بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

Sudan University of Science & Technology

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Laplace Transform

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In Mathematics

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الأيـــــه

(اللَّهُ لاَ إِلَٰهَ إِلاَّ هُوَ الْحَيُّ الْقَيُّومُ لاَ تَأْخُذُهُ سِنَةٌ وَلاَ نَوْمٌ لَّهُ مَا فِي السَّمَاوَاتِ وَمَا فِي الأَرْضِ مَن ذَا الَّذِي يَشْفَعُ عِنْدَهُ إِلاَّ بِإِذْنِهِ يَعْلَمُ مَا بَيْنَ أَيْدِيهِمْ وَمَا خَلْفَهُمْ وَلاَ يُحِيطُونَ بِشَيْءٍ مِّنْ عِلْمِهِ إِلاَّ بِمَا شَاء وَسِعَ كُرْسِيُّهُ السَّمَاوَاتِ وَالأَرْضَ وَلاَ يَؤُودُهُ حِفْظُهُمَا وَهُوَ الْعَلِيُّ الْعَظِيمُ)

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Dedication

To our mothers and fathers

To our family and my tribe

To our teachers

To our colleagues and my colleagues

To our burn candles that illuminate for others

He taught me to each the characters

But above all to my prophet MOHAMED

Acknowledgments

First I would like to thank without end to our greater ALLAH, then I would like to express about my appreciation and thanks to our

supervisor: Dr. Salwa Harfi

and thanks for everyone help me...

Researchers...

Abstract

We studied in this research the Laplace transformation of function. In chapter 1 we defined Laplace transform and we solved some examples and we proved some theorems.

In chapter 2 we gave the inverse of Laplace and solved some problems and we defined the convolution.

In chapter 3 we applied the Laplace transformation to solving the system of differential equations with constant coefficient.

الخلاصة

في هذا البحث تم دراسة تحويل لابلاس للدوال ، في الباب الاول تم تعريف تحويل لابلاس و تم حل بعض المسائل و برهنة بعض النظريات.

في الباب الثاني اعطينا تحويل لابلاس العكسي و تم حل بعض المسائل و ايضاً تعريف الالتفاف. اما في الباب الثالث فقد تم تطبيق تحويل لابلاس في حل منظومة معادلات تفاضلية ذات معاملات ثابتة.

Introduction

We shall study in our research the importance of existence of Laplace transforms of functions and also we interested by applying the convolution to Laplace inverse, and we also we are giving some applications to illustrate the importance of our theorems (Existence theorem) used in the applications.

Chapter 1

The Laplace Transform

In this chapter we shall introduce a concept which is especially useful in the solution of initial-value problems. This concept is the so-called Laplace transform, which transforms a suitable function F of order variable t into related function f of real variable a linear differential equation on unknown” function so t it transforms. The given initial-value problem into an algebraic problem in variable s, in section (1.3) we shall indicate. Just how this transformation is accomplish and how the resulting algebraic problem is then comploxed to find the solution of the given initial-value problem first however, in section (1.1) we shall introduce the Laplace transform itself and develop certain of it’s most basic and useful properties.

At First we define Improper Integral

We say that the integral $\int_a^b f(x)\,dx$ is improper integral:

i) If the integer and function $f(x)$ has some dis continuous points on $[a, b]$.

ii) If a or b is equal infinity.

In case 1 we have:

1)If $x = b$ is the point of discontinuity of $f(x)$ we put

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)\,dx$$

2)If $x = a$ is the point of discontinuity of $f(x)$, we put

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^{b} f(x)\,dx$$

3)If $a < c < b,\ c$ is the point of discontinuity of $f(x)$, we put

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon'\to 0}\int_{c+\epsilon'}^b f(x)\,dx$$

while case 2 we have

1) $\displaystyle\int_a^\infty f(x)\,dx = \lim_{u\to\infty}\int_a^u f(x)\,dx$

2) $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{u'\to-\infty}\int_{u'}^b f(x)\,dx$

3) $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \lim_{u\to\infty}\int_0^u f(x)\,dx + \lim_{u'\to-\infty}\int_{u'}^0 f(x)\,dx$

Example 1.1

(1) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ has discontinuity at $x = 3$

(2) $\int_0^2 \frac{dx}{2-x}$ has discontinuity at $x = 2$

(3) $\int_0^\infty \frac{dx}{x^2+4}$ improper integral at $b = \infty$

Note:

The improper integral has a value if the limits in every case above are exist.

Definition: **Bounded Functions**

We say that $f(x)$ is bounded if $|f(x)| < M,\ \ \forall M \in IN$, and then $-M < f(x) < M$

$-M$ is the lower bound of $f(x)$ and M is the upper bound of $f(x)$.

Definition, Existence, and Basic Properties of the Laplace Transform

Definition and Existence

Definition:-

Let F be a real-valued function of the real variable defined $t > 0$ let s be a variable which we shall assume to be real, and consider the function defined by

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad (1.1)$$

For all values of s for which this integral exists. The function f defined by the integral (1.1) is called the Laplace transform of the function F. We shall denote the Laplace transform f of F by $\mathscr{L}(F)$ and shall denote $f(s)$ by $\mathscr{L}\{F(t)\}$ in order to be certain that the integral (1.1) does exist for some range of values of s the must impose suitable restrictions upon the function F under consideration mental do this shortly; however, first we shall directly determine the Laplace transforms a few simple functions

Example 1.2

Consider the function F defined by

$$F(t) = 1, \qquad \text{for } t > 0$$

Then

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}.1\, dt = \lim_{R\to\infty} \int_0^R e^{-st}.1\, dt = \lim_{R\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^R$$
$$= \lim_{R\to\infty} \left[\frac{1}{s} - \frac{e^{-sR}}{s}\right] = \frac{1}{s}$$

For all $s > 0$. Thus we have

$$\mathscr{L}\{1\} = \frac{1}{s} \quad (s > 0).$$

Example 1.3

Consider function F defined by then

$F(t) = t, \qquad \text{for } \; t > 0$

then

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st}.\, t\, dt = \lim_{R\to\infty} \int_0^R e^{-st}.\, t\, dt$$

Let

$$u = t, \qquad dv = e^{-st}dt$$

$$du = 1, \qquad v = -\frac{1}{s}e^{-st}$$

Hence Integrating by parts we get

$$\mathscr{L}\{t\} = \lim_{R\to\infty} \int_0^R e^{-st}.\, t\, dt = u - v - \int_0^R v\, du$$

$$= \lim_{R\to\infty} \left(-\frac{1}{s}.\, te^{-st}\right) + \frac{1}{s}\int_0^R e^{-st}\, dt$$

$$\lim_{R\to\infty} \left[\left[-\frac{1}{s}.\, te^{-st}\right] \begin{matrix} R \\ 0 \end{matrix} + \left[-\frac{1}{s^2} e^{-st}\right] \begin{matrix} R \\ 0 \end{matrix}\right]$$

$$= \lim_{R\to\infty} \left[-\frac{e^{-st}}{s^2}(st+1)\right] \begin{matrix} R \\ 0 \end{matrix} = \lim_{R\to\infty} \left[\frac{1}{s^2} - \frac{e^{-sR}}{s^2}(sR+1)\right] = \frac{1}{s^2}$$

For all $s > 0$. Thus

$$\mathscr{L}\{t\} = \frac{1}{s^2} \quad (s > 0).$$

Example 1.4

Consider the function f defined by

$F(t) = e^{at}, \qquad \text{for } \; t > 0$

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \lim_{R\to\infty}\int_0^R e^{(a-s)t}\,dt = \lim_{R\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^R$$

$$= \lim_{R\to\infty}\left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s}\right] = -\frac{1}{a-s} = \frac{1}{s-a} \text{ for all } s > a$$

Thus

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \quad (s > a).$$

Example 1.5

Consider the function F defined by

$$F(t) = \sin bt\,, \qquad \text{for } \ t > 0$$

$$\mathscr{L}\{\sin bt\} = \int_0^\infty e^{-st}.\sin bt\,dt = \lim_{R\to\infty}\int_0^R e^{-st}.\sin bt\,dt$$

$$u = e^{-st} \Rightarrow du = -se^{-st}dt$$

$$v = \sin bt\,dt \ \Rightarrow v = -\frac{1}{b}\cos bt$$

$$I = \lim_{R\to\infty}\left[-\frac{1}{b}e^{-st}\cos bt\right]_0^R - \frac{s}{b}\int_0^R e^{-st}\cos bt\,dt$$

$$I = \lim_{R\to\infty}\left[-\frac{1}{b}e^{-st}\cos bt - \frac{s}{b}J\right] \longrightarrow (1)$$

$$J = \int_0^R e^{-st}\cos bt\,dt$$

$$u = e^{-st} \Rightarrow du = -se^{-st}dt$$

$$v = \cos bt\,dt \ \Rightarrow v = \frac{1}{b}\sin bt$$

$$J = \lim_{R\to\infty}\left[\frac{1}{b}e^{-st}\sin bt\right]_0^R + \frac{s}{b}\int_0^R e^{-st}\sin bt\,dt$$

$$J = \left[\frac{1}{b}e^{-st}\sin bt\right]_0^R + \frac{s}{b}J \longrightarrow (2)$$

form (2) in (1)

$$I = \lim_{R\to\infty}\left[-\frac{1}{b}e^{-st}\cos bt - \frac{s}{b^2}e^{-st}\sin bt - \frac{s^2}{b^2}I\right]$$

$$I + \frac{s^2}{b^2}I = \lim_{R\to\infty}\left[-\frac{1}{b}e^{-sR}\cos bR - \frac{s}{b^2}e^{-sR}\sin bR\right]$$

$$I\left[1 + \frac{s^2}{b^2}\right] = \lim_{R\to\infty}\left[-\frac{1}{b}e^{-sR}\cos bR - \frac{s}{b^2}e^{-sR}\sin bR\right]$$

$$I\left[1 + \frac{s^2}{b^2}\right] = \lim_{R\to\infty}[-be^{-sR}\cos bR - se^{-sR}\sin bR]$$

$$I[S^2 + b^2] = \lim_{R\to\infty}[-be^{-sR}\cos bR - se^{-sR}\sin bR]$$

$$I = \lim_{R\to\infty}\left[-\frac{b}{S^2+b^2}e^{-sR}\cos bR - \frac{S}{S^2+b^2}e^{-sR}\sin bR\right]_0^R$$

$$= \lim_{R\to\infty}\left[-\frac{b}{S^2+b^2}(S\cos bR - b\sin bR)\right]_0^R$$

$$= \lim_{R\to\infty}\left[-\frac{b}{S^2+b^2} - \frac{e^{-sR}}{S^2+b^2}(S\sin bR - b\cos bR)\right] = \frac{b}{s^2+b^2}$$

$$\mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \quad \text{for all } s > 0.$$

