1.1. Introduction

The renewal processes, the arrival time process and the counting process are inverses, in a sense. The arrival time process is the partial sum process for a sequence of independent, identically distributed variables (the interarrival times). Thus, it seems reasonable that the fundamental limit theorems for partial sum processes (the law of large numbers and the central limit theorem), should have analogs for the counting process. That is indeed the case, and the purpose of this section is to explore the limiting behavior of renewal processes. The main results that we will study, known appropriately enough as renewal theorems, are important for other stochastic processes, particularly Markov chains.

Thus, consider a renewal process with interarrival distribution F and mean interarrival time μ, with the assumptions and basic notation established in the introductory section. When μ=∞, we let. When μ<∞, we let σ denote the standard deviation of the interarrival distribution.

2.1. Renewal Process

A renewal process is used to model occurrence of events happening at random times, where the times between the occurrences are modeled using independent and identically distribution random variables. many of even the most complicated models have within them an embedded renewal process. The formal model is as follows.

We let $Y_n, n \geq 1$.be a sequence of independent and identically distributed random variables which take only non- negative values. We also let Y_0 be a non-negative random variable, independent from $Y_n, n \geq 1$ though not necessarily of the same distribution. The range of these random variables could be discrete, perhaps $\{0,1,3,\ldots\}$, or continuous $\{0,\infty\}$.The random variables Y_n will be the inter- event times of the occurrences. we assume throughout that for all $n \geq 1\,[34]$.

$$P\{Y_0 = 1\} < 1 \text{...(2.1)}$$

Next, for all $n \geq 0$,we define the process S_n as :

$$S_n = \sum_{i=1}^{n} Y_i \text{......................................(2.2)}$$

Where

$$, S_2 = Y_0 + Y_1 + Y_2 \text{, and } S_1 = Y_0 + Y_1 \ , S_0 = Y_0 .$$

Note that the random variable S_n gives the amount of "time'' that random unit passes the nth occurrence (we think of Y_0as the "zeroth" occurrence).

The sequence $\{S_n, n \geq 1\}$ is called a renewal sequence the time S_n are called renewal times. The occurrences are usually called the renewal. Not that if we define $\mu = E(Y_i)$, $i \geq 1$then for $n \geq 1$.

$$E(S_n) = E(Y_0) + ... + E(Y_n) = n\mu + E(Y_0) \text{..................(2.3)}$$

2.1.1. Type of renewal processes

If $P\{Y_0 > 0\} = 0$,then the process is called delayed. If on the other hand, we have that $P\{Y_0 = 0\} = 1$,in which case $S_0 = Y_0 = 0$ with probability of one, then the process is pure.

Recalling that the Indicator function, $1_A : R \rightarrow \{0,1\}$,satisfies[1]

$$I_A(x) = \begin{cases} 1 & if\ ..x \in A \\ 0 & if\ ..x \notin A \end{cases}$$

We define the counting process

$$N(t) = \sum_{n=0}^{\infty} 1[0,1]S_n ,............................(2.4)$$

Which yield the number of renewals in the time interval $[0,1]$ .Note that if the process is pure, then $Y_0 = 0$ and $N(0) = 1$ with probability one N is counting functions.

A few more definition is required. if $P\{Y_n < \infty\} = 1$ for $n \geq 1$,then the renewal process is called proper.

However, if $P\{Y_n < \infty\} < 1$,then the process is called defective. Note that in the defective case there will be a final renewal. In this case $N(t)$ remains bound with a probability of one, though the bound is a random variable. In fact, the bound is geometric random variable with parameter $P\{Y_n < \infty\}$ [34].

2.2. Renewal theory

Let $(X_r)_{r<N}$ be a sequence of independent identically distribution random variables with the property that $P[X_r > 0] > 0$ .put $S_n = \sum_{r=1}^{n} X_r , S_0 = 0$,and define the renewal process $N(t)$ by $N(t) = \max\{n : S_0 \leq t\}, t \geq 0$ .The mean $m(t) = E[N(t)]$is called the renewal function. we have $N(t) \geq n$ if and only if $S_n \leq t$, and hence[4].

$$P[N(t) = n] = P[S_n \leq t] - P[S_{n+1} \leq t]...................(2.5)$$

and

$$E[N(t)] = \sum_{r=1}^{\infty} P[N(t) \geq r] = \sum_{r=1}^{\infty} P[S_r \leq t]...................(2.6)$$

2.2.1. Theorem

If $E\left[X_r\right] > 0$ then $N(t)$ has finite moments for all $t < \infty$.

Proof: since $E\left[X_r\right] > 0$ there exists $\varepsilon > 0$ such that $P\left[X_r \geq \varepsilon\right] \geq \varepsilon.$.

