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Center Manifolds Theorem and Applications to Dynamical Systems

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الآيـة

بسم الله الرحمن الرحيم

قال تعالي :

{ يَا مَعْشَرَ الْجِنِّ وَالْإِنْسِ إِنِ اسْتَطَعْتُمْ أَنْ تَنْفُذُوا مِنْ أَقْطَارِ السَّمَاوَاتِ وَالْأَرْضِ فَانْفُذُوا لَا تَنْفُذُونَ إِلَّا بِسُلْطَانٍ * فَبِأَيِّ آلَاءِ رَبِّكُمَا تُكَذِّبَانِ }

صدق الله العظيم

سورة الرحمن

الايات 33-34

الإهداء

الي معلم البشرية الاول ... رسولنا الكريم

سيدنا محمد صلي الله عليه وسلم

من أسكنونا قلوبهم وعلمونا السلام قبل الخصام

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إلى أمهاتنا الرحيمات

الشموس التي ترسل لنا كل صباح أجمل وأرق كلمة ومن ينتظروننا مع مولود كل يوم بشوق وحنين

إلي أخواننا وأخواتنا

الشكر والعرفان

تتناثر الكلمات حبراً وحباً .... علي صفائح الأوراق

لكل من علمني ....ومن أزال غيمة جهل مررت بها

ولكل من أعاد رسم ملامحي........وتصحيح عثراتي .

*الشكر أجزله للدكتور الانسان **عماد الدين عبدالله عبدالرحيم***

الذي أشرف علي هذا العمل بالنصح والارشاد والتوجيه

ويمتد الشكر الي أساتذة قسم الرياضيات الذين أحسنوا صنعنا وتسليحنا بسلاح العلم والمعرفه

الي كل من اسهم في اخراج هذا العمل بصورته هذه.

الي أخواني وأخواتي وكل من جمعنا بهم الدرب ، الي صناع الفرح الدفعه 22

هكذا هي الحياة تجمع وتفرق ولكن سيظل حبل الود والوصل بيننا وسنبقي اسرى احترامكم ووفائكم الي الابد.

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Abstract

In this research we consider the most useful methods to study the flow near bifurcation point, that is the center manifolds theorem. Because of this way study the Dynamical Systems and there geometrical theory with some examples. Also we discuss the basic concepts of studying dynamical system with some applications. We discuss the stationary and periodic solutions and we illustrate how to determine relations between linear or nonlinear stability. Also we study the center manifolds and their prosperities. We discuss   the bifurcations involving a single or several eigenvalues with some applications and examples.

الملخص

في هذا البحث إعتبرنا الطريقة الاكثر إستخداماً لدلراسة الفيض (السريان) بالقرب من نقطة إضطراب ، وهي نظرية متعددات السطوح المركزية .ولأجل هذا درسنا الانظمة الديناميكية ونظريتها الهندسية مع بعض الأمثلة.أيضاً ناقشنا المفاهيم الاساسية لدراسة الانظمة الديناميكية مع بعض الطبيقات . ناقشنا الحلول الثابتة والدورية ووضحنا كيفية تحديد العلاقات بين الاستقرار الخطي واللاخطي .ايضاً درسنا متعددات السطوح المركزية وخواصها .ناقشنا الإضطرابات المحتوية على متجه ذاتي وحيد أو عدة متجهات ذاتية مع بعض التطبيقات والأمثلة.

Chapter 1
Basic concepts of Dynamical systems

Section (1.1): Dynamical Systems

The last 30 years have witnessed a renewed interest in dynamical systems, partly due to the ‘‘discovery’’ of chaotic Behavior,and ongoing research has brought many new insights in their behavior. What are dynamical systems, and what is their geometrical theory? Dynamical systems can be defined in a fairly abstract way, but we prefer to start with a few examples of historical importance before giving general definitions. This will allow us to specify the class of systems that we want to study, and to explain the differences between the geometrical approach and other approaches.

Example (1.1.1):(The Motion of the Moon)

The problem of the Moon's motion is a particular case of the N-body problem, which gives a nice illustration of the historical evolution that led to the development of the theory of dynamical systems .This section follows mainly Gutzwiller’s article [Gu98]. Everyone knows that the phases of the Moon follow a cycle of a bit less than 30 days. Other regularities in the Moon’s motion were known to the Babylonians as early as 1000 B.C.One can look, for instance, at the time interval between Sunset and Moonrise at Full Moon. This interval is not constant, but follows a cycle over 19 years, Including 235 Full Moons (the Medtronic Cycle).

$Solar\ and\ Lunar\ Eclipses\ also\ follow\ a\ cycle\ with\ aperiod\ of$ 18 years and 11 days , containing 223 Full Moons (the Saros cycle). Greek astronomy started in the 5th century A.C.and initiated developments culminating in the work of Ptolemy in the second century B.C.In contrast with the Babylonians, who looked for regularities in long rows of numbers, the Greeks introduced geometrical models for their astronomical observations.

To account for the various observed deviations' from periodicity, they invented the model of epicycles.

In modern notation, and assuming a planar motion with Cartesian coordinates(x,y)∈ R^2, the complex number

$z = x + iy \in C$ evolves as function of time t according to the law

$$z = ae^{iw_1 t}(1+\varepsilon e^{iw_2 t}), \quad (1.1)$$

Where a, ε, w_1 and w_2 are parameters which are fitted to experimental data. The epicycle model was refined in subsequent centuries, with more terms being included into the sum (1.1) to explain the various "inequalities" (periodic deviations from the uniform motion of the Moon). Four inequalities were discovered by Tycho Brahe alonc in the $16th$ centuey. These terms could be partly explained when Kepler discovered his three laws in 1609:

(1)-The trajectory of a planet follows an ellipse admitting the sun as a focus.

(2)- equal areas, measured with respect to the sun, are swept in equal time intervals,

(3)- when several planets orbit the sun, the period of the motion squared is proportional to the third power of the semi-major axis of the ellipse.

Expanding the solution into Fourier series produces sums for which (1.1) is a first approx- Imation. However, while these laws describe the motion of the planets quite accurately, they fail to fit the observations for the Moon in a satisfactory way. A decisive new point of view was introduced by Newton when he postulated his low of Universal Gravitation (published in his principia in 1687).

A system with N planets is described by a set of ordinary differential equations .

$$m_i \frac{d^2x}{dt^2} = \sum_{\substack{j=1,\ldots N \\ j\neq 1}} \frac{Gm_i m_j (x_{j-}x_i)}{||x_{i-}x_j||^3}, \quad \text{i= 1,.....,N.} \qquad (1.2)$$

Here the $x_i \in R^3$ are vectors specifying the position of the planets, the m_i are positive scalars giving the masses of the particles, and G is a universal constant.

Newton proved that for tow bodies (N=2), the equation (1.2) is equivalent to Kepler's first two laws.

With three or more bodies, however, there is no simple solution to the equations of motion, and Kepler's third low is only valid approximately, when the interaction between planets is neglected.

The three-body problem initiated a huge amount of research in the following two hundred years.

Newton himself invented several clever tricks allowing him to compute corrections to Kepler's lows in the motion of the Moon.

He failed, however, to explain all the anomalies.

Perturbation theory was subsequently systematized by mathematicians such as Laplace, Euler, Lagrange, Poisson and Hamilton, who developed the methods of analytical mechanics.

As a first step, one can introduce the Hamiltonian function

$$\text{H} : (R^3)^N \times (R^3)^N \rightarrow R$$

$$(\text{p,q}) \rightarrow \sum_{i=1}^{N} \frac{p_i^2}{2m_i} - \sum_{i<j} \frac{Gm_i m_j}{||q_{i-}p_j||}, \qquad (1.3)$$

Where $p_{i=}m_i v_i \in R^3$ are the momenta of the planets and $q_i = x_i \in x_i R^3$ for i= 1,....N.

The equation of motion (1.2) is then equivalent to the equation

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{1.4}$$

One advantage of this formulation is that all the information on the motion is contained in the scalar function H.

The main advantage, however, is that the structure (1.4) of the equations of motion is preserved under special changes of variables, called canonical transformations.

In the case of the two-body problem, a good set of coordinates, is given by the Delaunay variables $(I, \varphi) \in R^3 \times T^3$(actually, there are 6+6 variables, but 6 of them correspond to the tri vial motion of the center of mass of the system).

The action variables $I_{1,} I_2, I_3$ are related to the semi-major axis, eccentricity and inclination of the Kepler ellipse, wile the angle variables $\varphi_1, \varphi_2, \varphi_3$ describe the position of the planet and the spatial orientation of the ellipse.

The two-body Hamiltonian takes the form

$$H(I, \varphi) = -\frac{\mu}{2I_1^2}, \tag{1.5}$$

Where $\frac{m_1 m_2}{m_1+m_2}$. The equations of motion are

$$\frac{d\varphi_1}{dt} = \frac{\partial H}{\partial I_1} = \frac{\mu}{I_1^3}, \qquad \frac{dI_1}{dt} = -\frac{\partial H}{\partial \varphi_1} = 0$$

$$\frac{d\varphi_2}{dt} = \frac{\partial H}{\partial I_2} = 0, \qquad \frac{dI_2}{dt} = -\frac{\partial H}{\partial \varphi_2} = 0 \tag{1.6}$$

$$\frac{d\varphi_3}{dt} = \frac{\partial H}{\partial I_3} = 0, \qquad \frac{dI_3}{dt} = -\frac{\partial H}{\partial \varphi_3} = 0$$

Describing the fact that the planet moves on an elliptical orbit with fixed dimensions and orientation .