Example 1.6

Consider the function F defined by

$F(t) = \cos bt\,, \qquad$ for $\; t > 0$

$$\mathscr{L}\{\cos bt\} = \int_0^\infty e^{-st}.\cos bt\,dt = \lim_{R\to\infty}\int_0^R e^{-st}.\cos bt\,dt$$

by part integral

$$u = e^{-st} \Rightarrow du = -se^{-st}dt$$

$$dv = \sin bt\,dt \;\Rightarrow v = -\frac{1}{b}\cos bt$$

$$I = \lim_{R\to\infty}\left[\frac{1}{b}e^{-sR}\sin R + \frac{s}{b}\int_0^R e^{-st}\sin bt\,dt\right]$$

$$I = \lim_{R\to\infty}\left[\frac{1}{b}e^{-sR}\sin R + \frac{s}{b}J\right] \longrightarrow (1)$$

$$J = \int_0^R e^{-st}\sin bt\,dt$$

$$u = e^{-st} \Rightarrow du = -se^{-st}dt$$

$$v = \sin bt\,dt \;\Rightarrow v = -\frac{1}{b}\cos bt$$

$$J = \lim_{R\to\infty}\left[-\frac{e^{-st}}{b}\cos R - \frac{s}{b}\int_0^R e^{-st}\sin bt\,dt\right]$$

$$J = \left[-\frac{e^{-st}}{b}\cos bR - \frac{s}{b}I\right] \longrightarrow (2)$$

form (2) in (1)

$$I = \lim_{R\to\infty}\left[\frac{1}{b}e^{-st}\sin bR - \frac{se^{-st}}{b^2}\cos bR - \frac{s^2}{b^2}I\right]$$

$$I + \frac{s^2}{b^2} I = \lim_{R\to\infty} \left[\frac{1}{b} e^{-sR} \sin bR - \frac{s}{b^2} e^{-sR} \cos bR\right]$$

$$I\left[1 + \frac{s^2}{b^2}\right] = \lim_{R\to\infty} \left[-\frac{1}{b} e^{-sR} \sin bR - \frac{s}{b^2} e^{-sR} \cos bR\right]$$

$$I\left[1 + \frac{s^2}{b^2}\right] = \lim_{R\to\infty} \left[-\frac{1}{b} e^{-sR} \sin bR - \frac{s}{b^2} e^{-sR} \cos bR\right]$$

$$I[S^2 + b^2] = \lim_{R\to\infty} [be^{-sR} \sin bR - se^{-sR} \cos bR]$$

$$I = \lim_{R\to\infty} \left[-\frac{b}{S^2 + b^2} e^{-sR} \cos bR - \frac{S}{S^2 + b^2} e^{-sR} \sin bR\right]_0^R$$

$$= \lim_{R\to\infty} \left[-\frac{be^{-sR}}{S^2 + b^2} \sin bR - \frac{Se^{-sR}}{S^2 + b^2} \cos bR\right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{e^{-sR}}{S^2 + b^2} (b \sin bR - S \cos bR)\right]_0^R$$

$$\therefore \mathscr{L}\{\cos bt\} = \frac{s}{s^2 + b^2} \quad (s > 0).$$

In each the above examples we have seen directly that the integral (1.1) actually does exist for some range of values of s. We shall now determine a class of functions F for which this is always the case. To do so we consider certain properties of functions.

Definition :-

A function F is said to be piecewise continuous (or sectionally continuous) on a finite interval $a \leq t \leq b$ if this interval can be defined into a finite number of subintervals such that (1) F is continuous in the interior of each of these subintervals, and (2) $F(t)$ approaches finite limits $as\ t$ approaches either endpoint of each of the subintervals, from its interior.

Suppose F is piecewise continuous on $a \leq t \leq b$, and t_0, $a < t_0 < b$, is an endpoint of one of the subintervals of the above definition. Then the tinite limit approached by $F(t)$ as t approaches t_0 from the left (that is, through smaller values of) is called the left-hand limit of $F(t)$ as t approaches t_0, denoted by $\lim_{t \to t_0^-} F(t)$ or by (t_0^-). In like manner, the finite limit approached by $F(t)$ as t approaches t_0 from the right (through larger values) is called the right-hand limit of $F(t)$ as t approaches t_0 denoted by $\lim_{t \to t_0^+} F(t)$ or $F(t_0 +)$. We emphasize that at such a point t_0m both $F(t_0 -)$ and $F(t_0 +)$ are finite but they are not in general equal.

We point out that if F is continuous on $a \leq t \leq b$ it is necessarily piecewise continuous on this interval. Also, we note that if F is pieswise continuous on $a \leq t \leq b$, then F is integrable on $a \leq t \leq b$.

Example 1.7

Consider the function F defined by

$$F(t) = \begin{cases} -1, & 0 < t < 2, \\ 1, & t > 2. \end{cases}$$

F is piecewise continuous on every finite interval $0 \leq t \leq b$, for every positive number b. At $t = 2$, we have

$$F(2 -) = \lim_{t \to 2-} F(t) = -1,$$

$$F(2 +) = \lim_{t \to 2+} F(t) = +1,$$

The graph of F is show in figure (1)

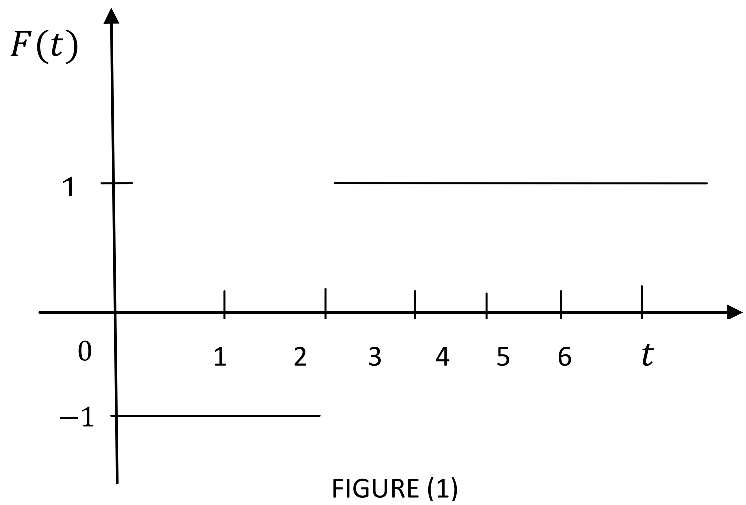

FIGURE (1)

Definition :-

A function F is said to be of exponential order if there exists a constant α and positive constants t_0 and M such that

$$e^{-\alpha t}|F(t)| < M$$

For all $t > t_0$ at which $F(t)$is defined. More explicitly, if F is of exponential order corresponding to some definite constant x in (1.7), then we say that F is of exponential order $e^{\alpha t}$.

In other words, we say that F is of exponential order if a constant α exist such that the product $e^{-\alpha t}|F(t)|$ is bounded for all sufficiently large values of t. From (1.7) we have

$$|F(t)| < Me^{\alpha t}$$

For all $t > t_0$ at which $F(t)$ is defined. Thus if F is of exponential order and the values $F(t)$ of F become infinite as $t \to \infty$, these values cannot become infinite more rapidly than a multiple M of the corresponding values $e^{\alpha t}$ of some exponential function. We note that if F is of exponential order $e^{\alpha t}$, then F is also of exponential order $e^{\beta t}$ for any $\beta > \alpha$.

Example 1.8

Every bounded function is of exponential order, with the constant $\alpha = 0$. Thus, for example, $\sin bt$ and $\cos bt$ are of exponential order.

Example 1.9

The function F such that $F(t) = e^{\alpha t} \sin bt$ is of exponential order, with the constant $\alpha = a$. For we then have

$$e^{-\alpha t}|F(t)| = e^{-\alpha t}e^{\alpha t}|\sin bt| = |\sin bt|,$$

Which is bounded for all t.

Example 1.10

Consider the function F such that $F(t) = t^n$, where $n > 0$. Then $e^{-\alpha t}|F(t)|$ is $e^{-\alpha t}t^n$. For any $\alpha > 0, \lim_{t\to\infty} e^{-\alpha t}t^n = 0$. Thus there exists $M > 0$ and $t_0 > 0$ such that

$$e^{-\alpha t}|F(t)| = e^{-\alpha t}t^n < M$$

For $t > t_0$. Hence $F(t) = t^n$ is exponential order, with the constant α equal to any positive number.

Example 1.11

The function F such that $F(t) = e^{t^2}$ is not of exponential order, for in this case $e^{-\alpha t}|F(t)|$ is $e^{t^2-\alpha t}$ and this becomes unbounded as $t \to \infty$, no matter what is the value of α.

We shall now proceed to obtain a theorem giving conditions on F which are sufficient for the integral (1.1) to exist. To obtain the desired result we shall need the following two theorems from advanced calculus, which we state without proof.

Theorem A. comparison Test for Improper Integerls

Hypothesis

1. Let g and G be real functions such that

$$0 \leq g(t) \leq G(t) \text{ on } a \leq t < \infty.$$

2. Suppose $\int_0^\infty G(t)\ dt$ exists.

3. Suppose g is integrable on every finite closed subinterval of $a \leq t < \infty$.

Conclusion. Then $\int_a^\infty g(t)\,dt$ exists.

