

CHAPTER TWO

ON THE ANGULAR MOMENTUM OF PHOTONS

Section (2.1): Introduction

Setting aside the postulation that the photon's spin is its intrinsic degree of freedom, a notion called correlation between the intrinsic degree of freedom and the momentum, the extrinsic degree of freedom, is introduced from the transversality condition. Any particular three component wavefunction f that is restricted by the transversality condition is expressed in terms of a correlation operator Π (where $\Pi = (u \ v)$) and a two-component wavefunction \check{f} . In this way, the correlation operator Π plays the role of connecting two different kinds of representations. In the so-called Maxwell representation, the wavefunction f carries the correlation; the operator of physical quantity does not. In the so-called Jones representation, the wavefunction \check{f} does not carry the correlation; the operator does. Not suffering from any restrictions, \check{f} appears to be the wavefunction about the intrinsic degree of freedom. Furthermore, the fact that the transversality condition cannot completely determine Π shows that the correlation operator possesses a kind of degree of freedom. So identified correlation degree of freedom may take the form of a unit vector I that is independent of the wavevector.

From the point of view of the correlation, it indicates a multiple-to-one correspondence between the Maxwell representation and the Jones representation. When expressed in the Jones representation, all the physical quantities, including the spin and orbital angular momentum (OAM), show up to carry the correlation. The spin lies exactly in the wavevector direction, with the helicity being the component of newly defined polarization operator in the wavevector direction. The OAM about the origin splits into two parts, the OAM of the barycenter about the origin and the OAM about the barycenter. The former is dependent on the helicity as well as I . The correlation degree of freedom I acts as a parameter to determine the helicity-dependent barycenter.

This chapter is concerned with the quantum-mechanical description of photon's angular momentum and related issues. It is known in the first quantization theory [1] that the total angular momentum J_M consists of the spin and the orbital angular momentum (OAM),

$$J_M = S_M + L_M \quad (2.1)$$

The operator of spin S_M and the operator of OAM L_M about the origin can be written as

$$S_M = \hbar \Gamma \quad (2.2a)$$

$$L_M = -\hbar k \times X_M \quad (2.2b)$$

respectively, where $(\Gamma_k)_{ij} = -i \epsilon_{ijk}$ with ϵ_{ijk} the Levi-Civita pseudo tensor, k is the wavevector,

$$X_M = i\nabla \quad (2.3)$$

is the position operator, and ∇ is the gradient operator in the momentum space (k-space). They act on the k-space vector wavefunction $f(k, t)$ which satisfies the Schrodinger equation,

$$i \frac{\partial f}{\partial t} = w f \quad (2.4)$$

and is restricted by the transversality condition,

$$f^\dagger = 0 \quad (2.5)$$

Where $w = ck$ is the angular frequency, $k = |k|$ is the wave number, $\hat{k} = \frac{k}{k}$ is the unit wavevector, superscript \dagger stands for the conjugate transpose, and the convention of matrix multiplication is used to denote the scalar product of two vectors. The angular frequency w plays the role of Hamiltonian. Equation (2.4) together with the transversality condition (2.5) is strictly equivalent to the system of free-space Maxwell's equations [1], [2]. The solution to the Maxwell's equations, $\epsilon(x, t)$ the real-valued electric field in the position space, is expressed in terms of $f(k, t)$ by [1], [2]

$$\epsilon(x, t) = \int N(\omega) f \exp(ik \cdot X) d^3k + c \quad (2.6)$$

Where $N(\omega) = \left[\frac{\hbar(\omega)}{2\epsilon_0(2\pi)^3} \right]^{1/2}$. What should be noted is that [1] it is this electric field and the corresponding magnetic field that determine the interaction of the photon with the electric charge in the position space.

Before the seminal work of Allen and his collaborators [3] on the OAM of photons, the above-mentioned separation of total angular momentum into spin and orbital parts was known to be physically meaningless [4]. A frequently mentioned reason is as follows. The operators of spin and OAM were believed to fulfill [5], [6] the following standard commutation relations,

$$S_M \times S_M = i\hbar S_M \quad (2.7a)$$

$$L_M \times L_M = i\hbar L_M \quad (2.7b)$$

and therefore to be associated [2], [4] with certain kinds of spatial rotation of the wave function. Upon considering [2], [5] that the rotation generated by either of them might ruin the transversality of the wavefunction, the separation of the spin from the OAM was thought to be physically meaningless.

Since the work of Allen et. al. [3], substantial experimental progresses have been achieved in distinguishing the difference between the spin and OAM. They were found [7], [8] to induce different effects in the interaction with tiny birefringent particles trapped off axis in optical tweezers. The spin angular momentum (SAM) makes the particle rotate about its own axis and the OAM makes the particle rotate about the axis of the optical beam. The conversion from spin to OAM was also observed in anisotropic [9], isotropic [10], and nonlinear [11] media. At the same time, much theoretical effort has also been made to explain the separation of the spin from the OAM. It showed with paraxial Laguerre-Gaussian beams that the OAM is carried by a helical wavefront and the spin is carried by the polarization that is denoted by the ellipticity. Nevertheless, taking this conclusion as a criterion [12], [13], the separation of the total angular momentum into spin and OAM parts was shown to be impossible beyond the paraxial approximation. Very recently [14], on the basis of quite a general criterion that the SAM is independent of the choice of reference point and the OAM is dependent on the choice of reference point, the total angular momentum of an arbitrary electromagnetic field was rigorously separated into spin and orbital parts. It was shown that [1] the SAM \bar{S} in a state f was given by the expectation value of spin operator (2.2a),

