

CHAPTER THREE

CANONICAL SEPARATION OF ANGULAR MOMENTUM OF LIGHT INTO ITS ORBITAL AND SPIN PARTS

Section (3.1): Quantum Mechanics of Photons

Many authors have emphasized the difficulties encountered in the separation of the total angular momentum of light into its orbital and spin parts. A popular formula expressing this separation, has the form

$$\int d^3r \mathbf{r} \times (\epsilon_0 \mathbf{E} \times \mathbf{B}) = \int d^3r \epsilon_0 E_i (\mathbf{r} \times \nabla) A_i + \int d^3r \epsilon_0 \mathbf{E} \times \mathbf{A} \quad (3.1)$$

This prescription is marred by a defect: the splitting is gauge dependent because it involves the vector potential \mathbf{A} . This problem has been resolved by an *ad hoc* postulate that the potential must be evaluated in the transverse gauge but this prescription lacks a deeper foundation. Most of the separation has been given only for monochromatic fields or in the paraxial approximation. An additional problem that has not been resolved to the satisfaction of many authors was caused by their wish to disentangle completely the orbital and spin degrees of freedom. This is possible for massive particles but not for massless particles. The direction of the spin for all massless particles is firmly locked onto the direction of momentum: it can only be parallel or antiparallel to momentum. In other words, the helicity of massless particles—the projection of its total angular momentum on the direction of momentum—can only take the values $\pm s$. This fact makes it impossible to independently rotate the orbital and spin degrees of freedom of photons.

(3.1.1): Darwin's Theory

Darwin's theory of evolution is the widely held notion that all life is related and has descended from a common ancestor: the birds, bananas and the flowers all related. Darwin's general theory presumes the development of life from non-life and stresses a purely naturalistic (undirected) "descent with modification". That is, complex creatures evolve from more simplistic ancestors naturally over time. In a nutshell, as a random genetic mutations occur within an organism's genetic code, the beneficial mutations are preserved because they aid survival—a process known as "natural selection". These beneficial mutations are passed on to the next generation. Over time, beneficial mutations accumulate and

the result is an entirely different organism (not just a variation of the original, but an entirely different creature).

(3.1.2): Fourier Transforms

Localized wave packets can be constructed by superposing, in the same region of space, waves of slightly different wavelengths, but with phases and amplitudes chosen to make the superposition constructive in the desired region and destructive outside it. Mathematically, we can carry out this superposition by means of Fourier transforms. For simplicity, we are going to consider a one-dimensional wave packet; this packet is intended to describe a "classical" particle confined to a one-dimensional region; for instance, a particle moving along the x-axis. We can construct the packet $\psi(x, t)$ by superposing plane waves (propagating along the x-axis) of different frequencies (or wavelengths):

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega t)} dk \quad (3.2)$$

$\phi(k)$ is the amplitude of the wave packet.

In what follows we want to look at the form of the packet at a given time; we will deal with the time evolution of wave packets later. Choosing this time to be $t = 0$ and abbreviating $\psi(x, 0)$ by $\psi_0(x)$, we can reduce (3.2) to

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk \quad (3.3)$$

Where $\phi(k)$ is the Fourier transform of $\psi_0(x)$,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi_0(x) e^{-ikx} dx \quad (3.4)$$

Definition (3.1.1): Normalizing the Wave function

When a wave function that solves the Schrödinger equation is multiplied by an undetermined constant A , we normalize the wave function by solving [92]:

$$\frac{1}{A^2} = \int_{-\infty}^{+\infty} |\psi(x, t)|^2 dx$$

The normalized wave function is then $A\psi(x, t)$. Normalization means that:

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = 1 \quad (3.5)$$

Example (3.1.2)

Find A and B so that:

$$\phi(x) = \begin{cases} A & \text{for } 0 \leq x \leq a \\ Bx & \text{for } a \leq x \leq b \end{cases}$$

Is normalized.