In the case of the three-body problem moon-Earth-sun, One can use two sets of Delaunay variables $(I, \varphi)\ and\ (J, \psi)$ describing, respectively, the motion of the system Moon-Earth, and the motion around the Sun of the center of mass of the the system Moon-Earth, The Hamiltonian takes the form

$$H(I, J, \varphi, \psi) = H_0(I, J) + H_1 + (I, J, \varphi, \psi). \qquad (1.7)$$

The unperturbed part of the motion is governed by the Hamiltonian

$$H_0(I, J) = -\frac{\mu}{2I_1^2} - \frac{\mu'}{2I_1^2}, \qquad (1.8)$$

Where $\mu' = \frac{(m_1+m_2)m_3}{m_1+m_2+m_3}$. Due to the special initial conditions of the system, the perturbing function H_1has a small amplitude. It depends several small parameters: the initial eccentricities $\varepsilon \simeq {}^1/_8$ of the Moon and $\varepsilon' \simeq {}^1/_{60}$ of the Earth, their inclinations i and i', and the ratio $a/a' \simeq 1/400$ of the semi-major axes of the two subsystems. All these quantities are functions of the actions of the actions $I\ and\ J$.The standard approach to expand H_1 in a trigonometric series

$$H_1 = -\frac{Gm_1m_2m_3}{m_1+m_2}\frac{a^2}{a'^3}\sum_{j\in Z^6} C_j e^i(j_1\varphi_1 + j_2\varphi_2 + j_3\varphi_3 + j_4\psi_1 + j_5\psi_2 + j_6\psi_3) \qquad (1.9)$$

The coefficients C_j are in turn expanded into Tyalor series of small parameters,

$$C_j = \sum_{k\epsilon N^5} c_{jk}(\frac{a}{a'})^{K1}\ \varepsilon^{k2}\varepsilon'^{k3}\gamma^{k4}\gamma^{k5}, \qquad (1.10)$$

Where $\gamma = sini/2\ snd\ \gamma' = sini'/2$. The solutions can then be expanded into similar series, thus yielding a Fouries expansion of the form (1.1) (in fact, it is better to simplify the Hamiltonian by successive canonical transformations, but the results are equivalent).

The most impressive achievement in this line of work is due to Delaunay, who published in 1860 and 1867 two volumes of over 900 pages .The contain expansions up to order 10. Which are simplified with 505 transformations .The main result for the trajectory of the Moon is a series containing 460 terms, filling 53 pages.

At the turn of the century, these perturbative calculations were criticized by Poincare , who questioned the convergence of the expansions . Indeed , although the magnitude of the first few order decreases, he showed that this magnitude may become extremely large at sufficiently high order . The phenomenon is related to the problem of small divisors appearing in the expansion, which we will discuss in simpler example in the next section. Poincare introduced a whole set of new methods attack the problem from a geometric point of view . Instead of trying to compute the solution for a given initial condition ,he wanted to understand the qualitative nature of solutions for all initial conditions ,or, as we would say nowadays, the geometric structure of phase space, He there by introduced concepts such as invariant points, curves and manifolds. He also provided examples where the solution cannot be written as a linear combination of periodic terms , a first encounter with chaotic motion. The question of convergence of the perturbation series continued nonetheless to be investigated, and was finally solved in a series of theorems by Kolmogorov, Arnol's and Moser (the so-called KAM theory) in the 1950s. They prove that the series converges for (very) small perturbations, for initial conditions living on a Cantor set. This did not solve the question of the motion of the Moon completely, although fairly accurate ephemerides can be computed for relatively short time spans of a few decades.

Using a combination of analytical and numerical methods, the existence of chaos in the Solar system was demonstrated by Laskar in 1989[La89], implying that exact positions of the planets cannot be predicted for times more then a few hundred thousand years in the future.

Example(1.1.2):(The standard map)

The standard map describes the motion of a " rotator" with one angular degree of freedom $q \in S^1(S^1 \text{denotes the circle } R/2\pi Z)$,which is periodically kicked by a pendulum-like force of intensity proportional to- sin q. If q_n denote the position and momentum just before the n^{th} kick, one has

$$q_{n+1} = q_n + p_{n+1} \qquad (mod\ 2\pi) \qquad (1.11)$$

$$p_{n+1} = p_n - \varepsilon \sin q_n$$

For $\varepsilon = 0$, the dynamics is very simple and one explicitly

$$q_n = q_0 + np_0 \qquad (mod\ 2\pi) \qquad (1.12)$$

$$p_n = p_0$$

Let us new analyse the iterated map (1.11) according to the perturbative method. The idea is to look for a change of variables (q , p) $\rightarrow (\varphi, I)$ transforming the system into a similar one, but without the term $\varepsilon \sin q_n$. Let us write

$$q = \varphi + f(\varphi, I) \qquad (1.13)$$

$$p = I + g(\varphi, I)$$

Where f and g are unknown functions, which are 2π-periodic in φ. We impose that this change of variables transforms the map (1.11) into map

$$\varphi_{n+1} = \varphi_n + I_{n+1} \qquad (mod\ 2\pi)$$

$$I_{n+1} = I_n = w \qquad (1.14)$$

This is equivalent to requiring that $f\ and\ g$ solve the functional equations

$$f(\varphi + w, w) = f(\varphi, w) + g(\varphi + w, w) \tag{1.15}$$

$$g(\varphi + w, w) = g(\varphi, w) - \varepsilon \sin\big(\varphi + f(\varphi, w)\big).$$

One can try solve these equations by expanding $f\ and\ g\ into\ Taylor\ series\ in\ \varepsilon$ and Fourier series in φ :

$$f(\varphi, w) = \sum_{j=1}^{\infty} \varepsilon^j f_j(\varphi, w) \qquad f_j(\varphi, w)$$

$$= \sum_{\substack{\infty \\ k=-\infty}}^{\infty} a_{kj}(w) e^{ik\varphi} \tag{1.16}$$

$$g(\varphi, w) = \sum_{j=1}^{\infty} \varepsilon^j g_j(\varphi, w) \qquad g_j(\varphi, w)$$

$$= \sum_{k=-\infty}^{\infty} b_{kj}(w) w^{ik\varphi} \tag{1.17}$$

We will use the expansions

$$\sin(\varphi + \varepsilon f) = sin\varphi + \varepsilon f_1 cos\varphi + \varepsilon^2 (f_2 cos\varphi - \tfrac{1}{2} f_1^2 sin\varphi) + \vartheta(\varepsilon^3). \tag{1.18}$$

At order ε, we have to solve the relations

$$f_1(\varphi + w, w) = f_1(\varphi, w) + g_1(\varphi + w, w) \tag{1.19}$$

$$g_1(\varphi + w, w) = g_1(\varphi, w) - sin\varphi,$$

Which become , in Fourier components,

$$a_{k,1} e^{ikw} = a_{k,1} + b_{k,1} e^{ikw}$$

$$b_{k,1} e^{ikw} = b_{k,1} - c_{k,1,} \tag{1.20}$$

Where $c_{k,1}$are the Fourier components of $\sin\varphi$, that is,

$c_{1,1} = -c_{-1,1} = 1/(2i)$ and all other components vanish. We thus get

$$g_1(\varphi, w) = \frac{e^{i\varphi}}{2i(1-e^{iw})} - \frac{e^{-i\varphi}}{2i(1-e^{-iw})} = \frac{\cos(\varphi - w/2)}{2\sin^2(w/2)} \qquad (1.21)$$

$$f_1(\varphi, w) = -\frac{e^{iw}e^{i\varphi}}{2i(1-e^{iw})^2} + \frac{e^{-iw}e^{-i\varphi}}{2i(1-e^{-iw})^2} = \frac{\cos(\varphi - w/2)}{4\sin^2(w/2)}.$$

Note that this is only possible for $e^{iw} \neq 1$,that ,$w \neq 0\ (mod\ 2\pi)$. At order ε^2 we obtain similar relations as (1.20), but now $c_{k,2}$ denotes the Fourier coefficients of $f_1(\varphi, w)\cos\varphi$, which are nonzero for $|k| = 2$. Thus g_2 *and* f_2 only exist if $e^{2iw} \neq 1$, or w$\neq 0, \pi$ (mod 2π):

Similarly, we will find that g_j *and* f_j only exist if $e^{jiw} \neq 1$, so the equations (1.15) can only be solved for irrational w/(2π) . Even then, the expansions of f *and* g will contain

Small terms of the $1 - e^{ikw}$in the denominators, so that the convergence of the series is not clear at all . In fact, the convergence has been proved by Moser for certain irrational w called Diophantine numbers [Mo73] .

Now let us turn to the geometric approach. We can consider (q,p) as coordinates in the plane (or on the cylinder because of the periodicity of q). For given (q_0, p_0), the set of points$\{(q_n, p_n)\}_{n\geq 0}$ is called the orbit with initial condition (q_0, p_0). We would like to know what the different orbits look like. The simplest case is the fixed point :if

$$q_{n+1} = q_n$$

$$p_{n+1} = p_n \qquad (1.22)$$

Then the orbit will consist of a single point . The fixed points of the standard map are (0,k) and (π ,k) with $k \in Z$.