Theorem B

Hypothesis

1. Suppose the real function g is integrable on over finite closed subinterval of $a \leq t < \infty$.
2. Suppose $\int_a^\infty |g(t)|\,dt$ exists.

Then $\int_a^\infty |g(t)|\,dt$ exist.

We now state and prove an existence theorem for Laplace transforms.

Theorem 1.1

Hypothesis. Let F be a real function which has the following properties:

1. F is piecewise continuous in every finite closed interval $0 \leq t \leq b \; (b > 0)$.
2. F is of exponential order; that is, there exist $\alpha, M > 0$ and $t_0 > 0$ such that

$$e^{-\alpha t}|F(t)| < M \;\; for \;\; t > t_0$$

The Laplace transform

$$\int_0^\infty e^{-st} F(t)\,dt$$

of F exists for $s > \alpha$

proof:

We have

$$\int_0^\infty e^{-st}F(t)\,dt = \int_0^{t_0} e^{-st}F(t)\,dt + \int_{t_0}^\infty e^{-st}F(t)\,dt$$

By Hypothesis 1, the first integral of the right member exists. By Hypothesis 2,

$$e^{-st}|F(t)| < e^{-st}Me^{\alpha t} = Me^{-(s-\alpha)t}$$

For $t > t_0$. Also

$$\int_{t_0}^\infty Me^{-(s-\alpha)t}\,dt = \lim_{R\to\infty}\int_{t_0}^R Me^{-(s-\alpha)t}\,dt = \lim_{R\to\infty}\left[-\frac{Me^{-(s-\alpha)t}}{s-\alpha}\right]_{t_0}^R$$

$$= \lim_{R\to\infty}\left[\frac{M}{s-\alpha}\right]\left[e^{-(s-\alpha)t_0} - e^{-(s-\alpha)R}\right]$$

$$= \left[\frac{M}{s-\alpha}\right]e^{-(s-\alpha)t_0} \quad \text{if} \quad s > \alpha$$

Thus

$$\int_{t_0}^\infty Me^{-(s-\alpha)t}\,dt \quad \text{exists for } s > \alpha$$

Finally, by Hypothesis 1, $e^{-st}|F(t)|$ is integrable on every finite closed subintegral of $t_0 \le t < \infty$. Thus, applying Theorem A with $g(t) = e^{-st}|F(t)|$ and $G(t) = Me^{-(s-t)}$ we see that

$$\int_{t_0}^\infty e^{-st}|F(t)|\,dt \quad \text{exists if } s > \alpha$$

In other words,

$$\int_{t_0}^\infty |e^{-st}F(t)|\,dt \quad \text{exists if } s > \alpha$$

and so Theorem B

$$\int_{t_0}^{\infty} e^{-st} F(t)dt$$

also exists if $s > \alpha$. Thus Laplace transform of F exist for $s > \alpha$.

Let us look back at this proof for a moment. Actually we showed that if F satisfies the hypothesis stated, then

$$\int_{t_0}^{\infty} e^{-st} |F(t)|\, dt \quad \text{exists if } s > \alpha$$

Further, Hypothesis 1 shows that

$$\int_{0}^{t_0} e^{-st} |F(t)|\, dt \quad \text{exists}$$

Thus

$$\int_{0}^{\infty} e^{-st} |F(t)|\, dt \quad \text{exists for } s > \alpha$$

In other words, if F satisfies the hypothesis of Theorem (1.1), then not only does $\mathscr{L}\{F\}$ exists for $s > \alpha$, but also $\mathscr{L}\{|F|\}$ exists for $s > \alpha$. That is,

$$\int_{0}^{\infty} e^{-st} |F(t)|\, dt \quad \text{is absolutely convergent for } s > \alpha$$

We point out that the condition on F described in the hypothesis of Theorem (1.1) we not necessary for the existence of $\mathscr{L}\{F\}$ exists. For distance, suppose we replace Hypothesis 1 by the following less restrictive condition. Let us suppose that F is piesewise continuous in every finite closed interval $a \leq t \leq b$, where $a > 0$, and is such that $|t^n F(t)|$ remains bounded as $t \to 0^+$ for some number where $0 < n < 1$. Then, provided Hypothesis 2 remains satisfied, it can be shown that $\mathscr{L}\{F\}$ still exists. Thus for example, if $F(x) = t^{-1/3}, t > 0, \mathscr{L}\{F\}$ exists. For though F does not satisfy Hypothesis 1 of Theorem (1.1)

$[F(t) \to \infty \;\; as \;\; t \to 0^+]$, it does satisfy the less restrictive requirement stated above (take $n = \frac{2}{3}$), and F is of exponential order.

Basic Properties of the Laplace Transform

Theorem 1.2

The Linear Property

Let F_1 and F_2 be functions whose Laplace transforms exist, and let c_1 and c_2 be contents.

then

$$\mathscr{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathscr{L}\{F_1(t)\} + c_2\mathscr{L}\{F_2(t)\}. \tag{1.9}$$

Proof.

$$\mathscr{L}[c_1F_1(t) + c_2F_2(t)] = \int_0^\infty [c_1F_1(t) + c_2F_2(t)]dt$$

$$\int_0^\infty c_2F_2(t)dt + \int_0^\infty c_2F_2(t)dt$$

$$\Rightarrow c_1\int_0^\infty F_1(t)dt + c_2\int_0^\infty F_2(t)dt$$

$$c_2\mathscr{L}F_1(t) + c_2\mathscr{L}F_2(t)$$

Example 1.12

Use Theorem (1.2) to find $\mathscr{L}\{\sin^2 at\}$.

Since $sin^2 at = (1 - \cos 2at)/2$, we have

$$\mathscr{L}\{\sin^2 at\} = \mathscr{L}\left\{\frac{1}{2} - \frac{1}{2}\cos 2at\right\}.$$

$$\mathscr{L}\left\{\frac{1}{2}-\frac{1}{2}\cos 2at\right\}=\frac{1}{2}\mathscr{L}\{1\}-\frac{1}{2}\mathscr{L}\{\cos 2at\}.$$

By Theorem (1.2), $\mathscr{L}\{1\} = 1/s$, and by (1.6), $\mathscr{L}\{\cos 2at\} = s/(s^2 + 4a^2)$. Thus

$$\mathscr{L}\{\sin^2 at\} = \frac{1}{2}.\frac{1}{s} - \frac{1}{2}.\frac{s}{s^2+4a^2} = \frac{2a^2}{s(s^2+4a^2)} \qquad (1.10)$$

Theorem 1.3

1. Let F be a real function which is continuous $t \geq 0$ and of exponential order e^{n}.
2. Let F' (the derivative of F) be piecewise continuous in every finite closed interval $0 \leq t \leq b$.

Then $\mathscr{L}\{F^{'}\}$ exist for $s > \alpha$; and

$$\mathscr{L}\{F^{'}(t)\} = s\mathscr{L}\{F(t)\} - F(0)$$

Proof.

By definition of the Laplace transform,

$$\mathscr{L}\{F^{'}(t)\} = \lim_{R\to\infty}\int_0^R e^{-st}F'(t)\,dt,$$

$$u = e^{-st} \;\Rightarrow du = -se^{-st}$$

$$dv = F'(t)dt \quad \Rightarrow v = F(t)$$

$$\lim_{R\to\infty}\left[e^{st}F(t)\int_0^R + s\int_0^R F(t)e^{-st}\,dt\right]$$

$$\lim_{R\to\infty}\left[e^{-Rs}F(R) - F(0) + s\mathscr{L}\ [F(t)]\right]$$

$$= -F(0) + s\mathscr{L}[F(t)]$$

$$\mathscr{L}\left(F'(t)\right) = s\mathscr{L}[f(t)] + F(0).$$

provided this limit exist. In any closed interval $0 \le t \le R,\ F'(t)$ has at most finite number of discontinuities; denote these by $t_1, t_2, \dots, t_n$, where

$$0 < t_1 < t_2 < \cdots < t_n \le R.$$

Then we may write

$$\int_0^R e^{-st} F'(t)\, dt = \int_0^{t_1} e^{-st} F'(t)\, dt$$
$$= \int_{t_1}^{t_2} e^{-st} F'(t)\, dt + \cdots + \int_{t_n}^{R} e^{-st} F'(t)\, dt.$$

Now the integrand of each of the integrals on the right is continuous. We must therefore integrate each by parts. Doing so, we obtain

$$\int_0^R e^{-st} F'(t)\, dt$$
$$= [e^{-st}F(t)]_0^{t_1-} + s\int_0^{t_1} e^{-st} F(t)\, dt + [e^{-st}F(t)]_{t_1+}^{t_2-}$$
$$+ s\int_{t_1}^{t_2} e^{-st} F(t)\, dt + \cdots + [e^{-st}F(t)]_{t_n+}^{R-}$$
$$+ s\int_{t_n}^{R} e^{-st} F(t)\, dt$$

By Hypothesis 1, F is continuous for $t \ge 0$. Thus

$$F(t_1 -) = F(t_1 +), F(t_2 -) = F(t_2 +), \dots, F(t_n -) = F(t_n +).$$

Thus all of the integrated "pieces" add out, except for $e^{-st}F(t)|_{\,t=0}$ and $e^{-st}F(t)|_{\,t=0}$ and there remains only

$$\int_0^R e^{-st} F'(t)\, dt = -F(0) + e^{-sR}F(R) + s\int_0^R e^{-st} F(t)\, dt.$$

But by Hypothesis 1 F is of exponential order $e^{\alpha t}$. Thus there exists $M > 0$ and $t_0 > 0$ such that $e^{-\alpha t}|F(t)| < M$ for $t > t_0$. Thus $|e^{-sR}F(R)| < Me^{-(s-\alpha)R}$ for $R > t_0$. Thus if $s > \alpha$,

$$\lim_{R\to\infty} e^{-sR}F(R) = 0$$

Further,

$$\lim_{R\to\infty} s\int_0^R e^{-st}F(t)\,dt = s\mathscr{L}\{F(t)\}.$$

Thus, we have

$$\lim_{R\to\infty} s\int_0^R e^{-st}F'(t)\,dt = -F(0) + s\mathscr{L}\{F(t)\}.$$

and so $\mathscr{L}\{F^{'}(t)\}$ exists for $s > \alpha$ and is given by (1.11).