Solution

$$\begin{aligned} \int_{-\infty}^{+\infty} |\phi(x)|^2 dx &= \int_0^a A^2 dx + \int_a^b B^2 x^2 dx \\ &= A^2 x \Big|_0^a + \frac{B^2 x^3}{3} \Big|_a^b = A^2 a + \frac{B^2(b^3 - a^3)}{3} \end{aligned} \quad (3.6)$$

using $\int_{-\infty}^{+\infty} |\phi(x)|^2 dx = 1$, we obtain

$$A^2 a + \frac{B^2(b^3 - a^3)}{3} = 1, \Rightarrow A^2 = \left(\frac{1}{a}\right) \left(1 - \frac{B^2(b^3 - a^3)}{3}\right) \quad (3.7)$$

As long as $\int_{-\infty}^{+\infty} |\phi(x)|^2 dx = 1$ is satisfied, we are free to arbitrarily choose one of the constants as long as it's not zero. So we set $B = 1$:

$$A^2 = \left(\frac{1}{a}\right) \left(1 - \frac{(b^3 - a^3)}{3}\right) \Rightarrow A = \sqrt{\left(\frac{1}{a}\right) \left(1 - \frac{(b^3 - a^3)}{3}\right)} \quad (3.8)$$

Example (3.1.3)

Find the Fourier transform for $\phi(k) = \begin{cases} A(a - |k|) & |k| \leq a \\ 0 & |k| > a \end{cases}$

Where a is a positive parameter and A is a normalization factor to be found.

Solution

The normalization factor A can be found at once

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} |\phi(k)|^2 dk = |A|^2 \int_{-a}^0 (a+k)^2 dk + |A|^2 \int_0^a (a-k)^2 dk \\ &= \frac{2a^3}{3} |A|^2 \end{aligned}$$

which yields $A = \sqrt{3/(2a^3)}$, $\phi(k) = \sqrt{3/(2a^3)} (a - |k|)$

Now the Fourier transform of $\phi(k)$ is

$$\begin{aligned} \psi_0(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} dk \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{3}{2a^3}} \left[\int_{-a}^0 (a+k) e^{ikx} dk + \int_0^a (a-k) e^{ikx} dk \right] \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{3}{2a^3}} \left[\int_{-a}^0 k e^{ikx} dk - \int_0^a k e^{ikx} dk + a \int_{-a}^a e^{ikx} dk \right] \end{aligned} \quad (3.10)$$

using the integrations

$$\begin{aligned} \int_{-a}^0 k e^{ikx} dk &= \frac{a}{ix} e^{-iax} + \frac{1}{x^2} (1 - e^{-ix}) \\ \int_0^a k e^{ikx} dk &= \frac{a}{ix} e^{iax} + \frac{1}{x^2} (e^{ix} - 1) \\ \int_{-a}^a e^{ikx} dk &= \frac{1}{ix} (e^{iax} - e^{-iax}) = \frac{2\sin(ax)}{x} \end{aligned}$$

and after some straight forward calculations, we end up with

$$\psi_0(x) = \frac{4}{x^2} \sin^2\left(\frac{ax}{2}\right) \quad (3.11)$$

The correct, gauge invariant separation of the total angular momentum into its orbital and spin parts was proposed a long time ago by Darwin. Darwin's classic work was cited without a comment. (The authors rederived his result, usually in a special case of monochromatic waves). The Darwin formula is based on the Fourier transforms of the electromagnetic field. Therefore, it does not suffer from gauge dependence. With slight changes of notation it reads:

$$\begin{aligned} & \int d^3r r \times (\epsilon_0 E \times B) \\ &= -2i \epsilon_0 \int \frac{d^3k}{c|k|} \times [E_i^*(k)(k \times \nabla_k)E_i(k) + E^*(k) \times E(k)] \end{aligned} \quad (3.12)$$

