Sudan University of Science and Technology
College of Graduate Studies

Solution of Linear and Nonlinear Differential Equations by Combind Homotopy Perturbation Method and Sumudu Transform

حل المعادلات التفاضلية الخطية وغير الخطية بدمج طريقة الإرتجاج الهوموتوبيا وتحويل سمودو

A Thesis Submitted in Fulfillment Requirements for the Degree of Doctor of philosophy in Mathematics

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Dedication

This thesis is dedicated to all of my family.

My wife…

*Everyone shares the interest in studying in the Differential Equations, and integral transforms***.**

Acknowledgement

All praise is to Almighty Allah for providing me the blessing and the strength to complete this work. I would like to express my sincere thanks to Dr. Tarig Mohyeldin Elzaki, who supervised this study, for his willing guidance and informative reviews and comments that contributed to the success of this work. I am truly grateful to him for his encouragement and help. I would like to thank Dr. Mohmed Hassan Mohmed, who cooperated in this study.

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Abstract

This study is mainly focusing on the application of the homotopy perturbation method and Sumudu transform of the linear and nonlinear partial differential equations.

It has established some theorems, definitions and properties of homotopy perturbation method and Sumudu transform. The study combines the homotopy perturbation method and Sumudu transform. Consequently, it gives the solution in series form and approximates components, and finds the exact solution. Then, it is applied to solve linear and nonlinear PDEs.

Finally, the solutions of linear and nonlinear PDEs by this method, and the other methods will be compared.

الخلاصة

تتمركز هذه الدراسة مجملا فى تطبيق طريقة الإرتجاج الهموتوبيا وتحويل سمودو فى حل المعادلات التفاضلية الجزئية الخطية وغير الخطية.

ستقوم الدراسة بتاسيس بعض النظريات , التعريفات والخصائص بالنسبة لطريقة الإرتجاج الهموتوبيا وتحويل سمودو. قامت الدراسة بدمج طريقة الإرتجاج الهموتوبيا وتحويل سمودو , مما ادى الى ايجاد الحل فى شكل متسلسلة وتقريب المكونات لايجاد الحل التام. من ثم طبقت الدراسة لحل المعادلات التفاضلية الجزئية الخطية وغير الخطية.

واخيرا تمت مقارنة حلول المعادلات التفاضلية الجزئية الخطية وغير الخطية بطرق اخرى.

Introduction:

In the last several years with the rapid development of nonlinear science, there appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations. Searching of exact and numerical solution of nonlinear equations in science and engineering is still quite problematic, that's needed new methods for finding the exact and approximate solutions. Most of new nonlinear equations do not have a precise analytic solution; so, numerical methods have largely been used to handle these equations.

There are also analytic techniques for nonlinear equations. Some of the classic analytic methods are Lyapunov's artificial small parameter method [36], δ-expansion method [37], perturbation techniques [38-40] and Hirota bilinear method [41, 42]. In recent years, many research workers have paid attention to study the solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition methods (ADM) [43], He's semi-inverse method [44], the tanh method, the homotopy perturbation method (HPM), the sinh – cosh method, the differential transform method and the variational iteration method (VIM) [45-52]. Several techniques including the Adomian decomposition method, the variational iteration method, the weighted finite difference techniques and the Laplace decomposition method have been used to handle advection equations [53-59]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results and huge computational work. He [60-68] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [73] and the Laplace decomposition algorithm [74-78].

Furthermore, the homotopy perturbation method is also combined with the well-known Sumudu transform method [69, 79] and the variational iteration method [80] to produce a highly effective technique for handling many nonlinear problems.

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CHAPTER ONE
Homotopy Perturbation Method and Sumudu Transform

1.1: Sumudu Transform

Ever since long time, differential equations have played an important role in all aspects of mathematics. With the invention of the computer and its programming, the role of mathematics has reached to its peak. In order to develop new technological processes, scientific computation is important and it helps in understanding and controlling our natural environment. Analysis of differential equations helps in a profound understanding of mathematical problems. Various techniques may be used to solve differential equations. Watugula [1] introduced a new integral transform and called it as Sumudu transform which is defined as:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u}\, e^{\left(-\frac{t}{u}\right)}\, f(t)\, dt\,; \tag{1}$$

Watugula [1] applied this transform to the solution of ordinary differential equations. Because of its useful properties, the Sumudu transform helps in solving complex problems in applied sciences and engineering mathematics. In spite of the usefulness of the new operator, only a few investigations were found in the literature. Henceforth, is the definition of the Sumudu transform and properties depicting the simplicity of the transform.

Definition (1.1.1): The Sumudu transform of the function $f(t)$, is defined by:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u}\, e^{\left(-\frac{t}{u}\right)}\, f(t)\, dt \tag{2}$$

Or

$$F(u) = S[f(t)] = \int_0^\infty f(u\,t)\, e^{-t}\, dt \tag{3}$$

For any function $f(t)$, and $-\tau_1 < u < \tau_2$.

Theorem (1.1.2) [2]: If $S[f(t)] = F(u)$ and

$$g(t) = \begin{cases} f(t-\tau) & , \quad t \geq \tau \\ 0 & , \quad t \leq \tau \end{cases}$$

Then

$$S[g(t)] = e^{\left(-\frac{\tau}{u}\right)} G(u)$$

Theorem (1.1.3) [2]: If $c_1 \geq 0$, $c_2 \geq 0$ and $c \geq 0$ are any constant, $f_1(t)$, $f_2(t)$ and $f(t)$ any functions having the Sumudu transform $G_1(u)$, $G_2(u)$ and $G(u)$ respectively then

i. $S[c_1\ f_1(t) + c_2\ f_2(t)] = c_1\ S[f_1(t)] + c_2\ S[f_2(t)]$
$= c_1 G_1(u) + c_2 G_2(u)$

ii. $S[f(ct)] = G(cu)$

iii. $\lim_{t \to 0} f(t) = f(0) = \lim_{u \to 0} G(u)$

Further are words more, for several functions $f(t)$ defined for $t \geq 0$ in the neighborhood of infinity (i.e. as $t \to \infty$)

$$\lim_{t \to \infty} f(t) = \lim_{u \to \infty} G(u)$$

1.1.1: The Relation Between Sumudu and Laplace Transform

The Sumudu transform $F_s(u)$ of a function $f(t)$ defined for all real numbers $t \geq 0$. The Sumudu transform is essentially identical with the Laplace transform. Given an initial $f(t)$ its Laplace transform $G(u)$ can be translated into the Sumudu transform $F_s(u)$ of f by means of the relation;

$$F(u) = \frac{G\left(\frac{1}{u}\right)}{u}$$

And it's inverse

$$G(s) = \frac{F_s\left(\frac{1}{s}\right)}{s}$$

Theorem (1.1.4): Let $f(t)$ with Laplace transform $G(s)$ then the Sumudu transform $F(u)$ of $f(t)$ is given by

$$F(u)=\frac{G\left(\frac{1}{u}\right)}{u}\ .$$

Proof:

Form definition (1.1.1) we get:

$$F(u)=\int_0^\infty e^{-t} f(u\,t)\, dt$$

If we set $w=u\,t$ and $dt=\frac{dw}{u}$ then

$$F(u)=\int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w)\frac{dw}{u}=\frac{1}{u}\int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w)dw$$

By definition of Laplace transform we get:

$$F(u)=\frac{G\left(\frac{1}{u}\right)}{u}$$

Theorem (1.1.5): It deals with the effect of the differentiation of the function $f(t)$, k times on the Sumudu transform $F(u)$ if $S[f(t)]=F(u)$then:

i. $S[f'(t)]\ =\frac{1}{u}[F(u)-\ f(0)]$

ii. $S[f''(t)]\ =\frac{1}{u^2}[F(u)]-\frac{1}{u^2}\ f(0)-\frac{1}{u}\ f'(u)$

iii. $S\left[f^{(n)}(t)\right]=\frac{1}{u^n}[F(u)]-\frac{1}{u^n}\sum_{k=0}^{n-1}u^k\ f^{(k)}(0)\ =\ u^{-n}\left[F(u)-\ \sum_{k=0}^{n-1}u^k\ f^{(k)}(0)\right]$

Where $f^{(0)}(0)=f(0)$, $f^{(k)}(0)$, $k=1,2,3,\cdots,n-1$ are the kth-order derivatives of the function $f(t)$ evaluated at $t=0$.

Proof:

i. Using integration by parts;

$$S[f'(t)] = \left[\frac{1}{u}\exp\left(-\frac{t}{u}\right) f(t)\right]_0^{\infty} + \frac{1}{u}\int_0^{\infty}\frac{1}{u}\exp\left(-\frac{t}{u}\right) f(t)dt$$

$$= -\frac{1}{u} f(0) + \frac{1}{u} F(u)$$

$$S[f'(t)] = \frac{1}{u}[F(u) - f(0)]$$

ii. Using integration by parts;

$$S[f''(t)] = \left[\frac{1}{u}e^{\left(-\frac{t}{u}\right)} f'(t)\right]_0^{\infty} + \frac{1}{u}\int_0^{\infty}\frac{1}{u}e^{\left(-\frac{t}{u}\right)} f'(t)dt$$

From (i) $$= -\frac{1}{u} f'(0) + \frac{1}{u} S[f'(t)]$$

$$S[f''(t)] = \frac{1}{u^2}[F(u)] - \frac{1}{u^2} f(0) - \frac{1}{u} f'(0)$$

iii. By definition the Laplace transform for $f^{(n)}(t)$ is given by;

$$G_n(s) = s^n G(s) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^{(k)}(0)$$

By using the relation between Sumudu and Laplace transform;

$$G_n\left(\frac{1}{u}\right) = \frac{G\left(\frac{1}{u}\right)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-(k+1)}}$$

Since $F_n(u) = \frac{G_n\left(\frac{1}{u}\right)}{u}$ we get:

$$u\ F_n(u) = \frac{u\,F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}\ u^{-1}}$$

$$F_n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}}$$

$$F_n(u) = u^{-n} F(u) - \sum_{k=0}^{n-1} u^{-n} u^{k} f^{(k)}(0)$$

$$S[f^{(n)}(t)] = F(u) = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^{k} f^{(k)}(0)\right]$$

Theorem (1.1.6): Let $f(t)$ be a function with the Sumudu transform $F(u)$ then;

$$S\left[e^{at}\ f(t)\right] = \frac{1}{1-au}\ F\left(\frac{u}{1-au}\right)$$

Proof:

$$S\left[e^{at} f(t)\right] = \int_0^\infty f(ut)e^{aut}\ e^{-t}\ dt = \int_0^\infty f(ut)e^{(-(1-au)t)}\ dt$$

Let $w = (1-au)t \quad \Rightarrow \quad dt = \dfrac{dw}{1-au}$

$$S\left[e^{at}\ f(t)\right] = \frac{1}{1-au}\int_0^\infty f\left(\frac{uw}{1-au}\right)e^{-w}\, dw$$

$$S\left[e^{at}\ f(t)\right] = \frac{1}{1-au}\ F\left(\frac{u}{1-au}\right)$$

Theorem (1.1.7) [3]: This theorem deals with multiplication of the function $f(t)$ by a power series of t , if:

i. $S[t\ f(t)] = u^2 \dfrac{d}{du}F(u) + u\,F(u)$

ii. $S\left[t^2\ f(t)\right] = u^4 \dfrac{d^2}{du^2}F(u) + 4u^3 \dfrac{d}{du}F(u) + 2u^2\,F(u)$

iii. $S\left[t^n\ f(t)\right] = u^n \displaystyle\sum_{k=0}^{n} a_k^n\, u^k\ F_k(u)$

iv. $S\left[t^{n+1}\ f(t)\right] = u^{n+1} \displaystyle\sum_{k=0}^{n+1} a_k^{n+1}\, u^k\ F_k(u)$

Theorem (1.1.8): Let $f(t)$ and $g(t)$ having Laplace transforms $F(s)$ and $G(s)$ respectively, and Sumudu transform $M(u)$ and $N(u)$, respectively.
Then the Sumudu transform of the convolution of f and g .

$$(f*g)(t) = \int_0^\infty f(t)g(t-\tau)d\tau$$

is given by:

$$S[(f*g)(t)] = uM(u)N(u)$$

Proof:

First, recall that the Laplace transforms of $(f*g)$ is given by:

$$L[(f*g)(t)] = F(s)G(s)$$

By using the relation between Sumudu and Laplace transform;

$$S[(f*g)(t)] = \frac{1}{u}\ L[(f*g)(t)]$$

And since $$M(u) = \frac{F\left(\frac{1}{u}\right)}{u}\ ,\ N(u) = \frac{G\left(\frac{1}{u}\right)}{u}$$

The Sumudu transform of $(f*g)$ is obtained as follows;

$$S[(f*g)(t)] = \frac{F\left(\frac{1}{u}\right)G\left(\frac{1}{u}\right)}{u} = u\,\frac{F\left(\frac{1}{u}\right)}{u}\ \frac{G\left(\frac{1}{u}\right)}{u} = u\ M(u)N(u)$$

$$S[(f*g)(t)] = u\ M(u)N(u)$$

Theorem (1.1.9): Let $G(u)$ denote the Sumudu transform of the function $f(t)$ let $f^{(n)}(t)$ denote the nth derivative of $f(t)$ with respect to t and let $F_n(u)$ denote the nth derivative of $F(u)$ with respect to u, then the Sumudu transform of the function $t^n f^{(n)}(t)$ is given by:

$$S\left[t^n f^{(n)}(t)\right] = u^n F_n(u)$$

Proof:

Let the Sumudu transform of $f(t)$;

$$F(u) = \int_0^\infty f(ut)e^{-t}\ dt$$

Therefore, for $n = 0, 1, 2, \ldots$ we get:

$$F_n(u) = \int_0^\infty \frac{d^n}{du^n} f(ut)e^{-t}\ dt = \int_0^\infty t^n f^n(ut)e^{-t}\ dt$$

$$F_n(u) = \frac{1}{u^n}\int_0^\infty (ut)^n f^n(ut)e^{-t}\ dt = \frac{1}{u^n}\ S\left[t^n\ f^n(t)\right]$$

$$\Rightarrow S\left[t^n\ f^n(t)\right] = u^n F_n(u)$$

Corollary (1.1.10) [2]:

Let $F_n(u)$ denote the nth derivative of $F_n(u) = S[f(t)]$, then

i. $S[t\, f'(t)] = u \dfrac{d\,F(u)}{du} = u\,F_1(u)$

ii. $S\left[t^2\ f'(t)\right] = u^2\left[2F_1(u) + u\,F_2(u)\right]$

iii. $S\left[t^3\ f'(t)\right] = u^3\left[6F_1(u) + 6u\,F_2(u) + u^2\ F_3(u)\right]$

iv. $S\left[t^4\ f'(t)\right] = u^4\left[12F_2(u) + 8u\,F_3(u) + u^2\ F_4(u)\right]$

Example (1.1.11): Consider the following inhomogeneous partial differential equation:

$$U_x(x,y) + U_y(x,y) = x + y; \tag{4}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \quad U(0,y) = 0;$$

Taking the Sumudu transform of Eq. (4), we get:

$$S[U_x(x,y)] + S[U_y(x,y)] = S[x + y] \tag{5}$$

$$\frac{d}{dx}U(x,u) + \frac{1}{u}[U(x,u) - U(x,0)] = x + u$$

$$\frac{d}{dx}U(x,u) + \frac{1}{u}U(x,u) = x + u$$

Thus we have the ordinary differential equation:

$$\frac{d}{dx}U(x,u) + \frac{1}{u}U(x,u) = x + u \tag{6}$$

The integrating factor is;

$$F = e^{\int \frac{1}{u} du} = u \tag{7}$$

Then

$$U(x,u) = \frac{1}{u}\left[\int u(x+u)dx + c\right] = \frac{x^2}{2} + xu + c \tag{8}$$

Since $U(x,0) = 0$ then $c = -\dfrac{x^2}{2}$, then

$$U(x,u) = \quad xu \tag{9}$$

Taking the inverse Sumudu transform;

$$U(x,y) = \ S^{-1}[xu] \tag{10}$$

$$U(x,y) = x\,y \tag{11}$$

1.2: Homotopy Perturbation Method

The homotopy Perturbation Method (HPM) was a result of some pioneering ideas beginning in 1999 by He [4]. Since then it has developed into a fully -fledged theory, which was the contribution from many researchers [5-12]. The HPM method was found to be a simple and accurate method to solve a large number of nonlinear problems.

It is well known about the main disadvantage of the Adomian method, that it is a complex and difficult method to perform calculation so called Adomian polynomials. There is an alternate approach to reduce the demerits of Adomian method, which involves a variational iteration method. On the Homotopy Perturbation Method (HPM) which is simple and straightforward may be employed to calculate Adomian Polynomials.

The homotopy perturbation method (HPM) may be used to solve the functional equations of the form:

$$u - N(u) = f , \tag{12}$$

Where N is a nonlinear operator from Hilbert space H to H , u is an unknown function, and f is a known function in H .

The homotopy perturbation method u as a series with components u_n, and $N(u)$ as a series with components H_n, homotopy polynomials, which can be calculated using the formula:

$$H_n = \frac{1}{n!} \frac{d^n}{dp^n} N\left(\sum_{i=0}^{\infty} u_i \, p^i \right)\Bigg|_{p=0} \tag{13}$$

1.2.1: Homotopy Perturbation Method and He polynomials

To illustrate the homotopy perturbation method (HPM), we consider (12) as;

$$L(v) = v(x) - f(x) - N(v) = 0 \tag{14}$$

With solution $u(x)$. As a possible remedy, we can define homotopy $H(v, p)$ as follows:

$$H(v,0) = F(v) \quad , \quad H(v,1) = L(v)$$

Where $F(v)$ is an integral operator with known solutions, v_0, which can be obtained easily. Typically, we may choose a convex homotopy in the form;

$$H(v,p)=(1-p)F(v)+p\,L(v)=0 \tag{15}$$

and continuously trace an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(u,0)$. The embedding parameter p monotonically increase from zero in the unit as the trivial problem $F(v)=0$ is continuously deformed to the original problem $L(v)=0$.

$$v=v_0+pv_1+p^2v_2+p^3v_3+\cdots \tag{16}$$

When $p\to 1$, Eq. (15) corresponds to Eqs. (14) and (16) becomes the approximate solution of Eq. (14), i.e.

$$v=\lim_{p\to 1} v_0+v_1+v_2+v_3+\cdots \tag{17}$$

Theorem (1.2.12): Suppose $N(v)$ is a nonlinear function, and $\sum_{k=0}^{\infty} p^k v_k$, then we get;

$$\frac{\partial^n}{\partial p^n}N(v)_{p=0}=\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{\infty}p^k v_k\right)_{p=0}=\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n}p^k v_k\right)_{p=0}$$

Proof: Since

$$v=\sum_{k=0}^{\infty}p^k v_k=\sum_{k=0}^{n}p^k v_k+\sum_{k=n+1}^{\infty}p^k v_k,$$

We have such result as follows:

$$\frac{\partial^n}{\partial p^n}N(v)_{p=0}=\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{\infty}p^k v_k\right)_{p=0}=\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n}p^k v_k+\sum_{k=n+1}^{\infty}p^k v_k\right)_{p=0}=\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n}p^k v_k\right)_{p=0}$$

Therefore, we obtain;

$$\frac{\partial^n}{\partial p^n}N(v)_{p=0}=\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{\infty}p^k v_k\right)_{p=0}=\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n}p^k v_k\right)_{p=0}$$

Taking, $F(v)=v(x)-f(x)-p\,N(v)=0$, and substituting (13) into (14), we get;

$$H(v,p)=v(x)-f(x)-p\,N(v)=0, \tag{18}$$

According to Maclaurin expansion of $N(v)$ with respect to p, we get;

$$N(v)=N(v)_{p=0}+p\left(\frac{\partial}{\partial p}N(v)_{p=0}\right)+p^2\left(\frac{1}{2!}\frac{\partial^2}{\partial p^2}N(v)_{p=0}\right)$$
$$+p^3\left(\frac{1}{3!}\frac{\partial^3}{\partial p^3}N(v)_{p=0}\right)+\cdots+p^n\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}N(v)_{p=0}\right)+\cdots$$

Substituting Eq. (16) into the above equation, we get;

$$N(v) = N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0} + p\left(\frac{\partial}{\partial p} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0}\right) + p^2\left(\frac{1}{2!}\frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0}\right)$$
$$+ p^3\left(\frac{1}{3!}\frac{\partial^3}{\partial p^3} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0}\right) + \cdots + p^n\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0}\right) + \cdots$$

According to Theorem (1.2.12)

$$N(v) = N(v_0) + p\left(\frac{\partial}{\partial p} N\left(\sum_{k=0}^{1} p^k v_k\right)_{p=0}\right) + p^2\left(\frac{1}{2!}\frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^{2} p^k v_k\right)_{p=0}\right)$$
$$+ p^3\left(\frac{1}{3!}\frac{\partial^3}{\partial p^3} N\left(\sum_{k=0}^{3} p^k v_k\right)_{p=0}\right) + \cdots + p^n\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{n} p^k v_k\right)_{p=0}\right) + \cdots \qquad (19)$$

Substituting Eqs. (16) and (19) into Eq. (18), and equating the terms with the identical powers of p , we get;

$$p^0 : v_0(x) - f(x) = 0 \Rightarrow v_0(x) = f(x)$$
$$p^1 : v_1(x) - N(v_0) = 0 \Rightarrow v_1(x) = N(v_0)$$
$$p^2 : v_2(x) - \frac{\partial}{\partial p} N\left(\sum_{k=0}^{1} p^k v_k\right)_{p=0} = 0 \Rightarrow v_2(x) = \frac{\partial}{\partial p} N\left(\sum_{k=0}^{1} p^k v_k\right)_{p=0}$$
$$\vdots$$
$$p^{n+1} : v_{n+1}(x) - \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{n1} p^k v_k\right)_{p=0} = 0 \Rightarrow v_{n+1}(x) = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{n} p^k v_k\right)_{p=0}$$

Definition (1.2.13): The He polynomials are defined as follows:

$$H_n(v_0, v_1, \cdots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{n} p^k v_k\right)_{p=0}, \quad n = 0, 1, 2, \cdots$$

Therefore, the approximate solution obtained by the homotopy perturbation method can be expressed in He polynomials:

$$u(x) = f(x) + \underbrace{N(v_0)}_{H_0} + \underbrace{\frac{\partial}{\partial p} N\left(\sum_{k=0}^{1} p^k v_k\right)_{p=0}}_{H_1} + \underbrace{\frac{1}{2!}\frac{\partial^2}{\partial p^2} N\left(\sum_{k=0}^{2} p^k v_k\right)_{p=0}}_{H_2}$$
$$+ \underbrace{\frac{1}{3!}\frac{\partial^3}{\partial p^3} N\left(\sum_{k=0}^{3} p^k v_k\right)_{p=0}}_{H_3} + \cdots + \underbrace{\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{n} p^k v_k\right)_{p=0}}_{H_n} + \cdots$$

The nonlinear term $N(u)$ can be also expressed in He polynomials:

$$N(u)=\sum_{n=0}^{\infty}H_n(v_0,v_1,\cdots,v_n)=H_0(v_0)+H_1(v_0,v_1)+\cdots+H_n(v_0,v_1,\cdots,v_n)+\cdots,$$

Where

$$H_n(v_0,v_1,\cdots,v_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}N\left(\sum_{k=0}^{n}p^k v_k\right)_{p=0},\ n=0,1,2,\cdots$$

Alternatively, the approximate solution can be expressed as following:

$$u(x)=f(x)+\sum_{n=0}^{\infty}H_n(v_0,v_1,\cdots,v_n).$$

This is very interesting and attractive to note that we can obtain He polynomial and its solution simultaneously.

Example (1.2.14): Consider the following inhomogeneous partial differential equation:

$$\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}=x+y; \tag{20}$$

With the initial conditions;

$$U(x,0)=0 \quad , \quad U(0,y)=0;$$

To solve Eq. (20) with initial condition, according to the homotopy perturbation technique, we construct the following homotopy

$$(1-p)\left(\frac{\partial v}{\partial y}-\frac{\partial u_0}{\partial y}\right)+p\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial x}-y-x\right)=0 \tag{21}$$

Or equivalently;

$$\frac{\partial v}{\partial y}-\frac{\partial u_0}{\partial y}+p\left(\frac{\partial u_0}{\partial y}+\frac{\partial v}{\partial x}-y-x\right)=0$$

Suppose the solution of Eq. (21) has the form;

$$v=v_0+pv_1+p^2v_2+p^3v_3+\cdots \tag{22}$$

Substituting Eq. (22) into Eq. (21) and comparing coefficients of terms with identical powers of p, leads to:

$$p^0:\ \frac{\partial v_0}{\partial y}-\frac{\partial u_0}{\partial y}=0$$

$$p^1:\frac{\partial v_1}{\partial y}+\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}-(x+y)=0\ ,\ v_1(x,0)=0$$

$$p^2 : \frac{\partial v_2}{\partial y} + \frac{\partial v_1}{\partial x} = 0 \quad , v_2(x , 0) = 0$$

$$p^3 : \frac{\partial v_3}{\partial y} + \frac{\partial v_2}{\partial x} = 0 \quad , v_3(x , 0) = 0$$

$$\vdots \qquad (23)$$

$$p^i : \frac{\partial v_i}{\partial y} + \frac{\partial v_{i-1}}{\partial x} = 0 \ , v_2(x , 0) = 0$$

For simplicity, we take, $v_0(x , y) = u_0(x , y) = 0$. So we derive the following recurrence relation;

$$v_i = \int_0^y \frac{\partial v_{i-1}}{\partial x} dy \quad , \quad i = 1 , 2 , 3 , \cdots \qquad (24)$$

Solving the above equations, we obtain;

$$U_0(x , y) = 0$$

$$U_1(x , y) = x y + \frac{y^2}{2}$$

$$U_2(x , y) = -\frac{y^2}{2} \qquad (25)$$

$$U_3(x , y) = 0$$

$$\vdots$$

And so on.

By setting, $p = 1$ in Equation (20) the solution of Equation (22) can be obtained, thus we get;

$$U(x , y) = 0 + x y + \frac{y^2}{2} - \frac{y^2}{2} + 0 + \cdots \qquad (26)$$

Equation (26) has the closed form;

$$U(x , y) = x y \qquad (27)$$

This is also the exact solution of the problem.

Example (1.2.15): Consider the following one-dimensional parabolic-like equation with variable coefficients,

$$U_t(x,t) - \frac{x^2}{2} U_{xx}(x,t) = 0 \quad , \tag{28}$$

Subject to the initial condition;

$$U(x,0) = x^2$$

According to the homotopy perturbation method, we can construct the homotopy $\Omega \times [0,1] \rightarrow \Re$ which satisfies;

$$\frac{\partial v}{\partial t} - \frac{\partial U_0}{\partial t} + p\left[\frac{\partial U_0}{\partial t} - \frac{x^2}{2}\frac{\partial^2 v}{\partial x^2}\right] = 0 \tag{29}$$

With the initial approximation

$$U_0 = U(x,0) = x^2;$$

Suppose that the solution of Eq. (28) can be represented as;

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots . \tag{30}$$

Substituting Eq. (30) into Eq. (29), and equating the terms of the same power of P, as following;

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial U_0}{\partial t} = 0 \quad , \quad U_0(x,0) = x^2$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial U_0}{\partial t} - \frac{x^2}{2}\frac{\partial^2 v_0}{\partial x^2} = 0 \quad , \quad U_1(x,0) = 0$$

$$p^2 : \frac{\partial v_2}{\partial t} - \frac{x^2}{2}\frac{\partial^2 v_1}{\partial x^2} = 0 \quad , \quad U_2(x,0) = 0$$

$$p^3 : \frac{\partial v_3}{\partial t} - \frac{x^2}{2}\frac{\partial^2 v_2}{\partial x^2} = 0 \quad , \quad U_3(x,0) = 0 \quad .$$

.

.

$$p^i : \frac{\partial v_n}{\partial t} - \frac{x^2}{2}\frac{\partial^2 v_{n-1}}{\partial x^2} = 0 \quad , \quad U_n(x,0) = 0$$

By choosing $U_0(x,t) = U(x,0)$, and solving the above equations, we obtain the following approximations;

$$
\begin{aligned}
U_0(x\,,\,t) &= x^2 \\
U_1(x\,,\,t) &= t\,x^2 \\
U_2(x\,,\,t) &= \frac{t^2}{2!}x^2 \\
&. \qquad\qquad . \\
&. \\
U_n(x\,,\,t) &= \frac{t^n}{n!}x^2
\end{aligned}
\tag{31}
$$

Then the exact solution of Eq. (28) is given by;

$$U(x\,,\,t) = \lim_{n\to\infty} x^2\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right) = x^2\ e^t \tag{32}$$

1.3: Homotopy Perturbation and Sumudu Transform Method for Solving of Partial Differential Equations

The homotopy Perturbation method proves to be powerful, effective and simple method which can be applied to a varied class of linear or nonlinear ordinary or partial differential equations, and linear and nonlinear integral equations. The method possesses several advantages, which is significant from the decomposition method. This method is a simple and direct way of solving linear or nonlinear ordinary or partial differential equations, and linear and nonlinear integral equations without the use of linearization, perturbation or any other bounded assumption.

The HPM was developed by Ariel et. al. [5]. Extensive research has been carried out by applying this method to a varied class of linear or nonlinear ordinary or partial differential equations, and linear and nonlinear integral equations.
The homotopy perturbation method involves decomposing the unknown function $U(x,y)$ of any equation into a sum of an infinite number of components defined by the decomposition series:

$$U(x,y) = \sum_{n=0}^{\infty} p^n U_n(x,y) \tag{33}$$

Where $U_n(x,y)$, $n \geq 0$ are to be determined in an iterative manner.

The decomposition method involves finding the components $U_0, U_1, U_2, \cdots$ individually. The decomposed component can be obtained by recursive relation who involves simple integrals.

To have a clear overview of the HPM, let us first consider the linear differential equation written in an operator form by:

$$LU + RU = g \tag{34}$$

Where S a lower order derivative which is invertible is, R is other linear differential operator, and *g* is a source term. If we apply the inverse operator S^{-1} to both sides of equation (34), we obtain linear differential equations.

$$U = f - S^{-1}[RU] \tag{35}$$

Where, the function f represents the terms arising from integrating the source term *g.* Using the homotopy perturbation method which defines the solution *u* by an infinite series of components given by:

$$U(x,y) = \sum_{n=0}^{\infty} p^n U_n(x,y); \tag{36}$$

where the components $U_0, U_1, U_2, \cdots$ are usually recurrently determined. Substituting Eq. (34) into both sides of Eq. (35) leads to;

$$\sum_{n=0}^{\infty} p^n U_n = f - p\, S^{-1}\left[R\left(\sum_{n=0}^{\infty} p^n U_n\right)\right] \tag{37}$$

For simplicity, Equation (37) can be rewritten as;

$$U_0 + p\,U_1 + p^2\,U_2 + p^3 U_3 + \cdots = f - p\,S^{-1}\left[R\left(U_0 + p\,U_1 + p^2\,U_2 + p^3 U_3 + \cdots\right)\right] \tag{38}$$

To construct the recursive relation needed for the determination of the components $U_0, U_1, U_2, \cdots$, it is important to note that the homotopy perturbation method suggests that the zeros component U_0 is usually defined by the function f described above, i.e. by all terms, that are not included under the inverse operator S^{-1}, which arise from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by;

$$\begin{aligned} U_0 &= f, \\ U_1 &= -S^{-1}[R(U_0)], \\ U_2 &= -S^{-1}[R(U_1)], \\ U_3 &= -S^{-1}[R(U_2)], \\ &\vdots \end{aligned} \tag{39}$$

It is evident from relation Eq. (39) that the reduced differential equation in terms of computable components. Again on substituting these components in equation (36), we obtain the solution in a series form.

Example (1.3.16): Consider the following inhomogeneous partial differential equation:

$$U_x(x,y)+U_y(x,y)=x+y; \tag{40}$$

With the initial conditions;

$$U(x,0)=0 \quad , \quad U(0,y)=0;$$

The $x-$solution:

Taking Sumudu transform of both sides of the equation (40) subject to the initial condition, we get;

$$S[U(x,y)]=u^2+u\,y-u\,S[U_y(x,y)] \tag{41}$$

The inverse of Sumudu transform implies that;

$$U(x,y)=\frac{x^2}{2}+x\,y-S^{-1}[u\,S[U_y(x,y)]] \tag{42}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty}p^n U_n(x,y)=\frac{x^2}{2}+x\,y-p\,S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty}p^n U_n(x,y)\right)_y\right]\right] \tag{43}$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} p^0 &: U_0(x,y)=\frac{x^2}{2}+x\,y \\ p^1 &: U_1(x,y)=S^{-1}[u\,S[(U_0)_y]]=-\frac{x^2}{2} \\ p^2 &: U_2(x,y)=S^{-1}[u\,S[(U_1)_y]]=0 \\ p^3 &: U_3(x,y)=S^{-1}[u\,S[(U_2)_y]]=0 \end{aligned} \tag{44}$$

Therefore the solution $U(x,t)$in series form is given by;

$$U(x,t)=U_0(x,t)+U_1(x,t)+U_2(x,t)+\cdots$$

$$U(x,t)=\frac{x^2}{2}+x\,y-\frac{x^2}{2}=x\,y \tag{45}$$

And in closed form given as;

$$U(x,t)=x\,y \tag{46}$$

The $y-$solution:

Taking Sumudu transform of both sides of the equation (40) subject to the initial condition, we get;

$$S[U(x,y)] = u^2 + u\,x - u\,S[U_x(x,y)] \tag{47}$$

The inverse of Sumudu transform implies that;

$$U(x,y) = \frac{y^2}{2} + x\,y - S^{-1}[u\,S[U_x(x,y)]] \tag{48}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y) = \frac{y^2}{2} + xy - p\,S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y)\right)_x\right]\right] \tag{49}$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned}
p^0 &: U_0(x,y) = \frac{y^2}{2} + x\,y \\
p^1 &: U_1(x,y) = S^{-1}[u\,S[(U_0)_x]] = -\frac{y^2}{2} \\
p^2 &: U_2(x,y) = S^{-1}[u\,S[(U_1)_x]] = 0 \\
p^3 &: U_3(x,y) = S^{-1}[u\,S[(U_2)_x]] = 0
\end{aligned} \tag{50}$$

Therefore the solution $U(x,t)$in series form is given by;

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$$

$$U(x,t) = \frac{y^2}{2} + x\,y - \frac{y^2}{2} = x\,y \tag{51}$$

And in closed form given as;

$$U(x,t) = x\,y \tag{52}$$

Example (1.3.17): Consider the following homogeneous partial differential equation;

$$U_t(x,t) - xU(x,t) = 0 \tag{53}$$

Initial condition as;

$$U(x,0) = 1$$

Taking Sumudu transform of both sides of the equation (53) subject to the initial condition, we get;

$$S[U(x,t)] = 1 + uS[xU] \tag{54}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = 1 + S^{-1}[uS[xU]] \tag{55}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 1 + p\left(S^{-1}\left[uS\left[x\sum_{n=0}^{\infty} p^n U_n(x,t)\right]\right]\right) \tag{56}$$

Or equivalently;

$$U_0 + pU_1 + p^2 U_2 + \cdots = 1 + p\left(S^{-1}\left[u\,S\left[x\left(U_0 + pU_1 + p^2 U_2 + \cdots\right)\right]\right]\right) \tag{57}$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = 1 \\ p^1 &: U_1(x,t) = S^{-1}[u\,S[xU_0]] = xt \\ p^2 &: U_2(x,t) = S^{-1}[u\,S[xU_1]] = x^2\frac{t^2}{2!} \end{aligned} \tag{58}$$

Proceeding in a similar manner, we obtain;

$$\begin{aligned} p^3 &: U_3(x,t) = x^3\frac{t^3}{3!} \\ p^4 &: U_4(x,t) = x^4\frac{t^4}{4!} \end{aligned} \tag{59}$$

Therefore the solution $U(x,t)$in series form is given by;

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$$

$$U(x,t) = 1 + xt + x^2\frac{t^2}{2!} + x^3\frac{t^3}{3!} + x^4\frac{t^4}{4!} + \cdots \tag{60}$$

And in closed form given as:

$$U(x,t) = e^{xt} \tag{61}$$

1.4: Homotopy Perturbation and Sumudu Transform Method for Solving System of Differential Equations

In the study of wave propagation, many researchers have been attracted by systems of linear or nonlinear partial differential equations, in order to study the chemical reaction diffusion model of Brusselator and shallow water waves. The commonly used methods are Riemann invariants and method of characteristics. The existing methods possess some difficulties in terms of computation and with the system of several partial differential equations.

In order to overcome the difficulties that arise from traditional methods, the homotopy perturbation method forms a basis for studying systems of partial differential equations. The homotopy perturbation method is more attractive as it generates quick convergent power series with each term computable. And, the method transforms the system of partial differential equations into a set of recursive relation, where each recursive relation can be easily computed and examine. Because of this simplicity of the homotopy perturbation method, we use this method.

1.4.1: Solving System of Differential Equations:

We first consider the system of partial differential equations written in an operator form;

$$\begin{aligned} U_t + V_x &= g_1 \\ V_t + U_x &= g_2 \end{aligned} \tag{62}$$

With the initial conditions;

$$\begin{aligned} U(x,0) &= f_1(x) \\ V(x,0) &= f_2(x) \end{aligned}$$

Using the differential operator property of the Sumudu transform and above initial conditions, we get;

$$\begin{aligned} S[U(x,t)] &= f_1(x) + u\,S[g_1 - V_x] \\ S[V(x,t)] &= f_2(x) + u\,S[g_2 - U_x] \end{aligned} \tag{63}$$

Now, applying the inverse Sumudu transform on both sides of Eq. (63), we get;

$$\begin{aligned} U(x,t) &= f_1(x) + S^{-1}\{u\,S[g_1 - V_x]\} \\ V(x,t) &= f_2(x) + S^{-1}\{u\,S[g_2 - U_x]\} \end{aligned} \tag{64}$$

Where $g_1(x,t), g_2(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We apply the homotopy perturbation method;

$$U(x,t) = \sum_{n=0}^{\infty} p^n \, U_n(x,t)$$
$$V(x,t) = \sum_{n=0}^{\infty} p^n \, V_n(x,t) \quad (65)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = f_1(x) + p\left(S^{-1}\left[uS\left[\sum_{n=0}^{\infty} p^n\left(g_{n_1} + V_n(x,t)\right)\right]\right]\right)$$
$$\sum_{n=0}^{\infty} p^n V_n(x,t) = f_2(x) + p\left(S^{-1}\left[uS\left[\sum_{n=0}^{\infty} p^n\left(g_{n_2} + U_n(x,t)\right)\right]\right]\right) \quad (66)$$

This is the coupling of the Sumudu transform and the homotopy perturbation method using He's polynomials.