$$\bar{S} = \int f^\dagger S_M f d^3k \quad (2.8)$$

and the OAM \bar{L} about the origin was given by the expectation value of OAM operator (2.2b),

$$\bar{L} = \int f^\dagger L_M f d^3k \quad (2.9)$$

However, up to now there has not been a well accepted quantum theory to explain the separation of the spin from the OAM. As early as in 1994, van Enk and Nienhuis [5] made a valuable attempt to distinguish the spin and OAM in a framework of second quantization. They showed that their second-quantization operators S_{EN} and L_{EN} for the spin and OAM, respectively, do not satisfy the standard commutation relations. Instead, they obtained the following commutation relations,

$$S_{EN} \times S_{EN} = 0 \quad (2.10a)$$

$$L_{EN} \times L_{EN} = ih(L_{EN} - S_{EN}) \quad (2.10b)$$

On the other hand, they made every effort to vindicate the commutation relations (2.7) for operators S_M and L_M . This is unacceptable upon observing that operators S_{EN} and L_{EN} are just the second-quantization counterparts of operators S_M and L_M , respectively. After all, commutation relations in quantum theory mean nothing but quantization conditions [15].

The purpose of this chapter is to show the develop a quantum theory for the angular momentum of photon. It was generally postulated [1, 4] that the spin of photon is its intrinsic degree of freedom (IDOF) and the three components of wavefunction f represent the "coordinates" of the spin, though the separation of the spin from the OAM was in question. But those postulations themselves are not without problems. Indeed, noticing that the momentum in the wavefunction f represents the photon's extrinsic degree of freedom, the transversality condition (2.5) implies that the three components of f cannot represent the coordinates of the photon's IDOF. This is because, as the notion of IDOF implies, a k -space wavefunction about the IDOF must not be subject to any restrictions. The main idea of this chapter is to uncover what is hidden beyond the transversality condition that is imposed on the wavefunction f .

Supposing that the three components of f are related somehow to the coordinates of the photon's IDOF, the transversality condition may imply that the

IDOF and momentum are not independent of each other. For clarity, such a non-independence is referred to as the correlation between the IDOF and the momentum. In view of this, the three-component wavefunction that is restricted by the transversality condition is said to carry the correlation between the IDOF and the momentum. If this is true, the concrete forms of the observable operators given by Eqs. (2.2) and (2.3) will be misleading, because the spin operator has nothing to do with the momentum and the operators of OAM and position depend only on the momentum. The primary task of this chapter is to investigate whether or not the so-called correlation exists and to explore its physical significance. In the remainder of this chapter, the term “correlation” refers only to that between the IDOF and the momentum.

The transversality condition (2.5) makes it possible to express the three-component wavefunction f in terms of a two-component wavefunction \check{f} and a quasi unitary matrix Π . Because \check{f} is not subject to any restrictions, its two components play the role of representing the coordinates of the photon's IDOF. The quasi unitary matrix Π is a 3-by-2 matrix, consisting of two mutually orthogonal unit vectors u and v that form a righthand Cartesian coordinate system with ω . It appears to be an operator, called correlation operator, to connect two different kinds of representations. In the so-called Maxwell representation, the wavefunction f carries the correlation; the observable operator does not. The so-called Jones representation does the opposite; the wavefunction \check{f} does not carry the correlation and the observable operator does.

Furthermore, the transversality condition itself cannot completely determine Π up to a rotation about ω . This shows that the correlation operator has some kind of degree of freedom, called correlation degree of freedom, which may take the form [16] of a wavevector independent unit vector I . What is hidden beyond the transversality condition is just this degree of freedom. It indicates a multiple-to-one correspondence between the Maxwell representation and the Jones representation from the point of view of the correlation. Each Maxwell representation is characterized by one specific value of the correlation degree of freedom [17]. The main results of this chapter are as follows.

The correlation operator Π is introduced in Section (2.2). The local Cartesian coordinate system uvw is shown in Section (2.3) to be the natural coordinate system to describe the IDOF that is represented in the Jones

representation by the Pauli matrices. On this basis, a matrix vector, called “polarization” operator, is introduced to explicitly express the correlation. The spin is not the IDOF. Carrying the correlation, it lies exactly in the wavevector direction, with the helicity being the component of the polarization vector in the wavevector direction. This is discussed in section (2.4) and is shown that the operator of position vector relative to the origin in the Jones representation splits into two parts, the position vector of the barycenter relative to the origin and the position vector relative to the barycenter. The former depends on the helicity, though the latter does not. The role of I comes out to define the helicity-dependent barycenter [16], [18]. Also, shown that the operator of OAM about the origin also splits into two parts, the OAM of the barycenter about the origin and the OAM about the barycenter. The former also depends on the helicity as well as l [14]. As an application, in addition discussed the effect of I on the eigenfunctions in the complete orthonormal set of Maxwell representation.