We can also have periodic orbits, consisting of m points, if

$$q_{n+m} = q_n$$

$$p_{n+m} = p_n. \qquad (1.23)$$

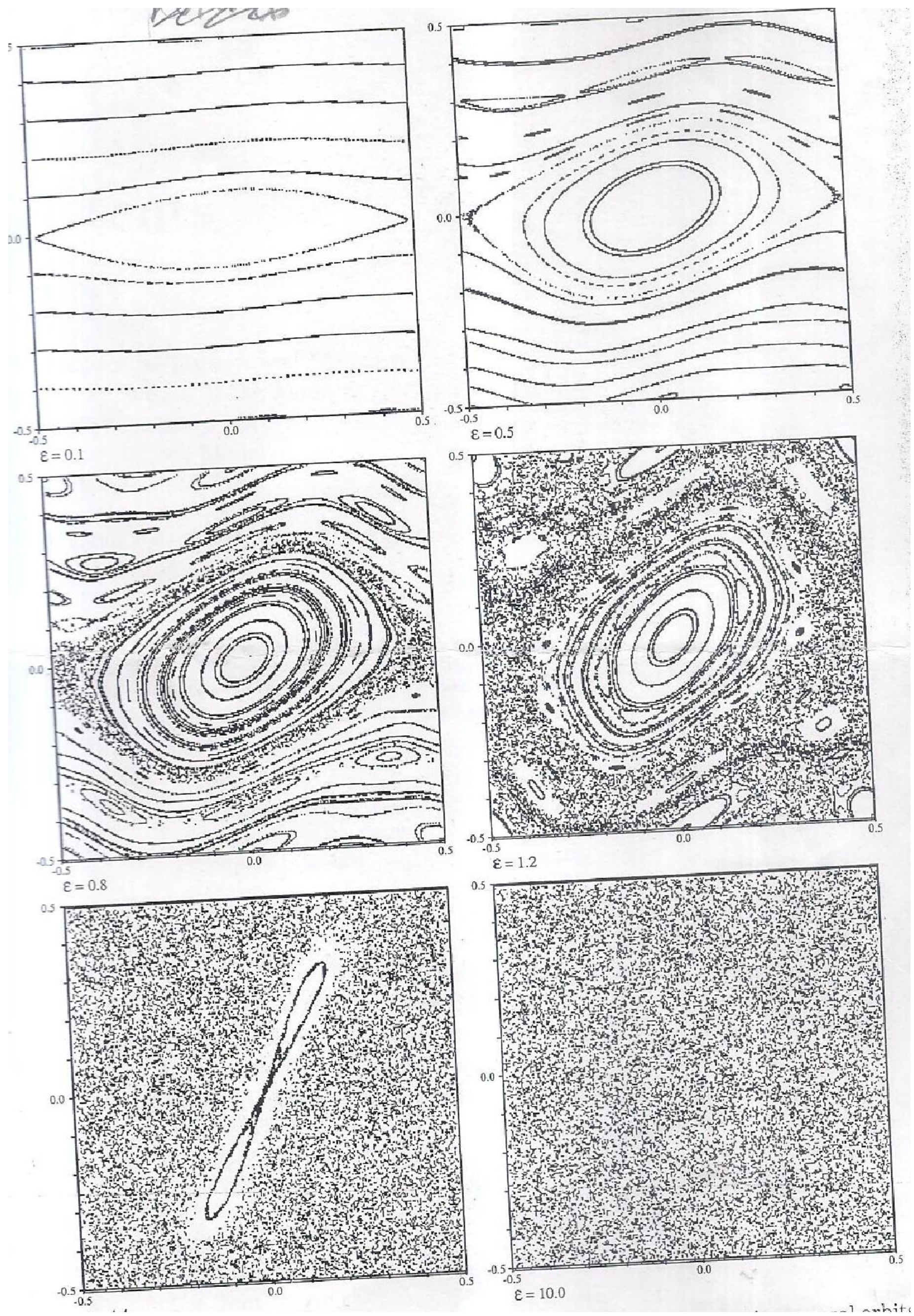


FIGURE (1.1) Another possible orbit is the invariant curve. For instance if the equations (1.15) admit a solution, we can write

$$q_n = \varphi_n + f(\varphi_{n,w}) \qquad (\text{mod}2\pi)$$

$$p_n = w + g(\varphi_{n,w}), \qquad (1.24)$$

Where $\varphi_n = \varphi_0 + n_w$. This is the parametric equation of a curve winding around the cylinder. Since w is irrational, the points fill the curve densely. One can also analyse the dynamics in the vicinity of periodic orbits. It turns out that this kind of map, most periodic orbits are of one two types: elliptic orbits are surrounded by invariant Curves, while hyperbolic orbits attract other orbits from one direction and expel them into another one. There are, however, much more exotic types of orbits. Some live on invariant Cantor sets, others densely fill regions of phase space with a positive surface. The aim of the geometrical theory of dynamical systems is to classify the possible behaviours and to find ways to determine the most important qualitative features of the system. An important advantage is that large classes of dynamical systems have a similar qualitative behavior. The does not mean that the perturbative approach useless. But it is in general preferable to start by analyzing the system from a qualitative point of view, and then, if necessary, use more sophisticated methods in order to obtain more detailed information. Now we study the Lorenz model convection is an important mechanism in the dynamics of the atmosphere: warm air has a lower density and therefore rises to higher altitudes, where it cools down and falls again, giving rise to patterns in the atmospheric currents .This mechanism can be modeled in the laboratory, an experiment known as $Raylegh - Be'nard\ conuection$. A fluid is contained between two horizontal plates, the upper one at temperature T_0 and the lower one at temperature $T_1 = T_0 + \Delta T > T_0$. the temperature difference ΔT *is the control parameter*, which can be modified.

For small values of Δt, the fluid remains at rest, and the temperature decreases linearly in the vertical direction, At slightly larger Δt, convection rolls appear (their shape depends on the geometry of the set-up). The flow is still stationary, that is, the fluid velocity at any give point does not change in time. For still larger Δt, the spatial arrangement of the rolls remains fixed, but their time dependences becomes more complex . Usually, it starts by getting periodic. Then different scenarios are observed, depending on the set-up. One of the them is the *period doubling casade* the time-dependence of the velocity field has period,P, 2P, 4P, ….., $2^n P$, ….., where the n^{th} period doubling occurs for a temperature difference ΔT_n satisfying

$$\lim_{n\to\infty} \frac{\Delta T_n - T_{n-1}}{\Delta T_{n+1} - T_n} = \delta \simeq 4.4 \qquad (1.25)$$

These ΔT_n accumulate at some finite ΔT_∞ for which the behavior is no longer periodic, but displays temporal chaos. In this situation, the direction of rotation of the rolls changes erratically in time. For very large ΔT, the behaviour can become *turbulent*:not only is the time depen - dence nonperiodic, but the spatial arrangement of the velocity filed also changes RB convection has been modeled in the following way . For simplicity, one considers the two-dimensional case, with an infinite extension in the horizontal

$x_1 - direction$,while the vertical $x_2 - direction$ is bounded between -$\frac{1}{2}$ and $\frac{1}{2}$. Let $D = R \times [-\frac{1}{2}, \frac{1}{2}]$. The state

$$v: D \times \mathbb{R} \to \mathbb{R}^2 \qquad \text{velocity,}$$

$$T: D \times \mathbb{R} \to \mathbb{R} \qquad \text{temperature,} \qquad (1.26)$$

$$p: D \times \mathbb{R} \to \mathbb{R} \qquad \text{Pressure.}$$

The deviation $\theta(x,t)$ from the linear temperature profile is by

$$(x,t) = T_0 + \Delta T\left(\tfrac{1}{2} - x_2\right) + T_1 \theta(x,t). \qquad (1.27)$$

The equations of hydrodynamics take the following form:

$$\frac{1}{\sigma}\left[\frac{\partial v}{\partial t}+(v.\Delta)v\right]=\Delta v-\nabla p+(0,\theta)^T$$

$$\frac{\partial\theta}{\partial t}+(v,\nabla)\theta=\Delta\theta+Rv_2 \qquad (1.28)$$

$$\nabla.v=0$$

Here σ, *the Prandtl number*, is a constant related to physical properties of the fluid, while R, *the Reynolds number*, is proportional to ΔT. Furthermore,

$$\nabla p=(\frac{\partial p}{\partial x_1},\frac{\partial p}{\partial x_2})^T$$

$$\Delta\theta=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial x_2^2}$$

$$\nabla.v=\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}$$

$$(v_{1.}\nabla)\theta=v_1\frac{\partial\theta}{\partial x_1}+v_2\frac{\partial\theta}{\partial x_2}.$$

The terms containing $(v.\nabla)$ introduce the nonlinearity into the system. the boundary conditions require that (θ,v_2) and $\partial v_1/\partial x_2$ should vanish for $x_2=\pm\frac{1}{2}$. We thus have to solve four coupled nonlinear partial differential equations for the four fields v_1,v_2,θ,p. The continuity equation $\nabla.v=0$ can be satisfied by introducing the *vorticity* $\psi: D\times\mathbb{R}\to\mathbb{R}$, such that

$$(v_1,v_2)=\left(-\frac{\partial\psi}{\partial x_2},\frac{\partial\psi}{\partial x_1}\right) \qquad (1.29)$$

It is also possible to eliminate the pressure from the two equations for $\partial v/\partial t$. we are left with two equations for ψ and θ. The problem can be further simplified by assuming a periodic dependence on x_1,of period $2\pi/q$. A possible approach (not the best one by modern standards, but historically important) is to expand the two fields into Fourier series (or "modes"):

$$\psi(x_1, x_2) = \sum_{k\epsilon\mathbb{Z}^2} a_k e^{ik_1 q x_1}\, e^{ik_2 \pi x_2}$$

$$\theta(x_1, x_2) = \sum_{k\in\mathbb{Z}^2} b_k e^{ik_1 q x_1}\, e^{ik_2 \pi x_2} \qquad (1.30)$$

Example(1.1.3):(The Lorenz Model)

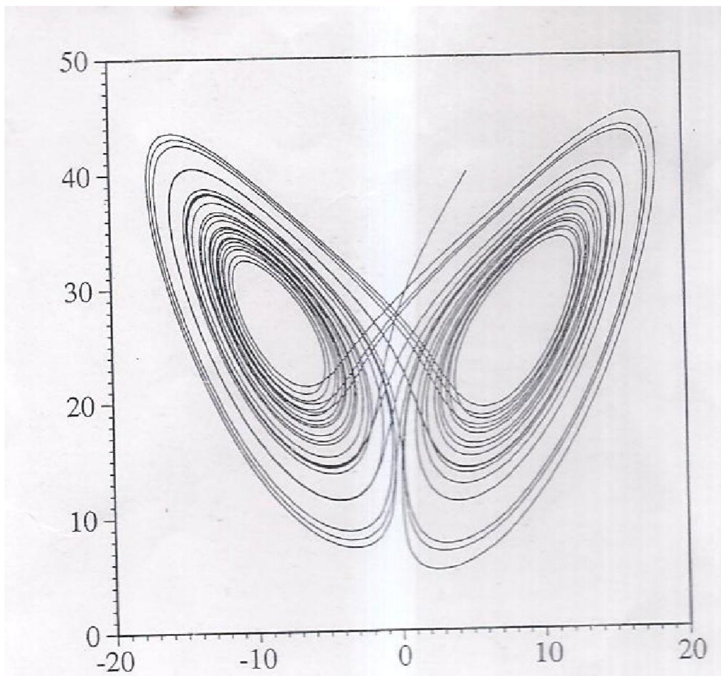


FIGURE (1.2). One trajectory of the Lorenz equations(1.33) for $\sigma = 10, b = \frac{8}{3}\, and\, r = 28$, projected (X,Z)-plane. (where the boundary conditionsimpose som relations between Fourier coefficients of the same $|k_2|$). Note that the terms of this sum are eigenfunctions of the linear operators in(1.28). Plugging these expansions into the equations, we obtain relation of the from

$$\frac{d}{dx}\binom{a_k(t)}{b_{k(t)}} = L_k \frac{d}{dx}\binom{a_k(t)}{b_{k(t)}} + N(\{a_{k'}, b_{k'}\}_{k'\in\mathbb{Z}}), \qquad (1.31)$$

Where $L_k$are 2×2 matrices and the $N(.)$ comes from the nonlinear terms in $(v.\nabla)$ and may depend on all other k'. Without these nonlinear terms the problem would be easy to solve.