Put $m(t) = \max\left\{n : \varepsilon \sum_{r=1}^{n} I\left\{X_r \geq \varepsilon\right\} \leq t\right\}$. Since $\varepsilon \sum_{r=1}^{n} I\left\{X_r \geq \varepsilon\right\} \leq \sum_{r=1}^{n} I\left\{X_r \leq \varepsilon\right\} \sum_{r=1}^{n} X_r$ it follows that $N(t) = m(t)$, and bence with $m = \left\lfloor t\varepsilon^{-1} \right\rfloor$,

$$E\left[N(t)\right] \leq E\left[\mu(t)\right] = \sum_{n=1}^{\infty} P\left[\mu(t) \geq n\right] = \sum_{n=1}^{\infty}\left[\varepsilon \sum 1_{\{X_r \geq \varepsilon\}} \leq t\right]$$

$$= \sum_{n=1}^{\infty} \sum_{\Lambda \subset [1,\ldots,n], \neq\Lambda \leq \mu} P\left[X_j \geq \varepsilon, j \in \Lambda, X_j < \varepsilon, j \notin \Lambda\right]$$

$$\sum_{n=1} \sum_{\Lambda \in [1,\ldots,n]\ \neq\Lambda \leq m} P\left[X_1 \geq \varepsilon\right]^{\neq\Lambda} \left(1 - P\left[X_1 \geq \varepsilon\right]\right)^{n-\neq\Lambda}$$

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n\Lambda m} C_k^n P\left[X_1 \geq \varepsilon\right]^k (1 - P\left[X_1 \geq \varepsilon\right])^{n-k}$$

$$\leq \sum_{k=0}^{m} P\left[X_1 \geq \varepsilon\right]^k \sum_{n=k}^{\infty} C_k^n)\left(1 - P\left[X_1 \geq \varepsilon\right]\right)^{n-k}$$

$$= \sum_{k=0}^{m} P\left[X_1 \geq \varepsilon\right]^k \sum_{n=0}^{\infty} C_n^{n+k})\left(1 - P\left[X_1 \geq \varepsilon\right]\right)^{n}$$

$$= \frac{1}{P\left[X_r \geq \varepsilon\right]^k}\left(\left\|\frac{1}{\varepsilon}\right| + 1\right) \qquad (2.7)$$

In final equality in (2.7) we used the equality:

$\sum_{n=0}^{\infty} C_n^{n+k})Z^k \leq \frac{1}{(1-Z)^{k+1}}$ for $|Z|<1$, putting the inequality in (2.7) follows theorem (2.2.1) it follows that $E[N(t)]$ is finite whenever $E[X_r]$ is strictly positive. This fact will be used in theorem(2.2.2).

2.2.2. Theorem

The following equality is valid:

$$E\left[S_{N(t)+1}\right] = E[X_1]E[N(t)+1] \quad \text{..........................(2.8)}$$

The equality in theorem (2.2.2) is called wald equality

Proof: the time $N(t)+1$ is stopping time with respect to filtration.

$$f_n = \sigma(X_r : 0 \leq r \leq n) = \sigma(S_r - rE[X_1] : 0 \leq r \leq n)$$

Notice that the process $n \to S_n - nE[X_1]$ is a martingale, and hence

$$E\left[S\left(N(t)+1\right)_{\Lambda n} - \left(\left(N(t)+1\right)_{\Lambda n}\right)E[X_1]\right]$$

$$= E\left[S\left(N(t)+1\right)_{\Lambda 0} - \left(\left(N(t)+1\right)_{\Lambda 0}\right)E[X_1]\right] = 0 \quad \text{..................(2.9)}$$

Since $E[N(t)]$ is finite, from (2.9) we get by letting n tend to ∞.

$$0 = \lim_{n\to\infty} E\left[S_{(N(t)+1)_{\Lambda n}} - ((N(t)+1)_{\Lambda n})E[X_1]\right]$$

$$= E\left[S_{(N(t)+1)} - ((N(t)+1)E[X_1]\right] \quad \text{...............................(2.10)}$$

Consequently, the conclusion in above theorem

2.2.3. Theorem

Let $(X_r)_{r\in N}$ be a sequence of independent, identically distributed random variables such that $P[X_r=0]=0$ put $S_0=0$ and $S_n=\sum_{r=1}^{n}X_r$. Let the process $N(t)$ be defined as in theorem (2.2.2) . let $F(t)$ be the distribution function of the variable X_r. Put $m(t)=E[N(t)]$,then $m(t)$ satisfied the renewal funcation:

$$m(t)=f(t)+\int_0^t m(t-s)dF(S)=\sum_{k=1}^{\infty}\left(\mu^*{}_F\right)^k[0,1]......................(2.11)$$

Where

$\mu_F(a,b)=F(b)-F(a)$,and $\mu_1*\mu_2[a,b]=\int_0^\infty\int_0^\infty I_{(a,b)}(S+t)d\mu_1(S)d\mu_2\ ,0\le a\le b$

i.e. convolution product of the measures μ_1 and μ_2.

Moreover, $(1-\int_0^\infty e^{-\lambda s}dF(S))+\lambda\int_0^\infty e^{-\lambda s}dF(s)$.

If X_r are independent exponentially distributed random variables, and thus the process $(N(t):t\ge 0)$ is Poisson of parameter $\lambda>0$, then $m(t)=\lambda t$.

