)Introduction

In this chapter, we use the homotopy perturbation method coupled with the Laplace transformation, called homotopy perturbation transform method (HPTM) for solving the linear and nonlinear PDEs. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Laplace decomposition algorithm [71-74] and the homotopy perturbation transform method (HPTM) [75–78] to produce highly effective techniques for solving many nonlinear problems. The basic motivation of this chapter to apply an effective modification of HPM to overcome the deficiency, it is worth mentioning that the (HPTM) is an elegant combination of the Laplace transformation, the homotopy perturbation method, and He’s polynomials, The (HPTM) algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for linear and nonlinear partial differential equations

(5.2)Analysis of (HPTM)

The HPTM is a combined of the HPM and Laplace transform method. We apply HPTM to the following general nonlinear partial differential equation with the initial conditions of the form,

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \quad (1)$$

$$u(x,0) = h(x), \quad u_t(x,0) = f(x). \quad (2)$$

Where D is the second order linear differential operator, $D = \frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D ; N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking the Laplace transform (denoted by L) on both sides of Eq. (1):

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \quad (3)$$

$$s^2 L[u(x,t)] - su(x,0) - u_t(x,0) + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \quad (4)$$

Using the initial conditions:

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2} L[g(x,t)] - \frac{1}{s^2} L[Ru(x,t)] - \frac{1}{s^2} L[Nu(x,t)] \quad (5)$$

Operating with Laplace inverse on both sides of Eq. (5) gives

$$u(x,t) = G(x,t) - L^{-1} \left[\frac{1}{s^2} L[Ru(x,t) + Nu(x,t)] \right] \quad (6)$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (7)$$

And the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (8)$$

Where $H_n(u)$ are He's polynomials given by;

$$H_n(u_0, u_1, \dots, u_n) = \left(\frac{1}{n!} \frac{\partial^n}{\partial p^n} N \left(\sum_{i=0}^n p^i u_i \right) \right)_{p=0}, \quad n = 0, 1, 2, \dots \quad (9)$$

Substituting Eq. (7) and (8) in Eq. (6), we get,

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - p \left(L^{-1} \left[\frac{1}{s^2} L \left[R \sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right). \quad (10)$$

Which is the coupling of the Laplace transform and the HPM using He's polynomials.

Comparing the coefficient of like powers of p , the following approximations are obtained.

$$\begin{aligned} p^0 : u_0(x,t) &= G(x,t), \\ p^1 : u_1(x,t) &= -L^{-1} \left[\frac{1}{s^2} L[Ru_0(x,t) + H_0(u)] \right], \\ p^2 : u_2(x,t) &= -L^{-1} \left[\frac{1}{s^2} L[Ru_1(x,t) + H_1(u)] \right], \\ p^3 : u_3(x,t) &= -L^{-1} \left[\frac{1}{s^2} L[Ru_2(x,t) + H_2(u)] \right], \\ &\vdots \end{aligned}$$

And so on

Example (5.2.1) Consider the linear Klein-Gordon equation [78]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u \quad (11)$$

Subject to the initial conditions

$$u(x,0) = 1 + \sin x, \quad \frac{\partial u}{\partial t}(x,0) = 0. \quad (12)$$

By applying the aforesaid method subject to the initial condition, we have

$$u(x,s) = \frac{1 + \sin x}{s} + \frac{1}{s^2} L \left[u - \frac{\partial^2 u}{\partial x^2} \right] \quad (13)$$

The inverse of the Laplace transform implies that

$$u(x,t) = 1 + \sin x + L^{-1} \left[\frac{1}{s^2} L \left[u + \frac{\partial^2 u}{\partial x^2} \right] \right] \quad (14)$$

Now, we apply the homotopy perturbation method; we have

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (15)$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + \sin x + p \left(L^{-1} \left[\frac{1}{s^2} L \left[\sum_{n=0}^{\infty} p^n u_n(x,t) + \frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) \right] \right] \right) \quad (16)$$

