Sudan University of Science& Technology

Faculty of Sciences

Approximations of analytical solutions for seepage flow Derivatives in porous media using fractional calculus

تقريبات الحلول التحليلية لمشتقات الجريان الناز في وسط مسامي   باستخدام الحسبان الكسري

A thesis submitted as fulfilment for the requirement of the PHD in applied mathematic

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DEDICATION

TO

MY PARENTS,

HUSBAND AND DAUGHTERS,

BROTHERS AND SISTERS,

FAMILY AND FRIENDS

ACKNOWLEDGEMENTS

I would like to thank every body who contributed to the success of this work especially my supervisor **Prof.Mohammed Ali Bashir** of Alneelain University. At the sametime I am grateful for all those that helped me from beginning. A thanks are also duo to co-supervisor**Dr.BelkeesAbdelaziz** of Sudan University for her help.

Also I would like to express my lot thanks to my husband.

ABSTRACT

In this thesis the variational iteration method is implemented to give exact solutions for seepage flow derivatives in porous media. A correction functional for the fractional partial equation is well constructed by a general Lagrange multipliers which can be identified optimally via variational theory. Some examples are given and comparisons are made with the Adomian Decomposition Method (ADM).The comparisons show that the method is very effective ,convenient and overcome the difficulty arising in calculating Adomian polynomials andwe have solved seepage flow derivatives in porous media by using the Adomian Decomposition method(ADM). Our solution proved rapid convergence to the exact solution.

الخلاصة

في هذا البحث تم تنفيذ طريقة تكرارالتغايرلإعطاء الحلول الدقيقة لمسالة تدفق السائل في الاوساط المسامية. معادلة التصحيح للكسر الجزئي تعطى بمضاريب لاجرانج ويمكن التعرف عليها على النحو الأمثل من خلال نظرية التغيير .اعطيت بعض الأمثلة وتمت مقارنة الحل مع طريقة ادوميان.تظهرالمقارنة أن الطريقة فعالة جدا ،مريحة.وتم حل مسألة تدفق السائل في الاوساط المسامية عن طريق ادوميان .ثبت لدينا التقارب السريع والدقيق في الحل.

Introduction

Calculus is a very important branch of mathematics. It was invented by European mathematicians, Isaac Newton and Gottfried Leibnitz in the seventieth century. The first thought in fractional calculus was introduced in the following question: L'Hospital asked Leibniz about the possibility that n be fractional. Liebniz (1695) replied; it will lead to a paradox." But he added prophetically, 'From this apparent paradox, one day useful consequences will be drawn'." In the years following, a little advancement was made in the development of fractional calculus. One of the earliest meaningful results given by Lacroix(1819) and Joseph Liouville (1832).

Fractional calculus represents more accurately some natural behaviour related to different areas of engineering and is applied to modern application of science, engineering and mathematics. Some of the areas where fractional calculus has made a profound impact include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, physics and control theory [1-5].

In recent years, it has turned out that many phenomena can be successfully modelled by the use of fractional derivatives and integrals. Several analytical and numerical methods have been proposed to solve fractional ordinary, integral and partial differential equations of physical interest. The most commonly methods used are: Adomian Decomposition Method (ADM) and Variational Iteration Method .

In this thesis , we introduced the method of solution of nonlinear ordinary and partial differential equation . The two methods studied are Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM), many illustrated examples was given.

We introduced the main topic needed throughout this work , including the definition of flow through porous and the equations that govern this flow , Also fractional calculus and its important properties where studied.

The Adomian Decomposition Method (ADM) was introduced . This method and the improvement made by the noise phenomenon and modified Decomposition Method [19] are reliable and effective techniques of promising results. This method provide the solution in an infinite series form.

The Varational Iteration Method (VIM), was investigated. This method provides the solution in an infinite series

We applied the Variational Iteration Method (VIM) in solving fractional three dimensional Darcy's law, and we obtained an exact solution.

Contents

Chapter One

Adomian Decomposition Method (ADM)

1.1 Adomian Decomposition Method

The Adomian Decomposition Method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solution in particular.The method proved tobe powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equation,and nonlinear integralequation .The decomposition method demonstrates fast convergence of the solution and therefore provides several significant advantages. The method will be successfully used to handle most types of partial differential equation that appear in several physical models and scientific applications.The method attacks the problem in a direct way and in a straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behaviour of the model under discussion.

The Adomian Decomposition Method was introduced and developed by George Adomian is well addressed in the literature. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equation, partial differential equation and integral equation as well.

The Adomian Decomposition Method consists of decomposing the unknown function $u(x,y)$ of any equation into a sum of an infinite number of components defined by decomposition serie

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \qquad (1.1)$$

Where the components $u_n(x,y),\ n \geq 0$ are to be determined in a recursive manner .

The decomposition method concerns itself with finding the components $u_0, u_1, u_2 \ldots$ individually .As will be seen through the table, the determination of these components can be achieved in an easy way through a recursive relation that usually involve simple integrals.

To give a clear overview of Adomian decomposition method, we first consider the linear differential equation written in an operator form by

$$Lu + Ru = g, \qquad (1.2)$$

where L is, mostly, the lower order derivative which is assumed to be invertible, R

Is other linear differential operator , and g is a source term. We apply the inverse operator L^{-1} to both sides of equation (1.2) and using given condition to obtain

$$u = f - L^{-1}(Ru) \quad (1.3)$$

Where the function f represents the terms arising from integrating the source term g andfrom using the given conditions that are assumed to be prescribed. As indicated before. Adomian decomposition method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n \quad (1.4)$$

Where the components $u_0, u_1, u_2 \ldots$ are usually

recurrently determined. Substituting (1.4) into both side of (1.3) leads to

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}(R(\sum_{n=0}^{\infty} u_n)) \qquad (1.5)$$

For simplicity . equation (1.5) can be rewritten as

$$u_0 + u_1 + u_2 + \ldots = f - L^{-1}(R(u_0 + u_1 + u_2 + \ldots)). \quad (1.6)$$

To construct the recursive relation needed for the components $u_0, u_1, u_2 \ldots$

It is important to note that Adomian decomposition method suggests that zeroth component u_0 is usually defined by the function f described above, i.e.by all terms, that are not included under the inverse operator L^{-1} , which arise from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by

$$u_0 = f,$$

$$u_{k+1} = -L^{-1}(R(u_k)), \; k \geq 0, \; (1.7)$$

Or equivalently

$$u_0 = f,$$

$$u_1 = -L^{-1}(R(u_0)),$$

$$u_2 = -L^{-1}(R(u_1)), \quad (1.8)$$

$$u_3 = -L^{-1}(R(u_2)),$$

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It is clearly seen that the relation (1.8) reduced the differential equation under consideration into an elegant determination of computable components. Having determined these components, we then substitute it into (1.4) to obtain the solution a series form.

It was formally that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated to confirm the rapid convergence of the resulting series.

However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. It was also shown by many that the series obtained by evaluating few terms gives an approximation of high degree of accuracy if compared with other numerical techniques.

It seems reasonable to give a brief outline about the works conducted by Adomian in applying Adomian's method. Adomian and in many other works introduced his method and applied it to many deterministic and stochastic problems. He implemented his method to solve frontier problems ofphysics The Adomian'sachievements in this regard are remarkable and of promising results.

Adomian'smethod has attracted a considerable amount of research work. A comparison between the decomposition method and the perturbation technique showed the efficiency of the decomposition method compared to the tedious work required by the perturbation method. A comparative study between Adomian's method and Taylor series method has been examined to show that the decomposition method requires less computational work if compared with Taylor series.

Other comparisons with traditional method such as finite difference method have been conducted in the literature.

It is to be noted that many studies have shown that few terms of the decomposition series provide a numerical result of a high degree of accuracy. Many other studies implement the decomposition method for differential equations, ordinary and partial, and for integral equation, linear and nonlinear.

It is normal in differential equations that we seek a closed form solution or a series solution with a proper number of terms.It seems reasonable to use the decomposition method to discuss two ordinary differential equation where an exact solution is obtained for the first equation and a series approximation is determined for the second equation. For the first problem we consider the equation.

$$u'(x) = u(x), \; u(0) = A \quad (1.9)$$

In an operator form, equation (1.9) becomes

$$Lu = u, \quad (1.10)$$

Where the differential operator L is given by

$$L = \frac{d}{dx}, \quad (1.11)$$

And therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x (.)\, dx, \quad (1.12)$$

Applying L^{-1} to both sides of (1.10) and using the initial condition we obtain

$$L^{-1}(Lu) = L^{-1}(u), \quad (1.13)$$

So that

$$u(x) - u(0) = L^{-1}(u), \quad (1.14)$$

Or equivalently

$$u(x) = A + L^{-1}(u), \quad (1.15)$$

Substituting the series assumption (1.5) into both sides of (1.15) gives

$$\sum_{n=0}^{\infty} u_n(x) = A + L^{-1}\left(\sum_{n=0}^{\infty} u_n(x)\right), \quad (1.16)$$

In view of (1.16) , the following recursive relation

$$u_0(x) = A \quad (1.17) \quad u_{k+1}(x) = L^{-1}(u_k(x)), \; k \geq 0,$$

Follows immediately. Consequently ,we obtain

$$u_0(x) = A$$

$$u_1(x) = L^{-1}(u_0(x)) = Ax,$$

$$u_2(x) = L^{-1}(u_1(x)) = A\frac{x^2}{2!}, \quad (1.18)$$

$$u_3(x) = L^{-1}(u_2(x)) = A\frac{x^3}{3!},$$

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Substituting (1.18) into (1.5) gives the solution in a series form by

$$u(x) = A\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right), \quad (1.19)$$

And it a closed form by

$$u(x) = Ae^{x}, \quad (1.20)$$

We next consider the well-Known Airy's equation

$$u''(x) = xu(x), u(0) = A, u'(0) = B, \quad (1.21)$$

In an operator form, equation (1.21) becomes

$$Lu = xu, \quad (1.22)$$

Where the differential operator L is given by

$$L = \frac{d^2}{dx^2}, \quad (1.23)$$

And therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x \int_0^x (.)\, dxdx \quad , (1.24)$$

Operating L^{-1} with on both sides of (1.21) and using the initial conditions we obtain

$$L^{-1}(Lu) = L^{-1}(xu), \quad (1.25)$$

So that

$$u(x) - xu'(0) - u(0) = L^{-1}(xu), \quad (1.26)$$

Or equivalently

$$u(x) = A + Bx + L^{-1}(xu), \quad (1.27)$$

Substituting the series assumption (1.5) into both sides of (1.27) yields

$$\sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} u_n(x)\right), \quad (1.28)$$

Following the decomposition method we obtain the following recursive relation

$$u_0(x) = A + Bx \,,$$

$$(1.29) \quad u_{k+1}(x) = L^{-1}(xu_k(x)), \; k \geq 0,$$

Consequently, we obtain

$$u_0(x) = A + Bx$$

$$u_1(x) = L^{-1}(xu_0(x)) = A\frac{x^3}{6} + B\frac{x^4}{12},$$

$$u_2(x) = L^{-1}(xu_1(x)) = A\frac{x^6}{180} + B\frac{x^7}{504}, \quad (1.30)$$

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Substituting (1.30) into (1.5) gives the solution in a series form

$$u(x) = A(1 + \frac{x^3}{6} + \frac{x^6}{180} + ...) + B(x + \frac{x^4}{12} + \frac{x^7}{504} + ...), \quad (1.31)$$

Other components can be easily computed to enhance the accuracy of the approximation.

We consider the inhomogeneous partial differential equation :

$$\begin{aligned} u_x + u_y &= f(x, y), \\ u(0, y) &= g(y), \; u(x, 0) = h(x), \end{aligned} \quad (1.32)$$

In an operator form, Eq (1.32) can be written as

$$L_x u + L_y u = f(x, y), \quad (1.33)$$

Where

$$L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, \quad (1.34)$$

Where each operator is assumed easily invertible, and thus the inverse operators L_x^{-1} and L_y^{-1} exist and given by

$$L_x^{-1}(.) = \int_0^x (.)\, dx, \quad (1.35) \quad L_y^{-1}(.) = \int_0^y (.)\, dy,$$

This mean that

$$L_x^{-1} L_x u(x, y) = u(x, y) - u(0, y), \quad (1.36)$$

Appling L_x^{-1} to both sides of (1.33) gives

$$L_x^{-1} L_x u = L_x^{-1}(f(x, y)) - L_x^{-1}(L_y u), \quad (1.37)$$

Or equivalently

$$u(x,y) = g(y) + L_x^{-1}(f(x,y)) - L_x^{-1}(L_y u), \quad (1.38)$$

Obtained by using (1.36) and by using the condition $u(0,y) = g(y)$ As stated Above the decomposition method sets

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \quad (1.39)$$

Substituting (1.39) into both sides of (1.38) we find

$$\sum_{n=0}^{\infty} u_n(x,y) = g(y) + L_x^{-1}(f(x,y)) - L_x^{-1}(L_y(\sum_{n=0}^{\infty} u_n(x,y)), \quad (1.40)$$

This can be rewritten as

$$u_0 + u_1 + u_2 + ... = g(y) + L_x^{-1}(f(x,y)) - L_x^{-1}L_y(u_0 + u_1 + u_2 + ...), \quad (1.41)$$

The zeroth component u_0 ,as suggested by Adomian method is always identified by the given initial condition and the terms arising from $L_x^{-1}(f(x,y)),$ both side of which are assumed to be known. Accordingly, we set

$$u_0(x,y) = g(y) + L_x^{-1}(f(x,y)), \quad (1.42)$$

Consequently, the other components $u_{k+1}, k \geq 0$ are defined by using the relation

$$u_{k+1}(x,y) = -L_x^{-1}L_y(u_k), \; k \geq 0, \quad (1.43)$$

Combining (1.42) and (1.43) we obtain the recursive scheme

$$u_0(x,y) = g(y) + L_x^{-1}(f(x,y)), \quad (1.44)$$

$$u_{k+1}(x,y) = -L_x^{-1}L_y(u_k), \; k \geq 0,$$

That form the basis for a complete determination of the components $u_0, u_1, u_2, ...$ therefore, the components can be easily obtained by

$$u_0(x,y) = g(y) + L_x^{-1}(f(x,y)),$$

$$u_1(x,y) = -L_x^{-1}(L_y u_0(x,y), \quad (1.45)$$

$$u_2(x,y) = -L_x^{-1}(L_y u_1(x,y)),$$

$$u_3(x,y) = -L_x^{-1}(L_y u_2(x,y)),$$

And so on. Thus the components u_n can be determined recursively as far as we like.

It is clear that the accuracy of the approximation can be significantly improved by simply determining more components. Having established the components of $u(x,y)$, the solution in a series form follows immediately. However, the expression

$$\phi_n = \sum_{r=0}^{n-1} u_r(x,y), \quad (1.46)$$

Is considered the n-term approximation to u. For concrete problems, where exact solution is not easily obtainable, we usually use the truncated series (1.46) for numerical purposes. As indicated earlier, the convergence of Adomian Decomposition Method has been established.

It is important to note that the solution can be obtained by finding the y - solution by applying the inverse operator L_y^{-1} to both sides of the equation

$$L_y = f(x,y) - L_x u, \quad (1.47)$$

The equality of the x - solution and the y - solution is formally justified and will be examined through the coming examples.

It found, as will be seen later, that very few terms of the series obtained in

(1.39) provide a high degree of accuracy level which makes the method powerful when compared with other existing numerical techniques. In many cases the series representation of $u(x,y)$ can be summed to yield the closed form solution. Several works in this direction have demonstrated the power of the method for analytical and numerical application.

The essential features of the decomposition method for linear and nonlinear equation, homogeneous and inhomogeneous, can be outlined as follows:

1.Express the partial differential equation, linear or nonlinear, in an operator form.

2.Apply the inverse operator to both sides of the equation written in an operator for.

3.Set the unknown function $u(x,y)$ into a decomposition series

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y), \quad (1.48)$$

Whose components are elegantly determined. We next substitute the series (1.48) into both sides of the resulting equation.

4.Identify the zeroth component $u_0(x,y)$ as the terms arising from the given condition and from integrating the source term $f(x,y)$, both are assumed to be known.

5.Determine the successive components of the series solution $u_k, k \geq 1$ by applying the recursive scheme (1.44), where each component u_k, can be completely determined by using the previous component u_{k-1}.

6.Substitute the determined components (1.48), to obtain the solution in a series form. An exact solution can be easily obtained in manyequation if such a closed form solution exists.

It is to be noted Adomian Decomposition Method approaches any equation, homogeneous or inhomogeneous, and linear or nonlinear in a straightforward manner without any need to restrictive assumptions such as linearization, discretization or perturbation. There is not need in using this method to convert in homogeneous condition to homogeneous condition as required by other techniques.

The essential steps of the Adomian Decomposition Method will be illustrated by discussing the following examples.

Example 1.

Use Adomian Decomposition Method to solve the following inhomogeneous PDE

$$u_x+u_y=x+y, u(0,y)=0, u(x,0)=0, \quad (1.49)$$

Solution.

In an operator form, Eq. (1.49), can be written as

$$L_x u=x+y-L_y u, \quad (1.50)$$

Where

$$L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, \quad (1.51)$$

It clear that L_x is invertible, hence L_x^{-1} exists and given by

$$L_x^{-1}(.) = \int_0^x (.)\, dx, \quad (1.52)$$

The x - solution:

This solution can be obtained by applying L_x^{-1} to both sides of (1.50), hence we find

$$L_x^{-1} L_x u = L_x^{-1}(x + y) - L_x^{-1} L_y u, \quad (1.53)$$

Or equivalently

$$\begin{aligned} u(x, y) &= u(0, y) + \frac{1}{2}x^2 + xy - L_x^{-1}(L_y u) \\ &= \frac{1}{2}x^2 + xy - L_x^{-1}(L_y u), \end{aligned} \quad (1.54)$$

Obtained upon using the given condition $u(0, y) = 0$, Eq. (1.36) and by integrating $f(x, y) = x + y$ with respect to x, As stated above, the decomposition method identifies the unknown function $u(x, y)$, as an infinite of components $u_n(x, y), n \geq 0$ given by

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \quad (1.55)$$

Substituting (1.55) in to both sides of (1.54), we find

$$\sum_{n=0}^{\infty} u_n(x, y) = \frac{1}{2}x^2 + xy - L_x^{-1}(L_y(\sum_{n=0}^{\infty} u_n(x, y))), \quad (1.56)$$

Using few terms of the decomposition (1.55), we obtain

$$u_0 + u_1 + u_2 + \ldots = \frac{1}{2}x^2 + xy - L_x^{-1}(L_y(u_0 + u_1 + u_2 + \ldots)), \quad (1.57)$$

As presented before, the decomposition method identifies the zeroth component u_0 by all term arising from the given condition and from integrating $f(x, y) = x + y$, therefore we set

$$u_0(x, y) = \frac{1}{2}x^2 + xy, \quad (1.58)$$

Consequently, the recursive scheme that will enable us to completely determine the successive components thus constructed by

$$u_0(x, y) = \frac{1}{2}x^2 + xy, \quad (1.59)$$

$$u_{k+1}(x, y) = -L_x^{-1}(L_y(u_k)), \; k \ge 0,$$

This in turn gives

$$u_1(x, y) = -L_x^{-1}(L_y(u_0))$$
$$= -L_x^{-1}(L_y(\frac{1}{2}x^2 + xy)) = \frac{1}{2}x^2, \quad (1.60)$$

$$u_2(x, y) = -L_x^{-1}(L_y(u_1)) = -L_x^{-1}(L_y(-\frac{1}{2}x^2)) = 0.$$

Accordingly, $u_k = 0, k \ge 2$. Having determined the components of $u(x, y)$, we find

$$u = u_0 + u_1 + u_2 + \ldots = \frac{1}{2}x^2 + xy - \frac{1}{2}x^2 = xy \quad (1.61)$$

The y - solution:

It is important to note that the exact solution can be finding the y - solution. In an operator form we can write equation by

$$L_y u = x + y - L_x u, \quad (1.62)$$

Assume that L_y^{-1} exists and given by

$$L_y^{-1}(.) = \int_0^y (.)\, dx, \quad (1.63)$$

Applying L_y^{-1} to both sides of the Eq. (1.62), gives

$$u(x, y) = xy + \frac{1}{2}y^2 - L_y^{-1}(L_x u). \quad (1.64)$$

As mentioned above, the decomposition method sets the solution $u(x, y)$ in an series form by

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \quad (1.65)$$

Inserting (1.65) into both sides of the (1.64), we obtain

$$\sum_{n=0}^{\infty} u_n(x, y) = xy + \frac{1}{2}y^2 - L_y^{-1}(L_x(\sum_{n=0}^{\infty} u_n(x, y))), \quad (1.66)$$

Using few terms only for simplicity reasons, we obtain

$$u_0 + u_1 + u_2 + \ldots = xy + \frac{1}{2}y^2 - L_y^{-1}(L_x(u_0 + u_1 + u_2 + \ldots)), \quad (1.67)$$

The decomposition method identifies the zeroth component u_0 by all term arising from the given condition and from integrating $f(x, y) = x + y$, therefore we set

$$u_0(x, y) = xy + \frac{1}{2}y^2, \quad (1.68)$$

Consequently, the recursive scheme that will enable us to completely determine the successive components thus constructed by

$$u_0(x, y) = xy + \frac{1}{2}y^2, \quad (1.69)$$

$$u_{k+1}(x,y) = -L_y^{-1}(L_x(u_k)),\ k \geq 0,$$

This gives

$$u_1(x,y) = -L_y^{-1}(L_x(u_0))$$
$$= -L_y^{-1}(L_x(xy + \frac{1}{2}y^2)) = -\frac{1}{2}y^2, \quad (1.70)$$

$$u_2(x,y) = -L_y^{-1}(L_x(u_1)) = -L_y^{-1}(L_x(-\frac{1}{2}y^2)) = 0.$$

Accordingly, $u_k = 0, k \geq 2$. Having determined the components of $u(x,y)$, we find

$$u(x,y) = u_0 + u_1 + u_2 + \ldots = xy + \frac{1}{2}y^2 - \frac{1}{2}y^2 = xy \quad (1.71)$$

Example 2.

Use Adomian Decomposition Method to solve the following homogeneous PDE

$$u_x - u_y = 0, u(0,y) = y, u(x,0) = x, \quad (1.72)$$

Solution.

In an operator form, Eq. (1.72), can be written as

$$L_x u(x,y) = L_y u(x,y), \quad (1.73)$$

Where

$$L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, \quad (1.74)$$

Applying the inverse operator L_x^{-1} to both sides of (1.73), and using the given condition $u(0,y) = y$ yields

$$u(x,y) = y + L_x^{-1}(L_y u). \quad (1.75)$$

As mentioned above, the decomposition method sets the solution $u(x, y)$ in an series form by

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \quad (1.76)$$

Inserting (1.76) into both sides of the (1.75), we obtain

$$\sum_{n=0}^{\infty} u_n(x, y) = y + L_x^{-1}(L_y(\sum_{n=0}^{\infty} u_n(x, y))). \quad (1.77)$$

Using few terms only for simplicity reasons, we obtain

$$u_0 + u_1 + u_2 + ... = y + L_x^{-1}(L_y(u_0 + u_1 + u_2 + ...)), \quad (1.78)$$

Proceeding as before, we identify the zeroth component $u_0(x, y)$, by

$$u_0(x, y) = y, \quad (1.79)$$

Having identifies the zeroth component $u_0(x, y)$, we obtain the recursive scheme

$$u_0(x, y) = y, \quad (1.80)$$

$$u_{k+1}(x, y) = L_x^{-1} L_y(u_k)), \; k \geq 0,$$

The components $u_0, u_1, u_2, ...$ are thus determined as follows:

$$u_0(x, y) = y, \quad (1.81)$$

$$u_1(x, y) = L_x^{-1} L_y u_0 = L_x^{-1} L_y(y) = x,$$

$$u_2(x, y) = L_x^{-1} L_y u_1 = L_x^{-1} L_y(x) = 0,$$

It is obvious that all components, $u_k = 0, k \geq 2$ .consequently, the solution is given by

$$u(x, y) = u_0 + u_1 + ... = u_0 + u_1 = x + y \quad (1.82)$$

The exact solution obtained by using the decomposition series (1.76) .