Comparing the coefficient of like power of p, the following approximation are obtained

$$p^0 : U_0(x,t) = f_1(x) \quad , \quad V_0(x,t) = f_2(x)$$
$$p^1 : U_1(x,t) = S^{-1}[u\,S[\,H_0(U)]]$$
$$p^1 : V_1(x,t) = S^{-1}[u\,S[\,H_0(V)]]$$
$$p^2 : U_2(x,t) = S^{-1}[u\,S[H_1(U)]]$$
$$p^2 : V_2(x,t) = S^{-1}[u\,S[H_1(V)]] \quad (67)$$
$$. \qquad . \qquad .$$

To have a clear overview, forthwith are several examples to demonstrate the efficiency of the method.

Example (1.4.18): Consider the following system of partial differential equations,

$$U_t + V_x = 0$$
$$V_t + U_x = 0 \quad (68)$$

With the initial conditions;

$$U(x,0) = e^x$$
$$V(x,0) = e^{-x}$$

Taking Sumudu transform of equations (68) subject to the initial conditions, we get;

$$\begin{aligned} S[U(x,t)] &= e^{x} - u\,S[V_x] \\ S[V(x,t)] &= e^{-x} - u\,S[U_x] \end{aligned} \qquad (69)$$

The inverse Sumudu transform implies that:

$$\begin{aligned} U(x,t) &= e^{x} - S^{-1}[u\,S[V_x]] \\ V(x,t) &= e^{-x} - S^{-1}[u\,S[U_x]] \end{aligned} \qquad (70)$$

Now applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,t) &= e^{x} - p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n V_n\right]_x\right]\right\} \\ \sum_{n=0}^{\infty} p^n V_n(x,t) &= e^{-x} - p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n U_n\right]_x\right]\right\} \end{aligned} \qquad (71)$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} &p^0 : U_0(x,t) = e^{x} \quad , \quad V_0(x,t) = e^{-x} \\ &p^1 : U_1(x,t) = t\,e^{-x} \quad , \quad V_1(x,t) = -t\,e^{x} \\ &p^2 : U_2(x,t) = \frac{t^2}{2!}e^{x} \; , \quad V_2(x,t) = \frac{t^2}{2!}e^{-x} \\ &p^3 : U_3(x,t) = \frac{t^3}{3!}e^{-x} , \quad V_3(x,t) = -\frac{t^3}{3!}e^{x} \end{aligned} \qquad (72)$$

And so on, using Eq. (72) we obtain;

$$\begin{aligned} U(x,t) &= e^{x}\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right) + e^{-x}\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right), \\ V(x,t) &= e^{-x}\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right) - e^{x}\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right) \end{aligned} \qquad (73)$$

This has an exact analytical solution of the form

$$(U,V) = \left(e^{x}\cosh t + e^{-x}\sinh t \, , \, e^{-x}\cosh t - e^{x}\sinh t\right) \qquad (74)$$

Example (1.4.19): Consider the following system of partial differential equations,

$$\begin{aligned} U_x + V_t &= 3x \\ 2U_t - 3\,V_x &= t \end{aligned} \qquad (75)$$

With the initial conditions;

$$\begin{aligned} U(x,0) &= x^2 \\ V(x,0) &= 0 \end{aligned}$$

Taking Sumudu transform of equations (75) subject to the initial conditions, we get;

$$S[V(x,t)] = 3xu - u\,S[U_x]$$
$$S[U(x,t)] = \frac{u^2}{2} + x^2 + \frac{3}{2}u\,S[V_x] \tag{76}$$

The inverse Sumudu transform implies that:

$$V(x,t) = 3xt - S^{-1}[u\,S[U_x]]$$
$$U(x,t) = \frac{t^2}{4} + x^2 + \frac{3}{2}S^{-1}[u\,S[V_x]] \tag{77}$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n V_n(x,t) = 3xt - p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n U_n\right]_x\right]\right\}$$
$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \frac{t^2}{4} + x^2 + \frac{3}{2}p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n V_n\right]_x\right]\right\} \tag{78}$$

Comparing the coefficients of like power p , we get;

$$p^0 : V_0(x,t) = 3xt \quad , \quad U_0(x,t) = \frac{t^2}{4} + x^2$$
$$p^1 : V_1(x,t) = -2xt \quad , \quad U_1(x,t) = \frac{9}{4}t^2$$
$$p^2 : V_2(x,t) = 0 \quad , \quad U_2(x,t) = -\frac{3}{2}t^2 \tag{79}$$
$$p^3 : V_3(x,t) = 0 \quad , \quad U_3(x,t) = 0$$

And so on, using Eq. (79) we obtain;

$$V(x,t) = 3xt - 2xt = xt,$$
$$U(x,t) = \frac{t^2}{4} + x^2 + \frac{9}{4}t^2 - \frac{3}{2}t^2 = t^2 + x^2 \tag{80}$$

This has an exact analytical solution of the form;

$$(U,V) = (t^2 + x^2 \ , \ xt) \tag{81}$$

Example (1.4.20): Consider the following system of partial differential equations,

$$U_{tt} - V_x = 2x^2 - e^t$$
$$V_t + U_{xx} = 2t^2 + x\,e^t \tag{82}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \quad U_t(x,0) = 0$$
$$V(x,0) = x$$

Taking Sumudu transform of equations (182) subject to the initial conditions, we get;

$$S[U(x,t)] = 2x^2u^2 - \frac{u^2}{1-u} + u^2\, S[V_x]$$
$$S[V(x,t)] = 4u^3 + \frac{x\,u}{1-u} + x - u\,S[U_{xx}] \quad (83)$$

The inverse Sumudu transform implies that:

$$U(x,t) = x^2t^2 + t + 1 - e^t + S^{-1}\left[u^2\, S[V_x]\right]$$
$$V(x,t) = \frac{4}{3}t^3 + x\,e^t - S^{-1}\left[u\,S[U_{xx}]\right] \quad (84)$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n\, U_n(x,t) = x^2t^2 + t + 1 - e^t + p\left\{S^{-1}\left[u^2\, S\left[\sum_{n=0}^{\infty} p^n\, V_n\right]_x\right]\right\}$$
$$\sum_{n=0}^{\infty} p^n\, V_n(x,t) = \frac{4}{3}t^3 + x\,e^t - p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n\, U_n\right]_{xx}\right]\right\} \quad (85)$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,t) = x^2t^2 + t + 1 - e^t \quad , \quad V_0(x,t) = \frac{4}{3}t^3 + x\,e^t$$
$$p^1 : U_1(x,t) = -t - 1 + e^t \quad , \quad V_1(x,t) = -\frac{4}{3}t^3 \quad (86)$$
$$p^2 : U_2(x,t) = 0 \quad , \quad V_2(x,t) = 0$$
$$p^3 : U_3(x,t) = 0 \quad , \quad V_3(x,t) = 0$$

And so on, using Eq. (86) we obtain;

$$U(x,t) = x^2t^2 + t + 1 - e^t - t - 1 + e^t = x^2t^2 ,$$
$$V(x,t) = \frac{4}{3}t^3 + xe^t - \frac{4}{3}t^3 = xe^t \quad (87)$$

This has an exact analytical solution of the form;

$$(U,V) = \left(x^2t^2 \;,\; xe^t\right) \quad (88)$$

CHAPTER TWO

Applcation of Homotopy Perturbation Method and Sumudu Transform for Solving Heat and Wave Equations

The integral part of applied sciences and engineering mathematics are heat and wave like models that arises from different physical phenomena. Various methods and techniques are available to solve these problems, but every method have inbuilt deficiencies. Some of the methods are spectral, characteristics, modified variational iteration, Adomian's decomposition method and He's polynomial [13]. He (1999, 2003 and 2004) developed the homotopy perturbation method (HPM) by combining the concepts of standard homotopy and perturbation for solving different physical phenomena.

It is important to note that the HPM is applied without any restrictive assumption or transformation, results in eliminating round off errors. The use of He's polynomial in the nonlinear system was first introduced by Ghorbani and Saberi-Nadjafi (2007) and Ghornabi (2009). They developed an elegant combination of the Sumudu transform method, the homotopy perturbation method and He's polynomial. Madani and Fathizadeh (2010) and Khan and Wu (2011) combined the homotopy perturbation method with Lapalace transformation method. In 2011, Singh, Kumar and Sushila introduced a new technique called homotopy perturbation Sumudu transform method (HPSTM) for solving nonlinear equations.

HPSTM gives the solution for nonlinear equations in the form of convergent series. The main advantage of this method is its potentiality of combining two powerful methods for deriving exact and approximate solution for nonlinear equations. This forms the motivation for us to apply HPSTM for solving nonlinear equations in understanding different physical phenomena. Numbers of examples are presented to assert the efficiency and reliability of the technique.

2.1: Homotopy Perturbation and Sumudu Transform Method (HPSTM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of form,

$$DU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \tag{1}$$
$$U(x,0) = h(x)\,, \qquad U_t(x,0) = f(x)$$

Where D is the second order linear differential operator $\left(D = \frac{\partial^2}{\partial t^2}\right)$, R is the linear differential operator of order less than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking the Sumudu transform on both sides of Eq. (1), we get,

$$S[DU(x,t)] + S[RU(x,t)] + S[NU(x,t)] = S[g(x,t)] \tag{2}$$

Using the differential operator property of the Sumudu transforms and above initial conditions, we get,

$$\begin{aligned} S[DU(x,t)] &= u^2 S[g(x,t)] + h(x) + u\, f(x) \\ &\quad - u^2 S[RU(x,t)] + S[NU(x,t)] \end{aligned} \tag{3}$$

Now, applying the inverse Sumudu transform of both sides of Eq. (3), we get,

$$U(x,t) = G(x,t) - S^{-1}\left[u^2\, S[RU(x,t) + NU(x,t)]\right] \tag{4}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We apply the homotopy perturbation method;

$$U(x,t) = \sum_{n=0}^{\infty} p^n\, U_n(x,t) \tag{5}$$

And the nonlinear term can be decomposed as;

$$NU(x,t) = \sum_{n=0}^{\infty} p^n\, H_n(x,t) \tag{6}$$

For some He's polynomials $H_n(U)$ that are given by;

$$H_n(U_0, U_1, U_2, \cdots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i\, U_i(x,t)\right)\right]_{p=0}, \quad n = 0, 1, 2, 3, \cdots \tag{7}$$

Substituting Eqs. (5) and (6) in Eq. (4) we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = G(x,t) - p\left(S^{-1}\left[u^2\, S\left[R\sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(x,t)\right]\right]\right) \tag{8}$$

This is the coupling of the Sumudu transform and the homotopy perturbation method using He's polynomials.

Comparing the coefficient of like power of p, the following approximation is obtained;

$$\begin{aligned} p^0 &: U_0(x,t) = G(x,t) \\ p^1 &: U_1(x,t) = -S^{-1}\left[u^2 S\left[RU_0(x,t) + H_0(U)\right]\right] \\ p^2 &: U_2(x,t) = -S^{-1}\left[u^2 S\left[RU_1(x,t) + H_1(U)\right]\right] \\ p^3 &: U_3(x,t) = -S^{-1}\left[u^2 S\left[RU_2(x,t) + H_2(U)\right]\right] \\ &\quad . \end{aligned} \tag{9}$$

2.2: Heat Equation

The homotopy Perturbation and Sumudu transform Method can be used to solving the heat equation;

$$U_t = kU_{xx}, \quad 0 < x < \pi, \ t > 0, \tag{10}$$

Where $U = U(x,t)$ represents the temperature of the rod at the position x at time t and k is the thermal diffusivity of the material that measures the rod ability to heat conduction.

Boundary Conditions

Boundary conditions (BC) are mainly of three types namely, Dirichlet boundary conditions, Neumann boundary conditions, and mixed boundary conditions. In addition, the boundary conditions may be homogeneous or inhomogeneous type.

Boundary condition (BC) that describe the temperature U at both ends of the rod. One form of the BC is given by the Dirichlet boundary conditions;

$$\begin{aligned} U(0,t) &= 0, \ t \geq 0, \\ U(l,t) &= 0, \ t \geq 0. \end{aligned} \tag{11}$$

It clearly indicates the ends of the rod are at $0^0 F$ temperatures.

Initial Condition (IC) describes the initial temperature u at time $t = 0$. The IC is usually defined by;

$$U(x,0) = f(x), \quad 0 < x < l \ . \tag{12}$$

Based on these definitions, the initial-boundary value problem that controls the heat conduction in a rod is given by;

$$\begin{aligned} &\text{PDE} \quad U_t = kU_{xx}, \quad 0 < x < l,\ t > 0, \\ &\text{BC} \quad U(0,t) = 0,\ t \ge 0 \\ &\qquad\ \ U(l,t) = 0,\ t \ge 0 \\ &\text{IC} \quad U(x,0) = f(x), \quad 0 < x < l \end{aligned} \tag{13}$$

As stated before let us focus our discussions on determining a particular solution of the heat equation (13).

2.2.1: One Dimensional Heat Flow

The distribution of heat flow in one dimensional space is governed by the following initial boundary value.

Example (2.2.1): Consider the following one-dimensional initial boundary value problem as heat-like models;

$$U_t = U_{xx}, \quad 0 < x < \pi,\ t > 0 \tag{14}$$

With boundary condition as;

$$U(0,t) = 0, \quad U(\pi,t) = 0 \tag{15}$$

And initial condition as;

$$U(x,0) = \sin x \tag{16}$$

Taking Sumudu transform on both sides of equation (14) subject to the initial condition, we get;

$$S[U(x,t)] = \sin x + uS[U_{xx}] \tag{17}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = \sin x + S^{-1}[uS[U_{xx}]] \tag{18}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \sin x + p\left(S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \tag{19}$$

Or equivalently;

$$\begin{aligned} &U_0 + pU_1 + p^2U_2 + p^3U_3 + \cdots \\ &\qquad = \sin x + p\left(S^{-1}\left[u\,S\left(U_0 + pU_1 + p^2U_2 + p^3U_3 + \cdots\right)_{xx}\right]\right) \end{aligned} \tag{20}$$

Comparing the coefficients of like power p , we get;

$$p^0 \ : U_0(x,t) = \ \sin x$$

$$p^1 \ : U_1(x,t) = -\ S^{-1}[u\,S[(U_0)_{xx}]] = -t\sin x \qquad (21)$$

$$p^2 \ : U_2(x,t) = -\ S^{-1}[u\,S[(U_1)_{xx}]] = \frac{t^2}{2!}\sin x$$

Proceeding in a similar manner, we obtain;

$$p^3 : U_3(x,t) = -\frac{t^3}{3!}\sin x$$

$$p^4 : U_4(x,t) = \frac{t^4}{4!}\sin x \qquad (22)$$

Therefore the solution $U(x,t)$in series form is given by;

$$U(x,t) = U_0(x,t)+U_1(x,t)+U_2(x,t)+\cdots$$

$$U(x,t) = \left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\cdots\right)\sin x \qquad (23)$$

And in closed form given as;

$$U(x,t) = e^{-t}\sin x \qquad (24)$$

Example (2.2.2): Consider the following one-dimensional initial boundary value problem as heat-like methods;

$$U_t = U_{xx}\,, \qquad 0 < x < \pi \ , \ t > 0 \qquad (25)$$

With boundary condition as;

$$U(0\ ,\ t) = e^{-t}\ , \quad U(\pi,t) = \pi - e^{-t} \qquad (26)$$

And the initial condition as;

$$U(x,0) = x + \cos x \qquad (27)$$

Taking Sumudu transform of both sides of the equation (25) subject to the initial condition, we get;

$$S[U(x,t)] = x+\cos x + uS[U_{xx}] \qquad (28)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = x + \cos x + S^{-1}[uS[U_{xx}]] \qquad (29)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n\,U_n(x,t) = x + \cos x + p\left(S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \qquad (30)$$

Or equivalently;

$$U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \cdots = x + \cos x + p\left(S^{-1}\left[u\, S\left(U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \cdots\right)_{xx}\right]\right) \quad (31)$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = x + \cos x \\ p^1 &: U_1(x,t) = S^{-1}\left[u\, S\left[(U_0)_{xx}\right]\right] = -t\cos x \\ p^2 &: U_2(x,t) = S^{-1}\left[u\, S\left[(U_1)_{xx}\right]\right] = \frac{t^2}{2!}\cos x \end{aligned} \quad (32)$$

Proceeding in similar manner, we obtain;

$$\begin{aligned} p^3 &: U_3(x,t) = -\frac{t^3}{3!}\cos x \\ p^4 &: U_4(x,t) = \frac{t^4}{4!}\cos x \end{aligned} \quad (33)$$

Therefore the solution $U(x,t)$ in series form is given by;

$$U(x,t) = x + \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots\right)\cos x \quad (34)$$

And in closed form given as;

$$U(x,t) = x + e^{-t}\cos x \quad (35)$$

Example (2.2.3): Consider the following one-dimensional initial boundary value problem as heat-like methods;

$$U_t = \frac{1}{2}x^2 U_{xx}, \qquad 0 < x < 1,\ t > 0 \quad (36)$$

With boundary condition as;

$$U(0,t) = 0, \quad U(1,t) = e^{-t} \quad (37)$$

And the initial condition as;

$$U(x,0) = x^2 \quad (38)$$

Taking Sumudu transform of both sides of the equation (36) subject to the initial condition, we get;

$$S[U(x,t)] = x^2 + \frac{1}{2}x^2 u\, S[U_{xx}] \quad (39)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = x^2 + \frac{1}{2}x^2\, S^{-1}\left[uS[U_{xx}]\right] \quad (40)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x^2 + p\left(\frac{1}{2}x^2 S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \tag{41}$$

Or equivalently;

$$U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \cdots$$
$$= x^2 + p\left(\frac{1}{2}x^2 S^{-1}\left[u\,S\left(U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \cdots\right)_{xx}\right]\right) \tag{42}$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,t) = x^2$$
$$p^1 : U_1(x,t) = \frac{1}{2}x^2 S^{-1}\left[u\,S\left[(U_0)_{xx}\right]\right] = x^2 t \tag{43}$$
$$p^2 : U_2(x,t) = \frac{1}{2}x^2 S^{-1}\left[u\,S\left[(U_1)_{xx}\right]\right] = x^2 \frac{t^2}{2!}$$

Proceeding in a similar manner, we obtain;

$$p^3 : U_3(x,t) = x^2 \frac{t^3}{3!}$$
$$p^4 : U_4(x,t) = x^2 \frac{t^4}{4!} \tag{44}$$

Therefore the solution $U(x,t)$in series form is given by;

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$$
$$U(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots\right) \tag{45}$$

And in closed form given as;

$$U(x,t) = x^2 e^t \tag{46}$$

2.2.2: Two Dimensional Heat Flow

The distribution of heat flow in a two dimensional space is governed by the following initial boundary value.

Example (2.2.4): Consider the following two-dimensional initial boundary value problem which describes the heat-like models;

$$U_t = U_{xx} + U_{yy}, \qquad 0 < x, y < \pi, \ t > 0 \tag{47}$$

With boundary conditions as;

$$\begin{aligned} U(0, y, t) = U(\pi, y, t) = 0 \\ U(x, 0, t) = U(x, \pi, t) = 0 \end{aligned} \tag{48}$$

And the initial condition as;

$$U(x, y, 0) = (\sin x)(\sin y) \tag{49}$$

Taking Sumudu transform of both sides of the equation (47) subject to the initial condition, we get;

$$S[U(x, y, t)] = (\sin x)(\sin y) + uS[U_{xx} + U_{yy}] \tag{50}$$

The inverse of Sumudu transform implies that;

$$U(x, y, t) = (\sin x)(\sin y) + S^{-1}[uS[U_{xx} + U_{yy}]] \tag{51}$$

The decomposition method defined the solution $U(x, y, t)$ as a series given by;

$$U(x, y, t) = \sum_{n=0}^{\infty} U_n(x, y, t)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x, y, t) = \sin x \sin y + p \Bigg(S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, t) \right)_{xx} \right]\right] \\ + S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, t) \right)_{yy} \right]\right] \Bigg) \end{aligned} \tag{52}$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned} p^0 &: U_0(x, y, t) = \sin x \sin y \\ p^1 &: U_1(x, y, t) = S^{-1}[u S[(U_0)_{xx}]] + S^{-1}[u S[(U_0)_{yy}]] = -2t \sin x \sin y \\ p^2 &: U_2(x, y, t) = S^{-1}[u S[(U_1)_{xx}]] + S^{-1}[u S[(U_1)_{yy}]] = \frac{(2t)^2}{2!} \sin x \sin y \end{aligned} \tag{55}$$

Proceeding in a similar manner, we obtain;

$$\begin{aligned} p^3 : U_3(x, y, t) &= -\frac{(2t)^2}{3!}\sin x \sin y \\ p^4 : U_4(x, y, t) &= \frac{(2t)^2}{4!}\sin x \sin y \end{aligned} \qquad (56)$$

Therefore the solution $U(x, y, t)$ in series form is given by;

$$U(x, y, t) = \left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \cdots\right)\sin x \sin y \qquad (57)$$

And in closed form given as;

$$U(x, y, t) = e^{-2t}\sin x \sin y \qquad (58)$$

Example (2.2.5): Consider the following two-dimensional initial boundary value problem which describes the heat-like models;

$$U_t = U_{xx} + U_{yy} - U, \qquad 0 < x, y < \pi, \; t > 0 \qquad (59)$$

With boundary conditions as;

$$\begin{aligned} U(0, y, t) &= U(\pi, y, t) = 0 \\ U(x, 0, t) &= U(x, \pi, t) = e^{-3t}\sin x \end{aligned} \qquad (60)$$

And the initial condition as;

$$U(x, y, 0) = \sin x \cos y \qquad (61)$$

Taking Sumudu transform of both sides of the equation (59) subject to the initial condition, we get;

$$S[U(x, y, t)] = \sin x \cos y + uS[U_{xx} + U_{yy} - U] \qquad (62)$$

The inverse of Sumudu transform implies that;

$$U(x, y, t) = \sin x \cos y + S^{-1}[uS[U_{xx} + U_{yy} - U]] \qquad (63)$$

The decomposition method defined the solution $U(x, y, t)$ as a series given by;

$$U(x, y, t) = \sum_{n=0}^{\infty} U_n(x, y, t)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x, y, t) &= \sin x \cos y + p\left(S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, t)\right)_{xx}\right]\right]\right. \\ &\left. + S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, t)\right)_{yy}\right]\right] + S^{-1}\left[uS\left[\sum_{n=0}^{\infty} p^n U_n(x, y, t)\right]\right]\right) \end{aligned} \qquad (64)$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,y,t) = \sin x \cos y$$
$$p^1 : U_1(x,y,t) = S^{-1}[u\,S[(U_0)_{xx}]] + S^{-1}[u\,S[(U_0)_{yy}]] + S^{-1}[u\,S[U_0]]$$
$$= -3t \sin x \cos y \tag{65}$$
$$p^2 : U_2(x,y,t) = S^{-1}[u\,S[(U_1)_{xx}]] + S^{-1}[u\,S[(U_1)_{yy}]] + S^{-1}[u\,S[U_1]]$$
$$= \frac{(3t)^2}{2!} \sin x \cos y$$

Proceeding in a manner, we obtain;

$$p^3 : U_3(x, y, t) = -\frac{(3t)^2}{3!} \sin x \cos y$$
$$p^4 : U_4(x, y, t) = \frac{(3t)^2}{4!} \sin x \cos y \tag{66}$$

Therefore the solution $U(x,y,t)$in series form is given by;

$$U(x,y\ ,t) = \left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!} - \cdots\right) \sin x \cos y \tag{67}$$

And in closed form given as;

$$U(x,y,t) = e^{-3t} \sin x \cos y \tag{68}$$

Example (2.2.6): Consider the following two-dimensional initial boundary value problem which describes the heat-like models;

$$U_t = \frac{1}{2}\left(x^2 U_{xx} + y^2 U_{yy}\right), \qquad 0 < x, y < 1 \ , \ t > 0 \tag{69}$$

With boundary conditions as;

$$U(0,y,t) = 0 \ , \quad U(1,y,t) = 2\sinh t$$
$$U(x,0,t) = 0 \ , \quad U(x,1,t) = 2\cosh t \tag{70}$$

And the initial condition as;

$$U(x,y,0) = y^2 \tag{71}$$

Taking Sumudu transform of both sides of the equation (69) subject to the initial condition, we get;

$$S[U(x,y,t)] = y^2 + \frac{1}{2} y^2 u S[U_{xx}] + \frac{1}{2} x^2 u S[U_{yy}] \tag{72}$$

The inverse of Sumudu transform implies that;

$$U(x,y,t) = y^2 + \frac{1}{2} y^2 S^{-1}[uS[U_{xx}]] + \frac{1}{2} x^2 S^{-1}[uS[U_{yy}]] \tag{73}$$

The decomposition method defined the solution $U(x,y,t)$ as a series given by;

$$U(x,y,t) = \sum_{n=0}^{\infty} U_n(x,y,t)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y,t) = y^2 + p\left(\frac{1}{2} y^2 S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{xx}\right]\right] + \frac{1}{2} x^2 S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{yy}\right]\right]\right) \tag{74}$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned} p^0 &: U_0(x,y,t) = y^2 \\ p^1 &: U_1(x,y,t) = \frac{1}{2} y^2 S^{-1}[u\,S[(U_0)_{xx}]] + \frac{1}{2} x^2 S^{-1}[u\,S[(U_0)_{yy}]] = x^2 t \\ p^2 &: U_2(x,y,t) = \frac{1}{2} y^2 S^{-1}[u\,S[(U_1)_{xx}]] + \frac{1}{2} x^2 S^{-1}[u\,S[(U_1)_{yy}]] = y^2 \frac{t^2}{2!} \end{aligned} \tag{75}$$

Proceeding in a similar manner, we obtain;

$$\begin{aligned} p^3 &: U_3(x,y,t) = x^2 \frac{t^3}{3!} \\ p^4 &: U_4(x,y,t) = y^2 \frac{t^4}{4!} \end{aligned} \tag{76}$$

Therefore the solution $U(x,y,t)$ in series form is given by;

$$U(x,y,t) = x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) + y^2\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) \tag{77}$$

And in closed form given as;

$$U(x,y,t) = x^2 \sinh t + y^2 \cosh t \tag{78}$$

2.2.3: Three Dimensional Heat Flow

The distribution of heat flow in a two dimensional space is governed by the following initial boundary value problem.

Example (2.2.7): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models as;

$$U_t = U_{xx} + U_{yy} + U_{zz}, \qquad 0 < x, y, z < \pi \ , t > 0 \tag{79}$$

With boundary conditions as;

$$\begin{aligned} U(0, y, z, t) &= U(\pi, y, z, t) = 0 \\ U(x, 0, z, t) &= U(x, \pi, z, t) = 0 \\ U(x, y, 0, t) &= U(x, y, \pi, t) = 0 \end{aligned} \tag{80}$$

And the initial condition as;

$$U(x, y, z, 0) = 2\sin x \sin y \sin z$$

Taking Sumudu transform of both sides of the equation (79) subject to the initial condition, we get;

$$S[U(x, y, z, t)] = 2\sin x \sin y \sin z + uS[U_{xx} + U_{yy} + U_{zz}] \tag{81}$$

The inverse of Sumudu transform implies that;

$$U(x, y, z, t) = 2\sin x \sin y \sin z + S^{-1}[uS[U_{xx} + U_{yy} + U_{zz}]] \tag{82}$$

The decomposition method defined the solution $U(x, y, z, t)$ as a series given by;

$$U(x, y, z, t) = \sum_{n=0}^{\infty} U_n(x, y, z, t)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x, y, z, t) = 2\sin x \sin y \sin z + p\left(S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{xx}\right]\right]\right. \\ \left. + S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{yy}\right]\right] + S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{zz}\right]\right]\right) \end{aligned} \tag{83}$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned} p^0 &: U_0(x, y, z, t) = 2\sin x \sin y \sin z \\ p^1 &: U_1(x, y, z, t) = S^{-1}[u\,S[(U_0)_{xx}]] + S^{-1}[u\,S[(U_0)_{yy}]] + S^{-1}[u\,S[(U_0)_{zz}]] \\ &= -2(3t)\sin x \sin y \sin z \end{aligned}$$

$$p^2\ : U_2(x,y,z,t) = S^{-1}[u\,S[(U_1)_{xx}]] + S^{-1}[u\,S[(U_1)_{yy}]] + S^{-1}[u\,S[(U_1)_{zz}]] \\ = -\frac{2(3t)^2}{2!}\sin x \sin y \sin z \tag{84}$$

Proceeding in similar manner, we obtain;

$$p^3 : U_3(x,y,z,\ t) = -\frac{2(3t)^3}{3!}\sin x \sin y \sin z \\ p^4 : U_4(x,y,z,\ t) = \frac{2(3t)^4}{4!}\sin x \sin y \sin z \tag{85}$$

Therefore the solution $U(x,y,z,t)$in series form is given by;

$$U(x,y\ ,z,\ t) = 2\sin x \sin y \sin z\left(1-3t+\frac{(3t)^2}{2!}-\frac{(3t)^3}{3!}+\frac{(3t)^4}{4!}-\cdots\right) \tag{86}$$

And in closed form given as;

$$U(x,y,z,t) = 2e^{-3t}\sin x \sin y \sin z \tag{87}$$

Example (2.2.8): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models as;

$$U_t = U_{xx} + U_{yy} + U_{zz} - 2U, \qquad 0 < x,y,z < \pi\ ,\ t > 0 \tag{88}$$

With boundary conditions as;

$$U(0,y,z,t) = U(\pi,y,z,\ t) = 0 \\ U(x,0,z,t) = U(x,\pi,z,t) = 0 \\ U(x,y,0,t) = U(x,y,\pi,t) = 0 \tag{89}$$

And the initial condition as;

$$U(x,y,z,\ 0) = \sin x \sin y \sin z$$

Taking Sumudu transform of both sides of the equation (88) subject to the initial condition, we get;

$$S[U(x,y,z,t)] = \sin x \sin y \sin z + uS[U_{xx} + U_{yy} + U_{zz} - 2U] \tag{90}$$

The inverse of Sumudu transform implies that:

$$U(x,y,z,t) = \sin x \sin y \sin z + S^{-1}[uS[U_{xx} + U_{yy} + U_{zz} - 2U]] \tag{91}$$

The decomposition method defined the solution $U(x,y,z,t)$ as a series given by;

$$U(x,y,z,t) = \sum_{n=0}^{\infty} U_n(x,y,z,t)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) = \sin x \sin y \sin z + p\left(S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{xx} \right]\right] + S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{yy} \right]\right] + S^{-1}\left[uS\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{zz} \right]\right] - 2S^{-1}\left[uS\left[\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right]\right] \right) \tag{92}$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} p^0 &: U_0(x,y,z,t) = \sin x \sin y \sin z \\ p^1 &: U_1(x,y,z,t) = S^{-1}\left[u S\left[(U_0)_{xx}\right]\right] + S^{-1}\left[u S\left[(U_0)_{yy}\right]\right] \\ &\qquad + S^{-1}\left[u S\left[(U_0)_{zz}\right]\right] - 2S^{-1}\left[u S\left[U_0\right]\right] \\ &\qquad = -5t \sin x \sin y \sin z \\ p^2 &: U_2(x,y,z,t) = S^{-1}\left[u S\left[(U_1)_{xx}\right]\right] + S^{-1}\left[u S\left[(U_1)_{yy}\right]\right] \\ &\qquad + S^{-1}\left[u S\left[(U_1)_{zz}\right]\right] - 2S^{-1}\left[u S\left[U_1\right]\right] \\ &\qquad = \frac{(5t)^2}{2!} \sin x \sin y \sin z \end{aligned} \tag{93}$$

Proceeding in a similar manner, we obtain;

$$\begin{aligned} p^3 &: U_3(x,y,z,t) = -\frac{(5t)^3}{3!} \sin x \sin y \sin z \\ p^4 &: U_4(x,y,z,t) = \frac{(5t)^4}{4!} \sin x \sin y \sin z \end{aligned} \tag{94}$$

Therefore the solution $U(x,y,z,t)$ in series form is given by;

$$U(x,y,z,t) = \sin x \sin y \sin z \left(1 - 5t + \frac{(5t)^2}{2!} - \frac{(5t)^3}{3!} + \frac{(5t)^4}{4!} - \cdots \right) \tag{95}$$

And in closed form given as;

$$U(x,y,z,t) = e^{-5t} \sin x \sin y \sin z \tag{96}$$

Example (2.2.9): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models as;

$$U_t = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right), \qquad 0 < x, y, z < 1, \ t > 0 \qquad (97)$$

With boundary conditions as;

$$\begin{aligned} U(0, y, z, t) &= 0, \quad U(\pi, y, z, t) = y^4 z^4 \left(e^t - 1\right) \\ U(x, 0, z, t) &= 0, \quad U(x, \pi, z, t) = x^4 z^4 \left(e^t - 1\right) \\ U(x, y, 0, t) &= 0, \quad U(x, y, \pi, t) = x^4 y^4 \left(e^t - 1\right) \end{aligned} \qquad (98)$$

And the initial condition as;

$$U(x, y, z, 0) = 0 \qquad (99)$$

Taking Sumudu transform of both sides of the equation (97) subject to the initial condition, we get;

$$S[U(x, y, z, t)] = x^4 y^4 z^4 t + \frac{1}{36} u S\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right] \qquad (100)$$

The inverse of Sumudu transform implies that:

$$U(x, y, z, t) = x^4 y^4 z^4 t + \frac{1}{36} S^{-1}\left[u S\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right]\right] \qquad (101)$$

The decomposition method defined the solution $U(x, y, z, t)$ as a series given by;

$$U(x, y, z, t) = \sum_{n=0}^{\infty} U_n(x, y, z, t)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x, y, z, t) &= x^4 y^4 z^4 t + p\left(\frac{1}{36} x^2 S^{-1}\left[u S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{xx}\right]\right]\right. \\ &\left. + \frac{1}{36} y^2 S^{-1}\left[u S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{yy}\right]\right] + \frac{1}{36} z^2 S^{-1}\left[u S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{zz}\right]\right]\right) \end{aligned} \qquad (102)$$

Comparing the coefficients of like power p, we get;

$$p^0 : U_0(x, y, z, t) = x^4 y^4 z^4 t$$

$$\begin{aligned} p^1 : U_1(x, y, z, t) &= \frac{1}{36} x^2 S^{-1}\left[u S\left[(U_0)_{xx}\right]\right] + \frac{1}{36} y^2 S^{-1}\left[u S\left[(U_0)_{yy}\right]\right] \\ &+ \frac{1}{36} z^2 S^{-1}\left[u S\left[(U_0)_{zz}\right]\right] = x^4 y^4 z^4 \frac{t^2}{2!} \end{aligned}$$

$$p^2\ : U_2(x,y,z,t) = \frac{1}{36}x^2\,S^{-1}\left[u\,S\left[(U_1)_{xx}\right]\right] + \frac{1}{36}y^2 S^{-1}\left[u\,S\left[(U_1)_{yy}\right]\right] + \frac{1}{36}z^2 S^{-1}\left[u\,S\left[(U_1)_{zz}\right]\right] = x^4y^4z^4\,\frac{t^3}{3!} \tag{103}$$

Proceeding in a similar manner, we obtain;

$$p^3 : U_3(x, y, z,\ t) = x^4y^4z^4\,\frac{t^4}{4!}$$
$$p^4 : U_4(x, y, z,\ t) = x^4y^4z^4\,\frac{t^5}{5!} \tag{104}$$

Therefore the solution $U(x, y, z, t)$in series form is given by;

$$U(x, y\ , z,\ t) = x^4y^4z^4\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots\right) \tag{105}$$

And in closed form given as;

$$U(x, y, z, t) = x^4y^4z^4\left(e^{-t} - 1\right) \tag{106}$$

2.3: Wave Equations

In this section, we will apply the newly developed homotopy perturbation method and Sumudu transform to handle the wave equation.

6.2.3: One Dimensional Wave Equations

The homotopy perturbation method will be illustrated by discussing the following typical wave model.

Without loss of generality, as a simple wave equation, consider the following initial-boundary value problem:

$$U_{tt} = c^2\, U_{xx}\,, \qquad 0 < x < l\;,\; t > 0$$

Subject to boundary conditions as;

$$U(0\;,\;t) = 0\,,\quad U(l\,,t) = 0\,, \tag{107}$$

And the initial condition as;

$$U(x\,,0) = f(x), \quad U_t(x\,,0) = g(x)$$

It is obvious the Eq. (107), that governs the wave displacement, contains the term U_{tt}. Consequently, two initial conditions should be given. The initial conditions describe the initial displacement and the initial velocity of any point at the starting time $t = 0$.

