3.1 Introduction:

In this chapter we recall the definition of the most common stochastic models which are used to repairable system model, reliability models, homogenous Poisson process and non homogenous process and families of lifetime distribution.

3.2: Reliability:

Reliability is defined as the probability of success or the probability that the system will perform as intended function under specified design limits[1].

More specific reliability is the probability that a product part will operate properly for specified period of time (design life) under the design operating condition without failure. In other words, reliability may be used as measure of the system success in providing, Its function properly. Reliability is once of quality characteristics that consumer require from the menufacture of products[1].

Mathematically: reliability R(t) is the probability that a system will be successful in the interval from time 0 time t:

$$R(t) = P(T > t) \qquad t > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.1)$$

Where T is a random variable denoting the time -to-failure or failure time .

Unreliability F(t), a measure of failure, is defined as the probability that the system will fail by time t:

$$F(t) = P(T \le t) \qquad \text{for} \quad t \ge 0$$

In other words, F(t) is the failure distribution function. If the time-to-failure random variable T has a density function $f(t)$. Then

$$R(t) = \int_{t}^{\infty} f(s)ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.2)$$

or , equivalently

$$f(t) = -\frac{d}{dt}\left[R(t)\right]$$

The density function can be mathematically described in terms of T.

$$\lim_{\Delta t \to 0} p(t < T \le t + \Delta t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3)$$

This can be interpreted as the probability that the failure time T will occur between the operating time t and the next interval of operation , $t + \Delta t$.

Consider new and successfully tested system that operates well when put into service at time $t = 0$, the system becomes less likely to remain successful as the time interval of course , is zero.

3.2.1: Fault Rate:

The possibility of fault machine in specific time period t_1, t_2 can be expressed with following non reliability, equation[24].

$$\int_{t_1}^{t_2} f(t)dt = \int_{\infty}^{t_1} f(t)dt - \int_{-\infty}^{t_2} f(t)dt = F(t_1) - F(t_2)$$

Or can be expressed with reliability

$$\int_{t_1}^{t_2} f(t)dt = \int_{t_1}^{\infty} f(t)dt - \int_{t_2}^{\infty} f(t)dt = R(t_1) - R(t_2)$$

The rate that a fault took place with in specific period of time is called as "Fault Rate" through out the period t_1 indicates for no fault in the beginning of period and therefore the equation can be expressed as follows:

$$\frac{R(t)-R(t_2)}{(t_2-t_1)R(t_1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.4)$$

It has been observed that the fault rate depend on time if the period t_1 denoted as $t+\Delta t$ the equation (3.4) stated as follow:

$$\frac{R(t)-R(t+\Delta t)}{\Delta t.R(t)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5)$$

and means with rate of number of faults in each unit time.

3.2.2: Hazard Rate:

Define as limits of rate of faults for a period of near-zero equation can be written in the form:

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t)-R(t+\Delta t)}{\Delta t.R(t)} = \frac{1}{R(t)}\left[-\frac{dR(t)}{dt}\right]$$

$$h(t) = \frac{f(t)}{R(t)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.6)$$

to find out possibility of fault machine it have age t in time period $[t, t + \Delta t]$ written as:

$$f_{pos} = h(t).dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.7)$$

The hazard rate refer to change in rate fault through age of machine. To find out hazard rate for the sample machines N (machine consisting of n elements), we will assume that $N_s(t)$ is a random variable denotes the number of machines working successfully at time t thus, the $N_s(t)$ is binomial distribution[24].

$$P\left[N_s(t) = n\right] = \frac{N}{N.(N-n)} = \left[R(t)\right]^n \left[1 - R(t)\right]^{N-n}$$

$$n = 0,1,\ldots,N$$

the expected value for $N_s(t)$:

$$E\left[N_s(t)\right] = N.\left[R(t)\right] = N(t)$$

Hence

$$R(t) = \frac{E(N_s(t))}{N} = \frac{\bar{N}(t)}{N} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.8)$$

and reliability in time t, it is arithmetic mean for rate success in t. Thus:

$$F(t) = 1 - R(t) = 1 - \frac{\bar{N}(t)}{N} = \frac{N - \bar{N}(t)}{N} \ \ldots\ldots\ldots\ldots\ldots\ldots (3.9)$$

and rate density fall equal

$$F(t)=\frac{dF(t)}{dt}=-\frac{1}{N}.\frac{d\bar{N}(t)}{dt}$$

3.2.2: System Mean Time To Failure:

Suppose that the reliability function for a system is given by R(b) , the expected failure time during which a component is expected to perform success fully , or the system mean time to failure (MTTF) , given by [1]:

$$MTTF=\int_0^\infty tf\ (t)dt \text{ ....(3.10)}$$

Substituting

$$f\ (t)=-\frac{d}{dt}\left[R(t)\right]$$

From equation (3.10) and performing integration by part, we obtain.

$$MTTF=-\int_0^\infty tdt\left[R(t)\right]=\left[-tR(t)\right]_0^\infty+\int_0^\infty R(t)dt \text{ ....(3.11)}$$

The first term on the right hand side of above equation equals zero at both limits, since the system must fail after a finite amount of operating time , therefore , we must have $tR(t)\rightarrow 0\quad as\ tR(t)\rightarrow 0$ this leaves the second term , which equals.

$$MTTF=\int_0^\infty R(t)dt \text{ ....(3.12)}$$

Thus, MTTF is the definite integral evalution of the reliability function. In general . if $\lambda(t)$ is defined as the failure rate function , then , by definition , MTTF is not equal to $1/\lambda(t)$.

The MTTF should be used when the failure time distribution function is specified because the reliability level implicit by the MTTF depends on the underlying failure time distribution . Although the MTTF measure is one of the most widely used reliability calculation , it also one of most missed calculations , it has been misinterpreted as " guaranteed minimum life time".

3.2.3 Maintainability:

When a system fails to perform satisfactory, repair is normally carried out to locate and corrected the fault. The system is restored to operational effectiveness by making an adjustment is defined as the probability that a failed system will be restored to specified conditions within a given period of time when maintenance is performed according to pre cribbed procedures resources, in other words , maintainability is the probability of isolating and repairing a fault in system within a given time [1].

Let T denote the random variable of the time to repair or the total downtime. If the repair time T has a repair time density function g (t) , then the maintainability ,v(t) , is defined as the probability that the failed system will back in service by time t.

$$V(t) = P(T \le t) = \int_0^t g(s)ds$$

For example , if $g(s) = \mu e^{-\mu t}$ where $\mu > 0$ is a constant repair rate , them

$$V(t) = 1 - e^{-\mu t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.13)$$

Which represents the exponential form of the maintainability function . An important measure of time used in maintenance studies is mean time to repair (MTTR) or the mean downtime . MTTR is the expected value of the random variable repair time , not failure time , and is given by:

$$MTTR = \int_0^\infty t g(t) dt \text{ ....(3.15)}$$

When the distribution has a repair time density given by , then from the above equation ,   when the repair time is T.

3.2.4 :Availability

Reliability is a measure that requires system success for an entire mission time . no failure or repairs are allowed.

The availability of a system is defined as the probability that the system is successful at time t , mathematically:

$$\text{Availability} = \frac{\text{system up time}}{\text{system up time+system down time}} \text{ ....(3.15)}$$

Availability is a measure of success used primarily for repairable system. For non-repairable system availability $A(t)$ equals reliability $R(t)$ .In repairable system $A(t)$ will be equal to or greater than $R(t)$. The mean time between failure (MTBF) is an important measure in repairable system. This implies that the system has MTBF is an expected value of the random variable time between failures mathematically[1].

$$\text{MTBF=MTTF+MTTR ....(3.16)}$$

3.5: Homogeneous Poisson Process (HPP):

If a system in service can be repaired to a good new condition following each failure, then the failure process is called a renewal process. For renewal process, the time between failure are independent and identically distributed[2].

A special case of this is the homogeneous Poisson process (HPP) . Which has Independent and exponential time between failures. A counting process is homogenous Poisson process with parameter $\lambda > 0$ if:

1. N(0)=0
2. The process has independent increments.
3. The number of failures in any interval of length t is distributed as a Poisson distribution with parameter λ .

There are several implications to this definition of the Poisson process. The distribution with parameter $\lambda(t_1,t_2)$ therefore , the probability mass function is:

$$p\left[N(t_2) - N(t_1) = n\right] = \frac{\left[\lambda\left(t_2 - t_1\right)\right]^{X} e^{-\lambda(t_2 - t_1)}}{n!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.17)$$

The expected number of failures by time t is $\Lambda = E\left[N(t)\right] = \lambda t$ where λ is often called the failure intensity or rate of occurrence of failure (ROCOF). $u(t) = \Lambda'(t) = \lambda$ .the intensity function is therefore if $X_1, X_2, \ldots$ are identically corresponds to Poisson process.

