

## CHAPTER FIVE

### AXIOMATIC GEOMETRICAL OPTICS

#### Section (5.1): Geometric Phases in Physics

Suppose a system undergoes an evolution so that after some time it returns to its original state. We shall call such an evolution a cyclic evolution. If the system is classical, then it is impossible to say from its initial and final states that it has undergone any evolution. However, the wave function of a quantal system retains a memory of its motion in the form of a geometric phase factor. This phase factor can be measured by interfering the wave function with another Coherent wave function enabling one to discern whether or not the system has undergone an evolution. Therefore geometric phase factors are 'signatures' of quantum motion. The adjective 'geometric' emphasizes that such phase factors depend only on the loop in the quantum mechanical state space the set of rays of the Hilbert space, sometimes called the projective Hilbert space. In particular, geometric phases are independent of parameterization of the path in the projective Hilbert space, and therefore of the speed at which it has been traversed.

As early as 1956 [37], in phase shifts in non-quantal polarized light [38], S. Pancharatnam anticipated the quantal geometric phases. He studied the problem of determining the phase change undergone by polarized light after it has passed through a sequence of polarizers such that its final polarization is the same as its initial polarization. To describe how the phase of polarized light changes under passage through a polarizer, Pancharatnam needed to define the phase difference between two different polarization states. He reasoned that the most natural way to accomplish this task is to ask what would happen if two such states were brought to interfere with each other, and accordingly he proposed the following definition: the polarization states of any two monochromatic beams of light with the same moment are in phase if the superposition of the two has the maximum possible intensity. Let  $|A\rangle$  and  $|B\rangle$  represent the polarization state-vectors of photons in the two beams. Since the intensity of their superposition is proportional to

$$(|A\rangle + |B\rangle)(\langle A| + \langle B|) = 2 + 2|\langle A|B\rangle| \cos\{ph\langle A|B\rangle\} \quad (5.1)$$

according to his convention  $|A\rangle$  and  $|B\rangle$  are in phase when their scalar product  $\langle A|B\rangle$  is real and positive, or equivalently, when  $ph\langle A|B\rangle = 0$ . Incidentally, since orthogonal states do not interfere, this convention breaks down for such states, and the phase difference between them remains undefined. In the general case of non orthogonal states, it is natural to identify the phase difference between  $|A\rangle$  and  $|B\rangle$  with the phase  $ph\langle A|B\rangle$  of their scalar product.

Pancharatnam used this definition of the phase difference to analyze an experiment involving a sequence of changes in polarization of a beam of classical light by sending it through suitable polarizers. His experiment consisted of three sequential changes in polarization, from  $|A\rangle$  to  $|B\rangle$  to  $|C\rangle$  and back to a state  $|A'\rangle$  of the initial polarization. It is easy to show that in such a scheme each successive state remains in phase with the previous one. Now, the label A used here to describe a state of a polarized wave of light represents a set of values (the eigenvalues of a complete set of commuting observables) required to specify this state uniquely. In Pancharatnam's experiment all but one of these values including the one that specifies the polarization were returned to their original values, with the phase of polarization being the only exception. Thus, Pancharatnam's evolution was not cyclic in the sense described above. Indeed, in what follows, the classical phase difference he observed will be shown to come from the quantum mechanical phase difference between the initial and final one-photon states,

$$\langle \psi | \psi' \rangle = e^{-\frac{i}{2}\alpha_{ABC}} \quad (5.2)$$

Where  $\alpha_{ABC}$  is the solid angle subtended by the geodesic triangle ABC on the Poincaré sphere (whose points, as is well-known, represent all conceivable forms of polarization states). For simplicity, we ignore the dynamical phase difference due to the fixed frequency of the photon. Remarkably enough, Pancharatnam not only anticipated the quantal geometric phases, but also was able to corroborate his theory experimentally.

Another geometric phase given by a solid-angle formula analogous to (5.2) was put forward in 1984 by M.V. Berry (who was unaware of Pancharatnam's work). He investigated the nonrelativistic Schrödinger evolution

$$i \frac{d}{dt} |\psi(t)\rangle = H(R(t)) |\psi(t)\rangle \quad (5.3)$$

of a quantal system in a slowly changing environment described by a set of  $N$  time-dependent parameters  $R(t) = (R_1(t), R_2(t), \dots, R_N(t))$ , with the initial state

$$|\psi(0)\rangle = |n; R(0)\rangle \quad (5.4)$$

being the stationary state given by the time-independent Schrödinger equation

$$H(R(0)) |n; R(0)\rangle = E_n(R(0)) |n; R(0)\rangle \quad (5.5)$$

If  $H(R(t))$  is non-degenerate and slowly varying, then it is known that the time-evolving Schrödinger state  $|\psi(t)\rangle$  remains an eigenstate of the instantaneous Hamiltonian  $H(R(t))$ . More precisely,

$$|\psi(t)\rangle = e^{-i \int_0^t ds E_n(R(s))} e^{ib[n; R(t)]} |E_n(R(t))\rangle \quad (5.6)$$

Where

$$b[n; R(t)] = \int_0^t ds \left\langle n; R(s) \left| i \frac{d}{ds} \right| n; R(s) \right\rangle \quad (5.7)$$

or equivalently,

$$b[n; R(t)] = \int_{R(0)}^{R(t)} dR' \cdot \langle n; R' | i \nabla_{R'} | n; R' \rangle \quad (5.7')$$

where  $\nabla_R$  is the gradient operator in the parameter space  $R$ . This is, of course, just the time-honored adiabatic theorem.

Berry's investigations, however, went beyond the usual formulation of the adiabatic theorem captured in (5.6) and (5.7). He considered the case of an adiabatic transport around a closed path

$$\rho_t = \{R(t) | R(T) = R(0); 0 < t < T\} \quad (5.8)$$

in the parameter space, and made the crucial observation that in such an adiabatic setup the phase factor  $e^{ib[n;R(t)]}$  is not integrable; i.e., in general it cannot be written as a function of  $R$ , and in particular is not single-valued under continuation around the loop:  $e^{ib[n;R(T)]} \neq e^{ib[n;R(0)]}$ . Moreover, it is easy to see that (5.7') can be reexpressed in the form

$$b[n; \rho] = \oint_{\rho} dR \cdot \langle n; R | i \nabla_R | n; R \rangle, \quad (5.9)$$

from which it is evident that  $b[n; R(T)]$ , or the Berry phase as it is now called, is independent of parameterization:  $b[n; R(T)] = b[n; \rho]$ , where  $\rho$  denotes the unparameterized loop corresponding to  $\rho_t$ . In particular, unlike the usual dynamical phase  $\{- \int ds E_n\}$ , the Berry phase  $b[n; \rho]$  is independent of the rate at which the state of the system traverses around  $\rho$ .

To illustrate his findings Berry analyzed the example of a spin- $s$  particle interacting with a magnetic field  $B$  through the Hamiltonian

$$H(B) = kB \cdot S \quad (5.10)$$

where  $k$  is a constant involving the gyro magnetic ratio, and  $S$  is the vector spin operator whose components have  $2s + 1$  eigenvalues  $n$  lying between  $-s$  and  $+s$  with integer spacing. The eigen values of  $H(B)$  are, of course,

$$E_n(B) = kBn \quad (5.11)$$

with  $B = |B|$ . Now, if one identifies the components of the external magnetic field  $B$  with the parameter space  $R$ , then Berry's formula is easily applicable to this case. In particular, (5.9) gives the geometric phase change of an eigenstate  $|n; B(t)\rangle$  of  $H(B(t))$  as  $B(t)$  is slowly transported and hence the spin is slowly processed around a loop  $\gamma$  in the  $B$ -space. Berry was able to show that, in that case,

$$e^{ib[n;B(T)]} = e^{-in\alpha_\gamma} \quad (5.12)$$

where  $\alpha_\gamma$  is the solid angle subtended by the loop  $\gamma$  at  $B = 0$ . In particular, when  $s = \frac{1}{2}$  and the initial state is  $|\frac{1}{2}; B(0)\rangle$  ('spin up' along B), then the right-hand side of (5.12) takes the form of the right-hand side of the observation (5.2) of Pacharatnam namely,  $e^{-\frac{i}{2}\alpha_\gamma}$ . To establish an analogy between (5.2) and (5.12) it suffices now to identify  $|\psi(0)\rangle = |\psi\rangle$  and  $|\psi(T)\rangle = |\psi'\rangle$ , and note that the left-hand side of (5.12) can be rewritten, after the dynamical phase is removed, in the form  $\langle\psi(0)|\psi(T)\rangle$ .

A simple explanation of the beautiful result (5.12) was given in 1987 by J. Anandan and L. Stodolsky [39]. They considered a sphere whose points represented the possible directions of the magnetic field B. In the above example of Berry, it is sufficient for the direction of B to trace a closed curve  $\hat{\gamma}$  on this sphere in order for each eigenstate to acquire a geometric phase. In other words, it is not necessary for  $B(t)$  to form a closed curve; it is sufficient if merely the directions of  $B(0)$  and  $B(T)$  coincide. Anandan and Stodolsky then considered a Cartesian triad with its origin on  $\hat{\gamma}(t)$  and its z-axis in the radial direction of the sphere (the direction of  $B(t)$  and the spin axis). If the triad is moved along  $\hat{\gamma}(t)$  so that the x,y-axes are parallel-transported along the surface of the sphere, then when the triad returns to the original point  $\hat{\gamma}(0) = \hat{\gamma}(T)$ , it will have rotated about its z-axis by the solid angle  $\alpha$  subtended by  $\hat{\gamma}$  at the center of the sphere. Now, relative to the triad, each eigenstate individually should acquire only the usual dynamical phase factor because the triad has no angular velocity about the spin axis. Consequently, the additional phase factor acquired by the eigenstate must be interpreted as the geometric phase factor due to the rotation of the triad given by  $e^{i\alpha J_z}|n\rangle = e^{i\alpha n}|n\rangle$ , where  $J_z$  generates rotation about the z-axis of the triad.

For an arbitrary cyclic evolution in any Hilbert space, the above angle generalizes to a set of angles  $\alpha_1, \dots, \alpha_N$ . These are the geometric quantum angles introduced by Anandan [40], which perhaps provides the deepest approach so far to the geometric phase. The geometric phases acquired by a complete set of orthogonal states  $\{|n\rangle\}$  are now obtained by the action on each  $|n\rangle$  by  $e^{i\sum_{k=1}^N \alpha_k J_k}$ , where the element of the set  $\{J_k\}$  commute among themselves. In

the classical limit the geometric angles  $\{\alpha_k\}$  reduce to the classical angles of J.H. Hannay [41], while the observables  $\{J_k\}$  become the corresponding action variables that are in involution with each other.