In 1962, Saltzmann considered approximations of the equations (1.31) with finitely many terms, and observed that the dynamics seemed to be dominated by three Fourier modes. In 1963,Lorenz decided to truncate the equations to these modes[Lo63], stting

$$\psi(x_{1,}x_2) = \alpha_1 X(t) sinqx_1 cos\pi x_2$$

$$\theta(x_{1,}x_2) = \alpha_2 Y(t) cosqx_1 cos\pi x_2 + \alpha_3 Z(t) sin2\pi x_{2.} \qquad (1.32)$$

Here $\alpha_1 = \sqrt{2}(\pi^2 + q^2)/\pi q\ , \alpha_2 = \sqrt{2}\alpha_3\ and\ \alpha_3 = (\pi^2 + q^2)^3/(\pi q^2)$ are constants introduced only in order to simplify the resulting equations. All other Fourier modes in the expansion (1.31) are set to Zero, a rather drastic approximation. After scaling time by factor $(\pi^2 + q^2)$, one gets the equations

$$dX/dT = \sigma(Y - X)$$

$$dY/dT = rX - Y - XZ \qquad (1.33)$$

$$dZ/dT = -bZ + XY,$$

Where $4\pi^2/(\pi^2 + q^2)$, and $Rq^2/(\pi^2 + q^2)^3$is proportional to the control paramter R, and thus to ∇T. These so-called $lorenz\ equation$ are a very cery approximation of the equations (1.28) , nevertheless they may exhibit very complicated dynamics. In fact, for $0 \le r \le 1$, all solutions are attracted by the origin $X = Y = Z = 0$, corresponding to the fluid at rest.For r>1, a pair of equilibria with $X \neq 0$ attracts the orbits , they correspond to convection rolls with the two possible directions of rotation . In-creasing r produces a very complicated sequence of bifurcations, including period doubling cascades[Sp82]. For certain values of the parameters, $a\ strange\ attraclly$ is formed, in which case the convection rolls change their diection of rotation very erratically(Fig,1.2), and the dynamics is very sensitive to small changes in the initial conditions(the $Butterfly\ effect$). The big surprise was that such a simple approximation, containing only three modes, could capture such complex behaviours.

Example(1.1.4) :(The Logistic Map)

Our last example is a famous map inspired by population dynamics . Consider a population of animals that repeoduce once ayear. Let $p_n$be the number of individuals in year number n. The offspring being usually proportional to the number of adults , the simplest model for the evolution of the population from one year to the population from one year to the next is the linear equation

$$p_{n+1} = \lambda p_n \qquad (1.34)$$

Where λ is the nataliy rate (minus the mortality rate). This low leads to an exponential growth of the form

$$p_n = \lambda^n p_0 = e^{nln\lambda} p_0 \qquad (1.35)$$

$the\ Malthus\ low$. This model becomes unealistic when the number of individals is so large that the limitation of resources become apparent. The simplest possibility to limit the growth is to introduce a quadratic term-βp_n^2, leading to the law

$$p_{n+1} = \lambda p_n - \beta p_n^2. \qquad (1.36)$$

The rescaled varible $x = \beta P$ then obeys the equation

$$x_{n+1} = f x_n(x_n) = \lambda x_n(1 - x_n) \qquad (1.37)$$

The map f_λ is called the $logistic\ map$. Observe that for

$0 \leq \lambda \leq 4$, $f_\lambda$maps the interval $[0,1]$ into itself. The dynamics of the sequence x_2 depends drastically on the value of λ.

For $0 \leq \lambda \leq 1$, all orbits converge to 0, which means that population becomes extinct. For $1 \leq \lambda \leq 3$,all orbits stating at $x_0 > 0$ converge to $1 - 1/\lambda$, and thus the population reaches a stable equilibrinm. For $3 < \lambda \leq 1$, the orbits converge to a cycle of period 2, so that the population asymptoticlly jumps back and forth between two values. For $\lambda > 1 + \sqrt{6}$, the system goes through a whole seqence of period doubings. Similarly as in RB convection, the values λ_n of the parameter for which the n^{th} period doubling occurs obey the low

$$\lim_{n-\infty} \frac{\lambda_n + \lambda_{n-1}}{\lambda_{n+1} + \lambda_n} \qquad (1.38)$$

Where δ is called the $feigenbaum\ constant$. In 1978, Feigenbaum as well as Coullet and Tresser independently outlined an argument showing that such period doubling cascades

Section(1.2):(Basic Concepts)

Restar by dis cussing the orbits an Flows let $D \subset \mathbb{R}^n$be an opan domain. One type of dnamical systems we will consider is given by a map $F: D \rightarrow D. D$ is called the phase space of the system. It is possible to consider more general differentiable manifolds as phase space, such as the circle, the cyliner or the torus, but we will limit the discussion to Euclidean domains. Generalizations to other manifolds are usually straightforward.

Definition(1.2.1): The (positive)orbit $of\ F\ through\ a\ point\ x_0 \in D$ $is\ the\ sequence\ (x_k)_{k\geq 0}\ defined\ by\ \ x_{k+1} = F(x_k)$

$for\ all\ integers\ k \geq 0. We\ have\ thus$

$$x_k = F^k(x) \qquad where\ F^k = \underbrace{FoFo \ldots oF}_{k\ times} \qquad (1.40)$$

$In\ case\ F\ is\ invertible, we\ can\ also\ define\ the\ ngative\ orbit\ of\ x_0$

By the relation

$$F(x_k) = x_{x+1} \; for \; all \; k < 0, which \; are \; equiualent \; to (1.2.1)$$

$if \; we \; set \; F^{-k} = (F^{-1})^k \; for \; all \; k > 0. The \; orbit \; of \;\; x_0 \;\; is$ then given by $(x_k)_{k \in \mathbb{Z}}$.

Note the trivial relation

$$F^{k+l}(x_0) = F^k\left(F^k(x_0)\right) \qquad (1.41)$$

For all positive integers k, l (and all integers if F is invcrtible), which will admit an analogue in the case of ODEs.As particular cases of maps F,

we have homeomorphisms, which are continuous maps admitting a continuous inverse, and $diffeomorphisms$, which are continuously differen-tiable maps admitting a continuously differentiable inverse Similarly, all $r \geq 1, a \quad C^r diffeomorphism \; is \; an \; invertible \; map \; F \; such$ that both F and F^{-1} admit continuous derivativesv up order r. The $ordinary \; differential \; equations$ we are going to consider are of the form

$$\dot{x} = f(x), \qquad (1.42)$$

Where $f: D \to \mathbb{R}^n, and \; \dot{x}$ denotes $\frac{dx}{dt}$. Equivalently, we can write(1.42) as a system of equations for the components of x,

$$\dot{x}_i = f_i(x), \;\; i, \ldots \ldots \ldots, n \qquad (1.43)$$

$D \subset \mathbb{R}^n$ is again called phasc space and f is called a vector field. To difine the orbits of f, we have to treat the problem of existence and uniqueness a bit more carefully.

The following results are assumed to be known from basic analysis(see for instance [Hal69],[Har64],[HS74]).

Let$f$ $be\ continuous. for\ every\ x_0 \epsilon D, there\ exists\ at$ least one local solution of (1.42).

Through x_0, that is, there is an open interval $I \ni 0$ and a function $x: I \to D\ such\ that\ x(0) = x_0\ and\ \dot{x}(t) = f\big(x(t)\big)\ for\ all\ t \epsilon I.$

Theorem (1.2.3): Every solution $x(t) with\ x(0) = x_0\ can\ be$ Continued to a maximal in terval of existence $(t_1, t_2) \ni 0$.

$if\ t_2 < \infty\ or\ t_1 > -\infty$, then for any compact $K \subset D$, there exists a time $\big(t_{1,} t_2\big)\ with\ x(t) \notin k$ (this means that solution will diverge or reach ∂D).

Theorm(1.2.4): (Picard-Lindel$\ddot{o}$f)

Assume $f\ is\ continuous\ and\ locally\ Lipschitzian, that\ is,$

For every compact $k \subset D$, there exists a constant L_k such that $\big||f(x) - f(y)|\big| \leq L_k\big||x - y|\big| for\ all\ x, y \in k$. then there is a unique solution $x(t)\ of$ (1.42) $with\ x(0) = x_0$ for every $x_0 \in D$. Note in particular that if f is continuously differentiable, then it is locally n Lipschitzian. We will usually consider vector fields which are at least once continuously differentiable.

Example (1.2.5): It is easy to give counter examples to global existence and uniqueness.For instance,

$$\dot{x} = x^2 \implies x(t) = \frac{1}{\frac{1}{x_0} - t} \qquad (1.44)$$

has a solution divrging for t=$\frac{1}{x_0}$. A phsically interesting counterexample to uniqueness is the leaky bucket equation

$$\dot{x} = -\sqrt{|x|}. \qquad (1.45)$$

Here x is proportional to the height of water in a buchet with a hole in the bottom, and (1.45) reflects the fact that the kinetic energy of the water inside. For every c, (1.45) the solution

$$x(t) = \begin{cases} \frac{1}{4}(t-c)^2 & for\ t < c \\ 0 & for\ t \geq c. \end{cases} \quad (1.46)$$

In particular , for any $c \leq 0$, (1.45) such that $x(0) = 0$.

This reflects the fact that if the bucket is empty at time 0, we do know at what time it was full.

For simplicity, we will henceforth assume that the ODE(1.42) admits a unique global solution for all $x_0 \epsilon D$. This allows to introduce the follwing definitions:[1]

Definition(1.2.6):): let $x_0 \in D\ and\ let\ x(t)\ be\ the\ unique\ \ solution\ of$ (1.42) $with$ initial condition $x(0) = x_0$.

1. The integral curve through x_0 is the set
$$\{(x,t) \in D \times \mathbb{R}: x = x(t)\}.$$
2. The orbit $through\ x_0 is\ the\ set$
$$\{x \in D: x = x(t), t \in \mathbb{R}\}$$

3. $the\ flow\ of\ the\ equation$ (1.42) $is\ the\ map$

$$\varphi : D \times \mathbb{R} \rightarrow \ D$$

$$\left(x_{0,}t\right) \mapsto \varphi_t(x_{\ 0}) = x(t) \quad (1.47)$$

Gemetrically speaking, the orbit is a curve in phase space containing x_0 such that the vector field $f(x)$ is tangent to the curve at any point x of the curve. Uniqueness of the solution means that there is only one orbit through any point in phase space .

By definition, we have $\varphi_0(x_0) = x_0$ for all $x_0 \in D$, uniqueness of solutions implies that $\varphi_t(\varphi_s(x_0) = \varphi_{t+s}(x_0)$.These properties can be rewritten as

$$\varphi_0 = id \qquad \varphi_t o \varphi_s = \varphi_{t+S} \tag{1.48}$$

Which means that the family $\{\varphi_t\}_t$ forms a group.