3.4:Non homogenous Poisson Process (NHPP).

The non-homogenous Poisson Process (NHPP) that represents the number of failures experienced up to time t is $\{N(t), t \geq 0\}$.

The main issue in the NHPP model is to determine an appropriate mean value function to denote the expected number of failures experienced up to a certain time with different assumptions , the model will end up with different functional forms of the mean value function. Note that in renewal time between failures is relaxed and in the NHPP ,the stationary assumption is relaxed [1].

The NHPP model is based on the following assumptions :

1. The failure process has an independent increment.

   The number of failures during the time interval $(t, t+s)$ depends on s, and does not depend on the past history of the process.

2. The failures rate of the process is given by :

   P{exactly one failure in $(t, t+\Delta t)$ }$= P\{N(t, t+\Delta t) - N(t) = 1\} = \lambda(t)\Delta t + 0\Delta t$

   Where λt is the intensity function.

3. During a small interval Δt , the probability of more than one failure negligible that is,

   P{two or more than failures in $(t, t+\Delta t)$ } = 0

On the basis of these assumptions the probability of exactly n failures occurring during the time interval $(0,t)$ for the NHPP is given by:

$$P_r\left[N(t)=n\right]=\frac{\left[m(t)\right]^n e^{-mt}}{n!} \quad \text{.....................................(3.18)}$$

Where $m(t)=E\left[N(t)\right]=\int_0^t \lambda(t)ds$ and λt is the intensity funcation, it can be easily show that mean value funcation $m(t)$ is non-decreasing.

3.5: Test for the Time Trend and Repair Effect:

First, the Laplace test is utilized in testing for the time trend, and the repair. The hypothesis is expressed as[2]:

H_o: No time trend exist (HPP)

H_1: Time trend exist (NHPP)

For Laplace test given r repair $T_1, T_2, ..., T_r$ and censoring time trend $> T_r$ we calculate the statistic:

$$T = 2\sum_{i=1}^{r} \ln \frac{Trend}{T_i} \quad \text{.............................(3.19)}$$

This test statistic follows a standard normal distribution, this test is recommended for the case when the choice is between trend exist (HPP) and trend not exist (NHPP).

3.6: Repair Rate:

A different approach is used for modeling the rate of occurrence of failure incidences for a repairable system. These rates are called repair rates. Time is measured by system power-on-hours from initial turn-on at time zero, to the end of system life. Failures occur as given system ages and the system is repaired to a state that may be the same as new, better, or worse.

The frequency of repairs may be increasing, decreasing, or staying at a roughly constant rate [24].

Let $N(t)$ be a counting function that keeps track of the cumulative number of failures of a given system has had from time zero to time t. $N(t)$ is a step function that jumps up one every time a failure occurs and stays at the new level until the next failure. Every system will have its own observed $N(t)$ function over time. If we observe the $N(t)$ curves for a large number of similar systems and "averaged" these curves, we would have an estimate of *M(t)*= the expected number (average number) of cumulative failures by time t for these systems. The derivative of *M(t)* denoted $m(t)$ is defined to be the Repair Rate or the Rate of Occurrence of Failures at t or ROCOF.

3.7: Common Distribution Functions

This section presents some of the common distribution functions and several hazard models that have applications in reliability engineering [1] .

3.7.1:Poisson Distribution

Although the Poisson distribution can be used in a manner similar to the binomial distribution, it is used to deal with events in which the sample size is unknown. This is also a discrete random variable distribution whose pdf is given by

$$P\left(X = x\right) = \frac{\left(\lambda t\right)^{x} e^{-\lambda t}}{x\,!} \quad \text{for} \quad x = 0,1,2,\ldots$$

where λ constant failure rate, x is the number of events. In other words, $P(X = x)$ is the probability of exactly x failures occurring in time *t*. Therefore, the reliability Poisson distribution, $p(k)$ (the probability of k or fewer failures) is given by

$$P(k) = \sum_{x=0}^{k} \frac{(\lambda t)^x e^{-\lambda t}}{x!} \quad \text{.............................(3.20)}$$

This distribution can be used to determine the number of spares required for the reliability of standby redundant systems during a given mission.