Example (2.3.10): Consider the following one-dimensional initial boundary value problem which describes the wave-like models as;

$$U_{tt} = U_{xx}\,, \qquad 0 < x < \pi\;,\; t > 0 \tag{108}$$

Subject to boundary conditions as;

$$U(0\;,\;t) = 0\,,\quad U(\pi\,,t) = 0 \tag{109}$$

And the initial condition as;

$$U(x\,,0) = 0\,,\quad U_t(x\,,0) = \sin x \tag{110}$$

Taking Sumudu transform of both sides of the equation (108) subject to the initial condition, we get;

$$S[U(x\,,t)] = u \sin x + u^2 S[U_{xx}] \tag{111}$$

The inverse of Sumudu transform implies that:

$$U(x\,,t) = t \sin x + S^{-1}\left[u^2 S[U_{xx}]\right] \tag{112}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = t\sin x + p\left(S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \quad (113)$$

Or equivalently;

$$U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \cdots = t\sin x + p\left(S^{-1}\left[u^2 S\left(U_0 + pU_1 + p^2 U_2 + p^3 U_3 + \cdots\right)_{xx}\right]\right) \quad (114)$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,t) = t\sin x$$

$$p^1 : U_1(x,t) = S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] = -\frac{t^3}{3!}\sin x \quad (115)$$

$$p^2 : U_2(x,t) = S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right] = \frac{t^5}{5!}\sin x$$

Proceeding in a similar manner, we obtain;

$$p^3 : U_3(x,t) = -\frac{t^7}{7!}\sin x$$

$$p^4 : U_4(x,t) = \frac{t^9}{9!}\sin x \quad (116)$$

Therefore the solution $U(x,t)$in series form is given by;

$$U(x,t) = U_0(x,t)+U_1(x,t)+U_2(x,t)+\cdots$$

$$U(x,t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} - \cdots\right)\sin x \quad (117)$$

And in closed form given as;

$$U(x,t) = \sin t \sin x \quad (118)$$

Example (2.3.11): Consider the following one-dimensional initial boundary value problem which describes the wave-like models as;

$$U_{tt} = U_{xx}, \quad 0 < x < \pi , \ t > 0 \quad (119)$$

Subject to boundary conditions as;

$$U(0 , t) = 1+\sin t, \quad U(\pi,t) = 1-\sin t \quad (120)$$

And the initial condition as;

$$U(x,0) = 1, \quad U_t(x,0) = \cos x \quad (121)$$

Taking Sumudu transform on both sides of equation (119) subject to the initial condition, we get;

$$S[U(x,t)] = 1 + u\cos x + u^2 S[U_{xx}] \qquad (122)$$

The inverse of Sumudu transform implies that:

$$U(x,t) = 1 + t\cos x + S^{-1}\left[u^2 S[U_{xx}]\right] \qquad (123)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 1 + t\cos x + p\left(S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \qquad (124)$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,t) = 1 + t\cos x$$

$$p^1 : U_1(x,t) = S^{-1}\left[u^2 S[(U_0)_{xx}]\right] = -\frac{t^3}{3!}\cos x \qquad (125)$$

$$p^2 : U_2(x,t) = S^{-1}\left[u^2 S[(U_1)_{xx}]\right] = \frac{t^5}{5!}\cos x$$

Proceeding in a similar manner, we get;

$$p^3 : U_3(x,t) = -\frac{t^7}{7!}\cos x$$

$$p^4 : U_4(x,t) = \frac{t^9}{9!}\cos x \qquad (126)$$

Therefore the solution $U(x,t)$ in series form is given by;

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$$

$$U(x,t) = 1 + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} - \cdots\right)\cos x \qquad (127)$$

And in closed form given as;

$$U(x,t) = 1 + \sin t\cos x \qquad (128)$$

Example (2.3.12): Consider the following one-dimensional initial boundary value problem which describes the wave-like models as;

$$U_{tt} = \frac{1}{2}x^2 U_{xx}, \qquad 0 < x < 1,\ t > 0 \qquad (129)$$

Subject to boundary conditions as;

$$U(0,t) = 0, \quad U(1,t) = 1 + \sinh t \qquad (130)$$

And the initial condition as;

$$U(x,0) = x, \quad U_t(x,0) = x^2 \tag{131}$$

Taking Sumudu transform of both sides of the equation (129) subject to the initial condition, we get;

$$S[U(x,t)] = x + u\,x^2 + \frac{1}{2}x^2u^2S[U_{xx}] \tag{132}$$

The inverse of Sumudu transform implies that:

$$U(x,t) = x + t\,x^2 + \frac{1}{2}x^2\,S^{-1}\left[u^2S[U_{xx}]\right] \tag{133}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n\,U_n(x,t) = x + t\,x^2 + \frac{1}{2}x^2\,p\left(S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty} p^nU_n(x,t)\right)_{xx}\right]\right]\right) \tag{134}$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = x + t\,x^2 \\ p^1 &: U_1(x,t) = S^{-1}\left[u^2\,S[(U_0)_{xx}]\right] = x^2\,\frac{t^3}{3!} \\ p^2 &: U_2(x,t) = S^{-1}\left[u^2\,S[(U_1)_{xx}]\right] = x^2\,\frac{t^5}{5!} \end{aligned} \tag{135}$$

Proceeding in a similar manner, we obtain;

$$\begin{aligned} p^3 &: U_3(x,t) = x^2\,\frac{t^7}{7!} \\ p^4 &: U_4(x,t) = x^2\,\frac{t^9}{9!} \end{aligned} \tag{136}$$

Therefore the solution $U(x,t)$in series form is given by;

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$$

$$U(x,t) = x + x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} - \cdots\right) \tag{137}$$

And in closed form given as;

$$U(x,t) = x + x^2\sinh t \tag{138}$$

2.3.2: Two Dimensional Wave Equation

The propagation of waves in a two dimensional vibrating membrane of length a and width b is governed by the following initial-boundary value problem;

$$U_{tt} = c^2\left(U_{xx} + U_{yy}\right), \qquad 0 < x < a \ , \ 0 < x < b \ , \ t > 0 \tag{139}$$

Subject to boundary conditions;

$$\begin{aligned} U(0, y, t) = U(a, y, t) = 0 \\ U(x, 0, t) = U(x, b, t) = 0 \end{aligned} \tag{140}$$

And the initial condition;

$$U(x, y, 0) = f(x, y) \ , \ U_t(x, y, 0) = g(x, y) \tag{141}$$

As discussed before, the solution in the t direction, in the $x-$space, or in the $y-$space will lead to identical results. However, the solution in the $t-$direction reduces the size of calculations compared with the other space solutions because it uses the initial conditions only. For this reason the solution in the t direction will be discussed in this chapter.

Example (2.3.13): Consider the following two-dimensional initial boundary value problem which describes the heat-like models as;

$$U_{tt} = 2\left(U_{xx} + U_{yy}\right), \qquad 0 < x, y < \pi \ , \ t > 0 \tag{142}$$

With boundary conditions as;

$$\begin{aligned} U(0, y, t) = U(\pi, y, t) = 0 \\ U(x, 0, t) = U(x, \pi, t) = 0 \end{aligned} \tag{143}$$

And the initial condition as;

$$U(x, y, 0) = \sin x \sin y, \quad U_t(x, y, 0) = 0 \tag{144}$$

Taking Sumudu transform of both sides of the equation (142) subject to the initial condition, we get;

$$S[U(x, y, t)] = \sin x \sin y + 2u^2 S\left[U_{xx} + U_{yy}\right] \tag{145}$$

The inverse of Sumudu transform implies that:

$$U(x, y, t) = \sin x \sin y + 2S^{-1}\left[u^2 S\left[U_{xx} + U_{yy}\right]\right] \tag{146}$$

The decomposition method defined the solution $U(x, y, t)$ as a series given by;

$$U(x, y, t) = \sum_{n=0}^{\infty} p^n U_n(x, y, t)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y,t) = \sin x \sin y + 2p\left(S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{xx}\right]\right] + S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{yy}\right]\right]\right) \quad (147)$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,y,t) = (\sin x)(\sin y)$$

$$p^1 : U_1(x,y,t) = 2S^{-1}\left[u^2 S[(U_0)_{xx}]\right] + 2S^{-1}\left[u^2 S[(U_0)_{yy}]\right] = -\frac{(2t)^2}{2!}(\sin x)(\sin y) \quad (148)$$

$$p^2 : U_2(x,y,t) = 2S^{-1}\left[u^2 S[(U_1)_{xx}]\right] + 2S^{-1}\left[u^2 S[(U_1)_{yy}]\right] = \frac{(2t)^4}{4!}(\sin x)(\sin y)$$

Proceeding in a similar manner, we obtain:

$$p^3 : U_3(x,y,t) = -\frac{(2t)^6}{6!}(\sin x)(\sin y)$$
$$p^4 : U_4(x,y,t) = \frac{(2t)^8}{8!}(\sin x)(\sin y) \quad (149)$$

Therefore the solution $U(x,y,t)$ in series form is given by;

$$U(x,y,t) = \left(1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \frac{(2t)^6}{6!} - \cdots\right)\sin x \sin y \quad (150)$$

And in closed form given as;

$$U(x,y,t) = \sin x \sin y \cos(2t) \quad (151)$$

Example (2.3.14): Consider the following two-dimensional initial boundary value problem which describes the wave-like models;

$$U_t = \frac{1}{2}(U_{xx} + U_{yy}), \qquad 0 < x, y < \pi, \ t > 0 \quad (152)$$

With boundary condition as;

$$U(0,y,t) = U(\pi,y,t) = 1$$
$$U(x,0,t) = 1 + \sin x \sin t, \quad U(x,\pi,t) = 1 - \sin x \sin t \quad (153)$$

And the initial condition as;

$$U(x,y,0) = 1, \quad U_t(x,y,0) = \sin x \cos y \quad (154)$$

Taking Sumudu transform on both sides of equation (152) subject to the initial condition, we get;

$$S[U(x,y,t)] = 1 + u\sin x\cos y + \frac{1}{2}u^2 S[U_{xx} + U_{yy}] \quad (155)$$

The inverse of Sumudu transform implies that:

$$U(x,y,t) = 1 + t\sin x\cos y + \frac{1}{2}S^{-1}\left[u^2 S[U_{xx} + U_{yy}]\right] \quad (156)$$

The decomposition method defined the solution $U(x,y,t)$ as a series given by;

$$U(x,y,t) = \sum_{n=0}^{\infty} p^n U_n(x,y,t)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y,t) = 1 + t\sin x\cos y + \frac{1}{2}p\left(S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{xx}\right]\right] + S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{yy}\right]\right]\right) \quad (157)$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned} p^0 &: U_0(x,y,t) = 1 + t(\sin x)(\cos y) \\ p^1 &: U_1(x,y,t) = \frac{1}{2}S^{-1}\left[u^2 S[(U_0)_{xx}]\right] + \frac{1}{2}S^{-1}\left[u^2 S[(U_0)_{yy}]\right] \\ &= -\frac{1}{3!}t^3(\sin x)(\cos y) \\ p^2 &: U_2(x,y,t) = \frac{1}{2}S^{-1}\left[u^2 S[(U_1)_{xx}]\right] + \frac{1}{2}S^{-1}\left[u^2 S[(U_1)_{yy}]\right] = \\ &= \frac{t^5}{5!}(\sin x)(\cos y) \end{aligned} \quad (158)$$

Proceeding in a similar manner, we obtain;

$$\begin{aligned} p^3 &: U_3(x,y,t) = -\frac{t^7}{7!}(\sin x)(\cos y) \\ p^4 &: U_4(x,y,t) = \frac{t^9}{9!}(\sin x)(\cos y) \end{aligned} \quad (159)$$

Therefore, the solution $U(x,y,t)$ in series form is given by;

$$U(x,y,t) = 1 + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} - \cdots\right)(\sin x)(\cos y) \quad (160)$$

And in closed form given as;

$$U(x,y,t) = 1 + (\sin x)(\cos y)(\sin t) \qquad (161)$$

Example (2.3.15): Use the homotopy perturbation method to solve the initial boundary value problem;

$$U_{tt} = \frac{1}{12}\left(x^2 U_{xx} + y^2 U_{yy}\right), \qquad 0 < x, y < 1 \,,\ t > 0 \qquad (162)$$

Subjected the Neumann boundary conditions as;

$$\begin{aligned} U_x(0,y,t) &= 0 \ , \quad U_x(1,y,t) = 4\cosh t \\ U_y(x,0,t) &= 0 \ , \quad U_y(x,\pi,t) = 4\sinh t \end{aligned} \qquad (163)$$

And the initial condition as;

$$U(x,y,0) = x^4 \ , \quad U_t(x,y,0) = y^4$$

Taking Sumudu transform of both sides of the equation (162) subject to the initial condition, we get;

$$S[U(x,y,t)] = x^4 + u\,y^4 + \frac{1}{12}x^2u^2S[U_{xx}] + \frac{1}{12}y^2u^2S[U_{yy}] \qquad (164)$$

The inverse of Sumudu transform implies that;

$$U(x,y,t) = x^4 + t\,y^4 + \frac{1}{12}x^2S^{-1}\left[u^2S[U_{xx}]\right] + \frac{1}{12}y^2S^{-1}\left[u^2S[U_{yy}]\right] \qquad (165)$$

The decomposition method defined the solution $U(x,y,t)$ as a series given by;

$$U(x,y,t) = \sum_{n=0}^{\infty} p^n U_n(x,y,t)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,y,t) = x^4 + t\,y^4 + p\Bigg(&\frac{1}{12}x^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{xx}\right]\right] \\ &+ \frac{1}{12}y^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)_{yy}\right]\right]\Bigg) \end{aligned} \qquad (166)$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned} p^0 &: U_0(x,y,t) = x^4 + t\,y^4 \\ p^1 &: U_1(x,y,t) = \frac{1}{12}x^2S^{-1}\left[u^2S[(U_0)_{xx}]\right] + \frac{1}{12}y^2S^{-1}\left[u^2S[(U_0)_{yy}]\right] \\ &\qquad = x^4\frac{t^2}{2!} + y^4\frac{t^3}{3!} \end{aligned} \qquad (167)$$

$$p^2 \ : U_2(x,y,t) = \frac{1}{12} x^2 S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right] + \frac{1}{12} y^2 S^{-1}\left[u^2 S\left[(U_1)_{yy}\right]\right]$$
$$= x^4 \frac{t^4}{4!} + y^4 \frac{t^5}{5!}$$

Proceeding in a similar manner, we obtain;

$$p^3 : U_3(x, y,\ t) = x^4 \frac{t^6}{6!} + y^4 \frac{t^7}{7!}$$
$$p^4 : U_4(x, y,\ t) = x^4 \frac{t^8}{8!} + y^4 \frac{t^9}{9!} \tag{168}$$

Therefore the solution $U(x, y,\ t)$ in series form is given by;

$$U(x, y\ , t) = x^4 \left(1 + \frac{t^2}{4!} + \frac{t^4}{4!} + \cdots\right) + y^4 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) \tag{169}$$

And in closed form given as;

$$U(x, y, t) = x^4 \cosh t + y^4 \sinh t \tag{170}$$

2.3.3: Three Dimensional Wave Equation

The propagation of waves in a three dimensional volume of length a, width b, and height d is governed by the following initial boundary value problem;

$$U_{tt} = c^2 \left(U_{xx} + U_{yy} + U_{zz}\right), \qquad t > 0 \tag{171}$$

With the following boundary conditions;

$$U(0, y, z, t) = U(a, y, z,\ t) = 0$$
$$U(x, 0, z, t) = U(x, b, z, t) = 0 \tag{172}$$
$$U(x, y, 0, t) = U(x, y, d, t) = 0$$

And the initial condition as;

$$U(x, y, z,\ 0) = f(x, y, z)\ ,\ \ U_t(x, y, z,\ 0) = g(x, y, z) \tag{174}$$

Where $0 < x < a$, $0 < y < b$, $0 < z < d$, and $U = U(x, y, z, t)$ is the displacement of any point located at the position (x, y, z) of a rectangular volume at any time t, and c is the velocity of a propagating wave.

Example (2.3.16): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the wave-like models;

$$U_{tt} = 3\left(U_{xx} + U_{yy} + U_{zz}\right), \qquad 0 < x, y, z < \pi \ , \ t > 0 \tag{175}$$

Subject to the following boundary conditions;

$$\begin{aligned} U(0, y, z, t) &= U(\pi, y, z, t) = 0 \\ U(x, 0, z, t) &= U(x, \pi, z, t) = 0 \\ U(x, y, 0, t) &= U(x, y, \pi, t) = 0 \end{aligned} \tag{176}$$

And the initial condition as;

$$U(x, y, z, 0) = 0 \quad , \quad U_t(x, y, z, 0) = 3\sin x \sin y \sin z$$

Taking Sumudu transform of both sides of the equation (175) subject to the initial condition, we get;

$$S[U(x, y, z, t)] = 3u \sin x \sin y \sin z + 3u^2 S[U_{xx} + U_{yy} + U_{zz}] \tag{177}$$

The inverse of Sumudu transform implies that:

$$U(x, y, z, t) = 3t \sin x \sin y \sin z + 3S^{-1}\left[u^2 S[U_{xx} + U_{yy} + U_{zz}]\right] \tag{178}$$

The decomposition method defined the solution $U(x, y, z, t)$ as a series given by;

$$U(x, y, z, t) = \sum_{n=0}^{\infty} p^n U_n(x, y, z, t)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x, y, z, t) = 3t\sin x \sin y \sin z + 3p\Bigg(S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{xx}\right]\right] \\ + S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{yy}\right]\right] + S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x, y, z, t)\right)_{zz}\right]\right]\Bigg) \end{aligned} \tag{179}$$

Comparing the coefficients of like power p , we have;

$$\begin{aligned} p^0 &: U_0(x, y, z, t) = 3t \sin x \sin y \sin z \\ p^1 &: U_1(x, y, z, t) = 3\, S^{-1}\left[u^2\, S[(U_0)_{xx}]\right] + 3S^{-1}\left[u^2\, S[(U_0)_{yy}]\right] \\ &\qquad + 3S^{-1}\left[u^2\, S[(U_0)_{zz}]\right] = -\frac{(3t)^3}{3!}\sin x \sin y \sin z \\ p^2 &: U_2(x, y, z, t) = 3S^{-1}\left[u^2\, S[(U_1)_{xx}]\right] + 3S^{-1}\left[u^2\, S[(U_1)_{yy}]\right] \\ &\qquad + 3S^{-1}\left[u^2\, S[(U_1)_{zz}]\right] = \frac{(3t)^5}{5!}\sin x \sin y \sin z \end{aligned} \tag{180}$$

Proceeding in a similar manner, we obtain;

$$p^3 : U_3(x, y, z, t) = -\frac{(3t)^7}{7!} \sin x \sin y \sin z$$
$$p^4 : U_4(x, y, z, t) = \frac{(3t)^9}{9!} \sin x \sin y \sin z \qquad (181)$$

Therefore, the solution $U(x, y, z, t)$ in series form is given by;

$$U(x, y, z, t) = 3 \sin x \sin y \sin z \left(3t - \frac{(3t)^3}{3!} + \frac{(3t)^5}{5!} - \frac{(3t)^7}{7!} - \cdots \right) \qquad (182)$$

And in closed form given as;

$$U(x, y, z, t) = \sin x \sin y \sin z \sin(3t) \qquad (183)$$

Example (2.3.17): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models;

$$U_{tt} = U_{xx} + U_{yy} + U_{zz} - U, \qquad 0 < x, y, z < \pi, \; t > 0 \qquad (184)$$

Subject to the following boundary conditions;

$$U(0, y, z, t) = U(\pi, y, z, t) = 0$$
$$U(x, 0, z, t) = U(x, \pi, z, t) = 0 \qquad (185)$$
$$U(x, y, 0, t) = U(x, y, \pi, t) = 0$$

And the initial condition as;

$$U(x, y, z, 0) = 0 \;, \quad U_t(x, y, z, 0) = 2 \sin x \sin y \sin z$$

Taking Sumudu transform of both sides of the equation (184) subject to the initial condition, we get;

$$S[U(x, y, z, t)] = 2u \sin x \sin y \sin z + u^2 S[U_{xx} + U_{yy} + U_{zz} - U] \qquad (186)$$

The inverse of Sumudu transform implies that;

$$U(x, y, z, t) = 2t \sin x \sin y \sin z + S^{-1}\left[u^2 S[U_{xx} + U_{yy} + U_{zz} - U]\right] \qquad (187)$$

The decomposition method defined the solution $U(x, y, z, t)$ as a series given by;

$$U(x, y, z, t) = \sum_{n=0}^{\infty} p^n U_n(x, y, z, t)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) = t\sin x\sin y\sin z + p\left(S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{xx} \right]\right] + S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{yy} \right]\right] + S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right)_{zz} \right]\right] - S^{-1}\left[u^2 S\left[\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) \right]\right]\right) \tag{188}$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} p^0 &: U_0(x,y,z,t) = 2t\,\sin x\sin y\sin z \\ p^1 &: U_1(x,y,z,t) = S^{-1}[u\,S[(U_0)_{xx}]] + S^{-1}[u\,S[(U_0)_{yy}]] \\ &\qquad + S^{-1}[u\,S[(U_0)_{zz}]] - 2S^{-1}[u\,S[U_0]] \\ &\qquad = -\frac{(2t)^3}{3!}\sin x\sin y\sin z \\ p^2 &: U_2(x,y,z,t) = S^{-1}[u\,S[(U_1)_{xx}]] + S^{-1}[u\,S[(U_1)_{yy}]] \\ &\qquad + S^{-1}[u\,S[(U_1)_{zz}]] - 2\,S^{-1}[u\,S[U_1]] \\ &\qquad = \frac{(2t)^5}{5!}\sin x\sin y\sin z \end{aligned} \tag{189}$$

Proceeding in a similar manner, we obtain;

$$\begin{aligned} p^3 &: U_3(x,y,z,t) = -\frac{(2t)^7}{7!}\sin x\sin y\sin z \\ p^4 &: U_4(x,y,z,t) = \frac{(2t)^9}{9!}\sin x\sin y\sin z \end{aligned} \tag{190}$$

Therefore the solution $U(x,y,z,t)$ in series form is given by;

$$U(x,y,z,t) = \sin x\sin y\sin z\left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} - \cdots \right) \tag{191}$$

And in closed form given as;

$$U(x,y,z,t) = \sin x\sin y\sin z\sin(2t) \tag{192}$$

Example (2.3.18): Consider the following three-dimensional inhomogeneous initial boundary value problem which describes the heat-like models;

$$U_{tt} = \left(x^2+y^2+z^2\right)+\frac{1}{2}\left(x^2U_{xx}+y^2U_{yy}+z^2U_{zz}\right), \quad 0<x,y,z<1\,,\ t>0 \qquad (193)$$

Subject to the following boundary conditions;

$$\begin{aligned} &U(0,y,z,t)=y^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right), \quad U(1,y,z,t)=\left(1+y^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right)\\ &U(x,0,z,t)=x^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right), \quad U(x,\pi,z,t)=\left(1+x^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right)\\ &U(x,y,0,t)=\left(x^2+z^2\right)\left(e^t-1\right) \quad, \quad U(x,y,\pi,t)=\left(x^2+y^2\right)\left(e^t-1\right)+\left(e^{-t}-1\right)\end{aligned} \qquad (194)$$

And the initial condition as;

$$U(x,y,z,0)=0 \quad,\quad U(x,y,z,0)_t = x^2+y^2-z^2 \qquad (195)$$

Taking Sumudu transform of both sides of the equation (193) subject to the initial condition, we get;

$$\begin{aligned} S[U(x,y,z,t)] &= \left(x^2+y^2+z^2\right)u^2+\left(x^2+y^2-z^2\right)u\\ &+\frac{1}{2}u^2S\left[x^2U_{xx}+y^2U_{yy}+z^2U_{zz}\right]\end{aligned} \qquad (196)$$

The inverse of Sumudu transform implies that:

$$\begin{aligned} U(x,y,z,t) &= \left(x^2+y^2+z^2\right)\frac{t^2}{2}+\left(x^2+y^2-z^2\right)t\\ &+\frac{1}{2}S^{-1}\left[u^2S\left[x^2U_{xx}+y^2U_{yy}+z^2U_{zz}\right]\right]\end{aligned} \qquad (197)$$

The decomposition method defined the solution $U(x,y,z,t)$ as a series given by;

$$U(x,y,z,t)=\sum_{n=0}^{\infty}p^n\,U_n(x,y,z,t)$$

Now, applying the homotopy perturbation method, we get;

$$\begin{aligned}\sum_{n=0}^{\infty}p^n\,U_n(x,y,z,t) &= \left(x^2+y^2+z^2\right)\frac{t^2}{2}+\left(x^2+y^2-z^2\right)t\\ &+p\left(\frac{1}{2}x^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty}p^nU_n(x,y,z,t)\right)_{xx}\right]\right]\right.\\ &+\frac{1}{2}y^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty}p^nU_n(x,y,z,t)\right)_{yy}\right]\right]+\left.\frac{1}{2}z^2S^{-1}\left[u^2S\left[\left(\sum_{n=0}^{\infty}p^nU_n(x,y,z,t)\right)_{zz}\right]\right]\right)\end{aligned} \qquad (198)$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^2}{2} + \left(x^2+y^2-z^2\right)t$$

$$p^1 : U_1(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^4}{4!} + \left(x^2+y^2-z^2\right)\frac{t^3}{3!}$$

$$p^2 : U_2(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^6}{6!} + \left(x^2+y^2-z^2\right)\frac{t^5}{5!}$$

$$p^3 : U_3(x,y,z,t) = \left(x^2+y^2+z^2\right)\frac{t^8}{8!} + \left(x^2+y^2-z^2\right)\frac{t^7}{7!} \tag{199}$$

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Therefore the solution $U(x, y, z, t)$ in series form is given by;

$$\begin{aligned} U(x,y,z,t) &= \left(x^2+y^2\right)\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}-\cdots\right) \\ &+ z^2\left(-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\cdots\right) \end{aligned} \tag{200}$$

And in closed form given as;

$$U(x,y,z,t) = \left(x^2+y^2\right)e^{t} + z^2 e^{-t} - \left(x^2+y^2+z^2\right) \tag{201}$$

CHAPTER THREE
Nonlinear Partial Differential Equations

3.1: Homotopy Perturbation Method

In the previous chapters, the homotopy perturbation method has been applied to a broad class of linear partial differential equations. It is evident that this method can be applied to homogeneous and inhomogeneous problems without any restriction or linearization. The method emphasizes on decomposing the unknown function, u into an infinite series of recursive components through iterations.

In this chapter, the homotopy perturbation method will be applied to nonlinear partial differential equations. This method involves a special representation for nonlinear terms such as u^2, u^3, u^4, $\sin u$, e^u, $u u_x$, u_x^2... . The method introduces a formal algorithm in representing nonlinear terms, and it is necessary to represent nonlinear terms in proper form [14].

In the following sections, representations of nonlinear terms are illustrated with examples, and an alternate algorithm for calculating homotopy polynomials will be outlined with examples.

3.1.1: Calculation of Homotopy Polynomials

It is well known now that homotopy perturbation method suggests that the unknown linear function u may be represented by the decomposition series

$$u = \sum_{n=0}^{\infty} p^n u_n \quad , \tag{1}$$

The nonlinear term $N(u)$, such as u^2, u^3, u^4, $\sin u$, e^u, $u u_x$, u_x^2, etc. can be expressed by an infinite series of the so-called homotopy polynomials $H_n(u)$ given in the form

$$N(u) = \sum_{n=0}^{\infty} H_n(u_0, u_1, u_2, \cdots, u_n), \tag{2}$$

where $H_n(u)$ the homotopy polynomials.

In literature, several strategies have been introduced to calculate homotopy polynomials. An alternate reliable method which employs only elementary operations

and does not require specific formulas has been reported. This alternate method that is based on algebraic & trigonometric identities and on Taylor series.
The homotopy polynomials $H_n(u)$ can be found by the following expression

$$H_n(u) = \frac{1}{n!}\,\frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{n} p^i\,u_i\right)\right]_{p=0}, \quad n = 0\,, 1\,, 2, \cdots. \qquad (3)$$

The general formula Eq. (3) can be formulated as following;

$$\begin{aligned}
H_0 &= N(u_0),\\
H_1 &= u_1 N'(u_0),\\
H_2 &= u_2 N'(u_0) + \frac{1}{2!}u_1^2\, N''(u_0),\\
H_3 &= u_3 N'(u_0) + u_1\, u_2\, N'(u_0) + \frac{1}{3!}u_1^3\, N'''(u_0),\\
H_4 &= u_4 N'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1\, u_3\right) N''(u_0) + \frac{1}{2!}u_1^2\, u_2\, N'''(u_0) + \frac{1}{4!}u_1^4\, N^{(4)}(u_0).
\end{aligned} \qquad (4)$$

Other polynomials can be generated in a similar manner.

Two important observations can be made here. First, H_0 depends only on u_0, H_1 depends only on u_0 and u_1, H_2 depends only on u_0, u_1 and u_2, and so on. Second, substituting Eq. (4) into Eq. (2) gives;

$$\begin{aligned}
N(u) &= H_0 + H_1 + H_2 + H_3 + \cdots\\
&= N(u_0) + (u_1 + u_2 + u_3 + \cdots)\, N'(u_0)\\
&\quad + \frac{1}{2!}\left(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + \cdots\right) N''(u_0)\\
&\quad + \frac{1}{3!}\left(u_1^3 + 3u_1^3u_2 + 3u_1^2u_3 + 6u_1u_2u_3 + \cdots\right) N'''(u_0) + \cdots\\
&= N(u_0) + (u - u_0)\, N'(u_0) + \frac{1}{2!}(u - u_0)^2\, N''(u_0) + \cdots
\end{aligned}.$$

The homotopy polynomials given above in Eq. (4) clearly show that the sum of the subscripts of the components of u of each term of $H_n(u)$ is equal to *n*. As stated before, it is clear that H_0 depends only on u_0, H_1 depends only u_0 and u_1, H_2 depends only on u_0, u_1 and u_2. The same conclusion holds for other polynomials.

In the following section, an attempt is made to calculate homotopy polynomials for different forms of nonlinearity that may arise in nonlinear ordinary or partial differential equations.

3.1.2: Calculation of Homotopy Polynomials H_n

I: Nonlinear Polynomials

Case 1: $N(u) = u^2$

The polynomials can be obtained as follows:

$$H_0 = N(u_0) = u_0^2,$$
$$H_1 = u_1 N'(u_0) = 2u_0 u_1,$$
$$H_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0) = 2u_0 u_2 + u_1^2,$$
$$H_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 F''''(u_0) = 2u_0 u_3 + 2u_1 u_2.$$

Case 2: $N(u) = u^3$

The polynomials are given by;

$$H_0 = N(u_0) = u_0^3,$$
$$H_1 = u_1 N'(u_0) = 3u_0^2 u_1,$$
$$H_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0) = 3u_0^2 u_2 + 3u_0 u_1^2,$$
$$H_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0) = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3.$$

Case 3: $N(u) = u^4$

Proceeding as before we find

$$H_0 = N(u_0) = u_0^4,$$
$$H_1 = u_1 N'(u_0) = 4u_0^4 u_1,$$
$$H_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0) = 4u_0^3 u_2 + 6u_0^2 u_1^2,$$
,
$$H_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0) = 4u_0^3 u_3 + 4u_0^3 u_0 + 12u_0^2 u_1 u_2.$$

In a parallel manner, homotopy polynomials can be calculated for nonlinear polynomials of higher degrees.

II: Nonlinear Derivatives

Case 1: $N(u) = (u_x)^2$

The homotopy polynomials are given by;

$$H_0 = u_{0_x}^2,$$
$$H_1 = 2u_{0_x}u_{1_x},$$
$$H_2 = 2u_{0_x}u_{2_x} + u_{1_x}^2,$$
$$H_3 = 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x}.$$

Case 2: $N(u) = u_x^3$

The homotopy polynomials are given by;

$$H_0 = u_{0_x}^3,$$
$$H_1 = 3u_{0_x}^2 u_{2_x} + 3u_{0_x}u_{1_x}^2,$$
$$H_2 = 3u_{0_x}^2 u_{2_x} + 3u_{0_x}u_{1_x}^2,$$
$$H_3 = 3u_{0_x}^2 u_{3_x} + 6u_{0_x}u_{1_x}u_{2_x} + u_{1_x}^3.$$

Case 3: $N(u) = u\,u_x = \frac{1}{2}L_x\left(u^2\right)$

The homotopy polynomials for this nonlinearity are given by;

$$H_0 = N(u_0) = u_0\,u_{0_x},$$
$$H_1 = \frac{1}{2}L_x(2u_0\,u_1) = u_{0_x}u_1 + u_0\,u_{1_x},$$
$$H_2 = \frac{1}{2}L_x\left(2u_0\,u_2 + u_1^2\right) = u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0,$$
$$H_3 = \frac{1}{2}L_x(2u_0\,u_3 + 2u_1\,u_2) = u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0.$$

III: Trigonometric Nonlinearity

Case 1: $N(u) = \sin u$

The homotopy polynomials of this form of nonlinearity are given by;

$$H_0 = \sin u_0,$$
$$H_1 = u_1 \cos u_0,$$
$$H_2 = u_2 \cos u_0 - \frac{1}{2!}u_1^2 \sin u_0,$$
$$H_3 = u_3 \cos u_0 - u_1\,u_2 \sin u_0 - \frac{1}{3!}u_1^3 \cos u_0.$$

Case 2: $N(u) = \cos u$

Proceeding as before giving;

$$H_0 = \cos u_0,$$
$$H_1 = -u_1 \sin u_0,$$
$$H_2 = -u_2 \sin u_0 - \frac{1}{2!} u_1^2 \cos u_0,$$
$$H_3 = -u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{3!} u_1^3 \sin u_0.$$

IV: Hyperbolic Nonlinearity

Case 1: $N(u) = \sinh u$

The H_n polynomials of this form of nonlinearity are given by;

$$H_0 = \sinh u_0,$$
$$H_1 = u_1 \cosh u_0,$$
$$H_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1^2 \sinh u_0,$$
$$H_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 - \frac{1}{3!} u_1^3 \cosh u_0.$$

Case 2: $N(u) = \cosh u$

The homotopy polynomials are given by;

$$H_0 = \cosh u_0,$$
$$H_1 = u_1 \sinh u_0,$$
$$H_2 = u_2 \sinh u_0 + \frac{1}{2!} u_1^2 \cosh u_0,$$
$$H_3 = u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!} u_1^3 \sinh u_0.$$

V: Exponential Nonlinearity

Case 1: $N(u) = e^u$

The homotopy polynomials of this form of nonlinearity are given by;

$$H_0 = e^{u_0},$$
$$H_1 = u_1 e^{u_0},$$
$$H_2 = \left(u_2 + \frac{1}{2!} u_1^2 \right) e^{u_0},$$
$$H_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1^3 \right) e^{u_0}.$$

Case 2: $N(u) = e^{-u}$

Proceeding as before gives;

$$H_0 = e^{-u_0},$$
$$H_1 = -u_1 e^{-u_0},$$
$$H_2 = \left(-u_2 + \frac{1}{2!}u_1^2\right) e^{-u_0},$$
$$H_3 = \left(-u_3 + u_1 u_2 - \frac{1}{3!}u_1^3\right) e^{-u_0}.$$

VI: Logarithmic Nonlinearity

Case 1: $N(u) = \ln u \;,\; u > 0$

The H_n polynomials for logarithmic nonlinearity are given by;

$$H_0 = \ln u_0,$$
$$H_1 = \frac{u_1}{u_0},$$
$$H_2 = \frac{u_2}{u_0} - \frac{u_1^2}{2\,u_0^2},$$
$$H_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{u_1^3}{3\,u_0^3}.$$

Case 2: $N(u) = \ln(1+u) \;,\; -1 < u \le 1$

The H_n polynomials are given by;

$$H_0 = \ln(1 + u_0),$$
$$H_1 = \frac{u_1}{1 + u_0},$$
$$H_2 = \frac{u_2}{u_0} - \frac{u_1^2}{2\,u_0^2},$$
$$H_3 = \frac{u_3}{1 + u_0} - \frac{u_1 u_2}{(1 + u_0)^2} + \frac{u_1^3}{3\,(1 + u_0)^3}.$$

3.1.3: Alternative Algorithm for Calculating Homotopy Polynomials

It is important to note that various practical techniques that may calculate homotopy polynomials in a practical way without the use of special formulae were attempted by many researchers. However, the methods developed so far in doing so are same as that of homotopy. Therefore, there is a need of simple and reliable technique for calculation.

In this section, alternate algorithms that may be used to calculate homotopy polynomials for nonlinear terms are presented in an easier way [14, 15]. The methods depend mainly on algebraic and trigonometric identities and on Taylor expansion.

Moreover, we should use the fact that the sum of subscripts of the components of u in each term of the polynomial H_n is equal to n. The alternative algorithm suggests that we substitute u as a sum of components u_n , $n \geq 0$ as defined by the decomposition method. It is clear that H_0 is always determined independent of the other polynomials H_n , $n \geq 1$, where H_0 is defined by;

$$H_0 = N(u_0). \tag{5}$$

The alternative method assumes that we first separate $H_0 = N(u_0)$ for every nonlinear term $N(u)$.With this separation done, the remaining components of $N(u)$ can be expanded by using algebraic operations, trigonometric identities, and Taylor series as well. We next collect all terms of the expansion obtained such that the sum of the subscripts of the components of u in each term is the same. Having collected these terms, the calculation of the homotopy polynomials is thus completed. Several examples have been tested, and the obtained results have shown that homotopy polynomials can be elegantly computed without any need to the formulas established by homotopy. The technique will be explained by discussing the following illustrative examples.