Proof: on the event $\{X_1>t\}$ we have $N(t)=0$, and hence by using conditional expectation we see [34]

$$m(t)=E[N(t)]=E\left[N(t)1\{X_1\le t\}\right]=E\left[E\left[N(t)1\{X_1\le t\}\sigma(X_1)\right]\right]$$

$$E\left[1\{X_1\le t\}E\left[N(t)-N(t)|\sigma(X_1)\right]\right]+E\left[1_{\{X_1\le t\}}\right]$$

$E\left[N(X_1)\middle|\sigma(X_1)\right]$ (on the event $\{X_1 \le t\}$ we have $N(X_1)=1$)

$$= E\left[1\{X_1 \le t\}E\left[N(t)-N(X_1)\middle|\sigma(X_1)\right]\right]+E\left[1_{\{X\,1\le t\}}E\left[1\middle|\sigma(X_1)\right]\right]$$

The distribution of $N(t)-N(S),t>S$,is the same as the distribution of $N(t-S)$.

$$= E\left[1_{\{X_1\le t\}}E\left[N(t-X_1)\middle|\sigma(X_1)\right]\right]+E\left[1_{\{X\,1\le t\}}E\left[1\middle|\sigma(X_1)\right]\right]$$

$$= E\left[N(t-S)1_{\{X_1\le t\}}\right]+E\left[1_{\{X_1\le t\}}\right]$$

$$= \int_0^t m(t-X)dF(X)+f(t) \qquad (2.12)$$

This completes the proof of theorem (2.2.3)

2.3. The behavior of $N(t)/t$ as $t\to\infty$

Our goal in this section is to characterize the limiting behavior of $N(t)/t$ **as** $t\to\infty$**.** where we preserve the notation from the previous section. We begin with a law of large result[4].

2.3.1.Theorem

Let $N(t)$ be the counting process associated with the renewal sequence $\{S_n, n\ge 0\}$ letting $\mu = E(Y_1)$,

$$\lim_{t\to\infty} = \frac{N(t)}{t} = \frac{1}{\mu} \qquad (2.13)$$

with a probability of one, where the right-hand side is interpreted as zero if $\mu = \infty$.

Proof. Note that if the process is defective, then (i) $\mu = \infty$, and (ii) N(t) is uniformly bounded in t. Using (ii) we have that $\lim_{t \to \infty} = \frac{N(t)}{t} = 0$ with probability one, and the result holds. Hence, we need only consider the case that the process is proper, in which case $N(t) \to \infty$ as $t \to \infty$ with a probability of one. we begin by recalling the strong law of large numbers. Suppose that $\{Z_n\}$ are i.i.d. non-negative random variables with $E[Z_n] \leq \infty$. Then

$$\frac{1}{n}(Z_1 + ... + Z_n) \to E(Z_1), \text{as } t \to \infty$$

almost surely. Note that the above allows for $E[Z_n] = \infty$.

Returning to our processes, the strong law of large numbers states that with a probability one.

$$\lim_{n \to \infty} \frac{Y_1 + ... + Y_n}{n} = \mu$$

Thus, we may conclude that with probability one,

$$\frac{S_n}{n} = \frac{Y_0}{n} + \frac{1}{n} \sum_{i=1}^{n} Y_i \to \mu, \text{as } n \to \infty$$

Noting that $N(t) \to \infty$ as $t \to \infty$, we can therefore conclude that

$$\frac{1}{N(t)} S_{N(t)} \to \mu, \text{as } t \to \infty \text{.......................................(2.14)}$$

Almost surely. By construction, for $N(t) \geq 1$ we have that

$$S_{N(t)-1} \le t \le S_{N(t)}$$

Therefore, so long as $N(t) \ge 2$,

$$\frac{S_{N(t)-1}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)}}{N(t)} \Rightarrow \frac{S_{N(t)-1}}{N(t)-1} \times \frac{t}{N(t)} < \frac{S_{N(t)}}{N(t)}$$

Applying (2.14), we see that as $t \to \infty$

$$\frac{t}{N(t)} \to \mu \Rightarrow \frac{N(t)}{t} \to \frac{t}{\mu}$$

Note that the above result is intuitively pleasing as it says that the shorter the wait between events, as characterized by μ, the more events you expect to see in a given time interval. Note that it also says that[34].

$$N(t) = \frac{1}{\mu} + 0(t) \text{ ,as } t \to \infty \text{...(2.15)}$$

2.4. Point Process

As the name suggests, the basic idea of a point process is to allow us to model a random distribution of points in a space, usually a subset of Euclidean space such as $R, [0,\infty]$, or R^d, for $d \ge 1$. Here are a few examples of renewal processes distributed points on $[0,\infty]$ so that gaps between points are i.i.d. random variables.

The Poisson process, which will be the main object of our focus, is a renewal process which distributes points oo gaps are i.i.d. exponential random variables.

We begin with an important mathematical notion, that of a measure.

2.4.1. Definition

Let E be a subset of Euclidean space, and let F be a $\sigma-$algebra of E. Then, $\mu : f \rightarrow R$ is a measure if the following three conditions hold:

1. For $A \in f$, i.e. for A a subset of E, $\mu \geq 0$.
2. If $\{A_i\}$ are disjoint sets of F, then

$$\mu(U_i A_i) = \sum_i \mu(A_i) \ldots\ldots\ldots\ldots\ldots\ldots(2.16)$$

3. $\mu(\varnothing) = 0$.