Where $E(k)$ is the plane-wave component of the electric field,

$$E(r, t) = \int \frac{d^3k}{(2\pi)^{3/2}} [E(k)e^{-i\omega kt + ikr} + c] \quad (3.13)$$

We follow in the footsteps of Darwin who wrote 'The main principle of the idea that, since matter and light both possess the dual characters of particle and wave, a similar mathematical treatment should be applied to both, and that this has not been yet done as fully as should be possible'. We show that, indeed, the wave particle duality enables one to determine the correct separation of total angular momentum. Namely, we shall show that the Darwin separation of the total angular momentum for an arbitrary electromagnetic field into two parts follows from the photon picture of the electromagnetic field. It is in essence the separation into the part perpendicular to the photon momentum and the part parallel to the photon momentum. The first part must be identified with the orbital angular momentum whereas the second part must be identified with spin—represented by helicity. In this way, by seamlessly joining the particle and the field aspect of electromagnetism, we complete the program started by Darwin. Our analysis of the angular momentum of light starts from the quantum mechanical description of photons.

There is no consensus as to what represents the photon wave function in the coordinate representation (cf, [27]) but there is no disagreement as to the meaning of the photon wave function in momentum space. This wave function was introduced in the early years of quantum electrodynamics and was used as a standard concept. Once we accept the existence of the photon wave function in

momentum space we should define then action of various operators representing physical quantities.

Definition (3.1.4):

Ordinary quantum mechanical systems have a fixed number of particles, with each particle having a finite number of degrees of freedom. In contrast, the excited states of QFT can represent any number of particles. This makes quantum field theories especially useful for describing systems where the particle count number may change over time, a crucial feature of relativistic dynamic.

Example (3.1.5):

Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

Solution

Recall that the energy and momentum of a relativistic particle are given by

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}, p = mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \quad (3.14)$$

Where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Squaring and adding the expressions of E and p , we obtain $E^2 = p^2 c^2 + m_0^2 c^4$, hence

$$E = c\sqrt{p^2 + m_0^2 c^2} \quad (3.15)$$

Using this relation along with $p^2 + m_0^2 c^2 = m_0^2 c^2 / (1 - v^2/c^2)$ we can show that the group velocity is given as follows

$$v_g = \frac{dE}{dp} = \frac{d}{dp} \left(c\sqrt{p^2 + m_0^2 c^2} \right) = \frac{pc}{\sqrt{p^2 + m_0^2 c^2}} = v \quad (3.16)$$

The group velocity is thus equal to the speed of the particle, $v_g = v$.

The phase velocity can be found from (3.15): $v_{ph} = E/p = c \sqrt{1 + m_0^2 c^2 / p^2}$

Which, when combined with $p = m_0 v / \sqrt{1 - v^2 / c^2}$, leads to

$$\sqrt{1 + m_0^2 c^2 / p^2} = c/v,$$

hence

$$v_{ph} = \frac{E}{P} = c \sqrt{1 + \frac{m_0^2 c^2}{p^2}} = \frac{c^2}{v} \quad (3.17)$$

This shows that the phase velocity of the wave corresponding to a relativistic particle with $m_0 \neq 0$ is larger than the speed of light, $v_{ph} = c^2/v > c$. This is indeed unphysical. The result $v_{ph} > c$ seems to violate the special theory of relativity, which states that the speed of material particles cannot exceed c . In fact, this principle is not violated because v_{ph} does not represent the velocity of the particle which is represented by the group velocity (3.16). As a result, the phase speed of a relativistic particle has no meaningful physical significance.

Finally, the product of the group and phase velocity is equal to c^2 , i.e., $v_g v_{ph} = c^2$.

Definition (3.1.6): (Poincaré Group)

The Poincaré group is the group of Minkowski spacetime isometries. It is a ten-dimensional noncompact Lie group. The abelian group of translations is a normal subgroup, while the Lorentz group is also a subgroup, the stabilizer of the origin. The Poincaré group itself is the minimal subgroup of the affine group which includes all translations and Lorentz transformations. More precisely, it is a semi-direct product of the translations and Lorentz group.