Example 1.13

Consider the function defined by $F(t) = sin^2 at$. This function satisfies the hypothesis of Theorem(1.3). Since $F'(t) = 2a\sin at\cos at$ and $F(0) = 0$, Equation(1.11) gives

$$\mathscr{L}\{2a\sin at\cos at\} = s\mathscr{L}\{\sin^2 at\}.$$

By (1.10).

$$\mathscr{L}\{\sin^2 at\} = \frac{2a^2}{s(s^2 + 4a^2)}.$$

Thus,

$$\mathscr{L}\{2a\sin at\cos at\} = \frac{2a^2}{s^2 + 4a^2}.$$

Since $2a\sin at\cos at = a\sin 2at$, we also have

$$\mathscr{L}\{\sin 2at\} = \frac{2a}{s^2 + 4a^2}$$

Observe that this is the result (1.5), obtained in Example (1.4), with $b = 2a$.

We now generalize Theorem (1.3) and obtain the following result:

Theorem 1.4

If

1. Let F be a real function having a continuous $(n-1)st$ derivative $F^{(n-1)}$ (and since $F, F', \ldots, F^{(n-2)}$ are also continuous) for $t \geq 0$; and assume that $F, F', \ldots, F^{(n-1)}$ are all of exponential order $e^{\alpha t}$. and Suppose $F^{(n)}$ is piecewise continuous in every finite closed interval $0 \leq t \leq b$.

Then

$\mathscr{L}\left\{F^{(n)}(t)\right\}$ exists for $s > \alpha$ and

$$\mathscr{L}\left\{F^{(n)}(t)\right\}$$

$$= s^n \mathscr{L}\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-3}F''(0) - \cdots - F^{(n-1)}(0). \qquad (1.12)$$

Proof

$\{F^n\}$ exist for all $S > \alpha$ and is given by

$$\mathscr{L}\{F^n\} = S\mathscr{L}\left\{F^{(n-1)} - F^{(n-1)}(0)\right\}$$

at $n = k$

$$\mathscr{L} \quad \left\{F^{(n)}(t)\right\} =$$

$$S^k \mathscr{L}\{F(t)\} - S^{k-1}F(0) - S^{k-2}F'(0) - S^{k-3}f''(0) - \cdots - f^{(k-1)}(0)$$

which is relation is true

at $n = k + 1$

$$\mathscr{L}\left\{F^{(k+1)}(t)\right\}$$
$$= S^{k+1}\mathscr{L}\{F(t)\} - S^{(k+1)-1}F(0) - S^{(k+1)-2}F(0) - \cdots$$
$$- F^{(k+1)-1}(0)$$

Outline of Proof.

One first proceeds as in the proof of Theorem (1.3) to show that $\mathscr{L}\left\{F^{(n)}\right\}$ exists for $s > \alpha$ and is given by

$$\mathscr{L}\left\{F^{(n)}\right\} = s\mathscr{L}\left\{F^{(n-1)}\right\} - F^{(n-1)}(0).$$

Then completes the proof by mathematical induction.

Example 1.14

We apply Theorem (1.4), with $n = 2$, to find $\mathscr{L}\{\sin bt\}$, which we are already found directly and given by (1.5). Clearly the function F defined by $F(t) = \sin bt$ satisfies the hypothesis of the theorem with $\alpha = 0$. For $n = 0$ Equation (1.12) becomes

$$\mathscr{L}\left\{F''(t)\right\} = s^2\mathscr{L}\{F(t)\} - sF(0) - F'(0). \qquad (1.13)$$

We have $F'(t) = b\cos bt$, $F''(t) = -b^2\sin bt$, $F(0) = 0, F'(0) = b$. substituting into Equation (1.13) we find

$$\mathscr{L}\{-b^2\sin bt\} = s^n\mathscr{L}\{\sin bt\} - b,$$

and so

$$(s^2 + b^2)\mathscr{L}\{\sin bt\} = b.$$

Thus,

$$\mathscr{L}\{\sin bt\} = \frac{b}{s^2 + b^2} \qquad (s > 0),$$

Which is the result (1.5), already found directly.

Theorem 1.5 **Translation Property**

Hypothesis.

Suppose F is such that $\mathscr{L}\{F\}$ exist for $s > \alpha$.

Then

For any constant a.

$$\mathscr{L}\{e^{at}F(t)\} = f(s-a)$$

for $s > \alpha + a$, where $f(s)$ denotes $\mathscr{L}\{F(t)\}$.

Proof.

$$f(s) = \mathscr{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt.$$

Replacing s, by $s-a$, we have

$$f(s-a) = \int_0^\infty e^{-(s-a)t}F(t)\,dt = \int_0^\infty e^{-st}[e^{at}F(t)]\,dt = \mathscr{L}\{e^{at}F(t)\}$$

Example 1.15

Find $\mathscr{L}\{e^{at}t\}$. We apply Theorem (1.5) with $F(t) = t$.

$$\mathscr{L}\{e^{at}t\} = f(s-a),$$

where $f(s) = \mathscr{L}\{F(t)\} = \mathscr{L}\{t\}$. By (2.3), $\mathscr{L}\{t\} = 1/s^2 \;\; (s > 0)$. That is $f(s) = 1/s^2$ and so $f(s-a) = 1/(s-a)^2$. thus

$$\mathscr{L}\{e^{at}t\} = \frac{1}{(s-a)^2} \quad (s > a). \qquad (1.15)$$

Example 1.16

Find $\mathscr{L}\{e^{at}\sin bt\}$. We let $F(t) = \sin bt$. Then $\mathscr{L}\{e^{at}\sin bt\} = f(s-a)$, where

$$f(s) = \mathscr{L}\{\sin bt\} = \frac{1}{s^2+b^2} \quad (s > a).$$

Thus

$$f(s-a) = \frac{1}{(s-a)^2 + b^2}$$

And so

$$\mathscr{L}\{e^{at} \sin bt\} = \frac{1}{(s-a)^2 + b^2} \quad (s > a).$$

Theorem 1.6

Hypothesis.

Suppose F is a function satisfying the hypothesis of Theorem (1.5) with Laplace transform f so that

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt;$$

and G is the function defined as follows:

$$G(t) = \begin{cases} 0, & 0 < t < a, \\ F(t-a), & t > a. \end{cases} \qquad (1.17)$$

Then

$$\mathscr{L}\{G(t)\} = e^{-as} f(s).$$

Proof:

$$\mathscr{L}\{G(t)\} = \int_0^\infty e^{-st} G(t)\, dt = \int_0^a e^{-st}.0\, dt + \int_a^\infty e^{-st} F(t-a)\, dt$$
$$= \int_a^\infty e^{-st} F(t-a)\, dt.$$

Letting $t - a = \tau$, we obtain

$$\int_a^\infty e^{-st} F(t-a)\, dt = \int_0^\infty e^{-s(\tau+a)} F(\tau)\, d\tau = e^{-as} \int_0^\infty e^{-s\tau} F(\tau)\, d\tau$$
$$= e^{-as} \mathscr{L}\{F(\tau)\}.$$

Thus

$$\mathscr{L}\{G(t)\} = e^{-as} f(s).$$

Example 1.17

Find $\mathscr{L}\{G(t)\}$ if

$$G(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \sin t, & t > \dfrac{\pi}{2}. \end{cases}$$

Since $\sin t = \cos(t - \pi/2)$, we may write

$$G(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \cos\left(t - \dfrac{\pi}{2}\right), & t > \dfrac{\pi}{2}. \end{cases}$$

Thus Theorem (1.6) applies with $F(t) = \cos t$ and

$$\mathscr{L}\{G(t)\} = e^{-(\pi/2)s} f(s), where \;\; f(s) = \mathscr{L}\{\cos t\} = \frac{s}{s^2 + 1}$$

(using (1.6) with $b = 1$). Therefore, we have

$$\mathscr{L}\{G(t)\} = \frac{s e^{-(\pi/2)s}}{s^2 + 1}$$

Theorem 1.7

Suppose F is a function satisfying the hypothesis of Theorem (1.6), with Laplace transform f,where

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt. \qquad (1.19)$$

Then

$$\mathscr{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}[f(s)]. \qquad (1.20)$$

Proof.

Differentiate both sides of Equation (1.19) n times with respect to s. Thus differentiation is justified in the present case and yields

$$f'(s) = (-1)^1 \int_0^\infty e^{-st} tF(t)\, dt,$$
$$f''(s) = (-1)^2 \int_0^\infty e^{-st} t^2 F(t)\, dt,$$
$$\vdots$$
$$f^{(n)}(s) = (-1)^n \int_0^\infty e^{-st} t^n F(t)\, dt,$$

from which the conclusion (1.20) is at once apperent.

Example 1.18

Find

$$\mathscr{L}\{t^2 \sin bt\}.$$

By Theorem (1.7),

$$\mathscr{L}\{t^2 \sin bt\} = (-1)^2 \frac{d^2}{ds^2}[f(s)],$$

Where

$$f(s) = \mathscr{L}\{\sin bt\} = \frac{b}{s^2 + b^2}$$

(using (1.5)). Form this,

$$\frac{d}{ds}[f(s)] = -\frac{2bs}{(s^2 + b^2)^2}$$

and

$$\frac{d^2}{ds^2}[f(s)] = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}.$$

Thus,

$$\mathscr{L}\{t^2 \sin bt\} = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}.$$

Chapter 2

The Inverse Transform and the Convolution

A. The Inverse Transform

Thus far in this chapter we have been concerned with the following problem: Give a function F, defined for $t > 0$, to find its Laplace transform, which we denoted by $\mathscr{L}\{F\}$ or f. Now consider the inverse problem: Given a function f to find a funcfe F whose Laplace transform is the given f. We introduce the notation $\mathscr{L}\{f\}$ to denote such a function F, denote $\mathscr{L}\{f(s)\}$ by $F(t)$, and call such a function inverse transform of f. That is,

$$F(t) = \mathscr{L}^{-1}\{f(s)\}$$

Means that $F(t)$ is such that

$$\mathscr{L}\{F(t)\} = f(s).$$

Theorem 2.1

Hypothesis. Let F and G be two functions which are continuous for have the same Laplace transform f.