We now illustrate the usefulness of geometric angles by providing a quantum-mechanical explanation of the above-mentioned experiment of Pancharatnam. For this purpose, we need to generalize his classical electromagnetic polarized wave, with fixed momentum  $p$ , passing through an arbitrary number of polarizers, such that the final polarization is the same as the initial polarization. A classical electromagnetic wave is an approximation of a coherent state in quantum electrodynamics. In the Coulomb gauge the quantized vector-potential for the electromagnetic field may be written as

$$A = \sum_k \sum_{\lambda=1}^2 [\alpha_{k,\lambda} e^{-i(k \cdot x - \omega t)} + \alpha_{k,\lambda}^\dagger e^{-i(k \cdot x - \omega t)}] e_{k,\lambda} \quad (5.13)$$

where  $k$  is the momentum vector,  $\omega = |K|$  is the frequency,  $e_{k,\lambda}$  are real orthogonal polarization vectors perpendicular to  $k$ , and  $\alpha_{k,\lambda}$ ,  $\alpha_{k,\lambda}^\dagger$  are the annihilation and creation operators for the mode  $(k, \lambda)$ . The electric and magnetic fields corresponding to  $A$  are  $E = -\frac{\partial A}{\partial t}$  and  $B = \nabla \times A$ , respectively. The coherent state corresponding to the electromagnetic wave considered by Pancharatnam is then

$$|z_1, z_2, p\rangle = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \exp(z_1 \alpha_{p,1}^\dagger + z_2 \alpha_{p,2}^\dagger) |0\rangle \quad (5.14)$$

which is an eigen state of  $\alpha_{p,\lambda}$ , with eigen values  $z_\lambda$ . Therefore,

$$\begin{aligned} \langle z_1, z_2, p | A | z_1, z_2, p \rangle \\ = 2 \{ |z_1| \cos(p \cdot x - \omega t + \theta_1) e_{p,1} + |z_2| \cos(p \cdot x - \omega t + \theta_2) e_{p,2} \} \end{aligned} \quad (5.15)$$

Where  $\theta_1$  and  $\theta_2$  are the phases of  $z_1$  and  $z_2$  respectively. It follows that  $|z_1|\omega$  and  $|z_2|\omega$  are the amplitudes of the electric field  $E$  in the directions of  $e_{p,1}$  and  $e_{p,2}$ , respectively.

We may represent the polarization state of a one-photon state

$(z_1\alpha_{p,1}^\dagger + z_2\alpha_{p,2}^\dagger)|0\rangle$  as a two-component Spinor  $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$  in a two dimensional vector space with the usual inner product, which makes it a Hilbert space. The corresponding projective Hilbert space is the Poincaré sphere.

**Definition (5.1.1): Poincaré Sphere**

("Considering  $x$  and  $y$  as the coordinates of a variable point, and  $t$  as the time, one seeks the motion of a point to which one gives the velocity as a function of the coordinates).

As each photon corresponding to the mode  $(p, \mu)$  passes through the polarizer it undergoes a transition to a state  $(p, \mu')$ . The new state is obtained by simply projecting the old state onto the state that passes through the polarizer. It can be shown that this corresponds to parallel-transporting the old state-vector along the shorter geodesic joining the two points on the Poincaré sphere representing the two polarization states [42]. Therefore, using arguments similar to those used by Anandan and Stodolsky [39], the final state obtained after a sequence of such polarization changes that return the photons to their initial polarization state is given by the action of the operator  $e^{i\alpha J}$  on the initial photon state. Here  $\alpha$  is the solid angle subtended by the geodesic polygon defined by the sequence of the polarization states on the Poincaré sphere, and  $J = \frac{N}{2}$ ,  $N$  being the number operator for the initial and final mode  $(p, \mu)$ . As a result, the final state of the electromagnetic field is

$$e^{i\alpha J}|z_1, z_2, p\rangle = |z_1 e^{i\frac{\alpha}{2}}, z_2 e^{i\frac{\alpha}{2}}, p\rangle \tag{5.16}$$

Finally, in this resultant state we have

$$\begin{aligned} &\langle z_1, z_2, p|A|z_1, z_2, p\rangle \\ &= 2 \left\{ |z_1| \cos\left(p \cdot x - \omega t + \theta_1 + \frac{\alpha}{2}\right) e_{p,1} + |z_2| \cos\left(p \cdot x - \omega t + \theta_2 + \frac{\alpha}{2}\right) e_{p,2} \right\} \end{aligned} \tag{5.17}$$

Comparison of this expression with equation (5.15) shows that  $\frac{\alpha}{2}$  is the phase Pancharatnam observed in his classical experiment. A similar explanation can be given to the experiment of Tomita and Chiao [43], except in their case we would have  $J = N$  for the photon since it is a spin-1 particle.

Having obtained these results using the operator  $\exp(i\alpha J)$ , which depends on the geometric angle  $\alpha$ , we may generalize them to an arbitrary superposition of number eigenstates with the same polarization. In general, such a state would not be a Coherent state and cannot therefore be represented by a classical electromagnetic wave. Nevertheless, the geometric part of the evolution may be obtained by taking the expectation value of  $\exp(i\alpha J)$  with respect to the initial state.

The above mentioned geometric treatment of Berry's phase by Anandan and Stodolsky suggest that the geometric phase is associated with the motion of a quantum system and not with the particular Hamiltonian used to achieve this motion. This is the basic idea used by Aharonov and Anandan [44] in obtaining a geometric phase, which, since it is associated with the motion of the quantum-mechanical state itself,

Does not require an adiabatically varying Hamiltonian (environment). However, if an adiabatically varying Hamiltonian is used to implement this motion, then this geometric phase is the same as Berry's phase. They defined the evolution of a normalized state  $|\psi(t)\rangle$  to be cyclic in the interval  $[0, T]$  if and only if

$$|\psi(T)\rangle = e^{i\phi(0,T)} |\psi(0)\rangle \quad (5.18)$$

where  $\phi(0, T)$  is a real number. Equivalently, this can be reexpressed with the help of the unitary time evolution operator  $U(0, T)$  in the form

$$U(0, T)|\psi(0)\rangle = e^{i\phi(0,T)} |\psi(0)\rangle \quad (5.19)$$

It follows from this equation that for an initial state to evolve cyclicly in the interval  $[0, T]$  under the time evolution operator  $U(0, T)$ , it is necessary and sufficient for it to be an eigenstate of the operator  $U(0, T)$ . Incidentally, this assures the existence of cyclic evolutions as defined above at least in the finite-dimensional case. According to Aharonov and Anandan, the geometric contribution to  $\phi(0, T)$ , denoted by  $\beta$ , is

$$e^{i\beta} = e^{i\phi(0,T) + \int_0^T ds \langle \psi(s) | i \frac{d}{ds} | \psi(s) \rangle} \quad (5.20a)$$



or equivalently,

$$e^{i\beta} = \langle \psi(0) | \psi(T) \rangle e^{i \int_0^T ds \langle \psi(s) | i \frac{d}{ds} | \psi(s) \rangle} \quad (5.20b)$$

which reduces to the Berry's phase factor in the adiabatic limit [44],[45]. What is more, just as Berry's phase, it is independent of the choice of parameterization or the speed at which the path  $|\psi(t)\rangle$  is traversed. More significantly, Aharonov and Anandan demonstrated that  $\beta$  is projective-geometric in nature; i.e., it is the same for all paths  $|\varphi(t)\rangle$  that project to the same path in the projective Hilbert space. In other words, it is the same for any two motions  $|\varphi(t)\rangle$  and  $|\psi(t)\rangle$  such that  $P_{\varphi(t)} = P_{\psi(t)}$ , where  $P_\alpha$  denotes a ray corresponding to a vector  $|\alpha\rangle$ , namely

$$P_\alpha = \{ |\beta\rangle \mid |\beta\rangle = z|\alpha\rangle; z \in \mathbb{C} \} \quad (5.21)$$

The above two properties imply that  $\beta = \beta[p_\psi]$ , and suggest that  $\beta$  may have a geometric interpretation in terms of paths in the projective Hilbert space. Indeed, it may be geometrically understood as the anholonomy with respect to the natural connection on the projective Hilbert space. This interpretation generalizes an earlier differential-geometric interpretation of the Berry phase given by B. Simon in 1983 [46]. In the same year F. Wilczek and A. Zee reported on how the theory can be generalized to include the adiabatic evolution of degenerate quantum states [47].

### Definition (5.1.2): Adiabatic and Sudden Approximations

In discussing the time-dependent perturbation theory, we have dealt with phenomena where the perturbation  $\hat{V}(t)$  is small, but we have paid no attention to the rate of change of the perturbation, approximation methods treating phenomena where  $\hat{V}(t)$  is not only small but also switched on either adiabatically (slowly) or suddenly (rapidly). We assume here that  $\hat{V}(t)$  is switched on at  $t = 0$  and off at a later time  $t$  (the turning on and off may be smooth or abrupt).

They showed that in the case of a  $d$ -fold degeneracy Berry's phase factor of the nondegenerate case,  $e^{ib[n;p]}$ , is generalized to a  $d \times d$  unitary matrix, which is now called the non-Abelian Berry phase or the Wilczek-Zee phase. More precisely, if the initial state is one of the eigen states belonging to an

orthonormal set of eigen states of  $H(R(0))$  with a d-fold degenerate eigen value  $E_n(R(0))$ , i.e.

$$H(R(0))|l; R(0)\rangle = E_n(R(0))|l; R(0)\rangle \quad (5.22)$$

With  $l = 1, 2, \dots, d$ , then

$$|\psi(t)\rangle = e^{-i \int_0^t ds E_n(R(s))} \sum_{l'=1}^d D_{l'l}(R(t)) |l'; R(t)\rangle \quad (5.23)$$

here, the matrix D is a path-ordered exponential integral

$$D[R(t)] = P \exp \left\{ i \int_{R(0)}^{R(t)} dR' A' \right\} \quad (5.24)$$

with

$$A_{l'l} R(t) = \langle l'; R | i \nabla_R | l; R \rangle \quad (5.25)$$

and P represents the path-ordering. The non-Abelian phase factor  $D[l; R(T)]$  is a unitary matrix, and may be denoted by  $D[l; \rho]$  because it too is independent of parameterization or the speed with which a particular path is traversed and is therefore geometric.

We conclude by remarking that, there are at least four different reasons for the phenomenal success of the concepts related to geometric phases. First, these concepts are exceptionally clear and have a very elegant geometric interpretation in terms of anholonomies and connections (gauge fields). Second, geometric phases have a certain unifying character that enables one to relate many apparently disparate phenomena. Third, these phases can be observed and, indeed, various predictions of the geometric phases have been amply corroborated. Finally, and perhaps most importantly, these concepts reassert the importance and fruitfulness of geometric ideas in physical theories.

## Section (5.2): Geometrical Representation of Angular Momentum Coherence and Squeezing

The harmonic oscillator phase space description of electromagnetic fields has had great success in understanding the semi classical and quantum theories of coherent. Coherent states are defined generally as minimum uncertainty states. The product of the uncertainties in the quadratures is minimum for these states and both the quadratures have equal uncertainties. Squeezing redistributes the uncertainties of the two quadratures resulting in one of them having less than its value for the coherent state at the expense of increasing the other [48]. The geometrical description of the Heisenberg uncertainty relation of two non commuting variables (quadratures) is well known to give a better understanding of the inherent fluctuation due to the quantum nature of the squeezed light. The quadratures of the phase space retain the minimum uncertainty property of the coherent states.

### Definition (5.2.2): SU(2)

SU(2) is a  $2 \times 2$  unitary matrix with determinant 1, and with elements as complex and used to describe and calculate internal rotations, and deals with systems with two states.