In a relativistic theory—and there is no nonrelativistic theory of photons—we should first of all define the operators representing ten generators of the

Poincaré group: the generators of translation in space (momentum \hat{P}), translation in time (energy \hat{H}), rotation (angular momentum), and Lorentz boosts (moment of energy \hat{K}). These operators must obey the following commutation relations appropriate for the Poincaré group [92]:

$$[\hat{H}, \hat{P}_i] = 0, [\hat{H}, \hat{J}_i] = 0, [\hat{H}, \hat{K}_i] = -ihc\hat{P}_i \quad (3.18a)$$

$$[\hat{P}_i, \hat{P}_j] = 0, [\hat{J}_i, \hat{J}_j] = ih\epsilon_{ijk}\hat{J}_k, [\hat{K}_i, \hat{K}_j] = -ihc^2\epsilon_{ijk}\hat{J}_k \quad (3.18b)$$

$$[\hat{J}_i, \hat{P}_j] = ih\epsilon_{ijk}\hat{P}_k, [\hat{J}_i, \hat{K}_j] = ih\epsilon_{ijk}\hat{K}_k, [\hat{K}_i, \hat{P}_j] = ih\delta_{ij}\hat{H} \quad (3.18c)$$

There are no problems with the construction of the generators for *massive* particles. The following set of operators was given long time ago by Foldy [28]

$$\hat{H} = E_p \quad (3.19a)$$

$$\hat{P} = p \quad (3.19b)$$

$$\hat{J} = h\nabla_p \times p + S \quad (3.19c)$$

$$\hat{K} = ihE_p\nabla_p - \frac{S \times p}{mc^2 + E_p} \quad (3.19d)$$

Where ∇_p denotes the gradient with respect to the components of momentum and the spin vector S is built from three $(2s + 1) \times (2s + 1)$ matrices that obey the commutation relations of angular momentum. The matrices S act on the $(2s + 1)$ component wavefunctions describing the states of a particle with spin s . In this case, the splitting of the angular momentum into its orbital and spin parts is quite obvious.

The representation of the generators of the Lorentz group for *massless* particles was given by Lomont and Mose [29]. We will use here a modified version of these generators for photons [30], [31] that exhibits its geometrical meaning. The momentum operator, by definition, acts on the wavefunctions in momentum representation as a multiplication by $h\mathbf{k}$. There is no question that the operator representing the energy of the photon (the Hamiltonian) must be the modulus of the momentum vector multiplied by c . The complete list of generators also contains the operator of angular momentum and the boost operator [92],

$$\hat{H} = hw_k \quad (3.20a)$$

$$\hat{P} = \hbar k \quad (3.20b)$$

$$\hat{J} = i\hbar D \times k + \hbar \hat{x} n_k \quad (3.20c)$$

$$\hat{K} = i\hbar \omega_k D \quad (3.20d)$$

Where $n_k = k/|k|$, the photon helicity operator \hat{x} has the eigenvalues ± 1 , and D stands for the covariant derivative on the light cone ($\nabla_k = \partial/\partial k$),

$$D = \nabla_k - i\hat{x} \alpha(k) \quad (3.21)$$

These operators act on the two-component photon wavefunctions

$$f(k) = \begin{pmatrix} f_L(k) \\ f_R(k) \end{pmatrix} \quad (3.22)$$

And satisfy the commutation relations (3.18a)–(3.18c) appropriate for the Poincaré group. The two components of the photon wavefunction correspond to two eigenvalues of \hat{x} ,

$$\hat{x} \begin{pmatrix} f_L(k) \\ f_R(k) \end{pmatrix} = \begin{pmatrix} f_L(k) \\ -f_R(k) \end{pmatrix} \quad (3.23)$$

