Sudan University of Science and Technology

College of Graduate Studies

Contraction with Defect Operators and Derivatives of Fractals with Norm of Hilbert Matrix on Bergman and Hardy Spaces

الانكماشات مع مؤثرات الإنحراف وإشتقاقات الغائمات مع نظيم مصفوفة هيلبرت على فضاءات بيرجمان وهاردي

A Thesis Submitted in Fulfillment Requirements for the Degree of Ph. D in Mathematics

By

Ehssan Mohammed Ahmed Abd Elrahim

Supervisor

Prof Dr: Shawgy Hussein Abd Alla

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بسم الله الرحمن الرحيم

Sudan University of Science & Technology

College of Graduate Studies

جامعة السودان للعلوم والتكنولوجيا

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Approval Page

Name of Candidate: Ehssan Mohammed Ahmed Abd Elrahim

Thesis title: Contraction with Defect Operators and Derivatives of Fractals with norm of Hilbert Matrix on Bergman and Hardy spaces

Approved by:

1. External Examiner

Name: Othman Mohamed EL Mekki

Signature: .... Date: 16/6/2015

2. Internal Examiner

Name: Mohd Arafa Ahmed Elkhatieb

Signature: Mohd Arafa Ahmed Elkhatieb Date: 16-6-2015

3. Supervisor

Name: Prof. Dr. Shawgy Hussein Abd Alla

Signature: .... Date: 16/6/2015

cgs @ sustech edu. البريد الالكتروني فاكس/ 83 769363 ص.ب 407

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College of Graduate Studies

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Candidate's name: Enssan Mohammed Ahmed Abd Elrahim

Candidate's signature: [signature] Date: 30/10/2015

إقرار

أنا الموقع أدناه أقر بأنني المؤلف الوحيد لرسالة الدكتوراه المعنونة الانكماشات مع مؤثرات الانحراف واشتقاقات الفاركتلات مع نظيم مصفوفة هيلبرت على فضاءات بيرجمان وهاردي

وهى منتج فكري أصيل . وباختياري أعطى حقوق طبع ونشر هذا العمل لكلية الدراسات العليا جامعه السودان للعلوم والتكنولوجيا ،عليه يحق للجامعه نشر هذا العمل للأغراض العلمية .

اسم الدارس : إحسان محمد أحمد عبد الرحيم

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Dedication

To my parents

husband

Brothers and sisters.

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Abstract

The Q-function of quasi-selfadjoint contractions extension of closed symmetric contractions are considered. We show the pure point spectrum of the Laplacians on Fractal graphs and what is not in the domain of the Laplacian on Sierpinski gasket type fractals. We study the m-function and some inverse problems and spectral analysis for finite and semi-infinite Jacobi matrices.The completely characterizations of nonunitary contractions with rank one defects operators and corresponding unitary colligations with truncated CMV matrices are discussed. The harmonic coordinates on Fractals with finitely ramified cell structure with the products of random matrices and methods of derivatives on p.c.f are established. Composition operators and Hilbert matrix on Bergman spaces are investigated. We give the norm of the Hilbert matrix on Bergman and Hardy spaces with a theorem of Nehari type.

الخلاصة

اعتبرنا الدالة – Q لتمديد انكماشات شبه – المرافق الذاتي للانكماشـــات المتماثلـــة المغلقة ، أوضحنا طيف النقطة البحث للابلسيان عن البيانات الغائمة وماهو ليس في مجال اللابلسيان عن غائمات نوع طوق سيربنسكي . درسنا الدوال – m وبعـــض مسائل الانعكاس والتحليل الطيفي لمصفوفات الجاكوبي المنتهية وشبه – اللانهائية .

تمت دراسة التشخيصات التامة للانكماشات غير الواحدية مع مؤثرات عيوب الرتبة الواحدة والضمامات الواحدية المقابلة مع اقتطـــاع مصـــفوفات CMV تـــم تأســـيس الاحداثيات التوافقية على الغائمات مع تشييد الخلايا المتشبعه المنتهية مـــع ضـــرب المصفوفات العشوائية وطرق الاشتقاقات علـــي p.c.f. ناقشـــنا مـــؤثرات الترتيـــب ومصفوفة هيلبرت على فضاءات بيرجمان . تم أعطاء نظيم مصفوفة هيلبرت علـــي فضاءات بيرجمان وهاردي مع مبرهنة نوع نيهاري .

Introduction:

A bounded everywhere defined operator T in a Hilbert space $\mathfrak{H}$ is said to be a quasi-selfadjoint contraction or (for short) a qsc-operator, if T is a contraction and ker $(T - T^*) \neq \{0\}$. For a closed linear subspace $\mathfrak{N}$ of $\mathfrak{H}$ containing ran $(T - T^*)$ the operator-valued function $Q_T(z) = P_{\mathfrak{N}} (T - zI)^{-1} \upharpoonright \mathfrak{N}$, $|z| > 1$, where $P_{\mathfrak{N}}$ is the orthogonal projector from $\mathfrak{H}$ onto $\mathfrak{N}$, is said to be a *Q-function* of T acting on the subspace $\mathfrak{N}$.

We establish the pure point spectrum of the Laplacians on two point self-similar fractal graphs . We consider the analog of the Laplacian on the Sierpinski gasket and related fractals, constructed by Kigami. A function f is said to belong to the domain of Δ if f is continuous and Δf is defined as continuous function. We show that if f is a nonconstant function in the domain of Δ, then f^2 is not in the domain of Δ. We give two proof of this fact. *The first is based on the analog of the pointwise identity* $\Delta f^2 - 2f\,\Delta f = |\nabla f|^2$, where we show that $|\nabla f|^2$ does not exist as a continuous function. We study inverse spectral analysis for finite and semi-infinite Jacobi matrices H. Our results include a new proof of the central result of the inverse theory (that the spectral measure determines H).We use a relation between products of matrices on $M^2(\mathbb{R}[x])$and Jacobi matrices to study some inverse problems on Jacobi matrices, including uniqueness and existence theorems.

The new models for completely nonunitary countractions with rank one defect operators acting on some Hilbert space of dimension $N \leq \infty$. These models complement nicyely the well known models of Livsic and Sz.-Nagy-Foias. We show that each such operator actin on some finite-midmensional (respectively, separable infinite-dimensional Hilbert space- is unitarily equivalent to some finite (respectively semi-infinite- truncated MCV matrix obtained from the "full" CMV matrix by deleting the first row and the first column and acting in $\mathbb{C}^N$ (respectively $\ell^2(\mathbb{N})$). This result can be viewd as a nonunitary version of the famous characterization of unitary operators with a simple spectrum due to Cantero, Moral and Velazques, as well as an analog for contraction operators .

We define sets with finitely reamified cell structure, which are generalizations of P.c.f. self- similar sets introduced by Kigami and of fractafolds introduced by Strichartz. In general, we do not assume even local self-similarity, and allow countably many cells connected at each junction point. In particular, we consider post –critically infinite fractals.

We define and study intrinsic first oreder derivatives on post critically finite fractals and show differentiability almost everywhere with respect to self-similar measures for certain classes of fractals and functions.

We find an upper bound for the norm of the induced operator. The Hilbert matrix induces a bounded operator on most Hardy and Bergman spaces, as was shown by Diamantopulos and SiSKaKis. We generalize this for any Hankel operator on Hardy spaces by using a result of Hollenbeck and Yerbitsky on the Riesz projection.

The Contents

Chapter 1

Quasi-selfadjoint Contractions Extension and Q-functions

The main properties of such Q-functions are studied, in particular the underlying operator-theoretical aspects are considered by using some block representations of the contraction T and analytical characterizations for such functions $Q_T(z)$ are established. Also a reproducing kernel space model for $Q_T(z)$ is constructed. In the special case where T is selfadjoint $Q_T(z)$ coincides with the Q-function of the symmetric operator $A \coloneqq T\restriction (\mathfrak{H} \ominus \mathfrak{N})$ and its selfadjoint extension $T = T^*$ in the usual sense.

Sec(1.1)Closed Symmetric Contractions:-

The concept of a Q-function was introduced by M.G. Krein for the case of adensely symmetric operator S in a Hilbert space $\mathfrak{H}$ with equal defect number by means of a selfadjoint extension A of S, cf. [27],[28],[34], and also [39],[31],[32].. Such a function belong to the class N of Nevanlinna (or Herglotz-Nevanlinna) function , i.e, $Q(z) \in N$ if it is holomorphic in the open upper and lower half – planes and satisfies the condition $Q(\tilde{z}) = Q(z)^* \, and \, (\operatorname{Im} Q(z)) \geq 0$, $z \in \mathbb{C}_+ \cup \mathbb{C}_-$, the Q- function plays an essential role in Krein's resolvent formula, whih describes all (generalized resolvent of) selfadjoint extensions of S.ln fsct,all generalized resolvents

(canonical as well as exit space) were first described independently by . M . A. Naimark [42] and M .G. Krein[27]; see slso [31] for further historical remarks. A characteristic property of a Q-function Q (z) in the class of Nevanlinna functions is that lm Q (z) is invertible (at some or equivalently at every point $z \in \mathbb{C}_+ \cup \mathbb{C}_-$): every Nevanlinna function with this propently is a Q –function of some simple symmetric operator S and a selfadjoint extension A of S in a Hilbert space $\hbar$. Moreover , the simple (completely non- selfadjoint) symmetric. operator S and its selfadjoint extension A are essentially unique in the sense that the Q- function of S determines S and A uniquely up to unitary equivalence . A nother approach for describing selfadjoint as well as non-selfadjoint intermediate extensions of a symmetric operator is via a boundary value spase and the corresponding Weyl function ,see[22],[20],[19] .

Two specil subclasses of Q-functions, consisting of the so-called Q_μ- and Q_M –functions , which belong to the class N of Nevanlinna functions were defined and investigated by M.G Krein and I.E Ovcharenko in[33].Here the underlying symmetric operator is a non – densely defined contraction .In a recent section[8] by contains also some extension of Q_μ - and Q_M – functions were introduced; in fact ,this section contains also some corrections to the result stated in [33]. some other type of Q-function associated to a non – densely defined symmetric contraction has been considered in[48], including the resolvent formulas for the selfadjoint (canonical and exit space) extensione.

In this section a class of operator – valued Q – function associated with a non – densely defined symmetric contraction A and its, in general , non – selfadjoint contractive extensive T is introduced . By definition abounded operator T in the Hilpert space $\mathfrak{H}$ is a quasi – selfadjoint contraction or , for short , a qsc-operator if dom T = $\mathfrak{H}$, $\|T\| \leq 1$ and ker (T – T*) $\neq$ {0}. Let T be a qsc-operator-valued function Q (z) as follows

$$Q(z) = P_{\mathfrak{N}}(T - zI)^{-1} \upharpoonright \mathfrak{N}, \quad |z| < 1. \tag{1}$$

In what follows the function Q in(1) will be called aQ-function of T with respect to the subspace $\mathfrak{N} \subset \mathfrak{H}$.observe , that if T is selfadjoint then the function Q defined by (1) is an ordinary Q-function associated with T and the symmetric restriction A : = $T \upharpoonright \mathfrak{H}_0$ of T , where $\mathfrak{H}_O = \mathfrak{H} \ominus \mathfrak{N}$ =. However , if T is not selfad-joint this function in general is not a Nevanlinna function. A qsc-operator T may be considered as a contractive, in general, non-selfadjoint extension of the symmetric contraction= $T \upharpoonright \mathfrak{H}_0$ which is also called a quasi-selfadjoint contractive extension of A; here A is symmetric due to $\mathfrak{H}_0 \subset$ ker (T – T*). Such kind of extension were parametrized and investigated by M .G . krein [28]and by M . G . krein and I . E . Ovcharenko [33] . In particular , in[33] two special Q-functions of the Nevanlinna class for the symmertric contraction were defined and studied and the resolvent formulas for selfadjoint contractive extensions (sc-extensions) were established. These formulas were extended in[11]and[13] for qsc-extensions . Aboundary value space approach for describing extensions of dual paris of densely defined operators appeaers in[38] and for dual pairs of linear relations and their canonical and generalized resolvent in[40] [41] see also In[35] the approach can be seen as a non – selfadjoint counterpart of the Q- function approach developed and systematically used in the papers of M.G. Krein and H . Langer , cf., e.g, [39]-[32]

The contents of this Section will be briefly described. In some preIiminary notions are introduced. The extension theory for closed symmetric contractions is developed . This includes a discussion of minimality of the underlying symmetric operator A and its contractive extensions. The Q-functions for intermediate contractive extensions as in(1) are introduced. where also a number of associated nonnegative kernels will appear . A resolvent formula for qsc-extensions of a symmetric contraction A is derived . It involoves a Q- function of the form(1) for a given qsc-extension T of A. a model for such Q-functions is constructed by means of a qsc-operator acting in a reproducing kernel Hilbert space and it is proved that two $\mathfrak{N}$ -minimal qsc-operators whose Q-functions in (1) coincide are unitary unitarily equivalent. This model is used to establish some characteristic properties of Q-functions of qsc-operators.linear fractional transformations of Q-functions are considered . The results can be connected with and augmented by the study of a certain class of passive systems. In particular, the Q-functions of quasi-selfadjoint operators investigated in the present section are in one-to-one correspondence with the transfer functions of so-called passive quasi-selfadjoint systems, which are introduced and investigated in[9] .

The class of all continuous linear operators defined on a complex Hilbert space $\mathfrak{H}_1$ and taking values in a complex Hilbert spase $\mathfrak{H}_2$is denoted by L($\mathfrak{H}_{1,}$ $\mathfrak{H}_2$) and L$(\mathfrak{H})$:= L $(\mathfrak{H},\mathfrak{H})$.The domain, the range,and the null-space of a linear operator Tare denoted by dom T , ran T,and ker T. For T $\in$L$(\mathfrak{H})$ the operators T_R= (T + T*)/2, T_I = (T – T*)/2i are said to be the real and the imaginary part of T. For a contraction T $\in$ L($\mathfrak{H}_{1,}$ $\mathfrak{H}_2$) the defect operator D_T of T is defined by .

$$D_T\colon (1 - T^*T)^{1/2} \tag{2}$$

It is a nonnegative contraction and satisfies the well-known commutation relation

$$TD_T = D_{T^*}T, \tag{3}$$

Cf. [47]. The closure of the range ran D_T is denoted by $\mathfrak{D}_T$ and $\rho(T)$stands for the set of all regular points of a closed operators T .$if\, R_l and R_r are$ two nonnegative operators in $L(\mathfrak{H})$ and $S_0 \in$ $L(\mathfrak{H})$ then the symbol B(S_0, R_ι, R_r) denotes the operator ball in $L(\mathfrak{H})$ wih the center S_0 and the left and right radii R_1 and R_r respectiveiy ,i.e.,the set of all operators in L$(\mathfrak{H})$ of the form $T = S_0 + R_I^{1/2} \mathrm{X} R_r^{1/2}$, where X is a contrction from ran R_r into ran R_r It is well known , see [44], [45]. That a necessary and sufficient condition for T$\in$ L $(\mathfrak{H})$ to B(S_0, R_ι, R_r) is the following :

$$\left|\left((T - S_0)f, g\right)\right|^2 \le (R_r f, f)(R_l g, g) \quad \text{for all } f, g \in \mathfrak{H}. \tag{4}$$

If $R_\iota = R_r = R$ the corresponding operator ball is denoted by $B(S_{0,}R)$.

Recall that T $\in L(\mathfrak{H})$ is a quasi-selfadjoint contraction (a qsc-operator) if

$$\text{dom T} = \mathfrak{H}, \quad \|T\| \le 1, \text{ and ker (T – T*) } \ne \{0\}.$$

A qsc-operator T is said to be a quasi-selfadjoint contractive extension or qsc-extnsion ot a closed symmetric contraction A if

$$\text{Dom A} \subset \text{ker (T- T*) or equivalently ran (T –T*)} \subset \text{(dom A)}^{\perp} ,$$

Cf [11],[13] .Clearly ,an operator T$\in L(\mathfrak{H})$is a qsc-extension of A if and only if

$$A \subset T \ and \ \ A \subset T^*$$

or, equivalently, if T is an intermediate extension of A. A qsc-operator T has always symmetric restrictions A for which T is a qsc-extension . Namely, with a subspace $\mathfrak{N} \supset ran(T - T^*)$ define

$$\text{Dom A} = \mathfrak{H} \ominus \mathfrak{N}, \quad \text{A}=T \restriction domA.$$

Then dom A $\subset$ ker (T – T*) .A qsc-operator T is called completely non-selfadjoint if there is no non-zero invariant subspace on which the resteiction of T is selfadjoint .

Lemma(1.1.1)[1]:[16] A qsc-operator T is completely non-selfadjoint if and only if

$$\overline{span} \ \{ranT^n(T - T^*) : n = 0,1,... \} = \mathfrak{H}.$$

Let $\alpha \in [0, \pi/2)$ and denote by $S(\alpha)$the following sector of the complex plane:

$$S(\alpha) = \{z \in \mathbb{C}: |arg\ z| \le \alpha\}.$$

A Linear operator S, in general unbounded, in a Hilbert space $\mathfrak{H}$ is said to be sectorial with vertex at the origin and semiangle α , if its numerical range

$$W(S) = \{(Sf, f) : \|f\| = 1, f \in \text{domS} \}$$

is contained in the sector $S(\alpha)$, cf. This condition is equivalent to

$$|\text{Im}(S\ f,\ f\ )\ | \le (\tan\alpha)\,\text{Re}(Sf, f) \quad \text{for all } f \in \text{dom S}.$$

If the resolvent set of S is not empty then S is called maximal sectorial.

A bounded operator T on a Hilbert spase $\mathfrak{H}$ is said to belong to the class C (α), $\alpha \in (0, \pi/2)$, if

$$\left\| T\sin\alpha \pm i\cos\alpha I \right\| \le 1, \qquad (5)$$

Cf.[4].Clearly, T belongs to C(α)if and only if T* belongs to C(α).Moreover, it follows from(5) that the operators belonging to C$(\alpha)$are contractive. The condition(5) is equivalent to each of following two conditions:

$$\left|(T_I f, f)\right| \le \frac{\tan\alpha}{2}\left\|D_T f\right\|^2 \text{ for all } f \in \mathfrak{H}: \qquad (6)$$

or

the operator $(I - T^*)(I + T)$ is sectorial with (7)

vertex at the origin and semiangle α,

Cf[5]. Note that the linear fractional transformation T = (I-S) (I+S)$^{-1}$ of a maximal sectorial operator S with vertex at the origin and semiangle α is an operator of the class C (α). Let

$$\tilde{C} = \cup\{C(\alpha) : \alpha \in [0, \pi/2)\}.$$

Some properties of the operators in the class $\tilde{C}$ were studied in[4].[5]. In particular, in[4] ,it was proved that T$\in$ $\tilde{C}$ implies that

$$ranD_{T^n} = ranD_{T^{*n}} = ranD_{T_R} \quad , n = 1,2,\ldots,$$

where T_R is the real part of T. Furthermore it was proved in [4] that the subspace $\mathfrak{D}_T$ reduces the operator T, that the operator $T \upharpoonright \mathfrak{D}_T$ ker D_T is selfadjoint and unitary, and that $T \upharpoonright \mathfrak{D}_T$ is a completely non-unitary contraction of the class C_{00}, i.e.,

$$\lim_{n\to\infty} \mathrm{T}^n f = \lim_{n\to\infty} \mathrm{T}^{*n} f = 0 \quad \text{for all} \quad f \in \mathfrak{D}_T,$$

Let the Hilbert space $\mathfrak{H}$ be decomposed as $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$and decompose $T \in L(\mathfrak{H})$ accordingly:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad T_{ij} \in L\,(\mathfrak{H}_i, \mathfrak{H}_j)\ . \qquad (8)$$

Define the operator-valued functions

$$V_T(z) = T_{21}(T_{11} - zI)^{-1}T_{12} - T_{22,} \qquad W_T(z) = -zI - V_T(z), \qquad z \in p(T_{11}). \tag{9}$$

By the Schur-Frobenius formula the resolvent $(T - z)^{-1}$ of T can be rewritten the block form

$$\begin{pmatrix} (T_{11} - zI)^{-1}(I + T_{12}W_T(z)^{-1}T_{21}(T_{11} - zI)^{-1}) & -(T_{11} - zI)^{-1}T_{12}W_T^{-1}(z) \\ -W_T^{-1}(z)T_{21}(T_{11} - zI)^{-1} & W_T^{-1}(z) \end{pmatrix}. \tag{10}$$

for $z \in \rho(T) \cap \rho(T_{11})$. In particular ,

$$P_{\mathfrak{H}_2}(T - zI)^{-1} \upharpoonright \mathfrak{H}_2 = -(V_T(z) + zI)^{-1}, \qquad z \in \rho(T) \cap \rho(T_{11}). \tag{11}$$

Let $\mathfrak{N}$ be a Hilbert space . An operator-valued $V(z), z \in C \setminus R,$ with values in $L(\mathfrak{N})$ is said to be a Nevanlinna function or an R-function,cf.[25]. if V (z) is holomorphic on C/R, V* $(z) \geq 0$ for all $z \in C/R$. The subclass of Nevanlinna functions V (z) which are holomorphic on the domain Ext $[-1,1] := C \setminus [-1.1]$ is denoted by $N_{\mathfrak{N}}[-1,1]$ By the general theory of Nevanlinna functions,cf.[25],[16]every functionV(z) in $N_{\mathfrak{N}}[-1,1]$ has an integral representation of the form

$$V(z) = \Gamma + \int_{-1}^{1} \frac{dG(t)}{t - z},$$

where Γ is a bounded selfadjoint operator on $\mathfrak{N}$ and the $L(\mathfrak{N})$ -valued function G(t) is nondecreasing, nonnegative, normalized by G (- 1 – 0)=0, and has finite total variation concentrated on $[-1,1]$. Clearly , V (∞) : =s - $\lim_{z \to \infty}$ V $(z) = \Gamma$. The next result is also well known, cf .[15].

Theorem(1.1.2)[1] Let $\mathfrak{N}$ be a Hilbert space and let V $(z) \in N_{\mathfrak{N}}[-1,1]$. Then then there erist a Hilbert space $\mathfrak{H}$, a selfadjoint contraction B on $\mathfrak{H}$, and F $\in$ L $(\mathfrak{N}, \mathfrak{H})$, such that

$$V(z) = V(\infty) + F^*(B - zI)^{-1}F, \qquad z \in \text{Ext } [-1,1]. \tag{12}$$

In what follows the subclass of functions V $(z) \in N_{\mathfrak{N}}[-1,1]$. which have the limit values V (± 1) in $L(\mathfrak{N})$ plays a central pole . ln this case Theorem(1.1.2) can be completed as follows.

Theorem(1.1.3)[1]: Let $\mathfrak{N}$ be a Hilbert space and let V $(z) \in N_{\mathfrak{N}}[-1,1]$. If for all $f \in \mathfrak{N}$ the limit values

$$\lim_{x\uparrow -1}(V(x)f, f), \qquad \lim_{x\downarrow 1}(V(x)f, f) \tag{13}$$

are finite, then there exist a Hilbert space $\mathfrak{H}$, a selfadjoint contraction B in $\mathfrak{H}$ and an operator G $\in L(\mathfrak{N}, \mathfrak{D}_B)$, such that

$$V(z) = V(\infty) + G^* D_B^2 (B - zI)^{-1} G, \qquad z \in Ext[-1,1] \ . \tag{14}$$

Conversely , for every function V$(z)$of the form (14) the limit values (13) exist for all $f \in \mathfrak{N}$ and are finite .

Proof. By Theorem (1.1.2)V(z)has the representation(12), where B is a selfadjoint contraction in a Hilbert space $\mathfrak{H}$ and F $\in L(\mathfrak{N}, \mathfrak{H})$. Since the limits in (13) exist for all $f \in \mathfrak{N}$, one concludes that .

$$ranF \subset ran(1-B)^{1/2} \cap ran(1+B)^{1/2}.$$

Consequently , ran F$\subset$ran D_B and this implies that F=$D_B G$ for some operator G$\in$L $(\mathfrak{N}, \mathfrak{D}_B)$,cf .[24].

Conversely, if $V(z)$ is of the form (14) then ran $D_B \subset ran(B \pm I)^{1/2}$ *and*

this implies the existence of thse limit values (13) for all $f \in \mathfrak{N}$, cf . [33].

It follows from Theorem (1.1.3)that

$$V(-1) := s - \lim_{x\uparrow -1} V(x) = V(\infty) + G^*(1-B)G \in L(\mathfrak{N}), \tag{15}$$

$$V(-1) := s - \lim_{x\downarrow 1} V(x) = V(\infty) + G^*(1-B)G \in L(\mathfrak{N}),$$

so that

$$V(-1) + V(1) = 2V(\infty) - 2G^* BG, \qquad V(-1) - V(1) = 2G^* G. \tag{16}$$

An operator-valued function K$(z,\xi) : \Omega \times \Omega \to L(\mathfrak{N}), \Omega \subset \mathbb{C}$ is said to be a nonnegative kernel[2],[14] ,[43] if

$$\sum_{i,j=1}^{n} (k(w_j, w_i) f_i, f_j)_n \geq 0$$

for every choice of points $\{w_i\}_{i=1}^n \subset \Omega$ *andvectors* $\{f_i\}_{i=1}^n \subset \mathfrak{N}$ with the kernel k$(z,\xi)$is associated a reproducing kernel Hilbert space $\mathcal{H}_K$ it is the kernel $k(z,\varepsilon)$ is associated a reproducing kernel Hilbert space$\mathcal{H}_K$.It is the completion of the linear space of vectors of the form .

$$\sum_{i=1}^{n} k(., w_i) f_{i,} \quad \{w_i\}_{i=1}^{n} = 1 \subset \Omega, \quad \{f_i\}_{i=1}^{n} = 1 \subset \mathfrak{N} \ , n \in N,$$

with respect to the inner product .

$$\left(\sum_{i=1}^{n} K(.\,, \omega_i) f_i, \sum_{j=1}^{n} K(.\,, \mu_j) g_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(K(\mu_j, \omega_i) f_i, g_j\right)_{\mathfrak{N}}.$$

Then the Hilbert space $\mathcal{H}_K$ consists of the $\mathfrak{N}$ -valued functions $f(\cdot)$ such that for every $h \in \mathfrak{N}$ the reproducing property holds:

$$(f(.), K(.\,, \omega) h)_{\mathcal{H}K} = f(\omega, h)_{\mathfrak{N}}, \ \ \omega \in \Omega.$$

Observe that an L($\mathfrak{N}$) -valued function V(z)belongs to the Nevanlinna class N($\mathfrak{N}$) if and only if the function.

$$k(z,\xi) = \frac{V(z) - V(\xi)^*}{z - \bar{\xi}}, z, \xi \in C \setminus R,$$

is a nonnegative kernel. Also note that the kernel associated with generalized resolvents (of selfadjoint exit space extensions) in a Hilbert space is given by

$$k(z,\xi) = \frac{V(z) - V(\xi)^*}{z - \bar{\xi}} - V(z), V(\xi)^*, \quad z, \xi \in C \setminus R$$

An operator-valued function $K(z,\xi) : \Omega \times \Omega \rightarrow L(\mathfrak{N}), \Omega \subset \mathbb{C}$ is said to be an α -sectorial kernel, if .

$$\sum_{i,j=1}^{n} (K(\omega_j, \omega_i) f_i, f_j)_{\mathfrak{N}} \quad \in S(\alpha)$$

For every choice of points $\{\omega_i\}_{i=1}^{n} \subset \Omega$ and vectors $\{f_i\}_{i=1i}^{n} \subset \mathfrak{H}$, [$i.e.$,

$$\left|\operatorname{Im} \sum_{i,j=1}^{n} (K(\omega_j, \omega_i) f_i, f_j)_{\mathfrak{N}}\right| \leq (\tan \alpha) \operatorname{Re} \left(\operatorname{Im} \sum_{i,j=1}^{n} (K(\omega_j, \omega_i) f_i, f_j)_{\mathfrak{N}}\right)$$

cf.[6].For $\alpha = 0$ the corresponding kernel is nonnegative.

Let A be a non- densely defined closed symmetric contraction in the Hilbert spase $\mathfrak{H}$ with the domain dom $A =: \mathfrak{H}_0$ $and\ let$ $\mathfrak{N} := \mathfrak{H} \ominus dom\ A. let\ P_0\ and\ P_{\mathfrak{N}}$ be the orthogonal projections in $\mathfrak{H}$ onto $\mathfrak{H}_0$and respectively . Then the operator $A_0 = P_0 A$ is contractive and self adjoint in the subspace $\mathfrak{H}_0$.Let $D_{A_0} = (I - A_0^2)^{\frac{1}{2}}$ be the defect operator determined by A_0. The operator $A_{21} = P_{\mathfrak{N}} A$ is also contractive. Moreover, it follows from $A^* A \leq I$. That $A_{21}^* A_{21} \leq D_{A_0}^2$.Therefore, the identity

$$K_0 D_{A_0} f = P_{\mathfrak{N}} Af, \quad f \in dom\, A$$

defines a contractive operator k_0 from $\mathfrak{D}_{A_0} := \overline{ran} D_{A_0}$ into $\mathfrak{N}$, cf, [21], [24].This gives the following decomposition for the symmetric contraction A

$$A = A_0 + K_0 D_{A_0} = \begin{pmatrix} A_0 \\ K_0 D_{A_0} \end{pmatrix}. \tag{17}$$

Let the closed symmetric contraction A be defined on the subspace $\mathfrak{H}_0 = dom A$ and decompose A according to $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{N}$ as in(17).Let T be a qsc-extension of A, so that $A \subset T$ and $A \subset T^*$,and decompose T=(T_{ij})also with respect to $\mathfrak{H}=\mathfrak{H}_0 \oplus \mathfrak{N}$,cf.(8) .Then clearly $T_{11} = A_0 T_{12}^* = T_{21} = K_0 D_{A_0}$. The next result gives a parametrization of all qsc – extensions of A and some of its subclasses by means of block formulas cf. [15],[18],[46],and [11],[13]. For completeness a short, simple proof presented.

Theorem(1.1.4)[1]:Let A be a closed symmetric contraction A in $\mathfrak{H}=\mathfrak{H}_0 \oplus \mathfrak{N}$ with dom A = $\mathfrak{H}_0$ and decompose A as in(17). Then:

(i)the formula

$$T = \begin{pmatrix} A_0 & D_{A_0} K_0^* \\ K_0 D_{A_0} & -K_0 A_0 K_0^* + D_{K_0^*} X D_{K_0^*} \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{N} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{N} \end{pmatrix} \tag{18}$$

gives a one – to – one correspondence between all qsc – extensions T of the symmetric contraction $A = A_0 + K_0 D_{A_0}$ and all contractions X in the subspace $\mathfrak{D}_{K_0^*} := \overline{ran} D_{K_0^*} \subset R$;

(ii) T in(18) belong to the class $C(\alpha)$ if and only if X belongs to the class $C(\alpha)$, $\alpha \in (0, \pi/2)$;

(iii) T is a selfadjoint contractive extension of A if and only if Xin (18) is a selfadjoint contraction in $\mathfrak{D}_{K_0^*}$

Proof: (i) Every operator $T \in L(\mathfrak{H})$ satisfying the conditions $A \subset T$ and $A \subset T^*$ admits the block matrix representation of the form

$$T=\begin{pmatrix} A_0 & D_{A_0}K_0^* \\ K_0D_{A_0} & D \end{pmatrix}:\begin{pmatrix}\mathfrak{H}_0\\ \mathfrak{N}\end{pmatrix}\longrightarrow\begin{pmatrix}\mathfrak{H}_0\\ \mathfrak{N}\end{pmatrix}$$

where $D\in L(\mathfrak{N})$ then I-T*T is given in the block form

$$I-T^*T=\begin{pmatrix} D_{A_0}^2-D_{A_0}K_0^*K_0D_{A_0} & -A_0D_{A_0}K_0^*-D_{A_0}K_0^*D \\ -K_0D_{A_0}A_0-D^*D_{A_0}A_0 & D_{K_0^*}^2-K_0A_0^2K_0^*-D^*D \end{pmatrix}.$$

Contractivity of T means that

$$0\le\left\|D_{A_0}f-A_0K_0^*h\right\|^2+\left\|D_{K_0^*}h\right\|^2-\left\|K_0D_{A_0}f+Dh\right\|^2 \tag{19}$$

for all $f\in\mathfrak{H}_0$ and $h\in\mathfrak{N}$ .Since $\operatorname{ran}K_0^*\subset\mathfrak{D}_{A_0}$ and $A_0\mathfrak{D}_{A_0}\subset\mathfrak{D}_{A_0}$,there exists a sequence $\{f_n\}_{n=1}^{\infty}\subset\mathfrak{D}_{A_0}$ such that for a given $h\in\mathfrak{N}$ the equality

$$\lim_{n\to\infty}D_{A_0}f_n=A_0K_0^*h$$

holds. Hence, it follows from(19) that $E=K_0A_0K_0^*+D$ satisfies

$$\|Eh\|^2\le\left\|D_{K_0^*}h\right\|^2,\quad \left\|E^*h\right\|^2\le\left\|D_{K_0^*}h\right\|^2,\qquad h\in\mathfrak{N}, \tag{20}$$

where the second inequality follows from the first one by taking into account that T* is a contraction, too. By the second inequality in (20) there exists a contraction $Z\in\mathfrak{N}(R,\mathfrak{D}_{K_0^*})$ such the $E=D_{K_0^*}Z$,i.e.,$D=-K_0A_0K_0^*+D_{K_0^*}Z$.

By substituting this into (19) one obtains

$$0\le\left\|D_{k_0}f-A_0K_0^*h)-K_0^*Zh\right\|^2+\left\|D_{K_0^*}h\right\|^2-\|Zh\|^2,\ f\in\mathfrak{H}_0,\ \ h\in\mathfrak{N} \tag{21}$$

since by means of (3) one has

$$-\left\|K_0(D_{A_0}f-A_0K_0^*h)+D_{K_0^*}Zh\right\|^2=-\left\|K_0(D_{A_0}f-A_0K_0^*h\right\|^2$$

$$-\|Zh\|^2+\left\|K_0^*Zh\right\|^2-2\operatorname{Re}(DK_0(D_{A_0}f-A_0K_0^*h),K_0^*Zh)$$

Due to the inclusion rank $Z\subset\mathfrak{D}_{A_0}$,one can choose a sequence $\{f_n\}_{n=1}^{\infty}\subset$ $\mathfrak{D}A0$ such that for a given $h\in\mathfrak{N}$ the equality

$$\lim_{n\to\infty}D_{K_0}D_{A_0}f_n=D_{K_0}A_0K_0^*Zh \tag{22}$$

Holds. Now (21) shows that $\|Zh\|^2 \leq \left\|D_{K_0^*}h\right\|^2$ for all $h \in \mathfrak{N}$ so that $Z = XD_{K_0^*}$ for some contraction X in $D_{K_0^*}$ Therefore

$$E = D_{K_0^*} X D_{K_0^*} \text{ and}$$

$$D = K_0 A_0 K_0^* + D_{K_0^*} X D_{K_0^*} \tag{23}$$

Conversely, let D be of the from (23) , where X is a contraction in $\mathfrak{D}_{K_0^*}$ .Then $D_X^2 \geq 0$ implies that T given by (18) satisfie

$$\left((1 - T^*T)\begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f \\ h \end{pmatrix}\right) = \left\|D_{k_0}\left(D_{A_0} f - A_0 K_0^* h\right) - K_0^* X D_{K_0^*} h\right\|^2 + \left\|D_X D_{K_0^*} h\right\|^2 \geq 0 . \tag{24}$$

Thus, every contraction X in $\mathfrak{D}_{K_0^*}$ defines a qsc- extension T of A via (18)(ii). It follow from (18) and (24) that T satisfies (6) and only if

$$\leq \frac{\tan\alpha}{2}\left(\left\|D_{K_0}\left(D_{A_0} f - A_0 K_0^* h\right) - K_0^* X D_{K_0^*} h\right\|^2 + \left\|D_X D_{K_0^*} h\right\|^2\right) \tag{25}$$

Holds for all $f \in \mathfrak{H}_0$, $h \in \mathfrak{N}$ in view of the condition (22) in equivalent to

$$\left|\left(X_1 h, h\right)\right| \leq \frac{\tan\alpha}{2}\left\|D_X h\right\|^2 \tag{26}$$

For all $h \in \mathfrak{D}_{K_0^*}$

(iii)The statement is clear since T in (18) in selfadjoint if and only if T is self adjoint in $\mathfrak{D}_{K_0^*}$

The class of all selfadjoint contractive (sc-) extensions of A in part (iii) of Theorem(1.1.4),forms an operator interval $|A_\mu, A_m|$. Using the block representation (18) the endpoints of $\{A_\mu, A_m\}$ are given by

$$A_\mu = \begin{pmatrix} A_0 & D_{A_0} K_0^* \\ K_0 D_{A_0} & K_0 A_0 K_0^* - D_{k_0^*}^2 \end{pmatrix} \tag{27}$$

and

$$A_M = \begin{pmatrix} A_0 & D_{A_0} K_0^* \\ K_0 D_{A_0} & K_0 A_0 K_0^* + D_{k_0^*}^2 \end{pmatrix} . \tag{28}$$

With $X = -I \upharpoonright \mathfrak{D}_{K_0^*}$ and $X = I \upharpoonright \mathfrak{D}_{K_0^*}$ respectively. From the formulas (27) and (28) it is seen that

$$\frac{A_\mu + A_M}{2} = \begin{pmatrix} A_0 & D_{A_0}K_0^* \\ K_0 D_{A_0} & K_0 A_0 K_0^* \end{pmatrix}, \frac{A_M - A_\mu}{2} = \begin{pmatrix} 0 & 0 \\ 0 & D_{K_0^*}^2 \end{pmatrix}.$$

This means that all qsc – extensions in (18) of the symmetric contraction A from an operator ball

$$B\left(\frac{A_\mu + A_M}{2}, \frac{A_M - A_\mu}{2}\right)$$

with center

$$(A_\mu + A_M)/2$$

and equal left and right radii

$$R_l = R_r = (A_M + A_\mu)^{1/2} / \sqrt{2}$$

The one – to – one correspondence between all qsc- extensions of A and all contractions X in Theorem (1.1.4) can be reformulated also as follow

$$T = \frac{A_\mu + A_M}{2} + \left(\frac{A_M + A_\mu}{2}\right)^{1/2} \times \left(\frac{A_M + A_\mu}{2}\right)^{1/2} \tag{29}$$

where the parameters X are contractions in the subspace $\overline{\operatorname{ran}}(A_m - A\mu$,cf.[11],[12],[13].It is easy to see from (18),(27)and(28), that if T is a qsc-extension of A such tha $T_R = (T - T^*)/2 = A_m(A_\mu$ then in fact $T = A_m(A_\mu)$.Namely, $X = X_R + iX_i$ satisfies

$$\begin{cases} 0 \le X^* X = X_R^2 + i(X_R X_I - X_I X_R) + X_I^2 \le I \\ 0 \le XX^* = X_R^2 - i(X_R X_I - X_I X_R) + X_I^2 \le I \end{cases}, \tag{30}$$

so that $0 \le X_R^2 + X_1^2 \le I$ and here clearly $X_R^2 = I$ implies $X_1 = 0$

The description of all contractive selfadjoint extensions of a symmetric contraction A as the operator interval $|A_\mu, A_M|$ is due to M.G. Krien[28]. In that section the notion of shorted operators was also introduced and used for instance to establish the following characterization for A_μ and A_M:

$$ran(I + A_\mu)^{1/2} \cap \mathfrak{N} = \{0\}, \qquad ran(I + A_M)^{1/2} \cap \mathfrak{N}\{0\}, \tag{31}$$

cf.[8],[23].Block formulas for describing all contractive extensions of a duel pair appear in[15],[18],[46], a description in Crandall's form in The one to one correspondence between all qsc- extensionsTof Athe class $C(\alpha)$ and all operators X in $\overline{ran}(A_M - A_\mu)$ belonging to the class $C(\alpha)$ by means of (29) was proved in a different way in another proof based on(18) was given in[39].

According to[33] a closed symmetric contraction A is said to be simple if there is no non- zero subspace in dom A which is invariant under A. Since A is symmetric simplicity of A is equivalent A being completey non- selfadjoint, i.e.,to A having no selfadjoint parts.

Lemma(1.1.5)[1]: Let the closed symmetric contraction $A = A_0 + K_0 D_{A_0}$ in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{N}$, $\mathfrak{H}_0 = \text{dom}A$ be decomposed as in (17) with $K_0 : \mathfrak{D}_{A_0} \longrightarrow \mathfrak{N}$ .is simple if and only if the subspace

$$\mathfrak{H}_0^s := \overline{span}\, \{(A_0 - zI)^{-1} K_0^* \mathfrak{N} : z \in \rho(A_0)\} \tag{32}$$

$$= \quad \overline{span}\, \{A_0^n K_0^* \mathfrak{N} : n = 0,1, \dots\}$$

Coincides with$\mathfrak{H}_0$. In this case$\mathfrak{D}_{A_0} = \mathfrak{H}_0, K_0 : \mathfrak{H}_0 \to \mathfrak{N}$,an $\|A_0 f\| < \|f\|$ for al $f \in \mathfrak{H}_0 / \{0\}$

Proof. Suppose that A is simple. Then clearly $\ker D_{A_0} = \{0\}$ or equivalently $\|A_0 f\| < \|f\|$ for all $f \in \mathfrak{H}_0 / \{0\}$ so that $\mathfrak{D}_{A_0} = \mathfrak{H}_0$ and $K_0 : \mathfrak{H}_0 \to \mathfrak{N}$ Observe that the subspace $\mathfrak{H}_0^s$in(22) and therefore also $\mathfrak{H}_0 \ominus \mathfrak{H}_0^s$ is invariant under $A_0 = A_0^*$ Then the subspace$\mathfrak{H}_0 \ominus \mathfrak{H}_0^s$is also invariant under D_{A_0} . Moreover,

$$\mathfrak{H}_0 \ominus \mathfrak{H}_0^* = \{f \in \mathfrak{H}_0 : K_0 A_*^n f = 0, n = 0,1, \dots\} \tag{33}$$

Is follow that $K_0 D_{A_0} f = 0$for all $f \in \mathfrak{H}_0 \ominus \mathfrak{H}_0^s$Hence, in view of (17)$Af = A_0 f$ for $f \in \mathfrak{H}_0 \ominus \mathfrak{H}_0^s$all This means that the subspace $\mathfrak{H}_0 \ominus \mathfrak{H}_0^s$is invariant under A since A is a simple, one concludes that$\mathfrak{H}_0^s = \mathfrak{H}_0$

Conversely, assume that$\mathfrak{H}_0^s = \mathfrak{H}_0$. Since $ran K_0^* \subset D_{A_0}$ and D_{A_0} is invariant underA_0, the definition of $\mathfrak{H}_0^s$in (22) shows that $\mathfrak{H}_0^s \subset \mathfrak{D}_{A_0}$. Hence, the assumption implies that $\mathfrak{H}_0^s = \mathfrak{D}_{A_0 = \overline{ran}\, D_{A_0}}$ so that ker D_{A_0} =\{0\}.Now suppose that $\widetilde{\mathfrak{H}}_0 \subset \mathfrak{H}_0$ is a subspace which is invariant under A . Then for every $f \in \tilde{\mathfrak{H}}_0$ one has $Af = A_0 f + K_0 D_{A_0} f \in \tilde{\mathfrak{H}}_0$ so that $K_0 D_{A_0} f = 0$ for all $f \in \tilde{\mathfrak{H}}_0$ and . Hence $D_{A_0} \tilde{\mathfrak{H}}_0$ is invariant under A_0 and D_{A_0} Moreover since $D_{A_0} = \{0\}$ the image $D_{A_0} \tilde{\mathfrak{H}}_0$ is dense in $\tilde{\mathfrak{H}}_0$. This implies that K_0 and since A_0^n one has $K_0 \tilde{\mathfrak{H}}_0 \subset \tilde{\mathfrak{H}}_0$ for all $n = 0,1\ldots, i.e.,$

$$\tilde{\mathfrak{H}}_0 \subset \{f \in \mathfrak{H}_0 : K_0 A_0^n f = 0, n = 0,1,\ldots\} \mathfrak{H}_0 \ominus \mathfrak{H}_0$$

c.f(33)Therefore A is simple.

Let T be a qsc- extension of A in the Hilbert space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{N}$ with $\mathfrak{H}_0 = domA$.. It is evident that the subspace

$$\tilde{\mathfrak{H}}_T = \overline{span}\{(T - zI)^{-1}\mathfrak{N}\} : |z| > 1 = \overline{span}\,\{T^n \mathfrak{N} : n = 1,2,\ldots\}, \qquad (34)$$

is invariant underT, and that the subspace

$$\mathfrak{H}''_T := \mathfrak{H} \ominus \mathfrak{H}'_T, \qquad (35)$$

is invariant underT* . Since $\mathfrak{N} \subset \mathfrak{H}'_T$, one obtains

$$\mathfrak{H}''_T \subset \mathfrak{N}^{\perp} = domA \subset \ker(T - T^*)$$

Therefore the restriction of T^* to $\mathfrak{H}''_T$ is a selfadjoint operator in $\mathfrak{H}''_T$ The restriction$\subset$ $T \upharpoonright \mathfrak{H}'_T \left(= P_{\mathfrak{H}'_T} \upharpoonright \mathfrak{H}'_T\right)$. is called the $\mathfrak{N}$ $-$minimal par of T Moreover T is said to be $\mathfrak{N} -$ minimal if the equality $\mathfrak{H} = \acute{\mathfrak{H}}_T$ holds. If T be a qsc- extension of A then its adjoint T is also a qsc extension of A and one can associate with it the subspace $\acute{\mathfrak{H}}_T$ and the corresponding $\mathfrak{N} -$ minimal part of T^*. The next result shows the $\mathfrak{N} -$ minimal parts of T and T^* are qsc- extensions of the simple part $A \upharpoonright \mathfrak{H}_0^S$ of A in the same subspace $\mathfrak{H}'_T = \mathfrak{H}'_{T^*}$

Proposition(1.1.6)[1]: let A be a symmetric contraction in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{N}$ with $\mathfrak{H}_0 = domA$. Let T be a qsc- extension of A in $\mathfrak{H}$ and letT* be its adjoint. Then the subspaces $\acute{\mathfrak{H}}_T, \acute{\mathfrak{H}}_{T^*}$ and $\mathfrak{H}_0^S$ of $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{N}$ as defined in (34) and (32) are connected by

$$(\mathfrak{H}' :=) \mathfrak{H}'_T = \mathfrak{H}'_{T^*} = \mathfrak{H}_0^S \oplus \mathfrak{N}. \qquad (36)$$

In particular, the symmetric contraction A is simple if and only if the qsc- extension T, or equivalently T*of A is $\mathfrak{N} -$ minimal.

Proof. It follows from the Schur – Frobenius formula (10) that

$$(T-z)^{-1}\mathfrak{N} = \begin{pmatrix} -(A_0 - z)^{-1} D_{A_0} K_n^* \\ \mathfrak{N} \end{pmatrix}, |z| > 1,$$

which implies that

$$\overline{\text{span}}\{(T - zI)^{-1}\mathfrak{N}\}: |z| > 1\}$$

$$= \overline{span}\ \{(A_0 - zI)^{-1} D_{A_0} K_n^* \mathfrak{N}: z \in \rho(A_0)\} \oplus \mathfrak{N}$$

$$= (clos D_{A_0}\, \overline{span}\{(A_0 - zI)^{-1} D_{A_0} K_n^* \mathfrak{N}: z \in \rho(A_0)\} \oplus \mathfrak{N}$$

This shows that

$$\mathfrak{H}'_T = (\text{clos}\, D_{A_0} \mathfrak{H}_0^S) \oplus \mathfrak{N}. \tag{37}$$

Since $K_0^* \subset \mathfrak{D}_{A_0}$ and $\mathfrak{D}_{A_0}$ is invariant under A_0 one has $\mathfrak{H}_0^S \subset \mathfrak{D}_{A_0}$.In particular, $\mathfrak{H}_0^S \cap \ker D_{A_0} = \{0\}$ which together with $D_{A_0}\, \mathfrak{H}_0^S \subset \mathfrak{H}_0^S$ implies that $D_{A_0}\, \mathfrak{H}_0^S = \mathfrak{H}_0^S$. Hence, (37) implies the equality $\acute{\mathfrak{H}}_T = \mathfrak{H}_0^S \oplus \mathfrak{N}$. It follows from

$$(T^* - zI)^{-1} - (T - zI)^{-1} = (T - zI)^{-1}[T - T^*](T - zI)^{-1}, \qquad |z| > 1,$$

and the inclusion ran $(T - T^*) \subset \mathfrak{N}$ that

$$(T^* - zI)^{-1}\ \mathfrak{N} \subset (T - zI)^{-1}\ \mathfrak{N} \subset \acute{\mathfrak{H}}_T, \quad |z| > 1.$$

Therefore , $\mathfrak{H}'_{T^*} \subset \mathfrak{H}'_T$ and the reverse inclusion follows by symmetry. This completes the proof of (36).

The last statement is clear from(36)

For selfadjoint extension of A the result in Proposition(1.1.6) has been given in the case of closed densely defined symmetric operators Athere is an equivalent criterion for the simplicity of A due to M.G. Krein based on the defect elements:

$$\overline{span}\{\ker(A^* - \lambda): \lambda \in C/R\} = \mathfrak{H},$$

cf. Lemma(1.1.5) .This characterization has been extended to non – densely defined symmetric operators in[37]

Sec(1.2)

Quasi-Self adjoint Contractions

Let T be a qsc – operator in a separable Hilbert space $\mathfrak{H}$ and let $\mathfrak{N}$ be subspace of $\mathfrak{H}$ such that $\mathfrak{N} \supset \operatorname{ran}(T - T^*)$. The operator – valued function

$$Q_T(z) = P_{\mathfrak{N}}(T - zI)^{-1} \upharpoonright \mathfrak{N}, \quad |z| < 1, \tag{38}$$

where $P_{\mathfrak{N}}$ is the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{H}$ is said to be Q – function associated withTand the subspace $\mathfrak{N}$ .Clearly, it has the limit value $Q_T(\infty) = 0$ and the Q – function of T and T* in $\mathfrak{N}$ are connected by

$$Q_{T^*}(z) = Q_T(\bar{z})^*, \qquad |z| > 1. \tag{39}$$

If T is a selfadjoint contraction then Q – function (38) is a Nevanlinna function of the class $N_{\mathfrak{N}}$[-1,1]. The next result contains some basic properties for the Q – function $Q_T(z)$ of a qsc- operator T as defined in (38)

Proposition(1.2.1)[1]: Let $Q_T(z)$ be a Q – function of a qsc – operator T as defined in (38)Then:

(i) $Q_T(z)$ has the following asymptotic expansion:

$$Q_T(z) = -\frac{1}{z} I + \frac{1}{z^2} F + 0\left(\frac{1}{z^2}\right), \qquad z \to \infty, \tag{40}$$

where $F = -P_n T \upharpoonright \mathfrak{N}$;

(ii) $Q_T^{-1}(z) \in L(\mathfrak{N})$ for all $|z| > 1$;

(iii) $Q_T^{-1}(z)$has strong limit values$Q_T^{-1}(\pm 1)$:

$$Q_T^{-1}(-1) = \lim_{z \uparrow -1} Q_T^{-1}(x), \qquad Q_T^{-1}(1) = \lim_{z \downarrow -1} Q_T^{-1}(x);$$

(iv)for all $f, g \in \mathfrak{N}$ thefollowing inequality holds:

$$\left|((Q_T^{-1}(-1) + Q_T^{-1}(1))f, g\right|^2 \le ((Q_T^{-1}(-1) - Q_T^{-1}(1))f, f)((Q_T^{-1}(-1) - Q_T^{-1}(1))g, g);$$

(v)the function $-Q_T^{-1}(z) - F - zI$ is an operator – valued Nevanlinna function:

(vi) $Q_T(z) \in N_{\mathfrak{N}}[-1,1]$ ifand onlyif F=F^*

Moreover, if T is decomposed as in (18) with $\mathfrak{H}_0(\mathfrak{H} \ominus \mathfrak{N})$ and $A = T \upharpoonright \mathfrak{H}_0$, *then*

$$F = K_0 A_0 K_0^* - D_{K_0^*} X D_{K_0^*}, \tag{41}$$

$$Q_T^{-1}(-1) = D_{K_0^*}(X+1)D_{K_0^*}, \qquad Q_T^{-1}(1) = D_{K_0^*}(X-1)D_{K_0^*} \tag{42}$$

$$-Q_T^{-1}(z) - F - zI = K_0\left(I - A_0^2\right)\left(A_0 - zI\right)^{-1} K_0^* \tag{43}$$

Proof.(i) Clearly $\lim_{z\to\infty} zQ_T(z)h = \lim_{z\to\infty} z P_{\mathfrak{N}}(T - zI)^{-1}h = -P_{\mathfrak{N}}h$ for all $h \in \mathfrak{N}$. Moreover, for all $h \in \mathfrak{N}$

$$\lim_{z\leftarrow\infty} z(1 + zQ_T(z))h = \lim_{z\leftarrow\infty} zP\mathfrak{N}T(T - zI)^{-1}h = -PTh. \tag{44}$$

Hence, $Q_T(z)$ admits the asymptotic expansion (40)

(ii)Let $|z| > 1$, *let* $f \in \mathfrak{N}$, and let $\varphi = (T, -zI)^{-1}f$. Then $\|f\| \le (1 + |z|)\|\varphi\|$

and

$$|(Q_T(z)f, f)| = \left|\left((T - zI)^{-1} f, f\right)\right| = |(\varphi, T - zI)\varphi|$$

$$= |(\varphi, T\varphi) - \bar{z}\|\varphi\|^2| \ge \frac{|z| - 1}{(|z| + 1)^2}\|f\|^2.$$

Since = $|Q_T(z)f, f| = |(Q_T(z)^* f, f)|$, this implies that

$$\|Q_T(z)f\| \ge \frac{|z|-1}{(|z|+1)^2}\|f\| \quad Q_T(z)^* f \ge \frac{|z|-1}{(|z|+1)^2}\|f\|.$$

Therefore $Q_T^{-1}(z) \in L(\mathfrak{N})$ *for all* $|z| > 1$.

(iii) Decompose $\mathfrak{H}(\mathfrak{H}_0 \oplus \mathfrak{N})$ and write T in block form as in(18) where $\mathfrak{H}_0(\mathfrak{H} \ominus \mathfrak{N}$, $A = T|\mathfrak{H}0, A0 = P0A$ isa selfadjoint contraction $\mathfrak{H}_0$, $D_{A_0} = (1 - A_0^2)^{1/2}$, $K_0 \in L(\mathfrak{D}_{A_0}, \mathfrak{N})$ is a contraction and X is a contraction in the subspace $\mathfrak{D}_{k_n^*}$ $\mathfrak{N}$.The formula (41) for F is immediate from (18). Write $Q_T^{-1}(z)$ as in (11)

$$Q_T^{-1}(z) = -V_T(z) - zI, \quad |z| > 1$$

where

$$V_T(z) = K_0\left[A_0 + (A_0 - zI)^{-1}\left(1 - A_0^2\right)\right]K_0^* - D_{K_0^*} X D_{K_0^*} \tag{45}$$

This shows that the limit values $Q_T^{-1}(\pm 1)$ exist and that they are given by (42).

(iv) It follows from (42)that

$$\frac{Q_T^{-1}(-1)+Q_T^{-1}(1)}{2} = D_{K_0^*} X D_{K_0^*}$$

$$\frac{Q_T^{-1}(-1)+Q_T^{-1}(1)}{2} = D_{K_0^*}^2 = 1 - K_0 K_0^* \tag{46}$$

It remains to apply the criterion (4) with S_0 =0 and $R_1 = R_r = D_{K_0^8}^2$.

(v) It follows from (41) and (45) that(43) holds. Clearly, the function in(43) is a Nevanlinna function.

(vi) if $Q_T(z) \in N_{\mathfrak{N}}[-1,1]$then $-Q_T(z)^{-1}$is a Nevanlinna function and now part (v) implies that $F = F^*$.Conversely, if $F = F^*$ then the function $V_T(z)$ in(45)and $-Q_T(z)^{-1} = V_T(z) + zI$ are Nevanlinna functions. Therefore$Q_T(z) \in N_{\mathfrak{N}}[-1,1]$.

Let T be a qsc – operator, let $Q_T(z)$be defined by (38) and let F be defined by $F = -P_{\mathfrak{N}} \upharpoonright \mathfrak{N}$. Associate with$Q_T(z)$ the following kernels:

$$G_T(z,\xi) := \frac{Q_T(z) - Q_T(\xi) \quad -Q_T(z)(F-F^*)Q_T(\xi) \quad Q_T^{-1}(1)}{z-\xi} \tag{47}$$

$$M_T(z,\xi): I + zQ_T(z) + \bar{\xi}Q_T(\xi)^* + z\bar{\xi}G_T(z,\xi) \tag{48}$$

$$L_T(z,\xi) := G_T(z,\xi) - M_T(z,\xi) \tag{49}$$

and

$$M_T(z,\xi) = L_T(z,\xi) + Q_T(z)(F-F^*)Q_T(\xi)^*, \tag{50}$$

with $z \neq \bar{\xi}, |z|, |\xi| < 1$. The insertion of the definition of $G_T(z,\xi)$ in $L_T(z,\xi)$ and $K_T(z,\xi)$ leads to the identities

$$(z-\bar{\xi})L_T(z,\xi) = (1-z^2)Q_T(z) - (1-\bar{\xi}^2)Q_T(\xi)^*$$
$$-(1-z\bar{\xi})Q_T(z)(F-F^*)Q_T(\xi)^* - (z-\bar{\xi})I,$$

and

$$(z-\bar{\xi})L_T(z,\xi) = (1-z^2)Q_T(z) - (1-\bar{\xi}^2)Q_T(\xi)^*$$

$$-(1+z)(1-\bar{\xi})Q_T(z)(F-F^*)Q_T(\xi)^* - (z-\bar{\xi})I.$$

Proposition(1.2.2)[1]: Let T be a qsc – operator, let $Q_T(z)$ be defined by(39), and let T be defined by $F = -P_{\mathfrak{N}} \upharpoonright \mathfrak{N}$. Let the kernels associated with $Q_T(z)$ be given by (47),(48),(49)and (50). Then the following equalities hold for every $z \neq \bar{\xi}$, $|z|, |\xi| > 1$:

$$G_T(z,\xi) = P_{\mathfrak{N}}(T-zI)^{-1}(T^*-\bar{\xi}I)^{-1} \upharpoonright \mathfrak{N}, \tag{51}$$

$$M_T(z,\xi) = P_{\mathfrak{N}}(T-zI)^{-1}TT^*(T^*-\bar{\xi}I)^{-1} \upharpoonright \mathfrak{N}, \tag{52}$$

and

$$L_T(z,\xi) = P_{\mathfrak{N}}(T-zI)^{-1}(T-T^*)\mathfrak{N}. \tag{53}$$

The operator- valued function $G_T(z,\xi), M_T(z,\xi)$, and $L_T(z,\xi)$ are nonnegative kernels. If in addition the operator T belongs to the class $C(\alpha)$ then the function.

$$K_T(z,\xi)=P_{\mathfrak{N}}(T-zI)^{-1}(1+T)(1-T^*)(T^*-\bar{\xi}I)^{-1} \upharpoonright \mathfrak{N} \tag{54}$$

with $|z|, |\xi| > 1$ is an α –sectorial kernel.

Proof. Note that $\operatorname{ran}(T-T^*) \subset \mathfrak{N}$ implies that $\mathfrak{N}^{\perp} \subset \ker(T-T^*)$, and hence $T-T^*=P_{\mathfrak{N}}(T-T^*)P_{\mathfrak{N}}$. Therefore for every $f, g \in \mathfrak{N}$,

$$((Q_T(z)-Q_T^*(\zeta))f,g) = (P_{\mathfrak{N}}(T-zI)^{-1}f - P_{\mathfrak{N}}(T^*-\bar{\xi}I)^{-1}f,g)$$

$$= (P_{\mathfrak{N}}(T-zI)^{-1}(T^*-T)(T^*-\bar{\xi}I)^{-1}f,g)$$

$$+(z-\bar{\xi})(P_{\mathfrak{N}}(T-zI)^{-1}(T^*\bar{\xi}I)^{-1}f,g)$$

$$=(Q_T(z)(F-F^*)Q_T(\xi)^*f,g)$$

$$+(z-\bar{\xi})(P_{\mathfrak{N}}(T-zI)^{-1}(T^*-\bar{\xi}I)^{-1}f,g)$$

Hence, it follows that

$$Q_T(z)-Q_T^*(\xi) = Q_T(z)(F-F^*)Q_T^*(\xi)+(z-\bar{\xi})(P_{\mathfrak{N}}(T-zI)^{-1}(T^*-\bar{\xi}I)^{-1} \upharpoonright \mathfrak{N},$$

and this proves (51). The identity (52) follows now from

$$(T^*(T^*-\bar{\xi}I)^{-1}f, T^*(T^*-\bar{z}I)^{-1}g) = (f-\bar{\xi}(T^*-\bar{\xi}I)^{-1}f, g+\bar{z}(T^*-\bar{z}I)^{-1}g)$$
$$= (f,g) + z(Q_T(z)f,g) + \bar{\xi}Q_T^*(\xi)f,g) + z\bar{\xi}(G_T(z,\xi)f,g) \quad f,g \in \mathfrak{N}.$$

Subtracting (53) from (51) gives immediately the identity (53).

It is clear from the given formulas (51),(52),and(53), that the functions $G_T(z,\xi)$, $M_T(z,\xi)$,and$L_T(z,\xi)$ are nonnegative kernels.

Since (T-T*)= $P_{\mathfrak{N}}$(T-T*)$P_{\mathfrak{N}}$, the definitions of $Q_T(z)$ and F in (38),(44) show that

$$-Q_T(z)(F-F^*)Q_T^*(\xi) == (P_{\mathfrak{N}}(T-zI)^{-1}(T^*-T)(T^*-\bar{\xi}I)^{-1}.$$

Combining this identity with (53) leads to (54).

It is a consequence of(7) that $+K_T(z-\xi)$ is an $\alpha-$sectorial kernel.

Proposition(1.2.3)[1]: Let T be a qsc – operator in a Hilbert space $\mathfrak{H}, \mathfrak{N} \subset \operatorname{ran}(T-T^*)$. Suppose that T is $\mathfrak{N}-$minimal,i.e., $\mathfrak{H} = \overline{\operatorname{span}}\{(T-z)^{-1}\mathfrak{N}: |z| > 1\}$. Then the following conditions are equivalent;

(i) $\mathfrak{N} = \mathfrak{H}$;

(ii) $G_T(z,z) = Q_T(z)Q_T(z)^*$ for at least one (and equivalently for every): with $|z| > 1$,

where $Q_T(z)$is Q- function of T defined by (38) and $Q_T(z,\xi)$ is defined by (47),

(iii) the operator- valued function $Q_T^{-1}(z) + zI$ is constant.

Proof: (i) $\Longrightarrow$ (ii)& (iii) if $\mathfrak{N} = \mathfrak{H}$ then $Q_T(z) = (T-T^*)^{-1}$ and the equality $G_T(z,z) = Q_T(z)Q_T(z)^*$ $all\ z, |z| > 1$,follows immediately from (51). Besides, $Q_T^{-1}(z) + zI + T$ for $all\ z, |z| > 1$

(ii) $\Longrightarrow$ (i) Now suppose that $Q_T(z,z) = Q_T(z)Q_T(z)^*$for some $z, |z| > 1$.

Then (38) and (51) yield

$$\|(T^*-\bar{z}I)^{-1}f\| = P_{\mathfrak{N}}\|(T^*-\bar{z}I)^{-1}f\| \qquad for\ every\ f \in \mathfrak{N}.$$

Therefore, $(T^*-\bar{z}I)^{-1}\mathfrak{N} \subset \mathfrak{N}$ which implies that the subspace $\mathfrak{N}$ is invariant underT*,and hence also under T, since ran($T-T^*$) $\subset \mathfrak{N}$. Because T is $\mathfrak{N}-$minimal, this leads to$\mathfrak{N} = \mathfrak{H}$.

(iii) $\Longrightarrow$ (ii) Suppose that $Q_T^{-1}(z) + zI$ is constant for $|z| > 1$. According to Proposition(1.1.3) the function $-Q_T^{-1}(z) + zI + F$ has a holomorphic continuation onto

Ext[-1,1]as a Nevanlinna function. Since $-Q_T^{-1}(z)+zI+F$ is constant for |z| > 1,one has

$$-Q_T^{-1}(z)-zI+Q_T(z)^{-*}+\bar{z}I+F^*=0, \qquad |z|>1$$

it follows that

$$\frac{-Q_T^{-1}(z)+Q_T(z)^{-*}-(F-F^*)}{z-\bar{z}}=1, \qquad |z|>1$$

and thus

$$\frac{Q_T(z)\left(-Q_T^{-1}(z)+Q_T(z)^{-*}-(F-F^*)\right)Q_T(z)^*}{z-\bar{\xi}}=Q_T(z)Q_T(z)^*, \quad |z|>1.$$

Therefore G (z,z)= $Q_T(z)Q_T(z)^*$ for all z,|z|>1.

Observe, that equality (51) can be rewritten in the following two equivalent forms:

$$\frac{-Q_T(z)^{-1}-F-(-Q_T(\xi)^{-1}-F)^*}{z-\bar{\xi}}$$

$$= Q_T(z)^{-1}P_{\mathfrak{N}}(T-zI)^{-1}-Q_T(\xi)^{-1} \tag{55}$$

and

$$\frac{-Q_T(z)^{-1}-F-zI-(-Q_T(\xi)^{-1}-F-\xi I)^*}{z-\bar{\xi}}$$

$$= Q_T(z)^{-1}P_{\mathfrak{N}}(T-zI)^{-1}\left(T^*-\bar{\xi}I\right)^{-1}Q_T(\xi)^{-*}. \tag{56}$$

These formulas show that $-Q_T(z)^{-1}-F$ and $-Q_T(z)^{-1}-F-zI$ ideed are Nevalinna functions. In particular, the conditions (i) – (iii) in Proposition(1.2.3)are equivalent to the right side of (56) to vanish.

Remark(1.2.4)[1]: The Q – function as defined in(38) can be interpreted as the Weyl function for a special kind of boundary value space of a duel pair of operators,cf [38],[40],[41]. To explain this. Let $A=A_0=K_0D_{A0}$ be a Hermitian contraction and let T be a qsc – extension of A,i.e., T is a contractive extension of a dual pair {A, A}. let A^*can be the adjoint linear relation of A in the Cartesian product $\mathfrak{H} \times \mathfrak{H}$. Then A* can be represented as follows:

$$A^* = \{\{f, Tf + \varphi\}: f \in \mathfrak{H}.\ \varphi \in \mathfrak{N}\} = \{\{f, T^*f + \psi\}: f \in \mathfrak{H},\ \psi \in \mathfrak{N}\}.$$

Define the following bounded linear operators acting from A^* into $\mathfrak{N}$:

$$\Gamma_0\{f, f'\} = P_{\mathfrak{N}}f, \Gamma_1\{f, f'\} = P_{\mathfrak{N}}T^*f - P_{\mathfrak{N}}f', \Gamma_2\{f, f'\} = P_{\mathfrak{N}}Tf - P_{\mathfrak{N}}f'$$

where $\{f, f'\} \in A^*$. *Then* $\{\mathfrak{N}, \Gamma_0, \Gamma_1, \Gamma_2\}$ *forms a boundary value space for* A^*.

In particular, for all $\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in A^*$ the following identity holds

$$(f', g) - (f, g') = (\Gamma_0\hat{f}, \Gamma_1\hat{g}) - (\Gamma_2\hat{f}, \Gamma_0\hat{g})$$

and moreover ker $\Gamma_1 = T^*$, ker $\Gamma_2 = T$, and

$$\ker \Gamma_0 = \{\{h, A_0h + \varphi\}: h \in \mathfrak{H}_0,\ \varphi \in \mathfrak{N}\}.$$

The corresponding γ −fields are the following operator functions

$$\begin{cases} \gamma_0(z)\varphi = -(A_0 - zI)^{-1}K_0^*D_{A_0}\varphi, \\ \gamma_1(z)\varphi = -(T^* - zI)^{-1}\varphi, \\ \gamma_2(z)\varphi = -(T - zI)^{-1}\varphi, \end{cases}$$

where $\varphi \in \mathfrak{N}$ and $|z| > 1$. It follows that $Q_T(z) = \Gamma_0\gamma_2(z)$ is given by

$$Q(z) = P_{\mathfrak{N}}(T - zI)^{-1} \upharpoonright \mathfrak{N},$$

and that $-Q_T^{-1}(z) = \Gamma_2\gamma_0(z)$ is given by

$$-Q_T^{-1}(z) = \left(K_0[A_0 + (A_0 - zI)^{-1}(I - A_0^2)]K_0^* - D_{K_0^*}XD_{K_0^*} + zI\right) \upharpoonright \mathfrak{N},$$

where T is decomposed as in (18) see also Proposition(1.1.7). In particular. this means that $Q_T(z)$ can be interpreted as the Weyl function corresponding to the boundary value space $\{\mathfrak{N}, \Gamma_0, \Gamma_1, \Gamma_2\}$ in the sense of [41],[42].

Let $A = A_0 + K_0D_{A_0}$ be a closed symmetric contraction in $\mathfrak{H}$ and let T be a qsc-extension of A given by the block matrix (18). If $Q(z) = P_{\mathfrak{N}}(T - zI)^{-1} \upharpoonright \mathfrak{N}$ is the Q- function of T, then by(46)the operator $\frac{\left(Q_T^{-1}(-1) - Q_T^{-1}(1)\right)}{2}$ is nonnegative on $\mathfrak{N}$ .Let

$$B_{Q_T} := B\left(-\frac{Q_T^{-1}(-1)+Q_T^{-1}(1)}{2}, \frac{Q_T^{-1}(-1)-Q_T^{-1}(1)}{2}\right) \tag{57}$$

be the operator ball $L(\mathfrak{N})$in with center

$$-\left(Q_T^{-1}(-1)-Q_T^{-1}(1)\right)/\,2 = -D_{K_0^*}XD_{K_0^*}$$

and equal left and right radii

$$-\left(Q_T^{-1}(-1)-Q_T^{-1}(1)\right)/\,2 = -D_{k_0^*}^2$$

Recall that it is the set of all operators in $\mathfrak{N}$ of the form

$$-\frac{Q_T^{-1}(-1)+Q_T^{-1}(1)}{2}+\left(\frac{Q_T^{-1}(-1)-Q_T^{-1}(1)}{2}\right)^{1/2} Y\left(\frac{Q_T^{-1}(-1)-Q_T^{-1}(1)}{2}\right)^{1/2}$$

$where \|Y\| \le 1.$

Thearem(1.2.5)[1]: Let A be a closed symmetric operator in aHilbert space $\mathfrak{H}$.Then the formula

$$\left(\widetilde{T}-zI\right)^{-1} = (T-zI)^{-1} - (T-zI)^{-1}\widetilde{B}\left(I+Q_T(z)\widetilde{B}\right)^{-1} P\,\mathfrak{N}\left(\widetilde{T}-zI\right)^{-1} \tag{58}$$

with $|z| \le 1$ gives a one – to – one correspondence between the resolvents of all qsc extensions $\widetilde{T}$ of A and all operators $\widetilde{B}$ belonging to the operator ball B_{Q_T} in (57)

Proof. By Theorem (1.1.4) every qsc – extension $\widetilde{T}$ of A can be written in the block form

$$\widetilde{T} = \begin{pmatrix} A_0 & D_{A_0}K_0^* \\ K_0^*D_{A_0} & K_0A_0K_0^* + D_{K_0^*}\widetilde{Y}D_{K_0^*} \end{pmatrix} \tag{59}$$

where $\|Y\| \le 1$. This together with(18) gives

$$B := \left(\widetilde{T}-T\right) \upharpoonright \mathfrak{N} = -D_{K_0^*}XD_0^* + D_{K_0^*}\widetilde{Y}D_{K_0^*} \tag{60}$$

which in view of (46) this means that $\widetilde{B} \in B_{Q_T}$ it follow from

$$\widetilde{T} - zI = T - zI + \widetilde{B}P_{\mathfrak{N}} \tag{61}$$

that

$$(T - z)^{-1} = \left(\tilde{T} - z\right)^{-1} + (T - z)^{-1}\tilde{B}P_{\mathfrak{N}}\left(\tilde{T} - z\right)^{-1}, \quad |z| > 1,$$

and compression to $\mathfrak{N}$ lead to

$$Q(z) = \tilde{Q}(z) + Q(z)\tilde{B}\tilde{Q}(z).$$

Since $Q_T(z)$ and $\tilde{Q}_T(z)$ are invertible by part (ii) of Proposition (1.1.7) one obtains

$$\tilde{Q}(z)^{-1} = Q(z)^{-1} + \tilde{B} = Q(z)^{-1}\left(I + Q(z)\tilde{B}\right) = \left(I + \tilde{B}Q(z)\right)Q(z)^{-1}.$$

Therefore, the operators $\mathfrak{N}$

$$1 + Q(z)\tilde{B} \text{ and } I + \tilde{B}Q(z), \quad |z| > 1$$

are invertible in $\mathfrak{N}$, too. Furthermore, by rewriting (61) in the form

$$\tilde{T} - zI = \left(I + \tilde{B}P_{\mathfrak{N}}(T - zI)^{-1}\right)(T - zI).$$

it is clear that $\left(I + \tilde{B}P_{\mathfrak{N}}(T - zI)^{-1}\right)^{-1} \in L(\mathfrak{H})$ for every $|z|>1$ and

$$\left(\tilde{T} - zI\right)^{-1} = (T - zI)^{-1}\left(I + \tilde{B}P_{\mathfrak{N}}(T - zI)^{-1}\right)^{-1}, \ |z| > 1 \quad (62)$$

It also follow from that

$$\left(\tilde{T} - zI\right)^{-1} - (T - zI)^{-1} = -\left(\tilde{T} - zI\right)^{-1}\tilde{B}P_{\mathfrak{N}}(T - zI)^{-1} \quad (63)$$

Now using the identities (61), (62) and

$$\left(I + \tilde{B}P_{\mathfrak{N}}(T - zI)^{-1}\right)^{-1}\tilde{B}P_{\mathfrak{N}} = \tilde{B}P_{\mathfrak{N}}\left(I + P_{\mathfrak{N}}(T - zI)^{-1}\tilde{B}P_{\mathfrak{N}}\right)^{-1}$$

one obtains

$$\left(\tilde{T} - zI\right)^{-1} - (T - zI)^{-1}$$

$$= (T - zI)^{-1}\left(I + \tilde{B}P_{\mathfrak{N}}(T - zI)^{-1}\right)^{-1}\tilde{B}P_{\mathfrak{N}}(T - zI)^{-1}$$

$$= (T - zI)^{-1}\tilde{B}P_{\mathfrak{N}}\left(1 + P_{\mathfrak{N}}(T - zI)^{-1}\tilde{B}P_{\mathfrak{N}}\right)^{-1}P_{\mathfrak{N}}(T - zI)^{-1}$$

$$= -(T - zI)^{-1}\tilde{B}\left(I + Q_T(z)\tilde{B}\right)^{-1}P_{\mathfrak{N}}(T - zI)^{-1},$$

which gives the required identity (58)

Conversely, assume that $\widetilde{B} \in B_{Q_T}$, that $\widetilde{B}$ is given by

$$-\frac{Q_T^{-1}(-1)+Q_T^{-1}(1)}{2}+\left(\frac{Q_T^{-1}(-1)-Q_T^{-1}(1)}{2}\right)^{1/2}\widetilde{Y}\left(\frac{Q_T^{-1}(-1)-Q_T^{-1}(1)}{2}\right)^{1/2}$$

for some $\|\widetilde{Y}\| \le 1$. By(46) one has $\widetilde{B} = -D_{K_0^*}XD_{K_0^*} + D_{K_0^*}\widetilde{Y}D_{K_0^*}$. Consider the qsc – extension $\widetilde{T}$ of A given by the block operator $\widetilde{T}$ of the form (59)which is determined by $\widetilde{Y}$. Then cleary $\widetilde{B} = (\widetilde{T} - T) \upharpoonright \mathfrak{N}$. As was shown above, the operator $1+Q_T(z)\widetilde{B}$ is invertible for all $|z| > 1$ and the resolvent of $\widetilde{T}$ takes the form(58).

The one – to – one correspondence is clear from the given arguments.

Observe that the Q – function $Q_T(z)$ of the operator $\widetilde{T}$ in(58) and the Q-function $Q_T(z)$ of T are connected via

$$Q_{\tilde{T}}(z) = P_{\mathfrak{N}}(T - zI)^{-1} \upharpoonright \mathfrak{N} = \left(I + Q_T(z)\widetilde{B}\right)^{-1}Q_T(z) = Q_T(z)\left(I + \widetilde{B}Q_T(z)\right)^{-1} = \left(\widetilde{B} + Q_T^{-1}(z)\right)^{-1} \tag{64}$$

Let $\mathfrak{N}$ be a Hilbert space. An operator valued function Q(z)with values in $L(\mathfrak{N})$ and holomorphic outside the unit disk is said to belong to the class $Q(\mathfrak{N})$ if:

(i) $Q(z)$ *has the expansion*

$$Q(z) = -\frac{1}{z}I + \frac{1}{z^2}F + 0\left(\frac{1}{z^2}\right), \quad z \to \infty; \tag{65}$$

(ii)the $L(\mathfrak{N})$ – valued function

$$G(z,\xi) = \frac{Q(z) - Q(\xi)^* - Q(z)(F - F^*)Q(\xi)^*}{z - \bar{\xi}}, \quad z \neq \bar{\xi},$$

with $|z|, |\xi| > 1$ is a nonnegative kernel;

(iii)the $L(\mathfrak{N})$ – valued function

$$L(z,\xi) = \frac{(1 - z^2)Q(z) - (1 - \bar{\xi}^2)Q(\xi)^* - (1 - z\bar{\xi})Q(z)(F - F^*)Q(\xi)^* - z(1 - \bar{\xi})I}{z - \bar{\xi}}$$

with $z \neq \bar{\xi}$, $|z|, |\xi| > 1$ is a nonnegative kernel:

(iv)there exit a complex number z_0, $|z_0| > 1$,and a vector $f \in \mathfrak{N}$ such that $G(z_0, z_0) f \neq Q(z_0) Q(z_0)^* f$.

If T is a qsc- operator in the Hilbert space $\mathfrak{H}$, $\mathfrak{N}$ is a subspace of $\mathfrak{H}$ such that ,$\mathfrak{N} \neq \mathfrak{H}$ and ran (T-T*)$\subset\mathfrak{N}$ and $Q_T(z)$ is its Q- function defined by (38) then according to Propositions (1.1.7),(1.2.1) and (1.2.2) the function $Q(z)$belongs to the class $Q(\mathfrak{N})$. *The converse statement is also true.*

Theorem(1.2.5)[1]: Let $Q(z)$be a function of the class $Q(\mathfrak{N})$. Then there exist a *Hilbert* space $\mathfrak{H} \supset \mathfrak{N}$, $\mathfrak{N} \neq \mathfrak{H}$, *and an* $\mathfrak{N}-$*minimal* qsc- operator T in $\mathfrak{H}$ such that $\mathfrak{N} \supset \mathrm{ran}(T - T^*)$ and

$$Q(z) = P_{\mathfrak{N}}(T - zI)^{-1} 1 \mathfrak{N} \quad \text{,for all } |z| > 1 \tag{66}$$

If, in addition, the $L(\mathfrak{N})-$valued function
$K(z,\xi): L(z,\xi) - Q(z)(F - F^*)Q(\xi)^*$

$$= \frac{(1 - z^2)Q(z) - (1 - \bar{\xi}^2)\, Q(\xi)^* - (1 + z)(1 - \bar{\xi})Q(z)(F - F^*)Q(\xi)^* - (z - \bar{\xi})I}{z - \bar{\xi}}$$

with $z \neq \bar{\xi}$, $|z|, |\varepsilon| > 1$ where F is given by(65),is an$\alpha-$ sectorial kernel with $\alpha \in [0, \pi/2)$,then the corresponding operator T belongs to the class $C(\alpha)$.