Angular momentum or SU(2) algebra describes the behavior of an ensemble of two quantum-level noninteracting systems. Examples of these systems include interferometers, non-interacting ensemble of two-level atoms or molecules. Sensitivity of interferometers, when described by SU(2) algebra, can be defined in terms of matrix elements of mean and variances of angular momentum operators [49]. Feynman et al. [50] have constructed a simple geometrical representation of the Schrodinger equation for such two-level systems to solve maser problems and radiation damping. In their approach the two-level quantum system is described by the state vector  $\vec{r}$  with components determined by the probability amplitude. The dynamics of the system is described by the differential equation of the state vector  $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$ , where  $\vec{\omega}$  is the angular velocity. However, in their approach there is no description of the uncertainty associated with the state vector. These uncertainties are important in describing the full quantum nature of the system e.g., the interaction of the

system with quantized radiation. Our section is an extension of the Feynman et al.'s description as we have included the uncertainties associated with the state vectors. Considerable amount of work has also been done in order to understand the coherent angular momentum states [51], [52]. Arecchi had constructed the angular momentum coherent states by rotating the angular momentum ground state [52] similarly as of displacement operator coherent states for electromagnetic waves. During last few years several authors have tried to construct squeezed angular momentum states and study their properties in terms of atomic systems [53], [54].

Schwinger developed an abstract algebra which perfectly describes the angular momentum systems [55]. The algebra assumes the angular momentum operators as a combination of two sets ( $\pm$ ) of uncorrelated creation and annihilation operators of spin  $\pm \frac{1}{2}$  which obey the bosonic commutation relations. Apart from the usual rotational angular momentum states, this algebra also describes the pseudo-angular momentum systems such as interferometry or a collection of two level systems proving to be quite a success. Atkins and Dobson (AD) had constructed angular momentum coherent (SAMC), (where SMAC is Schwinger Angular Momentum Squeezed) states using Schwinger representation. They had connected these states to two dimensional oscillator states quantized in orthogonal direction. The matrix elements of the components can be calculated easily from the expressions of their operator form. In the classical limit the SAMC states become the classical vector  $\vec{J}$  in the sense that they behave like their classical analogs and their variances are small compared to the absolute value of their averages. They have used these states to study elliptically polarized states successfully [56]. These SAMC states have also been used to describe the rotational levels of nuclear and molecular systems by Fonda et al. [57].

(Recently, we have proposed a technique for the generation of angular momentum squeezed states [58]). By combining the Schwinger angular momentum representation with boson operators and the concept of squeezing of bosons via Bogoliubov transformation, we were successful in producing states that exhibit squeezing of angular momentum operators. We have called these states Schwinger Angular Momentum Squeezed (SAMS) states. Our procedure

was an extension of the work of Atkins and Dobson [59] on angular momentum coherent (SAMC) states. We also found application of the SAMS states in enhancing the sensitivity of interferometers and in study of two level atoms [58]. In this section we have constructed an elegant geometrical representation of the concept of coherence and squeezing for the angular momentum operators. We present some new results for two mode squeezing and discuss their geometrical behavior and non-usefulness in this section. The results of SAMC states and single mode SAMS states are also considered for the geometrical phase space realization of these states. Our representation, combined with the picture given by Feynman et al., ensures to be an important tool to study the interaction between radiation and coherent or squeezed two-level systems.

The section is organized as follows. In subsection (5.2.1) we recapitulate some results of the Schwinger Angular Momentum Coherent (SAMC) states to describe them geometrically and compare them with the bosonic counterpart. In the next subsection (5.2.2) we present the new two mode squeezing results (5.2.2b) along with relevant single mode SAMS state results (5.2.2a) to understand the phase space geometry. Finally we conclude comparing our technique with other works on coherent angular momentum in context of geometrical understanding of the phase space. We also furnish the possible applications of the geometrical representation in the conclusion.

### **(5.2.1): ANGULAR MOMENTUM COHERENT STATES**

Schwinger [55] developed the entire angular momentum algebra in terms of two sets(up and down) of uncorrelated harmonic oscillator creation and annihilation operators constructing the angular momentum operators as

$$J_+ = J_x + iJ_y = a_+^\dagger a_- \quad (5.26a)$$

$$J_- = J_x - iJ_y = a_-^\dagger a_+ \quad (5.26b)$$

$$J_z = \frac{1}{2} (a_+^\dagger a_+ - a_-^\dagger a_-) \quad (5.26c)$$

The operators  $a_\pm^\dagger$  ( $a_\pm$ ) create (annihilate) a  $\pm \frac{1}{2}$  spin and follow the bosonic commutation relations which means that the whole system is considered as a

combination of two sets of boson states. This construction satisfies the standard angular momentum commutation relation  $[J_l, J_m] = i\epsilon_{lmn}J_n$ . The angular momentum basis states  $|j, m\rangle$  can thus be created by action of the oscillator operators on the vacuum spinor  $|0,0\rangle$

$$\begin{aligned} |j, m\rangle &= |j + m\rangle_+ \otimes |j + m\rangle_- \\ &= [(j + m)!(j - m)!]^{-\frac{1}{2}}(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}|0,0\rangle \end{aligned} \quad (5.27)$$

AD [59] constructed the Schwinger Angular Momentum Coherent (SAMC) states as the simultaneous eigenstates of the operators  $a_\pm$ . They have shown that the SAMC states are minimum uncertainty states for the angular momentum-angle uncertainty relation in the large  $N(N = n_+ + n_- \geq 10)$  limit [59]. According to their definition the angular momentum coherent states  $|\check{\alpha}\rangle = \check{D}|0,0\rangle$ , with  $\check{D} = D_+(\alpha_+)D_-(\alpha_-)$ , obey,

$$\alpha_\pm |\check{\alpha}\rangle = \alpha_\pm |\alpha_+, \alpha_-\rangle \quad (5.28)$$

The expansion of  $|\check{\alpha}\rangle$  in terms of angular momentum basis is given by

$$|\check{\alpha}\rangle = e^{-\frac{1}{2}N} \sum_j \sum_m (2j!)^{-\frac{1}{2}} \binom{2j}{j+m}^{\frac{1}{2}} \alpha_+^{j+m} \alpha_-^{j-m} |j, m\rangle \quad (5.29)$$

### Definition (5.2.3):SO(3)

SO(3) is a  $3 \times 3$  orthogonal matrix with determinant 1, and with elements as real, used to describe and calculate external rotations.

The  $\infty:1$  mapping of  $\check{\alpha}$  onto  $(\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$  is a consequence of the  $2:1$  homomorphism of SU(2) and SO(3) [15]. SU(2) is spanned by the subset of spinors of length  $\sqrt{N}$  and SO(3) is spanned by the subset of vectors of length  $\langle J \rangle$ . The spinor  $|\check{\alpha}\rangle$  is a vector sum of different vectors  $|j, m\rangle$  in the physical angular momentum space. Once the four parameters in  $\alpha_\pm$  are fixed, it automatically fixes the values of  $j$  and  $m$  in the angular momentum space. To calculate the mean and variances of the angular momentum components one has to use the expressions of them in SAMC basis and fix the corresponding parameters. The matrix elements can be expressed in terms of any set of these parameters ( $\alpha_\pm$  or  $j$  and

m). The expressions in terms of the angular momentum parameters give better understanding in the physical space. For this reason we have expressed the matrix elements in terms of angular momentum parameters throughout the chapter.

The mean of the angular momentum components are calculated as following

$$\langle J_x \rangle = \sqrt{j^2 - m^2} \cos \theta \quad (5.30a)$$

$$\langle J_y \rangle = \sqrt{j^2 - m^2} \sin \theta \quad (5.30b)$$

$$\langle J_z \rangle = m \quad (5.30c)$$

Where  $\theta = (\theta_+ - \theta_-)$  and the variances are

$$\Delta J_x^2 = \Delta J_y^2 = \frac{1}{2}j \quad (5.31)$$

The above results show that the average value of  $\vec{J}$  i.e. the tip of it lies on a sphere of radius  $j$ . This can easily be verified by squaring and adding the mean values of the angular momentum components. The fluctuations of the components are also same in the three directions. The equation of the region of uncertainty creates a sphere of radius  $\sqrt{\frac{3j}{2}}$  about the tip of the vector. This is shown in Fig. (5.1). The polar angle of the state vector in the three dimensional phase space is actually realized to be  $\theta$ , the difference between two phase angles of the coherence parameters.

We have shown the uncertainties of harmonic oscillator and a schematic of In three dimensional phase space we have shown the uncertainty sphere of the angular momentum vector in Fig. (5.1.c). This is also compared with the uncertainty circle of the harmonic oscillator in two dimensional phase space in ig. (5.1.a). For SAMC states the radii of these uncertainty spheres ( $= \sqrt{\frac{3j}{2}}$ ) depend only on the radii of the mean sphere ( $= j$ ) on which the tip of the vector lies. For a fixed  $j$  value the radii of the uncertainty spheres does not vary on its position on the sphere for the choice of the parameters  $m$  or  $\theta$ . The uncertainty spheres has

a circular projection (uncertainty circle) in the  $X - Y$  plane. We have shown some uncertainty spheres on it for some different positions (different values of  $m$  and  $\theta$ ) in Fig. (5.1.c). The positions for  $|m||j\rangle$  can be compared with the displacement of the ground state in the phase space picture for harmonic oscillator. The displacement of the uncertainty circle in the phase space of harmonic oscillator is performed by the rotation of the uncertainty sphere in corresponding three dimensional phase space for SAMC states. The position after rotation is governed by the values of the parameters  $m$  and  $(\theta_+ - \theta_-)$ .

The uncertainty relation corresponding to the commutation relation between the components of the angular momentum,  $[J_l, J_m] = i\epsilon_{lmn}J_n$ , is

$$\Delta J_x^2 \Delta J_y^2 \geq \frac{1}{4} |\langle J_z \rangle|^2 \quad (5.32)$$

Putting the expressions of the matrix elements in the last equation one can check that the equality occurs at  $m = \pm j$ . The two solutions for the equality show the  $SU(2)$  symmetry of the system. Other values of the angular momentum projection have uncertainties of both the quadratures equal and same as the extremum cases. Though the non-extremum cases does not violate the uncertainty relation in the last equation but the relation is an equality only for extremum cases. Anyway, for physical purposes we are interested in the absolute uncertainties in the quadratures which remain same for all the SAMC states with same  $j$  value.

### (5.2.2): ANGULAR MOMENTUM SQUEEZED STATES

Following the work of AD on angular momentum coherent states, we have generated squeezed angular momentum states by operating the squeezing operators of the bosonic states on the SAMC states [58]. For the two mode ( $\pm$ ) bosonic case the squeezing operators can be defined as

$$S_{\pm}(\xi_{\pm}) = \exp \left[ \frac{1}{2} (\xi_{\pm} \alpha_{\pm}^{\dagger 2} - \xi_{\pm}^* \alpha_{\pm}^2) \right], \quad \xi_{\pm} = r_{\pm} e^{i\phi_{\pm}} \quad (5.33)$$

We have created the general SAMS states by operating the two mode squeezing operator  $\mathcal{S} = S_+(\xi_+)S_-(\xi_-)$  on the SAMC states



$$|\psi\rangle = S_+(\xi_+)S_-(\xi_-)D_+(\gamma_+)D_-(\gamma_-)|0,0\rangle \quad (5.34)$$

For convenience in calculation we use the relation to interchange the order of the squeezing and displacement operators [16] and write the general SAMS states as

$$|\psi\rangle = D_+(\alpha_+)D_-(\alpha_-)S_+(\xi_+)S_-(\xi_-)|0,0\rangle \quad (5.35)$$

Where  $\gamma_{\pm} = \cosh \gamma_{\pm} \alpha_{\pm} + e^{i\phi_{\pm}} \sinh \gamma_{\pm} \alpha_{\pm}^*$

### (A).Single Mode Squeezing

The calculation of the expectation values and variances of the operators of our interest is cumbersome due to the dependence on the large number (eight) of parameters involved in the general SAMS states. First we consider the squeezing in one mode only. At this point we do this for simplicity though the utility of this choice will be clear in the following subsection. The SU(2) symmetry tells us that we can choose any one of the modes for squeezing. So we choose to squeeze in the + mode which reduces the expression of the basis state vectors of single mode SAMS states to

$$|\psi\rangle = \tilde{D}S(\xi)|0,0\rangle \quad (5.36)$$

Where we have dropped the unnecessary suffix +.