Comparing the coefficient of like powers of p , we have

$$\begin{aligned} p^0 : u_0(x,t) &= 1 + \sin x \\ p^1 : u_1(x,t) &= L^{-1} \left[\frac{1}{s^2} L \left[u_0 + \frac{\partial^2 u_0}{\partial x^2} \right] \right] = \frac{t^2}{2}, \\ p^2 : u_2(x,t) &= L^{-1} \left[\frac{1}{s^2} L \left[u_1 + \frac{\partial^2 u_1}{\partial x^2} \right] \right] = \frac{t^4}{24}, \\ p^3 : u_3(x,t) &= L^{-1} \left[\frac{1}{s^2} L \left[u_2 + \frac{\partial^2 u_2}{\partial x^2} \right] \right] = \frac{t^6}{720}, \\ &\vdots \end{aligned} \quad (17)$$

So that the solution is given by

$$u(x,t) = 1 + \sin x + \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720} + \dots \quad (18)$$

In series form, and

$$u(x,t) = \sin x + \cosh t \quad (19)$$

In closed form

Example (5.2.2) Consider the following diffusion-convection problem [76]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \quad (20)$$

With the initial condition

$$u(x,0) = x + e^{-x}. \quad (21)$$

Taking the Laplace transform on the both sides, subject to the initial condition, we get

$$u(x,s) = \frac{x + e^{-x}}{s} + \frac{1}{s} L \left[\frac{\partial^2 u}{\partial x^2} - u \right] \quad (22)$$

Taking inverse Laplace transform, we get

$$u(x,t) = x + e^{-x} + L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u}{\partial x^2} - u \right] \right] \quad (23)$$

Now, we apply the homotopy perturbation method; we have

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (24)$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + e^{-x} + p \left(L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n u_n(x,t) \right] \right] \right) \quad (25)$$

Comparing the coefficient of like powers of p , we have

$$\begin{aligned} p^0 : u_0(x,t) &= x + e^{-x}, \\ p^1 : u_1(x,t) &= L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u_0}{\partial x^2} - u_0 \right] \right] = -xt, \\ p^2 : u_2(x,t) &= L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u_1}{\partial x^2} - u_1 \right] \right] = x \frac{t^2}{2!}, \\ p^3 : u_3(x,t) &= L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u_2}{\partial x^2} - u_2 \right] \right] = -x \frac{t^3}{3!}, \\ &\vdots \end{aligned} \quad (26)$$

And so on. Therefore the series solution is given by

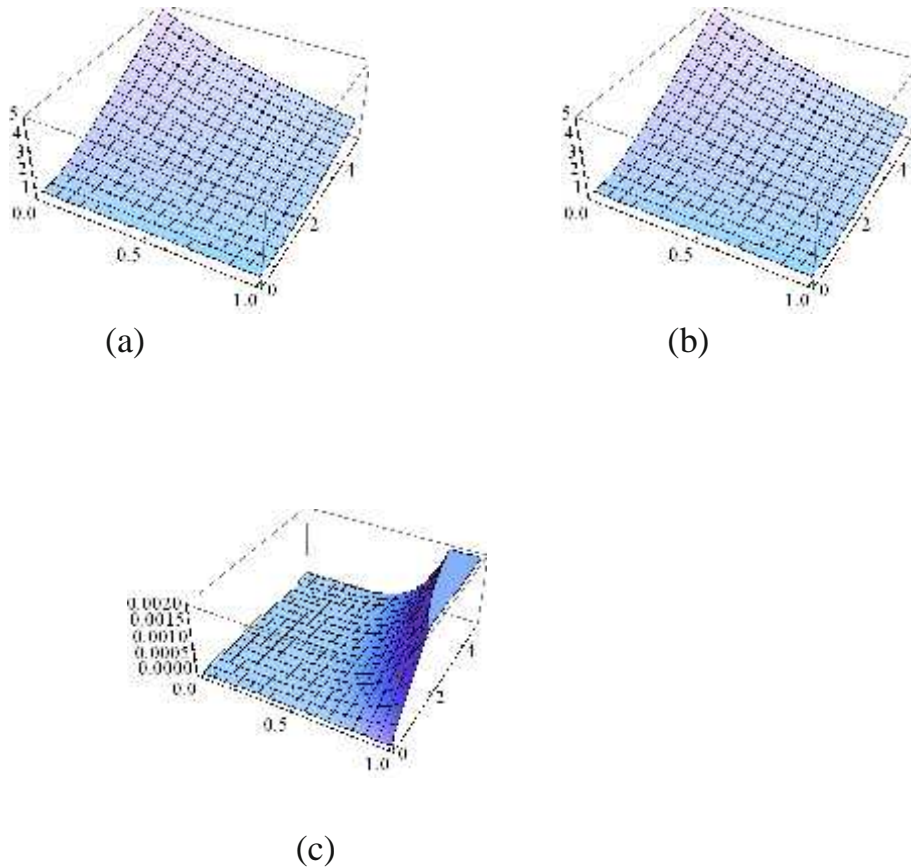
$$u(x,t) = e^{-x} + x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right) \quad (27)$$