We assume that $\{X_n, n \geq 0\}$ are random elements of E, which represent points in the state space E. Next, we define the discrete (random, as it depends upon the point X_n) measure by:

$$I_A(x) = \begin{bmatrix} 1 & if\ ..x \in A \\ 0 & if\ ..x \notin A \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots(2.17)$$

Note, therefore, that 1_{X_n} is a function whose domain is F, i.e. the subsets of E, and whose range is $[0,1]$, and that it takes the value one whenever X_n is in the subset of interest. Next, we note that by taking the sum over n, we find the total number of the points $\{X_n\}$ contained in the set A. Therefore, we define the counting measure
N by

$$N \stackrel{def}{=} \sum_n I_{X_n} \ldots\ldots\ldots\ldots\ldots\ldots(2.18)$$

So that for $A \subset E$,

$$N(A) = \sum_n I_{X_n}(A) \text{ ....}(2.19)$$

simply gives the total number of points in $A \subset E$.

2.4.2. Definition

The function N is called a point process, and $\{X_n\}$ are called the points. We note that as N depends explicitly on the values of the points, X_n, it is natural to call such an object a random measure since the points themselves are random. we will make the (technical) running assumption that bounded regions of A must always contain a finite number of points with a probability of one. That is, for any bounded set A[34],

$$P\{N(A) < \infty\} = 1 \text{ ....,}(2.20)$$

For a renewal process, we have $E = [0,1]$, and the points are the renewal times $\{S_n\}_{n=0}^{\infty}$. The point process is

$$N = \sum_{n=0}^{\infty} I_{S_n} \text{ ....}(2.21)$$

Note that the notation for the counting process has changed from $N(t)$ to $N([0,1])$.

An important statistic of a point process is the mean measure, or intensity, of the process, which is defined to be

$$\mu(A) = EN(A)$$

giving the expected number of points found in the region A. We note that the intensity is commonly referred to as the propensity in the biosciences.

2.5 The Poisson Process

Poisson processes will play a critical role in our modeling and understanding of continuous time Markov chains. We will eventually develop a general notion of a Poisson process, though we begin with the formulation of the one-dimensional model that most people see in their first introduction to probability course. We will refer to this as the first formulation.

We suppose that starting at some time, usually taken to be zero, we start counting events. For each t, we obtain a number, N(t), giving the number of events that has occurred up to time t. Note that we have reverted, for the time being, to our notation from renewal processes.

Recalling that a function f is said to be o(h), and written $f \in o(h)$ or $f = o(h)$, if $\frac{f(h)}{h} \to 0$ as $h \to 0$, we make the following modeling assumptions on the process $N(t)\ (= N([0; t]))$.

1. For some $\lambda > 0$, the probability of exactly one event occurring in a given time interval of length h is equal to $\lambda h + o(h)$. Mathematically, this assumption states that for any $t \geq 0$.

$$P\{N(t+N) - N(t) = 1\} = \lambda h + o(h), \text{as } h \to 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.22)$$

2. The probability that 2 or more events occur in an interval of length h is o(h) [34]:

$$P\{N(t+N)-N(t)=2\}=\lambda h+o(h),\text{as}\, h\to 0 \ldots\ldots\ldots\ldots\ldots(2.23)$$

3. The random variables $N(t_1)-N(s_1)$ and $N(t_2)-N(s_2)$) are independent for any choice of $s_1 \le t_1 \le s_2 \le t_2$. This is usually termed an independent interval assumption.

We then say that $N(t)$ is a homogeneous Poisson process with intensity, propensity, or rate, λ. The following proposition describes the distribution associated to the random variables $N(t)-N(s)$, and makes clear why the process $N(t)$ is termed a Poisson process.

Proposition (2.5.1):

Let $N(t)$ be a Poisson process satisfying the three assumptions above. Then, for any $t \ge S \ge 0$ and $k \in \{0,1,2,...\}$.

$$P\{N(t)-N(S)=K\}=e^{-\lambda(t-s)}\frac{\left(\lambda\left(t-s\right)\right)^k}{K!} \ldots\ldots\ldots\ldots(2.24)$$

Proof.

We will prove the proposition in the case $S=0$, with the more general case proved similarly. We must show that under the above assumptions, the number of events happening in any length of time t has a Poisson distribution with parameter λt. That is, we will show that:

$$P\{N(t)=K\}=e^{-\lambda t}\frac{(\lambda t)^k}{K!}$$

We begin by breaking the interval [0; t] up in to n subintervals of length $\frac{t}{n}$, where n should be thought of as large. We will eventually let $n\to\infty$. We define the two events[33]

$A_n \overset{def}{=} \{k$ of the subintervals contain exactly 1 event and other n - k contain zero $\}$

$B_n \overset{def}{=} \{k$ and at least one of the subintervals contains 2 or more eventsg $\}$