Calculating the mean of the angular momentum components [58]

$$\langle J_x \rangle = \sqrt{j^2 - m^2} \cos \theta \quad (5.37a)$$

$$\langle J_y \rangle = \sqrt{j^2 - m^2} \sin \theta \quad (5.37b)$$

$$\langle J_z \rangle = m + \frac{1}{2} \sinh^2 r \quad (5.37c)$$

we see that the traversing spherical surface of the mean of the tip of the angular momentum vector has been changed to a prolate ellipsoid with same axes in X and Y. The expression for the mean value of the angular momentum projection or the Z-axis of the ellipsoid show an increase as squeezing is increased. This is shown in Fig. (5.1.d). It is to be noted that squeezing in the other mode will

change the mean sphere to an oblate ellipsoid with  $\langle J_z \rangle = m - \frac{1}{2} \sinh^2 r$  instead of a prolate one. However, the other two axes will not change due to the choice of mode of squeezing. We calculated the variances to get a feel of the uncertainty nature of the SAMS states as

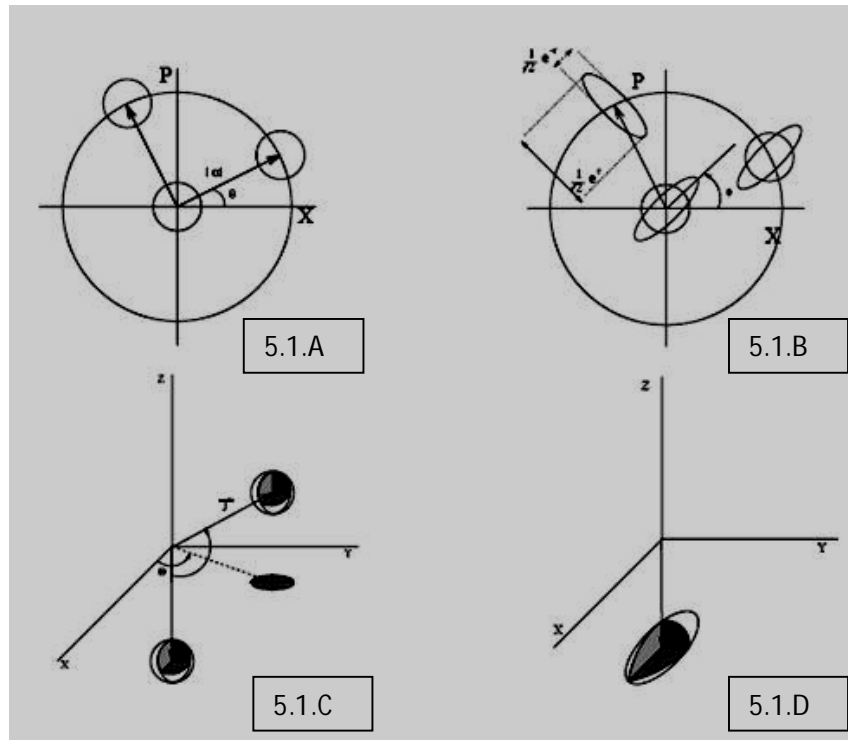
$$\Delta J_x^2 = \frac{1}{2}j + \frac{1}{2} \sinh r \left[ \frac{1}{2} \sinh r \{1 + 2(j - m)\} + (j - m) \cosh r \cos \delta \right] \quad (5.38a)$$

$$\Delta J_y^2 = \frac{1}{2}j + \frac{1}{2} \sinh r \left[ \frac{1}{2} \sinh r \{1 + 2(j - m)\} - (j - m) \cosh r \cos \delta \right] \quad (5.38b)$$

$$\Delta J_z^2 = \frac{j + m}{4} [e^{2r} \cos^2 \eta + e^{-2r} \sin^2 \eta] + \frac{1}{2} \sinh^2 r \cosh^2 r + \frac{j - m}{4} \quad (5.38c)$$

Where  $\delta = 2\theta_- - \phi$  and  $\eta = \theta_- - \phi$ .  $r$  and  $\phi$  the squeezing parameters of the + mode. The effect of squeezing in the other mode on the variances of the angular momentum components can be obtained from the last equation by interchanging the suffixes which is a consequence of the SU(2) symmetry. The uncertainty ellipsoid for  $m = -j$  have been shown in Fig. (5.1.d). It is to be noted that all the axes of this uncertainty ellipsoid are different. This results to an ellipsoidal projection on the X – Y plane. The squeezing of angular momentum can be compared with the harmonic oscillator squeezing, which is shown in Fig.(5.1.b).

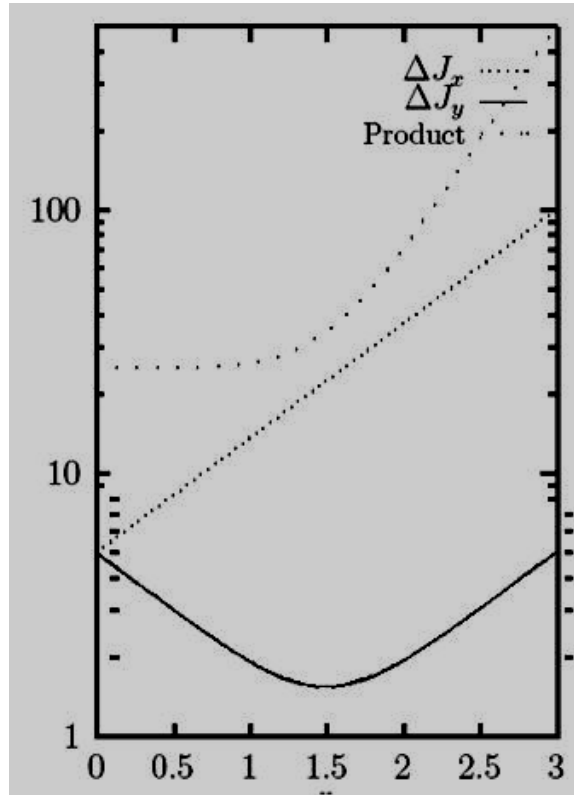
Fig (5.1)



In Fig. (5.2) we have plotted the dependence of the axes of the projected uncertainty ellipses on the squeezing parameter  $r$ . The projected uncertainty circle on the  $X - Y$  plane for the SAMC states are transformed to ellipses, but with greater area (uncertainty product). It is clear from the expressions that the maximum squeezing i.e. minimum fluctuation of the squeezed quadrature occurs to the minimum uncertainty circle at  $m = \pm j$  as expected physically. From Fig. (5.2) it is clear that the minimum uncertainty circle is squeezed (length of the semiminor axis is reduced) up to a critical value of  $r$  ( $= r_{\min}$ ) though its area (uncertainty product) is increased throughout. After that critical value of  $r$  the length of both the axes of the uncertainty ellipse increases.

It is interesting to note that squeezing in the  $+$  or  $-$  mode results squeezing of uncertainty in  $J_y$  and  $J_x$  respectively. This means that the squeezing in the angular momentum quadratures are directly related to the mode of squeezing. It will be interesting to express the squeezing operators in terms of operators in  $X - Y$  coordinates instead of  $\pm$  to identify the reason and exact mapping between them.

Fig (5.2)



### (B).Double Mode Squeezing

Now we consider the case of double mode squeezing. In the last subsection we have squeezed only in one mode for the sake of simplicity and promised to give the practical reasoning for this simplification in this subsection. Actually two mode squeezing does not help in reducing the uncertainty of any of the quadratures which we will show now. We can claim from the results of the last subsection that if we squeeze both the modes the uncertainties of the quadratures will be squeezed and expanded simultaneously. The squeezing of the second mode in effect reduce the amount of squeezing achieved by the first mode squeezing.

We have calculated the expectation values of the components of the angular momentum for double mode squeezing as

$$\langle J_x \rangle = \sqrt{j^2 - m^2} \cos \theta \quad (5.39a)$$

$$\langle J_y \rangle = \sqrt{j^2 - m^2} \sin \theta \quad (5.39b)$$

$$\langle J_z \rangle = m + \frac{1}{2} (\sinh^2 r_+ - \sinh^2 r_-) \quad (5.39c)$$

The expectations of the angular momentum components in X and Y direction are seen to be same as that of SAMC states with no effect of squeezing. The mean sphere is clearly seen to be transformed to an ellipsoid in general with same X and Y axes. The Z axis of the ellipsoid will increase or decrease as difference of squares of the hyperbolic sine functions of the two parameters  $r_{\pm}$ . This affects the shape of the mean spheroid to prolate or oblate. However, the expectation value of  $J_z$  can be made to be same as that of SAMC states by squeezing both the modes equally. This will make the mean ellipsoid to be same mean sphere as for SAMC states.

To show that the effect of double mode squeezing does not help in squeezing of the angular momentum quadratures we have calculated the uncertainties for some special choice of parameters. We have chosen the phases in the squeezing parameters to be equal to zero and the magnitudes of the squeezing parameters to be equal to  $r$ . This choice does not affect the basic motivation of representing the states geometrically or prove the disadvantage of double mode squeezing. We have calculated the uncertainties in the angular momentum components for this special [60] choice as

$$\begin{aligned} \Delta J_x^2 = & \frac{1}{2} [j + \sinh r \{ \sinh r + \cosh r \cos(2\theta_+) \} (j + m) \\ & + \sinh r \{ \sinh r + \cosh r \cos(2\theta_-) \} (j - m) + \sinh^2 r (1 + \cosh 2r)] \quad (5.40a) \end{aligned}$$

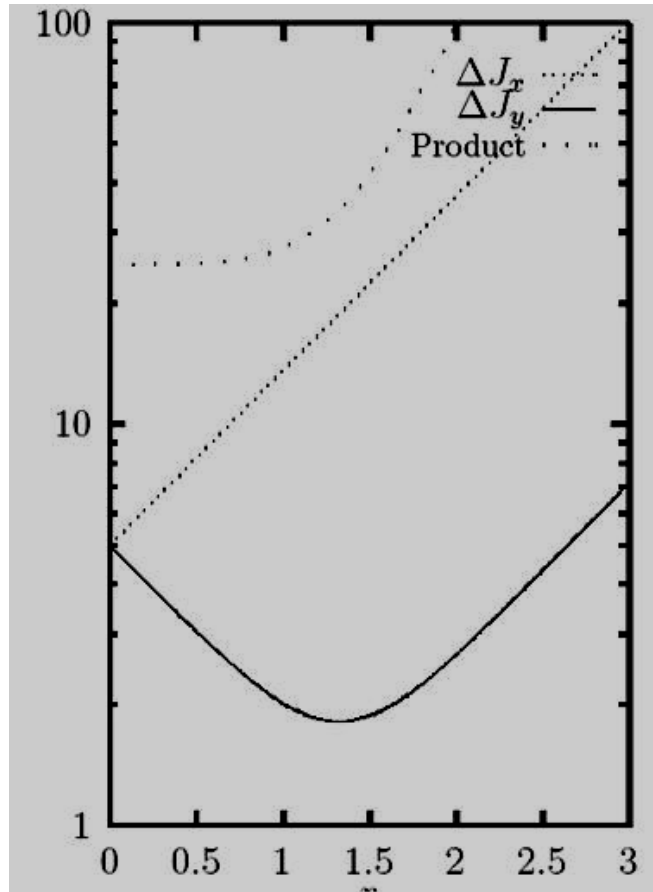
$$\begin{aligned} \Delta J_x^2 = & \frac{1}{2} [j + \sinh r \{ \sinh r - \cosh r \cos(2\theta_+) \} (j + m) \\ & + \sinh r \{ \sinh r - \cosh r \cos(2\theta_-) \} (j - m) + \sinh^2 r (1 + \cosh 2r)] \quad (5.40b) \end{aligned}$$

$$\Delta J_x^2 = \frac{j + m}{4} [e^{2r} \cos^2 \theta_+ + e^{-2r} \sin^2 \theta_+]$$

$$+ \frac{j - m}{4} [e^{2r} \cos^2 \theta_- + e^{-2r} \sin^2 \theta_-] + \sinh^2 r \cosh^2 r \quad (5.40c)$$