Proof. Step 1. Let $\tilde{\mathfrak{H}}$ the reproducing kernel Hilbert space associated with the nonnegative kernel $G(z,\varepsilon)$, *i.e.*, $\tilde{\mathfrak{H}}$ is the completion of

$$span\{G(.,\omega): f \in \mathfrak{N}, |\omega| > 1\}$$

with respect to the norm determined by the inner product

$$(G(.,\omega)f, G(.,\mu)g)_{\tilde{\mathfrak{H}}} = (G(\mu,\omega)f, g)_{\mathfrak{N}}.$$

For all $f \in \mathfrak{N}$ and $|\omega|, |\mu| > 1$

$$\| \bar{\omega}G(.,\omega)f, \bar{\mu}G(.,\mu)f\|^2_{\tilde{\mathfrak{H}}} = |\omega|^2(G(\omega,\omega)f,f)_{\mathfrak{N}}|\mu|^2(G(\mu,\mu)f,f)_{\mathfrak{N}}$$
$$-\mu\bar{\omega}(G(\mu,\omega)f,f)_{\mathfrak{N}} - \mu\omega(G(\mu,\omega)f,f)_{\mathfrak{N}}. \tag{67}$$

In view of (65) one has $Q(z) = (-1/z)I + °(1/z) as\ z \to \infty$, which implies that

$$\lim_{\omega \to \infty} \bar{\omega}G(z,\omega)f = -Q(z)f, \ |z| > 1, \tag{68}$$

and moreover that

$$\lim_{\mu,\,\omega \hat{\to} \infty} \overline{\omega} G(z,\omega) f = f, f \in \mathfrak{N}. \tag{69}$$

$(Here \hat{\to}$ *stands for the nontangental limit in a sector* $|arg\,(z) - \pi/2| \leq \alpha < \pi 2$

Hence(67) and(69)imply that the following limit exists in $\tilde{\mathfrak{H}}$

$$Kf := -\lim_{\omega \hat{\to} \infty} \overline{\omega} G(z,\omega) f \tag{70}$$

and defines a linear operatorK: $\mathfrak{N} \to \tilde{\mathfrak{H}}$ for which

$$\|Kf\|^2_{\tilde{\mathfrak{H}}} = lim_{\omega \hat{\to} \infty} \|\overline{\omega} G(\cdot,\omega) f\|^2_{\tilde{\mathfrak{H}}} = \lim_{\omega \hat{\to} \infty} |\omega|^2 (G(\omega,\omega) f, f)_{\mathfrak{N}} = \|f\|^2_{\mathfrak{N}} \tag{71}$$

Thus K is isometric. It follows from (68) that

$$(Kf, G(.,\mu) g)_{\tilde{\mathfrak{H}}} = -\lim_{\omega \hat{\to} \infty} \overline{\omega} (G(.,\omega) f, g)_{\tilde{\mathfrak{H}}}$$

$$= -\lim_{\omega \hat{\to} \infty} \overline{\omega} (G(\mu,\omega) f, G(.,\mu) g)_{\mathfrak{N}} = (Q(\mu) f, g)_{\mathfrak{N}},$$

which shows that

$$(K^* G(.,\mu) g) Q(\mu)^* g, \qquad g \in \mathfrak{N} \tag{72}$$

Step 2.Define the linear relation S in $\mathfrak{H}$ b

$$S = \{\{\textstyle\sum_{i=1}^n G(.,\omega_i) f_i + \sum_{i=1}^n k f_i + \sum_{i=1}^n \varpi_i G(.,\omega_i) f_i\} : f_i \in \mathfrak{N}, |\omega_i| > 1\} \tag{73}$$

By definition the doman of S isdense in $\tilde{\mathfrak{H}}$ *in fact S is a contractive linear operator in* $\tilde{\mathfrak{H}}$*, since*

$$\|\textstyle\sum_{i=1}^n G(.,\omega_i) f_i\|^2_{\tilde{\mathfrak{H}}} - \|\sum_{i=1}^n k f_i + \sum_{i=1}^n \overline{\omega}_i G(.,\omega_i) f_i\|^2_{\tilde{\mathfrak{H}}}$$

$$= \textstyle\sum_{i,j=1}^n (G(\omega_j,\omega_i) f_i, f_j)_{\mathfrak{N}} - \sum_{i,j=1}^n (f_i, f_j)_{\mathfrak{N}} - \sum_{i,j=1}^n \overline{\omega}_i (Q(\omega_j)^* f_i, f_j)_{\mathfrak{N}}$$

$$- \textstyle\sum_{i,j=1}^n \omega_j (Q(\omega_j)^* f_i, f_j)_{\mathfrak{N}} - \sum_{i,j=1}^n \omega_j \overline{\omega}_i G(\omega_j,\omega_i) f_i, f_j)_{\mathfrak{N}}$$

$$= \textstyle\sum_{i,j=1}^n (L(\omega_j,\omega_i) f_i, f_j)_{\mathfrak{N}} \geq 0,$$

where (71) and (72) have been used. Therefore, the operator S has a unique contractive continuation which is defined everywhere on $\widetilde{\mathfrak{H}}$ and for which the same notation S is preserved.

Step3. To calculate the imaginary part of S note that for $h = \sum_{i=1}^{n} G(.,\omega_i)f_i$ the the following identities holds

$$(Sh, h) = \left(\sum_{i=1}^{n} kf_i + \sum_{i=1}^{n} \overline{\omega}_i G(.,\omega_i)f_i + \sum_{j=1}^{n} G(.,\omega_j)f_j\right)_{\widetilde{\mathfrak{H}}}$$

$$= \sum_{i,j=1}^{n}(Q(\omega_j)f_i + \overline{\omega}_i G(\omega_j,\omega_i)f_i, f_j)_{\mathfrak{N}}.$$

Similarly one obtains

$$(Sh, h) = \sum_{i,j=1}^{n}(Q(\omega_j)^* f_i + G(\omega_j,\omega_i)f_i, f_j)_{\mathfrak{N}}$$

Since $Q(\omega_j) - Q(\omega_j)^* + (\overline{\overline{\omega_j}}, \omega_i)G(\omega_j,\omega_i) = Q(\omega_j)(F - F^*)(\omega_j)^*$,one obtain

$$\left((S - S^*)(\textstyle\sum_{i=1}^{n} G(.,\omega_i)f_i), \sum_{j=1}^{n} G(.,\omega_j)f_j\right)_{\widetilde{\mathfrak{H}}}$$

$$= \sum_{i,j=1}^{n}(Q(\omega_j)(F - F^*)Q(\omega_j)^* f_i, f_j)_{\mathfrak{N}}$$

$$= \sum_{i,j=1}^{n}((F - F^*)K^*G(.,\omega_i)\, f_i, K^*G(.,\omega_j)f_j)_{\mathfrak{N}}$$

$$= \left(K(F - F^*)K^*(\textstyle\sum_{i=1}^{n} G(.,\omega_i)f_i), \sum_{j=1}^{n} G(.,\omega_j)f_j\right)_{\widetilde{\mathfrak{H}}}$$

This implies that

$$S - S^* = K(F - F^*)K^*. \tag{74}$$

By the definition of (73)one has $(S - \overline{\omega}I)G(.,\omega_i)f = Kf$, so that

$$(S - \varpi I)^{-1}Kf = G(.,\omega)f, \quad f \in \mathfrak{N}, \ |\omega| > 1 \tag{75}$$

Step4 . Since K is isometric ran K is closed. Let $\mathfrak{H}_0 = kerK^*$ and define $\mathfrak{H} := \mathfrak{H}_0 \oplus \mathfrak{N}$. Observe,that according to(72) $h = \sum_{i=1}^{n} G(.,\omega_i)f_i$belongs to the subspace $\mathfrak{H}_0$of $\mathfrak{H}$if and only if $\sum_{i=1}^{n} Q(\omega_j)^* f_i = 0$. Now decompose $\widetilde{\mathfrak{H}} = \mathfrak{H}_0 \oplus$ ran K and define the operator $\mathfrak{A}: \widetilde{\mathfrak{H}} \rightarrow \mathfrak{H}$ by

$$\mathfrak{U}(x+y) = x + K^* y, \quad x \in \mathfrak{H}_0, \; y \in ran\ K.$$

Then $\mathfrak{U}$ if maps $\mathfrak{H}$ onto H and it is unitary. Hence, the operator T defined byT: $\mathfrak{U}S^*\mathfrak{U}$-1 is contractive in $\mathfrak{H}$ and(74) shows that $, ran(T - T^*) \subset \mathfrak{U}(ranK) =$ $\mathfrak{N}$. Furthermore for $f, g \in \mathfrak{N}$and|z|>1 the identities (72)and(75) yield

$$((T-z)^{-1} f, g)\mathfrak{H} = (S^* - zI)^{-1} \mathfrak{U}^{-1} f, \mathfrak{U}^{-1})_{\overline{\mathfrak{H}}}$$

$$= (S^* - zI)^{-1} Kf, Kg)_{\overline{\mathfrak{H}}}$$

$$= (Kf, (S^* - \bar{z}I)^{-1} Kg)_{\overline{\mathfrak{H}}}$$

$$= (Kf G(.,z) g)_{\overline{\mathfrak{H}}}$$

$$= (Q(z) f, g)_{\mathfrak{N}}$$

Thus

$$Q(z) = P_{\mathfrak{N}} (T - zI)^{-1} \upharpoonright \mathfrak{N}, \quad |z| > 1$$

Moreover, it follows from(75)that the operator T is $\mathfrak{N}$ −minimal.

Step 5.Finally it is shown that $\mathfrak{H}_0 \neq \{0\}$. If $\mathfrak{H}_0 = \{0\}$ then $\mathfrak{N} = \mathfrak{H}$ and by proposition (1.2.2) the equality $G(z,z) = Q(z)Q(z)^*$ holds for all |z|>1 . But this is impossible due to the condition (iv) of the definition of the class $Q(\mathfrak{N})$

Therefore $\mathfrak{H}_0 \neq 0, \mathfrak{N} \neq \mathfrak{H}$ and T is a qsc–operator whose Q– function $Q_T(z)$ coincides with $Q(z)$.

As to the last statement observe, that since $Q(z)$ is the form (66)the kernel $K(z, \xi)$ admits the operator representation(54) in Proposition (1.2.1) Since T is $\mathfrak{N}$ – minimal, it follows from(54)and that $T \in Q(\alpha)$.

The qsc-operator T constructed in Theorem(1.2.6) is $\mathfrak{N}$ -minimal. The next result shows that this model for functions $Q(z)$ belonging to the class $Q(\mathfrak{N})$ is essentially unique. Namely, the $\mathfrak{N}$ -minimal part of a qsc–operator T(and hence also of T*) is up to unitary equivalence uniquely determined by its Q-function; afact which is well known in the selfadjoint case.

Theorem(1.2.6)[1]: Let $\mathfrak{H}_1 = \mathfrak{H}_{01} \oplus \mathfrak{N}$ and $\mathfrak{H}_2 = \mathfrak{H}_{02} \oplus \mathfrak{N}$ be two Hilbert spaces, and let T_1 and T_2 be qas-operators in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, such that ran(T_1 −

$T_I^*) \subset \mathfrak{N}$ and $(T_2 - T_2^*) \subset \mathfrak{N}$ if $Q_{T_1}(z) \subset Q_{T_2}(z)$ in some neighborhood of infinity then the $\mathfrak{N}$ -minimal parts of T_1 and T_2 are unitarily equivalent.

Proof. Assume that $Q_{T_1}(z) = Q_{T_2}(z)$ holds in some neighborhood of infinity, say, for |z|, r >1. Then these functions coincide everywhere outside the unit disk. It follows from(40)and (44)that $F_1=F_2$, while (51) implies that

$$P_{\mathfrak{N}}(T_1^* - \xi I)^{-1}(T_1 - zI)^{-1} \upharpoonright \mathfrak{N} = P_{\mathfrak{N}}(T_2^* - \xi I)^{-1}(T_2 - zI)^{-1} \upharpoonright \mathfrak{N}$$

for all $|z|, |\xi| > 1; cf.$ (39)·Hence,for all f,g ∈ $\mathfrak{N}$

$$((T_1 - zI)^{-1}f, (T_1 - \xi I)^{-1}g) = (T_2 - zI)^{-1}f, (T_2 - \xi I)^{-1}g). \tag{76}$$

Now define the linear relation U from $\mathfrak{H}_1' = \{\overline{span}(T_1 - zI)^{-1}\mathfrak{N}: |z| > 1\}$ into $\mathfrak{H}_2' = \{\overline{span}(T_2 - zI)^{-1}\mathfrak{N}: |z| > 1\}$ by the formula

$$U = \left\{ \sum_{k=1}^{n} (T_1 - z_k I)^{-1} f_k, \sum_{k=1}^{n} (T_2 - z_k I)^{-1} f_k \right\}.$$

Then the identity(76) implies that U is a unitary operator from $\mathfrak{H}_1'$ onto $\mathfrak{H}_2'$ .In addition, $Uf = f$ *for* all $f \in \mathfrak{N}$,and

$$UT_1\left(\sum_{k=1}^{n} (T_1 - z_k I)^{-1} f_k \right) = \sum_{k=1}^{n} f_k + U\left(\sum_{k=1}^{n} z_k (T_1 - z_k I)^{-1} f_k \right)$$

$$= \sum_{k=1}^{n} f_k + \sum_{k=1}^{n} z_k (T_1 - z_k I)^{-1} f_k = T_2 U\left(\sum_{k=1}^{n} (T_1 - z_k I)^{-1} f_k \right)$$

Therefore, the simple parts of T_I and T_2 are unitarly equivalent.

The definition of the class $Q(\mathfrak{N})$ can be seen as an analytical characterization for Q-function of qas-operators T as defined in (38). Another characterization is established in the next theorem.

Theorem(1.2.7)[1]: Let $\mathfrak{N}$ be a Hilbert space. The following conditions are equivalent;

(i)the function $Q(\mathfrak{N})$belongs to the class$Q(\mathfrak{N})$;

(ii) (a) $Q(z) \in L(\mathfrak{N})$ is holomporphic in the domain $|z| > 1$ and with $F \in L(\mathfrak{N})$

it has the esymptotic expansion

$$Q(z) = -\frac{1}{z} I + \frac{1}{z^2} F + 0\left(\frac{1}{z^2}\right), \qquad z \to \infty;$$

(b)the function

$$-Q^{-1}(z) - zI - F$$

is not constant, it has holomorphic continuation onto Exit{-1,1} as a bounded Nevanlinna function, and the strong limits $Q^{-1}(\pm 1)$ exist;

(c) $Q^{-1}(-1) - Q^{-1}(1) \geq 0$ and for all $f, g \in \mathfrak{N}$ the following inequality holds:

$$\left|\left(\left(Q^{-1}(-1) - Q^{-1}(1)\right) f, g\right)\right|^2$$
$$\leq \left(\left(Q^{-1}(-1) - Q^{-1}(1)\right) f, f\right)\left(\left(Q^{-1}(-1) - Q^{-1}(1)\right) g, g\right).$$

Proof.(i) $\Rightarrow$ (ii)let the function $Q(z)$ belong the class $Q(\mathfrak{N})$. Then(a) holds by definition see (65). ByTheorem (1.2.5) the function $Q(z)$ has the operator representation $Q(z) = P_{\mathfrak{N}}(T - zI)^{-1} \upharpoonright \mathfrak{N}$, where T is a qas-operator in a Hilbert space $\mathfrak{H} \supset \mathfrak{N}$ such that $ran(T_1 - T_1^*) \subset \mathfrak{N}$.Now(b)follows from parts(ii)and (v) of Proposition (1.2.7) and Proposition (1.2.3) see also the identity (50). The inequality in(c) is obtained from part (iv)Proposition(1.2.6).

(ii)$\Rightarrow$ (i)Now assume that the function $Q(z)$has properties(a)-(c) . It follows from (a) and (c) that

$$Q^{-1}(z) = zI - F - G(z), \qquad G(z) = 0(1), z \to \infty$$

Here $G(z) \in N_{\mathfrak{N}}[-1,1]$ and $G(\infty) = 0$. Now it follows from Theorem(1.2.3)that $G(z)$has the representation

$$G(z) = K_0 (A_0 - zI)^{-1} \left(I - A_0^2\right) K_O^*$$

where A_0 is a selfadjoint contraction in some Hilbert space $\mathfrak{H}_0$ and$K_0 \in L(\mathfrak{H}_0, \mathfrak{N})$. Moreover, according to(15)

$$G(-1) = -Q^{-1}(-1) + I - F = K_0 (A_0 - z)^{-1} K_O^*$$

$$G(1) = -Q^{-1}(-1) + I - F = -K_0 (A_0 + z)^{-1} K_O^*.$$

This gives

$$\begin{cases} \dfrac{Q^{-1}(-1)-Q^{-1}(1)}{2} = I - K_0 K_0^*, \\ \dfrac{Q^{-1}(-1)+Q^{-1}(1)}{2} = K_0 A_0 K_0^* - F. \end{cases} \tag{77}$$

Now the assumption (c) implies that $I - K_0 K_0^* \geq 0$ and

$$\left|\left(\left(K_0 A_0 K_0^* - F\right) f, g\right)\right| \leq \left\|D_{K_0^*} f\right\| \left\|D_{K_0^*} f\right\|, \qquad f, g \in \mathfrak{N}$$

By (4)ther exists a contraction X in $\mathfrak{D}_{K_0^*}$suth that

$$-F = -K_0 A_0 K_0^* - D_{K_0^*} X D_0^*. \tag{78}$$

Consider the Hilbert space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{N}$and let the operator T in $\mathfrak{H}$ be given by the block form (18) .Then Tis a contraction and in fact,a qsc- extension of the closed symmetric contraction $A = A_0 + K_0 D_{A_0}$defined on $\mathfrak{H}_0$. According to Schur – Frobenius formula(see(7),(11))

$$P_{\mathfrak{N}}(\mathrm{T} - zI)^{-1} \upharpoonright \mathfrak{N} = -(G(z), \ |z|<1,$$

i.e., $Q(z)$ is the Q-function of T. Therefore $Q(z)$belongs to the class $Q(\mathfrak{N})$.

The model establish in Theorem(1.2.5)yields the following simple characterizations of Q-function corresponding to the extreme selfadjoint contractive extensions $A_\mu$and A_M of A within the class $Q(\mathfrak{N})$.

Proposition(1.2.8)[1]: Let $Q(z)$ belong to the class $Q(\mathfrak{N})$ and suppose that

$$\liminf_{x \uparrow -1} \left|\left(Q(x) f, f\right)\right|, \qquad for all f \in \mathfrak{N} \setminus \{0\} \tag{79}$$

$$\liminf_{x \downarrow 1} \left|\left(Q(x) f, f\right)\right| = \infty, \qquad for all f \in \mathfrak{N} \setminus \{0\} \tag{80}$$

Then$Q(z)$s a Nevanlinna function in $N_{\mathfrak{N}}[-1,1]$and it can be represented in the form $Q(z) = P_{\mathfrak{N}}(A_\mu - zI)^{-1} \upharpoonright \mathfrak{N}$ or $Q(z) = P_{\mathfrak{N}}(A_M - zI)^{-1} \upharpoonright \mathfrak{N}, z \in Ext[-1,1]$, respectively, where $A_\mu$and A_M are the left and right extreme sc-extension of some symmetric contraction A.

Proof. According to Theorem (1.2.5)the function $Q(z)$has the operator representation$Q(z) = P_{\mathfrak{N}}(\mathrm{T} - zI)^{-1} \upharpoonright \mathfrak{N}$, where T is a qsc – operator in a Hilbert space $\mathfrak{H} \supset \mathfrak{N}$, such that $ran(T - T^*) \subset \mathfrak{N}$. Moreover ,T is a qsc- entension of the closed symmetric contraction A defined by $A = T \upharpoonright domA$ with $domA = \mathfrak{H} \ominus \mathfrak{N}$. Let

$T_R = (T + T^*)/2$ and $T_1 = (T - T^*)/2$ be the real and the imaginary part of T, respectively, so that $T = T_R - iT_I^*$ .Then for $|x| > 1$

$$(T + xI)^{-1} = (T_R - xI)^{-1/2}(I + iB)^{-1}(T_R - xI)^{-1/2}$$

where

$$B = (T_R - xI)^{-1/2} T_I (T_R - xI)^{-1/2}$$

Is a bounded selfadjoint operator. This shows that for all $f \in \mathfrak{N}$

$$(Q(x)f, f) = (I + iB)^{-1}(T_R - xI)^{-1/2} f, T_I (T_R - xI)^{-1/2} f$$

Since $\left\|(I + iB)^{-1}\right\| \leq 1$, one obtains

$$|(Q(x)f, f)| \leq \left\| f, T_I (T_R - xI)^{-1/2} f \right\|^{2\ 2}$$

Now the assumption (79) implies that

$$\liminf_{x \uparrow -1} \left\|(T_R - xI) f\right\|^2 = \infty \quad \text{for all} \quad f \in \mathfrak{N} \setminus \{0\}.$$

This means that ran $(I + T_R)^{1/2} \cap \mathfrak{N} = \{0\}$,cf.,e.g.,[7].Since T_R is a sc-extensions of A one concludes from the characterization in (31) that $T_R = A_\mu$,cf.[28],[8],[23].Now, in view (30) $T_1 = 0$ and $T = A_\mu$ .The proof of the other statement is similar.

Some further characteristic properties of Q –functions in the selfadjoint case, in particular, of $Q_\mu -$ and $Q_M -$ *functions* corresponding to the sc- extensions A_μ and A_M have been estiablished in[8], including some corrections to result stated in[33]

The Krein formula(58) and the discussion following it concerning the formulas in(64) gives rise to a linear fractional transformation of Q- functions.

Theorem (1.2.10)[1]. Let Q(z)belong to the class $Q(\mathfrak{N})$. Then the function

$$Q(z) = (I + BQ(z)^{-1}, \qquad |Z| > 1$$

belongs to the class $Q(\mathfrak{N})$ if and only if

$$B \in B\left(\frac{Q^{-1}(-1)+Q^{-1}(1)}{2}, \frac{Q^{-1}(-1)-Q^{-1}(1)}{2}\right). \tag{81}$$

Moreover, $Q(z) = (I + BQ(z))^{-1}$ is a Nevanlinna function of the class $N_{\mathfrak{N}}[-1,1]$ if and only if B satisfies the conditions

$$B + Q^{-1}(1) \le 0, \qquad B + Q^{-1}(-1) \le 0. \tag{82}$$

Proof. First observe that, if $B \in L(\mathfrak{N})$ and $\mathfrak{N}(1 + BQ(z))^{-1} \in L(\mathfrak{N})$ for all $|z| > 1$, then it follows from(65) that

$$\tilde{Q}(z) = Q(z)(I + BQ(z))^{-1} = -\frac{1}{z}I + \frac{1}{z^2}(F - B) + 0\left(\frac{1}{z^2}\right), \qquad z \to \infty;$$

and clearly $\tilde{Q}^{-1}(z) = Q^{-1}(z) + B.$

Now assume that $\tilde{Q}(z) \in Q(\mathfrak{N})$.Then $\tilde{Q}(z) \in L(\mathfrak{N})$,for all $|z| < 1$, and since by Theorem(1.2.8). $Q(z)^{-1}$, $\tilde{Q}(z)^{-1} \in L(\mathfrak{N})$, $|z| < 1$,one has $B, (I + BQ(z))^{-1} \in L(\mathfrak{N})$ for all |z| > 1.Moreover, the limit values $\tilde{Q}^{-1}(\pm 1)$exist and satisfy

$$B + Q^{-1}(1) \le 0, \qquad B + Q^{-1}(-1) \le 0.$$

Now part(c) of Theorem (1.2.8)implies that

$$\left|\left(\left(B + \frac{Q^{-1}(-1)+Q^{-1}(1)}{2}\right)f, g\right)\right|^2$$

$$\le \left(\frac{Q^{-1}(-1)-Q^{-1}(1)}{2}f, f\right)\left(\frac{Q^{-1}(-1)-Q^{-1}(1)}{2}g, g\right) \tag{83}$$

holds for all $f, g \in \mathfrak{N}$ .Therefore, the condition (81) is satisfied.

Conversely. Let the operator $B \in L(\mathfrak{N})$satisfy the condition (81). By assumption Q(z) belongs to $Q(\mathfrak{N})$and Theorem (1.2.6) shows that $Q(z) = P_{\mathfrak{N}}(T - z)^{-1} \restriction \mathfrak{N}$, where T is qsc- operator in some Hilbert space $\mathfrak{H} \supset \mathfrak{N}$. Moreover,T is a qsc – extension of the symmetric contraction A=$T \restriction \mathfrak{H}_0$, $\mathfrak{H}_0 = \mathfrak{H} \ominus \mathfrak{N}$. Now by Theorem (1.2.4) the assumption (81) means that B defines a qsc- extensions $\tilde{T}$ of A whose resolvent is given by (58)with $\tilde{B} = B$. According to (64) the Q-function $Q_{\widetilde{T}}(z)$ is of the form $\widetilde{Q}(z) = Q(z)\left(I + BQ(z)\right)^{-1}$, $|z| < 1$ and as a Q-function belongs to the class $Q(\mathfrak{N})$; see the discussion preceding Theorem (1.2.6).

To prove the second part of the theorem, observe that in view of (42)

$$Q_{\widetilde{T}}^{-1}(-1) = B + Q^{-1}(-1) = D_{K_0^*}(Y + I)D_{K_0^*},$$

and

$$Q_{\widetilde{T}}^{-1}(1) = B + Q^{-1}(-1) = D_{K_0^*}(Y - I)D_{K_0^*},$$

where Y is a contraction in the supspace $\mathfrak{D}_{k_0^*} = \overline{\mathrm{ran}}\mathfrak{D}_{k_0^*}$. By Theorem (1.1.4) $\widetilde{T}$ is a selfadjoint contraction if and only if Y is a selfadjoint contraction in $\mathfrak{D}_{k_0^*}$ or equivalently ,B satisfies the conditions (82). Now , if (82) holds then $\widetilde{T}$ is selfadjoint and $Q(z)\left(I + BQ(z)\right)^{-1} = Q_{\widetilde{T}}(z) \in N_{\mathfrak{N}}[-1,1]$.

Conversely,if $Q_{\widetilde{T}}(z) \in N_{\mathfrak{N}}[-1,1]$then by part(vi) of Proposition (1.1.7) one has $\widetilde{F} = \widetilde{F}^*$and consequently $\widetilde{T} = \widetilde{T}^*$, *i.e.*,the conditions (82) are satisfied.

Chapter 2

Pure Point Spectrum of the Laplacian

All eigenvalues have infinite multiplicity and a countable system of orthonormal eigenfunictions with compact support is complete in the corresponding Hilbert space. In fact the correct interpretation of Δf^2 is as a singular measure, a result due to Kusuoka; we give a new proof of this fact. The second is based on a dichotomy for the local behavior of a function in the domain of Δ. At a junction point x_0 of the fractal: in the typical case (nonvanishing of the normal derivative) we have upper and lower bounds for $|f(x) - f(x_0)|$ in terms of $d(x, x_0)^\beta$ for a certain value β, and in the nontypical case (vanishing normal derivative) we have an upper bound with an exponent greater than 2. This method allows us to show that general nonlinear functions do not operate on the domain of Δ.

Sec(2.1) Fractal graphs

In the last decade, considerable attention has been paid by graph theorists to the study of spectra of the difference Laplacians infinite graphs. We refer separately of Mohar and Woess [61] Which is an excellent survey of this theory. Explicit computational results about the spectrum of the Laplacians are Known only when the graph under consideration satisfies certain kind of regularity property that leads to the existence of the absoulutely continuous spectrum ([see [61, 50]).

If we study fractal or disordered materials and the difference Laplacians are some discrete approximation, we should expect the spectrum to be pure point.

The first result of this type is the physics article [62] where the spectrum of the Laplacian on the Sierpinski lattice is considered. An application of the very interesting Renormalization Group method to this case was given by Bellissard in [52].

We study the spectrum of the Laplacians on so-called two-point self-similar fractal graphs (TPSG) (we mean the Laplacians which correspond to the adjacency matrix and the simple random walk). A good example of such a kind of graphs is the modified Koch graph which can be considered as the discrete approximation of the fractal set, namely the modified Koch curve [58].

Roughly speaking, we will prove that if the TPSG has an infinite number of cycles and the length of these cycle approaches infinity, then the spectrum of the Laplacians is pure point.

The problem of the description of the spectrum as a set in R is not trivial as shown by the example of the modified Koch graph. The spectrum for this graph is the union of two sets. The first set is the Julia set of the rational function.

$$R(z) = 9(z-1)\left(z-\frac{4}{3}\right)\left(z-\frac{5}{3}\right)\left(z-\frac{3}{2}\right)^{-1}.$$

This is a Cantor set of Lebesgue measure zero which may be obtained as a closure of a countable set of eigenvalues of the Laplacian with infinite multiplicity. The second set is a discrete countable set of eigenvalues with infinite multiplicity which has the limit points in the first set.

We note the new property of the eigenfunction of the Laplacians on TPSG: a countable system of orthonormal eigenfunction with compact support is complete in the Hilbert space where this operator is defined.

We consider in Theorem (2.1.5) the Anderson localization for the Schrodinger operator with Bernoulli potential on TPSG. It was proven that any eigenvalue of the Laplacian is an eigenvalue of infinite multiplicity of the Schrodinger operator for any coupling constant. Unfortunately, we cannot prove that the spectrum of such operator is pure point. However, we note that Aizenman and Molchanov [51] proved the localization of the spectrum in the standard Anderson model for suffiently large disorders on general graphs.

The two-point self-similar fractal graphs can be considered as nested pre-fractals with two essential fixed points introduced by Lindestrom [57].We also note that some questions about the integrated density of states of the Laplacian on fractal graphs were studied in [59 ,54].

Some special examples of TPSG were considered in physical models of the percolation theory (see [64, 53]).

Let G= (V,E) be a connected infinite locally finite graph, with vertex set V and edge set E. We suppose that the degree d_x of all vertices $x \in V$ is finite.

Let A=A(G) be the adjacency matrix of the graph G and P= P(G)= $(p_{u.v})$ $u, \in V$ be the transition matrix, where

$$P_{u.r} = a_{u.v} / d_u$$

and $a_{u.v}$ is the number of edges between u and v.

Associated with each of the preceding two matrices are the difference Laplacians.

$$\Delta_A = D(G) - A(G) \tag{1}$$

and

$$\Delta_p = I(G) - P(G), \tag{2}$$

where D(G) is the diagonal matrix of d_x, $x \epsilon V$ and I(G)is the identity matrix over V.

Let us introduce the spaces of functions on V.

$$l_2(V)=\{ f(x), x\epsilon v; \ , \sum_{x\epsilon v}|f(x)|^2 < \infty\} \tag{3}$$

with the inner product

$$(g, f) = \sum_{x\epsilon v} g(x)\bar{f}(x)$$

and

$$l_2^{\#}(V) = \{f(x), x\epsilon V; \sum_{x\epsilon v} d_x|f(x)|^2 < \infty\} \tag{4}$$

with inner product

$$(g, f) = \sum_{x\epsilon v} d_x g(x)\bar{f}(x)$$

We note that if the function deg(x) = d_x, xϵV is bounded, then the operators $\Delta_A$and $\Delta_P$are self-adjoint bounded operators in l_2 (v) and $l_2^{\#}(V)$), respectively.

Let us introduce so-called two point self-similar graphs.

Suppose M= (V_M, E_M) and G_0 = (V_0, E_0) are finite connected graphs and M is an odered graph. We fix some $e_0 \in E_M$, which is not aloop,and vertices $\alpha, \beta \in V_M$, *and* α_0, β_0. $\alpha \neq \beta$, $\alpha_0 \neq \beta_0$.

Informally speaking, the construction of a TPSG G is as fllows:to get G_1 from M and G_0we replace every edge (a,b) $\in E_M$,a,b $\in V_M$ by a copy of G_0 such that α_0 goes to a and β_0 to b. Then we take $\alpha_1 = \alpha$, $\beta_1 = \beta$ and proceed by induction. If a graph G_n= (V_n, E_n) with fixed vertices αn, $\beta_n \epsilon V_n$ is defined then the graph G_{n+1} is obtained by replacement of every edge (a,b) of M by the copy of G_n such that α_n

goes to a and β_n goes to b. The vertices α_{n+1}, β_{n+1} are the vertices α, β after this replacement.

We can assume that $G_n \subseteq G_{n+1}$ is the copy corresponding to e_0 and define infinite graph $G= \bigcup_{n=1}^{\alpha} G_n$.

Let us give a more formal definition.

Definition(2.1.1)[49]: A graph G is called TPSG with model graph M and initial graph G_0 if the following holds:

(i) There are finite subgraphs $G_0, G_1, G_2, \dots$ such that $G_n \subseteq G_{n+1}$, $n \geq 0$, and $G= \bigcup_{n\geq 0} G_n$.

(ii) For any $n \geq 0$and e $\in E_M$ there is a graph homomorphism $\psi_n^e : G_n \to G_{n+1}$ such that $G_{n+1}= \bigcup_{e \in E_M} \psi_n^e (G_n)$ and $\psi_n^{e_0}$ is the inclusion of G_n to G_{n+1}.

(iii) For all n $\geq$ 0 there are two vertices α_n , $\beta_n \in V_n$ such that ψ_n^e restricted to $G_n \backslash \{\alpha_n, \beta_n\}$ is a one-to-one mapping for every e $\in E_M$.
Moreover $\psi_n^{e_1} (V_n \backslash \{\alpha_n, \beta_n\}) \cap \psi_n^{e_2} V_n \backslash \{\alpha_n, \beta_n\}) = \emptyset$ if $e_1 \neq e_2$.

(iv) For n $\geq$ 1, there is an injection $K_n: V_M \to V_n$ such that $\alpha_n = K_n(\alpha)$,$_n$ $\beta_n = K_n(\beta)$ and for every edge e= (a,b) $\in E_M$, $\psi_{n-1}^e (\alpha_{n-1})= K_n(a)$, $\psi_{n-1}^e (\beta_{n-1})= K_n(b)$.

We say that the vertices α_n, β_n are the boundary vertices of G_n , i.e., $\partial G_n = \{\alpha_n , \beta_n\}$ and int $G_n = V_n \backslash \{\alpha_n , \beta_n\}$ are interior vertices of G_n.

Suppose M does not have loops and G_0 is just two vertices and one edge. Then two point self-similar graphs are in one-to-one correspondence to so-called post-critically finite (p.c.f) self-similar sets with the post-ciritical set consisting of two points. Namely the graphs G_n are isomorphic to so-called pre-fractals for such p.c.f. sets. However, G is not a p.c.f. set since the limiting procedures in these two cases are different. The definition of a p,c.f. set can be found in [55] or [56].

Definition (2.1.2)[49]: Two different vertices x and y of a graph Γ are equivalent if there is an automorphism φ of Γ such that φ(x)= y, φ(y)= x.

By induction it is easy to prove the following lemma.

Lemma (2.1.3)[49]: if the vertices $\alpha, \beta \in V_M$ and $\alpha_0, \beta_0 \in V_0$ are equivalent in M and G_0 , respectively, then vertices α_n, β_n are equivalent in G_n for all n.

Although our results are valid for nonsymmetric graphs (with some additional assumptions on the orientation of M) we do not consider such graphs for the sake of simplicity.

Let us introduce the graph $\widetilde{M}$ $(V_{\widetilde{M}}, E_{\widetilde{M}})$ which can be obtained in the same way as G_1 if we take the graph M instead of G_0 and the vertices α, β play the role of $\alpha_{0,}\, \beta_0$.

we define the graph $\tilde{G}_{n+2}$ by replacement of every edge of $\widetilde{M}$ by the copy of G_n such that for every edge (a,b) $\epsilon\ E_{\widetilde{M}}$, a, b$\epsilon V_{\widetilde{M}}$ we say α_n goes to a and β_n to b.

Lemma (2.1.4)[49]: .The graphs $\tilde{G}_{n+2}$ and G_{n+2} are isomorphic.

Proof. By definition $\tilde{G}_{n+2}$ can be written as

$$\tilde{G}_{n+2} = \cup_{e\in E_{\widetilde{M}})}\ \widetilde{\Psi}_n^e(Gn) \quad (5)$$

where the maps $\widetilde{\Psi}_n^e$ have the same properties as ψ_n^einDefinition(2.2.1).The proof follows by induction .

Let us introduce the space $l_2(x)$ by $l_2(X)$)= $\{f \in l_2(V): f(x) = 0\ for\ x \in V \setminus X\}$, where X $\subset$V. $l_2^*(X)$is defined analogously. By $\Delta_A(X)$, $\Delta_p(X)$ we denote the restriction of Δ_A, Δ_p to $l_2(X)$ $l_2^*(X)$more precisely, $\Delta_{A.P}(x)= P\Delta_{A.P}$ P, where P is the orthogonal projector to $l_2(x)$ or $l_2^*(X)$we will call these operators the Laplacians with zero boundary conditions on ∂G_n by $\Delta_A(n)$ and $\Delta_p(n)$.

By Lemma(2.1.3) there is isomorphism φ_n: $G_n \rightarrow G_n$ such that $\varphi_n(\alpha_n) = \beta_n$, $\varphi_n(\beta_n) = \alpha_n$. this is isomorphism induces unitary maps $\cup_n : l_2(G_n) \rightarrow l_2(G_n)$ and $\cup_n^*$: $l_2^*(G_n) \rightarrow : l_2^*(G_n)$ by formula :$\cup_n^*\, f = fo\varphi_n$.

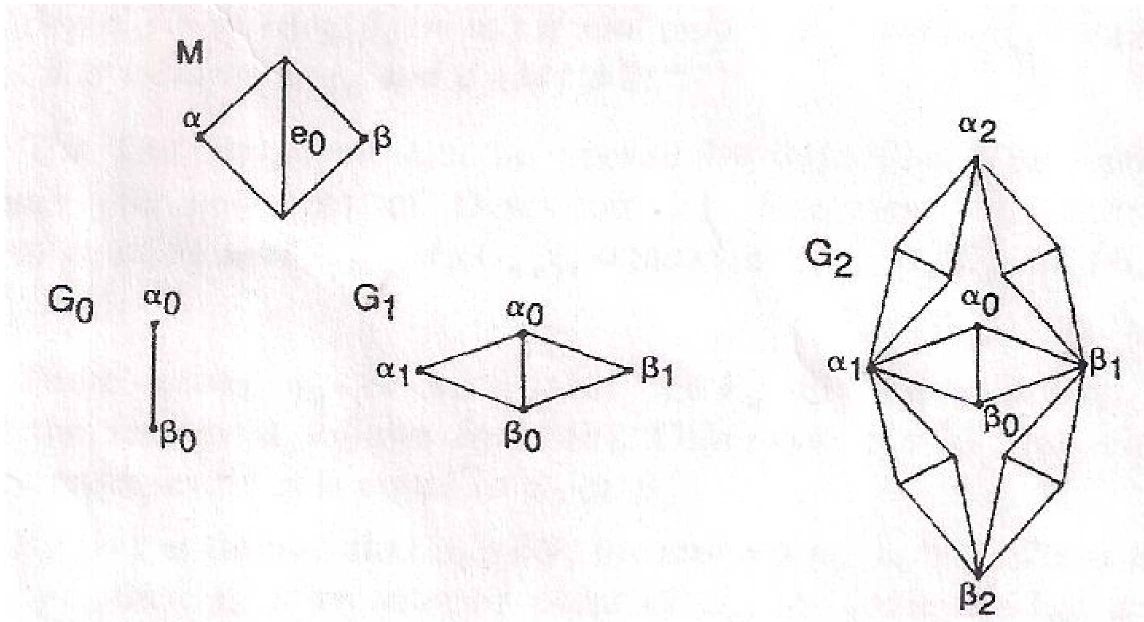


Figure 1

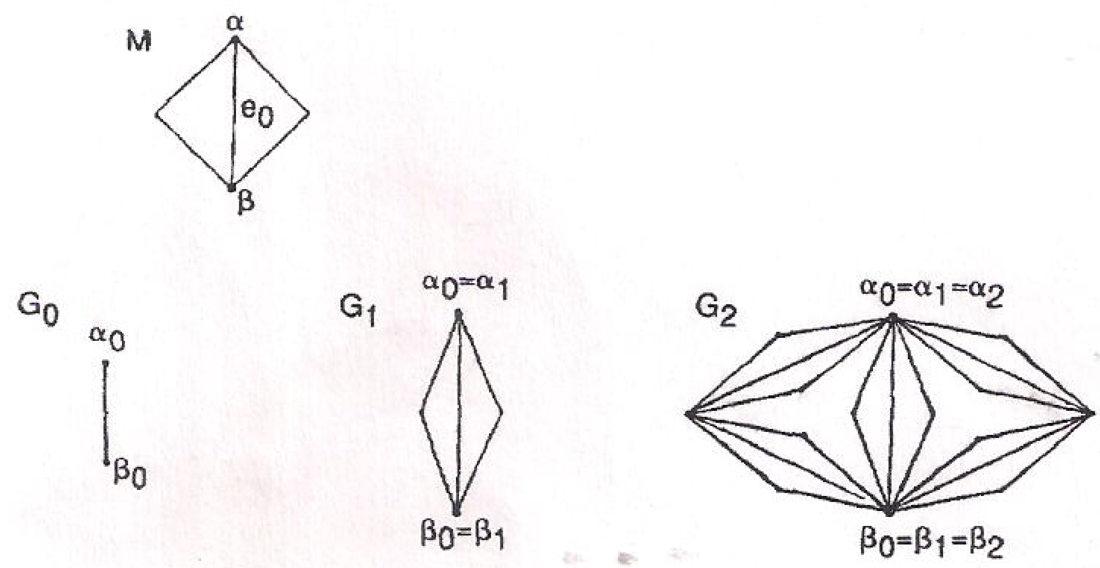


FIGURE 2

Lemma (2.1.5)[49]: $U_n(U_n^*)$ commutes with $\Delta_A(G_n) and\ \Delta_A(n)$ $(\Delta_P(G_n)$and $\Delta_P(n))$

Proof of this lemma immediately follows from the definition of Δ_A and Δ_p.

Let us consider the function deg(x)= d_x. It can occur that the function deg (.) is not bounded in general. Moreover, there can exit a point $x_0 \in V$ such that deg (x_0) =∞.

The next Lemma should be more clear from the following examples (see Figs. 1 and 2).

For an arbitrary graph $\tilde{G}$let us denote by d_α ($\tilde{G}$) the degree of the vertex x in $\tilde{G}$

Lemma (2.1.6)[12] :

(i) $d_{\alpha_n}(G_n)= d_{\alpha_0}((G_0).(d_\alpha((M))^n = d_{\alpha_{n-1}}(G_{n-1}). d_\alpha(M)$.

(ii) If $x \in$ int G_n ,then deg (x)= d_x (G_n)= d_x (G_{n+1})for every n≥ 1.

(iii) The function deg(x) is bounded if and only if d_α (M)=1.

(iv) If $x \in$V and $x \neq \alpha_0$, β_0 then deg (x) < ∞.

(v) Deg (α_0) =∞ (deg (β_0)= ∞) if and only if α is indicent to e_0 and d_α *(M)* ≥ 2 (β is incident to e_0 and d_β (M) ≥ 2).

Proof. The first statement can be proved by induction. The second follows from (ii) and (iii) of Definition (2.1.1) Statement (iii) follow from (i) and equality $max_{x\in G_{n+1}} d_x(G_{n+1}) = max\{max_{x\in G_n} d_x(G_n). d_{x_0}(G_n)\ max_{x\in M} d_x(M)\}$.

(iv) There exists $n_0 \in$N such that $x \in V_n$ for every n ≥ n_0. If x∈ int G_n, the ststement follows from (ii). Otherwise, X ∈ ∂G_n for every n≥ n_0 and consequently x is equal to α_0 or β_0.

(v) By (iv), it follows that $\alpha_0 \in \partial G_n$ for any n ≥ n_0, $n_0 \in$N. If α is not incident to e_0, then α_0 is an interior point of G_{n_1} for some n_1. Let α be incident to e_0 and d_α(M) ≥ 2. Then statement (v) follows from (i).

Definition (2.1.7)[49] . We denote by

$$\partial G=\{x, \deg(x)= \infty\}$$

the boundary of the graph G. if ∂G=∅, we say that G is a graph without boundary.

By Lemma (2.1.6) we obtain the following lemma.

Lemma(2.1.8)[49] (i) e_0 = (α , β) and d_α(M) ≥ 2, if and only if ∂G= {α_0, β_0}.

(ii) The boundary ∂G has only one point if and only if one of the points α or β is a vertex of e_0 and the degree of this vertex in M is not less than 2.

(iii) If conditions (i), (ii) are not satisfied for the graph G then ∂G= ∅.

Let us introduce the main results of this section the operator. We consider the operator Δ_p. if the graph G is without boundary, then the operator is self-adjoint because it is a linear symmetric bounded operator.

If G has the boundary, we define the operator Δ_p with zero boundary conditions, i.e.

$$\Delta_p^0 \colon l_2^\# (V^0) \to l_2^\# (V^0),$$

where

$$l_2^\# (V^0) \{f \in l_2^\# (V) \ \ f(x)=0, x \in \partial G\}.$$

The Δ_p^0 is a self-adjoint bounded operator, too.

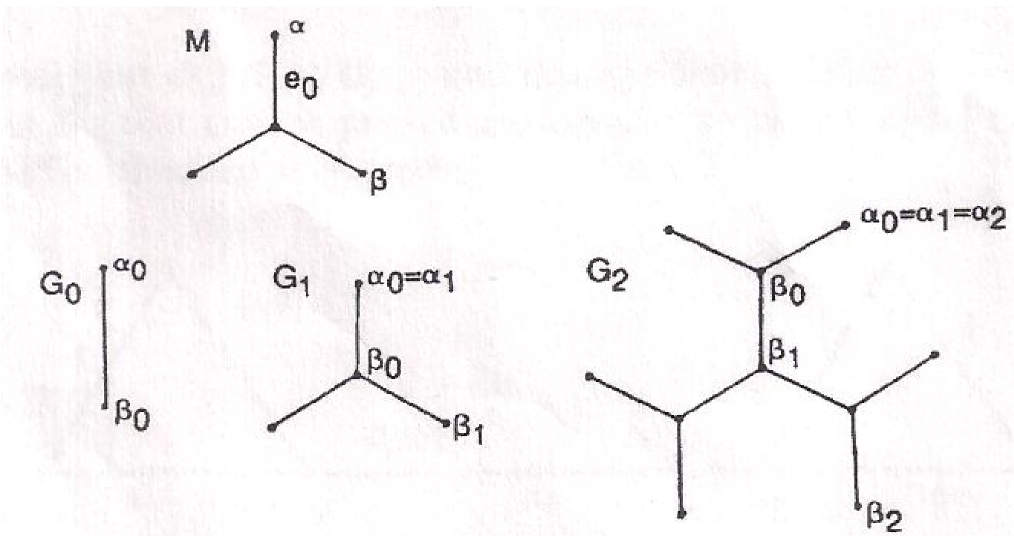


FIGURE 3

The simple example of a two-point self-similar graph such that the condition of Theorems(2.1.9),(2.1.10),(2.1.11),(2.1.12), (2.1.13) are not satisfied is the lattice Z. it is well known that the spectrum of the Laplacian in this case is absolutely continuous.

Condition (iv) in Definition (2.1.1) defines the structure of eignfunctions of Laplacians. It is easy to see that condition (i) – (iii) of Definition (2.1.1) are satisfied for Sierpinsky lattic but Theorems(2.1.10),(2.1.12),(2.1.11) ,(2.1.13). are not true in this case. By [52] it follows that there are such eigenvalues that if a function φ is an eigenfunction corresponding to one of them, then φ cannot have a compact support.

The problem of describing the spectrum as a set in R is hard enough as shown by the example of the operator Δ_p on the modified Koch graph in [58].

Let us introduce functions W:V→ R which do not change the nature of the spectrum of Laplacian ; i.e, the spectrum of the Schrodinger operator.

$$H= \Delta + W \tag{6}$$

will be pure point, too. Here we denote Δ_A and Δ_p by the same symbol Δ.

We note that periodic functions are potentials of this sort for the Schrodinger operator in l_2(Z") but only in the case of absolutely continuous spectrum.

Suppose that W_0: Vn_0→ R is a function such that $W_0(\varphi(x))= W_0(x)$, where φ: G_n→ G_n is an automorphism of G_n, $\varphi(x_n)= \beta_n$, $\varphi(\beta_n)= \alpha_n$. let us define the potential W:V→ R by induction. We denote by W_{m+1} the restriction of W on Vn_{0+m+1} and we suppose $W_{m+1}(x)= W_m(y)$, where x= $\psi^e_{n_0+m}$ (y), y$\in V_{n_0+m}$, e$\in E_M$ for every m ≥ 0.

Theorem (2.1.9)[49]: Let m∈N, δ >0 and c < ∞ be fixed numbers and for every n=1,2,…, there exists a linear operator Φ_n: $\mathcal{H}_n \to \mathcal{H}_{n+m}$ such that $\|\Phi_n\| \le$ c, (f, Φ_n(f)) ≥ $\delta\|f\|^2$ for any $f \in \mathcal{H}_n$ *and* $H\Phi_n(f)= \lambda^i_n \Phi n(f)$ for any $f \in \tilde{F}^i_n$, i=1, …, K(n).

Then the following statements hold:

(i) The operator H has only pure point spectrum. The set of eigenvalues is $\cup_{n\ge1} \cup_{1\le I \le K(n)} f\,\{\lambda^i_n\}$.

(ii) There is a countable set S$\subset \widetilde{\mathcal{H}}$ of orthonormal eigenfunctions of the operator H which is complete in $\mathcal{H}$.

(iii) If $\Phi_n(f) \notin \mathcal{H}_n$ for any nonzero $f \in \mathcal{H}_n$ and every $n \geq 1$, then each eigenvalue of H has infinite multiplicity.

(iv) H is a self-adjoint operator in $\mathcal{H}$.

Proof. At first we note from the definition of H_n that $\mathcal{H}_n = \oplus_{i=1}^{K(n)} \tilde{F}_n^i$.

Let

$$S_n = \{ f \in \mathcal{H}_n \ ; H f \in \mathcal{H}_n \}.$$

It is easy to see that $S_n \subset S_{n+1}$ for every $n \geq 1$.

We introduce the set S by the formula

$$S = \cup_{n\geq 1} \cup_{1\leq i\leq k(n)} \left(F_n^i \cap S_n \right)$$

and we note that the set $S_n \cap F_n^i$ is not empty for $n \geq m+1$ because $\Phi_n(f) \in \mathcal{H}_{n+m}$ for every $f \in \mathcal{H}_n$ and

$H_{n+m}\, \Phi_n(f)$

$$= P_{n+m}\Phi_n(f) = P_{n+m}(\lambda_n^i\ \Phi_n(f)) = \lambda_n^i\ \Phi_n(f),\ f \in F_n^i. \tag{7}$$

One can see from the condition of theorem (2.1.10) and (7) that if $\lambda \in \sigma(H_n)$ *then* λ *is* an eigenvalue of H. That gives us the inclusion

$$\cup_{n\geq 1} \cup_{1\leq i\leq K(n)} \{ \lambda_n^i \} \subset \sigma(H). \tag{8}$$

We will prove that the set S is complete in $\mathcal{H}$. Suppose that there exists $f \in \mathcal{H}$ such that $(f, g) = 0$ for any $g \in S$.

Let A be a subspace of $\mathcal{H}$ and P_A be the orthogonal projection to A.

Then

$$\|P_A f\| \geq \frac{1}{\|g\|} |(g, f)| \tag{9}$$

for every $g \in A$, $g \neq 0$, and $f \in \mathcal{H}$. This follows from the expression

$$|\ \|g\|^{-1} (g, f)| = \|g\|^{-1} |(P_A g, f)| = \|g\|^{-1} |(P_A^2 g, f)|$$

$$= \|g\|^{-1} (g, P_A f)| \leq \|g\|^{-1} \|g\| \|P_A f\| \leq \|P_A f\|.$$

Let us introduce the subspace A_n of $\mathcal{H}_n$ by the formula

$$A_n = \oplus_{i=1}^{K(n)} (\tilde{F}_n^i \cap S_n)$$

and let Q_n be the orthogonal projector to A_n.

If $f_n = P_n f$, n=1,2,…, by (9)and the condition of Theorem (2.1.9) we have

$$\|Q_{n+m} f_n\| \geq |(\Phi_n (f_n), f_n| \, \|\Phi_n(f_n)\|^{-1}$$

$$\geq (c \, \|f_n\|)^{-1} \, |(\Phi_n(f_n), f_n)| \geq c^{-1} \, \partial \, \|f_n\|. \quad (10)$$

Since $A_{n+m} \subset$ Span S we obtain $Q_{n+m} f = 0$. Hence

$$0 = \|Q_{n+m} f\| \geq \|Q_{n+m} f_n\| - \|f - f_n\| \geq c^{-1} \partial \|f_n\| - \|f - f_n\|.$$

This implies $f=0$ since $\|f - f_n\| \to 0$ as $n \to \infty$. Therefore S is complete in $\mathcal{H}$ and (i), (ii) is proved.

(iii) For arbitrary eigenvalue λ of H there exists a corresponding eigenfunction $f \in S$ and consequently there are such n_0, i that $f \in F^i_{n_0} \cap S_{n_0}$.

We denote $g_0 = \Phi n_0 (f)$ and $g_{K+1} = \Phi n_{0+Km}(g_K)$. then $\{gk\}_{k=0}^{\infty}$ is alinearly independent sequence of eigenfunctions of the operator H because, by the definition of Φ_n, $g_{k+1} \notin \mathcal{H}_{n_0+km}$.

(iv) It is enough to prove that Ran (H ± i) are complete sets in $\mathcal{H}$ (see [28]that follows from (ii) of our theorem.

Theorem(2.1.10)[49] : Suppose that the graph M has cycle and the edge e_0 belongs ro this cycle. Then the spectrum of the operator $\Delta_p(\Delta_p^0)$ is pure point. Moreover, a countable set of orthonormal eugenfunctions of $\Delta_p(\Delta_p^0)$ with compact support is complete in $l_2^{\#}$ (V)($l_2^{\#}$ (V^0)) and every eigenvalue has infinite multiplicty.

If e_0 does not belong to the cycle, we do not know the structure of the spectrum in general. However, there is the following theorem in a particular case.

Theorem (2.1.11) [49]: Suppose all conditions for the graph G in Theorem (2.1.10) hold. Then:

(i) The operator $\Delta_A(\Delta_A^0)$ is self-adjoint.
(ii) All statements of Theorem (2.1.10) are true.

Proof of Theorem (2.1.10) and Theorem (2.1.11)

By Theeorem (2.1.9)it is enough to construct the operaror Φ_n: $\mathcal{H}_n \rightarrow \mathcal{H}_{n+m}$, $m \geq 1$ with required properties. We will prove Theorem (2.1.10) for the operator Δ_p because the case of the Δ_A is the same.

Let $\mathcal{H}_n = l_2^{\#}$ (int G_n). We suppose that the cycle in M is defined bythe set of vertices $\{V_K\}_{K=0,}^{1}$ $v_i \in V_m$, $v_0=v_1$.

If l=2m, m∈N, we can introduce sets of edges

$$E^{+}= \{(v_{2k}, v_{2k+1})\}_{k=0}^{m} \subset E_M,$$

$$E^{-} = \{(v_{2k-1,}v_{2k})\}_{k=1}^{m} \subset E_M$$

We note that for any $x \in \psi_n^e$ ($V_n \backslash \delta G_n$) there is a unque $y \in V_n \backslash \partial G_n$ such that $x= \psi_n^e$ (y), $e \in E_M$.

The maps ψ_n^e, $e \in E^+ \cup E^-$ can be chosen such that if different edges e_1 and e_2 have a common vertex, then at least one of the following equalities holds

$$\Psi (\alpha_n)= \Psi_n^{e_2} (\alpha_n) \text{ or } \Psi_n^{e_1} (\beta_n)= \Psi_n^{e_2} (\beta_n). \qquad (11)$$

let us define operators $\Psi_{n,}^{e}$: $\mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$ For any $e \in E_M$ as follows:

$$\phi_n^e(f)(x) = \begin{cases} 0 & \text{if } x \notin \Psi_{n'}^{e} (V_n \backslash \partial G_n) \\ f(y) & \text{if } x = \Psi_{n'}^{e}(y), y \in V_n \backslash \partial G_n \end{cases}$$

Then we define the operator

$$\Phi_n = \sum_{e \in E+} \Phi_{n'}^{e} - \sum_{e \in E-} \Phi_{n'}^{e},$$

which maps $\mathcal{H}_n$ into $\mathcal{H}_{n+1}$. we will verify that it satisfies the conditions of Theorem (2.1.9)

We note that if e_1, $e_2 \in E_M$, and $e_1 \neq e_2$ then $\Phi_n^{e_1}(f)$ and $\Phi_n^{e_2}(f)$ have disjoint supports. Thus $\Phi_n^{e_1}(f)$ is orthogonal to $\Phi_n^{e_2}(f)$ and the bound $\|\Phi_n\| \leq c=1$ is obtained. By condition (ii) of Definition (2.1.1) we have $\Phi_n^{e_0}(f) = f$ and

$$(f, \Phi_n(f)) = \| f \|^2$$

for every $f \in \mathcal{H}_n$. Now if $f \in \tilde{F}_n^i$ then the equality

$$-\Delta_p \Phi_n(f)= \lambda_n^i \Phi_n(f)$$

follows from the definition of the operator Φ_n.

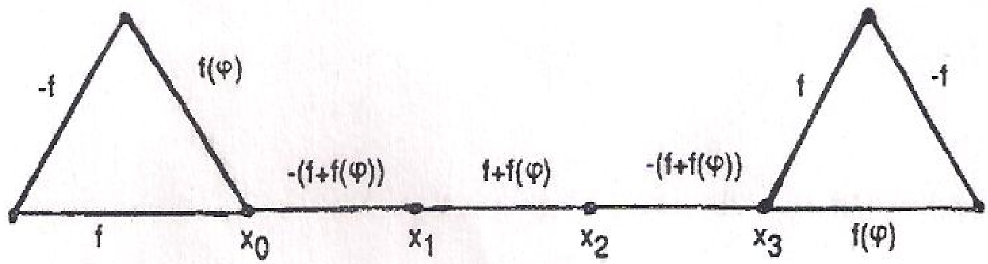


DIAGRAM 1

Since $\Phi_n(f)$ is an eigenfunction of the operator $-\Delta_p$ with compact support by the definition of the set S in the proof of Theorem (2.1.9) we find that S is a set of eigenfunctions with compact supports.

Let $l = 2m + 1, m \geq 1$.The construction of the operator Φ_n in this case is more delicate. In graph $\widetilde{M}$ (see Lemma (2.1.4) we have at least two cycles of length l, joining by a path, and e_0 belongs to one of these cycles.

Say these cycles are $\{v_k\}_{k=0}^{i}, \{u_k\}_{k=0}^{i}$, $v_0 = v_1, u_0 = u_1$ and they are joined by a path $v_0 = x_0, x_1, \ldots, x_r = u_0$.

Let $E_r^+ = \{(v_k, v_{k+1}), K \text{ is even}\}$, $E_r^- = \{(v_k, v_{k+1}), K \text{ is odd}\}$; $E_u^+, E_u^-, E_x^+, E_x^-$ are defined similarly. Also, we define operators $\widetilde{\Phi}_n^e$ analogously to Φ_n^e, using $\widetilde{\Psi}_n^e$ instead of Ψ_n^e (see Lemma(2.1.4)

Then

$$\Phi_n = \sum_{e\in E_r^+} \widetilde{\Phi}_n^e - \sum_{e\in E_r^-} \Phi_n^e - \sum_{e\in E_x^+}(\widetilde{\Phi}_n^e + \widetilde{\Phi}_n^e \text{ o } U_n^\#) + \sum_{e\in E_x^-}(\widetilde{\Phi}_n^e + \widetilde{\Phi}_n^e \text{ o} U_n^\#) + (-1)^{r+1} \left(\sum_{e\in E_u^+} \widetilde{\Phi}_n^e - \sum_{e\in E_u^-} \widetilde{\Phi}_n^e \right).$$

We suppose that condition (11) is satisfied in this case, too. This construction is sketched in Diagram 1 if r is odd and on Diagram 2 if r is even.

We note that Φ_n: $G_n \rightarrow G_{n+2}$ and this operator satisfies the conditions of Theorem (2.1.9)that can be proved analogously to case 1 using Lemmas (2.1.4)and (2.1.5) The theorem is proved.

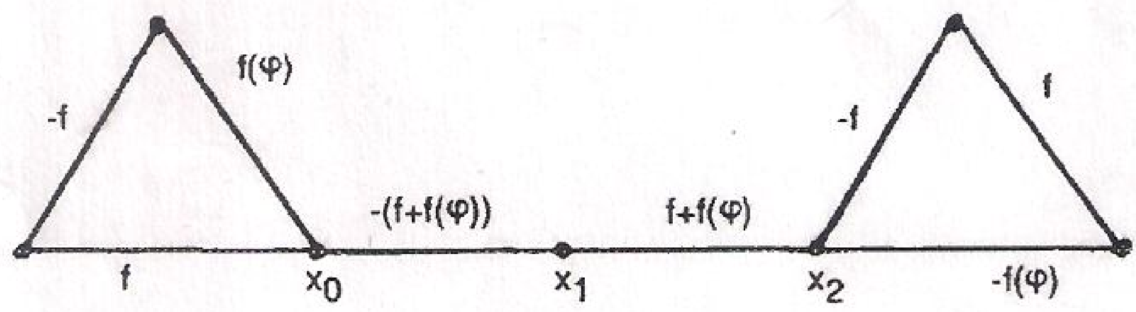


DIAGRAM 2

Theorem (2.1.12) [49]: Suppose that the graph M has an odd cycle and there is an isomorphism φ: M→ M such that φ(α)= β, φ(β)= α, and φ(e_0) ≠ e_0. If :

(i) the edge e_0 belongs to a path joining α and β or.

(ii) the edge e_0 belongs to a path joining α (or β) with the cycle then the conclusions of Theorem(2.1.10) hold for Δ_p and Δ_P^0.

Let us know consider the operator Δ_A. If the boundary of G is empty its action is well defined on all functions with compact support which form a dense subspace of l^2(V). If $\partial G \neq \phi$.

we define Δ_A^0 as an operator with zero boundary conditions (see above definition for Δ_P^0). This operator is symmetric and thus closable. We will denote its closure by the same symbol Δ_A(Δ_A^0).

Theorem (2.1.13)[49]: If all conditions of Theorem(2.1.10) are satisfied for the graph G, them the operator Δ_A(Δ_A^0.

We note that the operator Δ_A is not self-adjoint in general. An example of a locally finite graph with no unique self-adjoint extension of Δ_A was given in [26].

The condition of the existence of a cycle in the graph M is not a necessary condition for the spectrum to be pure point. Moreover the graph G may be a tree in this case (see Fig.3).

Proof of Theorem (2.1.12)and Theorem (2.2.13)

We will consider only operator Δ_p because the case of Δ_A is the same .

Also we assume that e_0 does not belong to a cycle. Otherwise it is a special case of Theorem (2.1.10)

We define

$$\mathcal{H}_n = \{f\in l_2^{\#} (Int\ G_n), \Delta_p f= \Delta_p(n)f \text{ or } U_2^{\#} f= f\}$$

We have $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. Let us show that $\widetilde{\mathcal{H}}= \cup_{n\geq 1}\mathcal{H}_n$ is complete in H = $ll_2^{\#}$ (V). For any $f\in$ H there is such n that $||f - f_n||\leq \frac{1}{4}||f||$, where f_n is the restriction of f to V_n. Since $\varphi(e_0) \neq e_0$ we have

$$|(f, f_n + U_{n+1}^{\#} f_n)| \geq |(f_n , f_n + U_{n+1}^{\#} f_n)| - ||f - f_n|| \,.\, ||f_n + U_{n+1}^{\#} f_n)||$$

$$\geq ||f_n||^2 - \frac{\sqrt{2}}{4}\ ||f_n||^2 \geq \frac{3}{16}\ ||f||^2$$

because $||f_n|| \geq \frac{3}{4}||f||$ and $||f_n + U_{n+1}^{\#} f_n|| = \sqrt{2}\ ||f_n||$.This implies that $\widetilde{\mathcal{H}}$ is complete since f is arbitrary and $f_n + U_{n+1}^{\#} f_n \in .\ \widetilde{\mathcal{H}}$.

Therefore we need only construct operator Φ_n which satisfies the conditions of Theorem (2.1.9)

(i) One can see that the graph $\widetilde{M}$ has two odd cycles joining by a path such that e_0 belongs to this path. In this case, Φ_n can be defined exactly the same way as in the proof of Theorem (2.1.11) for an odd cycle.

(ii)If, for example, α is incident to e_0, then there is a path $\alpha= x_0, \ldots, x_r= u_0$ and an odd cycle $\{u_k\}_{k=0}^{n}$, $u_0= u_u$, where $e_0= (x_0, x_1)$. Then Φ_n can be defined by

$$\Phi_n= \sum_{e\in E_u^-}(\Phi_n^e + \Phi_n^e \circ U_n^{\#}) - \sum_{e\in E_u^-}(\Phi_n^e + \Phi_n^e \circ U_n^{\#}) +(-1)^r \left(\sum_{e\in E_u^-} \Phi_n^e - \sum_{e\in E_u^-} \Phi_n^e\right).$$

where Φ_n^e, E_x^+, E_x^-, E_u^+, E_u^- are defined the same way as in the proof of Theorem (2.1.10)

If α_0 is not include with e_0 the proof is analogously (i). The theorem is proved.

Theorem (2.1.14)[49] :Suppose there exit different vertices y_0, y_1, y_2, $\in$ V(M) such that there are edges (y_0, y_1), (y_1, y_2) $\in$ E(M), $e_0= (y_0, y_1)$, $d y_0$ (M)= $d y_2$ (M)= 1 and the set $\{y_0, y_2\}$ does not coincide with the set $\{\alpha, \beta\}$.

The all result of Theorems (2.1.10) and (2.1.12) hold.

Proof . At first we suppose that $\propto$, β are not from the set $\{y_0, y_2\}$. Without loss of generality we can assume that $d_{x_0}(G_n) < d_{\beta_0}(G_{n+1})$ and $\Psi_n^{y_1,y_2}(\beta_n) = \beta_n$.
Let us define

$$\mathcal{H}_n = \{f\in l_n^{\#} (G): f(x)= 0 \text{ if } x\in V \setminus (V_n\setminus \beta_{n\,n})\}.$$

The operator Φ_n: $\mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$ can be given by the formula

$$\Phi_n(f)(x) = \begin{cases} f(x) & \text{if } x \in V_n \\ -f(x) & \text{if } x \in \Psi_n^{y_1,y_2}(y),\ y \in G_n \\ 0 & \text{otherwise} \end{cases} \qquad (11)$$

If $\alpha = y_0$ the definition of the operator Φ_n is the same.

Let $\propto = y_2$. Then we have to consider the graph $\widetilde{M}$(Lemma (2.1.4) instead of M which has the necessary properties to construct Φ_n by the formula (11).

Theorem (2.1.15)[49] . *if* the function W is defined as above, then all results of Theorems (2.1.10) ,(2.1.11) ,(2.1.12),(2.1.13) (2.1.14)(hold for the Schrodinger operator [6].

Let us consider the so-called Bernoulli potential $\{W(x),\ x \in V\}$ made of a sequence of i.i.d. random variables taking only two values 0 and 1.

We set

$$P\{W(x)= 0\}= P\{W(x)= 1\} = \frac{1}{2}, \quad x \in V.$$

We are interested in the random Schrodinger operator

$$H_\beta = \Delta + \beta W$$

with a coupling constant $\beta > 0$.

Proof .The proof is one –to-one to the proof Theorems(2.1.10),((2.1.11),(2.1.12) ,(2.1.13) ,(2.1.14)

Theorem(2.1.16)[49] :Let G satisfy conditions of one of the Theorems (2.1.10),(2.1.11) ,(2.1.12) ,(2.1.13), (2.1.14).Then for any $\beta > 0$ with probability one , every eigenvalue of Δ is an eigenvalue of H_β of infinite multiplicity.

Let $\mathcal{H}$ be a Hilbert space with the inner product(,) and $\mathcal{H}_n$, n= 1, 2, …, be a sequence of $\mathcal{H}$ such that $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and $\widetilde{\mathcal{H}} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ is dense in $\mathcal{H}$ We suppose that H is a closed symmetric operator on $\mathcal{H}$ such that $\widetilde{\mathcal{H}}$ belongs to the domain of definition of the operator H and $H_0 = P_n H P_n$, where P_n is the orthogonal projector on $\mathcal{H}_n$.

Then H_n: $\mathcal{H}_n \rightarrow \mathcal{H}_n$ and H_n is symmetric, too.

Let $\lambda_n^1, \ldots\ldots \lambda_n^{k(n)}$ be all distinct eigenvalues of the operator H_n (restricted to $\mathcal{H}_n$).

Let $\tilde{F}_n^i$ be the eigenspace corresponding to λ_n^i and let F_n^i be an orthnormal basis of $\tilde{F}_n^i$.

Proof . It is easy to see that if ψ is an eigenfunction of the operator Δ with compact support and supp $\psi \cap$ supp W= ϕ then the function ψ is an eigenfunction of the operator H_β.

Let Λ be a set of all eigenvalue of the Δ and let S a countable set of orthonormal eigenfunctions of the Δ with copmact support. For every $\lambda \in \Lambda$ there is an eigenfunction $f \in$ S and the integer n_0 such that supp $f \in G_{n_0}$.

We note that graph G can be written as the union of copies G_{n_0}. With probability one there is an infinity set of disjoint copies of G_{n_0} where W is zero. Consequently λ is an eigenvalue of the operator H_β of infinite multiplicity.

Sec(2.2) Sierpinski Gasket Type Fractals

There exists a well developed theory of Laplacians on a class of fractals including the familiar Sierpinski gasket. This theory may be obtained indirectly through the construction of probabilistic processes analogous to Brownian motion [68, 73, 74, 75, 83], or directly by taking renormalized limits of graph Laplacians, as in the work of Kigami [66, 69]. See [66, 69, 71, 76, 77, 78, 79, 82, 84, 85, 86, 87] for a sampling of works on this subject.

To define a Laplacian Δ on a fractal F, we need a Dirichlet from $\mathcal{E}(f, f)$, which is the analog of $\int |\nabla f|^2\, dx$, and a measure μ on F. The Dirichlet form determines the harmonic functions, which are minimizers of $\mathcal{E}(f, f)$ subject to boundary conditions. The Laplacian is determined by the analog of

$$\int \nabla f \cdot \nabla g \, dx = -\int g \Delta f \, dx + \text{boundary terms}, \tag{12}$$

with $\varepsilon(f, g)$ playing the role of the left hand side, and dμ substituting for dx on the right side. It is possible to interpret $\varepsilon(f, g)$ as the total mass of a signed $\nu_{f,g}$ defined by

$$\int h d\nu_{f,g} = \mathcal{E}(f\text{h}, \text{g}) + \mathcal{E}(f, \text{gh}) - \mathcal{E}(\text{h}, f\text{g}) \tag{13}$$

for h in the domain of ε[70] ,but the energy measures $v_{f,g}$may be unrelated to the measure μ used to define the Laplacian. In fact, Kusuoka [81] proves they are singular for many fractals. We will give a new proof of this fact that is considerably shorter, and that works for a larger class of examples. There is no immediate interpretation of the energy measure $v_{f,g}$ as an inner product of gradients. A theory of gradients is described in [85], but it is not clear yet if it can be related to energy measures.

The domain of the Laplacian is defined to be the set of continuous funictions f for which Δf is defined as a continuous funiction. This domain is well behaved in that it is dense in the continuous funictions in the uniform norm, and forms a core for defining - Δ as a self-adjoint positive definite operator on L^2 (dμ) with a discrete spectrum. we wish to point out that the domain is rather peculiar, however, in that it fails to have properties one might expect it to have by analog with the usual theory of Laplacians.We will show that the domain is not closed under multiplication; in fact, if f is any nontrivial function in the domain, then f^2 is not in the domain. We will also show that if we take a standard embedding of F into a Euclidean space, then the restriction to F of noncontant C^∞ functions are not in the domain.

One way to understand our results is to begin with the identity.

$$\Delta f^2 - 2f\ \Delta f|\nabla f|^2, \tag{14}$$

which holds pointwise for the usual Laplacian. There is an analogous result holding for a graph Laplacian. In our case we show that the right side blows up in the limit. Since $f\ \Delta\ f$ exits, this shows $\Delta\ f^2$cannot exist .in fact the identity(14) shows that nonexistence of Δf^2 is essentially equivalent to Kusuoka's singularity result for the energy measure $\nu_{f,f}$ .Our proof shows in more detail the divergence of Δf^2at specific points.

Another approach is to study the behavior of funiction in the domain of Δ in the neighborhood of a junction point on F (the junction points are the points in the graph approximations to F). We show that there is a dichotomy: either

$$c_1 d(x, x_0)^\beta \leq |f(x) - f(x_0)| \leq c_2 d(x, x_0)^\beta \tag{15}$$

for a certain $\beta < 1$, or

$$|f(x) - f(x_0)| \leq c d(x, x_0)^y \log d(x, x_0) \tag{16}$$

for a certain γ >2, with the first case holding if and only if the normal derivative of f at x_0 is nonzero. (This result was proved for harmonic funictions on the Sierpinki gasket in [69]. It is then simple to see that when the first case holds for f at x_0, neither case can hold for f^2 at x_0. The argument is then completed by observing that the normal derivative can vanish at every junction point only for a constant function. The same reasoning leads to the conclusion that essentially any nonlinear function, not just the square, will fail to act on the domain of Δ.

What are we to make of these negative result? One point of view is that they indicate certain natural limitations of the theory. For example, one might be tempted to develop a distribution theory on fractals with the role of the space of test functions played by the domain of all powers of Δ. Such a theory would not allow multiplication of a distribution by a test function.

Another point of view is that we need to broaden the definition functions to measures in such way that it is possible to define a Laplacian mapping functions to measures in such a way that $\Delta f^2 \Delta f^2$ is well defined. The drawback of this approach is that the domain and range of this Laplacian are not the same, so natural objects like Δ^2 would not be defined. Still another idea is that we need to pick the initial measure μ more carefully. In [75] a rather broad class of measures is allowed in the definition of Δ (in fact the notation Δ_μ is used there to indicate the independence of the Laplacian on the measure). In most detailed studies, however, the measure is assumed to be self-similar, and sometimes it is even required to be normalized Hausdorff measure (a specific self-similar measure). The rationale for this restriction is that all the energy measures $\nu_{f,\ g}$ are absolutely continuous with respect to v. This allows the definition of a carre du champs operator [67] Γ(f, g) via d$\nu_{f,g}$= Γ(f, g) dv. Thus if we use v in the definition of Δ, then all the problems disappear, and $\Delta f^2$is well defined. Of course, one must be wary of changing the problem in order to overcome difficulties. In this case there are sufficient doubts that we really know what constitutes "the natural measure" to use on fractals, that it would certainly be interesting to explore the properties of the Laplacian defined with this measure. Although v is not self-similar in the strict sense, it does satisfy identities of a self-similar nature (involving some negative coefficients and overlaps) that could be used to facilitate computations.

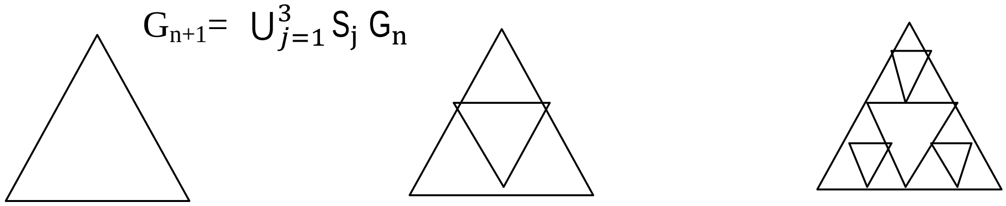
We will present our results in detail for the case of the symmetric Laplacian on the planar Sierpinski gasket. In this case it is very easy to give all definitions explicitly. The same arguments can be extended to many other examples of post critically finite (p.c.f.) self-similar fractals.

The Sierpinski gasket SG is the attractor of the iterated funiction system (i.f.s.) in the plane

$$S_j\, x = \frac{1}{2}(x - p_j) + p_j, \quad j=1,2,3,$$

where p_1, p_2, p_3 are vertices of a triangle T. We regard it as the limit of graph G_n, where G_0 is just the triangle T, and

$$G_{n+1} = \bigcup_{j=1}^{3} S_j\, G_n$$



G_0 G_1 G_2

FIG.1. The graphs G_0, G_1, G_2.

with the identification of the three junction points where the images S_jG_n meet (see Fig. 1). The three vertices of T will be regarded as boundary points of each graph G_n and SG. Note that every nonboundary vertex of G_n has exactly 4 neighboring vertices, so

$$-\Delta_n f(x) = f(x) - \frac{1}{4}\sum_{y\sim x} f(y) \tag{17}$$

defines a symmetric graph Laplacian on G_n, and

$$K_n(f, f) = \frac{1}{4}\sum_{x\sim y}(f(x) - f(y))^2 \tag{18}$$

the associated energy form. The Dirichlet form on SG is defined to be

$$\varepsilon(f, f) = \lim_{n\to\infty}\left(\frac{5}{3}\right)^n \varepsilon_n(f, f). \tag{19}$$

The choice of the renormalization factor $\left(\frac{5}{3}\right)^n$ is dictated by the fact

$$\left(\frac{5}{3}\right)^n \varepsilon_n(f, f) \geq \left(\frac{5}{3}\right)^{n-1} \varepsilon_{n-1}(f, f), \tag{20}$$

with equality holding if and only if $\Delta_n f(x) = 0$ at each vertex in G_n that is not in G_{n-1}. Thus the limit in (19) always exists as an extended real number.

A function on G_n is called harmonic if $\Delta_n f(x) = 0$ at every nonboundary vertex x of G_n; equivalently, f minimizes $\varepsilon_n(f, f)$ over all functions with the same boundary values. A function that is harmonic on G_{n-1} has a unique extension to a harmonic function on G_n, given by the following harmonic algorithm,

$$f(v_{12}) = \frac{2}{5} f(v_1) + \frac{2}{5} f(v_2) + \frac{1}{5} f(v_3) \tag{21}$$

If v_1, v_2, v_3 are the vertices of any small triangle in G_{n-1}, and v_{12} is the vertex in G_n between v_1 and v_2 (see Fig .2.2). A continuous function *f* on SG is called harmonic if its restriction to every G_n is harmonic. The space

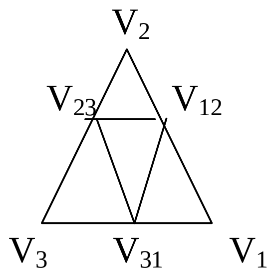


FIG 2. Labeling of vertices in G_n on a small tringle in G_{n-1}.

Of harmonic functions is 3-dimenional, and the values of *f* at the dense set of all junction points is determined by the boundary values $f(p_j)$ by successive applications of the harmonic algorithm.

We choose for the measure μ on SG the symmetric Bernoulli measure, which is the unique probability measure satisfying the self-similar identity

$$\mu = \frac{1}{3}\mu \circ S_1^{-1} + \frac{1}{3}\mu \circ S_2^{-1} + \frac{1}{3}\mu \circ S_3^{-1}. \tag{22}$$

This is simply the measure that assigns the weight $\left(\frac{1}{3}\right)^n$ to each of the 3^n small triangles in G_n (regarded as subsets of SG). With this choice of measure, the Laplacian on SG is just

$$\Delta f(x) = \lim_{n\to\infty} (3/2)5^n \, \Delta_n f(x). \tag{23}$$

This is interpreted in the following sense. Let *f* and g be continuos functions on SG. We say *f* belongs to the domain of Δ and Δ *f* = g provided $\lim_{n\to\infty} 5^n \Delta_n f(x) = g(x)$ for every non boundary junction point *x* (of course $\Delta_n f(x)$ is only defined for n large enough that x is a vertex of G_n).

The renormalization constant 5^n is explained as $3^n . \left(\frac{5}{3}\right)^n$, with 3^ncoming from the reciprocal of the measure and $\left(\frac{5}{3}\right)^n$ being the renormalization factor from the Dirichlet form. The definition is consistent with the definition of harmonic function, in that the harmonic functions are the solutions of Δ *f* = 0.