3.7.2: Exponential Distribution

The exponential distribution plays an essential role in reliability engineering because it has a constant failure rate. This distribution has been used to model the lifetime of electronic and electrical components and systems. This distribution is appropriate when a used component that has not failed is as good as a new component – a rather restrictive assumption. Therefore, it must be used diplomatically since numerous applications exist where the restriction of the memory less property may not apply[1].

$$f\left(t\right) = \frac{1}{\theta} e^{\frac{1}{\theta}} = \lambda e^{-\lambda t}, t \ge 0 \quad \text{.........................(3.21)}$$

Where $\theta = \frac{1}{\lambda} > 0$ is an MTTF's parameter and $\lambda \ge 0$ is a constant failure rate. The hazard function or failure rate for the exponential density function is constant, *i.e.*,

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\theta} e^{\frac{1}{\theta}}}{e^{-\frac{1}{\theta}}} = \frac{1}{\theta} = \lambda$$

The failure rate for this distribution is λ ,a constant, which is the main reason for this widely used distribution. Because of its constant failure rate property, the exponential is an excellent model for the long flat "intrinsic failure" portion of the 18 System Software Reliability bathtub curve. Since most parts and systems spend most of their lifetimes in this portion of the bathtub curve, this justifies frequent use of the exponential (when early failures or wear out is not a concern). The exponential model works well for inter-arrival times. When these events trigger failures, the exponential lifetime model can be used.

3.7.3: Normal Distribution

The normal distribution plays an important role in classical statistics owing to the *Central Limit Theorem*. In reliability engineering, the normal distribution primarily applies to measurements of product susceptibility and external stress. This two parameters distribution is used to describe systems in which a failure results due to some wear out effect for many mechanical systems.

The normal distribution takes the well-known bell shape. This distribution is symmetrical about the mean and the spread is measured by variance. The larger the value the flatter the distribution. The pdf is given by:

$$f\ (t) = \frac{1}{\sigma\sqrt{2\pi}^{e}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma/}\right)^2} \quad -\infty < t < \infty \quad \text{......................(3.22)}$$

Where μ is the mean value and σ is the standard deviation.

The reliability function is the cumulative distribution funcation i.e

$$R(t) = \int_{-t}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2} ds$$

There is no closed form solution for the above equation. However, tables for the standard normal density function are readily available and can be used to find probabilities for any normal distribution.

where Φ is a standard normal distribution function. Thus, for a normal random variable *T*, with mean μ and standard deviation σ.

$$P(T \le t) = P\left(z \le \frac{t-\mu}{\sigma}\right) = \phi\left(\frac{t-\mu}{\sigma}\right)$$

Where Φ yields the relationship necessary if standard normal tables are to be used.

The hazard function for a normal distribution is a monotonically increasing function of *t*. This can be easily shown by proving that $h(t) \ge 0$ for all *t*. Since

$$h(t) = \frac{f(t)}{R(t)}$$

$$h(t) = \frac{R(t)\, f(t) + f^2(t)}{R^2(t)} \ge 0 \quad \text{......................(3.23)}$$

One can try this proof by employing the basic definition of a normal density function *f*.

3.7.4: Log Normal Distribution

The log normal lifetime distribution is a very flexible model that can empirically fit many types of failure data. This distribution, with its applications in maintainability engineering, is able to model failure probabilities of repairable systems and to model the uncertainty in failure rate information. The log normal density function is given by:

$$f\ (t) = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2} t \geq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.24)$$

Where μ and σ are parameters such that $-\infty < \mu < \infty$ and, $\sigma > 0$ Note that μ and σ are not the mean and standard deviation of the distribution.

The relationship to the normal (just take natural logarithms of all the data and time points and you have “normal” data) makes it easy to work with many good software analysis programs available to treat normal data.