We used the indices L and R to denote the eigenfunctions of the helicity operator since they correspond to left-handed and right-handed circular polarization. The properties of the covariant derivative are obtained from the commutation relations for the angular momentum and they read:

$$[D_i, D_j] = i\hat{x} \epsilon_{ijl} n_l / |k|^2 \quad (3.24)$$

These conditions determine the vector $\alpha(k)$ up to a gauge transformation

$$\alpha(k) \rightarrow \alpha(k) + \nabla_k \varphi(k) \quad (3.25)$$

Which is connected to the change of the phase of the wavefunction, in analogy to the theory of charged particles coupled to an electromagnetic field. The generators (3.20a)–(3.20b) are Hermitian with respect to the following Lorentz-invariant scalar product

$$\langle f|g\rangle = \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot g(k) = \int \frac{d^3k}{hw_k} [f_L^*(k)g_L(k) + f_R^*(k)g_R(k)] \quad (3.26)$$

Section (3.2): Electromagnetic Field

In order to solve the problem of the total angular momentum separation into two parts for the classical electromagnetic field, we shall employ the correspondence between the fundamental physical quantities (energy, momentum, and angular momentum) in photon quantum mechanics and in Maxwell theory. In the quantum mechanics of photons these quantities are represented by the operators (3.20a)– (3.20d) .

(3.2.1): Maxwell's Theory

Maxwell's equations can be cast into covariant form. As Einstein expressed it: The general laws of nature are to be expressed by equations which holds good for all systems of coordinates that is are covariant with respect to any substitution whatever generally covariant".

Maxwell's theory of electromagnetism is alongside with Einstein's theory of gravitation, on of the most beautiful of classical field theories. Having chosen units in which $\mu_0 = \epsilon_0 = c = 1$, Maxwell's equations then take the form [92]:

$$\nabla \cdot E = \rho \quad (3.27)$$

$$\nabla \times B - \frac{\partial E}{\partial t} = J \quad (3.28)$$

$$\nabla \cdot B = 0 \quad (3.29)$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad (3.30)$$

Where E and B the electric and magnetic field. ρ and J are the charge and current densities.

In Maxwell theory these quantities are given as space integrals of corresponding densities built from quadratic expressions in field vectors. A very convenient tool in this construction is a complex vector F ,

$$F = \sqrt{\frac{\epsilon_0}{2}} (E + icB) \quad (3.31)$$

That was named the Riemann–Silberstein (RS) vector in [27].

The Maxwell equations expressed in terms of F are

$$\partial_t F(r, t) = -ic\nabla \times F(r, t), \nabla \cdot F(r, t) = 0 \quad (3.32)$$

The field energy H , the field momentum P , the field angular momentum J , and the field moment of energy K can all be constructed from the energy–momentum tensor of the electromagnetic field. These quantities expressed in terms of the RS vector are:

$$H = \frac{1}{2} \int d^3r \left[\epsilon_0 E^2 + B^2/\mu_0 \right] = \int d^3r F^* \cdot F \quad (3.33a)$$

$$P = \int d^3r [\epsilon_0 E \times B] = \frac{1}{2i} \int d^3r F^* \times F \quad (3.33b)$$

$$J = \int d^3r r \times [\epsilon_0 E(r) \times B(r)] = \frac{1}{2i} \int d^3r r \times (F^* \times F) \quad (3.33c)$$

$$K = \frac{1}{2} \int d^3r r r \left[\epsilon_0 E^2 + B^2/\mu_0 \right] = \int d^3r r r (F^* \cdot F) \quad (3.33d)$$

These quantities, like their counterparts in photon quantum mechanics (3.20a) – (3.20d) serve as the generators of Poincaré transformations of the electromagnetic field. They have analogous algebraic properties of the Poincaré group (3.18a)–(3.18c), with quantum commutators replaced by Poisson brackets, $[a, b]/i\hbar \rightarrow [a, b]$.