We also need the notion of normal derivative at the boundary points. Each boundary point has exactly 2 neighboring vertices in each graph G_n, so we define

$$(\partial_v)_n f(p)= \frac{1}{2} f(p) - \frac{1}{4} \sum_{y \sim p} f(y) \tag{24}$$

for the normal derivative in G_n, and

$$\partial_v f(p)= \lim_{n\to\infty} (\frac{5}{3})^n (\partial_v)_n (p) \tag{25}$$

for the normal derivative on SG, if the limit exists. On G_n we have the Gauss-Green formula

$$\varepsilon_n(f, g)= - \sum_x g(x) \Delta_n f(x) + \sum_p g(p)(\partial_v)_n f(p) \tag{26}$$

(the x -sum is over non boundary points, and the p-sum over the 3 boundary points). Multiplying by $\left(\frac{5}{3}\right)^n$ and taking the limit we obtain

$$\varepsilon(f, g)= - \int g\Delta f d\mu + \sum_p g(p) \partial_v f(p), \tag{27}$$

The Gauss- Green formula on SG. This makes sense provided f and g are in the domain of the Dirichlet form and f is in the domain of the Laplacian, and this argument proves that the normal derivatives exist for functions in the domain of the Laplacian, For f and g in the domain of Δ we can also obtain the symmetric variant

$$\int(g\Delta f - f\Delta g)d\mu = \sum_p(g(p) \partial_v f(p) - f(p) \partial_v g(p)) \tag{28}$$

by subtraction.

Now let $T_n= S_{j_1} \dots S_{j_n}T$ be any small triangle in G_n. For each vertex p of T_n we can define the outward normal derivative by

$$\partial_v f(p)= \lim_{n\to\infty} \left(\frac{5}{3}\right)^k \left(\frac{1}{2} f(p) - \frac{1}{4} \sum_{y \sim p} f(y)\right),$$

where the sum is over the 2 neghboring vertices of G_k that are in T_n. note that if we take the other triangle that has p as a vertex, the normal dertivative will change by a minus sign; and the normal derivative only depends on which side of p the triangle lies on. We then have the existence of normal derivatives at all junction points for functions in the domain of Δ, and the local Gauss-Green formula on T_n

$$\int_{T_n} (g \Delta f - f \Delta g) d\mu = \sum_{\partial T_n}(g(p) \partial_v f(p) - f(p) \partial_v g(p)). \tag{29}$$

Theorem (2.2.1) [65]: Let f be in the domain of Δ on SG, and let x be any junction point where $\partial_v f(x) \neq 0$. Then $\Delta f^2(x)$ is undefined, and in fact the limit in (51) is $+\infty$.

Proof. On G_n a simple computation yields

$$\Delta_n f^2(x) - 2 f(x)\, \Delta_n f(x) = \frac{1}{4} \sum_{y \sim x} (f(x) - f(y))^2. \tag{30}$$

We multiply by 5^n and try to take the limit. Since $5^n f(x)\, \Delta_n f(x) \to f(x)\, \Delta f(x)$ it suffices to show $5^n \sum_{y \sim x} (f(x) - f(y))^2 \to +\infty$. Now the assumption that $\partial_v f(x) \neq 0$ implies that there exists a sequence of neighboring vertices y_n in G_n (for large enough n) such that $| f(x) - f(y_n)|_2 \geq c(3/5)^n$, because otherwise $\partial_v f(x) = 0$ by(53). Thus $5^n \sum_{y \sim x} (f(x) - f(y))^2 \geq c((3/5))^2\, 5)^n$ which diverges because $(3/5)^2 . 5 = 9/5 > 1$.

Lemma (2.2.2) [65]: Let f be a nonconstant function in the domain of Δ. Then there exists a junction point where $\partial_v f(x) \neq 0$.

Proof. Apply the local Gauss-Green formula (57) with g ≡1, to obtain

$$\int_{T_n} \Delta f \, d\mu = \sum_{\partial T_n} \partial_v f(p) \, . \tag{31}$$

If we had $\partial_v f(x) = 0$ at every junction point, this would imply that the integral of Δf vanishes on every triangle T_n. Since Δf is continuous, this can only happen if f is harmonic. But it is easy to check that nonconstant harmonic functions have non-zero normal derivative at least at one vertex of every small triangle.

Corollary (2.2.3) [65]: if f is a nonconstant function in the domain of Δ, then f^2 is not in the domain of Δ.

Now we indicate how Δf^2 can be defined as a measure. First we observe that there is a positive energy measure $_{v\,f}$ obtained from the Dirichlet form.

If A is any polygonal set bounded by edges from one of the graphs G_k, then we let

$$v_f(A)\, f(A) = \lim_{n \to \infty} \left(\frac{5}{3}\right)^n \frac{1}{4} \sum_{\substack{x \backsim y \\ x, y \in A \cap G_n}} (f(x) - f(y))^2 \tag{32}$$

The existence of the limit follows from the same argument that gives the limit in (19). It is clear that v_f is finitely additive, and extends to a finite Borel measure by the Caratheodory extension theorem. It is easy to see that $_{v}f$ is non –atomic.In fact $v_f = v_{f,f}$ = defined by(14)

Now if we multiply (30) by $(5/3)^n$ and sum over all x in a polygonal set A, we can pass to the limit to obtain

$$\lim_{n \to \infty} 3^{-n} \sum_{x \in A \cap Gn} 5^n \Delta_n f^2(x) = 2\int_A f \Delta f d\mu + v_f (A). \qquad (33)$$

This suggests that we have

$$\Delta f^2 = 2f \Delta f d\mu + v_f \qquad (34)$$

for f in the domain of Δ, with the following definition for a statement ΔF=p where F is a continuous function and p a finite Borel measure.

Definition (2.2.4) [65]. We say a continuous function F is in the measure domain of Δ and ΔF= p if there exists a finite Borel measure ρ such that

$$\lim_{n \to \infty} 3^{-n} \sum_{x \in A \cap Gn} 5^n \Delta_n F(x) = p(A) \qquad (35)$$

for all polygonal sets A.

This definition is consistent with the function definition: if F is in the domain of Δ with ΔF= g then F is in the measure domain with ΔF= g dμ.

With this definition, (33) implies (34) .

We show next that v_f is singular with respect to μ. Because of the net structure of the triangles in SG, the analog of the Lebesgue differentiation of the integral Theorem holds for triangular sets. Thus, to show that v_f is singular with respect to μ, it suffices to show that for μ-a.e. x,

$$3^n v_f (T_n) \to 0 \qquad (36)$$

for T_n a sequence of triangles with $\mu(T_n) = 3^{-n}$ converging to x. For simplicity assume f is harmonic. Then we have simply

$$v_f (T_n) = \left(\frac{5}{3}\right)^n \frac{1}{4}((f(a_n) - f(b_n))^2 + f(b_n) - f(c_n))^2 + (f(c_n) - f(a_n))^2) , \qquad (37)$$

Where a_n, b_n, c_n are the vertices of T_n. The values $f(a_n)$, $f(b_n)$, $f(c_n)$ are derived from the values of f at the boundary points by applying a product of matrices determined by the harmonic algorithm (49). depending on the mappings that send T to T_n. Since constants do not contribute to the energy (37).it is convenient to factor out by the constants to obtain a 2-dimensional Hilbert space with energy norm. Taking n= 0 for simplicity, we have an orthonormal basis of the two harmonic functions h_1 and h_2 with boundary values (h_1(a), h_1(b), h_1(c))= (0, $\sqrt{2}$, $\sqrt{2}$) and (h_2(a), h_2(b), h_2(c))=(0, $\sqrt{2/3}$ - $\sqrt{2/3}$). With respect to this basis, the matrices have the form.

$$M_1=\begin{pmatrix} 3/5 & 0 \\ 0 & 1/5 \end{pmatrix}, \quad M_2=\begin{pmatrix} 3/10 & \sqrt{3}/10 \\ \sqrt{3}/10 & 1/2 \end{pmatrix}$$

$$M_3=\begin{pmatrix} 3/10 & -\sqrt{3}/10 \\ -\sqrt{3}/10 & 1/2 \end{pmatrix}.$$

We can then invoke the theory of products of random matrices, and

Furstenberg's theorem [70]: There exists an exponent $\alpha > \sqrt{3}/5$ such that

$$\text{Log}\|M_{j_n} \dots M_{j_1}\| \sim n \log\alpha \tag{38}$$

as $n\to\infty$ for a.e. choice of matrices. But this is exactly the same as μ-a.e. x in (36). To obtain the estimate (36).from (38).we need $\alpha <1/\sqrt{5}$. This inequality is proved in the next Theorem.

The next Theorem follows from a more general result proved by S.Kusuoka in [81]. Our proof seems to be shorter and more analytic in nature. Moreover, we show that our method can be applied to general finitely ramified fractals with fewer assumptions than are made in [81]. In the proof of Theorem (2.2.12) we avoid using Furstenberg's Theorem [72] although do use this Theorem in the proof of Theorem (2.2.5) in order to shorten the exposition.

In what follows the domain of the Dirichlet form ε is denoted by $\mathcal{F}$

Theorem (2.2.5) [65]. For any $f \in \mathcal{F}$ the measure ν_f is singular with respect to μ.Moreover, there exists a measure ν (singular to μ), such that all the energy measures are absolutely continuous with respect to ν.

Proof. For μ.a.e. point x we can define a unique sequence of matrices $A_n(x)= M_{j_n}$ as above. Then Furstenberg's Theorem implies that

$$\lim_{n\to\infty}\frac{1}{n}\log\|A_n(x)\dots A_1(x)\,v_0\|=\log\alpha$$

for μ.a.e. x. Here v_0 denotes the components of the harmonic function in the h_1, h_2 basis (mod constants), and ‖.‖ is now just the Euclidean norm on R^2. Since $M_1^2 + M_2^2 + M_3^2 = \frac{3}{5}I$, it follows that

$$\int_T A_n^*(x) A_n(x) d\mu(x) = \frac{1}{5} I.$$

Hence, by Jensen's inequality, for any nonzero vector v we have

$$\int_T \log\|A_n(x)v\| d\mu(x) < \frac{1}{2} \log \int_T \langle v, A_n^*(x) An(x)\, v \rangle d\mu(x) = \frac{1}{2}\log(\frac{1}{5}\|v\|).$$

Thus

$$\beta = A \sum_{\{v:\|v\|=1\}} \int_T \log||An(x)\, v\, ||d\mu(x) < \frac{1}{2} \log \frac{1}{5}. \qquad (39)$$

Denote $v_n(x) = A_n(x) \ldots A_1(x) v_0$. The matrices $A_n(x)$ are statistically independent with respect to μ, and so $A_n(x)$ is statistically independent of $v_{n-1}(x)$. Hence

$$\int_T \log v_n(x) d\mu(x) = \int_T \log \left\| A_n(x) \frac{v_{n-1}(x)}{\|v_{n-1}(x)\|} \right\| d\mu(x)$$

$$+ \int_T \log\|v_{n-1}(x)\| d\mu(x)$$

$$\leq \beta + \int_T \log\|v_{n-1}(x)\| d\mu(x)$$

By induction this implies $\log \alpha \leq \beta$ and so $\alpha < 1/\sqrt{5}$. Therefore v_h is singular with respect to μ for any harmonic function h.

Suppose now that $f \in \varepsilon$. Then f can be approximated by a sequence of functions $\{f_m\}$ that are continuous and piecewise harmonic on the triangles T_m[74, 75], The approximation is in energy norm, $\varepsilon(f - f_m - f_n) \to 0$ as $m \to \infty$, and also uniformly. Let $v = v_{h_1} + v_{h_2}$. Note that for any harmonic function h the measure v_h has a bounded density with respect to v since $v_{c_1 h_1 + c_2 h_2} \leq 2(c_1^2 v_{h_1} + c_2^2 v_{h_2})$. The same is true for the functions f_m. We claim that the measures v_{f_m} from a Cauchy sequence in the space of measures.

This will complete the proof that $v_f << v$ because $L^1(v)$ is already complete in the measure norm.

To see this we use the general estimate

$$| \nu_g(A)- \nu_{g'} (A)|^2 \leq \varepsilon(g+ g', g+ g')\, \varepsilon(g- g', g - g') \quad (40)$$

for any $g,g' \in \mathcal{F}$ and any polygonal subset A of SG. Taking g and g' to be f_m and f_K shows that $| \nu_{f_m}(A) - \nu_{f_K}(A)| \to 0$ uniformly in A as m, $K \to \infty$. This implies that $\{ \nu_{f_m} \}$ is a Cauchy sequence.

We prove (40). first in the case A= SG, when $\nu_g(SG)= \varepsilon(g, g)$ and $\nu_{g'}(SG)= \varepsilon(g', g')$, so(40). is just

$$\varepsilon(g, g)^2 + \varepsilon(g', g')^2 - 2\varepsilon(g, g)\, \varepsilon(g', g')$$

$$\leq(\varepsilon(g, g) + 2\varepsilon(g, g')+ \varepsilon(g', g'))(\varepsilon(g, g) - 2\varepsilon(g, g')+ \varepsilon(g', g')). \quad (41)$$

Multiplying out the right side of (41). and cancelling like terms reduces to

$$0\leq 4\varepsilon(g, g)\, \varepsilon(g', g') - 4\varepsilon(g, g')^2$$

which is just the Cauchy- Schwartz inequality. The modification of the argument for general A is simple. We just restrict all energies to A, to obtain $| \nu_g(A) - \nu_{g'}(A)|^2 \leq \nu_{g+g'}(A)$. Since $\nu_{g+g'}$ and $\nu_{g+g'}$ are positive measures, (40). follows.

It is clear by polarization that the energy measures ν_{fg} are also absolutely continuous with respect to ν.

The measure ν is independent of the choice of orthonormal basis (h_1, h_2),

and so it may be regarded as a natural measure associated to the Dirichlet form.

It is easy to see that the map $f \to (d\, \nu_f / d\, \nu)$ is a continuous quadratic map from the domain of ε to $L^1(\nu)$.

Theorem (2.2.6) [65]. For any $f \in \mathcal{F}$ the measure ν_f has no atoms.

Proof. In view of Theorem **(2.2.5).** it suffices to prove this when f is harmonic. In fact we will show

$$\nu_f (T_n)\leq (3/5)^n\, \varepsilon(f, f) \quad (42)$$

for any triangle of level n($T_n= S_{j_1} \ldots S_{j_n} T$). A simple computation shows that for any harmonic function f,

$$\nu_f (S_j T) \leq(3/5)\, \nu_f (T) \quad (43)$$

and in fact constant 3/5 is attained when $f(\nu_K)= \partial_{jk}$. We then obtain (42) by iterating (43), and it is clear that (42) implies ν_f has no atoms.

Let f belong to the domain of Δ on SG, and let x be any nonboundary junction point. Let T_n and T_n' denote the 2 small triangles in G_n that have x as a vertex, and let a_n, b_n and c_n, d_n denote the neighboring vertices to x in T_n and T_n'. We know

$$-\Delta f(x) = \lim_{n\to\infty} \frac{3}{2}5^n \left(f(x) - \frac{1}{4}\left(f(a_n) + f(b_n) + f(c_n) + f(d_n)\right)\right). \quad (44)$$

But what is the rate of convergence? To answer this question we first use the Gauss- Green formula to obtain an integral expression for the difference. Let h_n denote the piecewise harmonic function supported on the union $T_n \cup T'_n$ which takes the value 1 at x and 0 at a_n, b_n, c_n, d_n.

Lemma (2.2.7) [65]. We have

$$\frac{3}{2}5^n \left(f(x) - \frac{1}{4}f(a_n) + f(b_n) + f(c_n) + f(d_n))\right) + \Delta f(x)$$
$$= (3/2)3^n \int_{T_n \cup T_n'} h_n(y)\left(\Delta f(x) - \Delta f(y)\right) d\mu(y). \quad (45)$$

Proof . Apply (21) to T_n and T_n' and sum to obtain

$$\int_{T_n \cup T_n} h_n \Delta f \, d\mu = \sum_{\partial Tn} h_n \partial_v f - f\partial_v h_n + \sum_{\partial Tn} h_n \partial_v f - f\partial_v h_n.$$

Now the terms involving $\partial_v f$ cancel , because h_n is 0 except at x where the values of $\partial_v f$ differ by a minus sign. On the other hand we have of differ by a minus sign. On the other hand we have $\partial_v h_n(x) = \frac{1}{2}\left(\frac{5}{3}\right)^n$ and $\partial_v h_n(y) = -\frac{1}{4}\left(\frac{5}{3}\right)^n$ for $y = a_n, d_n, c_n, d_n$ for harmonic functions $\partial_v = (\frac{5}{3})^n (\partial_v)_n$ exactly). Thus we have

$$\int_{T_n \cup T_n} h_n \Delta f \, d\mu = \left(\frac{5}{3}\right)^n (f(x) - \frac{1}{4}(f(a_n) + f(b_n) - f(c_n) + f(d_n)))$$

and we obtain (45) by combining this with the fact that $3^n \int_{T_n \cup T_n} h_n \, d\mu = 2/3$.

It follows that the convergence in(41) is uniform, with the rate depending on the modulus of continuity of Δf. If Δf is Lipschitz, then the error is $O(2^{-n})$.

For the next result we consider any small triangle in G_{n-1} and label the vertices as in(21) We have the following extension of the harmonic algorithm:

Theorem (2.2.8) [65]. Let f be in the domain of Δ. Then

$$f(v_{12}) = \frac{2}{5} f(v_1) + \frac{2}{5} f(v_2) + \frac{1}{5} f(v_3)$$
$$+ \frac{2}{3}\frac{1}{5n}\left(\frac{6}{5}\Delta f(v_1) + \frac{2}{5}\Delta f(v_2) + \frac{2}{5}\Delta f(v_3)\right) + R_n, \quad (46)$$

where the remainder R_n satisfies

$$R_n = 0(5^{-n}) \quad (47)$$

uniformly depending only on the modulus of continuity of Δf. Moreover, if Δf is Lipschitz then

$$R_n = 0(10^{-n}) \quad (48)$$

Proof . Let $A_n = f(v_{12}) + f(v_{23}) + f(v_{31})$, $B_n = f(v_1) + f(v_2) + f(v_3)$ and

$C_n = \Delta f(v_{12}) + \Delta f(v_{23}) + \Delta f(v_{31})$. Apply (73) to each to the points v_{12}, v_{21} and v_{31} to obtain

$$f(v_{12}) - \frac{1}{4}(f(v_1) + f(v_2) + f(v_{31}) + f(v_{23})) = \frac{2}{3}5^{-n}\, \Delta f(v_{12}) + 0(5^{-n}) \qquad (49)$$

and so forth, and add to obtain

$$\frac{1}{2} A_n - \frac{1}{2} B_n = \frac{2}{3}5^{-n}\, C_n + 0(5^{-n}). \qquad (50)$$

Now the left side of (49) is just

$$\frac{5}{4} f(v_{12}) - \frac{1}{4}(f(v_1) + f(v_2) + A_n),$$

And we can substitute (50) to eliminate A_n, so

$$f(v_{12}) = \frac{1}{5}(f(v_1) + f(v_2) + B_n + \frac{4}{3}.5^{-n}\, C_n$$
$$+ 0(5^{-n})) + 5^{-n}(4/5)\, \Delta f(v_{12}) + 0(5^{-n})$$

which is (46)

Theorem (2.2.9) [65] . Let f be the domain of Δ and let x be any junction point

(a) If $\partial_v f(x) \neq 0$ then there exist positive constants c_1, c_2 such that

$$c_1(3/5)^n \leq |f(x) - f(a_n)| \leq c_2\,(3/5)^n \qquad (51)$$

(and the same for b_n, c_n, d_n).

(b) If $\partial_v f(x) = 0$

then

$$|f(x) - f(a_n)| \leq c_2 n 5^{-n} \qquad (52)$$

(and the same for b_n, c_n, d_n).

Proof. In either case we have

$$f(a_n) - f(b_n) = \frac{1}{5}(f(a_{n-1}) - f(b_{n-1})) + O(5^{-n})$$

by subtracting (46) and its analog. From this we obtain easily

$$|f(a_n) - f(b_n)| \leq c n 5^{-n} \qquad (53)$$

(we can eliminate the factor n from (53) and (52) if we assume that Δf is Lipschitz continuous).

By applying (46) twice and adding we obtain

$$f(x) - \frac{1}{2}(f(a_n) + f(b_n)) = \frac{3}{5}(f(x) - \frac{1}{2}(f(a_{n-1}) + f(b_{n-1}))) + O(5^{-n}).$$

if we write $v_n = (\frac{5}{3})^n\,(f(x) - \frac{1}{2}(f(a_n) + f(b_n)))$ this is just

$$v_n = v_{n-1} + O(3^{-n}), \qquad (54)$$

and since $O(3^{-n})$ is a convergent geometric series it follows that v_n is a Cauchy sequence, and the limit is a multiple of the normal derivative. In the case that the normal derivative is nonzero, we obtain $c_1 \leq v_n \leq c_2$ which yields (51) when combined with (53). On the other hand, if $v_n \to 0$ then (54) implies $v_n = O(3^{-n})$, which yields (53). when combined with (54).

Since $d(x, a_n) = 2^{-n}$, we can express (51)as

$$c_1\, d(x, y)^{\beta} \le |f(x) - f(y)| \le c_2\, d(x, y)^{\beta} \qquad (55)$$

for $\beta = \log(5/3)/\log 2 \approx .7369655$ and y equal to one of the points a_n, b_n, c_n, d_n. By using similar arguments it is easy to extend (83) to all points y. Similarly (80) becomes

$$|f(x) - f(y)| \le c d(x, y)^{y} \log d(x, y) \qquad (56)$$

for $\gamma =$ log 5/log 2≈ 2.3219281. This dichotomy was established in [38] for harmonic functions (Theorem (2.2.9),without the logarithm term in (56).

It is easy to give another proof of Corollary (2.2.3) , using this dichotomy, although we do not obtain Theorem(2.2.1) since we need to assume that a function belongs to the domain of the Laplacian in order to obtain the dichotomy at a single point. On the other hand, the dichotomy shows how difficult it is for a function to belong to the domain of the Laplacian, and allows us to deduce more general negative results.

Theorem (2.2.10) [65]. Let $\Phi: R \to R$ be any C^2 function such that Φ^n only has isolated zeroes. If f is any nonconstant function on SG in the domain of Δ, then $\Phi(f)$ is not in the domain of Δ.

Proof. By a simple extension of Lemma (2.2.2)we can find a junction point x_0 where $\partial_v f(x_0) \neq 0$ and also $f(x_0)$ is not a zero of Φ. Consider the function $g(x) = \Phi(f(x)) - \Phi'(f(x_0))\, f(x)$. If $\Phi(f)$ were in the domain of Δ.

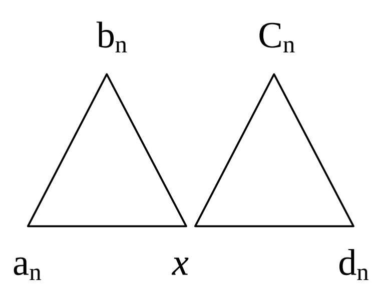


FIGURE. 3.

then g would be also.Theorem (2.2.8) would apply to g at x_0. But by Taylor's Theorem.

$$g(x) - g(x_0) = \Phi(f(x)) - \Phi(f(x_0)) - \Phi'(f(x_0))(f(x) - f(x_0)) = \frac{1}{2}\Phi''(z)(f(x) - f(x_0))^2 \qquad (57)$$

for z between $f(x_0)$ and $f(x)$. Since f is continuous, by taking x close enough to x_0 we can make $\Phi''(z)$ close to $\Phi''(f(x_0))$ which is not zero .Since f satisfies (51) at x_0, we obtain from (57) $c_1(3l5)^{2n} \le |g(x_0)-g(a_n)| \le c_2(3l5)^{2n}$ for large enough n, so g satisfies neither (51) nor (52).

Theorem(2.2.11)[65] Let f be any C^1 on R^2 with non constant restriction to SG Then f is not in the domain of Δ.

Proof. Suppose f were in the domain of Δ. By Lemma (2.2.2) there exists a junction point where $\partial_n f(x) \neq 0$. Then we are in part a) of Theorem (2.2.9), and (51) is inconsistent with f being C^1.

We can also observe directly that $\Delta f(x)$ is undefined at a junction point x if f is differentiable at x and the directional derivative in the direction perpendicular to the line segment containing x is non-zero. For example, if x lies on a horizontal line segment as in Fig. 4.1, then

$f(x)-\frac{1}{4}(f(a_n)+f(b_n)+f(c_n)+f(d_n)=\frac{\sqrt{3}}{4}\frac{\partial f}{\partial x_2}(x)2^{-n}+0(2^{-n})$. So if $(\partial f/\partial x_2)(x)\neq 0$, $\Delta f(x)$ is sundefined.

Let (K,S,$\{f_S\}_{S\in S}$)be a post critically finite self-similar structure and(D, r) be a harmonic structure as defined in [75]. Here K is a compact metric space, S=[1, 2, ...,N], f_S:K$\to K$ are continuous injections and r=(r_1, ...,r_N) is a collection of positive numbers. The reader may find all the definitions in [75]. This harmonic structure defines a Dirichlet form ε which satisfies a self-similarity relation

$$\varepsilon(f,f)=\lambda\sum_{i=1}^{N}\frac{1}{r_i}\varepsilon(f\circ F_i, f\circ F_i), \tag{58}$$

where λ is a constant associated with (D,r).

The p. c .f. self-similar set K has a finite boundary $V_0 \subset K$, and the bound-ary ofK $K_{\omega_1....\omega_n}=F_{\omega_1}....F_{\omega_n}(K)$ is $F_{\omega_1}...F_{\omega_n}(V_0)$.The important feature of a p.c.f. structure is that the intersection of the sets $K_{\omega_1....\omega_n}$ and $K_{\omega'_1....\omega'_n}$ contained in the boundary of these sets unless $\omega_i=\omega'_i, i=1,...,n.$

There are matricesM_1, ...,M_N such that the boundary values of harmonic function h on the boundary of $K_{\omega_1....\omega_n}$ are equal to $M_{\omega_n}....M_{\omega_1}\upsilon_0$ where υ_0 is the vector of the boundary values of h. For all $x\in K$, except a countable subset, there corresponds a unique sequence $\{\omega_m\}_{m\ge1}$ such that $\{x\}=\bigcap_{m\ge1}K_{\omega_1....\omega_n}$. Then we denote $A_m(x)=M_{\omega_n}$

Let μ be a Bernoulli measure on K such that $\mu(K_{\omega_1.....\omega_m}) = \mu_{\omega_1}....\mu_{\omega_m}$,where $\mu_i = \mu(K_i)$. Then matrices $A_m(x)$ are statistically independent with respect to μ with Prob{ $A_m(x) = M_i$ }= μ_i.

For any f from the domain $\mathcal{F}$ of ε we can define the measure ν_f in the same way as it was done for the Sierpinski gasket. Then there is a matrix $Q = (-D)^{1/2}$ such that for any harmonic function h[75].

$$\nu_h(K_{\omega_1}....{}_{\omega_m}) = \frac{\lambda^m}{r_{\omega_1}...r_{\omega_m}} \left\| QM_{\omega_m}...M_{\omega_1}\upsilon_0 \right\|^2, \tag{59}$$

where υ_0 is the vector of the boundary values of h

For the next Theorem we assume that

$$\mu_i = \frac{1}{r_i}.\mathrm{I} \tag{60}$$

The same assumption is made in [75]. Note that we have constants $r_1 = r_2 = r_3 = 1$ and $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$, the same as (60) up to a constant factor.

Theorem(2.2.12)[65]: Suppose that for any non constant harmonic function with boundary values υ_0 there exists m such that function $x \mapsto \|QA_m(x)....A_1(x)\upsilon_0\|$ is not constant.Then the measure υ_f is singular with respect to μ for any $f \in \mathcal{F}$

Proof. By (57) we have that

$$\|Q\upsilon_0\|^2 = \lambda\sum_{i=1}^{N}\frac{1}{r_i}\|QM_i\upsilon_0\|^2 = \lambda\int_K \|QA_1(x)\upsilon_0\|^2 d\mu(x)$$
$$= \lambda\int_K \|QA_m(x).......A_1(x)\upsilon_0\|^2 d\mu(x) \tag{61}$$

for any m. This relation is the same as[75]. The assumption of the Theorem implies, similar to (39), that for some m

$$\sup_{\{\upsilon_0;\|Q\upsilon_0\|=1\}} \int_K \log\|QV_m(x)\| \; d\mu(x) = \beta < -\frac{m}{2}\log\lambda, \tag{62}$$

where $v_m(x) = A_m(x).......A_1(x)\upsilon_0$.

In this proof for the sake of simplicity we assume that for any nonconstant harmonic function $\|Qv_m(x)\| \neq 0$ for all m and x. Otherwise one can change the expression under the integral in (62) to log($\|QV_m(x)\| + \delta$) If δ >0 is small then the inequality (62) still holds though with a larger β.Then, by induction,

$$\int_K \|Qv_{mn}(x)\| d\mu(x)$$

$$
\begin{aligned}
&= \int_k \log\left\| QA_{mn}(x)\ldots\ldots A_{m(n-1)}(x)\frac{v_{m(n-1)}(x)}{\left\|Qv_{m(n-1)}(x)\right\|}\right\| d\mu(x) \\
&+ \int_k \log\left\|Qv_{m(n-1)}(x)\right\| d\mu(x) \\
&\leq \beta + \int_k \log\left\|Qv_{m(n-1)}(x)\right\| d\mu(x) \leq n\beta
\end{aligned}
$$

if= $\|Qv_0\| = 1$. Moreover, one can see that for any sequence $\omega_1,\ldots,.\omega_k$ we have

$$\int_{k_{\omega_1,\ldots,.\omega_k}} \log \left\|Qv_{mn+k}(x)\right\| d\mu(x) \leq \mu_{\omega 1},\ldots,.\mu_{\omega k}\left(n\beta + \log\left\|Qv_k(x)\right\|\right).$$

This implies that (at least for a subsequence)

$$\limsup_{n\to\infty}\frac{1}{n}\log\left\|Qv_n(x)\right\| \leq \frac{1}{m}\beta < -\frac{1}{2}\log\lambda \tag{63}$$

for $\mu - a.e.x.$

Inequality (62) follows from the fact that the sequence $\left\{\log\left\|Qv_{mn}(x)\right\| - \beta n\right\}_{n=1}^{\infty}$ is a super martingale on the probability space (K, μ). To prove it in more elementary terms, define $f_k(x) = \log\left\|Qv_{mk}(x)\right\|, g_{k+1} = \left(\mu_{\omega 1},\ldots,.\mu_{\omega k}\right)^{-1} \times \int_{k_{\omega_1,\ldots,.\omega_k}} f_{k+1}(x)d\mu(x) forx \in K_{\omega_1}\ldots_{\omega_n}$ $andh_k(x) = f_k(x) - g_k(x).$

It is easy to see that $\{h_n\}_{n=1}^{\infty}$ is a bounded orthogonal sequence in L_μ^2 and so $\left\|(1/n)\sum_{k=1}^{n} h_n\right\|_{L_\mu^2} \to 0 asn \to \infty$ At the same time $g_{n+1}(x) \leq \beta + f_n(x).$ that is $f_{n+1}(x) \leq \beta + f_n(x) + h_{n+1}(x).$.

Then the L^2-convergence implies that(at least for a subsequence) inequality (63) holds for
$\mu - a.e.x.$

Thus by (58),(59),(60),(61),(62).fo $\mu - a.e$. sequence $\omega_1, \omega_2,\ldots$ we hav
for any harmonic function h.

To define the measure ν, let $\{h_1,\ldots,h_p\}$ be an orthonormal basis of the nonconstant harmonic functions in $\|Q.\|$-norm. Then $\nu = \nu_{h_1} + \ldots . \nu_{h_p}$ .However, if not all matricesM_1, ...,M_N are invertible, ν -measure of some open sets may not be positive.

The rest of the proof goes in the same way as in Theorem (2.2.6).

The singularity of the measures ν_f was proved in [81]under the assumption that the matrices{M_1, ...,M_N}are invertible and strongly irreducible, and an additional assumption on a certain index [81].

Theorem(2.2.13)[65]: Under the hypotheses of Theorem(2.2.12).,the measure ν_f has no atoms,for any $f \in \mathbb{F}$.

Proof . we claim that there is a constant ρ <1 and a positive integer n such that for any harmonic function f,

$$\nu_f\left(K_{w_1......w_n}\right) \le \rho \nu_f\left(K\right) \qquad (64)$$

for any choice of ($(w_1,....., w_n)$ Once we have (64), the proof is the same as Theorem (2.2.6), using (64) in place of (43). By a compactness argument.

Chapter 3
m-Function and Inverse Spectral Analysis

We show an extenstion of the theorem of Hochstadt (who proved the result in case n=N) that n eigenvalues of an N×N Jacobi matrix H can replace the first n matrix elements in determining H uniquely. We completely slove the inverse problem for $(\delta_{n,}(H-z)^{-1}\delta_n)$ in the case N<∞

Sec(3.1) Finite and Semi-Infinite Jacobi Matrices

There is an enormous literature on inverse spectral problems for- $d^2/dx^2 + V(x)$(see[89,120,147-151,155], but considerably less for their discrete analog, the infinite and semi-infinite Jacobi matrices (see e.g.,[91,92,94-96,101-110,113,116-119,121,123,128,129,133-135,141-143,152-154,157,158,160-162]) and even less for finite Jacobi matrices[97,98,112,115,130-132,136-139].Our in this section is to study the last two problems using one of the most powerful tools from spectral theory of $-d^2/dx^2 + V(x)$, the m- finctions of Weyl.

Explicitly, we study finite $N \times N$ matrices of the form

$$H = \begin{pmatrix} b_1 & a_1 & 0 & 0 & . & . & . \\ a_1 & b_2 & a_2 & 0 & . & . & . \\ 0 & a_2 & b_3 & a_3 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & 0 & a_{N-1} & b_N \end{pmatrix} \tag{1}$$

and the semi-infinite analog H defined on

$$\varrho^2(\mathbb{N}) = \left\{ u = (u(1), u(2), \dots) | \sum_{n-1}^{\infty} |u(n)|^2 < \infty \right\}$$

Given by:

$$(Hu)(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1), \quad n \geq 2, \tag{2}$$
$$= a_1 u(2) + b_1 u(1),$$

In both cases, the a's and b's are real numbers with $a_n > 0$

To avoid inessential technical complication, we only consider the case where $\sup_n[|a_n| + |b_n|] < \infty$, in which case H is a map from ϱ^2 to ϱ^2, and defines a bounded self-adjoint operator.

In the semi-infinite case, we set $N = \infty$. At times , to have unified notation, we use something like $1 \leq j < N+1$ to indicate $1 < j < N$ in the finite case and $1 \leq j < \infty$ in the semi-infinite case.

It will sometimes be useful to consider the b's and a's as a single sequence b_1 , a_1, b_2, ….=c_1,c_2,… that is

$$c_{2n-1} = b_n, \quad c_{2n} = a_n \quad n = 1,2,\dots. \tag{3}$$

What makes Jacobi ,matrices special among all matrices is that the eigenvalue condition $Hu = \lambda u$ is a second- order difference equation. The case n =1 of(2) can be thought of as forcing the Dirichlet boundary condition $u(0)=0$, Thus, any possible non- zero solution of $Hu = \lambda u$ must have $u(1) \neq 0$, which implies .

(i) Eigenvalues of H must be simple (otherwise, a linear combination would vanish at n=1).

(ii) Eigenfunctions must be non- vanishing at n =1.

Thus for $N < \infty, H$ has eigenvalues $\lambda_{\perp} < \cdots < \lambda_N$ and associated orthonormal eignvectors $\varphi_1, \dots .\varphi N$ with $\varphi_j(1) \neq 0$.For N $= \infty$, the proper way of encompassing (i), (ii) is that $\delta_1$is a cyclic vector for $H(\delta_j$is the vector in e^2 with $\delta_j(n) = 1$ (resp.0) if $n = j\ (resp. n \neq j))$

The spectral measure $d_p for\ the\ pair(H, \delta_1)$ is defined by $(\delta_1, H^e\delta_1) = \int \lambda^e\, dp(\lambda)$.

Since our H's are bounded , dp is a masure of bounded support . In case $N < \infty$,

$$dp(\lambda) = \sum_{j=1}^{N} |\varphi_j(1)|^2 \delta(\lambda - \lambda_j) d\lambda \qquad (\varphi_j, \varphi_k) = \delta_{j,k}. \tag{4}$$

The central fact of the inverse theory is that dp determines the a's and b's and any $d.p$ can occur for a unique H.(If $N < \infty), dp$ has support at exactly N points. If $N < \infty), dp$ must have infinite support). The usual proof of this central fact is via orthogonal polynomials and has been rediscovered by many people. For the readers convenience , we have a brief appendix presenting this approach.

One purpose of this section to present a new approach to the central result based on m-function and trace formla.Given $p\ one$ from $m(z) = \int dp(\lambda)(\lambda - z)^{-1}$. The function $m(z) has$ an asymptotic expansion at infinity given by

$$m(z) \sim -\frac{1}{z} - \frac{b_1}{z^2} - \frac{a_1^2 + b_1^2}{z^3} + O(z^{-4}). \tag{5}$$

Thus, one easily recovers b_1 and a_1(recall $a_1 > 0$) from $m(z)$. Now define $m_1(z)$ by.

$$(-m(z))^{-1} = z - b_1 + a_1^2 m_1(z) \tag{6}$$

It turns out that $m_1(z)$ is the spectral measure for the Jacobi matrix obtained by removing the top raw and left-most column of H. An obvious inductive procedure obtains b_2,a_2,…

The *m*-functions defined by this method , which we call $m_+(z,n)$ (so $m(z):=m_+(z,0),m_1(z):=m+(z,1)$,etc),form the class of m-functions defined by

$$m_+(z,n) = \delta_{n+1}\big(H_{[n+1,N]} - z\big)^{-1}\delta_{n+1}\big) \qquad (7)$$

where $H_{[n+1,N]}$ is the matrix with the top *n* rows and *n* left columns removed and thought of as acting on $\ell^2(n+1, n+2, \dots, N)$. There is a second m-function that plays a role.

$$m_-(z,n) = \delta_{n-1}\big(H_{[1,n-1]} - z)^{-1}\delta_{n+1}\big)^{-1}\delta_{n-1} \qquad (8)$$

Where $H_{[n+1]}$is the *n x n* upper left corner of *H*.

also related these m-functions to solutions of the second –order difference equation and obtains relations between $m_\pm(z,n)$ and $m_\pm(z,n+1)$ (of which (6) is a special case) . also contains some critical formulas expressing the diagonal Green's functions $G(z,n,n)$:= $(\delta_n, (H-z)^{-1}\delta_n)$ in terms of $m_+(z)$ and $m_-(z)$.

Also contains one of the most intriguing results of this section. In [139]Hochstadt proved the remarkable result that for a finite Jacobi matrix, a knowledge of all but the first *N c*'s and the N -eigenvalues, that is , of $c_{N+1},c_{N+2},\dots,c_{2N-1}$ and $\lambda_1, \dots, \lambda_N$, determines H uniquely. We extend this by showing that c_{N+1} ,…,c_{2N-1} and any *n* eigenvalues of *H* determine *H* uniquely for any $n=1,2,\dots N$,

After a brief interlude obtaining the straightforward analog of Borg's two-spectra theorem[99](see also[100,145,146,148,150]) first considered in the Jacobi context by Hochstadt[137,138](see also[10,30,43,44,48,51,72]) we turn to the question of determining H from a diagonal Green's function element $\delta_n, (H - z)^1\delta_n)$ when$N < \infty$. If $n = 1$ or *N*, the central inverse spectral theory result says *G(z, n, n)* uniquely determines *H*. For other n, there are always at least $\binom{N-1}{n-1}$ different *H's* compatible with a given *G(z, n, n)*. Generically, there are precisely that many *H's*. also has a complete analysis.

Finally we present some results and conjectures about the inverse problem when $a_n \equiv 1$.

Let *H* be a finite or semi-infinite Jacobi matrix of the type described. We begin by defining some special functions of a complex variable *z* which we will call $\{P(z,n)\}_{n=1}^{N+1} and \{\psi+(z,n)\}_{n=0}^{N}$. The *P(z,n)'s* are polynomials of degree *n - 1* defined by the pair of conditions

$$a_n P(z,n+1) + b_n P(z,n) + a_{n-1} P(z,n-1) = zP(z,n)$$
$$1 \le n < N+1, \quad P(z,0) = 0, \quad P(z,1) = 1 \tag{9}$$

For convenience , we define $a_N := 1$ in order to define,*P(z, N + 1)* in case $N < \infty$Cleary(9)define$p(z,n)$ that inductively as a polynomial of the claimed degrees again, inductively it is clear that:

$$P(z,j+1) = \frac{1}{a_1,\dots,a_j} z^j + \text{lower degree in z.} \tag{10}$$

As explained, the *P's* are intimately related to the intimately related to the spectral measure for *H*.

$$P(z,j+1) = (a_1 \dots a_j)^{-1} \det(z - H_{[1.j]}, \qquad j \ge 1, \tag{11}$$

Where $H_{[1.j]}$, is the $j \times j$ matrix in the upper left corner of *H*.

Proof. By (10), $a_1 \dots a_j P(z,j+1)$and $detH_{[1.j]}$ are monic polynomials of degree *j*. Thus, it suffices to show they have the same zeros and multiplicities. But $P(z.j+1)$ if and only if there is a vector $v = (v = v_1, \dots v_j)$1with$v_1 = 1$so that $(H_{[1.j]} - z)v = 0$. As explained, every eigenvector of $(H_{[1.j]})$ has $v_1 \neq 0$. , Thus , the zeros of $P(z,j+1)$ are precisely the eigenvalues of $H_{[1.j]}$ Since the eigenvalues are simple, the multiplicities are all one.

In case$N < \infty, \psi_+(z,n)$is defined via

$$a_n \psi_+(z,n+1) + b_n \psi_+(z,n) + a_{n-1} \psi_+(z,n-1) = z\psi_+(z,n)$$
$$n = 1, \dots, N-1, \quad \psi_+(z,n+1) \quad = 0, \tag{12}$$

where again for convenience we define a =1to enable us to define

$$\psi_+(z,N-j) = \frac{1}{a_{N-1} \dots a_{N-j}} \det\left(z - H_{[1.j+1,N]}\right) \tag{13}$$

is a polynomial of degree *j*.

In case $N = \infty, \psi_+(z,n)$ initially is only defined in the region $(z) \neq 0$ by requiring (12) and.

$$\psi_+(z,0) = 1, \quad \sum_{n=0}^{\infty n} |\psi_+(z,n)|^2 < \infty. \tag{14}$$

It is a standard argument that when H is bounded and self-adjoint, there is a solution that is ℓ^2 at infinity unique up to constant multiples (and everywhere 0nvanishing so one can normalize it by$\psi_+(z,n) = 1$.

Given any two sequences$u(n), v(n)$, define the (modified) Wronskian$W(u.v)$by

$$W(u,v)(n) = a_n[u(n)v(n+1) - v(n+1)v(n)]$$

For any two solutions of (9), W is constant. The Green's function is defined by$(1 < m,n < N+1)$

$$G(z,m,n) = (\delta_m,(H-z)^{-1}\delta_n) \quad (15)$$

For Im$(z) \neq 0$.We will also sometimes use $(j \leq m,n \leq k)$

Proposition (3.1.2) [88]:

$$G(z,m,n) = [W(P(z,.),\psi_+(z,.))]^{-1}P(z,\min(m,n))\psi_+(z.\max(m,n)) \quad (16)$$

Proof:

One easily checks that if $\psi_+ G(z,m,n)$is defined by(16), then

$$\sum_k (H_{m,k} - z\delta_{m,k})G(z,k,n) = \delta_{m,n}$$

In the finite case, the choice of P,ψ_+ensures that the equation holds at the points where n or m equals lot *N*. In the infinite case, the choice of *P* ensures the equation holds at n or m equals 1, and the choice ofψ_+ ensures that$\sum_n G(z,k,n)f_n \; is \; \ell^2$in k for any finite support sequence$\{f_n\}$. In either case, it follows that is indeed the matrix of the resolvent.

We can now define the most basic mfuncti0n (there will be more later),

$$m(z) = (\delta_n,(H-z)^{-1}\delta_n \quad (17)$$

We have, by(16)

Proposition(3.1.3)[88]:

$$m(z) = -\frac{\psi_+(z,1)}{a_0\psi_+(z,0)} \quad (18)$$

proof.$P(z,0) = 0, P(z,1) = 1$so (16) becomes

$$G(z,1,1) = \frac{\psi_+(z,1)}{-a_0\psi_+(z,0)}$$

In terms of the spectral measure *dp*,

$$m(z) = \int \frac{dp(\lambda)}{\lambda - z} \quad (19)$$

Theorem(3.1.4) [88] If N is finite, then

$$m(z) = -\frac{\prod_{\ell=1}^{(N-1}(z-v_\ell)}{\prod_{j=1}^{N}(z-\lambda_j)}, \quad (20)$$

where $\lambda_1 < \cdots < \lambda_n$ are the eigenvalues of H and $v_1 < \cdots < v_{N-1}$ are the eigenvalue of $H_{[2,N]}$.

Proof. by (12) and (17)

$$m(z) = -\frac{\det(z - H_{[2,n)})}{\det(z, H)}$$

This can be viewed as a cofactor formula for the matrix elements of $H - z^{-1}$

Corollary (3.1.5)[88]: If N isfinite,$\{\lambda_1\}j =_{j=1}^{N} \cup \{v_\ell\}_{\ell=1}^{N-1}$uniquely determine H. Any set of real λ' s and v's are allowed as long as

$$\lambda_1 < v_1 < \lambda_2 < v_2 < \cdots < \lambda_N \qquad (21)$$

Proof. By (19), the $\lambda' s$ and $v' s$ determine *m(z),* and then by (19), they determine *dp* the *a*'s and $b' s$. That any $v' s$, $\lambda' s$ are allowed follows from the fact that if

$$m(z) = \sum_{j=1}^{N} \frac{a_j}{\lambda_{j-z}}$$

then $a_j > 0$ for all j is equivalent to(21)

Definition $(\mathbf{3.1.6})[\mathbf{88}]$: $m_+(z, n) = \left(\delta_{n+1}, (H_{[n+1,N]^{-1}\delta_{n+1}}\right), n = 0,1, \dots, N - 1$, where$H_{[n+1,N]}$is interpreted as $H_{[n+1,\infty]}$ if *N* = ∞.

Thus, *m(z)* := *m+(z,* 0), and by the same calculation that led to (17),

$$-\psi_+(z, n + 1)/[a_{n\psi_+}(z, n)]m_+(z, n) = \qquad (22)$$

Equation (11) implies the following Ricatti equation (more precisely, an analog of what is a Ricatti equation in the continuum case),

$$a_n^2 m_+(z, n) + \frac{1}{m_+(z, n - 1)} = b_n - z \qquad (23)$$

It is also useful to have an analog of the m-function, but starting at 1 instead of at N or ∞.

Definition(3.1.7) [88]: $m_+(z, n) = (\delta_{n-1}, \left(H_{[1,n-1]^{-1}\delta_{n-1}}\right), n = 2,3 \dots, N + 1$

We immediately have analogs of (22) and (2.15), viz.,

$$m_-(z, n) = -P(z, n - 1)/[a_{n-1}P(z, n)] \qquad (24)$$

$$a_{N-1}^2 m_-(z, n) + \frac{1}{m_-(z, n + 1)} = b_n - 1 \qquad (25)$$

The usefulness of having both *m+(z)* and *m_(z)* is that we can use them to express *G(z, n, n).* We claim

Theorem (3.1.8) [88]:

$$G(z, n, n) = \frac{-1}{a_n^2 m_+(z, n) + a_n^2 m_-(z, n) + z - b_n} \qquad (26)$$

$$= \frac{-1}{a_{n-1}^2 m_-(z,n) - \frac{1}{m_+(z,n-1)}} \qquad (27)$$

$$= \frac{-1}{a_n^2 m_+(z,n) - \frac{1}{m_-(z,n+1)}}, \qquad n = 1,2,\ldots \qquad (28)$$

Proof : It suffices to prove (26), for then (24) follows from (23) and then (27) follows from (24).

To prove (26), use (15) evaluating the Wronskian at *n - 1* to see that

$$G(z,n,n) = \frac{-1}{a_{n-1}\left(\frac{P(z,n-1)}{P(z,n)} - \frac{\psi_+(z,n-1)}{\psi_+(z,n)}\right)}$$

$$= \frac{1}{-a_{n-1}^2 m_-(z,n) + (m_+(z,n-1)^{-1}}$$

By(21) and (23)

Theorem(3.1.9)[88]: .Let $N \in \mathbb{N}$. At any eigenvalue $\lambda_j$of H we infer that

$$m_-(\lambda_j, n+1) = [a_n^2 m_+(\lambda_j, n)]^{-1} \quad 1 \le n \le N, \qquad (29)$$

where equality in (29) includes the case that both sides equal infinity.

Proof.At first sight, this would seem to be a triviality. For *G(z, n, n)* has poles at λ_j and thus the denominator in (29) must vanish. But there is a subtlety. It can happen that at an eigenvalue λ_j of *H*, $P(\lambda_j$, n) = $\psi_+(\lambda_j$, n) = 0 and *G(z, n, n)* then also vanishes at λ_j.

Thus we consider two cases: First $\varphi_{j(n)} \neq 0$(φ_j the eigenvector of H associated with (λ_j). In that case *G(z, n, n)* has a pole as z⟶ λ_j and so by (28), (29) must hold (although both sides will be infinite if φ_j(n + 1) = 0).

In the second case, $\varphi_j(n)$ = 0. Then both sides of (29) are zero, and so (29)holds. (However, the denominator of (27) is $\infty - \infty$ and happens to be ∞ so that *G(z, n, n)* vanishes, but (29) still holds.)

In this section, we will use m-functions to show how to recover a Jacobi matrix from the spectral function *dp.* The more usual approach via orthogonal polynomials is sketched. Our approach is new, although iterated m-functions are equivalent to a continued fraction expansion of m(z), and so the work of Masson and Repka [152]is not unrelated to our approach. We begin with

Theorem(3.1.10)[88]:.*Near z =*∞

$$m(z) = -\frac{1}{z} - \frac{b_1}{z^2} - \frac{a_1^2 + b_1^2}{z^3} + O(z^{-4}) \qquad (30)$$

First proof.By the basic definition of m(z) (see (16)) and the norm convergent expansion (since H is bounded)

$$(H - z)^{-1} = -z^{-1}(1 - z^{-1}H)^{-1}$$
$$= -z^{-1} - z^{-2}H - z^{3}H^{2} + O(z^{-4}).$$

We have

$$m(z) - z^{-1} - z^{-2}(\delta_1, H\delta_1) - z^{-3}||H\delta_1||^2 + O(z^{-4})$$

Clearly, $(\delta_1, H\delta_1) = b_1$ *and* $\|H\delta_1\|^2 = \|a_1\delta_2 + b_1\delta_1\| = a_1^2 + b_1^2$

Second proof. By (23),

$$m(z) = \frac{1}{b_1 - z - a_1^2 m_+(z,1)}$$

But $m+(z, 1) = -1/z + O(z^{-2})$. Thus,

$$m(z) = -\frac{1}{z}\left(1 - \frac{b_1}{z} - \frac{a_1^2}{z^2} + O(z^{-3)}\right)^{-1}$$
$$= -\left(1 - \frac{b_1}{z} - \frac{a_1^2}{z^2} + \left(\frac{b_1}{z} + O(z^3)\right)\right)$$

In terms of the spectral measure *dp*, (30) becomes

$$b_1 = \int \lambda \, dp(\lambda), \tag{31}$$

$$a_1^2 = \lambda^2 dp(\lambda) - \left(\int \lambda \, dp\,(\lambda)\right)^2 , \tag{32}$$

formulas implicit in the orthogonal polynomial approach.

In case N <∞, there is a direct way to interpret (30) as generating trace formulas:

Theorem(3.1.11)[88]: Assume $N \in \mathbb{N}$, and let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of H and $v_1, \dots, v_{N-1}$ the eigenvalues of $H_{[2,N]}$. Then

$$b_1 = \sum_{j=1}^{N} \lambda_j - \sum_{\ell=1}^{N-1} v_\ell \tag{33}$$

$$2a_1^2 + b_1^2 = \sum_{j=1}^{N} \lambda_j^2 - \sum_{\ell=1}^{N-1} v_e^2 \tag{34}$$

Proof. Write (see (29))

$$m(z) = -\frac{\Pi_{e=1}^{N-1}(z - v_\ell)}{\Pi_{j=1}^{N}(z - \lambda_j)} = -\frac{1}{2}\prod_{\ell=1}^{N-1}\left(1 - \frac{v_\ell}{z}\right)\prod_{j=1}^{N}\left(1 - \frac{\lambda_j}{z}\right)^{-1}$$
$$= -\frac{1}{z} - \frac{\alpha}{z^2} - \frac{\beta}{z^3} + O(z^{-1})$$

Where

$$\alpha = \sum_{j=1}^{N} \lambda_j - \sum_{\ell=1}^{N-1} v_\ell \qquad (35)$$

$$\beta = \sum_{j=1}^{N} \lambda_j^2 + \sum_{j<k}^{N} \lambda_j \lambda_k \quad \sum_{\ell<m}^{N-1} v_\ell \, v_m - \sum_{j=1}^{N} \lambda_j \sum_{\ell=1}^{N-1} v_\ell \qquad (36)$$

(35) is just (33), and using (34), (55) becomes

$$\beta = \frac{1}{2}\sum_{j=1}^{N} \lambda_j^2 - \frac{1}{2} + \sum_{\ell=1}^{N-1} v_\ell^2 + \frac{1}{2}\alpha^2$$

Thus,

$$\sum_{j=1}^{N} \lambda_j^2 - \sum_{\ell=1}^{N-1} v_\ell^2 = 2\beta - \alpha^2 = 2a_1^2 + b_1^2$$

By (29)

of course, (33), (34) have direct proofs in terms of traces since they just say that

$$Tr\,(H) - Tr\big(H_{[2,N]}\big) = b_1 \qquad (37)$$

$$Tr(H^2) - Tr\big({H^2}_{[2,N]}\big) = 2a_1^2 + b_1^2 \qquad (38)$$

and is one reason why (30) should be thought of as generating trace formulas. In the case of periodic Jacobi matrices, this strategy has been employedin[153].

There is another way to write (30) that doesn't require us to analyze *m(z)* for large *z*. Define the ξ function [124] by

$$\xi(\lambda) = \frac{1}{\pi}\mathrm{Arg}\,(m(\lambda + i0)) \text{ for a.e } \lambda \in \mathbb{R} \qquad (39)$$

Then if supp *(dp)* = spec$(H) \subset [a.\beta]$ we infer that $\xi(\lambda) = 0$ for $\lambda < a$ and $\xi(\lambda) = 1$ for $\lambda \geq \beta$. We claim

Theorem(3.1.12) [88]:

$$b_1 = \alpha + \int_\alpha^\beta \big(1 - \xi(\lambda)\big) d\lambda \qquad (40)$$

$$2a_1^2 + b_1^2 = \alpha + \int_\alpha^\beta 2\lambda(1 - \xi(\lambda))d\lambda \qquad (41)$$

Proof. [124].By Theorem (3.1.10) , the function *-zm(z)* has the asymptotics near ∞

$$-zm(z) = 1 + \frac{b_1}{z} + \frac{a_1^2 + b_1^2}{z^2} + O(z^{-3})$$

Using ln$(1+x) = z - \frac{1}{2}x^2 + O(x^3)$for | x |sufficiently small, we see that

$$Q(z) = In(-zm)(z))$$

has the asymptotics

$$Q(z) = \frac{b_1}{z} + \frac{2a_1^2 + b_1^2}{2z^2} + O(z^{-2}) \quad as \quad z \longrightarrow \infty \qquad (42)$$

Notice that the right sides of (40), (41) are unchanged if β is increased or α is decreased (since$\xi(\lambda) = 1 \ if \lambda > a$) and, so we can assume that $0 \in (a, \beta)$. Then *Q(z)* is analytic in $\mathbb{C}/[\alpha, \beta]$and on (α, β):

$$\frac{1}{\pi} Im\big(Q(\lambda + i0)\big) = \xi(\lambda), \quad \lambda < 0,$$

$$\xi(\lambda) - 1 \qquad \lambda < 0.$$

By (42), for *R* sufficiently large,

$$b_1 = \frac{1}{2\pi i} \oint_{|z|=R} Q(z)dz) = -\int_\alpha^\beta \tfrac{1}{\pi} Im\big(Q(\lambda + i0)\big) d\lambda$$

$$= -\int_\alpha^0 d\lambda + \int_\alpha^\beta (1 - \xi(\lambda))d\lambda$$

which is expression (39), and

$$2a_1^2 + b_1^2 = \frac{1}{2\pi i} \oint_{|z|=R} 2zQ(z)dz = -\int_\alpha^\beta \frac{1}{\pi} 2\lambda im(Q(\lambda + i0)d\lambda$$

$$= -\int_\alpha^0 2\lambda d\lambda + \int_\alpha^\beta 2\lambda(1 - \xi(\lambda))d\lambda$$

which is expression (31)

Equations (31)-(34), (37), (38), (40), (41), etc., clearly underscore that one can derive an infinite sequence of such trace formulas which are precisely the well-known invariants of the hierarchy of Toda lattices. A systematic approach to these trace formulas can be found, for instance[101,107,122,160].

We can now describe the scheme for recovering H from *dp,* or equivalently,from *m(z)* =$\int dp(\lambda)(\lambda - z)^{-1}$.

(i) Use the trace formulas (via (30) or (40), (41)) to recover b_1 and a_1^2.

(ii) Use (23), viz.

$$m_+(z,1) = a_1^{-2}\left(b_1 - z - \frac{1}{m(z)}\right),$$

to find $m_+(z, 1)$, which is the m-function for$H_{[2,\infty]}$

(iii) Use the trace formulas to find $b_2, a_2^2$and then (22) to find *m+(z, 2)* , . . . , etc.

This clearly shows a given *dp* can come from at most one *H*, since we have just described how to compute the *bj* and $a_j^2$from *dp*. We want to prove existence via this method, that is, given any *dp* of compact support, this method yields an *H* which is bounded and whose spectral measure is precisely *dp*.

Lemma(3.1.13)[88]: Suppose that m(z) = $\int dp(\lambda)(\lambda - z)^{-1}$, where dρ is a probability measure on [-C, C] whose support contains more than one point. Define

$$b_1 = \lambda \int \lambda dp(\lambda), \quad a_1^2 \int \lambda^2 dp(\lambda) - b_1^2 \tag{43}$$

(a_1^2 is always strictly positive by the support hypothesis on dp). Define $m_1(z)$ *by*

$$m_1(z) = a_1^{-2}\left[b_1 - z - \frac{1}{m(z)}\right]$$

Then

$$m_1(z) = \int \frac{dp_1(\lambda)}{\lambda - z} \tag{44}$$

where dp_1 is a probability measure also supported on [-C, C]. Moreover, ρ is supported on exactly N points if and only if p_1 is supported on exactly (N - 1)points.

Proof. By (42) and an expansion of a geometric series, (29) holds, so

$$\widehat{m}(z) \coloneqq (-m)z))^{-1} = z - b_1 - \frac{a_1^2}{z} + O(z^{-2}) \tag{45}$$

Since m(z) has Im(m(z)) > 0 when Im (z) > 0 (we recall that m is a Herglotz. function), $\widehat{m}(z)$ =(-m(z)) $^{-1}$has the same property. Moreover, $\widehat{m}(z)$ is analytic on$\mathbb{C}\setminus[C, C]$since $m(\lambda) > 0$ for λ< -C and m(λ) < 0 for λ> C. Thus, by the Herglotz representation theorem,

$$\widehat{m}(z) = \hat{c} + \hat{d}z + \int \frac{d\hat{p}(\lambda)}{\lambda - z}$$

for a measure$d\hat{p}$ By (44), $\hat{c} = -b_1, \hat{d} = 1$, and, $\int d\hat{p}(\lambda) = a_1^2$

Thus,

$$\widehat{m}(z) = \hat{c} + \hat{d}z + \int \frac{dp_1(\lambda)}{\lambda - z}$$

and$dp_1 = a_1^{-2}\, d\hat{p}$is also a probability measure.

Since $dp$is supported on *N* points if and only if *m(z)* is a ratio P_{N-1} (z)/Q_N(z) of polynomials with deg(P_{N-l}(z)) = N - 1, deg(Q_N (z))= N, we obtain the last assertion.

Theorem(3.1.14) [88]. Every N-point probability measure arises as the spectral measure of a unique N × N Jacobi matrix. Every probability measure of bounded and infinite support arises as the spectral measure of a unique semi-infinite bounded Jacobi matrix.

Proof. By iterating the $\rho \to \rho_1$ procedure of the lemma, we can find suitable a_j^2, b_j inductively. If $d\rho$ has *N*-point support, the process terminates after N – 1 steps where dp_N has a single point, and we define b_N to be that point. If dp has infinite support, ρ the process continues indefinitely. Because sup $(dp_1) \subseteq [-C, C]$, $|a_1|$ and $|b_1|$ are bounded by C, and so *H* is bounded.

Let $d\bar{\rho}$ be the spectral measure for the H that has just been constructed. We will show $d\rho$, thereby completing the proof.

Let $\widetilde{m}(z) = \rho \int d\bar{p}(\lambda)(\lambda - z)^{-1}$. Then by construction,

$$\widetilde{m}(z) = \frac{-1}{z - b_1 + a_1^2 \left[\frac{-1}{z - b_2 + a_2^2 \ldots}\right]}$$

That is, m and $\widetilde{m}$ have identical partial fraction expansions although a priori theremainders could be different. This means that the Taylor series for $\widetilde{m}(z)$ near z = ∞ agrees with that for *m* near z = ∞ so *m(z)* = $\widetilde{m}$ *(z)*, and hence $d\rho = d\tilde{\rho}$

The continuum analog of the orthogonal polynomial approach of the Appendix is the Gel'fand-Levitan [120] inverse spectral theory which is a kind of continuum orthonormalization. It would be very interesting to find a continuum analog of the *m*-function approach to inverse problems that we discussed in this section. As an application of the m-function approach to inverse problems, we prove the following (which can also be obtained via orthogonal polynomials):

Theorem(3.1.15)[88].[93,120]Fix $N \in \mathbb{N}$.Consider the following parametrizations of N × N
Jacobi matrices."

(i) $\{a_n\}_{n=1}^{N-1} \cup \{b_n\}_{n=1}^{N} (a_n) > 0$.

(ii) $\{\lambda_j\}_{j=1}^{N} \cup \{v_e\}_{e=1}^{N-1} (\lambda_1 < v_1 < \lambda_2 < \cdots v_{N-1} < \lambda_N)$.

(iii) $\{\lambda_j\}_{j=1}^{N} \cup \{\alpha_j\}_{j=1}^{N} \left(\lambda_1 < \cdots < \lambda_N, \alpha_j < 0, \sum_{k=1}^{N} \alpha_k = 1\right)$

Here λ_j are the eigenvalues of H, v_e are the eigenvalues of $H_{[2,n]}$ and the α's are the residues of the poles in m so

m(z)=$\sum_{j=1}^{N} \alpha_j (\lambda_j - z)^{-1}$(or $dp(\lambda) = \sum \alpha_j \delta(\lambda - \lambda_j) d\lambda$).

The maps between these parameters are real bianalytic diffeomorphisms.

Proof. It is well known and elementary (the determinant of the Jacobian matrix is just $\pm\Pi_{j<k}(\lambda_j - \lambda_k)^{-1}$that the map from the *N* coefficients of amonic polynomial $P_N(\lambda)$ of degree *N* to the roots $\lambda_j, \ldots \lambda_N$of that polynomial is a bianalytic diffeomorphism in the region where the roots are all real and distinct. This

immediately implies that the map from (i) to (ii) is real analytic. The map from (ii) to (iii) is rational since $\alpha_j = \Pi_{\ell=1}^{N-1}(\lambda_j - v_\ell)\Pi_{k\neq j}^{N}(\lambda_j - \lambda_k)^{-1}$. That means we need only show that the map from (iii) to (i) is real analytic.

Since $b_1 = \sum_{j=1}^{N} \alpha_j\lambda_j$ and $\alpha_1^2 = (\sum_{j=1}^{N} \alpha_j\lambda_j^2) - b_1^2$, those are analytic functions. Moreover, the v_ℓ are the roots of the polynomial $\sum_{j=1}^{N} \alpha_j \Pi_{k\neq j}(z - \lambda k)$ and so real analytic in $(\lambda j, \alpha j)$ by the first sentence in this proof, $m+(z,$ 1) has the form $\sum_{\ell=1}^{N-1} \beta_\ell(v_\ell - z)^{-1}$,where$\beta_\ell = [a_1^2 m'(v_\ell)]^{-1}$is clearly analytic in the $\lambda's$ and a's. Thus following the m-function reconstruction shows that the $a's$ and b's are real analytic functions of the λ's and a's.

In[139],Hochstadt proved the following remarkable theorem (see (3)) for the definition of c_j):

Theorem (3.1.16) [88]. Let$\mathrm{N} \in \mathbb{N}$. Suppose that $\mathrm{c}_{\mathrm{N}+1}, \dots, \mathrm{c}_{2\mathrm{N}-1}$are known, as well as the eigenvalues $\lambda_1, \dots, \lambda_\mathrm{N}$of H. Then $\mathrm{c}_1, \dots, \mathrm{c}_\mathrm{N}$ are uniquely determined.

Hochstadt's proof is sketched in the appendix (but in "reflected"coordinates,$i.e.\, c_1, \dots, c_{N-1}$are assumed to be known). Our goal in this section is to prove.

Lemma(3.1.17)[88].[126,127,139,140]Suppose f_1 = P1/Q_1, $f_2 = P_2/Q_2$, where $\deg(P_1) = \deg(P_2)$and $\deg(Q_1) = \deg(Q_2)$, and $d = \deg(f_i)$,

(i) If f_1 and f_2 agree at d + 1 points in C, then $f_1 = f_2$.

(ii) If f_1 and f_2 are both monic and they agree at dpoints in C, then $f_1 = f_2$.

Proof. If $f_1(z) = f_2(z)$, then P_1 $(z)Q_2(z)$- $P_2(z)Q_1(Z) = 0$ (even if both values are infinite, since then Q_1 =Q2 = 0). In case (i), P_IQ2 - $Q_IP_2$has degree d. In case (ii), the leading terms cancel and PxQ_2 - Q_1P_2has degree d-1. The lemma follows from the fact that ifa polymonial R_{do} of degree do vanishes at d_0 +1 points,then $R_{do} \equiv 0$.

Theorme (3.1.18)[88]. Suppose that $1 \leq \mathrm{j} \leq \mathrm{N}$ and $\mathrm{c}_\mathrm{j}, \dots, \mathrm{c}_{2\mathrm{N}-1}$are known, as well as j of the eigenvalues. Then $\mathrm{c}_1, \dots, \mathrm{c}_\mathrm{j}$ are uniquely determined.

Proof. Suppose first that j is odd so j = 2n-1, andb_l, . . . ,a_n-l, b_u are unknown, but a_n, b_{n+l},..., b_N are known, as well as j eigenvalues which we will denote $\lambda_1, \dots, \lambda_{2n-1}$By(28)

$$-m - (\lambda_{j,} n + 1) + [-a_n^2 m_+(\lambda_{j,n})]^{-1}.$$

By definition, $m_+(z, n)$ is determined by $H_{[n+1,N]}$and so by b_{n+l},a_{n+l},..., b_N.

Thus, $[-a_n^2 m_+(\lambda_j, n)]^{-1}$I are known numbers.

By the analog of Theorem (3.1.4)(see also (23)), -m.(z,n + 1) is a ratio$P_{n-1}(z)/Q_n(z)$ of polynomials, where $\deg(P_{n-l}(z)) = n - 1$ and $\deg(Q_n(z)) = n$, and each is monic. By part (ii) of Lemma (3.1.17) the values of such a monic rational function of degree 2n- 1 is determined by its values at the 2n- 1 points $\lambda_1, \dots, \lambda_{2n-1}$

Once we know $m_-(z, n + 1)$, $b_l, a_l, \ldots, b_n$ are determined by Corollary (3.1.5).

Suppose next that j is even so $j = 2n$, and a_n moves from the known group to the unknown group. We can use

$$-a_n^2 m_-(\lambda_j, n + 1) = (-m_+(\lambda_j, n)^{-1}$$

to conclude that we know $f(z) := a_n^2 m_-(n + 1)$ at the $2n$ points $\lambda_1, \ldots, \lambda_{2n}$.The function $f(z)$ is no longer monic, but it is of degree $2n - 1$ and so its valuesat $2n$ points determine it uniquely by part (i) of Lemma (3.1.22). Once we know$-a_n^2 m_-(z, n + 1)$, we can obtain $a_n^2$by $a_n^2 \lim_{|z|\to\infty}[-zm_-(zm_-(z, n + 1) = 1$and then$b_l, a_l, \ldots, b_n$by Corollary (3.1.5).

Example(3.1.19) [88]. (j = 1) We use $m_-(z.n) = (\delta_1, (H_{[1,1]} - z)^{-1}\delta_1) = (b_1 - z)^{-1}$

Then

$$b_1 = \lambda_1 + a_1^2(\lambda_j, 1)$$

This has a solution as long as $m_+(\lambda_1 m1) \neq \infty$.The only forbidden values for $\lambda_1$are the obvious ones, namely, the eigenvalues ν_e of $H_{[2,N]}$which we know must be unequal to the$\lambda's$.

Example(3.1.20) [88]. ($j = 2$) We get

$$b_1 = \lambda_j + a_1^2(\lambda_j, 1) \quad j = 1,2$$

$m_+(\lambda_j, 1) \neq \infty$is still required, but we also need that

$$-\frac{m_+(\lambda_2, 1)m_+(\lambda_1, 1)}{\lambda_2 - \lambda_1}$$

which equals a_1^{-2}, must be positive. This avoids two eigenvalues between a single pair of eigenvalues of $H_{[2,N]}$ but requires a lot more. There are severe restrictions in the$\lambda_1's$for existence (see, e.g., the discussion in [112]). As j increases, these become more complicated.

Borg[99] proved a famous theorem that the spectra for two boundary conditions of a bounded interval regular Schrodinger operator uniquely determine the potential. Later refinements (see, e.g.,100,145,146,148,150]) imply that they even determine the two boundary conditions.

We consider analogs of this result for a finite Jacobi matrix. Such analogs were first considered by Hochstadt[137,138](see also[98,118,131,132,136,139]).In one sense, the fact that the eigenvalues of $H_{[1,N]}$and $H_{[2,N]}$ determine H is such a two-spectrum result and, indeed, it can be viewed as Theorem (3.1.28)below for $b = \infty$. Our results are straightforward adaptations of known results for the continuum or the semi-infinite case, but the ability to determine parameters by counting sheds light on facts like the one that the lowest eigenvalue in the Borg result is not needed under certain circumstances.

Given *H*, an $N \times N$ Jacobi matrix, define *H(b)* to be the Jacobi matrix where all a's and b's are the same as *H*, except b_l is replaced by b_l + b, that is,

$$H(b) = H + b(\delta_1, .)\delta_1. \tag{46}$$

Theorem(3.1.21) [88]. The eigenvalues $\lambda_1 + \lambda_N$ of H, together with b and N – 1 eigenvalues$(\lambda(b)_1 \ldots, \lambda(b)_{N-1}$,of H determine H uniquely.

Proof. Choosing a_0 = 1, we have

$$m(z) = -\psi_+(z,1)/\psi_+(z,0)$$

and

$$\psi_+(z,0) + (b_1 - z)\psi_+(z,1) + a_1\psi_+(z,2) = 0$$

It follows that *z* is an eigenvalue of *H(b)* if and only if

$$\psi_+(z,0) = b\psi_+(z,1),$$

that is, if and only if

$$m(z) = -\frac{1}{b}$$

(a standard result in the general theory of rank-one perturbations[156]).

Write *m(z)* =$-P_{N-1}(z)/Q_N(z)$, where $P_{N-I}(Z)$ and $Q_N(Z)$ are monic polynomials of degree *N - 1* and *N*, respectively. $Q_N(z) = \prod_{j=1}^{n}(z - \lambda_j)$ is known

and

$$P_{n-1}(\lambda(b)_k) = b^{-1}\prod_{j=1}^{N}(\lambda(b)_k - \lambda_j), \quad 1 \le k \le N-1$$

are also known. Since the values ofa monic polynomial $P_d(z)$ of degree *d* at *d* points uniquely determine $P_d(z)$ by Lagrange interpolation,

$\lambda(b)_1, \ldots, \lambda(b)_{N-1}$uniquely determine $P_{N-I}(z)$. The solution of the inverse problem, given$-P_{N-1}(z)/Q_N(z)$,and hence *m(z),* then determines *H* uniquely.

Theorem(3.1.22) [88]. The eigenvalues $\lambda_1, \ldots, \lambda_N$ of H, together with the N eigenvalues $\lambda(b)_1, \ldots, \lambda(b)_N$of some H(b) (with b unknown), determine H and b.

Proof. Following the proof of Theorem(3.1.22),we have a monic polynomial P_{N-1}(z), an unknown $\beta \coloneqq 1/b$, and

$$P_{N-1}(\lambda)(b)_k) = \beta\prod_{j=1}^{N}(\lambda)(b)_k - (\lambda_j).$$

Let:

$$R_N(z) = \beta\prod_{j=1}^{N}(z - \lambda_j) - P_{N-1}(z)$$

Since $R_N(z) = \beta z^N$ +lower-order terms and $R_N(\lambda)(b)_k) = 0,1\leq k \leq N$we have

$$R_N(z) = \beta \prod_{j=1}^{N} (z - \lambda_j + 1)$$

Since $R_{N-1}(z)$is monic of degree *N - 1,*

$$R_N(z) = \beta z^N - \left(\beta \sum_{j=1}^{N} (\lambda_j + 1) z^{N-1} + \cdots \right)$$

on the one hand and

$$R_N(z) = \beta z^N - \left(\beta \sum_{j=1}^{N} \lambda(b)_j \right) z^{N-1} + \cdots$$

on the other. It follows that

$$\beta = \frac{1}{\sum_{j=1}^{N} (\lambda(b)_j - \lambda_j)} = b^{-1}. \quad (48)$$

Once β is known, $R_N(z)$ determines $P_{N-1}(z)$, and thus *m(z)* and *H*.(48)then determines *b*.

The basic inverse spectral theorems show that $(\delta_1, (H - z)^{-1}\delta_1)$determines *H* uniquely.We take $N \in \mathbb{N}, 1 \leq n \leq N$, and ask whether $(\delta_1, (H - z) - 1\delta n$determines *H* uniquely. For notational convenience, we occasionally allude to*G(z, n, n)* as the *nn* Green's function in the remainder of this section. The n = 1 result can be summarized via:

Theorem(3.1.23)[88]: $(\delta_1, (H - z)^{-1}\delta_1)$ has the form $\sum_{j=1}^{N} \alpha_j (\lambda_j - z)^{-1}$ with $\lambda_1 < \cdots < \lambda_N, \sum_{j=1}^{N} \alpha_j = 1$and each α_j>0. Every such sum arises as the11 Green"s function of an H and of exactly one such H.

For general n, define fi =rain*(n, N + 1 - n)*. Then we will prove the following theorems:

Theorem(3.1.24)[88]:. $(\delta_1, (H - z)^{-1}\delta_n)$ has the form $\sum_{j=1}^{N} \alpha_j (\lambda_j - z)^{-1}$ with k one of N, N - 1, . . ., N - $\bar{n}$+ 1 and $\lambda_1 < \cdots < \lambda_k, \sum_{j=1}^{N} \alpha_j = 1$and each $\alpha_j > 0$. Every sum arises as the nn Green's function of at least one H.

Theorem(3.1.25)[88]:. If k = N, then precisely $\binom{N-1}{n-1}$H's yield the given nn Green's function.

Theorem(3.1.26)[88].if k < N, then infinitely many H's yield the given nn Green's function.Indeed,the inverse spectral family is a collection of $\binom{N-1}{N-k}\binom{k-1-N-k}{n-1-N-k}$disjoint manifolds, each of dimension N - k and diffeomorphic to an (N - k)-dimensional open ball.

Proof. Consider first the case k = N (which is generic; $k < N$ occurs in a set of Jacobi matrices of codimension 1). Let $\mu_1 < \cdots < \mu_{N-1}$, be the zeros of $G(z.n.n) := \sum_{j=1}^{N} \alpha_j(\lambda_j - z)^{-1}$.Then.

$$-G(z.n.n)^{-1} = z - b + \sum_{\ell=1}^{N-1} \frac{\beta_\ell}{\mu_\ell - z} \tag{49}$$

where $b, \mu_\ell \in \mathbb{R}$ and $\beta_\ell > 0$ are determined by the a's and λ's. By,

$$-G(z;n,n)^{-1} = z - b_n + a_n^2 m_+(z,n) + a_{n-1}^2 m_-(z,n) \tag{50}$$

$m_-(z,n) = (\delta_{n-1}, (H_{[n-1]} - z)^{-1}\delta_{n-1})$ determines $H_{[n-1]}$uniquely (by Theorem**(3.1.15)**.and has the form

$$m_-(z,n) = \sum_{j=1}^{n-1} \frac{\gamma_j}{e_{j-z}} \quad ,\gamma_j > 0, \tag{51}$$

where $\sum_{j=1}^{n-1} \gamma_j = 1$ and the $e_i's$are the eigenvalues of $H_{[n-1]}$. Similarly,$m_+(z,n) =$ *(δn−1,(Hn−1−z)−1δn−1)* determines $Hn+1,N$ uniquely and has the form

$$m_+(z,n) = \sum_{j=1}^{N-n} \frac{k_j}{f_{j-z}} \quad , \quad k_j > 0, \tag{52}$$

where$\sum_{j=1}^{N-n} k_j = 1$ and the $f_i's$ are the eigenvalues of$H_{[n+1,N]}$. Comparing (49)-(52), we see that$\{\mu_\ell\}_{\ell=1}^{N-1} = \{e_j\}_{j=1}^{n-1} \cup \{f_j\}_{j=1}^{N-n}$. We can choose which $\mu_\ell$are to be e_j in $\binom{N-1}{n-1}$ways. Once we make the choice,

$$a_{n-1}^2 = \sum_{\ell \text{ so } \mu_\ell \text{ is an } e_j} \beta_\ell \quad and \quad a_n^2 = \sum_{\ell \text{ so } \mu_\ell \text{ is an } f_j} \beta_\ell$$

and$m_+(z, n)$ are determined. But $H_{[1,n-1]}$, $H_{[n+1,N]}$anda_{n-1}, b_n, a_n, determine *H*. Thus for each choice, we can uniquely determine *H*. Moreover, since any sums of the form (51), (52) are legal for$m_\pm(z,n)$, we have existence for each of the $\binom{N-1}{n-1}$ choices.

$k = N$ if and only if all the eigenfunctions $\varphi_j(n)$ are non-vanishing at n.Eigenfunctions obey the boundary conditions at both ends, so if $\varphi_j(n)$vanishes, so do *P(z,n) and*$\psi_+(z,n)$, which are polynomials of degree *n* - 1 and N -*n*;so at most min(*n* - 1, N - *n*) := ñ- 1 eigenvalues of *H* can fail to contribute to *G(z, n, n),* that is, at least N - $\tilde{n}$ + 1 eigenvalues must contribute, that is, k is one of $N.N - 1, \ldots, -\tilde{n} + 1$. Eigenvalues that don't contribute are zeros of *G(z, n, n)*and simultaneously eigenvalues of $H_{[1,n-N]}$and $H_{[n+1,N]}$.