Mathematically, if a random variable x is defined as $X = \ln T$, then X is normally distributed with a mean of μ and a variance of σ^2 . That is,

$$E(X) = E(\ln T) = \mu$$

and

$$V(X) = v(\ln T) = \sigma^2$$

Since $T = e^x$, the mean of the log normal distribution can be found by using the normal distribution. Consider that

$$E(T) = E(e^x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\left[x-\frac{1}{2}(\sigma)^2\right]} dx$$

And by rearrangement of the exponent, this integral becomes

$$E(T) = e^{\mu+\frac{\sigma^2}{2}} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{2}{2\sigma^2}\left[x-\left(\mu+\sigma^2\right)\right]^2} dx$$

Thus, the mean of the log normal distribution is

$$E(T) = e^{\mu+\frac{\sigma^2}{2}}$$

Proceeding in a similar manner,

$$E\left(T^2\right) = E\left(e^{2X}\right) = e^{2\left(\mu+\sigma^2\right)}$$

Thus, the variance for the log normal is

$$V(T) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

The cumulative distribution function for the log normal is

$$F(t) = \int_0^t \frac{1}{\sigma 2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{ins-\mu}{\sigma}\right)^2} ds$$

And this can be related to the standard normal deviate Z by

$$F(t) = P[T \le t] = P(lnT \le Int)$$
$$= P\left[z \le \frac{lnt-\mu}{\sigma}\right]$$

Therefore, the reliability function is given by

$$R(t) = P\left[z > \frac{\ln t - \mu}{\sigma}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.25)$$

And the hazard function would be

$$h(t) = \frac{f(t)}{R(t)} = \frac{\Phi\left(\frac{\text{int} - \mu}{\sigma}\right)}{\sigma t R(T)}$$

Where Φ is a cdf of standard normal density.

Mechanisms. Some of these are: corrosion and crack growth, and in general, failures resulting from chemical reactions or processes.

3.7.5: Weibull Distribution

The exponential distribution is often limited in applicability owing to the Memoryless property. The Weibull distribution is a generalization of the exponential distribution and is commonly used to represent fatigue life, ball bearing life, and vacuum tube life. The Weibull distribution is extremely flexible and appropriate for modeling component lifetimes with fluctuating hazard rate functions and for representing various types of engineering applications. The three-parameters probability density function is

$$f(t) = \frac{\beta (t - y)^{\beta - 1}}{\theta \beta} e^{-\left(\frac{t - y}{\theta}\right)^{\beta}} \quad t \geq y \geq 0 \quad \ldots\ldots\ldots\ldots\ldots (3.26)$$

Where θ and β are known as the scale and shape parameters, respectively, and y is known as the location parameter. These parameters are always positive. By using different parameters, this distribution can follow the exponential distribution, the normal distribution, etc. It is clear that, for $t \leq y$, the reliability function $R(t)$ *is*

$$R(t)= e^{-\left(\frac{t-y}{\theta}\right)^{\beta}} \; for > y > 0, \beta > 0, \theta > 0$$

Hence,

$$R(t)= \frac{\beta\left(t-y\right)^{\beta-1}}{\theta^{\beta}} t > y > 0, \beta > 0, \theta > 0$$

It can be shown that the hazard function is decreasing for $\beta - 1$, increasing for $\beta \geq 1$, and constant when $\beta = 1$.

3.7.5.1:Other Forms of Weibull Distributions

The Weibull distribution again is widely used in engineering applications. It was originally proposed for representing the distribution of the breaking strength of materials. The Weibull model is very flexible and also has theoretical justification in many applications as a purely empirical model. Another form of Weibull probability density function is, for example,

$$f(x)= \lambda y x^{y-1} e^{-\lambda y} \qquad \text{……………………..(3.27)}$$

When $\gamma = 2$, the density function becomes a Rayleigh distribution.

It can easily be shown that the mean, variance and reliability of the above Weibull distribution are, respectively, as follows:

$$\text{Mean} \; = \lambda \frac{1}{y} \Gamma\left(1 + \frac{1}{y}\right)$$

$$\text{Variance} \; = \lambda \frac{1}{y}\left(\Gamma\left(1 + \frac{1}{y}\right) - \left(\Gamma\left(1 + \frac{1}{y}\right)^{2}\right)\right)$$

$$\text{Reliability} = e^{-\lambda t y}$$

3.7.6: Gamma Distribution

Gamma distribution can be used as a failure probability function for components whose distribution is skewed. The failure density function for a gamma distribution is

$$f\left(t\right)=\frac{t^{\alpha-1}}{\beta^{\alpha}\Gamma\left(\alpha\right)}e^{\frac{t}{B}} \quad t\geq 0,\alpha,\beta>0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.28)}$$