The uncertainty ellipsoids on the mean ellipsoid are similar as single mode squeezing case. The uncertainties of the quadratures and their product are plotted in Fig. (5.3) for  $j = 50, j = -50$  and  $\theta_+ = \theta_- = 0$ . Here we have plotted the results for the extremum projection states which drops out the first term in both the quadrature uncertainties. The squeezing in the uncertainty of  $J_y$  prove our claim that double mode squeezing deteriorates the effect of single mode squeezing which was expected from qualitative reasoning. Squeezing both the modes by same amount retains the SU(2) symmetry of the system but the choice of the phases and magnitudes of coherent parameters breaks it resulting different expressions and curves for the quadratures. This choice has been made to show the difference distinctly and the dependence on the coherence phases. With all the parameters same for the two modes one can show that the two quadratures will behave similarly. We do not present the geometrical pictures for the double mode squeezing as they are similar to the single mode squeezing.

Fig. (5.3)



### (5.2.3): Conclusions

We have represented the angular momentum coherent (SAMC) and squeezed (SAMS) states geometrically and studied their properties for a simple choice of parameters. These simplifications does not hamper the qualitative geometrical interpretation of these states. Actually, consideration of all the parameters makes the results complicated and not easily visible in the phase space picture. Due to this reason we have simplified the results by these choices. We have also shown that two mode squeezing deteriorates the squeezing effect in angular momentum quadratures. We applied the SAMS states in analyzing the sensitivity of interferometry [58]. The effect of two mode squeezing on interferometry can be seen from the results of the uncertainties in two mode squeezing. As two mode squeezing increases the uncertainty of the squeezed

quadrature it will also increase the value of minimum detectable phase difference ( $\Delta\Phi$ ) of any interferometer using beam splitters. The relation between them in the frame rotated by  $\frac{\pi}{2}$  about X axis can be written as  $\Delta\Phi = \frac{\Delta J_y}{|\langle J_x \rangle|}$  [58].

Any ensemble of two quantum-level system (e.g. atoms or molecules can be considered as a spin- $\frac{1}{2}$  particle is described by the SU(2) algebra of angular momentum systems [51],[52]. The number operators ( $\hat{n}_{\pm} = j \pm m$ ) in the case of interferometric representation correspond to the population or occupation numbers in the upper and lower states of the system. In fact the complete set to describe the system is achieved by adding a permutation group  $P_N$  to the SO(3) group. However, this does not change the basic essence of the formalism. Squeezed atomic states were constructed by preferential population distributions [58]. Feynman et al. [50] have shown that the components of the pseudo angular- momentum vector completely specifies the state of the system semiclassically. The power of the geometrical method developed by Feynman et al., lies in visualising and solving problems involving transition between two quantum levels. For example, the two classic problems discussed by them, the beam type maser oscillators and the radiation damping, could be visualised very clearly by the orientation of the state vector. Later application has led to elegant method for visualising and solving the photon echo problem also. Geometrical methods are found useful for problems that can be solved analytically. But it can also provide valuable insight into the behavior of the processes that are insolvable by analytical technique. Fermion interferometry can be considered by changing the bosonic commutation relation to the anticommutation relation for fermions. This is a totally new possibility as no fermion analog of the boson squeezed states has yet been found [61].

Arecchi [52] had developed the coherent atomic states by rotating the minimum uncertainty Dicke state (lowest projection state  $|j, -j\rangle$ ) in three dimensional phase space. Recently, the authors in Ref. [62] have described the Dicke states as a cap and annular surfaces. As these states are eigenstates of  $J_z$  they should not be described by cap or annular surfaces with nonzero  $\Delta J_z$  on the mean sphere of the angular momentum vector. Instead the different Dicke states ( $|j, m\rangle$ ) should be represented by an uncertainty circle. The Arecchi type atomic coherent states have some other geometrical representation problem which is



not present in our picture. Their coherent states are actually some rotated angular momentum ground states. The projections of these rotated states in  $X - Y$  plane is an ellipse and thus show squeezing. Physically, mere rotation should not change the status of the state. This question has been raised by Kitagawa et al [54] that if these Arecchi type coherent states describe squeezed angular momentum states under suitable choice of coordinates. Moreover, the area of the projected ellipse, which is a measure of the uncertainty product, is reduced from the area of the uncertainty circle of the ground state. This is a direct violation of Heisenberg uncertainty principle. In our definition of angular momentum coherent or squeezed states we have overcome this difficulty in representation and answered the question of Kitagawa et al.. The geometrical picture developed by us for SAMC or SAMS states does not have this ambiguity and thus is a better representation for the angular momentum coherent and squeezed states.

### **Section (5.3): Axiomatic geometrical optics, Minkowski controversy**

The discussion about how to define the momentum and the angular momentum of a photon in a dispersive medium (PDM), and even simply of a classical wave, has recurred in literature periodically during the last 100 years. The recent burst of theoretical [63] and experimental [64–66] publications indicates both an abiding interest in the problem and, apparently, a lack of consensus or certainty about what the correct answer is. Let us mention only briefly that two alternative forms of the PDM momentum are adopted most commonly:

$$P_M = \hbar\omega n_p/c, P_A = \hbar\omega/(n_g c), \quad (5.41)$$

which are known, respectively, as the Minkowski interpretation and the Abraham interpretation [64]. (Here,  $\omega$  is the frequency,  $c$  is the speed of light, and  $n_p = c/v_p$ , and  $n_g = c/v_g$ , are the refraction indexes associated with, correspondingly, the phase velocity  $v_p$  and the group velocity  $v_g$ ; for the two associated angular momenta, Ref. [67].) Since both have supporting theoretical and experimental evidence [1], the question about which of the two interpretations is “more correct” has been controversial.

A resolution to this Abraham-Minkowski controversy (AMC) was proposed recently in Ref. [68]. It was argued there that both interpretations are correct; namely,  $P_M$  can be attributed as the canonical momentum and  $P_A$  can be attributed as the kinetic momentum of a photon. Yet, strictly speaking, the argument of Ref. [68] applies only to the case of a nonrelativistic solid dielectric. The subsequent generalization in Ref. [69] is not quite complete either; for example, the latter neglects electrostriction and magnetostriction, kinetic effects, and spatial dispersion, and also attributes  $v_g$  entirely to the Poynting flux. Thus, a quantitative relativistic theory is still lacking that would correct the existing understanding of PDM, and Eqs. (5.41) in particular. The purpose of this section is to resolve these issues in a consistent manner and, through that, formulate a comprehensive asymptotic theory of linear waves of arbitrary nature.

To understand what the right framework is for such a calculation, notice that introducing a photon implies that the frequency  $\omega$  and the wave vector  $k$  are well defined. These are exactly the validity conditions of the asymptotic theory commonly known as geometrical optics (GO). (The term “optics” here means only that the theory deals with sufficiently large  $\omega$  and  $k$ ; i.e., waves need not be electromagnetic.) Although usually defined through rays and wave equations [70–72], the most fundamental, axiomatic GO is an abstract field theory that applies to any field having a Lagrangian density of a specific form [Eq. (5.51)]; dissipative effects can also be added. Just like Newton’s laws of particle motion hold, with obvious exceptions, independently of specific forces acting on particles, the basic GO equations are then invariant to the wave nature [80], and the wave properties can be derived in general. Hence, axiomatic GO should resolve the AMC automatically and transparently.

### **(5.3.1): NOTATION**

#### **Definition (5.3.4): Lorenz's system**

Although the Lorenz's system has no singular approximation, it has been numerically checked that its functional jacobian matrix possesses at least a large and negative real eigenvalue in a large domain of the phase space. So, it can be considered as a slow fast dynamical system but not as a singularly perturbed

system. Thus, Geometric Singular Perturbation Theory can not provide the slow invariant manifold associated with Lorenz system.

The following notation will be assumed below. We use the symbol  $\doteq$  for definitions. Greek indexes span from 0 to 3 and refer to coordinates in spacetime,  $x^\alpha$ . In particular, for the Minkowski spacetime, we adopt  $x^0 \doteq ct$ , where  $t$  is time. Hence the Lorentz transformation matrix,  $\Lambda_\beta^\alpha \doteq \partial x^\alpha / \partial x'^\beta$ , is given by

$$\begin{aligned}\Lambda_0^0 &= \gamma, & \Lambda_i^0 &= \gamma v_i / c, & \Lambda_0^i &= \gamma v^i / c, \\ \Lambda_j^i &= \delta_j^i + (\gamma - 1) v^i v_j / v^2\end{aligned}\quad (5.42)$$

Where  $v^i$  is the velocity of the “primed” reference frame with respect to the laboratory frame, and  $\gamma \doteq (1 - v^2/c^2)^{-1/2}$ . Latin indexes  $i, j$ , and  $l$  span from 1 to 3 and refer to spatial coordinates,  $x^i$ . Spatial vectors are denoted with bold,  $\mathbf{X}$ ; spatial tensors are also marked with hat,  $\hat{\mathbf{T}}$ ; symbols such as  $\mathbf{XY} \doteq \hat{\mathbf{Z}}$  stand for spatial dyadics,  $Z^{ij} = X^i Y^j$ ; the symbol  $\hat{\mathbf{1}}$  denotes the unit spatial tensor; and the three-tensor

$$\hat{\Lambda} \doteq \hat{\mathbf{1}} + \frac{\gamma - 1}{v^2} \mathbf{v}\mathbf{v} \quad (5.43)$$

is the spatial part of  $\Lambda_\beta^\alpha$ . Summation over repeating indexes will also be implied; e.g.,  $X^i Y_i = \sum_{i=1}^3 X^i Y_i$ .