The events are mutually exclusive and:

$$P\{N(t)=k\} = P\{A_n\} + P\{B_n\} \text{.....................}(2.25)$$

Note that the left hand side does not depend upon n. We will show that $P\{B_n\} \to 0$, as $n \to \infty$, hence proving that events happen one at a time. This is called orderliness .Using Boole's inequality, which states that

$$P\left\{\underset{i}{U} C_i\right\} \le \sum_i P\{C_i\},$$

for any set of events $P\{C_i\}$, we have

$= P\{B_n\} \le P\{\text{at least one subinterval has 2 or more events}\}$

$= P\left\{\overset{n}{\underset{i=1}{U}}\{\text{at least one subinterval has 2 or more events}\}\right.$

$\le \sum_{i=1}^{n} P\{\text{ith subinterval contains 2 or more}\}$

$$= \sum_{i=1}^{n} o\left(t/n\right) = no\left(t/n\right) = t\left[\frac{0\left(t/n\right)}{t/n}\right]$$

Thus, $P\{B_n\} \to 0$, as $n \to \infty$. Now we just need to understand the limiting behavior of $P\{A_n\}$. We see from assumptions 1 and 2 that

P{ 0 events occur in a given interval of length h }

Now using assumption 3, independence of intervals, we see that

$$P\{A_n\} = \binom{n}{k}\left(\lambda \frac{t}{n}\right)^k \left[1-\left(\frac{\lambda t}{n}\right)\right]^{n-k}$$

$$= (\lambda t)^k \frac{n!}{(n-k)!k!}\left(\frac{1}{n}\right)^k \left(1-\lambda t\left(\frac{1}{n}\right)\right)^n \left(1-\frac{\lambda t}{n}\right)^{-k}$$

Noting that

$$= \frac{n!}{(n-k)!}\left(\frac{1}{n}\right)^k = \frac{n.(n-1)...(n-k+1)}{n^k}$$

$$= 1.\left(1-\frac{1}{n}\right)^k ...\left(1-\frac{k}{n}+\frac{1}{n}\right) \overset{n\to\infty}{\to} 1$$

$$\left(1-\lambda t\left(\frac{1}{n}\right)\right)^n \overset{n\to\infty}{\to} e^{-\lambda t},$$

$$\left(1-\frac{\lambda t}{n}\right)^{-k} \overset{n\to\infty}{\to} 1,$$

we find that

$$\lim_{n\to\infty} P\{A_n\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Therefore, we see that:

$$P\left\{N(t)-k\right\}=\lim_{n\to\infty}P\left\{A_n\right\}+\lim_{n\to\infty}P\left\{B_n\right\}=\frac{e^{-\lambda t}\left(\lambda t\right)^k}{k!}\;\ldots\ldots\ldots\ldots(2.26)$$

2.6. Continuous renewal processes

A Poisson process can be defined as a counting process for which the Interarrival times are iid which an exponential distribution. A renewal process is more general counting process than Poisson process, which is defined as below [33].

2.6.1. Definition :

Renewal process a with counting process for which the Interarrival times are iid with an arbitrary distribution is said to be renewal process.

A is never is called a renewal if upon its occurrence every thing stints over again probabilistically. Let X_1 be the time to the first renewal and let X_n (n=2,3,...) be the time between (n-1) st renewal and n-th renewal. Assume that X_n (n=1,2,...) are iid random variable with distribution function F. to be non trivial, assume that:

$$F(o)=p\left[X_n=o\right]<1$$

Let

$$M=E\left[X_n\right]=\int_o^\infty XdF(x)\ldots\ldots\ldots\ldots\ldots\ldots(2.27)$$

Which will be positive. Define the time of the n-th renewal by :

$$S_n=\sum_{c=1}^{n}X_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.28)$$

Let N(t) be the number of renewals by time so that

$$N_{(t)} = \max\left[n : S_n \le t\right] \text{..........................(2.29)}$$

Then, the counting process $\left\{N_{(t)}, t \ge 0\right\}$ will be renewal process.

Consider a component that is used continuously with replacement. let y be the life time of the component, which is random with distribution function of G. the component is replaced by a new one upon failure or at affixed time period T which ever comes first (The replacement policy is called an age replacement) then each replacement will be a renewal so counting the number of replacements leads to renewal process. An Interarrival time X will be y or T depending on whether life time is shorter or not. That is,

$$X = \min\left(Y, T\right) = \left\{\begin{matrix} Y & IF & Y < T \\ T & IF & Y \ge T \\ & & \end{matrix}\right\}$$

Then mean interarrival time is obtained by

$$M = E\left[\min(Y, T)\right] = \int_0^\infty P(\min(Y, T) > X)dx$$

$$= \int_0^T P\left\{Y > X\right\}dx = \int_0^T G^c(x)dx \text{..........................(2.30)}$$

2.7. Distribution of the Number of Renewals

Suppose we are interested in the distribution of N(t). The following result holds.