It is important to note here that the exact solution given by (1.82) can be also be obtained by determining the y - solution as discussed above.

1.2 nonlinear partial differential equation

Adomian decomposition method has been I mentioned before and has been applied to a wide class of linear partial differential equation. The method has been applied directly and in a straightforward manner to homogeneous and inhomogeneous problems without any restrictive assumptions or linearization. The method usually decomposes the unknown function u into an infinite sum of components that will be determined recursively through iterations as discussed before.

The Adomian decomposition method will be applied on next part to handle nonlinear partial differential equations. An important remark should be made here concerning the representation of the nonlinear terms that appear in the equation. Although the linear term u is expressed as an infinite series of components, the Adomian decomposition method requires a special representation for the nonlinear terms such as $u^2, u^3, u^4, \sin u, e^u, uu_x, u_x^2, etc$. that appear in the equation. The method introduce a formal algorithm to establish a proper representation for all forms of nonlinear terms. The representation of the nonlinear teams is necessary to handle the nonlinear equation in an effective and successful way.

In the following, the Adomian scheme for calculating representation of nonlinear term will be introduced in details. The discussion will be supported by several illustrative examples that will cover a wide variety of forms of nonlinearity. In a like manner, an alternative algorithm for calculating Adomian polynomials will be outlined in details supported by illustrative examples.

1.3 Calculation of Adomian Polynomials

It is well known now that Adomian decomposition method suggest that the unknown linear function u may be represented by the decomposition series

$$u = \sum_{n=0}^{\infty} u_n, \quad (1.83)$$

Where the components $u_n, n \geq 0$ can be elegantly computed in a recursive way.

However, the nonlinear term $F(u)$, such as

$u^2, u^3, u^4, \sin u, e^u, uu_x, u_x^2, etc$. can be expressed by an infinite series of the so-called. Adomian polynomials A_n given in the form

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, n_n), \quad (1.84)$$

Where the so-called Adomian polynomials A_n can be evaluated for all forms of nonlinearity.

The Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2,\ldots \quad (1.85)$$

The general formula (1.85) can be simplified as follows. Assuming that the nonlinear function is $F(u)$, therefore by using (1.85), Adomian polynomials are given by

$$A_0 = F(u_0),$$

$$A_1 = u_1 F'(u_0),$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0), \quad (1.86)$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0),$$

$$A_4 = u_4 F'(u_0) + (\frac{1}{2!}u_2^2 + u_1 u_3) F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(4)}(u_0),$$

Other polynomials can be generated in a manner.

Two important observation can be made here. First, A_0 depends only on u_0, u_1 and u_2, and so on. Second substituting (1.86) into (1.84) gives

$$F(u) = A_0 + A_1 + A_2 + A_3 + \ldots,$$

$$= F(u_0) + (u_1 + u_2 + u_3 + \ldots)\, F'(u_0)$$

$$+ \frac{1}{2!}(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + \ldots)\, F''(u_0) + \ldots$$

$$= F(u_0) + (u - u_0 + \ldots) F'(u_0) + \frac{1}{2!}(u - u_0)^2 F''(u_0) + \ldots$$

The last expansion confirms a fact that the A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used. The few Adomian polynomials given in (1.86) clearly show that the sum of subscripts of the components of u of each term of A_n is equal to n. As stated before, it is clear that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2 .

In the following ,we will calculate Adomian polynomials for several forms of nonlinearity that may arise in nonlinear ordinary or partial differential equations.

Calculation of Adomian Polynomials A_n

1.Nonlinear polynomials

Case 1. $F(u) = u^2$

The polynomials can be obtained as follows:

$$A_0 = F(u_0) = u_0^2,$$

$$A_1 = u_1 F'(u_0) = 2u_0u_1,$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) = 2u_0u_2 + u_1^2,$$

$$A_3 = u_3 F'(u_0) + u_1u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0) = 2u_0u_3 + 2u_1u_2,$$

Case 2. $F(u) = u^3$

The polynomials are given by

$$A_0 = F(u_0) = u_0^3,$$

$$A_1 = u_1 F'(u_0) = 3u_0^2 u_1,$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = 3u_0^2 u_2 + 3u_0 u_1^2,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3,$$ **Case 3.**

$$F(u) = u^4$$

Proceeding as before we find

$$A_0 = F(u_0) = u_0^4,$$

$$A_1 = u_1 F'(u_0) = 4u_0^4 u_1,$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = 4u_0^3 u_2 + 6u_0^2 u_1^2,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = 4u_0^3 u_3 + 4u_1^3 u_0 + 12u_0^2 u_1 u_2,$$

In a parallel manner, Adomian polynomials can be calculated for nonlinear polynomials of higher degrees.

II. Nonlinear Derivatives

Case 1. $F(u) = (u_x)^2$

$$A_0 = F(u_0) = u_{0_x}^2,$$

$$A_1 = u_1 F'(u_0) = 2u_{0_x} u_{1_x},$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = 2u_{0_x} u_{2_x} + u_{1_x}^2,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x},$$

Case 2. $F(u) = (u_x)^3$

The Adomian polynomials are given by

$$A_0 = F(u_0) = u_{0_x}^3,$$

$$A_1 = u_1 F'(u_0) = 3u_{0_x}^2 u_{1_x},$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = 3u_{0_x}^2 u_{2_x} + 3u_{0_x} u_{1_x}^2,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = 3u_{0_x}^2 u_{3_x} + 6u_{0_x} u_{1_x} u_{2_x} + u_{1_x}^3,$$

Case 3. $F(u) = uu_x = \frac{1}{2} L_x(u^2)$

The Adomian polynomials for nonlinearity are given by

$$A_0 = F(u_0) = u_0 u_{0_x},$$

$$A_1 = \frac{1}{2} L_x(2u_0 u_1) = u_{0_x} u_1 + u_0 u_{1_x},$$

$$A_2 = \frac{1}{2} L_x(2u_0 u_2 + u_1^2) = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0,$$

$$A_3 = \frac{1}{2} L_x(2u_0 u_3 + 2u_1 u_2) = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0,$$

III. Trigonometric Nonlinearity

Case 1. $F(u) = \sin u$

The Adomian polynomials for nonlinearity are given by

$$A_0 = F(u_0) = \sin u_0,$$

$$A_1 = u_1 F'(u_0) = u_1 \cos u_0,$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = u_2 \cos u_0 - \frac{1}{2} u_1^2 \sin u_0,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0,$$

Case 2. $F(u) = \cos u$

Proceeding as before gives

$$A_0 = F(u_0) = \cos u_0,$$

$$A_1 = u_1 F'(u_0) = -u_1 \sin u_0,$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = -u_2 \sin u_0 - \frac{1}{2} u_1^2 \cos u_0,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0,$$

IV. Hyperbolic Nonlinearity

Case 1. $F(u) = \sinh u$

The Adomian polynomials for nonlinearity are given by

$$A_0 = F(u_0) = \sinh u_0,$$

$$A_1 = u_1 F'(u_0) = u_1 \cosh u_0,$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = u_2 \cosh u_0 + \frac{1}{2} u_1^2 \sinh u_0,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = u_3 \cos u_0 + u_1 u_2 \sin u_0 + \frac{1}{3!} u_1^3 \cos u_0,$$

Case 2. $F(u) = \cosh u$

Proceeding as before gives

$$A_0 = F(u_0) = \cosh u_0,$$

$$A_1 = u_1 F'(u_0) = u_1 \sinh u_0,$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = u_2 \sinh u_0 + \frac{1}{2} u_1^2 \cosh u_0,$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!} u_1^3 \cos u_0,$$

V. Exponential Nonlinearity

Case 1. $F(u) = e^u$

The Adomian polynomials for nonlinearity are given by

$$A_0 = F(u_0) = e^{u_0},$$

$$A_1 = u_1 F'(u_0) = u_1 e^{u_0},$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = (u_2 + \frac{1}{2} u_1^2) e^{u_0},$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = (u_3 + u_1 u_2 + \frac{1}{3!} u_1^3) e^{u_0},$$

Case 2. $F(u) = e^{-u}$

Proceeding as before gives

$$A_0 = F(u_0) = e^{-u_0},$$

$$A_1 = u_1 F'(u_0) = -u_1 e^{-u_0},$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = (-u_2 + \frac{1}{2} u_1^2) e^{-u_0},$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = (-u_3 + u_1 u_2 - \frac{1}{3!} u_1^3) e^{-u_0},$$

VI. Logarithmic Nonlinearity

Case 1. $F(u) = \ln u, u \succ 0$

The A_n polynomials for logarithmic nonlinearity are given by

$$A_0 = F(u_0) = \ln u_0,$$

$$A_1 = u_1 F'(u_0) = \frac{u_1}{u_0},$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = \frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0^2},$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{1}{3}\frac{u_1^3}{u_0^3},$$

Case 2. $F(u) = \ln(1+u), -1 \prec u \leq 1$

The A_n polynomials for logarithmic nonlinearity are given by

$$A_0 = F(u_0) = \ln(1+u_0),$$

$$A_1 = u_1 F'(u_0) = \frac{u_1}{1+u_0},$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = \frac{u_2}{1+u_0} - \frac{1}{2}\frac{u_1^2}{(1+u_0)^2},$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = \frac{u_3}{1+u_0} - \frac{u_1 u_2}{(1+u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1+u_0)^3},$$

1.4 Alternative Algorithm for Calculating Adomian Polynomials

It is worth noting that a considerable amount of research work has been invested to develop an alternative method to Adomian algorithm for calculating Adomian polynomials A_n. The aim was to develop a practical technique that will calculate Adomian polynomials in a practical way without any need to the formulae introduced before. However, the methods developed so far in this regard are identical to that used by Adomian.

We believe that a simple and reliable technique can be established to make the calculation less dependable on the formulae presented before.

In the following, we will introduce an alternative algorithm that can be used to calculate Adomian polynomials for nonlinear terms in an easy way. The newly developed method depends mainly on algebraic and trigonometric identities, and on Taylor expansions as well. Moreover, we should use the fact that the sum of subscripts of the components of u in each term of the polynomial A_n. is equal to n.

The alternative algorithm suggests that we substitute u as a sum of components $u_n, n \geq 0$ as defined by the decomposition method. It is clear that A_0 is always determined independent of the other polynomials $A_n, n \geq 1$ where A_0 is defined by

$$A_0 = F(u_0) \quad (1.87)$$

The alternative method assumes that we first separate $A_0 = F(u_0)$ for every nonlinear term $F(u)$. With this separation done, the remaining components of $F(u)$ can be expanded by using algebraic operation, trigonometric identities, and Taylor series as well. We next collect all terms of the expansion obtained such that the sum of the subscripts of the components of u in each term is the same. Having collected these terms, these terms, the calculation of the Adomian polynomials is thus completed. Several examples have been tested, and the obtained results have shown that Adomian polynomials can be elegantly computed without any need to the formulas established by Adomian. The technique will be explained by discussing the following illustrative examples.

Adomian Polynomials by Using the Alternative Method

1.Nonlinear polynomials

Case 1. $F(u) = u^2$

We first set

$$u = \sum_{n=0}^{\infty} u_n \quad (1.88)$$

Substituting (1.88) into $F(u) = u^2$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + \ldots)^2. \quad (1.89)$$

Expanding the expression at the right hand side gives

$$F(u) = u_0^2 + 2u_0u_1 + 2u_0u_2 + u_1^2 + 2u_0u_3 + 2u_1u_2 + ... \quad (1.90)$$

The expansion in (1.90) can be rearranged by grouping all terms with the sum of the subscripts is the same. This means that we can rewrite g (1.90) as

$$F(u) = \underbrace{u_0^2}_{A_0} + \underbrace{2u_0u_1}_{A_1} + \underbrace{2u_0u_2 + u_1^2}_{A_2} + \underbrace{2u_0u_3 + 2u_1u_2}_{A_3}$$
$$+ \underbrace{2u_0u_4 + 2u_1u_3 + u_2^2}_{A_4} + \underbrace{2u_0u_5 + 2u_1u_4 + 2u_2u_3}_{A_5} + ... \quad (1.91)$$

This completes the determination of Adomian polynomials given by

Case 1. $F(u) = u^2$

$A_0 = u_0^2,$

$A_1 = 2u_0u_1,$

$A_2 = 2u_0u_2 + u_1^2,$

$A_3 = 2u_0u_3 + 2u_1u_2,$

$A_4 = 2u_0u_4 + 2u_1u_3 + u_2^2,$

$A_5 = 2u_0u_5 + 2u_1u_4 + 2u_2u_3,$

Case 2. $F(u) = u^3$

Proceeding as before, we set

$$u = \sum_{n=0}^{\infty} u_n, \quad (1.92)$$

Substituting (1.92) into $F(u) = u^3$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + ...)^3. \quad (1.93)$$

Expanding the right hand side yields

$$F(u)=u_0^{\ 3}+3u_0^2u_1+3u_0^2u_2+3u_0u_1^2+3u_0^2u_3+6u_0u_1u_2+u_1^3$$
$$+3u_0^2u_4+3u_1^2u_2+3u_2^2u_0+6u_0u_1u_3\ldots \qquad (1.94)$$

The expansion in (1.94) can be rearranged by grouping all terms with the sum of the subscripts is the same. This means that we can rewrite g (1.94) as

$$F(u)=\underbrace{u_0^3}_{A_0}+\underbrace{3u_0^2u_1}_{A_1}+\underbrace{3u_0^2u_2+3u_0u_1^2}_{A_2}+\underbrace{3u_0^2u_3+6u_0u_1u_2+u_1^3}_{A_3}$$
$$+\underbrace{3u_0^2u_4+3u_1^2u_2+3u_2^2u_0+6u_0u_1u_3}_{A_4}+\ldots \qquad (1.95)$$

Consequently, Adomian polynomials can be written by

$A_0=u_0^3,$

$A_1=3u_0^3u_1,$

$A_2=3u_0^3u_2+3u_0u_1^2,$

$A_3=3u_0^2u_2+3u_0u_1^2,$

$A_4=3u_0^2u_3+6u_0u_1u_2+u_1^3,$

II. Nonlinear Derivatives

Case 1. $F(u)=(u_x)^2$

We first set

$$u=\sum_{n=0}^{\infty}u_n \quad (1.96)$$

Substituting (1.96) into $F(u)=u_x^2$ gives

$$F(u)=(u_{0_x}+u_{1_x}+u_{2_x}+u_{3_x}+\ldots)^2. \ (1.97)$$

Squaring the right side gives

$$F(u) = u_{0_x}{}^2 + 2u_{0_x}u_{1_x} + 2u_{0_x}u_{2_x} + u_{1_x}^2 + 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x} + ... \quad (1.98)$$

Grouping the terms as discussed above we find

$$F(u) = \underbrace{u_{0_x}^2}_{A_0} + \underbrace{2u_{0_x}u_{1_x}}_{A_1} + \underbrace{2u_{0_x}u_{2_x} + u_{1_x}^2}_{A_2} + \underbrace{2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x}}_{A_3} \\ + \underbrace{u_{2_x}^2 + 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x}}_{A_4} + ... \quad (1.99)$$

Adomian polynomials are given by

$$A_0 = u_{0_x}^2,$$

$$A_1 = 2u_{0_x}u_{1_x},$$

$$A_2 = 2u_{0_x}u_{2_x} + u_{1_x}^2,$$

$$A_3 = 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x},$$

$$A_4 = 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x} + u_{2_x}^2,$$

Case 2. $F(u) = uu_x$

We first set

$$u = \sum_{n=0}^{\infty} u_n \quad (1.200)$$

$$u_x = \sum_{n=0}^{\infty} u_{n_x}$$

Substituting (2.200) into $F(u) = uu_x$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + ...) \times \\ (u_{0_x} + u_{1_x} + u_{2_x} + u_{3_x} + ...) \quad (1.201)$$

Multiplying the two factors gives

$$F(u) = u_0 u_{0_x} + u_{0_x} u_1 + u_0 u_{1_x} + u_{1_x}^2$$
$$+ u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0 + u_{0_x} u_3$$
$$+ u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0 \qquad (1.202)$$
$$+ u_{0_x} u_4 + u_0 u_{4_x} + u_{1_x} u_3 + u_1 u_{3_x} + u_2 u_{2_x} + \ldots.$$

Proceeding with grouping the terms we obtain

$$F(u) = \underbrace{u_{0_x} u_0}_{A_0} + \underbrace{u_{0_x} u_1 + u_{1_x} u_0}_{A_1}$$
$$+ \underbrace{u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0}_{A_2}$$
$$+ \underbrace{u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0}_{A_3} \qquad (1.203)$$
$$+ \underbrace{u_{0_x} u_4 + u_{1_x} u_3 + u_{2_x} u_2 + u_{3_x} u_1}_{A_4} + u_{4_x} u_0 + \ldots$$

It then follows that Adomian polynomials are given by

$$A_0 = u_{0_x} u_0,$$

$$A_1 = u_{0_x} u_1 + u_{1_x} u_0,$$

$$A_2 = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0,$$

$$A_3 = 2u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0,$$

$$A_4 = u_{0_x} u_4 + u_{1_x} u_{3_x} + u_{2_x} u_2 + u_{3_x} u_1 + u_{4_x} u_0,$$

III. Trigonometric Nonlinearity

Case 1. $F(u) = \sin u$

Note that algebraic operations cannot be applied here. Therefore, our main aim is to separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute

$$u = \sum_{n=0}^{\infty} u_n \quad (1.204)$$

Into $F(u) = \sin u$ to obtain

$$F(u) = \sin\left[u_0 + (u_1 + u_2 + u_3 + u_4 + \ldots)\right] \quad (1.205)$$

To calculate A_0, recall the trigonometric identity

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi. \quad (1.206)$$

Accordingly, Equation (1.205) becomes

$$\begin{aligned} F(u) &= \sin u_0 \cos(u_1 + u_2 + u_3 + \ldots) \\ &+ \cos u_0 \sin(u_1 + u_2 + u_3 + \ldots) \end{aligned} \quad (1.207)$$

Separating $F(u_0) = \sin u_0$ from other factors and using Taylor expansions for $\cos(u_1 + u_2 + u_3 + \ldots)$ and $\sin(u_1 + u_2 + u_3 + \ldots)$ give

$$\begin{aligned} F(u) &= \sin u_0 \left(1 - \frac{1}{2!}(u_1 + u_2 + \ldots)^2 + \frac{1}{4!}(u_1 + u_2 + \ldots)^4 - \ldots\right) \\ &+ \cos u_0 \left((u_1 + u_2 + \ldots) - \frac{1}{3!}(u_1 + u_2 + \ldots)^3 + \ldots\right), \end{aligned} \quad (1.208)$$

So that

$$\begin{aligned} F(u) &= \sin u_0 \left(1 - \frac{1}{2!}(u_1^2 + 2u_1u_2 + \ldots)\right) \\ &+ \cos u_0 \left((u_1 + u_2 + \ldots) - \frac{1}{3!}u_1^3 + \ldots\right), \end{aligned} \quad (1.209)$$

Note that we expanded the algebraic terms; then few terms of each expansion are listed. The last expansion can be rearranged by grouping all term with the same sum of subscripts. This means that Eq. (1.209) can be rewritten in the form

$$F(u)=\underbrace{\sin u_0}_{A_0}+\underbrace{u_1\cos u_0}_{A_1}+\underbrace{u_2\cos u_0-\frac{1}{2!}u_1^2\sin u_0}_{A_2}+$$

$$\underbrace{u_3\cos u_0-u_1u_2\sin u_0-\frac{1}{3!}u_1^3\cos u_0}_{A_3}+\ldots \qquad (1.210)$$

Case 2. $F(u) = \cos u$

Proceeding as before gives

$$F(u)=\underbrace{\cos u_0}_{A_0}+\underbrace{u_1\sin u_0}_{A_1}+\underbrace{(-u_2\sin u_0-\frac{1}{2!}u_1^2\cos u_0)}_{A_2}$$

$$+\underbrace{(-u_3\cos u_0-u_1u_2\sin u_0+\frac{1}{3!}u_1^3\cos u_0)}_{A_3}+\ldots \qquad (1.211)$$

IV. Hyperbolic Nonlinearity

Case 1. $F(u) = \sinh u$

To calculate the A_n polynomials for $F(u) = \sinh u$, we first substitute

$$u = \sum_{n=0}^{\infty} u_n \quad (1.212)$$

Into $F(u) = \sinh u$ to obtain

$$F(u) = \sinh(u_0 + (u_1 + u_2 + u_3 + \ldots)). \quad (1.213)$$

To calculate A_n recall the hyperbolic identity

$$\sinh(\theta + \phi) = \sinh\theta \cosh\phi + \cosh\theta \sinh\phi. \quad (1.214)$$

Accordingly, Eq. (1.213) becomes

$$F(u) = \sinh u_0 \cosh(u_1 + u_2 + u_3 + ...) + \cosh u_0 \sinh(u_1 + u_2 + u_3 + ...). \qquad (1.215)$$

Separating $F(u_0) = \sinh u_0$ from other factors and using Taylor expansions for $\cosh(u_1 + u_2 + u_3 + ...)$ and $\sinh(u_1 + u_2 + u_3 + ...)$ give

$$F(u) = \sinh u_0 (1 + \frac{1}{2!}(u_1 + u_2 + ...)^2 + \frac{1}{4!}(u_1 + u_2 + ...)^4 + ...) + \cosh u_0 ((u_1 + u_2 + ...) + \frac{1}{3!}(u_1 + u_2 + ...)^3 + ...), \qquad (1.216)$$

So that

$$F(u) = \sinh u_0 (1 + \frac{1}{2!}(u_1^2 + 2u_1u_2 + ...)) + \cosh u_0 ((u_1 + u_2 + ...) + \frac{1}{3!}u_1^3 + ...), \qquad (1.217)$$

By grouping all term with the same sum of subscripts

$$F(u) = \underbrace{\sinh u_0}_{A_0} + \underbrace{u_1 \cosh u_0}_{A_1} + \underbrace{u_2 \cosh u_0 + \frac{1}{2!}u_1^2 \sinh u_0}_{A_2} + \underbrace{u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!}u_1^3 \cosh u_0}_{A_3} + ... \qquad (1.218)$$

Case2. $F(u) = \cosh u$

Proceeding as in $\sinh u$ we find

$$F(u) = \underbrace{\cosh u_0}_{A_0} + \underbrace{u_1 \sinh u_0}_{A_1}$$

$$+ \underbrace{u_2 \sinh u_0 + \frac{1}{2!} u_1^2 \cos u_0}_{A_2} \qquad (1.219)$$

$$+ \underbrace{u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!} u_1^3 \sinh u_0)}_{A_3} + \ldots$$

V. Exponential Nonlinearity

Case 1. $F(u) = e^u$

Substituting

$$u = \sum_{n=0}^{\infty} u_n, \quad (1.220)$$

Into $F(u) = e^u$ gives

$$F(u) = e^{(u_0 + u_1 + u_2 + u_3 + \ldots)}, \quad (1.221)$$

Or equivalently

$$F(u) = e^{u_0} e^{(u_1 + u_2 + u_3 + \ldots)}, \quad (1.222)$$

Keeping the term e^{u_0} and using the Taylor expansion for the other factor we obtain

$$F(u) = e^{u_0}(1 + (u_1 + u_2 + \ldots) + \frac{1}{2!}(u_1 + u_2 + \ldots)^2 + \ldots), \quad (1.223)$$

By grouping all term with the same sum of subscript

$$F(u) = \underbrace{e^{u_0}}_{A_0} + \underbrace{u_1 e^{u_0}}_{A_1} + \underbrace{(u_2 + \frac{1}{2!}u_1^2)e^{u_0}}_{A_2}$$

$$+ \underbrace{(u_3 + u_1 u_2 + \frac{1}{3!}u_1^3)e^{u_0}}_{A_3} \qquad (1.224)$$

$$+ \underbrace{(u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 + \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4)e^{u_0}}_{A_4} + ...$$

Case 1. $F(u) = e^{-u}$

Proceeding as before we find

$$F(u) = \underbrace{e^{-u_0}}_{A_0} + \underbrace{(-u_1)e^{-u_0}}_{A_1} + \underbrace{(-u_2 + \frac{1}{2!}u_1^2)e^{-u_0}}_{A_2}$$

$$+ \underbrace{(-u_3 + u_1 u_2 - \frac{1}{3!}u_1^3)e^{-u_0}}_{A_3} \qquad (1.225)$$

$$+ \underbrace{(-u_4 + u_1 u_3 + \frac{1}{2!}u_2^2 - \frac{1}{2!}u_1^2 u_2 + \frac{1}{4!}u_1^4)e^{-u_0}}_{A_4} + ...$$

VI. Logarithmic Nonlinearity

Case 1. $F(u) = \ln u, u \succ 0$

Substituting

$$u = \sum_{n=0}^{\infty} u_n, \quad (1.226)$$

Into $F(u) = \ln u$ gives

$$F(u) = \ln(u_0 + u_1 + u_2 + u_3 + ...) \quad (1.227)$$

Equation (1.227) can be written as

$$F(u) = \ln(u_0(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \ldots)) \quad (1.228)$$

Using the fact that $\ln(\alpha\beta) = \ln\alpha + \ln\beta$, Equation (1.228) can be written as

$$F(u) = \underbrace{\ln u_0}_{A_0} + \underbrace{\frac{u_1}{u_0}}_{A_1} + \underbrace{\frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0}}_{A_2} + \underbrace{\frac{u_3}{u_0} + \frac{u_1 u_2}{u_0^2} + \frac{1}{3}\frac{u_1^3}{u_0^3}}_{A_3} + \ldots \quad (1.229)$$

Case 1. $F(u) = \ln(1 + u), -1 \prec u \leq 1$

In a like manner we obtain

$$F(u) = \underbrace{\ln(1 + u_0)}_{A_0} + \underbrace{\frac{u_1}{1 + u_0}}_{A_1} + \underbrace{\frac{u_2}{1 + u_0} - \frac{1}{2}\frac{u_1^2}{(1 + u_0)^2}}_{A_2} + \underbrace{\frac{u_3}{1 + u_0} + \frac{u_1 u_2}{(1 + u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1 + u_0)^3}}_{A_3} + \ldots \quad (1.230)$$

As stated before, there are other methods that can be used to evaluate Adomian polynomials. However, these method suffer from the huge size of calculation. For this reason, the most commonly used methods are presented in this chapter.