Conclusion.

$$F(t) = G(t) \ \ for \ \ all \ \ t \geq 0.$$

Thus if it is known that a given function f has a continuous inverse transform F, then $F$is the only continuous inverse transform of f. Let us consider the following example

Example. 2.1

By Equation (1.1), $\mathscr{L}\{1\} = 1/s$. Thus an inverse transform of the function $f$defined by $f(s) = 1/s$ is the continuous function F defined for all t by $F(t) = 1$. Thus by Theorem (2.1) there is no other continuous inverse transform of the function $f$such that $f(s) = 1/s$. However, discontinuous inverse transforms of the function $f$exist. For example, consider the function G defined as follows:

$$G(t) = \begin{cases} 1, & 0 < t < 3, \\ 2, & t = 3, \\ 1, & t > 3. \end{cases}$$

Then

$$\mathscr{L}\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt = \int_0^3 e^{-st}\,dt + \int_3^\infty e^{-st}\,dt$$
$$= \left[-\frac{e^{-st}}{s}\right]_0^3 + \lim_{R\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^R = \frac{1}{s} \quad \text{if } s > 0$$

Thus this discontinuous function G is also an inverse transform of f defined by $f(s) = 1/s$. However, we again emphasize that the only continuous inverse transform of f defined by $f(s) = 1/s$ is$F$ defined for all t by $F(t) = 1$. Indeed we write

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

Example 2.2

Using Table (1), find $\mathscr{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\}$

Solution.

Looking in the $f(s)$ column of Table (1), we would first look for $f(s) = \frac{1}{s^2+bs+c}$. However, we find no such $f(s)$; but we do find $f(s) = \frac{b}{(s+a)^2+b^2}$ (number 11). We can put the given expression $\frac{1}{s^2+6s+13}$ in this form as follows:

$$\frac{1}{s^2+6s+13} = \frac{1}{(s+3)^2+4} = \frac{1}{2}.\frac{2}{(s+3)^2+2^2}.$$

Thus, using number 11 of Table (1), we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\} = \frac{1}{2}e^{-3t}\sin 2t.$$

Example 2.3

Using Table (1), find $\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$.

Solution.

No enter of this form appears in the $f(s)$ column of Table (1). We employ the method of partial fractions. We have

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

and hence

$$1 = (A+B)s^2 + Cs + A.$$

Thus

$$A+B=0, \qquad C=0, \qquad \text{and } \; A=1.$$

Form these equations, we have the partial fractions decomposition

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

By number 1 of Table(1), $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ and by number 4, $\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$.

Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

B. The Convolution

Another important procedure in connection with the use of tables of transforms is that furnished by the so-called convolution theorem which we shall state below. We first define the convolution of two functions F and G.

Definition :-

Let F and G be two functions which are piecewise continuous on every finite closed interval $0 \leq t \leq b$ and of exponential order. The function denoted by $F * G$ and defined by

$$F(t) * G(t) = \int_0^1 F(t)G(t-\tau)\,d\tau \qquad (2.1)$$

is called the convolution of the function F and G.

Let us change the variable of integration in (2.1) by means of the substitution $u = t - \tau$. We have

$$F(t) * G(t) = \int_0^t F(\tau)G(t-\tau)\,d\tau = -\int_t^0 F(t-u)G(u)\,du$$
$$= \int_0^t G(u)F(t-u)\,du = G(t) * F(t).$$

Thus we have shown that

$$F * G = G * F \qquad (2.2)$$

Suppose that both $F$and G are piecewise continuous on every finite closed interval . $0 \leq t \leq b$ and of exponential order e^{at}. Then it can be shown that $F * G$ is also piecewise continuous on every finite closed interval $0 \leq t \leq b$ and of exponential large where $e^{(a+\epsilon)t}$ is any positive number. Thus $\mathscr{L}\{F * G\}$ exists for s sufficiently large. More explicitly, it can be shown that $\mathscr{L}\{F * G\}$ exists for $s > a$.

We now prove the following important theorem concerning $\mathscr{L}\{F * G\}$.

Theorem 2.2

Let the functions F and G be piecewise continuous on every finite closed interval $0 \leq t \leq b$ and of exponential order e^{at}.

Then

$$\mathscr{L}\{F * G\} = \mathscr{L}\{F\}\mathscr{L}\{G\} \tag{2.3}$$

for $s > a$.

Proof.

By definition of the Laplace transform, {F * G) is the function defined by

$$\int_0^{\infty} e^{-st}\left[\int_0^t F(\tau)G(t-\tau)\,d\tau\right]dt. \tag{2.4}$$

The integral (2.4) may be expressed as the iterated integral

$$\int_0^{\infty}\int_0^t e^{-st}F(\tau)G(t-\tau)\,d\tau\,dt. \tag{2.5}$$

Further, the iterated integral (2.5) is equal to the double integral

$$\iint_{R_1} e^{-st}F(\tau)G(t-\tau)\,d\tau\,dt, \tag{2.6}$$

where R_1 is the 45° wedge bounded by the lines $\tau = 0$ and $t = \tau$ (see Figure(2)).

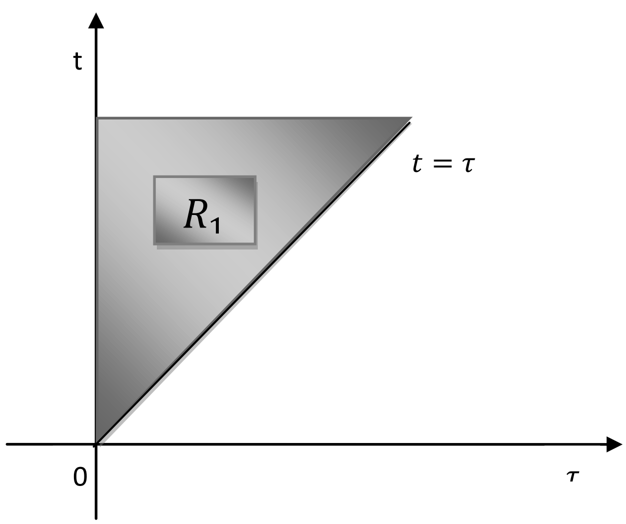

FIGURE.(2)

We now make the change of variable

$$u = t - \tau,$$
$$v = \tau, \tag{2.7}$$

to transform the double integral (2.6). The change of variables (2.7) has Jacobian and transforms the region R_1 in the τ, t plane into the first quadrant of the u, v plane. Thus the double integral (2.6) transforms into the double integral

$$\iint_{R_2} e^{-s(u+v)} F(v) G(u)\, du\, dv, \tag{2.8}$$

where R_2 is the quarter plane defined by $u > 0, v > 0$ (see Figure(3)).

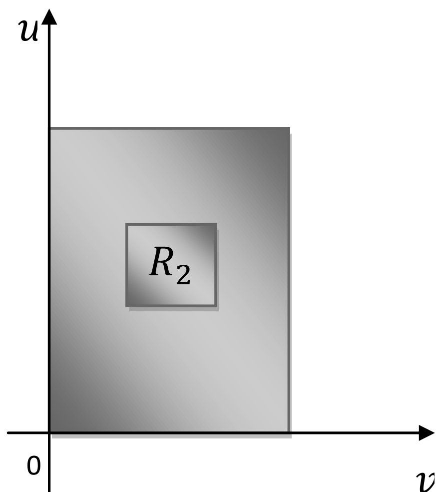

FIGURE.(3)

The double integral (2.8) is equal to the iterated integral

$$\int_0^{\infty} \int_0^{t} e^{-s(u+v)} F(v) G(u)\, du\, dv. \tag{2.9}$$

But the iterated integral in (2.9) can be expressed in the form

$$\int_0^{\infty} e^{-st} F(v)\, dv \int_0^{\infty} e^{-su} G(u)\, du \tag{2.10}$$

But the left-hand integral in (2.10) defines $\mathscr{L}\{F\}$ and the right-hand integral defines $\mathscr{L}\{G\}$. Therefore the expression (2.10) is precisely $\mathscr{L}\{F\}\mathscr{L}\{G\}$.

We not that since the integrals involved are absolutely convergent for $s > a$, the operations performed are indeed legitimate for $s > a$. Therefore we have shown that

$$\mathscr{L}\{F * G\} = \mathscr{L}\{F\}\mathscr{L}\{G\} \text{ for } s > a$$

Denoting $\mathscr{L}\{F\}$ by f and $\mathscr{L}\{G\}$ by g, we may write the convolution (2.3) in the form

$$\mathscr{L}\{F(t) * G(t)\} = f(s)g(s).$$

Hence, we have

$$\mathscr{L}^{-1}\{f(s)g(s)\} = F(t) * G(t) = \int_0^t F(\tau)G(t-\tau)\,d\tau \quad (3.11)$$

and using(2.2), we also have

$$\mathscr{L}^{-1}\{f(s)g(s)\} = G(t) * F(t) = \int_0^t G(\tau)F(t-\tau)\,d\tau \quad (3.12)$$

Suppose we are given a function h and are required to determine $\mathscr{L}^{-1}\{h(s)\}$. If we can express $h(s)$ as a product $f(s)g(s)$, where $\mathscr{L}^{-1}\{f(s)\} = F(t)$ and $\mathscr{L}^{-1}\{g(s)\} = G(t)$ are known, then we can apply either (2.11) or (2.12) to determine $\mathscr{L}^{-1}\{h(s)\}$.

Example 2.4

Fined $\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ using the convolution and Table (1).