All solutions of Maxwell equations in vacuum can be decomposed into plane waves with positive and negative frequencies. This decomposition gives the following Fourier representation of $F(r, t)$:

$$\int F(r, t) = \sqrt{N} \int \frac{d^3k}{(2\pi)^{3/2}} e(k) [f_L(k) e^{-i\omega_k t + ik \cdot r} + f_R^*(k) e^{i\omega_k t - ik \cdot r}] \quad (3.34)$$

Where the complex polarization vector $e(k) = [l_1(k) + il_2(k)]/\sqrt{2}$ has the following properties:

$$ck \times e(k) = -i\omega_k e(k) \quad (3.35a)$$

$$e(k) \cdot e(k) = 0 \quad (3.35b)$$

$$e^*(k) \cdot e(k) = 1 \quad (3.35c)$$

$$e^*(k) \times e(k) = in_k \quad (3.35d)$$

$$e^*(k) \cdot e(-k) = 0 \quad (3.35e)$$

$$e(k) \times e(k) = 0 \quad (3.35f)$$

$$e_i^*(k)e_j(k) = \frac{1}{2} \left(\delta_{ij} + i\epsilon_{ijl} \frac{k_l}{|k|} \right) \quad (3.35h)$$

The identification of the Fourier coefficients with the components of the photon wave function in the formula (3.34) will be justified in the next section where we will unify the field picture and the photon picture. The second term in (3.34) involves complex conjugation. This is dictated by the fact that the photon energy is always positive. Therefore, the time evolution of the wave function is given by the factor $\exp(-i\omega_k t)$. Therefore, the reversal of the sign in the exponent requires complex conjugation. We pulled out the factor \sqrt{N} to assure the normalization of f .

Section (3.3): Separation of Angular Momentum

We shall combine now the field picture and the photon picture to obtain the decomposition of the total angular momentum of the field. To this end, we substitute the Fourier representation of the field into the formulas (3.33a), (3.33d)[92].

$$H = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot hw_k f(k) \quad (3.36a)$$

$$P = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot hk f(k) \quad (3.36b)$$

$$J = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot [ihD \times k + h\hat{x}n_k] f(k) \quad (3.36c)$$

$$K = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot ihw_k D f(k) \quad (3.36d)$$

Note, that the resulting expressions have the form of quantum mechanical expectation values

$$H = N\langle f|\hat{H}|f\rangle \quad (3.37a)$$

$$P = N\langle f|\hat{P}|f\rangle \quad (3.37b)$$

$$J = N\langle f|\hat{J}|f\rangle \quad (3.37c)$$

$$K = N\langle f|\hat{K}|f\rangle \quad (3.37d)$$

These formulas exhibit a perfect agreement between the results obtained from the particle picture and from the field picture, as Darwin had anticipated. Every value calculated for the total electromagnetic field is a product of the quantum mechanical average value per one photon, multiplied by N . That means that the normalization factor N is the total number of photons. We may now unambiguously split the total angular momentum of the electromagnetic field (3.36c) into two parts. The vector J_0 whose integrand is perpendicular to the wave vector is the orbital part and the vector J_s whose integrand is parallel to the wavevector is the spin part represented by helicity.

$$J_0 = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot [ihD \times k] f(k) \quad (3.38a)$$

$$J_s = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot h\hat{x}n_k f(k) = N \int \frac{d^3k}{w_k} n_k [|f_L(k)|^2 - |f_R(k)|^2] \quad (3.38b)$$

The final step of our analysis is the proof that the expressions for J_0 and J_s coincide with those obtained by Darwin. To this end, we employ the relation between $E(k)$ and $f(k)$ that follows from the formulas (3.13) and (3.34)

$$E(k) = \sqrt{\frac{N}{2\epsilon_0}} [e(k)f_L(k) + e^*(k)f_R(k)] \quad (3.39)$$

Upon substituting this relation into the second term in (3.12), with the use of the properties of the polarization vectors (3.35d) and (3.35f), we obtain

$$-2i\epsilon_0 \int \frac{d^3k}{c|k|} E^*(k) \times E(k)$$

$$= -iN \int \frac{d^3}{c|k|} [e^*(k) \times e(k) |f_L(k)|^2 + e(k) \times e^*(k) |f_R(k)|^2] = J_s \quad (3.40)$$

In the same way we may establish the equality of the orbital part in the Darwin form and in the quantum mechanics of photons. Note that the separation of the total angular momentum into its orbital and spin parts is conserved in time since both parts are separately time independent.