Example 3.

Use Adomian Decomposition Method to solve the following homogeneous PDE

$$xu_x + u_y = 3u, u(0, y) = 0, u(x, 0) = x^2, \quad (1.231)$$

Solution.

In an operator form, Eq. (1.231), can be written as

$$L_y u(x, y) = 3u(x, y) - xL_x u(x, y), \quad (1.232)$$

Where

$$L_x = \frac{\partial}{\partial x},\ L_y = \frac{\partial}{\partial y},\quad (1.233)$$

Applying the inverse operator L_y^{-1} to both sides of (1.232), and using the given condition $u(x,0) = x^2$ yields

$$u(x,y) = x^2 + L_y^{-1}(3u - xL_x u).\quad (1.234)$$

As mentioned above, the decomposition method sets the solution $u(x,y)$ in an series form by

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)\quad (1.235)$$

Inserting (1.235) into both sides of the (1.234), we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = x^2 + L_y^{-1}\left(3\left(\sum_{n=0}^{\infty} u_n(x,u)\right) - xL_x\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right)\quad (1.236)$$

By considering few term of the decomposition of $u(x,y)$, Eq. (1.236), becomes

$$\begin{aligned} u_0 + u_1 + u_2 + \ldots = x^2 + L_y^{-1}(3(u_0 + u_1 + u_2 + \ldots) \\ - xL_x(u_0 + u_1 + u_2 + \ldots)), \end{aligned}\quad (1.237)$$

Proceeding as before, we identify the zeroth component $u_0(x,y)$, by

$$u_0(x,y) = x^2,\ (1.238)$$

Having identifies the zeroth component $u_0(x,y)$, we obtain the recursive scheme

$$u_0(x,y) = x^2,\ (1.239)$$

$$u_{k+1}(x,y) = L_y^{-1}(3u_k - xL_x u_k),\ k \geq 0,$$

The components $u_0, u_1, u_2, \ldots$ are thus determined as follows:

$$u_0(x,y) = x^2,\ (1.240)$$

$$u_1(x,y) = L_y^{-1}(3u_0 - xL_x u_0) = L_y^{-1}(x^2) = x^2 y,$$

$$u_2(x,y) = L_y^{-1}(3u_1 - xL_x u_1) = L_y^{-1}(x^2 y) = \frac{x^2 y^2}{2!},$$

.

.

Consequently,the solution is given by

$$u(x,y) = u_0 + u_1 + u_2 \ldots = x^2\left(1 + y + \frac{y^2}{21} + \ldots\right) = x^2 e^x \quad (1.241)$$

Example 4.

Use Adomian Decomposition Method to solve the following homogeneous PDE

$$u_t + cu_x = 0, u(x,0) = x, \quad (1.242)$$

where c is constant

Solution.

In an operator form, Eq. (1.242), can be written as

$$L_t u(x,t) = -cL_x u(x,t), \quad (1.243)$$

Where

$$L_x = \frac{\partial}{\partial x}, L_t = \frac{\partial}{\partial t}, \quad (1.244)$$

It is clear that operator L_t is invertible, and the inverse operator L_t^{-1} is an indefinite integral from 0 to t. Applying the inverse operator L_t^{-1} to both sides of (1.243), and using the given condition $u(x,0) = x$ yields

$$u(x,t) = x - cL_t^{-1}(L_x u(x,t). \quad (1.245)$$

As mentioned above, the decomposition method sets the solution $u(x,y)$ in an series form by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (1.246)$$

Inserting (1.246) into both sides of the (1.245), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x - cL_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right) \quad (1.247)$$

By considering few term of the decomposition of $u(x,t)$, Eq. (1.247), becomes

$$u_0 + u_1 + u_2 + ... = x - cL_t^{-1}(L_x(u_0 + u_1 + u_2 + ...)), \quad (1.248)$$

Proceeding as before, we identify the zeroth component $u_0(x,t)$, by

$$u_0(x,t) = x, \quad (1.249)$$

Having identifies the zeroth component $u_0(x,t)$, we obtain the recursive scheme

$$u_0(x,t) = x, \quad (1.250)$$

$$u_{k+1}(x,t) = -cL_t^{-1}(L_x u_k), \; k \geq 0,$$

The components $u_0, u_1, u_2, ...$ are thus determined as follows:

$$u_0(x,t) = x, \quad (1.251)$$

$$u_1(x,t) = -cL_t^{-1}(L_x u_0) = -cL_t^{-1}(1) = -ct,$$

$$u_2(x,y) = -cL_t^{-1}(L_x u_1) = L_y^{-1}(0) = 0,$$

We can easily observe that $u_k = 0, k \geq 2$ .It follows that the solution in a closed form is given by

$$u(x,y) = x - ct \quad (1.252)$$

Example 5.

Use Adomian Decomposition Method to solve the following PDE

$$u_x + u_y + u_z = u, u(0,y,z) = 1 + e^y + e^z$$
$$u(x,0,z) = 1 + e^x + e^z, u(x,y,0) = 1 + e^x + e^y \quad (1.253)$$

where

$u = u(x,y,z)$

Solution.

In an operator form, Eq. (1.253), can be written as

$$L_x u(x,y,z) = u - L_y u - L_z u, \quad (1.254)$$

Where

$$L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, L_z = \frac{\partial}{\partial z} \quad (1.255)$$

Assume that the operator L_x is invertible, and the inverse operator L_x^{-1} is an indefinite integral from 0 to x. Applying the inverse operator L_x^{-1} to both sides of (1.254), and using the given condition $u(0,y,z) = 1 + e^y + e^z$ yields

$$u(x,y,z) = 1 + e^y + e^z + L_x^{-1}(u - L_y u - L_z u), \quad (1.256)$$

As mentioned above, the decomposition method sets the solution $u(x,y,z)$ in an series form by

$$u(x,y,z) = \sum_{n=0}^{\infty} u_n(x,y,z) \quad (1.257)$$

Inserting (1.257) into both sides of the (1.256), we obtain

$$\sum_{n=0}^{\infty} u_n(x,y,z) = 1 + e^y + e^z + L_x^{-1}\left(\sum_{n=0}^{\infty} u_n - L_y\left(\sum_{n=0}^{\infty} u_n\right) - L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (1.258)$$

By considering few term of the decomposition of $u(x,y,z)$, Eq. (1.258), becomes

$$u_0 + u_1 + u_2 + ... = 1 + e^y + e^z + L_x^{-1}(u_0 + u_1 + u_2 + ...)$$
$$- L_x^{-1}(L_y(u_0 + u_1 + u_2 + ...)) - L_x^{-1}(L_z(u_0 + u_1 + u_2 + ...)) \quad (1.259)$$

Proceeding as before, we identify the zeroth component $u_0(x, y, z)$, by

$$u_0(x, y, z) = 1 + e^y + e^z, \quad (1.260)$$

Having identifies the zeroth component $u_0(x, y, z)$, we obtain the recursive scheme

$$u_0(x, y, z) = 1 + e^y + e^z, \quad (1.261)$$

$$u_{k+1}(x, y, z) = L_x^{-1}(u_k) - L_x^{-1}(L_y(u_k)) - L_x^{-1}(L_z(u_k))$$

The components $u_0, u_1, u_2, ...$ are thus determined as follows:

$$\begin{aligned}
u_0(x,y,z) &= 1 + e^y + e^z, \\
u_1(x,y,z) &= L_x^{-1}(u_0 - L_y u_0 - L_z u_0) = L_x^{-1}(1 + e^y + e^z - e^y - e^z) = L_x^{-1}(1) = x, \\
u_2(x,y,z) &= L_x^{-1}(u_1 - L_y u_1 - L_z u_1) = L_x^{-1}(x) = \frac{x^2}{2!}, \qquad (1.262) \\
u_3(x,y,z) &= L_x^{-1}(u_2 - L_y u_2 - L_z u_2) = L_x^{-1}\left(\frac{x^2}{2!}\right) = \frac{x^3}{3!}
\end{aligned}$$

And so on. Consequently, the solution in a series form is given by

$$u(x, y, z) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right) + e^y + e^z, \quad (1.263)$$

And in a closed form

$$u(x, y, z) = e^x + e^y + e^z, \quad (1.264)$$

1.5.Homogeneous and inhomogeneous Heat Equations

1.5.1 one Dimensional Heat Flow

The Adomian decomposition method will be used to solve the following homogeneous heat equation[6] where the boundary conditions are also homogenous.

Example 6.

Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_t = u_{xx}, 0 \prec x \prec \pi, t \succ 0$

BC $\quad \begin{aligned} u(0,t) &= 0, t \geq 0, \\ u(\pi,t) &= 0, t \geq 0, \end{aligned} \qquad (1.265)$

IC $\quad u(x,t) = \sin x,$

Solution:

In an operator form, Equation (1.265) can be written as

$$L_t u(x,t) = L_{xx} u(x,t), \qquad (1.266)$$

Applying the inverse operator L_t^{-1} to both sides of (1.266), and using the initial condition we find

$$u(x,t) = \sin x + L_t^{-1}(L_{xx} u(x,t)), \quad (1.267)$$

We next define the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (1.268)$$

substituting the decomposition (1.268) into both sides of the (1.267), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin x + L_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty} u_n(x,t) \right)\right), \quad (1.269)$$

Or equivalently

$$u_0 + u_1 + u_2 + ... = \sin x + L_t^{-1}(L_{xx}(u_0 + u_1 + u_2 + ...)), \quad (1.270)$$

Identifying the zeroth component $u_0(x,t)$, as assumed before we obtain

$$u_0(x,t) = \sin x,$$
$$u_1(x,t) = L_t^{-1}(L_{xx}(u_0)) = L_t^{-1}(-\sin x) = -t\sin x,$$
$$u_2(x,t) = L_t^{-1}(L_{xx}(u_1)) = L_t^{-1}(-t\sin x) = \frac{t^2}{2!}\sin x, \qquad (1.271)$$

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Consequently, the solution $u(x,t)$ in a series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \qquad (1.272)$$
$$= \sin x\left(1 - t + \frac{t^2}{2!} - \ldots\right),$$

and in a closed form by

$$u(x,t) = e^{-t}\sin x, \qquad (1.273)$$

Obtained upon using the Taylor expansion of e^{-t}.The solution (1.273) satisfies the PDE, the boundary conditions and the initial condition.

Example 7.

Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_t = u_{xx}, (0 \prec x \prec \pi), (t \succ 0)$

BC $\quad u(0,t) = e^{-t}, t \geq 0,$
$\quad u(\pi,t) = \pi - e^{-t}, t \geq 0, \qquad (1.274)$

IC $\quad u(x,t) = x + \cos x,$

Solution:

It is important to note that the boundary conditions in this example are inhomogeneous. The decomposition method does not require any restrictive assumption on boundary conditions when approaching the problem in the t .direction or in the x direction.

In an operator form, Equation (1.274) can be written as

$$L_t u(x,t) = L_{xx} u(x,t), \qquad (1.275)$$

Applying the inverse operator L_t^{-1} to both sides of (1.275), and using the initial condition we find

$$u(x,t) = x + \cos x + L_t^{-1}(L_{xx} u(x,t)), \quad (1.276)$$

We next define the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (1.277)$$

substituting the decomposition (1.277) into both sides of the (1.276), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x + \cos x + L_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty} u_n(x,t) \right)\right), \quad (1.278)$$

Or equivalently

$$u_0 + u_1 + u_2 + \ldots = x + \cos x + L_t^{-1}(L_{xx}(u_0 + u_1 + u_2 + \ldots)), \quad (1.279)$$

Identifying the zeroth component $u_0(x,t)$, as assumed before we obtain

$$\begin{aligned}
u_0(x,t) &= x + \cos x, \\
u_1(x,t) &= L_t^{-1}(L_{xx}(u_0)) = L_t^{-1}(-\cos x) = -t \cos x, \\
u_2(x,t) &= L_t^{-1}(L_{xx}(u_1)) = L_t^{-1}(t \cos x) = \frac{t^2}{2!} \cos x, \\
u_3(x,t) &= L_t^{-1}(L_x(u_2)) = L_t^{-1}\left(-\frac{t^2}{2!} \cos x\right) = \frac{t^3}{3!} \cos x, \\
&\;\;. \\
&\;\;. \\
&\;\;.
\end{aligned} \qquad (1.280)$$

Consequently, the solution $u(x,t)$ in a series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \qquad (1.281)$$

$$= x + \cos x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right),$$

and in a closed form by

$$u(x,t) = e^{-t} \sin x, \qquad (1.282)$$

Obtained upon using the Taylor expansion of e^{-t}.

It is important to point out that the decomposition method has been used in the last two examples in the $t-$ dimension by using the differential operator L_t and by operating with the inverse operator L_t^{-1}. However, the method can also be used in the $x-$ dimension. Although the he $x-$ solution can be obtained in a similar fashion, however it requires more computational work if compared with the solution in the $t-$ dimension. This can be attributed to the fact that we used the initial condition IC only in using He $t-$ dimension, where as a boundary condition and an initial condition are used to obtain the solution in He $x-$ direction. This can be clearly illustrated by discussing the following examples.

Example 8:

Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\qquad u_t = u_{xx}, (0 \prec x \prec \pi), (t \succ 0)$

BC $\qquad u(0,t) = 0, t \geq 0,$

$\qquad\quad u(\pi,t) = 0, t \geq 0, \qquad (1.283)$

IC $\qquad u(x,0) = \sin x.$

Solution:

In an operator form, Equation (1.283) can be written as

$$L_x u(x,t) = L_t u(x,t), \qquad (1.284)$$

Where

$$L_x = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial}{\partial t} \quad (1.285)$$

so that L_x^{-1} is a to two-fold integral operator defined by

$$L_x^{-1}(.) = \int_0^x \int_0^x (.)\, dxdx \quad (1.286)$$

This means that

$$L_x^{-1} L_x u = u(x,t) - u(0,t) - xu_x(0,t) = u(x,t) - xu_x(0,t) \qquad (1.287)$$

Applying the inverse operator L_x^{-1} to both sides of (1.284), and using the proper boundary condition we obtain

$$\begin{aligned} u(x,t) &= xu_x(0,t) + L_x^{-1}(L_t u(x,t)), \\ &= xh(t) + L_x^{-1}(L_t u(x,t)) \end{aligned} \qquad (1.288)$$

Where

$$h(t) = u_x(0,t). \qquad (1.289)$$

We next define the unknown function $u(x,t)$ by a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (1.290)$$

substituting the decomposition (1.290) into both sides of the (1.288), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = xh(t) + L_x^{-1}\left(L_t\left(\sum_{n=0}^{\infty} u_n(x,t) \right)\right), \quad (1.291)$$

Or equivalently

$$u_0 + u_1 + u_2 + \ldots = xh(t) + L_x^{-1}(L_t(u_0 + u_1 + u_2 + \ldots)), \quad (1.292)$$

Identifying the zeroth component $u_0(x,t)$, as assumed before we obtain

$$u_0(x,t) = xh(t),$$

$$u_1(x,t) = L_x^{-1}(L_t(u_0)) = L_x^{-1}\left(xh'(t)\right) = \frac{1}{3!}x^3 h'(t),$$

$$u_2(x,t) = L_x^{-1}(L_t(u_1)) = L_x^{-1}\left(\frac{1}{3!}x^3 h''(t)\right) = \frac{1}{5!}x^5 h''(t),$$

.

. (1.293)

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Accordingly, the solution $u(x,t)$ in a series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \quad (1.294)$$

$$= xh(t) + \frac{1}{3!}x^3 h'(t) + \frac{1}{5!}x^5 h''(t) + \ldots$$

The unknown function $h(t)$ should be derived so that the solution $u(x,t)$ is completely determined. This can be achieved by using the initial condition

$$u(x,0) = \sin x, \quad (1.295)$$

Substituting $t = 0$ into (1.294) using the initial condition (1.295), and using the Taylor expansion of $\sin x$ we find

$$= xh(0) + \frac{1}{3!}x^3 h'(0) + \frac{1}{5!}x^5 h''(0) + ... = \sin x = \quad (1.296)$$
$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - ...$$

Equating the coefficients of like powers of x in both sides gives

$$h(0) = 0, h'(0) = -1, h''(0) = 1,... \quad (1.297)$$

Using th Taylor expansion of $h(t)$ and the result (1.297) in

$$h(t) = h(0) + h'(0)t + \frac{1}{2!}h''(0)t^2 - \frac{1}{3!}h'''t^3 + ...$$
$$= 1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + ... \quad (1.298)$$
$$= e^{-t}$$

Combining (1.294) and (1.298) the solution $u(x,t)$ in a series is

$$u(x,t) = e^{-t}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + ...\right), \quad (1.299)$$

and in a closed form is given by

$$u(x,t) = e^{-t}\sin x, \quad (1.300)$$

1.5.2Two Dimensional Heat Flow

The Adomian decomposition method will be used to solve the following homogeneous heat equation in two dimensions with homogeneous or inhomogeneous boundary conditions [4,6,9]

Example 9.

Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_t = (u_{xx} + u_{yy}), (0 \prec x, y \prec \pi), (t \succ 0)$

BC
$$u(0,y,t) = u(\pi,y,t) = 0,$$
$$u(x,0,t) = u(x,\pi,t) = 0,, \qquad (1.301)$$

IC $\qquad u(x,y,0) = \sin x \sin y,$

Solution:

In an operator form, Equation (1.301) can be written as

$$L_t u = L_x u + L_y u, \qquad (1.302)$$

Where

$$L_x = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial}{\partial t}, L_y = \frac{\partial^2}{\partial y^2} \qquad (1.303)$$

Applying the inverse operator L_t^{-1} to both sides of (1.302), and using the initial condition we find

$$u(x,y,t) = \sin x \sin y + L_t^{-1}(L_x u(x,y,t) + L_y u(x,y,t)), \qquad (1.304)$$

We next define the unknown function $u(x,y,t)$ by a sum of components defined by the series

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) \qquad (1.305)$$

substituting the decomposition (1.305) into both sides of the (1.304), we obtain

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right)\right), \qquad (1.306)$$

Or equivalently

$$u_0 + u_1 + u_2 + \ldots = \sin x \sin y$$
$$+ L_t^{-1}(L_x(u_0 + u_1 + u_2 + \ldots) + L_y(u_0 + u_1 + u_2 + \ldots)), \qquad (1.307)$$

Having identifies the zeroth component $u_0(x,y,t)$, we obtain the recursive scheme

$u_0(x,y,t) = \sin x \sin y$, (1.308)

$u_{k+1}(x,y,t) = L_t^{-1}(L_x u_k + L_y u_k)$

With u_0 defined as shown above ,the first few terms of the decomposition (1.305) are given by

$$\begin{aligned} u_0(x,y,t) &= \sin x \sin y, \\ u_1(x,y,t) &= L_t^{-1}(L_x u_0 + L_y u_0) = L_t^{-1}\left(-2\sin x \sin y\right) = -2t \sin x \sin y, \\ u_2(x,y,t) &= L_t^{-1}(L_x u_1 + L_y u_1) = L_t^{-1}(4\sin x \sin y) = \frac{(2t)^2}{2!}\sin x \sin y, \\ u_3(x,y,t) &= L_t^{-1}(L_x u_2 + L_y u_2) = L_t^{-1}(-8\sin x \sin y) = -\frac{(2t)^3}{3!}\sin x \sin y, \end{aligned} \quad (1.309)$$

And so on.Combining (1.305) and (1.309) the solution $u(x,t)$ in a series is

$$u(x,y,t) = \sin x \sin y\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \ldots\right), \quad (1.310)$$

and in a closed form by

$$u(x,y,t) = e^{-2t}\sin x \sin y, \quad (1.311)$$

Example 10.

Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_t = \left(u_{xx} + u_{yy} - u\right), (0 \prec x, y \prec \pi), (t \succ 0)$

BC $\quad \begin{aligned} &u(0,y,t) = u(\pi,y,t) = 0, \\ &u(x,0,t) = -u(x,\pi,t) = e^{-3t}\sin x, \end{aligned} \quad (1.312)$

IC

Solution:

In an operator form, Equation (1.312) can be written as

$$L_t u = L_x u + L_y u - u, \qquad (1.313)$$

Where

$$L_x = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial}{\partial t}, L_y = \frac{\partial^2}{\partial y^2} \quad (1.314)$$

Applying the inverse operator L_t^{-1} to both sides of (1.313), and using the initial condition we find

$$u(x, y, t) = \sin x \cos y + L_t^{-1}(L_x u(x, y, t) + L_y u(x, y, t) - u(x, y, t)), \quad (1.315)$$

We next define the unknown function $u(x, y, t)$ by a sum of components defined by the series

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) \qquad (1.316)$$

substituting the decomposition (1.316) into both sides of the (1.315), we obtain

$$\sum_{n=0}^{\infty} u_n = \sin x \cos y + L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n \right) + L_y\left(\sum_{n=0}^{\infty} u_n \right) + \sum_{n=0}^{\infty} u_n \right), \quad (1.317)$$

Or equivalently

$$u_0 + u_1 + u_2 + \ldots = \sin \cos y + L_t^{-1}(L_x(u_0 + u_1 + u_2 + \ldots) + L_y(u_0 + u_1 + u_2 + \ldots) + L(u_0 + u_1 + u_2 + \ldots)), \quad (1.318)$$

Having identifies the zeroth component $u_0(x, y, t)$, we obtain the recursive scheme

$$u_0(x,y,t) = \sin x \cos y, \quad (1.319)$$

$$u_{k+1}(x,y,t) = L_t^{-1}(L_x u_k + L_y u_k - u_k)$$

With u_0 defined as shown above ,the first few terms of the decomposition (1.319) are given by

$$\begin{aligned} u_0(x,y,t) &= \sin x \cos y, \\ u_1(x,y,t) &= L_t^{-1}(L_x u_0 + L_y u_0 - u_0) = L_t^{-1}(-3\sin x \cos y) \\ &= -3t \sin x \cos y, \\ u_2(x,y,t) &= L_t^{-1}(L_x u_1 + L_y u_1 - u_1) = L_t^{-1}(9t \sin x \cos y) \\ &= \frac{(3t)^2}{2!} \sin x \cos y, \\ u_3(x,y,t) &= L_t^{-1}(L_x u_2 + L_y u_2 - u_2) = L_t^{-1}(-27\frac{t^2}{2} \sin x \cos y) \\ &= \frac{(3t)^3}{3!} \sin x \cos y, \end{aligned} \quad (1.320)$$

And so on.Combining (1.316) and (1.320) the solution $u(x,y,t)$ in a series is

$$u(x,y,t) = \sin x \cos y\left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \ldots\right),$$

(1.321)

and in a closed form by

$$u(x,y,t) = e^{-3t} \sin x \cos y. \quad (1.322)$$

1.5.3 Three Dimensional Heat Flow

The Adomian decomposition method will be used to solve the following homogeneous heat equation in three dimensions with homogeneous or inhomogeneous boundary conditions .