Which converge very rapidly to the exact solution

$$u(x,t) = e^{-x} + xe^{-t} \quad (28)$$

The numerical results of $u(x,t)$ for the approximate solution (27) obtained by using HPTM, the exact solution and the absolute error $E_7(u) = |u_{ex} - u_{app}|$ for various values of t and x are shown by Fig. (5.2.3)(a)–(c). It is observed from Fig. (5.2.3) (a) and (b) that $u(x,t)$ increases with the increase in x and decrease in t . Fig. (5.2.3) (a)–(c) clearly show that the approximate solution (27) obtained by HPTM is very near to the exact solution. It is to be noted that only the seventh order term of the HPTM was used in evaluating the approximate solutions for Fig. (5.2.3). It is evident that the efficiency of the HPTM can be dramatically enhanced by computing further terms of $u(x,t)$ when the HPTM is used.

Fig (5.2.3): The surface shows the solution $u(x,t)$ for Eq. (20): (a) exact solution; (b) approximate solution (27); (c) $|u_{ex} - u_{app}|$



Example (5.2.4) Consider the following diffusion-convection problem [76]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (-1 + \cos x - \sin^2 x)u, \quad (29)$$

With the initial condition

$$u(x,0) = \frac{1}{10} e^{\cos x - 1}. \quad (30)$$

Taking the Laplace transform on the both sides, subject to the initial condition, we get

$$u(x,s) = \frac{1}{s} \frac{1}{10} e^{\cos x - 1} + \frac{1}{s} L \left[\frac{\partial^2 u}{\partial x^2} + (-1 + \cos x - \sin^2 x)u \right] \quad (31)$$

Taking inverse Laplace transform, we get

$$u(x,t) = \frac{1}{10} e^{\cos x - 1} + L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u}{\partial x^2} + (-1 + \cos x - \sin^2 x)u \right] \right] \quad (32)$$

Now, we apply the homotopy perturbation method; we have

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (33)$$

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x,t) = \frac{1}{10} e^{\cos x - 1} + p \left(L^{-1} \left[\frac{1}{s^2} L \left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) \right. \right. \right. \\ \left. \left. \left. + (-1 + \cos x - \sin^2 x) \sum_{n=0}^{\infty} p^n u_n(x,t) \right] \right] \right) \end{aligned} \quad (34)$$

Comparing the coefficient of like powers of p , we have

$$\begin{aligned} p^0 : u_0(x,t) &= \frac{1}{10} e^{\cos x - 1}, \\ p^1 : u_1(x,t) &= L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u_0}{\partial x^2} + (-1 + \cos x - \sin^2 x)u_0 \right] \right] = \frac{1}{10} e^{\cos x - 1} (-t) \\ p^2 : u_2(x,t) &= L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u_1}{\partial x^2} + (-1 + \cos x - \sin^2 x)u_1 \right] \right] = \frac{1}{10} e^{\cos x - 1} \left(\frac{t^2}{2!} \right), \\ p^3 : u_3(x,t) &= L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u_2}{\partial x^2} + (-1 + \cos x - \sin^2 x)u_2 \right] \right] = \frac{1}{10} e^{\cos x - 1} \left(-\frac{t^3}{3!} \right), \\ &\vdots \end{aligned} \quad (35)$$