Latin indexes (excluding  $i, j, l$ , and nonbold roman, as in  $n_p$ ) denote partial derivatives with respect to the corresponding variables; e.g., for  $f \doteq f(a, \omega, \mathbf{k}, t, \mathbf{x})$ , the symbol  $f_x$  denotes the derivative (gradient) with respect to the last argument,  $X$ . In addition to those, we also introduce “full” temporal and spatial derivatives,  $\partial_t X \equiv \partial X / \partial t$  and  $\partial_i X \equiv \partial X / \partial x^i$ , which treat all arguments of (any)  $X$  as functions of, correspondingly,  $t$  and  $x^i$ . For instance, for the above  $f$ , one has

$$\partial_t f = f_a \partial_t a + f_\omega \partial_t \omega + f_{k_i} \partial_t k_i + f_t, \quad (5.44)$$

$$\partial_i f = f_a \partial_i a + f_\omega \partial_i \omega + f_{k_j} \partial_i k_j + f_{x_i}, \quad (5.45)$$

While  $\partial_t a(t, x) = a_t(t, x)$ , etc. The symbol  $\nabla$  denotes the associated full covariant derivative; e.g.,  $\nabla_i f = \partial_i f$  is the full gradient of the scalar  $f$ , and  $\nabla \cdot F$  is the full divergence of the vector  $F$ ,

$$\nabla \cdot F = \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial x^i} (\sqrt{\eta} F^i) \quad (5.46)$$

Where  $\eta \doteq \det \eta_{ij}$ , and  $\eta_{ij} = \eta_{ji}$  is the spatial metric. [In Cartesian coordinates, Euclidean space has  $\eta_{ij} = \eta^{ij} = \text{diag}(1,1,1)$ , so  $\eta = 1$ ] The symbol  $\nabla_\alpha$  denotes the analogous (to  $\partial_i$ ) full derivative with respect to  $x^\alpha$ , and  $F_{;\alpha}$  denotes the analogous full covariant derivative. For example, the four-divergence is

$$F_{;\alpha}^\alpha = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} (\sqrt{g} F^\alpha), \quad (5.47)$$

Where  $g \doteq -\det \eta_{\mu\nu}$ , and  $g_{\mu\nu} = g_{\nu\mu}$  is the space-time metric. For an introduction to the tensor notation and index manipulation rules in particular

### (5.3.2): GENERAL WAVES (Covariant formulation)

First, let us consider a general nondissipative wave described by some action integral  $S = \int \mathcal{L} \sqrt{g} d^4x$ , where

$$\sqrt{g} d^4x \equiv \sqrt{g} dx^1 dx^2 dx^3 dx^4 \quad (5.48)$$

is an invariant volume element in spacetime, and the four scalar  $\mathcal{L}$  is the Lagrangian density. Since the action of the underlying medium is not included here, no invariance requirements on  $\mathcal{L}$  are imposed. Instead, we assume that the wave structure remains fixed (albeit not necessarily sinusoidal), so the wave is fully described by some canonical phase  $\theta$ , which will be understood as a scalar field  $\theta = \theta(x^\nu)$ , and  $a = a(x^\nu)$ , which is an arbitrary measure of the wave local amplitude [74]. We also assume that the envelope evolves on spacetime scales that are large compared to those of local oscillations. On such time scales, it is only the average Lagrangian density that contributes to  $S$ , so one can adopt that  $\mathcal{L}$  does not depend on  $\theta$  explicitly. Instead,  $\mathcal{L}$  must depend on the phase four-gradient,

$$k_\mu \doteq \theta_\mu \quad (5.49)$$

Which is the generalized "wave vector" (actually, a four covector here), obviously having zero four-curl,

$$k_{\mu;\nu} - k_{\nu;\mu} = k_{\mu,\nu} - k_{\nu,\mu} = \theta_{\mu,\nu} - \theta_{\nu,\mu} = 0. \quad (5.50)$$

[Equation (5.50) is known as the consistency relation.] Besides that,  $\mathfrak{L}$  must depend on  $a_\nu$ ; yet the dependence on the amplitude gradients  $a_{,\nu}$  is negligible in the GO limit. Thus, allowing also for slow parametric dependence on the spacetime coordinates  $x^\nu$ , we postulate

$$\mathfrak{L} = \mathfrak{L}(a, k_\mu, x^\nu) \quad (5.51)$$

Which as well can be considered as the definition of the GO approximation. Hence wave equations are inferred using the least action principle, namely, as follows.

First, let us consider the variation of  $S$  with respect to the wave amplitude  $a$ . Since  $\delta_a S = \int \mathfrak{L}_a \delta_a \sqrt{g} d^4x$  for any  $\delta_a$ , the requirement  $\delta_a S = 0$  leads to

$$\mathfrak{L}_a = 0. \quad (5.52)$$

Equation (5.52) can be understood as the wave dispersion relation, and it is generally nonlinear, i.e., it may retain essential dependence on [75,77].

Second, let us consider the variation of  $S$  with respect to the wave phase  $\theta$ [78]. Due to Eq. (5.49) and the fact that  $\mathfrak{L}$  does not depend on  $\theta$  explicitly, for any  $\delta_\theta$  one has

$$\begin{aligned} \delta_\theta S &= \int \mathfrak{L}_{k_\mu} \delta\theta_\mu \sqrt{g} d^4x \\ &= \left[ (\sqrt{g} \mathfrak{L}_{k_\mu} \delta\theta)_{,\mu} - (\sqrt{g} \mathfrak{L}_{k_\mu})_{,\mu} \delta\theta \right] d^4x = - \int (\mathfrak{L}_{k_\mu})_{,\mu} \delta\theta \sqrt{g} d^4x \end{aligned}$$

Where we used the fact that the wave field vanishes at infinity, so  $\int (\dots)_\mu d^4x = 0$ . Thus, the requirement  $\delta_\theta S = 0$  yields that the four-divergence of the action flux density  $\mathcal{J}^\mu \doteq -\mathfrak{L}_{k_\mu}$  is zero [79]

$$\mathcal{J}^\mu{}_\mu = 0. \quad (5.53)$$

Which is called the action conservation theorem (ACT). Since the ACT has the form of a continuity equation, one can treat  $\mathcal{G}^\mu \doteq \mathcal{J}^\mu/\hbar$  as the flux density of some fictitious quasiparticles, or “photons.” (In application to specific waves, one can as well think of plasmons, phonons, polaritons, or any other elementary excitations instead). However, remember that within our classical description, it is only the product  $\hbar\mathcal{G}^\mu$  that has an explicit physical meaning, so the actual value of  $\hbar$  will be irrelevant for our purposes.

Finally, let us also introduce the wave EMT as follows. Consider the (generally asymmetric) tensor

$$\mathcal{T}_\alpha{}^\beta \doteq k_\alpha \mathcal{J}^\beta + \delta^\beta{}_\alpha \mathfrak{L}. \quad (5.54)$$

The divergence of  $\mathcal{T}_\alpha{}^\beta$  equals

$$\begin{aligned} \mathcal{T}_\alpha{}^\beta{}_{;\beta} &= k_{\alpha;\beta} \mathcal{J}^\beta + k_\alpha \mathcal{J}^\beta{}_{;\beta} + \delta_\alpha^\beta (\mathfrak{L}_\alpha a_{;\beta} + \mathfrak{L}_{k\lambda} k_{\lambda;\beta} + \mathfrak{L}_{x^\beta}) \\ &= k_{\alpha;\beta} \mathcal{J}^\beta + \delta_\alpha^\beta (\mathfrak{L}_{k\lambda} k_{\lambda;\beta} + \mathfrak{L}_{x^\beta}) \\ &= k_{\alpha;\beta} \mathcal{J}^\beta - k_{\lambda;\alpha} \mathcal{J}^\lambda + \mathfrak{L}_{x^\alpha} \\ &= k_{\alpha;\beta} \mathcal{J}^\beta - k_{\alpha;\lambda} \mathcal{J}^\lambda + \mathfrak{L}_{x^\alpha} \\ &= \mathfrak{L}_{x^\alpha}. \end{aligned} \quad (5.55)$$

Where we used Eqs. (5.51)–(5.53). This tensor is then associated with the conservation law,  $\mathcal{T}_\alpha{}^\beta{}_{;\beta} = 0$ , yielded when the system is translationally invariant in spacetime (i.e., when the fourforce is zero,  $\mathfrak{L}_{x^\alpha} = 0$ ). Hence,  $\mathcal{T}_\alpha{}^\beta$  is a true canonical EMT [80], as one could also infer from the standard definition that is based on Noether’s theorem [81]. However, notice that in contrast to the fundamental theorem of the vacuum field theory,  $\mathcal{T}^{\alpha\beta}$  does not permit the usual

[82] symmetrization, since  $L$  is not restricted by any invariance requirements [83]. For symmetrization via adopting an effective, "optical" metric. In particular, the very fact that a scalar field such as  $\theta(x^\nu)$  yields an asymmetric EMT already proves the lack of Lorentz invariance [84].

### (5.3.3): Application to the Minkowski spacetime

From now on, we will assume the Minkowski spacetime with metric signature  $(-, +, +, +)$ ; hence,

$$g_{00} = g^{00} = -1, \quad \eta_{ij} \doteq g_{ij}, \quad \eta = g. \quad (5.56)$$

(Although the space is Euclidean, we will allow for curvilinear coordinates; thus, albeit flat, the spatial metric  $\eta_{ij}$  can otherwise be arbitrary.) In this case,  $k_\alpha = (-\omega/c, \mathbf{k})$ , and  $k^\alpha = (\omega/c, \mathbf{k})$ , where

$$\omega \doteq -\partial_t \theta, \quad \mathbf{k} \doteq \nabla \theta. \quad (5.57)$$

Then Eq. (5.50) turns into the following set of equations:

$$\partial_t \mathbf{k} + \nabla \omega = 0, \quad \nabla \times \mathbf{k} = 0. \quad (5.58)$$

One may notice also that the latter equation here can be considered as the initial condition for the former one, taking curl of which readily  $\partial_t (\nabla \times \mathbf{k}) = 0$ . Accordingly, Eq. (5.51) becomes

$$\mathcal{L} = \mathcal{L}(a, \omega, \mathbf{k}; t, \mathbf{x}). \quad (5.59)$$

The dispersion relation hence holds in the form (5.52). The ACT can be rederived from Eq. (5.59) or it can be deduced from Eq. (5.53) by substituting  $\mathcal{J}^\beta = (c\mathcal{T}, \mathcal{J})$  either way, one gets

$$\partial_t \mathcal{T} + \nabla \cdot \mathcal{J} = 0. \quad (5.60)$$

where  $\mathcal{T}$  is the action density, and  $\mathcal{J}$  is the action spatial flux density, introduced as follows:

$$\mathcal{T} \doteq \mathcal{L}_\omega, \quad \mathcal{J} \doteq -\mathcal{L}_k \quad (5.61)$$

In particular integrating of Eq.(5.60) over the volume  $dV = \sqrt{\eta}d^3x$  yields conservation of the integral action,

$$I = \int \mathcal{T}dV = \text{const} \quad (5.62)$$

Introducing the photon density  $N = \mathcal{T}/h$  and the photon spatial flux density  $\mathcal{G} = \mathcal{T}/\hbar$ , one can further rewrite Eq.(5.60) as  $\partial_t N + \nabla \cdot \mathcal{G} = 0$ . and Eq.(5.62) will yield the photon conservation  $N = \int NdV = \text{const}$ . Also notice that both  $I$  and  $N$  are Lorentz invariants, as are well known to flow from the general properties of the continuity equation[85]. The elements of the (contra variant) EMT are now

$$\mathcal{T}^{00} = \omega I - \mathcal{Q}, \quad \mathcal{T}^{0i} = \omega \mathcal{J}^i/c \quad (5.63)$$

$$\mathcal{T}^{i0} = ck^i \mathcal{T}, \quad \mathcal{T}^{ij} = k^i \mathcal{J}^j + \eta^{ij} \mathcal{Q}.$$

$$\frac{\partial \mathcal{T}^{00}}{\partial t} + \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial x^i} (c \mathcal{T}^{0i} \sqrt{\eta}) = w, \quad (5.64)$$

Which is continuity equation for  $\mathcal{T}^{00}$  with the right -hand side being  $\omega = \mathcal{G}^{00} c \mathcal{Q}_{x0} = -\mathcal{Q}_t$ . since the latter has the meaning of the canonical power source,  $\mathcal{E} = \mathcal{T}^{00}$  must be the wave canonical energy density ,and  $\mathcal{Q}^i = c \mathcal{T}^{0i}$  must be the canonical energy flux density .similarly,

$$\frac{1}{c} \frac{\partial \mathcal{T}^{i0}}{\partial t} + \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial x^j} (\mathcal{T}^{ij} \sqrt{\eta}) = f^i \quad (5.65)$$

Which is a continuity equation for the three-vector  $\mathcal{T}^{i0}/c$  with the right-hand side being  $f = \mathcal{Q}_x$ . Since the latter has the meaning of the canonical momentum source,  $p^i = \mathcal{T}^{i0}/c$  must be the wave canonical momentum density, and the (generally asymmetric) three-tensor  $\Pi^{ij} = \mathcal{T}^{ij}$  must be the canonical momentum flux density [86].