Thus if $k < N$, the $k- 1$ poles of-$G(z, n, n)^{-1}$ are in three sets. $n_0 := N- k$ are eigenvalues of both $H_{[1,n-N]}$and $H_{[n+1,N]}$$n_1 := n - 1 - (N - k)$, are eigenvalues of $H_{[1,n-N]}$alone, and $n_2 := (N - n) - (N - k) = k - n$ are eigenvalues of $H_{[1,n-1]}$alone. Notice that $N > k \geq N - \tilde{n} + 1$ implies $n_0 > 0$, $n_1 \geq 0$, $n_2 \geq 0$ and that $n_0 + n_1 + n_2 = k - 1$, $n_0 + n_1 = N - n$.To reconstruct $m_+(z, n)$ given $-G(z, n, n)^{-1}$, we have to make two sets of choices:

(i) Figure out which of$\mu_1, \dots, \mu_{k-1}$lie in each of the three sets. This yields

$$\binom{k-1}{n_0}\binom{k-1-n_0}{n_1} = \frac{(k-1)!}{n_0!\,n_1!\,n_2!}$$

discrete choices.

(ii) For each of the no $n_0 \mu_\ell{}'s$ in the set of common eigenvalues, we must pick a decomposition

$$\beta_\ell = \beta_\ell^{(1)} + \beta_\ell^{(2)}, \quad \beta_\ell^{(i)} < 0$$

and then take

$$a_n^2 m_+(z.n) = \sum_{\substack{\ell \text{ so that} \\ \mu_e \text{ is solely an} \\ H_{[1,n-1]} \text{eigenvalue}}} \frac{\beta_\ell}{\mu_e - z} \sum_{\substack{\ell \text{ so that} \\ \mu_e \text{ is a} \\ \text{common eigenvalue}}} \frac{{\beta_\ell}^{(1)}}{\mu_e - z}$$

and

$$a_n^2 m_-(z, n) = \sum_{\substack{\ell \text{ so that} \\ \mu_e \text{ is solely an} \\ H_{[1,n-1]} \text{eigenvalue}}} \frac{\beta_\ell}{\mu_e - z} + \sum_{\substack{\ell \text{ so that} \\ \mu_e \text{ is a} \\ \text{common eigenvalue}}} \frac{\beta_\ell^{(2)}}{\mu_e - z}$$

Every such choice yields an acceptable H. Since the map from poles and residues to matrices is a diffeomorphism (Theorem (3.1.15),the $\frac{(k-1)!}{n_0!n_1!n_2!}$disjoint sets of poles and $\times_{n_0\,\ell's}(0, \beta_\ell)$ residues lead to that number of manifolds diffeomorphic to the n_0-dimensional open ball.

A Jacobi matrix with all $a_n = 1$ is called a discrete Schrrdinger operator. The inverse problem for such operators is open, that is, there are no effective conditions on a spectral measure dp that tell us that its associated Jacobi matrix has all a_n=1. (The isospectral manifold of general Jacobi matrices with $a_n \in \mathbb{R}$is discussed in [161], see also[111],[114],and[117].)

Consider the finite case,$N \epsilon \mathbb{N}$. The number N of free parameters $\{b_n\}_{n=1}^N$equals exactly the number of eigenvalues.$\{\lambda_j\}_{j=1}^N$ The natural inverse problem is from λ's to b's. We do not have a complete solution, but have a number of conjectures and comments which we make in this section. $\lambda_1 < \lambda_2 < \cdots < \lambda_N$are the eigenvalues of H. For any $b = (b_l, \dots b_N) \in \mathbb{R}$, define $\wedge(b) = (\lambda_1, \dots, \lambda_N) \epsilon \mathbb{R}^N$as the eigenvalues. Let S_N= Ran (∧).

Conjecture(3.1.27) [88]: (Main Conjecture). S_N is a closed set in $\mathbb{R}^N$ whose interior S_N^{sin} is dense in S_N. For any $\lambda \in S_N^{sin}$ Λ^{-1} (λ) contains N! points. For any, $\lambda \in \partial S_N$, Λ^{-1} (λ) contains fewer than N! points.

Thus, we believe that Λ^{-1} $[S_N^{sin}]$ is an *N!*-fold cover of S_N^{sin}, but it is likely anuninteresting one.

Conjecture(3.1.28) [88] $\Lambda^{-1} S_N^{sin}$ is a union of *N!* disjoint sets. On each of them, Λ is a diffeomorphism to S_N^{sin}

In the complex domain, things are more interesting. There is a small neighborhood,D, of $\mathbb{R}^N$ in $\mathbb{C}^N$ to which Λ can be analytically continued and on which $\lambda_j \neq \lambda_k$ still holds. Introduce

$\bar{S}_n = \Lambda\,[D]$ and B = $\{\lambda \in \bar{S}_n | \Lambda^{-1}[\lambda]\}$ has ordinality less than *N!*$\}$.

Conjecture(3.1.29)[88]. B has real codimension 2. Is $\Lambda^{-1}[\bar{S}_n \backslash B]$ connected and is an *N!*-cover of $\bar{S}_n \backslash B$.

Thus, Λ^{-1} is a ramified cover of $\bar{S}_n$. We begin with an analysis of the case N=2, so H = $\begin{pmatrix} b_1 & 1 \\ 1 & b_2 \end{pmatrix}$ Then

$$\Lambda(\mathrm{b}) = \left(\frac{b_1+b_2}{2} - \sqrt{\left(\frac{b_1+b_2}{2}\right)^2 + 1}, \frac{b_1+b_2}{2} + \sqrt{\left(\frac{b_1+b_2}{2}\right)^2 + 1} \right) \qquad (54)$$

$Thus\ S_2 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_2 \geq \lambda_1 + 2\}. \partial S_2 = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_2 \geq \lambda_1 + 2\}$

.$\Lambda^{-1}(\alpha - 1, \alpha + 1) = \left\{\begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix}\right\}$, otherwise $\Lambda^{-1}\big((\lambda_1, \lambda_2)\big)$ has two points $\begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}$ and $\begin{pmatrix} y & 1 \\ 1 & x \end{pmatrix}$ $\Lambda^{-1}(S_2^{int})$ has two connected components where $b_l > b_2$ and where $b_2 > b_l$.

If one continues into the complex domain, $\Lambda^{-1}[\tilde{S}_2 \backslash B]$ is connected.

Thus, our conjectures are true in the not quite trivial case $N = 2$.

At first sight, it may seem surprising that S_N is closed. After all, the eigenvalueimage of all Jacobi matrices $\{\lambda \in \mathbb{R}^N | \lambda_1 < \lambda_2 < \cdots < \lambda_N\}$ is open and notclosed. The existence of strict inequalities is a reflection of the condition $a_n > 0$. Once $a_n \equiv 1$, they disappear.

Theorem(3.1.30) [88]. *S_N is closed.*

Proof. Let $\lambda_m \in S_N$ and pick $b_m \in \mathbb{R}^N$ so that $\Lambda\,(b_m) = \lambda_m$. Suppose $\lambda_m \to \lambda_\infty \in \mathbb{R}^2$. as $m \to \infty$. Let *H(b)* be the N × N Schrodinger matrix with the components of b along the diagonal. Then

$$|\Lambda(\mathrm{b})|^2 = Tr(H(b)^2) = 2(N-1) + ||b||^2,$$

so $\{b_m\}$ is a bounded subset of $\mathbb{R}^N$. Thus, we can find a subsequence $\{m_p\}$ such that $b_{m_p} \to b_\infty$.as $p \to b_\infty$.By continuity of Λ, $\Lambda(b_\infty) = \lambda_\infty thatis, \lambda_\infty \in S_n$.

This theorem implies that if $\|b\| \leq R$,, then there is a minimum distance between eigenvalues. One might think there are global bounds on eigenvalue splittings (i.e., N-dependent but independent of R), but that is false if $N \geq 3$, as is seen by the following example motivated by tunneling considerations. Let $H(\beta)$ be the N x N Schrodinger matrix with $b_l = b_N = \beta$ *and* $b_2 = \ldots\ b_{N-1} = 0$. Then for β large, the two largest eigenvalues $E_{\pm}(\beta)$ satisfy

$$E_{\pm}(\beta) = \beta \pm O(\beta^{-(N-2)} \tag{54}$$

and if $N \geq 3$, $|E_{+}(\beta) - E - (\beta) \to 0$ as $\beta \to \infty$

An important open question is finding some kind of effective description of S_N.

We note that if

$$\varphi_{+} = \left(\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}\right) \text{and} \varphi_{-} = \left(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \ldots, \frac{(-1)^{N+1}}{\sqrt{N}}\right),$$

$$\text{then} (\varphi_{+}, H\varphi_{+}) - (\varphi_{-}, H\varphi_{-}) = 4(1/N)) \text{so} \lambda_N - \lambda_1 \geq 4\left(1 - \left(\frac{1}{N}\right)\right).$$

The N! in our main conjecture comes from the following

Theorem(3.1.31)[88]. For β large, $\lambda_\beta := (\beta, 2\beta, 3, \ldots, N\beta) \in S_N$ and $\Lambda^{-1}(\lambda_\beta)$ has N! points.

Proof. Consider the *N!* Hamiltonians

$$H_\pi(\beta) = \beta \begin{pmatrix} \pi(1) & & 0 \\ & \ddots & \\ 0 & & \pi(N) \end{pmatrix} + \begin{pmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & & 1 \\ 0 & & 1 & 0 \end{pmatrix} \tag{55}$$

where π is an arbitrary permutation on {1,..., N}. Then $A(\beta) = \beta^{-1} H_\pi(\beta)$ at $\beta = 0$ has N eigenvalues (1, 2 ,...,N) and it is easy to see that for β small, the Jacobian of Λ is invertible. It follows by the inverse function theorem that for β.

large, there is a unique $\widetilde{H}_\pi(\beta) = H_\pi(\beta) + O(\beta)^{-1}$ so that the eigenvalues of $\widetilde{H}_\pi(\beta)$ are precisely $(\beta, 2\beta, \ldots N\beta)$

A separate and easy argument shows that for β large, any Schrodinger matrix with eigenvalues (β,..., Nβ) must have $bn = \beta_\pi(n) + O(\beta^{-1})$ for some permutation, π and so must be one of the $\widetilde{H}_\pi(\beta)$.

The evidence for the strong forms of the conjectures here is not overwhelming. We make them as much to stimulate further research as because we are certain theyare true.

Sec (3.2) Inverse Problems on Jacobi Matrices:

The study of inverse eigenvalue problems for Jacobi matrices is not purely of mathematical interest, actually, in aplications, it is related to vebariting systems see[169] and the classical moment problems see[164] of a jacobi matrix

$$J_n\vec{v} = \lambda\vec{v}, \tag{56}$$

can be iwed as a discretization of the one- dimensional Schrödinger equation

$$y''(x) + (\lambda - p(x))y(x) = 0 \qquad 0 < x < 1, \tag{57}$$

where q(x) is acounuous function defined on(0,1).Hence, it is not surprising that there are several analogies between the inverse eigenvalue problems for Jacobi matrices and the inverse spectral problems for sturm- liouville equations. For example, for a given pair (h, H, q) $R^2 \times$ c(0,1), let $Q_{h.H(q)}$ denote the spectrum of the equation

$$y''(x) + (\lambda - q(x))y(x) = 0 \qquad 0 < x < 1, \tag{58}$$

with the boundary conditions

$$\begin{cases} y'(0) - hy(0) = 0 \\ y'(1) - Hy(1) = 0 \end{cases} \tag{59}$$

where (h,H) is in R^2.Borg[2] showed that if $\sigma_{h,H}(q_2)$ and $\sigma_{h,H1}(q_1)$ for some $H \neq H_1$,then $q_1(x) = q_2(x)$ on $[0,1]$On the other hand, denote

$$J_n[a_1, a_2, \ldots. a_n; b_1, b_2, \ldots, b_{n-1}] = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & \ldots & 0 \\ b_1 & a_2 & b_2 & 0 & 0 & \ldots & 0 \\ . & b_1 & a_3 & b_3 & 0 & \ldots & 0 \\ . & . & . & . & . & \ldots & . \\ & & & & . & & \\ . & . & . & . & . & . & . \\ & & & & . & & \\ \ldots & \ldots & \ldots & \ldots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & b_{n-1} & a_n \end{pmatrix} \tag{60}$$

Hochstart[170] proved that an irreducible Jacobi matrix.

$$J_n[a_1, a_{21}, \ldots, a_n; b_1, b_{21}, \ldots, b_{n-1}]$$

Is uniquely determined by its eigenvalues(corresponding to the dirichelt-Neumann spectrum of (58) and the eigenvalues of its truncated matrix $J_{n-1}[a_1, a_{21},..., a_{n-1}; b_1, b_{21},..., b_{n-2}]$ (corresponding to the dirichlet spectrum of(58) if we require that $b_i > 0$ for I = 1, 2,….., n-1. In 1973, Hochstadt[171] showed that if $q(x) = q(x-1)$q for or x in(0,1) then one spectrum set $\sigma_{h,H}$(q)can determine q(x) uniquely; later , in 1974, a discretized version of the following theorem was also proved by him; he showed that the eigenvalues of an irreducible persymmetric Jacobi matrix

$$J_n[a_1, a_{21},..., a_n; b_1, b_{21},..., b_{n-1}]$$

$(i.e, a_1 = a_n, b_1 = b_{n-1,} a_2 = a_{n-1}, b_2 = b_{n-2},...)$ determine this matrix uniquely with the requirement $b_i > 0$ for i= 1,2,……….. n-1 until 1978, Hochstadt and liberman[172]proved that.

Theorem(3.2.1)[163]. let $q(x) = \bar{q}(x)$ be two summable functions in(0,1).Suppose that $q(x) = \bar{q}(x)$ for all $x \in (1/2,1)$ and $\sigma_{h,H}(q) = \sigma_{h,H}(\bar{q})$ then $q(x) = \bar{q}$ (1-x) almost everywhere in (0,1).

They named the pair $(q(x))_{(1,1/2)}$, $\sigma_{h,H}(q)$ with the term` mixed data, Afterwards, Hochstardt [173] immediately proved that

Theorem (3.2.2)[163]. Let $J_n[a_1, a_{21},..., a_n; b_1, b_{21},..., b_{n-1}]$ be a Jacobi matrix with $b_i >$ 0 for i= 1,2,……..,n-1 suppose we are given its n distinct eigenvalues $\lambda_1, \lambda_{21},.... \lambda_{n1}$, as well as the n-1 entries a_1, a_2,………..$a_{(n/2)}$, b_1, b_2,…………b$[(n-1/2)]$ then these data determine a unique Jacobi matrix.

So far, most of the theorems are concerned with the , uniqueness, there are not many papers that discuss the existence of the inverse eigenvalue problems, In 1984, Deift and Nanda[166] provided sufficient conditions for the solvability of theorem (3.2.1) they also gave a description for the solution set. Finally ,I have to mention one more result.

Theorem (3.2.2)[163]. fix $c,d \in \mathbb{R}$ with $c < d$ and $q \in L^1((c,d))$ real –value let $S(c,d;q)$ denote the set of eigenvalues $-\frac{d^2}{dx^2} + q$ on $L^2((c,d))$ with the boundary conditions u(c) = u(d)= 0. Suppose $q_1, q_2 \in L^1((0,1)$ are real –valued and there is some $a \subset (0,1)$ so that

(i) $S(0,1;q_1)=S(0,1;q_2)$, $S(0,a;q_1)=S(0,a;q_2)$ and $S(a,1;q_1)=S(a,1;q_2)$

(ii) the sets $S(0,1;q_1)$ $S(0,a;q_1)$ and $S(a,1;q_1)$ are pairwisely dis joint.

Then $q_1=q_2$ a.e. on (0,1). In particular, if a = 1/2 the condition(ii) can be dropped.

This section was partially motivated by Theorems (3.2.1)and(3.2.3).We stady some inverse problems for Jacobi matrices. We give a brief introduction, some preliminary results.

We will review some connections among continued fractions, Mobius transforms and Jacobi matrices that play core roles for our main theorems, The readers who are interested in this topic may refer to[174]

Let ($\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be tow sequences of intigeres with $a_0 \in \mathbb{Z}$, $a_i > 0$, and $b_i > 0$ for I $\geq$ 1. Denote

$$\frac{p_n}{Q_n} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{. + \cfrac{.}{a_{n-1} + \cfrac{b_n}{a_n}}}}}$$

$$\equiv a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \ldots \frac{b_n}{+ a_n} \qquad (61)$$

For example,

$$\frac{P_0}{Q_0} = \frac{a_0}{1},$$

$$\frac{P_1}{Q_1} = a_0 + \frac{b_1}{a_1} = \frac{a_0 a_1 + b_1}{a_1}$$

$$\frac{P_2}{Q_2} = q_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2}} = \frac{a_2(a_1 a_0 + b_2) + b_2 a_0}{a_1 a_2 + b_2}$$

On the other hand, we denote

$$\mathrm{T}_0(Z) = \frac{a_0 z + 1}{z} \equiv: \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{T}_\mathrm{n}(Z) = \frac{a_{n0} z + 1}{b_n z} :\equiv \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$T_0 \circ T_1(z) = a_0 + \frac{bz}{a_1 z + 1} = a_0 + \frac{b_1}{a_1 + z},$$

$$T_0 \circ T_1(z) \circ T_2(z) = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2 z}{a_2 z + 1}}$$

$$= a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{1}{z}}}$$

Hence we have

$$\lim_{z \to \infty} T_0 \circ T_1(z) \circ T_2(z) = \frac{P_2}{Q_2} \text{ and } \lim_{z \to \infty} T_0 \circ T_1(z) \circ T_2(z) = \frac{P_1}{Q_1}$$

In general, we have

$$\frac{P_n}{Q_n} = \lim_{z \to \infty} T_0 \circ T_1 \ldots\ldots T_n(z), \tag{62}$$

$$\frac{P_{n-1}}{Q_{n-1}} = \lim_{z \to \infty} T_0 \circ T_1 \ldots\ldots T_n(z), \tag{63}$$

Hence

$$T_0 \circ T_1 \ldots\ldots\ldots T_n(z) : \equiv \begin{pmatrix} P_n & P_{n-1} \\ Q_n & q_{n-1} \end{pmatrix} \tag{64}$$

$$= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ b_k & 0 \end{pmatrix} \tag{65}$$

Note that

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}$$

$$= \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix} \tag{66}$$

holds with the initial conditions

$$\begin{cases} P_0 = a_0, & P_{-1} = 1, \\ Q_0 = 1 & Q - 1 = 0, \end{cases} \tag{67}$$

that is,

$$\begin{cases} P_K = a_K P_{k-1} + b_k P_{K-2}, \\ Q_K = a_k Q_{k-1} + b_k Q_{K-2}. \end{cases} \tag{68}$$

The readers can refer to[174] for more details.Conversely, if we have the pair (p_n, p_{n-1})(or the pair(P_n, Q_n)), then we can reconstruct $\{a_k\}_{k=0}^{n}$ and $\{b_j\}_{k=1}^{n}$ from p_n andp_{n-1} by

$$\frac{P_n}{P_{n-1}} = \frac{a_n P_{n-1} + b_n P_{n-2}}{P_{n-1}} = a_n + \frac{b_n}{\frac{P_{n-1}}{P_{n-2}}}$$

$$= a_n + \cfrac{b_n}{a_{n-1} + \cfrac{b_{n-1}}{. + \cfrac{.}{a_n + \cfrac{b_1}{a_0}}}} \tag{69}$$

Let J_n denote an irreducible Jacobi matrix $J_n[a_1, a_{21}, ..., a_n; b_1, b_{21}, ..., b_{n-1}]$, i.e. $b_1 \neq 0$ for i = 1, …., n – 1, and

$$J_{j,k} = J_{K-J+1}[a_J ...., a_{K;} b_1, b_J, ..., b_{K-1}],$$

denote the (k –j + 1) × (k – j + 1) principal minor submatrix of $J_n, P_k(x)$ the characteristic polynomial of $J_{1,k}$ and $P_{j,k}(x)$ the characteristic polynomial of $J_{1,k}$ then we have

$$P_k(x) = (x - ak)P_{k-1}(x) - b_{k-1}^2 P_{k-2}(x) \qquad k = 2,3,...,n, \tag{70}$$

with $P_1(x) = x - a_1$ and $P_0(x) = 1$, similarly,

$$P_{k,n}(x) = (x - a_k)P_{k+1,n}(x) - b_{k-1}^2 P_{k-2}(x) \qquad k = 1,....,n-1. \tag{71}$$

with $P_{n,n}(x) = x - a_n$ and $P_{n+1,n}(x) = 1$By the recursive relation (84), formally, we can reconstruct J_n from $P_n(x)$and $P_{k-1}(x)$ or($P_{1,n}(x)$and $P_{2,n}(x)$Moreover, if we denote $Q_k(x)$the solution of(84) with initial condition Q_1 = 1, Q_0 = 0. Then we have

$$\begin{pmatrix} P_k(x) & P_{k-1}(x) \\ Q_k(x) & Q_{k-1}(x) \end{pmatrix} = \begin{pmatrix} P_{k-1}(x) & P_{k-2}(x) \\ Q_{k-1}(x) & Q_{k-2}(x) \end{pmatrix} \begin{pmatrix} x - a_k & 1 \\ -b_{k-1}^2 & 0 \end{pmatrix} \tag{72}$$

Comparing with (61) – (68), we can build one corresponding relation between Jacobi matrices and products of 2 × 2 nonsingular matrices, more precisely , we denote

$$J_n[a_1, a_{21},..., a_n; b_1, b_{21},..., b_{n-1}] \cong \begin{pmatrix} x - a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x - a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x - a_n & 1 \\ -b_{n-1}^2 & 0 \end{pmatrix} \tag{73}$$

One important result for the inverse problems of Jacobi matrices is the uniqueness theorem which is stated as follows:

Theorem (3.2.4)[163]. (Hochstated [170]). For two given real sequences $\{\lambda_i\}_{j=1}^{n}$ (the eigenvalues of J_n) and $\{\mu_j\}_{j=1}^{n-1}$(the eigenvalues of $J_{1,n-1}$ with

$$\lambda_1 \angle \mu_i \angle \lambda_{i+1}, \qquad i = 1,2,..., n-1$$

Then $\{\lambda_i\}_{j=1}^{n}$ determine $J_n = J_n[a_1, a_{21},..., a_n; b_1, b_{21},..., b_{n-1}]$uniquely if we require

b_i > 0 for i = 1,………,n-1.

In other words, $P_n(x)$ and $P_{n-1}(x)$ (or $(Q_n(x))$ determine a Jacobi matrix with positive off- diagonals uniquely. The readers can refer to [169] for more complete comprehension. Next, the author is going to provide an example to show how Theorem (3.2.4) and(74) work for the inverse problems of Jacobi matrices.

Theorem(3.2.5)[163]. (Hochstadt [173]). Let $J = J_n[a_1, a_{21},..., a_n; b_1, b_{21},..., b_{n-1}]$ be a Jacobi matrix with all a_i,b_i real and b_i positive. Suppose we are given n distinct real eigenvalues $\lambda_1, \lambda_2,......., \lambda_n$as well the $n-1$ entries $a_1, a_2,....., a_{[n/2]}, b_1, b_2,....., b_{[n-1/2]}$, then these data determine a unique Jacobi matrix.

Proof. We may treat the case for n beign even, the argument for n being odd is similar. Let n= 2k, k ∈ ℕ. *suppose that there are two Jacobi matrix*

$$J_n = J_{2k}\left[a_1, a_{21}, ..., a_k, a_{k+1}; b_1, b_{21}, ..., b_{k-1}, b_k, ..., b_{2k-1}\right]$$

and

$$\tilde{J}_n = J_{2k}\left[a_1, a_{21}, ..., \tilde{a}_k, \tilde{a}_{k+1}; b_1, b_{21}, ..., b_{k-1}, \tilde{b}_k, ..., \tilde{b}_{2k-1}\right]$$

Which satisfy the assumptions. Then we can write

$$J \cong \begin{pmatrix} x-a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix}\cdots\begin{pmatrix} x-a_k & 1 \\ -b_{k-1}^2 & 0 \end{pmatrix}\begin{pmatrix} x-a_{k+1} & 1 \\ -b_k^2 & 0 \end{pmatrix}\cdots\begin{pmatrix} x-a_{2k} & 1 \\ -b_{2k-1}^2 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} P_k(x) & P_{k-1}(x) \\ Q_k(x) & Q_{k-1}(x) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -b_k^2 \end{pmatrix}\begin{pmatrix} x-a_{k+2} & 1 \\ -b_{k+1}^2 & 0 \end{pmatrix}\cdots\begin{pmatrix} x-a_{2k} & 1 \\ -b_{2k-1}^2 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} P_k(x) & P_{k-1}(x) \\ Q_k(x) & Q_{k-1}(x) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -b_k^2 \end{pmatrix}\begin{pmatrix} P_{k+2,n}(x) & P_{k+2,n-1}(x) \\ Q_{k+2,n}(x) & Q_{k+2,n-1}(x) \end{pmatrix}$$

$$=\begin{pmatrix} P_{2k}(x) & P_{2k-1}(x) \\ Q_{2k}(x) & Q_{2k-1}(x) \end{pmatrix}$$

Similary,

$$\tilde{J} \cong \begin{pmatrix} x-a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix}\cdots\begin{pmatrix} x-a_k & 1 \\ -b_{k-1}^2 & 0 \end{pmatrix}\begin{pmatrix} x-\tilde{a}_{k+1} & 1 \\ -b_k^2 & 0 \end{pmatrix}\cdots\begin{pmatrix} x-\tilde{a}_{2k} & 1 \\ -b_{2k-1}^2 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} P_k(x) & P_{k-1}(x) \\ Q_k(x) & Q_{k-1}(x) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\tilde{b}_k^2 \end{pmatrix}\begin{pmatrix} \tilde{P}_{k+2,n}(x) & \tilde{P}_{k+2,n-1}(x) \\ \tilde{Q}_{k+2,n}(x) & \tilde{Q}_{k+2,n-1}(x) \end{pmatrix}$$

$$=\begin{pmatrix} \tilde{P}_{2k}(x) & \tilde{P}_{2k-1}(x) \\ \tilde{Q}_{2k}(x) & \tilde{Q}_{2k-1}(x) \end{pmatrix}$$

Hence

$$P_{2k}(x) = P_k(x)P_{k+2}(x) - b_k^2 P_{k-2}(x)Q_{k+2,n}(x)$$

$$\tilde{P}_{2k}(x) = P_k(x)\tilde{P}_{k+2}(x) - \tilde{b}_k^2 P_{k-2}(x)\tilde{Q}_{k+2,n}(x)$$

By the assumption $P_k(x) = \tilde{P}_{2k}(x)$ we have

$$P_k(x)\left[P_{k+2,n}(x) - \tilde{P}_{k+2,n}(x)\right] = P_{k-1}(x)\left[b_k^2 Q_{k+2,N}(x) - \tilde{b}_k^2 \tilde{Q}_{k+2,n}(x)\right]$$

Note that the zeros of $P_{2k}(x)$ and $P_{k-1}(x)$ are interlacing and that deg $[b_k^2 Q_{k+2,N}(x) - \tilde{b}_k^2 \tilde{Q}_{k+2,n}(x)]$ $n-k-2=k-2$ We conclude that

$$b_k^2 Q_{k+2,N}(x) + \tilde{b}_k^2 \tilde{Q}_{k+2,n}(x)$$

Moreover, both $Q_{k+2,N}(x)$ and $\tilde{Q}_{k+2,N}(x)$ are monic and b_k and $\tilde{b}_k$ are positive,hence $b_k = \tilde{b}_k$, $Q_{k+2,N}(x) = \tilde{Q}_{k+2,N}(x)$ and $P_{k+2,n}(x)$)= $\tilde{P}_{k+2,n}(x)$ this implies that $J_{k+2,n} = \tilde{J}_{k+2,n}$,and $a_{k+2} = trace\ J - trace\ J_{1,K} - - trace\ J_{,K+2,N} = = \tilde{a}_{k+1,}$ *i.e.*

We are going to use(87) to investigate some inverse problem for Jacobi matrices, including existence and uniqueness.The next theorem concerns uniqueness of a mixed data problem.

Theorem(3.2.6)[163]. Denote

$$J = J_n[a_1,..., a_n; b_1,..., b_{n-1}]$$

and

$$\tilde{J} = J_n[\tilde{a}_1,...,\tilde{a}_n; \tilde{b}_1,...,\tilde{b}_{n-1}]$$

with $b_1 > 0, \bar{b}_1 > 0$ >for i=1,2,...,n-1for two given natural numbers $0 < m_1 < m_2 \leq n$ .Suppose that

(i) $J_{m1} + 2, n = \tilde{J}_{m1} + 2, n4$ where $J_{i,j}$ and $\tilde{J}_{i,j}$ are as defined.

(ii) $b_{m1} + 1 = \tilde{b}_{m1} + 1$. Note that if $m_1+1 = m_2 = n$, this condition can be dropped.

(iii) σ ($J_{1,mj}$) = σ ($\tilde{J}_{1,mj}$) for j = 1,2.

Then $J_n = \tilde{J}_n$

proof. For the case $m_2 = m_1 + 1$, the theorem follows directly from theorem(3.2.4) hence we may assume that $m_2 \geq m_1 + 2$. We write

$$J_n \cong \begin{pmatrix} x-a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-a_n & 1 \\ -b_{n-1}^2 & 0 \end{pmatrix} \quad (74)$$

$$= \begin{pmatrix} P_n(x) & P_{n-1}(x) \\ Q_n(x) & Q_{n-1}(x) \end{pmatrix}$$

$$\tilde{J}_n \cong \begin{pmatrix} x-\tilde{a}_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-\tilde{a}_2 & 1 \\ -\tilde{b}_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-\tilde{a}_n & 1 \\ -\tilde{b}_{n-1}^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{P}_n(x) & \tilde{P}_{n-1}(x) \\ \tilde{Q}_n(x) & \tilde{Q}_{n-1}(x) \end{pmatrix} \tag{75}$$

On the other hand, denote

$$\begin{pmatrix} P_k(x) & P_{k-1}(x) \\ Q_k(x) & Q_{k-1}(x) \end{pmatrix} = \begin{pmatrix} x-a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-a_k & 1 \\ -b_{k-1}^2 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{P}_k(x) & \tilde{P}_{k-1}(x) \\ \tilde{Q}_k(x) & \tilde{Q}_{k-1}(x) \end{pmatrix} = \begin{pmatrix} x-\tilde{a}_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-\tilde{a}_2 & 1 \\ -\tilde{b}_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-\tilde{a}_k & 1 \\ -\tilde{b}_{k-1}^2 & 0 \end{pmatrix}$$

where $P_K(x)$ $Q_K(x)$), $\tilde{P}_K(x)$and $\tilde{Q}_K(x)$are is defined.Actually, $P_K(x)$is characteristic polynomial of $J_{1,k}$, $P_K(x)$is the characteristic polynomial of $j_{2,k}$ and $\tilde{P}_K(x)$) is the characteristic polynomial of $\tilde{J}_{2,k}$, let.

$$\begin{pmatrix} A_{i,j}(x) & B_{i,j}(x) \\ C_{I,J}(x) & D_{i,j}(x) \end{pmatrix} = \begin{pmatrix} x-a_i & 1 \\ -b_I^2 & 0 \end{pmatrix} \begin{pmatrix} x-a_2 & 1 \\ -b_I^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-a_J & 1 \\ -b_{J-1}^2 & 0 \end{pmatrix} \tag{76}$$

and

$$\begin{pmatrix} \tilde{A}_{i,j}(x) & \tilde{B}_{i,j}(x) \\ \tilde{C}_{I,J}(x) & \tilde{D}_{i,j}(x) \end{pmatrix} = \begin{pmatrix} x-\tilde{a}_i & 1 \\ -b_I^2 & 0 \end{pmatrix} \begin{pmatrix} x-\tilde{a}_2 & 1 \\ -\tilde{b}_I^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-\tilde{a}_J & 1 \\ -\tilde{b}_{J-1}^2 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} A_{j+1,j}(x) & B_{j+1,j}(x) \\ C_{j+1,j}(x) & D_{j+1,j}(x) \end{pmatrix} = \begin{pmatrix} \tilde{A}_{j+1,j}(x) & \tilde{B}_{j+1,j}(x) \\ \tilde{C}_{j+1,j}(x) & \tilde{D}_{j+1,j}(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since

$$\begin{pmatrix} P_{m_2}(x) & P_{m_2-1}(x) \\ Q_{m_2}(x) & Q_{m_2-1}(x) \end{pmatrix} = \begin{pmatrix} P_{m_1+1}(x) & P_{m_1}(x) \\ Q_{m_1+1}(x) & P_{m_1}(x) \end{pmatrix} \begin{pmatrix} A_{m_1+2,m_2}(x) & B_{m_1+2,m_2}(x) \\ C_{m_1+2,m_2}(x) & D_{m_1+2,m_2}(x) \end{pmatrix}$$

We have

$$P_{m_2}(x) = P_{m_1+1}(x)\, A_{m_1+2,m_2}(x) + (x) C_{m_1+2,m_2}(x) \tag{77}$$

Similarly,

$$\tilde{P}_{m_2}(x) = \tilde{P}_{m_1+1}(x)\,\tilde{A}_{m_1+2,m_2}(x) + (x)\tilde{C}_{m_1+2,m_2}(x) \tag{78}$$

By our assumptions, we have

$$\begin{pmatrix} A_{m_1+2,m_2}(x) & B_{m_1+2,m_2}(x) \\ C_{m_1+2,m_2}(x) & D_{m_1+2,m_2}(x) \end{pmatrix} = \begin{pmatrix} \tilde{\tilde{A}}_{m_1+2,m_2}(x) & \tilde{B}_{m_1+2,m_2}(x) \\ \tilde{C}_{m_1+2,m_2}(x) & \tilde{D}_{m_1+2,m_2}(x) \end{pmatrix}$$

$P_{m_1}(x) = \tilde{P}_{m_1}(x)$ and $P_{m_2}(x) = \tilde{P}_{m_2}(x)$ hence $P_{m_1+1}(x) = \tilde{P}_{m_1+1}(x)$ Hence,by Theorem(3.2.4). $J_{1,m1+1} = \tilde{J}_{1,m1+1}$ .With assumption(i) again, $J_n = \tilde{J}_n$.

By similar arguments, we have the conditions of existence for Theorem (3.2.5).

corollary(3.2.7)[163]. Let $m_1$and $m_2$be tow natural numbers with $0< m_1 < m_2 < n$, and [$\mu_1 < \mu_2 < \ldots \quad \mu_{m1}$] and [$\lambda_1 < \lambda_2 < \ldots < \lambda_{m2}$ be two sequences of real numbers corresponding to m_1 and m_2, respectively. For a given $(n-m_1-1)x(n-m_1-1)$ Jacobi matrix

$$T = J_{n-m1-1}\left[a_{m1+2} ...., a_{n;} b_{m1+2} ,..., b_{n-1}\right],$$

$$\begin{pmatrix} A_{m_1+2,n}(x) & B_{m_1+2,n}(x) \\ C_{m_1+2,n}(x) & D_{m_1+2,n}(x) \end{pmatrix}$$

$$= \begin{pmatrix} x - a_{m_1+2} & 1 \\ -b_{m+1}^2 & 0 \end{pmatrix} \begin{pmatrix} x - a_{m_1+3} & 1 \\ -b_{m+2}^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x - a_n & 1 \\ -b_{m-1}^2 & 0 \end{pmatrix}$$

Suppose that

(i) $Q(x)$ is ammonic polynomial of degree m_1+1.

(ii)The zeros of $Q(x)$ are all real and simple ,say{ $\tilde{\mu}_1 \quad \ldots < \tilde{\mu}_{m_{1+1}}$ },which are interlacing with the set { $\tilde{\mu}_1 \quad \ldots < \tilde{\mu}_{m_1}$ }.

Then we can reconstruct a unique n n Jacobi matrix J_n with positive off-diagonal elements such that σ ($J_{1,m1}$) = { $\mu_1 < \mu_2 < .... < \mu_{m1}$ }) = $\{\lambda_1 ,.... \lambda_{m2}\}$ and $J_{m_{1+1}}$= T.

Proof. Since $J_{m_{1+1}}$ the (m_1+1,m_1+2) entry of J_n are pre-determent, it is sufficient to determine, $J_{m_{1+1}}$Suppose such a Jacobi matrix exists, denoted by

$$J_n = J_n\left[a_1 ,..., a_n ; b_1 ,..., b_{n-1}\right]$$

then

$$J_{1,m_2} \cong \begin{pmatrix} x-a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-a_{m_2} & 1 \\ -b_{m_2-1}^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} P_{m_1+1}(x) & P_{m_2}(x) \\ Q_{m_1+1}(x) & Q_{m-1}(x) \end{pmatrix} \begin{pmatrix} A_{m_1+2,m_2}(x) & B_{m_1+2,m_2}(x) \\ C_{m_1+2,m_2}(x) & D_{m_1+2,m_2}(x) \end{pmatrix}$$

Since

$$\begin{pmatrix} A_{m_1+2,m_2}(x) & B_{m_1+2,m_2}(x) \\ C_{m_1+2,m_2}(x) & D_{m_1+2,m_2}(x) \end{pmatrix}$$

is pre- determined,

$$P_{m_1+1}(x)(x) = \{\prod_{i=1}^{m_2}(x-\mu_i) - [\prod_{i=1}^{m_1}(x-\mu_i)]C_{m_1+2,m_2}(x)/A_{m_1+2,m_2}(x)\}$$

Hence if $P_{m_1+1}(x)$satisfies assumption(i) and (ii), we can reconstruct $J_{m_1+1}(x)$), henceforth, J_n (x) could be reconstructed as required.

Example(3.2.8)[163]. There does not exist a 4 x 4 irreducible Jacobi matrix J= J_4[a_1,a_2,a_3,a_4: b_1,b_2,b_3] withσ ($J_{1,2}$) = {2,4},σ (J) ={1,3,5,6}, b_3 =1 and a_4 =2. Since in this case, $A_{4,4}(x)$ =-1, hence

$$Q(x) = [(x-1)(x-3)(x-5)(x-6) - (x-2)(x-4)(-1)]/(x-2)$$

$$= \frac{x^4 - 15x^3 + 78x^2 - 159x + 98}{x-2}$$

cannot be reduced to a polynomial.

Example(3.2.9)[163]. Reconstruct a 4 x4 Jacobi matrix J_4 = J_4[a_1, a_2, a_3, a_4: b_1, b_2, b_3] with σ($J_{1,2}$) = {1,3},σ(j_4) = { 1-$\sqrt{3}$, 1, 1+$\sqrt{3}$,4}, $j_{3,4}$ = J_2[2,1:2] and b_i 0, I = 1,2,3.

Solution. Let b_3^2 = 4 and

$$Q(x) = [(x^2 - 2x - 2)(x-1)(x-4) - (x-1)(x-3)(-4)]/(x-2)$$

$$= x^3 - 6x^2 + 10x + 4.$$

Then the zeros of Q(x) are 2- $\sqrt{2}$,2 and 2+ $\sqrt{2}$, more over we have

$$\frac{Q(x)}{(x-1)(x-3)} = x-2-\frac{1}{x-2-\frac{1}{x-2}}.$$

Hence, we can take

$$J_4 = J_4[2,2,2,1:1,1,2].$$

With the same techniques given above, we will provide an alternative approach to the existence theorem for an inverse Jacobi matrix problem which was promoted by Deift and Nanda see[166] let J_n be an n n Jacobi matrix with positive off-diagonals (J_n is uniquely determined by $\sigma(J_n)$ = s_2 and $\sigma(J_{1,n-1})$ (= s_3), the question is, under what conditions can J_n be completed to a 2n x2n Jacobi matrix J_{2n} with apre- given spectral set σ (J_{2n}) = s_1?

Lemma(3.2.10)[163]. let $s_1 = \{\lambda_1, \lambda_2, \lambda_3, .., \lambda_{2n}\}$,$s_2$ = { $\mu_1, \mu_2, \ldots, \mu_n$} and s_3 = { $\nu_1, \nu_2, \ldots, \nu_{n-1}$ }

Be three sets of real numbers with

$$\mu_1 < \mu_2 < \lambda_2 < \lambda_3 < \mu_2 < \lambda_4 < \ldots \lambda_{2n-2} < \lambda_{2n-1} < \mu_n < \lambda_{2n}, \tag{79}$$

and

$$\mu_1 < \nu_2 < \mu_2 < \cdots < \mu_k < \nu_k < \mu_{k+1} < \cdots < \nu_{n-1} < \mu_n . \tag{80}$$

Denote

$$P_{2n}(x) = \prod_{i=1}^{2n}(x-\lambda_i), \; P_n(x) = \prod_{j=1}^{n}(x-\mu_j) \; P_{n-1}(x) = \prod_{I=1}^{N-1}(x-\nu_I)$$

$C_{n+1,2n}(x)$ be a polynomial with deg $C_{n+1,2n}(x) \le n-1$ with

$$C_{n+1,2n}(\mu_i) = P_{2n}(\mu_i)/P_{n-1}(\mu_i), \; for i = 1,2,\ldots,n, and$$

Then

$$A_{n+1,2n}(x) = [P_{2n}(x) - P_{n-1}(x)C_{n+1,2n} / P_n(x).$$

(i) $C_{n+1,2n}(x)$ is a polynomial of degree n-1 with negative leading coefficient.

(ii) $A_{n+1,2n}(x)$ is a monic polynomial of degree n.

Proof. By the assumption, we have $\frac{P_{2n}(\mu_i)P_{2n}(\mu_{i+1})}{P_{n-1}(\mu_i)P_{n-1}(\mu_{i+1})}$<0, hence $C_{n+1,2n}(x)$has a zero in($\mu_i, \mu_i + 1$) for i= 1,2,3,…,n. since $C_{n+1,2n}(x)$is a polynomial with deg $C_{n+1,2n}(x)$< n-1 and $P_{2n}(\mu_n)/P_{n-1}(\mu_n)<$ 0,we conclude assertion(i). to show assertion(ii), we observe that

$$P_{2n}(\mu_i) \quad - P_{n-1}(\mu_i)\, C_{n+1,2n}(\mu_i) = P_{2n}(\mu_i) - P_{n-1}(\mu_i)\, \frac{P_{2n}(\mu_i)}{P_{n-1}(\mu_i)} = 0 = P_n(\mu_i)$$

and deg[$P_{2n}(x) - P_{n-1}(x)\, C_{n+1,2n}(x)$]= 2n and deg $P_n(x)$) =n. These lead to assertion(ii).

Corollary (3.2.11)[163]. let S_1,S_2,S_3, $P_{2n}(x)$, $P_{n-1}(x)$ $A_{n+1,2n}(x)$ and $C_{n+1,2n}(x)$ be as given in Lemma (3.2.10) we denote by- a^2 the leading coefficient of $C_{n+1,2n}(x)$ .suppose the zeros (t_1, t_2,…,t_{n-1}) of $C_{n+1,2n}(x)$are interlacing with the zeros{ $s_1,s_2,\ldots,s_n$} of $A_{n+1,2n}(x)$ $(i,e, si < ti < si+1,\ fori = 1,2,\ldots, n-1)$are interlacing. Then $A_{n+1,2n}(x)$ and $(1/-\alpha^2)C_{n+1,2n}(x)$ determine a Jacobi matrix

$$\tilde{J}_n = J_n\left[\tilde{a}_1, \tilde{a}_2 \ldots, \tilde{a}_n; \tilde{b}_1, \ldots, \tilde{b}_{n-1}\right]$$

with positive off –diagonal elements suth that the characteristic polynomial of $\tilde{J}_n$ is $A_{n+1,2n}(x)$ and the characteristic polynomial of

$$\tilde{J}_{n-1} = J_n\left[\tilde{a}_1, \tilde{a}_2 \ldots, \tilde{a}_{-1n}; \tilde{b}_{21}, \ldots, \tilde{b}_{n-1}\right]$$

is $(1/-\alpha^2)C_{n+1,2n}(x)$

Theorem(3.2.12)[163]. $n \in N$ .let s_1,s_2,s_3 and $P_{2n}(x)$be as given in Lemma(3.2.10) Supp0se that $J_n = J_n[a_1, a_2 \ldots, a_n; b_1, \ldots, b_{n-1}]$be an $n \times n$ Irreducible Jacobi matrix with $b_1 > 0$ for i=1,2,..,n-1, $\sigma(J_n)$

$$\begin{pmatrix} P_k(x) & P_{k-1}(x) \\ Q_k(x) & Q_{k-1}(x) \end{pmatrix} = . \begin{pmatrix} x-a_1 & 1 \\ -b_1^2 & 0 \end{pmatrix} \begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix} . . \begin{pmatrix} x-a_n & 1 \\ -b_{n-1}^2 & 0 \end{pmatrix}$$

$C_{n+1,2n}(x)$ is the unique polynomial of degree n-1 with

$$C_{n+1,2n}(\mu_i) = P_{2n}(\mu_i)/P_{n-1}(\mu_i)0 \tag{81}$$

for i=1,2,3,…,n,

$$A_{n+1,2n}(x) = \left[P_{2n}(x) - P_{n-1}(x)C_{n+1,2n}\ / P_n(x)\right. \tag{82}$$

Suppose zeros of $A_{n+1,2n}(x)$ are interlacing with the zeros of $C_{n+1,2n}(x)$,J can be completed to a2nx2n Jacobi matrix J_{2n} with

$$\sigma(J_{2n}) = \{\lambda_1, \lambda_2, ... \lambda_{2n}\}$$

Proof. By the assumption and corollary (3.2.11).we can reconstruct an nxn Jacopi matrix $\tilde{J} = J_n\left[\tilde{a}_1, \tilde{a}_2 ..., \tilde{a}_{-1n}; \tilde{b}_{21}, ..., \tilde{b}_{n-1}\right]$by $A_{n+1,2n}(x)$ and $\left(1/-\alpha^2\right)C_{n+1,2n}(x)0$.We may writ

$$\tilde{J} \cong \begin{pmatrix} x-a_{N+1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-a_{n+2} & 1 \\ -b_{n+1}^2 & 0 \end{pmatrix}..\begin{pmatrix} x-a_{2n} & 1 \\ -b_{2n-1}^2 & 0 \end{pmatrix} \tag{83}$$

$$\begin{pmatrix} A_{n+1,2n}(x) & B_{n+1,2n}(x) \\ -1/\alpha^2 C_{n+1,2n}(x) & D_{n+1,2n}(x) \end{pmatrix}$$

Henes

$$\begin{pmatrix} 1 & 1 \\ 1 & -\alpha^2 \end{pmatrix}\begin{pmatrix} x-a_{N+1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-a_{n+2} & 1 \\ -b_{n+1}^2 & 0 \end{pmatrix}..\begin{pmatrix} x-a_{2n} & 1 \\ -b_{2n-1}^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x-a_{N+1} & 1 \\ -\alpha^2 & 0 \end{pmatrix}\begin{pmatrix} x-a_{n+2} & 1 \\ -b_{n+1}^2 & 0 \end{pmatrix}..\begin{pmatrix} x-a_{2n} & 1 \\ -b_{2n-1}^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} A_{n+1,2n}(x) & B_{n+1,2n}(x) \\ C_{n+1,2n}(x) & -\alpha^2 D_{n+1,2n}(x) \end{pmatrix}$$

Moreover

$$\begin{pmatrix} P_n(x) & P_{n-1}(x) \\ Q_n(x) & Q_{n-1}(x) \end{pmatrix} = \begin{pmatrix} A_{n+1,2n}(x) & B_{n+1,2n}(x) \\ C_{n+1,2n}(x) & -\alpha^2 D_{n+1,2n}(x) \end{pmatrix}$$

$$= \begin{pmatrix} x-a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix}..\begin{pmatrix} x-a_n & 1 \\ -b_{n-1}^2 & 0 \end{pmatrix}\begin{pmatrix} x-a_{N+1} & 1 \\ -\alpha^2 & 0 \end{pmatrix}\begin{pmatrix} x-a_{n+2} & 1 \\ -b_{n+1}^2 & 0 \end{pmatrix}..\begin{pmatrix} x-a_{2n} & 1 \\ -b_{2n-1}^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} P_{2N}(x) & P_n(x)B_{n+1,2n}(x) - \alpha^2 P_{n-1}(x)D_{n+1,2n}(x) \\ Q_n(x)A_{n+1,2n}(x) + Q_{n-1}(x)C_{n+1,2n}(x) & Q_N(x)B_{n+1,2n}(x) - \alpha^2 Q_{n-1}(x)D_{n+1,2n}(x) \end{pmatrix}$$

$\cong J_{2N}\left[a_{m1+2} ...., a_{2N;} b_{1+2}, ..., b_{2N-1}, \alpha, b_{n+1}, ..., b_{2n-1}\right] \equiv J_{2N}$,

with $\sigma(J_{2n}) = \{\lambda_1, \lambda_2, ... \lambda_{2n}\}$This completes the proof.

Thorem(3.2.14)[163]. let

$$J_n = J_n[a_1,...,a_n; b_1,...,b_{n-1}]$$

and

denote tow Jacobe matrices with $b_1 > 0$, $\tilde{b}_1 > 0$. for i=1,2,,n-1. Suppose that

$\sigma(J) = \sigma(\tilde{J})$

$\sigma(J_{1,k}) = \sigma(\tilde{J}_{1,k})$ and $\sigma(J_{k+2,n}) = \sigma(\tilde{J}_{k+2,n})$ for some $1 < k \le n\text{-}2, k \in N.$

$\sigma(J) = \sigma(J_{1,k}) and (\sigma J_{k+2,n})$ are pairwisely disjoint.

Then J=j̃

Proof. It is sufficient show that $\sigma(J_{1,n-1}) = \sigma(\tilde{J}_{1,n-1})$.We write

$$J \cong \begin{pmatrix} x-a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-a_2 & 1 \\ -b_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-a_{K+1} & 1 \\ -b_K^2 & 0 \end{pmatrix} \begin{pmatrix} x-a_N & 1 \\ -b_{N-1}^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} P_{K+1}(x) & P_K(x) \\ Q_{K+1}(x) & Q_K(x) \end{pmatrix} \cdots \begin{pmatrix} x-a_{K+1} & 1 \\ -b_{K+1}^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-a_N & 1 \\ -b_{N-1}^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} P_{K+1}(x) & P_K(x) \\ Q_{K+1}(x) & Q_K(x) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -b_{K+1}^2 \end{pmatrix} \begin{pmatrix} x-a_{K+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-a_N & 1 \\ -b_{n-1}^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} P_{K+1}(x) & P_K(x) \\ Q_{K+1}(x) & Q_K(x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -b_{K+1}^2 \end{pmatrix} \begin{pmatrix} P_{K+2,N}(x) & P_{K+2,N-1}(x) \\ Q_{K+2,N}(x) & Q_{K+2,N-1}(x) \end{pmatrix}$$

$$\begin{pmatrix} P_n(x) & P_{n-1}(x) \\ Q_n(x) & Q_{n-1}(x) \end{pmatrix}$$

Similarly ,we have

$$\tilde{J} \cong \begin{pmatrix} x-\tilde{a}_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-\tilde{a}_2 & 1 \\ -\tilde{b}_1^2 & 0 \end{pmatrix} \cdots \begin{pmatrix} x-\tilde{a}_{K+1} & 1 \\ -\tilde{b}_K^2 & 0 \end{pmatrix} \begin{pmatrix} x-\tilde{a}_N & 1 \\ -\tilde{b}_{N-1}^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{P}_{K+1}(x) & \tilde{P}_K(x) \\ \tilde{Q}_{K+1}(x) & \tilde{Q}_K(x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{b}_{K+1}^2 \end{pmatrix} \begin{pmatrix} \tilde{P}_{K+2,N}(x) & \tilde{P}_{K+2,N-1}(x) \\ \tilde{Q}_{K+2,N}(x) & \tilde{Q}_{K+2,N-1}(x) \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{P}_n(x) & \tilde{P}_{n-1}(x) \\ \tilde{Q}_n(x) & \tilde{Q}_{n-1}(x) \end{pmatrix}.$$

Hence

$$P_k(x) = \frac{-P_n(x)P_{k+2,n-1}(x) + P_{n-1}(x)P_{k+2,n}(x)}{b_{k+1}^2 b_{k+2}^2 \ldots b_{n-1}^2}, \tag{84}$$

$$\widetilde{P}_k(x) = \frac{-\widetilde{P}_n(x)\widetilde{P}_{k+2,n-1}(x) + \widetilde{P}_{n-1}(x)\widetilde{P}_{k+2,n}(x)}{\widetilde{b}_{k+1}^2 \widetilde{b}_{k+2}^2 \ldots \widetilde{b}_{n-1}^2}, \tag{85}$$

Since $P_n(x) = \widetilde{P}_n(x), P_{1,k}(x) = \widetilde{P}_{1,k}(x) and P_{k+2,n}(x) = \widetilde{P}_{k+2}(x), we have$

$$P_n(x)\left[\frac{P_{k+2,n-1}(x)}{b_{k+1}^2 \ldots b_{n-1}^2} - \frac{\widetilde{P}_{k+2,n-1}(x)}{\widetilde{b}_{k+1}^2 \ldots \widetilde{b}_{n-1}^2}\right] = P_{k+2,n}(x)\left[\frac{P_{n-1}(x)}{b_{k+1}^2 \ldots b_{n-1}^2} - \frac{\widetilde{P}_{n-1}(x)}{\widetilde{b}_{k+1}^2 \ldots \widetilde{b}_{n-1}^2}\right]$$

Note that deg,we $\left[\frac{P_{n-1}(x)}{b_{k+1}^2 \ldots b_{n-1}^2} - \frac{\widetilde{P}_{n-1}(x)}{\widetilde{b}_{k+1}^2 \ldots \widetilde{b}_{n-1}^2}\right]$

$\leq n-2 and \quad that P_{n-1}(x) and \widetilde{P}_{n-1}(x) are \quad both$

moinc , by assumption (iii) we obtain $P_{n-1}(x) = \widetilde{P}_{n-1}(x)$ This completes the proof.

Chapter 4

Completely Nonunitary Contractions with Rank One Defect Operator

It shown that another functional model for contractions with rank one defect operators takes the form of the compression $f(\zeta) \rightarrow p_{\kappa}(\zeta f(\zeta))$ on the Hilbert space $L^2(\mathbf{T}, d\mu)$ with a prodbabilty measure μ onto the subspace $K = L^2(\mathbb{T}, d\mu)\Theta\mathbb{C}$. The relationship between characteristic functions of sub-matrices of the truncated CMV matrix with rank one defect operators and the corresponding Schur iterates is established. We develop direct and inverse spectral analysis for finite and semi-infinite truncated CMV matrices. In particular, we study the problem of reconstruction of such matrices from their spectrum or the mixed.

Sec(4.1) Rank One Defects Operator and Corresponding is Unitarily Colligations

It is well known [176] that every self- adjoint or unitary operator with a simple spectrum acting on some separable Hilbert space is unitarily equivalent to the operator of multiplication by the independent variable on the Hilbert space $L^2(R, d\mu)$ or $L^2(T, d\mu)$ respectively, where dμ is a probability measure on the real line R or on the until circle $T = \{\xi \in C : |\zeta| = 1\}$. The matrix representation of self – adjoint operators with sample spectrum was established for the first time by stone[176] , He proved that every self- adjoin operator with a simple spectrum is unitarily equivalent to certain Jacobi (tri- diagonal) matrix of form

$$j = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & . & . \\ a_1 & b_2 & a_2 & 0 & 0 & . & . \\ 0 & a_2 & b_3 & a_3 & 0 & . & . \\ . & . & . & . & . & . & . \end{pmatrix} \quad \textbf{(1)}$$

where $a_k > 0$, and b_k are real numbers for all $k \in$ N.The non-self-adjoint version of the Stone theorem has been recently obtained in [178] for dissipative non-self-adjoint operators with rank one imaginary part. It turned out that the matrix representation of such operators is a non-self-adjoint Jacobi matrix of the form (1) with only nonreal first entry b_1 satisfying Im$b_1 > 0$.

The problem of the canonical matrix representation of a unitary operator with a simple spectrum has been recently solved by M. Cantero, L. Moral and L. Velázquez in [188].

They introduced and studied five-diagonal unitary matrices of the form

$$C = C(|a_n|) = \begin{pmatrix} \alpha_n & \tilde{\alpha}_1 po & p_1 po & 0 & 0 & \ldots \\ po & -\tilde{\alpha}_1\alpha_0 & p_1\alpha_0 & 0 & 0 & \ldots \\ 0 & \tilde{\alpha}_2 p_1 & -\tilde{\alpha}_2\alpha_1 & \tilde{\alpha}_3 p_2 & p_3 p_2 & \ldots \\ 0 & p_2 p_1 & -p_2\alpha_1 & -\alpha_3\alpha_2 & p_3 p_2 & \ldots \\ 0 & 0 & 0 & \tilde{\alpha}_4 p_3 & -\alpha_4\alpha_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \quad (2)$$

Such matrix appears as a matrix representation of the unitary operator $(Uf)(\xi) = \xi f(\zeta)$ in $L^2(T, d\mu)$ with respect to the orthonormal system $\{\chi n\}$ obtained by orthonormalization of the sequence

$$\{1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \ldots\}.$$

The so called Schur parameters or Verblunsky coefficients $\{\alpha n\}$, $|\alpha n| < 1$, arise in the Szeg˝o recurrence formula

$$\zeta\phi_n(\zeta) = \phi_{n+1}(\zeta) + \overline{\alpha}\zeta^n\overline{\phi_n(1/\overline{\zeta})}, \qquad n = 0,1,\ldots$$

for monic orthogonal with respect to $d\mu$ polynomials $\{\Phi_n\}$, and $p_n := \sqrt{1-|\alpha_n|^2}$. The matrices $(\{\Phi_n\})$are called the *CMV* matrices. The spectral analysis of unitary CMV matrices has recently attracted much attention, and we refer on this matter to the [188,189,197,198,213-215] .

As pointed out by Simon in a recent section[215], the actual history of CMV matrices is more involved as it started in 1991 with Bunse-Gerstner and Elsner [187], and then with Watkins in 1993 [215], before Cantero, Moral, and Velázquez (CMV) re-discovered them in 2003. In a context different from orthogonal polynomials on the unit circle, Bourget, Howland, and Joye [183] introduced a set of doubly infinite matrices with three sets of parameters which for special choices of the parameters reduces to two-sided CMV matrices on $\ell^2(Z)$.

The spectral theory of non-self-adjoint and nonunitary operators and their models is based on the concept of *characteristic function* of the corresponding operator or the operator colligation[180,185,186,203-210,216].

In this section we employ the Sz.-Nagy–Foias theory [216], and the Brodskiˇı–Livšic unitary colligations approach [185] to the spectral analysis of contractions acting on Hilbert spaces. The corresponding characteristic function belongs to the Schur class of operator-valued functions holomorphic in the open unit disk D. By Sz.-Nagy–Foias theorem [216] each completely nonunitary contraction T with rank one defect operators $D_T = (1 - T^*T)^{1/2}$ and $D_{T^*} = (1 - TT^*)^{1/2}$ (shortly, with rank one *defects*) is unitarily equivalent to the operator (functional model) of the form

$$\mathfrak{H}_\Theta = (H^2 \oplus \operatorname{clos}\Delta L^2(\mathbb{T})) \ominus \{\Theta u \oplus \Delta u : u \in H^2\}$$

$$= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in H^2 . g \in clos\Delta L^2(T) . p_{H^2}(\overline{\Theta} f + \Delta g) = 0 \right\}$$

$$\mathfrak{I}_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = P_{\mathfrak{H}_\Theta} \zeta \begin{pmatrix} f \\ g \end{pmatrix}, \qquad \mathfrak{I}^*_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{\zeta}(f - f(0)) \\ \bar{\zeta} g \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{H}$$

where H^2 is the Hardy space.

$$\Theta = \Theta_T(z) = \left(-T + zD_{T*}(1 - zT*)^{-1} D_T\right) D_T$$

is the characteristics function of T, $\Delta^2 = 1 - |\Theta|^2$, P_{H^2} is the orthogonal projection onto H^2 in $L^2(\mathbb{T})$, and $p_{\mathfrak{H}_\Theta}$ is the orthogonal projection onto the model space $\mathfrak{H}_\Theta$.

We obtain a new functional model that complements the above mentioned Sz.-Nage- Foias functional model, and show that every completely nonunitary contraction T with rank one defects is unitarily equivalent to the compression $f(\zeta) \to p_k(\zeta f(\zeta))$ on the Hilbert space $L^2(T, d\mu)$ with a probability measure μ onto subspace $K = L^2(\mathbb{T}, d\mu) \ominus C$

We study the so called truncated CMV matrix T obtained from the “full”CMV matrix $C = C(\{\alpha_n\})$ (82)by deleting the first row and first column.

$$T = T(\{\alpha_n\}) = \begin{pmatrix} \tilde{\alpha}_1\alpha_0 & p_1\alpha_n & o & 0 & \dots \\ \tilde{\alpha}_2 p_1 & \tilde{\alpha}_2\alpha_1 & \tilde{\alpha}_3 p_2 & p_3 p_2 & \dots \\ p_2 p_1 & -p_2\alpha_1 & -\tilde{\alpha}_3\alpha_2 & -p_3\alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

In the semi- infinite case T takes on the block- matrix from

$$T = \begin{pmatrix} B_1 & C_1 & 0 & 0 & 0\dots \\ A_1 & B_2 & C_2 & 0 & 0\dots \\ 0 & A_2 & B_3 & C_3 & 0\dots \\ \dots & \dots & \dots & \dots & \dots\dots \end{pmatrix}$$

It turned out that the truncated CMV matrix $T^{(k)}(\{\alpha_n\})$ is a contraction with rank one defects and the Sz.- Nagy- Foias characteristic function agrees with the Schur function which has $\{\alpha\}$ as its Schur parameters.Moreover, we show that the sub- matrix $T^{(k)}(\{\alpha_n\})$ obtained from $T(\{\alpha_n\})$ by deleting the first k rows and columns is also a contraction with rank one defects, and characteristics function agrees with the well- known kth Schur iterate.

$$f_k(z) = \frac{f_{k-1}(z)-\alpha_{k-1}}{z(1-\alpha_{k-1})f_{k-1}(z)} \qquad f_0(z)$$

This relation is an analog of the corresponding relation between the m- function of a Jacobi matrix and the m- function of its sub- matrix [298].

Our main result states that an arbitrary completely nonunitary contraction T with rank one defects unitarily equivalent to any operator from the one- parameter family $T(\|e^{it}\alpha_n\|)$,where $\{\alpha_n\}$ are the Schur parameters of the SZ- Nagy- Foias characteristic function of T. We develop direct and inverse spectral analysis finite and semi- infinite truncated CMV matrices.

It is shown that given an arbitrary set of N not necessarily distinct numbers from D there is a one- parameter family of unitarity equivalent N x N truncated CMV matrices having those numbers as the eigen values counting algebraic multiplicity. We prove the uniqueness of N x N truncated CMV matrix T with given not necessary distinct eigenvalues $z_1.....z_r$, and given first $\mathrm{N-r+1}$ Schur parameters $\alpha_0(T).....\alpha_{N-r}(T)$. This result on inverse spectral analysis of finite truncated CMV matrices is an analog of the Hochstadi [302] and Gesztesy- Simon [298] uniqueness Theorem for finite self-adjoint Jacobe matrices as well as for established in[178] uniqueness theorem for finite non-self-adjiont jacobi matrices with rank one imaginary part. We obtain the existence of N×N truncated CMV matrix T when its eigenvalues $z_1.....z_m$ and the last Schur parameters $\alpha_m(T)......\alpha_N(T)$ are known.

Here is a summary of the rest of the section. We discuss some basics from the Sz.- Nagy-Foias theory and the unitary colligations with the focus upon the characteristics function and its properties, we provides a brief overview of the theory of orthogonal polynomials on the unit circle and CMV matrices. The main results concerning truncated CMV matrices and the models of completely nonuitariy contractions with rank one defects are presented ,the inverse spectral analysis for truncated CMV matrices .

Let H be a separable Hilbert space with the inner product (.,.) Abounded linear operator T in H is called a contraction if $\|T\|\leq 1$ (for the basic properties of contractions see[217]),if T is a contraction then the operators.

$$D_T := (1-T^*T)^{1/2} \qquad D_{T^*} := (1-TT^*)^{1/2}$$

are called the defect operators of T or, shortly, defects and the subspaces $D_T = \overline{ran} D_T, D_{T^*} = \overline{ran} D_{T^*}$ the defect subspaces of T . The dimensions $\dim D_T, \dim D_{T^*}$, are known as defect numbers of T. . Given a Pair of numbers $n, n^* = 0,1,\ldots\infty$, it is easy to construct a contraction with $n = \dim D_{T^*}, n^* = \dim D_{T^*}$ Eash contraction T acting on a finite. dimensional Hilbert space has equal defect numbers $n = n^*$

The defect operators satisfy the following intertwining relations.

$$TD_T = D_{T^*}T, T^*D_{T^*} = D_T T^* \tag{3}$$

and the block- operators

$$\begin{pmatrix} -T^* & D_T \\ D_{T^*} & T \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_{T^*} \\ H \end{pmatrix} \to \begin{pmatrix} \mathfrak{D}_T \\ H \end{pmatrix}, \qquad \begin{pmatrix} -T & D_{T^*} \\ D_{T^*} & T^* \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_T \\ H \end{pmatrix} \to \begin{pmatrix} \mathfrak{D}_{T^*} \\ H \end{pmatrix}$$

are unitary operators in the corresponding orthogonal sums of the spaces it follows from(3) that

$\subset \mathfrak{D}_{T^*}, T^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$ and $T(\ker D_T) = \ker D_{T^*}, T^*(\ker D_{T^*} = \ker D_T$: Moreover $T \restriction \ker D_T$ and $T^* \restriction \ker D_{T^*}$ are isometric operators. It follows that T is a quasi-unitary extension [204] of the isometric operator $V = T \restriction \ker D_T$

A contraction T is called completely nonuitary if there is no nontrivial reducing subspace of T, on which T generates a unitary operator. One of the fundamental results of the contractions theory[217]reads that, given a contraction T in H, these is acanonical orthogonal decomposition

$$H = H_0 \oplus H_1, \qquad T = T_0 \oplus T_1, T_j = T \restriction H_j \quad J = 0,1$$

where H_0 and H_1 reduce T,T_0 is a completely nonuitary contraction and $T_1$is a unitary operator. Moreover,

so $$H_1 = \left(\bigcap_{n\ge1} \ker D_{T^n}\right) \cap \left(\bigcap_{n\ge1} \ker D_{T^{*n}}\right),$$

T is completely nounitary

$$\Leftrightarrow \left(\bigcap_{n\ge1} \ker D_{Tn}\right) \cap \left(\bigcap_{n\ge1} \ker D_{T^*n}\right) = \{0\} \tag{4}$$

Clearly

$$\bigcap_{n\geq 1} \ker D_{T^n} = H\theta\overline{span}\{T^{*n} D_T H, n = 0,1,....\} \quad (5)$$

$$\bigcap_{n\geq 1} \ker D_{T^{*n}} = H\theta\overline{span}\{T^{n} D_{T*} H, n = 0,1,....\}$$

Let V be an isometry in H.A subspace Ω in H is called wandering for V if $V^p\Omega \perp V^q\Omega$ for all p.q $\in Z_+$, p $\neq$ q .SinceV is an isometry, the latter is equivalent to $V^n\Omega \perp \Omega$ for all $n \in N$ if $H = \oplus_{n=0}^{\infty} V^n\Omega$ then V is called a unilateral shift and Ω is called the generating subspace. The dimension of $\Omega$is called the multiplicity of the unilateral shift V. It is well known [216] that V is a unilateral shift if and only if $\bigcap_{n=0}^{\infty} V^n H = \{0\}$. Clearly, if an isometry V is the unilateral shift in H,B then Ω = H $\ominus$ VH is the generating subspace forV.

Given a contraction T in H and asubspace $\mathfrak{H} \subset$ H, the unilateral shift V:$\mathfrak{H} \to \mathfrak{H}$ is said to be contained in T if $\mathfrak{H}$ is invariant forT, and.The subspaces $\bigcap_{n\geq 1} \ker D_{T^n}$ and $\bigcap_{n\geq 1} \ker D_{T^{*n}}$ are invariant for T and T* respectively, and the operators $V_T: T \upharpoonright \bigcap_{n\geq 1} \ker D_{T^n}$ and $V_{T^*}: T^* \upharpoonright \bigcap_{n\geq 1} \ker D_{T^{*n}}$are unilateral shift Moreover $V_T$andV_{T^*} are the maximal unilateral shifts contained in T and The multiplicities of the shifts V_T and V_{T^*} do exceed the defect numbers $\dim\mathfrak{D}_{T^*}$and$\dim\mathfrak{D}_T$,respectively [192] if T is a completely nonunitary contraction with rank one defects.then(see[190],[192]).

Tdoes not contain the unilateral shift

$\Leftrightarrow T*$ Does not contain the unilateral shift

$$\Leftrightarrow \bigcap_{n\geq 1} \ker D_{T^n} = \{0\} \qquad \Leftrightarrow \bigcap_{n\geq 1} \ker D_{T^{*n}} = \{0\} \quad (6)$$

The function[217].

$$\Theta_T(z) = \left(-T + zD_{T*}(1 - zT*)^{-1} D_T\right)|D_T$$

is known as the characteristic function of the Sz- Nager- Foias type of a contraction T. This function belong to the Schur class $S(D_T, D_{T*})$of $L(D_T D_{T*})$ -valued holomorphic in the unit disk D operator- functions, i.e., $\|\Theta_T(0)f\| < \|f\|$ for all $f \in D_T \setminus \{0\}$. The characteristic function of T and T^* are connected by the relation

$$\Theta_{T*}(z) = \Theta_T^*(\bar{z}), \qquad z \in \mathbb{D}$$

Two operator- valued functions $\Theta_1 \in S(m_1, n_1)$ and $\Theta_2 \in S(m_2, n_2)$ are said to agree if there are two unitary operator $V : n_1 \to n_2$ and $W : m_2 \to m_1$ such that

$$V\Theta_1(z)\, W = \Theta_2(z) \qquad z \in \mathbb{D}$$

It is well known[217] that two completely nonunitary contractions T_1 and T_2 are unitarily equivalent if and only if their characteristic functions Θ_{T_1} and Θ_{T_2} agree. Every operator- valued function Θ from the Schur class $S(m,n)$ has almost everywhere nontangential strong limit values $\Theta(\zeta), \zeta \in T$. A function $\Theta \in S(m,n)$ is called inner if $\Theta^*(\zeta)\Theta(\zeta) = 1m$ for a.e., $\zeta \in T$. A function $\Theta \in S(m,n)$ is called bi- inner, if it is both inner and co- inner. A contraction T on a Hilbert space $\mathfrak{H}$ belong to the classes C0.(C.0),if

$$s - \lim_{n\to\infty} T^n = 0 \qquad \left(s - \lim_{n\to\infty} T^{*n} = 0\right)$$

respectively. By definition $C_{00} := C_{0.} \cap C_{.0}$. The completely nonunitary part of a contraction T belong to the class C_0, C_0 or C_{00} if and only its characteristics function $\Theta_T(z)$ is inner. or bi- inner,respectively[217].

In the following statement[217] the spectrum of completely nonuitary contraction is described.

Theorem (4.1.1)[175]: letT be a completely nonunitary contraction onH. Denote by S_T the set of points $z \in D$ for which the operator $\Theta_T(z)$ is not boundedly invertible, together with those $z \in T$ not lying on any of the open arcs of T on which Θ_T is a unitary operator valued analytic function.Furthermore, denote by S_T^0 the set of points $z \in D$ for which $\Theta_T(z)$ is not invertible at all. Then the spectrum $\sigma(T)$ of T agrees with S_T,and the point spectrum $\sigma(T)$ with S_T^0.

It T is completely nonunitary contraction with rank one defects, and if z_0 is an eigenvalue of T, then the geometric multiplicity of z_0 is one, the algebraic multiplicity is finite, and the characteristic function Θ_T admits the following factorization.

$$\Theta_T(z) = c \prod \frac{zk}{zk} \frac{zk - z}{1 - zkz} \exp\left(\int_0^{2\pi} \frac{e^{ie} + z}{e^{ie} - z} \, d\mu(t) \right)$$

$$\times \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ie} + z}{e^{ie} - z} Ink(t)dt \right)$$

where $|c| = 1, k(t) \geq 0. Ink(t) \in L_1[0,2\pi], \mu$ is a finite nonnegative measure singular with respect to the Lebesgue measure, and $\{zk\}$ are the eignvalues of T.In addition, if $\dim H = N < \infty$,and T is a completely noninitary contraction in H with defects, then its characteristic function is the linite Blaschhke product of order N of the form

$$b(z) = e^{i\varphi} \prod_{k=1}^{m} \left(\frac{z - z_k}{1 - \bar{z} z_k} \right)^{l_k}.$$

where $z_1, ...z_m$ are distinct eigenvalues of Twith the algebraic multiplicityies $l_1 + ... + l_m$ respectively $l_1 + ... + l_m = N$, and $\varphi \in [0,2\pi]$. Hence a finite- dimensional completely nonunitary contractionT with rank one defects belongs to the class C_{00}, and $\lim_{n\to\infty} \|T^n\| = 0$ it is easily seen from Theorem(4.1.1). that the point spectrum of a contractionT with rank one defects agrees with D if and only if $\Theta_T \equiv 0$.

Every contractionT acting on Hilbert space H can be included into the unitary operator colligation[11]1

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathfrak{M}, \mathfrak{N}, H \right\},$$

where m and n are separable Hilbert spaces and

$$U = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} : \begin{pmatrix} \mathfrak{M} \\ H \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ H \end{pmatrix} \right\}$$

is a unitary operator.T is called the basic operator of the unitary colligation Δ. The spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the left outer space and right outer space, respesctively. The unitarly of means

$$U^*U = \begin{pmatrix} I_{\mathfrak{M}} & 0 \\ 0 & I_H \end{pmatrix}, \quad UU^* = \begin{pmatrix} I_{\mathfrak{N}} & 0 \\ 0 & I_H \end{pmatrix}$$

or equivalently,

$$T^*T + G^*G1_H \qquad F^*F + S^*S = 1mT^*F + G^*S = 0 \tag{7}$$

$$TT^* + FF^* = 1_H GG + SS^* = 1n \qquad TG^* + FS^* = 0$$

The colligation

$$\Delta = \left\{\begin{pmatrix} -T^* & D_T \\ D_{T^*} & T \end{pmatrix}; \mathfrak{D}_T, \mathfrak{D}_{T^*}, H\right\}, \tag{8}$$

provides an example of the unitary colligation with give basic operatorT

Let $\Delta = \left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathfrak{M}, \mathfrak{N}, H\right\}$,be a unitary colligation. Define the following subspaces in H

$$\{T^* F\mathfrak{M}, n = 0,1, \dots\}, H^{(c)} = \overline{\text{span}}$$

$$\overline{\text{span}}\{T^{*n} G^* \mathfrak{N}, n = 0,1, \dots\}. \ H^{(0)} = \tag{9}$$

The subspaces $H^{(c)}$ and $H^{(0)}$ are called the controllable and the observable subspaces, respectively. Let

$$\left(H^{(c)}\right)^{\perp} := H\theta H^{(c)}.\left(H^{(c)}\right)^{\perp} := \left(H^{(c)}\right)^{\perp} := H\theta H^{(0)} \tag{10}$$

A unitary colligation Δ is called prime if $\overline{H^{(c)} + H}^{(0)} = H$. Clearly, the latter condition is equivalent to

$$\left(H^{(c)}\right)^{\perp} \cap \left(H^{(0)}\right)^{\perp} = \{0\}$$

From**(7**) and (10) we get

$$\left(H^{(c)}\right)^{\perp} = \bigcap_{n\geq 0} \ker\left(F^* T^{*n}\right) = \bigcap_{n\geq 0} \ker\left(D_{T*} T^{*n}\right) = \bigcap_{n\geq 0} \ker(D_{T^{*n}})$$

$$\left(H^{(0)}\right)^{\perp} = \bigcap_{n\geq 0} \ker\left(G T^{n}\right) = \bigcap_{n\geq 0} \ker\left(D_T T^{n}\right) = \bigcap_{n\geq 0} \ker(D_{T^n}) \tag{11}$$

If follows now from (4) that the unitary colligation

$$\Delta = \left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathfrak{M}, \mathfrak{N}, H\right\},$$

is prime if and only if T is a completely nonunitary operator.

Given a unitary colligation

$$\Delta = \left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathfrak{M}, \mathfrak{N}, H\right\},$$

its characteristic functions[2] [286] is defined by

$$\Theta_{\Delta}(z) = S + zG(1_H - zT)^{-1} F, \qquad z \in D$$

This function belong to the Schur class $S(\mathfrak{M},\mathfrak{N})$ of $\mathcal{L}(\mathfrak{M},\mathfrak{N})$ −valued holomorphic in the unit disk D operator- functions. In particular, the characteristic function of the unitray colligationΔ_0 (8)

$$\Theta_0(z) = (-T^* + zD_T(1 - zT)^{-1}D_{T^*}) \upharpoonright \mathfrak{D}_{T^*}$$

is in fact the Sz- Nagy- Fioas characteristic function of the operatorT^*

Two prime unitary colligations

$$\Delta_1 = \left\{\begin{pmatrix} S & G_1 \\ F_1 & T_1 \end{pmatrix}; \mathfrak{M}, \mathfrak{N}, H_1\right\} \text{and} \left\{\begin{pmatrix} S & G_2 \\ F_2 & T_2 \end{pmatrix}; \mathfrak{M}, \mathfrak{N}, H_2\right\} \Delta_2 =$$

Which have equal characteristic function are unitarily equivalent in the following sense [286] there exists a unitary operator $V : H_1 \to H_2$ such that

$$VT_1 = T_2V, \qquad VF_1F_2, \quad G_2V = G_1$$

$$\Leftrightarrow \begin{pmatrix} 1n & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} S & G_1 \\ F_1 & T_1 \end{pmatrix} = \begin{pmatrix} S & G_2 \\ F_2 & T_2 \end{pmatrix} \begin{pmatrix} 1n & 0 \\ 0 & V \end{pmatrix}$$

Besides given$\Theta \in S(\mathfrak{M},\mathfrak{N})$,there exists a prime unitary colligation

$$\Delta = \left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathfrak{M}, \mathfrak{N}, H\right\}$$

such that $\Theta_\Delta = \Theta$ in D [286].

Theorem(4.1.2)[175] Let T be a contraction with finite defect numbers acting on Hilbert space H. Suppose that m and n are two given Hilbert space such that $\dim\mathfrak{N} = \dim\mathfrak{D}_T$,and $\dim \mathfrak{M} = \dim\mathfrak{D}_{T^*}$. Then all unitary colligation with the basic operator T and outer sunspaces $\mathfrak{M}$ and $\mathfrak{N}$ take the form.

(12)

$$\Delta = \left\{\begin{pmatrix} -KT^*M & KD_T \\ D_{T^*}M & T \end{pmatrix} : m, n, H\right\}$$

where $K : D_T \to n$ and $M : m \to D_{T^*}$ are unitary operators, The characteristic function of Δ is

$$\Theta_\Delta(z) = K\Theta_{T^*}(z)M, \qquad z \in D,$$

i.e., Θ_Δ agrees with the characteristic function Θ_{T^*} of T^*

proof. Let $\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} : m, n, H \right\}$ be a unitary colligation. From the relation

$G^*G + T^*T = 1_H$ it follows that

$$\|Gf\|^2 = \|D_T f\|^2. \qquad f \in H$$

Hence, the operator $K: \mathfrak{D}_T \to \mathfrak{N}$ defined by

$$KD_T f = Gf, \qquad f \in H$$

is isometric, and ran $K = \mathfrak{N}$. Similarly, the relation $FF^* + TT^* = 1_H$ yield than the operator $K: \mathfrak{D}_{T^*} \to \mathfrak{M}$ given by the relation

$$ND_{T^*} f = F^* f, \qquad f \in H$$

Is isometric, and ran $N = \mathfrak{M}$ so $M = N^*: \mathfrak{M} \to \mathfrak{D}_{T^*}$ is unitary, and $F = D_{T^*}M$.

From the relation $T^*F + G^*S = 0$ we get $T^*D_{T^*}M + D_T K^* S = 0$ Hence by. $T^*M + K^*S = 0$ As $\operatorname{ranM} = D_{T^*}$ $\operatorname{ranK}^* = D_T$ and $TD_{T^*} \subset D_T$ we have

$$S = KT^*M$$

Observe also that

$$TG^* + FS^* = TD_T K^* - D_{T^*}MM^*TK = 0$$

$$SS^* + GG^* = KT^*MM^*TK^* + KD_T^2 K$$

$$= K(T^*T + 1 - 1T^*T)k^* = 1n$$

$$S^*S + F^*F = M^*TK^*KT^*M + M^*D_{T^*}T$$

$$= M^*(TT^* + 1 - 1TT^*)M = 1m$$

Thus, all conditions(7) are satisfied, i.e, the colligation Δ is of the form(12).