3.1.4: Homotopy Polynomials by Using the Alternative Method

I: Nonlinear Polynomials

Case 1: $N(u) = u^2$

We first set

$$u = \sum_{n=0}^{\infty} p^n u_n \,. \tag{6}$$

Substituting Eq. (6) into $N(u) = u^2$ gives;

$$N(u) = (u_0 + u_1 + u_2 + u_3 + \cdots)^2 \,. \tag{7}$$

Expanding the expression at the right hand side gives;

$$N(u) = u_0^2 + 2u_0 u_1 + 2u_0 u_2 + u_1^2 + 2u_0 u_3 + 2u_1 u_2 + \cdots \,. \tag{8}$$

The expansion in Eq. (8) can be rearranged by grouping all terms with the sum of the Subscripts are the same. This means that we can rewrite Eq. (8) as;

$$N(u) = \underbrace{u_0^2}_{H_0} + \underbrace{2u_0 u_1}_{H_1} + \underbrace{2u_0 u_2 + u_1^2}_{H_2} + \underbrace{2u_0 u_3 + 2u_1 u_2}_{H_3} + \underbrace{2u_0 u_4 + 2u_1 u_3 + u_2^2 +}_{H_4} \cdots \tag{9}$$

This completes the determination of homotopy polynomials given by

$$\begin{aligned} H_0 &= u_0^2, \\ H_1 &= 2u_0 u_1, \\ H_2 &= 2u_0 u_2 + u_1^2, \\ H_3 &= 2u_0 u_3 + 2u_1 u_2. \end{aligned} \quad ,$$

Case 2: $N(u) = u^3$

Proceeding as before, we set;

$$u = \sum_{n=0}^{\infty} p^n u_n \,. \tag{10}$$

Substituting Eq. (10) into $N(u) = u^3$ gives;

$$N(u) = (u_0 + u_1 + u_2 + u_3 + \cdots)^3 . \tag{11}$$

Expanding the right hand side yields

$$\begin{aligned} N(u) = {} & u_0^3 + 3u_0^2 u_1 + 3u_0^2 u_2 + 3u_0 u_1^2 + 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3 \\ & + 3u_0^2 u_4 + 3u_1^2 u_2 + 3u_2^2 u_0 + 6u_0 u_1 u_3 + \cdots \end{aligned} \tag{12}$$

We can rewrite Eq. (12) as;

$$\begin{aligned} N(u) = {} & \underbrace{u_0^3}_{H_0} + \underbrace{3u_0^2 u_1}_{H_1} + \underbrace{3u_0^2 u_2 + 3u_0 u_1^2}_{H_2} + \underbrace{3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3}_{H_3} \\ & + \underbrace{3u_0^2 u_4 + 3u_1^2 u_2 + 3u_2^2 u_0 + 6u_0 u_1 u_3}_{H_4} + \cdots \end{aligned} \,. \tag{13}$$

Consequently, homotopy polynomials can be written by;

$$\begin{aligned} H_0 &= u_0^3, \\ H_1 &= 3u_0^2 u_1, \\ H_2 &= 3u_0^2 u_2 + 3u_0 u_1^2, \\ H_3 &= 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3. \end{aligned},$$

II: Nonlinear Derivatives

Case 1: $N(u) = u_x^2$

We first set

$$u_x = \sum_{n=0}^{\infty} p^n u_{n_x} \quad . \tag{14}$$

Substituting Eq. (14) into $N(u) = u_x^2$ gives;

$$N(u) = (u_{0_x} + u_{1_x} + u_{2x} + u_{3x} + \cdots)^2 . \tag{15}$$

Squaring the right side gives;

$$N(u) = u_{0x}^2 + 2u_{0x} u_{1x} + 2u_{0x} u_{2x} + u_{1x}^2 + 2u_{0x} u_{3x} + 2u_{1x} u_{2x} + \cdots. \tag{16}$$

Grouping the terms as discussed above we find

$$N(u) = \underbrace{u_{0_x}^2}_{H_0} + \underbrace{2u_{0_x} u_{1_x}}_{H_1} + \underbrace{2u_{0_x} u_{2_x} + u_{1_x}^2}_{H_2} + \underbrace{2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x}}_{H_3} + \underbrace{2u_{0x} u_{4_x} + 2u_{1_x} u_{3_x} + u_{2_x}^2}_{H_4} + \cdots \tag{17}$$

Homotopy polynomials are given by;

$$\begin{aligned} H_0 &= u_{0_x}^2, \\ H_1 &= 2u_{0_x} u_{1_x}^2, \\ H_2 &= 2u_{0_x} u_{2_x} + u_{1_x}^2, \\ H_3 &= 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x}, \\ H_4 &= 2u_{0_x} u_{4_x} + 2u_{1_x} u_{3_x} + u_{2_x}^2. \end{aligned}$$

Case 2: $N(u) = u\, u_x$

We first set

$$\begin{aligned} u &= \sum_{n=0}^{\infty} p^n u_n \\ u_x &= \sum_{n=0}^{\infty} p^n u_{n_x} \end{aligned} \tag{18}$$

Substituting Eq. (18) into $N(u) = u\, u_x$ yields;

$$N(u) = (u_0 + u_1 + u_2 + u_3 + \cdots) \times (u_{0_x} + u_{1_x} + u_{2x} + u_{3x} + \cdots) . \tag{19}$$

Multiplying the two factors gives;

$$\begin{aligned} N(u) &= u_0 u_{0_x} + u_{0_x} u_1 + u_0 u_{1_x} + u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0 + u_{0_x} u_3 \\ &+ u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0 + u_{0_x} u_4 + u_0 u_{4_x} + u_{1_x} u_3 \\ &+ u_1 u_{3_x} + u_2 u_{2_x} + \cdots \end{aligned} \quad . \tag{20}$$

Proceeding with grouping the terms we obtain;

$$\begin{aligned} N(u) &= \underbrace{u_0 u_{0_x}}_{H_0} + \underbrace{u_{0_x} u_1 + u_0 u_{1_x}}_{H_1} + \underbrace{u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0}_{H_2} \\ &+ \underbrace{u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0}_{H_3} \\ &+ \underbrace{u_{0_x} u_4 + u_0 u_{4_x} + u_{1_x} u_3 + u_1 u_{3_x} + u_2 u_{2_x}}_{H_4} + \cdots \end{aligned} \tag{21}$$

It then follows that homotopy polynomials are given by;

$$\begin{aligned} H_0 &= u_0 u_{0_x}, \\ H_1 &= u_{0_x} u_1 + u_0 u_{1_x}, \\ H_2 &= u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \\ H_3 &= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0. \end{aligned}$$

III: Trigonometric Nonlinearity

Case 1: $N(u) = \sin u$

Note that algebraic operations cannot be applied here. Therefore, our main aim is to separate $H_0 = N(u_0)$ from other terms. To achieve this goal, we first substitute

$$u = \sum_{n=0}^{\infty} p^n u_n \ , \tag{22}$$

Into $N(u) = \sin u$ to obtain;

$$N(u) = \sin[u_0 + (u_1 + u_2 + u_3 + u_4 + \cdots)] \quad . \tag{23}$$

To calculate H_0, recall the trigonometric identity;

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \quad . \tag{24}$$

Accordingly, Equation (23) becomes;

$$\begin{aligned} N(u) &= \sin u_0 \cos(u_1 + u_2 + u_3 + u_4 + \cdots) \\ &+ \cos u_0 \sin(u_1 + u_2 + u_3 + u_4 + \cdots) \end{aligned} \quad . \tag{25}$$

Separating $N(u_0) = \sin u_0$ from other factors and using Taylor expansions for $\cos(u_1 + u_2 \cdots)$ and $\sin(u_1 + u_2 \cdots)$ give;

$$\begin{aligned} N(u) &= \sin u_0 \left(1 - \frac{1}{2!}(u_1 + u_2 + \cdots)^2 + \frac{1}{4!}(u_1 + u_2 + \cdots)^4 - \cdots\right) \\ &+ \cos u_0 \left((u_1 + u_2 + \cdots) - \frac{1}{3!}(u_1 + u_2 + \cdots)^3 + \cdots \right) \end{aligned} \tag{26}$$

So that

$$N(u) = \sin u_0 \left(1 - \frac{1}{2!}\left(u_1^2 + 2u_1u_2 + \cdots\right) + \cdots\right) + \cos u_0 \left((u_1 + u_2 + \cdots) - \frac{1}{3!}u_1^3 + \cdots\right) \tag{27}$$

On expanding the terms algebraically, few terms of each expansion are below. The last expansion can be rearranged by grouping all terms with the same subscripts. Equation (27) can be rewritten as:

$$N(u) = \underbrace{\sin u_0}_{H_0} + \underbrace{u_1 \cos u_0}_{H_1} + \underbrace{(u_2 \cos u_0 - \frac{1}{2!}u_1^2 \sin u_0)}_{H_2} + \underbrace{(u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!}u_1^3 \sin u_0)}_{H_4} + \cdots \tag{28}$$

Case 2: $N(u) = \sin u$

Proceeding as before we obtain;

$$N(u) = \underbrace{\cos u_0}_{H_0} - \underbrace{u_1 \sin u_0}_{H_1} + \underbrace{(- u_2 \sin u_0 - \frac{1}{2!}u_1^2 \cos u_0)}_{H_2} + \underbrace{(- u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{3!}u_1^3 \sin u_0)}_{H_4} + \cdots \tag{29}$$

IV: Hyperbolic Nonlinearity

Case 1: $N(u) = \sinh u$

To calculate the H_n polynomials for $N(u) = \sinh u$, we first substitute;

$$u = \sum_{n=0}^{\infty} p^n u_n \quad , \tag{30}$$

Into $N(u) = \sinh u$ to obtain;

$$N(u) = \sinh\left[u_0 + (u_1 + u_2 + u_3 + u_4 + \cdots)\right] \quad . \tag{31}$$

To calculate H_0, recall the hyperbolic identity;

$$\sinh(\theta + \phi) = \sinh\theta \cos\phi + \cosh\theta \sinh\phi \quad . \tag{32}$$

Accordingly, Eq. (31) becomes;

$$N(u) = \sinh u_0 \cosh(u_1 + u_2 + u_3 + u_4 + \cdots) + \cosh u_0 \sinh(u_1 + u_2 + u_3 + u_4 + \cdots) \quad . \tag{33}$$

Separating $N(u_0)=\sinh u_0$ from other factors and using Taylor expansions for $\cosh(u_1+u_2\cdots)$and $\sinh(u_1+u_2\cdots)$ give;

$$\begin{aligned} N(u) &= \sinh u_0\left(1-\frac{1}{2!}(u_1+u_2+\cdots)^2+\frac{1}{4!}(u_1+u_2+\cdots)^4-\cdots\right) \\ &+ \cosh u_0\left((u_1+u_2+\cdots)+\frac{1}{3!}(u_1+u_2+\cdots)^3+\cdots\right) \\ &= \sinh u_0\left(1-\frac{1}{2!}(u_1^2+2u_1u_2+\cdots)+\cdots\right) \\ &+ cohsu_0\left((u_1+u_2+\cdots)+\frac{1}{3!}u_1^3+\cdots\right) \end{aligned}.$$

By grouping all terms with the same sum of subscripts we find

$$\begin{aligned} N(u) &= \underbrace{\sinh u_0}_{H_0}+\underbrace{u_1\cosh u_0}_{H_1}+\underbrace{(u_2\cosh u_0+\frac{1}{2!}u_1^2\sinh u_0)}_{H_2} \\ &+ \underbrace{(u_3\cosh u_0+u_1u_2\sinh u_0+\frac{1}{3!}u_1^3\sinh u_0)}_{H_4}+\cdots \end{aligned}. \qquad (34)$$

Case 2: $N(u)=\cosh u$

Proceeding as in $\sinh x$ we find

$$\begin{aligned} N(u) &= \underbrace{\cosh u_0}_{H_0}+\underbrace{u_1\sinh u_0}_{H_1}+\underbrace{(u_2\sinh u_0+\frac{1}{2!}u_1^2\cosh u_0)}_{H_2} \\ &+ \underbrace{(u_3\sinh u_0+u_1u_2\cosh u_0+\frac{1}{3!}u_1^3\sinh u_0)}_{H_4}+\cdots \end{aligned}. \qquad (35)$$

V: Exponential Nonlinearity

Case 1: $N(u)=e^u$

$$u=\sum_{n=0}^{\infty}p^n u_n \quad , \qquad (36)$$

Into $N(u)=e^u$ gives;

$$N(u)=e^{(u_0+u_1+u_2+u_3+u_4+\cdots)} \quad , \qquad (37)$$

Or equivalently;

$$N(u)=e^{u_0}e^{(u_1+u_2+u_3+u_4+\cdots)} \quad . \qquad (38)$$

Keeping the term e^{u_0} and using the Taylor expansion for the other factor we obtain;

$$N(u) = e^{u_0} \times \left(1 + (u_1 + u_2 + u_3 + u_4 + \cdots) + \frac{1}{2!}(u_1 + u_2 + u_3 + u_4 + \cdots)^2\right). \quad (39)$$

By grouping all terms with identical sum of subscripts we find;

$$\begin{aligned} N(u) = &\underbrace{e^{u_0}}_{H_0} + \underbrace{u_1 e^{u_0}}_{H_1} + \underbrace{\left(u_2 + \frac{1}{2!}u_1^2\right) e^{u_0}}_{H_3} + \underbrace{\left(u_3 + u_1 u_2 + \frac{1}{3!}u_1^3\right) e^{u_0}}_{H_3} \\ &+ \underbrace{\left(u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 + \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4\right) e^{u_0}}_{H_4} + \cdots \end{aligned} . \quad (40)$$

Case 2: $N(u) = e^{-u}$

Proceeding as before we find

$$\begin{aligned} N(u) = &\underbrace{e^{-u_0}}_{H_0} + \underbrace{(-u_1)e^{-u_0}}_{H_1} + \underbrace{\left(-u_2 + \frac{1}{2!}u_1^2\right) e^{-u_0}}_{H_3} + \underbrace{\left(-u_3 + u_1 u_2 - \frac{1}{3!}u_1^3\right) e^{-u_0}}_{H_3} \\ &+ \underbrace{\left(-u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 - \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4\right) e^{-u_0}}_{H_4} + \cdots \end{aligned} . \quad (41)$$

VI: Logarithmic Nonlinearity

Case 1: $N(u) = \ln u \;, \; u > 0$

Substituting

$$u = \sum_{n=0}^{\infty} p^n u_n \,, \quad (42)$$

Into $N(u) = \ln u$ gives;

$$N(u) = \ln(u_0 + u_1 + u_2 + u_3 + u_4 + \cdots) \quad . \quad (43)$$

Equation (43) can be written as;

$$N(u) = \ln\left(u_0 \cdot \left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right)\right) \quad . \quad (44)$$

Using the fact that $\ln(\alpha\beta) = \ln\alpha + \ln\beta$, Equation (44) becomes;

$$N(u) = \ln u_0 + \ln\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots\right) \quad . \quad (45)$$

Separating $N(u_0) = \ln u_0$ and using the Taylor expansion of the remaining term we obtain;

$$N(u) = \ln u_0 + \left\{ \left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots \right) - \frac{1}{2} \left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots \right)^2 \right.$$
$$\left. + \frac{1}{3} \left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots \right)^3 - \frac{1}{4} \left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \cdots \right)^4 + \cdots \right\} . \quad (46)$$

Proceeding as before, Equation (46) can be written as;

$$N(u) = \underbrace{\ln u_0}_{H_0} + \underbrace{\frac{u_1}{u_0}}_{H_1} + \underbrace{\frac{u_2}{u_0} - \frac{1}{2} \frac{u_1^2}{u_0^2}}_{H_2} + \underbrace{\frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{1}{3} \frac{u_1^3}{u_0^3}}_{H_3} + \cdots \quad . \quad (47)$$

Case 2: $N(u) = \ln(1+u) \ , \quad -1 < u \leq 1$

In a like manner we obtain;

$$N(u) = \underbrace{\ln(1+u_0)}_{H_0} + \underbrace{\frac{u_1}{1+u_0}}_{H_1} + \underbrace{\frac{u_2}{1+u_0} - \frac{1}{2} \frac{u_1^2}{(1+u_0)^2}}_{H_2}$$
$$+ \underbrace{\frac{u_3}{1+u_0} - \frac{u_1 u_2}{(1+u_0)^2} + \frac{1}{3} \frac{u_1^3}{(1+u_0)^3}}_{H_3} + \cdots \quad . \quad (48)$$

As mentioned before, there are other methods to evaluate homotopy polynomials, but disadvantage of methods is prolonged calculations. For this reason, the most commonly used methods are presented.

3.2: Homotopy perturbation and Sumudu Transform Method (HPSTM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of form

$$DU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \tag{49}$$
$$U(x,0) = h(x), \qquad U_t(x,0) = f(x)$$

Where D is the second order linear differential operator, $D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term [15].

Taking Sumudu transform of both sides of Eq. (49), we get;

$$S[DU(x,t)] + S[RU(x,t)] + S[NU(x,t)] = S[g(x,t)] \tag{50}$$

Using the differential operator property of the Sumudu transform and above initial conditions, we get;

$$\begin{aligned} S[DU(x,t)] &= u^2 S[g(x,t)] + h(x) + u\, f(x) \\ &\quad - u^2 S[RU(x,t)] + S[NU(x,t)] \end{aligned} \tag{51}$$

Now, applying the inverse Sumudu transform of both sides of Esq. (51), we get;

$$U(x,t) = G(x,t) - S^{-1}\left[u^2\, S[RU(x,t) + NU(x,t)]\right] \tag{52}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We apply the homotopy perturbation method;

$$U(x,t) = \sum_{n=0}^{\infty} p^n\, U_n(x,t) \tag{53}$$

And the nonlinear term can be decomposed as;

$$NU(x,t) = \sum_{n=0}^{\infty} p^n\, H_n(x,t) \tag{54}$$

For some He's polynomials $H_n(U)$ that are given by;

$$H_n(U_0, U_1, U_2, \cdots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i\, U_i(x,t)\right)\right]_{p=0}, \quad n = 0, 1, 2, 3, \cdots \tag{55}$$

Substituting Eqs. (53) and (54) in Eq. (52) we get;

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x,t) &= G(x,t) \\ &\quad - p\left(S^{-1}\left[u^2\, S\left[R\sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(x,t)\right]\right]\right) \end{aligned} \tag{56}$$

This is the coupling of the Sumudu transform and the homotopy perturbation method using He's polynomials.

Comparing the coefficient of like power of p, the following approximation is obtained;

$$\begin{aligned}
p^0 &: U_0(x,t) = G(x,t) \\
p^1 &: U_1(x,t) = -S^{-1}\left[u^2 S\left[RU_0(x,t) + H_0(U)\right]\right] \\
p^2 &: U_2(x,t) = -S^{-1}\left[u^2 S\left[RU_1(x,t) + H_1(U)\right]\right] \\
p^3 &: U_3(x,t) = -S^{-1}\left[u^2 S\left[RU_2(x,t) + H_2(U)\right]\right] \\
&\;\;\vdots
\end{aligned} \tag{57}$$

Example (3.2.1): Consider the following nonlinear advection problem

$$U_t + U U_x = 0 \tag{58}$$

$$U(x,0) = -x.$$

Taking Sumudu transform of both sides of Eq. (58) subject to the initial Condition, we get;

$$S[U(x,t)] = -x - u\,S[U U_x] \tag{59}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = -x - S^{-1}\left[u\,S[U U_x]\right] \tag{60}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = -x - p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{61}$$

Where $H_n(U)$ are He's polynomials that represents the nonlinear terms.

The first few components of He's polynomials, are given by;

$$\begin{aligned}
H_0(U) &= U_0 U_{0x} \\
H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\
H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \\
H_3(U) &= U_0 U_{3x} + U_1 U_{2x} + U_2 U_{1x} + U_3 U_{0x} \\
&\;\;\vdots
\end{aligned} \tag{62}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = -x \\ p^1 &: U_1(x,t) = -S^{-1}[u\, S[H_0(U)]] = -xt \\ p^2 &: U_2(x,t) = -S^{-1}[u\, S[H_1(U)]] = -xt^2 \\ p^3 &: U_3(x,t) = -S^{-1}[u\, S[H_2(U)]] = -xt^3 \end{aligned} \tag{63}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = -x\left(1 + t + t^2 + t^3 + \cdots\right) \tag{64}$$

And in a closed form by;

$$U(x,t) = \frac{x}{t-1} \tag{65}$$

Example (3.2.2): Consider a nonlinear partial differential equation

$$U_t = x^2 + \frac{1}{4}U_x^2 \tag{66}$$

$$U(x,0) = 0.$$

Taking Sumudu transform of both sides of Eq. (66) subject to the initial Condition, we get;

$$S[U(x,t)] = x^2 u + \frac{1}{4}u\, S\left[U_x^2\right] \tag{67}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = x^2 t + \frac{1}{4}S^{-1}\left[u\, S\left[U_x^2\right]\right] \tag{68}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x^2 t + \frac{1}{4}p\left(S^{-1}\left[u\, S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{69}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by

$$\begin{aligned} H_0(U) &= U_{0x}^2 \\ H_1(U) &= 2U_{0x}U_{1x} \\ H_2(U) &= 2U_{0x}U_{2x} + U_{1x}^2 \\ H_3(U) &= 2U_{0x}U_{3x} + 2U_{1x}U_{2x} \end{aligned} \tag{70}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = x^2 t \\ p^1 &: U_1(x,t) = \frac{1}{4} S^{-1}[u\, S[H_0(U)]] = \frac{1}{3} x^2 t^3 \\ p^2 &: U_2(x,t) = \frac{1}{4} S^{-1}[u\, S[H_1(U)]] = \frac{2}{15} x^2 t^5 \\ p^3 &: U_3(x,t) = \frac{1}{4} S^{-1}[u\, S[H_2(U)]] = \frac{17}{315} x^2 t^7 \end{aligned} \tag{71}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = x^2 \left(t + \frac{1}{3} t^3 + \frac{2}{15} t^5 + \frac{17}{315} t^7 + \cdots \right) \tag{72}$$

And in a closed form of;

$$U(x,t) = x^2 \tan t \tag{73}$$

Example (3.2.3): Consider a nonlinear partial differential equation,

$$U_{tt} - U_t U_{xx} = -t + U \tag{74}$$

$$\begin{aligned} U(x,0) &= \sin x \\ U_t(x,0) &= 1 \end{aligned}.$$

Taking Sumudu transform of both sides of Eq. (74) subject to the initial Condition, we get;

$$S[U(x,t)] = u + \sin x - u^3 + u^2\, S[U + U_t U_{xx}] \tag{75}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = t + \sin x - \frac{t^3}{6} + S^{-1}[u^2\, S[U + U_t U_{xx}]] \tag{76}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = t + \sin x - \frac{t^3}{6} + p\left(S^{-1}\left[u^2\, S\left[\sum_{n=0}^{\infty} p^n (U_n + B_n(U)) \right]\right]\right) \tag{77}$$

Where $B_n(U)$ the homotopy polynomials are represents the nonlinear term $U_t U_{xx}$.

To use the modified decomposition method, we identify the component U_0 by $U_0 = t + \sin x + -\frac{t^3}{6}$, and remaining term $\frac{t^3}{6}$ will be assigned $U_1(x,t)$ among other terms. Consequently, we obtain the recursive relation;

$$U_0(x,t) = t + \sin x - \frac{t^3}{6}$$
$$U_1(x,t) = S^{-1}\left[u^2 S\left[U_0 + B_0(U)\right]\right]$$
$$U_{n+1}(x,t) = S^{-1}\left[u^2 S\left[U_n + B_n(U)\right]\right], \qquad n \geq 1$$

Consequently, we obtain;

$$U_0(x,t) = t + \sin x - \frac{t^3}{6}$$
$$U_1(x,t) = S^{-1}\left[u^2 S\left[U_0 + B_0(U)\right]\right] = \frac{t^3}{6}$$

The exact solution is;

$$U(x,t) = t + \sin x \tag{78}$$

Example (3.2.4): Consider a nonlinear partial differential equation,

$$U_{tt} + \frac{1}{4}U_x^2 = U \tag{79}$$
$$U(x,0) = 1 + x^2$$
$$U_t(x,0) = 1 \quad .$$

Taking Sumudu transform of both sides of Eq. (79) subject to the initial Condition, we get;

$$S[U(x,u)] = x^2 + 1 + u + u^2 S\left[U - \frac{1}{4}U_x^2\right] \tag{80}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = x^2 + 1 + u + S^{-1}\left[u^2 S\left[U - \frac{1}{4}U_x^2\right]\right] \tag{81}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x^2 + 1 + u + p\left(S^{-1}\left[u^2 S\left[\sum_{n=0}^{\infty} p^n\left(U - \frac{1}{4}H_n(U)\right)\right]\right]\right) \tag{82}$$

Where $H_n(U)$ the homotopy polynomials are represented the nonlinear term U_x^2.
The decomposition method admits the use recursive relation;

$$U_0(x,t) = x^2 + 1 + t$$
$$U_{k+1}(x,t) = S^{-1}\left[u^2 S\left[U_k - \frac{1}{4}H_k(U)\right]\right], \qquad k \geq 1$$

The homotopy polynomials are given by;

$$H_0 = u_{0_x}^2,$$
$$H_1 = 2u_{0_x}u_{1_x}^2,$$
$$H_2 = 2u_{0_x}u_{2_x} + u_{1_x}^2,$$
$$H_3 = 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x},$$
$$H_4 = 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x} + u_{2_x}^2.$$

Comparing the coefficients of like powers of p, we get;

$$p^0 \ : U_0(x,t) = x^2 + 1 + t$$
$$p^1 \ : U_1(x,t) = S^{-1}\left[u^2 S\left[U_0 - \frac{1}{4}H_0(U)\right]\right] = \frac{t^2}{2!} + \frac{t^3}{3!}$$
$$p^2 \ : U_2(x,t) = S^{-1}\left[u^2 S\left[U_1 - \frac{1}{4}H_1(U)\right]\right] = \frac{t^4}{4!} + \frac{t^5}{5!}$$
$$p^3 \ : U_3(x,t) = S^{-1}\left[u^2 S\left[U_2 - \frac{1}{4}H_2(U)\right]\right] = \frac{t^6}{6!} + \frac{t^7}{7!}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = x^2 + \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)$$

And in closed form given as;

$$U(x,t) = x^2 + e^t \tag{83}$$

Example (3.2.5): Consider a nonlinear partial differential equation,

$$U_{tt} + U^2 - U_x^2 = 0 \tag{84}$$
$$U(x,0) = 0$$
$$U_t(x,0) = e^x.$$

Taking Sumudu transform of both sides of Eq. (84) subject to the initial Condition, we get;

$$S[U(x,u)] = u\ e^x + u^2 S\left[U_x^2 - U^2\right] \tag{85}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = t\,e^x + S^{-1}\left[u^2 S\left[U_x^2 - U^2\right]\right] \tag{86}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = t\,e^x + p\left(S^{-1}\left[u^2\, S\left[\sum_{n=0}^{\infty} p^n \left(B_n(U) - A_n(U)\right)\right]\right]\right) \quad (87)$$

Where $B_n(U)$, $A_n(U)$ are the homotopy polynomials that represents the nonlinear term U_x^2 and U^2 respectively. We next set the recursive relation

$$U_0(x,t) = t\,e^x$$

$$U_{k+1}(x,t) = S^{-1}\left[u^2\, S\left[B_k(U) - A_k(U)\right]\right], \qquad k \geq 1$$

The homotopy polynomials are given by;

$$B_0 = u_{0_x}^2 \quad , \quad A_0 = u_0^2 ,$$

$$B_1 = 2u_{0_x} u_{1_x}^2 \quad , \quad A_1 = 2u_0 u_1^2 ,$$

$$B_2 = 2u_{0_x} u_{2_x} + u_{1_x}^2 \quad , \quad A_2 = 2u_0 u_2 + u_1^2 ,$$

$$B_3 = 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x} \quad , \quad A_3 = 2u_0 u_3 + 2u_1 u_2$$

The first few components of the solution $U(x,t)$ are given by;

$$p^0 : U_0(x,t) = t\, e^x$$

$$p^1 : U_1(x,t) = S^{-1}\left[u^2\, S\left[B_0(U) - A_0(U)\right]\right] = 0$$

And therefore other components vanish. Consequently, the exact solution is given

$$U(x,t) = t\, e^x \quad (88)$$

Example (3.2.6): consider the following non homogenous advection problem,

$$U_t + \frac{1}{36} x U_{xx}^2 = x^3 \quad (89)$$

$$U(x,0) = 0$$

Taking Sumudu transform of both sides of Eq. (89) subject to the initial condition, we get;

$$S[U(x,t)] = x3u - \frac{1}{36} u\, S\left[x U_{xx}^2\right] \quad (90)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = x^3 t - \frac{1}{36} S^{-1}\left[u\, S\left[x U_{xx}^2\right]\right] \quad (91)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x^3 t - \frac{1}{36} p\left(S^{-1}\left[u\, S\left[x \sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \quad (92)$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by

$$\begin{aligned}
H_0(U) &= U_{0xx}^2 \\
H_1(U) &= 2U_{0xx}U_{1xx} \\
H_2(U) &= 2U_{0xx}U_{2xx} + U_{1xx}^2 \\
&\vdots
\end{aligned} \tag{93}$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = x^3 t \\
p^1 &: U_1(x,t) = -\frac{1}{36}S^{-1}[u\,S[x\,H_0(U)]] = -\frac{1}{3}x^3t^3 \\
p^2 &: U_2(x,t) = -\frac{1}{36}S^{-1}[u\,S[x\,H_1(U)]] = \frac{2}{15}x^3t^5 \\
p^3 &: U_3(x,t) = -\frac{1}{36}S^{-1}[u\,S[x\,H_2(U)]] = -\frac{17}{315}x^3t^7
\end{aligned} \tag{94}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = x^3\left(t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots\right) \tag{95}$$

And in a closed form of,

$$U(x,t) = x^2 \tanh t \tag{96}$$

Example (3.2.7): Consider the following homogenous advection problem,

$$U_t + U^2U_x = 0 \tag{97}$$

$$U(x,0) = 2x.$$

Taking Sumudu transform of both sides of Eq. (97) subject to the initial Condition, we get;

$$S[U(x,t)] = 2x - u\,S[U^2U_x] \tag{98}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = 2x - S^{-1}[u\,S[U^2U_x]] \tag{99}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 2x - p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{100}$$

Where $H_n(U)$ the homotopy polynomials are represented the nonlinear term, U^2U_x. This gives the recursive relation,

$$U_0(x,t) = 2x$$
$$U_{k+1}(x,t) = -S^{-1}[u\,S\,[H_k(U)]], \qquad k \geq 1 \tag{101}$$

This gives the first few components of $U(x,t)$ as,

$$p^0 \ : U_0(x,t) = 2x$$
$$p^1 \ : U_1(x,t) = -S^{-1}[u\,S[\,H_0(U)]] = -8x^2 t$$
$$p^2 \ : U_2(x,t) = -S^{-1}[u\,S[\,H_1(U)]] = 64x^3 t^2 \tag{102}$$
$$p^3 \ : U_3(x,t) = -S^{-1}[u\,S[\,H_2(U)]] = -640x^4 t^3$$

And so on. It follows that the solution in a series form is given by,

$$U(x,t) = 2x - 8x^2t + 64x^3t^2 - 640x^4t^3 + \cdots \tag{103}$$

Two observations can be made here. First, we can easily observe that

$$U(x,t) = 2x \quad , \qquad t = 0 \tag{104}$$

That satisfies the initial condition. We next observe that for $t > 0$, the series solution in Eq. (103) can be formally expressed in a closed form by:

$$U(x,t) = \frac{1}{4t}\left(\sqrt{1+16xt} - 1\right) \tag{105}$$

Combining Eq. (104) and Eq. (19) gives the solution in the form;

$$U(x,t) = \begin{cases} 2x & t = 0 \\ \dfrac{1}{4t}\left(\sqrt{1+16xt} - 1\right), & t > 0 \end{cases} \tag{106}$$

Example (3.2.8): Consider the following homogenous advection problem,

$$U_t + U\,U_x = 0 \tag{107}$$
$$U(x,0) = \sin x\,.$$

Taking Sumudu transform of both sides of Eq. (107) subject to the initial Condition, we get;

$$S[U(x,t)] = \sin x - u\,S[U\,U_x] \tag{108}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = \sin x - S^{-1}[u\,S[U\,U_x]] \tag{109}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \sin x - p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{110}$$

Where $H_n(U)$ are He's polynomials that represented the nonlinear terms.

The first few components of He's polynomials, are given by

$$\begin{aligned} H_0(U) &= U_0\,U_{0x} \\ H_1(U) &= U_0\,U_{1x} + U_1\,U_{0x} \\ H_2(U) &= U_0\,U_{2x} + U_1\,U_{1x} + U_2\,U_{0x} \\ &\vdots \end{aligned} \tag{111}$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned} p^0 &: U_0(x,t) = \sin x \\ p^1 &: U_1(x,t) = -\,S^{-1}[u\,S[H_0(U)]] = -\,t\sin x\cos x \\ p^2 &: U_2(x,t) = -\,S^{-1}[u\,S[H_1(U)]] = \left(\sin x\cos^2 x - \frac{1}{2}\sin^3 x\right)t^2 \end{aligned} \tag{112}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = \sin x - t\sin x\cos x + \left(\sin x\cos^2 x - \frac{1}{2}\sin^3 x\right)t^2 + \cdots \tag{113}$$

However, by using the traditional method of characteristics, we can show that the solution can be expressed in the parametric form:

$$\begin{aligned} U(x,t) &= \sin\zeta, \\ \zeta &= x - t\sin\zeta \end{aligned} \tag{114}$$

For numerical approximations, the series solution obtained above is more effective and practical compared to the parametric form solution given in Eq. (114).

3.3: Solving System of Nonlinear Partial Differential Equations

Example (3.3.9): Consider the following system of partial differential equations;

$$U_t + U_x + 2V = 0$$
$$V_t + V_x - 2U = 0 \tag{115}$$

With the initial conditions;

$$U(x,0) = \cos x$$
$$V(x,0) = \sin x$$

Taking Sumudu transform of Eq. (115) subject to the initial conditions, we get;

$$S[U(x,t)] = \cos x - u\,S[2V + U_x]$$
$$S[V(x,t)] = \sin x + u\,S[2U - V_x] \tag{116}$$

The inverse Sumudu transform implies that:

$$U(x,t) = \cos x - S^{-1}[u\,S[2V + U_x]]$$
$$V(x,t) = \sin x + S^{-1}[u\,S[2U + V_x]] \tag{117}$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \cos x - p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n \left(2V_n + [U_n]_x\right)\right]\right]\right\}$$
$$\sum_{n=0}^{\infty} p^n V_n(x,t) = \sin x + p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n \left(2U_n - [V_n]_x\right)\right]\right]\right\} \tag{118}$$

Comparing the coefficients of like power p , we get;

$$p^0 : U_0(x,t) = \cos x \quad , \quad V_0(x,t) = \sin x$$
$$p^1 : U_1(x,t) = -t\sin x \quad , \quad V_1(x,t) = t\cos x$$
$$p^2 : U_2(x,t) = -\frac{t^2}{2!}\cos x \ , \quad V_2(x,t) = -\frac{t^2}{2!}\sin x \tag{119}$$
$$p^3 : U_3(x,t) = \frac{t^3}{3!}\sin x \quad , \quad V_3(x,t) = -\frac{t^3}{3!}\cos x$$

And so on, using (119) we obtain;

$$U(x,t) = \cos x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right),$$
$$V(x,t) = \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + \cos x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) \tag{120}$$

This has an exact analytical solution of the form

$$(U,V) = (\cos(x+t)\,,\,\sin(x+t)) \tag{121}$$

Example (3.3. 10): Consider the following system of nonlinear partial differential equations,

$$U_t + VU_x + U = 1$$
$$V_t + UV_x - V = 1 \qquad (122)$$

With the initial conditions;

$$U(x,0) = e^x$$
$$V(x,0) = e^{-x}$$

Taking Sumudu transform of Eq. (122) subject to the initial conditions, we get;

$$S[U(x,t)] = e^x - u\,S[VU_x + U - 1]$$
$$S[V(x,t)] = e^{-x} - u\,S[UV_x - V - 1] \qquad (123)$$

The inverse Sumudu transform implies that:

$$U(x,t) = e^x - S^{-1}[u\,S[VU_x + U - 1]]$$
$$V(x,t) = e^{-x} - S^{-1}[u\,S[UV_x - V - 1]] \qquad (124)$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = e^x - p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right\}$$
$$\sum_{n=0}^{\infty} p^n V_n(x,t) = e^{-x} - p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(V)\right]\right]\right\} \qquad (125)$$

Where, $H_n(U)$, $H_n(V)$ are He's polynomials that represented the nonlinear terms.

$$H_n(U) = p[VU_x + U - 1]$$
$$H_n(V) = p[UV_x - V - 1]$$

Where,

$$U = U_0 + pU_1 + p^2U_2 + \cdots$$
$$V = V_0 + pV_1 + p^2V_2 + \cdots$$

The first few components of He's polynomials, are given by,

$$H_0(U) = V_0U_{0x} + U_0 - 1 \quad , \quad H_0(V) = U_0V_{0x} - V_0 - 1$$
$$H_1(U) = V_0U_{1x} + V_1U_{0x} + U_1 \quad , \quad H_1(V) = U_0V_{1x} + U_1V_{0x} - V_1 \qquad (126)$$
$$H_2(U) = V_0U_{2x} + V_1U_{1x} + V_2U_{0x} + U_2 \quad , \quad H_2(V) = U_0V_{2x} + U_1V_{1x} + U_2V_{0x} - V_2$$

Comparing the coefficients of the same powers of *p*, we get;

$$p^0 : U_0(x,t) = e^x \qquad , V_0(x,t) = e^{-x}$$

$$p^1 : U_1(x,t) = -S^{-1}[u\,S[H_0(U)]] = -t\,e^x \;, V_1(x,t) = -S^{-1}[u\,S[H_0(V)]] = t\,e^{-x}$$

$$p^2 : U_2(x,t) = -S^{-1}[u\,S[H_1(U)]] = \frac{t^2}{2!}e^x \;, V_2(x,t) = -S^{-1}[u\,S[H_1(V)]] = \frac{t^2}{2!}e^{-x} \qquad (127)$$

$$p^3 : U_3(x,t) = -S^{-1}[u\,S[H_2(U)]] = -\frac{t^3}{3!}e^x \;, V_3(x,t) = -S^{-1}[u\,S[H2(V)]] = \frac{t^3}{3!}e^{-x}$$

And so on, using (127) we obtain;

$$U(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = e^{x-t} \;,$$
$$V(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = e^{t-x} \qquad (128)$$

This has an exact analytical solution of the form:

$$(U,V) = \left(e^{x-t} \;,\; e^{t-x}\right) \qquad (129)$$

Example (3.3.11): Consider the following system of nonlinear partial differential equations,

$$U_t + V_x W_y - V_y W_x = -U$$
$$V_t + W_x U_y + W_y V_x = V \qquad (130)$$
$$W_t + U_x V_y + U_y V_x = W$$

With the initial conditions;

$$U(x, y, 0) = e^{x+y}$$
$$V(x, y, 0) = e^{x-y}$$
$$W(x, y, 0) = e^{-x+y}$$

Taking Sumudu transform of Eq. (130) subject to the initial conditions, we get;

$$S[U(x, y, t)] = e^{x+y} - u\,S[V_y W_x - V_x W_y - U]$$
$$S[V(x, y, t)] = e^{x-y} - u\,S[V - W_x U_y - W_y U_x] \qquad (131)$$
$$S[W(x, y, t)] = e^{-x+y} - u\,S[W - U_x V_y - U_y V_x]$$

The inverse Sumudu transform implies that:

$$U(x, y, t) = e^{x+y} - S^{-1}[u\,S[V_y W_x - V_x W_y - U]]$$
$$V(x, y, t) = e^{x-y} - S^{-1}[u\,S[V - W_x U_y - W_y U_x]] \qquad (132)$$
$$W(x, y, t) = e^{-x+y} - S^{-1}[u\,S[W - U_x V_y - U_y V_x]]$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x, y, t) = e^{x+y} - p\left\{S^{-1}\left[u S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right\}$$
$$\sum_{n=0}^{\infty} p^n V_n(x, y, t) = e^{x-y} - p\left\{S^{-1}\left[u S\left[\sum_{n=0}^{\infty} p^n H_n(V)\right]\right]\right\} \quad (133)$$
$$\sum_{n=0}^{\infty} p^n W_n(x, y, t) = e^{-x+y} - p\left\{S^{-1}\left[u S\left[\sum_{n=0}^{\infty} p^n H_n(W)\right]\right]\right\}$$