And so on. Therefore the series solution is given by

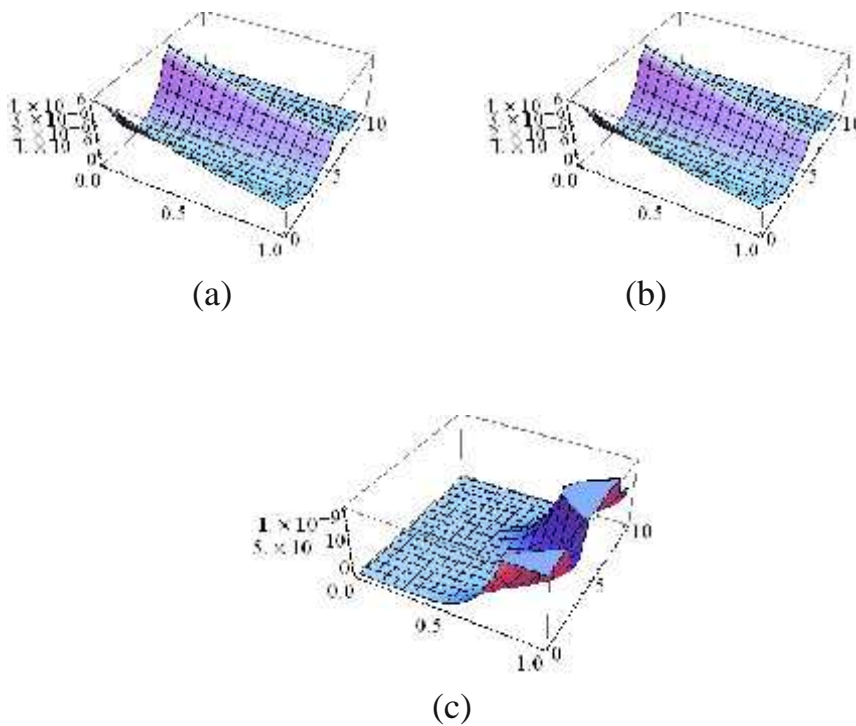
$$u(x,t) = \frac{1}{10} e^{\cos x - 1} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right) \quad (36)$$

Which converge very rapidly to the exact solution

$$u(x,t) = \frac{1}{10} e^{\cos x - 11t} \quad (37)$$

The numerical results of $u(x,t)$ for the approximate solution (36) obtained with the help of HPTM, the exact solution and the absolute error $E_7(u) = |u_{ex} - u_{app}|$ for various values of t and x are shown by Fig. (5.2.5)(a)–(c), we observed that the approximate solution (36) obtained by the HPTM is very near to the exact solution.

Fig (5.2.5): The surface shows the solution $u(x,t)$ for Eq. (20): (a) exact solution; (b) approximate solution (27); (c) $|u_{ex} - u_{app}|$



Example (5.2.6) Consider the following Advection problem [75]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 2t + x + t^3 + xt^2, \quad (38)$$

With the initial condition

$$u(x,0) = 0. \quad (39)$$

Taking the Laplace transform on the both sides, subject to the initial condition, we get

$$u(x,s) = \frac{2}{s^3} + \frac{x}{s^2} + \frac{6}{s^5} + \frac{2x}{s^4} - \frac{1}{s} L \left[u \frac{\partial u}{\partial x} \right] \quad (40)$$

Taking inverse Laplace transform, we get

$$u(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - L^{-1} \left[\frac{1}{s} L \left[u \frac{\partial u}{\partial x} \right] \right] \quad (41)$$

Now, we apply the homotopy perturbation method; we have

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (42)$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - p \left(L^{-1} \left[\frac{1}{s} L \left[\sum_{n=0}^{\infty} p^n H_n \right] \right] \right) \quad (43)$$