Note the similarity between this relation and the relation (1.41) for iterated maps.

Example (1.2.7): In the case $f(x) = -x, x \in \mathbb{R}, we\ have$

$$\varphi_t(x_0) = x_0 e^{-t} \tag{1.49}$$

The system admits three distinct of phase space .We can define its volume by a usual Riemann integral:

$$vol(m) = \int_{m} dx, \quad dx = dx_1 \dots \dots dx_n. \tag{1.50}$$

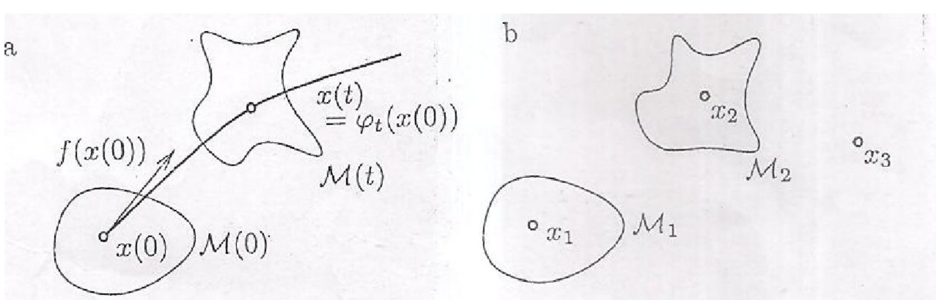
The set m will evolve under the influence of the dynamics: we can define the sets $m_k = F^k(m) or\ m(t) = \varphi_t(m)$. How does their volume evolve with time? The answer is actually quite simple.

Consider first the case of a map F. We assume that F is continuously differetible and denote by

$$\frac{\partial F}{\partial x}(x) \tag{1.51}$$

The Jacobian matric of F, which is the $n \times n\ matrix\ A\ with$

$$elementss\ a_{ij} = \frac{\partial F_i}{\partial x_j}(x).$$ [1]



FIGURE(1.3):(a)Evolution of a volume with the flow,(b) evolution with an iterated map.

Proposition (1.2.8): Assume F is a diffeomorphism and let

$V_k = Vol(m_k)$. then

$$V_{k+1} = \int_{\mathcal{M}k} \left| det \frac{\partial F}{\partial x}(x) \right| dx. \tag{1.52}$$

Proof: This is a simple application of the formula for a change of variables in an integral:

$$V_{k+1} = \int_{\mathcal{M}k+1} dy = \int_{\mathcal{M}k} \left| det \frac{\partial y}{\partial x}(x) \right| dx = \int_{\mathcal{M}k} \left| det \frac{\partial F}{\partial x}(x) \right| dx. \tag{1.53}$$

Definition(1.2.9): *The map F is calld dissipative if*

$$\left| \det \left(\frac{\partial F}{\partial x}(x) \right) \right| < 1 \ \forall x \epsilon D \tag{1.54}$$

Proposition (1.2.8) implies that $V_{k+1} = V_k$ *if F is conservative and*

$V_{k+1} < V_k$ *if F is dissipative* .

More generally ,if $\left| det \frac{\partial F}{\partial x}(x) \right| \leq \lambda$ for some constant λ and all $x \epsilon D$, then $V_k \leq \lambda^k V_0$.

For differential equations, the result is the following .

Proposition (1.2.10): Assume f is continuously differentiable and $let\ V(t) = Vol(\mathcal{M}(t))$ then

$$\frac{d}{dx} V(t) = \int_{\mathcal{M}(t)} \nabla \cdot f(x) dx \tag{1.55}$$

$$where\ \nabla \cdot f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}\ is\ divergence.$$

Proof : We have

$$V(t) = \int_{\mathcal{M}(t)} dy = \int_{m} \left| det \frac{\partial}{\partial x} \varphi_t(x) \right| dx$$

Let us fix $x \in \mathcal{M}, let\ y(t) = \varphi_t(x)\ and\ set$

$$J(t) = \frac{\partial}{\partial x} \varphi_t(x), \qquad A(t) = \frac{\partial}{\partial x} \big(y(t)\big).$$

Note that by definition of $\varphi_t, J(0) = ¶$ is the identily matrix . Now we can compute

$$\frac{d}{dt} J(t) = \frac{\partial}{\partial x} y(t) = \frac{\partial}{\partial x} f\big(\varphi_t(x)\big) = \frac{\partial f}{\partial x} (\varphi_t(x) \frac{\partial}{\partial x} \varphi_t(x),$$

And thus

$$d/dx\, J(t) = A(t)J(t), \;\; J(0) = \Pi.$$

This is linear, time-dependent differential equation for J(t), which is known to admit a unique global solution. This implies in particular that $detJ(t) \neq 0$ ∀t, since otherwise $J(t)$ would not be surjective, contradicting uniqueness. Since $det\, J(0) = 1$, continuity implies that $detJ(t) > 0\ \forall t$. Now let us determine the evolution of $detJ(t)$. By Taylor's formula, there exists $\theta \in [0,1] such\ that$

$$J(t+\varepsilon) = J(t) + \varepsilon \frac{d}{dx} J(t+\theta\epsilon) = J(t)[¶ + \varepsilon J(t)^{-1} A(t+\theta\varepsilon)].$$

Form linear algebra, we know that for any $n \times n$ matrix B,

$$\det(¶ + \varepsilon B) = 1 + \varepsilon TrB + r(\varepsilon)$$

With $lim_{\varepsilon \to 0} r(\varepsilon)/\varepsilon = 0$ (this is a consequence of the definition of the determinant as a sum over permutations).

Using Tr(AB)=Tr(BA), this leads to

$$detJ(t+\varepsilon) = detJ(t)[1 + \varepsilon Tr(A(t+\theta\varepsilon)J(t+\theta\varepsilon)J(t)^{-1} + r(\varepsilon)],$$

And thus

$$\frac{d}{dt}detJ(t) = \lim_{\varepsilon \to 0} \frac{detJ(t+\varepsilon) - detJ(t)}{\varepsilon} = Tr\big(A(t)\big)detJ(t).$$

Taking the derivative of $V(t)$ we get

$$\begin{aligned}\frac{d}{dt}V(t) &= \int_{\mathcal{M}} \frac{d}{dt}detJ(t)dx \\ &= \int_{\mathcal{M}} Tr\left(\frac{\partial f}{\partial x}(y)\right)detJ(t)\,\mathrm{det} \\ &= \int_{\mathcal{M}t} Tr\left(\frac{\partial f}{\partial x}(y)\right)dy\end{aligned}$$

And the conclusion follows from the fact that $Tr\frac{\partial f}{\partial x} = \nabla \cdot f$.

Definition (1.2.11):

(1). The vector fild f is called conservative if

$$\nabla . f(x) = 0, \forall x \in D \tag{1.56}$$

(2) . $The\ vector\ f\ is\ called\ dissipative\ if$

$$\nabla \cdot f(x) < 0 \;\; \forall x \epsilon D. \tag{1.57}$$

Proposition(1.2.10) implies that V(t) is constant if f is conservative, and monotonously decreasing when f is dissipative.

More generally, if $\nabla \cdot f(x) \leq c\ \forall x \in D, then\ V(t) \leq V(0)e^{ct}$.

Of course, one can easily write down dynamical systems which are neither conservatives nor dissipative, but the conservative and dissipative situations are very common in applications.

Example (1.2.12): Consider a Hamiltonian system, with Hamiltonian $H \in C^2(\mathbb{R}^{2m}, \mathbb{R})$. Then $x = (q, p) \epsilon \mathbb{R}^m \times \mathbb{R}^m$ and the equations(1.4) take the from

$$f_i(x) = \begin{cases} \frac{\partial H}{\partial p_i} & i = 1 \dots\dots m \\ -\frac{\partial H}{\partial q_{i-m}} & i = m+1, \dots .2m \end{cases} \quad (1.58)$$

This implies that

$$\nabla \cdot f = \sum_{i=1}^{m} \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \sum_{i=1}^{m} \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0 \quad (1.59)$$

Thus all (sufficiently smooth) Hamiltonian systems are conservative.

Chapter 2
Stationary and periodic solutions

Section (2.1): Stationary solutions
Definition (2.1.1):

(1). A fixed point of the map F is appoint $x^* \in D$ such that

$$F(x^*) = x^* \tag{2.1}$$

(2) . A singular point of the vector field f is point $x^* \in D$ such that

$$f(x^*) = 0 \tag{2.2}$$

In both cases , x^* is also called equilibrium point. It is orbit is simply $\{ x^*\}$ and it is called stationary orbit .

Note that a singular point of is also fixed point in the flow ,and there for some times abusively called "a fixed point of f" we a are now interested in the behavior near an equilibrium point .In this section , we with always assume that f and F are twice continuously differentiable .If x^* is a singular point of f ,the change of variables

$x = x^* + y$ leads to the equation

$$\dot{y} = f(x^* + y) = Ay + g(y) \tag{2.3}$$

Where we have introduced the jacobian matrix

$$A = \frac{\partial f}{\partial x}(x^*) \tag{2.4}$$

Taylor's formula implies that there exists a neighborhood N of 0 and a constant $M < 0$ such that

$$||g(y)|| \leq M||y||^2 \quad \forall\, y \in N \tag{2.5}$$

Similarly, the change of variables $xk = x * +yk$ transform an interated map into

$$y_k + 1 = F(x* \ +y_k) - x* \qquad (2.6)$$

$$=By_k + G(y_k)$$

Where

$$B=\frac{\partial F}{\partial x}(x*), ||G(y)|| \le M\|y\|^2 \quad \forall\ y \in N \qquad (2.7)$$

Now we discussing the linear case , and let us start by analyzing the equation (2.3) and (2.6) in the linear case that without the terms g(x) and G(x) .consider first the ODE

$$\dot{y} = Ay \qquad (2.8)$$

The solution can be written as

$$y(t)=e^{At}y(0), \qquad (2.9)$$

Where the exponential of A is defined by the absolutely convergent series

$$e^{At} \equiv \exp(At) = \sum_{K=0}^{\infty} \frac{t^k}{k!} A^K \qquad (2.10)$$

In order to understand the behavior of e^{At}, let us recall some facts from linear algebra. we can write the characteristic polynomial of A as

$$c_A(\lambda) = \det(\lambda\, \mathbb{I} - A) = \prod_{J=1}^{M} (\lambda - aj)^{mj} \qquad (2.11)$$

Where a_1 , … … … … … . . , $a_m \in$ C are distinct eigenvalues of A, and m_j are their algebraic multiplicities .The geometric multiplicity g_j of a_j is defined as the number independent eigenvectors associated with a_j and satisfies $1 \le g_j \le m_j$.