Section (2.2): From Transversality to the Operator of Correlation

Based on the transversality condition (2.5), the wavefunction f can be expanded in terms of two time-independent mutually orthogonal base vectors as

$$f = f_1 u + f_2 v \quad (2.11)$$

Where for simplicity the base vectors u and v are supposed to be real-valued and to form a right hand Cartesian coordinate system with w , satisfying the following requirements,

$$V^T u = W^T v = u^T w = 0 \quad (2.12a)$$

$$|u| = |v| = 1 \quad (2.12b)$$

$$u \times v = w \quad (2.12c)$$

and the superscript T stands for the transpose. Convert Eq. (2.11) into a compact form,

$$f = \Pi \check{f} \quad (2.13)$$

Where

$$\Pi = (u \ v) \quad (2.14)$$

is a 3-by-2 matrix upon noticing that u and v are column vectors of three components. Remembering that f is perpendicular to w , the two-component wavefunction $\check{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is the mapping of f on the local coordinate system uvw . Substituting Eq. (2.13) into Eq. (2.4), we arrive at

$$i \frac{\partial \check{f}}{\partial t} = \omega \check{f} \quad (2.15)$$

Which is the Schrödinger equation for the two-component wavefunction \check{f} . Obviously, Eq. (2.15) together with Eq. (2.13) is equivalent to the system of free-space Maxwell's equations. The matrix Π in Eq. (2.13) performs a quasi unitary transformation in the following sense.

Firstly, it fulfills

$$\Pi^\dagger \Pi = I_2 \quad (2.16)$$

So that keeps unchanged the norm of a wavefunction under the transformation,

$$f^\dagger f = \check{f}^\dagger \check{f}$$

Where I_2 stands for the 2-by-2 unit matrix. Secondly, we readily obtain from Eqs. (2.13) and (2.16)

$$\check{f} = \Pi^\dagger f \quad (2.17)$$

meaning that Π^\dagger transforms a three-component wavefunction into a two-component wavefunction. Substituting Eq. (2.17) into Eq. (2.13) and taking into account the arbitrariness of wavefunction, we have

$$\Pi \Pi^\dagger = I_3 \quad (2.18)$$

Where I_3 stands for the 3-by-3 unit matrix. Eqs. (2.16) and (2.18) express the quasi unitarity [19] of the transformation matrix Π . Π^\dagger is the Moore-Penrose pseudo inverse of Π , and vice versa.

Distinct from the three-component wave function f , the two-component wave function \check{f} does not suffer from any restrictions. Therefore, it does not carry the correlation. Its components, f_1 and f_2 play the role of representing the coordinates of the IDOF. It is worth emphasizing that \check{f} is the mapping of f onto the local coordinate system uvw . That is to say, \check{f} is defined on the local

coordinate system. f_1 and f_2 cannot be the Cartesian components of f in the laboratory coordinate system. The quasi unitarity of Π^\dagger guarantees that all the two-component wave functions obtained via Eq. (2.17) form a representation. Noticing that the two-component wavefunction is just the Jones vector [20] for the plane-wave component of f at wave vector k , this representation will be referred to as the Jones representation, and the two-component wave function will be referred to as the Jones wave function.

That the Jones wave function does not carry the correlation would not mean that the IDOF is independent of the momentum in the Jones representation. As a matter of fact, substituting Eq. (2.13) into Eq. (2.8) and noticing Eq. (2.2a), we express the expectation value of the SAM in terms of the Jones wave function as

$$\bar{S} = \int \check{f}^\dagger S_J \check{f} d^3k$$

Where

$$S_J = \Pi^\dagger S_M \Pi = h \Pi^\dagger \Gamma \Pi \quad (2.19)$$

is interpreted as the operator of spin in the Jones representation. Considering that the operator S_M given by Eq. (2.2a) is independent of the momentum, the operator S_J must not be. In other words, it must carry the correlation. For the sake of clarity, the three-component wave function f will be referred to as the Maxwell wave function, and the representation consisting of all the Maxwell wave functions will be referred to as the Maxwell representation. In view of this, the operator S_M is interpreted as the operator of spin in the Maxwell representation. Just the same, the operator L_M given by (2.2b) and the operator X_M given by Eq. (2.3) are interpreted as the operators of OAM and position, respectively, in the Maxwell representation. Substituting Eq. (2.13) into Eq. (2.9), we have

$$\bar{L} = \int \check{f}^\dagger L_J \check{f} d^3k$$

Where $L_J = \Pi^\dagger L_M \Pi$ is the operator of OAM in the Jones representation. Since the momentum operator itself in k-space appears to be a c number, L_J can be expressed as

$$L_J = -\hbar k \times X_J \quad (2.20)$$

In terms of the operator of position in the Jones representation,

$$X_J = \Pi^\dagger X_M \Pi \quad (2.21)$$

Because both L_M and X_M depend only on the momentum, it follows that both L_J and X_J must depend on the momentum as well as the IDOF. That is, they must carry the correlation.

In a word, the quasi unitary matrix in Eq. (2.13) acts as the operator of correlation to connect two different kinds of representations. In the Maxwell representation, the wave function carries the correlation; the operator of physical observable does not. The Jones representation does just the opposite; the wave function does not carry the correlation and the operator does.