Example 11.

Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_t = (u_{xx} + u_{yy} + u_{zz}), (0 \prec x, y \prec \pi), (t \succ 0)$

BC
$$u(0, y, z, t) = u(\pi, y, z, t) = 0,$$
$$u(x, 0, z, t) = u(x, \pi, z, t) = 0,, \qquad (1.323)$$

IC $\quad u(x, y, z, 0) = 2\sin x \sin y \sin z.$

Solution:

In an operator form, Equation (1.323) can be written as

$$L_t u = L_x u + L_y u + L_z u, \qquad (1.324)$$

Where

$$L_x = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial}{\partial t}, L_y = \frac{\partial^2}{\partial y^2}, L_z = \frac{\partial^2}{\partial z^2} \quad (1.325)$$

Applying the inverse operator L_t^{-1} to both sides of (1.324), and using the initial condition we find

$$u(x, y, z, t) = 2\sin x \sin y \sin z + L_t^{-1}(L_x u + L_y u + L_z u), \quad (1.326)$$

We next define the unknown function $u(x, y, z, t)$ by a sum of defined by the series

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, z, y, t) \qquad (1.327)$$

substituting the decomposition (1.327) into both sides of the (1.326), we obtain

$$\sum_{n=0}^{\infty} u_n = 2\sin x \sin y \sin z$$
$$+ L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right), \quad (1.328)$$

Or equivalently

$$u_0 + u_1 + u_2 + \ldots = \sin x \sin y \sin z$$
$$+ L_t^{-1}(L_x(u_0 + u_1 + u_2 + \ldots) + L_y(u_0 + u_1 + u_2 + \ldots) + L_z(u_0 + u_1 + u_2 + \ldots)), \quad (1.329)$$

Having identifies the zeroth component $u_0(x, y, z, t)$, we obtain the recursive scheme

$$u_0(x, y, z, t) = 2 \sin x \sin y \sin z, \quad (1.330)$$

$$u_{k+1}(x, y, z, t) = L_t^{-1}(L_x u_k + L_y u_k + L u_k)$$

It follows that the first few terms of the decomposition series of $u(x, y, z, t)$ are given by

$$\begin{aligned}
&u_0(x, y, z, t) = 2 \sin x \sin y \sin z, \\
&u_1(x, y, t) = L_t^{-1}(L_x u_0 + L_y u_0 + L_z u_0) = L_t^{-1}(-6 \sin x \sin y \sin z) \\
&= -2(3t) \sin x \sin y \sin z, \\
&u_2(x, y, t) = L_t^{-1}(L_x u_1 + L_y u_1 + L_z u_1) = L_t^{-1}(18 \sin x \sin y \sin z) \\
&= \frac{2(3t)^2}{2!} \sin x \sin y \sin z, \\
&u_3(x, y, t) = L_t^{-1}(L_x u_2 + L_y u_2 + L_z u_2) = L_t^{-1}(-54 \sin x \sin y \sin z) \\
&= -\frac{2(3t)^3}{3!} \sin x \sin y \sin z,
\end{aligned} \quad (1.331)$$

And so on.Combining (1.327) and (1.331) the solution $u(x, y, z, t)$ in a series is

$$u(x, y, t) = 2 \sin x \sin y \sin z \left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \ldots\right), \quad (1.332)$$

and in a closed form by

$$u(x, y, z, t) = 2e^{-3t} \sin x \sin y \sin z, \quad (1.333)$$

Example 12.

Use the Adomian decomposition method to solve the inhomogeneous initial-boundary value problem

PDE $u_t = (u_{xx} + u_{yy} + u_{zz}) + \sin z, 0 \prec x, y \prec \pi, t \succ 0$

$$u(0,y,z,t) = \sin z + e^{-2t}\sin y,$$
$$u(\pi,y,z,t) = \sin z - e^{-2t}\sin y,$$
$$u(x,0,z,t) = \sin z + e^{-2t}\sin x,$$
BC $$u(x,\pi,z,t) = \sin z - e^{-2t}\sin x, \qquad (1.334)$$
$$u(x,y,0,t) = u(x,y,\pi,t) = e^{-2t}\sin(x+y)$$

IC $\quad u(x,y,z,0) = \sin(x+y) + \sin z.$

Solution:

In an operator form, Equation (1.334) can be written as

$$L_t u = L_x u + L_y u + L_z u + \sin z, \qquad (1.335)$$

Where

$$L_x = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial}{\partial t}, L_y = \frac{\partial^2}{\partial y^2}, L_z = \frac{\partial^2}{\partial z^2} \quad (1.336)$$

Applying the inverse operator L_t^{-1} to both sides of (1.335), and using the initial condition we find

$$u(x,y,z,t) = \sin(x+y) + \sin z + t\sin z$$
$$+ L_t^{-1}(L_x u + L_y u + L_z u), \qquad (1.337)$$

We next define the unknown function $u(x, y, z, t)$ by a sum of defined by the series

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, z, y, t) \qquad (1.338)$$

substituting the decomposition (1.338) into both sides of the (1.337), we obtain

$$\sum_{n=0}^{\infty} u_n = \sin(x+y) + \sin z + t \sin z$$
$$+ L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n \right) + L_y\left(\sum_{n=0}^{\infty} u_n \right) + L_z\left(\sum_{n=0}^{\infty} u_n \right) \right), \qquad (1.339)$$

Or equivalently

$$u_0 + u_1 + u_2 + \ldots = \sin(x+y) + \sin z + t\sin z$$
$$+ L_t^{-1}(L_x(u_0 + u_1 + u_2 + \ldots) + L_y(u_0 + u_1 + u_2 + \ldots) + L_z(u_0 + u_1 + u_2 + \ldots)), \qquad (1.340)$$

Having identifies the zeroth component $u_0(x, y, t)$, we obtain the recursive scheme

$$u_0(x, y, z, t) = \sin(x+y) + \sin z + t \sin z, \quad (1.341)$$

$$u_{k+1}(x, y, z, t) = L_t^{-1}(L_x u_k + L_y u_k + L u_k), k \geq 0$$

It follows that the first few terms of the decomposition series of $u(x, y, z, t)$ are given by

$$u_0(x, y, z, t) = \sin(x+y) + \sin z + t \sin z,$$
$$u_1(x, y, t) = L_t^{-1}(L_x u_0 + L_y u_0 + L_z u_0) = L_t^{-1}(-2\sin(x+y) - \sin z - t \sin z)$$
$$= -2t \sin(x+y) - t \sin z - \frac{t^2}{2!}\sin z, \qquad (1.342)$$
$$u_2(x, y, t) = L_t^{-1}(L_x u_1 + L_y u_1 + L_z u_1) = L_t^{-1}(4t \sin(x+y) + t \sin z + \frac{t^2}{2}\sin z)$$
$$= \frac{(2t)^2}{2!}\sin(x+y) + \frac{t^2}{2!}\sin z + \frac{t^3}{3!}\sin z,$$

And so on.Combining (1.338) and (1.342) the solution $u(x, y, z, t)$ in a series is

$$u(x, y, z, t) = \sin z$$
$$+ \sin(x + y)\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \ldots\right) \qquad (1.343)$$
$$+ \left(t \sin z - t \sin z - \frac{t^2}{2!}\sin z + \frac{t^2}{2!}\sin z + \ldots\right),$$

and in a closed form by

$$u(x, y, z, t) = \sin z + e^{-2t}\sin(x + y), \qquad (1.344)$$

1.6. Nonlinear PDEs Systems by Adomian Decomposition Method(ADM)

Systems of nonlinear partial differential equations will be examined by using (ADM). Systems of nonlinear partial differential equations arise in many scientific models such as the propagation of shallow water waves and model of chemical reaction-diffusion model. To achieve our goal in handling systems of nonlinear partial differential equations, we write a system in an operator for by

$$\begin{aligned} L_t u + L_x v + N_1(u, v) &= g_1, \\ L_t v + L_x u + N_2(u, v) &= g_2, \end{aligned} \qquad (1.345)$$

With initial data

$$\begin{aligned} u(x, 0) &= f_1(x), \\ v(x, 0) &= f_2(x) \end{aligned} \qquad (1.346)$$

Where L_t and L_x are considered, without loss generality, first order partial differential operators, N_1 and N_2 are nonlinear operators, and g_1 and g_2 are source terms. Operating with the integral operator L_t^{-1} to the system (1.345) and using initial data (1.346) yields

$$\begin{aligned} u(x, t) &= f_1(x) + L_t^{-1}g_1 - L_t^{-1}L_x v - L_t^{-1}N_1(u, v), \\ v(x, t) &= f_2(x) + L_t^{-1}g_2 - L_t^{-1}L_x u - L_t^{-1}N_2(u, v), \end{aligned} \qquad (1.347)$$

The linear unknown functions $u(x,t)$ and $v(x,t)$ can be decomposed by infinite series of components

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$
$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \qquad (1.348)$$

However, the nonlinear operators, $N_1(u,v)$ and $N_2(u,v)$ should be represented by using the infinite series of the so-called Adomian polynomials A_n and B_n as follows:

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n,$$
$$N_2(u,v) = \sum_{n=0}^{\infty} B_n, \qquad (1.349)$$

Where $u_n(x,t)$ and $v_n(x,t), n \geq 0$ are the components of e $u(x,t)$ and $v(x,t), n \geq 0$ respectively that will be recurrently determined, and A_n and $B_n, n \geq 0$ are Adomian polynomials that can be generated for all forms of nonlinearity. The algorithms for calculating Adomian polynomials were introduced in 1.3 and 1.4. Substituting (1.349) into (1.347) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f_1(x) + L_t^{-1} g_1 - L_t^{-1} L_x \left(\sum_{n=0}^{\infty} v_n \right) - L_t^{-1} \left(\sum_{n=0}^{\infty} A_n \right),$$
$$\sum_{n=0}^{\infty} v_n(x,t) = f_2(x) + L_t^{-1} g_2 - L_t^{-1} L_x \left(\sum_{n=0}^{\infty} u_n \right) - L_t^{-1} \left(\sum_{n=0}^{\infty} B_n \right), \qquad (1.350)$$

Two recursive relations can be constructed from (1.350) given by

$$u_0(x,t) = f_1(x) + L_t^{-1} g_1,$$
$$u_{k+1}(x,t) = -L_t^{-1}(L_x v_k) - L_t^{-1}(A_k), k \geq 0, \qquad (1.351)$$

And

$$v_0(x,t) = f_2(x) + L_t^{-1} g_2,$$
$$v_{k+1}(x,t) = -L_t^{-1}(L_x u_k) - L_t^{-1}(B_k),\ k \geq 0, \qquad (1.352)$$

It is an essential feature of the decomposition method that the zeroth components $u_0(x,t)$ and $v_0(x,t)$ are defined always by all terms that arise from initial data and from integrating the source terms. Having defined the zeroth pair (u_0, v_0) the remaining pair e (u_k, v_k), $k \geq 1$, can be obtained in a recurrent manner by using (1.351) and (1.352). Additional pairs for the decomposition series solution normally account for higher accuracy. Having determined the components of $u(x,t)$ and $v(x,t)$ the solution (u, v) of the system follows immediately in the form a power series expansion upon using (1.348).

Example 13.

Consider the nonlinear system:

$$u_t + vu_x + u = 1,$$
$$v_t - uv_x - v = 1, \qquad (1.353)$$

With the condition

$$u(x,0) = e^{x},$$
$$v(x,0) = e^{-x} \qquad (1.354)$$

Solution :

Operating with L_t^{-1} on (2.353) we obtain

$$u(x,t) = e^{x} + t - L_t^{-1}(vu_x + u),$$
$$v(x,t) = e^{-x} + t + L_t^{-1}(uv_x + v), \qquad (1.355)$$

The linear term $u(x,t)$ and $v(x,t)$ can be represented by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$
$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \quad (1.356)$$

And the nonlinear term vu_x and uv_x by an infinite series of Adomian polynomials

$$vu_x = \sum_{n=0}^{\infty} A_n,$$
$$uv_x = \sum_{n=0}^{\infty} B_n, \quad (1.357)$$

Wher A_n and B_n are the Adomian polynomials that can be generated for any forms of nonlinearity. Substituting (1.356) and (1.357) into (1.355) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = e^x + t - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} u_n\right),$$
$$\sum_{n=0}^{\infty} v_n(x,t) = e^{-x} + t + L_t^{-1}\left(\sum_{n=0}^{\infty} B_n\right) + L_t^{-1}\left(\sum_{n=0}^{\infty} u_n\right), \quad (1.358)$$

To accelerate the convergence of the solution, the modified decomposition method will be applied here. The modified decomposition method defines the recursive relations in the form

$$u_0(x,t) = e^x,$$
$$u_1(x,t) = t - L_t^{-1}(A_0 + u_0), \quad (1.359)$$
$$u_{k+1}(x,t) = -L_t^{-1}(A_k + u_k), \; k \geq 1,$$

And

$$v_0(x,t) = e^{-x},$$
$$v_1(x,t) = t + L_t^{-1}(B_0 + v_0), \quad (1.360)$$
$$v_{k+1}(x,t) = L_t^{-1}(B_k + v_k), \; k \geq 1,$$

The Adomian polynomials for the nonlinear term vu_x are given by

$$A_0 = v_0 u_{0_x},$$
$$A_1 = v_1 u_{0_x} + v_0 u_{1_x},$$
$$A_2 = v_2 u_{0_x} + v_1 u_{1_x} + v_0 u_{2_x},$$
$$A_3 = v_3 u_{0_x} + v_2 u_{1_x} + v_1 u_{2_x} + v_0 u_{3_x},$$

The Adomian polynomials for the nonlinear term uv_x are given by

$$B_0 = u_0 v_{0_x},$$
$$B_1 = u_1 v_{0_x} + u_0 v_{1_x},$$
$$B_2 = u_2 v_{0_x} + u_1 v_{1_x} + u_0 v_{2_x},$$
$$B_3 = u_3 v_{0_x} + u_2 v_{1_x} + u_1 v_{2_x} + u_0 v_{3_x},$$

Using the derived Adomian polynomials into (1.359) and (1.360) we obtain the following pairs of components

$$\begin{aligned}
(u_0, v_0) &= \left(e^{x}, e^{-x}\right), \\
(u_1, v_1) &= \left(-te^{x}, te^{-x}\right), \\
(u_2, v_2) &= \left(\frac{t^2}{2!}e^{x}, \frac{t^2}{2!}e^{-x}\right), \qquad (1.361) \\
(u_3, v_3) &= \left(-\frac{t^3}{3!}e^{x}, \frac{t^3}{3!}e^{-x}\right),
\end{aligned}$$

Accordingly, the solution of the system in a series form is given by

$$(u, v) = \left(e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right), e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)\right), \quad (1.362)$$

And in a closed form by

$$(u, v) = \left(e^{x-t}, e^{-x+t}\right), \quad (1.363)$$

Example 14.

Consider the nonlinear system:

$$\begin{aligned} u_t - v_x w_y &= 1, \\ v_t - w_x u_y &= 5, \\ w_t - u_x v_y &= 5 \end{aligned} \qquad (1.364)$$

With the condition

$$\begin{aligned} u(x,y,0) &= x + 2y, \\ v(x,y,0) &= x - 2y, \\ w(x,y,0) &= -x + 2y, \end{aligned} \qquad (1.365)$$

Solution :

Operating with L_t^{-1} on (1.364) we obtain

$$\begin{aligned} u(x,y,t) &= (x + 2y + t) + L_t^{-1}(v_x w_y), \\ v(x,y,t) &= (x - 2y + 5t) + L_t^{-1}(w_x u_y), \\ w(x,y,t) &= (-x + 2y + 5t) + L_t^{-1}(u_x v_y), \end{aligned} \qquad (1.366)$$

The linear term $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ can be represented by the decomposition series

$$\begin{aligned} u(x,y,t) &= \sum_{n=0}^{\infty} u_n(x,y,t), \\ v(x,y,t) &= \sum_{n=0}^{\infty} v_n(x,y,t), \\ w(x,y,t) &= \sum_{n=0}^{\infty} w_n(x,y,t) \end{aligned} \qquad (1.367)$$

And the nonlinear term $v_x w_y$, $w_x u_y$ and $u_x v_y$ by an infinite series of Adomian polynomials

$$v_x w_y = \sum_{n=0}^{\infty} A_n,$$
$$w_x u_y = \sum_{n=0}^{\infty} B_n, \qquad (1.368)$$
$$u_x v_y = \sum_{n=0}^{\infty} C_n$$

Wher A_n, B_n and C_n are the Adomian polynomials that can be generated for any forms of nonlinearity. Substituting (1.368) and (1.367) into (1.366) gives

$$\sum_{n=0}^{\infty} u_n(x,y,t) = (x+2y+t) + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right),$$
$$\sum_{n=0}^{\infty} v_n(x,y,t) = (x-2y+5t) + L_t^{-1}\left(\sum_{n=0}^{\infty} B_n\right), \qquad (1.369)$$
$$\sum_{n=0}^{\infty} w_n(x,y,t) = (-x+2y+5t) + L_t^{-1}\left(\sum_{n=0}^{\infty} C_n\right),$$

To accelerate the convergence of the solution, the modified decomposition method will be applied here. The modified decomposition method defines the recursive relations in the form

$$u_0(x,y,t) = x + 2y + t,$$
$$u_{k+1}(x,y,t) = L_t^{-1}(A_k),\ k \geq 0, \qquad (1.370)$$

And

$$v_0(x,y,t) = x - 2y + 5t,$$
$$v_{k+1}(x,y,t) = L_t^{-1}(B_k),\ k \geq 0, \qquad (1.371)$$

And

$$w_0(x,y,t) = -x + 2y + 5t,$$
$$v_{k+1}(x,y,t) = L_t^{-1}(C_k),\ k \geq 0, \quad (1.372)$$

The Adomian polynomials for the nonlinear term $v_x w_y$ are given by

$$A_0 = v_{0_x} w_{0_y},$$
$$A_1 = v_{1_x} w_{0_y} + v_{0_x} w_{1_y},$$
$$A_2 = v_{2_x} w_{0_y} + v_{1_x} w_{1_y} + v_{0_x} w_{2_y},$$
$$A_3 = v_{3_x} w_{0_y} + v_{2_x} w_{1_y} + v_{1_x} w_{2_y} + v_{0_x} w_{3_y},$$

The Adomian polynomials for the nonlinear term $w_x u_y$ are given by

$$B_0 = w_{0_x} u_{0_y},$$
$$B_1 = w_{1_x} u_{0_y} + w_{0_x} u_{1_y},$$
$$B_2 = w_{2_x} u_{0_y} + w_{1_x} u_{1_y} + w_{0_x} u_{2_y},$$
$$B_3 = w_{3_x} u_{0_y} + w_{2_x} u_{1_y} + w_{1_x} u_{2_y} + w_{0_x} u_{3_y},$$

The Adomian polynomials for the nonlinear term $u_x v_y$ are given by

$$C_0 = u_{0_x} v_{0_y},$$
$$C_1 = u_{1_x} v_{0_y} + u_{0_x} v_{1_y},$$
$$C_2 = u_{2_x} v_{0_y} + u_{1_x} v_{1_y} + u_{0_x} v_{2_y},$$
$$C_3 = u_{3_x} v_{0_y} + u_{2_x} v_{1_y} + u_{1_x} v_{2_y} + u_{0_x} v_{3_y},$$

Substituting these polynomials into the appropriate recursive relations we find

$$
\begin{aligned}
(u_0, v_0, w_0) &= (x + 2y + t, x - 2y + 5t, -x + 2y + 5t), \\
(u_1, v_1, w_1) &= (2t, -2t, -2t), \\
(u_k, v_k) &= (0,0,0), k \geq 2.
\end{aligned}
\quad (1.373)
$$

Consequently, the exact solution of the system of nonlinear partial differential equations is given by

$$
(u, v, w) = (x + 2y + 3t, x - 2y + 3t, -x + 2y + 3t) \quad (1.374)
$$

Chapter Two

Varational Iteration Method (VIM).

2.1The Variational Iteration Method (VIM)

It was stated before that Adomian Decomposition Method, with its modified form and the noise terms phenomenon, and some of the traditional methods will be used in this chapter. The other well-known methods, such as the inverse scattering method, the pseudo spectral method,

In addition to Adomian Decomposition Method, the newly developed variational iteration method will be applied. The variational iteration method(VIM) is thoroughly used by mathematicians to handle a wide variety of scientific and engineering applications: linear and nonlinear, and homogeneous and inhomogeneous as well. It was shown that this method is effective and reliable for analytic and numerical purposes. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. The (VIM) does not require specific treatments for nonlinear problems as in Adomian method, perturbation techniques, etc. in what follows, we present the main steps of the method.

Consider the differential equation

$$Lu + Nu = g(t), \quad (2.1)$$

Where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.

The variational iteration method presents a correction functional for Eq. (2.1) in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))d\xi, \quad (2.2)$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that main steps of the He's variational iteration method require first the determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally. Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words we can use

$$\int \lambda(\xi)u_n'(\xi)d\xi = \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)d\xi$$

$$\int \lambda(\xi)u_n''(\xi)d\xi = \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)d\xi \quad (2.3)$$

And so on. The last two identities can be obtained by integrating by parts.

Having determined the Lagrange multiplier $\lambda(\xi)$ the successive approximations $u_{n+1}, n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution

$$u = \lim_{n \to \infty} u_n. \quad (2.4)$$

In other words, the correction functional (2.2) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

The variational iteration method will be used now to study the same examples used before in chapter1 to help for comparison reasons.