The results on decomposition of matrices leading to the Jordan canonical from can be formulated as follows [HS74].The matrix A can be decomposed as

$$A=S+N, \qquad SN=NS \tag{2.12}$$

Here S, the semi simple part, can be written as

$$S=\sum_{j=1}^{m} a_j P_j \tag{2.13}$$

Where the P_j are projectors on the eigen spaces of A, satisfying

$P_j P_k = \delta_{jk} P_j, \sum_j P_j = 1$ and $m_j = \dim\left(p_j R^n\right)$. The nilpotent part N can be written as

$$N=\sum_{j=1}^{m} N_j, \tag{2.14}$$

Where the Nj satisfy the relations

$$N_j^{mj} = 0, \quad N_j \ N_k = 0 \quad \text{for } P_j \ N_k = N_k \ P_j \ = \delta_{jk} \ N_j \tag{2.15}$$

In an appropriate basis, each Nj is block-diagonal, with g_j blocks of the from

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \tag{2.16}$$

In fact $N_j = 0$ unless $g_j < mj$

Lemma (2.1.2): With the above notations

$$e^{At}=\sum_{j=1}^{m} {}_{e}^{aj}t \, pj \left(1 + Nj + \cdots \ldots \ldots . + \frac{1}{(mj-1)!} N_j^{mj-1} t^{mj-1}\right) \tag{2.17}$$

Proof: We use the fact that $e^{At}e^{Bt} = e^{(A+B)t}$ when ever

$AB = BA$,which can be checked by adirectcalculationthene $^{At}=e^{st}e^{Nt}$ with

$$e^{st}=\prod_{j=1}^{m} e^{ajPjt} = \prod_{j=1}^{m}(1 + ({}_{e}^{aj}t - 1)Pj = 1 + \sum_{j=1}^{m} +({}_{e}^{aj}t - 1)Pj = \sum_{j=1}^{m} {}_{e}^{aj}tPj \, , e^{Nt} = \prod_{j=1}^{m} e^{Njt} = 1 + \sum_{j=1}^{m}(e^{Njt} - 1)$$

The result follows from the facts that Pj $(e^{N_k t}-1)=0$ for $j \neq k$,and that $e^{N_j t}$ contains only finitely the expression (2.17) shows that the long-time behavior is determined by the real parts of the eigenvalues a_j, while the nilpotent terms, when present, influence the short parts behaviour. This motivates the following terminology.

Definition (2.1.3): The unstable, stable and center subspace of the singular point x* are defined, respectively, by

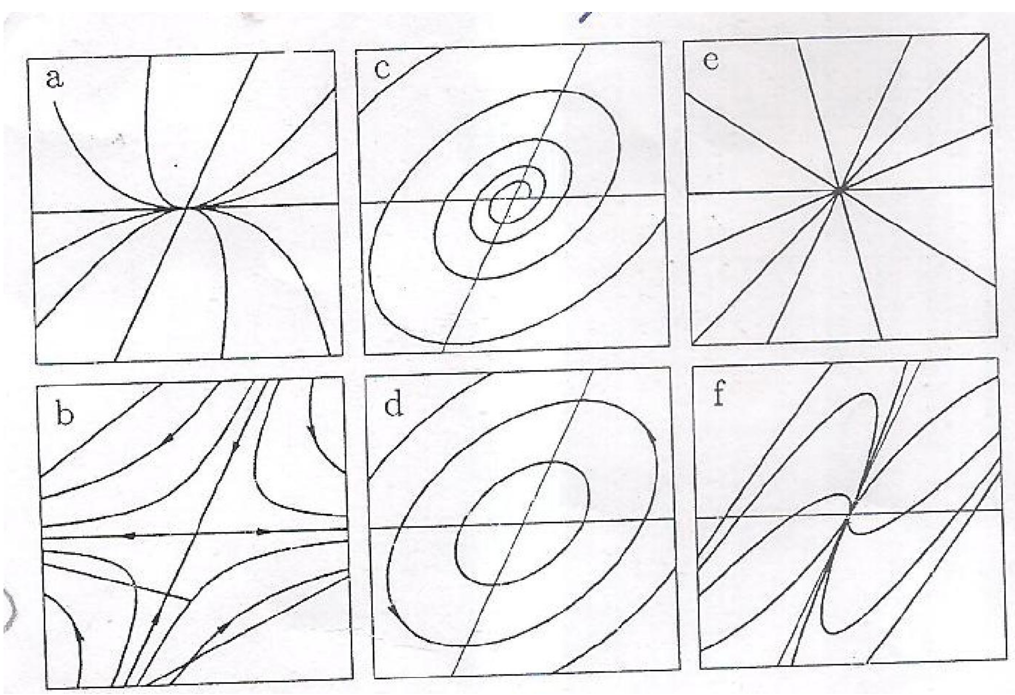
$$E^{+} := p^{+} R^{n}=\{y:\lim_{t\to-\infty} e^{At}y=0\},$$
$$p^{+}:=\sum_{j:\mathrm{Re}a_j>0} Pj,$$

$$\overline{E} := p^{-}R^{n}=\{ y:\lim_{t\to+\infty} e^{At}y=0\},$$
$$\overline{E}:= \sum_{j:\mathrm{Re}a_j<0} Pj, \qquad (2.18)$$

$$E^{0} := p^{0}R^{n}, \qquad p^{0}:= \sum_{j:\mathrm{Re}a_j=0} Pj,$$

The subspaces are invariant subspaces of e^{At}, that is

$$e^{At}E^{+} \subset E^{+}, e^{At}E^{-} \subset E^{-} \text{and } e^{At}E^{0} \subset E^{0}.$$



FIGURE(2.1):phase port of a linear two _dimensional system (a)node,(b)saddle ,(c)focus,(d)center ,(e)degenerate node,(f)improper node.

- a sink if $E^{+}=E^{0}=\{0\}$,
- a source if $E^{-}=E^{0}=\{0\}$,
- a hyperbolic point if $E^{0}=\{0\}$,
- an elliptic point if $E^{+}=E^{-}=\{0\}$,

Example (2.1.4): Let n=2, and let A be in Jordan canonical from, with $\det A \neq 0$ .Then we can distinguish between the following behaviours, depending of the eigenvaluse a_1, a_2 of A (see fig(2.1)).

(1). $a_1 \neq a_2$

(a). if $a_1, a_2 \in R$,then $A=\begin{pmatrix} a1 & 0 \\ 0 & a2 \end{pmatrix}$ and

$$e^{At}=\begin{pmatrix} e^{a1t} & 0 \\ 0 & e^{a2t} \end{pmatrix} \Rightarrow \begin{matrix} y_1(t) = & e^{a1(t)} & y_1(0) \\ y_2(t) = & e^{a2(t)} & y_2(0) \end{matrix}$$

The orbits are curves of the from $y_2 = cy1^{a2/a1}$. x^* is called a node if $a_1 a_2 > 0$, and a saddle if $a_1 a_2 < 0$.

(b). if $a_1 = \overline{a_2}$=a+iw ∈ C, then then the real canonical from of A is

$$A=\begin{pmatrix} a & -w \\ w & a \end{pmatrix} \qquad \text{and}$$

$$e^{At}=e^{at}\begin{pmatrix} \cos wt & -\sin wt \\ \sin wt & \cos wt \end{pmatrix}$$

$$\Rightarrow \begin{matrix} y1(t) = e^{at}(y1(0)\cos wt + - y2(0)\sin wt) \\ y2(t) = e^{at}(y1(0)\sin wt + y2(0)\cos wt) \end{matrix}$$

X* is called focus if $a \neq 0$, and a center is a=0 . The 0rbits are spirals or ellipses.

(2). $a_1 = a_2 =: a$

(a). If a has geometric multiplicity 2 ,then A= a_1 and $e^{At} = e^{at}1$; X*is called a degenerate node .

(b). If a has geometric multiplicity 1, then A= $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ and $e^{At} = e^{at} \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \Rightarrow \begin{matrix} y_1(t) = & e^{a1(t)} & (y_1(0) + y_2(0)t) \\ y_2(t) = & e^{a2(t)} & y_2(0) \end{matrix}$

X* is called an improper node.

Let us now turn to the case of the linear iterated map

$$y_k +1 = B\, y_k \tag{2.19}$$

Which admits the solution

$$y_k = B^k y0 \tag{2.20}$$

Using a similar decomposition B = S + N into the semisimple and nilpotent part, we arrive at

Lemma (2.1.5): Let b_i be the eigen values of B, and $P_{i,}$ N_i the associated projectors and nilpotent – matrixes then

$$B^k = \sum_{i=1}^{m} Pi \sum_{j=0}^{\min\{k,mi-1\}} \binom{k}{j} b_i^{k-j} N_i^j \tag{2.21}$$

Proof: The main point is to observe that

$$B^k = \Big(\sum_{i=1}^{m} (bi\ Pi + Ni)\Big)^k = \sum_{i=1}^{m} (bi\ Pi + Ni)^k$$

Because all cross –terms vanish .Then one applies the binomial formula.

For large k, the behavior of B^k is dictated by the terms b_i^{k-mi+1} . this leads to the following equivalent of definition (2.1.3).

Definition (2.1.6): The unstable, stable and center subspaces of the fixed point x^* are defined, respectively, by

$$E^+ \coloneqq p^+R^n = \{y : \lim_{k\to-\infty} B^k y = 0\}, \qquad p^+ =: \sum_{j:|bj|>1} pj \ ,$$
$$\bar{E} := \bar{p}R^n = \{ y : \lim_{k\to+\infty} B^k y = 0\}, \qquad p^- =: \sum_{j:|bj|<1} pj \qquad (2.22)$$
$$E^0 := p^0R^n, \qquad p^0 =: \sum_{j:|bj|=1} pj \ ,$$

These subspaces are invariant under B .the remaining terminology or sinks, sources, hyperbolic and elliptic points is unchanged.

Definition (2.1.7): Let X* be an equilibrium point of the system

$$x^{\cdot} = f(x).$$

(1). X* is called stable if for any $\in > 0$, one can find a $\delta = \delta(\epsilon)0$ such that whenever $||x_{0-} x^*|| < \delta$, one has $||\alpha(x_0)\text{-}\ x^*|| < \varepsilon$ for all $t \geq 0$.