Section (2.3): IDOF and its Correlation with the Momentum

A well known two-component notion in classical optics is the Jones vector [20], from which the Stokes parameters [21] are defined in terms of the Pauli matrices to describe the state of polarization of a vector plane wave. The Stokes parameters constitute a vector that corresponds to one point on the Poincaré sphere. But little attention has been paid to the question of what is meant by the direction of that vector. In this section we will show, by generalizing the concept of Stokes parameters from a vector plane wave to any Maxwell wave function f , that the photon's IDOF in the Jones representation is represented by the Pauli matrices but should be understood in the local coordinate system uvw . Due to this reason, the local coordinate system is regarded as the inner coordinate system for the IDOF. In addition, a new kind of degree of freedom is needed to characterize the inner coordinate system in the laboratory coordinate system. This kind of degree of freedom, representable as a unit vector, determines how the IDOF is correlated with the momentum.

(2.3.1): IDOF is a Notion that is Defined in the Inner Coordinate System

IDOF is represented by the Pauli matrices in the Jones representation. As mentioned before, the Jones wave function $\check{f}(k)$ given by Eq. (2.17) is the mapping of the Maxwell wave function f onto the local coordinate system uvw . It is nothing but the Jones vector for the plane-wave component of f at wave

vector k . Generalizing the Stoke parameters of a vector plane wave [20], [21], we define the Stokes parameters of any Maxwell wavefunction f in terms of the corresponding Jones wave function \check{f} and the Pauli matrices as

$$s_i = \check{f}^\dagger \sigma_i \check{f}, i = 1,2,3 \quad (2.22)$$

Where σ_i 's are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.23)$$

Considering that the Jones wave function does not carry the correlation and that the Pauli matrices have nothing to do with the momentum, it is reasonable to think that the Pauli matrices (2.23) represent the IDOF of the photon in the Jones representation.

The correlation operator Π given by Eq. (2.14) consists of the unit vectors along the transverse axes of the local coordinate system $uv\omega$. But this coordinate system cannot be completely determined by the requirements (2.12) that originate from the transversality condition, up to a rotation about ω [22]. Consider a new correlation operator,

$$\Pi' = (u' v') \quad (2.24)$$

The base vectors of which are assumed to be related with those of the old one by a rotation about w through an angle ϕ that is w -dependent,

$$u' = u \cos \phi + v \sin \phi \quad (2.25a)$$

$$v' = -u \sin \phi + v \cos \phi \quad (2.25b)$$

These two equations may be integrated into a single equation of the following form,

$$\Pi' = \Pi D \quad (2.26)$$

by a 2-by-2 rotation matrix

$$D = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

So the correlation operator has some kind of degree of freedom. It is the degree of freedom to choose the local coordinate system. What is hidden beyond the

transversality condition is just this kind of degree of freedom, which will be referred to as the correlation degree of freedom.

(2.3.2): The Maxwell Representation is Characterized by the Correlation Degree of Freedom

Now that the Jones representation is connected with the Maxwell representation by the correlation operator, the existence of the correlation degree of freedom means that we cannot have a one-to-one correspondence between the Maxwell wavefunction and the Jones wavefunction. Either one Jones representation corresponds via Eq. (2.13) to multiple Maxwell representations, or one Maxwell representation corresponds via Eq. (2.17) to multiple Jones representations. Remembering that the Jones wavefunction does not carry the correlation, the Jones representation should be regarded as onefold from the point of view of correlation. As a result, the corresponding Maxwell representation is multi-fold. Any specific correlation operator that is determined by the correlation degree of freedom will transform, via Eq. (2.13), the Jones representation into a corresponding Maxwell representation. In other words, the multi-fold Maxwell representation is characterized by the correlation degree of freedom.

On the other hand, Eq. (2.13) also means that a given Maxwell wave function can be expressed in terms of different correlation operators. That is to say, a given Maxwell wave function can be expressed in different Maxwell representations. In view of this, different Maxwell representations are said to be equivalent. It is worth noting that when expressed in different Maxwell representations, a given Maxwell wave function will manifest different correlation characterizations and thus correspond to different Jones wave functions and different sets of Stokes parameters. Indeed, the pseudo inverse of the new correlation operator (2.24) transforms the same Maxwell wave function f into a new Jones wave function,

$$\check{f}' = \Pi'^{\dagger} f \quad (2.27)$$

Substituting Eqs. (2.26) and (2.13) and making use of Eq. (2.16), we obtain

$$\check{f}' = D^T \check{f} \quad (2.28)$$

Since rotation matrix D is in general dependent on the wavevector, \check{f}' is essentially different from \check{f} . By definition (2.22), this new Jones wavefunction gives a new set of Stokes parameters,

$$S'_i = \check{f}'^\dagger \sigma_i \check{f}' \quad (2.29)$$

Substituting Eq. (2.28) into Eq. (2.29) and making use of Eqs. (2.22), we have

$$S'_1 = s_1 \cos 2\emptyset + s_2 \sin 2\emptyset \quad (2.30a)$$

$$S'_2 = -s_1 \sin 2\emptyset + s_2 \cos 2\emptyset \quad (2.30b)$$

$$S'_3 = s_3 \quad (2.30c)$$

These relations imply that the local coordinate system uvw in association with a particular Maxwell representation acts as the natural coordinate system for the IDOF, as we will show below.