Example 1:

Use variational iteration method to solve the following inhomogeneous PDE

$$u_x + u_y = x + y, u(0, y) = 0, u(x,0) = 0. \quad (2.5)$$

Solution:

The correction functional for equation (2.5) is

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi)\left(\frac{\partial u_n(\xi,y)}{\partial \xi} + \frac{\partial \tilde{u}_n(\xi,y)}{\partial y} - \xi - y\right)d\xi. \quad (2.6)$$

using (2.3) the stationary conditions

$$1 + \lambda \downarrow_{\xi = x} = 0, \quad (2.7)$$

$$\lambda' \downarrow_{\xi = x} = 0,$$

Follow immediately. This in turn gives

$$\lambda = -1, \quad (2.8)$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.6) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \left(\frac{\partial u_n(\xi,y)}{\partial \xi} + \frac{\partial u_n(\xi,y)}{\partial y} - \xi - y \right) d\xi., n \geq 0 \quad (2.9)$$

As stated, we can select $u_0(x,y) = u(0,y) = 0$ from the given conditions. Using this selection into (2.9) we obtain the following successive approximations

$$u_0(x,y) = 0,$$

$$u_1(x,y) = 0 - \int_0^x \left(\frac{\partial u_0(\xi,y)}{\partial \xi} + \frac{\partial u_0(\xi,y)}{\partial y} - \xi - y \right) d\xi = \frac{1}{2}x^2 + xy,$$

$$u_2(x,y) = \frac{1}{2}x^2 + xy - \int_0^x \left(\frac{\partial u_1(\xi,y)}{\partial \xi} + \frac{\partial u_1(\xi,y)}{\partial y} - \xi - y \right) d\xi = xy, \quad (2.10)$$

$$u_3(x,y) = xy - \int_0^x \left(\frac{\partial u_2(\xi,y)}{\partial \xi} + \frac{\partial u_2(\xi,y)}{\partial y} - \xi - y \right) d\xi = xy,$$

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$$u_n(x,y) = xy,$$

The VIM admits the use of

$$u(x,y) = \lim_{n \to \infty} u_n(x,y). \quad (2.11)$$

That gives the exact solution by

$u(x,y) = xy. (2.12)$

Example 2:

Solve the following homogeneous partial differential equation by the variational iteration method

$$u_x - u_y = 0, u(0,y) = y, u(x,0) = x. (2.13)$$

Solution:

The correction functional for equation (2.13) is

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi)\left(\frac{\partial u_n(\xi,y)}{\partial \xi} - \frac{\partial u_n(\xi,y)}{\partial y}\right)d\xi. (2.14)$$

This gives the stationary condition

$$1 + \lambda\downarrow_{\xi = x} = 0, (2.15)$$

$$\lambda'\downarrow_{\xi = x} = 0,$$

This gives

$$\lambda = -1, (2.16)$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.15) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^x \left(\frac{\partial u_n(\xi,y)}{\partial \xi} - \frac{\partial u_n(\xi,y)}{\partial y}\right)d\xi, n \geq 0. (2.17)$$

We now select $u_0(x,y) = u(0,y) = y$ from the given conditions. Using this selection into (2.17) we obtain the following successive approximations

$$u_0(x,y) = y,$$

$$u_1(x,y) = y - \int_0^x \left(\frac{\partial u_0(\xi,y)}{\partial \xi} - \frac{\partial u_0(\xi,y)}{\partial y} \right) d\xi = x + y,$$

$$u_2(x,y) = x + y - \int_0^x \left(\frac{\partial u_1(\xi,y)}{\partial \xi} - \frac{\partial u_1(\xi,y)}{\partial y} - \xi - y \right) d\xi = x + y, \; (2.18)$$

$$u_3(x,y) = x + y - \int_0^x \left(\frac{\partial u_2(\xi,y)}{\partial \xi} - \frac{\partial u_2(\xi,y)}{\partial y} - \xi - y \right) d\xi = x + y,$$

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$$u_n(x,y) = x + y,$$

The VIM gives the exact solution by

$$u(x,y) = x + y. \; (2.18)$$

Example 3

Usethe variational iteration method to solve the following homogeneous partial differential equation

$$u_y + xu_x = 3u, u(0,y) = 0, u(x,0) = x^2. \; (2.19)$$

Solution:

The correction functional for equation (2.19) is

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi) \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + x \frac{\partial \tilde{u}_n(x,\xi)}{\partial x} \right) d\xi. \; (2.20)$$

As presented before the stationary conditions

$$1 + \lambda \downarrow_{\xi = x} = 0, \; (2.21)$$

$$\lambda' \downarrow_{\xi = x} = 0,$$

And this gives

$$\lambda = -1, \quad (2.22)$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.20) gives the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^x \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + x\frac{\partial u_n(x,\xi)}{\partial x} - 3u_n \right) d\xi, n \geq 0. \quad (2.23)$$

We can select $u_0(x,y) = x^2$ from the given conditions. Using this selection into (2.23) we obtain the following successive approximations

$$u_0(x,y) = x^2,$$

$$u_1(x,y) = x^2 - \int_0^y \left(\frac{\partial u_0(x,\xi)}{\partial \xi} + x\frac{\partial u_0(x,\xi)}{\partial x} - 3u_0(x,\xi) \right) d\xi = x^2 + x^2 y,$$

$$u_2(x,y) = x^2 + x^2 y - \int_0^y \left(\frac{\partial u_1(x,\xi)}{\partial \xi} + x\frac{\partial u_1(x,\xi)}{\partial x} - 3u_1(x,\xi) \right) d\xi$$

$$= x^2 + x^2 y + \frac{1}{2!}x^2 y^2, \quad (2.24)$$

$$u_3(x,y) = x^2 + x^2 y - \int_0^y \left(\frac{\partial u_2(x,\xi)}{\partial \xi} + x\frac{\partial u_2(x,\xi)}{\partial x} - 3u_2(x,\xi) \right) d\xi$$

$$= x^2 + x^2 y + \frac{1}{2!}x^2 y^2 + \frac{1}{3!}x^2 y^3,$$

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$$u_n(x,y) = x^2(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + ...).$$

The VIM gives the exact solution by

$$u(x,y) = x^2 e^y. \quad (2.25)$$

Example 4:

Solve the following partial differential equation by the variational iteration metho

$$u_x + u_y + u_z = u,$$
$$u(0,y,z) = 1 + e^y + e^z,$$
$$u(x,0,z) = 1 + e^x + e^z, \quad (2.26)$$
$$u(x,y,0) = 1 + e^x + e^y,$$

where

$$u = u(x,y,z).$$

Solution:

The correction functional for equation (2.26) is

$$u_{n+1}(x,y,z) = u_n(x,y,z)$$
$$+\int_0^x \lambda(\xi)\left(\frac{\partial u_n(\xi,y,z)}{\partial \xi} + \frac{\partial \tilde{u}_n(\xi,y,z)}{\partial y} + \frac{\partial \tilde{u}_n(\xi,y,z)}{\partial z} - u_n(\xi,y,z)\right)d\xi. \quad (2.27)$$

This gives the stationary condition

$$1+\lambda\downarrow_{\xi=x}=0,\ (2.28)$$

$$\lambda'\downarrow_{\xi=x}=0,$$

This gives

$$\lambda=-1,\ (2.29)$$

Substituting this value of the Lagrange multiplier $\lambda=-1$ into the functional (2.27) gives the iteration formula

$$u_{n+1}(x,y,z)=u_n(x,y,z)$$
$$+\int_0^x\lambda(\xi)\left(\frac{\partial u_n(\xi,y,z)}{\partial\xi}+\frac{\partial\tilde{u}_n(\xi,y,z)}{\partial y}+\frac{\partial\tilde{u}_n(\xi,y,z)}{\partial z}-u_n(\xi,y,z)\right)d\xi, n\geq 0.\ (2.30)$$

We now select $u_0(x,y,z)=1+e^y+e^z$ from the given conditions. Using this selection into (2.30) we obtain the following successive approximations

$$u_0(x,y,z)=1+e^y+e^z,$$

$$u_1(x,y)=1+e^y+e^z$$
$$-\int_0^x\left(\frac{\partial u_0(\xi,y,z)}{\partial\xi}+\frac{\partial u_0(\xi,y,z)}{\partial y}+\frac{\partial u_0(\xi,y,z)}{\partial z}-u_0(\xi,y,z)\right)d\xi$$
$$=1+x+e^y+e^z,$$

$$u_2(x,y)=1+x+e^y+e^z$$
$$-\int_0^x\left(\frac{\partial u_1(\xi,y,z)}{\partial\xi}+\frac{\partial u_1(\xi,y,z)}{\partial y}+\frac{\partial(\xi,y,z)}{\partial z}-u_1(\xi,y,z)\right)d\xi\quad(2.31)$$
$$=1+x+\frac{1}{2!}x^2+e^y+e^z$$

$$u_3(x,y) = 1 + x + \frac{1}{2!}x^2 + e^y + e^z$$

$$-\int_0^x \left(\frac{\partial u_2(\xi,y,z)}{\partial \xi} + \frac{\partial u_2(\xi,y,z)}{\partial y} + \frac{\partial u_2(\xi,y,z)}{\partial z} - u_2(\xi,y,z) \right) d\xi$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + e^y + e^z,$$

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$$u_n(x,y,z) = (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...) + e^y + e^z.$$

As a result, the exact solution is given by

$$u(x,y) = e^x + e^y + e^z. \quad (2.32)$$

2.2.Homogeneous and inhomogeneous Heat Equations

Example5:

Use the (VIM)to solve the initial-boundary value problem

PDE $u_t = u_{xx}, (0 \prec x \prec \pi), (t \succ 0),$

BC $u(0,t) = 0, u(\pi,t) = 0, t \geq 0, \quad (2.33)$

IC $u(x,0) = \sin x.$

Solution:

The correction functional for equation (2.33) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} \right) d\xi. \quad (2.34)$$

As presented before the stationary conditions

$$1+\lambda\downarrow_{\xi=x}=0, \quad (2.35)$$

$$\lambda'\downarrow_{\xi=x}=0,$$

And this gives

$$\lambda=-1, \quad (2.36)$$

Substituting this value of the Lagrange multiplier $\lambda=-1$ into the functional (2.34) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,\xi)}{\partial\xi}-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}-3u_n\right)d\xi, n\geq 0. \quad (2.37)$$

We can select $u_0(x,0)=\sin x$ from the given conditions. Using this selection into (2.37) we obtain the following successive approximations

$$u_0(x,y)=\sin x,$$

$$u_1(x,y)=\sin x-t\sin x$$

$$u_2(x,y)=\sin x-t\sin x+\frac{1}{2!}t^2\sin x \quad (2.38)$$

$$u_3(x,y)=\sin x-t\sin x+\frac{1}{2!}t^2\sin x-\frac{1}{3!}t^3\sin x$$

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$$u_n(x,y)=\sin x(1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\ldots).$$

The VIM gives the exact solution by

$$u(x,y)=e^{-t}\sin x. \quad (2.39)$$

Obtained upon using the Taylor expansion of e^{-t}.

Example6:

Use the (VIM)to solve the initial-boundary value problem

PDE $\quad u_t = u_{xx} + \sin x, (0 \prec x \prec \pi), (t \succ 0),$

BC $\quad \begin{array}{l} u(0,t) = e^{-t}, t \geq 0, \\ u(\pi,0) = -e^{-t}, t \geq 0, \end{array} \quad (2.40)$ IC $\quad u(x,0) = \cos x.$

Solution:

The correction functional for equation (2.40) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - \sin x\right)d\xi. \quad (2.41)$$

As presented before the stationary condition

$$1 + \lambda \downarrow_{\xi = t} = 0, \quad (2.42)$$

$$\lambda' \downarrow_{\xi = t} = 0,$$

And this gives

$$\lambda = -1, \quad (2.43)$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (2.41) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - \sin x\right)d\xi, n \geq 0. \quad (2.44)$$

We can select $u_0(x,0) = \cos x$ from the given conditions. Using this selection into (2.44) we obtain the following successive approximations

$$u_0(x,y) = \cos x,$$

$$u_1(x,y) = \cos x - t\cos x + t\sin x,$$

$$u_2(x,y) = \cos x - t\cos x + t\sin x + \frac{1}{2!}t^2 \sin x + \frac{1}{2!}t^2 \cos x, \quad (2.45)$$

$$u_3(x,y) = \cos x(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3) + \sin x(t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3),$$

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$$u_n(x,y) = \cos x(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \ldots) + \sin x(t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \ldots).$$

Accordingly, the exact solution

$$u(x,y) = e^{-t}\cos x + (1 - e^{-t})\sin x. \quad (2.46)$$

2.3 Homogeneous and inhomogeneous Wave Equations

As stated before the variational iteration method (VIM) gives rapidly convergent successive approximations of the exact solution if an exact solution exists. Otherwise, the method provides an approximation of high accuracy level by using only few iterations. In what follows,The variational iteration method will be used in the following wave equations.

Example7:

Use the (VIM)to solve the initial-boundary value problem

PDE $\quad u_{tt} = u_{xx}, (0 \prec x \prec \pi), (t \succ 0),$

BC $\quad u(0,t) = 0, u(\pi,t) = 0, t \geq 0, \quad (2.46)$

IC $\quad u(x,0) = \sin x.$

Solution:

The correction functional for equation (2.46) is

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}-\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right)d\xi. \quad (2.47)$$

As presented before the stationary conditions

$$1+\lambda\downarrow_{\xi=t}=0,$$

$$\lambda'\downarrow_{\xi=t}=0, \quad (2.48)$$

$$\lambda''\downarrow_{\xi=t}=0,$$

And this gives

$$\lambda=\xi-t, \quad (2.49)$$

Substituting this value of the Lagrange multiplier $\lambda=\xi-t$ into the functional (2.47) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)d\xi, n\geq 0. \quad (2.50)$$

We can select $u_0(x,t)=\sin x$ from the given conditions. Using this selection into (2.50) we obtain the following successive approximations

$$u_0(x,t)=\sin x,$$

$$u_1(x,t)=\sin x-\frac{1}{2!}t^2\sin x,$$

$$u_2(x,y)=\sin x-\frac{1}{2!}t^2\sin x\frac{1}{4!}t^4\sin x, \quad (2.51)$$

$$u_3(x,y)=\sin x-\frac{1}{2!}t^2\sin x+\frac{1}{4!}t^4\sin x-\frac{1}{6!}t^6\sin x,$$

.

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.

$$u_n(x,y) = \sin x(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + ...).$$

This gives the exact solution by

$u(x,y) = \sin x \cos t.$ (2.52)

By noting that $u(x,t) = \lim_{n\to\infty} u_n$

Example 8:

Use the (VIM)to solve the initial-boundary value problem

PDE $\quad u_{tt} = u_{xx} - 2, (0 \prec x \prec \pi), (t \succ 0),$

BC $\quad u(0,t) = 0, u(\pi,t) = \pi^2, t \geq 0,$ (2.53)

IC $\quad u(x,0) = x^2, u_t(x,0) = \sin x.$

Solution:

The correction functional for equation (2.53) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + 2\right)d\xi. \quad (2.54)$$

As presented before the stationary conditions

$1 + \lambda \downarrow_{\xi = t} = 0,$

$\lambda' \downarrow_{\xi = t} = 0,$ (2.55)

$\lambda'' \downarrow_{\xi = t} = 0,$

And this gives

$$\lambda = \xi - t, \quad (2.56)$$

Substituting this value of the Lagrange multiplier $\lambda = \xi - t$ into the functional (2.54) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 2\right) d\xi, n \geq 0. \quad (2.57)$$

We can select $u_0(x,t) = x^2 + t\sin x$ from the given conditions. Using this selection into (2.57) we obtain the following successive approximations

$$u_0(x,t) = x^2 + t\sin x,$$

$$u_1(x,t) = x^2 + t\sin x - \frac{1}{3!}t^3\sin x,$$

$$u_2(x,y) = x^2 + t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x, \quad (2.58)$$

$$u_3(x,y) = x^2 + t\sin x - \frac{1}{3!}t^3\sin x + \frac{1}{5!}t^5\sin x - \frac{1}{7!}t^7\sin x,$$

.

.

.

$$u_n(x,y) = x^2 + \sin x(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \ldots).$$

This gives the exact solution by

$$u(x,y) = x^2 + \sin x \sin t. \quad (2.59)$$

By using Taylor series for $\sin t$ and by noting that $u(x,t) = \lim_{n \to \infty} u_n$

Example 9:

Use the (VIM)to solve the initial value problem

PD $u_{tt} = u_{xx} + e^{-t}, (-\infty \prec x \prec \infty), (t \succ 0), \quad (2.60)$

IC $\quad u(x,0) = 1, u_t(x,0) = -1 + \sin x$

Solution:

Note that the initial value problem is inhomogeneous.

The correction functional for equation (2.61) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - e^{-\xi}\right) d\xi. \quad (2.62)$$

As presented before the stationary condition

$$1 + \lambda \downarrow_{\xi = t} = 0,$$

$$\lambda' \downarrow_{\xi = t} = 0, \quad (2.63)$$

$$\lambda'' \downarrow_{\xi = t} = 0,$$

And this gives

$$\lambda = \xi - t, \quad (2.64)$$

Substituting this value of the Lagrange multiplier $\lambda = \xi - t$ into the functional (2.62) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - e^{-\xi}\right) d\xi, n \geq 0. \quad (2.65)$$

We can select $u_0(x,t) = 1 - t + t \sin x$ from the given conditions. Using this selection into (2.65) we obtain the following successive approximations

$$u_0(x,t) = 1 - t + t\sin x,$$

$$u_1(x,t) = e^{-t} + t\sin x - \frac{1}{3!}t^3 \sin x,$$

$$u_2(x,y) = e^{-t} + t\sin x - \frac{1}{3!}t^3 \sin x + \frac{1}{5!}t^5 \sin x, \quad (2.66)$$

$$u_3(x,y) = \sin x - \frac{1}{2!}t^2 \sin x + \frac{1}{4!}t^4 \sin x - \frac{1}{6!}t^6 \sin x,$$

.

.

.

$$u_n(x,y) = e^{-t} + \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \ldots\right).$$

This gives the exact solution by

$$u(x,y) = e^{-t} \sin x \sin t. \quad (2.67)$$

By noting that $u(x,t) = \lim_{n \to \infty} u_n$

2.4 Nonlinear PDEs by Variational Iteration Method (VIM)

Systems of nonlinear partial differential equation arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of chemical reaction-diffusion model. To use the (VIM), we write a system in an operator form by

$$\begin{aligned} L_t u + R_1(u,v,w) + N_1 u(u,v,w) &= g_1, \\ L_t u + R_2(u,v,w) + N_2(u,v,w) &= g_2, \\ L_t u + R_3(u,v,w) + N_3(u,v,w) &= g_3, \end{aligned} \quad (2.68)$$

With initial data

$$\begin{aligned} u(x,0) &= f_1(x), \\ v(x,0) &= f_2(x), \\ w(x,0) &= f_3(x), \end{aligned} \quad (2.69)$$

Where L_t is considered a first order partial differential operator, $R_j, 1 \le j \le 3$ and $N_j, 1 \le j \le 3$ are linear and nonlinear operators respectively, and g_1, g_2 and g_3 are source terms. The correction functionals for equations of the system (2.68) can be written as

$$\begin{aligned} &u_{n+1}(x,t) = u_n(x,t) \\ &+\int_0^t \lambda_1(Lu_n(x,\xi)) + R_1(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_1(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_1(\xi)d\xi, \\ &v_{n+1}(x,t) = v_n(x,t) \\ &+\int_0^t \lambda_2(Lv_n(x,\xi) + R_2(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_2(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_2(\xi))d\xi, \\ &w_{n+1}(x,t) = w_n(x,t) \\ &+\int_0^t \lambda_3(Lw_n(x,\xi) + R_3(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) + N_3(\tilde{u}_n,\tilde{v}_n,\tilde{w}_n) - g_3(\xi))d\xi, \end{aligned} \quad (2.70)$$

Where $\lambda_j, 1 \leq j \leq 3$ are general Lagrange's multipliers, which can be identified optimally via the variational theory, and $\tilde{u}_n, \tilde{v}_n$, and $\tilde{w}_n$ as restricted variations which means $\delta\tilde{u}_n = 0, \delta\tilde{v}_n = 0$ and $\delta\tilde{w}_n = 0$. It is required first to determine the Lagrange's multipliers λ_j that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), v_{n+1}(x,t), w_{n+1}(x,t), n \geq 0$ of the solutions $u(x,t), v(x,t)$ and $w(x,t)$ will follow immediately upon using the obtained Lagrange's multipliers and by using selective functions u_0, v_0 and w_0 . The initial values are usually used for the selective zeroth approximations. Whit the Lagrange's multipliers λ_j determined, then several approximations

$u_j(x,t), v_j(x,t), w_j(x,t), j \geq 0$ can be determined [18]. Consequently, the solutions are given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t), \\ v(x,t) &= \lim_{n\to\infty} v_2(x,t), \\ w(x,t) &= \lim_{n\to\infty} w_3(x,t), \end{aligned} \qquad (2.71)$$

Example 10:

Use the (VIM)to solve inhomogeneous nonlinear system

PDE $\begin{aligned} u_t + vu_x + v &= 1, \\ v_t - uv_x - v &= 1, \end{aligned} \quad (2.72)$ IC $\quad u(x,0) = e^x, v(x,0) = e^{-x}.$

Solution:

The correction functional for equation (2.61) is

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{v}_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} + \tilde{u}_n(x,\xi) - 1\right)d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n(x,\xi)}{\partial \xi} - \tilde{u}_n(x,\xi)\frac{\partial v_n(x,\xi)}{\partial x} - \tilde{v}_n(x,\xi) - 1\right)d\xi, \end{aligned} \qquad (2.73)$$

The stationary conditions are given by

$$1+\lambda_1=0,\lambda_1'(\xi=t)=0,$$
$$1+\lambda_2=0,\lambda_2'(\xi=t)=0, \quad (2.74)$$

So that

$$\lambda_1=\lambda_2=-1, \quad (2.75)$$

Substituting this value of the Lagrange multiplier $\lambda_1=\lambda_2=-1$ into the functional (2.73) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,\xi)}{\partial\xi}+\tilde{v}_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}+\tilde{u}_n(x,\xi)-1\right)d\xi,$$
$$v_{n+1}(x,t)=v_n(x,t)-\int_0^t\left(\frac{\partial v_n(x,\xi)}{\partial\xi}-\tilde{u}_n(x,\xi)\frac{\partial v_n(x,\xi)}{\partial x}-\tilde{v}_n(x,\xi)-1\right)d\xi, \quad (2.76)$$

The zeroth approximations $u_0(x,t)=e^x$, and $v_0(x,t)=e^{-x}$ are selected by using the given initial conditions. Therefore, we obtain the following successive approximations

$$u_0(x,t)=e^x, v_0(x,t)=e^{-x},$$
$$u_1(x,t)=e^x-te^x, v_1(x,t)=e^{-x}+te^{-x},$$
$$u_2(x,t)=e^x-te^x+\frac{t^2}{2!}e^x+noiseterms \quad ,$$
$$v_2(x,t)=e^{-x}+te^{-x}+\frac{t^2}{2!}e^{-x}+noiseterms \quad , \quad (2.77)$$

.

.

By cancelling the noise terms between $u_2, u_3, \ldots$ and between $v_2, v_3, \ldots$ we find

$$u_n(x,t) = e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right),$$
$$v_n(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right), \quad (2.78)$$

And as a result, the exact solution are given by

$$u(x,t) = e^{x-t},$$
$$v(x,t) = e^{-x+t}, \quad (2.79)$$

Obtained upon using the Taylor expansion for e^{-t} and e^{t} . It is obvious that we did not use any transformation formulas or linearization assumptions for handling the nonlinear terms.

In what follows, a system of three nonlinear partial differential equations in three unknown functions n $u(x, y, t)$, $v(x, y, t)$ and $w(x, y, t)$ will be studied. It is worth that noting that handling this system by traditional methods is quiet complicated.