Where H_n are He's polynomials that represent the nonlinear terms

The first few components of He's polynomials, for example, are given by

$$\begin{aligned} H_0(u) &= u_0 u_{0x}, \\ H_1(u) &= u_0 u_{1x} + u_1 u_{0x}, \\ H_2(u) &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \\ &\vdots \end{aligned}$$

Comparing the coefficient of like powers of p , we have

$$\begin{aligned} p^0 : u_0(x,t) &= t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\ p^1 : u_1(x,t) &= -L^{-1} \left[\frac{1}{s} L[H_0(u)] \right] = -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2xt^5}{15} + \frac{7t^6}{72} - \frac{1}{63} xt^7 - \frac{1}{98} t^8, \\ p^2 : u_2(x,t) &= -L^{-1} \left[\frac{1}{s} L[H_1(u)] \right] = \frac{5t^{12}}{8064} + \frac{2xt^{11}}{2079} + \frac{2783t^{10}}{302400} + \frac{38xt^9}{2835} + \frac{143t^8}{2880} \\ &\quad + \frac{22xt^7}{315} + \frac{7t^6}{12} + \frac{2xt^5}{15}, \\ &\vdots \end{aligned} \quad (44)$$

It is important to recall here that the noise terms appear between the components $u_0(x,t)$ and $u_1(x,t)$, more precisely, the noise terms $\pm \frac{t^4}{4} \pm \frac{xt^3}{3}$

Between the components $u_0(x,t)$ and $u_1(x,t)$ can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation.

The exact solution is therefore

$$u(x,t) = t^2 + xt \quad (45)$$

Example (5.2.7) Consider the following homogeneous nonlinear PDE [77]

$$\frac{\partial u}{\partial t} - u - u \frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x} \right)^2 = 0, \quad (46)$$

With the initial condition

$$u(x,0) = \sqrt{x}. \quad (47)$$

Taking the Laplace transform on the both sides, subject to the initial condition, we get

$$u(x,s) = \frac{\sqrt{x}}{s} + \frac{1}{s} L \left[u + u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 \right] \quad (48)$$

Taking inverse Laplace transform, we get

$$u(x,t) = \sqrt{x} + L^{-1} \left[\frac{1}{s} L \left[u + u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 \right] \right] \quad (49)$$

Now, we apply the homotopy perturbation method; we have

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (50)$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sqrt{x} + p \left(L^{-1} \left[\frac{1}{s} L[u] + \frac{1}{s} L \left[\sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right) \quad (51)$$

Where H_n are He's polynomials that represent the nonlinear terms

The first few components of He's polynomials, for example, are given by

$$H_0(u) = u_0 \frac{\partial^2 u_0}{\partial x^2} + \left(\frac{\partial u_0}{\partial x} \right)^2 = 0,$$

$$H_1(u) = u_0 \frac{\partial^2 u_1}{\partial x^2} + u_1 \frac{\partial^2 u_0}{\partial x^2} + 2 \frac{\partial u_0}{\partial x} \frac{\partial u_1}{\partial x} = 0,$$

$$H_2(u) = u_0 \frac{\partial^2 u_2}{\partial x^2} + u_1 \frac{\partial^2 u_1}{\partial x^2} + u_2 \frac{\partial^2 u_0}{\partial x^2} + \left(\frac{\partial u_1}{\partial x} \right)^2 + 2 \frac{\partial u_0}{\partial x} \frac{\partial u_2}{\partial x} = 0,$$

⋮

Comparing the coefficient of like powers of p , we have

$$\begin{aligned}
 p^0 : u_0(x,t) &= \sqrt{x}, \\
 p^1 : u_1(x,t) &= L^{-1} \left[\frac{1}{s} (L[u_0] + L[H_0(u)]) \right] = \sqrt{x}t, \\
 p^2 : u_2(x,t) &= L^{-1} \left[\frac{1}{s} (L[u_1] + L[H_1(u)]) \right] = \sqrt{x} \frac{t^2}{2!}, \\
 p^3 : u_3(x,t) &= L^{-1} \left[\frac{1}{s} (L[u_2] + L[H_2(u)]) \right] = \sqrt{x} \frac{t^3}{3!}, \\
 &\vdots
 \end{aligned} \tag{52}$$