(2). x^* is called asymptotically stable if it is stable, and there is a $\delta_0 > 0$ such that $\lim_{t\to\infty} \alpha_t(x_0) = x^*$ for all x_0 such that $||x_{0-} x^*|| < \delta_0$.

(3). The basin of attraction of asymptotically stable equilibrium x^* is the set

$$\{x \in D : \lim_{t\to\infty} \alpha_t(x) = x^*\} \qquad (2.23)$$

(4). x^* is called unstable if it is not stable. If x^* is affixed point of the map F, similar definition hold with $\alpha_t(.)$ replaced by $F^k\ (.)$.

The linearization of the system $\dot{x} = f(x)$ around an equilibrium x^* is the equation $\dot{y} = Ay$ with $A = \frac{\partial f}{\partial x}\ (x^*)$. x^* is called linearly stable if y=0 is stable equilibrium of it is linearization, and similarly in the asymptotically stable and unstable cases. Lemma (2.1.2) show that

(5). x^* is linearly asymptotically stable if and only if all eigenvalues of A have a strictly negative real part;

(6). x^* is linearly asymptotically stable if and only no eigenvalues of A have has positive real part, and all purely imaginary eigenvalues have equal algebraic and geometric multiplicities. The problem is now to determine relations between linear or nonlinear stability. A useful method to do this is due to Liapunov. Here we will limit the discussion to differential equation, although similar results can be obtained for the maps.

Theorem (2.1.8):(Liapunov)

Let x^* be singular point of f, let U be a neighborhoods continuously differentiable on U_{0}, such that

(1). $V(x) > V(x^*)$ for all $x \in U_0$;

(2). The derivative of V along orbits is negative in U_0,that is ,

$$\dot{V}(x) := \frac{d}{dt} V(\alpha_t(x))|_{t=0} = \nabla V(x).f(x) \leq 0 \ \forall x \in U_0. \tag{2.24}$$

Then x^* is s table. If, furthermore,

(3). The derivative of V along orbits is strictly negative,

$$\dot{V}(x) = \nabla V(x).f(x) < 0 \qquad \forall x \in U_0 \tag{2.25}$$

Then x^* is asymptotically stable .

Proof: Pick $\varepsilon > 0$ small enough that the closed ball $\overline{B} = (x^*, \varepsilon)$ with center x^* and radius ε, is contained in U. Let $S = \partial\overline{B}(x^*, \varepsilon)$ be the sphere of radius ε centered in x^* . S being compact, V admits a minimum on S, that we call β. consider the open set

$W = \{x \in \overline{B} : V(x) < \beta\}$.

$x^* \in W$ by condition 1. And thus there exists $\delta > 0$ such that the open ball $B(x^*, \delta)$ is centered in W. For any $x_0 \in B(x^*, \delta)$, we have $V(\alpha_t(x_0)) < \beta$ for all $t \geq 0$, and thus $\alpha_t(x_0) \in W$ for all $t \geq 0$ by condition 2.Which proves that x^* is stable.

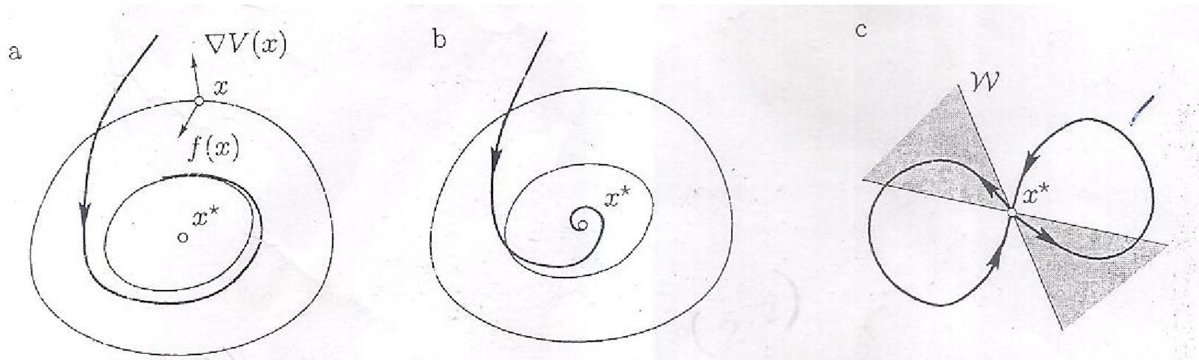


FIGURE (2.2) , (a) stable fixed point with level curves a Liapunov function . (b)Asymptotically stable fixed point, here the vector field must cross all level curves in the same direction.(c) Example of an unstable fixed point with the set W of cetaev's theorem .assume now that (2.25) holds . since the positive orbit of $x_0 \in W$ is bounded it admits a convergent subsequence

$$(x_n)n \geq 0 = \left(\alpha t_n(x_0)\right)n \geq 0 \rightarrow y^* \in \overline{W}, tn \rightarrow \infty .$$

Consider then function $t \mapsto V(\alpha_{tn}(x_0))$ .It is continuously differentiable ,monotonously decreasing ,and admits sub sequence converging to $V(y^*)$ .Thus $V(\alpha_{tn}(x_0))$ must converge to $V(y^*)$ as $t \rightarrow \infty$ .Let $\delta > 0$ be small constant and define the compact set

$$K=\{x \in \overline{W}: V(y^*) \leq V(x) \leq V(y^*)+\delta\}.$$

If $y^* \neq x^*$, then $x^* \in k$, and thus maximum of $\dot{V}$ on K is a strictly negative constant c. take n large enough that $x_n \in K$ .Then $\alpha_t(x_n) \in K$ for all $t \geq 0$ .But this implies that $V(\alpha_t(x_n)) \leq V(x_n)+ct$.

Which becomes smaller than $V(y^*)$ for t large enough . which is impossible.thus $y^* = x^*$, and all orbits starting in W converge to x^*.

The interpretation of (2.24) is that the vector field crosses all level sets of V in the same direction (fig 2.2) .V is called Liapunov function for x^* ,and astrict liapunov function if (2.25) holds . In fact ,the proof also shows that if V is a strict Liapunov function of U

,and W is asset of the from W $=\{x:V(x) < \beta\}$ contained in U ,then W is contained in the basin of attraction of x^* .

Thus liapunov function can be used to estimate such basins of attraction.

Corollary (2.1.9): Assume x^* is linearly asymptotically stable equilibrium , that is, all eigenvalues of $A = \frac{\partial f}{\partial x}(x^*)$ have a strictly negative real parts. Then x^* is asymptotically stable .

Proof: There are many different construction of strict liapunov function .We will give one of them .In order to satisfy condition 1.of the theorem, we will look for quadratic from

$$v(x)=(x- x^*) .Q(x- x^*)$$

where Q is asymmetric, positive definite matrix. By assumption, there constant

$a_0 > 0$ such that $\text{Re}a_i \leq -a_0$ for all eigenvalues a_i of A . Thus lemma (2.1.2) implies that

$$||e^{At} y|| \leq p(t)e^{a_0 t} ||y|| \ \forall y,$$

Where P is polynomial of degree less than n. this implies that the function

$$V(x) = \int_0^\infty ||e^{As}(x - x^*)||^2 \, ds$$

exists. V(x) is of the above from with

$$Q=\int_0^\infty e^{A^T s} e^{As} \, ds .$$

Q is clearly symmetric, positive definite and bounded, thus there is a $K > 0$ such that $||Q y|| \leq K||y||$ for all y. Now we calculate the following expression in tow different ways

$$\int_0^{\infty} \frac{d}{ds}(e^{A^T s} e^{As}) \, ds = \lim_{t\to\infty} e^{A^T t} - \square = -\square$$

$$\int_0^{\infty} \frac{d}{ds}(e^{A^T s} e^{As}) \, ds \int_0^{\infty} (A^T e^{A^T s} e^{As} + e^{A^T s} e^{As} A) \quad ds = A^T Q + Q A .$$

We have thus proved that

$$A^T Q + Q A = -\square.$$

Now if $y(t) = \alpha_t(x) - x^*$, we have

$$\dot{V} = \frac{d}{dt} V(\alpha_t(x)) = \frac{d}{dt}\big(y(t). Qy(t)\big) = \dot{y}(t). Qy(t) + y(t). Q\dot{y}(t).$$

Inserting $\dot{y} = Ay + g(y)$ produces tow terms. The first one is

$$g(y). Q(y) + y. QAy = y. (A^T Q + Q A)y = -\big||y|\big|,$$

And the second one gives

$$g(y). Q(y) + Qg(y) = 2g(y). Qy \le 2\big||g(y)|\big| \, \big||Qy|\big| \le 2MK\big||y|\big|^3.$$

Hence $\dot{V} \le -\big||y|\big|^2 + 2MK\big||y|\big|^3$, which shows that V is strict liapunov function for $||(x- x^*)|| < \frac{1}{2MK}$, and the result is proved. There exists acharacterization of unstable equilibria based on similar ideals:

Theorem (2.1.10): (cetaev)

Let x^* be a singular point of f, U aneighbourhood of x^* and $U_0 = {}^{U}/_{\{x^*\}}$. Assume there exists and open W, containing x^* in it is closure, and containuous function $V: U \to R$, which is containuously differentiable on U_0 and satisfies

1. $V(x) > 0$ *for all* $x \in U_0 \cap W$;

2. $\dot{V}(x) > 0$ *for all* $x \in U_0 \cap W$;

3. $V(x) = 0$ for all $x \in U_0 \cap W$;

Then x^* is unstable.

proof: first observe that the diffinition of an unstable point can be stated of follows.There exists $\varepsilon > 0$ *such that , for any* $\delta > 0$,

when can find x_0 with $||(x_0 - x^*)|| < \delta$ and $T< 0$ *such that*

$$\|\alpha t(x_0) - x^* \| \geq \varepsilon.$$

Now take $\varepsilon > 0$ *sufficietly small that* $B(x^*, \varepsilon) \subset U$. for any $\delta > 0$ $B(x^*, \delta) \cap W \neq 0$. we can thus take an $x_0 \in U_0 \cap W$ such that $\|\alpha t(x_0) - x^* \| < \delta$, *and by condition* . $v(x_0) > 0$

now assume by contradiction that $\|\alpha t(x_0) - x^* \|\varepsilon$ for all $t \geq 0$. $\alpha t(x_0)$ must say in $U_0 \cap W$ for all t , be cause it cannot

reash the boundary of W where V 0 . Thus there exists asequence

$x_n = \alpha t_n(x_0)$convergening some $x_1 \in U_0 \cap W$. But this contradics,

the fact that . $\dot{V}(x_1) > 0$, *an in the proof* of theorem (2.2.8) and

thus $\|\alpha t(x_0) - x^* \|$ must become larger that ε.