In summary, one then has

$$\mathcal{T}^{\alpha\beta} = \begin{pmatrix} \varepsilon & \mathcal{Q}/c \\ cp & \hat{\Pi} \end{pmatrix} \quad (5.66)$$



$$\mathcal{E} = \omega\mathcal{T} - \mathcal{Q}, \quad \mathcal{Q} = \omega\mathcal{J}, \quad p = k\mathcal{T}, \quad \hat{\Pi} = k\mathcal{J} + \mathcal{Q}\hat{I} \quad (5.67)$$

and Eqs. (5.64) and (5.65) can be written as follows

$$\partial_t \mathcal{E} + \nabla \cdot \mathcal{Q} = \omega, \quad \partial_t p + \nabla \cdot \hat{\Pi} = f. \quad (5.68)$$

It is hence seen that the wave energy propagates at velocity  $\mathcal{Q}/\mathcal{E}$  that is generally different from the action flow velocity  $\mathcal{J}/\mathcal{T}$  Eq. (5.60), and similarly for the momentum flow velocity. Moreover, those three turn out to be different from the velocities of information, or the nonlinear group velocities, of which there can also be more than one.

### (5.3.4): LINEAR WAVES: MINKOWSKI REPRESENTATION Basic equations

Now let us consider a linear wave, i.e., such that has  $\omega(k; t, x)$  independent of  $a$ . In this case, from Eq. (5.52) it is seen that  $\mathcal{Q}_a$  must be separable  $\mathcal{Q}_a = \mathcal{D}(\omega, k)A_a$ , where  $A(a, \omega, k)$  is some function such that  $A_a$  is nonzero. [Parametric dependence of functions like  $\mathcal{Q}$ ,  $\mathcal{D}$ , and  $A$  on  $(t, x)$  is also assumed, but will be omitted for the sake of brevity.] Then,

$$\mathcal{Q} = \mathcal{D}(\omega, k)A \quad (5.69)$$

It will hence be convenient to think of  $a$  as a linear measure of the oscillating field amplitude. Then, most commonly, one will have  $A \propto a^2$ ; yet for our purposes, the actual dependence need not be specified.

Equation (5.52) now yields

$$\mathcal{D}(\omega, k) = 0. \quad (5.70)$$

Thus Eqs. (5.61) become

$$\mathcal{T} = \mathcal{D}_\omega A, \quad \mathcal{J} = -\mathcal{D}_k A, \quad (5.71)$$

and Eqs. (5.67) take the form

$$\mathcal{E} = \omega \mathcal{J}, \quad \mathcal{Q} = \omega \mathcal{J}, \quad p = k \mathcal{J}, \quad \hat{\Pi} = k \mathcal{J}. \quad (5.72)$$

Hence the photon canonical energy  $H = \mathcal{E}/N$  and the photon canonical momentum  $P = p/N$

$$H = \hbar\omega, \quad P = \hbar k, \quad (5.73)$$

matching the Minkowski interpretation exactly and independently of the wave nature. (In fact,  $P = \hbar k$  holds even for nonlinear waves [Eqs. (5.67)].) In particular,  $P^\alpha \doteq (H/c, P) = \hbar k^\alpha$  happens to be a true four-vector, by definition of  $k^\alpha$ , so  $P^\alpha P_\alpha$  is a Lorentz invariant. The latter can also be understood as a measure of the photon canonical mass  $\mathfrak{M}$ , defined via

$$\mathfrak{M}^2 \doteq -P^\alpha P_\alpha / c^2 \quad (5.74)$$

Further, differentiating Eq. (5.70) with respect to  $k$  [with  $\omega = \omega(k; t, x)$ ] also gives  $\mathcal{D}_\omega v_g + \mathcal{D}_k = 0$ , where we introduced the linear group velocity  $v_g \doteq \omega_g$ ; therefore,

$$v_g = -\mathcal{D}_k / \mathcal{D}_\omega = \mathcal{J} / \mathcal{T} \quad (5.75)$$

Hence, Eq. (5.66) yields  $\mathcal{T}^{\alpha\beta} = \mathcal{N} T^{\alpha\beta}$ , where

$$T^{\alpha\beta} = \begin{pmatrix} \hbar\omega & \hbar\omega v_g / c \\ c\hbar k & \hbar k v_g \end{pmatrix} \quad (5.76)$$

is the canonical EMT per photon. Alternatively, one can also exclude  $\mathcal{N}$  and rewrite Eqs. (5.72) as

$$p = k\mathcal{E}/\omega, \quad \mathcal{Q} = \mathcal{E}v_g, \quad \hat{\Pi} = pv_g \quad (5.77)$$

It is seen, from here and Eqs. (5.68), that the canonical action, energy, and momentum are all transported at the same velocity  $v_g$ . However, keep in mind that the full, or kinetic, energy and momentum densities carried by the wave generally do not have this property.

Finally, let us introduce photon trajectories  $d_t x = v_g$ , also known as GO rays. Along those trajectories,

$$d_t = \partial_t + v_g \cdot \nabla \quad (5.78)$$

Then Eqs. (5.58) yield

$$d_t x = v_g, \quad d_t k = -\omega_x, \quad d_t \omega = \omega_t \quad (5.79)$$

[Remember that the derivatives  $\omega_x$  and  $\omega_t$  of  $\omega(k; t, x)$  are taken at fixed  $k$ .] In particular, the ACT can hence be written as

$$d_t \ln \mathcal{J} = -\nabla \cdot v_g \quad (5.80)$$

Also notice that Eqs. (5.79) can be understood as canonical equations for the photon motion governed by the Hamiltonian  $H(X, P; t)$ . In this form, i.e.,

$$d_t X = H_p, \quad d_t P = -H_x, \quad d_t H = H_t, \quad (5.81)$$

they are identical to the motion of a true classical particle such as an electron, which supports the well-known analogy between GO and classical mechanics. Reverting to Eqs. (5.51) and (5.59), it is seen then that not just waves, but classical particles too can be described in terms of phases and amplitudes [87].

## Section (5.4): LINEAR WAVES (ABRAHAM REPRESENTATION)

### (5.4.1): Basic definitions

In addition to the wave canonical, or Minkowski, EMT that we discussed so far, one can also introduce the corresponding so-called kinetic, or Abraham, EMT,

$$\tau^{\alpha\beta} = \begin{pmatrix} \varepsilon & \vartheta/c \\ c\rho & \hat{\pi} \end{pmatrix}. \quad (5.82)$$

It is defined such that being a part of the complete EMT that describes the “wave + medium” system (WMS),  $\tau^{\alpha\beta}$  includes all the wave-related (i.e.,  $a$ -dependent) dynamics of the medium and fields. We hence express it as  $\tau^{\alpha\beta} = \mathcal{T}^{\alpha\beta} + \Delta\tau^{\alpha\beta}$ , where  $\Delta\tau^{\alpha\beta}$  is the “ponderomotive” part that is stored in the medium, and, similarly,

$$\varepsilon = \mathcal{E} + \Delta\varepsilon, \quad \rho = p + \Delta\rho, \quad \mu = \mathcal{M} + \Delta\mu. \quad (5.83)$$

In particular, notice the following. Since the WMS is closed and thus Lorentz-invariant, its complete EMT is symmetrizable [82]. Yet its unperturbed part is symmetrizable by itself (because it describes a closed system too, namely, the medium absent a wave), so  $\tau^{\alpha\beta}$  is also symmetrizable separately. On the other hand, since  $\tau^{\alpha\beta}$  is proportional to the wave intensity, it is defined uniquely and, therefore, must be symmetric. This yields  $\rho = \vartheta/c^2$ , and

$$\mu = x \times \rho \quad (5.84)$$

Also, since the integral energy-momentum of the whole WMS is defined uniquely, and its a-dependent part is defined uniquely too, one can find  $(\varepsilon/c, \rho)$  as the a-dependent part of the WMS canonical energy-momentum density. Given the WMS Lagrangian density, the latter can, in principle, be found straight forwardly in any specific problem [81]. However, the general answer is not informative (meaning that  $\tau^{\alpha\beta}$  is by itself a somewhat artificial construct; [88]). Thus, below, we consider only the particular model of an isotropic medium, most popular in the AMC context, yet still refrain from specifying the wave nature.

#### (5.4.2): Wave energy-momentum in an isotropic medium

**General case:** Consider an isotropic medium (such as gas, fluid, or plasma) comprised of elementary particles or fluid elements whose dynamics absent a wave is described by some aggregate Lagrangian  $L$ . In the presence of a wave, the WMS Lagrangian is hence  $L + \mathcal{L}$ , where  $\mathcal{L} = \int \mathcal{L}dV$  is the wave Lagrangian. Assuming that particles contribute to  $\mathcal{L}$  additively, the latter can be written as  $\mathcal{L} = \mathcal{L}^{(0)} - \sum_{\ell} \Phi^{\ell}$ , where  $\mathcal{L}^{(0)}$  is independent of all particle velocities  $u^{(\ell)}$ , and each of the so-called ponderomotive potentials  $\Phi^{(\ell)}$  [75,76], or dipole potentials, depends on the specific  $u^{(\ell)}$  but not on other velocities. Omitting the index  $\ell$ , we can write the canonical momentum of each particle as the sum of the mechanical part  $\partial_u L$  and the ponderomotive part  $-\partial_u \Phi$ , also yielding the ponderomotive contribution to the particle canonical energy,  $-u \cdot \partial_u \Phi$ . (This energy should not be confused with the ponder motive potential  $\Phi$  itself, which is a part of the wave canonical energy [89].) Thus, the densities of the ponderomotive momentum and energy stored in particles can be written as follows:

$$\Delta\rho = - \sum_s n^s \langle \partial_u \Phi \rangle^{(s)}, \quad (5.85)$$

$$\Delta\varepsilon = - \sum_s n^s \langle u \cdot \partial_u \Phi \rangle^{(s)} \quad (5.86)$$

Where the summation is taken over different species,  $n^s$  are the (locally averaged) densities of those species, and angular brackets denote averaging over velocities within the corresponding ensembles.