And so on. Therefore the series solution is given by

$$u(x,t) = \sqrt{x} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \tag{53}$$

Which converge very rapidly to the exact solution

$$u(x,t) = \sqrt{x}e^t \tag{54}$$

(5.3) Comparison of Rate of Convergence of HPM and HPTM

Example (5.3.8) Consider the inhomogeneous Advection problem [18]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial t} = -\sin(x+t) - \frac{1}{2} \sin 2(x+t), \quad u(x,0) = \cos x. \tag{55}$$

Standard HPM: According to homotopy Eq. (35) in example (3.3.6) we have

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p \left(v \frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial t} + \sin(x+t) + \frac{1}{2} \sin 2(x+t) \right) = 0 \tag{56}$$

And the solution for first few steps reads:

$$\begin{aligned}
 v_0 &= \cos x, \\
 v_1 &= \frac{1}{2}t \sin 2x + \cos(x+t) - \cos x + \frac{1}{4} \cos 2(x+t) - \frac{1}{4} \cos 2x, \\
 v_2 &= -\frac{1}{4}t^2 \sin x \sin 2x + \frac{1}{2}t^2 \cos x \cos 2x - \sin x \sin(x+t) + \sin^2 x \\
 &\quad + \cos x \cos(x+t) + \cos^2 x + t \sin 2x - \frac{1}{8} \sin x \sin 2(x+t) + \frac{1}{8} \sin x \sin 2x \\
 &\quad + \frac{1}{4} \cos x \cos 2(x+t) - \frac{1}{4} \cos x \cos 2x + \frac{1}{4}t \sin x \cos 2x + \frac{1}{2}t \cos x \sin 2x \\
 &\quad \vdots
 \end{aligned}$$

Therefore the approximate solution of Eq. (55) can be written as

$$u = \frac{1}{16} (\cos x - 2t^2 \cos x + 12 \cos 2x + 3 \cos 3x - 6t^2 \cos 3x + 16 \cos(x+t) - \cos(2t+x) \\ 16 \cos(2x+4) + 4 \cos(2x+2t) - 3 \cos(3x+2t) - 2t \sin x - 8t \sin 2x - 6t \sin 3x) + \dots \quad (57)$$

HPTM: to solve Eq. (55) by *MPTM*, taking the Laplace transform on the both sides, subject to the initial condition, we get

$$u(x, s) = \frac{\cos x}{s} + \frac{1}{s} \left[\left(\frac{-\cos x - s \sin x}{1 + s^2} \right) - \frac{1}{2} \left(\frac{2 \cos 2x + s \sin 2x}{4 + s^2} \right) \right] - \frac{1}{s} L \left[u \frac{\partial u}{\partial x} \right] \quad (58)$$

Taking inverse Laplace transform, we get

$$u(x, t) = \cos(x+t) + \frac{1}{4} \cos 2(x+t) - \frac{1}{4} \cos 2x - L^{-1} \left[\frac{1}{s} L \left[u \frac{\partial u}{\partial x} \right] \right] \quad (59)$$

Now, we apply the homotopy perturbation method; we have

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) \quad (60)$$

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = \cos(x+t) + \frac{1}{4} \cos 2(x+t) - \frac{1}{4} \cos 2x - p \left(L^{-1} \left[\frac{1}{s} L \left[\sum_{n=0}^{\infty} p^n H_n \right] \right] \right) \quad (61)$$

Where H_n are He's polynomials that represent the nonlinear terms

The first few components of He's polynomials, for example, are given by

$$H_0(u) = u_0 u_{0x}, \\ H_1(u) = u_0 u_{1x} + u_1 u_{0x}, \\ \vdots$$

Comparing the coefficient of like powers of p , we have

$$p^0 : u_0(x, t) = \cos(x+t) + \frac{1}{4} \cos 2(x+t) - \frac{1}{4} \cos 2x, \\ p^1 : u_1(x, t) = -L^{-1} \left[\frac{1}{s} L [H_0(u)] \right] = -\frac{1}{4} \cos 2(x+t) + \frac{1}{4} \cos x + \frac{1}{64} \cos 4x + \dots, \\ \vdots \quad (62)$$

It is important to recall here that the noise terms appear between the components $u_0(x, t)$ and $u_1(x, t)$, more precisely, the noise terms $\pm \frac{1}{4} \cos 2(x+t) \pm \frac{1}{4} \cos 2x$ between the components $u_0(x, t)$ and $u_1(x, t)$ can be cancelled and the remaining terms of $u_0(x, t)$ still satisfy the equation.