Solution.

We write $\frac{1}{s(s^2+1)}$ as the product $f(s)g(s)$, where $f(s) = \frac{1}{s}$ and $g(s) = \frac{1}{s^2+1}$. By Table(1), number 1, $F(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$, and by number 3, $G(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$. Thus by (3.11),

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = F(t) * G(t) = \int_0^t 1. \sin(t-\tau)\, d\tau,$$

and by (2.12),

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = G(t) * F(t) = \int_0^t \sin\tau\,.\,1\, d\tau,$$

The second of these two integrals it slightly more simple. Evaluating it, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

Observe that we obtained this result in Example (2.4) by means of partial fractions.

TABLE.(1). LAPLACE TRANSFORMS

	$F(t)$	$f(s)$
1	1	$\frac{1}{s}$
2	e^{at}	$\frac{1}{s-a}$
3	$\sin bt$	$\frac{b}{s^2+b^2}$
4	$\cos bt$	$\frac{s}{s^2+b^2}$
5	$\sinh bt$	$\frac{b}{s^2-b^2}$
6	$\cosh bt$	$\frac{s}{s^2-b^2}$
7	$t^n\ (n=1,2,3,..)$	$\frac{n!}{s^{n+1}}$
8	$t^n e^{at}\ (n=1,2,3,\dots)$	$\frac{n!}{(s-a)^{n+1}}$
9	$t\ sin\ bt$	$\frac{2bs}{(s^2+b^2)^2}$
10	$t\ cos\ bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
11	$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
12	$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$
13	$\frac{\sin bt - bt\cos bt}{2b^3}$	$\frac{1}{(s^2+b^2)^2}$
14	$\frac{t\sin bt}{2b}$	$\frac{s}{(s^2+b^2)^2}$

Chapter 3

Laplace Transform Solution of Linear Differential Equations with Constant Coefficients

A. The Method

We now consider how the Laplace transform may be applied to solve the initial-value problem consisting of the mth-order linear differential equation with consider coefficients

$$a_0 \frac{d^n Y}{dt^n} + a_1 \frac{d^{n-1} Y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dY}{dt} + a_n Y = B(t), \qquad (3.1)$$

Plus the initial conditions

$$Y(0) = c_0, Y'(0) = c_1, \ldots Y^{(n-1)}(0) = c_{n-1} \qquad (3.2)$$

Theorem 3.1

assures us that this problem has a unique solution.

We now take the Laplace transform of both members of Equation (3.1). By Theorem (3.1), we have

$$a_0 \mathscr{L}\left\{\frac{d^n Y}{dt^n}\right\} + a_1 \mathscr{L}\left\{\frac{d^{n-1} Y}{dt^{n-1}}\right\} + \cdots + a_{n-1} \mathscr{L}\left\{\frac{dY}{dt}\right\} + a_n \mathscr{L}\{Y\} = \mathscr{L}\{B(t)\},$$

We now apply Theorem (3.1) to

$$\mathscr{L}\left\{\frac{d^n Y}{dt^n}\right\}, \mathscr{L}\left\{\frac{d^{n-1} Y}{dt^{n-1}}\right\}, \ldots, \mathscr{L}\left\{\frac{dY}{dt}\right\} \qquad (3.3)$$

in the left member of Equations(3.3). Using the initial conditions (3.2), we have

$$\mathscr{L}\left\{\frac{d^nY}{dt^n}\right\} = s^n\mathscr{L}\{Y(t)\} - s^{n-1}Y(0) - s^{n-2}Y'(0) - \cdots - Y^{(n-1)}(0)$$
$$= s^n\mathscr{L}\{Y(t)\} - c_0s^{n-1} - c_1s^{n-2} - \cdots - c_{n-1},$$
$$\mathscr{L}\left\{\frac{d^{n-1}Y}{dt^{n-1}}\right\} = s^{n-1}\mathscr{L}\{Y(t)\} - s^{n-2}Y(0) - s^{n-3}Y'(0) - .. - Y^{(n-2)}(0)$$
$$= s^{n-1}\mathscr{L}\{Y(t)\} - c_0s^{n-2} - c_1s^{n-3} - \cdots - c_{n-2},$$
$$\vdots$$
$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = s\mathscr{L}\{Y(t)\} - Y(0) = s\mathscr{L}\{Y(t)\} - c_0.$$

Thus, letting $y(s)$ denote $\mathscr{L}\{Y(t)\}$ and $b(s)$ denote $\mathscr{L}\{B(t)\}$, Equation (4.3) becomes

$$\begin{aligned}&[a_0s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n]y(s)\\ &\quad - c_0[a_0s^{n-1} + a_1s^{n-2} + \cdots + a_{n-1}]\\ &\quad - c_1[a_0s^{n-2} + a_1s^{n-3} + \cdots + a_{n-2}] - \cdots\\ &\quad - c_{n-2}[a_0s + a_1] - c_{n-1}a_0 = b(s).\end{aligned} \tag{3.4}$$

Since B is known function of t, then b, assuming it exists and can be determined is a known function of s. Thus Equation (3.4) is an algebraic equation in the "unknown" $y(s)$. We now solve the algebraic equation (3.4) to determine $y(s)$. Once $y(s)$ has been found, we then find the unique solution

$$Y(t) = \mathscr{L}^{-1}\{y(s)\}$$

of the given initial-value problem using the table of transforms.

1. Take the Laplace transform of both sides of the differential equation (3.1) applying Theorem(3.1) and using the initial conditions (3.2) in the process, and equate the result to obtain the algebraic equation (3.4) in the "unknown" $y(s)$.
2. Solve the algebraic equation (3.4) thus obtained to determine $y(s)$.

3. Having found $y(s)$, employ the table of transforms to determine the solution $y(s) = \mathscr{L}^{-1}\{y(s)\}$ of the given initial-value problem.

B. Examples

We shall now consider several detailed examples which will illustrate the procedure outlined above.

Example 3.1

Solve the initial-value problem

$$\frac{dY}{dt} - 2Y = e^{5t} \qquad (3.5)$$

$$Y(0) = 3 \qquad (3.6)$$

Step 1. Taking the Laplace transform of both sides of the differential equation (3.5), we have

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} - 2\mathscr{L}\{Y(t)\} = \mathscr{L}\{e^{5t}\}. \qquad (3.7)$$

Using Theorem (3.1) with $n = 1$ and denoting $\mathscr{L}\{Y(t)\}$ by $y(s)$, we may express $\mathscr{L}\{dY/dt\}$ in terms of $Y(0)$ as follows:

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = sy(s) - Y(0).$$

Applying the initial condition (3.6), this becomes

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = sy(s) - 3.$$

Using this, the left member of Equation (3.7) becomes $sy(s) - 3 - 2y(s)$. From Table (1), number 2, $\mathscr{L}\{e^{5t}\} = 1/(s-5)$. Thus Equation (4.7) reduces to the algebraic equation

$$[s-2]y(s) - 3 = \frac{1}{s-5} \qquad (3.8)$$

in the unknown $y(s)$.

Step 2. We now solve Equation (3.8) for $y(s)$. We have

$$[s-2]y(s)=\frac{3s-14}{s-5}$$

and so

$$y(s)=\frac{3s-14}{(s-2)(s-5)}$$

Step 3. We must now determine

$$\mathscr{L}^{-1}\left\{\frac{3s-14}{(s-2)(s-5)}\right\}.$$

We empty partial fractions. We have

$$\frac{3s-14}{(s-2)(s-5)}=\frac{A}{s-2}+\frac{B}{s-5},$$

and so $3s-14=A(s-5)+B(s-2)$. From this we find that

$$A=\frac{8}{3}\ \ and\ \ B=\frac{1}{3},$$

and so

$$\mathscr{L}^{-1}\left\{\frac{3s-14}{(s-2)(s-5)}\right\}=\frac{8}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\}.$$

Using number 2 of Table (1),

$$\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}\ \text{ and }\ \mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\}=e^{5t}.$$

Thus the solution of the given initial-value problem is

$$Y=\frac{8}{3}e^{2t}+\frac{1}{3}e^{5t}.$$

Example 3.2

Solve the initial-value problem

$$\frac{d^2Y}{dt^2} - 2\frac{dY}{dt} - 8Y = 0. \qquad (3.9)$$

$$Y(0) = 3, \qquad (3.10)$$

$$Y'(0) = 6, \qquad (3.11)$$

Step 1. Taking the Laplace transform of both sides of the differential equation(4.9), we have

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} - 2\mathscr{L}\left\{\frac{dY}{dt}\right\} - 8\mathscr{L}\{Y(t)\} = \mathscr{L}\{0\}. \qquad (3.12)$$

Since $\mathscr{L}\{0\} = 0$, the right member of equation(3.12) is simply 0. Denote $\mathscr{L}\{Y(t)\}$ by $y(s)$. Then, applying Theorem (3.4), we have the following expressions for $\mathscr{L}\{d^2Y/dt^2\}$ and $\mathscr{L}\{dY/dt\}$ in terms of $y(s), Y(0)$, and $Y'(0)$;

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2y(s) - sY(0) - Y'(0),$$

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = sy(s) - Y(0).$$

Applying the initial conditions (3.10) and (3.11) to these expressions, they become:

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2y(s) - 3s - 6,$$

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = sy(s) - 3.$$

Now, using these expressions, Equation (3.12) becomes

$$s^2y(s) - 3s - 6 - 2sy(s) + 6 - 8y(s) = 0$$

or

$$[s^2 - 2s - 8]y(s) - 3s = 0. \qquad (3.13)$$

Step 2. We now solve Equation (3.13) for $y(s)$. We have at once

$$y(s) = \frac{3s}{(s-4)(s+2)}.$$

Step 3. We must now determine

$$\mathscr{L}^{-1}\left\{\frac{3s}{(s-4)(s+2)}\right\}.$$

We shall again employ partial fractions. From

$$\frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

we find that $A = 2, B = 1$. Thus

$$\mathscr{L}^{-1}\left\{\frac{3s}{(s-4)(s+2)}\right\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