Where $H_n(U),\ H_n(V),\ H_n(W)$ are He's polynomials that represents the nonlinear terms,

$$H_n(U) = p\left[V_y W_x - V_x W_y - U\right]$$
$$H_n(V) = p\left[V - W_x U_y - W_y U_x\right]$$
$$H(W) = p\left[W - U_x V_y - U_y V_x\right]$$

Where,

$$U = U_0 + pU_1 + p^2 U_2 + \cdots$$
$$V = V_0 + pV_1 + p^2 V_2 + \cdots$$
$$W = W_0 + pW_1 + p^2 W_2 + \cdots$$

Comparing the coefficients of the same powers of *p*, we get;

$$p^0 : U_0(x, y, t) = e^{x+y}, V_0(x, y, t) = e^{x-y}, W_0(x, y, t) = e^{-x+y}$$
$$H_0(U) = e^{x+y}, \quad H_0(V) = e^{x-y}, \quad H_0(W) = e^{-x+y}$$
$$p^1 : U_1(x, y, t) = -S^{-1}\left[u S\left[H_0(U)\right]\right] = -t\, e^{x+y};$$
$$p^1 : V_1(x, y, t) = -S^{-1}\left[u S\left[H_0(U)\right]\right] = t\, e^{x-y};$$
$$p^1 : W_1(x, y, t) = -S^{-1}\left[u S\left[H_0(U)\right]\right] = t\, e^{-x+y};$$
$$p^2 : U_2(x, y, t) = -S^{-1}\left[u S\left[H_1(U)\right]\right] = \frac{t^2}{2!} e^{x+y}; \quad (134)$$
$$p^2 : V_2(x, y, t) = -S^{-1}\left[u S\left[H_1(U)\right]\right] = \frac{t^2}{2!} e^{x-y};$$
$$p^2 : W_2(x, y, t) = -S^{-1}\left[u S\left[H_1(U)\right]\right] = \frac{t^2}{2!} e^{-x+y};$$

And so on, using (134) we obtain;

$$U(x,y,t)=e^{x+y}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right)=e^{x+y-t},$$
$$V(x,y,t)=e^{x-y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)=e^{x-y+t} \tag{135}$$
$$W(x,y,t)=e^{-x+y}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)=e^{-x+y+t}$$

This has an exact analytical solution of the form;

$$(U,V,W)=\left(e^{x+y-t},\ e^{x-y+t},\ e^{-x+y+t}\right) \tag{136}$$

Example (3.3.12): Consider the following system of nonlinear partial differential equations,

$$\begin{aligned} U_t-U_{xx}-2UU_x+(UV)_x&=0\\ V_t-V_{xx}-2VV_x+(UV)_x&=0 \end{aligned} \tag{137}$$

With the initial conditions;

$$U(x,0)=\sin x$$
$$V(x,0)=\sin x$$

Taking Sumudu transform of Eq. (137) subject to the initial conditions, we get;

$$\begin{aligned} S[U(x,t)]&=\sin x+u\,S[U_{xx}+2UU_x-UV_x-VU_x]\\ S[V(x,t)]&=\sin x+u\,S[V_{xx}+2VV_x-UV_x-VU_x] \end{aligned} \tag{138}$$

The inverse Sumudu transform implies that:

$$\begin{aligned} U(x,t)&=\sin x+S^{-1}\{u\,S[U_{xx}+2UU_x-UV_x-VU_x]\}\\ V(x,t)&=\sin x+S^{-1}\{u\,S[V_{xx}+2VV_x-UV_x-VU_x]\} \end{aligned} \tag{139}$$

Now applying the homotopy perturbation method, we get;

$$\begin{aligned} \sum_{n=0}^{\infty}p^n U_n(x,t)&=\sin x+p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty}p^n H_n(U)\right]\right]\right\}\\ \sum_{n=0}^{\infty}p^n V_n(x,t)&=\sin x+p\left\{S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty}p^n H_n(V)\right]\right]\right\} \end{aligned} \tag{140}$$

Where $H_n(U),\ H_n(V)$ are He's polynomials that represents the nonlinear terms.

$$H_n(U)=p[U_{xx}+2UU_x-UV_x-VU_x]$$
$$H_n(V)=p[V_{xx}+2VV_x-UV_x-VU_x]$$

Where

$$U = U_0 + pU_1 + p^2 U_2 + \cdots$$
$$V = V_0 + pV_1 + p^2 V_2 + \cdots$$

The first few components of He's polynomials, are given by,

$$\begin{aligned}
H_0(U) &= U_{0xx} + 2U_0 U_{0x} - U_0 V_{0x} - V_0 U_{0x} \\
H_0(V) &= V_{0xx} + 2V_0 V_{0x} - U_0 V_{0x} - V_0 U_{0x} \\
H_1(U) &= U_{1xx} + 2U_0 U_{1x} + 2U_1 U_{0x} - U_0 V_{1x} - U_1 V_{0x} - V_1 U_{0x} - V_0 U_{1x} \\
H_1(U) &= V_{1xx} + 2V_0 V_{1x} + 2V_1 V_{0x} - U_0 V_{1x} - U_1 V_{0x} - V_1 U_{0x} - V_0 U_{1x} \\
&\;\;\vdots
\end{aligned} \tag{141}$$

Comparing the coefficients of the same powers of *p*, we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = \sin x \quad , \quad V_0(x,t) = \sin x \\
p^1 &: U_1(x,t) = S^{-1}[u\,S[H_0(U)]] = -t \sin x; \\
p^1 &: V_1(x,t) = S^{-1}[u\,S[H_0(V)]] = -t \sin x; \\
p^2 &: U_2(x,t) = S^{-1}[u\,S[H_1(U)]] = \frac{t^2}{2!} \sin x; \\
p^2 &: U_2(x,t) = S^{-1}[u\,S[H_1(U)]] = \frac{t^2}{2!} \sin x; \\
p^3 &: U_3(x,t) = -S^{-1}[u\,S[H_2(U)]] = -\frac{t^3}{3!} \sin x; \\
p^3 &: V_3(x,t) = -S^{-1}[u\,S[H_2(V)]] = -\frac{t^3}{3!} \sin x; \\
&\;\;\vdots
\end{aligned} \tag{142}$$

And so on, using (142) we obtain;

$$\begin{aligned}
U(x,t) &= \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \sin x = e^{-t} \sin x \,; \\
V(x,t) &= \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \sin x = e^{-t} \sin x \,;
\end{aligned} \tag{143}$$

This has an exact analytical solution of the form:

$$(U,V) = \left(e^{-t} \sin x \,,\; e^{-t} \sin x\right) \tag{144}$$

CHAPTER FOUR

Linear and Nonlinear Fractional Differential Equations and Sumudu Transform

4.1: Linear Fractional Differential Equations

Fractional calculus provides an efficient and an excellent way of describing many dynamical phenomena in scientific and engineering areas such as physics, chemistry, and economics [19]. This feature of fractional calculus has appealed many researchers in the past. In this chapter, a new method called homotopy perturbation Sumudu transform method (HPSTM) is introduced for solving the linear and initial value problems. This method is a combination of Sumudu transform, homotopy perturbation method.

The following section offers the effectiveness of the homotopy perturbation Sumudu transform method (HPSTM) in solving fractional initial boundary value (FIBVP).

4.1.1: Preliminaries and Notations

In this section, we give some basic definitions and properties of the fractional calculus theory which are further used in this chapter.

Definition (4.1.1) [16]:

A real function $f(x), x>0$ is said to be in space $C_\mu, \mu \in \Re$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$.

Definition (4.1.2) [16]:

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as;

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)du \quad , \quad \alpha > 0 \tag{1}$$

$$J^0 f(t) = f(t).$$

Some properties of the operator J^{α}, which are needed here, are as following:

For $f^{n} \in C_{\mu}, n \in N$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$:

i. $J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$

ii. $J^{\alpha} t^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}$ (2)

Definition (4.1.3) [18]:

The Sumudu transform of the Caputo fractional derivative is defined as follows:

$$S\left[D_t^{\alpha} f(t)\right] = u^{-\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0^{+}), \quad m-1 < \alpha \leq m$$

Definition (4.1.4) [16]:

The fractional derivative of $f(t)$ in the Caputo sense is defined as;

$$D^{\alpha} f(t) = J^{m-\alpha} D^{m} f(t) \tag{3}$$

For $m-1 < \alpha \leq m, m \in N, t > 0$ and $f \in C_{-1}^{m}$

Caputo fractional derivative initially calculates an ordinary derivative and then followed by fractional integration to a desired order of fractional derivative.

Similar to the integer-order integration, the Riemann-Liouville fractional integral operator is a linear operation:

$$J^{\alpha}\left(\sum_{i=1}^{n} c_i f_i(t)\right) = \sum_{i=1}^{n} c_i J^{\alpha} f_i(t) \tag{4}$$

Where $\{c_i\}_{i=1}^{n}$ are constants.

In the present work, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition, as pointed by [35], is as follows: to therefore familiar to us. In contrast, for the Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives solve differential equations (both classical and fractional); we need to specify additional conditions in order to produce a unique solution. For the case of the Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are (and/or integrals) of the unknown solution at the initial point $x = 0$, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details see [35].

4.1.2: Sumudu Transform

The Sumudu transform is a powerful tool in applied mathematics and engineering. Virtually every beginning course in differential equations at the undergraduate level introduces this technique for solving linear differential equations. The Sumudu transform is indispensable in certain areas of control theory.

Given a function $f(x)$ defined for $0 < x < \infty$, the Sumudu transform $F(u)$ is defined as;

$$F(u) = \int_0^\infty f(ux)\, e^{-x}\, dx \tag{5}$$

At least for those s for which the integral converges.

Let $f(x)$ be a continuous function on the interval $[0\,,\,\infty)$ which is of exponential order, that is, for some $c \in \Re$ and $x > 0$

$$\sup \frac{|f(x)|}{e^{cx}} < \infty .$$

In this case the Sumudu transform Eq. (5), exists for all $\frac{1}{u} > c$.

Some of the useful Sumudu transforms which are applied in this section are as follows:

For $S[f(x)] = F(u)$ and $S[g(x)] = G(u)$

$$S[f(x) + g(x)] = F(u) + G(u),$$

$$S[x^\beta] = u^\beta\, \Gamma(\beta+1) \quad , \quad \beta > -1,$$

$$S[f^{(n)}(x)] = \frac{F(u)}{u^n} - \frac{f(0)}{u^n} - \frac{f'(0)}{u^{n-1}} - \cdots - \frac{f^{(n-1)}(0)}{u} \tag{6}$$

$$S\left[\int_0^x f(t)dt\right] = u\,F(u)$$

$$S\left[\int_0^x f(x-t)\, g(t)\, dt\right] = u\,F(u)G(u). \tag{7}$$

Lemma (4.1.5):

The Sumudu transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ can be obtained in the form of:

$$S\left[J^{\alpha} f(x)\right] = u^{\alpha} F(u).$$

Proof:

The Sumudu transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ is:

$$S\left[J^{\alpha} f(x)\right] = S\left[\frac{1}{\Gamma(\alpha)}\int_0^x (x-\alpha)^{\alpha-1} f(t)\, dt\right] = \frac{1}{\Gamma(\alpha)} u F(u) G(u),$$

Where

$$G(u) = S\left[x^{\alpha-1}\right] = u^{\alpha-1}\Gamma(\alpha)$$

Lemma (4.1.6):

The Sumudu transform of Caputo fractional derivative for $m-1 < \alpha \leq m$, $m \in N$, can be obtained in the form of:

$$S\left[D^{\alpha} f(x)\right] = u^{m-\alpha}\left[\frac{F(u)}{u^m} - \frac{f(0)}{u^m} - \frac{f'(0)}{u^{m-1}} - \cdots - \frac{f^{(m-1)}(0)}{u}\right]$$

Proof:

The Sumudu transform of the Caputo fractional derivative of order $\alpha > 0$ is:

$$S\left[D^{\alpha} f(x)\right] = S\left[J^{m-\alpha} f^{(m)}(x)\right] = u^{m-\alpha} S\left[f^{(m)}(x)\right].$$

Using equation (64). Now, we can transform fractional differential equations into algebraic equations and then by solving this algebraic equation, we can obtain the unknown Sumudu function $F(u)$.

4.1.3: Inverse Sumudu Transform

The function $f(x)$ in Eq. (5), is called the inverse Sumudu transform of $F(u)$ and will be denoted by $f(x) = S^{-1}\left[F(u)\right]$ in the section. In practice, when one uses the Sumudu transform to, for example, solve a differential equation, one has to at some Point invert the Sumudu transform by finding the function $f(x)$ which corresponds to some specified $F(u)$[17].

The Inverse Sumudu transform of $F(u)$ is defined as:

$$f(x) = S^{-1}[F(u)] = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} u\,F(u)e^{\frac{x}{u}}\,du\,,$$

Where, σ large enough that is $F(u)$ is defined for the real part of $\frac{1}{u} \geq \sigma$ surprisingly, this formula isn't really useful. Therefore, in this section some useful function $f(x)$ is obtained from their Sumudu transform. In the first we define the most important special functions used in fractional calculus the Mittag-Leffler functions and the generalized Mittag-Leffler functions.

For $\alpha\,,\ \beta > 0$ and $z \in C$

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha + 1)}\,,$$

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha + \beta)}\,.$$

Now, we prove some Lemmas which are useful for finding the function $f(x)$ from its Sumudu transform.

Lemma (4.1.7):

For $a \in C$ and $\frac{1}{u^{\alpha}} > |a|$ we have the following inverse Sumudu transform formula,

$$S^{-1}\left[\frac{u^{\beta-1}}{1 + a u^{\alpha}}\right] = x^{\beta-1}E_{\alpha,\beta}\left(-a x^{\alpha}\right)\,.$$

Proof:

By using the series expansion can be rewritten $\frac{u^{\beta}}{1 + a u^{\alpha}}$ as;

$$\frac{u^{\beta}}{1 + a u^{\alpha}} = u^{\beta}\ \frac{1}{1 + a u^{\alpha}} = u^{\beta}\sum_{n=0}^{\infty}\left(-a u^{\alpha}\right) = \sum_{n=0}^{\infty}(-a)^n\ u^{n\alpha+\beta-1}\,.$$

The inverse Sumudu transform of the above function is;

$$\sum_{n=0}^{\infty}\frac{(-a)^n\ u^{n\alpha+\beta-1}}{\Gamma(n\alpha + \beta)} = x^{\beta-1}\sum_{n=0}^{\infty}\frac{\left(-a x^{\alpha}\right)^n}{\Gamma(n\alpha + \beta)} = x^{\beta-1}E_{\alpha,\beta}\left(-\ a\ x^{\alpha}\right).$$

Lemma (4.1.8):

For $\alpha \geq \beta > 0$, $a \in \Re$ and α , $\beta > 0$ we get;

$$S^{-1}\left[\frac{u^{\alpha(n+1)-1}}{(1+a\,u^{\alpha-\beta})^{n+1}}\right] = x^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}\; x^{k(\alpha-\beta)} \;.$$

Proof:

Using the series expansion of $(1+x)^{-n-1}$ of the form:

$$\frac{1}{(1+x)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k}{k}(-x)^k$$

We get;

$$\frac{u^{\alpha(n+1)-1}}{(1+a u^{\alpha-\beta})^{n+1}} = u^{\alpha(n+1)-1}\frac{1}{(1+a u^{\alpha-\beta})^{n+1}} = u^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(-a\,u^{\alpha-\beta}\right)^k$$

Giving the inverse Sumudu transform of above function can prove the Lemma.

Lemma (4.1.9):

For $\alpha \geq \beta$, $\alpha > \gamma$, $a \in \Re$ and $\left|\dfrac{u^{\alpha-1}}{1+a u^{\alpha-\beta}}\right| > |b|$ we get;

$$S^{-1}\left[\frac{u^{\alpha+\beta-\gamma-1}}{u^{\beta}+a u^{\alpha}+b u^{\alpha+\beta}}\right] = x^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}\; x^{k(\alpha-\beta)+n\alpha}$$

Proof:

$\dfrac{u^{\alpha+\beta-\gamma-1}}{u^{\beta}+a u^{\alpha}+b u^{\alpha+\beta}}$ By using the series expansion can be rewritten as;

$$\frac{u^{\alpha+\beta-\gamma-1}}{u^{\beta}+a u^{\alpha}+b u^{\alpha+\beta}} = \frac{u^{\alpha+\beta-\gamma-1}}{u^{\beta}+a u^{\alpha}}\;\frac{1}{1+\dfrac{b u^{\alpha+\beta}}{u^{\beta}+a u^{\alpha}}} = \sum_{n=0}^{\infty}\frac{(-b)^n\,u^{\alpha+\beta-\gamma-1}}{u^{\beta}+a u^{\alpha}+b u^{\alpha+\beta}}$$

Now by using Lemma (4.1.8) the Lemma can be proved.

4.1.4: The Homotopy Perturbation Sumudu Transform Method

In order to elucidate the solution procedure of this method, we consider a general fractional nonlinear partial differential equation of the form:

$$D_t^{\alpha} U(x,t) = LU(x,t) + NU(x,t) + q(x,t) \tag{8}$$

With $n-1 < \alpha \le n$, and subject to the initial condition;

$$\frac{\partial^{(r)} U(x,0)}{\partial t^{r}} = U^{(r)}(x,0) = f_r(x) \tag{9}$$

$$r = 0, 1, \ldots, n-1$$

Where $D_t^{\alpha} U(x,t)$ is the Caputo fractional derivative, $q(x,t)$ is the source term, **L** is the linear operator and N is the general nonlinear operator.

Taking the Sumudu transform (denoted throughout this section by S) on both sides of Eq. (8), we get;

$$S\left[D_t^{\alpha} U(x,t)\right] = S\left[LU(x,t) + NU(x,t) + q(x,t)\right] \tag{10}$$

Using the property of the Sumudu transform and the initial conditions in Eq. (9), we get;

$$u^{-\alpha} S\left[U(x,t)\right] - \sum_{k=0}^{n-1} u^{-(\alpha-k)} U^{(k)}(x,0) = S\left[LU(x,t) + NU(x,t) + q(x,t)\right] \tag{11}$$

And

$$S\left[U(x,t)\right] = \sum_{k=0}^{n-1} u^{k} f_k(x) + u^{\alpha} S\left[LU(x,t) + NU(x,t) + q(x,t)\right] \tag{12}$$

Operating with the Sumudu inverse on both sides of Eq. (12) we get;

$$U(x,t) = S^{-1} \sum_{k=0}^{n-1} u^{k} f_k(x) + S^{-1}\left\{u^{\alpha} S\left[LU(x,t) + NU(x,t) + q(x,t)\right]\right\} \tag{13}$$

Now, applying the classical perturbation technique. And assuming that the solution of Eq. (13) is in the form;

$$U(x,t) = \sum_{n=0}^{\infty} p^{n} U_n(x,t) \tag{14}$$

Where $p \in [0,1]$ is the homotopy parameter.

The nonlinear term of Eq. (13) can be decomposed

$$NU(x,t) = \sum_{n=0}^{\infty} p^{n} H_n(U) \tag{15}$$

Where H_i are He's polynomials, which can be calculated with the formula:

$$H_n(U_0, U_1, U_2, \ldots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i U_i\right)\right]_{p=0}, \quad n = 0,1,2,... \tag{16}$$

Substituting Eqs. (14) and (15) in Eq. (13), we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = S^{-1}\left[\sum_{k=0}^{n-1} u^k f_k(x)\right] + p S^{-1}\left[u^{\alpha} S\left[L\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right) + \sum_{n=0}^{\infty} p^n H_n(U) + q(x,t)\right]\right] \tag{17}$$

Equating the terms with identical powers of p ; we can obtain a series of equations as the follows:

$$\begin{aligned} p^0 &: U_0(x,t) = S^{-1}\left[\sum_{k=0}^{n-1} u^k f_k(x)\right] \\ &\quad \vdots \\ p^n &: U_n(x,t) = S^{-1}\left[u^{\alpha} S\left[L\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right) + \sum_{n=0}^{\infty} p^n H_n(U) + q(x,t)\right]\right] \end{aligned} \tag{18}$$

By utilizing the results in Eq. (18), and substituting them into Eq. (13) then the solution of Eq. (8), can be expressed as a power series in p. The best approximation for the solution of Eq. (9), is:

$$U(x,t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n U_n(x,t) = U_0 + U_1 + U_2 + \cdots \tag{19}$$

The solutions of Eq. (19) generally converge very rapidly.

4.1.5: Illustrative Examples

In this section is applied the method presented and give an exact solution of some linear fractional differential equations.

Example (4.1.10):

As the first example, we consider the following initial value problem in the case of the inhomogeneous Bagley-Torvik equation,

$$D^2 y(x) + D^{\frac{3}{2}} y(x) + y(x) = 1 + x, \tag{20}$$
$$y(0) = y'(0) = 1.$$

This equation by using Sumudu transform is converted to,

$$\frac{1}{u^2}\left[F(u) - y(0) - u\,y'(0)\right] + u^{2-\alpha}\left[\frac{F(u)}{u^2} - \frac{y(0)}{u^2} - \frac{y'(0)}{u}\right] + F(u) = 1 + u$$

$$\frac{1}{u^2}\left[F(u) - 1 - u\right] + u^{2-\alpha}\left[\frac{F(u)}{u^2} - \frac{1}{u^2} - \frac{1}{u}\right] + F(u) = 1 + u$$

$$F(u)\left[\frac{u^{\alpha} + u^2 + u^{\alpha+2}}{u^{\alpha+2}}\right] = \left[\frac{u^{\alpha} + u^2 + u^{\alpha+2}}{u^{\alpha+2}}\right] + u\left[\frac{u^{\alpha} + u^2 + u^{\alpha+2}}{u^{\alpha+2}}\right]$$

$$F(u) = 1 + u.$$

Using the inverse Sumudu transform the exact solution of this problem $y(x) = 1 + x$ can be obtained.

Example (4.1.11):

Our second example covers the inhomogeneous linear equation,

$$D^{\alpha} y(x) + y(x) = \frac{2\,x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x \tag{21}$$
$$y(0) = 0, \quad 0 < \alpha \le 1.$$

Using the Sumudu transform $F(u)$ is obtained as follows:

$$u^{1-\alpha}\left[\frac{F(u)}{u} - \frac{y(0)}{u}\right] + F(u) = 2u^{2-\alpha} - u^{1-\alpha} + 2u^2 - u$$

$$F(u)\left(u^{-\alpha} + 1\right) = 2u^2\left(u^{-\alpha} + 1\right) - u\left(u^{-\alpha} + 1\right)$$

$$F(u) = 2u^2 - u.$$

Then $y(x) = x^2 - x$ is obtained by using the inverse Sumudu transform.

Example (4.1.12):

Consider the following linear initial value problem,

$$D^{\alpha} y(x) + y(x) = 0 \tag{22}$$

$$y(0) = 1 \quad , \quad y'(0) = 0 .$$

The second initial condition is for $\alpha > 0$ only.

In two cases of α , $S[D^{\alpha} y(x)]$ is obtained as;

i. For $\alpha < 1$ $\quad S[D^{\alpha} y(x)] = u^{2-\alpha}\left[\frac{F(u)}{u^2} - \frac{1}{u^2}\right] = \frac{F(u)-1}{u^{\alpha}}$

ii. For $\alpha > 1$ $\quad S[D^{\alpha} y(x)] = u^{2-\alpha}\left[\frac{F(u)}{u^2} - \frac{1}{u^2}\right] = \frac{F(u)-1}{u^{\alpha}}$

Which are the same. Now the Sumudu transform $F(u)$ is obtained as;

$$\frac{F(u)-1}{u^{\alpha}} + F(u) = 0$$

$$F(u) = \frac{1}{1+u^{\alpha}}$$

Using the lemma (4.1.7), the exact solution of this problem can be obtained as:

$$y(x) = E_{\alpha}\left(-x^{\alpha}\right)$$

Example (4.1.13):

Consider the following linear initial value problem;

$$D^{\alpha} y(x) = y(x) + 1 \quad , \quad 0 < \alpha \le 1 \tag{23}$$

$$y(0) = 0 .$$

Using the Sumudu transform $F(u)$ is obtained as follows;

$$\frac{F(u)}{u^{\alpha}} = F(u) + 1$$

$$F(u) = \frac{u^{\alpha}}{1-u^{\alpha}}$$

Using the Lemma (4.1.7) the exact solution of this problem can be obtained as:

$$y(x) = x^{\alpha} E_{\alpha,\alpha+1}\left(x^{\alpha}\right)$$

Example (4.1.14):

Consider the composite fractional oscillation equation;

$$y''(x) - a D^{\alpha} y(x) - b\, y(x) = 8 \quad , \quad 1 < \alpha \le 2 \tag{24}$$

$$y(0) = y'(0) = 0 .$$

Using the Sumudu transform, $F(u)$ is obtained as follows;

$$\frac{F(u)}{u^2} - a u^{2-\alpha} \frac{F(u)}{u^2} - b F(u) = 8$$

$$F(u) = \frac{8\, u^{\alpha+2}}{u^{\alpha} - a u^2 - b u^{\alpha+2}}.$$

Using the lemma (4.1.9) the exact solution of this problem can be obtained as:

$$y(x) = 8 x^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^n\ a^k \binom{n+k}{k}}{\Gamma(k(2-\alpha) + 2(n+1) + 1)}\ x^{k(2-\alpha)+2n}$$

Example (4.1.15):

Consider the following system of fractional algebraic-differential equations,

$$D^{\alpha} x(t) - t\, y'(t) + x(t) - (1+t) y(x) = 0 \qquad , \quad 0 < \alpha \le 1 \tag{25}$$

$$y(t) - \sin t = 0,$$

Subject to the initial conditions;

$$x(0) = 1 \qquad , \qquad y(0) = 0.$$

Using the Sumudu transform $F(u) = S^{-1}[y(t)]$ and $G(u) = S^{-1}[x(t)]$ is obtained as follows

$$u^{1-\alpha}\left[\frac{G(u)}{u} - \frac{1}{u}\right] - u \frac{d}{du} F(u) + F(u) - G(u) - F(u) - u^2 \frac{d}{du} F(u) - u F(u) = 0$$

$$F(u) = \frac{u}{1+u^2} \quad , \quad F'(u) = \frac{1-u^2}{(1+u^2)^2}$$

$$G(u)\left(\frac{1+u^{\alpha}}{u^{\alpha}}\right) = \frac{2u\,(1+u)}{(1+u^2)^2} + u^{1-\alpha}$$

$$G(u) = \frac{2u^{\alpha+1}}{1+u^{\alpha}} \cdot \frac{1+u}{(1+u^2)^2} + \frac{u}{1+u^{\alpha}}.$$

The exact solution for $\alpha = 1$ is $x(t) = t \sin t + e^{-t}$. Using the Lemma (4.1.7) and (4.1.8) the exact solution for $0 < \alpha \le 1$ can be obtained as:

$$x(t) = 2x^{\alpha+1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n+k} (k+1)^k\, t^{2k}$$

$$\left(\frac{t^{n\alpha+1}}{\Gamma((n+1)\alpha + 2k + 3)} + \frac{t^{n\alpha}}{\Gamma((n+1)\alpha + 2k + 2)}\right) + E_{\alpha}(-t^{\alpha})$$

$$= 2\, t^{\alpha+1} \sum_{k=0}^{\infty} (-1)^{\left[\frac{k}{2}\right]} \left(\left[\frac{k}{2}\right] + 1\right) t^k\ E_{\alpha,\,\alpha+k+2}(-t^{\alpha}) + E_{\alpha}(-t^{\alpha})$$

$$y(t) = \sin t$$

Example (4.1.16): Consider the two-dimensional fractional wave equation of the form:

$$D_t^{\alpha}U(x,y,t) = 2\left(\frac{\partial^2 U(x,y,t)}{\partial x^2} + \frac{\partial^2 U(x,y,t)}{\partial y^2}\right) \tag{26}$$

Where $1 < \alpha < 2$, $-\infty < x$, $y < \infty$; subject to the initial condition;

$$U(x,y,0) = \sin x \sin y \ , \ \frac{\partial U(x,y,0)}{\partial t} = 0$$

Taking the Sumudu transform of both sides of Eq. (26), thus we get;

$$S[D_t^{\alpha}U(x,y,t)] = S\left[2\left(D_x^2 + D_y^2\right)U(x,y,t)\ \right] \tag{27}$$

Using the property of the Sumudu transform and the initial condition in Eq. (27), we get;

$$\begin{aligned} &u^{-\alpha}S[U(x,y,t)] - u^{-\alpha}U(x,y,0) \\ &\qquad + u^{1-\alpha}\frac{\partial U(x,y,0)}{\partial t} = S\left[2\left(D_x^2 + D_y^2\right)U(x,y,t)\ \right] \end{aligned} \tag{28}$$

And

$$S[U(x,y,t)] = \sin x \sin y + u^{\alpha}S\left[2\left(D_x^2 + D_y^2\right)U\ \right]$$

Operating with the Sumudu inverse on both sides of Eq. (28) we get;

$$U(x,y,t) = \sin x \sin y + S^{-1}\left[u^{\alpha}S\left[2\left(D_x^2 + D_y^2\right)U\ \right]\right] \tag{29}$$

By applying the homotopy perturbation method in Eq. (29) we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y,t) = \sin x \sin y + p S^{-1}\left\{u^{\alpha}S\left[2\left(D_x^2 + D_y^2\right)\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)\right]\right\} \tag{30}$$

Equating the terms with identical powers of p, we get;

$$\begin{aligned} &p^0:\ U_0(x,t) = \sin x \sin y \\ &p^1:\ U_1(x,t) = S^{-1}\left\{u^{\alpha}S\left[2\left(D_x^2 + D_y^2\right)U_1\right\} = \frac{-4\,t^{\alpha}}{\Gamma(\alpha+1)}\sin x \sin y \\ &p^2:\ U_2(x,t) = S^{-1}\left\{u^{\alpha}S\left[2\left(D_x^2 + D_y^2\right)U_2\right\} = \frac{4^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x \sin y \\ &\quad . \qquad\qquad\qquad . \qquad\qquad\qquad . \\ &p^n:\ U_n(x,t) = S^{-1}\left\{u^{\alpha}S\left[2\left(D_x^2 + D_y^2\right)U_n\right\} = \frac{(-1)^n 4^n t^{n\alpha}}{\Gamma(n\alpha+1)}\sin x \sin y \end{aligned} \tag{31}$$

Thus the solution of Eq. (26) is given by

$$
\begin{aligned}
U(x,t) &= \lim_{p \to 1} \sum_{n=0}^{\infty} p^n U_n(x,y,t) \\
&= (\sin x \sin y)\left(1 - \frac{4t^{\alpha}}{\Gamma(\alpha+1)} + \frac{4^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right) \\
&= \sin x \sin y \sum_{n=0}^{\infty} \frac{(-1)^n 4^n t^{n\alpha}}{\Gamma(n\alpha+1)} \\
&= \sin x \sin y \, E_{\alpha}(-4t^{\alpha})
\end{aligned}
\tag{32}
$$

If we put $\alpha \to 2$ in Eq. (32) or solve Eq. (26) with $\alpha = 2$, we obtain the exact solution

$$
\begin{aligned}
U(x,t) &= \sin x \sin y \sum_{n=0}^{\infty} \frac{(-1)^n (2t)^n}{\Gamma(n\alpha+1)} \\
&= \sin x \sin y \cos(2t)
\end{aligned}
$$

Example (4.1.17): Consider the following three-dimensional fractional heat-like equation:

$$
D_t^{\alpha} U(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right] \tag{33}
$$

Where $0 < x, y, z < 1,\ 0 < \alpha \le 1;$

Subject to the initial condition;

$$
U(x,y,z,0) = 0
$$

Taking the Sumudu transform of both sides of Eq. (33), thus we get;

$$
S\left[D_t^{\alpha} U(x,y,z,t)\right] = S\left[x^4 y^4 z^4 \frac{1}{36}\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right]\right] \tag{34}
$$

Using the property of the Sumudu transform and the initial condition in Eq. (34), we get;

$$
S[U(x,y,z,t)] = x^4 y^4 z^4 + u^{\alpha} S\left[\frac{1}{36}\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right]\right] \tag{35}
$$

Operating with the Sumudu inverse on both sides of Eq. (35) we get;

$$
U(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36} S^{-1}\left\{u^{\alpha} S\left[x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}\right]\right\} \tag{36}
$$

By applying the homotopy perturbation method in Eq. (36) we get;

$$
\begin{aligned}
\sum_{n=0}^{\infty} p^n U_n(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36} p\, S^{-1}\Bigg\{u^{\alpha} S\Bigg[x^2 \sum_{n=0}^{\infty} p^n (U_n)_{xx} \\
+ y^2 \sum_{n=0}^{\infty} p^n (U_n)_{yy} + z^2 \sum_{n=0}^{\infty} p^n (U_n)_{zz}\Bigg]\Bigg\}
\end{aligned}
\tag{37}
$$

Equating the terms with identical powers of p, we get;

$$p^0: \ U_0(x,t) = x^4 y^4 z^4$$

$$p^1: \ U_1(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{36}\left(x^2 (U_0)_{xx} + y^2 (U_0)_{yy} + z^2 (U_0)_{zz}\right)\right)\right]$$

$$= x^4 y^4 z^4 \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$p^2: \ U_2(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{36}\left(x^2 (U_1)_{xx} + y^2 (U_1)_{yy} + z^2 (U_1)_{zz}\right)\right)\right]$$

$$= x^4 y^4 z^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{38}$$

. .

. .

$$p^n: \ U_n(x,t) = S^{-1}\left[u^\alpha S\left(\frac{1}{36}\left(x^2 (U_{n-1})_{xx} + y^2 (U_{n-1})_{yy} + z^2 (U_{n-1})_{zz}\right)\right)\right]$$

$$= x^4 y^4 z^4 \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

Thus the solution of Eq. (33) is given by:

$$U(x,t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n U_n(x,t) = x^4 y^4 z^4 \left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right) \tag{39}$$

$$= x^4 y^4 z^4 E_\alpha(t^\alpha)$$

If we put $\alpha \to 1$ in Eq. (39) or solve Eq. (33) with $\alpha = 1$, we obtain the exact solution;

$$U(x,t) = x^4 y^4 z^4 e^t$$

Example (4.1.18): Consider the linear inhomogeneous fractional KdV equation,

$$D_t^\alpha U(x,t) + U_x(x,t) + U_{xxx}(x,t) = 2t \cos x \ , \ \ t > 0, \ \ 0 < \alpha \le 1 \tag{40}$$

Subject to the initial condition;

$$U(x,0) = 0$$

We can solve Eq. (40) by HPSTM by applying the Sumudu transform of both sides of Eq. (40), we obtain:

$$S\left[D_t^\alpha U(x,t)\right] + S\left[U_x(x,t) + U_{xxx}(x,t)\right] = S\left[2t \cos x\right] \tag{41}$$

Using the property of the Sumudu transform, we get;

$$S\left[U(x,t)\right] = U(x,0) - u^\alpha S\left[U_x(x,t) + U_{xxx}(x,t) - 2t \cos x\right] \tag{42}$$

Now applying the Sumudu inverse on both sides of Eq. (42) we obtain:

$$U(x,t) = 2\cos x \, \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - S^{-1}\left[u^{\alpha} \, S\left[U_x(x,t) + U_{xxx}(x,t)\right]\right] \qquad (43)$$

Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in P as given below:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \qquad (44)$$

Where the homotopy parameter p is considered as a small parameter $p \in [0\ ,\ 1]$. By substituting from Eq. (44) into Eq. (43) and using HPM we get:

$$\begin{aligned} U(x,t) &= 2\cos x \, \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &- p\, S^{-1}\left[u^{\alpha} \, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_x + \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xxx}\right]\right] \end{aligned} \qquad (45)$$

By equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as:

$$\begin{aligned} p^0 &: \ U_0(x,t) = 2\cos x \, \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ p^1 &: \ U_1(x,t) = 0 \\ p^2 &: \ U_2(x,t) = 0 \\ p^3 &: \ U_3(x,t) = 0 \\ &. \qquad\qquad . \end{aligned} \qquad (46)$$

The HPSTM series solution is;

$$U(x,t) = 2\cos x \, \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \qquad (47)$$

For the special case $\alpha = 1$, we obtain;

$$U(x,t) = t^2 \cos x \qquad (48)$$

Example (4.1.19): Consider the following system of linear FPDEs;

$$\begin{cases} D_t^{\alpha} U - V_x + V + U = 0 \\ D_t^{\alpha} V - U_x + V + U = 0 \end{cases} \qquad (49)$$

Where $0 < \alpha \ , \ \beta < 1$; subject to the initial condition;

$$U(x,0) = \sinh x \ , \ V(x,0) = \cosh x$$

Taking the Sumudu transform of both sides of Eq. (49), thus we get;

$$\begin{cases} S\left[D_t^{\alpha} U(x,t)\right] = S[V_x - V - U] \\ S\left[D_t^{\alpha} V(x,t)\right] = S[U_x - V - U] \end{cases} \tag{50}$$

Using the property of the Sumudu transform and the initial condition in Eq. (50), we get;

$$\begin{cases} u^{-\alpha}\, S[U(x,t)] + u^{-\alpha}\, U(x,0) = S[V_x - V - U] \\ u^{-\alpha}\, S[V(x,t)] + u^{-\alpha}\, V(x,0) = S[U_x - V - U] \end{cases} \tag{51}$$

And

$$\begin{cases} S[U(x,t)] = \sinh x + u^{\alpha}\, S[V_x - V - U] \\ S[V(x,t)] = \cosh x + u^{\alpha}\, S[U_x - V - U] \end{cases}$$

Operating with the Sumudu inverse on both sides of Eq. (51) we get;

$$\begin{cases} U(x,t) = \sinh x + S^{-1}\left[u^{\alpha}\, S[V_x - V - U]\right] \\ V(x,t) = \cosh x + S^{-1}\left[u^{\alpha}\, S[U_x - V - U]\right] \end{cases} \tag{52}$$

By applying the homotopy perturbation method in Eq. (52) we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \sinh x + p\, S^{-1}\left\{u^{\alpha} S\left[\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} q^n V_n(x,t)\right) - \left(\sum_{n=0}^{\infty} q^n V_n(x,t)\right) - \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)\right]\right\}$$

And (53)

$$\sum_{n=0}^{\infty} q^n V_n(x,t) = \cosh x + p\, S^{-1}\left\{u^{\alpha} S\left[\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right) - \left(\sum_{n=0}^{\infty} q^n V_n(x,t)\right) - \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)\right]\right\}$$

Equating the terms with identical powers of p, we get;

$$p^0: \ \{U_0(x,t) = \sinh x \quad , \quad V_0(x,t) = \cosh x$$

$$p^1: \ \begin{cases} U_1(x,t) = \dfrac{-t^{\alpha}}{\Gamma(\alpha+1)} \cosh x \ , \quad V_1(x,t) = \dfrac{-t^{\beta}}{\Gamma(\beta+1)} \sinh x \end{cases} \tag{54}$$

$$p^2: \ \begin{cases} U_2(x,t) = \dfrac{-t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \cosh x + \dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \sinh x + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \cosh x \\ V_2(x,t) = \dfrac{-t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \sinh x + \dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \cosh x + \dfrac{t^{2\beta}}{\Gamma(2\beta+1)} \sinh x \end{cases}$$

Thus the solution of Eq. (49) is given by;

$$\begin{aligned} U(x,t) &= \sinh x\left(1+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\cdots\right) - \cosh x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right) \\ V(x,t) &= \cosh x\left(1+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\cdots\right) - \sinh x\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\cdots\right) \end{aligned} \quad (55)$$

Setting $\alpha = \beta$ in Eq. (55) we reproduce the solution

$$\begin{aligned} U(x,t) &= \sinh x\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right) - \cosh x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(3\alpha+1)}+\cdots\right) \\ V(x,t) &= \cosh x\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right) - \sinh x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(3\alpha+1)}+\cdots\right) \end{aligned} \quad (56)$$

If we put $\alpha \to 1$ in Eq. (56) or solve Eq. (49) with $\alpha = \beta = 1$, we obtain the exact solution

$$\begin{aligned} U(x,t) &= \sinh x\left(1+\frac{t^{2}}{2!}+\frac{t^{4}}{4!}+\cdots\right) - \cosh x\left(t+\frac{t^{3}}{3!}+\frac{t^{5}}{5!}+\cdots\right) = \sinh(x-t) \\ V(x,t) &= \cosh x\left(1+\frac{t^{2}}{2!}+\frac{t^{4}}{4!}+\cdots\right) - \sinh x\left(t+\frac{t^{3}}{3!}+\frac{t^{5}}{5!}+\cdots\right) = \cosh(x-t) \end{aligned}$$

4.2: Nonlinear Fractional Differential Equations

In this section, the homotopy perturbation Sumudu transform method (HPSTM) is used to evaluate the exact analytical solution of nonlinear fractional partial differential equations [20].