The exact solution is therefore

$$u(x, t) = \cos(x+t) \quad (62)$$

Example (5.3.9) Consider the inhomogeneous nonlinear Klein Gordon equation [42]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x \cos t + x^2 \cos^2 t, \quad (63)$$

Subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = 0 \quad (64)$$

Standard HPM: According to homotopy Eq. (65) in example (3.4.10), we have

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} + v^2 + x \cos t - x^2 \cos^2 t \right) = 0 \quad (66)$$

And the solution for first few steps reads:

$$\begin{aligned} v_0 &= x, \\ v_1 &= -x + \frac{1}{8}x^2 - \frac{3}{4}x^2t^2 + x \cos t - \frac{1}{8}x^2 \cos 2x, \\ v_2 &= -\frac{1}{16}x^2 + \frac{1}{8}t^2 - \frac{t^4}{24} - 2x^2 + t^2x^2 + \frac{x^3}{16} - \frac{t^2x^3}{8} + \frac{t^4x^3}{24} + 2x^2 \cos t \\ &\quad - \frac{1}{16} \cos 2x - \frac{1}{16}x^3 \cos 2x, \\ &\vdots \end{aligned}$$

Therefore the approximate solution of Eq. (63) can be written as

$$\begin{aligned} u(x,t) &= -\frac{31}{16}x^2 - \frac{3}{4}x^2t^2 + \frac{1}{8}t^2 - \frac{t^4}{24} + t^2x^2 + \frac{x^3}{16} - \frac{t^2x^3}{8} + \frac{t^4x^3}{24} + x \cos t \\ &\quad - \frac{1}{8}x^2 \cos 2x + 2x^2 \cos t - \frac{1}{16} \cos 2x - \frac{1}{16}x^3 \cos 2x + \dots \end{aligned} \quad (67)$$

HPTM: To solve Eq. (55) by MPTM, taking the Laplace transform on the both sides, subject to the initial condition, we get

$$u(x,s) = \frac{x}{s} - \frac{x}{s(1+s^2)} + \frac{(2+s^2)x^2}{s^3(4+s^2)} + \frac{1}{s} L \left[\frac{\partial^2 u}{\partial x^2} - u^2 \right] \quad (68)$$

Taking inverse Laplace transform, we get

$$u(x,t) = x \cos t - \frac{1}{8}x^2 \cos t + \frac{x^2t^2}{4} + \frac{x^2}{8} + L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u}{\partial x^2} - u^2 \right] \right] \quad (69)$$

Now, we apply the homotopy perturbation method; we have

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (70)$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x \cos t - \frac{1}{8} x^2 \cos t + \frac{x^2 t^2}{4} + \frac{x^2}{8} + p \left(L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \left[\sum_{n=0}^{\infty} p^n H_n \right] \right] \right] \right) \quad (71)$$

Where H_n are He's polynomials, the first few components of He's polynomials, for example, are given by

$$\begin{aligned} H_0(u) &= u_0^2, \\ H_1(u) &= 2u_0 u_1, \\ &\vdots \end{aligned}$$

Comparing the coefficient of like powers of p , we have

$$\begin{aligned} p^0 : u_0(x,t) &= x \cos t - \frac{1}{8} x^2 \cos t + \frac{x^2 t^2}{4} + \frac{x^2}{8}, \\ p^1 : u_1(x,t) &= L^{-1} \left[\frac{1}{s} L \left[\frac{\partial^2 u_0}{\partial x^2} - H_0(u) \right] \right] = \frac{1}{8} x^2 \cos t - \frac{x^2 t^2}{4} - \frac{x^2}{8} + \frac{1}{64} x^4 \cos 2t + \dots \\ &\vdots \end{aligned} \quad (72)$$

The noise terms $\pm \frac{1}{8} x^2 \cos t \pm \frac{x^2 t^2}{4} \pm \frac{x^2}{8}$ between the components $u_0(x,t)$ and $u_1(x,t)$ can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation.