(2.3.3): Local Coordinate System uvw as the Natural Coordinate System for the IDOF

To this end, let us first recall that the three Pauli matrices σ_i form a vector quantity [23]. That is to say, the IDOF that is represented by the Pauli matrices in the Jones representation is a vector. Then we observe that the matrix of rotation about w can be expressed in terms of the Pauli matrix σ_3 as

$$D = \exp(-i\sigma_3\emptyset) \quad (2.31)$$

This means that σ_3 is the component of the IDOF in the wave vector direction, which is in consistency with Eq. (2.30c). Furthermore, we remember that the three Pauli matrices fulfill the commutation relations

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \quad (2.32)$$

A comparison of Eq. (2.32) with Eq. (2.12c) shows that σ_1 and σ_2 are the components of the IDOF along the axes u and v of the coordinate system uvw , respectively. In other words, the local coordinate system uvw appears to be the natural Cartesian coordinate system for the IDOF no matter how the local coordinate system is chosen. In view of this, it is reasonable to regard the local coordinate system as the inner coordinate system for the IDOF.

After figuring out the property of the IDOF, we are in a position to explain the meaning of the correlation and the origin of the correlation degree of freedom.

(2.3.4): Polarization as a Notion to Express the Correlation

Now that the local coordinate system uvw is the inner coordinate system to describe the IDOF, when described in the laboratory coordinate system, the vector of the IDOF will assume the following form in the Jones representation,

$$\sigma = \sigma_1 u + \sigma_2 v + \sigma_3 w. \quad (2.33)$$

Clearly, it is indeterminate unless a particular inner coordinate system is chosen in the laboratory coordinate system, or equivalently, a particular Maxwell representation is chosen. This matrix vector explicitly expresses the correlation of the IDOF with the momentum. Given a Jones wavefunction \check{f} that corresponds to a Maxwell wavefunction f in a particular Maxwell representation, we introduce the so-called Stokes vector [20] as follows,

$$s = \check{f}^\dagger \sigma \check{f} = s_1 u + s_2 v + s_3 w \quad (2.34)$$

Where s_i 's are the Stokes parameters given by Eqs. (2.22). Remembering that the role of the Stokes vector in classical optics is to describe the polarization state of a plane wave, the matrix vector (2.33) is simply referred to as the operator of "polarization" [24]. It is worth pointing out that the Stokes vector is also dependent on the choice of the Maxwell representation, except for the component in the wavevector direction. In fact, in the new Maxwell representation that is associated with the new inner coordinate system $u'v'\omega$, the polarization vector (2.33) reads

$$\sigma' = \sigma_1 u' + \sigma_2 v' + \sigma_3 w$$

the Stokes vector for the same Maxwell wavefunction is given by

$$s' = \check{f}'^\dagger \sigma' \check{f}' = s'_1 u' + s'_2 v' + s'_3 w$$

Substituting Eqs. (2.25) and (2.30) into it, we find

$$s' = (s_1 \cos \emptyset + s_2 \sin \emptyset)u + (s_2 \cos \emptyset - s_1 \sin \emptyset)v + s_3 w$$

So far we have explained why the IDOF is correlated with the momentum and have shown that the polarization vector introduced here is dependent on the choice of the Maxwell representation, or the correlation degree of freedom. Of

course, once the correlation degree of freedom is specified, the polarization vector as well as the Stokes parameters is well defined. In order to explore the physical significance of the correlation, it is necessary to represent the correlation degree of freedom explicitly.

(2.3.5): Representation of Correlation Degree of Freedom in Terms of a Unit Vector

As observed previously, the correlation degree of freedom is the degree of freedom to choose the inner coordinate system uvw in the laboratory coordinate system. In an effort [16, 17] to formulate a representation theory for vector electromagnetic beams, it was once shown that the transverse axes u and v of the inner coordinate system can be completely determined by a wave vector independent unit vector I in the following way,

$$u(I) = v \times \frac{K}{k}, v(I) = \frac{I \times K}{|I \times K|} \quad (2.35)$$

This shows that so introduced unit vector I plays the role of representing the correlation degree of freedom. Now we will consider only this correlation degree of freedom and denote the correlation operator explicitly by $\Pi(I)$. Once the Maxwell representation is specified by one particular value of the correlation degree of freedom I , to each Jones wavefunction \check{f} that is defined in the inner coordinate system, there will correspond a Maxwell wavefunction that is defined in the laboratory coordinate system through the following quasi unitary transformation,

$$f_1 = \Pi(I)\check{f} \quad (2.36)$$

It is worth mentioning that even when the correlation degree of freedom I is specified, we still have a degree of freedom to choose the correlation operator. This is because the base vectors that make up of the correlation operator are not necessarily real valued. It is needless to say that the concrete appearance of the Pauli matrices (2.23) depends on the specific form of the correlation operator (2.14). If we change the form of the correlation operator by altering its base vectors, the appearance of the Pauli matrices are expected to change accordingly.

Now we are ready to examine the physical significance of the correlation in the Jones representation.