Where α is the shape parameter and β is the scale parameter. Hence,

$$R\left(t\right)=\int_{t}^{\beta}\frac{t^{\alpha-1}}{\beta^{\alpha}\Gamma\left(\alpha\right)}s^{\alpha-1}e^{-\frac{t}{\beta}}ds$$

If t is an integer, it can be shown by successive integration by parts that

$$\boldsymbol{R(t)=e^{-\frac{t}{\beta}}\sum_{i=0}^{\alpha-1}\frac{\left(\frac{t}{\beta}\right)^{i}}{i\,!}}$$

and

$$h\left(t\right)=\frac{f\left(t\right)}{R\left(t\right)}=\frac{\frac{1}{\beta^{\alpha}}t^{\alpha-1}e^{-\frac{t}{\beta}}}{-\frac{1}{\beta}\sum_{i=0}^{\alpha-1}\frac{\left(\frac{t}{\beta}\right)^{i}}{i\,!}}$$

The gamma density function has shapes that are very similar to the Weibull distribution. At $\alpha=1$, the gamma distribution becomes the exponential distribution with the constant failure rate $1/\beta$. The gamma distribution can also be used to model the time to the n^{th} failure of a system if the underlying failure distribution is exponential. Thus, if X_i is exponentially distributed with parameter $\theta = 1/\beta$, then $T = X_1 + X_2 + ... + X_n$, is gamma distributed with parameters β and n .

The other form of the gamma probability density function can be written as follows:

$$f\,(\text{x}) = \frac{\beta^{\alpha} t^{\alpha-1}}{\Gamma(\alpha)} e^{-t\beta} \quad \text{for } t > 0 \text{........................(3.29)}$$

This pdf is characterized by two parameters: shape parameter α and scale parameter β. When $0 < \alpha < 1$, the failure rate monotonically decreases; when $1 > \alpha$, the failure rate monotonically increase; when $\alpha = 1$ the failure rate is constant.

The mean, variance and reliability of the density function in equation (3.27) are, respectively,

$$\text{Mean(MTTF)} = \alpha\beta$$

$$\text{Variance} = \alpha\beta^2$$

$$\text{Reliability} = \int_t^{\infty} \frac{t^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} e^{-x\beta} dx$$

The gamma model is a flexible lifetime model that may offer a good fit to some sets of failure data. It is not, however, widely used as a lifetime distribution model for common failure mechanisms. A common use of the gamma lifetime model occurs in Bayesian reliability applications.

3.7.8: Beta Distribution

The two-parameter Beta density function, $f(t)$ is given by

$$f(t)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}t^{\alpha-1}(1-t)^{\beta-1}\ 0<t<1, \alpha>0, \beta>0 \quad \text{.........(3.30)}$$

Where α and β are the distribution parameters. This two-parameter distribution is commonly used in many reliability engineering applications.

3.7.9: Pareto Distribution

The Pareto distribution was originally developed to model income in a population. Phenomena such as city population size, stock price fluctuations, and personal incomes have distributions with very long right tails. The probability density function of the Pareto distribution is given by

$$f(t)=\frac{\alpha k^{\alpha}}{t^{\alpha+1}}\ k\le t\le,\infty \quad \text{....................(3.31)}$$

The mean, variance and reliability of the Pareto distribution are, respectively,

Mean = $k/(\alpha\text{-}1)$ for $\alpha>1$

Variance = $\dfrac{\alpha k^2}{(\alpha\text{-}1)^2(\alpha-2)}$ for $\alpha>2$

Reliability $=\left(\dfrac{k}{t}\right)^{\alpha}$

The Pareto and log normal distributions have been commonly used to model the population size and economical incomes. The Pareto is used to fit the tail of the distribution, and the log normal is used to fit the rest of the distribution.