Definition (3.3.1): Bessel Beam

A Bessel beam is a field of electromagnetic, acoustic or even gravitational radiation whose amplitude is described by a Bessel function of the first kind.

A true Bessel beam is non-diffractive. This means that as it propagates, it does not diffract and spread out, this is in contrast to the usual behavior of light (or sound), which spreads out after being focused down to small spot. Bessel beams are also self-healing, meaning that the beam can be partially obstructed at one point, but will re-form at a point further down the beam axis.

As an illustration, we consider the Bessel beam characterized by the frequency $c|k|$, the z -component of the total angular momentum m , the component k_z of the wave vector in the z -direction, and the helicity ± 1 . In this case, the Darwin vector (3.39) (up to a normalization factor) as given in [32] has the for

$$E_{k,m,k_z}(K', \phi, K'_z) = \begin{pmatrix} -(k_z/k) \cos \phi \pm i \sin \phi \\ -(k_z/k) \sin \phi \mp i \cos \phi \\ (k_\perp/k)^2 \end{pmatrix} e^{im\phi} \delta(k_\perp - k'_\perp) \delta(k_z - K'_z) \quad (3.41)$$

Since the Bessel beam has an infinite extension in space, both parts of the total angular momentum are infinite. However, their ratio is well defined. For the components in the beam direction, the ratio of the orbital to spin parts equals to $mk/k_z \mp 1$.

In order to express J_o and J_s as integrals in coordinate space we have to invert the Fourier transformation in (3.13) for $t = 0$ as follows:

$$E(k) = \int \frac{d^3r}{2(2\pi)^{3/2}} e^{-ik \cdot r} \left[E(r) + \frac{ic}{|k|} \nabla \times B(r) \right] \quad (3.42)$$

Where we made use of Maxwell equations. Inserting this formula into the Darwin expression for the spin part J_s , after the integration over k , we obtain

$$J_s = \epsilon_0 \int d^3r \int \frac{d^3r'}{4\pi} E(r) \times \frac{\nabla \times B(r')}{|r - r'|} \quad (3.43)$$

Where we used the formula

$$\int \frac{d^3r}{(2\pi)^3} \frac{e^{ik \cdot r}}{|k|^2} = \frac{1}{4\pi|r|} \quad (3.44)$$

This gauge invariant integral representation of J_s becomes equal to the last term in (3.1) if the vector potential is identified with the following integral:

$$A(r) = \int \frac{d^3r'}{4\pi} \frac{\nabla \times B(r')}{|r - r'|} \quad (3.45)$$

This representation of the vector potential is valid in the transverse gauge, as has been anticipated. Note that the seemingly local form of the formula (3.1) is misleading because the gauge invariant vector potential is a nonlocal function of the magnetic field.

Section (3.4): Conclusions

We have shown that the separation of the total angular momentum of the electromagnetic field into its orbital and spin parts dictated by quantum mechanics of photons reproduces the results derived from the properties of Maxwell fields by Darwin. This separation, when expressed in the form of coordinate-space integrals, coincides with the results derived heuristically by many authors, provided the vector potential is related to the magnetic field by the integral formula (3.45). In contrast to energy, momentum, and the total angular momentum of the electromagnetic field, the orbital angular momentum and the spin parts cannot be expressed as integrals of local densities: they are intrinsically nonlocal objects.