Conversely,if $\dim\mathfrak{N} = \dim \mathfrak{D}_T < \infty$, $\dim\mathfrak{M} = \dim \mathfrak{D}_{T^*} < \infty$, and $k: \mathfrak{D}_T \to \mathfrak{N}$ and $M: \mathfrak{M} \to \mathfrak{D}_{T^*}$ are unitary operators, then one can easily see that

$$U = \begin{pmatrix} -K^*M & KD_T \\ D_{T^*}M & T \end{pmatrix} : \begin{pmatrix} \mathfrak{M} \\ H \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ H \end{pmatrix}$$

is a unitary operator , i.e, the relation(7) are satisfied. It follows that

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix}; m, n, H \right\}$$

is a unitary colligation, where $G = KD_T, F = D_{T*}M, S = -kT*M$

For the characteristic function Θ_Δ we obtain for all $z \in D$

$$\Theta_\Delta(z) = S + zG(1 - zt)^{-1}F$$

$$= -kT*M + zkD_T(1 - zT)^{-1}D_{T*}M = K\Theta_{T*}(z)M$$

Corollary(4.1.3)[175]: Let T be a contraction with finite defect numbers, $\dim\mathfrak{N} = \dim\mathfrak{D}_T$,and $\dim\mathfrak{M} = \dim\mathfrak{D}_{T^*}$. and let.

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix}; m, n, H \right\}$$

be a unitary colligation. Then all other unitary colligations with the operator T and outer subspace m and n take the form

$$\tilde{\Delta} = \left\{ \begin{pmatrix} C_1SC_2 & G \\ FC_2 & T \end{pmatrix}; m, n, H \right\}$$

where C_1 and C_2 are unitary operators in n and m, respectively

Proof. by Theorem (4.12) we have

$$K : D_T, \qquad F = D_{T*}M, \qquad S = KT*M$$

Where $k : \mathfrak{D}_T \to \mathfrak{M}$ andM: $\mathfrak{M} \to \mathfrak{D}_{T^*}$ are unitary operators. If $\tilde{\Delta} = \left\{ \begin{pmatrix} \tilde{S} & \tilde{G} \\ \tilde{F} & T \end{pmatrix}; m, n, H \right\}$

is some other unitary colligation then $\tilde{G} = \tilde{K}D_T, \tilde{F} = D_{T*}\tilde{M}\tilde{S} = -\tilde{K}T*\tilde{M}$where $K : D_T \to m$ and $\tilde{M} : n \to D_{T*}$ are unitary operators let $C_1 := \tilde{K}K^{-1}, C_2 := M^{-1}\tilde{M}$ then C_1 and C_2 are unitary operators in $\mathfrak{N}$ n and $\mathfrak{M}$ respectively, and

$$\tilde{G} = C_1G, \qquad \tilde{F} = FC_2, \qquad \tilde{S} = C_1SC_2$$

as needed

Theorem (4.1.4)[175]. Each contraction T with rank one defects on the Hilbert space H can be included into the unitary colligation

$$\Delta=\left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}: C, C, H\right\}$$

Let $\vec{I}=\binom{1}{0}\in C\oplus H$ and let the subspace $\left(H^{(c)}\right)^{\perp}$ in H be defined by(10). Then

$$\left(H^{(c)}\right)^{\perp}=(C\oplus H)\theta\,\overline{span}\{U^{n}\vec{1}; n=0,1\ldots.\}$$

$$\left(H^{(0)}\right)^{\perp}=(C\oplus H)\theta\,\overline{span}\{U^{*n}\vec{1}; n=0,1\ldots.\} \tag{13}$$

And so the following conditions are equivalent:

(i) the unitary colligation $\Delta=\left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; C, C, H\right\}$ is primer;

(ii) T is completely nonunitary contracting;

(iii) $\vec{1}$ is the cyclic vector for $U:\overline{span}\{U^{n}\vec{1}, n\in Z\}=C\oplus H$.

All other unitary colligations with basic operatorT and the outer spacesC Care the form

$$\tilde{\Delta}=\left\{\begin{pmatrix} c_1c_2S & c_1G \\ c_2F & T \end{pmatrix}; C, C, H\right\} \tag{14}$$

where $|c_1|=|c_2|=1$

Proof. Since $\dim D_T=\dim D_{T*}=1$ by Theorem (4.1.2) we can choose unitary colligation $\Delta=\left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; C, C, H\right\}$ of the form (12),i.e, $S=-KT*M, G=KD_{T*}, F=D_{T*}M$ and $K:ranD_T\to C, M:C\to ranD_{T*}$ are isometric operators. So, $U=\begin{pmatrix} S & G \\ F & T \end{pmatrix}:\begin{pmatrix} C \\ H \end{pmatrix}\to\begin{pmatrix} C \\ H \end{pmatrix}$ is the unitary operator.

To prove (13), suppose that the vector $h=\begin{pmatrix} z \\ h \end{pmatrix}\in C\oplus H$ is orthogonal to the subsp

$\overline{span}\{U^{n}\vec{1}n=0,1,\ldots..\}$. Then $U^{*n}\vec{h}\perp,\vec{1}, n=0,1\ldots\ldots,$ so $z=0$ and $\vec{h}=\begin{pmatrix} 0 \\ h \end{pmatrix}$.By using

$U^{*}=\begin{pmatrix} S* & F* \\ G & T* \end{pmatrix}$. we get consequently

$$F^*h=0, F^*T^*h=0 F^*T^{*2}h=0......F^*T^{*k}h=0,.........$$

it follows from (11) that $h\in\left(H^{(c)}\right)^{\perp}$. Conversely, if $h\in\left(H^{(c)}\right)^{\perp}$ then $h\perp\overline{span}\{U^n\vec{1}, n=0,1,......\}$. Similarly, $\left(H^{(0)}\right)^{\perp}=(C\oplus H)\theta\left(\overline{span}\{U^{*n}\vec{1}, n=0,1,.....\}\right)$, as needed.

We arrive at the following conlusion:

$\vec{1}$ is a cyclic vectorfor U

$\Leftrightarrow \left(H^{(c)}\right)^{\perp}\cap\left(H^{(0)}\right)^{\perp}=\{0\}$

$\Leftrightarrow$ The unitary colligation $\Delta=\left\{\begin{pmatrix} S & G \\ F & T\end{pmatrix}; C, C, H\right\}$ is prime

$\Leftrightarrow$ The operator T is completely nonunitary.

By Corollary (4.1.3) all other unitary colligations with basic operatorT and the outer subspace C are given by (14) with $|c_1| = |c_2| = 1$.

Let us give more precise expressions for the operatorsF,G, and S Let $\tilde{\varphi}_1\in D_T, \tilde{\varphi}_2\in D_{T^*}$ put

$$\varphi_1=\frac{\hat{\varphi}_1}{\|\hat{\varphi}_1\|},\qquad \varphi_2=\frac{\hat{\varphi}_2}{\|\hat{\varphi}_2\|}.$$

Then

$$Kh=b_1(h,\varphi_1),\qquad h\in \operatorname{ran} D_T,$$

$$M^*g=b_2(h,\varphi_2),\qquad g\in \operatorname{ran} D_{T^*},$$

where $|b_1|=|b_2|=1$ observe that $T\varphi_1=-\alpha_0\varphi_2$ and $T^*\varphi_2=-\tilde{\alpha}_0\varphi_1$ where α_0 is a complex number from D.It follows that

$$D_T^2\varphi_1=\left(1-|\alpha_0|^2\right)\varphi_1 \qquad D_{T^*}^2\varphi_2=\left(1-|\alpha_0|^2\right)\varphi_2$$

Let $p_0=\sqrt{1-|\alpha_0|^2}$. Since $\dim\left(ranD_T^2\right)=\dim\left(ranD_{T^*}^2\right)=1$ the number is a unique positive eigenvalue of $D_T(D_{T^*})$. Next,

$$Gh=b_1\left(D_T h,\varphi_1\right)=b_1\left(h,D_T\varphi_1\right)=b_1 p_0\left(h,\varphi_1\right)$$

$$F^*h=b_2\left(D_{T^*}h,\varphi_2\right)=b_2\left(h,D_{T^*}\varphi_2\right)=b_2 p_o\left(h,\varphi_2\right) h\in H$$

Hence $F_1 = p_o b_2 \varphi_2$ Since $S = KT * M$, we get

$$S_1 = -b_1 b_2 (T * \varphi_2, \varphi_1) = b_1 b_2 \tilde{\alpha}_0$$

In the case $\dim H = N < \infty$ the operator T can be given by the $N \times N$ matrix with respect to some orthonormal basic we can choose $\hat{\varphi}_1$ (respectively, $\hat{\varphi}_2$) as one the nonzero columns of the matrix $1 - T * T(1 - TT *)$ in addition.

$$\mathrm{Trace}(1 - \mathrm{T} * \mathrm{T}) = Trace(1 - TT*) = p_0^2$$

Thus, if

$$\varphi_2 = \begin{pmatrix} \varphi_2^{(1)} \\ \varphi_2^{(2)} \\ . \\ . \\ . \\ \varphi_2^{(N)} \end{pmatrix}$$

then the column F takes the form

$$F = \bar{b}_2 \rho_0 \begin{pmatrix} \varphi_2^{(1)} \\ \varphi_2^{(2)} \\ . \\ . \\ . \\ \varphi_2^{(N)} \end{pmatrix}$$

If

$$\varphi_1 = \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_1^{(2)} \\ . \\ . \\ . \\ \varphi_1^{(N)} \end{pmatrix}$$

then the row G take the form $G = b_1 p_o \begin{pmatrix} (1) & (2) & & & & -(N) \\ \varphi_1 & \varphi_1 & . & . & . & \varphi_1 \end{pmatrix}$.Finally, the numbers S is given by $G = b_1 b_2 (T^* \varphi_2, \varphi_1)$

If dim H=N and T is a completely nonunitary contraction with rank one defects then Θ_Δ is a finite Blaschke product

$$\Theta_\Delta(z) = e^{i\varphi} \prod_{k=1}^{N} \frac{z - \tilde{z}k}{1 - zkz}$$

Where the numbers $z_1, ...z_N$ are the eigenvalues of T Since all other colligations are of the form (14), for the characteristic function $\Theta_{\tilde{\Delta}}(z)$ we get $\Theta_{\tilde{\Delta}}(z) = c_1 c_2 \Theta_\Delta(z) = e^{it}\Theta_\Delta(z), z \in D$ and $t \in [0, 2\pi)$.

Let U be a unitary operator with a cyclic vector e, acting on the Hilbert space H. The spectral measure μ associated with U and e provides the relation

$$(\mathrm{F(U)e,e}) = \int_T \mathrm{F}(\zeta)\mathrm{d}\mu(\zeta)$$

which the spectral Theorem for unitaries. For instance,

$$F(z) = \left((U + z_1)(U - z_1)^{-1} e, e\right) = \int_T \frac{\zeta + z}{\zeta - z} d\mu, z \in D \qquad (15)$$

is the Caratheodory function(28) i.e.,F is holomorphic in the unit disc D. Re $F > 0$ in D, and $\mathrm{F}(0) = 1$

Theorem (4.1.5)[175]: Let T be a completely nonunitary contraction with rank one defects, $\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} : C, C, H \right\}$ be the prime colligation, and Θ_Δ be its characteristic function. Put

$$F(z)=\left((U+z_1)(U-z_1)^{-1}\vec{1}\,\vec{1}\right), z\in D \qquad (16)$$

where $\Delta=\left\{\begin{pmatrix} S & G \\ F & T\end{pmatrix}:\begin{pmatrix} C \\ H\end{pmatrix}\to\begin{pmatrix} C \\ H\end{pmatrix}\right\}$. Then

$$\overline{\Theta_\Delta(z)}=\frac{1}{z}\frac{F(z)-1}{F(z)+1}, F(z)\frac{1+z\overline{\Theta_\Delta(\tilde{z})}}{1-z\overline{\Theta_\Delta(\tilde{z})}}, z\in D \qquad (17)$$

Proof. We use the well- known schur- Frobenius formula for the inverse of block operators(see[193,194]). Let $\mathfrak{H}_1$ and$\mathfrak{H}_2$ be two Hilbert spaces, and Φ an operator in $\mathfrak{H}_1\oplus\mathfrak{H}_2$given by the block operator matrix

$$\phi=\begin{pmatrix} A & B \\ C & D\end{pmatrix}:\begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2\end{pmatrix}\to\begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2\end{pmatrix}$$

Suppose that$D^{-1}\in\mathcal{L}(\mathfrak{H}_2)and(A-BD^{-1}C)^{-1}\in\mathcal{L}(\mathfrak{H}_1)$Then$\emptyset^{-1}\in\mathcal{L}(\mathfrak{H}_1\oplus\mathfrak{H}_2,\mathfrak{H}_1\oplus\mathfrak{H}_2)$and

$$\phi^{-1}=\begin{pmatrix} K^{-1} & -K^{-1}BD^{-1} \\ -D^{-1}CK^{-1} & D^{-1}+D^{-1}CK^{-1}CK^{-1}BD^{-1}\end{pmatrix}$$

where $K=A-BD^{-1}C$

Applying this formula for

$$\phi=1-zU=\begin{pmatrix} 1-zS & -zG \\ -zF & 1-zT\end{pmatrix}:\begin{pmatrix} C \\ H\end{pmatrix}\to\begin{pmatrix} C \\ H\end{pmatrix}\in D$$

we get $K=1-zS-z^2G(1-zT)^{-1}F=1-z\Theta_\Delta(z)$Therefore

$$\left((1-zU)^{-1}1,1\right)=\frac{1}{1-z\Theta_\Delta(z)}, z\in D$$

Let

$$\psi(z)=\left((1+zU)(1-zU)^{-1}\vec{1},\vec{1}\right)z\Theta_\Delta$$

Clearly, the equality $F(z)=\overline{\psi(z)}$holds, which yield (17)

[191].It is well recognized now that the , theory of orthogonal Polynomials on the real plays an important role in the spectral theory of self- adjoint operators (and

close to such operators) acting on Hilbert spaces. Likewise,the theory of orthogonal polynomials on the unit circle (OPUC) appears in the same fashion in the study of unitary operators and close to such operators. Here we recall some rudiments and advances of the OPUC theory.

If μ is a nontrivial probability measure on T (that is , not supposed on a finite set), the monic orthogonal polynomials $\Phi_n(z,\mu)$ are uniquely determined by

$$\Phi_n(z)=\prod_{j=1}^{n}\left(z-z_{n,j}\right)\ \int_k \zeta^{-j}\Phi_n(\zeta)d\mu=0, \qquad j=0,1......n-1 \tag{18}$$

so on the Hilbert space $L^2(T,d\mu),(\Phi_n,\Phi_m)=0, n\neq m$ .We also consider the orthonormal polynomials ϕ_n of the form $\phi_n/\|\phi_n\|$

In case when μ is supported on a finite set, that is,

$$\mu=\sum_{k=1}^{N}\mu_k\delta(\zeta_k), \qquad \zeta_k\in T\ , \tag{19}$$

a finite number of orthogonal polynomials $\{\Phi_k\}_{k=0}^{N-1}$ can be defined in the same manner.

Clearly, (18) and the fact that the space of polynomials of degree at most n has dimension n+1 imply

$$\deg(P)=n, \qquad P\perp\zeta^j,,\ j=0,1,.......,n-1\Rightarrow \quad P=c\Phi_n \tag{20}$$

On $L^2(T,d\mu)$ the anti- unitary map $f^*(\zeta):=\zeta^n\overline{f(\zeta)}$ which depends on n) is naturally defined. The set of polynomials of degree at most n is left invariant:

$$P(z)=\sum_{j=0}^{n}P_j z^j \tag{21}$$

(20) now implies

$$\deg(P)\leq n, \quad P\perp\zeta^j, \qquad j=1,......,n \quad \Rightarrow P=c\Phi_n^* \tag{22}$$

A key feature of the unit circle is that is that the multiplication $Uf=zf$ in $L^2(T,d\mu)$ is a unitary operator, So the difference $\Phi_{n+1}(z)-z\Phi_n(z)$ is of degree n and orthogonal to z^j for $j=1.2......,n$ and by(22).

$$\Phi_{n+1}(z)=z\Phi_n(z)-\tilde{\alpha}_n(\mu)\Phi_n^*(z) \tag{23}$$

with some complex numbers $\tilde{\alpha}_n(\mu)$ called the Verblunsky coefficients [214]. (23). is known as the Szego recurrences after its first occurrence in the celebrated book of G.szego (20) at $z=0$ imply

$$\alpha_n(\mu)=\alpha_n=-\overline{\Phi_n+1(0)} \tag{24}$$

It is Known that for nontrivial measure $|\alpha_n|<1$ for all n=0,1,2,…. ,and for trivial measures(19)one has a finite set of Verblunsky coefficients $\{\alpha_n\}_{n=0}^{N-1}$ with $|\alpha_n|<1, n=0,1,.......N-2$ and $|\alpha_{N-1}|=1$. Since it arises often, define

$$p_j=\sqrt{1-|\alpha_j|^2}, 0<p_j\le 1, |\alpha_j|^2+p_j^2=1 \tag{25}$$

The inverse Szego recurrences are also of interest[214].

$$z\Phi_n(z)=p_n^{-2}\left(\Phi_{n+1}(z)+\tilde{\alpha}_n\Phi_{n+1}^*(z)\right) \tag{26}$$

Let D^∞ be set of complex sequences $\{\alpha_j\}_{j=0}^{\infty}$ with $|\alpha_j|<1$ .The map S from $\mu\to\{\alpha_j(\mu)\}_{j=0}^{\infty}$ is a well- defined map from the set P of nontrivial probability measures onT to D^∞. It was S . Verblunsky who proved that S is a bijection. As a matter of fact,S is a homeomorphism, provided P is equipped with the weak*-topology, and D^∞ with the topology of component convergence. Moreover, it follows directly from (23) that for two measures μ_1 and μ_2

$$\alpha_j(\mu_1)=\alpha_j(\mu_2) \quad j=0,1,.....n-1$$

$$\Rightarrow \Phi_j(z,\mu_1)=\Phi_j(z,\mu_2) \quad j=0,1,.....n$$

Conversely, by (26)

$$\Phi_n(z,\mu_1)=\Phi_n(z,\mu_2)\Rightarrow\alpha_j(\mu_1)=\alpha_j(\mu_2) \quad j=0,1,.....n-1$$

The orthogonal set $\{\phi_n\}_n \ge 0$ does not necessarily form a basis in $L^2(T,d\mu)$if $d\mu=dm$ is the normalized Lebesgue measure on T then $\phi_n=\zeta^n$ and $\zeta^{-1}$is orthogonal to all ϕ_n

A celebrated result of Szego- Komogorov- Krein reads that $\{\phi_n\}$is basis in $L^2(T,d\mu)$ if and only if $\log\mu' \notin L^1(T)$ where μ' is the Radon- Nikodym derivative of μ with respect to dm. In addition, the following result holds true [215].

Theorem (4.1.6)[175]: For any nontrivial probability measure μ on the unit circle, the following are equivalent.

(i) $\lim_{n\to\infty}\|\Phi_n\| = 0$

(ii) $\sum_{n=0}^{\infty}|\alpha_n|^2 = \infty$

(iii)the system $\{\phi_n\}_{n=0}^{\infty}$ is the orthonormal basic in $L^2(T,d\mu)$

Note that if $\sum_{n=0}^{\infty}|\alpha_n|^2 < \infty$ and P is the orthogonal projection in $L^2(T,d\mu)$ onto $\overline{span}\{\zeta^n, n=0,1,....\}$then(see[214].)

$$\|(1-P)\bar{\zeta}\| = \prod_{n=0}^{\infty} P_O \tag{27}$$

Let us now turn to the basic properties of zero $\{z_{n,j}\}_{j=1}^{n}$ of OPUC. It is will known[215] that $|z_{n,j}| < 1$for all n and j. Moreover, a result of Geronimus[215] reads that given a monic polynomial $P_n$of degree $n$with all its zeros inside D, there is a (nontrivial) probabiltity measure μ onT such that $P_n = \Phi_n(\mu)$.Actually, there are infinitely many such measure, all of them have the same Verblunsky coefficients up to the order $n-1$,and the same same moments up to the order n. Given a monic polynomial P_n with all its zeros inside the disk, let us call a monic polynomial Q_{n+m} an extension of $P_n$if there is a measure μ such that

$$P_n = \Phi_n(\mu), \qquad Q_{n,m} = \Phi_{n+m}(\mu)$$

To obtain all such extensions one just has to extend a sequence of Verblunsky coefficients $\alpha_n,.....\alpha_{n-1}$ which are completely determined by P_n by a sequences $\beta_0,.....\beta_{m-1}$with are bitrary $\beta_j \in D$and then apply (23).

One of the most recent advances in the study of zeros of OPUC is the theorem of Simon and Totik [215]. Which claims that given a polynomial P_n as , and an arbitrary set of point $z_1,.......z_m$ in the unit disk, not necessarily distinct, there is an extension Q_{n+m} of P_0 such that $Q_{n+m}(z_j)=0, j=1,2....,m$ counting the multiplicity. The latter as usual means that

$$z_k = z_{k+1} = ... = z_{k+p} \Rightarrow Q_{n+m}(z_k) = Q'_{n+m}(z_k) = ... = Q^{(p)}_{n+m}(z_k) = 0$$

The uniqueness of such extension is an open problem. A particular case $m=1$ appeared earlier in [178]. Now $\beta_0 = \alpha_n$ is defined uniquely from (23) by

$$0 = Q_{n+1}(z_1) = z_1 P_n(z_1) - \tilde{\alpha}_n P_n^*(z_1)$$

There is an important analytic aspect of the OPUC theory which was developed by Geronimus[195,196].

Given a probability measure μ on T. define the caratheodory function by

$$f(z) = F(z,\mu) := \int_T \frac{\zeta + z}{\zeta - z} d\mu(\zeta) = 1 + 2\sum_{n=1}^{\infty} \beta_n z^n, \beta_n = \int_T \zeta^{-n} d\mu \qquad (28)$$

the moments of $\mu.F$ is an analytic function in D which obeys $\operatorname{Re} F > 0, F(0) = 1$. The Schur function is then defined by

$$f(z) = f(z,\mu) := \frac{1}{z}\frac{F(z)-1}{F(z)+1}, f(z) = \frac{1+zf(z)}{1-zf(z)} \qquad (29)$$

so it is an analytic function in D with $\sup_D |f(z)| \le 1$ A one - to – one correspondence can be easily set up between the three classes (probability measures, Caratheodory and Schur functions). Under this correspondence μ is trivial, that is, supported on a finite set, if an only if the associate Schur function is a finite Blaschke product. Moreover, this Blaschke product has order $N-1$ for measures (19).

We proceed with the Schur algorithm. Given a Schur function $f = f_0$ Which is not a finite Blachke product, define inductively

$$f_{n+1}(z) = \frac{f_n(z) - \gamma_n}{z(1-\gamma_n f_n(z))}, \gamma_n = f_n(0) \qquad (30)$$

It is clear that sequence $\{f_n\}$ is an infinite sequence of Schur function (called the nth Schur iterates) and neither of its terms is a finite Baschke product. The numbers $\{\gamma_n\}$ are called the Schur parameters.

$$Sf = \{\gamma_0, \gamma_1, ...\}$$

In case when

$$f(z)=e^{i\gamma}\prod_{k=1}^{N}\frac{z-z_k}{1-z_k z}$$

Is a finite Blaschke product of order N, the Schur algorithm terminates al the Nth step. The sequence of Schur parameters $\{\gamma_k\}_{k=0}^{N}$ is finite, $|\gamma_k|<1$ for $k=0,1.....N-1$ and $|\gamma_N|=1$.

If a Schur function f is not a finite balaschke product, the connection between the nontangential limit values $f(\zeta)$ and its Schur parameters $\{\gamma_n\}$ is given by the formula

$$\prod_{n=0}^{\infty}\left(1-|\gamma_n|^2\right)=\exp\left\{\int_T In\left(1-|f(\zeta)|^2\right)dm\right\} \qquad (31)$$

(see[284])It follows that

$$\sum_{n=0}^{\infty}|\gamma_n|^2=\infty \Leftrightarrow In\left(1-|f(\zeta)|^2\right)\notin L^2(T)$$

In addition, if one conditions

(i) $\text{Lim sup}_{n\to\infty}|\gamma_n|=1$

(ii) $\text{Lim}_{n\to\infty}\gamma_n\gamma_{n+m}=0$ for each $m=1,2,\ldots$ but $\text{Lim sup}_{n\to\infty}|\gamma_n|>1$

is fulfilled then f is the inner function(see[202],[212]).

We will make use of the following fundamental result of Suchur [213]: the set of all Schur function f with prescribed first Schur parameters $\gamma_0,.....,\gamma_n$ Given by linner fractional transformation

$$f(z)=\frac{A(z)+zB^*(z)s(z)}{B(z)+zA^*(z)s(z)} \qquad (32)$$

Where s is an arbitrary Schur function, and A,B are polynomials of degree at most n Moreover,

$$sf=\{\gamma_0,.....\gamma_n,\gamma_0(s),\gamma_1(s),....\}$$

The pair (A,B),known as the Wall pair, is completely determined by $\{\gamma_j\}_{j=0}^{n}$ .Specifically.

$$W(z) := \begin{pmatrix} zB^*(z) & A(z) \\ zA^*(z) & B(z) \end{pmatrix} = Q_{\gamma 0}(z)Q_{\gamma 1}(z)....,Q_{\gamma n}(z)$$

where

$$Q_{\omega}(z) = \frac{1}{\sqrt{1-|\omega|^2}} \begin{pmatrix} z & \omega \\ z\tilde{\omega} & 1 \end{pmatrix} \omega \in D$$

By conputing determinants, we see that

$$B^*(z)B(z) - A^*(z)A(z) = z^n \prod_{j=0}^{n} \left(1-|\gamma_j|^2\right)^{1/2}$$

so A and B have no common zero in C/{0}.In fact they have no common zero at all since B(0)=1. It is known also that $B \neq 0$ *in* $\tilde{D}$, and both $AB^{-1}$and $A^*B^{-1}$are Schur functions.

A straightforward computation shows that Q_w (and hence W) are j- inner matrix functions:

$$W^*(z)jW(z) \geq j \quad for \ z \in D$$

$$W^*(z)jW(z) \geq j \quad for \ z \in T$$

with the signature matrix

$$j = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

For further properties of the Wall pairs see[202],[215].

A curious situation when the Schur parameters for a finite Blaschke product can be computed explicitly was found by Khrushchev[303].Let μ be a nontrivial probability measure (or measure of the form (19) with big enough N) with Verblunsky coefficients n $\{a_k\}$,and Φ_n be its nthmontic orthogonal polynomial. Consider the following Blaschke produucet of order n:

$$b_0(z) := \frac{\Phi_n(z)}{\Phi_n^*(z)} = \prod_{j=1}^{n} \frac{z - z_{n,j}}{1-\tilde{z}_{n,j}z} b_0(0) = -\tilde{\alpha}_{n-1}$$

It is a matter of a simple compution based on (56) to make sure that

$$b_1(z) := \frac{b_0(z) - b_0(0)}{z(1 - b_0(0)b_0(z))\Phi_n^*(z)} = \frac{\Phi_{n-1}(z)}{\Phi_{n-1}^*(z)}$$

Hence the Schur parameters of b_0 are of the form

$$Sb_0 = \{-\tilde{\alpha}_{n-1}, -\tilde{\alpha}_{n-2}, ...\tilde{\alpha}_0, 1\} \ . \tag{33}$$

Theorem(4.1.7)[175]: Let μ be nontrivial probability measure on Tand f its Schur with the Schur parameters $\gamma_n(f)$ then $\gamma_n(f) = \alpha_n(\mu)$. For measures (19) the latter equality holds for $n = 0, 1, ...N - 1$

It is clear now why a minus and conjugate is taken in (23)

Theorem (4.1.8)[175]: Given two sets $\alpha_0, ...\alpha_{n-1}$ and $z_1, ...z_m$ of complex numbers in D and $\gamma \in$ T there exists a finite Blaschke products b of order $n + m$ such that

(i) $Sb = \{\omega_0, \omega_{m-1}, \tilde{\alpha}_0, ..., \tilde{\alpha}_{n-1}, \gamma\}$

(ii) $b(z_j) = 0, \ j = 1, ..., m$ counting multiplicity

Proof. Denote μ $\beta_k := -\gamma\tilde{\alpha}_{n-k-1}, k = 0, 1, ..., n-1$ and construct a system monic

Orthogonal polynomials $\{\Phi_k(z, \beta)\}_{k=0}^n$ by (23). The theorem of Simon Totik claims that there is a measure μ with

$$\Phi_n(z, \mu) = \Phi_n(z, \beta), \qquad \Phi_{n+m}(zj, \mu) = 0 \quad j = 1, ..., m$$

counting the multiplicity. The first equality means that $\alpha_k(\mu) = \beta_k, k = 1, ...n-1$ Finally , put

$$b(z) := \gamma \frac{\Phi_{n+m}(z, \mu)}{\Phi_{n+m}^*(z, \mu)}$$

The result now follows from Khrushechev's formula (33).

Note that for m=1the Blaschke producet uniquely determined.

Sec(4.2) Truncated CMV Matrices

One of the most interesting developments in the OPUC theory in recent years is the discovery by Cantero, Moral, and Velázquez [188,189] of à matrix realization for the operator of multiplication by ζ on $L^2(\mathbb{T}, d\mu)$ which is a unitary matrix of finite band size (i.e., $|\langle\zeta\chi_{m.}\chi_{n.}\rangle|$ =0 if $|m\text{-}n|<k$ for some k); in this case, $k = 2$ to be compared with $k = I$ for the Jacobi matrices, which correspond to the real line case. The CMV basis (complete, orthonormal system) $\{\chi_{m.}\}$ is obtained by orthonormalizing the sequence $1, \zeta^{-1}, \zeta^{-2}, \zeta^{-2}, ...$ and the matrix, called the CMV matrix,

$$C = C(u) = ||c_{n,m}||_{m,n=0}^{\infty} ||\zeta\chi_{m.}\chi_{n}||, \quad m, n \in \mathbb{Z}_{+}$$

is five –diagonal. Remarkably, the $\chi's$ can be expressed in terms of $\emptyset's$ and $\emptyset^{*}'s$:

$$\chi_{2n}(z) - z^{-n}\emptyset_{2n}^{*}(z), \qquad \chi_{2n} + 1(z) = z^{-n}\emptyset_{2n+1}(z), \quad n \in \mathbb{Z}_{+}$$

and the matrix elements in terms of $\alpha's$ and $\rho's$:

$$C = C(\{a_n\}) = \begin{pmatrix} \overline{\alpha}_0 & \tilde{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \ldots \\ \rho_0 & -\tilde{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\ 0 & \tilde{\alpha}_2\alpha_1 & -\overline{\alpha}_2\alpha_1 & \tilde{\alpha}_3\rho_2 & \rho_3\rho_2 & \ldots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_4 & -\rho_3\alpha_2 & \ldots \\ 0 & 0 & 0 & -\tilde{\alpha}_4\rho_3 & -\tilde{\alpha}_4\alpha_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \tag{34}$$

$\alpha's$ are the Verblunsky coefficients and $\rho's$ are given in (25).

It is not hard to write down a general formula for the matrix entries C_{ij}See[200]).Let$2\epsilon_m \coloneqq 1 - (-1)^m. m \in \mathbb{Z}_{+}, and\ \epsilon_{-1} = 1, so\ \{\epsilon_m\}_{m\geq 0} = \{0,1,0,1, \ldots,\},$

$$\epsilon_m + \epsilon_{m+1} = 0, \quad \epsilon_m\epsilon_{m+1} = 0\ .\ \epsilon_m - \epsilon_{m+1} = (-1)^{m+1}.$$

Then

$$\begin{aligned} c_{mm} &= -\bar{\alpha}_m\alpha_{m-1} \\ c_m + 2.m &= \rho_m\rho_m + 1\epsilon_{m,} \\ c_{m.m+2} &= \rho_m\rho_{m+1}\epsilon_{m+1}\ , \end{aligned} \tag{35}$$

and

$$\begin{aligned} c_{m+1.m} &+ \bar{\alpha}_{m+1}\rho_m 1\epsilon_{m,} - \alpha_{m-1}\rho_m\epsilon_{m+1,} \\ c_{m.m+1} &= \bar{\alpha}_{m+1}\rho_m 1\epsilon_{m+1} - \alpha_{m-1}\rho_m\epsilon_m. \end{aligned} \tag{36}$$

It is clear (cf. [182]), that any semi-infinite CMV matrix C (34) can be written in the three-diagonal block-matrix form

$$C = \begin{pmatrix} B_0 & C_0 & 0 & 0 & 0 & . & . \\ A_0 & B_1 & C_1 & 0 & 0 & . & . \\ 0 & A_1 & B_2 & C_2 & 0 & . & . \\ . & . & . & . & . & . & . \end{pmatrix} \tag{37}$$

With

$$B_0 = (\bar{\alpha}_0), \quad C_0 = (\bar{\alpha}_1\rho_0 \quad \rho_1\rho_0), \quad A_0 = \begin{pmatrix}\rho_0\\ 0\end{pmatrix},$$

$$A_n = \begin{pmatrix}\rho_{2_n}\rho_{2_n-1} & -\rho_{2_n}\alpha_{2_n-1}\\ 0 & 0\end{pmatrix}, \quad B_n = \begin{pmatrix}-\bar{\alpha}_{2_{n-1}}\alpha_{2_n-2} & -\rho_{2_n-1}\alpha_{2_n-2}\\ \alpha_{2_n}\rho_{2_n-1} & -\bar{\alpha}_{2_{n-1}}\alpha_{2_n-1}\end{pmatrix}$$

$$C_n = \begin{pmatrix}0 & 0\\ -\bar{\alpha}_{2_{n-1}\rho_{2_n}} & \rho_{2_n+1}\rho_{2_n}\end{pmatrix}, \quad n = 1,2,\ldots \qquad (38)$$

There is a nice multiplicative structure of the *CMV* matrices. In the semi-infinite case *C* is the product of two matrices: *C* = *ℒM*, where

$$\mathcal{L} = \psi(a_0) \oplus \psi(a_2) \oplus \ldots \oplus \psi(a_{2_m}) \oplus \ldots,$$

$$M = 1_{1\times 1} \oplus \psi(a_1) \oplus \psi(a_3) \oplus \ldots \oplus \psi(a_{2_{m+1}}) \oplus \ldots, \qquad (39)$$

and$\psi(\alpha) = \begin{pmatrix}\tilde{\alpha} & \rho\\ \rho & \alpha\end{pmatrix}$ The finite *(N + 1)* x *(N +* 1) CMV matrix *C* obeys $a_0, a_1, \ldots a_{N-1} \in \mathbb{D}. |a_N| = 1$, and is also the product *C* = *ℒM*, where in this case $\psi(a_N) = (\bar{a}_N)$.

It is just natural to take the ordered set $1, \zeta^{-1}, \zeta, \zeta^{-2}, \zeta^2, \ldots$instead of $1, \zeta^{-1}, , \zeta^2, \zeta^{-2}, \ldots$

that leads to the alternate CMV basis $\{\chi_n\}$and the alternate CMV matrix

$$\tilde{C} = \|\langle\zeta\chi_m, \chi_n\rangle\| = \begin{pmatrix}\bar{\alpha}_0 & \rho_0 & 0 & 0 & 0 & \ldots\\ \tilde{\alpha}_1\rho_0 & -\tilde{\alpha}_1\alpha_0 & \tilde{\alpha}_2\rho_1 & \rho_2\rho_1 & 0 & \ldots\\ \rho_1\rho_0 & -\rho_1\alpha_0 & -\bar{\alpha}_2\alpha_1 & -\rho_2\alpha_1 & 0 & \ldots\\ 0 & 0 & \tilde{\alpha}_3\rho_2 & -\bar{\alpha}_4 & \tilde{\alpha}_4\rho_3 & \ldots\\ 0 & 0 & \rho_3\rho_2 & -\rho_3\alpha_2 & \tilde{\alpha}_4\alpha_3 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\end{pmatrix}$$

which turns out to be the transpose of *C* (see [215]). Furthermore ,*ℒ*= $\mathcal{L}^1$ and $M = M^1$imply $\tilde{C} = C^1 = M\mathcal{L}$.

An important relation between *CMV* matrices and monic orthogonal polynomials is similar to the well-known property of orthogonal polynomials on the real line

$$\phi_n(z) = \det\,(zI_n - C^{(n)})$$

holds, where $C^{(n)}$ is the principal n x n block of *C*.

One of the most important results of Cantero, Moral, and Velázquez [138] states that each unitary operator U with the simple spectrum (i.e., having a cyclic vector e_1) acting on some infinite-dimensional separable Hilbert space (respectively, finite-dimensional Hilbert space) is unitarily equivalent to a certain *CMV* matrix in $\ell^2(\mathbb{Z}_+)$(respectively, in $(\mathbb{C}^n)^n$). The corresponding a*'s* come up as the Verblunsky coefficients of the spectral measure $d\mu$ of *U* associated withℓ_1 . This is the analog of Stone's self-adjoint cyclic model Theorem. To be more precise, let us, following [216], call a cyclic unitary model a unitary operator U acting on a separable Hubert space $\mathcal{H}$ with the distinguished cyclic unit vector v_o. Two cyclic unitary models, $(\mathcal{H}, U, v_0)$and $(\bar{\mathcal{H}}, \bar{U}, \bar{v}_0)$ are called equivalent if there is a unitary operator W from $\mathcal{H}$ onto $\bar{\mathcal{H}}$ such that$Wv_0 = \bar{v}_0$and $WUW^{-1} = \bar{U}$. It is clear that $\delta_0 = (1,0,0,\dots)^t$is cyclic for any *CMV* matrix *C*.

Moreover, every class of equivalent unitary models contains exactly one CMV model (ℓ^2, C, δ_0).

Theorem(4.2.1)[175] . Let T be *a* completely nonunitaty contraction with rank one defects. Then there exists a probability measure μ on$\mathbb{T}$ such that T is unitarily equivalent to the following operator

$$\mathfrak{T}h(\xi) = P_{\mathfrak{H}}(\xi h(\xi)), \quad h \in \mathfrak{H} := L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}. \tag{41}$$

where$P_{\mathfrak{H}}$is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$ onto$\mathfrak{H}$. The Schur function associated withμ is exactly the characteristic function of T..

Proof. Include T into a prime unitary colligation

$$\Delta = \left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix} : \mathbb{C}, \mathbb{C}, \mathbb{H}\right\}$$

The characteristic function $\Theta_\Delta$agrees with the characteristic function of T^*. By Theorem(4.1.4) the vector $\vec{1} = \binom{1}{0}$is cyclic for the unitary operator $U = \begin{pmatrix} S & G \\ F & T \end{pmatrix}$.

Let $E_U(\zeta)$ be the resolution of identity for U. Define $d_\mu(\zeta) := (dE_U(\zeta)\vec{1}, \vec{1})$ and put

$$uf(\zeta) = \zeta f(\zeta)$$

the unitary multiplication operator in $L^2(\mathbb{T}, d\mu)$. By the spectral Theorem for unitaries with cyclic vectors (cf. [215]) there exists a unitary operator $W: \mathbb{C} \oplus H \to$ $L2 \mathbb{T}, d\mu$such that

$$U = W^{-1}\mathcal{U}W \text{and} = W\vec{1} = 1$$

It follows that W takes the block-operator form

$$W = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} : \begin{pmatrix} C \\ H \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathfrak{H} \end{pmatrix}$$

where $\mathfrak{H} = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}, V: H \to L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$ is a unitary operator. If $\mathfrak{T}$is given by (41),

then

$$\mathfrak{T} := P_{\mathfrak{H}}\mathcal{U} \restriction \mathfrak{H} = VTV^{-1}$$

i.e., T is unitarily equivalent to $\mathfrak{T}$. Clearly, $\mathcal{U}$ has the block form

$$\mathcal{U} = \begin{pmatrix} P_{\mathbb{C}}\mathcal{U} \restriction \mathbb{C} & P_{\mathfrak{H}}\mathcal{U} \restriction \mathfrak{H} \\ P_{\mathfrak{H}}\mathcal{U} \restriction \mathfrak{H} & \mathfrak{T} \end{pmatrix}$$

where $P_{\mathbb{C}}$ is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$ onto the subspace $\mathbb{C}$ of the constant functions in $L^2(\mathbb{T}, d\mu)$. The unitary colligation Δ t is unitarily equivalent to the unitary colligation

$$\left\{\begin{pmatrix} P_{\mathbb{C}}\mathcal{U} \restriction \mathbb{C} & P_{\mathfrak{H}}\mathcal{U} \restriction \mathfrak{H} \\ P_{\mathfrak{H}}\mathcal{U} \restriction \mathfrak{H} & \mathfrak{T} \end{pmatrix}, \mathbb{C}, \mathbb{C}, \mathfrak{H}\right\}. \tag{42}$$

Note that

$$P_{\mathbb{C}}(\mathcal{U} \restriction) = \int_{\mathbb{T}} \xi d\mu. \quad P_{\mathfrak{H}}(\mathcal{U} \restriction) = \xi - \int_{\mathbb{T}} \xi d\mu. \quad P_{\mathbb{C}}(\mathcal{U}^*1) = \bar{\xi} - \int_{\mathbb{T}} \bar{\xi} d\mu.$$

Le $F(\mathcal{Z}) = ((U + \mathcal{Z}I)(U - \mathcal{Z}I)^{-1}\vec{1}, \vec{1})$ .Then

$$F(\mathcal{Z}) = (\mathcal{U} + zI)^{-1}1,1) = \int_{\mathbb{T}} \frac{\xi + \mathcal{Z}}{\xi - \mathcal{Z}} . d\mu\,(\xi)$$

i.e., F is the Caratherodory function associated with μ. From Theorem (4.1.7) we conclude

$$\overline{\Theta_\Delta(\bar{\mathcal{Z}})} = \frac{1}{\mathcal{Z}} \frac{F(\mathcal{Z}) - 1}{\mathcal{Z}F(\mathcal{Z}) + 1}$$

and so by (38) $\overline{\Theta_\Delta(\bar{\mathcal{Z}})}$ agrees with the Schur function associated with μ.

Let $\{\emptyset_n\}$ be the system of monic polynomials orthogonal with respect to μ, and let $\{\alpha_n\}$ be the corresponding Verblunsky coefficients. By Geronimus' theorem $\{\alpha_n\}$ are the Schur parameters of *f*. Let $\mathfrak{H}^{(c)}$ be the controllable subspace of the unitary colligation (42). From (13) it follows that.

$$(\mathfrak{H})^{(c)})^{-1} L^2(\mathbb{T}, d\mu) \ominus \overline{span}\{\xi^n, n = 0, \dots\}\mathfrak{H}$$

If μ is a nontrivial measure, then in view of (27) we obtain

$$\left\| P_{(\mathfrak{H})^{(c)})} \bar{\xi} \right\| = \coprod_{n=0}^{\infty} (1 - |a_n|^2)^{1/2}$$

The latter is equivalent to

$$\left\| P_{(\mathfrak{H})^{(c)\perp})} P_{\mathbb{C}}(\mathcal{U}^*1) \right\| = \coprod_{n=0}^{\infty} (1 - |a_n|^2)^{1/2}$$

Hence, from (12) and (8) we have the equivalence

$$\overline{span}\{\mathfrak{T}^n \mathfrak{D}_{\mathfrak{T}^*}, n = 0,1, \dots\} = \mathfrak{H} \Leftrightarrow \sum_{n=0}^{\infty} |a_n|^2 = \infty \tag{43}$$

Remark(4.2.2)[175]. By the construction of Theorem (4.1.5) the Schur function f associated with μ is exactly $\overline{\Theta_\Delta(\bar{\mathcal{Z}})}$ .Another (unitary equivalent) models of T are connected with the operators $U_\lambda = \begin{pmatrix} \bar{\lambda} S & G \\ \bar{\lambda} F & T \end{pmatrix}$,where $|\lambda| = 1$.The characteristic function of the unitary colligation

$$\Delta_\lambda = \left\{ \begin{pmatrix} \bar{\lambda} S & G \\ \bar{\lambda} F & T \end{pmatrix} . \mathbb{C}, \mathbb{C}, H \right\}$$

is $\bar{\lambda}\Theta_\Delta$.The model operator$\mathfrak{T}_\lambda$ takes the form

$$\mathfrak{H}_\lambda = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}, \quad \mathfrak{T}_\lambda h(\xi) = P_{\mathfrak{H}'}(\xi h(\xi)), \quad h(\xi) \in \mathfrak{H}_\lambda$$

The Schur function f_λ associated with μ_λ is $f_\lambda = \lambda f$ .The connection between the Caratheodory functions $F_\lambda(z) = ((U + z1)(U - z1)^{-1}\vec{1}, \vec{1})$ and F given by

$$F_\lambda(z) \frac{(1-\lambda) + (1+\lambda)F(z)}{(1+\lambda) + (1-\lambda)F9z)}$$

The measures $\mu_\lambda$are known as the *Aleksandrov* measures associated with μ [215].

Let $C = C(\{\alpha_n\})$be the *CMV* matrix given by (34). Recall that $C(\{\alpha_n\})$is the matrix representation of the unitary operator *u* of multiplication by ζin $L^2(\mathbb{T}, d\mu)$, where μ is the probabilitymeasure with Verblunsky coefficients $\{\alpha_n\}$. By the Geronimus Theorem the Schur parameters of the Schur function (29) associated with μ are $\{\alpha_n\}$.

The matrix *C* determines the unitary operator in the space $\ell^2(\mathbb{Z}_+)$ are (respectively $\mathbb{C}^{N+1}$in inthe case of *(N + 1) x (N + 1)* matrix). The vector S_0 = (1,0,0, ...)^{1}‘ is cyclic for *C*. Consider the matrix

$$\mathcal{T} = \mathcal{T}(\{\alpha_n\}) = \begin{pmatrix} -\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & . & . & . \\ \alpha_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_1\rho_2 & . & . & . \\ \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & . & . & . \\ 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & . & . & . \end{pmatrix} \qquad (44)$$

obtained from *C* by deleting the first row and the first column. It is clear from (37) that a semi-infinite $\mathcal{T}$takes on the three-diagonal *2 x 2* block-matrix form

$$\mathcal{T} = \begin{pmatrix} B_1 & C_1 & 0 & 0 & 0 & . & . \\ A_1 & B_2 & C_2 & 0 & 0 & . & . \\ 0 & A_2 & B_3 & C_3 & 0 & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

WhereA_n , B_n and C_n are defined in (38). Henceforth $\mathcal{T}$ is called a truncatedCMV matrix $\mathcal{T}$ is the matrix of the operator $\mathfrak{T} = P_{\mathfrak{H}}\mathcal{U} \restriction \mathfrak{H}$,where $P_{\mathfrak{H}}$is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$onto the subspace $=\mathfrak{H} L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$.

It is easy to see that given $\mathcal{T}(44)$, the values α_n are uniquely determined. Indeed, from (4) and (14) entries we have by (25)$|\alpha_1|^2 = |\bar{\alpha}_2\alpha_1|^2 + \rho_2^2|\alpha_1|^2$, so $|\alpha_1|$and $\rho_1 > 0$ are known,and we find $\alpha_0, \alpha_2$from (2) and (3) entries of (44). From (3) and (4) entries we get$\rho_2 > 0$, then, α_1 , $\alpha_3$etc. We call $\alpha_n = \alpha_n(\mathcal{T})$the parameters of$\mathcal{T}$ (44).

As it was mentioned inthis Section , $\mathcal{L}M$, $\mathcal{L}$ and M are defined in (39). Given a matrix A, we denote by $Ar(Ac)$ the matrix obtained from A by deleting the first row (column).

Clearly, $A_{rc}= (A_r)_c$. So we have$\mathcal{T} = C_n = \mathcal{L}_r\mathcal{M}_c, \mathcal{M}$. M is isometric with dim ran $(1 - \mathcal{M}_c\mathcal{M}_c^*) = 1$, whereas$\mathcal{L}_r$ is coisometric with dim ran. $(1 - \mathcal{L}_r^*\mathcal{L}_r) = 1$.

Let $P_{\delta_0}\restriction$be the orthogonal projection in$\ell^2(\mathbb{Z}_+)(\mathbb{C}^{N+1})$ onto the subspace $\delta_0 \perp\cong \ell^2(\mathbb{N}(\mathbb{C}^N)$. Then the matrix $\mathcal{T}$ determines on the Hilbert space $\delta_0^\perp$ the operator $\mathcal{T} = P_{\delta_0^\perp}C \restriction \delta_0 \perp$ Let the operators (matrices) $S: \mathbb{C} \to C, \mathcal{F}: \mathbb{C} \to \delta_0^\perp \to \mathbb{C}$ be given by

$$S1 = \bar{\alpha}, \mathcal{F}1 = \begin{pmatrix} \rho_0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}, \quad \mathcal{G}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} = \bar{\alpha}_1\rho_0 h_1 + \rho_1 h_2$$

Hence, the matrix C takes the block form

$$C = \begin{pmatrix} S & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix}$$

From (12) it follows that

$$\left\| \mathcal{G}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\|^2 = \left\| D_{\mathcal{T}}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\| = \rho_0^2|\bar{\alpha}_1 h_1 + \rho_1 h_2|^2,$$

$$\mathfrak{D}_{\mathcal{T}} = \{\lambda(\alpha_1\delta_1 + \rho_1\delta_1), \lambda \in \mathbb{C}\}$$

$$\left\| \mathcal{F}^*\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\|^2 = \left\| D_{\mathcal{T}^*}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\| = \rho_0^2|\bar{\alpha}_1 h_1|^2, \qquad \mathfrak{D}_{\mathcal{T}^*} = \{\lambda\delta_1, \lambda \in \mathbb{C}\}$$

$D_T h = \rho_0(h, \alpha_1\delta_1 + \rho_1\delta_2)(\alpha_1\delta_1 + \rho_1\delta_2)$. $\quad D_{T^*} h = \rho_0(h, \delta_1)\delta_1, \quad h \in$ $\ell 2($ NNCN. $\quad T\alpha 1\delta 1+\rho 1\delta 2=-\alpha 1\delta 1.$ 45)

Since δ_0 is the cyclic vector for C, then by Theorem (4.1.5) the unitary colligation

$$\Delta_C = \left\{\begin{pmatrix} S & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix} : C, C, \delta_0\right\} \tag{46}$$

is prime, and $\mathcal{T}$ is a completely nonunitary operator with rank one defects on the Hilbert spaces $\ell_2(N) or C^N$

Let

$$F(z) = ((C + zI)(C - zI)^{-1}\delta_0, \delta_0), \; f(z) = \frac{1}{z}\frac{F(z)-1}{F(z)+1} \tag{47}$$

Proposition(4.2.3)[175].

(i)For a semi-infinite truncated CMV matrix $\mathcal{T} = \mathcal{T}(\{\alpha_n\})$ the following statements are equivalent,

(a) the matrix $\mathcal{T}$ does not contain a unilateral shift ;

(b) the matrix $\mathcal{T}^*$ does not contain a unilateral shift ;

(c) $\overline{span}[\mathcal{T}^n\delta_1, n = 0,1, \dots] = \ell_2(N)$;

(d) $\overline{span}[\mathcal{T}^{*n}(\alpha_1\delta_1 + \beta_1\delta_2), n = 0,1, \dots] = \ell_2(N)$;

(e) $\sum_{n=0}^{\infty}|\alpha_n|^2 = \infty$;

(f) ln(1-$|f(e^{It})|^2 \notin L^1|-\pi, \pi|$.

(ii) If $\mathcal{T}$ is a semi-infinite truncated CMV matrix

(a) $\lim sup_{n\to\infty}\{\alpha_n\} = 1$.

(b) $lim_{n\to\infty}\alpha_n\alpha_{n+m} = 0 \; for \; m = 1.2, \dots \; but$

$$\lim sup_{n\to\infty}|\alpha_n| > 0$$

is fulfilled, then

$$\text{s-}lim_{n\to\infty}\mathcal{T}^n = \text{s-}lim_{n\to\infty}\mathcal{T}^{*n}$$

(iii) If T is a finite truncated CMV matrix, then $lim_{n\to\infty}\|\mathcal{T}^n\| = 0$

Proof.

(i) Since $\{\alpha_n\}$ are the Schur parameters of the Schur function f associated with the full CMV matrix C $(\{\alpha_n\})$, and f agrees with the characteristic function of $\mathcal{T}(\{\alpha_n\})$, the equivalence of the statements (a)–(f) follows from (5), (6), (9), (11), (31), (45),(43), and Theorems (4.1.4) and (4.1.8)

(ii) Each condition (a) or (b) implies f is inner .Hence $\mathcal{T}$ belongs to the class C_{00}, i.e., s-$lim_{n\to\infty}\mathcal{T}^n$ = s-$lim_{n\to\infty}\mathcal{T}^{*n}$ = 0

(iii) The function f is a finite Blaschke product and so inner. Since $\mathcal{T}$ is finite-dimensional,we get $lim_{n\to\infty}\|\mathcal{T}^n\| = 0$.

Proposition(4.2.4)[275]

.Let $\mathcal{T}(\{\alpha_n\})$, and $\mathcal{T}(\{\beta_n\})$ be truncated CMV matrices. Then $\mathcal{T}(\{\alpha_n\})$and $\mathcal{T}(\{\beta_n\})$are unitarily equivalent if and only if$\beta_n = e^{It}\alpha_n$for all n and t∈[0,2π).Moreover, if V is the diagonal unitary matrix of the form

$$\mathcal{V}= \text{diag}(e^{it},1, e^{it},1,\ldots) \tag{48}$$

then

$$\mathcal{V}\mathcal{T}(\{\alpha_n\})\mathcal{V}^{-1} = \mathcal{T}\left(\{e^{it}\alpha_n\}\right). \tag{49}$$

Proof.

Consider two CMV matricesC$(\{\alpha_n\}) and C(\{\beta_n\})$ and associated with them Schur functions$f_0$andf_β. Since these functions agree with the characteristic functions of $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{\beta_n\})$, respectively, the operators $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{\beta_n\})$ are unitarily equivalent if and only if$f_0$and$f_\beta$differ by a scalar unimodular factor, which in turn yields $\beta_n = e^{it}\alpha_n$for all n and t∈[0,2π).

Equality (49)wish$\mathcal{V}$ (48) can be verified by the direct calculation based on (35), (36).So $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}\left(\{e^{it}\alpha_n\}\right)$. are unitarily equivalent.

From (49) it follows that

$$\mathcal{T}\left(\{e^{it}\alpha_n\}\right) = e^{itA}\mathcal{T}(\{\alpha_n\})e^{-itA}.$$

where A is a self-adjoint diagonal matrixA= diag(1,0,1,0. . .).Hence the matrix $\mathcal{T}\left(\{e^{it}\alpha_n\}\right)$satisfies the differential equation

$$\frac{d\mathcal{T}(t)}{d\mathcal{T}} = i(A\mathcal{T}(t) - \mathcal{T}(t)A), \quad t \in R$$

and $\mathcal{T}(0) = \mathcal{T}(|\alpha_n|)$.

The next Theorem states that truncated CMV matrices are mode ls of completely nonunitary contractions with rank one defects.

Theorem (4.2.5)[175]: Let T be a completely nonunitary contraction with rank one defects acting on infinite-dimensional separable Hilbert space H (respectively,finite-dimensional Hilbert space).Then $\mathcal{T}$is unitarily equivalent to the operator acting on $\ell_2(N)$(respectively, on C^Nin the case dim H=N) determined by the truncated CMV matrix $\mathcal{T}=\mathcal{T}(\{\alpha_n\})$, where$\{\alpha_n\}$are the Schur parameters of the characteristic function of $\mathcal{T}$. In particular, every completely nonunitary contraction with rank one defects is a product of co-isometric and isometric operators with rank one defects.

Proof.Include $\mathcal{T}$into a prime unitary colligation

$$\Delta=\left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}: C, C, H\right\}.$$

. By Theorem(4.1.4) the vector$\bar{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$is a cyclic for the unitaryoperato $U = \begin{pmatrix} S & G \\ F & T \end{pmatrix}$. From the results of [188, 187] (see also [213, 214]) there exists a unique CMV matrixC such that

$$U = W^{-1}\text{CW}, \quad \delta_0 = W\bar{1},$$

where W is a unitary operator fromC⊕H onto $\ell^2(Z_+)(C^{N+1})$ and $\delta_0 = (1,0,0,\ldots)^t$. It follows that the operatorW takes the block-operator form

$$\text{W}=\begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix}: \begin{pmatrix} C \\ H \end{pmatrix} \to \begin{pmatrix} C \\ \delta_0^{\perp} \end{pmatrix}.$$

where $\chi: \text{H} \to \delta_0^{\perp}$ is a unitary operator. Hence $\mathcal{T} = \chi T \chi^{-1}$, i.e., the operator T is unitarily equivalent to the operator in$l_2(N)(C^N)$ given by the truncated CMV matrix $\mathcal{T} = \mathcal{T}(\{\alpha_n\})$ .From representation (28) of $F(z) = ((U + zI)(U - zI-11,1$.and Theorem (4.1.5) it follows that$\alpha n$are the Schur parameters of the function $\overline{\Theta_\Delta(\bar{z})}$ that agrees with the characteristic function of T.

Let Q be an arbitrary unitary operator in $\delta_0^{\perp}$. SinceT=$\mathcal{L}_r M_c$, we get

$$\mathcal{T} = \chi^{-1}T\chi = \chi^{-1}\mathcal{L}_r M_c \chi = \chi^{-1}\mathcal{L}_r Q Q^{-1}$$

Where M=$Q^{-1}M_c\chi$ is an isometric operator with rank one defect, and. L=$\chi^{-1}\mathcal{L}_r Q$ is a co-isometric operator with rank one defect.

Note that the unitary colligation (46) is unitary equivalent to the unitary colligation (42).

Let V be an isometric operator acting on some Hilbert space H with the domain dom V and the range ranV. The numbers dim(H⊖domV) and dim(H⊖ ran V) are called the defect indices of V. The isometric operator V is called prime if there is no nontrivial subspace on which V is unitary. In [203, 204] M. Livˇsic developed the spectral theory of isometric operators with equal defect indices, and their quasi-unitary extensions. A nonunitary operator S on H is called a quasi-unitary extension of the isometric operator V with the defect indices (n, n),if S agrees with V on dom V and maps H ⊖dom V into H⊖ranV.

Let $\vec{U}$ be the bilateral shift in $\ell_2(Z)$, i.e., $\vec{U}_{\delta_k}$=$\delta_{k-1}, k \in Z$, where$\{\delta_k = k \in Z\}$is the canonical orthonormal basis in$\ell_2(Z). Define,$$\vec{V}_0$by

$$dom\vec{V}_0 = \delta_0^{\perp}, \ \vec{V}_0 \restriction dom\vec{V}_0$$

Then ran $\vec{V}_0 = \delta_{-1}^{\perp}$. Let the quasi-unitary extension $\vec{S}_0$ of $\overrightarrow{V_0}$ be given $\vec{S}_0\delta_0 = 0, \vec{S}_0 \restriction dom\, \vec{V}_0 = \vec{V}_0$. Then each point of D is the eigenvalue of $\vec{S}_0$. So the spectrum of $\overrightarrow{S}_0$ agrees with D. The following result is essentially due to M. Livˇsic [203].

Theorem (4.2.6)[175]. Let S be a quasi-unitary contractive extension of a prime isometric operator V with the defect indices(1). If the whole open disk D consists of the point spectrum of S, then V and S are unitarily equivalent to $\vec{V}_0$ and $\vec{S}_0$, respectively.

Clearly, the rank of the defect operators $(I - \vec{S}_0^*\vec{S}_0)^{1/2}$and $(I - \vec{S}_0\vec{S}_0^*)^{1/2}$ is equal to one.Since the point spectrum of $\vec{S}_0$ $is\ D$,the Sz.-Nagy–Foias characteristic function Θ of$\vec{S}_0$ is identically equal to zero. On the other hand, one can easily show (and it is well known) thata completely nonunitary contraction with rank one defects and zero characteristic function is unitarily equivalent to the operator S⊕S^*, where S is the unilateral shift in$\ell_2(N)$. So the operators$\overrightarrow{S}_0$ and S $\oplus$ S^* are unitarily equivalent. Since all Schur parameters of the function Θ = 0 are zeros, the corresponding truncated CMV matrix $\mathcal{T}_0$=$\|t_0(i,j)\|$ takes the form

$$\mathcal{T}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

i.e.,$t_0(2k, 2k+2) = t_0(2k+1, 2k-2) = k \geq 1$, and the rest t_0(i,j) = 0. The matrix $\mathcal{T}_0$ Is a submatrix of the free CMV matrix C_0corresponding to zero Schur parameters. Each point z of D is the eigenvalue of $\mathcal{T}_0$. The corresponding eigensubspace is

$$\mathfrak{N}_z = \{\lambda(0,1,0,z,0,z^2,0,z^3,\dots), \lambda \in C\}$$

Hence, the spectrum of $\mathcal{T}_0$ is the closed unit disk $\overline{D}$.

Let $\mathcal{V}_0$ be the operator in $\ell_2(N)$.

$$\mathrm{dom}\mathcal{V}_0 = \ell_2(N) \ominus \{c\delta_1\} = kerD_{T_{0,}} \mathcal{V}_0 = \mathcal{T}_0 \upharpoonright \mathrm{dom}\mathcal{V}_0. \tag{50}$$

Then ran $\mathcal{V}_0 = \ell_2(N) \ominus \{c\delta_1\} = kerD_{T_0^*}$, and $\mathcal{V}_0$ is isometric with the defect indices (1).The contraction $\mathcal{T}_0$ is the quasi-unitary extension of $\mathcal{V}_0$ with the zero characteristic function.Therefore, the truncated CMV matrix $\mathcal{T}_0$ is unitarily equivalent to the operator $\vec{S}_0$, and by

Livšic Theorem [204] the isometric operator $\mathcal{V}_0$ is unitarily equivalent to $\vec{V}_0$.

All other quasi-unitary contractive extensions of V_0 are given by the truncated CMVmatrices

$\mathcal{T} = \|t(i,j)\|$

$$\mathcal{T} = \begin{pmatrix} 0 & -re^{i\varphi} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{51}$$

i.e., $t(2k, 2k+2) = t(2k+1, 2k-2) = k \geq 1, t(1,2) = -re^{i\varphi}, r \in (0,1), \varphi$ r is an arbitrary number from the interval [0,2π), and the restt(i, j) = 0. The characteristic function of $\mathcal{T}$ is the constant function $\Theta = re^{i\varphi}$. The spectrum of each such matrix is the unit circle $\mathcal{T}$. Because $|\Theta^{-1}| = r^{-1}$, each of such matrix is similar to unitary matrix [216].

The matrices $\mathcal{T}_0$ *and* $\mathcal{T}$ contain the shift

$$\mathrm{dom}\mathcal{W} = \overline{span}\{\delta_1, \delta_3, \dots, \delta_{2n-1}, \dots\}, \mathcal{W}\left(\sum_{n=1}^{\infty} h_n\delta_{2n-1}\right) = \sum_{n=1}^{\infty} h_n\delta_{2n+1}$$

The matrices T_0^* and T contain the shift

$$\mathrm{dom}\mathcal{W}_* = \overline{span}\{\delta_2, \delta_4, \dots, \delta_{2n-1}, \dots\}, \mathcal{W}_*\left(\sum_{n=1}^{\infty} h_n\delta_{2n+1}\right) = \sum_{n=1}^{\infty} h_n\delta_{2n+1}$$

Let T be a completely nonunitary contraction with rank one defects and the constant characteristic function Θ, 0<|Θ(z)|=r <1. Then by Theorem (4.2.5) T is unitarily equivalent to the truncated CMV matrices (51).

Along with truncated CMV matrices $\mathcal{T}(\{\alpha_n\})$ (44), we consider here truncated CMV matrices $\widetilde{\mathcal{T}}(\{\alpha_n\})$obtained from the alternate CMV matrix $\widetilde{C}(\{\alpha_n\})$ (40) by the same procedure. The matrix $\widetilde{\mathcal{T}}(\{\alpha_n\})$is the transpose of $\mathcal{T}(\{\alpha_n\})$

$$\widetilde{\mathcal{T}} = \begin{pmatrix} -\overline{\alpha}_1\alpha_0 & \overline{\alpha}_2\rho_1 & \rho_2\alpha_0 & 0 & \dots \\ -\rho_1\alpha_0 & -\overline{\alpha}_2\alpha_1 & -\rho_2\alpha_1 & 0 & \dots \\ 0 & \overline{\alpha}_3\rho_2 & -\overline{\alpha}_3\alpha_2 & \overline{\alpha}_4\rho_3 & \dots \\ 0 & \rho_3\rho_2 & -\rho_3\alpha_2 & -\overline{\alpha}_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{52}$$

and

$$\widetilde{\mathcal{T}}(\{\alpha_n\}) = \acute{\mathcal{T}}(\{\alpha_n\}) = (\acute{M}_c)(\acute{\mathcal{L}}_r)M_c$$

It is not hard to show that $\widetilde{\mathcal{T}}(\{\alpha_n\})$ is a completely nonunitary contraction with rank one defects, and its characteristic function $\widetilde{f}$ agrees with the Schur functionassociated with Verblunsky coefficients (Schur parameters) $\{\alpha_n\}$. Indeed (cf. 47))

$$(\tilde{C} + zI)(\tilde{C} - zI)^{-1} = (C^1 + zI)(C^1 - zI)^{-1} = ((C + zI)(C + zI)^{-1})^1$$

and so $\tilde{F}(z) := \left((\tilde{C} + zI)(\tilde{C} - zI)^{-1}\delta_0, \delta_0\right) = F(z), \widetilde{f} = f.$as claimed.So.the matrices $\mathcal{T}(\{\alpha_n\})$ *and* $\widetilde{\mathcal{T}}(\{\alpha_n\})$ are unitarily equivalent.

Denote by $\mathcal{T}^{(k)}(\tilde{\mathcal{T}}^{(\tilde{k})})$ the matrix obtained from $\mathcal{T}(\tilde{\mathcal{T}})$ by deleting the first k rows and columns. The following result provides the characteristic function of $\mathcal{T}^{(k)}$.

Theorem (4.2.7)[175]..Let μbe a probability measure on$\mathbb{T}$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^{N}, N \le \infty$, and let$f, C\{\alpha_n\}, \widetilde{C}\{\alpha_n\}, \mathcal{T}(\{\alpha_n\}), \widetilde{\mathcal{T}}(\{\alpha_n\})$ be the corresponding Schur function, CMV and truncated CMV matrices, respectively. Then $\mathcal{T}^{(k)}, \tilde{\mathcal{T}}^{(k)}$ are completely nonunitary contractions with rank one defects, and the following relations hold:

$$\mathcal{T}^{(2m-1)}\{\alpha_n\}_{n=0}^{N} = \widetilde{\mathcal{T}}(\{\alpha_n\}_{n=2m-1}^{N}),$$
$$\mathcal{T}^{(2m)}\{\alpha_n\}_{n=0}^{N} = \mathcal{T}\{\alpha_n\}_{n=2m}^{N}, \qquad m = 1,2, \dots.$$

So, the characteristic function of $\mathcal{T}^{(k)}$agrees with the kth Schur iterate of f.

Proof. The relations

$$\mathcal{T}^{(1)}\{\alpha_n\}_{n=0}^{N} = \tilde{\mathcal{T}}\{\alpha_n\}_{n=1}^{N}. \quad \tilde{\mathcal{T}}^{(1)}\{\alpha_n\}_{n=1}^{N} = \mathcal{T}\{\alpha_n\}_{n=2}^{N}$$

follows directly from (44) and (52). The rest is a matter of simple induction and the definition of the kth Schur iterates.

The relation between characteristic functions of the sub-matrices $\mathcal{T}^{(k)}(\{\alpha_n\}_{n=0}^{N})$ and the kth Schur iterates established in the above Theorem is a complete analog of the result concerning the connections between m-functions of a Jacobimatrix and its sub-matrice [127] .

Theorem (4.2.8)[175]..Let μ be a probability measure on T with Verblunsky coefficients $\{\alpha_n\}_{n=0}^{N}, N \leq \infty$.

Consider three subspaces in $L^2(T,\mu)$:

$$\mathcal{H}_{2m} = span\{1,\zeta,\bar{\zeta},\zeta^2,\bar{\zeta}^2,\dots,\zeta^{\mathrm{m}},\bar{\zeta}^{\mathrm{m}}\},$$
$$\mathcal{H}_{2m-1} = span\{1,\zeta,\bar{\zeta},\zeta^2,\bar{\zeta}^2,\dots,\bar{\zeta}^{\mathrm{m}-1},\bar{\zeta}^{\mathrm{m}}\},$$
$$\widetilde{\mathcal{H}}_{2m-1} = span\{1,\bar{\zeta},\zeta,\bar{\zeta}^2,\zeta^2,,\dots,\zeta^{\mathrm{m}-1},\bar{\zeta}^{\mathrm{m}}\}.$$

Denote by $\mathfrak{H}_{2m}(\mathfrak{H}_{2m-1},\widetilde{\mathfrak{H}}_{2m-1})$ their orthogonal complements in $L^2(T,\mu)$, and by $P_{2m}(P_{2m-1},\tilde{P}_{2m-1})$ the orthogonal projections onto $\mathfrak{H}_{2m}(\mathfrak{H}_{2m-1},\widetilde{\mathfrak{H}}_{2m-1})$, respectively. Then the operators

$$\mathfrak{T}_k h(\zeta) = P_K(\zeta \mathrm{h}(\zeta)). \quad h(\zeta) \in \mathfrak{H}_k. \tag{53}$$

$$\widetilde{\mathfrak{T}}_{2m-1} h(\zeta) = \tilde{P}_K(\zeta \mathrm{h}(\zeta)). \quad h(\zeta) \in \widetilde{\mathfrak{H}}_{2m-1}.$$

are completely nonunitary contractions with rank one defects. The characteristic function of $\mathfrak{T}_k$ agrees with the kth Schur iterate of the Schur function $f(\mu)$, the characteristic function $\widetilde{\mathfrak{T}}_{2m-1}$ agrees with (2m-1)th Schur iterate of $f(\mu)$. So, the operator $\mathfrak{T}_k$ is unitarily equivalent to the operator

$$h(\zeta) = P_0^{(k)}(\zeta \mathrm{h}(\zeta)). \quad \mathrm{h}(\zeta) \in L^2(T,d\mu(\{\alpha_n\}_{n=k}^{N})) \ominus C. \tag{54}$$

where $P_0^{(k)} is$ the orthogonal projection onto $L^2(T,d\mu(\{\alpha_n\}_{n=k}^{N})) \ominus C$. In addition $\mathfrak{T}_{2m-1}$ is unitarily equivalent to $\widetilde{\mathfrak{T}}_{2m-1}$

Proof. Recall that CMV matrices $C(\{\alpha_n\},\tilde{C}(\{\alpha_n\})$ represent the unitary operator $\mathrm{U}h(\zeta) = \zeta \mathrm{h}(\zeta) \mathrm{in} L^2(T,d\mu\{\alpha_n\}))$ with respent to the complete orthonormal systems $\{\chi_n\}$ and $\{x_n\}$, reprectively.Moreove

$$\mathcal{H}_{2m} = span\{\chi_0,\chi_1,\dots,\chi_{2m}\} = span\{x_0,x_1,\dots,x_{2m}\}$$
$$\mathcal{H}_{2m-1} = span\{\chi_0,\chi_1,\dots,\chi_{2m-1}\}$$

$$\widetilde{\mathcal{H}}_{2m-1} = span\{x_0, x_1, \dots, x_{2m-1}\}$$

Since $\mathcal{T}(\{\alpha_n\}_{n=0}^{N})(\tilde{\mathcal{T}}(\{\alpha_n\}_{n=0}^{N})$ is the matrix of $\mathfrak{T}$ (41) with respect to the basis $\{\chi_n\}_{n=0}^{N}$, the operators $\mathfrak{T}_{2m}$, $\mathfrak{T}_{2m-1} and\ \widetilde{\mathfrak{T}}_{2m-1}$ have the matrices $\mathcal{T}^{(2m)}$, $\mathcal{T}^{(2m-1)} and\ \tilde{\mathcal{T}}^{(2m-1)}$, respectively. From Theorem (4.2.8)it follows that $\mathfrak{T}_k$ are completely nonunitary contractions with rank one defects for all k, and their characteristic functions agree with the kth Schur iterates of f. By Theorems (4.2.8)and (4.2.1) the operator $\mathfrak{T}_k$ is unitarily equivalent tothe operator given by (54). We also have

$$\tilde{\mathcal{T}}^{(2m-1)}(\{\alpha_n\}_{n=0}^{N}) = \mathcal{T}(\{\alpha_n\}_{n=2m-1}^{N})$$

Therefore, the characteristic function of $\widetilde{\mathfrak{T}}^{2m-1}(\{\alpha_n\}_{n=0}^{N})$ agrees with (2m-1)th iterate $f_{2m-1} 0f\ f$, and hence the operators $\widetilde{\mathfrak{T}}^{2m-1}(\{\alpha_n\}_{n=0}^{N}) and \mathfrak{T}^{2m-1}(\{\alpha_n\}_{n=0}^{N})$ are unitarily equivalent.

We complete the section with the general result from the contractions theory which is proved with the help of the truncated CMV model.

Theorem (4.2.9)[175]. Let T be a completely nonunitary contraction with rank one defects in a separable Hilbert space H, dimH ≥2, and let $P\ kerD_{T^*}$, $P\ kerD_T$ be the orthogonal projections onto $kerD_{T^*}$ and $kerD_T$ in H, respectively. Then the operators

$$T_1 = P_{kerD_{T^*}} T \upharpoonright kerD_{T^*}, \quad \widetilde{T}_1 = P_{kerD_T} T \upharpoonright kerD_T$$

are unitarily equivalent completely nonunitary contractions with rank one defects, and their characteristic functions agree with the function

$$h_1(z) := \frac{1}{z} \frac{h(z) - h(0)}{1 - \overline{h(0)} h(z)}$$

where h is the characteristic function of T.

Proof. By Theorem (4.2.5) the operator T is unitarily equivalent to the truncated CMV matrices $\mathcal{T} = \mathcal{T}(\{\alpha_n\}_{n=0}^{N})$ and $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}(\{\alpha_n\}_{n=0}^{N})$, where $\{\alpha_n\}_{n=0}^{N}$ are the Schur parameters of h, N ≤ ∞. So, there exists a unitary operators $V, \widetilde{V}$: $\delta_0^{\perp} \to H$ such that

$$VTV^{-1} = \tilde{V}\tilde{T}\tilde{V}^{-1} = T$$

It follows that

$$VD_{T^*}V^{-1} = D_{T^*}, \quad \tilde{V}D_{\tilde{T}}\tilde{V}^{-1} = D_T$$

and hence $V_{kerD_{T^*}} = kerD_{T^*}, \tilde{V}_{kerD_{\tilde{T}}} = kerD_T$. Due to (45) we have

$$\mathfrak{D}_{T^*} = \mathfrak{D}_{\tilde{T}} = span\{\delta_1\}$$

and

$$\mathcal{T}^{(1)} = P_{kerD_{\mathcal{T}^*}}\mathcal{T} \upharpoonright kerD_{\mathcal{T}^*},\ \tilde{\mathcal{T}}^{(1)} = P_{kerD_{\mathcal{T}}}\mathcal{T} \upharpoonright kerD_{\tilde{\mathcal{T}}}$$

Hence

$$V\mathcal{T}^{(1)}V^{-1} = T_1, \qquad \tilde{V}\tilde{\mathcal{T}}^{(1)}\tilde{V}^{-1} = \widetilde{T_1}$$

Now from Theorem (4.2.8) it follows that $T_1 and \widetilde{T_1}$ are completely nonunitary contractions with rank one defects, and their characteristic functions agree with the first Schur iterate $h_1 of\ h$. Hence $T_1\ and\ \widetilde{T_1}$ are unitarily equivalent.

Consider a N×N truncated CMV matrix

$$\mathcal{T} = \mathcal{T}(\{\alpha_n\}) = \begin{pmatrix} -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & \dots & 0 \\ \bar{\alpha}_2\rho_3 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \dots & 0 \\ \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \bar{\alpha}_N\rho_{N-1} \\ \dots & \dots & \dots & -\rho_{N-1}\alpha_{N-2} & -\bar{\alpha}_N\alpha_{N-1} \end{pmatrix} \qquad (55)$$

(for even N it looks a bit different). The problem under investigation in the present section in the reconstruction of the matrix $\mathcal{T}$ (55) from either the complete set of its eigenvalues or from the mixed spectral data: the part of the spectrum and the part of the parameters $\alpha_n(\mathcal{T})$

Theorem(4.2.10)[175]. Let $z_1, z_{1,}, \dots, z_N$ be not necessarily distinct numbers from the open unit disk. Then there exists a truncatedN×NCMV matrix $\mathcal{T}$(55)which has eigenvalues $z_1, z_{1,}, \dots, z_N$, counting their algebraic multiplicities. Such matrix is determined uniquely up to multiplication of its parameters $\alpha_n(\mathcal{T})$ by the same unimodular factor.

Proof. Let

$$b(z) = e^{i\psi}\prod_{k=1}^{N}\frac{1-zk}{1-\bar{z}_k z},\quad z \in D, \varphi \in [0,2\pi) \qquad (56)$$

we want to show that b is the characteristic function of a truncated CMV matrixT(55).Put

$$F(z) = \frac{1 + zb(z)}{1 - zb(z)}$$

which is a rational function with N+ 1 distinct simple poles lying on $\mathbb{T}$, Re $F(z) > 0, Z \in \mathbb{D}$, and $F(0) = 1$. It follows that there exists a probability measure $d\mu$ on the unit circle supported at those poles, so that

$$F(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta).$$

Let $\{\alpha_0, \dots, \alpha_{N-1}, \alpha_N\}$be the Schur parameters of b, that is the same as the Verblunsky coefficients of μ. Construct the (N+1) ×(N+1) unitary CMV matrix C of the form (34).Then

$$F(z) = ((C + zI)(C - zI)^{-1}\delta_0, \delta_0), \ |z| < 1,$$

where $\delta_0(1,0, \dots, 0) \in C^{N+1}$.Let $\mathcal{T}$ be N×N be truncated CMV matrix of the form (55) C has the block form

$$C = \begin{pmatrix} S & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix}$$

Where $S = \bar{\alpha}_0, \mathcal{G} = (\bar{\alpha}_0\rho_0, \rho_1\rho_0, 0, \dots, 0), and$

$$\mathcal{F} = \begin{pmatrix} \rho_0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

Theorem(4.2.11)[175]. Let $z_1, \dots, z_m$ be distinct nonzero points in D , $l_1, \dots, l_m$ be positive integers, and r=$l_1, \dots, l_m \le N$ and. Let$\alpha_0, \dots, \alpha_{N-r} \in D$. If there exists a N×N truncated CMV matrix $\mathcal{T}$(55)such that $z_1, \dots, z_m$are eigenvalues of $\mathcal{T}$with the algebraic multiplicities$l_1, \dots, l_m$, and$\alpha_j(\mathcal{T})$=$\alpha_j, j = 0, \dots, N - r$, then this matrix is unique.

Proof. If the required $\mathcal{T}$ exists then its characteristic function$\Theta_{\mathcal{T}}(z)$ is the Blaschke product of order N and of the form

$$b(z) = e^{it} \prod_{k=1}^{m} \left(\frac{z-zk}{I-\bar{z}_k z}\right)^{I_k} \prod_{j=1}^{N-1} \frac{z-v_j}{I-\bar{v}_j z}, \tag{57}$$

with the given first N−r+ 1 Schur parameters $\alpha_0(b), \dots, \alpha_{N-r}(b)$. Our goal is to prove the uniqueness of such function b.