If a medium can be modeled as a single fluid (in particular meaning that kinetic effects are inessential, unlike, e.g., in warm plasma), one can simplify Eqs. (5.86) and (5.85) further as follows. First of all, notice that the velocities  $u$  of fluid elements are all equal to a single velocity  $v$ , so Eqs. (5.85) and (5.86) become

$$\Delta\rho = -n \partial_v \Phi, \quad \Delta\varepsilon = v \cdot \Delta\rho. \quad (5.87)$$

It is hence convenient to rewrite Eqs. (5.87) in terms of Lorentz invariant proper parameters of the medium [90]. Since  $\Phi$  that enters here depends on the wave intensity, it must be gaugeinvariant; thus, being (minus) the interaction Lagrangian of a single element, it transforms as  $\Phi = \Phi' / \gamma$ , with primes in this section denoting the medium rest frame, and  $\gamma = (1 - v^2/c^2)^{-1/2}$ . Also,  $n = \gamma n'$ , where  $n'$  is the proper density, correspondingly. Since the latter does not depend on  $v$ , we then get  $\Delta\rho = -\partial_v (n' \Phi') + \gamma^2 v n' \Phi' / c^2$ . Further, let us denote

$$n' \Phi' = \mathfrak{L}' - \mathfrak{L}'^{(0)} \doteq \mathfrak{u}', \quad (5.88)$$

Where  $\mathfrak{L}'^{(0)}$  is  $L'^{(0)}$  per unit volume, and introduce

$$\mathfrak{R} \doteq \frac{\gamma^2 v}{c^2} \mathfrak{u}', \quad (5.89)$$

understood as the striction contribution. Since  $\mathfrak{L}'^{(0)}$  is also independent of  $v$ , one then can write

$$\Delta\rho = \partial_v \mathfrak{L}' + \mathfrak{R}. \quad (5.90)$$

Due to the fact that a Lagrangian density is a four-scalar,  $L''$  that enters Eq. (5.90) can also be replaced with  $\mathfrak{L}$ . However, using  $\mathfrak{L}'(a', k'_\mu)$  is preferable because it cannot depend on  $v$  explicitly, but rather depends on it solely through  $a'$  and  $k'_\mu$ . [Remember that the velocity derivative in Eq. (5.90) must be taken at fixed  $a$  and  $k_\mu$ .] Due to  $\mathfrak{L}'_{a'} = 0$ , we then get

$$\partial_\nu \mathfrak{L}' = -(\partial_\nu \Lambda^\nu_\mu) k_\nu \mathcal{J}'^\mu, \quad (5.91)$$

Where we substituted the (covector) Lorentz transformation (5.42),  $k'_\mu = \Lambda^\nu_\mu k_\nu$ . On the other hand,  $k_\nu = (\Lambda^{-1})^\lambda_\nu k'_\lambda$ , so Eq. (81) can also be written as

$$\partial_\nu \mathfrak{L}' = -\gamma G^\lambda_\mu \mathcal{J}'^\mu / c \quad (5.92)$$

Where we introduced a dimensionless matrix function

$$G^\lambda_\mu(v) \doteq (c/\gamma) (\Lambda^{-1})^\lambda_\nu (\partial_\nu \Lambda^\nu_\mu). \quad (5.93)$$

As shown in the Appendix, Eq. (82) is also equivalent to

$$\partial_\nu \mathfrak{L}' = \gamma \text{Tr}(G \mathcal{J}') / c = \mathfrak{B} + \mathfrak{R}, \quad (5.94)$$

Where the terms on the right-hand side are defined as

$$\mathfrak{B} = \gamma \hat{\Lambda} \cdot \left( \frac{\mathfrak{L}' v'_g}{c^2} - \mathcal{P}' \right), \quad (5.95)$$

$$\mathfrak{R} = \frac{\gamma^2}{\gamma + 1} \left[ \frac{v}{c} \times \left( \frac{v'_g}{c} \times \mathcal{P}' \right) \right]. \quad (5.96)$$

Yet,  $v'_g$  is parallel to  $k'$  in an isotropic medium, so  $\mathfrak{B}$  vanishes, and we finally get

$$\Delta \rho = \mathfrak{B} + \mathfrak{R}, \quad \Delta \varepsilon = v \cdot (\mathfrak{B} + \mathfrak{R}). \quad (5.97)$$

### (5.4.3): LINEAR ELECTROMAGNETIC WAVES(Wave Lagrangian)

Finally, let us apply the above results to illustrate how the properties of linear electromagnetic waves can be calculated explicitly within our general approach, without using Maxwell's equations for the wave envelope. Note also that similar calculations can be performed for nonlinear waves too, for which  $L$  can be constructed from first principles as well [91].

First, let us consider a nondissipative wave, as usual. The wave Lagrangian density (derived independently, [82]) can be expected in the form  $\mathcal{L} = \mathcal{L}^{(0)} - \mathcal{U}$ , where

$$\mathcal{L}^{(0)} \doteq \frac{1}{16\pi} (\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{B}}^* \cdot \tilde{\mathbf{B}}) \quad (5.98)$$

$\tilde{\mathbf{E}}$  and  $\tilde{\mathbf{B}}$  are the electric and magnetic field envelopes, and  $\mathcal{U}$  is the potential-energy density of the wave-medium interaction. For linear, i.e., dipolar interaction, we can take

$$\mathcal{U} = -\frac{1}{4} \text{Re}(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{P}} - \tilde{\mathbf{B}}^* \cdot \tilde{\mathbf{M}}) \quad (5.99)$$

Here,  $\tilde{\mathbf{P}}$  is the electric dipole moment density (i.e., the polarization), and  $\tilde{\mathbf{M}}$  is the magnetic dipole moment density (i.e., the magnetization); also, one factor 1/2 comes from the time averaging and the other 1/2 comes from the fact that  $\tilde{\mathbf{P}}$  and  $\tilde{\mathbf{M}}$  are linear functions of  $\tilde{\mathbf{E}}$  and  $\tilde{\mathbf{B}}$ , correspondingly. Now let us introduce  $\tilde{\mathbf{D}}$  and  $\tilde{\mathbf{H}}$  via

$$\tilde{\mathbf{D}} \doteq \tilde{\mathbf{E}} + 4\pi\tilde{\mathbf{P}} \doteq \hat{\epsilon} \cdot \tilde{\mathbf{E}}, \quad (5.100)$$

$$\tilde{\mathbf{B}} \doteq \tilde{\mathbf{H}} + 4\pi\tilde{\mathbf{M}} \doteq \hat{\mu} \cdot \tilde{\mathbf{H}}, \quad (5.101)$$

assuming that the permittivity tensor  $\hat{\epsilon}$  and the permeability tensor  $\hat{\mu}$  (not to be confused with the kinetic angular momentum density  $\mu$ ) are Hermitian so the assumption of zero dissipation is satisfied. One gets then

$$\mathcal{L} = \frac{1}{16\pi} (\tilde{\mathbf{E}}^* \cdot \hat{\epsilon} \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{B}}^* \cdot \hat{\mu}^{-1} \cdot \tilde{\mathbf{B}}) \quad (5.102)$$

(Here  $\hat{\mu}^{-1}$  is the tensor inverse to  $\hat{\mu}$ ), also meaning that

$$\mathcal{U} = -\frac{1}{16\pi} [\tilde{\mathbf{E}}^* \cdot (\hat{\epsilon} - \hat{1}) \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{B}}^* \cdot (\hat{\mu}^{-1} - \hat{1}) \cdot \tilde{\mathbf{B}}] \quad (5.103)$$

this implies assigning the following ponderomotive potentials to particles (or fluid elements) comprising the medium:

$$\Phi = -\tilde{\mathbf{E}}^* \cdot \hat{\alpha} \cdot \tilde{\mathbf{E}}/4 - \tilde{\mathbf{B}}^* \cdot \hat{\beta} \cdot \tilde{\mathbf{B}}/4, \quad (5.104)$$

Where  $\hat{\alpha}$  and  $\hat{\beta}$  are the particle electric and magnetic polarizabilities [68], and

$$\hat{\epsilon} = \hat{1} + \sum_s 4\pi n^{(s)} \langle \hat{\alpha} \rangle^{(s)}, \quad (5.105)$$

$$\hat{\mu}^{-1} = \hat{1} - \sum_s 4\pi n^{(s)} \langle \hat{\beta} \rangle^{(s)}. \quad (5.106)$$

Parametrization and dispersion. We remember that there is a freedom in defining  $\mathbf{a}$ , so there are various options for how to parametrize the wave Lagrangian density. First, let us consider  $\tilde{\mathbf{E}}$  and  $\tilde{\mathbf{E}}^*$  as independent vector fields, i.e.,  $\mathbf{a} = (\tilde{\mathbf{E}}, \tilde{\mathbf{E}}^*)$ . In this case, it is convenient to write

$$\mathcal{L}^{(0)} = \frac{1}{16\pi} \left( \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}} - \frac{c^2}{\omega^2} |\mathbf{k} \times \tilde{\mathbf{E}}|^2 \right) \quad (5.107)$$

(where we used that  $\tilde{\mathbf{B}} = c\mathbf{k} \times \tilde{\mathbf{E}}/\omega$ ) and

$$\mathcal{L} = \left[ \frac{1}{16\pi} \left( \tilde{\mathbf{E}}^* \cdot \hat{\epsilon} \cdot \tilde{\mathbf{E}} - \frac{c^2}{\omega^2} (\mathbf{k} \times \tilde{\mathbf{E}}^*) \cdot \hat{\mu}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) \right) \right] \quad (5.108)$$

correspondingly. Using that

$$\begin{aligned} (\mathbf{k} \times \tilde{\mathbf{E}}^*) \cdot \hat{\mu}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) &= -(\tilde{\mathbf{E}}^* \times \mathbf{k}) \cdot \hat{\mu}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) \\ &= -\tilde{\mathbf{E}}^* \cdot \{ \mathbf{k} \times [\hat{\mu}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}})] \} \end{aligned} \quad (5.109)$$

one can further rewrite Eq. (5.108) as follows



$$\mathfrak{L} = \frac{\tilde{\mathbf{E}}^*}{16\pi} \cdot \left[ \hat{\mathbf{e}} \cdot \tilde{\mathbf{E}} + \frac{c^2}{\omega^2} \mathbf{k} \times [\hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}})] \right] \quad (5.110)$$

Then, varying  $\mathfrak{L}$  with respect to  $\tilde{\mathbf{E}}^*$  yields the following dispersion relation

$$\hat{\mathbf{e}} \cdot \tilde{\mathbf{E}} + \frac{c^2}{\omega^2} \mathbf{k} \times [\hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}})] = 0, \quad (5.111)$$

in agreement with Maxwell's equations [75]. Similarly, varying  $\mathfrak{L}$  with respect to  $\tilde{\mathbf{E}}$  yields the complex conjugate equation.

Alternatively, if the polarization vector  $\mathbf{e}$  is prescribed (or considered as an independent field), then one can as well introduce a scalar amplitude instead, say,  $a = |\tilde{\mathbf{E}}|$ . This yields  $\mathfrak{L} = \mathfrak{D}(\omega, \mathbf{k})a^2$ , with  $\mathfrak{D}(\omega, \mathbf{k})$  given by

$$\mathfrak{D} = \frac{1}{16\pi} \cdot \left[ \mathbf{e}^* \cdot \hat{\mathbf{e}} \cdot \mathbf{e} - \frac{c^2}{\omega^2} (\mathbf{k} \times \mathbf{e}^*) \cdot \hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \mathbf{e}) \right]. \quad (5.112)$$

The dispersion relation that follows [Eq. (5.70)] is Eq. (5.111) multiplied by  $\mathbf{e}^*$ .