Example 11:

Use the (VIM)to solve inhomogeneous nonlinear system

PDE
$$u_t - v_x w_y = 1,$$
$$v_t - w_x u_y = 5, \quad (2.80)$$
$$w_t - u_x v_y = 5,$$

With the initial conditions

IC
$$u(x, y, 0) = x + 2y,$$
$$v(x, y, 0) = x - 2y, \quad (2.81)$$
$$w(x, y, 0) = -x + 2y.$$

Solution:

The correction functional for equation (2.61) is

$$u_{n+1}(x,y,t)=u_n(x,y,t)+\int_0^t \lambda_1(\xi)\left(\frac{\partial u_n(x,y,\xi)}{\partial \xi}-\frac{\partial v_n(x,y,\xi)}{\partial x}*\frac{\partial w_n(x,y,\xi)}{\partial y}-1\right)d\xi,$$
$$v_{n+1}(x,y,t)=v_n(x,y,t)+\int_0^t \lambda_2(\xi)\left(\frac{\partial v_n(x,y,\xi)}{\partial \xi}-\frac{\partial w_n(x,y,\xi)}{\partial x}*\frac{\partial u_n(x,y,\xi)}{\partial y}-5\right)d\xi, \quad (2.82)$$
$$w_{n+1}(x,y,t)=w_{n+1}(x,y,t)+\int_0^t \lambda_3(\xi)\left(\frac{\partial w_n(x,y,\xi)}{\partial \xi}-\frac{\partial u_n(x,y,\xi)}{\partial x}*\frac{\partial v_n(x,y,\xi)}{\partial y}-5\right)d\xi,$$

The stationary conditions are given by

$$1+\lambda_1=0,\lambda_1'(\xi=t)=0,$$
$$1+\lambda_2=0,\lambda_2'(\xi=t)=0, \quad (2.83)$$
$$1+\lambda_3=0,\lambda_3'(\xi=t)=0,$$

So that

$$\lambda_1=\lambda_2=\lambda_3=-1, \quad (2.84)$$

Substituting this value of the Lagrange multiplier $\lambda_1=\lambda_2=\lambda_3=-1$ into the functional (1.82) gives the iteration formula

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t \left(\frac{\partial u_n(x,y,\xi)}{\partial \xi}-\frac{\partial v_n(x,y,\xi)}{\partial x}*\frac{\partial w_n(x,y,\xi)}{\partial y}-1\right)d\xi,$$
$$v_{n+1}(x,y,t)=v_n(x,y,t)-\int_0^t \left(\frac{\partial v_n(x,y,\xi)}{\partial \xi}-\frac{\partial w_n(x,y,\xi)}{\partial x}*\frac{\partial u_n(x,y,\xi)}{\partial y}-5\right)d\xi, \quad (2.85)$$
$$w_{n+1}(x,y,t)=w_{n+1}(x,y,t)-\int_0^t \left(\frac{\partial w_n(x,y,\xi)}{\partial \xi}-\frac{\partial u_n(x,y,\xi)}{\partial x}*\frac{\partial v_n(x,y,\xi)}{\partial y}-5\right)d\xi,$$

The zeroth approximations

$$
\begin{aligned}
u_0(x, y, t) &= x + 2y, \\
v_0(x, y, t) &= x - 2y, \qquad (2.86) \\
w_0(x, y, t) &= -x + 2y,
\end{aligned}
$$

are selected by using the given initial conditions. Therefore, we obtain the following successive approximations

$$\begin{cases} u_0(x,y,t) = x + 2y, \\ v_0(x,y,t) = x - 2y, \\ w_0(x,y,t) = -x + 2y, \end{cases}$$

$$\begin{cases} u_1(x,y,t) = x + 2y + 3t, \\ v_1(x,y,t) = x - 2y + 3t, \\ w_1(x,y,t) = -x + 2y + 3t, \end{cases}$$

$$\begin{cases} . \\ . \\ . \end{cases} \qquad (2.87)$$

$$\begin{cases} u_n(x,y,t) = x + 2y + 3t, \\ v_n(x,y,t) = x - 2y + 3t, \\ w_n(x,y,t) = -x + 2y + 3t, \end{cases}$$

Are readily obtained .Notice that the successive approximations become the same for u after obtaining the first approximation .The same conclusion can be made for v and w .Based on this, the exact solutions are given by

$$\begin{aligned} u(x,y,t) &= x + 2y + 3t, \\ v(x,y,t) &= x - 2y + 3t, \\ w(x,y,t) &= -x + 2y + 3t. \end{aligned} \qquad (2.88)$$

Chapter Three

Seepage flow Derivatives in porous media.

3.1 Definition of a porous media

In order to study the flow of fluid through porous media, it is first of all necessary to clarify what is understood by the terms that denote the two materials involved fluids and porous media.

We define(porous media) as solid bodies that contain pores, (pores) are void spaces which must be distributed more or less frequently through the material if it is to be called(porous).

3.2 Darc'y law

Darc'y law derived experimentally and was thus considered an empirical law based on volume average of the Navier Stock momentum equation. The assumption needed for derivation of Darc'y law include low flow speeds and that porous fluid direction is a dominating actiong on the fluid.

3.2.1 Single-Phase Flow

The differential form for the Darc'y law in single-phase is

$$\overline{u} = \frac{k}{\mu}\left(\nabla p + \rho \frac{g}{g_c}\right) \quad (3.1)$$

Where

k is absolute permeability tensor of the porous medium.

μ is fluid viscosity.

g is gravitational.

g_c is conversion constant.

ρ is fluid density.

$\overline{u}$ is fluid velocity.

3.2.2 Permeability

The hydraulic conductivity tensor λ describes the influence of the fluid and rock properties on the volumetric flow density(flow velocity), and given as

$$\lambda = \frac{k}{\mu} \quad (3.2)$$

Where

k represents the absolute permeability of the given porous medium.

3.3 Introduction to Fractional Calculus

The calculus was the first achievement of modern mathematic, Isaac Newton & Leibniz discover calculus in the seventieth century.

Leibniz first introduced the idea of symbolic method and based the symbol.

$$\frac{d^n y}{dx^n} = D^n y \quad (3.3)$$

For the nth derivative, where n is non-negative. L'Hospital asked Leibniz about the possibility that n be fractional Leibniz replied it well lead to paradox.

3.3.1 Lacrox formula

In 1819 Lacrox developed the formula for the nth derivative of $y = x^m$ where m is positive integer

$$D^n y = \left(\frac{m!}{(m-n)!} \right) x^{m-n}, m \geq n \quad (3.4)$$

Replacement of the factorial symbol by the gamma function goes

$$D^n y = \left(\frac{\Gamma(m+1)}{\Gamma(m-n+1)} \right) x^{m-n}, m \geq n \quad (3.5)$$

Now of (3.5) is define for other n integer or not (arbitrary number)

3.3.2 Lioville's Formula For Derivative

3.3.2.1Lioville'sFirst Formula

For any integer n we have

$$D^n e^{ax} = a^n e^{ax} \quad (3.6)$$

Lioville's replace of n by an arbitrary order α (rational ,irrational ,or complex) it is clear that the Right Hand Side (RHS) of (3.6) is well define case, that obtained the following formula

$$D^\alpha e^{ax} = a^\alpha e^{ax} \quad (3.7)$$

This formula is called first Lioville's formula.

In series expansion of (x) ,Lioville's formula is given by

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x} \quad (3.8)$$

Where

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x} \quad (3.9)$$

3.3.2.2 Lioville's second formula:

Lioville's formulated another definition of (second formula) fractional derivative based on the gamma function to extendLacriox formula.

$$\Gamma \beta = \int_0^\infty t^{\beta-1} e^{-xt}\, dt$$

$$\Gamma(\beta) x^{-\beta} = \int_0^\infty t^{-\beta} x^{-\beta} e^{-t} dt \,,\ \beta \succ 0$$

Let $t = xt \Rightarrow dt = xdt$

$$\Gamma(\beta)x^{-\beta} = \int_0^\infty t^{\beta-\beta-1} t^{\beta-1} xe^{-tx} xdt \ ,$$

$$\Gamma(\beta)x^{-\beta} = \int_0^\infty t^{\beta-1} e^{-tx} dt \ , \beta \succ 0$$

$$\Gamma(\beta) D^{\alpha} x^{-\beta} = \int_0^\infty t^{\beta-1} e^{-tx} dt \ , \beta \succ 0$$

$$= (-1)^{\alpha} \int_0^\infty t^{\alpha+\beta-1} e^{-xt} dt$$

$$D^{\alpha} x^{-\beta} = \frac{(-1)^{\alpha}}{\Gamma \beta} \int_0^\infty t^{\alpha+\beta-1} e^{-tx} dt \ ,$$

$$= \frac{(-1)^{\alpha}}{\Gamma \beta} \Gamma(\beta + \alpha) x^{-\beta-\alpha}$$

$$D^{\alpha} x^{-\beta} = \frac{(-1)^{\alpha}}{\Gamma \beta} \Gamma(\beta + \alpha) x^{-\beta-\alpha} \qquad (3.10)$$

This is called Lioville's second definition of fractional derivative according to Lioville's derivative of a constant $\beta = 0$

But the derivative of a constant function to Lacroix formula is

$$D^{\alpha} 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \neq 0 \quad (3.11)$$

This lead to a discrepancy between the two definition of fractional derivative. But other mathematicians preferred Lioville's definition.

In 1822.Fourier obtained the following integral representation for $f(x)$ and it is derivatives

$$f(x) = \frac{1}{2\pi}\int_a^x f(\xi)d\xi \int_a^x t^n \cos t(t-\xi)dt, \quad (3.12)$$

And

$$D^n(f(x)) = \frac{1}{2\pi}\int_a^x f(\xi)d\xi \int_a^x t^n \cos\left\{t(x-\xi)+\frac{n\pi}{2}\right\}dt, \quad (3.13)$$

Replacing integer n by arbitrary real α

$$D^\alpha(f(x)) = \frac{1}{2\pi}\int_a^x f(\xi)d\xi \int_a^x t^\alpha \cos\left\{t(x-\xi)+\frac{\alpha\pi}{2}\right\}dt, \quad (3.14)$$

Greer derived formula for the fractional derivative of trigonometric

$$D^\alpha(e^{iax}) = i^\alpha a^\alpha e^{iax} = i^\alpha a^\alpha(\cos ax + i\sin ax)$$

$$= a^\alpha\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)(\cos ax + i\sin ax)$$

$$\Rightarrow D^\alpha \cos ax = a^\alpha\left(\cos\frac{\alpha\pi}{2}\cos ax - \sin\frac{\alpha\pi}{2}\sin ax\right)$$

$$D^\alpha \cos ax = a^\alpha \cos\left(\frac{\alpha\pi}{2} + ax\right) \quad (3.15)$$

$$D^\alpha \sin ax = a^\alpha\left(\cos\frac{\alpha\pi}{2}\cos ax + \cos\frac{\alpha\pi}{2}\cos ax\right)$$

$$D^\alpha \sin ax = a^\alpha \sin\left(\frac{\alpha\pi}{2} + ax\right)$$

3.3.3 Fractional Derivatives And Integral

The idea of fractional derivative or fractional integral can be described in different ways. We consider a linear homogeneous nth order ordinary differential equation

$$\begin{aligned} & D^{n} y = 0 \\ & y^{(k)}(a) = 0, 0 \leq k \leq n-1 \end{aligned} \quad (3.16)$$

The solution is fundamental set $P\{1, x, x^{2}, \ldots x^{n-1}\}$ i.e

$$y = \sum_{i=0}^{n-1} c_i x^{i} = \sum_{r=0}^{n-1} c_r x^{r} \quad (3.17)$$

3.3.4 Non homogeneous O.D.E

$$\begin{aligned} & D^{n} y = f(x) \\ & y^{(k)}(0) = 0, b \leq x \leq c \end{aligned} \quad (3.18)$$

Solution:

We use Laplace transform ℓ

$$\begin{aligned} & \ell y^{(k)}(0) = \ell(0) \Rightarrow y^{(k)}(0) = 0 \\ & \ell D^{n}(y) = \ell f(x) \Rightarrow s^{n} \bar{y} = \bar{f}(s), \end{aligned} \quad (3.19)$$

where

$$\begin{aligned} & \bar{y} = \ell\{y\} \text{and } \bar{f}(s) = \ell\{f\} \\ & \bar{y} = s^{-n} \bar{f}(s), \\ & \ell^{-1}\{\bar{y}\} = \ell^{-1} s^{-1} \bar{f}(s) \Rightarrow \\ & y(x) = \ell^{-1} s^{-1} \bar{f}(s) \end{aligned} \quad (3.20)$$

Used convolution

$$y(x) = \frac{1}{\Gamma n}\int_a^x (x-t)^{n-1} f(t)\,dt \qquad (3.21)$$

This formula is called Rimman integral.

In general

$$y(x) = \frac{1}{\Gamma n}\int_a^x (x-t)^{n-1} f(t)\,dt \qquad (3.22)$$

Is called Rimman-Lioville's.

Replacing n by real α gives the Rimman-Lioville's fractional integral

$$y(x) = {}_{\alpha}D_x^{\alpha}(f(x)) = J^{-\alpha} = \frac{1}{\alpha}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \qquad (3.23)$$

Where

${}_{\alpha}D_x^{\alpha}(f(x)) = J^{-\alpha}$ Is the Rimman-Lioville's operator.

If

$a = 0$ Is called Rimman fractional integral and

If

$a = -\infty$ called Lioville's fractional integral.

Chapter Four

The application of Varational Iteration Method (VIM).

4.1 Liouville'sFirst Formula:

4.1.1 Exponential Function:

With the know result $D^n e^{ax} = a^n e^{ax}$ where $D = \frac{d}{dx}, n \in N$, and extended it at first in the particular case $\alpha = \frac{1}{2}, a = 2$ and then to arbitrary order α (rational, irrational or complex) by

$$D^{\alpha} e^{ax} = a^{\alpha} e^{ax} \quad (4.1)$$

He assumed the series representation for $f(x)$ as $f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}$ and defined the derivative of arbitrary order α by

$$D^{\alpha} f(x) = \sum_{k=0}^{\infty} c_k a_k^{\alpha} e^{a_k x} \quad (4.2)$$

4.2Liouville’sSecond Formula:

4.2.1Power Function:

Where this isLiouville's first approach, his second method was applied to the explicit function $x^{-\alpha}$. He considered the integral

$$I = \int_0^{\infty} u^{\beta - 1} e^{-xu} \, du \quad (4.3)$$

Substituting $xu = t$ gives t he result

$$I = x^{-\beta} \int_0^{\infty} t^{\beta - 1} e^{-t} dt = x^{-\beta} \Gamma(\beta), , , , , \operatorname{Re} \alpha \succ 0 \quad (4.4)$$

Operating on both sides of $x^{-\beta} = \frac{1}{\Gamma(\beta)}$ whit D^{α} with respect to x heobtained

$$\Gamma(\beta) D^{\alpha} x^{-\beta} = \int_{0}^{\infty} u^{\beta-1} D^{\alpha} e^{-xu} du$$

$$D^{\alpha}(e^{-xu}) = (-1)^{\alpha} u^{\alpha} e^{-xu}$$

$$D^{\alpha} x^{-\beta} = (-1)^{\alpha} \frac{\Gamma \alpha + \beta}{\Gamma \beta} x^{-\alpha-\beta} \quad (4.5)$$

Liouville used the latter in this investigation of potential theory

That lead to

$$D^{\alpha} x^{\beta} = \frac{\Gamma \beta + 1}{\Gamma \beta - \alpha + 1} x^{\beta-\alpha}, (\beta \succ -1) \quad (4.6)$$

4.3The Partial Differential Equation Of SeepageFlowThrough Porous Media

The partial differential equation for incompressible single phase percolation flow under the hypotheses for continuity and Darcy low can be written general as follows

$$\frac{\partial}{\partial x}(k_x \frac{\partial p}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial p}{\partial y}) + \frac{\partial}{\partial z}(k_z \frac{\partial p}{\partial z}) = \frac{1}{v} \frac{\partial p}{\partial t}, \quad (4.7)$$

$$(x, y, z) \in \Omega$$

$$p(t, x, y, z) s_1 = \varphi_1$$
$$\frac{\partial p(t, x, y, z)}{\partial n} s_2 = \varphi_2 \quad (4.8)$$
$$p(t, x, y, z) = \varphi_0(x, y, z)$$

Where k_x, k_y and k_z are the percolation coefficients along the x, y and z direction respectively , p is the pressure, and Ω denotes the percolation domain , and $s_1 + s_2$ covers all its percolation domain.

4.4 The Fractional Partial Differential Equation Of Seepage

The above percolation is (seepage) equation under the assumptions of continuity of seepage flow and Darcy low .Generally these two assumptions are not valid for real seepage flow.

We proposes the following modified Darc'ysLow or generalized Darc'ysLow with Riemman-Lioville's fractional derivatives

$$q_x = k_x \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}, (0 \prec \alpha_1 \prec 1)$$

$$q_y = k_y \frac{\partial^{\alpha_2} p}{\partial x^{\alpha_2}}, (0 \prec \alpha_2 \prec 1) \qquad (4.9)$$

$$q_z = k_z \frac{\partial^{\alpha_3} p}{\partial x^{\alpha_3}}, (0 \prec \alpha_3 \prec 1)$$

In case of $\alpha_1 = \alpha_2 = \alpha_3 = 1$ (4.9) correspond Darcy low, the Riemmanlioville's fractional derivatives generally

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}} = \frac{1}{\Gamma 1 - \alpha} \frac{d}{dx} \int_0^x (x - t)^{-\alpha} u(t) dt \qquad (4.10)$$

Under the assumptions of continuity of seepage flow we have following fractional differential equation

$$\frac{\partial}{\partial x}(k_x \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}) + \frac{\partial}{\partial y}(k_y \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}) + \frac{\partial}{\partial z}(k_z \frac{\partial^{\alpha_3} p}{\partial z^{\alpha_3}}) = \frac{1}{r}\frac{\partial p}{\partial t}, \qquad (4.11)$$

$$(x, y, z) \in \Omega$$

From (4.9)

If seepage flow is considered as rigid body motion the continuity equation can be written as follows

$$\frac{\partial^0}{\partial x^0}(k_x \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}) + \frac{\partial^0}{\partial y^0}(k_y \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}) + \frac{\partial^0}{\partial z^0}(k_z \frac{\partial^{\alpha_3} p}{\partial z^{\alpha_3}}) = \frac{1}{r}\frac{\partial p}{\partial t}, \qquad (4.12)$$

$$(x, y, z) \in \Omega$$

Actually the seepage flow is neither continued or rigid, so the more general equation for seepage can be expressed as follows

$$\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}(k_x \frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}) + \frac{\partial^{\beta_2}}{\partial y^{\beta_2}}(k_y \frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}) + \frac{\partial^{\beta_3}}{\partial z^{\beta_3}}(k_z \frac{\partial^{\alpha_3} p}{\partial z^{\alpha_3}}) = \frac{1}{r}\frac{\partial p}{\partial t}, \quad (4.13)$$

$$(x, y, z) \in \Omega$$

Where $(0 \prec \beta_1, \beta_2, \beta_3 \prec 1)$

It is exciting that thevariatianaliterantial method (VIM) is also valid for such fractional differential equation , which are very difficult to solve even with numerical simulation. Duo to the fact that fractional differential equation can be excellently describe the natural phenomena approximation approach to it has caught attention by numerous mathematician

We will apply the variatianaliterantial method (VIM) to obtain an analytical solution for a fractional differential equation.

Consider first the following system

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}} = f(x, u), \ (1 \prec \alpha \prec 2) \quad (3.14)$$
$$u(a) = b$$

Solution:

According to the variatianliterantial method (VIM), we construct the following correction functional

$$u_{n+1}(x) = u_n(x) + I^{\alpha} F(x) \quad (4.15)$$

Where I^{α} the Riemman-lioville's fractional integrate defined as follows

$$I^{\alpha} F(x) = \frac{1}{\Gamma \alpha} \int_0^x (x - t)^{\alpha - 1} F(t) dt \quad (4.16)$$

For example, from Lacrox

$$D^{-\frac{1}{2}} t^n = \frac{\Gamma n + 1}{\Gamma n + 1 + \frac{1}{2}} t^{n + \frac{1}{2}}, n \succ -1 \quad (4.17)$$

To identify approximately the Lagrange multiplier we apply restricted variations to nonlinear term and also to $\frac{\partial^{\alpha} u}{\partial x^{\alpha}}$ when there exist derivative with integer order.

But where there exists no derivative with integer order, as far as there exists no way stationary conditions directly from a functional with the Riemman-lioville's fractional integrateso the correction functional can be approximately expressed as follow

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda_1 \left[\frac{\partial u_n}{\partial x} - f(x, u_n) \right] dx \quad (4.18)$$

Substituting the identified Lagrange multipliers in also to (4.14) results in the following iteration procedures

$$u_{n+1}(x) = u_n(x) + I^{\alpha} \left[\lambda_1 \left(\frac{\partial^{\alpha} u_n}{\partial x^{\alpha}} - f(x, u_n) \right) \right] dx \quad (4.19)$$

Example 1:

Solve

$$D^{\frac{1}{2}} u = \frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} (u+1)^2, \; u(0) = 0 \;, (0 \prec x \prec 1) \; (4.20)$$

By using (VIM)

Solution:

According to the variatianliterantial method (VIM), we have following correction functional

$$u_{n+1}(x) = u_n(x) + I^{\alpha} F(x) \qquad (4.21)$$

Then

$$u_{n+1}(x) = u_n(x) + I^{\frac{1}{2}}\left\{\lambda\left[D^{\frac{1}{2}}u_n - \frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}(u_n+1)^2\right]\right\} \quad (4.22)$$

And it is stationary conditions can be readily obtained

$$\lambda'(\tau) = 0$$
$$\lambda(\tau) + 1 = 0,,,,,at,,,,,,,,t = \tau \quad (4.23)$$

So the multiplier can be identified by $\lambda = -1$ substitute in (4.22) yield

$$u_{n+1}(x) = u_n(x) - I^{\frac{1}{2}}\left\{\left[D^{\frac{1}{2}}u_n - \frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}(u_n+1)^2\right]\right\} \quad (4.24)$$

We start with $u_0(0) = 0$, by the variatianal iteration formula (4.24), we have

$$u_0(0) = 0,$$

$$u_1(x) = u_0(x) - I^{\frac{1}{2}}\left\{\left[D^{\frac{1}{2}}u_0 - \frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}(u_0+1)^2\right]\right\}$$

$$= 0 - I^{\frac{1}{2}}\left\{\left[0 - \frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}(0+1)^2\right]\right\}$$

$$= I^{\frac{1}{2}}\left\{\left[\frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\right]\right\} = \frac{1}{\pi^{\frac{1}{2}}} I^{\frac{1}{2}}\left[x^{\frac{1}{2}}\right] = \frac{1}{\pi^{\frac{1}{2}}} \frac{\Gamma\frac{1}{2}+1}{\Gamma\frac{1}{2}+1+\frac{1}{2}} x^{\frac{1}{2}+\frac{1}{2}}$$

$$= \frac{1}{\pi^{\frac{1}{2}}} \frac{\Gamma\frac{3}{2}}{\Gamma 2} x = \frac{x}{2}$$

$$u_2(x) = u_1(x) - I^{\frac{1}{2}}\left\{\left[D^{\frac{1}{2}} u_1 - \frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}(u_1+1)^2\right]\right\}$$

$$= \frac{x}{2} - I^{\frac{1}{2}}\left\{\left[D^{\frac{1}{2}}\left(\frac{x}{2}\right) - \frac{x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\left(\frac{x}{2}+1\right)^2\right]\right\} \quad (4.25)$$

The exact solution my obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \quad (4.26)$$

Then the exact solution is

$$u(x) = \frac{x}{2} + \frac{3}{8}x^2 + \ldots \quad (4.27)$$

Example 2:

Solve homogeneous partial differential equation by (VIM)

$$\frac{\partial^\alpha u(x,y)}{\partial y^\alpha} + x\frac{\partial u(x,y)}{\partial x} = 3u(x,y) = 0 \ , u(x,0) = x^2, u(0,y) = 0 \quad (4.28)$$

Solution:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda\left(\frac{\partial^\alpha u(x,\xi)}{\partial \xi^\alpha} + x\frac{\partial u(x,\xi)}{\partial x} - 3u(x,\xi)\right)d\xi \quad (4.29)$$

As presented before, the stationary conditions are

$$1 + \lambda|_{\xi=x} = 0$$
$$\lambda`|_{\xi=x} = 0 \quad (4.30)$$

This gives $\lambda = -1$

Substituting this value of the Lagrange multiplier $\lambda = -1$ in to the functional (4.29) gives

the iteration formula

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^y \left(\frac{\partial^\alpha u_n(x,\xi)}{\partial \xi^\alpha} + x\frac{\partial u_n(x,\xi)}{\partial x} - 3u_n(x,\xi) \right) d\xi \,, n \geq 0 \quad (4.31)$$

We can select $u_0(x,y) = u(x,0) = x^2$ from the given conditions. Using this selection into