According to the result of Schur [213] the set of all Schur functions b with given first N−r+ 1 Schur parameters is parametrized by

$$b(z)=\frac{A(z)+zB^*(z)s(z)}{B(z)+zA^*(z)s(z)} \quad (58)$$

where $s(z)$ is an arbitrary Schur function, and A, B are polynomials of degree at most N-r Since b is the Blaschke product of order N, it is clear that so is $s(z)$, deg $s(z) = r-1$, and

$$Sb = \{\alpha_0, \dots, \alpha_{N-r}, \alpha_0(s), \dots, \alpha_{N-r}(s)\}$$

Let us solve (58) for s:

$$s(z)=\frac{A(z)-B(z)b(z)}{-zB^*(z)+zA^*(z)b(z)}$$

so $s(z)$ satisfies the Nevanlinna-Pick interpolation problem(57), where $w_k^{(z)}$ are completely determined from the given nonzero zk's and α_j's . There is at most one such $s(z)$, and the uniqueness of b is proved.

Remark(4.2.12)[175].. Suppose that $z_1, \dots, z_m$are distinct nonzero points in D, and $l_1, \dots, l_m$=N ,so the only $\alpha_0$is prescribed. It is clear thatis completely determined by the choice of z_j and their multiplicities l_j:

$$b(z) = e^{it}\prod_{k=1}^{m}\left(\frac{z-zk}{I-\bar{z}_k z}\right)^{I_k}, \quad \alpha_0 = b(0) = e^{it}\prod_{k=1}^{m}\left(-z_k^{I_k}\right)$$

So for all other α_0 the inverse problem has no solution.

In the case when one of the eigenvalues is zero, all three possibilities (no solution, unique solution, and infinitely many solutions) may occur for the inverse problem in question. For instance, there is no solution at all as long as $z_1 = 0, \alpha_0 \neq 0$. Assume next, that r =$l_1 = 1, z_1 = 0$, and the points $\alpha_0, \alpha_1, \dots, \alpha_{N-r}$ are taken in D, with the only restriction $\alpha_0 = 0, \alpha_1 \neq 0$. The Blaschke products $b_\gamma$with the Schur parameters$\{\alpha_0, \alpha_1, \dots, \alpha_{N-r}; \gamma\}$ and arbitrary $\gamma \in \mathbb{T}$are of the form

$$b_\gamma(z) = e^{it}z\prod_{j=1}^{N-1}\frac{z-v_j}{I-\bar{v}_j z},$$

and the corresponding N×N truncated CMV matrices $\mathcal{T}_\gamma$,solve the problem. Finally, assume that except for the zero eigenvalue of multiplicity $k(z_1 = z_2 = \dots = z_k = 0)$, a few more nonzero (and not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_r$are given, as well as the points $\alpha_0 = \cdots \alpha_{k-1} = 0, \alpha_{N-r}$in D. If the solution of the corresponding mixed inverse problem $\mathcal{T}$ exists, its characteristic function takes the form

$$b(z) = e^{it} z^k \prod_{j=1}^{r} \frac{z - \lambda_j}{I - \bar{\lambda}_j z} g(z).$$

Where g is the Blaschke product of order N-k-1, $g(0) \neq 0.$, and the first N-k-1 +1Schur parameters of $h = z^{-k} b$ are given numbers $\alpha_k = \cdots \alpha_{N-1}$. Clearly ,h is exactly the kth Schur iterate of b. If the required truncated CMV matrix $\mathcal{T}$exists, then by Theorem(4.2.7) the characteristic function of $\mathcal{T}^{(k)}$agrees with h. It follows now from Theorem (4.2.11)that $\mathcal{T}^{(k)}$is unique, and since $\alpha_k \, \mathcal{T} = 0, \dots, k-1$, the matrix $\mathcal{T}$ is unique as well. The situation changes dramatically if we assume that the last parameters of $\mathcal{T}$ (55) are known. In this case we can prove the existence, but not the uniqueness of the solution.

Theorem (4.2.13)[175]. Let $z_1, \dots, z_m$ and $\alpha_m, \dots, \alpha_{N-r}$ be two collections of arbitrary complex numbers from the open unit disk, and let $\alpha_N \in \mathbb{T}$. Then there exists a N x N truncated CMVmatrix $\mathcal{T}$of the form(55) such that:

(i) $z_1, \dots, z_m$ are eigenvalues of $\mathcal{T}$, counting the algebraic multiplicity,

(ii) $\alpha_n(\mathcal{T}) = \alpha_n, n = m, m+1, \dots, N.$=

Proof.

By Theorem (4.1.8)there exists a Blaschke product $b(z)$ of order N such that $b(z_k) = 0, k = 1, \dots, m$, with the Schur parameters

$$\alpha_n(b) = \alpha_n, \quad n = m, m+1 \dots N.$$

Take now the matrix $\mathcal{T}$ (55) with $\alpha_n(\mathcal{T}) = \alpha_n, n = 0,1, \dots, N$. By Theorem (4.2.14) the characteristic function of $(\mathcal{T})$ agrees with $b(z)$, that completes the proof.

Theorem (4.2.13) thereby says that a N×N truncated CMV matrix $\mathcal{T}$ can be reconstructed from its m eigenvalues and the lower principal block of order N−m. The latter is either the truncated CMV matrix $\mathcal{T}, (\{\alpha_n\}_{n=m}^{N})$ or its transpose $\tilde{\mathcal{T}}$

In this section we consider the criterion when given complex numbers $z_n = , n = 1,2, \dots$from D are the eigenvalues counting algebraic multiplicity of some semi-infinite truncated CMV matrix.

Proposition (4.2.14)[175]..Given complex numbers $z_n =, n = 1,2, ...$ are eigenvalues counting algebraic multiplicity of some semi-infinite truncated CMV matrix if and only if

$$\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$$

.

Proof.

The convergence of the sum is equivalent to the convergence of the Blaschke product

$$b(z) = \prod_{k=1}^{\infty} \frac{\bar{z}_k}{z_k} \frac{z_k - z}{z_k - \bar{z}_k z},$$

Let $\{\alpha_n\}$ be the Schur parameters of b. The characteristic function of the truncated CMV matrix $\mathcal{T}(\{\alpha_n\}$ agrees with b. Hence the eigenvalues of $\mathcal{T}(\{\alpha_n\}$ are precisely the complex numbers$\{z_n\}$.

Chapter 5

Harmonic Cooridinates and Products of Random Matrices

We show that if Kigami's resistance form satisfies certain assumptions, then there exists a waek Riemannian metric such that the energy can be expressed as the integral of the norm squared of a weak gradient with respect to an energy measure. Furthermore, we show that if such a set can be homeomorphically represented in harmonic coordinates, then for smooth functions the weak gradient can be replaced by the usual gradient. We also show a simple formula for the energy measure Laplacian in harmonic coordinates.We apply our results to extend the geography is desting principle to these cases, and lso obtain results on the pointwise behavior of local eccentricities on the sierpinski gasket, previously studied by Oberg, stricharta and Yingst, and the authors. We also establish the relation of the derivatives to the tangents and gradients previously studied by strichartz and the authors. Our main tool is the Furstenberg-Kesten theory of products of random matrices.

Sec(5.1)Fractals with Finitely Ramified Cell Structure

There is a well developed theory of Dirichlet (energy, resistance) forms, and corresponding random processes, on the class of post-critically finite (p.c.f. for short) self-similar sets, which are finitely ramified [220, 237,240, 255, 258]. Also, many piecewise and stochastically self-similar fractals have been considered [225, 229, 230, 256]. The general non self-similar energy forms on the Sierpinski gasket were studied in [253]. In all the mentioned works the fractals considered have finitely ramified cell structure. We will extend some aspects of this theory for a class of space, which may have no self-similarity in any sense, and may have infinitely many cells connected at every junction point.Throughout this section we extensively and substantially use the general theory of resistance forms developed in [241]. The existence of such forms is a delicate question even in the self-similar p.c.f. case [231, 241, 251] and references therein]. To prove our results we use some methods introduced in [260]. We give the basic background information, and the reader may find all the details in [241, 260].

We give the definition of a resistance form in the sense of Kigami [241]. We define sets with finitely ramified cell structures. Examples of such fractals are p.c.f. self-similar sets introduced by Kigami in [237, 240]. Fractafolds introduced by Strichartz in [257], random fractals [225, 229, 230] and references therein, and non self-similar Sierpinski gaskets [253, 261]. The key topological assumption is that there is a cell structure such that every cell has finite boundary, but we do not assume any self-similarity.

The terminology we use can be explained as follows. The term "post-critically infinte", means that every junction point can be an intersection of countably infinite number of cells with pairwise disjoint interior, that is every cell can be linked to countably many other cells. The term "finitely ramified" means that every cell is joined with its complement in a finite number of points. A good example of an infinitely ramified fractal is the Sierpinski carpet. There exists a self-similar diffusion and corresponding Dirichlet form on the Sierpinski carpet [221, 222, 223, 249], but its uniqueness has not been proved.

We prove that Kigami's resistance form is a local regular Dirichlet form under appropriate conditions. We prove that if the resistance form satisfies certain non degeneracy assumptions, then there exists a weak Riemannian metric,defined almost everywhere such that the energy can be expressed as the integral of the norm of weak gradient with respect to an energy measure. This generalizes earlier results by Kusuoka [248] and the author [260]. We prove that if the finitely ramified fractal can be homemorphically represented in harmonic coordinates, then the weak gradient can be replaced by the usual gradient for smooth functions, which generalizes an earlier result by Kigamiin[238]. We prove a simple formula for the energy measure Laplacian in harmonic coordinates. This formula was announced, in the case of the standard energy form on the Sierpinski gasket, in [261] without a proof. In a sense, the generalized . Riemannian metric. In the case of the standard energy form on the Sierpinski gasket, it is proved by Kusuoka in [247] that this generalized Rimannian metric has rank one almost everywhere. This can be interpreted as that in harmonic coordinates on the Sierpinski gasket the energy Laplacian is the one dimensional second derivative in the tangential direction. We conjecture that this is the case for any finitely ramified fractal considered. The main tool we use in this Theorem is approximating the finitely ramified fractal by a sequence of so called quantum graphs [245, 246]. We discuss self-similar finitely ramified fractals, and existence of self-similar resistance forms in particular. We give several examples of finitely ramified fractals for which our theory can be applied . Among them are factor-spaces of p.c.f. self-similar sets, and post-critically infinite analogs of the Sierpinski gasket.

In the case of the standard energy form on the Sierpinski gasket, it is proved by Kigami in [244] that the heat Kernel with respect to the energy measure has Gaussian asymptotics in harmonic coordinates (a weaker version was obtain in

[252]. Recently a powerful machinery was developed to obtain heat Kernel estimates on various "rough" spaces, including many fractals [224,243]. It is not unlikely that this theory is applicable to many, if not all, finitely ramified fractals in harmonic coordinates. Also, some results about the singularity of the energy measure with respect to product measures [226, 232, 233] are valid in the case of finitely ramified self-similar fractals under suitable extra assumptions.

Definitions(5.1.1) 218]. A pair (ε,Dom ε) is called a resistance form on a countable set V_* if it satisfies the following conditions.

(i) Dom ε is a linear subspace of $\ell(V_*)$ containing constants, ε is a nonnegative symmetric quadratic form on Dom ε, and ε (u, u)= 0 if and only if u is constant on V_*

(ii) Let ~ be the equivalence relation on Dom ε defined by u ~ v if and only if u – v is constant on V_*. Then (ε /~, Dom ε) is a Hilbert space.

(iii) For any finite subset $V \subset V_*$ and for any $\upsilon \in \ell(V)$ there exists u $\in$ Dom ε such that $u|_v= v$.

(iv) For any p, q $\in V_*$

$$\text{Sup}\left\{ \frac{(u(p)-u(q))2}{\varepsilon(u,u)} : u \in\in \text{Dom}\,\varepsilon,\ \varepsilon(u, u) > 0\right\} < \infty.$$

This supremum is denoted by R(p, q) and called the resistance between p and q.

(iiv) for any u $\in$ Dom ε we have the ε(u-, u-) $\leq$ ε(u, u), where

$$\bar{u}(p)= \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 1. \end{cases}$$

Property (iiv) is called the Markov property.

Note that the effective resistance R is a metric on V_*, and that any function in Dom ε is R-continuous. Let Ω be the R-completion of V_*. Then any u $\in$Dom ε has a unique R-continuous extension to Ω.

For any finite subset $U \subset V_*$ the finite dimensional Dirichlet form ε_U on U is defined by

$$\varepsilon_U(f, f) = \inf\{\varepsilon(g, g) : g \in \text{Dom}\, \varepsilon,\ g|_U = f\},$$

which exists by [84], and moreover there is a unique g for which the inf is attained.

The Dirichlet form ε_U is called the trace of ε on U, and denoted. By the definition, if $U_1 \subset U_2$ then $\mathcal{E}_{U_1}$ is the trace of $\mathcal{E}_{U_2}$ on U_1, that is $\mathcal{E}_{U_1} = \text{Trace}_{U1}(\mathcal{E}_{U_2})$.

Theorem(5.1.2)[218] . (Kigami [241]). Suppose that V_n are finite subsets of V_* and that $\cup_{n=0}^{\infty} V_n$ is R-dense in V_*. Then

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \mathcal{E}_{V_n}(f, f)$$

for any $f \in$ Dom $\mathcal{E}$, where the limit is actually non-decreasing. Is particular, $\mathcal{E}$ is uniquely defined by the sequence of its finite dimensional traces $\mathcal{E}_{V_n}$ on V_n.

Theorem(5.1.3)[218] . (Kigami[241]). Suppose that V_n are finite sets, for each n there is a resistance form $\mathcal{E}_{V_n}$ on V_n , and this sequence of finite dimensional forms is compatible in the sense that each $\mathcal{E}_{V_n}$ is the trace of $\mathcal{E}_{V_{n+1}}$ on V_n, $were\ n = 0, 1, 2, \ldots\ then\ there\ exists\ a\ resistance\ form\ \ \mathcal{E}\ on\ V_* = U_{n=1}^{\infty}$ V_n such that

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \mathcal{E}_{V_n}(f, f)$$

for any $f \in$ Dom $\mathcal{E}$, and the limit is actually non-decreasing.

Definition(5.1.4)[218] . A finitely ramified fractal F is a compact metric space with a cell structure $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}$ and a boundary (vertex) structure $\nu = \{V_\alpha\}_{\alpha \in A}$ such that the following conditions hold.

(i) A is a countable index set;

(ii) each F_α is a distinct compact connected subset of F;

(iii) each V_α is a finite subset of F_α with at least two elements;

(iv) if $F_\alpha = \cup_{j=1}^{k} F_{\alpha_j}$; then $V_\alpha \subset \cup_{j=1}^{k} V_{\alpha_j}$;

(iiv) there exists a filtration $\{An\}_{n=0}^{\infty}$ such that

(a) A_n are finite subsets of A, $A_0 = \{0\}$, and $F_0 = F$;
(b) $A_n \cap A_m = \phi$ if $n \neq m$;
(c) For any $\alpha \in A_n$ there are $\alpha_1, \ldots, \alpha_k \in A_{n+1}$ such that $F_\alpha = \cup_{j=1}^{k} F_{\alpha_j}$;
(d) $F_{\alpha'} \cap F_\alpha = V_{\alpha'} \cap V_\alpha$ for any two distinct $\alpha, \alpha' \in A_n$;

(e) for any strictly decreasing infinite cell sequence $F_{\alpha_1} \supseteq F_{\alpha_2} \supseteq \ldots$ there exists $x \in$ F such that $\bigcap_{n\geq 1} F_{\alpha_n} = \{x\}$.

If these conditions are satisfied, then

$$(F, \mathcal{F}, \nu) = (F, \{F_\alpha\}_{\alpha\in A}, \{V_\alpha\}_{\alpha\in A})$$

Is called a finitely ramified cell structure.

Notation(5.1.5)[73] . We denote $V_n = \bigcup_{\alpha\in A_n} V_\alpha$. Note that $V_n \subset V_{n+1}$ for all $n \geq 0$ by Definition(5.2.4). We say that F_α is an n-cell if $\alpha \in A_n$.

Proposition(5.1.5)[218]:[237],[239],[240]. For any $x \in$ F there is a strictly decreasing infinite sequence of cells satisfying condition (G) of the definition. The diameter of cells in any such sequence tend to zero.

Proof. Suppose $x \in$ F is given. We choose F_{α_1} = F. Then, if F_{α_n} is chosen, we choose $F_{\alpha_{n+1}}$to be a proper sub-cell of F_{α_n} which contains x. Suppose for a moment that the diameter of cells in such a sequence does not tend to zero. Then for each n there is $x_n \in F_{\alpha_n}$ such that lim $\inf_{n\to\infty} \alpha_n$ d((x_n, x) = ε > 0. By compactness there is y $\in \bigcap_{n\geq 1} F_{\alpha_n}$ such that d((y, x) ≥ ε. This is a contradiction with the property (G) of Definition (5.1.4)

Proposition(5.1.6)[218] . The toplogical boundary of F_α is contained in V_α for any $\alpha \in A$.

Proof. For any closed set A we have ∂A= A ∩ Closure (A^c), where A^c is the complement of A. If A= F_α is an n-cell, then Closure (A^c) is the union of all n-cells except F_α. Then the proof follows from property (F) of Definition (5.1.4)

Proposition(5.1.7)[218]. The set $V_* = \bigcup_{\alpha\in A} V_\alpha$ is countably infinite, and F is uncountable.

Proof. The set V_* is a countable union of finite sets, and every cell is a union of at least two smaller sub-cells. Then each cell is uncountable by properties (B) and (C) of Definition (5.1.4)

Proposition(5.1.8)[218]. For any distinct x, y $\in$ F there is n(x, y) such that if m ≥ n (x, y) then any m-cell can not contain both x and y.

Proof. Let B_m (x, y) be the collection of all m-cells that contain both x and y. By definition any cell in B_{m+1}(x, y) is contained in a cell which belongs to B_m(x, y).

Therefore, if there are infinitely many nonempty collections $B_m(x, y)$, then there is an infinite decreasing sequence of cells that contains both x and y .

Proposition (5.1.9)[218]. For any $x \in$ F and n ≥ 0, let $U_n(x)$ denote the union of all n-cells that contain x. Then the collection of open sets $\mathcal{U} =\{ U_n(x)^0\}_{x\in F,\, n\geq 0}$ is a countable fundamental sequence of neighborhoods. Here B^0 denotes the topological interior of a set B.

Moreover, for any $x \in$ F and open neighborhood U of x there exist y ∈ V_* and n such that $x \in U_n(x) \subset U_n(y) \subset U$. In particular, the smaller collection of open sets $\mathcal{U}' = \{Un(x)^o\}_{X\in V_*, n\geq 0}$ is a countable fundamental sequence of neighborhoods.

Proof. Note that the collection $\mathcal{U}'$ is countable because V_* is countable by Propostion (5.1.16). The collection $\mathcal{U}$ is countable because if x and y belong to the interior of the same n-cell, then $U_n(x) = U_n(y)$.

First , suppose $x \in V_*$. Then we have to show that for any open neighborhood U of x there exists n ≥ 0 such that $U_n(x) \subset U$. Suppose for a moment that such n does not exist. Then for any n the set $U_n(x) \setminus U$ is a nonempty compact set.

Moreover, the sequence of sets $\{U_n(x)\setminus U\}_{n\geq 0}$ is decreasing and so has a nonempty intersection. Then we can choose $z \bigcap_{n\geq 0} U_n(x)\setminus U$. and for any n there is an n-cell that contains bothbx and y. This is a contradiction with Proposition (5.1.12)

Now suppose $x \notin V_*$. Then for any n > 0 there exists $y_n \in V_n$ such that $x \in U_n(y_n) \subset U_{n-1}(x)$. Moreover, we can assume also that $U_n(y_n) \cup U_{n-1}(y_{n-1})$ for any n >1. Then we have to show that any open neighborhood U of x there exist n > 0 such that $U_n(y_n) \subset U$. Suppose for a moment that such n does not exist. Then the set $U_n(y_n)\setminus U$ is a nonempty compact set. Moreover, the sequence of sets $\{U_n(y_n)\setminus U\}_{n\geq 1}$is decreasing and so has a nonempty intersection. Then we can choose $z \in \bigcap_{n\geq 1} U_n(y_n)\setminus U$, and for any n >1 there is an (n-1) – cell that contains both x and z. This is a contradiction with Proposition (5.1.12).

We assume that there is a resistance form on V_* in the sense of Kigami [76, 84]. See Definition (5.1.1)For convenience we will denote $\mathcal{E}_n(f, f) = \mathcal{E}_{V_n}(f, f)$. Recall that $\mathcal{E}(f, f) = \lim_n \mathcal{E} \to \mathcal{E}_n(f, f)$ for any $f \in$ Dom $\mathcal{E}$, where the limit is actually non-decreasing.

Definition(5.1.10)[218]. A function is harmonic if it minimizes the energy for the given set of boundary values.

Note that any harmonic function is uniquely defined by its restriction to V_0. Moreover, any function on V_0 has a unique continuation to a harmonic function. For any harmonic function h we have $\mathcal{E}\,\varepsilon(h, h) = \mathcal{E}_n(h, h)$ for all n by [84]. Also note that for any function $g \in \text{Dom}\,\varepsilon$ we have $\mathcal{E}_0(g, g) \le \mathcal{E}\varepsilon(g, g)$, and a function h is harmonic if and only if $\mathcal{E}_0(h, h) = \mathcal{E}(h, h)$.

Let $\mathcal{E}_\alpha(f, f) = (\mathcal{E}_\alpha)_{V\alpha}(f, f)$, where $\mathcal{E}_\alpha$ is the restriction of $\mathcal{E}$ to F_n. Then

$$\mathcal{E}_n = \sum_{\alpha\in A_n} \varepsilon_{V_\alpha}$$

Lemma(5.1.11)[218]. If h is harmonic and continuous then

$$\lim_{n\to\infty} \sum_{\alpha\in A_n, x\in F} \mathcal{E}_\alpha(h|_{V\alpha}, h|_{V\alpha}) = 0$$

Proof. Let $\mathcal{E}(h, h) = e > 0$. It is easy to see that the limit under consideration is decreasing and so it exists. Suppose for a moment this limit is equal to $c > 0$.

Without loss of generality we can assume that $h(x) = 0$ and that $|h(y)| \ge 1$ for any $y \in V_0 \setminus \{x\}$. By Proposition (5.1.5) for any $\varepsilon > 0$ there are cells $F_{\alpha_1}, \ldots, F_{\alpha_1}$ such that $|h(x) - h(y)| < \varepsilon$ for any $y \in \bigcup_{j=1}^{l} F_{\alpha_j}$, and $\bigcup_{j=1}^{l} F_{\alpha_j}$ contains a neighborhood of x. Without loss of generality we can assume that $V_0 \cap (\bigcup_{j=1}^{l} F_{\alpha_j} \setminus \{x\}) = \emptyset$.

Let $V' = \bigcup_{j=1}^{l} V_{\alpha_j}$ and consider the trace of the resistance form on $V_0 \cup V'$. Obviously if ε is small then there is a uniform bound for conductances between point in $V_0 \setminus \{x\}$ and V'. Then consider changing the values of h on V' to zero. Inside of $\bigcup_{j=1}^{l} F_{\alpha_j}$ the energy will be reduced by at least C, since the function is now constant there. On the other hand, outside of $\bigcup_{j=1}^{l} F_{\alpha_j}$ the energy increase will be bounded by a constant times εe. So the total energy will decrease if ε is small enough. This is a contradiction with the definition of a harmonic function, and so $c=0$.

Note that the proof works even if V' is an infinite set and so it is applicable to connected spaces with cell structure, such as the Sierpinski carpet, which is not a finitely ramified fractal.

Corollary(5.1.12)[218]. If h is harmonic and continuous then there is a unique continuous energy measure ν_h on F defined by $\nu_h(F_\alpha) = \mathcal{E}_\alpha(h|_{V\alpha}, h|_{V\alpha})$ for all $\alpha \in A$.

Definition(5.1.13)[218]. We fix a complete, up to constant functions, energy orthonormal set of harmonic functions $h_1, \ldots, h_k = |V_0| - 1$, and define the Kusuoka energy measure by

$$v = v_{h_1}. + \ldots + v_{h_k}.$$

If $F_{\alpha'} \subset F_\alpha$, then

$$M_{\alpha,\alpha'} : \ell(V_\alpha) \to \ell(V_{\alpha'})$$

is the linear map which is define as follows. If f_α is a function on V_α then let h_{f_α} be the unique harmonic function on F_α that coincides with f_α on V_α . Then we define

$$M_{\alpha,\alpha'} f_\alpha = h_{f_\alpha}|_{V\alpha'} .$$

Thus $M_{\alpha,\alpha'}$ transforms the (vertex) boundary values of a harmonic function on F_α into the values of this harmonic function on $V_{\alpha'}$ .We denote $M_\alpha = M_{0,\alpha..}$ We denote D_α the matrix of the Dirichlet form $\mathcal{E}_\alpha$ on V_α. By elementary linear algebra we have the following Lemma (see [260] and also [237, 240, 247].

Lemma(5.1.14)[218]: If $F_\alpha = \text{U}F_\alpha$ then

$$D_\alpha = \sum M^*_{\alpha,\alpha_j} D_{\alpha_j} M_{\alpha_j}$$

and

$$v(F_\alpha) = \text{Tr } M^*_\alpha D_\alpha M_\alpha.$$

In particular y is defined uniquely in the sense that it does not denend on the choice.

We denote $$Z_\alpha = \frac{M^*_\alpha D_\alpha M_\alpha}{v(F\alpha)}$$

if $v(F_\alpha) \neq 0$. Then we define matrix valued functions

$$Z_n(x) = Z_\alpha$$

If $v(F_\alpha) \neq 0$, $\alpha \in A_n$ and $x \in F_\alpha \setminus V_\alpha$. Note that $\text{Tr } Z_n(x) = 1$ by definition.

Theorem(5.1.15)[73]. For v -almost all x there is a limit

$$Z(x) = \lim_{n\to\infty} Z_n(x).$$

Proof. One can see, following the original Kusuoka's idea [95, 94], that Z_n is a bounded v-martingale.

One can see that the energy measures v_h are the same as the energy measures in the general theory of Dirichlet forms [100, 106]. One can also define the matrix Z as the matrix whose cntrics are the densities

$$Z_{ij} = \frac{dv_{h_i,h_j}}{dv}$$

Using the general theory of Dirichlet forms in [227, 228]. However we give a different description because the pointwise approximation using the cell structure is important in this Theorem.

Definition(5.1.16)[218]. A function is n-harmonic if it minimizes the energy for the given set of values on V_n.

Note that any n-harmonic function is uniquely defined by its restriction to $V_{n.}$ Moreover, any function on V_n has a unique continuation to an n-harmonic function. Also note that for any function g ∊ Dom$\mathcal{E}$ we have $\mathcal{E}_n(g, g) \leq \mathcal{E}(g, g)$, and a function f is n-harmonic if and only if $\mathcal{E}_n(f, f) = \mathcal{E}(f, f)$.

Recall that R is the effective resistance metric on V_*, and that any function in Dom $\mathcal{E}$ is R-continuous. Let **Ω** be the R-completion of V_*Then any u ∊ Dom $\mathcal{E}$ has a unique R-continuous extension to **Ω**. The next Theorem generalizes [240]for possibly non self-similar finitely ramified fractals.

Theorem(5.1.17)[218]. Suppose that all n-harmonic functions are conditions. Then any continuous function is R-continuous, and any R-Cauchy sequence converges in the topology of F. Also, there is a continuous injective map θ: **Ω** → F which is the identity on V_*.

Proof. It is easy to see from the maximum principle that any continuous function can be uniformly approximated by n-harmonic functions, which implies that any continuous. Suppose for a moment that $\{x_k\}$ is an R-Cauchy sequence in V_* which does not converge. By compactness, it must have a limit point say x . There is n and two disigint of $\{x_k\}$,say $\{y_k\}$.

Theorem(5.1. 18)[218]. Suppose that all n-harmonic functions are continuous. Then $\mathcal{E}$ is a local regular Dirichlet form on **Ω** (with respect any measure that charges every nonempty open set).

Proof. The regularity of $\mathcal{E}$ is proved in [241]. In particular, Dom $\mathcal{E}$ mod (constants)is a Hilbert space in the energy norm. Note that the set of n-harmonic functions is a core of ε in both the original and R-topologies. Also note that if a set is R-compact then it is compact in the original topology of f by Theorem (5.1.17) Suppose now f and g are two functions in Dom $\mathcal{E}$ with disjoint compact supports. Then, there is n and a finite number of n-cells $F_{\alpha 1}$, …, $F_{\alpha k}$ such that $\bigcup_{i=1}^{k} F_{\alpha_i}$ contains the support of f but is disjoint with the support of g. Then it is easy to see that for any m ≥ n we have $\mathcal{E}_m(f, g) = 0$ and so $\mathcal{E}(f, g) = 0$.

Definition (5.1.19)[218]. We say that $f \in$ Dom $\mathcal{E}$ is n- piecewise harmonic if for any $\alpha \in A_n$ there is a (globally) harmonic function h_α that coincides with f on F_α.

Note that, by definition, the notion of n-piecewise harmonic functions in general is more restrictive than the more commonly used notion of n-harmonic functions defined in the pervious section.

Definition(5.1.20)[218]. We say that the resistance form on a finitely ramified fractal is weakly non degenerate if the space of piecewise harmonic functions is dense in Dom ε.

The notion of weakly nondegenerate harmonic structures was studied in [87] in the case of p.c.f. self-similar sets.

Assumption (WN). In what follows we assume that the resistance form is weakly nondegenerate.

Proposition (5.1.21)[218]: The (WN) assumption implies supp(v) = F.

Proof. Our definitions imply that for any cell F_α there is a function of finite energy with support in this cell. If it can be approximated by piecewise harmonic functions, then $v(F_\alpha) > 0$.

Theorem(5.1.22)[218]. Let F_v be the factor-space (quotient) of F obtained by collapsing all cells of zero v-measure. Then F_v is a finitely ramified fractal with the cell and vertex structures naturally inherited from F.

Proof. The only nontrivial condition to verify is that any cell of F_v has at least two boundary points. The maximum principle implies that a cell F_α has a positive v-measure if and only if there is a harmonic function which is non constant on V_α.

Definition(5.1.23)[218]. If f is n-piecewise harmonic then we define its tangent Tan_α f for $\alpha \in A_n$ as the unique element of $\ell(V_0)$ that satisfies two conditions:

(i) if $h_{\alpha, Tan}$ is the harmonic function with boundary values $Tan_\alpha f$ then $h_{\alpha,Tan}$ coincides with f on F_α;

(ii)$h_{\alpha, Tan}$ has the smallest energy among all harmonic functions h_α such that h_α coincides with f on F_α.

We define L^2_Z as the Hilbert space of $\ell(V_0)$-valued functions on F with the norm defined by

$$\|u\|^2_{L^2_Z} = \int_F \langle u, Zu\rangle dv.$$

Definition(5.1.24))[218]. If f is n-piecewise harmonic then we define its gradient Grad f as the element of L^2_Z if $x \in F_\alpha$ and $\alpha \in A_n$.

Lemma(5.1.25))[218] . If f is n-piecewise harmonic then $\mathcal{E}(f, f) - |\|\text{Grad } f\|^2_{L^2_Z}$.

Proof. Follows from Lemma (4.1.14).

Theorem(5.2.26)[218] . Under the (WN) assumption Grad can be extended from the space of piecewise harmonic functions to an isometry

$$\text{Grad: Dom } \mathcal{E} \to \mathcal{L}^2_Z,$$

which is called the weak gradient.

Proof. The statement follows from Lemma (4.1.25).and the (WN) assumption.

Corollary(2.1.27))[218]. Under the (WN) assumption we have

$$v_f << v$$

for any $f \in$ Dom $\mathcal{E}$.

Proof. The statement follows from Theorem (5.2.26 **).**It can be obtained directly from the (WN) assumption, or the general theory of Dirichlet forms [100, 106].

Conjecture(5.1.28)[218] . We conjecture that the assumption supp (v) = F is equivalent to the (WN) assumption for all finitely ramified fractals.

Conjecture(5.1.29))[218] . We conjecture that for any finitely ramified fractal t all x.

The next Proposition follows easily from our definitions. It means, in particular, that Conjecture (5.1.29)) implies Conjecture (5.1.28).

Proposition(5.1.30))[218] . If supp(v) = F and rank Z(x) = 1 for v-almost all x then the To define harmonic coordinates one needs to chose a complete, up to constant functions, set of harmonic functions $h_1, \ldots, h_k$ and define the coordinate map ψ: F → R^k by $\psi(x)$= ($h_1(x)$, …, $h_k(x)$). A particular choice of harmonic coordinates is not important since they are equivalent up to a linear change of variables. Below we fix the most standard coordinares which make the computations simpler.

Definition(5.1.31)[218]. Let $V_0 = \{v_1,\ldots, v_m\}$ and let h_j be the unique harmonic function with boundary values $h_j(v_i) = \delta_{i,j}$. Kigami's harmonic coordinate map ψ:F →R^m is defined by $\psi(x) = (h_1(x), \ldots, h_m(x))$.

Lemma(5.1.32)[218].

(i) Any set $\psi(F_\alpha)$ is contained in the conver hull of $\psi(V_0)$.

(ii) A set $\psi(F_\alpha)$ has at least two points if and only if $\psi(V_\alpha)$ has at least two points.

(iii) (iii) If on $F_H = \psi(F)$ we define a cell structure that consists of all sets $\psi(F_\alpha)$ that have at least two points, then conditions (A) (E) and (G) of Definition (5.1.4) are satisfied.

(iV) If for all n and for any two distinct α, α' ∈ A we have

$$\psi(F_{\alpha'}) \cap \psi(F_\alpha) = \psi(V_{\alpha'}) \cap \psi(V_\alpha),$$

then $F_H = \psi(F)$ is a finitely ramified fractal with the cell structure defined in Item (iii) of this Lemma.

Proof. The maximum principle implies that $\psi(F_\alpha)$ is contained in the convex hull of $\psi(V_\alpha)$, which implies the other statements.

Theorem(5.1.40)[218] . ψ: F → $F_H = \psi(F)$ is a homeomorphism if and only if for any α ∈ A the map $\psi|_{V_\alpha}$ is an injection, and

$$\psi(F_{\alpha'} \cap F_\alpha) = \psi(F_{\alpha'}) \cap \psi(F_\alpha)$$

for all α, α' ∈ A.

Assumption (HC). In what follows we assume that ψ: F → $F_H = \psi(F)$ is a homeomorphism.

Proposition(5.1.33)[218]. The (HC) assumption implies the (WN) assumption.

Proof. It is easy to see that under the (HC) assumption any cell has positive measure, and that any continuous function can be uniformly approximated by piecewise harmonic functions. The latter is true because all harmonic functions are linear in harmonic coordinates, and the maximum principle implies that $\psi(F_\alpha)$ is contained in the convex hull of $\psi(V_\alpha)$.

Theorem(5.1.34)[218]. Under the (HC) assumption we have that if f is the restriction to F of a $C^1(R^m)$ function then $f \in \text{Dom}\,\mathcal{E}$, and such functions are dense in Dom $\mathcal{E}$.

Moreover, if $f \in C^1(R^m)$ then

in the sense of the Hilbert space L_z^2. In particular we have the Kigami formula

$$\mathcal{E}(f, f)\ \|\nabla f\|_{L_z^2}^2 = = \int_F \langle \nabla f, z\nabla f\rangle\, dv$$

for any $f \in C^1(R^m)$.

Proof. In fact, we will prove this result for a somewhat larger space of functions.

We say that f is a piecewise C^1 – function if for some n and for all $\alpha \in A_n$ there is $f_\alpha \in C^1(R^m)$ such that $f_\alpha|_{F\alpha} = f|_{F\alpha}$. In particular, a piecewise harmonic function is piecewise C^1.

If g is a linear function in R^m then $g|_{V0} = \nabla g$ since we identify $\ell(V_0)$ with R^m in the natural way. Therefore for any piecewise harmonic function f we have Grad f $= \nabla f$ in the sense of the Hilbert space L_z^2.

Any C^1- function is a piecewise C^1- function, and any piecewise C^1-function can be approximated by piecewise harmonic (that is, piecewise linear) functions in C^1 norm. Thus, to complete the proof we need an estimate of the energy of a function in terms of its C^1 norm, provided by the next simple Lemma (5.2.44)

Lemma(5.1.35)[218]. If f is the restriction to F of a $C^1(R^m)$ function then

$$\mathcal{E}_n(f, f) \le v(F)\|f\|_{C^1}^2(\mathbb{R}^m) \tag{1}$$

and the same estimate holds for $|\varepsilon(f, f)|$.

Proof. By Definition [237, 240] of $\mathcal{E}_n$ we have that

$$\mathcal{E}_n(f, f) = \sum_{x,y\in Vn} C_{n,x,y}(f(x) - f(y))^2 \le$$

$$\|f\|_{C^1}^2(\mathbb{R}^m) \sum_{x,y\in V_n} C_{n,x,y}|x - y|^2 - \|f\|_{C^1}^2(\mathbb{R}^m)v(f)\,. \tag{2}$$

[227].[228] The energy measure Laplacian can be defined as follows. We say that $f \in$ Dom Δ_v if there exists a function $\Delta_v f \in L_v^2$ such that

$$\mathcal{E}(f, g) = - \int_F g\Delta_v f dv \qquad (3)$$

for any function $g \in$ Dom $\mathcal{E}$ vanishing on the boundary V_0. By [84]. The Laplacian Δ_v is a uniquely defined linear operator with Dom $\Delta_v \subset$ Dom ε. In fact Dom Δ_v is $\mathcal{E}$ -dense in Dom ε, and is also dense in L_v^2 . The Laplacian Δ_v is self-adjoint with, say, Dirichlet or Neumann boundary conditions. Formula (3) is often called the Geuss-Green formula Extensive information on the relation of a Dirichlet form.

Theorem(5.1.36)[218].. Under the (HC) assumption we have that if f is the restriction to F of a C^2 (R^m) function then $f \in$ Dom Δ_v, and such functions are $\mathcal{E}$ -dense in Dom Δ_v. Moreover, v-almost everywhere

$$\Delta_v f = \mathrm{Tr}\,(ZD^2 f)$$

where $D^2 f$ is the matrix of the second derivatives of f.

Proof. We start with defining a different sequence of approximating energy forms.

In various situations these forms are associated with so called quantum graphs, photonic crystals and cable systems. If $f \in C^1$ (R^m) then we define

$$\mathcal{E}_n^Q\,(f, g) = \sum_{x,y \in Vn} C_{n,x,y}\, \mathcal{E}_{x,y}^Q\,(f, f)$$

where

$$\mathcal{E}_{x,y}^Q(f, f) = \int_0^1 \left(\frac{d}{dt} f\,(x(1-t) + ty)\right)^2 dt$$

is the integral of the square of the derivative

$$\frac{d}{dt} f\,(x(1-t) + ty) = \langle \nabla f\,(x(1-t) + ty), y - x\rangle$$

0f f along the straight line segment connecting x and y. Thus $\mathcal{E}_{x,y}^Q(f, f)$ is the usual one dimensional energy of a function on a straight line segment. If f is linear then $\mathcal{E}_{x,y}^Q(f, f) = (f(x) - f(y))^2$. Therefore if f is piecewise harmonic then $\mathcal{E}_{x,y}^Q(f, f) = \mathcal{E}_{x,y}^Q(f, f)$ for all large enough n. Also $\mathcal{E}_{x,y}^Q$ satisfies estimate (1) Therefore for any $C^1(R^m)$ – function we have

$$\lim_{n\to\infty} \mathcal{E}_{x,y}^Q(f, f) = \mathcal{E}(f, f)$$

by Theorem (5.2.34)

It is easy to see that if g is a $C^1(R^m)$ – function vanishing on V_0 and f is a $C^2(R^m)$ – function then

$$\varepsilon^{Q}_{x,y}(f, g) = \sum_{x,y \in Vn} C_{n,x,y} \int_0^1 g(x(1-t) + ty) \left(\frac{d^2}{dt^2} f(x(1-t) + ty)\right) dt$$

because after integration by parts all the boundary terms are canceled. Then if $\alpha \in A_n$ then

$$\sum_{x,y \in V\alpha} C_{n,x,y} \frac{d^2}{dt^2} f(x(1-t) + ty) =$$

$$\sum_{x,y \in V\alpha} C_{n,x,y} \sum_{i,j=1}^{m} D^2_{ij} f(x(1-t) + ty)(y_i - x_i)(y_j - x_j)$$

$$= \mathrm{Tr}(M^*_\alpha D_\alpha M_\alpha(D^2 f(x_\alpha) + R_n(x, y, t, f, \alpha, x_\alpha)))$$

where $x_\alpha \in V_\alpha$ and

$$\lim_{n\to\infty} |R_n(x, y, t, f, \alpha, x_\alpha)| = 0$$

Uniformly in $\alpha \in A_n$, x, y, $x_\alpha \in F_\alpha$ and $t \in [0, 1]$, which completes the proof. Note also that one can obtain an estimate similar to (1). as in Corollary (5.1.37)

Corollary(5.1.37)[218]. Under the (HC) assumption, $\Delta_v f \in L^\infty(F)$ for any $f \in C^2(R^m)$.

Corollary (5.1.38)[218]. If f is the restriction to F of a $C^2(R^m)$ function, and g is the restriction to F of a $C^1(R^m)$ function vanishing on the boundary, then

$$|\varepsilon_n(f, g)| \le v(F)\|g\|_{C(Rm)}\|f\|_{C_2(\mathbb{R}m)}$$

And the same estimate holds for $|\varepsilon(f, g)|$.

Proof. This estimate follows from the proof of Theorem (5.2.46)

Definition(5.1.39)[218].[234],[235],237],240]. A compact connected metric space F is called a finitely ramified self-similar set if there are injective contraction maps

$$\Psi_1. \ldots, \psi_m: F \to F$$

and a finite set $V_0 \in F$ such that

$$F = \bigcup_{i=1}^{m} \psi_i(F) \cup$$

and for any n and for any two distinct words w, w' $\in$ Wn = $\{1, \ldots, m\}^n$ we have

$$F_\omega \cap F_{\omega'} = V_\omega \cap V_{\omega'},$$

where $F_\omega = \psi(F)$ and $V_\omega = \psi_\omega(V_0)$. Here for a finite word $\omega = \omega_1 \ldots \omega_n \in W_n$ we denote

$$\psi_\omega = \psi_{w_1} \text{o} \ldots \text{o}\ \psi_{w_n}$$

The set V_0 is called the vertex boundary of F.

Proposition(5.1.40)[218]. A finitely ramified self-similar set is a finitely ramified fractal provided V_0 has at least two elements.

We have $A_n = W_n$ for $n \geq 1$ and $A = \{0\} \cup W_*$, where $W_* = \cup_{n\geq 1} W_n$.

Proof. All items in Definition (5.1.4) are self-evident. Note that item (B) holds because each cell is connected and has at least two elements, and the intersection of two cells is finite. Item (G) holds because ψ_i are contractions.

Definition(5.1.41)[218]: A resistance form ε on V_*, is self-similar with energy renormalization factors $\rho = (\rho_1, \ldots, \rho_m)$ if for any $f \in$ Dom ε we have

$$\mathcal{E}(f, f) = \sum_{i=1}^{m} \rho_i \varepsilon(f_i, f_i). \qquad (4)$$

Here we use the notation $f_\omega = f \text{o}\ \psi_\omega$ for any $\omega \in W^*$.

The energy renormalization factors, or weights, $\rho = (\rho_1, \ldots, \rho_m)$ are often also called conductance scaling factors because of the relation of resistance forms and electrical networks. They are reciprocals of the resistance scaling factors $r_j = \frac{1}{p_j}$.

Definition (5.1.42)[218] . For a set of energy renormalization factors $\rho = (\rho_1, \ldots, \rho_m)$ and any resistance form ε_0 on V_0 define the resistance form $\Psi_\rho(\varepsilon_0)$ on V_1 by

$$\Psi_p(\varepsilon_0)(f, f) = \sum_{i=1}^{m} p_i\, \varepsilon_0(g_i, g_i),$$

where

$$g_i = \int |_{\psi_i(V_0)}\, O\psi_i^{-1}.$$

Then $A(\varepsilon_0)$ is defined as the trace of $\Psi_p(\varepsilon_0)$ on V_0 :

$$A(\varepsilon_0) = \text{Trac}_{V0}\, \Psi_\rho(\varepsilon_0).$$

The next two Propositions are essentially proved in [76, 84, 86].

Proposition(5.1.43)[218]. If ε is self-similar then $\varepsilon_0 = A(\varepsilon_0)$.

Proposition(5.1.44)[218]. If ε_0 is such that $\varepsilon_0 = A(\varepsilon_0)$ then there is a self- similar resistance form ε such that ε_0 is the Trace of ε on V_0.

Theorem(5.1.45) [218]. On any self-similar finitely ramified fractal with a self-similar continuous. Since all ψ_i are contractions, there is n such that any n-cell contains for any $\omega \in W_n$ and any harmonic function h we have

$$\left|\max_{x\in F} h(x) - \min_{x\in F} h(x)\right| \geq (1-\varepsilon)\left|\max_{x\in F_w} h(x) - \min_{x\in F_w} h(x)\right|$$

Then for any positive integer m and any $w \in W_{mn}$ we have

$$\left|\max_{x\in F} h(x) - \min_{x\in F} h(x)\right| \geq (1-\varepsilon)^m\left|\max_{x\in F_w} h(x) - \min_{x\in F_w} h(x)\right|$$

We conjecture that the many other results of [76, 84] on the topology and analysis on p.c.f. self-similar set hold for finitely ramified self-similar sets as well. The next Theorem is one of these results. Following [75, 84], we say that the self-similar resistance form is regular if $\rho_i > 1$ for all i.

Theorem(5.1.46)[218]. If a self-similar resistance form on a self-similar finitely ramified.

Proof. If diam $_R$(.) denotes the diameter of a set in the effective resistive metric R, and $\rho_w = \rho_{w_1}\ldots\rho_{w_n}$ for any finite word $w3 = w_1 \ldots w_n \in W_n$ then

diam $R(F) \geq \rho_w$diam $R(F_\omega)$

by the self-similarity of the resistance form and the Definition of the metric R.

Definition(5.1.47)[73]. The group G is said to act on a finitely ramified fractal F if each $g \in G$ is a homeomorphism of F such that $g(V_n) = V_n$ for all $n \geq 0$.

Proposition (5.1.48)[73] . If a group G acts on a finitely ramified fractal F then for each $g \in G$ and each n-cell F_α, $g(F_\alpha)$ is an n-cell.

Proof. We have that n-cells are connected, have pair wise disjoint interiors, and their topological boundaries are contained in V_n, which is preserved by g by definition.

Theorem(5.1.49)[218]. Suppose a group G acts on a self-similar finitely ramified fractal F and G restricted to V_0 is the whole permulation group of V_0. Then there exists a unique, up to a constant, G-invariant self-similar resistance form ε with equal energy renormalization weights and

$$\mathcal{E}_0 (f, f) = \sum_{x,y \in V_0} (f (x) - f (y))^2. \quad (5)$$

Proof. It is easy to see that, up to a constant, E_0 is the only G-invariant resistance form on V_0. Let $\rho_1 = (1, \ldots, 1)$. Then $A(\varepsilon_0)$ is also G-invariant and so $\varepsilon_0 = c\mathrm{Trace}_{V0}\, \Psi_{\rho 1} (\varepsilon_0)$ for some c. Then the result holds for $\rho = c\rho_1$ by Proposition (5.1.43) and Proposition (5.1.44)

An n-cell is called a boundary cell if it intersects V_0. Other wise it is called an interior cell. We say that F has connected interior if the set of interior 1-cell is connected, any boundary 1-cell contains exactly one point of V_0, and the intersection of two different boundary 1-cells is contained in an interior 1-cell. The following theorem is proved in [85] for the p.c.f. case, but the proof applies for self-similar finitely ramified fractal without any changes.

Theorem(5.1.50)[73]. [231]. Suppose that F has connected interior, and a group G avts on a self-similar finitely ramified fractal F such that its action on V_0 is transitive. Then there exists a G-invariant self-similar resistance form ε.

Other results in [231] also apply for self-similar finitely ramified fractal.

Example(5.1.51)[218]. (Unit interval). The usual unit interval is a finitely ramified fractal. In this case V_* can be countable dense subset of {0, 1}. The usual energy form

$$\varepsilon(f, f) = \int_0^1 | f' (t)|^2 \, dt$$

satisfies all the assumptions of our paper. The energy measure is the Lebesgue.

Example (5.1.52)[218] . (Quantum graphs). A quantum graph, a collection of finite number of point in R^m joined by weighted straight line segments (see [245, 246] and also the proof of Theorem (5.1.36) is a finitely ramified fractal. The usual energy form on a quantum graph, which is the sum of weighted standard one dimensional forms on each segment, satisfies all the assumptions of our Section.

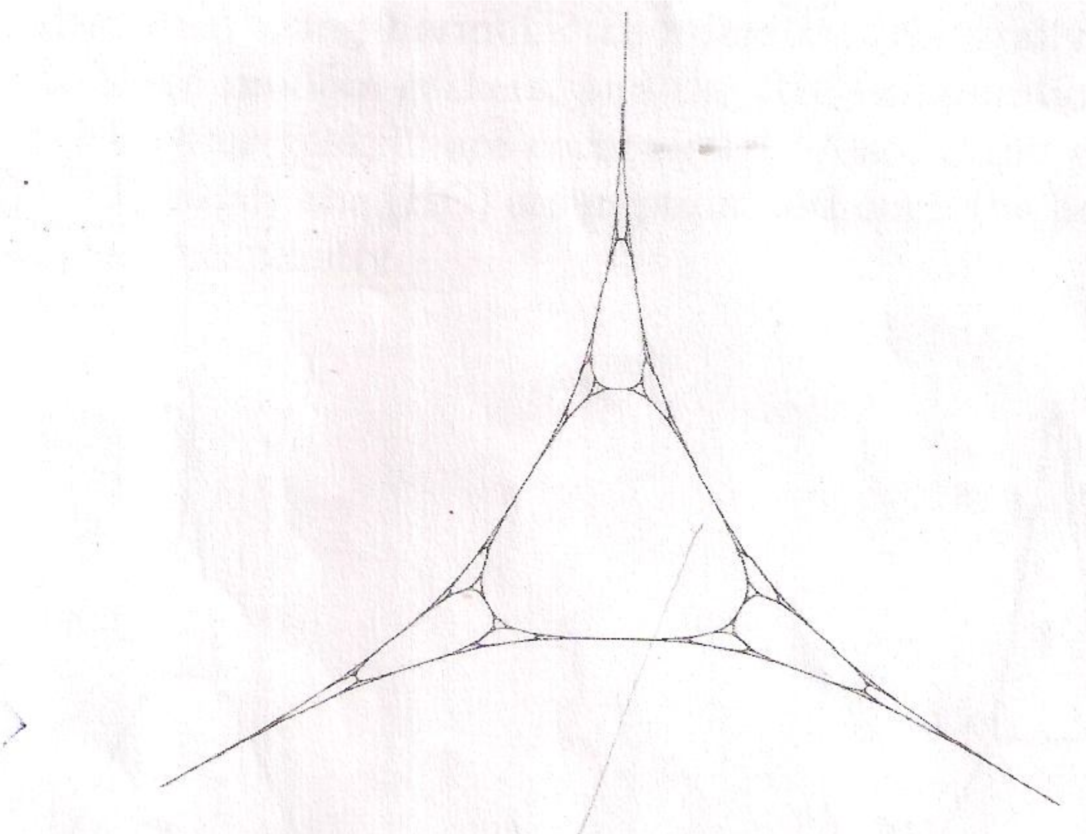


FIGURE 1. Sierpinski gasket in the standard harmonic coordinateski

Example (5.1.53)[73].(Sierpinski gasket). The Sierpinski gasket is a finitely ramified fractal. The standard energy form [236, 237, 240] on the Sierpinski gasket satisfies all the assumptions of our section. The Sierpinski gasket in harmonic coordinates, see Figure 1, was first considered in [238], where the statement of Theorem (5.1.34) was proved in this case. The statement of Theorem (5.1.36) was announced in [261]. without a proof. In the case of the standard energy form in the Sierpinski gasket Conjecture (5.1.29) was proved in [247]. The fact that the energy measure is singular with respect to any product (Bernoulli) measure was proved in [247, 226, 232, 233].

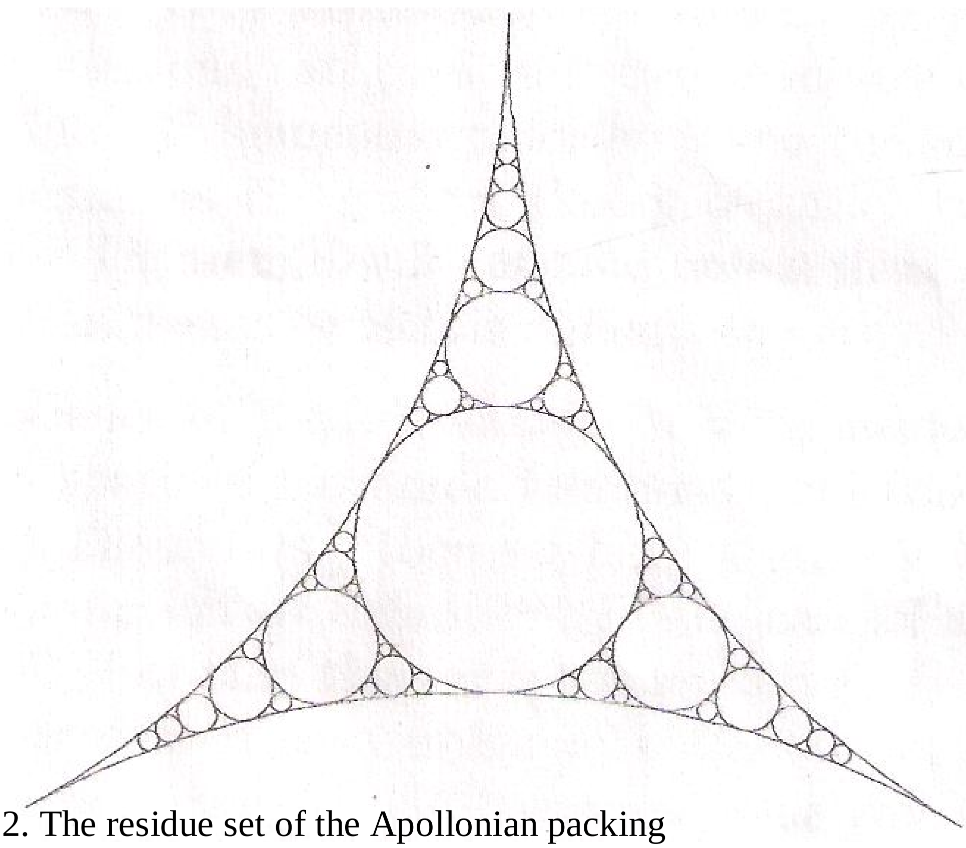


FIGURE 2. The residue set of the Apollonian packing

Example(5.1.54)[218] . (The residue set of the Apollonian packing). It was proved in [261] that the residue set of the Apollonian packing, see Figure 2. is the Sierpinski gasket in harmonic coordinates defined by a non self-similar resistance form. This resistance form satisfies all the assumptions of our section, including the (HC) assumption.

Example(5.1.55)[218] . (Random Sierpinski gasket). In [253] a family of random Sierpinski gasket was described using harmonic coordinates. Naturally, the results of this section apply to these random gaskets, and the (HC) assumption is satisfied due to the way in which these gaskets are constructed. Also, many examples of random fractals in [80, 81] satisfy the (HC) assumption, although the harmonic coordinates were not considered explicitly.

Example(5.1.56)[218] (Hexagasket). According to [260], the Hexagasket satisfies the (WN) assumption but not the (HC) assumption. However, by small perturbations of the harmonic coordinates one can construct two functions of finite energy which map the hexagasket into R^2 homeomorphically. Then the conclusion of Theorems (5.1.15) and (5.1.34)will hold because of the general theory of Dirichlet forms in [227, 228] However Theorem (5.1.36)will not hold unless these coordinates are in the domain of the domain of the energy Laplacian, which is difficult to verify.

Example(5.1.57)[218]. (Quotients of p.c.f. fractals). If we consider quotient of a p.c.f. fractal defined by its space of harmonic functions, and conditions of Theorem (5.1.32) are satisfied (see also Theorem (5.1.18)then we have a finitely ramified fractal which satisfies the (HC) assumption by Definition.Note that this set is not self-affine. In harmonic coordinates the Hexagasket is represented as a union of a Cantor set and a disjoint union of countably many closed straight line intervals. One can show that the energy measure of this Cantor set is zero, and in fact the energy measure is proportional to the Lebesque measure an each segment. Note that in the limit no two intervals graph. In this case a three point boundary, see [258, is chosen so that the resulting fractal can be embedded in R^2. For a different choice of the boundary the local structure of the fractal in harmonic coordinates is the same.

Example(5.1.58)[218] . (Vicsek set). Vicsek set (see, for instance, [89] is a finitely ramified fractal which does not satisfy the (WN) and (HC) assumptions. In harmonic coordinates it is represented by four straight line segments graph with five vertices and four edges, which is not homeomorphic to the Vicsek set.

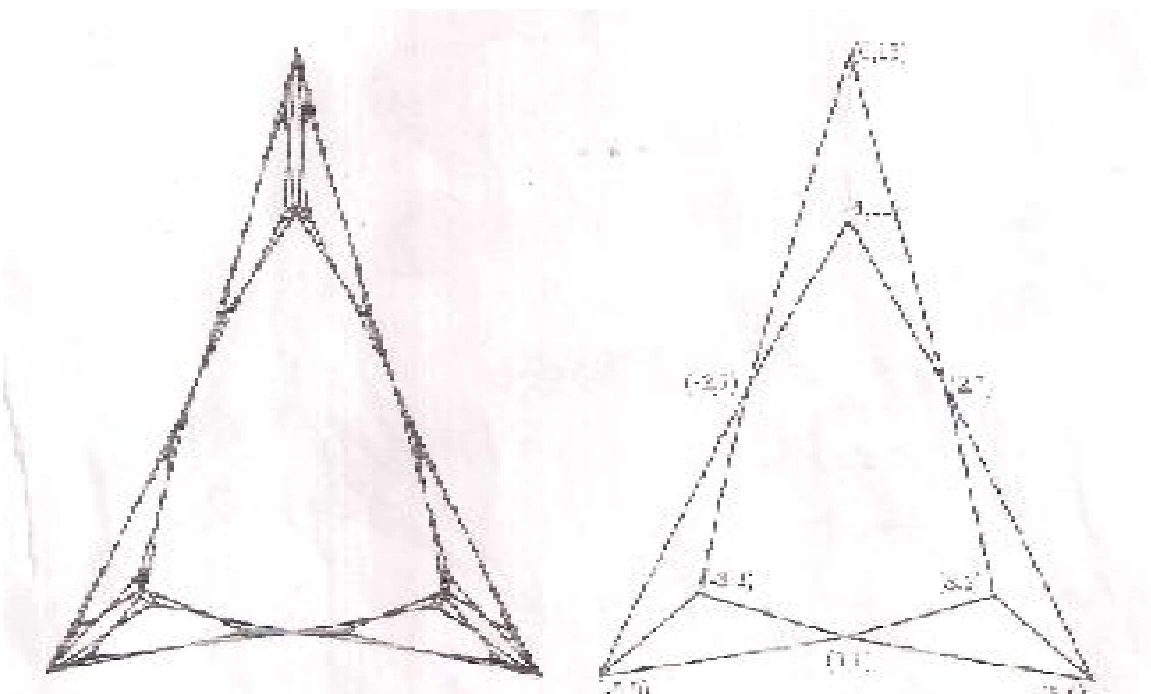


FIGURE 3. A regular post-critically infinite fractal and its first approximation.

Example(5.1.59)[218] . (Post-critically infinite Sierpinnski gasket). The post-critically infinite Sierpunski gasket, but is not a p.c.f. self-similar set. More exactly, its post-critical set defined in [237, 241] is countably infinite, and each vertex $v \in V_*$ is an intersection of countably many cells with pairwise disjoint interior. This fractal satisfies Definition (5.1.39) and can be constructed as a self-affine fractal in R^2 using nine contractions, we also sketch the first approximation to it in harmonic coordinates. In particular, shows the values of a symmetric and a skew-symmetric harmonic functions. By Theorem (5.1.49) one can easily construct a resistance form such that for any n the resistance are equal to $(50/53)^n$ in each triangle with vertices in V_n. The energy renormalization factor is $53/50 = \rho_1 = \ldots = \rho_9$. The fact that this factor is larger than one is significant because it implies that the harmonic structure is regular by Theorem (5.1.46), that is $\Omega = F$. By Theorem (5.1.32), this resistance form satisfies all the assumptions, including the (HC) assumption.

Example(5.1.60)[218]. In the end we describe two more examples of post-critically infinite finitely ramified fractals, which are shown in Figures 3 and 4. In these examples for any n there are n-cells which are joined in two points. Both fractals satisfy Definition (5.1.39).And can be constructed as a self-affine fractal in R^2 using six contractions. In particular, one can see the values of symmetric and skew-symmetric harmonic functions on each fractal. By Theorem (5.1.49) one can easily construct resistance forms such that Fg is given by (52)By Theorem (5.1.32), these re an elementary shows that the common energy renormalization factor in (51) is 5/4, and so the resistance form is regular. In the case of the fractal in Figure 4., the calculation shows that the common energy renormalization factor in (51)is 4/5, and so the resistance form is non regular.

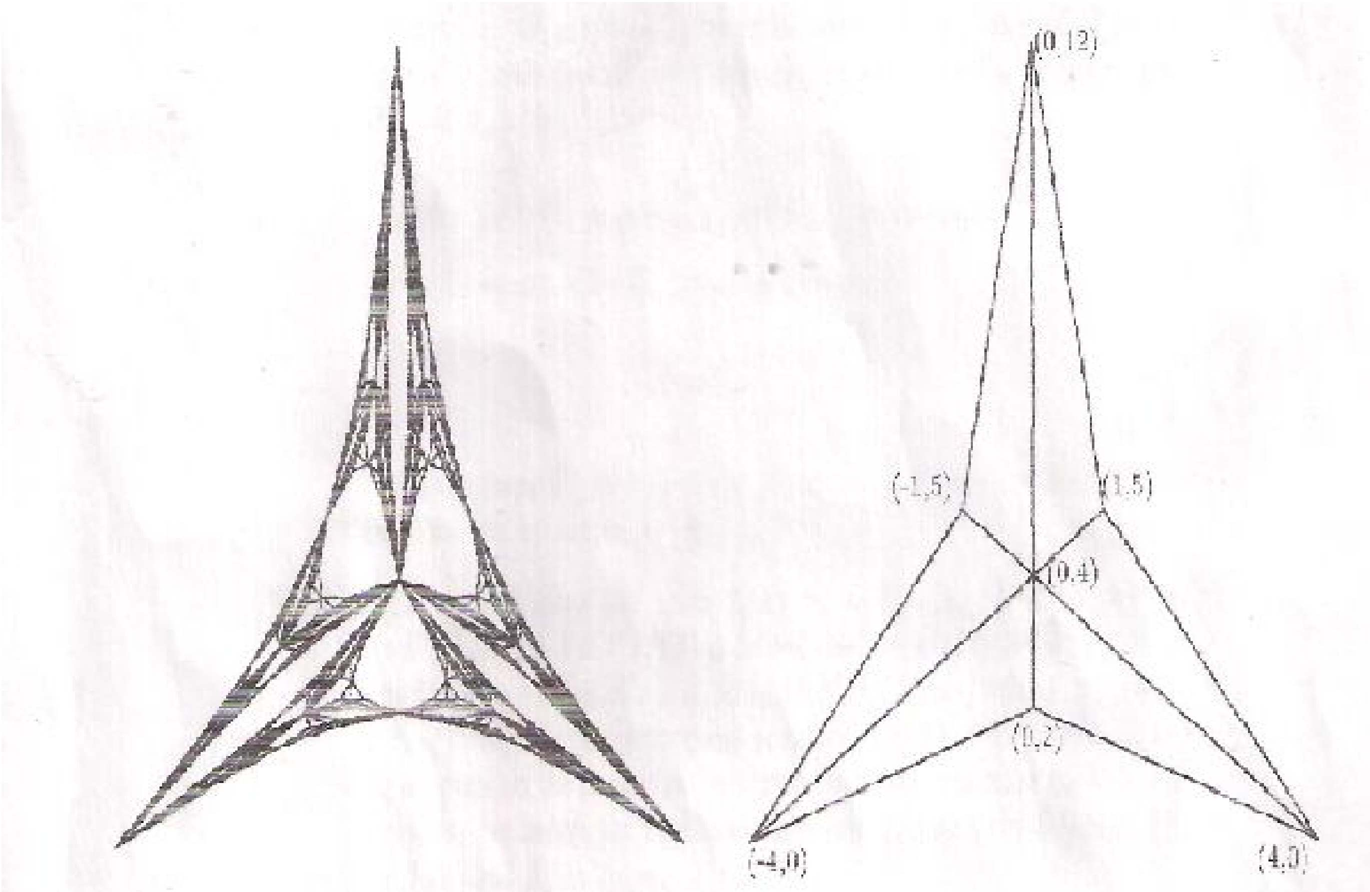


FIGURE 4. A non regular post-critically infinite fractal and its first approximation.

Sec(5.2) Derivatives on p.c.f Fractals

For the last twenty years a theory of analysis on fractals has evolved, with the construction of Laplacians and Dirichlet forms as cornerstones. There is both a probabilistic approach, where the Laplacian is constructed as an infinitesimal generator of a diffiusion process, and an analytic approach where the Laplacian can be defined as a limit of difference operators. In this section we will work in the context of post critically finite (p.c.f.) fractals, for which Kigami laid the foundations of an analytic theory[236,237,238,239].

We consider one of the most fundamental topics in analysis; the local structure of smooth functions.This is not only an interesting matterbas such, it also shed light on an important phenomenon that does not occur when the underlying set is smooth.

In classical analysis any two points in the interior of the considered set have homeomorphic neighborhoods. This is not the case in analysis on fractals. Some points, called junction points, are boundary points of several copies of the self-similar set and neighborhoods of such points are different from those at nonjunction points that have a canonical basis of neighborhoods consisting of copies of the self-similar set. However, although two nonjunction points x, x' have bases of homeomorphic neighborhoods, the homeomorphisms do not in general map x onto x'.

It turns out that, as a consequence of the above, the local behavior of functions depend on the point under consideration. This *geography is destiny* principle, that has no analog whatsoever in analysis on smooth sets, were proven for harmonic functions on the Sierpinki gasket by Oberg, Strichartz and Yingst in [267]. Restriction to the canonical neighborhoods will, for most harmonic functions, line up in the same direction, a direction that depends on the point, or rather the neighborhood. This property follows from theorems on products of random matrices since the restrictions to the canonical neighborhoods are given by linear mappings.

We will show that the geography is destiny principle extends to order fractals and to larger classes of functions with certain smoothness properties.

Generally speaking, the notion of smoothness of function addresses the degree of differentiability of the function and its derivatives. Since the basic differential operator in analysis on fractals is the Laplacian, the term smooth has mostly been used for a function f in the domain of the Laplacian, It has also been used to refer to those f for which $\Delta^K f$ is continuous for some or all k.

On the other hand, in the classical calculus a differentiable function locally behaves like an affine linear mapping. In fractal analysis the analogs of such mappings are the harmonic functions, and from this point of view we make a natural definition of a derivative, and thus a concept of differentiability, of a functions with respect to a harmonic function. This gives us wider classes of functions with some degree of smoothness for which we can prove geography is destiny. We also relate this derivative to the gradient defined by the second author [260].

Our results concerns generic, with respect to a self-similar measure, properties of the local behavior of smooth functions at nonjunction points. It would be interesting to know if the same properties hold generically with respect to the Kusuoka energy measure [247, 260]. Local behavior at junction points were studied in [256].

It is likely that our results can be extended to the category of self-similar finitely ramified fractals in [218].

We need to fix some notation, and at the same time recall some of the basic results of the theory. We refer to the books by Kigami [240] and Strichartz [258] for the whole story.

Positive constants in estimates will be denoted by C. The value of C might thus change from to line.

F will denote a, p.c.f. self-similar fractal, or post critically finite self-similar set, as defined in [240]. By is a compact connected metric space and there are contractions $\psi_{1,\dots,}\psi_2 : F \to F$ such that

$$F=\cup_{i=1}^{m} \psi_i\,(F), \tag{6}$$

and a finite set $V_0 \subset F$ such that for any n and for any two distinct words $w, w' \in W_n = \{1,\dots, m\}^n$ we have

$$F_w \cap F_{w'} = V_w \cap V_{w'} \tag{7}$$

Where $F_w = \psi_w(F)$ and $V_w = \psi_w(V_0)$. Here for a finite word $w = w_1 \dots w_n \in W_n$

We denote

$$\psi_w = \psi_{w_1} \text{o} \dots \psi_{w_n} = \tag{8}$$

We call $F_w, w \in W_n$ a cell of level n. If f is any function defined on F we use notation $f_w = f \text{ o } \psi_w$ for its restriction to F_w.

The set V_0 is called the boundary of F and consequently points in V_0 are referred to as boundary points. The fractal F is p.c.f. self- similar fractal if every boundary point is contained in only one 1-cell. We denote the number of boundary points by N_o and will assume that $N_o \geq 2$. A point $x \in F$ is called a junction point if $x \in F_w \cap \in F_{w'}$ for two distinct $w, w' \in W_n$.

Define $V_n = \cup_{w \in W_n} V_w$ $V_* = \cup_{n\geq 1} V_n$ and $W_* = \cup_{n\geq 1} W_n$. If $w = w_1 \dots w_k \in W_*$. we say that $|w| = K$ is the length of w. It is easy to see that V_* is dense in F. Note that, by definition, each ψ_i maps V_* into itself injectively.

Let $\Omega = \{1, \dots m\}^N$ be the space of infinite sequences $w = w_1 w_2 \dots$ and $W_n = \{1, \dots, m\}^n$ the set of finite words in letters $w \in W_1 = \{1, \dots, m\}$. For any $w \in \Omega$ let

$[w]_n = w_1 \dots w_n \in$ w_n and $[w]_{n,K} = w_{n+1} \dots w_K \in W_{K-n}$, K > n, These notations will be used also for $w \in W_*$ and $K < n \leq |w|$.

There is a natural continuous projection $\pi: \Omega \longrightarrow F$ defined by

$$\pi(w) = \bigcap_{n \geq 0} F_{[w]n} , \tag{9}$$

and $\pi^{-1}\{x\}$ is finite for any x by the p.c.f. assumption. Moreover, $\pi^{-1}\{x\}$ consists of more than one element if and only if x is a junction point . In case x is not a junction point we can therefore define = $\{x\}_n$ $[w]_n$ and $[x]_{n,K} = [w]_{n,K}$ if $x = \pi(w)$.

In particular, $\{x\}_n$ is well defined for any $x \notin V_*$.

We assume that a harmonic structure, as defined in [12], is fixed on the p.c.f. self-similar structure. This will give rise to a self-similar Dirichlet (resistance, energy) form

$$\varepsilon(f) = \sum_{i=1}^{m} p_i \varepsilon(f, f) = \sum_{w \in Wn} p_w \varepsilon(f_w, f_w). \tag{10}$$

Here $p_w = p_{w_1}, \dots, p_{w_n}$ where $p = (p_1, \dots, p_m)$ are the energy renormalization factors. The energy renormalization factors, or weights, are often called conductance scaling factors because of the relation of resistance forms and electrical networks. They are reciprocals of the resistance scaling factors $r_j = 1/p_j$. We will always assume that the resistance form is regular, i.e. $p_j > 1$, j=1, …, m.

The domain, Dom ε, of ε consists of continuous functions such the energy, $\varepsilon(f) = \varepsilon(f, f) < \infty$.

A function on F is harmonic if it minimizes the energy for the given set of boundary values.

Harmonic functions are uniquely defined by their restrictions to V_0 and we often, for convenience, identify the space of harmonic functions with the N_0-dimensional space $l(V_0)$ of functions on V_0.

The restrictions of a harmonic function to cells of level 1 give rise to linear mappings A_i, i= 1, …, m on $l(V_0)$ through $A_i h = h_i \circ \psi_i$. The restrictions to smaller cells are given by products of these matrices since $h_w = h \circ \psi_w = A_w h$, where $A_w = A_{w_n} \dots A_{w_1}$ for $w \in W_n$.

Constant functions are harmonic so constant functions on $l(V_0)$ will be eigenvectors of all the mappings A_i, i= 1, …, m with the corresponding eigen value equal to 1. To study the local behavior of harmonic functions it is therefore useful to factor out the constant functions. Denote by $\mathcal{H}$ the space of harmonic functions such that $\sum_{q\in V_0} h(q)$ =0 and define operators A'_I, i= 1, …, m on $\mathcal{H}$ by A'_i= $P_{\mathcal{H}}A_iP\,P_{\mathcal{H}}^*$, where $P_{\mathcal{H}}$ is the projection of $l(V_0)$ onto $\mathcal{H}$ given by $P_{\mathcal{H}}$h= h-$\sum_{q\epsilon V0} h(q)$. Note that each A_j commutes with $P_{\mathcal{H}}$.

We will from now on assume that the matrices A_i are invertible, which implies that A'_I are invertible. This is an underlying assumption in the theory of product of random matrices that we will use. It is equivalent to that the restriction of a nonconstant harmonic function to any cell is itself nonconstant. Harmonic structures with this property are called nondegenerate. To see what the local behavior of harmonic functions on a degenerate harmonic structure might be like, there is an interesting study in [267] on the case of the hexagasket.

For any function f defined on Fwe will denote by Hf the unique harmonic function that coincides with f on the boundary.

Given a finite nonatomic measure μ on F with the property that μ(O) >0 for any nonempty open set O there is a Laplacian Δ_μ, that is an unbounded operator defined on a dense set of continuous functions by

$$\varepsilon(u, v)= - \int_F u\Delta_\mu v d\mu \qquad (11)$$

for any $\mu \in$ Dom ε with $u|v_0$= 0. In this section we will always assume that $\Delta_\mu v \in L^\infty(F)$. Functions with this property is denoted Dom $_{L\infty}\Delta\mu$ but we will in what follows omit the index L^∞. We will also always assume that μ is self-similar, i.e. that there are real numbers μ_i , i= 1, …, m such that $\mu(F_w)$=1.

Harmonic functions are exactly those for which $\Delta_\mu h$= 0. It should be noted that even though the Laplacian depends on the measure μ, the set of harmonic functions only depend on the harmonic structure.

There is a Green's operator

$$Gu(x)= \int_F g(x, y)u(y)du(y) \qquad (12)$$

acting on $L^\infty(F)$ such that -ΔGu= u, and $Gu|V_0$= 0. Thus, any function $f\in$ Dom$\Delta_\mu$can be written f = Hf – Gu. The Green's function g(x, y) is continuous for regular harmonic structures.

We next define some regularity classes of functions on F.

Definition(5.2.1)[262]. We say that $f \in C^K(\mathcal{H})$ if there are harmonic functions $h_1, \ldots, h_l \in \mathcal{H}$ and $u \in C^K(R^l)$ such that $f = u(h_1, \ldots, h_l)$. We say that $f \in C^K(\text{Dom}\,\Delta_\mu)$, if there are $g_1, \ldots, g_l \in \text{Dom}\,\Delta_\mu$ and $u \in C^K(R^l)$ such that $f = u(g_1, \ldots, g_i)$.

Note that whereas $C^K(\text{Dom}\,\Delta_\mu)$ and $C^K(\mathcal{H})$ are multiplication domains, in general $\text{Dom}\,\Delta_\mu$ is not by [264, 232, 233] . Also note that by definition $C^K(\mathcal{H}) \cup \text{Dom}\,\Delta_\mu \subset C^K(\text{Dom}\,\Delta_\mu)$.

There are several approaches to define derivatives on a p.c.f. fractal F. A weak gradient was studied by KusuoKa in [247, 248]. A stronger notion of gradients and tangents was considered in [256, 260] by Strichartz and the second author. In this section we introduce the following definition.

Definition (5.2.2)[262]. Let f and h be real valued functions on a p.c.f. fractal F, and suppose h is continuous at $x \in$ F. For S ⊆ F let $\text{Osc}_s h = \sup_{x,y\in S}|h(y) - h(x)|$.

Then we say that f is differentiable with respect to h at a nonjunction point x if there is a real number $\frac{df}{dh}(x)$ such that

$$f(y) = f(x) + \frac{df}{dh}(x)(h(y) - h(x)) + o\,sc_{F_{[x]_n}} h \tag{13}$$

where n is such that $y \in F_{[x]n}$, and at a junction point x if

$$f(y) = f(x) + \frac{df}{dh}(x)(h(y) - h(x)) + o\,sc_{U_n(x)}(h)\ {}_{y\to x}\ , \tag{14}$$

where $U_n(x)$ is a canonical basis of neighborhoods and n is such that $y \in U_n(x)$. Naturally, $\frac{df}{dh}(x)$ is called the derivative of f at x with respect to h.

It is easy to show usual properties of the derivative $\frac{df}{dh}(x)$, such as sum, product, ratio and chain rules. Also if f is differentiable with respect to h at x, then f is continuous at x. For later use we formulate the following version of the chain rule.

Proposition (5.2.3)[262]. Suppose f_j: F → R, j= 1, …, l are differentiable with respect to h at x and that g: $R^i \to R$ is in $C^1(R^i)$. then $g(f_1, \ldots, f_i)$ is differentiable with respect to h at x and

$$\frac{d(g(f1,\ldots,fi))}{dh}(x) = \sum_{j=1}^{i} \frac{\partial g}{\partial f_j} f_1, \ldots, f_i)\, \frac{\partial f_j}{dh}(x). \tag{15}$$

We will only use Definition(5.2.2) for h harmonic. Harmonic functions are the natural choice with respect to which one should differentiate since they are, in a sense, the analogues of linear functions on the interval. In fact, we will only differentiate with respect to h ∈ $\mathcal{H}$ since $\frac{df}{d(h+c)} = \frac{df}{dh}$ for any constant c. The maximum and minimum of a harmonic function is always attained on the boundary and we can therefore replace $osc_{F_{[x]_n}} h_{[x]_n}$ by $\|A'_{[x]_n} h\|$ in(13)

We state the results on products of random matrices that will be used subsequently and we formulate a condition on the harmonic structure that is necessary to apply most of these results. We also state two main assumptions, a weak and a strong, on the self-similar measure. Each of these is precisely the condition, the weak one for the derivative and the strong one for the gradient, that allows one to say that on sufficiently small cells the influence of $Hf_{[x]_n}$dominates the term from the Green's function μ a.e. . This is the basis of essentially all of the results that do not follow directly of the theory on products of random matrices.

We prove that a function $f \in C^1(\mathcal{H})$ is differentiable with respect to arbitrary nonconstant harmonic functions μ. a.e. (see Theorem (5.2.23) Then, according to Definition (5.2.2) the function f behaves as a function of one variable up to smaller order terms. This means, in a sense, that the space F is essentially one dimensional. We then prove, under the weak main assumption, the same result for any function $f \in C^1(\text{Dom } \Delta u)$ in Theorem (5.2.24) We also prove an analog of Fermat's theorem on stationary points and discuss the relationship between our derivative and local derivatives defined at periodic points in [263, 265].

We prove the “geography is destiny” principle for smooth functions on the set where the derivative is different from zero and then use this to prove a result on the local behavior of the eccentricity for functions defined on fractals with three boundary points. The concept of eccentricity was introduced and studied for harmonic functions on the Sierpinski gasket in [267] and were studied for larger classes of functions in [254].

We relate the derivative to the gradient defined in [256, 260] under the strong main assumption. Using this relation and technical results from the theory of products of random matrices we are also able to show geography is destiny on the set where the gradient is different from zero.

Since our aim is to describe the local behavior of functions with certain smoothness properties with that of harmonic functions it is essential to understand their local structure.

If $x \in F$ is a nonjunction point it is contained in a unique sequence of cell F $x_{|n}$, and the local behavior of harmonic functions at x is given by the properties of the products $A'_{[x]_n|}$. The generic local behavior of harmonic functions with respect to a self-similar measure μ will thus be governed by the product of i.i.d. random matrices. We define random matrices.

$$M_n(x)= A'_{[x]_n}$$

on the probability space (F, μ) with the Borel sigma-field. Note that we have

$$\mathbb{P}[M_n= A'_w] = \mu_w,$$

and the random matrices M_n are products of i.i.d. random matrices with a common Bernoulli distribution given by

$$\mathbb{P}[M_1= A'_i]= \mu_i \text{ , } i=1, \dots ,m. \quad (16)$$

In the 60s and 70s a theory of products of random matrices, as a natural generalization of the classical limit theorems to products of i.i.d. invertible matrices, was developed by Furstenberg, Kesten, Guivarch, le page, Raugi, Osseledec et al.