Section (2.4): Sam is Aligned with the Wave Vector Direction

The spin operator in the Jones representation is given by Eq. (2.19). Let us decompose the vector operator Γ in the inner coordinate system uvw as

$$\Gamma = u(u^T T) + v(v^T T) + w(w^T T)$$

Substituting it into Eq. (2.19) and making use of properties (2.12), we get

$$S_J = wh\sigma_3 \quad (2.37)$$

Where σ_3 is given in Eqs. (23) and is related to Γ through

$$\sigma_3 = \Pi^\dagger(w^T T)\Pi \quad (2.38)$$

Eq. (2.37) shows that the spin is neither purely intrinsic nor purely extrinsic degree of freedom. It manifests the correlation of the IDOF with the momentum in such a way that it lies entirely along the direction of wavevector, with σ_3 being the operator of helicity in the Jones representation. But it does not depend on the correlation degree of freedom.

From Eq. (2.37) it follows that different Cartesian component of the spin commute,

$$S_J \times S_J = 0 \quad (2.39)$$

This is in agreement with the observation of van Enk and Nienhuis [5]. Besides, the spin is a constant of motion, because the momentum and helicity are constants of motion.

Being a quantization condition, a commutation relation should not be dependent on the choice of representation. So it is expected from Eq. (2.39) that the spin operator S_M in the Maxwell representation satisfies

$$S_M \times S_M = 0 \quad (2.40)$$

Indeed, when the quasi unitarities (2.16) and (2.18) are taken into account, it is not difficult to find from Eqs. (2.19) and (2.37) that $S_M = hw\Pi\sigma_3\Pi^\dagger$. Since

$$\Pi\sigma_3\Pi^\dagger = W^T T \quad (2.41)$$

as can be seen from Eq. (2.38), we finally have

$$S_M = \hbar \omega (W^T T) \quad (2.42)$$

Obviously, it fulfills commutation relation (2.40). A comparison of Eq. (2.42) with Eq. (2.2a) shows that the transversality condition makes the transverse components of T vanishes completely [6].

Next, we turn our attention to the OAM. Since the momentum operator in k -space is a c -number, we start with the position operator.

(2.4.1): Correlation Degree of Freedom Defines Helicity Dependent Barycenter

Substituting Eqs. (2.14) and (35) into Eq. (2.21) yields, after straightforward calculation,

$$X_j = \mathcal{E} + \xi \quad (2.43)$$

Where

$$\mathcal{E} = \frac{I \cdot k}{k|I \times k|} v \sigma_3 \equiv A(I, k) \sigma_3 \quad (2.44)$$

$A = \frac{I \cdot k}{k|I \times k|} v$, and $\xi = i\nabla$ that is defined in the Jones representation. The position operator in the Jones representation splits into two terms. The first term \mathcal{E} is neither purely extrinsic nor purely intrinsic degree of freedom. It is perpendicular to the wavevector. Its Cartesian components commute with one another, $\mathcal{E} \times \mathcal{E} = 0$. Moreover, it commutes with the Hamiltonian ω . So this term can be regarded as denoting some reference point relative to the origin. The second term ξ is an ordinary gradient operator in k -space. Because the wavefunction in the Jones representation is not subject to any restrictions, this term fulfills the canonical commutation relations,

$$\xi \times \xi = 0 \quad (2.45)$$

Obviously, it has nothing to do with the IDOF, having the meaning of position vector relative to the reference point. Because the momentum p also fulfills the canonical commutation relations,

$$p \times p = 0 \quad (2.46)$$

ξ appears to be the generalized coordinate that is conjugate to the momentum and fulfills the following canonical commutation relations with the momentum,

$$[\xi_i, P_j] = i\hbar\delta_{ij} \quad (2.47)$$

To understand the meaning of the reference point denoted by \mathcal{E} , let us examine its eigen state that is described by wavefunction

$$\check{f}_{\gamma, k_0}(k) = \check{\alpha}_{\gamma} \delta^3(k - k_0) \exp(-i\omega_0 t) \quad (2.48)$$

Where $\gamma = \pm 1$ are the eigenvalues of helicity operator σ_3 having eigenfunctions

$$\check{\alpha}_{+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \check{\alpha}_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad (2.49)$$

respectively, k_0 is the eigenvalue of the momentum, $\omega_0 = ck_0$, and $k_0 = |k_0|$. On one hand, the expectation value of ξ in the eigen state (2.48) vanishes,

$$\langle \xi \rangle = \int \check{f}_{\gamma, k_0}^{\dagger} \xi \check{f}_{\gamma, k_0} d^3k = 0 \quad (2.50)$$

On the other hand, to each specific eigen state indexed by γ and k_0 , there corresponds an eigenvalue of \mathcal{E} ,

$$\mathcal{E}_{\gamma, k_0} = \frac{\gamma I \cdot k_0}{k_0 |I \times k_0|} v_0 \quad (2.51)$$

Where $v_0 = \frac{I \times k_0}{|I \times k_0|}$. We see that the eigenvalue (2.51) represents the position of the center of mass [6], [25] or the position of the barycenter. In the following we will refer to the reference point denoted by \mathcal{E} explicitly as the barycenter. It can be understood as the manifestation of the correlation between the IDOF and the momentum in the position operator. But different from the spin operator (2.37), so defined barycenter (2.44) is not determinable solely by the IDOF and the momentum. It also depends on the way the IDOF is correlated with the momentum. So it has an unambiguous dependence on the correlation degree of freedom. This explains why the so-called spin Hall effect of photon [26] can be expounded [18] in terms of this degree of freedom.