(4.31) we obtain the following successive approximations

$$u_0(x,y) = x^2$$

$$u_1(x,y) = u_0(x,y) - \int_0^y \left(\frac{\partial^\alpha u_0(x,\xi)}{\partial \xi^\alpha} + x\frac{\partial u_0(x,\xi)}{\partial x} - 3u_0(x,\xi) \right) d\xi$$

$$= x^2 - \int_0^y \left(\frac{\partial^\alpha}{\partial \xi^\alpha}(x^2) + x(2x) - 3x^2 \right) d\xi = x^2 - \int_0^y (0 - x^2) d\xi = x^2 + x^2 y$$

$$u_2(x,y) = u_1(x,y) - \int_0^y \left(\frac{\partial^\alpha u_1(x,\xi)}{\partial \xi^\alpha} + x\frac{\partial u_1(x,\xi)}{\partial x} - 3u_1(x,\xi) \right) d\xi$$

$$= x^2 + x^2 y - \int_0^y \left(\frac{\partial^\alpha}{\partial \xi^\alpha}(x^2 + x^2\xi) + x(2x + 2x\xi) - 3(x^2 + x^2\xi) \right) d\xi$$

$$= x^2 + x^2 y - \int_0^y \left(0 + \frac{x^2 \xi^{1-\alpha}}{\Gamma(2-\alpha)} - x^2 - x^2\xi \right) d\xi = x^2 + x^2 y + \frac{x^2 y^2}{2!} + \left(-\frac{x^2 y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2 y \right)$$

$$u_3(x,y) = u_2(x,y) - \int_0^y \left(\frac{\partial^\alpha u_2(x,\xi)}{\partial \xi^\alpha} + x\frac{\partial u_2(x,\xi)}{\partial x} - 3u_2(x,\xi) \right) d\xi$$

$$= x^2 + x^2y + \frac{x^2y^2}{2!} + \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right)$$

$$- \int_0^y \left(\frac{\partial^\alpha}{\partial \xi^\alpha} \left(x^2 + x^2\xi + \frac{x^2\xi^2}{2!} + \left(-\frac{x^2\xi^{2-\alpha}}{\Gamma(3-\alpha)} + x^2\xi \right) \right) \right.$$

$$+ x \left(2x + 2x\xi + \frac{2x\xi^2}{2!} + \left(-\frac{2x\xi^{2-\alpha}}{\Gamma(3-\alpha)} + 2x\xi \right) \right)$$

$$\left. - 3 \left(x^2 + x^2\xi + \frac{x^2\xi^2}{2!} + \left(-\frac{x^2\xi^{2-\alpha}}{\Gamma(3-\alpha)} + x^2\xi \right) \right) \right) d\xi$$

$$= x^2 + x^2y + \frac{x^2y^2}{2!} + \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right)$$

$$- \int_0^y \left(\frac{x^2\xi^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{x^2\Gamma(3)\xi^{2-\alpha}}{2!\,\Gamma(3-\alpha)} + \left(-\frac{x^2\Gamma(3-\alpha)\xi^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{x^2\xi^{1-\alpha}}{\Gamma(2-\alpha)} \right) \right.$$

$$\left. - x^2 - x^2\xi - \frac{x^2\xi^2}{2!} + \frac{x^2\xi^{2-\alpha}}{\Gamma(3-\alpha)} - x^2\xi \right) d\xi$$

$$= x^2 + x^2y + \frac{x^2y^2}{2!} + \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right) - \frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^2y^{3-\alpha}}{\Gamma(4-\alpha)}$$

$$+ \frac{x^2\Gamma(3-\alpha)y^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y + \frac{x^2y^2}{2!} + \frac{x^2y^3}{3!} - \frac{x^2y^{3-\alpha}}{\Gamma(4-\alpha)}$$

$$+ \frac{x^2y^2}{2!}$$

$$= x^2 + x^2y + \frac{x^2y^2}{2!} + \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right) + \left(-\frac{2x^2y^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2x^2y^2}{2!}\right)$$

$$+ \left(-\frac{2x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y + \frac{x^2\Gamma(3-\alpha)y^{3-2\alpha}}{\Gamma(4-2\alpha)}\right) + \frac{x^2y^3}{3!}$$

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$u_n(x,y)$

$$= x^2 + x^2y + \frac{x^2y^2}{2!} + \frac{x^2y^3}{3!} + \cdots + \frac{x^ny^n}{n!} + \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right)$$

$$+ \left(-\frac{2x^2y^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2x^2y^2}{2!}\right) + \left(-\frac{2x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y + \frac{x^2\Gamma(3-\alpha)y^{3-2\alpha}}{\Gamma(4-2\alpha)}\right)$$

$$+ \cdots$$

$$u(x,t) = \lim_{n\to\infty} u_n(x,y) = \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right) + \left(-\frac{2x^2y^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2x^2y^2}{2!}\right) + \left(-\frac{2x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y + \frac{x^2\Gamma(3-\alpha)y^{3-2\alpha}}{\Gamma(4-2\alpha)}\right) + \cdots + \sum_{n=0}^{\infty}\frac{x^ny^n}{n!} = \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right) + \left(-\frac{2x^2y^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2x^2y^2}{2!}\right) + \left(-\frac{2x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y + \frac{x^2\Gamma(3-\alpha)y^{3-2\alpha}}{\Gamma(4-2\alpha)}\right) + \cdots = \left(-\frac{x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y\right) + \left(-\frac{2y^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2x^2y^2}{2!}\right) + \left(-\frac{2x^2y^{2-\alpha}}{\Gamma(3-\alpha)} + x^2y + \frac{x^2\Gamma(3-\alpha)y^{3-2\alpha}}{\Gamma(4-2\alpha)}\right) + \cdots + x^2e^x \quad (3.32)$$

When $\alpha = 1$ then the exact solution is x^2e^x

4.5 A Fractional Model Of Fluid Flow Through Porous MediaIn TwoDimensionBy (VIM):

If seepage flow is considered as rigid body motion the continuity equation can be written as follows in one dimension

$$\left(k\frac{\partial^{\alpha_1} p}{\partial x^{\alpha}}\right) - \frac{1}{v}\frac{\partial p}{\partial t} = 0, \qquad (4.33)$$

$x \in \Omega$

4.6 Solution AFractional Model Of Fluid Flow Through Porous Media In Two dimension by (VIM):

Solve FPDE BY (VIM)

$$\frac{\partial^{\alpha} p(x,t)}{\partial x^{\alpha}} - \frac{1}{v}\frac{\partial p(x,t)}{\partial t} = 0$$

$$p(0,t) = t$$

$$p(x,0) = \frac{x}{v} \qquad (4.34)$$

$$p_0(x,t) = t$$

Solution:

$$p_{n+1}(x,t) = p_n(x,t) + \lambda \int_0^x \frac{\partial^\alpha p_n(\xi,t)}{\partial \xi^\alpha} - \frac{1}{v}\frac{\partial p_n(\xi,t)}{\partial t} d\xi \qquad (4.35)$$

$n \geq 0$

As presented before, the stationary conditions are

$1 + \lambda|_{\xi=x} = 0$
$\lambda`|_{\xi=x} = 0$ (4.36)

This gives $\lambda = -1$

Substituting this value of the Lagrange multiplier $\lambda = -1$ in to the functional (4.36)

gives the iteration formula

$$p_{n+1}(x,t) = p_n(x,t) - \int_0^x \frac{\partial^\alpha p_n(\xi,t)}{\partial \xi^\alpha} - \frac{1}{v}\frac{\partial p_n(\xi,t)}{\partial t} d\xi \qquad (4.37)$$

$p_0(x,t) = t$

$$p_1(x,t) = p_0(x,t) - \int_0^x \frac{\partial^\alpha p_0(\xi,t)}{\partial \xi^\alpha} - \frac{1}{v}\frac{\partial p_0(\xi,t)}{\partial t} d\xi$$

$$= t - \int_0^x \frac{\partial^\alpha}{\partial \xi^\alpha}(t) - \frac{1}{v}\frac{\partial}{\partial t}(t) d\xi$$

$$= t - \int_0^x 0 - \frac{1}{v} d\xi = t - (-\frac{1}{v}x) = t + \frac{1}{v}x$$

$$p_1(x,t) = t + \frac{x}{v}$$

$$p_2(x,t) = p_1(x,t) - \int_0^x \frac{\partial^\alpha p_1(\xi,t)}{\partial \xi^\alpha} - \frac{1}{v}\frac{\partial p_1(\xi,t)}{\partial t} d\xi$$

$$= t + \frac{1}{v}x - \int_0^x \frac{\partial^\alpha}{\partial \xi^\alpha}(t + \frac{1}{v}\xi) - \frac{1}{v}\frac{\partial}{\partial t}(t + \frac{1}{v}\xi) d\xi$$

$$= t + \frac{1}{v}x - \int_0^x \frac{\xi^{1-\alpha}}{v\Gamma 2-\alpha} - \frac{1}{v} d\xi$$

$$= t + \frac{x}{v} - \left(\frac{x^{2-\alpha}}{v\Gamma 3-\alpha} - \frac{1}{v}x\right) = t + \frac{1}{v}x - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha} + \frac{1}{v}x$$

$$p_2(x,t) = t + \frac{x}{v} + \left[\frac{x}{v} - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]$$

$$p_3(x,t) = p_2(x,t) - \int_0^x \frac{\partial^\alpha p_2(\xi,t)}{\partial \xi^\alpha} - \frac{1}{v}\frac{\partial p_2(\xi,t)}{\partial t} d\xi$$

$$= t + \frac{2}{v}x + \left[-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right] -$$

$$\int_0^x \frac{\partial^\alpha}{\partial \xi^\alpha}(t + \frac{2}{v}\xi - \frac{\xi^{2-\alpha}}{v\Gamma 3-\alpha}) - \frac{1}{v}\frac{\partial}{\partial t}(t + \frac{2}{v}\xi - \frac{\xi^{2-\alpha}}{v\Gamma 3-\alpha}) d\xi$$

$$= t + \frac{2}{v}x - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha} - \int_0^x \frac{2\xi^{1-\alpha}}{v\Gamma 2-\alpha} - \frac{\xi^{2-2\alpha}}{v\Gamma 3-\alpha\,\Gamma 3-2\alpha} - \frac{1}{v}(1) d\xi$$

$$= t + \frac{2}{v} x - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha} - \left[\frac{2x^{2-\alpha}}{v\Gamma 3-\alpha} - \frac{x^{3-2\alpha}}{v\Gamma 4-2\alpha \Gamma 3-\alpha} - \frac{1}{v} x \right]$$

$$= t + \frac{2}{v} x - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha} - \frac{2x^{2-\alpha}}{v\Gamma 3-\alpha} + \frac{x^{3-2\alpha}}{v\Gamma 4-2\alpha \Gamma 3-\alpha} + \frac{1}{v} x$$

$$= t + \frac{3}{v} x - \frac{3x^{2-\alpha}}{v\Gamma 3-\alpha} + \frac{x^{3-2\alpha}}{v\Gamma 3-\alpha \Gamma 4-2\alpha}$$

$$p_3(x,t) = t + \frac{x}{v} + \left[\frac{2}{v} - \frac{3x^{2-\alpha}}{v\Gamma 3-\alpha} + \frac{x^{3-2\alpha}}{v\Gamma 3-\alpha \Gamma 4-2\alpha} \right]$$

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$$p_n(x,t) = t + \frac{2}{v} x + \frac{3}{v} x + \frac{4}{v} x + \ldots + \frac{n}{v} x \qquad (4.38)$$

$$+ \left[\frac{x}{v} - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha} \right] + \left[\frac{2x}{v} - \frac{3x^{2-\alpha}}{v\Gamma 3-\alpha} + \frac{x^{3-2\alpha}}{v\Gamma 3-\alpha \Gamma 4-2\alpha} \right] + \ldots$$

The exact solution my obtained by using

$$p(x,t) = \lim_{n \to \infty} p_n(x,t) \quad (4.39)$$

Then the exact solution is

$$\lim_{n \to \infty} p_n(x,t) = \lim_{n \to \infty} t + \frac{1}{v} x + \frac{2}{v} x + \frac{3}{v} x + \frac{4}{v} x + \ldots + \frac{n}{v} x$$

$$+ \left[- \frac{x^{2-\alpha}}{v\Gamma 3-\alpha} \right] - \frac{3x^{2-\alpha}}{v\Gamma 3-\alpha} + \frac{x^{3-2\alpha}}{v\Gamma 3-\alpha \Gamma 4-2\alpha} + \ldots$$

$$= t + \left[- \frac{x^{2-\alpha}}{v\Gamma 3-\alpha} + \frac{4}{v} x \right] + \left[- \frac{x^{3-2\alpha}}{v\Gamma 3-\alpha \Gamma 4-2\alpha} + \frac{1}{v} x \right] + \ldots + \sum_{n=1}^{\infty} \frac{n}{v} x$$

$$\left[-\frac{4x^{2-\alpha}}{v\Gamma 3-\alpha}+\frac{4}{v}x\right]+\left[-\frac{x^{3-2\alpha}}{v\Gamma 3-\alpha\Gamma 4-2\alpha}+\frac{1}{v}x\right]+\ldots+t+\frac{1}{v}x \quad (4.40)$$

When $\alpha = 1$ then the exact solution is $t+\frac{1}{v}x$

4.7Solution AFractional Model Of Fluid Flow Through Porous Media In ThreeDimensionBy (VIM)

If seepage flow is considered as rigid body motion the continuity equation can be written as follows in one dimension

$$(k_x \frac{\partial^{\alpha_1} p(x,y,t)}{\partial x^{\alpha_1}})+(k_y \frac{\partial^{\alpha_2} p(x,y,t)}{\partial y^{\alpha_2}})-\frac{1}{v}\frac{\partial p(x,y,t)}{\partial t}=0, \quad (4.41)$$

If $\alpha_1 = \alpha_2 = \alpha$ and $k_x = k_y = k = 1$ then we have

$$\frac{\partial^{\alpha} p(x,y,t)}{\partial x^{\alpha}}+\frac{\partial^{\alpha} p(x,y,t)}{\partial y^{\alpha}}-\frac{1}{v}\frac{\partial p(x,y,t)}{\partial t}=0, \quad (4.42)$$

$(x, y, z) \in \Omega$

Solve FPDE by(VIM):

$$\frac{\partial^{\alpha} p(x,y,t)}{\partial x^{\alpha}}+\frac{\partial^{\alpha} p(x,y,t)}{\partial y^{\alpha}}-\frac{1}{v}\frac{\partial p(x,y,t)}{\partial t}=0$$

$$p(0,y,,t)=1+e^{y}+e^{t}$$

$$p(x,0,t)=1+e^{x\left(\frac{1}{v}-1\right)}+e^{t} \quad (4.43)$$

$$p(x,y,0)=1+e^{x\left(\frac{1}{v}-1\right)}+e^{y}$$

$$p_0(x,y,t)=1+e^{y}+e^{z}$$

Solution:

$$p_{n+1}(x,y,t) = p_n(x,y,t)$$
$$+\lambda\int_0^x \frac{\partial^\alpha p_n(\xi,y,t)}{\partial \xi^\alpha} + \frac{\partial^\alpha p_n(\xi,y,t)}{\partial y^\alpha} - \frac{1}{v}\frac{\partial p_n(\xi,y,t)}{\partial t}\, d\xi$$
$$n \ge 0 \qquad (4.44)$$

As presented before, the stationary conditions are

$$1 + \lambda|_{\xi=x} = 0$$
$$\lambda`|_{\xi=x} = 0 \quad (4.45)$$

This gives $\lambda = -1$

Substituting this value of the Lagrange multiplier $\lambda = -1$ in to the functional (4.44)

gives the iteration formula

$$p_{n+1}(x,y,t) = p_n(x,y,t)$$
$$-\int_0^x \frac{\partial^\alpha p_n(\xi,y,t)}{\partial \xi^\alpha} + \frac{\partial^\alpha p_n(\xi,y,t)}{\partial y^\alpha} - \frac{1}{v}\frac{\partial p_n(\xi,y,t)}{\partial t}\, d\xi \quad (4.46)$$
$$n \ge 0$$

$$p_0(x,y,t) = 1 + e^y + e^z$$

$$p_1(x,y,t) = p_0(x,y,t) - \int_0^x \frac{\partial^\alpha p_0(\xi,y,t)}{\partial \xi^\alpha} + \frac{\partial^\alpha p_0(\xi,y,t)}{\partial y^\alpha} - \frac{1}{v}\frac{\partial p_0(\xi,y,t)}{\partial t} d\xi$$

$$= 1 + e^y + e^t - \int_0^x \frac{\partial^\alpha (1+e^y+e^t)}{\partial \xi^\alpha} + \frac{\partial^\alpha (1+e^y+e^t)}{\partial y^\alpha} - \frac{1}{v}\frac{\partial (1+e^y+e^t)}{\partial t} d\xi$$

$$= 1 + e^y + e^t - \int_0^x 0 + e^y - \frac{1}{v} e^t d\xi$$

$$= 1 + e^y + e^t - xe^y + \frac{x}{v}e^t = 1 + [1-x]e^y + \left[1 + \frac{x}{v}\right]e^t$$

$$p_1(x,y,t) = 1 + \left[1 - \frac{x}{1!}\right]e^y + \left[1 + \frac{x}{1!\,v}\right]e^t$$

$$p_2(x,y,t) = p_1(x,y,t) - \int_0^x \frac{\partial^\alpha p_1(\xi,y,t)}{\partial \xi^\alpha} + \frac{\partial^\alpha p_1(\xi,y,t)}{\partial y^\alpha} - \frac{1}{v}\frac{\partial p_1(\xi,y,t)}{\partial t} d\xi$$

$$= 1 + [1-x]e^y + \left[1 + \frac{x}{v}\right]e^t$$

$$- \int_0^x \frac{\partial^\alpha \left[1 + [1-\xi]e^y + \left[1+\frac{\xi}{v}\right]e^t\right]}{\partial \xi^\alpha} + \frac{\partial^\alpha \left[1 + [1-\xi]e^y + \left[1+\frac{\xi}{v}\right]e^t\right]}{\partial y^\alpha} - \frac{1}{v}\frac{\partial \left[1 + [1-\xi]e^y + \left[1+\frac{\xi}{v}\right]e^t\right]}{\partial t} d\xi$$

$$= 1 + [1-x]e^y + \left[1 + \frac{x}{v}\right]e^t - \int_0^x -\frac{\xi^{1-\alpha}}{\Gamma 2-\alpha}e^y + \frac{\xi^{1-\alpha}}{v\Gamma 2-\alpha}e^t + [1-\xi]e^y - \frac{1}{v}\left[1+\frac{\xi}{v}\right]e^t d\xi$$

$$= 1 + [1-x]e^y + \left[1 + \frac{x}{v}\right]e^t - \left[\left[-\frac{\xi^{2-\alpha}}{(2-\alpha)\Gamma 2-\alpha}e^y + \frac{\xi^{2-\alpha}}{v(2-\alpha)\Gamma 2-\alpha}e^t + \xi e^y - \frac{\xi^2 e^y}{2} - \frac{\xi e^t}{v} - \frac{\xi^2 e^t}{2v^2}\right]\right]$$

$$= 1 + [1-x]e^y + \left[1 + \frac{x}{v}\right]e^t - \left[-\frac{x^{2-\alpha}}{\Gamma 3-\alpha}e^y + \frac{x^{2-\alpha}}{v\Gamma 3-\alpha}e^t + xe^y - \frac{x^2 e^y}{2} - \frac{xe^t}{v} - \frac{x^2 e^t}{2v^2}\right]$$

$$p_2(x,y,t) = 1 + \left[1 - 2x + \frac{x^2}{2!} + \frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y + \left[1 + \frac{2}{v}x + \frac{1}{2!\,v^2}x^2 - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t$$

$$p_2(x,y,t) = \left[1 - \frac{x}{1!} + \frac{x^2}{2!} + \left[-x + \frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y + \left[1 + \frac{x}{1!\,v} + \frac{x^2}{2!\,v^2} + \left[\frac{x}{v} - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t\right]\right]$$

$$p_3(x,y,t) = p_2(x,y,t) - \int_0^x \frac{\partial^\alpha p_2(\xi,y,t)}{\partial \xi^\alpha} + \frac{\partial^\alpha p_2(\xi,y,t)}{\partial y^\alpha} - \frac{1}{v}\frac{\partial p_2(\xi,y,t)}{\partial t} d\xi$$

$$= 1 + \left[1 - 2x + \frac{x^2}{2} + \frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y + \left[1 + \frac{2}{v}x + \frac{x^2}{2!v^2} - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t$$

$$- \int_0^x \frac{\partial^\alpha \left[1 + \left[1 - 2\xi + \frac{\xi^2}{2} + \frac{\xi^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y + \left[1 + \frac{2}{v}\xi + \frac{\xi^2}{2!v^2} - \frac{\xi^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t\right]}{\partial \xi^\alpha}$$

$$+ \frac{\partial^\alpha \left[1 + \left[1 - 2\xi + \frac{\xi^2}{2} + \frac{\xi^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y + \left[1 + \frac{2}{v}\xi + \frac{\xi^2}{2!v^2} - \frac{\xi^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t\right]}{\partial y^\alpha}$$

$$- \frac{1}{v}\frac{\partial \left[1 + \left[1 - 2\xi + \frac{\xi^2}{2} + \frac{\xi^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y + \left[1 + \frac{2}{v}\xi + \frac{\xi^2}{2!v^2} - \frac{\xi^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t\right]}{\partial t} d\xi$$

$$p_3(x,y,t) =$$

$$= 1 + \left[1 - 2x + \frac{x^2}{2} + \frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y + \left[1 + \frac{2}{v}x + \frac{x^2}{2!v^2} - \frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t$$

$$- \int_0^x \left[\frac{-2\xi^{1-\alpha}}{\Gamma 2-\alpha} + \frac{\Gamma 3\xi^{2-\alpha}}{2!\Gamma 3-\alpha} + \frac{\Gamma 3-\alpha\xi^{2-2\alpha}}{\Gamma 3-\alpha\Gamma 3-2\alpha}\right]e^y + \left[\frac{2\xi^{1-\alpha}}{v\Gamma 2-\alpha} + \frac{\Gamma 3\xi^{2-\alpha}}{2v^2\Gamma 3-\alpha} - \frac{\Gamma 3\xi^{2-2\alpha}}{v\Gamma 3-\alpha\Gamma 3-2\alpha}\right]e^t$$

$$+ \left[1 - \xi + \frac{\xi^2}{2} + \frac{\xi^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y - \frac{1}{v}\left[1 + \frac{2}{v}\xi + \frac{\xi^2}{2v^2} - \frac{\xi^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t d\xi$$

$$p_3(x,y,t)=$$

$$=1+\left[1-2x+\frac{x^2}{2}+\frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y+\left[1+\frac{2}{v}x+\frac{x^2}{2!v^2}-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t$$

$$-\left[\begin{array}{l}\left[-\frac{2\xi^{2-\alpha}}{(2-\alpha)\Gamma 2-\alpha}+\frac{\xi^{3-\alpha}}{(3-\alpha)\Gamma 3-\alpha}+\frac{\xi^{3-2\alpha}}{(3-2\alpha\Gamma 3-2\alpha)}\right]e^y\\ +\left[\frac{2\xi^{2-\alpha}}{v(2-\alpha)\Gamma 2-\alpha}+\frac{\xi^{3-\alpha}}{v^2(3-\alpha)\Gamma 3-\alpha}-\frac{2\xi^{3-2\alpha}}{v(3-2\alpha)\Gamma 3-2\alpha}\right]e^t\\ +\left[\xi-\frac{\xi^2}{2!}+\frac{\xi^3}{3!}+\frac{\xi^{3-\alpha}}{(3-\alpha)\Gamma 3-\alpha}\right]e^y\\ -\frac{1}{v}\left[\xi+\frac{\xi^2}{v}+\frac{\xi^3}{3!v^2}-\frac{\xi^{3-\alpha}}{v(3-\alpha)\Gamma 3-\alpha}\right]e^t\end{array}\right]$$

$$p_3(x,y,t)=$$

$$=1+\left[1-2x+\frac{x^2}{2}+\frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]e^y+\left[1+\frac{2}{v}x+\frac{x^2}{2!v^2}-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]e^t$$

$$\left[\begin{array}{l}\left[\frac{2x^{2-\alpha}}{\Gamma 3-\alpha}-\frac{x^{3-\alpha}}{\Gamma 4-\alpha}-\frac{x^{3-2\alpha}}{\Gamma 4-2\alpha}\right]e^y\\ +\left[-\frac{2x^{2-\alpha}}{v\Gamma 3-\alpha}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}+\frac{2x^{3-2\alpha}}{v\Gamma 4-2\alpha}\right]e^t\\ +\left[-x+\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^{3-\alpha}}{\Gamma 4-\alpha}\right]e^y\\ +\left[\frac{x}{v}+\frac{x^2}{v^2}+\frac{x^3}{3!v^3}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}\right]e^t\end{array}\right]$$