Example (4.2.20): Consider the nonlinear time fractional FPE:

$$D_t^{\alpha} U(x,t) = \left[-\frac{\partial}{\partial x}\left(\frac{4}{x}U(x,t) - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}U(x,t)\right]U(x,t) \tag{57}$$

Where $t > 0;\ x \in \Re,\ 0 < \alpha \le 1$; subject to the initial condition;

$$U(x,0) = x^2$$

Taking the Sumudu transform of both sides of Eq. (57), thus we get;

$$S\left[D_t^{\alpha} U(x,t)\right] = S\left[\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}U(x,t) - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}U(x,t)\right]U(x,t)\right] \tag{58}$$

Using the property of the Sumudu transform and the initial condition in (58), we get;

$$S[U(x,t)] = x^2 + u^{\alpha} S\left[\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}U(x,t) - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}U(x,t)\right]U(x,t)\right] \tag{59}$$

Operating with the Sumudu inverse on both sides of Eq. (59) we get;

$$U(x,t) = x^2 + S^{-1}\left\{u^{\alpha} S\left[\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}U(x,t) - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}U(x,t)\right]U(x,t)\right]\right\} \tag{60}$$

By applying the homotopy perturbation method in Eq. (60) we get;

$$\begin{aligned}\sum_{n=0}^{\infty} p^n U_n(x,t) = x^2 + p\, S^{-1}\Bigg\{ u^{\alpha} S\Bigg[-4\sum_{n=0}^{\infty} p^n H_n(x,t) \\ + \frac{1}{3}\sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n B_n(x,t)\Bigg]\Bigg\}\end{aligned} \tag{61}$$

Where

$$H_n = \frac{\partial}{\partial x}\left(\frac{1}{x}U_n(x,t)\right)U_n(x,t)$$

$$B_n = \frac{\partial^2}{\partial x^2}U_n^2(x,t)$$

Equating the terms with identical powers of p, we get;

$$p^0:\ U_0(x,t) = x^2$$

$$p^1:\ U_1(x,t) = S^{-1}\left\{u^{\alpha} S\left[-4\ H_0 + \frac{1}{3}U_0 + B_0\right]\right\} \quad = x^2 \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2:\ U_2(x,t) = S^{-1}\left\{u^{\alpha} S\left[-4\ H_1 + \frac{1}{3}U_1 + B_1\right]\right\} \quad = x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{62}$$

$$\cdot \qquad\qquad \cdot \qquad\qquad \cdot$$

$$p^n:\ U_n(x,t) = S^{-1}\left\{u^{\alpha} S\left[-4\ H_{n-1} + \frac{1}{3}U_{n-1} + B_{n-1}\right]\right\} = x^2 \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

Thus the solution of Eq. (26) is given by:

$$\begin{aligned} U(x,t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n U_n(x,t) \\ &= x^2 + x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \\ &= x^2 E_{\alpha}(t^{\alpha}) \end{aligned} \tag{63}$$

If we put $\alpha \to 1$ in Eq. (63) or solve Eq. (57) with $\alpha = 1$, we obtain the exact solution;

$$U(x,t) = x^2 e^t$$

Example (4.2.21): Consider the following generalized nonlinear time fractional-order biological population model:

$$\begin{aligned} D_t^{\alpha}U(x,y,t) = &\left(\frac{\partial^2 U^2(x,y,t)}{\partial x^2} + \frac{\partial^2 U^2(x,y,t)}{\partial y^2}\right) \\ &+ U(x,y,t)(1 - rU(x,y,t)) \end{aligned} \tag{64}$$

Where $0 < \alpha \le 1$; subject to the initial condition;

$$U(x,y,0) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}$$

Taking the Sumudu transform of both sides of Eq. (64), thus we get;

$$S\left[D_t^{\alpha}U(x,y,t)\right] = S\left[2\left(D_x^2 + D_y^2\right)U^2(x,y,t) + U(x,y,t)(1 - rU(x,y,t))\right] \tag{65}$$

Using the property of the Sumudu transform and the initial condition in Eq. (65), we get;

$$\begin{aligned} &u^{-\alpha}S[U(x,y,t)] - u^{-\alpha}U(x,y,0) = \\ &\qquad S\left[2\left(D_x^2 + D_y^2\right)U^2(x,y,t) + U(x,y,t)(1 - rU(x,y,t))\right] \end{aligned} \tag{66}$$

And

$$S[U(x,y,t)] = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} + u^{\alpha} S\left[2\left(D_x^2+D_y^2\right)U^2(x,y,t) + U(x,y,t)\left(1-rU(x,y,t)\right)\right]$$

Operating with the Sumudu inverse on both sides of Eq. (66) we get;

$$U(x,y,t) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} + S^{-1}\left[u^{\alpha} S\left[2\left(D_x^2+D_y^2\right)U^2(x,y,t) + U(x,y,t)\left(1-rU(x,y,t)\right)\right]\right] \tag{67}$$

By applying the homotopy perturbation method in (67) we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,y,t) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} + pS^{-1}\left\{u^{\alpha} S\left[2\left(D_x^2+D_y^2\right)\left(\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)^2 + \left(1-r\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right)\sum_{n=0}^{\infty} p^n U_n(x,y,t)\right]\right\} \tag{68}$$

Equating the terms with identical powers of p, we get;

$$\begin{aligned}
p^0:&\ U_0(x,t) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}\\
p^1:&\ U_1(x,t) = S^{-1}\left\{u^{\alpha} S\left[2\left(D_x^2+D_y^2\right)U_1^2 + U_1\left(1-rU_1\right)\right]\right\} = \frac{t^{\alpha}}{\Gamma(\alpha+1)} e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}\\
p^2:&\ U_2(x,t) = S^{-1}\left\{u^{\alpha} S\left[2\left(D_x^2+D_y^2\right)U_2^2 + U_2\left(1-rU_2\right)\right]\right\} = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}\\
&\quad . \qquad\qquad\qquad .\\
p^n:&\ U_n(x,t) = S^{-1}\left\{u^{\alpha} S\left[2\left(D_x^2+D_y^2\right)U_n^2 + U_n\left(1-rU_n\right)\right]\right\} = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}
\end{aligned} \tag{69}$$

Thus the solution of Eq. (64) is given by

$$\begin{aligned}
U(x,t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n U_n(x,y,t)\\
&= e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right)\\
&= e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} E_{\alpha}\left(t^{\alpha}\right)
\end{aligned} \tag{70}$$

If we put $\alpha \to 1$ in Eq. (70) or solve Eq. (64) with $\alpha = 1$, we obtain the exact solution

$$U(x,t) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} e^{t} = e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)+t}$$

Example (4.2.22): Consider the nonlinear no homogenous time-fractional invicid Burgers equation

$$D_t^{\alpha} U(x,t) + U(x,t)\, U_x(x,t) = 1 + x + t \qquad (71)$$

Where $0 < \alpha \le 1$; subject to the initial condition;

$$U(x,0) = x$$

Taking the Sumudu transform of both sides of Eq. (71), thus we get;

$$S\left[D_t^{\alpha} U(x,t)\right] = S\left[1 + x + t - U(x,t)\, U_x(x,t)\right] \qquad (72)$$

Using the property of the Sumudu transform and the initial condition in Eq. (72), we get;

$$U(x,t) = x + S^{-1}\left[u^{\alpha} S\left[1 + x + t - U(x,t)\, U_x(x,t)\right]\right] \qquad (73)$$

By applying the homotopy perturbation method in Eq. (73) we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x + p\, S^{-1}\left[u^{\alpha}\left[S\left(1 + x + t\right) - S\left(\sum_{n=0}^{\infty} p^n H_n(U(x,t))\right)\right]\right]$$

$$= x + p\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}(1+x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - S^{-1}\left[u^{\alpha} S\left(\sum_{n=0}^{\infty} p^n H_n(U(x,t))\right)\right]\right] \qquad (74)$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_0 U_{0x} \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \\ &\;\; . \qquad . \qquad\qquad . \end{aligned} \qquad (75)$$

The coefficients of like powers of p , we get;

$$p^0: \; U_0(x,t) = x$$

$$p^1: \; U_1(x,t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}(1+x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - s^{-1}\left[u^{\alpha} S(H_0(U))\right]$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$p^2:\ U_2(x,t) = -s^{-1}\left[u^{\alpha}\, S(H_1(U))\right] = -s^{-1}\left[u^{2\alpha} + u^{2\alpha+1}\right]$$
$$= -\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]$$
$$p^3:\ U_3(x,t) = -s^{-1}\left[u^{\alpha}\, S(H_2(U))\right] = -s^{-1}\left[-u^{3\alpha} - u^{3\alpha+1}\right] \tag{76}$$
$$= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

Thus the solution of Eq. (71) is given by;

$$U(x,t) = x + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$
$$+ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$
$$= x + \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right)$$
$$+ \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) \tag{77}$$
$$= x - \sum_{n=1}^{\infty}\frac{(-1)^n\, t^{n\alpha}}{\Gamma(n\alpha+1)} - t\sum_{n=1}^{\infty}\frac{(-1)^n\, t^{n\alpha}}{\Gamma(n\alpha+2)}$$
$$= x + t + 1 - E_{\alpha,1}\left(-t^{\alpha}\right) - t\,E_{\alpha,}\left(-t^{\alpha}\right)$$

If we put $\alpha \to \frac{1}{2}$ in Eq. (77) or solve (71) with $\alpha = \frac{1}{2}$, we obtain the exact solution,

$$U(x,t) = x + t + 1 - E_{\frac{1}{2},1}\left(-t^{\frac{1}{2}}\right) - t\,E_{\frac{1}{2},2}\left(-t^{\frac{1}{2}}\right)$$
$$= x + t + 2 - 2\,e^{t}\,erfc\left(\sqrt{t}\right) - 2\sqrt{\frac{t}{\pi}}$$

If $\alpha = 1$ then, $U(x,t) = x + t + 1 - E_{1,1}(-t) - t\,E_{1,2}(-t) = x + t$

Example (4.2.23): Consider the following nonlinear time-fractional equation,

$$D_t^{\alpha}U(x,t) + U(x,t)\,U_x(x,t) = 2U - x \tag{78}$$

Where $1 < \alpha \le 2$; subject to the initial condition;

$$U(x,0) = x + 1 \quad , \quad U_t(x,0) = 1$$

Taking the Sumudu transform of both sides of Eq. (78), thus we get;

$$S\left[D_t^{\alpha}U(x,t)\right] = S\left[2U + x - U(x,t)\,U_x(x,t)\right] \tag{79}$$

Using the property of the Sumudu transform and the initial condition in Eq. (79), we get;

$$U(x,t) = x + S^{-1}\left[u^{\alpha} S\left[-x + 2U - U(x,t)\,U_x(x,t)\right]\right] \tag{80}$$

By applying the homotopy perturbation method in Eq. (80) we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 1 + x + pt + pS^{-1}\left[u^{\alpha}\left[-x + S\left(2\sum_{n=0}^{\infty} p^n U_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(U(x,t))\right)\right]\right] \tag{81}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by

$$\begin{aligned} H_0(U) &= U_0 U_{0x} \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \end{aligned} \tag{82}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = 1 + x \\ p^1 &: U_1(x,t) = t + s^{-1}\left[u^{\alpha}\left(-x + S\left(2U_0 - H_0(U)\right)\right)\right] \\ &= t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ p^2 &: U_2(x,t) = s^{-1}\left[u^{\alpha}\left(-x + S\left(2U_1 - H_1(U)\right)\right)\right] \\ &= S^{-1}\left[u^{\alpha+1} + u^{2\alpha}\right] + \left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\ p^3 &: U_3(x,t) = s^{-1}\left[u^{\alpha}\left(-x + S\left(2U_2 - H_2(U)\right)\right)\right] \\ &= S^{-1}\left[u^{2\alpha+1} + u^{3\alpha}\right] + \left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right] \end{aligned} \tag{83}$$

Thus the solution Eq. (78) is given by;

$$U(x,t) = 1 + x + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \tag{84}$$

$$= \; x \; + \left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right)$$
$$+ \left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots\right)$$
$$= x + \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - t \sum_{n=0}^{\infty} \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}$$
$$= \; x + \; E_{\alpha,1}\left(t^{\alpha}\right) + t\,E_{\alpha,2}\left(t^{\alpha}\right)$$

As a special case if we take $\alpha = 2$, we get;

$$U(x,t) = \; x \; + E_{2,1}\left(t^{2}\right) + \; t\,E_{2,2}\left(t^{2}\right)$$
$$= \; x + \cosh t + \sinh t$$
$$= x + e^{t}$$

Example (4.2.24): Consider the time-fractional fifth order KdV equation,

$$D_t^{\alpha} U(x,t) + U\;U_x - U\;U_{xxx} \; + U_{xxxxx} = \; 0 \qquad (85)$$

Where $0 < \alpha \le 1$; subject to the initial condition;

$$U(x,0) = e^{x}$$

Taking the Sumudu transform of both sides of Eq. (85), thus we get;

$$S\left[D_t^{\alpha} U(x,t)\right] = S\left[\;U\;U_{xxx} - U\;U_x - U_{xxxxx}\;\right] \qquad (86)$$

Using the property of the Sumudu transform and the initial condition in Eq. (86), we get;

$$U(x,t) = e^{x} \; + \; S^{-1}\left[u^{\alpha} S\left[U\;U_{xxx} - U\;U_x - U_{xxxxx}\right]\right] \qquad (87)$$

By applying the homotopy perturbation method in Eq. (87) we get;

$$\sum_{n=0}^{\infty} p^{n} U_n(x,t) = \; x \; + \; p\,S^{-1}\left[u^{\alpha}\left[-S\left(\sum_{n=0}^{\infty} p^{n} (U_n)_{xxxxx}\right) + S\left(\sum_{n=0}^{\infty} p^{n} H_n(U(x,t))\right)\right]\right] \qquad (88)$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by;

$$\begin{aligned} H_0(U) &= U_0\left[U_{0x} - U_{0xxx}\right] \\ H_1(U) &= U_0\left[U_{1x} - U_{1xxx}\right] + U_1\left[U_{0x} - U_{0xxx}\right] \\ H_2(U) &= U_0\left[U_{2x} - U_{2xxx}\right] + U_1\left[U_{1x} - U_{1xxx}\right] + U_2\left[U_{0x} - U_{0xxx}\right] \\ &\cdot \qquad \cdot \qquad \cdot \end{aligned} \qquad (89)$$

Comparing the coefficients of like powers of p , we get;

$$
\begin{aligned}
p^0:\ & U_0(x,t) = e^x \\
p^1:\ & U_1(x,t) = -s^{-1}\left[u^{\alpha} S\left((U_0)_{xxxxx} + H_0(U)\right)\right] \\
& = -s^{-1}\left[u^{\alpha}\left(e^x + e^x\left(e^x - e^x\right)\right)\right] = \frac{-e^x t^{\alpha}}{\Gamma(\alpha+1)} \\
p^2:\ & U_2(x,t) = -s^{-1}\left[u^{\alpha} S\left((U_1)_{xxxxx} + H_1(U)\right)\right] \\
& = \frac{e^x t^{2\alpha}}{\Gamma(2\alpha+1)} \\
p^3:\ & U_3(x,t) = -s^{-1}\left[u^{\alpha} S\left((U_2)_{xxxxx} + H_2(U)\right)\right] \\
& = \frac{-e^x t^{3\alpha}}{\Gamma(3\alpha+1)} \\
& \ \ \vdots
\end{aligned}
\tag{90}
$$

Thus the solution of Eq. (85) is given by;

$$
\begin{aligned}
U(x,t) &= e^x - \frac{e^x t^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^x t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{e^x t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \\
&= e^x E_{\alpha,1}\left(-t^{\alpha}\right)
\end{aligned}
\tag{91}
$$

As a special case if we take $\alpha = \frac{1}{2}$, we get;

$$U(x,t) = e^x E_{\frac{1}{2},1}\left(-t^{\frac{1}{2}}\right) = e^{t+x}\, erfc\left(\sqrt{t}\right)$$

If $\alpha = 1$ then, we get the solution of the classical equation as;

$$U(x,t) = e^{x-t}$$

CHAPTER FIVE
Linear and Nonlinear Physical Models

This chapter presents linear and nonlinear application in applied sciences [15]. In the last few decades, extensive studies were carried out in modeling linear and nonlinear partial differential equations. Several approaches such as characteristic methods, spectral methods and perturbation techniques have been used in studying these problems.

The following section offers the effectiveness of the homotopy perturbation Sumudu transform method (HPSTM) in solving linear and nonlinear physical models.

5.1: The Nonlinear Advection Problems

The nonlinear partial differential equation of the advection problem is of the form

$$U_t(x,t) + U(x,t)U_x(x,t) = f(x,t) \qquad , \quad U(x,0) = g(x) \tag{1}$$

The problem is solved by using the characteristic method, and by applying numerical methods such as Fourier series and Runge-Kutta method. In this section, the advection problem [16] is studied by utilizing homotopy perturbation method and Sumudu transform method.

On applying the Sumudu transform of both sides of Eq. (1),

$$S[U_t] + \frac{1}{2} S\left[\left(U^2\right)_x\right] = S[f(x,t)] \tag{2}$$

Using the differential operator property of the Sumudu transform and above initial conditions, we get;

$$S[U(x,t)] = g(x) + uS\left[\ f(x,t) - \frac{1}{2}\left(U^2\right)_x \right] \tag{3}$$

Now, applying the inverse Sumudu transform of both sides of Eq. (3), we get

$$U(x,t) = g(x) + S^{-1}\left[u\, S\left[\ f(x,t) - \frac{1}{2}\left(U^2\right)_x \right]\right] \tag{4}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We apply the homotopy perturbation method;

$$U(x,t) = \sum_{n=0}^{\infty} p^n\, U_n(x,t) \tag{5}$$

And the nonlinear term can be decomposed as;

$$U^2 = \sum_{n=0}^{\infty} p^n \; H_n(U) \tag{6}$$

For some He's polynomials $H_n(U)$ that are given by;

$$H_n(U_0, U_1, U_2, \cdots, U_n) = \frac{1}{n\,!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i \; U_i(x,t)\right)\right]_{p=0} , \quad n = 0\,,1\,,2\,,3\,,\cdots \tag{7}$$

Substituting Eqs. (5), and (6) in Eq. (4), we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = g(x) + S^{-1}[u\,S\,[\,f\,(x,t)]] - \frac{1}{2}\,p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n H_n(U)\right)_x\right]\right]\right) \tag{8}$$

This is the coupling of the Sumudu transform and the homotopy perturbation method using He's polynomials.

Comparing the coefficient of like power of p, the following approximation is obtained;

$$\begin{aligned}
p^0 &: U_0(x,t) = \; g(x) + S^{-1}[u\,S[\,f\,(x,t)]] \\
p^1 &: U_1(x,t) = -\frac{1}{2}S^{-1}[u\,S[\;(H_0(U))_x\,]] \\
p^2 &: U_2(x,t) = -\frac{1}{2}S^{-1}[u\,S[\;(H_1(U))_x\,]] \\
p^3 &: U_3(x,t) = \; -\frac{1}{2}S^{-1}[u\,S[\;(H_2(U))_x\,]] \\
&\;\;.
\end{aligned} \tag{9}$$

Thus, the exact solution is given by;

$$U(x,t) = \sum_{n=0}^{\infty} p^n \, U_n(x,t)$$

Example (5.1.1): Consider the inhomogeneous advection problem;

$$U_t + \frac{1}{2}\left(U^2\right)_x \; = e^x + t^2 \, e^{2x} \tag{10}$$

And the initial condition;

$$U(x,0) = 0\,.$$

Taking the Sumudu transform of both sides of Eq. (10), subject to the initial Condition, we get;

$$S[U(x,t)] = u\,e^x + 2\,u^3 e^{2x} \; - \frac{1}{2}u\,S\left[\left(U^2\right)_x\right] \tag{11}$$

The inverse of Sumudu transform implies that;

$$U(x,t)= t\,e^{x} + \frac{1}{3}t^{3}e^{2x} - \frac{1}{2}S^{-1}\left[u\,S\left[\left(U^{2}\right)_{x}\right]\right] \tag{12}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^{n}\,U_{n}(x,t) = t\,e^{x} + \frac{1}{3}t^{3}e^{2x} - \frac{1}{2}\,p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^{n}H_{n}(U)\right]\right]\right) \tag{13}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_{0x}^{2} \\ H_1(U) &= 2U_{0x}U_{1x} \\ H_2(U) &= 2U_{0x}U_{2x} + U_{1x}^{2} \\ H_3(U) &= 2U_{0x}U_{3x} + 2U_{1x}U_{2x} \\ &\;\;\vdots \end{aligned} \tag{14}$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned} p^{0} &: U_{0}(x,t) = t\,e^{x} + \frac{1}{3}t^{3}e^{2x} \\ p^{1} &: U_{1}(x,t) = -\frac{1}{2}S^{-1}\left[u\,S\left[H_{0}(U)\right]\right] \\ &\qquad = -\frac{1}{3}t^{3}e^{2x} - \frac{1}{5}t^{5}e^{3x} - \frac{2}{63}t^{7}e^{4x} \\ p^{2} &: U_{2}(x,t) = -\frac{1}{2}S^{-1}\left[u\,S\left[H_{0}(U)\right]\right] \\ &\qquad = \frac{1}{5}t^{5}e^{3x} + \frac{56}{315}t^{7}e^{4x} + \frac{31}{567}t^{9}e^{5x} + \frac{4}{756}t^{12}e^{6x} \\ &\;\;\vdots \end{aligned} \tag{15}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$\begin{aligned} U(x,t) = {} & t\,e^{x} + \frac{1}{3}t^{3}e^{2x} - \frac{1}{3}t^{3}e^{2x} - \frac{1}{5}t^{5}e^{3x} - \frac{2}{63}t^{7}e^{4x} \\ & + \frac{1}{5}t^{5}e^{3x} + \frac{56}{315}t^{7}e^{4x} + \frac{31}{567}t^{9}e^{5x} + \frac{4}{756}t^{12}e^{6x} \end{aligned} \tag{16}$$

And in a closed form by;

$$U(x,t) = t\,e^{x} \tag{17}$$

Example (5.1.2): Consider the inhomogeneous advection problem,

$$U_t + \frac{1}{2}\left(U^2\right)_x = -\sin(x+t) - \frac{1}{2}\sin 2(x+t) \tag{18}$$

And the initial condition as;

$$U(x,0) = \cos x.$$

Taking the Sumudu transform of both sides of Eq. (18) subject to the initial Condition, we get;

$$S[U(x,t)] = \cos x + u\,S\left[-\sin(x+t) - \frac{1}{4}\sin 2(x+t)\right] - \frac{1}{2}u\,S\left[\left(U^2\right)_x\right] \tag{19}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x - \frac{1}{2}S^{-1}\left[u\,S\left[\left(U^2\right)_x\right]\right] \tag{20}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x - \frac{1}{2}p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{21}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_{0x}^2 \\ H_1(U) &= 2U_{0x}U_{1x} \\ H_2(U) &= 2U_{0x}U_{2x} + U_{1x}^2 \\ H_3(U) &= 2U_{0x}U_{3x} + 2U_{1x}U_{2x} \\ &\vdots \end{aligned} \tag{22}$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned} p^0 &: U_0(x,t) = \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x \\ p^1 &: U_1(x,t) = -\frac{1}{2}S^{-1}\left[u\,S\left[H_0(U)\right]\right] \\ &\qquad = -\frac{1}{4}\cos 2(x+t) + \frac{1}{4}\cos 2x + \cdots \end{aligned} \tag{23}$$

It is noted that two noise terms appears in the components $U_0(x,t)$ and $U_1(x,t)$.

By removing these noise terms from U_0, the remaining terms of U_0 provides the exact solution. The exact solution is given by:

$$U(x,t) = \cos(x+t) \tag{24}$$

Example (5.1.3): Consider the homogeneous nonlinear problem,

$$U_t + U^2 U_x = 0 \tag{25}$$

And the initial condition as;

$$U(x,0) = 3x .$$

Taking the Sumudu transform of both sides of Eq. (25) subject to the initial Condition, we get;

$$S[U(x,t)] = 3x - u\,S[U^2 U_x] \tag{26}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = 3x - S^{-1}\left[u\,S[U^2 U_x]\right] \tag{27}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 3x - p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{28}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by:

$$\begin{aligned} U_0(x,t) &= 3x \\ U_{k+1}(x,t) &= -S^{-1}[u[H_k(U)]] \ , \ k \ge 0 \end{aligned} \tag{29}$$

This gives;

$$\begin{aligned} p^0 &: U_0(x,t) = t\,e^x + \frac{1}{3}t^3 e^{2x} \\ p^1 &: U_1(x,t) = -S^{-1}[u\,S[H_0(U)]] = -27x^2 t \\ p^2 &: U_2(x,t) = -S^{-1}[u\,S[H_1(U)]] = 486x^3 t^2 \\ p^3 &: U_3(x,t) = -S^{-1}[u\,S[H_2(U)]] = -10935x^4 t^3 \end{aligned} \tag{30}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = 3x - 27x^2 t + 486x^3 t^2 - 10935x^4 t^3 + \cdots \tag{31}$$

Based on this, the solution can be expressed in the form;

$$U(x,t) = \begin{cases} 3x & , \quad t = 0 \\ \dfrac{1}{6t}\left(\sqrt{1+36xt} - 1\right) & , \quad t > 0 \end{cases} \tag{32}$$

5.2: The Klein-Gordon Equation

In this section, the homotopy perturbation Sumudu transform method (HPSTM) has been applied to obtain the solution of the linear and nonlinear Klein-Gordon equations. The homotopy perturbation Sumudu transform method is a combined form of the Sumudu transform method with the homotopy perturbation method. This scheme finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The fact that this technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. The results reveal that the proposed algorithm is very efficient, simple and can be applied to other nonlinear problems.

5.2.1: Linear Klein-Gordon Equation

The linear Klein-Gordon equation in its standard form is given by;

$$U_{tt}(x,t) - U_{xx}(x,t) + a\,U(x,t) = h(x,t) \tag{33}$$

Subject to the initial conditions,

$$U(x,0) = f(x) \quad , \quad U_t(x,0) = g(x)$$

Where a is a constant and $h(x,t)$ is the source term. It is interesting to note that if $a = 0$; equation (33) becomes inhomogeneous wave equation.

Applying the Sumudu transform of both sides the equation (33) subject to the initial condition, we get,

$$S[U(x,t)] = f(x) + u\,g(x) + u^2 S[h(x,t) + U_{xx}(x,t) - aU(x,t)] \tag{34}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = f(x) + t\,g(x) + S^{-1}\left[u^2 S[h(x,t) + U_{xx}(x,t) - aU(x,t)]\right] \tag{35}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = f(x) + t\,g(x) + u^2 S\left[h(x,t)\right] + p\,S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} - a\sum_{n=0}^{\infty} p^n U_n(x,t)\right]\right] \tag{36}$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned} p^0 &: U_0(x,t) = f(x) + t\,g(x) + u^2 S[h(x,t)] \\ p^1 &: U_1(x,t) = S^{-1}\left[u^2 S[(U_0)_{xx} - aU_0]\right] \\ p^2 &: U_2(x,t) = S^{-1}\left[u^2 S[(U_1)_{xx} - aU_1]\right] \end{aligned} \tag{37}$$

Thus, the exact solution is given by;

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \tag{38}$$

Example (5.2.4): Consider the following linear Klein-Gordon equation,

$$U_{tt}(x,t) - U_{xx}(x,t) + U(x,t) = 0; \tag{39}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \quad U_t(x,0) = x;$$

Taking the Sumudu transform on both sides of equation (39) subject to the initial condition, we get;

$$S[U(x,t)] = u\,x + u^2 S[U_{xx}(x,t) - U(x,t)] \tag{40}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = xt + S^{-1}\left[u^2 S[U_{xx}(x,t) - U(x,t)]\right] \tag{41}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = xt + p\,S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n U_n(x,t)\right]\right] \tag{42}$$

Comparing the coefficients of like power p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = xt \\ p^1 &: U_1(x,t) = S^{-1}\left[u^2 S[(U_0)_{xx} - U_0]\right] = -\frac{xt^3}{3!} \\ p^2 &: U_2(x,t) = S^{-1}\left[u^2 S[(U_1)_{xx} - U_1]\right] = \frac{xt^5}{5!} \end{aligned} \tag{43}$$

Therefore the solution $U(x,t)$ in series form is given by;

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots$$

$$U(x,t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} - \cdots \right) x \tag{44}$$

And in closed form given as;

$$U(x,t) = x \sin t \tag{45}$$

Example (5.2.5): Consider the following linear Klein-Gordon equation:

$$U_{tt}(x,t)-U_{xx}(x,t)+U(x,t)=2\sin x; \tag{46}$$

With the initial conditions;

$$U(x,0)=\sin x \quad , \quad U_t(x,0)=1;$$

Taking the Sumudu transform of both sides of the equation (46) subject to the initial condition, we get;

$$S[U(x,t)]=\sin x+u+2u^2\sin x+u^2 S[U_{xx}(x,t)-U(x,t)] \tag{47}$$

The inverse of Sumudu transform implies that;

$$U(x,t)=\sin x+t+t^2\sin x+S^{-1}\left[u^2 S[U_{xx}(x,t)-U(x,t)]\right] \tag{48}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t)=\sin x+t+t^2\sin x$$

$$+pS^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}-\sum_{n=0}^{\infty} p^n U_n(x,t)\right]\right] \tag{49}$$

Comparing the coefficients of like power p , we get;

$$p^0 \ : U_0(x,t)=\sin x+t+t^2\sin x$$

$$p^1 \ : U_1(x,t)=S^{-1}\left[u^2 S[(U_0)_{xx}-U_0]\right]=-t^2\sin x-\frac{t^3}{3!}-\frac{t^4}{3!}\sin x$$

$$p^2 \ : U_2(x,t)=S^{-1}\left[u^2 S[(U_1)_{xx}-U_1]\right]=\frac{t^6}{90}\sin x+\frac{t^5}{5!}+\frac{t^4}{3!}\sin x \tag{50}$$

$$p3 \ : U_3(x,t)=S^{-1}\left[u^2 S[(U_2)_{xx}-U_2]\right]=-\frac{t^6}{90}\sin x-\frac{t^7}{7!}-\frac{2t^8}{7!}\sin x$$

Therefore the solution $U(x,t)$ in series form is given by;

$$U(x,t)=U_0(x,t)+U_1(x,t)+U_2(x,t)+\cdots$$

$$U(x,t)=\sin x+\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}-\cdots\right) \tag{51}$$

And in closed form given as;

$$U(x,t)=\sin x+\sin t \tag{52}$$

5.2.2: Nonlinear Klein-Gordon Equation

The nonlinear Klien-Gordon equation [15] describes nonlinear wave interaction, which is given by:

$$DU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \tag{53}$$

Subject to the initial conditions;

$$U(x,0) = f(x) \quad , \quad U_t(x,0) = g(x);$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Applying the Sumudu transform of both sides of the equation (53) subject to the initial condition, we get:

$$S[U(x,t)] = f(x) + u\,g(x) + u^2 S[h(x,t) + U_{xx}(x,t) - aU(x,t) - F(U(x,t))] \tag{54}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = f(x) + t\,g(x) + S^{-1}\left[u^2 S[h(x,t) + U_{xx}(x,t) - aU(x,t) - F(U(x,t))]\right] \tag{55}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = f(x) + t\,g(x) + S^{-1}\left[u^2 S[h(x,t)]\right] + p S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} - a\sum_{n=0}^{\infty} p^n U_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(U)\right]\right] \tag{56}$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = f(x) + t\,g(x) + S^{-1}\left[u^2 S[h(x,t)]\right] \\
p^1 &: U_1(x,t) = S^{-1}\left[u^2 S[(U_0)_{xx} - aU_0 - H_0(U)]\right] \\
p^2 &: U_2(x,t) = S^{-1}\left[u^2 S[(U_1)_{xx} - aU_1 - - H_1(U)]\right] \\
&. \qquad\qquad . \qquad\qquad .
\end{aligned} \tag{57}$$

Example (5.2.6): Consider the following nonlinear Klein-Gordon equation,

$$U_{tt}(x,t) - U_{xx}(x,t) + U^2(x,t) = x^2 t^2 \tag{58}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \qquad U_t(x,0) = x .$$

Taking the Sumudu transform of both sides of Eq. (58) subject to the initial Condition, we get:

$$S[U(x,t)] = xu + 2x^2 u^4 + u^2 S\left[U_{xx}(x,t) - U^2(x,t)\right] \tag{59}$$

The inverse of Sumudu transform implies that:

$$U(x,t) = xt + \frac{x^2 t^4}{12} + S^{-1}\left[u^2 S\left[U_{xx}(x,t) - U^2(x,t)\right]\right] \tag{60}$$

Now, applying the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = xt + \frac{x^2 t^4}{12} + \frac{1}{4} p\left(S^{-1}\left[u S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t) \right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(U) \right]\right]\right) \tag{61}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_0^2 \\ H_1(U) &= 2U_0 U_1 \\ H_2(U) &= 2U_0 U_2 + U_1^2 \\ H_3(U) &= 2U_0 U_3 + 2U_1 U_2 \end{aligned} \tag{62}$$

Comparing the coefficients of like powers of p , we get:

$$\begin{aligned} p^0 &: U_0(x,t) = xt + \frac{x^2 t^4}{12} \\ p^1 &: U_1(x,t) = S^{-1}\left[u^2 S\left[(U_0)_{xx} - H_0(U)\right]\right] \\ &\qquad = \frac{t^2}{180} - \frac{x^4 t^{10}}{12960} - \frac{x^3 t^7}{252} - \frac{x^2 t^4}{12} \\ p^2 &: U_2(x,t) = S^{-1}\left[u^2 S\left[(U_1)_{xx} - H_1(U)\right]\right] \\ &\qquad = \frac{x^2 t^{12}}{71280} - \frac{x t^9}{22680} - \frac{t^6}{180} + \frac{x^6 t^{16}}{18662400} \\ &\qquad - \frac{11 x^4 t^{10}}{45360} - \frac{383 x^5 t^{13}}{15921360} + \frac{x^3 t^7}{252} \end{aligned} \tag{63}$$

And so on. Combining the results obtained for the components, the solution in a form:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) = xt \tag{64}$$

Example (5.2.7): Consider the following nonlinear Klein-Gordon equation,

$$U_{tt}(x,t) - U_{xx}(x,t) + U^2(x,t) = 2x^2 - 2t^2 + x^4 t^4 \tag{65}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \quad U_t(x,0) = 0 .$$

Taking the Sumudu transform of both sides of Eq. (65) subject to the initial Condition, we get:

$$S[U(x,t)] = 2x^2 u^2 - 4u^4 + 24x^4 u^6 + u^2 S[U_{xx}(x,t) - U^2(x,t)] \tag{66}$$

The inverse of Sumudu transform implies that:

$$U(x,t) = x^2 t^2 - \frac{2}{6} t^4 + \frac{1}{30} x^4 t^6 + S^{-1}[u^2 S[U_{xx}(x,t) - U^2(x,t)]] \tag{67}$$

Now, applying the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x^2 t^2 - \frac{2}{6} t^4 + \frac{1}{30} x^4 t^6 + p\left(S^{-1}\left[u\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t) \right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(U) \right]\right]\right) \tag{68}$$

Where $H_n(U)$ are He’s polynomials that represent the nonlinear terms.