Due to the correlation of the IDOF with the momentum, the notion of position for the photon is no longer a purely extrinsic degree of freedom. It is no

wonder why its Cartesian components do not commute with one another [6], [27]. Straightforward calculations give

$$\hat{X}_J \times \hat{X}_J = i(\nabla \times A)\sigma_3 \quad (2.52)$$

So it cannot be regarded as the generalized coordinate that is conjugate to the momentum [2].

(2.4.2): OAM is Dependent on Helicity

Substituting Eq. (2.43) into Eq. (2.20), we find that the OAM in the Jones representation splits into two parts,

$$L_J = \Lambda + \lambda \quad (2.53)$$

The first part is $= h\mathcal{E} \times k$, which has the meaning of the OAM of the barycenter about the origin. With the help of Eq. (2.44), it becomes

$$\Lambda = h \frac{I \cdot k}{|I \times k|} u\sigma_3 \quad (2.54)$$

Clearly, it is perpendicular to the wavevector. It depends not only on the extrinsic and intrinsic degrees of freedom but also on the correlation degree of freedom. Like \mathcal{E} , it commutes with the Hamiltonian,

$$[\Lambda, \omega] = 0 \quad (2.55)$$

and its Cartesian components in the laboratory coordinate system commute with one another,

$$\Lambda \times \Lambda = 0 \quad (2.56)$$

The second part is $\lambda = -hk \times \xi$ which is interpreted as the OAM about the barycenter. Being commuting with the Hamiltonian

$$[\lambda, \omega] = 0 \quad (2.57)$$

It is a constant of motion. Since ξ and the momentum fulfill the canonical commutation relations (2.47), this part satisfies the standard commutation relations,

$$\lambda \times \lambda = ih\lambda \quad (2.58)$$

The entire OAM about the origin is thus the OAM of the barycenter about the origin plus the OAM about the barycenter. According to Eqs. (2.55) and (2.57), the entire OAM commutes with the Hamiltonian,

$$[L_J, \omega] = 0$$

But a straightforward calculation yields

$$L_J \times L_J = ih(L_J - S_J)$$

where S_J is the SAM given by Eq. (2.37). Contrary to van Enk and Nienhuis' claim [5], the entire OAM in the first quantization theory does not satisfy the standard commutation relations (2.7b). Furthermore, the entire OAM and the SAM do not commute. They fulfill the following commutation relations,

$$L_J \times S_J = ihS_J \quad (2.59)$$

At last, let us have a look at the total angular momentum in the Jones representation, $J_J = S_J + L_J$. Clearly, it commutes with the Hamiltonian. According to Eqs. (2.37), (2.53), and (2.54), it has the form of

$$J_J = h\sigma_3 \frac{I \times v}{I \cdot v} + \lambda \quad (2.60)$$

which shows a very interesting property that the component of J_J in the direction of I does not depend on the IDOF and is equal to the component of λ in the same direction. In addition, it follows from Eqs. (2.56), (2.58), and (2.59) that the total angular momentum satisfies the standard commutation relations,

$$J_J \times J_J = ihJ_J$$

(2.4.3): Implication of the Correlation Degree of Freedom on the Complete

Orthonormal Set of Maxwell Representation

In quantum mechanics, any state function can be expanded in terms of a complete orthonormal set of eigenfunctions. According to Eq. (2.13), the complete orthonormal set for the Maxwell representation can be obtained from the complete orthonormal set for the Jones representation. On the basis of commutation relations (2.32) and the properties of observable operators

discussed in Section (2.3), the elements of the complete orthonormal set for the Jones representation can be written as

$$\tilde{f}_{\gamma,e} = \tilde{\alpha}_{\gamma} f_e \quad (2.61)$$

where $\tilde{\alpha}_{\gamma}$'s are given by Eqs. (2.49), f_e 's are the eigenfunctions of the maximal set of commuting observables that depend only on the extrinsic degree of freedom, and the suffix e stands for the collection of relevant quantum numbers.

However, the elements of the complete orthonormal set for the Maxwell representation are not so simple, due to the multiple-to-one correspondence between the Maxwell representation and the Jones representation. In accordance with Eq. (2.36), each Maxwell representation has its own complete orthonormal set, the elements of which have to be characterized, in addition to the above-mentioned quantum numbers γ and e , by the correlation degree of freedom,

$$f_{I,\gamma,e} = \Pi(I) \tilde{f}_{\gamma,e} = A_{I,\gamma} f_e \quad (2.62)$$

where $A_{I,\gamma} = \Pi(I) \tilde{\alpha}_{\gamma}$. It is remarked that though $A_{I,\gamma}$ is the eigen function of the helicity operator $w^T T$ in the Maxwell representation regardless of how the correlation degree of freedom is chosen, the correlation degree of freedom determines the helicity-dependent barycenter as was revealed in Section (2.4). Here let us appreciate the observable effect of the correlation degree of freedom from the point of view of the Maxwell wave function.