3.7.10: Rayleigh Distribution

The Rayleigh function is a flexible lifetime distribution that can apply to many degradation process failure modes. The Rayleigh probability density function is:

$$f(t) = \frac{t}{\sigma^2} e^{\left(\frac{-t^2}{2\sigma^2}\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.32)$$

The mean, variance, and reliability of Rayleigh function are, respectively,

Mean = $\sigma\left(\frac{\pi}{2}\right)^{\frac{1}{2}}$

Variance = $\left(2-\frac{\pi}{2}\right)\sigma^2$

Reliability = 2

3.10: Maximum Likelihood Estimation Method

The method of maximum likelihood estimation (MLE) is one of the most useful techniques for deriving point estimators. in general, one deals with a sample density[1]:

$$f(x_1, x_2, \ldots x_n) = f(x_1, \theta)\, f(x_2, \theta) \ldots\ldots f(x_n, \theta)$$

Where $X_1, X_2, \ldots, X_n$ are random, independent observation from a population with density function *f*(*x*). For the general case, it is desired to find an estimate or estimates, $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ (if such exist) where:

$$X = x_1, x_2, \ldots x_n$$

Notation $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m$ refers to any other estimates different than. $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m$ Let us now discuss the method of MLE. Consider a random sample $X_1, X_2, ..., X_n$ from a distribution having pdf $f(x, \theta)$. This distribution has a vector' $\theta = \hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m$ of unknown parameters associated with it, where *m is* the number of unknown parameters. Assuming that the random variables are independent, then the likelihood function, $L(X, \theta)$, is the product of the probability density function evaluated at each sample point:

$$L(X, \theta) = \prod_{i=1}^{n} f(X_i . \theta) \qquad (3.33)$$

Where $x_1, x_2, ...x_n$ The maximum likelihood estimator $\hat{\theta}$ is found by maximizing $L(X, \theta)$ with respect to θ. In practice, it is often easier to maximize $\ln L(X, \theta)$ to find the vector of MLEs, which is valid because the logarithm function is monotonic. The log likelihood function is given by

$$U_i(\theta) = \frac{\partial \left[\log L(X, \theta) \right]}{\partial \theta_i} \; for i = 1, 2, ..., m \qquad (3.34)$$

And is asymptotically normally distributed since it consists of the sum of *n* independent variables and the implication of the central limit theorem. Since $L(X, \theta)$ is a joint probability density function for $x_1, x_2, ...x_n$, it must integrate equal to1, that is,

$$\int_0^\infty \int_0^\infty ... \int_0^\infty L(X, \theta) dX = 1$$

Assuming that the likelihood is continuous, the partial derivative of the left-hand side with respect to one of the parameters, θ_i , yields

$$\frac{\partial}{\partial\theta_i}\int_0^\infty\int_0^\infty\cdots\int_0^\infty L(X,\theta)dX = \frac{\partial}{\partial\theta_i}\int_0^\infty\int_0^\infty\cdots\int_0^\infty L(X,\theta)dX$$

$$=\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\partial \log L(X,\theta)}{\partial\theta_i}L(X,\theta)\,dX$$

$$=E\left[\frac{\partial \log L(X,\theta)}{\partial\theta_i}\right]$$

$$=E\left[U_i(\theta)\right]\ fori=1,2,\ldots,m$$

Where $U(\theta)=(U_1(\theta),U_2(\theta),...,U_n(\theta))'$ is often called the score vector and the vector $U(\theta)$ has components

$$U_i(\theta)=\frac{\partial\left[\log L(X,\theta)\right]}{\partial\theta_i}\ fori=1,2,\ldots,m \quad \ldots\ldots\ldots\ldots\ldots(3.35)$$

Which, when equated to zero and solved, yields the MLE vector θ.

Suppose that we can obtain a non-trivial function of $X_1,X_2,....,X_n$, say $h(X_1,X_2,....,X_n)$, such that, when θ is replaced by $h(X_1,X_2,....,X_n)$, the likelihood function *L* will achieve a maximum. In other words,

$$L(X,h(X))L(X,\theta)$$

For every θ. The statistic $h(X_1,X_2,....,X_n)$ is called a maximum likelihood estimator of θ and will be denoted as

$$\hat{\theta} = h\left(X_1, X_2, ..., X_n\right) \quad ....(3.36)$$

The observed value of $\hat{\theta}$ is called the MLE of θ. In general, in an exponential censored case , the non conditional joint pdf of that items have failed is given by:

$$L(X,\lambda) = \lambda^n e^{-\lambda}\sum_{i=1}^{n} x_i \quad \text{(r failed items)} \quad .... \quad (3.37)$$

And the probability distribution that (n-r) items will survive is:

$$P\left(X_{r+1} > t_1, X_{r+2} > t_2, ..., X_n > t_{n-r}\right) = e^{-\lambda\sum_{i=1}^{n} t_i}$$