Proposition: for a renewal process $\left\{N(t), t \ge 0\right\}$ with interarrival

$$P\left\{N(t) = n\right\} = F^n(t) - F^{n+1}(t), n = o, 1, ...$$

Where F^n is no fold correlation of F with $F^{(o)} = l$

Proof :

The event $\{N(t) \geq n\}$ is equivalent to the event $\{S_n, \leq t\}$ so.

$$P\{N(t) \geq n\} = P\{S_n, \leq t\} = F^n(t)$$

Therefore **,**

$$P\{N_{(t)} = n\}P = \{N_{(t)} \geq n\} - P\{N_{(t)} \geq n-1\} = F^n(t) - F^{n+1}(t)$$

Proposition (2.7.1):

Let $m(t) = E[N(t)]$ then m(t) is called a renewal funcation

Proposition : the renewal function is given by

$$m(t) = \sum_{n=1}^{\infty} F^n(t) .... (2.31)$$

Proof

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} P\{N(t) \geq n\} = \sum_{n=1}^{\infty} F^n(t).$$

The second equality holds since $N(t)$ is a non- negative

2.8 Random stopping times

Visualize performing an experiment repeatedly by successive sample out puts of given random variable (i.e. ,observing an out come of $X_1, X_2, ...,$ were the X_i'S are iid). The experiment is stopped when enough data has been accumulated for the par poses at hand [33].

This type of situation occurs frequently in application for example, we might be required [3].

Definition: stopping An integer –value N is said to be a stopping time for the sequence of independent variables, $X_1, X_2, ...,$ if the event {N=n} is independent of $X_{n+1}, X_{n+2}, ...,$ for all n=1,2,…

Example

(a) For a renewal process $N(t), t \le o, N(t)+1$, is a stopping time for interarrival times X_i'S .It can be seen that following events are equivalent :

$[N(t)+1=n]=[N(t)=1-n]=[X_1+...+X_n-1 \le t]$ *and* $\{X_1+...+X_n>t\}$ *so* $[N(t)+1=n]$ depends only on $X_1,...X_n$ and independent of $X_{n+1}, X_{n+2},...$

(b) For a renewal process $\{N(t), t \le o\} N(t)$ is not a stopping time for interarrival times X_i'S The reasoning may be similar to (a)

2.9. Wald's Equation:

If $X_1, X_2,...,$ are iid random variables with $E[X]<\infty$ and if the stopping time for $X_1, X_2,..$ such that $E[N]<\infty$

$$E\left[\sum_{C=1}^{N} X_i\right]=E[N]E[X] \qquad (2.32)$$

Proof

Let for n=1,2,…

$$I_n = \begin{cases} 1 & if \;\; n \le N \\ 0 & if \;\; n \succ N \end{cases}$$

Then

$$\sum_{n=1}^{n} X_n = \sum_{n=1}^{\infty} X_n I_n$$

Hence

$$E = \left[\sum_{n=1}^{n} X_n\right] = E\left[\sum_{n=1}^{\infty} X_n I_n\right] = \sum_{n=1}^{\infty} P[X_n I_n]$$

Since N is stopping time the event {N=n} or $\{I_n=1\}$ depends only on $X_1,...X_{n-1}$ and is independent of X_n so, I_n is independent of X_n therefore,

$$E=\left[\sum_{n=1}^{N} X_n\right]=\sum_{n=1}^{\infty} E[X_n]E[\mathrm{I}n]=E[X]\sum_{n=1}^{\infty} P\{N \geq n\}=E[X]E[N].$$

2.9.1. Proposition

The expected time of the first renewal after t is given by:

$$E\left[S_{N(t)+1}\right]=M_{(M(t)+1)}\text{................................}(2.32)$$

Proof :

The result immediately follows from wald's equation since is $N(t)+1$ stopping time for X_i'S .

Not that we cannot apply weld's equation to obtain $E\lfloor S_{n(t)}\rfloor$ since N(t) is not a stopping time. In fact, the interarrival time containing time , $X_{N(t)+1}$.has a different distribution form the usual ones. We will consider this quality (called spread at t) later[33].

Suppose that $\{N(t),t\leq 0\}$ is a Poisson process having rate λ .then,

$$E\left[S_{N(t)+1}\right]=\mu(m(t)+1)=\frac{1}{\lambda}(\lambda t+1)=t+\frac{1}{\lambda}$$

Which is also intuitively derived since $S_{N(t)+1}$ is t plus time to next event. But,

$$E\left[S_{N(t)}\right]=\left[E\left[S_{N(t)}|N(1)\right]\right]=\left[t\,\frac{N(t)}{N(t)+1}\right]$$

The second equality holds since $S_{N(t)}|N(t)=n$ is the largest one among a sample of size n from unif(0,t) using the property that:

$$E\left[\frac{y}{y+1}\right]=E\left[\frac{y-1}{\alpha}\right]\text{for y}\rightarrow\text{poi}(\alpha)$$

We have

$$E\left[S_{N(t)}\right] = E = \left[t\frac{N_{(t)-1}}{\lambda t}\right] = t - \frac{1}{\lambda}$$