In this section results and concepts from this theory that we will rely upon are summarized. They can all be found in [266], where the reader will find references to the original sources. However, we start by introducing the following notation.

The next Lemma collects some properties of the notion $\emptyset(a^n)$. As the proof is elementary we omit it.

Lemma (5.2.4)[262]. Suppose $C_n= \emptyset(a^n)$ and $d_n= \emptyset(b^n)$. Then the following properties hold.

(i) $1/c_n= \emptyset((1/a)^n)$

(ii) $c_n d_n= \emptyset((ab)^n)$

(iii) $\sum_{n\le N} c_n$ is $\emptyset(a^N)$ if a>1, O(1) if a<1 and $\emptyset(1)$ if a=1.

(iv) $\sum_{n>N} c_n = \emptyset(a^N)$ if a<1.

Moreover, $c_n=\emptyset(a^n)$ if and only if $c_n= o((a+\epsilon)^n= o((a-\epsilon)^n = o(c_n)$ for any $\epsilon>0$ but $C_n= \emptyset(a^n)$ is not equivalent to $C_n= O(a^n)$.

Throughout the rest of this section $Y_n\in Gl(R, d)$, n≥1, will be any sequence of i.i.d. invertible d x d random matrices with common distribution M and $S_n= Y_n \dots Y_1$.We also suppose the support of M is finite since this obviously holds for M_n with distribution given by (16). It should be noted that the results we present do not depend on the particular norms chosen on R^d and $Gl(R, d)$.

Theorem(5.2.5)[262]: [266] Let $a_1(n) \geq a_2(n) \geq \dots \geq a_d(n) > 0$ be the square roots of the eigenvalues of $(Y_n \dots Y_1)^* (Y_n \dots Y_1)$.

Then there are numbers $\alpha_+= \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d = \alpha_- > 0$ such that with probability one

$$a_p(n)= \emptyset(\alpha_p^n), \quad p= 1, \dots, d \tag{17}$$

and moreover

$$\|S_n\|= \|Y_n \dots Y_1\|= \emptyset(\alpha_+^n) \tag{18}$$

Definition(5.2.6)[262]: Let $\alpha_+= \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d = \alpha_- > 0$ be as in Theorem (5.3.5).The numbers $\log \alpha_p$, p=1, …, d are called the Lyapunov exponents associated to Y_n. The upper, respectively lower, Lyapunov exponents are $\log \alpha_+$ respectively $\log \alpha_-$.

It is clear that the Lyapunov exponents of y_n^{-1} are $-\log \alpha_{d-1} \geq \dots \geq -\log \alpha_+$. It should also be remarked that in general some Lyapunov exponents can be $-\infty$, however this possibility is excluded by the assumption that M has finite support.

Our interest lies in $h_{[x]_n}$, i.e. in the long term behavior of $S_n v$, $v\in\mathbb{R}^d$ and to apply the results on products of random matrices it is then necessary to make additional assumption M, i.e. on the matrices A'_i in the fractal setting. We need the following definition, with are[266].

Definition(5.2.7)[262]: A subset S of $Gl(d, R)$ is strongly irreducible if there does not exist a finite family $\{L_1, \dots, L_K\}$ of proper linear subspaces of $\mathbb{R}^d$ such that

$$M(L_1 \cup L_2 \cup \dots \cup L_K)= L_1 \cup L_2 \cup \dots \cup L_K , \tag{19}$$

For any $M \in S$.

Definition(5.2.8)[262] :The index of a subset T of Gl(d,R) is the least integer p such that there exists a sequence M_n in T for which $\|M\|_n^{-1}$ M_n converges to a rank p matrix. T is contracting if its index is one.

Denote by T_M the smallest closed semigroup that contains the support of M.

Theorem(5.2.9)[262]: Suppose T_m is strong irreducible, then for any $v \in R^d$, $v \neq 0$, with probability one

$$\|S_n v\| = \emptyset(\alpha_+^n). \tag{20}$$

Moreover, if T_m also is contracting then the two first Lyapunov exponents are distinct, i.e.,

$$\alpha_+ > \alpha_2. \tag{21}$$

For $\nu \in R^d$, $\nu \neq 0$, denote by ν ' the corresponding element in the real projective space $P(R^d)$, and let δ be the natural angular distance in $P(R^d)$. For $Y \in Gl(R, d)$ let $Y . \bar{v} = \overline{Yv} \in P(\mathbb{R}^d)$.

Theorem(5.2.10)[262]:[266]. Suppose T_M is strongly irreducible and contracting. Then, there is a random direction Z' (depending on S_n), such that for any$\bar{v}, \bar{w} \in \mathbb{P}(\mathbb{R}^d)$

$$S_+^n . \bar{\nu} \rightarrow \bar{Z} , \tag{22}$$

with probability one. If $\bar{\nu}$ is not orthogonal to $\bar{Z}$, then

$$\|S_n \nu\| = \emptyset(\alpha_+^n), \tag{23}$$

And if $\bar{\nu}$ is orthogonal to $\bar{Z}$ then

$$\lim\sup \frac{1}{n} \log \|S_n \nu\| \leq \log \alpha_2 \tag{24}$$

Moreover, for any nonzero $\nu \in R^d$ the probability that ν is orthogonal to $\bar{Z}$ is zero.

The next theorem formulates the contraction property that is the basis for the Geography is destiny principle.

Theorem(5.2.11)[262]:[266].Suppose T_m is strongly irreducible and contracting. Then for any $\bar{\nu}.\bar{w} \in P(R^d)$,

$$\frac{\delta(Sn.\bar{\nu}\ \ S_n.\bar{w})}{\delta(\nu\ ,.\bar{w})} = \emptyset((\alpha_2/\ \alpha_+)^n), \tag{25}$$

With probability one.

In section 6 we will make use of the following.

Theorem(5.2.12)[262]:[266]. Suppose T_M is strongly irreducible and contracting. For any unit vector $v \in R^d$ there is $\alpha > 0$ so that

$$\mathbb{E}(\log \|S_n\ \nu\ \| - n \log \alpha_+)^2 - na \tag{26}$$

Converges to a finite limit. Moreover, there exists $b > 0$ such that for any $\varepsilon > 0$

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}[|\log\|S_n\| - n \log \alpha_+| > n\varepsilon] < - b, \tag{27}$$

where E denotes expectation and P probability.

Definition(5.2.13)[262]: We say that F satisfies the SC- assumption if the semigroup generated by the A_i' , i= 1, … , m is strongly irreducible and contracting.

The index of a set is in general difficult to determine, however in the case of semigroups there is a useful result in [266].Recall that an eigenvalue λ of a matrix M is a simple if Ker $(M- \lambda Id)$ has dimension one and equals Ker $(M- \lambda Id)^2$ and it is dominating if $|\lambda| > |\lambda'|$ for any other eigenvalue λ'.

Proposition(5.2.14)[262]: A semigroup T in G*l*(d, R) which contains a matrix with a simple dominating eigenvalue is contracting.

Suppose a matrix $M \in Gl\ (2, R)$ has two distinct real eigenvalues. A finite union of lines invariant under M consists of either one or both of the eigenspaces, so we have the following.

Proposition(5.2.15)[262]: If the boundary V_0 consists of there points, then F satisfies the SC-assumption if there is some M_v with a simple dominating eigenvalue and there are two matrices M_w , $M_{w'}$ both with two distinct real eigenvalues and no eigenvector in common.

It is readily verified that for instance the standard harmonic structures on the Sierpinski gasket, as noted in [267, 256] and the level 3 Sierpinski gasket satisfies the SC- assumption. In fact, any nondegenerate structure with D_3 symmetry considered in[268] satisfies the SC-assumption satisfies if $\alpha \neq b$ where

$$\begin{pmatrix} 1 & 0 & 0 \\ 1-a-b & a & b \\ 1-a-b & b & a \end{pmatrix} \quad (28)$$

is the matrix corresponding to the restriction to a level 1 cell containing one of the boundary points.

With the SC- assumption one can obtain differentiability results for $C^1(\mathcal{H})$. For the same results on $C^1(\text{Dom}\ \Delta_\mu)$ an additional assumption on the measure μ is needed. we will use another, stronger, assumption on μ to have a.e. existence of the gradient. To this end, we define γ by

$$\log \gamma = \sum_{j=1}^{m} \mu_j \log(r_j\ \mu_j). \quad (29)$$

Then

$$r_{[x]_n} \mu_{[x]_n}{}^n = \emptyset(\gamma^n) \quad (30)$$

for μ a.e. x, essentially because the probability of occurrence of the scaling factor r_j μ_j. One can see that $\log \gamma$ is the analog of the Lyapunov exponent for the Laplacian scaling factor $r_{[x]_n}\ \mu_{[x]_n}$, which in turn is the product of energy and measure scaling factors.

Definition(5.2.16)[262]: We will say that (F, μ) satisfies the weak main assumption respectively the strong main assumption if F satisfies the SC-assumption and

$$\gamma < \alpha_+ . \quad (31)$$

respectively

$$\gamma < \alpha_- . \quad (32)$$

Essentially the weak main assumption says that, μ, a.e. , restrictions of harmonic functions to small cells scale to zero exponentially more slowly than the Laplacian scale, while the strong main assumption says that extensions of harmonic functions from smaller to larger cells scale to infinity exponentially faster than the Laplacian scales.

It is Known that the Sierpinski gasket with the standard harmonic structure and uniform self-similar measure satisfies the weak main assumption. It also holds for the level 3 Sierpinski gasket with the uniform self-similar measure and standard harmonic structure, which is discussed in detail in [256, 258]. In this case $\gamma = 7/90$ and of the six restriction matrices three have determinant $7/15^2$ and three have determinant $8/15^2$. It is Known that if all determinants equal one, then $\alpha_+ > 1$. It follows that for the level 3 Sierpinski gasket $\alpha_+ > \frac{\sqrt{7}}{15} > \gamma$.

It has been shown [270,256] that the Sierpinski gasket with standard harmonic structure and uniform self-similar measure satisfies the inequality,

$$\gamma\alpha_+ < \alpha_-^2 \qquad (33)$$

which is even stronger than (32)

for the standard harmonic structure on the Sierpinski gasket the resistance scaling factors are all 3/5. Sabot showed in [268] that for small perturbations of these factors there is a unique harmonic structure on the Sierpinski gasket, see also [18]. Since the harmonic restriction mappings depend continuously on the resistances, (33) implies that for small enough perturbations of the harmonic structure the Sierpinski gasket, with a self-similar measure not far from being uniform, will still satisfy the strong main assumption.

The following propositions are interpretations of Theorems (5.2.5)-(5.3.10) in terms of analysis on fractals.

Proposition(5.2.17)[262]: For μ, a.e. nonjunction point x,

$$\left\|M_{[x]_n}h\right\| = \emptyset(\alpha_+^n). \qquad (34)$$

Proposition(5.3.18)[262]: Suppose F satisfies the SC-assumption and h $\in \mathcal{H}$, $h \neq 0$. Then $\alpha_+ > \alpha_2$ and

$$\left\|h_{[x]_n}\right\| = \left\|M_{[x]_n}h\right\| = \emptyset(\alpha_+^n), \qquad (35)$$

For μ, a.e. nonjunction point x.

Proposition (5.2.19)[262] ; For μ, a.e. non junction point x there exists a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of condimension one such that

$$\left\|h_{[x]_n}\right\| = \emptyset(\alpha_+^n \alpha_+^n), \tag{36}$$

for h $\notin \mathcal{H}_x^-$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\|M_{[x]_n} h\right\| \leq \alpha_2 , \tag{37}$$

for $h^- \in \mathcal{H}_x^-$. For any non zero h $\in \mathcal{H}$, h $\notin \mathcal{H}_x^-$, μ, a. e. .

The subspace $\mathcal{H}x$ corresponds to the orthogonal complement of Z' in Theorem (5.2.10) we will denote by $\mathcal{H}x$ the orthogonal complement of $\mathcal{H}_x^-$ and by P_x^- and P_x^+ the orthogonal projections onto $\mathcal{H}_x^-$ and $\mathcal{H}_x^+$ respectively. Also denote by h_x^+ h_x^+ and element of $\mathcal{H}_x^+$of norm one. The property in Proposition (5.2.19)is what we will use to prove differentiability so we make the following definition.

Definition(5.2.20) [262]: We say that $x \in$ F is weakly generic if there is a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of co-dimension one such that

$$\left\|M_{[x]_n} h^-\right\| = o\|M_{[x]_n}\|_{n\to\infty} \tag{38}$$

for any $h^- \in \mathcal{H}_x^-$

Proposition(5.2.21)[262] : $x \in$ F is weakly generic if and only if there is a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of co-dimension one such that

$$\left\|M_{[x]_n} h^-\right\| = o M_{[x]_n}\|_{n\to\infty} \tag{39}$$

For any $h^- \in \mathcal{H}_x^-$ and h $\notin \mathcal{H}_x^-$.

Proof. Necessarily $\|M_{[x]_n} h_x^+\| = O\|M_{[x]_n}\|_{n\to\infty}$, since if not $\left\|M_{[x]_n} h\right\| = o(\|M_{[x]_n}]_n\|)$ for any h $\in \mathcal{H}$. The proposition follows immediately since if h $\notin \mathcal{H}_x^-$ then P_x^+ h $\neq$ 0.

Clearly μ. a.e. x is weakly generic if F satisfies the SC-assumption.

Proposition (5.3.22)[262]: If $x \in$ F is weakly generic and f = u(h_1, …, h_i) $\in$ $C^1(\mathcal{H})$ then $\frac{df}{dh}$ exists for any h $\notin \mathcal{H}_x^-$ with

$$\frac{df}{dh} = \sum_{j=1}^{l} \frac{\partial u}{\partial f_j} \frac{df_j}{dh}. \tag{40}$$

If h' ∈ $\mathcal{H}$ then

$$\frac{dh'}{dh} = \frac{<h', h_x^+>}{<h, h_x^+>}, \qquad (41)$$

And in particular h'∈ $\mathcal{H}_x^-$ if and only if $\frac{dh'}{dh_x^+} = 0$.

Proof. Because of Proposition (5.2.3) it is enough to show that $\frac{dh'}{dh}$ exists foe any h' ∈ $\mathcal{H}$. Write h'= $a_x h + h^-$ with $h^- \in \mathcal{H}_x^-$. Then since

$$(h'(y) - h'(x))|F_{[x]_n} = a\, x\, (h(y) - h(x)) + (M_{[x]_n} h^- \psi_{[x]_n}^{-1} y) - M_{[x]_n} h^- (\psi_{[x]_n}^{-1} x)) \quad (42)$$

it is clear from Proposition (5.3.22)that $\frac{dh'}{dh}(x) = a_x = \frac{<h' h_x^+>}{<h, h_x^+>}$ and (41)) follows.

Theorem (5.2.23)[262] Suppose F satisfies the SC-assumption. Then for any nonzero h ∈ $\mathcal{H}$ and any f = u(h$_1$,… , h$_l$) ∈ $C^1(\mathcal{H})$ we have that $\frac{df}{dh}(x)$ exists for μ. a.e. x and is given by (40)

Proof. This follows immediately from Proposition (5.3.20) since μ. a.e. x is weakly generic.

Lemma(5.2.24)[262]: Suppose u ∈L^∞(F) has support in a cell F_w. Then

$$Osc F_{[w]_k}\ Gu \le C(K+1) r_{[w]_k} ||\mu||_\infty , \qquad (43)$$

for k= 0, 1, …, n= |w|.

Proof . It will be enough to show that

$$|Gu(x) - Gu(x_0)| \le C(k+1)\, r_{[w]_k} \mu_w\, ||\mu||_\infty \qquad (44)$$

for $x \in F_{[w]_k}$ and $x_0 \in V_{[w]_k}$, This can be done by using properties of the Green's function

$$g(x, y) = \sum_{v \in \emptyset \cup W*} r_v\, \Psi(\psi_v^{-1}(x), \psi_v^{-1}(y)). \qquad (45)$$

For the exact definition of Ψ, see [240]. We only need that it is continuous and harmonic on 1-cells.

Since we consider points in $F_{[w]_k}$ and u has support in F_w we are only concerned about x and y in $F_{[w]_k}$, For those, $\Psi(\psi_v^{-1}(x), \psi_v^{-1}(y))$= 0 in case |v| ≥ k and $[v]_k \neq [w]_k$, and in case $|v| < k$ and $|w|_{[v]} \neq v$. The properties of Ψ also makes $\Psi(\psi_v^{-1}(x_0), \psi_v^{-1}y)) = 0$ for all $|v| \geq k$. In all

$$|g(x_0, y) - g(x, y)| \leq \sum_{m=0}^{k-1} r_{[w]m} |\Psi(\psi_{[w]_m}^{-1} x_0), \psi_{[w]_m}^{-1} y)) - \Psi(\psi_{[w]_m}^{-1} x), \quad \psi_{[w]_m}^{-1} (y))| + |\sum_{v \in \emptyset \cup W*} r_v r_{[W]k} \Psi(\psi_{vw}^{-1}(x), \psi_{vw}^{-1}(y))|. \quad (46)$$

The difference in the first term is, by the definition of Ψ, bounded by a constant times the difference of the value of 1-harmonic functions at $\psi_{[w]_m}^{-1}(x_0)$the points *and* $\psi_{[w]_m}^{-1}(x)$ .Both points lie in the cell $F_{[w]_{m,k}}$, and the difference is thus bounded by a constant times $r r_{[w]_{m,k}\,[W]m,k}$ since the largest eigenvalue of A'_I is less or equal to r_i , see [240], and the first term is bounded by $CKr[w]_k$. The second term is $r_{[w]_k} g(\psi_{[w]_k}^{-1} x, \psi_{[w]_k}^{-1} y) \leq r_{[w]_k} \|g\|_\infty$ and we conclude that

$$|Gu(x) - Gu(x_0)| \leq \int_F |g(x, y) - g(x_0, y)||u(y)|du(y)$$

$$\leq C(k+1) r_{[w]k} \int_{Fw} |u(y)|\, du(y) \leq C(k+1)\, r_{[w]_k} \mu_w \|u\|_\infty. \quad (47)$$

Lemma(5.2.25)[262] . Suppose F satisfies the SC-assumption. Given any non constant h. h' ∈ $\mathcal{H}$, we have for μ, a.e. $x \in F$ that

$$\sup_{v \in F_{[x]n}} |h'(y) - h'(x) - \frac{dh'}{dh}(x)(h(y) - h(x))| \leq c_{n,\, x} \frac{\|h\|\|h'\|}{|<h, h_x^+>|}, \quad (48)$$

where

$$\lim \sup \frac{1}{n} \log C_{n,x} \leq \log \alpha_2 . \quad (49)$$

Proof. Let x be such that $h \notin \mathcal{H}_x^-$. This holds for μ, a.e. x. Since, in the proof of Proposition (5.2.22) $h- = P_x^- h' - \frac{<h', h_\infty^+>}{<h, h_\infty^+>} P_x^- h$, it follows from[94] that for $y \in f_{[x]_n}$

$$|h'(y) - h'(x) - \frac{dh'}{dh}(h(y) - h(x))| \leq \|M_{[x]_n} h^-\| \quad (50)$$

$$\leq \frac{\|h\|\|h'\|}{|<h, h_x^+>|} \left(\frac{\|M_{[x]_n} P_x^- h'\|}{|<h,, h_x^+>|} + \frac{\|M_{[x]_n} P_x^- h\|}{\|h\|}\right).$$

Now, by Proposition (5.2.19)

$$\mathrm{Lim}_n \sup \frac{1}{n} \log\|M_{[x]_n} h-\| \leq \log \alpha_2 \quad (51)$$

for any h- $\in \mathcal{H}_x^-$. Thus

$$c_{n,x} = 2 \sup_{h_- \in \mathcal{H}_x^-} \frac{||M_{[x]_n} h-||}{||h-||} \tag{52}$$

satisfies (49) and (48) follows from (50)

Theorem(5.2.26) [262]: Suppose (F, μ) satisfies the weak main assumption and h is a nonconstant harmonic function. Then for μ-almost every x the derivative $\frac{df}{dh}(x)$ exists for any function f = u(g_1, …, g_l) ∈ C^1(Dom Δu) and is given by

$$\frac{df}{dh} = \sum_{j=1}^{1} \frac{\partial u}{\partial g_j} \frac{dg_j}{dh} \tag{53}$$

Moreover, there exists C such that if f ∈ Dom Δu, then for u, a.e. x

$$|\frac{df}{dh}| \le |\frac{d(H.f)}{dh}| + c \frac{||\Delta f||\infty}{|<h,h_x^+>} \sum_{n=0}^{\infty} (n+1)^r [x]_n \, \mu_{[x]_n} ||M||_{[x]_n}^- * h_n^+||. \tag{54}$$

Proof. In view of Proposition (5.2.3)it is enough to suppose f ∈ Dom Δμ. It is clear from Theorem (5.2.23) that we can suppose f = G_u. We also assume x ∈ F is weakly generic, $r_{[x]_n}\mu_{[x]_n} = \emptyset(\gamma^n)$ and h$\notin \mathcal{H}_x^-$ with $||M_{[x]_n}h|| = \emptyset(\alpha_+^n)$.

Denote $B_{[X]_n} = F_{[X]_{n-1}} \setminus F_{[X]_n}$ and let $u^{[x]}n$ be the restriction of u to $B_{[X]_n}$ so that

$$f = \sum_{n=1}^{\infty} Gu^{[x]}n. \tag{55}$$

Since $u^{[x]_n} = 0$ on $F_{[X]_n}$, $Gu^{[X]n}$ is harmonic on$F_{[X]_n}$, and thus $\frac{d(Gu^{[x]}n)}{dh}$ exists and our aim is to show that

$$\frac{df}{dh} = \sum_{n=1}^{\infty} \frac{d(Gu^{[x]}n)}{dh} . \tag{56}$$

To prove convergence of the right hand side of (56) we show that

$$|\frac{d(Gu^{[x]}n)}{dh} = \emptyset((\gamma/\alpha_+)^n) \tag{57}$$

Which is enough by Lemma (5.2.4) Let $v^{[x]}n$ be the function in $\mathcal{H}$ that corresponds to $(Gu^{[x]_n})_{[x]_n}$ and note that

$$\frac{d(Gu^{[x]}n)}{dh}(x) = \frac{d(v^{[x]}n)}{d(M_{[x]_n}h)}[x]_n\ (\psi^{-1}_{[x]_n}(x)) = \frac{\langle V_{[x]_n}, h^{+}{}_{-1}{}_{\psi[x]n}(x)\rangle}{M_{[x]_n}, h^{+}_{h^{+}(\psi^{-1}_{[x]_n}(x))}}, \tag{58}$$

Where the last equality follows from (41) According to Lemma(5.2.4)we obtain (57) by showing that the denominator of the right hand side of (58) is $\emptyset(\alpha_{+}^{n})$ and that the absolute value of the numerator is $\emptyset(\gamma^{n})$.

From Theorem (5.2.10)it follows that there is $\tilde{h}\in\mathcal{H}$ such that

$$h_x^{+} = \lim_{n\to\infty}\frac{M^{*}_{[x]_n}\tilde{h}}{||M^{*}_{[x]_n}\tilde{h}||} \tag{59}$$

and

$$h^{+}_{\psi_w(x)} = \lim_{n\to\infty}\frac{M^{*}_{[x]_n}\tilde{h}}{||M^{*}_{[x]_n}\tilde{h}} \tag{60}$$

consequently

$$h^{+}_{\psi^{-1}_{[x]_n}}(x) = \frac{M^{-1*}_{[x]_n}h^{+}_{x}}{||M^{-1*}_{[x]_n}h^{+}_{x}} \tag{61}$$

Note that

$$|| M^{-1*}_{[x]_n}h^{+}_{x}|| = \sup_{||h||=1} < h, M^{-1*}_{[x]_n}h^{+}_{x} >= \sup_{||k||=1}<\frac{M_{[x]_{n,k}\cdot}}{||M[x]n\ K||}, M^{-1*}_{[x]n}\ h^{+}_{x}>$$

$$= \sup_{||k||=1}\frac{< K, h^{+}_{x}>}{||M[x]n\ K||} = \frac{<K, h^{+}_{x}>}{||M[x]n\ K||} \tag{62}$$

for some K $\notin \bar{\mathcal{H}}_{x}$. Since $||M_{[x]_n}|| = \emptyset(\alpha_{+}^{n})$ it then follows by Lemma (5.2.4)that

$$||M||^{-1*}_{[x]_n}h^{+}_{x}|| = \emptyset((1/\alpha_{+})^{n}). \tag{63}$$

and

$$|\langle M_{[x]_n}h, h^{+}_{\psi^{-1}_{[x]_n}}(x)\rangle| = \frac{|<h,h^{+}_{x}>|}{|\left|M^{-1*}_{[x]_n}h^{+}_{x}\right||} = \emptyset(\alpha^{n}_{+}). \tag{64}$$

The numerator has the bound

$$|< v^{[x]_n}, h^{+}_{\psi^{-1}_{[x]_n}(x)} >| \leq C\ Osc(v^{[x]_n}) \leq C(n+1)\ r_{[x]_n}\mu_{[x]_n}\|\mu\|_{\infty} = \emptyset(\gamma^n), \qquad (65)$$

where the last inequality follows from Lemma (5.2.24)and the last equality follows from Lemma(5.2.4) Thus, the right hand side of (56) converges and (44) follows from (64) and (65) as soon as we have shown (56)

For y ∈ $F_{[x]n}$ we must show

$$| Gu(y) - Gu(x) - \sum_{n=1}^{\infty} \frac{d(Gu^{[x]}n)}{dh} (h(y) - h(x))| = 0(\|M_{[x]_k} h\|). \qquad (66)$$

We write

$$| Gu(y) - Gu(x) - \sum_{n=1}^{\infty} \frac{d(Gu^{[x]}n)}{dh} (h(y) - h(x))|$$

$$\leq |\sum_{n=1}^{k} (Gu^{[x]n}(y) - Gu^{[x]n}(x)) - \sum_{n=1}^{k} \frac{d(Gu^{[x]}n)}{dh} (h(y) - h(x))|$$

$$+ | \sum_{n=k+1}^{\infty} (Gu^{[x]}ny) - Gu^{[x]}nx)) |$$

$$+ | \sum_{n=k+1}^{\infty} \frac{d(Gu^{[x]}n)}{dh} (h(y) - h(x)) | \qquad (67)$$

Lemma(5.2.26) and Lemma (5.2.5) implies that the second term is estimated from above by

$$C(k+1)\ r_{[x]_k} = \emptyset(\gamma^k) = o(\|M_{[x]_k} h\|). \qquad (68)$$

The third term is $\emptyset(\gamma^k) = 0(\|M_{[x]_n} h\|$ since $|h(y) - h(x)| = \emptyset(\alpha_{+}^{\mathbf{k}})$ and

$$\sum_{n=k+1}^{\infty} \frac{d(Gu^{[x]}n}{dh} = \emptyset((\gamma/\alpha_{+})^k)$$

By lemma (5.2.5) and (57) Remains the first term which we write

$$| \sum_{n=1}^{k} Gu^{[x]}n(y) - Gu^{[x]}n(x) - \frac{dGu^{[x]}n}{dh} (h(y) - h(x))| \qquad (69)$$

Suppose that we fix a (large) constant M, which is to be chosen later, and that the integers from 1 to k are divided into M subintervals [jK/M, (j+1)k/M]. From the arguments below it is evident that without loss of generality we can assume that k is an integer multiple of M, say k= Mm. So we write the sum in (69) as M sums of m= k/M addends each, and have to show that for each j=1, …, M we have

$$|\sum_{n=m(j-1)+1}^{jm} Gu^{[x]_n}(\mathbf{y}) - {}^{[}Gu^{[x]_n}(\boldsymbol{x}) - \frac{\mathbf{d}(Gu^{[x]_n})}{\mathbf{dh}}(\mathrm{h(y)} - \mathrm{h}(\boldsymbol{x}))| = 0(\|M_{[x]_k} \quad (70)$$

If we denote

$$\mathrm{hj} = \sum_{n=m(j-1)+1}^{jm} Gu^{[x]_n}(\mathbf{y}) \quad . \quad (71)$$

then we have to show

$$|\sum_{n=m(j-1)+1}^{jm} \boldsymbol{h}_{\mathrm{j}}(\mathrm{y}) - h_j(\boldsymbol{x}) - \frac{\mathbf{dh_j}}{\mathbf{dh}}(\mathrm{h(y)} - \mathrm{h}(\boldsymbol{x}))| = \mathrm{o}(\|\boldsymbol{M}_{[x]_k}\boldsymbol{h}\|). \quad (72)$$

Note that h_j is harmonic on $F_{[x]jm}$. By Lemma (5.3.24) we have $\|h_j\|= \emptyset(\gamma^{m(j-1)})$ and Lemma (5.2.25)then implies that the left hand side of (72) is bounded by $\emptyset(\gamma^{m(j-1)}\alpha^{m(M-j)})$. Let $\tilde{\boldsymbol{\alpha}} = \max\{\gamma, \alpha_2\}$ and $\varepsilon= \frac{1}{2}(\alpha_+ - \tilde{\boldsymbol{\alpha}}) > 0$. If we have that

$$M > \frac{\log\gamma}{\log\tilde{\alpha} - \log(\tilde{\alpha} + \varepsilon)} \quad (73)$$

then

$$\gamma^{j-1}\alpha_2^{M-j} \le \tilde{\boldsymbol{\alpha}}^M\gamma^{-1} < (\tilde{\boldsymbol{\alpha}} + \varepsilon)^M = (\alpha_+ - \varepsilon)^M \quad (74)$$

which implies

$$\emptyset(\gamma^{m(j-1)}\alpha_2^{m(M-j)}) = 0((\alpha_+ - \varepsilon)^K)_{k\to\infty} \quad (75)$$

and this completes the proof.

Corollary(5.2.27) [262]: Suppose (F, μ) satisfies the weak main assumption. Then for any nonconstant harmonic function h there exists a set F' of full μ-measure such that if $f= u(g_1, \ldots, g_i) \in C^1(\mathrm{Dom}\,\Delta_\mu)$ has a local maximum at x ∈ F' , then $\frac{\mathbf{df}}{\mathbf{dh}}(x)= 0$.

Proof. Let F'' be the set of full μ-measure such that, according to Theorem **(5.2.25)** the derivative $\frac{\mathbf{df}}{\mathbf{dh}}(x)$ exists for any $f \in C^1(\mathrm{Dom}\,\Delta_\mu))$. There exists w ∈ W* such that the cell F_w does not contain any boundary points. We define F' as the set of all x such that x ∈ F'' and there are infinitely many n such that $[x]_{n,\, n+k}= w$, $|w|= k$. Obviously F' is a set of full μ-measure.

Non-negative harmonic functions satisfy a harmonic inequality [240], on F_w,

$$\max_{y \in F_w} h(y) \leq c \min_{y \in F_w} h(y), \qquad (76)$$

for some c >1. Suppose h is a harmonic function with a zero in F_w. Applying (76) on max_F h – h and h- min_F h gives

$$\max_F h \geq \frac{1}{c-1} Osc_{F_w}(h) \qquad (77)$$

and

$$\min_F h \leq \frac{1}{c-1} Osc_{F_w}(h) \qquad (78)$$

Suppose f ∈ $C^1(\mathrm{Dom}\ \Delta_u)$ has a local maximum at x ∈ F'. Since x ∈ F' we can choose a subsequence n_1 for which $[x]_{ni,\ ni+k} = w$. Then, for l large enough, we have for y ∈ $F_{[X]ni}$ that

$$F(y) - f(x)= \frac{df}{dh}(x)(h(y) - (h(x)) + 0\ (\|M_{[x]_{ni}}h\|) \leq 0 \qquad (79)$$

Using (77) on $h_{[x]_{ni}}$ (y) – h(x) we get

$$\max_{y\in F[x]ni}(h(y)- h(x))= \max_{y\in F}(h_{[x]ni}(y)-h(x)) \geq \frac{1}{c-1} Osc_{F_W}(h_{[x]ni})$$

$$= \frac{1}{c-1} Osc_{F_{w_i}+K}(h) \geq C\| \quad _{M_{W_I}+K}h\| \geq \frac{c}{\|M_w^{-1}\|} \|M_{[x]_{ni}}\ h\|. \qquad (80)$$

So that by (79) we must have $\frac{df}{dh}(x) \leq 0$. In the same way (78) implies

$$miny \in_{F_{[X]_I}} (h(y) - h(x)) \leq -\frac{c}{\|m_w^{-1}\|} \|M_{[x]_j}\ h\|. \qquad (81)$$

which together with (79)implies $\frac{df}{dh}(x) \geq 0$.

For the next theorem recall that a point x ∈ F is called periodic if it is a fixed point of some ψ_w, w ∈ W*.

Theorem(5.2.28)[262] : Let $x = \psi_w(x) \in F$ be a periodic point. Suppose M_w has a dominating eigenvalue λ and the corresponding eigenvector is denoted by h_λ. If $|\lambda| > r_w\mu_w$ then the local derivative $\frac{df}{dh\lambda}(x)$ exists for any f ∈ $C^1(\mathrm{Dom}\ \Delta_\mu)$. In particular, if x is a boundary fixed point then the normal derivative $\partial_N f(x)$ exists for any f ∈ $C^1(\mathrm{Dom}\ \Delta_\mu)$.

Proof. In order to prove this one can adapt the proof of Theorem (5.2.24)defining $B_{w^n} = F_{w^{n-1}} \backslash F_{w^n}$, where $w^n = \underbrace{w \dots w}_{n\ times}$ and use

$$f = \sum_{n=1}^{\infty} Gu^{w^n}. \tag{82}$$

The condition $|\lambda| > r_w \mu_w$ is necessary to have convergence of $\sum_{n=1}^{\infty} \frac{d(Gu^{w^n})}{dh_\lambda}$.

Corollary(5.2.29)[262]: If x is a non-boundary periodic point, the assumptions of Theorem (5.2.28) hold, and $f = u(g_1,\dots, g_l) \in C^1(\mathrm{Dom}\ \Delta_\mu)$ has a local maximum at x, then $\frac{\mathbf{df}}{\mathbf{d}h_\lambda}(x) = 0$.

Proof. The proof is the same as that of Corollary (5.2.27) and uses Theorem (5.2.24) and Theorem (5.2.28).

The result of Theorem (5.2.28) .partially improves in [265] where it was shown in the case of the Sierpinki gasket that $\partial_2 f$ and $\partial_3 f$ exist for any $f \in \mathrm{Dom}\Delta$.

Namely, under the assumption that M_w has two real eigenvalues $\lambda_2 > \lambda_3$, two local derivatives at periodic points of the Sierpinki gasket were defined in [265]. If h_2, $h_3 \in \mathcal{H}$ are any harmonic functions corresponding to these eigenvalues and

$$H f_{[X]_n} = \alpha_{1n} + \alpha_{2n} h_{2,[X]_n} + \alpha_{3n} h_{3,[X]_n} \tag{83}$$

then

$$\partial_2 f(x) = \lim_{n\to\infty} \alpha_{2n} \text{ and } \partial_3 f(x) = \lim_{n\to\infty} \alpha_{3n} \tag{84}$$

If the limit exists. Note that the notation λ_2 for the loading eigenvalue is used in [265] because $\lambda_1=1$ denotes the leading eigenvalue of the matrix A_w.

For arbitrary p.c.f. fractals. Local derivatives $\partial_2, \dots, \partial \partial_{N_0}$ can be defined analogously to (84) at any periodic point $x = \psi_w(x)$ such that M_w has distinct real eigenvalues $|\lambda_2| > \dots > |\lambda_{N_0}|$ with corresponding harmonic functions $h_2, \dots, h_{N_0}$.

Periodic points of this type are weakly generic and $\mathcal{H}_x^-$ is spanned by $h_3, \dots, h_{N_0}$.,, but the rate of decrease for $h \notin \mathcal{H}_x^-$ is $\|M_{[x]_{ni}} h\| = \emptyset(\sigma^n)$ for $\sigma = \lambda_2^{1/|w|}$ instead of $\emptyset(\alpha_+^n)$.

It should be noted that if x= $\psi_i(x)$ is a boundary point then ∂_2 equals, for an appropriate choice of h_2, the normal derivative ∂_N. For the Sierpinki gasket, ∂_3 equals the tangential derivative ∂_T, for an appropriate choice of h_3. For periodic points on the Sierpinki gasket where M_w has two complex conjugate eigenvalues local derivatives ∂^+ and ∂- were defined in [263] using the eigenvectors. It was also shown that there are infinitely many periodic points with this property.Such periodic points are not weakly generic. Actually for any nonconstant h $\in \mathcal{H}$, $\|M_{[x]n}h\|= O((\sqrt{3}/5)^n)$ and h is only differentiable with respect to harmonic functions that are proportional to h. The local behavior at such points is thus truly different from the generic behavior.

In this section we prove the geography is destiny principle for large classes of functions and use it to obtain a result on the pointwise behavior of the principle . It was formulated for the first time in [264] for harmonic functions on the Sierpinski gasket. For harmonic functions it holds under the SC-assumption.

For any h$\in l(V_0)$, h $\neq$0 we define the direction Dirh as the element in the projective space $P(\mathcal{H})$ corresponding to $P_{\mathcal{H}}h$. This definition extends to any function f defined on F, and nonconstant on the boundary, through Dir f = Dir $f|_{V0}$. $P(\mathcal{H})$.

Proposition(5.2.30)[262] Suppose F satisfies the SC-assumption. Then for any nonconstant harmonic functions h_1, $h_2 \in \mathcal{H}$

$$\lim_{n\to\infty} \rho\ (Dirh_1|_{F_{[X]_n}=}, Dirh_2|_{F_{[X]_n}=}\ )= 0 \tag{85}$$

for μ, a.e. x.

Proof. This follows from Theorem(5.2.11)

In fact, the convergence in (85) is even exponential by (25).

If f is differentiabe with respect to h with nonzero derivative at a point x, then the difference in direction of $f_{[X]_n}=$ and $f_{[X]_n}=$ will tend to zero. Note that by definition of the derivative, Dir $f_{[X]_n}=$ exists for n large enough if $\frac{\mathbf{df}}{\mathbf{dh}}(x) \neq 0$.

Proposition (5.2.31)[262]: Suppose $\frac{\mathbf{df}}{\mathbf{dh}}(x)$ exists and is different from zero. Then

$$\lim_{n\to\infty} \rho\ (\mathrm{Dir}\ f_{[x]_n}, \mathrm{Dir}\ h_{[x]_n}) = 0 \tag{86}$$

Proof. This is clear since f(y) – f(x)= c(h(y) – h(x))+o $(\|M_{[x]_n}\ h\|)$implies

$$\rho(\text{Dir } f_{[x]_n}, \text{Dir } h_{[x]_n}) = \rho\ (\text{Dir}(ch_{[x]_n} + 0((\|M_{[x]_n}\ h\|))\text{Dir } h_{[x]_n}) \to 0 \quad (87)$$

The above Proposition together with Theorem (5.2.25) immediately gives the following broad extension of the geography is destiny principle.

Theorem(5.2.32)[262]: Suppose (F, μ) satisfies the weak main assumption and that f ∈ C^1 (Dom Δ_u) and h ∈ $\mathcal{H}$ is a non constant harmonic function. Then

$$\lim_{n\to\infty} \text{p}(\text{Dir } f_{[x]_n}, \text{Dir } h_{[x]_n}) = 0 \quad (88)$$

for u, a.e. x outside the set where $\frac{\mathbf{df}}{\mathbf{dh}}$ (x)=0.

$$\{x : \frac{\mathbf{df}}{\mathbf{dh}}(x) = 0\} \subset \{x : |< \text{Hf}, h_x^+ >| < \text{C}'\varepsilon\} \quad (89)$$

for any f = H f + GΔf with $\|\Delta f\|_\infty < \varepsilon$ and ||h||= 1. Note that

$$\mu\{\text{x} : < \text{H } f, h_x^+ >= 0\} = 0$$

and so informally one can write $\mu\{x : \frac{\mathbf{df}}{\mathbf{dh}}(x) = 0\} \to 0$ as $\|\Delta f\|_\infty \to 0$. This can be restated as follows. Given any H $f \neq 0$ and ε > 0, there is $\delta(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, such that

$$\mu\{x : \frac{\mathbf{df}}{\mathbf{dh}}(x) = 0\} < \delta(\varepsilon)$$

for any f = H f + G Δ f with $\|\Delta f\|_\infty < \varepsilon$ and ||h||= 1.

In [267] the eccentricity e(h) of a nonconstant harmonic function h on the Sierpinski gasket were defined as

$$\text{e(h)} = \frac{\mathbf{h(q_1) - h(q_0)}}{\mathbf{h(q_2) - h(q_0)}}, \quad (90)$$

where q_i, i= 0, 1, 2 are the boundary points labeled so that $h(q_0) \le h(q_2)$.

Note that the eccentricity is the same for harmonic functions corresponding to the same element in $\mathcal{H}$. The concept of eccentricity F and nonconstant on the boundary.

It was shown in [267] that there is a measure on [0,1] such that for any nonconstant harmonic function, the distribution of eccentricities of the restrictions

h_w to cells of a fixed level |w|= n converges in the Wasserstein metric to this measure. This result was extended to functions with Holder continuous Laplacian in [254].

If, instead of the global distribution of local eccentricities, we look at the behavior of the eccentricities on neighborhoods of a point, the geography is destiny principle applies. Since e(-f)= 1-e(f) we define an equivalence relation on [0, 1] by $\boldsymbol{e}$ ~ e' if and only if e= e' or e=1-e'. We denote by e^{-} the equivalence class of e and let $d(e^{-}, e^{-\prime})= \min_{x\sim e,\, x'\sim e'}|x- x'|$ be the natural distance on [0, 1]/~.

Corollary(5.2.33)[262]: If F satisfies the SC-assumption then for any nonconstant harmonic functions h, h'

$$\lim_{n\to\infty} d(\bar{\boldsymbol{e}}(h_{[x]_n}n),\bar{\boldsymbol{e}}_{[x]_n}))= 0, \tag{91}$$

for μ a.e. x. If (F, u) satisfies the weak main assumption then for any f, f' ∈ $C^1(\mathrm{Dom}\ \Delta_\mu)$ and nonconstant h ∈ $\mathcal{H}$ we have

$$\lim_{n\to\infty} d(\bar{\boldsymbol{e}}\ (f_{[x]_n}),\ \bar{\boldsymbol{e}}(f'_{[x]_n}))= 0 \tag{92}$$

for μ, a.e. x outside the set where $\frac{\mathbf{df}}{\mathbf{dh}}$ or $\frac{\mathbf{df'}}{\mathbf{dh}}$ are zero.

Proof. Since $\bar{e}$ depends continuously on the direction these results follow immediately from Theorem(5.2.32).

We clarify the relation between the derivative and the gradient of a function on F defined in [260]. We will restrict attention to cases where (F, μ) satisfies the strong main assumption.

For a nonjunction point x ∈ F, let $\mathrm{Grad}_{[X]_n} = = M_{[x]_n}^{-1} P_{\mathcal{H}} H f_{[x]_n}$. The gradient of f at x is defined as

$$\mathrm{Grad}_x f= \lim_{n\to\infty} \mathrm{Grad}_{[x]_n} f, \tag{93}$$

If the limit exists. In [260] the gradient was defined for sequences w ∈Ω, so at junction points there are several "directional"gradients defined, but for nonjunction points $\mathrm{Grad}_x f$ is defined unambiguously.

Immediately from the definition we have.

Proposition(5.2.34)[262]. If h ∈ $\mathcal{H}$ then $\mathrm{Grad}_x h$ exists for all x and $\mathrm{Grad}_x h$= h.

In [260] the following estimate was proved for any harmonic structure on a, p.c.f. fractal.

$$\left\|Grad_{[x]_{n+1}f}f - Grad_{[x]_nf}\right\| \le C\|\Delta f\|_\infty r_{[x]_n}\mu_{[x]_n}\left\|M^{-1}_{[x]_n}\right\|. \tag{94}$$

It implies the following theorem.

Theorem(5.2.35)[262]. There exists a constant C such that for any $f \in \mathrm{Dom}\Delta$ with $\|\Delta f\|_\infty<\infty$ and any $x \in F\backslash V_*$ with

$$\sum_{n\ge1} r_{[x]_n}\,\mu_{[x]_n}\left\|M^{-1}_{[x]_n}\right\| < \infty, \tag{95}$$

$\mathrm{Grad}_x f$ exists and

$$\|P_{\mathcal{H}}Hf - \mathrm{Grad}_x f\| \le C\|\Delta f\|_\infty \sum_{n\ge1} r_{[x]_n}),\ \mu_{[x]_n}\left\|M^{-1}_{[x]_n}\right\|. \tag{96}$$

Also, for any n> 0

$$\|P_{\mathcal{H}}Hf - \mathrm{Grad}_x f\| \le C\|\Delta f\|_\infty \sum_{k=1}^{n} r_{[x]_n}\ \mu_{[x]_k}\left\|M^{-1}_{[x]_k}\right\|\ . \tag{97}$$

From Theorem (5.2.35)we can immediately deuce the following Lemma.

Lemma (5.2.36)[262] If(F, μ) satisfies the strong main assumption, then for any function $f \in \mathrm{Dom}\,\Delta_\mu$, $\mathrm{Grad}_x f$ exists for μ-almost all $x \in F$.

Proof. The upper Lyapunov exponent of the matrices M^{-1}_j with respect to the measure μ is 1/α – and so the series (95) converges exponentially μ -almost everywhere.

The next Lemma uses the central limit Theorem and large deviations results for products of random matrices. We will use it to show that $\mathrm{Grad}_x f$ is the unique function in $\mathcal{H}$ that best approximates f in neighborhoods of *x*.

Lemma(5.2.37)[262]: Suppose (F, μ) satisfies the strong main assumption. Then for any ε> 0.

$$\sum_{k\ge n} r_{[x]_k}\,\mu_{[x]_k}\left\|M^{-1}_{[x]_{k,n}}\right\| = \sigma((\gamma+\varepsilon)^n)_{n\to\infty} \tag{98}$$

For μ, a.e. x.

Proof. By the Borel-Cantelli Lemma this follows if for any $\delta > 0$

$$\sum_{n=1}^{\infty} \mu\{x: (\gamma+\varepsilon)^{-n} \sum_{K\geq n} r_{[x]_k}\, \mu_{[x]_k} \left\| M^{-1}_{[x]_{k,n}} \right\| > \delta\} < \infty. \tag{99}$$

Since $r_{[x]_n}\mu_{[x]_n} = \emptyset(\gamma^n)$ for μ, a.e. x it is then enough, by Lemma (5.2.5) (i): to show that

$$\sum_{n=1}^{\infty} \mu\left\{x: (\frac{\gamma-\varepsilon/2}{\gamma+\varepsilon})n \sum_{K\geq n} r_{[x]_k}\, \mu_{[x]_k} \left\| M^{-1}_{[x]_{k,n}} \right\| > \delta\right\}$$

$$= \sum_{n=1}^{\infty} \mu\{x: (\frac{\gamma-\varepsilon/2}{\gamma+\varepsilon})^n \sum_{K=1}^{\infty} r_{[x]_k}\, \mu_{[x]_k} \left\| M^{-1}_{[x]_k} \right\| > \delta\} \tag{100}$$

$$= \sum_{n=1}^{\infty} \mu\ \{x: \sum_{K=1}^{\infty} r_{[x]_k} \left\| M^{-1}_{[x]_{k,n}} \right\| > \partial\ (\frac{\gamma+\varepsilon}{\gamma-\varepsilon/2})^n (\frac{1-\beta}{\beta}) \sum_{K=1}^{\infty} \beta^K\} < \infty,$$

where the first equality follows from self-similarity and $1 > \beta > \frac{\gamma}{\alpha-}$ is a fixed number. Thus, it is enough to show that

$$\sum_{n=1}^{\infty} \sum_{K=1}^{\infty} \mu \left\{x: r_{[x]_k}\, \mu_{[x]_k} \left\| M^{-1}_{[x]_k} \right\| > \delta\ (\frac{\gamma+\varepsilon}{\gamma-\varepsilon/2})^n (\frac{1-\beta}{\beta})\, \beta K\right\}$$

$$= \sum_{n=1}^{\infty} \sum_{K=1}^{\infty} \mu \left\{x: \log (r_{[x]_k}\, \mu_{[x]_k} \left\| M^{-1}_{[x]_k} \right\|) - k \log (\frac{\gamma}{\alpha-}) > c_0 + nc_1 + kc_2\right\} < \infty, \tag{101}$$

where $c_1, c_2 > 0$. Assuming $1-\beta > \beta - \frac{\gamma}{\alpha-}$ we have $c_0 + kc_2 > 0$ and the last inner sum can then be estimated from above by

$$\frac{1}{c_1} \int_{BK} b_k(x)\, d\mu(x) \leq \frac{1}{c1} \sqrt{\mu(B_k)}\ ||b_K(x)||_{L^2_\mu} \tag{102}$$

where

$$B_k(x) = \log(r_{[x]_k}\, \mu_{[x]_k} \left\| M^{-1}_{[x]_k} \right\|) - k \log(\frac{\gamma}{\alpha-}) \tag{103}$$

and

$$B_k = \{x: b_k(x) > c_0 + kc_2\}. \tag{104}$$

By Theorem (5.2.12) the L^2_μ- norm of $b_k(x)$ grows polynomially while $\mu(B_k)$ decreases exponentially, which completes the proof.

Theorem(5.2.38)[262]: Suppose (F, μ) satisfies the strong main assumption and f ∈ Dom Δ_μ. Then for any ε > 0 and μ. a.e. x

$$f(y)= f(x)+ \mathrm{Grad}_x f(y) - \mathrm{Grad}_x f(x) + \sigma((\gamma + \varepsilon)^n)_{y\to x}\,, \qquad (105)$$

where y ∈ $F_{[x]_n}$.

Proof. The proof follows the same ideas as the proof of Theorem (5.2.24) but is actually simpler. We assume that f = Gu and let u_n be u multiplied by the indicator function of $F_{[x]_n}$. For y ∈ $F_{[x]_n}$ we have that

$$G(u- u_n)(y)- G(u- u_n)(x)- (\mathrm{Grad}_x\, G(u-u_n)(y) - \mathrm{Grad}_x G(u- u_n)(x))= 0 \qquad (106)$$

since G(u- u_n) is harmonic on $F_{[x]_n}$. Thus, we have to show that, for y ∈ $F_{[x]_n}$,

$$G\, u_n(y)- Gu_n(x) - (\mathrm{Grad}_x Gu_n\,(y)- \mathrm{Grad}_x\, Gu_n(x))= \sigma((\gamma + \varepsilon)^n). \qquad (107)$$

Lemma (5.2.24)implies

$$\|Gu_n\,(y) - Gu_n\,(x)\|L^\infty F_{[x]_n} = \sigma((\gamma + \varepsilon)^n), \qquad (108)$$

and it follows that

$$\|Grad_{[x]_n}\, Gu_n(y) - Grad_{[x]_n}\; Gu_n(x)\|\; L^\infty F_{[x]_n)} = \sigma((\gamma+ \varepsilon)^n) \qquad (109)$$

by the maximum principle applied to the harmonic function $(Grad_{[x]_n}\ (Gu_n)_{[x]_n}$, because its boundary values coincide with those of $(Gu_n)_{[x]_n}$. Hence it suffices to bound

$\|\mathrm{Grad}_{[x]_n}Gu_n(y) - \mathrm{Grad}_{[x]_n}\, Gu_n(x) - (\mathrm{Grad}_{\,x}\; Gu_n(y) - \mathrm{Grad}_{\,x}\; Gu_n(x))\|\; L^\infty F_{[x]_n)} \le$
$2\|\mathrm{Grad}_{[x]_n}Gu_n - \mathrm{Grad}_{\,x}\; Gu_n\|_{L\infty}F_{[x]_n)}$

$$\le 2\sum_{K=n}^{\infty} ||\,\mathrm{Grad}_{[x]_n}\; Gu_n - \mathrm{Grad}_{[x]_{k+1}}Gu_n\|L^\infty(F_{[x]_n})$$

$$= 2\sum_{k=n}^{\infty} ||\,\mathrm{Grad}_{[x]_{n,k}}(Gu n)_{[x]_n} - \mathrm{Grad}_{[x]_{n,k+1}}(\,Gu_n)_{[x]_n}\|L^\infty(F)$$

$$\le C\,\sum_{K=n}^{\infty} ||\Delta(Gu_n)_{[x]n}\|_\infty r_{[\mathbf{x}]_n}\mu_{[\mathbf{x}]_n\,[X]n.k}\,\left\|M^{-1}_{[x]_{n,k}}\right\|)$$

$$\le C\|u\|_\infty\, \sum_{K=n}^{\infty} r_{[\mathbf{x}]_n}\,\mu_{[\mathbf{x}]_n} r_{[\mathbf{x}]_{n,k}}\mu_{[\mathbf{x}]_{n,k}}\,\left\|M^{-1}_{[x]_{n,k}}\right\|= \sigma((\gamma+ \varepsilon)^n),$$

where we used that $(\text{Grad}_{[x]_k}\ \text{Gu}_n)_{[x]_n} = \text{Grad}_{[x]_{n,k}}\ \text{Gu}_n)_{[x]_n}$, the estimate (94) and Lemma (5.2.37).

As an immediate consequence we obtain the following Corollary, which makes it straightforward to prove μ, a.e. differentiability at points where $\text{Grad}_x f$ exists.

Corollary(5.2.39)[262]: Suppose (F, μ) satisfies the strong main assumption and f ∈ Dom Δμ . Then for μ, a.e. x

$$f(y)= f(x) + \text{Grad}_x f(y) - \text{Grad}_x f(x) + \sigma(\|M_{[X]_n} h\|)_{y\to x}, \tag{110}$$

for any nonconstant h ∈ $\mathcal{H}$.

The same result for $\text{Grad}_x f$, or rather the tangent $T_1(f)$, on the Sierpinski gasket was proved in[281] under the stronger assumption (33).

We can now state the relations between the derivative and the gradient.

Proposition(5.2.40)[262]: Suppose (F, μ) satisfies the strong main assumption, f ∈ Dom Δ_μ and h is a nonconstant harmonic function. Then the following assertions hold.

(i) For μ, a.e. x such that $\text{Grad}_x f = 0$, we have that $\frac{\mathbf{df}}{\mathbf{dh}}(x) = 0$.

(ii) For μ, a.e. x such that $\text{Grad}_x f \neq 0$, we have that $\frac{\mathbf{df}}{\mathbf{dGrad_x f}}(x) = 1$.

(iii) For μ. A.e. x

$$\frac{\mathbf{df}}{\mathbf{dh}}(x) = \frac{<\mathbf{Grad_x f}, \boldsymbol{h_x^+}>}{<\boldsymbol{h}, \boldsymbol{h_x^+}>}. \tag{111}$$

In particular for μ, a.e. x we have

$$\frac{\mathbf{df}}{\mathbf{dh_x^+}}(x) = <\text{Grad}_x f, \boldsymbol{h_x^+}>, \tag{112}$$

$$\left|\frac{df}{dh}(x)\right| = \frac{\|\boldsymbol{P_x^+}\ \mathbf{Grad_x f}\|}{\|\boldsymbol{P_x^+}\ \mathbf{h}\|} \tag{113}$$

and $\frac{\mathbf{df}}{\mathbf{dh}}(x) = 0$ if and only if $\text{Grad}_x f \in \mathcal{H}_x^-$.

Proof. The first two statements are obvious Corollary(5.2.39) for the third, we Know h $\notin \mathcal{H}_x^-$ for μ, a.e. x, and in that case

$$F(y) - f(x) = \text{Grad}_x f(y) - \text{Grad}_x f(x) + \sigma(\|M_{[X]_n} h\|)_{y\to x}$$

$$= \frac{<\text{Grad}_x f, h_x^+>}{<h, h_x^+>}(h(y) - h(x)) + \sigma(\|M_{[X]_n} h\|)_{y\to x.} \quad (114)$$

As formulated, Theorem(5.2.32)on geography is destiny, raises the question about where the derivative is different from zero. Our next results relates this to the same question on the gradient.

Lemma(5.2.41)[262]: Suppose (F, μ) satisfies the strong assumption. Then for any ε > 0 there is $\delta(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ such that if

$$\frac{\|\Delta f\|_\infty}{\|P_{\mathcal{H}} H f\|} < \varepsilon, \quad (115)$$

Then

$$\mu\{x: \text{Grad}_x f \in \mathcal{H}_x^-\} < \delta(\varepsilon). \quad (116)$$

In particular, $\mu\{x: \text{Grad}_x f \neq 0\} > 1 - \delta(\varepsilon)$.

Proof. For simplicity assume $\|P_{\mathcal{H}} Hf\| = 1$ and $\|\Delta f\|_\infty < \varepsilon < \frac{1}{4}$. Define

$$F_\varepsilon = \{x: C \sum_{n\geq 1} r_{[x]_n} \mu_{[x]_n} \|M_{[x]_n}^{-1}\| =< \varepsilon^{-\frac{1}{2}}\}, \quad (117)$$

where C is the constant in the estimate (93). Note that $\lim_{\varepsilon\to 0} \mu(F_\varepsilon) = 1$ by the strong main assumption. From (96) we have for any $x \in F_\varepsilon$ that

$$\|P_{\mathcal{H}} H f - \text{Grad}_x f\| \leq \sqrt{\varepsilon}, \quad (118)$$

So $\text{Grad}_x f \neq 0$ and

$$\rho(\text{Dir } P_{\mathcal{H}} H f, \text{Dir Grad}_x f) < 2\sqrt{\varepsilon} \quad (119)$$

for all $x \in F_\varepsilon$. Let $V \subset P(\mathcal{H})$ be the set of directions orthogonal to $P_{\mathcal{H}} H f$, and let $V_\varepsilon = \{v_0 \in P(\mathcal{H}): \inf_{v\in V} \rho(v_0, v) < \varepsilon\}$. If $x \in F_\varepsilon$ and $\text{Grad}_x f \in \mathcal{H}_x^-$ then by (118) we see that $\rho(\text{Dir } h_x^+, v) < 2\sqrt{\varepsilon}$ for all $v \in V$. It follows that

$$\mu\{x: \text{Grad}_x f \in \mathcal{H}_x^-\} \leq \mu\{x \in F_\varepsilon: \text{Grad}_x f \in \mathcal{H}_x^-\} + 1 - \mu(F_\varepsilon)$$

$$\leq \mu\{x: \text{Dir } h_x^+ \in V_{2\sqrt{\varepsilon}}\} + 1- \mu(F_\varepsilon) \tag{120}$$

$$= v\,(V_{2\sqrt{\varepsilon}}) + 1 - \mu(F_\varepsilon)$$

where the measure v is a μ-invariant measure on $P(\mathcal{H})$, which means that

$$v(A)= \sum_{i=1}^{m} \int_{p(\mathcal{H})} 1A(\text{Dir}(A_i'h))\, dv(\text{Dir } h), \tag{121}$$

for any Borel set A in $P(\mathcal{H})$. A theorem of product of random matrices says that if μ is supported on a strongly irreducible semigroup such measure v has the property that hyperplanes have zero v-measure [266]. Thus $\lim_{\varepsilon\to 0} v(V_{2\sqrt{\varepsilon}}\}\ v(V)= 0$.

Theorem(5.2.42)[262]: If(F, μ) satisfies the strong main assumption, then for any $f \in \text{Dom } \Delta_{\mu,}$

$$\text{Grad}_x f \notin \mathcal{H}_x^-\} \tag{122}$$

for μ, a.e. x with $\text{Grad}_x f \neq 0$.

Proof. For simplicity assume $\|\Delta f\|_\infty < 1$. Define

$$F_\varepsilon= \{x: \|\text{Grad}_x f\| > \varepsilon\} \tag{123}$$

and

$$F_{n,e}= \{x: \|grad_{[X]_n} f\| > \varepsilon \text{ and } r_{[\mathbf{x}]_n}\mu_{[\mathbf{x}]_n}\left\|M_{[\mathbf{x}]_\mathbf{n}}^{-1}\right\| < \varepsilon^2\}. \tag{124}$$

Clearly

$$\lim_{n\to\infty} \mu(F_\varepsilon \setminus F_{n.\varepsilon})= 0 \tag{125}$$

and

$$\lim_{\varepsilon\to 0} \mu(F_0 \setminus F_\varepsilon)= 0 \tag{126}$$

Then for any $x \in F_{n,\varepsilon}$ we have

$$\frac{\|\Delta f_{[\mathrm{x}]_\mathrm{n}}\|_\infty}{\|\mathbf{P}_{\mathcal{H}}\mathbf{H} f_{[\mathrm{x}]_\mathrm{n}}\|} = \frac{\left\|M_{[\mathrm{x}]_\mathbf{n}}^{-1}\right\| \|\Delta f_{[\mathrm{x}]_\mathrm{n}}\|_\infty}{\left\|M_{[\mathrm{x}]_\mathbf{n}}^{-1}\right\| \|M_{[\mathrm{x}]_\mathrm{n}}\, \text{Grad}_{[\mathrm{x}]_\mathrm{n}} f\|} \leq \frac{r_{[\mathrm{x}]_\mathrm{n}}\mu_{[\mathrm{x}]_\mathrm{n}} \left\|M_{[\mathrm{x}]_\mathbf{n}}^{-1}\right\|}{\|\text{Grad}_{[\mathrm{x}]_\mathrm{n}} f\|} < \varepsilon. \tag{127}$$

Here we can use Lemma (5.2.41) for each $f_{[\mathbf{x}]_\mathrm{n}}$ together with

$$\text{Grad}_{\mathrm{x}[\mathbf{x}]_\mathrm{n}} f = M_{[\mathbf{x}]_\mathrm{n}}\, Grad_{\psi_{[\mathrm{x}]_\mathrm{n}}(\mathrm{x})} f$$

and $M^{-1}_{[x]_n}\mathcal{H}^-_x\ \mathcal{H}^-_{\psi_{[x]_n}(x)}$, to obtain that

$$\delta(\varepsilon) > \mu\{x\text{: } \mathrm{Grad}_{x[x]_n} f \in \mathcal{H}^-_x\}$$

$$= \mu\{x\text{: } M_{[X]n}\ \mathrm{Grad}\psi_{[x]_n}(x)\ f \in \mathcal{H}^-_x\}$$

$$= \mu\{x\text{: } \mathrm{Grad}\psi_{[x]_n}(x)\ f \in M^{-1}_{[x]_n}\mathcal{H}^-_x\}$$

$$= \mu\{x\text{: } \mathrm{Grad}\psi_{[x]_n}(x)\ f \in \mathcal{H}^-_{\psi_{[x]_n}(x)},$$

$$= \mu_w^{-1}\ \mu\{y \in F_w\text{: } \mathrm{Grad}_y\ f \in \mathcal{H}^-_y\}. \tag{128}$$

Therefore,

$$\mu\ \{x \in F_{n.e}\text{: } \mathrm{Grad}_x f \in \mathcal{H}^-_x\} \tag{129}$$

$$= \sum \boldsymbol{\mu}\{\ x \in F_w\text{: } \mathrm{Grad}_x\ f \in \mathcal{H}^-_x\} < \sum \boldsymbol{\mu}_w \delta(\varepsilon) = \mu(F_{n,\ \varepsilon})\delta(\varepsilon),$$

where the sum is over all $w \in W_n$ such that $F_w \subset F_{n,\varepsilon}$. Thus,

$$M\{x \in F_\varepsilon\text{: } \mathrm{Grad}_x\ f \in \mathcal{H}^-_x\} < \limsup \mu(F_\varepsilon \backslash F_{n,\varepsilon}) + \mu(F_{n,\varepsilon})\partial(\varepsilon) < \partial(\varepsilon)\ \ \backslash \tag{130}$$

and

$$\mu\{x \in F_0\text{: } \mathrm{Grad}_x\ f \in \mathcal{H}^-_x\} = 0. \tag{131}$$

We can now formulate geography is destiny with conditions on the gradient.

Corollary(5.2.43)[262]: Suppose (F, μ) satisfies the strong main assumption, $f \in \mathrm{Dom}\ \Delta_\mu$ and h is a nonconstant harmonic function. Then

$$\lim_{n\to\infty} \rho(\mathrm{Dir}\ f_{[X]_n}, \mathrm{Dir}\ h_{[X]_n}) = 0 \tag{132}$$

for μ, a.e. x where $\mathrm{Grad}_x\ f \neq 0$

Proof . Theorem (5.2.42) Proposition (5.2.40) and Theorem (5.2.32)

The next corollary is one more analog of Fermat's Theorem.

Corollary(5.2.44)[262]. Suppose (F, μ) satisfies the strong main assumption. Then there exists a set F' of full μ-measure such that if $f = u(g_1, \ldots, g_l) \in C^1(\mathrm{Dom}\ \Delta_\mu)$ has a local maximum at $x \in F'$, then $\mathrm{Grad}_x f = 0$.

Proof. The proof is the same as that of Corollary (5.2.27) and uses Theorem (5.2.38).

Similarly to Corollary (5.2.29) we can obtain an analogous corollary for nonboundary periodic points under the assumption $r_w\ \mu_w\|M_w^{-1}\| < 1$. The existence of the gradient in such a case is guaranteed by Theorem (5.2.35).

Chapter 6

Composition Operator and Norm of the Hilbert Matrix

We find an upper bound for the norm of the induced operators. We compute the exact value of the norm of the Hilbert matrix. Using a new technique, we determine the norm of the Hilbert matrix on a wide range of Bergman spaces.

Sec (6-1) The Hilbert Matrix and Composition Operator

The classical Hilbert inequality

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{k^a}{n+k+1}\right|^p\right)^{\frac{1}{p}} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=0}^{\infty}|ak|^p\right)^{\frac{1}{p}} \qquad (1)$$

is valid for sequences a=$\{a_n\}$ in the sequence spaces L^p for $1 < p < \infty$, and the constant $\pi/sin(\pi/p)$ is best possible[275] Thus the Hilbert matrix

$$H= \frac{1}{i+j+1} \qquad i,j=1,2,...$$

acting by multiplication on sequences induces a bounded linear operator

$$\mathcal{H}_a = b \qquad b=\sum_{k=0}^{\infty}\frac{k^a}{n+k+1}$$

on the L^p space with norm$\|H\|_{l^p \to l^p} = \pi/sin(\pi/p)$ for $1 < p < \infty$.

The Hilbert matrix also induces an operator $\mathcal{H}$ on Hardy spaces H^P as explained below ,by its action on Taylor coefficients. In this article we prove an analogue of the inequality(1) on hardy space . More precisely we show

Theorem(6.1.1)[271]: (i) $If\, 2 < p \le \infty$ then

$$\|H(f)\|_{H^p} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|H(f)\|_{H^p}$$

for each $f \in H^p$

(ii) if 1< p < 2 then

$$\|H(f)\|_{H^p} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|H(f)\|_{H^p}$$

for each $f \in H^p$ with $f(0)=0$.

The proof will be given and involves an expression of $\mathcal{H}$ in terms of weighted composition operators of which we can estimate the Hardy space norms .

Recall that the Hardy space $H^p, 1 \le p \le \infty$ of the unite disc D is the Banach space of analytic function $f : D \to C$ for which

$$\|f\|_{H^p} = \sup_{r<1}\left(\int_0^{2\pi}\left|f(re^{i\theta})\right|^p \frac{d\theta}{2\pi}\right)^{1/p} < \infty, \qquad (2)$$

for finite p, and $\|f\|_\infty = \sup_{z\in D}|f(z)|$. For $1 \le p \le q \le \infty$ we have $H^1 \supset H^p \supset H^q \supset H^\infty$ and H^p is embedded as a closed subspace in $L^p(T)$, the Lebesgue space on the unit circle ,by identifying H^p with the closure of analytic polynomials in $L^p(T)$. Additional properties of Hardy space can be found in [273].

To study the effect of Hilbert matrix on Hardy space let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belong to H^1 Hardy's inequality says.

$$\sum_{n\ge 0}\frac{|a_n|}{n+1} \le \pi\|f\|_{H^1},$$

and it follow that the power series

$$F(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)$$

has bounded coefficients hence its radius of convergence is ≥ 1. In this way we obtain a well defined analytic function $F = H(f)$ on the disc for each $f \in H^1$. A calculation shows that we can write

$$\mathcal{H}(f)(z) = \int_0^1 f(1)\frac{1}{1-tz}dt. \qquad (3)$$

where the convergence of the integral is guaranteed by the Fejer –Riesz inequality [273]and the fact that $1/(1-tz)$ is bounded in t for each $z \in \mathbb{D}$.

The correspondence $f \to H(f)$ is clearly linear and we consider the restriction of this mapping to the space H^p for $p \geq 1$. For $p = 2$, the isometric identification of H^2 with L^2 gives.

$$\|\mathcal{H}\|_{H^2 \to H^2 =.} \pi.$$

On the other hand $\mathcal{H}$ is not bounded on the space H^1 and H^∞. For H^∞ this is because the constant function 1 is mapped to

$$\mathcal{H}\,(1)(z) = \frac{1}{z} \log \frac{1}{1-z}$$

which is not a bounded function. For H^1, let $\boldsymbol{\varepsilon > 0}$ and let

$$f_\varepsilon(z) = \frac{1}{(1-z)\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{1+\varepsilon}}$$

a function which belongs to H^1[273]and is positive on $[0,1]$.We assert that the analytic function $\mathcal{H}\,(f_\varepsilon)$ does not belong to H^1 for small values of ε . Indeed using(3) we find

$$\mathcal{H}\,(f_\varepsilon)(z) = \sum_{n=0}^{\infty}\left(\int_0^1 t^n f_\varepsilon(t)dt\right)z^n$$

and if we assume $\mathcal{H}\,(f_\varepsilon) \in H^1$ then Hardy's inequality implies that the quantity

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 t^n f_\varepsilon(t)dt = \int_0^1 f_\varepsilon(t)\sum_{n=0}^{\infty}\frac{t^n}{n+1}dt$$

$$= \int_0^1 f_\varepsilon(t)\left(\frac{1}{t}\log\frac{1}{1-t}\right)dt$$

$$= \int_0^1 \frac{1}{(1-t)\left(\frac{1}{t}\log\frac{1}{1-t}\right)^{\varepsilon}}dt$$

is finite. For $\varepsilon \leq 1$ this is a contradiction .

The operator $\mathcal{H}$ is however bounded on H^p for all $1 < p < \infty$. This is known and a quick way to see this is to view $\mathcal{H}$ as a Hankel operator. In fact $\mathcal{H}$ is a prototype for Hankel operators see[276]. We will not pursue this aspect further expect to note that a Hankel operator is bounded on H^2 if and only if it is bounded on each H^p for $1 < p < \infty$ see[272]. The results of[272] also imply that $\mathcal{H}$ is not bounded on H^1 a fact that we obtained by a direct argument above.

we indicate how $\mathcal{H}$ can be written as an average of certain weighted composition operators.

Every analytic function $\phi : \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator

$$C_\phi : f \to f o \phi$$

on H^p for $1 \le p \le \infty$ see[273]. In addition if $\omega(z)$ is a bounded analytic function then the weighted composition operator.

$$C_{\omega,\phi}(f)(z) = \omega(z) f(\phi(z))$$

is bounded on each H^p More information about these operator can be found in[274]or[277]. We will not need here any of their properties expect from the fact that they are bounded.

The connection of the Hilbert matrix with composition operator comes as follow. For $f \in H^1$ the Fejer – Riesz theorem, which guarantees convergence , along with analyticity shows that the integral in (3) is independent of the path of integration. For $z \in \mathrm{D}$ we can choose the path.

$$\varsigma(t) = \varsigma_z(t) = \frac{1}{(t-1)_z + 1}, \qquad 0 \le t \le 1 \tag{4}$$

i.e.a circular are in D joining 0 to 1. The change of variable in (3)gives

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{(t-1)z+1} \int \left(\frac{t}{(t-1)z+1}\right) dt \tag{5}$$

This expression says that the transformation $\mathcal{H}$ is an average

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f)(z)\,dt$$

of the weighted composition operators

$$T_t(f)(z) = \omega_t(z) f(\phi_t(z)). \tag{6}$$

where

$$\omega_t(z) = \frac{1}{(t-1)z+1} \quad \text{and} \quad \phi_t(z) = \frac{t}{(t-1)z+1}$$

It is easy to see that ϕ_t is a self map of the disc hence $f \to fo\phi_t$ is bounded on H^p, and that for each $\mathbf{0 < t < 1}$, $\omega_t(z)$ is a bounded analytic function. Thus $T_t : H^p \to H^p, 1 \le p \le \infty$, is bounded for $0 < t < 1$.

Proof.

We first obtain estimates for the norms of the weighted composition operator T_t. The estimates are achieved by transferring T_t to operators $\tilde{T}_t$ acting on Hardy spaces of the right half plane, which are isometric to Hardy spaces of the disc. The form of $\tilde{T}_t$ permits estimates of its norm, there by estimate the for the norm of T_t follows.

Lemma(6.1.2)[271] if $p \ge 2$, then.

$$\left\|T_t(f)\right\|_{H^p} \le \frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{p}}} \|f\|_{H^p}. \quad 0 < t < 1. \tag{7}$$

for each $f \in H^p$.

Proof. The Hardy space $H^p(\prod)$ of the right half plane $\prod = \{z : R(z) > 0\}$ consists of analytic function $f : \prod \to C$ such that

$$\|f\|^p{}_{H^p(\prod)} = \sup_{0<x<\infty} \int_{-\infty}^{\infty} \left|f(x+iy)\right|^p dy < \infty \tag{8}$$

These are Banach space for $1 \le p < \infty$.

Let $\mu(z) = 1 + z/1 - z$ be the conformal map of D onto $\prod$ with inverse $\mu^{-1}(z) = 1 - z/1 + z$ and let.

$$V(f)(z) = \frac{4\pi^{1/p}}{(1-z)^{2/p}} g(\mu(z)). \qquad f \in H^p(\prod).$$

It can be checked that this map is a Linear isometry from $H^p(II)$ onto H^1 with inverse given by

$$V^{-1}(g)(z) = \frac{1}{\pi^{1/p}(1+z)^{2/p}} g(\mu^{-1}(z)). \qquad g \in H^p$$

Let $\tilde{T}_t : H^p(\prod) \to H^p(\prod)$ be the operators defined by

$$\tilde{T}_t = V^{-1} T_t V$$

and suppose $h \in H^p(\prod)$. A calculation shows that $\tilde{T}_t$ are weighted composition operators given by

$$\tilde{T}_t(h)(z) = \frac{1}{(1+t)^{2/p}} \left(\frac{1}{(t-1)\mu^{-1}(z)+1} \right)^{1-\frac{2}{p}} h(\Phi_t(z)), \quad 0 < t < 1. \qquad (9)$$

where

$$\Phi_t(z) = \mu o \phi_t o \mu^{-1}(z) = \frac{t}{1-t} z + \frac{1}{1-t}$$

is an analytic function mapping $\prod$ into itself. By an elementary argument we see that if $z \in \prod$ then $|(t-1)\mu^{-1}(z)+1| \geq t$ and since $1 - 2/p \geq 0$ we have

$$\left|\tilde{T}_t(h)(z)\right| \leq \frac{t^{\frac{2}{p}-1}}{(1-t)\frac{2}{p}} |h(\Phi_t(z))|.$$

Integrating for the norm we have

$$\left\|\tilde{T}_t(h)\right\|_{H^p(H)} = \sup_{0<x<\infty} \left(\int_{-\infty}^{\infty} \left|\tilde{T}_t(h)(z)\right|^p dy \right)^{1/p}$$

$$\leq \frac{t^{\frac{2}{p}-1}}{1-t^{\frac{2}{p}}} \sup_{0<x<\infty} \left(\int_{-\infty}^{\infty} \left| h\left(\frac{t}{1-t}(x+iy) + \frac{1}{1-t} \right) \right|^p dy \right)^{1/p}$$

$$=\frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}}\sup_{1/(1-t)<X<\infty}\left(\int_{-\infty}^{\infty}\left|h(X+iY)\right|^{p}\frac{1-t}{t}dy\right)^{1/p}$$

Where we have changed the variables $X=\frac{1}{1-t}x+\frac{1}{1-t}$ and $Y=\frac{t}{1-t}y$,to obtain

$$\leq\frac{t^{\frac{1}{p}}}{(1-t)^{\frac{1}{p}}}\sup_{0<X<\infty}\left(\int_{-\infty}^{\infty}\left|h(X+iY)\right|^{p}dy\right)^{1/p}$$

$$=\frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{2}}}\left\|h\right\|_{H^{p}(H)}$$

The conclusion follows.

For the final step of the proof we will need some classical identities about the Gamma and Beta functions, see for example[278]. The Beta function is defined by

$$B(s,t)=\int_{0}^{1}x^{s-1}(1-x)^{t-1}dx$$

for each s,t with $R(s)>0$, $R(t)>0$. The value $B(s,t)$ can be expressed in terms of the Gamma function as $B(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+1)}$. We are also going to the functional equation for the Gamma function

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}.$$

which is valid for non-integer complex z

Now suppose $p\geq 2$ and $f\in H^{p}$ with $\left\|f\right\|_{H^{p}}=1$. Then

$$\left\|H(f)\right\|_{H^{p}}=\sup_{r<1}\left(\int_{0}^{2\pi}\left|H(f)(re^{i\theta})\right|^{p}\frac{d\theta}{2\pi}\right)^{1/p}.$$

$$=\sup_{r<1}\left(\int_{0}^{2\pi}\left|\int_{0}^{1}T_{t}(f)(re^{i\theta})dt\right|^{p}\frac{d\theta}{2\pi}\right)^{1/p}$$

$$\leq \int_0^1 \sup_{r<1}\left(\int_0^{2\pi}\left|T_t(f)(re^{\theta i})\right|^p \frac{d\theta}{2\pi}\right)^{1/p} dt$$

(by the continuous version of Minkowski's inequality)

$$= \int_0^1 \|T_t(f)\|_{H^p} dt \quad \leq \int_0^1 t^{1/p-1}(1-t)^{-1/p} dt$$

$$= B\left(\frac{1}{p}, 1-\frac{1}{p}\right) = \Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$$

and this give the assertion for $p \geq 2$.

Suppose now $1 < p < 2$ and $f \in H^p$ with $f(0) = 0$. Then $f(z) = zf_0(z)$ with $\|f_1\|_{H^p} = \|f_0\|_{H^p}$. Writing $\mathcal{H}$ in the integral from (5) we see that

$$H(f)(z) = \int_0^1 T_t(f0)(z)dt$$

where T_t are the weighted composition operators

$$T_t(g)(z) = \frac{t}{((t-1)z+1)^2} g\left(\frac{g}{(t-1)z+1}\right)$$

We now follow the proof (with same notation) of Lemma (6.2.2) to estimate the norms of T_t letting $\tilde{T}_t = V^{-1}T_tV : H^p(\prod) \to H^p(\prod)$ we find

$$\tilde{T}_t(h)(z) = \frac{t}{(1-t)^{\frac{2}{p}}}\left(\frac{1}{(t-1)\mu^{-1}(z)+1}\right)^{2-\frac{2}{p}} h(\Phi_t(z)) \quad 0 < t < 1 \tag{10}$$

for each $h \in H^p(\prod)$ .Because $2 - \dfrac{2}{p} > 0$ for $p > 1$, the rest of the calculation in Lemma(6.1.2) goes through and we conclude

$$\|T_t(g)\|_{H^p} \le \frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{p}}}\|g\|H^p \qquad 0<t<1,$$

for each $g \in H^p(\prod)$. Using this norm estimate we can repeat the final step of the proof of the case $p \ge 2$ to obtain

$$\|H(f)\|_{H^p} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|f_0\|_{H^p} = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|f\|_{H^p}$$

and this finishes the proof of the theorem .