By Table (1), number 2, we find

$$\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t} \text{ and } \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

Thus the solution of the given initial-value problem is

$$Y = 2e^{4t} + e^{-2t}.$$

Example 3.3

Solve the initial-value problem

$$\frac{d^2Y}{dt^2} + Y = e^{-2t}\sin t, \qquad (3.14)$$

$$Y(0) = 0, \qquad (3.15)$$

$$Y'(0) = 0, \qquad (3.16)$$

Step 1. Taking the Laplace transform of both sides of the differential equation (4.14), we have

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} + \mathscr{L}\{Y(t)\} = \mathscr{L}\{e^{-2t}\sin t\}. \qquad (3.17)$$

Denoting $\mathscr{L}\{Y(t)\}$ by $y(s)$ and applying Theorem (1.4), we express $\mathscr{L}\{d^2Y/dt^2\}$ in terms of $y(s), Y(0)$, and $Y'(0)$ as follows:

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2y(s) - sY(0) - Y'(0).$$

Applying the initial conditions (3.15) and (3.16) to this expression, it becomes simply

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2y(s);$$

and thus the left member of Equation(3.17) becomes $s^2y(s) + y(s)$. By number 11, Table (1), the right member of Equation (3.17) becomes

$$\frac{1}{(s+2)^2+1}$$

Thus Equation (3.17) reduces to the algebraic equation

$$(s^2+1)y(s) = \frac{1}{(s+2)^2+1} \qquad (3.18)$$

In the unknown $y(s)$.

Step 2. Solving Equation (3.18) for $y(s)$, we have

$$y(s) = \frac{1}{(s^2+1)[(s+2)^2+1]}.$$

Sep 3. We must now determine

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)[(s+2)^2+1]}\right\}.$$

We may use either partial fractions or the convolution. We illustrate both methods.

1. Use of Partial Fractions. We have

$$\frac{1}{(s^2+1)(s^2+4s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5}.$$

From this we find

$$\begin{aligned} 1 &= (As+B)(s^2+4s+5) + (Cs+D)(s^2+1) \\ &= (A+C)s^3 + (4A+B+D)s^2 + (5A+4B+C)s \\ &\quad + (5B+D). \end{aligned}$$

Thus we obtain the equations

$$A + C = 0,$$

$$4A + B + D = 0,$$

$$5A + 4B + C = 0,$$

$$5B + D = 1.$$

From these equations we find that

$$A = -\frac{1}{8}, \quad B = \frac{1}{8}, \quad C = \frac{1}{8}, \quad D = \frac{3}{8},$$

and so

$$\begin{aligned} &\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\} \\ &\quad = -\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &\quad + \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} + \frac{3}{8}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}. \end{aligned} \tag{3.19}$$

In order to determine

$$\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} + \frac{3}{8}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}, \tag{3.20}$$

we write

$$\frac{s}{s^2+4s+5}=\frac{s+2}{(s+2)^2+1}-\frac{2}{(s+2)^2+1}.$$

Thus the expression (3.20) becomes

$$\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}+\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\},$$

and so (3.19) may be written

$$\begin{aligned}\mathscr{L}^{-1}&\left\{\frac{1}{(s^2+1)(s^2+4s+5)}\right\}\\ &=-\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &+\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}+\frac{1}{8}\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}.\end{aligned}$$

Now using Table (1), numbers 4,3,12, and 11, respectively, we obtain the solution

$$Y(t)=-\frac{1}{8}\cos t+\frac{1}{8}\sin t+\frac{1}{8}e^{-2t}\cos t+\frac{1}{8}e^{-2t}\sin t$$

or

$$Y(t)=\frac{1}{8}(\sin t-\cos t)+\frac{e^{-2t}}{8}(\sin t+\cos t). \qquad (3.21)$$

2. Use of the Convolution. We write $\frac{1}{(s^2+1)[(s+2)^2+1]}$ as the product $f(s)g(s)$, where $f(s)=\frac{1}{s^2+1}$ and $g(s)=\frac{1}{(s+2)^2+1}$. By Table (1), number 3, $F(t)=\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$, and by number 11, $G(t)=\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}=e^{-2t}\sin t$. Thus by Theorem (2.9) using (3.20) or (3.21), we have, respectively,

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)[(s+2)^2+1]}\right\} = F(t) * G(t)$$
$$= \int_0^t \sin\tau \,.\, e^{-2(t-\tau)} \sin(t-\tau)\, d\tau$$

or

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)[(s+2)^2+1]}\right\} = G(t) * F(t)$$
$$= \int_0^t e^{-2(t-\tau)} \sin\tau \,.\, \sin(t-\tau)\, d\tau$$

The second of these integrals is slightly more simple; it reduces to

$$(\sin t)\int_0^t e^{-2\tau} \sin\tau \,.\, \cos\tau\, d\tau - (\cos t)\int_0^t e^{-2\tau} \sin^2\tau\, d\tau.$$

Introducing double-angle formulas this becomes

$$\frac{\sin t}{2}\int_0^t e^{-2\tau} \sin 2\tau\, d\tau - \frac{\cos t}{2}\int_0^t e^{-2\tau}\, d\tau + \frac{\cos t}{2}\int_0^t e^{-2\tau} \cos 2\tau\, d\tau.$$

Carrying out the indicated integrations we find that this becomes

$$-\sin t\left[\frac{e^{-2\tau}}{8}(\sin 2\tau + \cos 2\tau)\right]_0^{\tau} + \frac{\cos t}{4}\left[e^{-2\tau}\right]_0^{\tau}$$
$$+ \cos t\left[\frac{e^{-2\tau}}{8}(\sin 2\tau - \cos 2\tau)\right]_0^{\tau}$$
$$= -\frac{e^{-2\tau}}{8}(\sin t \sin 2t + \sin 2t \cos 2t) + \frac{\cos t}{8}.$$

Using double-angle formula and simplifying, this reduces to

$$\frac{1}{8}(\sin t - \cos t) + \frac{e^{-2\tau}}{8}(\sin t + \cos t),$$

Which is the solution (3.21) obtained above using partial fractions.

Example 3.4

Solve the initial-value problem

$$\frac{d^3Y}{dt^3} + 4\frac{d^2Y}{dt^2} + 5\frac{dY}{dt} + 2Y = 10\cos t, \qquad (3.22)$$

$$Y(0) = 0,$$

$$Y'(0) = 0,$$

$$Y''(0) = 3,$$

Step 1. Taking the Laplace transform of both sides of the differential equation (3.22), we have

$$\mathscr{L}\left\{\frac{d^3Y}{dt^3}\right\} + 4\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} + 5\mathscr{L}\{Y(t)\} = 10\mathscr{L}\{\cos t\}. \quad (4.23)$$

We denote $\mathscr{L}\{Y(t)\}$ by $y(s)$ and then apply to express

$$\mathscr{L}\left\{\frac{d^3Y}{d3}\right\}, \qquad \mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\}, \qquad and \quad \mathscr{L}\left\{\frac{dY}{dt}\right\}$$

In terms of $y(s)$, $Y(0)$, $Y'(0)$, and $Y''(0)$. We thus obtain

$$\mathscr{L}\left\{\frac{d^3Y}{d3}\right\} = s^3y(s) - s^2Y(0) - sY(0) - Y''(0),$$

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2y(s) - sY(s) - Y'(0),$$

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = sy(s) - Y(0).$$

Applying the initial conditions (3.22), these expressions become

$$\mathscr{L}\left\{\frac{d^3Y}{d3}\right\} = s^3y(s) - 3,$$

$$\mathscr{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2y(s),$$

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = sy(s).$$

Thus the left member of Equation (3.23) becomes

$$s^3y(s) - 3 + as^2y(s) + 5sy(s) + 2y(s)$$

or

$$[s^3 + 4s^2 + 5s + 2]y(s) - 3.$$

By number 4, Table (1),

$$10\mathscr{L}\{\cos t\} = \frac{10s}{s^2+1}.$$

Thus Equation (3.23) reduces to the algebraic equation

$$(s^3 + 4s^2 + 5s + 2)y(s) - 3 = \frac{10s}{s^2+1} \qquad (3.24)$$

in the unknown $y(s)$.

Step 2. We now solve Equation (3.23) for $y(s)$. We have

$$(s^3 + 4s^2 + 5s + 2)y(s) = \frac{3s^2 + 10s + 3}{s^2+1}$$

$$y(s) = \frac{3s^2 + 10s + 3}{(s^2+1)(s^3 + 4s^2 + 5s + 2)}.$$

Step 3. We must now determine

$$\mathscr{L}^{-1}\left\{\frac{3s^2 + 10s + 3}{(s^2+1)(s^3 + 4s^2 + 5s + 2)}\right\}.$$

Let us not despair! We can again employ partial fractions to put expression for $y(s)$ into a form where Table (1) can be used, but the work will be rather involved.