Which is different from $\mu m(t) = 1$

2.9.2. Definition

Suppose that $\{N(t), \mathrm{t} \le 0\}$ is renewal process

(a) The at t of the renewal process defined by:

$$A_{(t)} = t - S_{N(t)}$$

(b) The excess at t of the renewal process is defined by:

$$y_{(t)} = S_{N(t)+1} - t$$

(c) The spread at t of the renewal process is defined by:

$$y_{N(t)+1} = A_{(t)} + y_{(t)}$$

2.10. Long –run Renewal Rate

This section deals with the average number of renewal (permit time) in the long run, which will be called a long-run renewal rate

Proposition (2.10.1):

For a renewal process $\{N(t), \mathrm{t} \le 0\}$ having df F for interarrival times,

$$\lim_{\mathrm{t}\to\infty} \frac{\mathrm{N(t)}}{\mathrm{t}} = \frac{1}{\mu} \quad \text{w.p.1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.33)$$

Where

$$\mu = \int_{\circ}^{\infty} x dF(x)$$

Proof

Since $S_{N(t)}$ is the last renewal time prior to t and $S_{N(t)+1}$ is the first renewal time after t,

$$S_{N(t)} \le t \le S_{N(t)+1} \quad \text{or} \quad \frac{S_{N(t)}}{N_{(t)}} \le \frac{t}{N_{(t)}} \le \frac{S_{N(t)+1}}{N_{(t)}}$$

But

$$\lim_{t\to\infty} \frac{S_{\mathrm{N(t)}}}{\mathrm{N}_{(t)}} = \lim_{t\to\infty} \frac{\mathrm{X}_1 + ... + X_{N(t)}}{\mathrm{N(t)}} = E\left[X\right] = \mu$$

and

$$\lim_{t\to\infty} \frac{S_{\mathrm{N(t)+1}}}{\mathrm{N}_{(t)}} = \lim_{t\to\infty} \frac{S_{\mathrm{N(t)+1}}}{\mathrm{N}_{(t)+1}} \frac{N_{(t)+1}}{N_{(t)}} = E\left[X\right] = M$$

For a renewal process $\{N(t), \mathrm{t} \le 0\}$ having M of mean interarrival time.

2.11 Renewal Reward Processes

Consider a renewal process $\{N(t), \mathrm{t} \le 0\}$ let us assume that reward will be earned at time of renewal. There can be lost or profit attached to the renewal. let R_n denote the reward earned at the time of n-th renewal $(n = 1, 2, ...)$,which are iid random variables having a common mean $\mathrm{E}\{\mathrm{R}\}$. R_n may depend on $\mathrm{X_n}$. Then, the total reward earned by t, $\mathrm{R(t)}$ will be $[33]$

$$R_{(t)} = \sum_{n=1}^{N(t)} R_n \text{..................................(2.33)}$$

The new process $\{N_{(t)}, \mathrm{t} \le 0\}$ is called a renewal reward

Proposition (2.11.1):

Suppose that $\{R(t), \mathrm{t} \le 0\}$ is called a renewal reward process having $E\left[R\right] < \infty$ and $E\left[X\right] < \infty$. then, the long –run rate or long-run average reward is given by :

$$\lim_{\mathrm{t}\to\infty} \frac{R(t)}{\mathrm{t}} = \frac{E(R(t))}{E(X)} \quad \text{...........................(2.34)}$$

Proof : since

$$\frac{R(t)}{\text{t}} = \frac{\sum_{n=1}^{N(t)} R_n}{\text{N(t)}} \frac{\text{N(t)}}{\text{t}}$$

The result follows if SLLN is applied to each the interarrival time the a renewal process is often as renewal cycles. the above result says that the long-run reward rate is obtained by:

$$long - run \text{ reward rate } = \frac{\text{E}\left[\text{reward during acycle}\right]}{\text{E}\left[\text{reward acycle length}\right]}$$

Proposition (2.11.2):

Suppose that $\{R_{(t)}, \text{t} \leq 0\}$ is called a renewal reward process having $E[R] < \infty$ and $E[X] < \infty$. then

$$\lim_{\text{t}\to\infty} \frac{E[(R)]}{\text{t}} = \frac{E(R)}{E(X)}$$

2.12. Key Renewal Theorem

If F is not lattice and Q(t) is directly Riemann integrable, then**:**

$$\lim_{t\to\infty} \int_0^t Q(t-t)dm(x) = \frac{1}{\mu}\int_0^\infty Q(t)dt \text{ ....(2.35)}$$

$$u(t) = \int_0^\infty x dF(x)$$

Sufficient conditions for Q(t) to be directly Riemann integrable are:

1. $Q(t) \geq 0$

2. $Q(t)$ is nonincreasing

3. $\int_0^\infty Q(t)dt < \infty$

Limiting mean excess:

let Y(t) be the excess at t. The mean excess is obtained in example by

$$E[Y(t)] = h(t) + \int_0^t h(t-x)dm(x) \text{......................(2.36)}$$

Where

$$h(t) = \int_t^\infty (x-t)dF(x)$$

So,

$$\lim_{t\to\infty} E[Y(t)] = \lim_{t\to\infty} h(t) + \lim_{t\to\infty} \int_0^t h(t-x)dm(x) = \lim_{t\to\infty} \int_0^t h(t-x)dm(x)$$