The first few components of He’s polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_0^2 \\ H_1(U) &= 2U_0 U_1 \\ H_2(U) &= 2U_0 U_2 + U_1^2 \\ H_3(U) &= 2U_0 U_3 + 2U_1 U_2 \end{aligned} \tag{69}$$

Comparing the coefficients of like powers of p , we get:

$$\begin{aligned} p^0 &: U_0(x,t) = x^2 t^2 - \frac{2}{6} t^4 + \frac{1}{30} x^4 t^6 \\ p^1 &: U_1(x,t) = S^{-1}[u^2 S[(U_0)_{xx} - H_0(U)]] \\ &= -\frac{x^8 t^{14}}{163800} + \frac{x^4 t^{14}}{11880} - \frac{x^6 t^{10}}{1350} - \frac{x^6 t^{16}}{18662400} \\ &- \frac{t^{10}}{3240} + \frac{11 x^2 t^8}{840} - \frac{x^4 t^6}{30} + \frac{t^4}{6} \end{aligned} \tag{70}$$

And so on. Combining the results obtained for the components, the solution in a form:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) = x^2 t^2 \tag{71}$$

5.3: The Burgers Equation

The Burger's equation [23] is one of the fundamental equations in fluid mechanics. Burger's equation describes the coupling between diffusion and convection processes.

The standard form of Burgers' equation is given by:

$$U_t + U U_x = V U_{xx} \quad , \quad t > 0 \tag{72}$$

Where V is a constant that defines the kinematic viscosity. If the viscosity, $V = 0$ the equation is called in viscid Burgers equation. The in viscid Burgers equation governing gas dynamics. In the viscid Burgers equation has been discussed before as a homogeneous case of the advection problem.

Nonlinear Burger's equation is considered as a simple nonlinear partial differential equations [15] involving both convection and diffusion in fluid dynamics. Burger introduced this equation [23] in order to study the interaction of the opposite effects of convection and diffusion in turbulent fluid in a channel. This equation also describes the structure of shock waves, traffic flow and acoustic transmission. A lot of research has been carried out on Burger's equation.

The Cole-Hopf transformation is the commonly used approach. The solution $U(x,t)$ was replaced by ψ_x in Eq. 72) to obtain;

$$\psi_{xt} + \psi_x \psi_{xx} = V \psi_{xxx} \tag{73}$$

Where by integrating this equation with respect to x we find:

$$\psi_t + \frac{1}{2}\psi_x^2 = V \psi_{xx} \tag{74}$$

Using the Cole-Hopf transformation:

$$\psi = -2V \ln \phi \tag{75}$$

So that:

$$U(x,t) = \psi_x = -2V \frac{\phi_x}{\phi} \tag{76}$$

Transforms the nonlinear equation into the heat flow equation:

$$\phi_t = V \phi_{xx} \tag{77}$$

It is clear that nonlinear Burger's equation (72) has been converted to an easily solvable linear equation.

Let us consider the Burgers equation:

$$U_t + UU_x = U_{xx} \tag{78}$$

And the initial condition as;

$$U(x,0) = f(x).$$

Taking the Sumudu transform of both sides of Eq. (78) subject to the initial Condition, we get;

$$S[U(x,t)] = f(x) + u\,S[U_{xx} - U\,U_x] \tag{79}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = f(x) + S^{-1}[u\,S[U_{xx} - U\,U_x]] \tag{80}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = f(x) + p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} + \sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{81}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_0 U_{0x} \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \\ H_3(U) &= U_0 U_{3x} + U_1 U_{2x} + U_2 U_{1x} + U_3 U_{0x} \\ &\quad . \qquad . \qquad . \end{aligned} \tag{82}$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned} p^0 &: U_0(x,t) = f(x) \\ p^1 &: U_1(x,t) = S^{-1}[u\,S[(U_0)_{xx} + H_0(U)]] \\ p^2 &: U_2(x,t) = S^{-1}[u\,S[(U_1)_{xx} + H_1(U)]] \\ p^3 &: U_3(x,t) = S^{-1}[u\,S[(U_2)_{xx} + H_2(U)]] \\ &\quad . \qquad . \qquad . \end{aligned} \tag{83}$$

Additional components may be computed to increase the accuracy level.

The solution in a series form is as follows. However, the $n-$term approximant ϕ_n can be determined by:

$$\phi_n = \sum_{k=0}^{n-1} p^n\, U_k(x,t) \tag{84}$$

In the following we list some of the derived exact solutions of Burgers equation derived by many researchers:

$$U(x,t) = 2\tan x \ , -2\cot x \ , -2\tanh x$$

$$U(x,t) = \frac{x}{t} \ , \frac{x}{t} + \frac{2}{x+t} \ , \frac{x+t}{2t^2 - t} \tag{85}$$

$$U(x,t) = \frac{-2e^{-t}\cos x}{1 + e^{-t}\sin x} \ , \ \frac{-2e^{-t}\sin x}{1 + e^{-t}\cos x}$$

The following examples will illustrate the discussion carried out above by using homotopy perturbation method.

Example (5.3.8): Consider the following Burgers equation,

$$U_t + UU_x = U_{xx} \tag{86}$$

And the initial condition as;

$$U(x,0) = x \, .$$

Taking the Sumudu transform of both sides of Eq. (86) subject to the initial Condition, we get;

$$S[U(x,t)] = x + u\,S[U_{xx} - U\,U_x] \tag{87}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = x + S^{-1}[u\,S[U_{xx} - U\,U_x]] \tag{88}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = x + p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{89}$$

Where $H_n(U)$ are He’s polynomials that represent the nonlinear terms.

The first few components of He’s polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_0 U_{0x} \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \\ H_3(U) &= U_0 U_{3x} + U_1 U_{2x} + U_2 U_{1x} + U_3 U_{0x} \\ &\;\;. \qquad . \qquad . \end{aligned} \tag{90}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = x \\
p^1 &: U_1(x,t) = S^{-1}\left[u\,S\left[(U_0)_{xx} - H_0(U)\right]\right] = -xt \\
p^2 &: U_2(x,t) = S^{-1}\left[u\,S\left[(U_1)_{xx} - H_1(U)\right]\right] = xt^2 \\
p^3 &: U_3(x,t) = S^{-1}\left[u\,S\left[(U_2)_{xx} - H_2(U)\right]\right] = -xt^3 \\
&\quad \cdot \qquad \cdot \qquad \cdot
\end{aligned} \tag{91}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = x\left(1 - t + t^2 - t^3 + \cdots\right) \tag{92}$$

Consequently, the exact solution is given by;

$$U(x,t) = \frac{x}{1+t} \quad , \quad |t| < 1 \tag{93}$$

Example (5.3.9): Consider the following Burgers equation,

$$U_t + UU_x = U_{xx} \tag{94}$$

And the initial condition as;

$$U(x,0) = 1 - \frac{2}{x} \quad , \quad x > 0 .$$

Taking the Sumudu transform of both sides of Eq. (94) subject to the initial Condition, we get;

$$S[U(x,t)] = 1 - \frac{2}{x} + u\,S[U_{xx} - U\,U_x] \tag{95}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = 1 - \frac{2}{x} + S^{-1}\left[u\,S[U_{xx} - U\,U_x]\right] \tag{96}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 1 - \frac{2}{x} + p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t) \right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(U) \right]\right]\right) \tag{97}$$

Where $H_n(U)$ are He's polynomials that represents the nonlinear terms.

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(U) &= U_0\,U_{0x} \\ H_1(U) &= U_0\,U_{1x} + U_1\,U_{0x} \\ H_2(U) &= U_0\,U_{2x} + U_1\,U_{1x} + U_2\,U_{0x} \\ H_3(U) &= U_0\,U_{3x} + U_1\,U_{2x} + U_2\,U_{1x} + U_3\,U_{0x} \\ &\;\;\vdots \end{aligned} \tag{98}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = 1 - \frac{2}{x} \\ p^1 &: U_1(x,t) = S^{-1}\left[u\,S\left[(U_0)_{xx} - H_0(U)\right]\right] = -\frac{2}{x^2}t \\ p^2 &: U_2(x,t) = S^{-1}\left[u\,S\left[(U_1)_{xx} - H_1(U)\right]\right] = -\frac{2}{x^3}t^2 \\ p^3 &: U_3(x,t) = S^{-1}\left[u\,S\left[(U_2)_{xx} - H_2(U)\right]\right] = -\frac{2}{x^4}t^3 \\ &\;\;\vdots \end{aligned} \tag{99}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = 1 - \frac{2}{x} - \frac{2}{x^2}\,t - \frac{2}{x^3}t^2 - \frac{2}{x^4}t^3 - \cdots \tag{100}$$

is readily obtained. To determine the exact solution, Eq. (100) can be rewritten as;

$$U(x,t) = 1 - \frac{2}{x}\left(1 + \frac{t}{x} + \frac{t^2}{x^2} + \frac{t^3}{x^3} + \cdots\right) = 1 - \frac{2}{x-t} \tag{101}$$

Example (5.3.10): Consider the following Burgers equation,

$$U_t + UU_x = U_{xx} \tag{102}$$

And the initial condition as;

$$U(x,0) = 2\tan x\,.$$

Taking the Sumudu transform of both sides of Eq. (102) subject to the initial Condition, we get;

$$S[U(x,t)] = 2\tan x + u\,S\left[U_{xx} - U\,U_x\right] \tag{103}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = 2\tan x + S^{-1}\left[u\,S\left[U_{xx} - U\,U_x\right]\right] \tag{104}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 2\tan x + p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t) \right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(U) \right]\right]\right) \quad (105)$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by,

$$\begin{aligned} H_0(U) &= U_0 U_{0x} \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \\ H_3(U) &= U_0 U_{3x} + U_1 U_{2x} + U_2 U_{1x} + U_3 U_{0x} \\ &. \quad . \quad . \end{aligned} \quad (106)$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned} p^0 &: U_0(x,t) = 2\tan x \\ p^1 &: U_1(x,t) = S^{-1}\left[u\,S\left[(U_0)_{xx} - H_0(U)\right]\right] = 0 \\ p^2 &: U_2(x,t) = S^{-1}\left[u\,S\left[(U_1)_{xx} - H_1(U)\right]\right] = 0 \\ p^3 &: U_3(x,t) = S^{-1}\left[u\,S\left[(U_2)_{xx} - H_2(U)\right]\right] = 0 \\ &. \quad . \quad . \end{aligned} \quad (107)$$

Thus, the exact solution is given by;

$$U(x,t) = 2\tan x \quad (108)$$

Example (5.3.11): Consider the following Burgers equation,

$$U_t + UU_x = U_{xx} \quad (109)$$

And the initial condition as;

$$U(0,t) = -\frac{2}{t} \quad , \quad U_x(0,t) = \frac{1}{t} + \frac{2}{t^2} \ .$$

Applying the Sumudu transform of both sides of Eq. (109) subject to the initial Condition, we get:

$$S[U(x,t)] = -\frac{2}{t} + u\left(\frac{1}{t} + \frac{2}{t^2}\right) + u^2\, S\left[U_t + U U_x\right] \quad (110)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = -\frac{2}{t} + x\left(\frac{1}{t} + \frac{2}{t^2}\right) + S^{-1}\left[u^2\, S\left[U_t + U U_x\right]\right] \quad (111)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = -\frac{2}{t} + x\left(\frac{1}{t} + \frac{2}{t^2}\right) + p\left(S^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_t + \sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \quad (112)$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by,

$$\begin{aligned} H_0(U) &= U_0 U_{0x} \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \\ H_3(U) &= U_0 U_{3x} + U_1 U_{2x} + U_2 U_{1x} + U_3 U_{0x} \\ &\vdots \end{aligned} \quad (113)$$

Comparing the coefficients of like powers of p, we get;

$$p^0 : U_0(x,t) = -\frac{2}{t} + x\left(\frac{1}{t} + \frac{2}{t^2}\right)$$

$$p^1 : U_1(x,t) = S^{-1}\left[u^2 S\left[(U_0)_{xx} - H_0(U)\right]\right] = -2\frac{x^2}{t^3} + \frac{2x^3}{3t^4} \quad (114)$$

$$p^2 : U_2(x,t) = S^{-1}\left[u^2 S\left[(U_1)_{xx} - H_1(U)\right]\right] = \frac{4x^3}{3t^4} + \cdots$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = \frac{x}{t} - \frac{2}{t}\left(1 - \frac{x}{t} + \frac{x^2}{t^2} - \frac{x^3}{t^3} + \cdots\right) \quad (115)$$

Consequently, the exact solution is given by;

$$U(x,t) = \frac{x}{t} - \frac{2}{x+t} \quad , \quad (116)$$

5.4: The Telegraph Equation

The standard form of the telegraph equation [15] is given by:

$$U_{xx} - aU_{tt} + bU_t + cU \tag{117}$$

Where $U = U(x,t)$ is the resistance, and a , b and c are constants related to the inductance, capacitance and conductance of the cable respectively.

Note that the telegraph equation is a linear partial differential equation. The telegraph equation arises in the propagation of electrical signals along a telegraph line. Assuming $a = 0$ and $c = 0$, because of electrical properties of the cable, then we obtain:

$$U_{xx} = bU_t \tag{118}$$

Which is the standard linear heat equation mentioned before in Chapter 2.
On the other hand, the electrical properties may lead to $b = 0$ and $c = 0$. Hence we obtain:

$$U_{xx} = aU_{tt} \tag{119}$$

Which is the standard linear wave equation presented in Chapter 2.

$$U_{xx} = U_{tt} + U_t + U \quad , \quad 0 < x < L \tag{120}$$

With the boundary and initial conditions;

$$BC \quad U(0,t) = f(t) \ , \ U_t(0,t) = g(t)$$
$$IC \quad U(x,0) = h(x) \ , \ U_t(x,0) = v(x).$$

Taking the Sumudu transform of both sides of Eq. (120) subject to the initial Condition, we get;

$$S[U(x,t)] = f(t) + u\,g(t) + u^2\, S[U_{tt} + U_t + U] \tag{121}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = f(t) + x\,g(t) + S^{-1}\left[u^2\, S[U_{tt} + U_t + U]\right] \tag{122}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 2\tan x + p\left[S^{-1}\left[u^2\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{tt}\right.\right.\right. \\ \left.\left.\left. + \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_t + \sum_{n=0}^{\infty} p^n U_n(x,t)\right]\right]\right] \tag{123}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = f(t) + x\,g(t) \\
p^1 &: U_1(x,t) = S^{-1}\left[u^2\, S\left[(U_0)_{tt} + (U_0)_t + U_0\right]\right] \\
p^2 &: U_2(x,t) = S^{-1}\left[u^2\, S\left[(U_1)_{tt} + (U_1)_t + U_1\right]\right] \\
p^3 &: U_3(x,t) = S^{-1}\left[u^2\, S\left[(U_2)_{tt} + (U_2)_t + U_2\right]\right] \\
&\quad\vdots
\end{aligned} \tag{124}$$

Thus, the exact solution is given by;

$$U(x,t) = \sum_{n=0}^{\infty} p^n\, U_n(x,t) \tag{125}$$

Example (5.4.12): Consider the following homogeneous telegraph equation:

$$U_{xx} = U_{tt} + U_t - U \tag{126}$$

With the boundary and initial conditions;

$$\begin{aligned}
BC \quad & U(0,t) = e^{-2t} \;,\; U_t(0,t) = e^{-2t} \\
IC \quad & U(x,0) = e^{x} \;,\; U_t(x,0) = -2e^{x}.
\end{aligned}$$

Taking the Sumudu transform of both sides of Eq. (126) subject to the initial Condition, we get;

$$S[U(x,t)] = e^{-2t} + u\,e^{-2t} + u^2\, S\left[U_{tt} + U_t - U\right] \tag{127}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = e^{-2t} + xe^{-2t} + S^{-1}\left[u^2\, S\left[U_{tt} + U_t - U\right]\right] \tag{128}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n\, U_n(x,t) = e^{-2t} + xe^{-2t} + p\left[S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{tt} + \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_t - \sum_{n=0}^{\infty} p^n U_n(x,t)\right]\right]\right] \tag{129}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = e^{-2t} + xe^{-2t} \\
p^1 &: U_1(x,t) = S^{-1}\left[u^2\, S\left[(U_0)_{tt} + (U_0)_t - U_0\right]\right] = \frac{1}{2!}x^2 e^{-2t} + \frac{1}{3!}x^3 e^{-2t} \\
p^2 &: U_2(x,t) = S^{-1}\left[u^2\, S\left[(U_1)_{tt} + (U_1)_t - U_1\right]\right] = \frac{1}{4!}x^4 e^{-2t} + \frac{1}{5!}x^5 e^{-2t} \\
p^3 &: U_3(x,t) = S^{-1}\left[u^2\, S\left[(U_2)_{tt} + (U_2)_t - U_2\right]\right] = \frac{1}{6!}x^6 e^{-2t} + \frac{1}{7!}x^7 e^{-2t}
\end{aligned} \tag{130}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = e^{-2t}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right) \quad (131)$$

Consequently, the exact solution is given by;

$$U(x,t) = e^{x-2t} \quad (132)$$

Example (5.4.13): Consider the following homogeneous telegraph equation:

$$U_{xx} = U_{tt} + 4U_t + 4U \quad (133)$$

With the boundary and initial conditions;

$$\begin{array}{ll} BC & U(0,t) = 1 + e^{-2t} \;,\; U_t(0,t) = 2 \\ IC & U(x,0) = 1 + e^{2x} \;,\; U_t(x,0) = -2 \end{array}.$$

Taking the Sumudu transform of both sides of Eq. (133) subject to the initial Condition, we get;

$$S[U(x,t)] = 1 + e^{-2t} + 2u + u^2\, S[U_{tt} + 4U_t + 4U] \quad (134)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = 1 + e^{-2t} + 2x + S^{-1}[u^2\, S[U_{tt} + 4U_t + 4U]] \quad (135)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 1 + e^{-2t} + 2x + p\left[S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{tt} + 4\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_t + 4\sum_{n=0}^{\infty} p^n U_n(x,t)\right]\right]\right] \quad (136)$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned} p^0 &: U_0(x,t) = 1 + e^{-2t} + 2x \\ p^1 &: U_1(x,t) = S^{-1}[u^2\, S[(U_0)_{tt} + 4(U_0)_t + 4U_0]] = 2x^2 + \frac{4}{3}x^3 \\ p^2 &: U_2(x,t) = S^{-1}[u^2\, S[(U_1)_{tt} + 4(U_1)_t + 4U_1]] = \frac{2}{3}x^4 + \frac{4}{15}x^5 \;\cdot\cdot \end{aligned} \quad (137)$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots\right) \qquad (138)$$

Consequently, the exact solution is given by;

$$U(x,t) = e^{2x} + e^{-2t} \qquad (139)$$

5.5: Schrodinger Equation

In this section, linear and nonlinear Schrodinger equations [24, 25] will be discussed and investigated. It is well known that Schrodinger equations arise in the study of the time evolution of the wave function.

5.5.1: The Linear Schrodinger Equation

The initial value problem for the linear Schrodinger equation for a free particle with mass *m* is given by the following standard form;

$$U_t = i\,U_{xx} \quad , \quad i^2 = -1\,,\ t > 0 \qquad (140)$$

And the initial condition as;

$$U(x,0) = f(x).$$

Where $f(x)$ is continuous & square integrable. It is to be noted that the Schrodinger equation (140) discusses the time evolution of a free particle. Moreover, the function $U(x,t)$ is complex, and Eq. (140) is a first order Schrodinger differential equation in t.

The homotopy perturbation method will be applied to handle the linear and the nonlinear Schrodinger equations. In order to achieve this, applying the Sumudu transform of both sides of Eq. (140) subject to the initial condition, we get:

$$S[U(x,t)] = f(x) + i\,u\,S[U_{xx}] \qquad (141)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = f(x) + i\,S^{-1}[u\,S[U_{xx}]] \qquad (142)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = f(x) + i\,p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \qquad (143)$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = f(x) \\ p^1 &: U_1(x,t) = i\,S^{-1}\left[u\,S\left[(U_0)_{xx}\right]\right] \\ p^2 &: U_2(x,t) = i\,S^{-1}\left[u\,S\left[(U_1)_{xx}\right]\right] \\ p^3 &: U_3(x,t) = i\,S^{-1}\left[u\,S\left[(U_2)_{xx}\right]\right] \\ &. \quad . \quad . \end{aligned} \tag{144}$$

Thus, the exact solution is given by;

$$U(x,t) = \sum_{n=0}^{\infty} p^n\, U_n(x,t) \tag{145}$$

Example (5.5.14): Consider the linear Schrodinger equation,

$$U_t = i\, U_{xx} \tag{146}$$

And the initial condition as;

$$U(x,0) = e^{ix}$$

Taking the Sumudu transform of both sides of Eq. (146) subject to the initial Condition, we get;

$$S[U(x,t)] = e^{ix} + i\,u\,S[U_{xx}] \tag{147}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = e^{ix} + i\,S^{-1}\left[u\,S[U_{xx}]\right] \tag{148}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n\, U_n(x,t) = e^{ix} + i\,p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \tag{149}$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned} p^0 &: U_0(x,t) = e^{ix} \\ p^1 &: U_1(x,t) = i\,S^{-1}\left[u\,S\left[(U_0)_{xx}\right]\right] = -\,i\,t\,e^{ix} \\ p^2 &: U_2(x,t) = i\,S^{-1}\left[u\,S\left[(U_1)_{xx}\right]\right] = -\frac{1}{2!}t^2\, e^{ix} \\ p^3 &: U_3(x,t) = i\,S^{-1}\left[u\,S\left[(U_2)_{xx}\right]\right] = \frac{1}{3!}\,i\,t^3\, e^{ix} \end{aligned} \tag{150}$$

Summing these iterations yields the series solution;

$$U(x,t) = e^{ix}\left(1 - it + \frac{1}{2!}(it)^2 - \frac{1}{3!}(it)^3 + \cdots\right) \tag{151}$$

That leads to the exact solution;

$$U(x,t) = e^{i(x-t)} \tag{152}$$

Example (5.5.15): Consider the linear Schrodinger equation,

$$U_t = i\, U_{xx} \tag{153}$$

And the initial condition as;

$$U(x,0) = \sinh x$$

Taking the Sumudu transform of both sides of Eq. (153) subject to the initial Condition, we get;

$$S[U(x,t)] = \sinh x + i\, u\, S[U_{xx}] \tag{154}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = \sinh x + i\, S^{-1}[u\, S[U_{xx}]] \tag{155}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \sinh x + i\, p\left(S^{-1}\left[u\, S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx}\right]\right]\right) \tag{156}$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = \sinh x \\
p^1 &: U_1(x,t) = i\, S^{-1}[u\, S[(U_0)_{xx}]] = i\, t \sinh x \\
p^2 &: U_2(x,t) = i\, S^{-1}[u\, S[(U_1)_{xx}]] = -\frac{1}{2!} t^2 \sinh x \\
p^3 &: U_3(x,t) = i\, S^{-1}[u\, S[(U_2)_{xx}]] = -\frac{1}{3!} i\, t^3 \sinh x \\
&\quad \vdots
\end{aligned} \tag{157}$$

Summing these iterations yields the series solution;

$$U(x,t) = \sinh x\left(1 + (it) + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \cdots\right) \tag{158}$$

That leads to the exact solution;

$$U(x,t) = e^{it} \sinh x \tag{159}$$

5.5.2: The Nonlinear Schrodinger Equation

The nonlinear Schrodinger equation (NLS) in standard form is defined as

$$iU_t + U_{xx} + \gamma |U|^2 U = 0 \tag{160}$$

Where γ is a constant and $U(x,t)$ is complex. Equation (160) represents solitary type solutions. A solitary wave is a wave where the speed of propagation is independent of the amplitude of the wave.

The nonlinear Schrodinger equations are given by;

$$iU_t + U_{xx} + 2|U|^2 U = 0 \tag{161}$$

And

$$iU_t + U_{xx} - 2|U|^2 U = 0 \tag{162}$$

Let us begin our analysis by considering the initial value problem;

$$iU_t + U_{xx} + \gamma |U|^2 U = 0 \quad , \quad U(x,0) = f(x) \tag{163}$$

Taking the Sumudu transform of both sides of Eq. (163) subject to the initial Condition, we get;

$$S[U(x,t)] = f(x) + iu\,S\left[U_{xx} + \gamma |U|^2 U\right] \tag{164}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = f(x) + iS^{-1}\left[u\,S\left[U_{xx} + \gamma |U|^2 U\right]\right] \tag{165}$$

Recall from complex analysis that:

$$|U|^2 = U\overline{U} \tag{166}$$

Where $\overline{U}$ is the conjugate of U. Where the nonlinear term $N(U(x,t))$ is given by:

$$N(U) = U^2\overline{U} \tag{167}$$

In view of (167), and following the formal techniques used before to derive the homotopy polynomials, we can easily derive that $N(U)$ has the following polynomials representation.

$$\begin{aligned}
&H_0(U) = U_0^2\overline{U}_0 \\
&H_1(U) = 2U_0U_1\overline{U}_0 + U_0^2\overline{U}_1 \\
&H_2(U) = 2U_0U_2\overline{U}_0 + U_1^2\overline{U}_0 + 2U_0U_1\overline{U}_1 + U_0^2\overline{U}_2 \\
&H_3(U) = 2U_0U_3\overline{U}_0 + 2U_1U_2\overline{U}_0 + 2U_0U_2\overline{U}_1 + U_1^2\overline{U}_1 + 2U_0U_1\overline{U}_2 + U_0^2\overline{U}_3 \\
&\qquad . \qquad . \qquad .
\end{aligned} \tag{168}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = f(x) + i\,p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} + \gamma \sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \quad (169)$$

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned} p^0 &: U_0(x,t) = f(x) \\ p^1 &: U_1(x,t) = i\,S^{-1}\left[u\,S\left[(U_0)_{xx} + \gamma\, H_0(U)\right]\right] \\ p^2 &: U_2(x,t) = i\,S^{-1}\left[u\,S\left[(U_1)_{xx} + \gamma\, H_1(U)\right]\right] \\ p^3 &: U_3(x,t) = i\,S^{-1}\left[u\,S\left[(U_2)_{xx} + \gamma\, H_2(U)\right]\right] \\ &\quad . \qquad . \qquad . \end{aligned} \quad (170)$$

Thus, the exact solution is given by;

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \quad (171)$$

The analysis introduced above will be illustrated by discussing the following examples.

Example (5.5.16): Consider the following nonlinear Schrodinger equation,

$$i\,U_t + U_{xx} + 2|U|^2 U = 0 \quad (172)$$

And the initial condition as;

$$U(x,0) = e^{ix}$$

Taking the Sumudu transform of both sides of Eq. (172) subject to the initial Condition, we get;

$$S[U(x,t)] = e^{ix} + i\,u\,S\left[U_{xx} + 2|U|^2 U\right] \quad (173)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = e^{ix} + i\,S^{-1}\left[u\,S\left[U_{xx} + 2|U|^2 U\right]\right] \quad (174)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = e^{ix} + i\,p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} + 2\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \quad (175)$$

Comparing the coefficients of like powers of p , we get;

$$
\begin{aligned}
p^0 &: U_0(x,t) = e^{ix} \\
p^1 &: U_1(x,t) = i\,S^{-1}\left[u\,S\left[(U_0)_{xx} + 2\,H_0(U)\right]\right] = i\,t\,e^{ix} \\
p^2 &: U_2(x,t) = i\,S^{-1}\left[u\,S\left[(U_1)_{xx} + 2\,H_1(U)\right]\right] = -\frac{1}{2!}t^2\,e^{ix} \\
p^3 &: U_3(x,t) = i\,S^{-1}\left[u\,S\left[(U_2)_{xx} + 2\,H_2(U)\right]\right] = -\frac{1}{3!}i\,t^3\,e^{ix}
\end{aligned} \tag{176}
$$

Summing these iterations yields the series solution;

$$U(x,t) = e^{ix}\left(1 + i\,t + \frac{1}{2!}(i\,t)^2 + \frac{1}{3!}(i\,t)^3 + \cdots\right) \tag{177}$$

That leads to the exact solution;

$$U(x,t) = e^{i\,(x+t)} \tag{178}$$

Example (5.5.17): Consider the following nonlinear Schrodinger equation,

$$i\,U_t + U_{xx} - 2|U|^2 U = 0 \tag{179}$$

And the initial condition as;

$$U(x,0) = e^{ix}$$

Taking the Sumudu transform of both sides of Eq. (179) subject to the initial Condition, we get;

$$S[U(x,t)] = e^{ix} + i\,u\,S\left[U_{xx} - 2|U|^2 U\right] \tag{180}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = e^{ix} + i\,S^{-1}\left[u\,S\left[U_{xx} - 2|U|^2 U\right]\right] \tag{181}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = e^{ix} + i\,p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} - 2\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{182}$$

Comparing the coefficients of like powers of p , we get;

$$
\begin{aligned}
p^0 &: U_0(x,t) = e^{ix} \\
p^1 &: U_1(x,t) = i\,S^{-1}\left[u\,S\left[(U_0)_{xx} - 2\,H_0(U)\right]\right] = -3\,i\,t\,e^{ix} \\
p^2 &: U_2(x,t) = i\,S^{-1}\left[u\,S\left[(U_1)_{xx} - 2\,H_1(U)\right]\right] = \frac{1}{2!}(3\,i\,t)^2\,e^{ix} \\
p^3 &: U_3(x,t) = i\,S^{-1}\left[u\,S\left[(U_2)_{xx} - 2\,H_2(U)\right]\right] = -\frac{1}{3!}(3\,i\,t)^3\,e^{ix}
\end{aligned} \tag{183}
$$

Summing these iterations yields the series solution;

$$U(x,t) = e^{ix}\left(1 - (3it) + \frac{1}{2!}(3it)^2 - \frac{1}{3!}(3it)^3 + \cdots\right) \quad (184)$$

That leads to the exact solution;

$$U(x,t) = e^{i(x-3t)} \quad (185)$$

5.6: Korteweg-deVaries Equation

The Korteweg-deVaries (KdV) equation in simplest form [26- 28] is given by:

$$U_t + aUU_x + U_{xxx} = 0 \quad (186)$$

Let us first consider the initial value problem

$$U_t + aUU_x + bU_{xxx} = 0 \quad , \quad U(x,0) = f(x) \quad (187)$$

Where a and b are constants.

Taking the Sumudu transform of both sides of Eq. (187) subject to the initial Condition, we get;

$$S[U(x,t)] = f(x) - u\,S[aUU_x + bU_{xxx}] \quad (188)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = f(x) - S^{-1}[u\,S[aUU_x + bU_{xxx}]] \quad (189)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = f(x) - p\left(S^{-1}\left[u\,S\left[b\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xxx} + a\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \quad (190)$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by;

$$\begin{aligned} H_0(U) &= U_0 U_{0x} \\ H_1(U) &= U_0 U_{1x} + U_1 U_{0x} \\ H_2(U) &= U_0 U_{2x} + U_1 U_{1x} + U_2 U_{0x} \\ H_3(U) &= U_0 U_{3x} + U_1 U_{2x} + U_2 U_{1x} + U_3 U_{0x} \\ &\quad . \qquad . \qquad . \end{aligned} \quad (191)$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = f(x) \\
p^1 &: U_1(x,t) = -S^{-1}\left[u\,S\left[b(U_0)_{xxx} + a\,H_0(U)\right]\right] \\
p^2 &: U_2(x,t) = -S^{-1}\left[u\,S\left[b(U_1)_{xxx} + a\,H_1(U)\right]\right] \\
p^3 &: U_3(x,t) = -S^{-1}\left[u\,S\left[b(U_2)_{xxx} + a\,H_2(U)\right]\right]
\end{aligned} \tag{192}$$

Thus, the exact solution is given by;

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \tag{193}$$

Example (5.6.18): Consider the following homogeneous KdV equation;

$$U_t - 6U\,U_x + U_{xxx} = 0 \tag{194}$$

And the initial condition as;

$$U(x,0) = 6x$$

Taking the Sumudu transform of both sides of Eq. (194) subject to the initial Condition, we get;

$$S[U(x,t)] = 6x - u\,S[6U\,U_x - U_{xxx}] \tag{195}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = 6x - S^{-1}\left[u\,S[6U\,U_x - U_{xxx}]\right] \tag{196}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 6x - p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xxx} - 6\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{197}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by;

$$\begin{aligned}
H_0(U) &= U_0\,U_{0x} \\
H_1(U) &= U_0\,U_{1x} + U_1\,U_{0x} \\
H_2(U) &= U_0\,U_{2x} + U_1\,U_{1x} + U_2\,U_{0x} \\
H_3(U) &= U_0\,U_{3x} + U_1\,U_{2x} + U_2\,U_{1x} + U_3\,U_{0x} \\
&\;\; . \qquad . \qquad\qquad .
\end{aligned} \tag{198}$$

Comparing the coefficients of like powers of p , we get;

$$
\begin{aligned}
p^0 &: U_0(x,t) = 6x \\
p^1 &: U_1(x,t) = -S^{-1}\left[u\,S\left[(U_0)_{xxx} - 6\,H_0(U)\right]\right] = 6^3\,xt \\
p^2 &: U_2(x,t) = -S^{-1}\left[u\,S\left[(U_1)_{xxx} - 6\,H_1(U)\right]\right] = 6^5\,xt^2 \\
p^3 &: U_3(x,t) = -S^{-1}\left[u\,S\left[(U_2)_{xxx} - 6\,H_2(U)\right]\right] = 6^7\,xt^3
\end{aligned}
\tag{199}
$$

Summing these iterations yields the series solution;

$$U(x,t) = 6x\left(1 + 36t + (36t)^2 + (36t)^3 + \cdots\right) \tag{200}$$

That leads to the exact solution;

$$U(x,t) = \frac{6x}{1-36t} \quad , \quad |36t| < 1 \tag{201}$$

Example (5.6.19): Consider the following homogeneous KdV equation:

$$U_t - 6U\,U_x + U_{xxx} = 0 \tag{202}$$

And the initial condition as;

$$U(x,0) = \frac{1}{6}\,(x-1)$$

Taking the Sumudu transform on both sides of Eq. (202) subject to the initial Condition, we get;

$$S[U(x,t)] = \frac{1}{6}\,(x-1) - u\,S\left[6U\,U_x - U_{xxx}\right] \tag{203}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = \frac{1}{6}\,(x-1) - S^{-1}\left[u\,S\left[6U\,U_x - U_{xxx}\right]\right] \tag{204}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n\,U_n(x,t) = \frac{1}{6}\,(x-1) - p\left(S^{-1}\left[u\,S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xxx} - 6\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{205}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by;

$$
\begin{aligned}
H_0(U) &= U_0\,U_{0x} \\
H_1(U) &= U_0\,U_{1x} + U_1\,U_{0x} \\
H_2(U) &= U_0\,U_{2x} + U_1\,U_{1x} + U_2\,U_{0x} \\
H_3(U) &= U_0\,U_{3x} + U_1\,U_{2x} + U_2\,U_{1x} + U_3\,U_{0x}
\end{aligned}
\tag{206}
$$

Comparing the coefficients of like powers of p , we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = \frac{1}{6}(x-1) \\
p^1 &: U_1(x,t) = -S^{-1}\left[u\,S\left[(U_0)_{xxx} - 6\,H_0(U)\right]\right] = \frac{1}{6}(x-1)t \\
p^2 &: U_2(x,t) = -S^{-1}\left[u\,S\left[(U_1)_{xxx} - 6\,H_1(U)\right]\right] = \frac{1}{6}(x-1)t^2 \\
p^3 &: U_3(x,t) = -S^{-1}\left[u\,S\left[(U_2)_{xxx} - 6\,H_2(U)\right]\right] = \frac{1}{6}(x-1)t^3
\end{aligned} \tag{207}$$

Summing these iterations yields the series solution;

$$U(x,t) = \frac{1}{6}(x-1)\left(1 + t + t^2 + t^3 + \cdots\right) \tag{208}$$

That leads to the exact solution;

$$U(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right) \quad , \quad |t| < 1 \tag{209}$$

Example (5.6.21):

Consider an equation with initial condition is given by

$$U_t + UU_x - UU_{xxx} + U_{xxxxx} = 0 \quad , \quad U(x,0) = e^x \tag{210}$$

Taking Sumudu Transform on both sides of Eq. (210) subject to the initial Condition, we get;

$$S[U(x,t)] = e^x - u\,S\left[UU_x - UU_{xxx} + U_{xxxxx}\right] \tag{211}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = e^x - S^{-1}\left[u\,S\left[UU_x - UU_{xxx} + U_{xxxxx}\right]\right] \tag{212}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = e^x - p\left(\left[u\,S\left[\sum_{n=0}^{\infty} p^n\left[H_n - B_n\right] + \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xxxxx}\right]\right]\right) \tag{213}$$

Where $H_n(U)$ and $B_n(U)$ are He's polynomials that represents the nonlinear terms.

The first few components of He's polynomials, are given by;

$$\begin{aligned}
&H_0(U) = U_0 U_{0x} \quad , \quad H_1(U) = U_1 U_{0x} + U_0 U_{1x} \\
&H_2(U) = U_2 U_{0x} + U_1 U_{1x} + U_0 U_{2x} \\
&B_0(U) = U_0 U_{0xxx} \quad , \quad B_1(U) = U_1 U_{0xxx} + U_0 U_{1xxx} \\
&B_2(U) = U_2 U_{0xxx} + U_1 U_{1xxx} + U_0 U_{2xxx}
\end{aligned} \tag{214}$$

Comparing the coefficients of like powers of p , we get;

$$p^0 \ : U_0(x,t) = e^x$$
$$p^1 \ : U_1(x,t) = -\ S^{-1}\left[u\,S\left[\,20\,H_0 - B_0 - (U_0)_{xxxxx}\,\right]\right] = \ -t\,e^x$$
$$p^2 \ : U_2(x,t) = -S^{-1}\left[u\,S\left[\,20\,H_1 - B_1 - (U_1)_{xxxxx}\,\right]\right] = \ \frac{t^2}{2!}\,e^x \tag{215}$$
$$p^3 \ : U_3(x,t) = -S^{-1}\left[u\,S\left[\,20\,H_2 - B_2 - (U_2)_{xxxxx}\,\right]\right] = -\frac{t^3}{3!}\,e^x$$

Summing these iterations yields the series solution;

$$U(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \tag{216}$$

That leads to the exact solution;

$$U(x,t) = \ e^x \ e^{-t} \tag{217}$$

Example (5.6.22):

Consider the following FKdV Equation

$$U_t + U_x + U^2 U_{xx} + U_x U_{xx} - 20U^2 U_{xxx} + U_{xxxxx} = 0 \tag{218}$$

With the initial condition,

$$U(x,0) = \frac{1}{x}$$

Taking Sumudu transform of both sides of Eq. (218) subject to the initial Condition, we get;

$$S[U(x,t)] = \frac{1}{x} + u\,S\left[20U^2\,U_{xxx} - U_{xxxxx} - U_x - U^2 U_{xx} - U_x U_{xx}\right] \tag{219}$$

The inverse of Sumudu transform implies that;

$$U(x,t) = \frac{1}{x} + S^{-1}\left[u\,S\left[20U^2\,U_{xxx} - U_{xxxxx} - U_x - U^2 U_{xx} - U_x U_{xx}\right]\right] \tag{220}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n\,U_n(x,t) = \frac{1}{x} + p\left(\left[u\,S\left[\sum_{n=0}^{\infty} p^n\left[20\,H_n - B_n - R_n\right]\right.\right.\right.$$
$$\left.\left.\left. - \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xxxxx} - \left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_x\right]\right]\right) \tag{221}$$

Where $H_n(U)$, $B_n(U)$ and $R_n(U)$ are He's polynomials that represent the nonlinear terms.