The exact solution is therefore

$$u(x,t) = x \cos t \quad (73)$$

Example (5.2.10) Consider the following inhomogeneous nonlinear PDE [77]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 = 2x + t^4, \quad (74)$$

With the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = a. \quad (75)$$

Standard HPM: According to homotopy perturbation method we have:

$$\frac{\partial v^2}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} + p \left(\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x} \right)^2 + \frac{\partial^2 u_0}{\partial x^2} - 2x + t^4 \right) = 0 \quad (76)$$

Let's ignore the first few steps and start from determining v_i

$$\begin{aligned}
v_0 &= at, \\
v_1 &= xt^2 + \frac{1}{30}t^6, \\
v_2 &= 0, \\
v_3 &= \frac{1}{30}t^6, \\
&\vdots \\
v_k &= 0, \quad k \geq 4.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
v_0 &= v_0 + v_1 + v_2 + v_3 + \dots \\
&= at + xt^2.
\end{aligned} \tag{77}$$

HPTM: To solve Eq. (74) by *MPTM*, taking the Laplace transform on the both sides, subject to the initial condition, we get

$$u(x, s) = \frac{a}{s^2} + \frac{2x}{s^3} + \frac{4!}{s^7} - \frac{1}{s^2} L \left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 \right] \tag{78}$$

Taking inverse Laplace transform, we get

$$u(x, t) = at + xt^2 + \frac{t^6}{30} - L^{-1} \left[\frac{1}{s^2} L \left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 \right] \right] \tag{79}$$

Now, we apply the homotopy perturbation method; we have

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) \tag{80}$$

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = at + xt^2 + \frac{t^6}{30} - p \left(L^{-1} \left[\frac{1}{s^2} L \left[\frac{\partial^2 u}{\partial x^2} + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right) \tag{81}$$

The first few components of He's polynomials, for example, are given by

$$\begin{aligned}
H_0(u) &= \left(\frac{\partial u_0}{\partial x} \right)^2 = t^4, \\
H_1(u) &= 2 \frac{\partial u_0}{\partial x} \frac{\partial u_1}{\partial x} = 0, \\
H_2(u) &= \left(\frac{\partial u_1}{\partial x} \right)^2 + 2 \frac{\partial u_0}{\partial x} \frac{\partial u_2}{\partial x} = 0, \\
&\vdots
\end{aligned}$$

Comparing the coefficient of like powers of p , we have

$$\begin{aligned}
p^0 : u_0(x,t) &= at + xt^2 + \frac{t^6}{30}, \\
p^1 : u_1(x,t) &= -L^{-1} \left[\frac{1}{s^2} \left(L \left[\frac{\partial^2 u_0}{\partial x^2} \right] + L[H_0(u)] \right) \right] = -\frac{t^6}{30}, \\
p^2 : u_2(x,t) &= L^{-1} \left[\frac{1}{s} \left(L \left[\frac{\partial^2 u_1}{\partial x^2} \right] + L[H_1(u)] \right) \right] = 0, \\
&\vdots \\
&u_k(x,t) = 0, \quad k \geq 2
\end{aligned} \tag{82}$$

Therefore the exact solution is given by

$$u(x,t) = at + xt^2 \tag{83}$$

Remark (5.3.11)

From comparison, it is clear that the rate of convergence of HPTM is faster than homotopy perturbation method (HPM).

Furthermore, the exact solution can easily be obtained by using HPTM in comparison to HPM in some equation.

The HPTM usually result in the exact solution for the inhomogeneous problem, even for the problem which HPM leads to an approximate solution

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