$p_3(x,y,t)=$

$$=1+\left[1-3x+\frac{2x^2}{2!}-\frac{x^3}{3!}+\frac{3x^{2-\alpha}}{\Gamma 3-\alpha}-\frac{2x^{3-\alpha}}{\Gamma 4-\alpha}-\frac{x^{3-2\alpha}}{\Gamma 4-2\alpha}\right]e^y$$

$$+\left[1+\frac{3x}{1!v}+\frac{x^2}{2!v^2}+\frac{x^2}{v^2}+\frac{x^3}{3!v^3}-\frac{3x^{2-\alpha}}{v\Gamma 3-\alpha}-\frac{3x^{3-\alpha}}{v^2\Gamma 4-\alpha}+\frac{2x^{3-2\alpha}}{v\Gamma 4-2\alpha}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}\right]e^t$$

$$p_3(x,y,t)=1+\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\left[-2x+\frac{3x^{2-\alpha}}{\Gamma 3-\alpha}-\frac{x^{3-2\alpha}}{\Gamma 4-2\alpha}\right]+\left[x^2-\frac{2x^{3-\alpha}}{\Gamma 4-\alpha}\right]\right]e^y$$

$$+\left[1+\frac{x}{1!v}+\frac{x^2}{2!v^2}+\frac{x^3}{3!v^3}+\left[\frac{x}{v}-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]+\left[\frac{-2x^{2-\alpha}}{v\Gamma 3-\alpha}+\frac{2x^{3-2\alpha}}{v\Gamma 4-2\alpha}\right]+\left[\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}\right]\right]$$

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$$p_n(x,y,t)=1+\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\left[-x+\frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]+\left[-2x+\frac{3x^{2-\alpha}}{\Gamma 3-\alpha}-\frac{x^{3-2\alpha}}{\Gamma 4-\alpha}\right]+\left[x^2-\frac{2x^{3-\alpha}}{\Gamma 4-\alpha}\right]\right]e^y$$

$$+\left[1+\frac{x}{1!v}+\frac{x^2}{2!v^2}+\frac{x^3}{3!v^3}+\ldots+\frac{x^n}{n!v^n}+\left[\frac{x}{v}-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]+\left[\frac{-2x^{2-\alpha}}{v\Gamma 3-\alpha}+\frac{2x^{3-2\alpha}}{v\Gamma 4-2\alpha}\right]+\left[\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}\right]\right]e^t$$

$+\ldots$

The exact solution may obtained by using

$$p(x,y,t)=\lim_{n\to\infty} p_n(x,y,t)$$

Then the exact solution is

$$\lim_{n\to\infty} p_n(x,y,t)=1+\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\left[-x+\frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]+\left[-2x+\frac{3x^{2-\alpha}}{\Gamma 3-\alpha}-\frac{x^{3-2\alpha}}{\Gamma 4-\alpha}\right]+\left[x^2-\frac{2x^{3-\alpha}}{\Gamma 4-\alpha}\right]\right]e^y$$

$$+\left[1+\frac{x}{1!v}+\frac{x^2}{2!v^2}+\frac{x^3}{3!v^3}+\ldots+\frac{x^n}{n!v^n}+\left[\frac{x}{v}-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]+\left[\frac{-2x^{2-\alpha}}{v\Gamma 3-\alpha}+\frac{2x^{3-2\alpha}}{v\Gamma 4-2\alpha}\right]+\left[\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}\right]\right]e^t$$

$+\ldots$

$$p(x,y,t)=\lim_{n\to\infty} 1+\left[\left[-x+\frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]+\left[-2x+\frac{3x^{2-\alpha}}{\Gamma 3-\alpha}-\frac{x^{3-2\alpha}}{\Gamma 4-\alpha}\right]+\left[x^2-\frac{2x^{3-\alpha}}{\Gamma 4-\alpha}\right]+\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right]e^y$$

$$+\left[\left[\frac{x}{v}-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]+\left[\frac{-2x^{2-\alpha}}{v\Gamma 3-\alpha}+\frac{2x^{3-2\alpha}}{v\Gamma 4-2\alpha}\right]+\left[\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}\right]+\sum_{n=0}^{\infty}\frac{x^n}{n!v^n}\right]e^t$$

$$+\dots$$

$$p(x,y,t)=\left[\left[-x+\frac{x^{2-\alpha}}{\Gamma 3-\alpha}\right]+\left[-2x+\frac{3x^{2-\alpha}}{\Gamma 3-\alpha}-\frac{x^{3-2\alpha}}{\Gamma 4-\alpha}\right]+\left[x^2-\frac{2x^{3-\alpha}}{\Gamma 4-\alpha}\right]\right]e^y$$

$$+\left[\left[\frac{x}{v}-\frac{x^{2-\alpha}}{v\Gamma 3-\alpha}\right]+\left[\frac{-2x^{2-\alpha}}{v\Gamma 3-\alpha}+\frac{2x^{3-2\alpha}}{v\Gamma 4-2\alpha}\right]+\left[\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}-\frac{x^{3-\alpha}}{v^2\Gamma 4-\alpha}\right]\right]e^t$$

$$+\dots+1+e^{-x}e^y+e^t e^{\frac{x}{v}}$$

$$+\dots \qquad (4.47)$$

When $\alpha = 1$ then the exact solution is $p(x,y,t) = 1 + e^{y-x} + e^{t+\frac{x}{v}}$

Chapter Five

The application of Adomian Decomposition Method (ADM)

5.1Theorem 1:

The Riemann-Liouville fractional integration of polynomial function of $f(t) = t^n$ is defined as following [28];

$$J^{\alpha} f(t) = J^{\alpha}\left[t^n\right] = \frac{\Gamma[1+n]}{\Gamma 1+n+\alpha} t^{n+\alpha} \quad (5.1)$$

5.2Theorem 2:

The Riemann-Liouville fractional derivative of polynomial function of $f(t) = t^n$ is defined as following [28];

$$D^{\alpha} f(t) = D^{\alpha}\left[t^n\right] = \frac{\Gamma[1+n]}{\Gamma 1+n-\alpha} t^{n-\alpha} \quad (5.2)$$

Example 1.

Use Adomian Decomposition Method to solve the following homogeneous FPDE

$$\frac{\partial^{\alpha} u}{\partial y^{\alpha}} + x\frac{\partial u}{\partial x} = 3u, u(0,y) = 0, u(x,0) = x^2, \quad (5.3)$$

Solution.

In an operator form, Eq. (5.3), can be written as

$$L_y^{\alpha} u(x,y) = 3u(x,y) - xL_x u(x,y), \quad (5.4)$$

Where

$$L_x = \frac{\partial}{\partial x}, L_y^{\alpha} = \frac{\partial^{\alpha}}{\partial y^{\alpha}}, \quad (5.6)$$

We assume that the inverse of the operator exists as the $L_y^{-\alpha} = J_y^{\alpha}$

Applying the inverse operator J_y^{α} to both sides of (5.4), and using the given condition $u(x,0) = x^2$ yield

$$u(x,y) = x^2 + J_y^\alpha (3u(x,y) - xL_x u(x,y)). \quad (5.7)$$

As mentioned above, the decomposition method sets the solution $u(x,y)$ in an series form by

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (5.8)$$

Inserting (5.8) into both sides of the (5.7), we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = x^2 + J_y^\alpha \left(3\sum_{n=0}^{\infty} u_n(x,y) - xL_x \left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right). \quad (5.9)$$

Using few terms only for simplicity reasons, we obtain

$$u_0 + u_1 + u_2 + \ldots = x^2 + J_y^\alpha (3(u_0 + u_1 + u_2 + \ldots) - xL_x(u_0 + u_1 + u_2 + \ldots)), \quad (5.10)$$

Proceeding as before, we identify the zeroth component $u_0(x,y)$, by

$$u_0(x,y) = x^2, \quad (5.11)$$

Having identifies the zeroth component $u_0(x,y)$, we obtain the recursive scheme

$$u_0(x,y) = x^2, \quad (5.12)$$

$$u_{k+1}(x,y) = J_y^\alpha (3u_k - xL_x(u_k)), \quad k \geq 0,$$

The components $u_0, u_1, u_2, \ldots$ are thus determined as follows:

$$u_0(x,y)=x^2,\quad u_1(x,y)=J_y^{\alpha}\left(3u_0-xL_xu_0\right)=J_y^{\alpha}\left(3x^2-xL_x(x^2)\right)$$
$$=J_y^{\alpha}(3x^2-2x^2)=J_y^{\alpha}(x^2)=\frac{x^2y^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,y)=J_y^{\alpha}\left(3u_1-xL_xu_1\right)=J_y^{\alpha}\left(\frac{3x^2y^{\alpha}}{\Gamma(\alpha+1)}-xL_x\left(\frac{x^2y^{\alpha}}{\Gamma(\alpha+1)}\right)\right)$$
$$=J_y^{\alpha}\left(\frac{3x^2y^{\alpha}}{\Gamma(\alpha+1)}-\frac{2x^2y^{\alpha}}{\Gamma(\alpha+1)}\right)=\frac{3x^2y^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}$$
$$u_2=\frac{x^2y^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3(x,y)=J_y^{\alpha}\left(3u_2-xL_xu_2\right)=J_y^{\alpha}\left(\frac{3x^2y^{2\alpha}}{\Gamma(2\alpha+1)}-xL_x\left(\frac{x^2y^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right)$$
$$=J_y^{\alpha}\left(\frac{3x^2y^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2x^2y^{2\alpha}}{\Gamma(2\alpha+1)}\right)=J_y^{\alpha}\left(\frac{x^2y^{2\alpha}}{\Gamma(2\alpha+1)}\right)$$
$$u_3=\frac{x^2y^{3\alpha}}{\Gamma(2\alpha+2)},$$

It is obvious that all components $u_k=0, k\geq 1$ .Consequently, the solution is given by

$$u(x,y)=u_0+u_1+u_3\ldots$$

$$u(x,y)=x^2+\left(\frac{y^{\alpha}}{\Gamma(\alpha+1)}x^2+\frac{y^{2\alpha}}{\Gamma(2\alpha+1)}x^2+\frac{y^{3\alpha}}{\Gamma(3\alpha+1)}x^2+\ldots\right)$$
$$u(x,y)=x^2\left(1+\frac{y^{\alpha}}{\Gamma(\alpha+1)}+\frac{y^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{y^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right),\qquad(5.13)$$

The exact solution obtained by $\alpha=1$.

$$u(x,y) = x^2\left(1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots\right) \quad (5.14)$$

$$u(x,y) = x^2 e^x,$$

5.3 A fractional Model Of Fluid Flow Through Porous Media In TwoDimension

If seepage flow is considered as rigid body motion the continuity equation can be written as follows in one dimension

$$(k\frac{\partial^{\alpha_1} p}{\partial x^{\alpha}}) - \frac{1}{v}\frac{\partial p}{\partial t} = 0, \quad (5.15)$$

$x \in \Omega$

5.4 Solution of a fractional model of fluid flow through porous media in Two dimension by(ADM)

Use Adomian Decomposition Method to solve the following homogeneous FPDE

$$\frac{\partial^{\alpha} p}{\partial x^{\alpha}} = \frac{1}{v}\frac{\partial p}{\partial t}, \; p_0(0,t) = t, \; p(x,0) = \frac{x}{v}, \quad (5.16)$$

Solution.

In an operator form, Eq. (5.16), can be written as

$$L_x^{\alpha} p(x,t) = \frac{1}{v} L_t p(x,t), \quad (5.17)$$

Where

$$L_t = \frac{\partial}{\partial t}, L_x^{\alpha} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}, \quad (5.18)$$

We assume that the inverse of the operator exists as the $L_x^{-\alpha} = J_x^{\alpha}$

Applying the inverse operator J_x^{α} to both sides of (5.17), and using the given condition $p(0,t) = t$ yield

$$p(x,t) = p(0,t) + J_x^{\alpha}\left(\frac{1}{v}L_t(p(x,t))\right).$$

$$p(x,t) = t + J_x^{\alpha}\left(\frac{1}{v}L_t(p(x,t))\right) \qquad (5.19)$$

As mentioned above, the decomposition method sets the solution $u(x, y)$ in an series form by

$$p(x,t) = \sum_{n=0}^{\infty} p_n(x,t) \quad (5.20)$$

Inserting (5.20) into both sides of the (5.19), we obtain

$$\sum_{n=0}^{\infty} p_n(x,t) = t + J_x^{\alpha}\left(\frac{1}{v}L_t(p_n(x,t))\right). \quad (5.21)$$

Using few terms only for simplicity reasons, we obtain

$$p_0 + p_1 + p_2 + ... = t + J_x^{\alpha}(\frac{1}{v}L_t(p_0 + p_1 + p_2 + ...)), \quad (5.22)$$

Proceeding as before, we identify the zeroth component $p_0(x,t)$, by

$$p_0(x,t) = t, \quad (5.23)$$

Having identifies the zeroth component $p_0(x,t)$, we obtain the recursive scheme

$$p_0(x,t) = t, \quad (5.24)$$

$$p_{k+1}(x,t) = J_x^{\alpha}\left(\frac{1}{v}L_t(p_k(x,t))\right), k \geq 0,$$

The components $p_0, p_1, p_2, ...$ are thus determined as follows:

$$p_0(x,t) = t, \quad p_1(x,t) = J_x^{\alpha}\left(\frac{1}{v}L_t(p_0)\right) = J_x^{\alpha}\left(\frac{1}{v}L_t(t)\right)$$

$$p_1(x,t) = \frac{x^{\alpha}}{v\Gamma(1+\alpha)},$$

$$p_2(x,t) = J_x^{\alpha}\left(\frac{1}{v}L_t(p_1)\right) = J_x^{\alpha}\left(\frac{1}{v}L_t\left(\frac{x^{\alpha}}{v\Gamma(1+\alpha)}\right)\right) = 0$$

$$p_2(x,t) = 0,$$

We can easily observe that $p_k = 0, k \geq 2$ .It follows that the solution in a closed form is given by

$$p(x,t) = p_0 + p_1$$

$$p(x,t) = \left(t + \frac{x^{\alpha}}{v\Gamma(\alpha+1)}\right), \quad (5.25)$$

The exact solution obtained by $\alpha = 1$.

$$p(x,t) = \left(t + \frac{1}{v}x\right),$$

5.5 Solution Of Fractional Model Of Fluid Flow Through Porous Media In Three Dimension By (ADM)

If seepage flow is considered as rigid body motion the continuity equation can be written as follows in one dimension

$$\left(k_x \frac{\partial^{\alpha_1} p(x,y,t)}{\partial x^{\alpha_1}}\right) + \left(k_y \frac{\partial^{\alpha_2} p(x,y,t)}{\partial y^{\alpha_2}}\right) - \frac{1}{v}\frac{\partial p(x,y,t)}{\partial t} = 0, \quad (5.26)$$

If $\alpha_1 = \alpha_2 = \alpha$ and $k_x = k_y = k = 1$ then we have

$$\frac{\partial^{\alpha} p(x,y,t)}{\partial x^{\alpha}}+\frac{\partial^{\alpha} p(x,y,t)}{\partial y^{\alpha}}-\frac{1}{v}\frac{\partial p(x,y,t)}{\partial t}=0, \qquad (5.27)$$

$(x, y, z) \in \Omega$

Solution of FPDE by(ADM):

$$\frac{\partial^{\alpha} p(x,y,t)}{\partial x^{\alpha}}+\frac{\partial^{\alpha} p(x,y,t)}{\partial y^{\alpha}}-\frac{1}{v}\frac{\partial p(x,y,t)}{\partial t}=0$$

$$p(0,y,,t)=1+e^{y}+e^{t}$$

$$p(x,0,t)=1+e^{x\left(\frac{1}{v}-1\right)}+e^{t} \qquad (5.28)$$

$$p(x,y,0)=1+e^{x\left(\frac{1}{v}-1\right)}+e^{y}$$

$$p_0(x,y,t)=1+e^{y}+e^{z}$$

Solution:

$$L_x^{\alpha} p(x,y,t)=\frac{1}{v}L_t(p(x,y,t)-L_y^{\alpha}(p(x,y,t)), \quad (5.29)$$

Where

$$L_t=\frac{\partial}{\partial t}, L_x^{\alpha}=\frac{\partial^{\alpha}}{\partial x^{\alpha}}, L_y^{\alpha}=\frac{\partial^{\alpha}}{\partial y^{\alpha}} \quad (5.30)$$

We assume that the inverse of the operator exists as the $L_x^{-\alpha}=J_x^{\alpha}$

Applying the inverse operator J_x^{α} to both sides of (5.29), and using the given condition $p_0(x,y,t)=1+e^{y}+e^{t}$ yield

$$p(x,y,t)=p_0(x,y,t)+J_x^{\alpha}\left(\frac{1}{v}L_t(p(x,y,t))-L_y^{\alpha}(p(x,y,t))\right).$$
$$p(x,y,t)=1+e^{y}+e^{t}+J_x^{\alpha}\left(\frac{1}{v}L_t(p(x,y,t))-L_y^{\alpha}(p(x,y,t))\right) \quad (5.31)$$

As mentioned above, the decomposition method sets the solution $p(x,y,t)$ in an series form by

$$p(x,y,t)=\sum_{n=0}^{\infty} p_n(x,y,t) \quad (5.32)$$

Inserting (5.32) into both sides of the (5.31), we obtain

$$\sum_{n=0}^{\infty} p_n(x,y,t)=1+e^y+e^t+$$
$$J_x^{\alpha}\left(\frac{1}{v}L_t\left(\sum_{n=0}^{\infty} p_n(x,y,t)\right)-L_y^{\alpha}\left(\sum_{n=0}^{\infty} p_n(x,y,t)\right)\right). \quad (5.33)$$

Using few terms only for simplicity reasons, we obtain

$$p_0+p_1+p_2+\ldots=1+e^y+e^t$$
$$+J_x^{\alpha}\left(\frac{1}{v}L_t(p_0+p_1+p_2+\ldots)-L_y^{\alpha}(p_0+p_1+p_2+\ldots)\right), \quad (5.34)$$

Proceeding as before, we identify the zeroth component $p_0(x,y,t)$, by

$$p_0(x,y,t)=1+e^y+e^t, \quad (5.35)$$

Having identifies the zeroth component $p_0(x,y,t)$, we obtain the recursive scheme

$$p_0(x,y,t)=1+e^y+e^t, \quad (5.36)$$
$$p_{k+1}(x,y,t)=$$
$$J_x^{\alpha}\left(\frac{1}{v}L_t(p_k(x,y,t))-L_y^{\alpha}(p(x,y,t))\right), k\geq 0,$$

The components $p_0, p_1, p_2, \ldots$ are thus determined as follows:

$$p_0(x,y,t) = 1 + e^y + e^t,$$

$$p_1(x,y,t) = J_x^\alpha \left(\frac{1}{v} L_t(p_0(x,y,t)) - L_y^\alpha(p_0(x,y,t)) \right)$$

$$= J_x^\alpha \left(\frac{1}{v} L_t(1 + e^y + e^t) - L_y^\alpha(1 + e^y + e^t) \right)$$

$$= J_x^\alpha \left(\frac{1}{v}(e^t) - e^y \right) = \frac{x^\alpha e^t}{v\Gamma(\alpha+1)} - \frac{x^\alpha e^y}{\Gamma(\alpha+1)}$$

$$p_1(x,t) = x^\alpha \left(\frac{1}{v\Gamma(\alpha+1)} e^t - \frac{1}{\Gamma(\alpha+1)} e^y \right),$$

$$p_2(x,y,t) = J_x^\alpha \left(\frac{1}{v} L_t(p_1(x,y,t)) - L_y^\alpha(p_1(x,y,t)) \right)$$

$$= J_x^\alpha \left(\begin{array}{l} \frac{1}{v} L_t\left(x^\alpha \left(\frac{1}{v\Gamma(\alpha+1)} e^t - \frac{1}{\Gamma(\alpha+1)} e^y \right) \right) - \\ L_y^\alpha \left(x^\alpha \left(\frac{1}{v\Gamma(\alpha+1)} e^t - \frac{1}{\Gamma(\alpha+1)} e^y \right) \right) \end{array} \right)$$

$$= J_x^\alpha \left(\frac{1}{v} \left(\frac{x^\alpha}{v\Gamma(\alpha+1)} e^t \right) - \left(- \frac{x^\alpha}{\Gamma(\alpha+1)} e^y \right) \right)$$

$$= \left(\frac{1}{v} \left(\frac{x^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+)\Gamma(2\alpha+1)} e^t \right) + \left(\frac{x^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} e^y \right) \right)$$

$$p_2(x,y,t) = x^{2\alpha} \left(\frac{1}{v^2\Gamma(2\alpha+1)} e^t + \frac{1}{\Gamma(2\alpha+1)} e^y \right),$$

$$p_3(x,y,t) = J_x^\alpha\left(\frac{1}{v}L_t(p_2(x,y,t)) - L_y^\alpha(p_2(x,y,t))\right)$$

$$= J_x^\alpha\left(\begin{array}{l}\frac{1}{v}L_t\left(x^{2\alpha}\left(\frac{1}{v^2\Gamma(2\alpha+1)}e^t + \frac{1}{\Gamma(2\alpha+1)}e^y\right)\right) - \\ L_y^\alpha\left(x^{2\alpha}\left(\frac{1}{v^2\Gamma(2\alpha+1)}e^t + \frac{1}{\Gamma(2\alpha+1)}e^y\right)\right)\end{array}\right)$$

$$= J_x^\alpha\left(\frac{1}{v}\left(\frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)}e^t\right) - \left(\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^y\right)\right)$$

$$= \left(\frac{1}{v}\left(\frac{x^{3\alpha}\Gamma(\alpha+1)}{v^2\Gamma(\alpha+)\Gamma(2\alpha+1)}e^t\right) - \left(\frac{x^{3\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}e^y\right)\right)$$

$$= \left(\frac{x^{3\alpha}}{v^3\Gamma(2\alpha+2)}e^t - \left(\frac{x^{3\alpha}}{\Gamma(2\alpha+2)}e^y\right)\right)$$

$$p_3(x,y,t) = x^{3\alpha}\left(\frac{1}{v^3\Gamma(2\alpha+2)}e^t - \frac{1}{\Gamma(2\alpha+2)}e^y\right),$$

It is obvious that all components, $p_k = 0, k \geq 1$ .Consequently, the solution is given by

$$p(x,y,t) = p_0 + p_1 + p_2 \ldots$$

$$p(x,y,t) = p_0 + p_1 + p_2 + \ldots$$

$$p(x,y,t) = 1 + e^y + e^t + \left(\frac{x^\alpha}{v\Gamma(\alpha+1)}e^t - \frac{x^\alpha}{\Gamma(\alpha+1)}e^y\right) +$$

$$\left(\frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)}e^t + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^y\right) + \left(\frac{x^{3\alpha}}{v^3\Gamma(2\alpha+2)}e^t - \frac{x^{3\alpha}}{\Gamma(2\alpha+2)}e^y\right),$$

$$p(x,y,t) = 1 + \left[1 - \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(2\alpha+2)} + \ldots\right]e^y$$

$$+ \left[1 + \frac{x^\alpha}{v\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)} + \frac{x^{3\alpha}}{v^3\Gamma(2\alpha+2)} + \ldots\right]e^t$$

The exact solution obtained by $\alpha = 1$.

$$p(x,y,t) = 1 + \left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right]e^y + \left[1 + \frac{x}{v1!} + \frac{x^2}{v^2 2!} + \frac{x^3}{v^3 3!} + \dots\right]e^t$$

$$p(x,y,t) = 1 + \left[e^{-x}e^y\right] + \left[e^{\frac{x}{v}}e^t\right] = 1 + e^{y-x} + e^{t+\frac{x}{v}}$$

conclusion

The fundamental goal of this work has been to construct an approximate solution of seepage flow derivatives in porous media. The goal has been achieved by using the (ADM) and (VIM). The methods was used in a direct way without using linearization, perturbation or restrictive assumptions. Comparing this method with others, we consider this method to be more effective.

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