We proceed by writing

$$\frac{3s^2+10s+3}{(s^2+1)(s^3+4s^2+5s+2)} = \frac{3s^2+10s+3}{(s^2+1)(s+1)^2(s+2)}$$
$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{Ds+E}{s^2+1}. \quad (3.25)$$

From this find

$$3s^2+10s+3$$
$$= A(s+1)^2(s^2+1) + B(s+2)(s+1)(s^2+1)$$
$$+ C(s+2)(s^2+1) + (Ds+E)(s+2)(s+1)^3, \quad (3.26)$$

or

$$3s^2+10s+3$$
$$= (A+B+D)s^4 + (2A+3B+C+4D+E)s^3$$
$$+ (2A+3B+2C+5D+4E)s^2$$
$$+ (2A+B+C+2D+5E)s + (A+2B+C+2E)$$

From this we obtain the system of equations

$$A+B+D=0,$$

$$2A+3B+C+4D+E=0,$$

$$2A+3B+2C+5D+4E=3,$$

$$2A+B+C+2D+5E=10,$$

$$A+2B+C+2E=3. \quad (3.27)$$

Letting $s=-1$ in Equation (3.27), we find that $C=-2$; and letting $s=-2$ in this same equation results in $A=-1$. Using these values for A and C we find from the system (3.28) that

$$B=2, \qquad D=-1, \qquad and\ \ E=2.$$

Substituting these values thus found for A, B, C, D, and E into Equation (3.25), we see that

$$\mathscr{L}^{-1}\left\{\frac{3s^2+10s+3}{(s^2+1)(s^3+4s^2+5s+2)}\right\}$$
$$= -\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$
$$- \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Using Table (1), numbers 2, 2, 8, 4, and 3, respectively, we obtain the solution

$$Y(t) = -e^{-2t} + 2e^{-t} - 2te^{-t} - \cos t + 2\sin t.$$

Example 3.5

Solve the initial-value problem

$$\frac{d^2Y}{dt^2} + 2\frac{dY}{dt} + 5Y = H(t). \qquad (3.28)$$

where

$$H(t) = \begin{cases} 1, 0 < t < \pi \\ 0, t > \pi \end{cases} \qquad (3.29)$$

$$Y(0) = 0, \qquad (3.30)$$

$$Y'(0) = 0. \qquad (3.31)$$

Step 1. We take the Laplace transform of both sides of the differential equation (3.28) to obtain

$$\mathscr{L}\left\{\frac{d^3Y}{d3}\right\} + 2\mathscr{L}\left\{\frac{dY}{dt}\right\} + 5\mathscr{L}\{Y(t)\} = \mathscr{L}\{H(t)\}. \qquad (3.32)$$

Denoting $\mathscr{L}\{Y(t)\}$ by $y(s)$, using Theorem 31 as in the previous examples, and then applying the initial conditions (3.30) and (3.31), we see that the left member of Equation (3.32) becomes $[s^2 + 2s +$

$5]y(s)$. By the definition of the Laplace transform, from (3.29) we have

$$\mathscr{L}\{H(t)\} = \int_0^\infty e^{-st} H(t)\, dt = \int_0^\pi e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\pi = \frac{1 - e^{-\pi s}}{s}.$$

Thus Equation (32) becomes

$$[s^2 + 2s + 5]y(s) = \frac{1 - e^{-\pi s}}{s}. \tag{3.33}$$

Step 2. We solve the algebraic equation (3.33) for $y(s)$ to obtain

$$y(s) = \frac{1 - e^{-\pi s}}{s(s^2 + 2s + 5)}.$$

Step 3. We must now determine

$$\mathscr{L}^{-1}\left\{\frac{1 - e^{-\pi s}}{s(s^2 + 2s + 5)}\right\}.$$

Let us write this as

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2 + 2s + 5)}\right\} - \mathscr{L}^{-1}\left\{\frac{-e^{-\pi s}}{s(s^2 + 2s + 5)}\right\},$$

and apply partial fractions to determine the first of these two inverse transforms.

Writing

$$\frac{1}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5},$$

we find at once that $A = \frac{1}{5},\ B = -\frac{1}{5},\ C = -\frac{2}{5}$, Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+2s+5)}\right\} = \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+1)^2+4}\right\}$$

$$= \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{s+21}{(s+1)^2+4}\right\}$$

$$- \frac{1}{10}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$= \frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t,$$

using Table (1), numbers 1,12, and 11, respectively. Letting

$$f(s) = \frac{1}{s(s^2+2s+5)}$$

and

$$F(t) = \frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t,$$

we thus have

$$\mathscr{L}^{-1}\{f(s)\} = F(t).$$

We now consider

$$\mathscr{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+2s+5)}\right\} = \mathscr{L}^{-1}\{e^{-\pi s}f(s)\}.$$

By Theorem (3.2),

$$\mathscr{L}^{-1}\{e^{-\pi s}f(s)\} = G(t),$$

where

$$G(t) = \begin{cases} 0, & 0 < t < \pi \\ F(t-\pi), & t > \pi' \end{cases}$$

Thus

$$\mathscr{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+2s+5)}\right\}$$

$$= \begin{cases} 0, & 0 < t < \pi \\ \frac{1}{5} - \frac{1}{5}e^{-(t-\pi)}\cos 2(t+\pi) - \frac{1}{10}e^{-(t-\pi)}\sin 2(t-\pi), & t > \pi \end{cases}$$

$$= \begin{cases} 0, & 0 < t < \pi \\ \frac{1}{5} - \frac{1}{5}e^{-(t-\pi)}\cos 2t - \frac{1}{10}e^{-(t-\pi)}\sin 2t, & t > \pi \end{cases}$$

Thus the solution is given by

$$Y(t) = \mathscr{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+2s+5)}\right\} = F(t) - G(t)$$

$$= \begin{cases} \frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t - 0, & 0 < t < \pi \\ \frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t - \frac{1}{5} + \frac{1}{5}e^{-(t-\pi)}\cos 2t - \frac{1}{10}e^{-(t-\pi)}\sin 2t, & t > \pi \end{cases}$$

or

$$Y(t) = \begin{cases} \frac{1}{5}\left[1 - e^{-t}\left(\cos 2t - \frac{1}{2}\sin 2t\right)\right], & 0 < t < \pi \\ \frac{e^{-t}}{5}\left[(e^{\pi}-1) + \cos 2t + \left(\frac{e^{\pi}-1}{2}\right)\sin 2t\right], & t > \pi \end{cases}.$$

Laplace Transform Solution of Linear Systems

A. The Method

We apply the Laplace transform method to find the solution of linear system

$$a_1\frac{dX}{dt} + a_2\frac{dY}{dt} + a_3X + a_4Y = B_1(t),$$

$$b_1\frac{dX}{dt} + b_2\frac{dY}{dt} + b_3X + b_4Y = B_2(t), \qquad (3.34)$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$, and b_4 are constants and B_1 and B_2 are known functions, which satisfies the initial conditions

$$X(0) = c_1 \qquad and \qquad Y(0) = c_2, \tag{3.35}$$

where c_1 and c_2 are constants.

Example 3.7

Use a Laplace transform to find the solution of the system

$$\frac{dX}{dt} - 6X + 3Y = 8e^t,$$

$$\frac{dY}{dt} = -2X - Y = ae^t, \tag{3.36}$$

which satisfies the initial conditions

$$X(0) = -1,$$

$$Y(0) = 0. \tag{3.37}$$

Step 1. Taking the Laplace transform of both sides of each differential equations of system (3.36), we have

$$\mathscr{L}\left\{\frac{dX}{dt}\right\} - 6\mathscr{L}\{X(t)\} + 3\mathscr{L}\{Y(t)\} = \mathscr{L}\{8e^t\}.$$

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} - 2\mathscr{L}\{X(t)\} - \mathscr{L}\{Y(t)\} = \mathscr{L}\{4e^t\}. \tag{3.38}$$

Denote $\mathscr{L}\{X(t)\}$ by $x(s)$ and $\mathscr{L}\{Y(t)\}$ by $y(s)$. Then applying Theorem and the initial conditions (3.37), we have

$$\mathscr{L}\left\{\frac{dX}{dt}\right\} = sx(s) - X(0) = sx(s) + 1, \tag{3.39}$$

$$\mathscr{L}\left\{\frac{dY}{dt}\right\} = s(y)s - Y(0) = s(y)s.$$

Using the Table (1) number 2, we find

$$\mathscr{L}\{8e^t\} = \frac{8}{s-1} \text{ and } \mathscr{L}\{4e^t\} = \frac{4}{s-1}. \qquad (3.40)$$

Thus, from (3.39) and (3.40), we see that Equation (3.38) become

$$sx(s) + 1 - 6x(s) + 3y(s) = \frac{8}{s-1},$$

$$sy(s) - 2x(s) - y(s) = \frac{4}{s-1},$$

Which simplify to the form

$$(s-6)x(s) + 3y(s) = \frac{8}{s-1} - 1,$$

$$-2x(s) + (s-1)y(s) = \frac{4}{s-1},$$

$$(s-6)x(s) + 3y(s) = \frac{-s+9}{s-1},$$

$$-2x(s) + (s-1)y(s) = \frac{4}{s-1}. \qquad (3.41)$$

Step 2. We solve the linear algebraic system of the two equations (3.41) in two "unknown" $x(s)$ and $y(s)$. We have

$$(s-1)(s-6)x(s) + 3(s-1)y(s) = -s + 9,$$

$$-6x(s) + 3(s-1)y(s) = \frac{12}{s-1}.$$

Subtracting we obtain

$$(s^2 - 7s + 12)x(s) = -s + 9 - \frac{12}{s-1},$$

From which we find

$$x(s) = \frac{-s^2 + 10s - 21}{(s-1)(s-3)(s-4)} = \frac{-s+7}{(s-1)(s-4)}.$$

In like manner, we find

$$y(s) = \frac{2s-6}{(s-1)(s-3)(s-4)} = \frac{2}{(s-1)(s-4)}.$$

Step 3. We must now determine

$$X(t) = \mathscr{L}^{-1}\{x(s)\} = \mathscr{L}^{-1}\left\{\frac{-s+7}{(s-1)(s-4)}\right\}$$

and

$$Y(t) = \mathscr{L}^{-1}\{y(s)\} = \mathscr{L}^{-1}\left\{\frac{2}{(s-1)(s-4)}\right\}.$$

We first find $X(s)$. We use partial fractions and write

$$\frac{-s+7}{(s-1)(s-4)} = \frac{A}{s-1} + \frac{B}{s-4}.$$

From this we find

$$A = -2 \text{ and } B = 1.$$

Thus

$$x(t) = -2\,\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\},$$

And using Table (1), number 2, w obtain

$$X(t) = -2e^{t} + e^{4t}. \tag{3.42}$$

In like manner, we find $Y(s)$. Doing so, we obtain

$$Y(t) = -\frac{2}{3}e^{t} + \frac{2}{3}e^{4t}. \tag{3.43}$$

The pair define by (3.42) and (3.43) constitute the solution of the given system (3.36) which satisfies the given initial conditions (3.37).

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