If we assume that the second moment of an interarrival time (X, say) is finite, then the function h(t) is directly Riemann integrable. If we assume that F is not lattice in addition, we can apply key Renewal Theorem to obtain the limiting mean excess as follows:

$$\lim_{t\to\infty} E[Y(t)] = \lim_{t\to\infty} \int_0^t h(t-x)dm(x)$$

$$\frac{1}{\mu}\int_0^\infty h(t)dt = \frac{1}{\mu}\int_0^\infty \int_t^\infty (x-t)dF(x)dt$$

If we change the order of two integrals, we have

$$\lim_{t\to\infty} E[Y(t)] = \frac{1}{\mu}\int_0^\infty \int_t^x (x-t)dtdF(x)dt$$

$$= \frac{1}{2\mu}\int_0^\infty x^2 dx = \frac{E[X^2]}{2E[X]}$$

Note that the result is same as the average excess

2.13. Blackwell's Theorem:

1. If F is not lattice, then

$$\lim_{t\to\infty}\left[m(t+a)-m(t)\right]=\frac{a}{\mu}$$

2. If F is lattice with period d, then

$$\lim_{n\to\infty}E\left[number..of\ ..renewals..at..nd\right]=\frac{d}{\mu}$$

Note that Blackwell's Theorem for lattice case states

$$\lim_{n\to\infty}P\left[renewals..occurs..nd\right]=\frac{d}{\mu}$$

since the number of renewals at time nd will be 1 or 0.

3.14. Alternating Renewal Processes

Consider a renewal process whose interarrival times are N_s having the distribution F. Suppose that an interarrival time consists of an ON period and Y_n an OFF period such that $X_n=Z_n+Y_n$ where Z_n is the n-th ON period Z_n and is the n-th OFF period. Suppose also that are iid as H and are iid as G. and may not be independent[33].

Let

$$I(t)=\begin{bmatrix} 1 & if\ .the..system..ON\ .t \\ 0 & otherwise \end{bmatrix}$$

We are interested in

1. What is the long-run proportion of time that the system is ON ?

$$\lim_{t\to\infty}\frac{\int_0^t I(X)dx}{t}=?$$

2. What is the limiting probability that the system is ON ?

$$\lim_{t\to\infty}P\{I(t)=1\}=?$$

The long-run proportion of time that the system is On is given by

$$\lim_{t\to\infty}\frac{\int_0^t I(X)dx}{t}=\frac{E[Z_n]}{E[X_n]}$$

Obvious from the result from the renewal-reward process.

and F is non-lattice, then $E[Z_n+X_n]<\infty$ If

$$\lim_{t\to\infty}P\{I(t)=1\}=\frac{E[Z_n]}{E[Z_n]+E[Y_n]} \qquad (2.37)$$

2.15. Delayed Renewal Processes

Consider a renewal process having distribution F for the interarrival times. Suppose we start observing the process from a certain time point.

Let

X_1: time to the first renewal after observing X_1

X_n :(n=2,3,…): time between (n-1)st and n-th renewal

Then, X_1 has the different distribution G, say, from F for other interarrival times. Let us define the time of the n-th event as before (n=1,2,…):

$$S_n = \sum_{i=1}^{n} X_i \quad \text{……………………..(2.38)}$$

Let

$$N_D(t) = \max\{n : S_n \le t\}$$

Then, $\{N_D(t), t \ge 0\}$ is said to be a delayed (or general) renewal process. Note that if G=F it will be an ordinary renewal process

2.16 Expected number of renewals

The purpose of this section is to evaluate $E[N(t)]$, denoted $m(t)$, as a function of $t > 0$ for arbitrary renewal processes. We first find an exact expression, in the form of an integral equation, for $m(t)$. This can be easily solved by Laplace transform methods in special cases. For the general case, however, $m(t)$ becomes increasingly messy for large t, so we then find the asymptotic behavior of $m(t)$. Since $N(t)/t$ approaches $1/X$ with probability 1, we might expect $m(t)$ to grow with a derivative $m'(t)$ that asymptotically approaches $1/\bar{X}$. This is not true in general. Two somewhat weaker results, however, are true. The first, called the elementary renewal theorem, states that $\lim_{t \to \infty} m(t)/t = 1/\bar{X}$. The second result, called Blackwell's theorem, states that, subject to some limitations on $\delta > 0, \lim_{t \to \infty}[m(t+\delta) - m(t)] = \delta / \bar{X}$. This says essentially that the expected renewal rate approaches steady state as $t \to 1$. We will find a number of applications of Blackwell's theorem throughout the remainder of the text[33].

The exact calculation of $m(t)$ makes use of the fact that the expectation of a nonnegative random variable is defined as the integral of its complementary distribution function,

$$m(t) = E\left[N(t)\right] = \sum_{n=1}^{\infty} P\left\{N(t) \geq n\right\}$$

Since the event $\{N(t) \geq n\}$ is the same as $\{S_n \leq t\}$, $m(t)$ is expressed in terms of the distribution functions of S_n, $n \geq 1$, as follows.

$$m(t) = \sum_{n=1}^{\infty} P\left\{S_n \leq t\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2.39)$$