Sec(6.2) Bergman Spaces and Hilbert Matrix

The Hilbert matrix H with entries $a_{i,j} = \frac{1}{i+j+1}$ for i and j positive integers induces an operator by multiplication on sequences.

$$H:(a_n)_{n\ge 0} \to \left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)_{n\ge 0}$$

For $1 < p < \infty$, Hilbert's inequality[275]

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{\frac{1}{p}} \le \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{n=0}^{\infty}|a_n|^p\right)^{\frac{1}{P}} \tag{11}$$

implies that H induces a bounded operator l^P spaces of P-summable sequences. Moreover, the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is best-possible and the norm of His

$$\|H\|_{l^p \to l^p} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \qquad 1 < p < \infty$$

The Hilbert matrix also induces a transformation $\mathcal{H}$ on spaces of analytic functions by its action on Taylor coefficients defined by

$$\mathcal{H}: : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_k}{n+k+1} z^n$$

for those analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which the coefficient

$$A_n = \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \quad , \qquad n = 0,1,\ldots$$

Converge.

The operatorHhas been studied on Hardy spaces. [271] proved that H is a bounded operator on the Hardy spaces H^p, p > 1 , and for $1 < p \leq \infty$ we found the following upper bound for its norm see:

$$\|\mathcal{H}\|_{\mathbf{H^p}\to\mathbf{H^p}} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \tag{12}$$

where $dm(z) = (1/\pi)dxdy$ is the normalized Lebesgue measure on unite disc. We also.We also proved that for function s such that f(0)=0 the same estimate holds for 1< p < 2.

In this article we prove that $\mathcal{H}$ is a bounded operator on the Bergman spaces $A^p, 2 < p < +\infty$, of analytic function f on the disc for which

$$\|f\|^p_{A^p} = \int_D |f(z)|^p dm(z) < +\infty$$

disc We also provide norm estimates on those spaces . More precisely we show:

Theorem(6.2.1)[279]: The operator H is bounded on Bergman spaces $A^P, 2 < p < +\infty$, and satisfies :

(*i*) *If* $4 \leq p < \infty$ and $f \in A^p$, *then*

$$\|H(f)\|_{A^p} \leq \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}.$$

(*ii*) *If* $4 \leq p < \infty$ and $f \in A^p$, *then*

$$\|H(f)\|_{A^p} \leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p} \right)^{1/p} \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}$$

(iii) *If* $2 < p < 4$ and $f \in A^p$ with $f(0) = 0$ then

$$\|H(f)\|_{A^p} \leq \left(\frac{p}{2}+1\right)^{1/p} \frac{\pi}{\sin(2\pi/p)}\|f\|_{A^p}$$

The proof of this result will given, involves the representation, of $\mathcal{H}$ used in[271] to prove(12), in terms of weighted composition operators for which we can estimate the Bergman space norms. It uses a representation similar to one developed by A . G. Siskakis to prove that the Cesaro operator is bounded on the Hardy and Bergman spaces' respectively[285],[286] p. Galanopoulos [281]exploited the same representation to prove that the Cesaro operator is bounded on Dirichlet spaces.

We consider the operator

$$S(f)(z) = \int_0^1 \frac{f(t)}{1-tz}\,dt \tag{13}$$

This operator is well defined on Bergman spaces. Indeed, using,[287], we have

$$|f(z)| \leq \left(\frac{1}{1-|z|^2}\right)^{2/p} \|f\|_{A^p} \tag{14}$$

for $p > 2$ and $f \in A^p$ and hence

$$|s(f)(z)| \leq \frac{\int_0^1 \frac{1}{(1-t)^{2/p}}\,dt}{1-|z|}\|f\|_{A^p} < +\infty$$

Now given $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in A^p let $f_N(z) = \sum_{n=0}^{N} a_n z^n$. We see that

$$H(f_N)(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{N} \frac{a_k}{n+k+1} z^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{N} \int_0^1 t^{n+k} dt a_K z^n$$

$$= \sum_{n=0}^{\infty} f_N(t)(tz)^n dt$$

$$= S(f_N)(z) .$$

so $\mathcal{H}$ well defined on polynomials. Also, for $z \in D$ and $p > 2$ we see that

$$\left| S(f)(z) - \sum_{n=0}^{\infty} \sum_{k=0}^{N} \frac{a^k}{n+k+1} z^n \right| \le \frac{\int_0^1 |f(t) - f_N(t)| dt}{1-|z|}$$

$$\le \frac{\int_0^1 \frac{1}{(1-t)^{2/p}} dt}{1-|z|} \|f - f_N\|_{A^p}$$

Thus, as $N \to \infty$, the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{N} \frac{a_k}{n+k+1} z^n$$

converge and defines an analytic function

$$\mathcal{H}(f)(z) = S(f)(z) = \int_0^1 \frac{f(t)}{1-tz} dt$$

which is in the Bergman spaces $A^p, p > 2$.

We derive the expression of $\mathcal{H}$ in terms of weighted, composition operators mentioned above. Also , we prove that $\mathcal{H}$ is bounded on Bergman spaces A^P for P > 2and we give norm estimate Finally using the natural isometric isomorphism between A^2 and Dirichlet space D, we prove that $\mathcal{H}$ is not bounded on A^2.

We show how Hcan be written as an average of certain weighted composition operators.

Every analytic function $\phi : D \to D$ induces abounded composition operator $C_\phi : f \to f \circ \phi$ on A^p for $1 \le p \le +\infty$; the norm of this operator satisfies [244].

$$\|C_\phi\|_{A^p} \le \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)^{2/p}. \tag{15}$$

In addition, if $\omega(z)$ is a bounded analytic function , then the weighted composition operator

$$C_{\omega.\phi}(f)(z) = \omega(z) f(\phi(z))$$

is bounded on each A^p. This is the property of this operator that we will use.

The connection between the Hilbert matrix and composition operators arises as follows .For $z \in$ D and $0 < r < 1$ we define

$$C_r(f)(z) = \int_0^r f(t) \frac{1}{1-tz} dt \tag{16}$$

and we see that

$$H(f)(z) = \lim_{r \to 1} C_r(f)(z).$$

Given $z \in \mathbb{D}$ we choose the path of integration

$$t(s) = t_z(s) = \frac{rs}{r(s-1)z+1} \qquad 0 \le s \le 1$$

and changing variables in(16) we obtain

$$C_r(f)(z) = \int_0^r f(t) \frac{1}{1-ts} dt$$

$$= \int_0^1 f(t(s)) \frac{1}{1-t(s)z} t'(s)ds$$

$$= \int_0^1 \frac{r}{r(z-1)z+1} f\left(\frac{rs}{r(s-1)z+1}\right) ds$$

Now let $f \in A^p, \ \ p > 2.$ and $z \in D$ and $0 \le s \le 1$ let

$$h_r(s) = \frac{r}{r(s-1)z+1} f\left(\frac{rs}{r(s-1)z+1}\right)$$

$$= \frac{r}{r(s-1)z+1} f(\phi_{r,s}(z))$$

where $\phi_{r,s}(z) = rs/(r(s-1)z+1)$ is an analytic self – map of the unite disc.

Since

$$|r(s-1)z+1| \ge 1-|z| \qquad\qquad 0 \le s, r \le 1,$$

we have

$$\frac{r}{|r(s-1)z+1|} \le \frac{1}{1-|z|} \le \frac{2}{1-|z|^2}.$$

By (14)we have

$$|fo\phi_{r,s}(z)| \le \left(\frac{1}{1-|z|^2}\right)^{2/p} \|fo\phi_{r,s}(z)\|_{A^p}$$

and using(15) we obtain

$$\| fo\phi_{r.s} \|_{A^p} \leq \left(\frac{1+|\phi_{r,s}(0)|}{1-|\phi_{r,s}(0)|} \right)^{2/p} \| f \|_{A^p}$$

$$= \left(\frac{1+rs}{1-rs} \right)^{2/p} \|A\|_{A^p}$$

$$\leq \left(\frac{1+s}{1-s} \right)^{2/p} \| f \|_{A^p}$$

The above estimates give

$$|h_r(s)| \leq \frac{2}{(1-|z|^2)^{1+2/p}} \left(\frac{1+s}{1-s} \right)^{2/p} \| f \|_{A^p}.$$

For p >2 the right – hand side of the latter inequality is an integrable function of s .By Lebesgue's dominated convergence theorem we conclude that

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{(s-1)z+1} f\left(\frac{s}{(s-1)z+1} \right) ds,$$

that is , we can express $\mathcal{H}$ as an integral mean

$$H(f)(z) = \int_0^1 T_t(f)(z)dt$$

of the family of weighted composition operators

$$T_t(f)(z) = \omega_t(z) f(\phi_t(z))$$

where

$$\omega(z) = \frac{1}{(t-1)z+1}$$

and

$$\phi_t(z) = \frac{1}{(t-1)z+1}$$

It is easy to see that ω_t is a bounded function for 0 < t < 1, and that ϕ_t is a self – map of the disc.Thus the operator $T_t : A^p \to A^p, 1 \le p < +\infty$ Bounded on A^p forevery $0 < t < 1$.

We first obtain estimates for the norms of the weighted composition operators T_t

Lemma (6.2.2)[276]. Let $2 < p < +\infty$. Then :

(i) $If\ 4 \le p < +\infty$ and $f \in A^p$, then

$$\|T_t(f)\|_{A^p} \le \frac{t^{2/p-1}}{(1-t)^{2/p}} \|f\|_{A^p}$$

(ii)if 2 < p < +∞and $f \in A^p$, then

$$\|T_t(f)\|_{A^p} \le \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right)^{1/p} \frac{t^{2/p-1}}{(1-t)^{2/p}} \|f\|_{A^p}$$

Proof. We can easily check that

$$\omega_t(z)^2 = \frac{1}{t(1-t)} \phi'_t(\boldsymbol{z})$$

Let $f \in A^p$, $p > \mathbf{2}$. Using the last equation we obtain

$$\|T_t(f)\|_{A^p}^p = \int_D |\omega_t(z)|^p |f(\emptyset_{(t)}(z))|^{2+\epsilon} dm(z)$$

$$= \int_D |\omega_t(z)|^{p-4} |\omega_t(z)|^4 |f(\emptyset_t(z))|^p dm(z)$$

$$= \frac{1}{(t(1-t))^2} \int_D |\omega_t(z)|^{p-4} |f(\emptyset_{(t)}(z))|^2 dm(z)$$

$$= \frac{1}{t(1-t)^2} \int_{\emptyset_t D} |\omega_t\left(\emptyset_t^{-1}(z)\right)|^{p-4} |f(z)|^p dm(z)$$

$$=I.$$

We now consider two cases.

First suppose that. $p \geq 4$. We compute

$$\emptyset_t^{-1}(z) = \frac{\mathbf{z} - t}{(\mathbf{1} - t)\mathbf{z}}$$

and

$$\omega_t\left(\emptyset_t^{-1}(z)\right) = \frac{\mathbf{1}}{(t - \mathbf{1})\emptyset_{\mathbf{t}}^{-\mathbf{1}}(\mathbf{z}) + \mathbf{1}} = \frac{\mathbf{z}}{\mathbf{t}}$$

Hence

$$1 \leq \frac{\|f\|_{A^p}^p}{t^{p-2}(1-t)^2}.$$

Next assume that $2 < p < 4$. then

$$I = \frac{1}{t^2(1-t)^2} \int_{\emptyset_t(D)} |\omega_t\left(\emptyset_t^{-1}(\omega)\right)|^{p-4} |f(\omega)|^p dm$$

$$= \frac{1}{t^2(1-t)^2} \int_{\emptyset_t(D)} \frac{\omega}{t} |^{p-4} |f(\omega)|^p dm(\omega)$$

$$= \frac{1}{t^{p-2}(1-t)^2} \int_{\emptyset_t(D)} |\omega|^{p-4} |f(\omega)|^p dm(\omega)$$

$$\leq \frac{1}{t^{p-2}(1-t)^2} \int_D |\omega|^{p-4} |f(\omega)|^p dm(\omega).$$

The last integral is well defend near the origin since

$$\int_{D} |\omega|^{p-4}\, dm(\omega) = \frac{2}{p-2} < \infty, \quad p > \mathbf{2}.$$

We write

$$\int_{|\omega|^{p-4}} |f(\omega)|^{p} dm(\omega) = \int_{|\omega|<1/2} + \int_{1/2|\omega|<1} |\omega|^{p-4}|f(\omega)|^{p} dm(\omega)\,|$$

and we estimate

$$\int_{1/2|\omega|<1} |\omega|^{p-4}|f(\omega)|^{p} dm(\omega) \leq \int_{|\omega|<1/2} \frac{|\omega|^{p-4}|}{(1-|\omega|^{2})^{2}} f(\omega)|^{p} dm(\omega)\, \|f\|_{A^p}^{p}$$

$$\leq \frac{1}{(1-((1/2)^{2})^{2}} \int_{|\omega|<1/2} |\omega|^{p-4}| dm(\omega)\, \|f\|_{A^p}^{p}$$

$$= \frac{2^{7-p}}{9(p-2)} \|f\|_{A^p}^{p}$$

and

$$\int_{1/2\leq|\omega|<1} |\omega|^{p-4}|f(\omega)|^{p} dm(\omega) \leq \left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{p-4}} \int_{1/2\leq|\omega|<1} |\omega|^{p-4} dm(\omega)$$

$$\leq 2^{4-p} \int_{D} |f(\omega)|^{p}\, dm(\omega)$$

$$= 2^{4-p} \|f\|_{A^p}^{p}$$

We conclude that for $\mathbf{2} < \boldsymbol{p} < \mathbf{4}$,

$$I \leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right) \frac{t^{2-p}}{(1-t)^{2}} \|f\|_{A^p}^{p}$$

which is the desired result.

For the proof of the Theorem we need some classical identities for the Beta and Gamma function see. For example[278].The Beta function is defined

by

$$B(u,v) = \int_0^{+\infty} \frac{x^{u-1}}{(x+1)^{u+v}}\,dx \int_0^1 s^{u-1}(1-s)^{v-1}\,ds$$

For u,v such that $\Re(u) > 0, \Re(\upsilon) > 0$. The value $B(u,v)$ can be expressed in terms of Gamma function as

$$B(u,\upsilon) = \frac{\Gamma(u)\Gamma(\upsilon)}{\Gamma(u,\upsilon)}.$$

Moreover , the Gamma function satisfies the function equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

For non-integral complex numbers z.

Now we can complexthe proof of the Theorem(6.2.1). Let $f \in A^p$. We have from the continuous version of Minkowski's inequality

$$\|\mathcal{H}(f)\|_{A^p} = \left(\int_D |\mathcal{H}(f)(z)|^p \; dm(z)\right)^{1/p}$$

$$= \left(\int_D \left|\int_D^1 T_t(f)(z)dt\right|^p \; dm(z)\right)^{1/p}$$

$$\leq \int_0^1 \left(\left(\int_D |T_t(f)(z)| \right)^p \left(dm(z) \right) \right)^{1/p} dt$$

$$= \int_0^1 \|T_t(f)\|_{A^p}\, dt.$$

Using Lemma(6.2.2) for $p \geq 4$ we conclude

$$\|\mathcal{H}(f)\|_{A^p} \leq \int_0^1 t^{2/p-1}(1-t)^{-2/p} dt \|f\|_{A^p}$$

$$= B\left(\frac{2}{p}, 1-\frac{2}{p}\right) \|f\|_{A^p}$$

$$= \Gamma\left(\frac{2}{p}\right) \Gamma\left(1-\frac{2}{p}\right) \|f\|_{A^p}$$

$$= \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}$$

Analogously, $2 < p < 4$, and $f \in A^p$ we have

$$\|\mathcal{H}(f)\|_{A^p} \leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right)^{1/p} \frac{t^{2/p}}{(1-t)^{2/p}}\, dt \|f\|_{A^p}$$

$$= \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right)^{1/p} \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}$$

Now, consider $f \in A^p$, $2 < p < 4$ with f(0)=0 and write $f(z) = z f_o(z)$.

The function fo is a Bergman space function and satisfies

$$\|f_0\|_{A^p} \leq \left(\frac{p}{2} + 1\right)^{1/p} \|f\|_{A^p}$$

Indeed, this estimate is a special case of a result on A^p -inner function [281].However, it is also possible to give an elementary proof

Lemma(6.2.3)[279]. For every analytic function**f**,

$$\int_D |f(z)|^p \, dm(z) \le \left(\frac{p}{2}+1\right) \int_D |zf(z)|^p \, dm(z).$$

Proof. Let c >1. We compute.

$$\int_D |f(z)|^p \, dm - C\int_D |zf(z)|^p \, dm(z) = \int_0^1 (1 - Cr^{p+1}) \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right| d\theta dr$$

$$= \int_0^1 (Cr^{p+1}) M_p^p\,(f,r)dr.$$

$$=D.$$

The real function $\sigma(r) = (r) - C(r)^{p+1}$ is positive for $r \in \left(0, C^{-1/p}\right)$and negative for $r \in \left(\mathbf{C^{-1/p}}, \mathbf{I}\right)$. In addition, it is well known that $M_p^p\,(f,r)$is a nondecreasing functionof r [283]. Hence in order for $D$to be ≤ 0, it is enough to choose C such that the following inequality holds:

$$-\int_{C^{-1/p}}^1 (r - C(r)^{p+1})\, dr \ge \int_1^{C^{-1/p}} r - Cr^{p+1}\, dr$$

or equivalently,

$$\int_0^1 r - Cr^{p+1}\, dr \le 0.$$

From the last inequality we get the condition C$\ge \frac{\mathbf{p}}{\mathbf{2}} + \mathbf{1}$.

Now we compute

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{(t-1)z+1} f\left(\frac{t}{(t-1)z+1}\right) dt$$

$$= \int_0^1 \frac{1}{(t-1)z+1} f_0\left(\frac{t}{(t-1)z+1}\right) dt$$

$$= \int_0^1 \frac{1}{t} f \phi_t(z)^2 f_0(\phi_t(z)) dt$$

$$= \int_0^1 S_t f_0(z) dt,$$

where

$$S_t(g)(z) = \frac{1}{t} \phi_t(z)^2 g(\phi_t(z)), \qquad g \in A^p,$$

and $\phi_t(z) = t/(t-1)z+1)$.An easy computation show that

$$\phi_t(z)^2 = \frac{t}{1-t} \acute{\emptyset}_t(z), \quad z \in D, \qquad 0 < t < 1.$$

It follows that

$$\|S_t(g)\|_{A^p}^p = \frac{1}{t^p} \int_D |\phi_t(z)|^{2p} \left|g(\phi_t(z))\right|^p dm(z)$$

$$= \frac{1}{t^p} \int_D |\phi_t(z)|^{2p-4} \left|(\phi_t(z))\right|^4 \left|g(\phi_t(z))\right|^p dm(z)$$

$$\leq \frac{t^{2-p}}{(1-t)^2} \int_D |\phi_t(z)|^{2p-4} \left|g(\phi_t(z))\right|^p \left|(\emptyset'_t(z))\right|^2 dm(z)$$

$$= \frac{t^{2-p}}{(1-t)^2} \int_{\phi_t(\mathbb{D})} |\omega|^{2p-4} |g(\omega)|^p dm(\omega)$$

$$\leq \frac{t^{2-p}}{(1-t)^2} \int_{\phi_t(\mathbb{D})} |g(\omega)|^{2+\epsilon} dm(\omega)$$

$$\leq \frac{t^{2-p}}{(1-t)^2} \int_{\mathbb{D}} |g(\omega)|^p dm(\omega)$$

$$= \frac{t^{2-p}}{(1-t)^2} \int_{\mathbb{D}} \|g\|_{A^p}^p.$$

Hence

$$\|S_t(g)\|_{A^p} \leq \frac{t^{2/p-1}}{(1-t)^{2/p}} \|g\|_{A^p}$$

For the norm of $\mathcal{H}$ we compute

$$\|\mathcal{H}(f)\|_{A^p} \leq \left(\int_0^1 \frac{t^{2/p-1}}{(1-t)^{2/p}}\, dt\right) \|f_0\|_{A^p}$$

$$= \frac{\pi}{\sin(2\pi/p)} \|f_0\|_{A^p}$$

$$= \left(\frac{p}{2}+1\right)^{1/p} \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p},$$

Let D be the usual Dirichlet space of analytic function on the unit disc with square summable derivative. The following result is well known .

Lemma(6.2.4)[279]. Each bounded linear functional on the Bergman A^2 can be associated to a function $g \in D$ (by the pairing $\langle f, g\rangle = \sum_{n=0}^{\infty} a_n (2+\epsilon)_n$) and the association is an isometric isomorphism of the spaces.

This yields the following result

Proposition(6.2.5)[279].There is no bounded linear operator $T : A^2 \to A^2$ satisfying.

$$T(\xi_n)(0) = \frac{1}{n+1}, \qquad n = 0,1,2,\ldots$$

Where $\xi_n(z) = z^n$.

Proof. Suppose to the contrary. that there exists such an operator T. Using pairing that defines an isometric isomorphism between $(A^2)^*$ and $\mathcal{D}$,we find that the adjoint operator$T^*: \mathcal{D} \to \mathcal{D}$

$$\langle T(f), g\rangle = \langle f, T^*(g)\rangle \tag{17}$$

for every $f \in A^2, g \in \mathcal{D}$. We choose $g \equiv 1$ and write

$$T^*(1)(z) = \sum_{n=0}^{\infty} C_n z^n$$

as the Taylor series of$T^*(1) \in \mathcal{D}$.Using (7) for$f = \xi_n$ and $g \equiv 1$ we have

$$\frac{1}{n+1} = T(\xi_n)(0)$$

$$= \langle T(\xi_n), 1\rangle$$
$$= \langle \xi_n, T^*(1)\rangle$$
$$= c_n$$

For every n=0,1,2,....Hence

$$T^*(1)(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n$$

but this function is not in $\mathcal{D}$.

Now we consider the integral

$$\mathcal{H}(f) = \int_0^1 f(t) \frac{1}{1-tz} d(t).$$

This integral is well defined for polynomials are dense in A^2 . It is not known if the last integral is well defined for all $f \in A^2$.In any case, from Proposition (6.2.5) we obtain:

Corollary(6.2.6)[279]. $\mathcal{H}$ is not bounded on A^2.

Proof . We apply Proposition (6.2.5) and note that

$$H(\xi_n)(0) = \frac{1}{n+1} \qquad n = 0,1,2,...$$

Lemma(6.2.7)[297]. Let $0 \le \epsilon < +\infty$. Then

(i)if $0 \le \epsilon < +\infty$ and $f \in A^{4+\epsilon}$then

$$\left\|T_{1-\epsilon_3}(f)\right\|_{A^{4+\epsilon}} \le \frac{(1-\epsilon_3)^{\frac{2}{2+\epsilon}}}{\epsilon_3^{\frac{2}{4+\epsilon}}} \|f\|_{A^{4+\epsilon}}$$

(ii) if $0 < \epsilon < 2$ and $f \in A^{2+\epsilon}$,then

$$\left\|T_{1-\epsilon_3}(f)\right\|_{A^{2+\epsilon}} \le \left(\left(\frac{8}{9\epsilon} + 1\right) 2^{2-\epsilon}\right)^{\frac{1}{2+\epsilon}} \frac{(1-\epsilon_3)^{\frac{2}{1+\epsilon}}}{(\epsilon_3)^{\frac{2}{2+\epsilon}}} \|f\|_{A^{2+\epsilon}}$$

Proof. We can easily check that

$$\boldsymbol{\omega_{1-\epsilon_3}(z)^2} = \frac{1}{(1-\epsilon_3)\epsilon_3} \emptyset'_{(1-\epsilon_3)}(z)$$

Let $f \in A^{2+\epsilon}$, $\epsilon > 0$. using the last equation we obtain

$$\left\|T_{1-\epsilon_3}(f)\right\|_{A^{2+\epsilon}}^{2+\epsilon} = \int_D |\omega_{1-\epsilon_3}(z)|^{2+\epsilon} \left|f\left(\emptyset_{(1-\epsilon_3)}(z)\right)\right|^{2+\epsilon} dm(z)$$
$$= \int_D |\omega_{1-\epsilon_3}(z)|^{\epsilon-2} |\omega_{1-\epsilon_3}(z)|^4 \left|f\left(\emptyset_{(1-\epsilon_3)}(z)\right)\right|^{2+\epsilon} dm(z)$$
$$= \frac{1}{\epsilon_3^2(1-\epsilon_3)} \int_D |\omega_{1-\epsilon_3}(z)|^{\epsilon-2} \left|f\left(\emptyset_{(1-\epsilon_3)}(z)\right)\right|^{2+\epsilon} \left|\emptyset'_{1-\epsilon_3}\right|^2 dm(z)$$
$$= \frac{1}{\epsilon_3^2(1-\epsilon_3)} \int_{\emptyset_{(1-\epsilon_3)}D} \left|\omega_{1-\epsilon_3}\left(\emptyset_{1-\epsilon_3}^{-1}(z)\right)\right|^{\epsilon-2} |f(z)|^{2+\epsilon} dm(z)$$

$$=\mathrm{I}.$$

We now consider two cases.

First,suppose that $\epsilon \geq 0$. We compute

$$\emptyset_{1-\epsilon_3}^{-1}(z) = \frac{z+\epsilon_3-1}{\epsilon_3 z}$$

and

$$\omega_{1-\epsilon_3}\left(\emptyset_{1-\epsilon_3}^{-1}(z)\right) = \frac{1}{1-\epsilon_3\emptyset_{1-\epsilon_3}^{-1}(z)} = \frac{z}{1-\epsilon_3}$$

Hence

$$\mathrm{I} \leq \frac{\|f\|_{A^{4+\epsilon}}^{4+\epsilon}}{\epsilon_3^2(1-\epsilon_3)^{2+\epsilon}}.$$

Next, assume that $0 < \epsilon < 2$. Then

$$\mathrm{I} = \frac{1}{\epsilon_3^2\,(1-\epsilon_3)^2}\int_{\emptyset_{(1-\epsilon_3)}(D)} |\omega_{1-\epsilon_3}\left(\emptyset_{1-\epsilon_3}^{-1}(\omega)\right)|^{\epsilon-2}|f(\omega)|^{2+\epsilon}dm(\omega)$$

$$= \frac{1}{\epsilon_3^2\,(1-\epsilon_3)^2}\int_{\emptyset_{(1-\epsilon_3)}(D)} \frac{\omega}{1-\epsilon_3}|^{\epsilon-2}|f(\omega)|^{2+\epsilon}dm(\omega)$$

$$= \frac{1}{\epsilon_3^2\,(1-\epsilon_3)^2}\int_{\emptyset_{(1-\epsilon_3)}(D)}|\omega|^{\epsilon-2}\quad|f(\omega)|^{2+\epsilon}dm(\omega)$$

$$\leq \frac{1}{\epsilon_3^2\,(1-\epsilon_3)^2}\int_D |\omega|^{\epsilon-2}\quad|f(\omega)|^{2+\epsilon}dm(\omega).$$

The last integral is well defend near the origin, since

$$\int_D |\omega|^{\epsilon-2}\,dm(\omega) = \frac{2}{\epsilon} < \infty,\quad \epsilon > 0.$$

We write

$$\int_D |\omega|^{\epsilon-2}\,|f(\omega)|^{2+\epsilon}dm(\omega) = \int_{|\omega|<\frac{1}{2}} + \int_{\frac{1}{2}\leq|\omega|<1}|\omega|^{\epsilon-2}\,|f(\omega)|^{2+\epsilon}dm(\omega)$$

and we estimate

$$\int_{|\omega|<\frac{1}{2}}|\omega|^{\epsilon-2}\,|f(\omega)|^{2+\epsilon}dm(\omega) \leq \int_{|\omega|<\frac{1}{2}}\frac{|\omega|^{\epsilon-2}}{(1-|\omega|^2)^2}dm(\omega)\|f\|_{A^{2+\epsilon}}^{2+\epsilon}$$

$$\leq \frac{1}{\left(1-\left(\frac{1}{2}\right)^2\right)^2}\int_{|\omega|<\frac{1}{2}}|\omega|^{\epsilon-2}\,dm(\omega)\|f\|_{A^{2+\epsilon}}^{2+\epsilon}$$

$$= \frac{2^{5-\epsilon}}{9\epsilon}\|f\|_{A^{4+\epsilon}}^{4+\epsilon}$$

and

$$\int_{\frac{1}{2}\leq|\omega|<1}|\omega|^{\epsilon-2}|f(\omega)|^{2+\epsilon}dm(\omega) \leq \left(\frac{1}{2}\right)^{\epsilon-2}\int_{\frac{1}{2}\leq|\omega|<1}|f(\omega)|^{2+\epsilon}dm(\omega)$$

$$\leq 2^{2-\epsilon}\int_D |f(\omega)|^{\epsilon-2}\,dm(\omega)$$

$$= 2^{2-\epsilon}\|f\|_{A^{4+\epsilon}}^{4+\epsilon}.$$

We conclude that for $0 < \epsilon < 2$,

$$I < \left(\left(\frac{8}{9\epsilon}+1\right)2^{2-\epsilon}\right) \frac{(1-\epsilon_3)^{-\epsilon}}{\epsilon_3^2} \|f\|_{A^{2+\epsilon}}^{2+\epsilon}$$

Theorem(6.2.8)[297]. The operator $\mathcal{H}$ is bounded on Bergman spaces $A^{2+\epsilon}$ $0 < \epsilon < \infty$, and satisfies:

(i) if $0 \le \epsilon < \infty$ and $f \in A^{4+\epsilon}$, then

$$\|\mathcal{H}(f)\|_{A^{4+\epsilon}} \le \frac{\pi}{\sin(\frac{2\pi}{4+\epsilon})} \|f\|_{A^{4+\epsilon}}$$

(ii) if $0 < \epsilon < 2$ and $f \in A^{4+\epsilon}$, then

$$\|\mathcal{H}(f)\|_{A^{2+\epsilon}} \le \left(\left(\frac{8}{9\epsilon}+1\right)2^{2-\epsilon}\right)^{\frac{1}{2+\epsilon}} \frac{\pi}{\sin(\frac{2\pi}{2+\epsilon})} \|f\|_{A^{2+\epsilon}}$$

(iii) if $0 < \epsilon < 2$ and $f \in A^{2+\epsilon}$, then

$$\|\mathcal{H}(f)\|_{A^{2+\epsilon}} \le \left(\frac{4+\epsilon}{2}\right)^{\frac{1}{2+\epsilon}} \frac{\pi}{\sin(\frac{2\pi}{2+\epsilon})} \|f\|_{A^{2+\epsilon}}$$

Proof.

we need some classical identities for the Beta and Gamma function see. For example [283]. The Beta function is defined

by

$$B(u,v) = \int_0^{+\infty} \frac{x^{u-1}}{(x+1)^{u+v}} dx \int_0^1 (1-\epsilon_2)^{u-1}(\epsilon_2)^{v-1}\, d(1-\epsilon_2)$$

for u, v such that $\Re(u) > 0, \Re(\upsilon) > 0$.The value B(u, v) can be expressed in terms of Gamma function as

$$B(u,\upsilon) = \frac{\Gamma(u)\Gamma(\upsilon)}{\Gamma(u,\upsilon)}.$$

Moreover , the Gamma function satisfies the function equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

for non-integral complex numbers z.

Now we can complexthe proof of the Theorem(6.2.8) (see[13]). Let $f \in A^{2+\epsilon}$.

We have from the continuous version of Minkowski's inequality

$$\|\mathcal{H}(f)\|_{A^{\epsilon+2}} = \left(\int_D |\mathcal{H}(f)(z)|^{\epsilon+2}\, dm(z)\right)^{\frac{1}{\epsilon+2}}$$

$$= \left(\int_D \left|\int_0^1 |T_{1-\epsilon_3}(f)(z)|^{\epsilon+2} d(1-\epsilon_3)\right| dm(z)\right)^{\frac{1}{\epsilon+2}}$$

$$\leq \int_0^1 \left(\int_D \left| T_{1-\epsilon_3}(f)(z) \right| dm(z) \right)^{\frac{1}{2+\epsilon}} \quad d(1-\epsilon_3)$$
$$= \int_0^1 \left\| T_{1-\epsilon_3}(f) \right\|_{A^{2+\epsilon}} d(1-\epsilon_3).$$

Using Corollary(6.2.7) for$\epsilon \geq 0$we conclude

$$\|\mathcal{H}(f)\|_{A^{4+\epsilon}} \leq \int_0^1 (1-\epsilon_3)^{\frac{2}{\epsilon+3}} \epsilon_3^{\frac{-2}{\epsilon+4}} d(1-\epsilon_3) \|f\|_{A^{4+\epsilon}}$$
$$= B\left(\frac{2}{4+\epsilon}, \frac{2+\epsilon}{4+\epsilon}\right) \|f\|_{A^{4+\epsilon}}$$
$$= \Gamma\left(\frac{2}{4+\epsilon}\right) \Gamma\left(\frac{2+\epsilon}{4+\epsilon}\right) \|f\|_{A^{4+\epsilon}} \quad (\Gamma(1) = 1)$$
$$= \frac{\pi}{\sin\left(\frac{2\pi}{4+\epsilon}\right)} \|f\|_{A^{4+\epsilon}}$$

Analogously, $\mathbf{0 < \epsilon < 2}$, and $\boldsymbol{f} \in \mathbf{A^{2+\epsilon}}$ we have

$$\|\mathcal{H}(f)\|_{A^{2+\epsilon}} \leq \left(\left(\frac{8}{9\epsilon} + 1 \right) 2^{2-\epsilon} \right)^{\frac{1}{2+\epsilon}} \int_0^1 \frac{(1-\epsilon_3)^{\frac{2}{1+\epsilon}}}{\epsilon_3^{\frac{2}{1+\epsilon}}} \, d(1-\epsilon_3) \|f\|_{A^{2+\epsilon}}$$

$$= \left(\left(\frac{8}{9\epsilon} + 1 \right) 2^{2-\epsilon} \right)^{\frac{1}{2+\epsilon}} \frac{\pi}{\sin\left(\frac{2\pi}{2+\epsilon}\right)} \|f\|_{A^{2+\epsilon}}$$

Now, consider $f \in A^{2+\epsilon}$, $0 < \epsilon < 2$ with f(0) = 0 and write $f(z) = zf_0(z)$.The function $f_0$is a Bergman space function and satisfies

$$\|f_0\|_{A^{2+\epsilon}} \leq \left(\frac{4+\epsilon}{2}\right)^{\frac{1}{2+\epsilon}} \|f\|_{A^{2+\epsilon}}$$

Indeed, this estimate is a special case of a result on $A^{2+\epsilon}$ -inner function [282].However, it is also possible to give an elementary proof .

Lemma(6.2.9)[297]. For every analytic function$f$,

$$\int_D |f(z)|^{2+\epsilon} dm(z) \leq \left(\frac{4+\epsilon}{2}\right) \int_D |zf(z)|^{2+\epsilon} dm(z).$$

Proof. Let C >1. we compute

$$\int_D |f(z)|^{2+\epsilon} dm(z) - C \int_D |zf(z)|^{2+\epsilon} dm(z)$$
$$= \int_0^1 \left((1-\epsilon_1) - C(1-\epsilon_1)^{3+\epsilon} \right) \int_0^{2\pi} \left| f(1-\epsilon_1)e^{i\theta} \right|^{2+\epsilon} d\theta d(1-\epsilon_1)$$
$$= \int_0^1 \left((1-\epsilon_1) - C(1-\epsilon_1)^{3+\epsilon} \right) M_{2+\epsilon}^{2+\epsilon}(f, 1-\epsilon_1) d(1-\epsilon_1).$$
$$=D$$

The real function $\sigma(1-\epsilon_1) = (1-\epsilon_1) - C(1-\epsilon_1)^{3+\epsilon}$ is positivefor $(\mathbf{1} - \boldsymbol{\epsilon_1}) \in \left(\mathbf{0}, C^{\frac{-1}{2+\epsilon}}\right)$and negative for $(1-\epsilon_1) \in \left(C^{\frac{-1}{2+\epsilon}}, 1\right)$ in addition, it is well known that $M_{2+\epsilon}^{2+\epsilon}(f, 1-\epsilon_1)$is a non decreasing functionof$(\mathbf{1} - \boldsymbol{\epsilon_1})$ [3]. Hence in order for D to be $\leq \mathbf{0}$, it is enough to choose C such that the following inequality holds:

$$-\int_{C^{\frac{-1}{2+\epsilon}}}^{1}(1-\epsilon_1-C(1-\epsilon_1)^{3+\epsilon})\,d(1-\epsilon_1)\geq$$

$$\int_0^{C^{\frac{-1}{2+\epsilon}}}(1-\epsilon_1-C(1-\epsilon_1)^{3+\epsilon})\,d(1-\epsilon_1)\geq$$

or equivalently ,

$$\int_0^1(1-\epsilon_1-C(1-\epsilon_1)^{3+\epsilon})\,d(1-\epsilon_1)\leq 0.$$

From the last inequality we get the condition C$\geq \frac{4+\epsilon}{2}$.

Now we compute

$$\begin{aligned}\mathcal{H}(f)(z) &= \int_0^1 \frac{1}{1-\epsilon_3 z} f\left(\frac{1-\epsilon_3}{1-\epsilon_3 z}\right) d(1-\epsilon_3)\\ &= \int_0^1 \frac{1}{1-\epsilon_3 z} f_0\left(\frac{1-\epsilon_3}{1-\epsilon_3 z}\right) d(1-\epsilon_3)\\ &= \int_0^1 \frac{1}{1-\epsilon_3} f\phi_{(1-\epsilon_3)}(z)^2 f_0\left(\phi_{(1-\epsilon_3)}(z)\right) d(1-\epsilon_3)\\ &= \int_0^1 S_{(1-\epsilon_3)} f_0(z)d(1-\epsilon_3),\end{aligned}$$

where

$$S_{1-\epsilon_3(g)(z)=}\frac{1}{1-\epsilon_3}\phi_{(1-\epsilon_3)}(z)^2 g\left(\phi_{(1-\epsilon_3)}(z)\right),\qquad g\in A^{2+\epsilon}$$

and $\phi_{(1-\epsilon_3)}(z) = \frac{1-\epsilon_3}{1-\epsilon_3 z)}$.An easy computationshowthat

$$\phi_{(1-\epsilon_3)}(z)^2 = \frac{1-\epsilon_3}{\epsilon_3}\emptyset'_{1-\epsilon_3}(z),\ \ z\in D,\ \ 0<\epsilon_3<1.$$

It follows that

$$\begin{aligned}\left\|S_{1-\epsilon_3}(g)\right\|_{A^{2+\epsilon}}^{2+\epsilon} &= \frac{1}{(1-\epsilon_3)^{2+\epsilon}}\int_D \left|\phi_{(1-\epsilon_3)}(z)\right|^{2\epsilon+4}\left|g\left(\phi_{(1-\epsilon_3)}(z)\right)\right|^{2+\epsilon} dm(z)\\ &= \frac{1}{(1-\epsilon_3)^{2+\epsilon}}\int_D \left|\phi_{(1-\epsilon_3)}(z)\right|^{2\epsilon}\left|\left(\phi_{(1-\epsilon_3)}(z)\right)\right|^{4}\left|g\left(\phi_{(1-\epsilon_3)}(z)\right)\right|^{2+\epsilon} dm(z)\\ &\leq \frac{(1-\epsilon_3)^{-\epsilon}}{\epsilon_3^2}\int_D \left|\phi_{(1-\epsilon_3)}(z)\right|^{2\epsilon}\left|g\left(\phi_{(1-\epsilon_3)}(z)\right)\right|^{2+\epsilon}\left|\acute{\emptyset}_{(1-\epsilon_3)}(z)\right|^{2} dm(z)\\ &= \frac{(1-\epsilon_3)^{-\epsilon}}{\epsilon_3^2}\int_{\phi_{(1-\epsilon_3)}(\mathbb{D})}|\omega|^{2\epsilon}\,|g(\omega)|^{2+\epsilon}dm(\omega)\\ &\leq \frac{\mathbf{(1-\epsilon_3)^{-\epsilon}}}{{\epsilon_3}^2}\int_{\phi_{(1-\epsilon_3)}(\mathbb{D})}|g(\omega)|^{2+\epsilon}dm(\omega)\\ &\leq \frac{(1-\epsilon_3)^{-\epsilon}}{\epsilon_3^2}\int_{\mathbb{D}}|g(\omega)|^{2+\epsilon}dm(\omega)\\ &= \frac{(1-\epsilon_3)^{-\epsilon}}{\epsilon_3^2}\|g\|_{\mathrm{A}^{2+\epsilon}}^{2+\epsilon}.\end{aligned}$$

Hence

$$\left\|S_{1-\epsilon_3}(g)\right\|_{\mathrm{A}^{2+\epsilon}}\leq (1-\epsilon_3)^{\frac{2}{\epsilon+1}}\left(\epsilon_3^{\frac{2}{2+\epsilon}}\right)^{-1}\|g\|_{\mathrm{A}^{2+\epsilon}}$$

For the norm of $\mathcal{H}$ we compute

$$\|\mathcal{H}(f)\|_{A^{2+\epsilon}} \leq \left(\int_0^1 (1-\epsilon_3)^{\frac{2}{\epsilon+1}} \left(\epsilon_3^{\frac{2}{2+\epsilon}}\right)^{-1} \, d(1-\epsilon_3)\right) \|f_0\|_{A^{2+\epsilon}}$$

$$= \frac{\pi}{\sin\left(\frac{2\pi}{2+\epsilon}\right)} \|f_0\|_{A^{2+\epsilon}}$$

$$= \left(\frac{4+\epsilon}{2}\right)^{\frac{1}{2+\epsilon}} \frac{\pi}{\sin\left(\frac{2\pi}{2+\epsilon}\right)} \|f\|_{A^{2+\epsilon}},$$

Corollary(6.2.10)[297]. let $0 < \epsilon < \infty$. Then

(i) If $0 \leq \epsilon < \infty$ and $f \in A^{4+\epsilon}$ then

$$\left\|T_{1-\epsilon_{k+1}}(f)\right\|_{A^{4+\epsilon}} \leq \frac{(1-\epsilon_{k+1})^{\frac{2}{2+\epsilon}}}{\epsilon_{k+1}^{\frac{2}{4+\epsilon}}} \|f\|_{A^{4+\epsilon}}.$$

(ii) If $0 < \epsilon < 2$ and $f \in A^{2+\epsilon}$ then

$$\left\|T_{1-\epsilon_{k+1}}(f)\right\|_{A^{2+\epsilon}} \leq \left(\left(\frac{8}{9\epsilon} + 1\right) 2^{2-\epsilon}\right)^{\frac{1}{2+\epsilon}} \frac{(1-\epsilon_{k+1})^{\frac{2}{2+\epsilon}}}{\epsilon_{k+1}^{\frac{2}{2+\epsilon}}} \|f\|_{A^{2+\epsilon}}.$$

Proposition(6.2.11)[297]. There is no bounded linear operator $T : A^2 \to A^2$ satisfying.

$$T(\xi_n)(0) = \frac{1}{n+1}, \qquad n = 0,1,2,\ldots$$

Where $\xi_n(z) = z^n$.

Proof. Suppose to the contrary. that there exists such an operator T. Using pairing that defines an isometric isomorphism between $(A^2)^*$ and $\mathcal{D}$, we find that the adjoint operator $T^*: \mathcal{D} \to \mathcal{D}$

$$\langle T(f), g\rangle = \langle f, T^*(g)\rangle$$

for every $f \in A^2, g \in \mathcal{D}$. We choose $g \equiv 1$ and write

$$T^*(1)(z) = \sum_{n=0}^{\infty} C_n z^n$$

as the Taylor series of $T^*(1) \in \mathcal{D}$. Using (7) for $f = \xi_n$ and $g \equiv 1$ we have

$$\frac{1}{n+1} = T(\xi_n)(0)$$
$$= \langle T(\xi_n), 1\rangle$$
$$= \langle \xi_n, T^*(1)\rangle$$

$$= c_n$$

For every n=0,1,2,....Hence

$$T^*(1)(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n$$

but this function is not in $\mathcal{D}$.

Now we consider the integral

$$\mathcal{H}(f) = \int_0^1 f(1-\epsilon_{k+1}) \frac{1}{\epsilon_{k+1} z} d(1-\epsilon_{k+1}).$$

This integral is well defined for polynomials are dense in A^2 . It is not known if the last integral is well defined for all $f \in A^2$.In any case, from Proposition (6.2.11) we obtain:

Sec (6-3) Bergman and Hardy Spaces with a theorem of Nehari type

A Hankel operator on the space $l^p$of all square –summable complex sequences in an operator defined by a matrix whose entries $a_{n,k}$ depend only on the sum of the coordinates $a_{n,k} = c_{n+k}$ some sequence $(C_n)_{n=0}^{\infty}$. Hankel operator on different spaces are related to many areas such as the theory of moment sequence, orthogonal polynomials , Toeplitz operators ,or analytic Besov spaces .

Nehari's classical theorem states that every Hankel operator S on l^pcan be represented by an essentially bounded function g on the circle T, in the sense that $c_n = \hat{g}(n)$ for all $n \geq 0$;moreover ,a function g can always be chosen so that $\|g\|_{L^\infty(T)} = \|S\|_{l^2 \to l^2}$ see[295] ,[298] or[299]. A typical Hankel operator is the Hilbert matrix H whose entries are $a_{n,k} = (n+k+1)^{-1}, n,k \geq 0$. It is relevant in many fields ranging from number theory or linear algebra to numerical analysis and operator theory. For this operator, the following choice: $g(t) = ie^{-it}(\pi - t), 0 \leq t < 2\pi$ in Nehari's theorem yields $\|g\|_{L^\infty(T)} = \pi = \|H\|_{l^2 \to l^2}$. Several interesting facts about the Hilbert matrix are described in[290] and[293] problems and further results about the spectrum of H can be found in[298] .

The Hilbert matrix can be viewed as an operator on other spaces and it is a basic question to determine its operator norm.One from of Hilbert's classical inequality[271],[275].

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)\leq\frac{\pi}{\sin\pi/p}\left(\sum_{n=0}^{\infty}|a_k|^p\right)^{1/p}$$

can be used to compute the norm of H on the space l^2 all p- summable sequences:

$$\|H\|_{l^2\to l^2}=\frac{\pi}{\sin(\pi/p)}, \quad , \quad 1< p<\infty$$

The Toylor coefficients of the function in the Hardy spaces H^P are closely related to l^P spaces. Thus , it is natural to consider the Hilbert matrix as an operator defined on H^P by its action on the coefficients:

$$\hat{f}(n)\mapsto\sum_{k=0}^{\infty}\frac{\hat{f}(k)}{n+k+1}:$$

that is , by defining

$$Hf(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{\hat{f}(k)}{n+k+1}\right)z^n, \qquad f\in H^p, z\in D. \qquad (18)$$

It is possible to write Hf , $f\in H^p$ in other forms which are convenient for analyzing this operator see[271] for example :

$$Hf(z)=\int_0^1\frac{f(r)}{1-rz}dr. \qquad z\in D \qquad (19)$$

The equality of the expressions in(18) and (19) can be verified in a straightforward way from the Toylor series expansion of f .

The most basic question is:on which Hardy spaces is H bounded? Diamantopoulos and Siskakis [271] showed its boundedness on any H^p with 1< P <∞. By establishing another useful representation of H as an average of weighted composition operator and integrating over semi-circular paths, they obtained the following upper bound:

$$\|H\|_{H^P \to H^P} \leq \frac{\pi}{\sin(\pi/p)}, \qquad 2 \leq P < \infty.$$

In view of Nehari's l^P Theorem,this result is sharp when P=2.

In the case $1< P < \infty$. it was also shown in[271] that the above estimate continues to hold for the restricition of the operator to the subspace $\{f \in H^P : f(0) = 0\}$. Two natural question come to mind:

(a) Can the above norm estimate for H be extended to the case $1< P < \infty$. without restrictions?

(b)What is the actual value of the norm of H as an H^P operator $1 < p < \infty$?

We give a more general answer to the above question (a) by deducing the following Nehari-type result: an arbitrary Hankel operator H_g associated with a function $g \in L^\infty(T)$ is bounded H^P, $1 < P < \infty$:

$$\|H_g\|_{H^p \to H^p} \leq \frac{\|g\|_\infty}{\sin \pi/p}$$

The key point is that every Hankel operator on H^P has representations as a composition of a (non-analytic) isometru and a multiplication . followed by the Rizez(szego) projection P_+ from $L^p(T)$ onto its closed subspace H^P. It is well know that this projection is bounded for $1 < P < \infty$. In 1968 Gohberg and Krupnik[262] showed that

$$\|p_+\|_{L^p(T) H^p} \geq \frac{1}{\sin(\pi/p)}, 1 < p < \infty$$

and conjectured that equality should hold . Hollenbeck and Verbitsky [267]proved this conjecture in 2000. Their result allows us to deduce the estimate for $\|H_g\|$ above.

Using some Hardy spaces techniques and splitting H into a difference of two operators we also get a lower bound which yields

$$\|H\|_{H^p \to H^p} = \frac{\pi}{\sin(\pi/p)}, \qquad 1 < p < \infty,$$

thus answering the above question (b) for all admissible values of p.

The behavior of the Hilbert matrix as an operator defined by (18) turns out to be similar in the classical Bergman spaces A^P of functions P-integral in D with respect to the area measure. Diamantopoulos [258] recenntly proved that H is bounded on A^P if and only $P > 2$. In the case

$4 \leq P < \infty$ he obtained the estimate

$$\|H\|_{A^p \to A^p} \leq \frac{\pi}{\sin(2\pi/p)},$$

(This is what one may expect by the "rule of thumb" that say for many operators and functionals defined on both H^P and A^P their norm when acting on A^P is obtained by doubling an appropriate quantity in the norm when acting on H^P.) A less precise estimate for the norm of H on A^P when $2 < P < 4$ was also obtained in[279] .

We optain a lower bound valid for all $P > 2$ which coincides with the upper bound from [279] when $P \leq 4$, thus yielding the exact value of the norm for these exponents:

$$\|H\|_{A^p \to A^p} \leq \frac{\pi}{\sin(2\pi/p)}, \qquad 4 \leq p < \infty$$

In the case $2 < p < 4$ although we are currently not able to identify the exact value of the norm, we do improve the bound obtained in[279]. We also point out that the Hilbert matrix has an integral representation with respect to the area measure with a kernel rather different from the usual Bergman space kernels.

$D = \{z \in C: |z| < 1\}$ Will denote the unit disk in the complex plane C and H(D) will signify the algebra of holomorphic functions in D. For f in H(D) and $0 < r < 1$, the integral means $M_p(r,f)$ are defined by

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta \right)^{1/p}$$

and are increasing with r. The Hardy space H^P $(0 < P < \infty)$ is the space of all f in H(D) for which $\|f\|_{H^p} = \lim_{r\to 1} -M_p(r,f) < \infty$, and H^∞ is the space of all bounded f in H(D) we will denote by T the unit circle . The standard Lebesgue space $L^P(T)$ of the circle is to be considered with respect to the normalized measure $dm(z) = (2\pi)^{-1} dt$ where $z = e^{it}$, $0 \leq t < 2\pi$ .It is a well known fact that the space

H^p is the closed subspace of $L^P(T)$ consisting of all function whose fourier coefficient with the negative index vanish. The Riesz (szego) projection p_+ from $L^P(T)$ onto H^p is defined by

$$p_{+U(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{u(1)}{1-ze^{-ie}}dt \qquad z\in D \tag{20}$$

For more details , the reader is referred to [290] among other sources .

One can define Hankel operator on any space $H^P, 1 < P < \infty$. Given an arbitrary $g \in L^\infty (T)$,consider its Fourier coefficients with non- negative indices :

$$\hat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\text{int}} g(t)dt, \qquad n \geq 0$$

We can formally define the associated Hankel operator H_g by

$$H_g f(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{2\pi}\hat{g}(n+k)\hat{f}(k)\right)z^n \tag{21}$$

foran analytic function f with the Taylor series $f(z) = \sum_{n=0}^{\infty}\hat{f}(n)z^n$ in D. In particular,when $g(t) = ie^{it}(\pi - t), 0 \leq t < 2\pi$, a straightforward calculation shows that

$$\hat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\text{int}} g(t)dt = \frac{1}{n+1}, \qquad n \geq 0$$

hence $H_g = H$, the Hilbert matrix . This is well known; see[280],[295],or[298].

We will compute the norm of Hilbert matrix H as an H^P operator,$1 < P < \infty$ as a consequence of an upper bound for the norm valid for an arbitrary operator H_g as above. To this end , we consider the isometric conjugation operator (also called the flip operator) for the function on the unit circle T as $C f(e^{it}) = f(e^{-it})$. It is obvious that C is an isometry from H^P into $L^P(T)$. Next, let M_g denote the operator of multiplication by the essentially bounded function g: $M_g u = gu$; this is clearly bounded by $\|g\|L^\infty$ as an operator acting on $L^P(T)$ We will now establish an equality $H_g = p_+ M_g C$ which is known to hold in l^P context(see[295], thus obtaining a Nehari – type theorem for Hankel operators on Hardy spaces .

Theorem(6-3-1)[289]: let $1 < P < \infty$ and $g \in L^{\infty}(0{,}2\pi)$ The operator H_g defined as in (21) is bounded on H^P the equality $H_g = p_+ M_g C$ holds and consequently,

$$\|H_g\|_{H^p \to H^p} \le \frac{\|g\|_\infty}{\sin(\pi/p)}$$

In particular, when $g(t) = ie^{it}(\pi - t), 0 \le t < 2\pi$, we get $H_g = H$ and

$$\|H\|_{H^p \to H^p} \le \frac{\pi}{\sin(\pi/p)} \tag{22}$$

Proof. Given $f \in H^P$, denote by f_m its mth Taylor polynomial $f_m(z) = \sum_{k=0}^{m} \hat{f}(k) z^k$ the following result[293] will be useful: if $1 < P < \infty$ and then $\|f_m f\|_{H^p} \to 0$ as $m \to \infty$.

Given $f \in H^P$, we first verify that the power series for $H_g f$ converges in D.To this end, it suffices to show that

$$\left|\sum_{k=0}^{\infty} \hat{g}(n+k)\hat{f}(k)\right| \le \|g\|\infty\|f\|_{H^p} \tag{23}$$

For f_m instead of f, this follows immediately by recalling that C is an isometry of H^P into $L^P(T)$ and applying Holders inequality:

A similar argument applied to the difference $f_m - f_n$ shows that $(\sum_{k=0}^{\infty} \hat{g}(n + kf(k))m{=}0\infty$is a Cauchy sequence uniformly in n , so it is legitimate to let m→∞ obtain (23)

We will now establish the formula $H_g f = p_+ M_g C f$ for all f in $H^P, 1 < p < \infty$. By the theorem of Hollenbeck and Verbitsky this will immediately imply that H_g bounded and, moreover,(22) holds:

$$\|H_g\|_{H^p \to H^p} \le \|p+\|_{L^p(T) \to H^p} \|M_g\|_{L^p(T) \to L^p(T)} \le \frac{\|g\|_\infty}{\sin \pi/p}$$

Given $f \in H_g$ we get the identity $\|H_g f_m\|_{H_g} = p_+ M_g C f_m$ and the bound

$$\|H_g f_m\|_{H^p} \le \frac{\|g\|_\infty}{\sin \pi/p} \|f_m\|_{H^p}$$

for the mth Toylor polynomial f_m of f by an easy computation involving (21)and(20):

$$H_g f_m(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{m} \hat{f}(k)\int_0^{2\pi} e^{-i(n+k)t} g(t)\frac{dt}{2\pi} z^n = \int_0^{2\pi} \frac{g(t) f_m(e^{-it})}{1-e^{-it}z}\frac{dt}{2\pi} \tag{24}$$

The interchange of the series and the integral is justified by uniform convergence of the geometric series $\sum_{k=0}^{\infty}|z|^n$ on compact sets in D.

To extend the identity $H_g f_m = p_+ M_g C f_m$ and(24) for arbitrary f in H^P, note that $(H_g f_m)_{n=0}^{\infty}$ is a Cauchy sequence in H^P in view of

$$\|H_g(f_m \to f_n)\|_{H^p} \leq \frac{\|g\|_\infty}{\sin \pi/p}\|f_m \to f_n\|_{H^p}$$

so the standard H^P piontwise estimate $f(z) \leq (1-\|z\|^{2-1/p}) f_{H^P}$ [280] implies uniform convergence of $H_g f_m$ on compact sets. Next our earlier observation that

$$\left(\sum_{k=0}^{m} \hat{g}(n+k)\hat{f}(k)\right)_{m=0}^{\infty}$$

is a Cauchy sequence uniformly in n and standard estimates for the nth Toylor coefficients based on the Cauchy integral formula allow us to conclude that actually $H_g f_m \to H_g f$ uniformly on compact sets. Finally . the statement follow by Falou's Lemma after taking the limit as $n \to \infty$ in the inequality(24).

The main theorem of this section gives the Lower bound for the norm.

Theorem(6-3-2)[289]; Let as $1 < p < \infty$. Then the norm of the Hilbert matrix as an operator acting on H^p satisfies the Lower estimate

$$\|H\|_{H^p \to H^p} \geq \frac{\pi}{\sin \pi/p} \tag{25}$$

Proof . We break up the argument into four key steps.

Step1. We begin by selecting a family of test functions. Let ε be fixed $0 < \varepsilon < 1$ and choose an arbitrary γ such that $\varepsilon < \gamma < 1$. It is a standard exercise to check that the function $f_\gamma(z) \leq (1-z)^{-\gamma/p}$ belongs to H^P it is also easy to see that

$$\lim_{\gamma \to 1}\|f_\gamma\|_{H^p} = \infty \tag{26}$$

Step 2 . set $f = f_\gamma$ in the representation formula(19). The change of variable $1-r=x$ yields

$$Hf_\gamma(z)=\int_0^1 \frac{(1-r)^{-\gamma/p}}{1-rz}dr=\int_0^1 \frac{x^{-\gamma/p}}{1-z+xz}dx$$

Now define

$$g(z)=\int_0^\infty \frac{x^{-\gamma/p}}{1-z+xz}dx \qquad R(z)=\int_1^\infty \frac{x^{-\gamma/p}}{1-z+xz}dx \tag{27}$$

so that obviously

$$Hf_\gamma(z)=g(z)-R(z) \tag{28}$$

where each of the three function in(28) makes sense almost everywhere on T thus we can consider their $L^P(t)$ norms.

Step3. Note that $z^{1-\gamma/p}g(z)$ can be defined as an analytic function in the complex plane minus two slits : One along the positive part of the real axis from 1 to ∞ and another along the negative part of the real axis from 0 to ∞These value of z will always avoid the real value(1-x)$^{-1}$.

Now, whenever z is a real number such that 0 < z < 1 , after the change of variable xz/(1-z)=u we get

$$z^{1-\gamma/p}g(z)=\frac{z^{1-\gamma/p}}{1-z}\int_0^\infty \frac{x^{-\gamma/p}}{1+x\frac{z}{1-z}}dx=(1-z)^{-\gamma/p}\int_0^\infty \frac{u^{-\gamma/p}}{1+u}du$$

$$=r(\gamma/p)r(1-\gamma/p)(1-z)^{-\gamma/p}=\frac{\pi}{\sin(\pi\gamma/p)}(1-z)^{-\gamma/p}$$

by a well – know identity for the Gamma function[273,268],270].Hence

$$z^{1-\gamma/p}g(z)=(1-z)^{-\gamma/p}\frac{\pi}{\sin(\pi/p)}$$

holds throughout the silt disk D\(-1,0]. Both sides are defined almost everywhere on T, hence their $L^P(t)$ norms make sense and

$$\|g(z)\|_{L^p(T)} = \left\|z^{1-\gamma/p} g(z)\right\|_{L^p(T)} = \frac{\pi}{\sin \pi\gamma/p}\|f_\gamma\|_{H^p} \tag{29}$$

whenever ε < γ < 1.

Step4. We now obtain an upper bound for the $L^P(t)$-norm of the remaining integral R in(27). Note that R can be defined as analytic function in the plane minus a slit from 0 to ∞ along the negative part of the real axis , so it also makes sense almost everywhere on T it follows from the definition of the operator norm and by(28), the triangle inequality, and(29) that

$$\|H\|_{H^p \to H^P}\|f_\gamma\|_{H^p} \geq \|Hf_\gamma\|_{L^p(T)} \geq \left|\|g\|_{L^p(T)} - \|R\|_{L^p(T)}\right|$$

$$= \left|\frac{\pi}{\sin(\pi\,\gamma/p)}\|f_\gamma\|_{H^p} - \|R\|_{L^p(T)}\right|$$

Hence

$$\|H\|_{H^p \to H^p} \geq \left|\frac{\pi}{\sin(\pi\,\gamma/p)} - \frac{\|R\|_{L^p(T)}}{\|f_\gamma\|_{H^p}}\right| \tag{30}$$

Minkowski's inequality in its integral from (see[280,275], followed by a change of variable x-1=u and some simple estimate yields

$$\|R\|_{L^p(T)} = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_1^{\infty}\frac{x^{-\gamma/p}}{1+(x-1)e^{it}}\,dx\right|^p dt\right)^{1/p}$$

$$\leq \int_1^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{x^{-\gamma}}{\left|1+(x-1)e^{it}\right|^p}\,dt\right)^{1/p} dx$$

$$= \int_1^{\infty} x^{-\gamma/p}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{\left|1+(x-1)e^{it}\right|^p}\right)^{1/p} dx$$

$$= \int_0^{2\pi}(1-u)^{-\gamma/p}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{\left|1+ue^{it}\right|^p}\right)^{1/p} du$$

$$\leq \int_0^2 \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\left|1+ue^{it}\right|^p} \right)^{1/p} du$$

$$+ \int_2^\infty (1+u)^{-\varepsilon/p} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\left|1+ue^{it}\right|^p} \right)^{1/p} du$$

where ε was the number fixed in the first step of the proof .

An easy modification of a standard lemma: $\frac{1}{2\pi}\int_0^{2\pi} \left|1+ue^{it}\right|^{-p} dt = o\left(\left|u-1\right|^{1-p}\right)$ as $u \to 1$

[280],Both from below and from above, justifies the convergence of the integral

$$\int_0^2 \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\left|1+ue^{it}\right|^p} \right)^{1/P} du$$

On the other hand $\frac{1}{2\pi}\int_0^{2\pi} \left|1+ue^{it}\right|^{-p} dt \leq (u-1)^{-p}$ for $u > 2$ *so*

$$\int_2^\infty (1+u)^{-\varepsilon/p} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\left|1+ue^{it}\right|^p} \right)^{1/p} du \leq \int_2^\infty \frac{(1+u)^{-\varepsilon/p}}{u-1} du$$

This shows that $\|R\|_{L^P(t)}$ is bounded by a constant independent of our choice of $\gamma \in (\varepsilon,1)$.Now by(26) we get $\|R\|_{L^P(t)} / \|f_\gamma\|_{H^p} \to 0$ as $\gamma \nearrow 1$ and taking the limit in(30) ,we finally obtain(25).

Corollary(6-3-3)[289]: Let $1 < p < \infty$. The norm of the Hilbert matrix as an operator acting on H^p equals

$$\|H\|_{H^p \to H^p} = \frac{\pi}{\sin(\pi/p)}$$

For $g \in L^\infty(0,2\pi)$, let $H_g$be the operator defined by i.e.

$$H_g f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \hat{g}(n+k) \hat{f}(k) \right) z^n$$

and A_g: $H^1 \to l^1$ be the coefficient multiplier operator defined by

$$A_g f = (\hat{f}(n)\hat{g}(n))_{n-0}^{\infty}$$

We refer the reader to[260]for a detailed account of the theory of coefficient multipliers on Hardy spaces.

Hedlund[294] showed that if $\hat{g}(n) \geq 0$ whenever $n \geq 0$, then the norm of the operator H_g viewed as an l^1 operator (which is equivalent to begin an H^2 operator) equals the norm of the coefficient multiplier operator A_g from H^1 to the space l^1 of absolutely summable sequences .

This is implicit in the proof of Theorem (6.3.1) [294]. Thus.

$$\sum_{k=0}^{\infty}\left|\hat{f}(k)\hat{g}(k)\right| \leq \left\|H_g\right\|_{H^2 H^2}\|f\|_{H^1} \tag{31}$$

The standard choice $g(t) = ie^{-it}(\pi - t), 0 \leq t < 2\pi$ yield as a corollary Hardy's classical inequality see[290]or[264]:

$$\sum_{k=0}^{\infty}\frac{\left|\hat{f}(k)\right|}{k+1} \leq \pi\|f\|_{H^1} \qquad \text{for every } f \in H^2 \tag{32}$$

There is a slight improvement which is also sharp and canbe found

in[182]:

$$\sum_{k=0}^{\infty}\frac{\left|\hat{f}(k)\right|}{k+1/2} \leq \pi\|f\|_{H^1} \qquad \text{for every } f \in H^1. \tag{33}$$

This result can also be obtained from our Theorem(6.3.1) and by(31) choosing $g(t) = \pi e^{i\left(\frac{\pi-1}{2}\right)}, 0 \leq t < 2\pi$. Since$\|g\|_{\infty} = \pi$, a straightforward calculation shows that.

$$g(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\text{int}} g(t)dt = \frac{1}{n+1/2} \qquad n \geq 0$$

and (33) follow. It is interesting to notice that the constant π is best possible in both inequalities (32) and(33) even though this may look paradoxical at a first glance.

Let $A(z) = \pi^{-1}dxdy = \pi^{-1}rdrdt$ denote the normalized Lebesgue area measure on D. $z = x + yi = re^{it}$. Recall that the Bergman space $A^p$is the set of all f in H(D) for which

$$\|f\|_{A^p} = \left(\int_D |f(z)|^p dA(z)\right) \qquad < \infty$$

It is known that $H^p \subset A^{2p}$. Actually , the function in Bergman spaces exhibit a behavior some-what similar to that Hardy spaces functions but often a bit more complicated For more about these spaces, the reader may consult[291]or[282].

It was shown in[258]that in that the Hilbert matrix operator is unbounded on A^2.The situation is actually even worse : there exist a function f in A^2 such that not only $Hf \notin A^2$ but even the series defining $Hf(0)$ is divergent. Indeed, consider the function f defined by

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{\log(n+1)} z^n.$$

Then $f \in A^2$ since $\|f\|_{A^2}^2 = \sum_{n=1}^{\infty} (n+1)^{-1} \log^{-2}(n+1) < \infty$ .However,

$$Hf(0) = \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} = \infty.$$

It is well known that there exists a constant such that C > 0 such that

$$\sum_{K=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \le C\|f\|_{A^p}$$

for every $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ that belongs to $A^p, 2 < p < \infty$. This is a result of Nakamura Ohya, and Watanabe[269]; a proof can also be found in [291]. Therefore if f belongs to A^2, $2 < p < \infty$, and $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ then the power series

$$Hf(z) = \sum_{n=0}^{\infty} \left(\sum_{K=0}^{\infty} \frac{\hat{f}(k)}{n+k+1}\right) z^n$$

has bounded coefficients, hence its radius of convergence is ≥ 1. in this way we obtain a well defined analytic function Hf on D for each $f \in A^p, 2 < p < \infty$ .It actually turns out as was proved in[279] that H maps A^2into itself in a bounded fashion whenever $2 < p < \infty$. In order to show this, Diamantopoulos again used formula(19) in which the convergence of the integral is guaranteed by the poinwise estimates on A^P function and by the fact that $1/(1-rz)$ abounded function of f for each, $\mathbf{z} \in D$ (see[279].

The following formula shows that the Hilbert matrix operator has a different integral representation on the Bergman space. The representation below should be compared with our Theorem (6.3.1)for H^{p} applied to the Hilbert matrix for the Hardy spaces in order to appreciate the difference between the two situations.

Theorem(6-3-4)[289]. Let $2 < p < \infty$. Then the operator H can be written as follows:

$$Hf(z) = \int_D \frac{f(\overline{\omega})}{(1-\omega)(1-\overline{\omega}z)} dA(\omega) \tag{34}$$

for any $f \in A^{p}$.

Proof. writing

$$f(z) = \sum_{k=0}^{\infty} akz^k \qquad \frac{1}{1-\omega}\sum_{j=0}^{\infty}\omega^j \qquad \frac{1}{1-\omega z} = \sum_{n=0}^{\infty}\overline{\omega}^n z^n$$

and recalling that

$$\int_D \omega^m \overline{\omega} dA(\omega) = \begin{cases} \frac{1}{n+1}, & ifm = n \\ 0, & ifm \neq n \end{cases}$$

we see that

$$Hf(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{ak}{n+k+1}\right)z^n = \sum_{k=0}^{\infty}\left(\sum_{k=0}^{\infty}\int_D \omega^j \overline{\omega}^{n+k} dA(\omega)\right)z^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\int_D \frac{\overline{\omega}^k}{1-\omega} dAt(\omega)\right)(\overline{\omega}z)^n = \int_D^1 \frac{f(\overline{\omega})}{(1-\omega)(1-\overline{\omega}z)} dA(\omega)$$

The interchange of integrals and sums is again easily justified by a geometric series argument.

It should be observed that the representing kernel lacks the usual “symmetry” in two variables.

Our next result is analogous to Theorem(6.3.2)The key idea of the approach below is again the observation that our function f_γ are “ not far from begin eigenvectors’’ of the Hilbert matrix H. The proof below can also be adapted to the Hardy space case while the earlier proof of Theorem(6.3.2)with its typical “ Hardy space flavor ” cannot be made to work for A^P spaces.

Theorem(6.3.5)[290]: Let $2 < p < \infty$. Then the norm of the Hilbert matrix as an operator acting on A^P satisfies the lower estimate

$$\|H\|_{A^p \to A^p} \geq \frac{\pi}{\sin(2\pi/p)}$$

Proof : We use the same function f_γ as in the proof of Theorem(6.3.2). Note that $f_\gamma \in A^p$ if and only if $\gamma < 2$;this is well known and will be quantified below. Aplying H to f_γ and making the change of variable $\omega = (1-rz)/(1-r)$, a direct computation shows that $Hf_\gamma = \phi_\gamma f_\gamma$, where for every z in D we define

$$\phi_\gamma(z) = \int_1^\infty \frac{d\omega}{\omega(\omega - z)^{1-\gamma/p}} \tag{35}$$

Here is how the above formula should be understood. As r traverses the interval [0,1), the point ω runs long a ray L_z from 1 to the point at infinity. This ray is contained entirely in the half- plane to the right of the point 1 since

$$\operatorname{Re}\omega = \frac{1 - \operatorname{Re} z}{1-r} > 1$$

It is also important to observe that the integration in(35) can always be performed over the ray$[1,\infty)$ of the positive real semi-axis instead of over $L_z = \{(1-rz)/(1-r) : 0 \le r < 1\}$, Since for anys fixed z in D the integrals over the two paths coincide. This can be seen by a typical argument involving the Cauchy integral theorem and integrating over the triangle with the vertices $1,(1-rz)/(1-r)$ and $\operatorname{Re}(1-rz)/(1-r)$ and Letting $r \to 1$. Namely, writing $z = x + yi,$ we see that on the vertical line segment S_z from $\operatorname{Re}(1-rz)/(1-r) = (1-rx)/(1-r)$ to $(1-rx-ryi)/(1-r)$ every ω point satisfies

$$|\omega - z| \geq \operatorname{Re}\frac{1-rz}{1-r} - 1 = \frac{r(1-x)}{1-r}, \qquad |\omega| \geq \frac{1-rx}{1-r}$$

and the length of the segment S_z is $\left|1m\frac{1-rz}{1-r}\right| = \frac{r|y|}{1-r}$ Thus.

$$\left|\int_{S_t} \frac{d\omega}{\omega|\omega - z^{1-\gamma/p}|}\right| \leq \int_{S_t} \frac{|d\omega|}{|\omega||\omega - z|^{1-\gamma/p}} \leq \frac{\frac{r|y|}{1-r}}{\frac{1-rx}{1-r}\left(\frac{r(1-x)}{1-r}\right)^{1-\gamma/p}}$$

$$\leq \frac{r|y|}{1-rx}\left(\frac{1-r}{r(1-x)}\right)^{1-\gamma/p} \to 0 \text{ as } r \nearrow 1$$

By letting $r \nearrow 1$ it follow that

$$\int_{L_z} \frac{d\omega}{\omega(\omega-z)^{1-\gamma/p}} = \int_1^\infty \frac{d\omega}{\omega(\omega-z)^{1-\gamma/p}}$$

Knowing that in the definition (35)of the function $\emptyset_\gamma$ we can take to be a real numbers ≥ 1, it is immediate that $\emptyset_\gamma$ belongs the disk algebra wheneve $\gamma \leq 2$r since p >2 now (the case $\gamma = 2$ will also be useful to us although $f_2 \notin A^p$). Indeed $\emptyset_\gamma$ is clearly well defined as an analytic function of z for all $z \in \overline{D}\setminus\{1\}$as $1 - \gamma/p > 0$.The inequality The $s - 1 \leq |s - z|$ obviously holds for $s > 1$ and all z in $\overline{D}$, hence the function $\emptyset_\gamma$ attains its maximum modulus a t z=1 and

$$\phi_\gamma(1) = \int_1^\infty \frac{ds}{s(s-1)^{1-\gamma/p}} = \int_0^\infty \frac{dx}{(1+x)x^{1-\gamma/p}} = \frac{\pi}{\sin(\pi\gamma/p)} < \infty$$

whenever $\gamma \leq 2 < p$.

Set $C_\gamma = \|f_\gamma\|_{A^p}$ .By integrating in polar coordinates centered at z=1 rather than at the origin, one easily chechs that

$$C_\gamma^p = \int_D \frac{1}{|1-z|} dA(z) = 2\int_0^{\pi/2}\int_0^{2\cos tt} r^{1-\gamma} drdt$$

$$= \frac{2^{3-\gamma}}{2-\gamma}\int_0^{\pi/2} \cos^{2-\gamma} tdt = \frac{2^{3-\gamma}}{2-\gamma} B(3-\gamma, 3/2) \to \infty$$

as $\gamma \nearrow 2$. Defining $g_\gamma = f_\gamma C_\gamma$,it is clear that $Hg_\gamma = \phi_\gamma g_\gamma$ and the family of functions $\left\{|g_\gamma(z)|^p : 0 \leq \gamma \leq 2, z \in D\right\}$ has all the properties of an approximate identity:

(a) $|g_\gamma(z)|^p \geq 0$

(b) $\int_D |g_\gamma|^p dA = 1$

(c) $|g_\gamma(z)|^p \to 0$ on any compact subset of $\overline{D}\setminus\{1\}$,as $\gamma \to 2$ s

Using the usual procedure of splitting the disk into two domains $D_\varepsilon = \{z \in D: |z-1| < \varepsilon\}$ and D/D_ε and estimatinsg the difference

$$\int_D \left|Hg_\gamma(z)\right|^p dA(z) - \left|\phi_2(1)\right|^p = \int_D \left(\left|\phi_\gamma(z)\right|^p - -\left|\phi_2(1)\right|^p\right)\left|g_\gamma(z)\right|^p dA(z)$$

separately over each one the two regions, we see that difference tends to zero as $\gamma \to 2$ because the function $\phi_\gamma(z)$ is continuous on the compact set $\{(z,\gamma) \in \overline{D} \times [0,2]\}$ and is, hence, uniformly continuous there. It is also uniformly bounded on $\overline{D}_\varepsilon \times [0,2]$,a fact used also in one of the two estimates. This allows us to conclude that

$$\lim_{\gamma \to 2} \left\|Hg_y\right\|_{A^p} = \lim_{\gamma \to 2} \left\|\phi_\gamma g_\gamma\right\|_{A^p} = \left\|\phi_2\right\|_\infty = \phi_2(1) = \frac{\pi}{\sin(2\pi/p)}$$

Which gives the desired lower bound for the norm of H on A^p

By combining Theorem(6.3.5). with the upper bound proved in[279] $for\ 4 \le p < \infty$. we get the following consequence.

Corollary(6.3.6)[289]. Where $4 \le p < \infty$, the norm of the Hilbert matrix as an operator acting on A^p equals

$$\left\|H\right\|_{A^p \to A^p} = \frac{\pi}{\sin(2\pi/p)}$$

It should be remarked that the assumption $p-4 \ge 0$ is fundamental in obtaining the upper bound by Diamantopoulos' method[279].Let us new recall his estimates when $2 < p < 4$ .One is as follows:

$$\left\|Hf\right\|_{A^p} \le C_p \frac{\pi}{\sin 2\pi/p} \left\|f\right\|_{A^p} \quad \text{for every } f \in A^p \tag{36}$$

where $C_p \to \infty$ as $p \to 2$ The other is:

$$\left\|Hf\right\|_{A^p} \le \left(p/2+1\right)^{1/p} \frac{\pi}{\sin 2\pi/p} \left\|f\right\|_{A^p} \tag{37}$$

whenever $f \in A^p$ and $f(0)=0$ (again,2 < p< 4). Although the present time we are not able to extend Corollary(6.3.6) to the entire range $2 < p < \infty$, we do have a reasonable improvement of the upper bound(36) and our result is also closer to the estimate for $p \ge 4$.

Theorem(6.3.7)[289]. Let $2 < p < 4$ then there exists an absolute constant C independent of $p, 1 < C < \infty$ such that

$$\|Hf\|_{A^p} \le C \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p} \qquad \text{for every} \quad f \in A^p$$

Proof. Let $f \in A^p$ be a function whose Taylor series is $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ .Write $\hat{f}$=f_0+f where $f_0(z) = \hat{f}(0)$ and $f_1(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ .Then using we find that

$$\|Hf_1\|_{A^p} \le (p/2+1)^{1/p} \frac{\pi}{\sin(2\pi/p)} \|f_1\|_{A^p} \le \sqrt{3} \frac{\pi}{\sin(2\pi/p)} \|f_1\|_{A^p} \tag{38}$$

From

$$Hf_o(z) = \sum_{k=0}^{\infty} \frac{\hat{f}(0)}{n+1} z^n = \frac{\hat{f}(0)}{z} \log \frac{1}{1-z}$$

we obtain

$$\|Hf_0\|_{A^p} = \left|\hat{f}(0)\right| \left\| \frac{1}{z} \log \frac{1}{1-z} \right\|_{A^p}$$

It is easy to see that $C_p := \left\| \frac{1}{z} \log \frac{1}{1-z} \right\|_{A^p} \le C_4 < \infty$ From the version of the mean-value equality $\hat{f}(0) = \int_D f(z) dA(z)$ we find that $\left|\hat{f}(0)\right| \le \|f\|_{A^p}$ Thus.

$$\|Hf_0\|_{A^p} \le C_4 \|f\|_{A^p} \le C_4 \frac{\pi}{\sin 2\pi/p} \|f\|_{A^p} \tag{39}$$

Since

$$\|f_1\|_{A^p} = \|f - f_0\|_{A^p} \le \|f\|_{A^p} + \|f_0\|_{A^p} \le 2\|f\|_{A^p}.$$

Using(38)and(39)we get

$$\|Hf\|_{A^p} \le \left(2\sqrt{3} + C_4\right) \frac{\pi}{\sin(2\pi/p)} \|f\|_{A^p}.$$

The exact computation norm of the Hilbert matrix as an operator on A^p by the methods employed here might be a more difficult problem than its Hardy space counterpart perhaps because integral of H is more involved.The case $2 < p < 4$ well reguire a further stady.

List of Symbols

Symbol		Page
$\ominus$	:direct deference	1
Im	: Imaginary	1
Ker	: Kernel	2
dom	: Domain	4
ran	: range	4
arg	: argument	4
Re	: Real	4
$\oplus$	: orthogonal Sum	5
Ext	: Exterior	6
clos	: closure	15
qsc	: quasi self adjont contraction	29
TPSG	: tow-point self-similar fractal graph	36
p.c.f	: post- critical finite	39
deg	: degree	41
max	: maximum	42
supp	: Support	52
SG	: Sierpinski Gaskef	55
i.f.s	: iterated function system	55
a. e	: Almost Everywhere	60
L^p	: lebesgue measure on the real line	61
Prob	: probability	67
sup	: Supremum	68
det	: determinant	73
min	: Minimum	74
Tr	: Trace	78
Spec	: spectrum	78
L^2	: Hilbert Space	91
CMV	: Contero Moral and Velázquez	107
ℓ^2	: Hilbert Space	107
A^2	: Hardy spaces	108
dim	: Dimension	110
OPUC	: Orthogonal Pelynomials on the Unit Circle	123
diag	: diagonal	137
WN	: weakly non degenerate	158
HC	: harmonic coordinates	166
OSC	: Oscillation	176
H^p	: Hardy spaces	204
A^p	: Bergman Space	213
ℓ^p	: all sequence –summable complex	233
H^∞	: Essential Hardy spaces	236
L^∞	: Essential lebesgue spaces	237

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