The eigen function in a specific Maxwell representation that corresponds to a particular eigen function $\tilde{f}_{\gamma,e}$ in the Jones representation is given by Eq. (2.62). In a different Maxwell representation that is denoted by I' the eigenfunction that corresponds to the same eigenfunction in the Jones representation reads

$$f_{I',\gamma,e} = \Pi(I') \tilde{f}_{\gamma,e}$$

Making use of Eqs. (2.26) and (2.31) and noticing that $\tilde{f}_{\gamma,e}$ is the eigen function of σ_3 with eigenvalue γ , we get

$$f_{I',\gamma,e} = \exp(-i\gamma\Phi) f_{I,\gamma,e} \quad (2.63)$$

$f_{I',\gamma,e}$ is different from $f_{I,\gamma,e}$ by a wave vector-dependent phase factor. Upon taking Eq. (2.6) into account, we see that they are physically different.

As for the factor f_e that depends only on the extrinsic degree of freedom, we may have different choices on the basis of the discussions in subsections (2.4.1) and (2.4.2). Three different kinds of schemes are presented below.

(1) **Plane waves**

In the first place, according to commutation relations (2.46), we may choose $p_1, p_2,$ and p_3 as the maximal set of commuting observables. Denoting by k_0 the eigen momentum, $e = k_0$, we have for the eigenfunction,

$$f_e = \delta(k - k_0) \exp(-i\omega_0 t)$$

This is the case that we discussed in Section (2.4).

(2) **Spherical surface harmonics:**

According to commutation relations (2.57) and (2.58), a second choice for the maximal set of commuting observables is to select $\omega, \lambda^2,$ and $\lambda_z = -i\hbar \frac{\partial}{\partial \varphi}$. It is well known that the common normalized eigen functions of λ^2 and λ_z in k -space are the spherical harmonic functions [2],

$$Y_{lm}(w) = \left\{ \frac{2l+1(l-m)!}{4\pi(l+m)!} \right\}^{1/2} = P_l^m(\cos \vartheta) e^{im\varphi}$$

Which satisfy the following eigen value equations,

$$\lambda^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}, l = 0, 1, 2 \dots \quad (2.64a)$$

$$\lambda_z Y_{lm} = m\hbar Y_{lm}, m = \pm 1, \pm 2, \dots \pm l \quad (2.64b)$$

Their orthonormal relations assume the form

$$\int_{\vartheta=0}^{2\pi} \int_{\varphi=0}^{2\pi} Y_{l'm'}^* Y_{lm} d\Omega = \delta_{l'l} \delta_{m'm}$$

Letting be $\omega_0 = ck_0$ the eigen energy, it is easy to show that the expected eigenfunction has the form of

$$f_e = \frac{\delta(k - k_0)}{k_0} Y_{lm}(w) \exp(-i\omega_0 t) \quad (2.65)$$

Where $e = \{\omega_0, l, m\}$. They constitute a complete set and have the following orthonormality

$$\int_{k=0}^{\infty} \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\pi} f_{e'}^* f_e d^3k = \delta(k'_0 - k_0) \delta_{l'l} \delta_{m'm}$$

Where $e' = \{\omega'_0, l', m'\}$ and $\omega'_0 = ck'_0$.

(3) Non-diffraction beams

Noticing that λ_z and p_z commute and they are constants of motion, a third choice is to select ω , p_z , and λ_z as the maximal set of commuting observables. It is known that p_z and λ_z have the following normalized eigenfunctions in cylindrical coordinates,

$$X_{K_z, om} = \frac{1}{\sqrt{2\pi}} \delta(k_z - k_{z0}) e^{im\varphi}, m = \pm 1, \pm 2 \dots$$

With eigenvalues k_{z0} and mh , respectively. The common eigenfunctions of ω , p_z , and λ_z are given by

$$f_e = \frac{\sqrt{k_0}}{kp_0} \delta(k_p - k_{p0}) X_{z0\mu} \exp(-i\omega_0 t) \quad (2.66)$$

Where $e = \{\omega_0, k_{z0}, m\}$ and $k_{p0} = (k_0^2 - k_{z0}^2)^{1/2}$. They satisfy the following orthonormal relations,

$$\int f_e^* f_e k_p dk_p d\varphi dk_z = \delta(k'_0 - k_0) \delta(k'_{z0} - k_{z0}) \delta_{m'm}$$

The complete orthonormal set for each Maxwell representation that is obtained by substituting Eq. (2.66) into Eq. (2.62) is basically the complete orthonormal set of vector diffraction-free beams in position space [17].

Section (2.5): Conclusions

In conclusion, the transversality condition that is imposed on the Maxwell wave function not only indicates the correlation of photon's IDOF with its momentum but also implies a correlation degree of freedom that has observable effects. In the Jones representation in which the wave function does not carry the correlation, the IDOF is represented by the Pauli matrices. To explicitly express

the correlation, the operator of polarization vector is introduced. It reduces to the known meaning of polarization in the case of a single plane wave. It is shown in the Jones representation that all the physical quantities, including the SAM, carry the correlation. The role of the correlation degree of freedom is to determine the photon's barycenter. The spin is aligned exactly with the wave vector direction. The helicity is the component of the polarization vector in the wave vector direction. The OAM about the origin splits into two parts. One is the OAM of the barycenter about the origin. The other is the OAM about the barycenter.

Both the spin and OAM do not fulfill the standard commutation relations, even in the first quantization theory. So neither of them can be regarded as a generator of spatial rotation. Obviously, they are separable, though they both depend on the helicity. The spin has nothing to do with the correlation degree of freedom. The OAM has one term that depends on the correlation degree of freedom.