The first few components of He's polynomials, are given by;

$$
\begin{aligned}
&B_0(U)=U_0^2U_{0\,xx}\ \ ,\ \ B_1(U)=2U_0U_1U_{0\,xx}+U_0^2U_{1xx}\\
&B_2(U)=2U_0U_2U_{0xx}+U_1^2U_{0\,xx}+2U_0U_1U_{1xx}+U_0^2U_{2xx}\\
&B_3(U)=2U_0U_3U_{0xx}+2U_1U_2U_{0xx}+U_1^2U_{1xx}\\
&\qquad +2U_0U_2U_{1xx}+2U_0U_1U_{2xx}+U_0^2U_{3xx}\\
&R_0(U)=U_{0x}U_{0\,xx}\ \ ,\ \ R_1(U)=U_{0x}U_{1xx}+U_{0xx}U_{1x}\\
&R_2(U)=U_{2x}U_{0xx}+U_{1x}U_{1xx}+U_{0x}U_{2xx}\\
&R_3(U)=U_{0xx}U_{3x}+U_{1xx}U_{2x}+U_{1x}U_{2xx}+U_{0x}U_{3xx}\\
&H_0(U)=U_0^2U_{0\,xxx}\ \ ,\ \ H_1(U)=2U_0U_1U_{0xxx}+U_0^2U_{1xxx}\\
&H_2(U)=2U_0U_2U_{0xxx}+U_1^2U_{0\,xxx}+2U_0U_1U_{1xxx}+U_0^2U_{2xxx}\\
&H_3(U)=2U_0U_3U_{0xxx}+2U_1U_2U_{0xxx}+U_1^2U_{1xxx}\\
&\qquad +2U_0U_2U_{1xxx}+2U_0U_1U_{2xxx}+U_0^2U_{3xxx}
\end{aligned}
\tag{222}
$$

Comparing the coefficients of like powers of p, we get;

$$
\begin{aligned}
&p^0\ :U_0(x,t)=\frac{1}{x}\\
&p^1\ :U_1(x,t)=S^{-1}\left[u\,S\left[\,20H_0-B_0-R_0-(U_0)_{xxxxx}-(U_0)_x\right]\right]=\frac{t}{x^2}\\
&p^2\ :U_2(x,t)=S^{-1}\left[u\,S\left[\,20H_1-B_1-R_1-(U_1)_{xxxxx}-(U_1)_x\right]\right]=\frac{t^2}{x^3}\\
&p^3\ :U_3(x,t)=S^{-1}\left[u\,S\left[\,20H_2-B_2-R_2-(U_2)_{xxxxx}-(U_2)_x\right]\right]=\frac{t^3}{x^4}
\end{aligned}
\tag{223}
$$

Summing these iterations yields the series solution;

$$
U(x,t)=\frac{1}{x}\left(1+\frac{t}{x}+\left(\frac{t}{x}\right)^2+\left(\frac{t}{x}\right)^3+\cdots\right)
\tag{224}
$$

That leads to the exact solution;

$$
U(x,t)=\frac{1}{x-t}
\tag{225}
$$

CHAPTER SIX
Comparison of Homotopy Perturbation Method and Sumudu Transform and Another Method

The homotopy perturbation Sumudu transform method (HPSTM) is the combination of Sumudu transform and the homotopy perturbation method. This method does not require any additional polynomial such as Adomian polynomial. One may visualize that all these three methods HPSTM, ADM and STM are powerful and accurate for solving different kinds of linear and nonlinear fractional differential equations. The features of HPSTM are: it is very simple, straightforward and user friendly, it can be used for solving nonlinear problems, which is not possible using ADM and STM, and it does not require polynomial such as Adomian polynomial.

6.1: Comparison of Homotopy Perturbation Sumudu Transform Method and Sumudu Transform for Solving Linear Partial Differential Equations

6.1.1: Basic Idea of HPSTM

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of form

$$DU(x,t)+RU(x,t)+NU(x,t)=g(x,t) \tag{1}$$

$$U(x,0)=h(x)\,, \qquad U_t(x,0)=f(x)$$

Where D is the second order linear differential operator $D=\frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking the Sumudu Transform on both sides of Eq. (1), we get

$$S[DU(x,t)]+S[RU(x,t)]+S[NU(x,t)]=S[g(x,t)] \tag{2}$$

Using the differential operator property of the Sumudu Transform and above initial conditions, we get;

$$\begin{aligned} S[DU(x,t)] &= u^2\,S[g(x,t)]+h(x)+u\,f(x) \\ &\quad -u^2 S[RU(x,t)]+S[NU(x,t)] \end{aligned} \tag{3}$$

Now, applying the inverse Sumudu Transform on both sides of Eq(3), we get

$$U(x,t) = G(x,t) - S^{-1}\left[u^2\ S\left[RU(x,t) + NU(x,t)\right]\right] \tag{4}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We apply the Homotopy perturbation method;

$$U(x,t) = \sum_{n=0}^{\infty} p^n\ U_n(x,t) \tag{5}$$

And the nonlinear term can be decomposed as;

$$NU(x,t) = \sum_{n=0}^{\infty} p^n\ H_n(x,t) \tag{6}$$

For some He's polynomials $H_n(U)$ that are given by;

$$H_n(U_0, U_1, U_2, \cdots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i\ U_i(x,t)\right)\right]_{p=0}, \quad n = 0, 1, 2, 3, \cdots \tag{7}$$

Substituting Eqs. (5) and (6) in Eq. (4), we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = G(x,t) - p\left(S^{-1}\left[u^2\ S\left[R\sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(x,t)\right]\right]\right) \tag{8}$$

This is the coupling of the Sumudu Transform and the Homotopy perturbation method using He's polynomials.

Comparing the coefficient of like power of p, the following approximation is obtained;

$$\begin{aligned}
p^0 &: U_0(x,t) = G(x,t) \\
p^1 &: U_1(x,t) = -S^{-1}\left[u^2\ S\left[RU_0(x,t) + H_0(U)\right]\right] \\
p^2 &: U_2(x,t) = -S^{-1}\left[u^2\ S\left[RU_1(x,t) + H_1(U)\right]\right] \\
p^3 &: U_3(x,t) = -S^{-1}\left[u^2\ S\left[RU_2(x,t) + H_2(U)\right]\right] \\
&\ \ .
\end{aligned} \tag{9}$$

6.1.2: Method of Solution of the Problem

Consider the following linear Klein-Gordon equation,

$$U_{tt}(x,t) - U_{xx}(x,t) - 2\,U(x,t) = -2\sin x \sin t\,; \tag{10}$$

With the initial conditions;

$$U(x,0) = 0 \quad , \quad U_t(x,0) = \sin x\,;$$

Taking the Sumudu transform of both sides of Eq. (10), subject to the initial condition, we get;

$$S[U(x,t)] = u\sin x + u^2 S[U_{xx}(x,t) + 2U(x,t) - 2\sin x \sin t\ ] \qquad (11)$$

The inverse of Sumudu transform implies that;

$$U(x,t) = t\sin x + S^{-1}[u^2 S[U_{xx}(x,t) - U(x,t)]] \qquad (12)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = t\sin x + pS^{-1}\left[u^2 S\left[\left(\sum_{n=0}^{\infty} p^n U_n(x,t)\right)_{xx} + 2\sum_{n=0}^{\infty} p^n U_n(x,t) - 2\sin x \sin t\right]\right] \qquad (13)$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = t\sin x \\
p^1 &: U_1(x,t) = S^{-1}[u^2 S[(U_0)_{xx} + 2U_0 - 2\sin x \sin t]] \\
&= \left(\frac{t^3}{3!} - 2t + 2\sin t\right)\sin x \\
p^2 &: U_2(x,t) = S^{-1}[u^2 S[(U_1)_{xx} + 2U_1]] \\
&= \left(\frac{t^5}{5!} - 6\sin t - \frac{1}{3}t^3 + 2t\right)\sin x \\
p^3 &: U_3(x,t) = S^{-1}[u^2 S[(U_2)_{xx} + 2U_2]] \\
&= \left(\frac{t^7}{7!} + 2\sin t - \frac{1}{60}t^5 + \frac{t^3}{3} - 2t\right)\sin x
\end{aligned} \qquad (14)$$

Therefore the solution $U(x,t)$ in series form is given by;

$$U(x,t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} - \cdots\right)\sin x \qquad (15)$$

And in closed form given as;

$$U(x,t) = \sin t\ \sin x \qquad (16)$$

6.1.3: Basic Idea of STM (Definitions and Theorems)

The Sumudu transform is an integral transform similar to the Laplace transform, introduced in the early 1990s by Watugala [29] to solve linear differential equations and control engineering problems.

Note that these the theorems and definitions will use in this section.

Definition (6.1.1): The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by:

$$S[f(t)] = F_s(u) = \int_0^\infty \frac{1}{u} \exp\left[-\frac{t}{u}\right] f(t)\, dt \tag{17}$$

Definition (6.1.2): The double Sumudu transform of a function $f(x,t)$, defined for all real numbers $(t \geq 0\ ,\ x \geq 0)$, is defined by:

$$S[f(x,t)] = \frac{1}{u}\int_0^\infty \exp\left[-\frac{t}{u}\right] f(x,t)\, dt \tag{18}$$

In the same line of ideas, the Sumudu transform of the second partial derivative with respect to t is of the form [30],

$$S\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{u}F(x,u) - \frac{1}{u}F(x,0)$$

$$S\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{u^2}F(x,u) - \frac{1}{u^2}F(x,0) - \frac{1}{u}\frac{\partial F(x,0)}{\partial t} \tag{19}$$

Similarly, the Sumudu transform of the second partial derivative with respect to x is of form [30],

$$S\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}F(x,u)$$

$$S\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}F(x,u) \tag{20}$$

Theorem (6.1.3) [29]: Let $G(u)$ be the Sumudu transform of $f(t)$ such that

i. $G(1/s)/s$ is a meromorphic function, with singularities having $\operatorname{Re}[s] \le \gamma$ and

ii. there exist a circular region Γ with radius R and positive constants M and K with $|G(1/s)/s| < M R^{-K}$, then the function $f(t)$ is given by;

$$S^{-1}[G(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp[st]\, G\left(\frac{1}{s}\right)\frac{ds}{s} = \sum residual\left[\exp[st]\,\frac{G(1/s)}{s}\right] \quad (21)$$

6.1.4: Method of Solution of the Problem

Applying the Sumudu transform of Eq. (10), we get:

$$S[U_{tt}] - S[U_{xx}] - 2S[U] = -2\, S[\sin x \sin t] \quad (22)$$

$$\frac{1}{u^2}\left[U(x,u) - U(x,0) - u\, U_t(x,0)\right] - \frac{d^2}{dx^2}U(x,u) - 2U(x,u) = \frac{-2u}{1+u^2}\sin x$$

Then

$$\frac{d^2U}{dx^2} - \frac{1}{u^2}U + 2U = \frac{1}{u}\sin x + \frac{-2u}{1+u^2}\sin x$$

Thus we have the ordinary differential equation:

$$\frac{d^2U}{dx^2} - \left[\frac{1-2u^2}{u^2}\right]U = \frac{u^2-1}{u\left(1+u^2\right)}\sin x \quad (23)$$

$$U_c = A\exp\left(\sqrt{\frac{1-2u^2}{u^2}}\; x\right) + B\exp\left(-\sqrt{\frac{1-2u^2}{u^2}}\; x\right) \quad (24)$$

The initial conditions gives;

$$A = B = 0$$

Then

$$U_c = 0 \quad (25)$$

And

$$U_p(x,u) = \frac{u}{1+u^2}\sin x \quad (26)$$

The solution is;

$$U(x,u) = U_c(x,u) + U_p(x,u)$$

$$U(x,u) = \frac{u}{1+u^2}\sin x \quad (27)$$

Taking the inverse of Sumudu transform:

$$U(x,t)= S^{-1}\left[\frac{u}{1+u^2}\right]\sin x \tag{28}$$

$$U(x,t)= \sin t\ \sin x \tag{29}$$

6.2: Comparison of Homotopy Perturbation Sumudu Transform Method and Homotopy-Perturbation Method for Solving Nonlinear Partial Differential Equations

6.2.1: Basic Idea of HPM

We consider the following general nonlinear differential equation,

$$L u + N u = f(x,t) \tag{30}$$

With the initial conditions,

$$u(x,0)=c_1 \quad , \quad u_t(x,0)=c_2$$

Where u is a function of x and t and c_1; c_2 , are constants or functions of x, and L , and N are respectively, the linear and nonlinear operators.

According to HPM, we construct a homotopy which satisfies the following relation,

$$H(u,p)= Lu - Lv_0 + p\,L v_0 + p[N u - f(x,t)] = 0 \tag{31}$$

Where $p\in[0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions. When we put p^{-0} and p^{-1} in Eq. (32), we obtain:

$$H(u,0) = Lu - Lv_0 = 0 \ \text{, and} \quad H(u,1) = Lu + Nu - f(x,t) = 0 \tag{32}$$

Which are the linear and nonlinear original equations, respectively. In topology, this is called deformation and $Lu - Lv_0$, and $Lu + Nu - f(x,t)$ are called hemitropic. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small parameter for $0 \le p \le 1$.

We introduce in this work an alternative way of choosing the initial approximations, that is,

$$v_0 = u(x,0) + t\,u_t(x,0) + L^1 f(x,t) = c_1 + t\,c_1 + L^1 f(x,t) \tag{33}$$

Where $L^{1}(\cdot) - \int_0^t \int_0^t \dots \int_0^t (\cdot)\, dt \dots dt\, dt$ depends on the order of the linear operator. We assume that the initial approximation v_0 given in Eq. (33) can be decomposed into two parts, namely $v_{0,1}$ and $v_{0,2}$ such that:

$$v_0 = v_{0,1} + v_{0,2} \tag{34}$$

In HPM, the solution of Eq. (31) is expressed as:

$$u(x,t) = u_0(x,t) + p\,u_1(x,t) + p^2 u_2(x,t) + \cdots \tag{35}$$

Hence, the approximate solution of Eq. (30) can be expressed as a series of the powers of p, i.e.

$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \cdots \tag{36}$$

6.2.2: Methods of Solution Problems

i. Homotopy Perturbation Sumudu Transforms Method

Consider the following homogenous advection problem,

$$U_t + U U_x = 0 \tag{37}$$
$$U(x,0) = -x .$$

Taking the Sumudu Transform on both sides of Eq. (37) subject to the initial Condition, we get;

$$S[U(x,t)] = -x - u\,S[U U_x] \tag{38}$$

The inverse of Sumudu Transform implies that;

$$U(x,t) = -x - S^{-1}[u\,S[U U_x]] \tag{39}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = -x - p\left(S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} p^n H_n(U) \right]\right]\right) \tag{40}$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

Comparing the coefficients of like powers of p, we get;

$$\begin{aligned}
p^0 &: U_0(x,t) = -x \\
p^1 &: U_1(x,t) = -S^{-1}[u\,S[H_0(U)]] = -xt \\
p^2 &: U_2(x,t) = -S^{-1}[u\,S[H_1(U)]] = -xt^2 \\
p^3 &: U_3(x,t) = -S^{-1}[u\,S[H_2(U)]] = -xt^3
\end{aligned} \tag{41}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = -x\left(1 + t + t^2 + t^3 + \cdots\right) \tag{42}$$

And in a closed form by;

$$U(x,t) = \frac{x}{t-1} \tag{43}$$

ii. Homotopy perturbation Method

To solve Eq. (37) with initial condition, according to the homotopy perturbation technique, we construct the following homotopy;

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial U_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = 0 \tag{44}$$

or equivalently;

$$\frac{\partial v}{\partial t} - \frac{\partial U_0}{\partial t} + p\left[\frac{\partial U_0}{\partial t} + v\,\frac{\partial v}{\partial x}\right] = 0;$$

Suppose that the solution of Eq. (44) can be represented as;

$$v = v_0 + p\,v_1 + p^2 v_2 + p^3 v_3 + \cdots . \tag{45}$$

Substituting Eq. (45) into Eq. (44), and equating the terms of the same power, of P, it follows that;

$$\begin{aligned}
p^0 &: \frac{\partial v_0}{\partial t} - \frac{\partial U_0}{\partial t} = 0 \\
p^1 &: \frac{\partial v_1}{\partial t} + \frac{\partial U_0}{\partial t} + v_0\frac{\partial v_0}{\partial x} = 0 \\
p^2 &: \frac{\partial v_2}{\partial t} + v_0\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_0}{\partial x} = 0 \\
&.\quad .\quad .
\end{aligned} \tag{46}$$

By choosing $U_0(x,t) = U(x,0) = -x$, and solving the above equations, we obtain the following approximations;

$$\begin{aligned}
U_0(x,t) &= -x \\
U_1(x,t) &= -x\,t \\
U_2(x,t) &= -x\,t^2 \\
U_3(x,t) &= -x\,t^3 \\
&.\quad .\quad .
\end{aligned} \tag{47}$$

Then the exact solution of Eq. (37) is given by:

$$U(x,t) = -x\left(1 + t + t^2 + t^3 + \cdots\right) \tag{48}$$

Or in a closed form by:

$$U(x,t) = \frac{x}{t-1} \tag{49}$$

6.3: Comparison of Homotopy Perturbation Sumudu Transform Method and Adomian Decomposition Method for Solving Nonlinear Partial Differential Equations

6.3.1: Basic Idea of ADM

The principle of the Adomian decomposition method (ADM) when applied to a general nonlinear equation is in the following form:

$$LU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \tag{50}$$

$$U(x,0) = h(x)\,, \qquad U_t(x,0) = f(x)$$

inverse operator, L , with $L^{-1}(\cdot) = \int_0^t (\cdot)\,dt$ Equation (50) can be hence as;

$$U(x,t) = L^{-1}[g(x,t)] - L^{-1}[RU(x,t)] - L^{-1}[NU(x,t)] \tag{51}$$

The decomposition method represents the solution of equation (50) as the following infinite series:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \tag{52}$$

The nonlinear operator $NU = \psi(U)$ is decomposed as:

$$NU(x,t) = \sum_{n=0}^{\infty} A_n \tag{53}$$

Where, A_n are Adomian's polynomials, which are defined as [33]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\psi\left(\sum_{i=0}^{\infty}\lambda^i\, U_i(x,t)\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3, \cdots \tag{54}$$

Substituting equations Eqs. (52) and (53) into equation (50), we have

$$U = \sum_{n=0}^{\infty} U_n = U_0 - L^{-1}\left[R\left(\sum_{n=0}^{\infty} U_n\right)\right] - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{55}$$

Consequently, it can be written as:

$$\begin{aligned} U_0(x,t) &= \Phi - L^{-1}g(x,t) \\ U_1(x,t) &= -L^{-1}[R(U_0)] - L^{-1}(A_0) \\ U_2(x,t) &= -L^{-1}[R(U_1)] - L^{-1}(A_1) \\ &\ \ \vdots \\ U_n(x,t) &= -L^{-1}[R(U_{n-1})] - L^{-1}(A_{n-1}) \end{aligned} \qquad (56)$$

where ϕ is the initial condition,

Hence all the terms of U are calculated and the general solution obtained according to ADM as $U = \sum_{n=0}^{\infty} U_n$. The convergence of this series has been proven in [33].

However, for some problems [32] this series can't be determined, so we use an approximation of the solution from truncated series

$$U_M = \sum_{n=0}^{M} U_n \text{ , with, } \quad \lim_{M \to \infty} U_M = U \qquad (57)$$

5.4.2: Method of Solution of the Problem

i. Homotopy Perturbation Sumudu Transforms Method

Consider the following homogenous advection problem [31],

$$U_t + U U_x = 2t + x + t^3 + x t^2 \qquad (58)$$

$$U(x,0) = 0.$$

Taking the Sumudu Transform on both sides of Eq. (58) subject to the initial Condition, we get;

$$S[U(x,t)] = 2u^2 + xu + 3u^4 + 2xu^3 - u S[U U_x] \qquad (59)$$

The inverse of Sumudu Transform implies that;

$$U(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3 - S^{-1}[u S[U U_x]] \qquad (60)$$

Now, applying the Homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3 - p\left(S^{-1}\left[u S\left[\sum_{n=0}^{\infty} p^n H_n(U) \right]\right]\right) \qquad (61)$$

Where $H_n(U)$ are He's polynomials that represent the nonlinear terms.

Comparing the coefficients of like powers of p, we get;

$$p^0 : U_0(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3$$

$$p^1 : U_1(x,t) = -S^{-1}[u\,S[H_0(U)]] \tag{62}$$

$$= -\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{2}{15}xt^5 - \frac{1}{63}xt^7 - \frac{1}{96}t^8$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$U(x,t) = t^2 + xt \tag{63}$$

ii. Adomian decomposition method

We first rewrite Eq. (58) in an operational form;

$$LU = 2t + x + t^3 + xt^2 - UU_x$$
$$U(x,0) = 0 \tag{64}$$

Where the differential operator L is;

$$L = \frac{\partial}{\partial t} \tag{65}$$

The inverse L^{-1} is assumed as an integral operator given by:

$$L^{-1}(\cdot) = \int_0^t (\cdot)\,dt \tag{66}$$

Applying the inverse operator L^{-1} on both sides of Eq. (64) and using the initial condition we find:

$$U(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3 - L^{-1}(UU_x) \tag{67}$$

Substituting Eqs. (52) and (53) into the functional equation (64) gives:

$$\sum_{n=0}^{\infty} U_n(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3 - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{68}$$

Where A_n are the so-called Adomian polynomials, identifying the zeroth component $U_0(x,t)$ by $t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3$, the remaining components $U_n(x,t)$, $n \geq 1$, can be determined by using the recurrence relation:

$$U_0(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3$$
$$U_{k+1}(x,t) = -L^{-1}(A_k) \quad , \quad k \geq 0 \tag{69}$$

Where A_k are Adomian polynomials that were evaluated before in the homogeneous case. This in turn gives the components:

$$U_0(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3$$
$$U_1(x,t) = -\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{2}{15}xt^5 - \frac{1}{63}xt^7 - \frac{1}{96}t^8 \tag{70}$$
$$\cdot \quad \cdot \quad \cdot$$

It is important to recall here that the noise terms appear between the two components U_0 and U_1. The noise terms are identified as the identical terms with opposite signs. We then cancel the noise terms $\pm\frac{1}{4}t^4 \pm \frac{1}{3}xt^3$ between the components U_0 and U_1, and justify that the remaining terms of u0 satisfy the equation. Consequently, the exact solution is:

$$U(x,t) = t^2 + xt \tag{71}$$

6.4: Comparison of Homotopy Perturbation Sumudu Transform Method and Adomian Decomposition Method for Solving Nonlinear Fractional Partial Differential Equations

6.4.1: Basic Idea of HPSTM

To illustrate the basic idea of this method, we consider a general fractional nonlinear no homogeneous partial differential equation with the initial condition of the form,

$$D_t^{\alpha}U(x,t) + RU(x,t) + NU(x,t) = g(x,t) \tag{72}$$
$$U(x,0) = f(x)$$

Where $D_t^{\alpha}U(x,t)$ is the Caputo fractional derivative of the function $U(x,t)$, R is the linear differential operator, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Applying the Sumudu Transform (denoted in this section by S) on both sides of Eq. (72), we get:

$$S\left[D_t^{\alpha}U(x,t)\right] + S[LU(x,t) + NU(x,t)] = S[g(x,t)] \tag{73}$$

Using the property of the Sumudu transform, we get;

$$\begin{aligned} S[U(x,t)] = f(x) + u^{\alpha} S[g(x,t)] \\ - u^{\alpha} S[RU(x,t) + NU(x,t)] \end{aligned} \tag{74}$$

Operating with the Sumudu inverse on both sides of Eq. (73) gives;

$$U(x,t) = G(x,t) - S^{-1}\left[u^{\alpha} S[RU(x,t) + NU(x,t)]\right] \tag{75}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We apply the Homotopy perturbation method;

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \tag{76}$$

And the nonlinear term can be decomposed as;

$$NU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U) \tag{77}$$

For some He's polynomials $H_n(U)$ [26, 45] that are given by;

$$H_n(U_0, U_1, U_2, \ldots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i U_i\right)\right]_{p=0}, \quad n = 0,1,2,\ldots \tag{78}$$

Substituting Eqs. (76) and (77) in Eq. (75), we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = G(x,t) - p S^{-1}\left[u^{\alpha} S\left[R\sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(U)\right]\right] \tag{79}$$

This is the coupling of the Sumudu Transform and the HPM using He’s polynomials. Comparing the coefficients of like powers of p, the following approximations are obtained:

$$\begin{aligned} p^0 &: U_0(x,t) = G(x,t) \\ p^1 &: U_1(x,t) = -S^{-1}\left[u^{\alpha} S[RU_0 + H_0(U)]\right] \\ p^2 &: U_2(x,t) = -S^{-1}\left[u^{\alpha} S[RU_1 + H_1(U)]\right] \\ p^3 &: U_3(x,t) = -S^{-1}\left[u^{\alpha} S[RU_2 + H_2(U)]\right] \\ &\quad . \qquad . \qquad\qquad . \end{aligned} \tag{80}$$

By utilizing the results in Eq. (80), and substituting them into Eq. (75) then the solution of Eq. (72) can be expressed as a power series in p. The best approximation for the solution of initial condition is:

$$U(x,t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n U_n(x,t) = U_0 + U_1 + U_2 + \cdots \tag{81}$$

The solutions of Eq. (81) generally converge very rapidly

6.4.2: Methods of Solution of the Problems

Consider the following nonlinear time-fractional Harry Dym equation,

$$D_t^{\alpha} U(x,t) = U^3(x,t) U_x^3(x,t) \quad , \quad 0 < \alpha \le 1 \tag{82}$$

With the initial condition;

$$U(x,0) = \left(a - \frac{3\sqrt{b}}{2} x\right)^{2/3}$$

Applying the Sumudu Transform on both sides of Eq. (82), subject to initial conditions, we get;

$$S[U(x,t)] = \left(a - \frac{3\sqrt{b}}{2} x\right)^{2/3} + u^{\alpha} S\left[U^3(x,t) U_x^3(x,t)\right] \tag{83}$$

The inverse Sumudu Transform implies that;

$$U(x,t) = \left(a - \frac{3\sqrt{b}}{2} x\right)^{2/3} + S^{-1}\left[u^{\alpha} S\left[U^3(x,t) U_x^3(x,t)\right]\right] \tag{84}$$

Now applying the HPM, we get;

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \left(a - \frac{3\sqrt{b}}{2} x\right)^{2/3} + p\, S^{-1}\left[u^{\alpha} S\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right] \tag{85}$$

Where are He's polynomials that represent the nonlinear terms. So, then He's polynomials are given by;

$$\sum_{n=0}^{\infty} p^n H_n(U) = U^3 D_x^3 U \tag{86}$$

The first few components of He's polynomials are given by;

$$\begin{aligned} H_0(U) &= U_0^3 D_x^3 U_0 \\ H_1(U) &= U_0^3 D_x^3 U_1 + 3U_0^2 U_1 D_x^3 U_0 \\ H_2(U) &= U_0^3 D_x^3 U_2 + 3U_0^2 U_1 D_x^3 U_1 + \left(3U_0 U_1^2 + 3U_0^2 U_2\right) D_x^3 U_0 \\ &\quad . \qquad . \qquad . \end{aligned} \tag{87}$$

Comparing the coefficients of like powers of p, we get;

$$p^0: U_0(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3}$$

$$p^1: U_1(x,t) = S^{-1}\left[u^\alpha S\left[H_0(U)\right]\right] = -b^{3/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3} \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$p^2: U_2(x,t) = S^{-1}\left[u^\alpha S\left[H_1(U)\right]\right] = -\frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-4/3} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{88}$$

$$p^3: U_3(x,t) = S^{-1}\left[u^\alpha S\left[H_2(U)\right]\right]$$

$$= b^{9/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-7/3}\left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2\left(\Gamma(\alpha+1)\right)^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

. . .

In this manner the rest of components of the HPSTM solution can be obtained. Thus, the solution $U(x,t)$ of the Eq. (82) is given as;

$$U(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right) - b^{3/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-4/3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+ b^{9/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-7/3}\left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2\left(\Gamma(\alpha+1)\right)^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \tag{89}$$

6.4.3: Basic Idea of ADM

To illustrate the basic idea of Adomian decomposition method, we consider a general fractional nonlinear no homogeneous partial differential equation with the initial condition of the form,

$$D_t^\alpha U(x,t) + RU(x,t) + NU(x,t) = g(x,t) \tag{90}$$

Where $D_t^\alpha U(x,t)$ is the Caputo fractional derivative of the function $U(x,t)$, R is the linear differential operator, N represents the general nonlinear differential operator, and $g(x,t)$ is the source term.

Applying the operator J_t^α on both sides of Eq. (90), we get;

$$U(x,t) = \sum_{k=0}^{m-1}\left(\frac{\partial^k U}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^\alpha g(x,t) - J_t^\alpha\left[RU(x,t) + NU(x,t)\right] \tag{91}$$

Next, we decompose the unknown function into sum of an infinite number of components given by the decomposition series;

$$U = \sum_{n=0}^{\infty} U_n \tag{92}$$

And the nonlinear term can be decomposed as;

$$NU = \sum_{n=0}^{\infty} A_n \tag{93}$$

Where A_n are Adomian polynomials that are given by;

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i U_i\right)\right]_{\lambda=0} \ , \ n = 0,1,2,... \tag{94}$$

The components $U_0, U_1, U_2, \ldots$ are determined recursively by substituting Eqs. (92) and (93) into Eq. (91) leading to;

$$\sum_{n=0}^{\infty} U_n = \sum_{k=0}^{m-1}\left(\frac{\partial^k U}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) - J_t^{\alpha}\left[R\left(\sum_{n=0}^{\infty} U_n\right) + \sum_{n=0}^{\infty} A_n\right] \tag{95}$$

This can be written as;

$$\begin{aligned} U_0 + U_1 + U_2 + \ldots &= \sum_{k=0}^{m-1}\left(\frac{\partial^k U}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) \\ &- J_t^{\alpha}\left[R(U_0 + U_1 + U_2 + \ldots) + (A_0 + A_1 + A_2 + \ldots)\right] \end{aligned} \tag{96}$$

Adomian method uses the formal recursive relations as;

$$\begin{aligned} U_0 &= \sum_{k=0}^{m-1}\left(\frac{\partial^k U}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) \\ U_{n+1} &= -J_t^{\alpha}\left[R(U_n) + (A_n)\right] \ , \quad n \geq 0 \end{aligned} \tag{97}$$

6.4.4: Methods of Solution of The Problems

To solve the nonlinear time-fractional Harry Dym equation (82), we apply the operator on both sides of Eq. (82) we get;

$$U = \sum_{k=0}^{1-1}\frac{t^k}{k!}\left(D_t^k U\right)_{t=0} + J_t^{\alpha}\left[U^3 D_x^3 U\right] \tag{98}$$

This gives the following recursive relations using Eq. (97):

$$U_0 = \sum_{k=0}^{0} \frac{t^k}{k!} \left(D_t^k U\right)_{t=0}$$
$$U_{n+1} = J_t^{\alpha}\left[A_n\right], \quad n = 0, 1, 2, \cdots \tag{99}$$

Where

$$\sum_{n=0}^{\infty} A_n = U^3 D_x^3 U \tag{100}$$

The first few components of Adomian polynomials are given by

$$A_0(U) = U_0^3 D_x^3 U_0$$
$$A_1(U) = U_0^3 D_x^3 U_1 + 3U_0^2 U_1 D_x^3 U_0$$
$$A_2(U) = U_0^3 D_x^3 U_2 + 3U_0^2 U_1 D_x^3 U_1 + \left(3U_0 U_1^2 + 3U_0^2 U_2\right) D_x^3 U_0 \tag{101}$$
$$\cdot \qquad \cdot \qquad \cdot$$

The components of the solution can be easily found by using the previous recursive relations as;

$$U_0(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3}$$

$$U_1(x,t) = S^{-1}\left[u^{\alpha} S\left[H_0(U)\right]\right] = -b^{3/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3} \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$U_2(x,t) = S^{-1}\left[u^{\alpha} S\left[H_1(U)\right]\right] = -\frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-4/3} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{102}$$

$$U_3(x,t) = S^{-1}\left[u^{\alpha} S\left[H_2(U)\right]\right] = b^{9/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-7/3} \left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2\left(\Gamma(\alpha+1)\right)^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

And so on. In this manner the rest of the components of the decomposition solution can be obtained. Thus, the ADM solution $U(x,t)$ of Eq. (82) is given as;

$$U(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right) - b^{3/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3} \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-4/3} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+ b^{9/2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-7/3} \left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2\left(\Gamma(\alpha+1)\right)^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \tag{103}$$

This is the same solution as obtained by using HPSTM.

Table: Laplace and Sumudu transform of some function

$f(t)$	$F(s)=L[f(t)]$	$F(u)=S[f(t)]$
1	$\frac{1}{s}$	1
t	$\frac{1}{s^2}$	u
$\frac{t^{n-1}}{(n-1)!} \quad n=1,2,\cdots$	$\frac{1}{s^n}$	u^{n-1}
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{u}}$
$2\sqrt{\frac{t}{\pi}}$	$1/s^{\frac{3}{2}}$	$\sqrt{u}$
$\frac{t^{a-1}}{\Gamma(a)}, a>0$	$\frac{1}{s^a}$	u^{a-1}
e^{at}	$\frac{1}{s-a}$	$\frac{1}{1-au}$
$t e^{at}$	$\frac{1}{(s-a)^2}$	$\frac{u}{(1-au)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{at}$, $n=1,2,\ldots$	$\frac{1}{(s-a)^n}$	$\frac{u^{n-1}}{(1-au)^n}$
$\frac{1}{\Gamma(k)}t^{k-1}e^{at}$, $k>0$	$\frac{1}{(s-a)^k}$	$\frac{u^{k-1}}{(1-au)^k}$
$\frac{1}{(a-b)}\left(e^{at}-e^{bt}\right)$, $a\neq b$	$\frac{1}{(s-a)(s-b)}$	$\frac{u}{(1-au)(1-bu)}$
$\frac{1}{(a-b)}\left(a e^{at}-b e^{bt}\right)$, $a\neq b$	$\frac{s}{(s-a)(s-b)}$	$\frac{1}{(1-au)(1-bu)}$

$\frac{1}{w}\sin wt$	$\frac{1}{s^2+w^2}$	$\frac{u}{1+u^2w^2}$
$\cos wt$	$\frac{s}{s^2+w^2}$	$\frac{1}{1+w^2u^2}$
$\frac{1}{a}\sinh at$	$\frac{1}{s^2-a^2}$	$\frac{u}{1-a^2u^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$	$\frac{1}{1-a^2u^2}$
$\frac{1}{w}e^{at}\sin wt$	$\frac{1}{(s-a)^2+w^2}$	$\frac{u}{(1-au)^2+w^2u^2}$
$e^{at}\cos wt$	$\frac{s-a}{(s-a)^2+w^2}$	$\frac{1-au}{(1-au)^2+w^2u^2}$
$\frac{1}{w^2}(1-\cos wt)$	$\frac{1}{s(s^2+w^2)}$	$\frac{u^2}{1+w^2u^2}$
$\frac{1}{w^3}(wt-\sin wt)$	$\frac{1}{s^2(s^2+w^2)}$	$\frac{u^3}{1+w^2u^2}$
$\frac{1}{2w^3}(\sin wt-wt\ \cos wt)$	$\frac{1}{(s^2+w^2)^2}$	$\frac{u^3}{(1+w^2u^2)^2}$
$\frac{t}{2w}\sin wt$	$\frac{s}{(s^2+w^2)^2}$	$\frac{u^2}{(1+w^2u^2)^2}$
$\frac{1}{2w}(\sin wt+wt\cos wt)$	$\frac{s^2}{(s^2+w^2)^2}$	$\frac{u}{(1+w^2u^2)^2}$
$\frac{1}{2k^2}\sin kt\sinh kt$	$\frac{s}{s^4+4k^4}$	$\frac{u^2}{1+4k^4u^4}$
$\frac{1}{2k^3}(\sinh kt-\sin kt)$	$\frac{1}{s^4-k^4}$	$\frac{u^3}{1-k^4u^4}$

$\frac{1}{2k^2}(\cosh kt - \cos kt)$	$\frac{s}{s^4 - k^4}$	$\frac{u^2}{1 - k^4 u^4}$
$j_0(at)$	$\frac{1}{\sqrt{s^2 + a^2}}$	$\frac{1}{\sqrt{1 + a^2 u^2}}$
$H(t-a)$	$\frac{1}{s} e^{-as}$	$e^{-\frac{a}{u}}$
$\delta(t-a)$	e^{-as}	$\frac{1}{u} e^{-\frac{a}{u}}$
$\frac{2}{t}(1 - \cos wt)$	$\ln\left(\frac{s^2 + w^2}{s^2}\right)$	$\frac{1}{u}\ln(1 + w^2 u^2)$
$\frac{2}{t}(1 - \cosh at)$	$\ln \frac{s^2 - a^2}{s^2}$	$\frac{1}{u}\ln(1 - a^2 u^2)$
$\frac{1}{t}\sin wt$	$\tan^{-1}\frac{w}{s}$	$\frac{1}{u}\tan^{-1} wu$

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