Thus, the joint density function is

$$L\left(X,\lambda\right) = f\left(x_1, x_2, ..., x_r\right) P\left(X_{r+1} > t_1, ..., X_n > t_{n-r}\right)$$

$$\frac{n!}{(n-r)!}\lambda^r e^{-\lambda(\sum_{i=1}^{r} x_i + \sum_{j=1}^{n=r} t_i)}$$

$$T = -\lambda\sum_{i=1}^{n} x_i + \sum_{j=1}^{n=r} t_i \quad ....(3.38)$$

$$lnL = In\left(\frac{n!}{(n-r)!}\right) + rIn\lambda - \lambda T$$

and

$$\frac{\partial InL}{\partial \lambda} = \frac{r}{\lambda} - T = 0$$

Hence,

$$\hat{\lambda} = \frac{r}{T} \quad \text{..........................(3.37)}$$

Note that with the exponential, regardless of the censoring type or lack of censoring, the MLE of λ is the number of failures divided by the total operating time.

Definition **:** Let $X_1, X_2, ..., X_n$ represent a random sample from the Weibull distribution with pdf

$$f(x, \alpha, \lambda) = \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}} \quad \text{....................(3.39)}$$

The likelihood function is

$$L(x, \alpha, \lambda) = \alpha^n \lambda^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\lambda \sum_{i=1}^{n} x_i^{\alpha}}$$

Then

$$InL = n \ln + n \ln \lambda + (\alpha - 1)\sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} x_i^{\alpha}$$

$$\frac{\partial InL}{\partial \alpha} = \frac{n}{\alpha} - \lambda \sum_{i=1}^{n} x_i^{\alpha} = 0$$

As noted, solutions of the above two equations for α and λ are extremely difficult and require either graphical or numerical methods.

3.10: Goodness of fit Techniques

The problem at hand is to compare some observed distribution with theoretical distribution .Two common techniques that will be discussed are the χ^2 good ness- of- fit test and the Kolmogorov-Smirnov "d" test [1].

3.11.1: Chi-squared test:

The following statistic**:**

$$\chi^2 = \sum_{i=1}^{k} \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2 \quad \text{.........................(3.40)}$$

Has chi-square χ^2 distribution with K degree of freedom. The steps of chi-square test are as follows:

1. Divide the sample data into the mutually exclusive cells such the range of random variable is covered.
2. Determine the frequency f_i , of sample observation is each cell.
3. Determine the theoretical frequency, F_i for each cell.(are a under density funcation between cell boundaries X_n total sample size).
4. Form the statistic

$$S = \frac{\sum_{i=1}^{k} (f_i - F_i)}{F_i} \quad \text{.........................(3.41)}$$

5. Form the χ^2 table, choose a value of χ^2 with the desired significance level and degrees of freedom $(k-1-r)$, where r is number of population parameters estimated.
6. Reject the hypothesis that the sample distribution is the same as theoretical distribution if [1].

$$S > \chi^2_{1-\alpha,k-1-r}$$

3.11.2: Kolmogorov-Simrnov Test:

Both the (χ^2) and (d) test are non-parametric .However, the χ^2 assumes large sample normality of the observed frequency about its mean while "d" only assumes a continuous distribution let $X_1 \le X_2 \le X_3 \le ... \le X_n$ denote the ordered sample value .Define the observed distribution function, F_n , as follows [1]:

$$F_n(x) = \begin{cases} 0 & for \quad x \le x_1 \\ \frac{i}{n} & for \quad x_i < x \le x_{i+1} \\ 1 & for \quad x < x_n \end{cases} \quad ....(3.42)$$

Assume the testing hypothesis:

$$Ho : F(x) = F_0(x)$$

Where $F_0(x)$ is a given continuous distribution and $F(x)$ is an unknown distribution let.

$$d_n = \underset{-\infty<x<\infty}{Sup} |F(x) = F_0(x)| \quad ....(3.43)$$

Since $F_0(x)$ is a continuous increasing funcation, we can evaluated $|F(x) = F_0(x)|$ for each n. if $d_n \le d_{n,\alpha}$ then we would not reject the hypothesis that; otherwise, we would reject it when $d_n > d_{n,\alpha}$. The value $d_{n,\alpha}$ can be found from table. where n is the sample size and a is the level of significance.