Corollary (2.1.11): Assume x^* is an equilibrium such that

$A = \frac{\partial f}{\partial x} (x^*)$ has at least one eigenvalue with positive real park . then x^* is unstable .

Proof: consider first the case of A having no purely imaginary eigenvalues We can choose acoordinate system

along the unstable and subspaces of x^*, in which the dynamics is describeby the eqution

$$\dot{y}+=A+y+ +g+(y)$$

$$\dot{y}-= A - y - +g - (y),$$

where all eigenvalue of A+ have strictly positive, all eigenvalue of $A -$ have strictly negative real park , $y = (y+,y-)$ and the tearms g_{-}^{+} arebounded in norm by positive constant M times $||y||^2$. we can defined the matrices

$$Q_- = \int_0^\infty e^{A^T - s} \; e^{A - s} \, ds \, , Q_+ = \int_0^\infty e^{-A^T + s} \; e^{-A + s} \, ds \, ,$$

Which are bounded , symetric , positive definite , and satisfy

$$A^T - Q_- + Q - A - = - []$$

and

$$A^{T} + Q_+ + Q_+ A_+ = [] .$$ define the quadraticfrom

$$V(y) = y_+ . Q_+ y_+ - y_- . Q_- y_- \, .$$

The cone $W = \{y: V(y) > 0\}$ is non − empty , because it contains an eigenvector of A corresp − onding to an eigenvalue with positive real park . proceeding similary as in the proof of Corollary (2.1.9), we find

$$\dot{V}(y) = ||y_+||^2 + 2g_+(y). Q_+ y_+ + ||y_-||^2 - 2g_-(y). Q_- y_-$$
$$\geq ||y||^2 - 2KM||y||^3 .$$

Thus Cetaev, s theorem can be applied to show that x^* is unstable .

If A also has j purely imaginary eigenvalues , we obtainthe additional equation

$$\dot{y}0 = A_0 \, y_0 + g_0 \, (y),$$

Where all eigenvalue of A_0 are purely imaginary . Let S be an invertible complex matric of the Same dimension as A_0

and consider the function

$$V(y)= y_+ . Q_+ y_+ - y_- . Q_- y_- - \|Sy_0\|^2,$$

Where $u.v = \sum \bar{u}_1 v_i$ Type equation here.
for complex vectors u, v . Proceeding as above , weobtain that

$$\dot{V}(y) \geq \|y_+\|^2 + \|y_-\|^2 - 2KM \|y\|^2(\|y_+\|+\|y_-\|)$$

$$-2Re(Sy_0 . A_0 Sy_0) - 2Re(Sg_0(y). Sy_0).$$

We shall prove below that for any $\varepsilon > 0$, *one can construct* a matric

$S(\varepsilon)$ such that

$$Re(Sy_0 . A_0 Sy_0) \leq \varepsilon \|Sy_0\|^2.$$

We now take $U = \{y: \|y\| < \delta\}$, where $\delta > 0$ *has to be determined* , $W = \{y: V(y) > 0\}$. If $y \in W$, we have

$$Re(Sy_0 . A_0 Sy_0) \leq \varepsilon \|Sy_0\|^2 < \varepsilon y+. Q + y+ < \varepsilon K\|y_+\|^2$$

We introduce the constant

$$C(\varepsilon) = \sup_{y_0 \neq 0} \frac{\|Sy_0\|}{\|y_0\|} ,c(\varepsilon)= \sup_{y_0 \neq 0} \frac{\|y_0\|}{\|s\, y_0\|} .$$

Then we have

$$Re(Sg_0(y). Sy_0) \leq C \|g_0(y)\| \|Sy_0\| \leq CMC \|y\|^2\|y_+\|$$

$$\|y\|^2 \leq \|y_+\|^2 + \|y_-\|^2 + c^2\|Sy_0\|^2$$
$$< (1 + c^2 K)\left(\|y_+\|^2 + \|y_-\|^2\right).$$

Putting everything together , we obtain for all $y \in U \cap W$

$$\dot{V}(y) > (1 - \varepsilon K)\left(\|y_+\|^2 + \|y_-\|^2\right) - 2KM\|y\|^2$$

$$\left[\left(1 + CK^{-1/2}\right)\left|\left|y_{+}\right|\right| + \left|\left|y_{-}\right|\right|\right] >$$

$$\left(\left|\left|y_{+}\right|\right|^{2} + \left|\left|y_{-}\right|\right|^{2}\right)[1 - \varepsilon K - 2KM\left(1 + c(\varepsilon)^{2}K\right)\left(2 + C(\varepsilon)K^{-1/2}\right)\delta\Big]$$

Taking $\varepsilon > 1/_{K^2}$ and then δ small enough , we can guarantee that this quantity is positive for all $y \in U_0 \cap W$, and the corollary is proved . In the proof we have used the following result.

Lemma (2. 1. 12): Assume all the eigenvalues of the $m \times m$ matric A_0 are purely imaginary . for every $\varepsilon > 0$ there exists a complex invertible matrix S such that

$$|\mathrm{Re}(Sz . SA_0 z)| \leq \varepsilon\left|\left|Sz\right|\right|^{2} \qquad \forall z \in C^{m}. \qquad (2.26)$$

Proof:

Let S_0 be such that$B = S_0 A_0 S_0{}^{-1}$ is in complex Jordan canonical from , that is , the diagona lelement $b_{jj} = i\lambda_j$of B are purely imaginary .

$b_{jj} + 1\sigma j$ is either zero or one , and all otherelements of B are zero .

Let S_1be the diagonal matric with enteres $1 , \varepsilon^{-1} , \ldots \ldots \ldots \ldots \ldots \ldots \ldots . . , \varepsilon^{1-m}$

$$\text{Then} \quad C := S_1 B S_1{}^{-1} = \begin{pmatrix} i\lambda_1 & \varepsilon\sigma_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon\sigma_{m-1} \\ & & & i\lambda_m \end{pmatrix}$$

Let $S = S_1 S_0$ and $u = Sz$. Then

$$\left| \mathrm{Re}\, \frac{Sz\, SA_0}{\left|\left|Sz\right|\right|^{2}} \right| = \left| \mathrm{Re}\, \frac{u.Cu}{\left|\left|u\right|\right|^{2}} = \right| \mathrm{Re}\, \frac{\sum i\lambda j |u\, j\, |^{2} + \varepsilon \sum \sigma_j \overline{u_J}\, uj{+}1}{\sum |u|^{2}} \leq \varepsilon \frac{\sum |u||u{+}j|}{\sum \left|\left|u\right|\right|^{2}} \leq \varepsilon,$$

where used the fact that the upper sum has $m - 1$ terms

and the lower one m terms . The two corollaries of this section can be stated as follows if x^{*} is as ahyperbolic equilibrium , then it has the same type stability as the linearized system. The same properties are valid for maps (see Table 2.1). This is in genaral not ture for non − hyperbolic equilibria . This situation will be studied in more detail chapter 3.

	ODE $\dot{x}=f(x)$	Map $x_k + 1 = F(x_k)$
Conservative Dissipative	$\nabla . f = 0$ $\nabla . f < 0$	$\left\|\det \frac{\partial F}{\partial x}\right\| = 1$ $\left\|\det \frac{\partial F}{\partial x}\right\| < 1$
Equilibrium	$f(x^*) = 0$	$F(x^*) = x^*$
A sympt.stable if Unstable if	$\mathrm{Re}\, a_i < 0 \ \forall i$ $\exists i: \mathrm{Re}\, a_i > 0$	$\|b_i\| < 1 \ \forall i$ $\exists i: \|b_i\| > 1$

Table (2.1): Comparison of some propties of ordinary differentail equations and iterated maps. Here a_i and b_i are eigenvalues of the matrices

$$A = \frac{\partial f}{\partial x}(x^*) \text{ and } B = \frac{\partial F}{\partial x}\ (x^*)\ .$$

Now we discussing the invariant manifolds for hyperbolic equilibria points, the mplicated analogiesbetween non − linear systems can bepushed further. One of them has to do with invariant manifolds , which generalize the invariant subspaces of the linear case .

We assume in this section that f is of classC^r with $r \geq 2$.

Definition (2.1.13): Let x^* be asingular point of the system $\dot{x} = f(x)$, and let U be aneighbourhood of x^* .the local stable and unstable manifolds of x^* in U are defined, respectively, by

$$W^s_{loc}(x^*) := \left\{x \in U: \lim_{t\to\infty} \alpha t(x) = x^* \text{ and } \alpha t(x) \in U \ \forall t \geq 0\right\}$$
$$W^u_{loc}(x^*) := \left\{x \in U: \lim_{t\to\infty} \alpha t(x) = x^* \text{ and } \alpha t(x) \in U \ \forall t \leq 0\right\} \quad (2.27)$$

The global stable and unstable manifolds of x^* are defined by

$$W^s(x^*) = \bigcup_{t\leq 0} \alpha t\left(W^s_{loc}(x^*)\right),$$
$$W^u(x^*) = \bigcup_{t\geq 0} \alpha t\left(W^u_{loc}(x^*)\right). \qquad (2.28)$$

similar definition can be made for maps , Global in variant manifolds can have a very comp − licated structure , and may return infinitely often to neighbourhood of the equilibrium point .

This why one prefers to define separately local and global invariant

manifolds . The following theorem states that local invariant have anice structure.

Theorem(2.1.14): (stable manifold theorem)

Let x^* be ahyberbolic equilibrium point of the system $\dot{x} = f(x)$, such

that the matric $\frac{\partial f}{\partial x}(x^*)$ has n_+ eigenvalues with positive real parts and n_-eigenvalues with negative real parts, with $n_+ , n_- \geq 1$.

Then x^* admits , in aneighbourhood U,

- A local stable manifold $W^s(x^*)$. which is adifferentable manifold of class C^r and dimension n_- , tanget to the stable subspace E^- at x^* , and which can be represented as agraph ;
- A local unstable manifold $W^u(x^*)$, which is a differentable manifold of class $C^r$and dimension n_+, tanget to the unstable subspace E^+ at x^*, and which can rep − resented as agraph ;

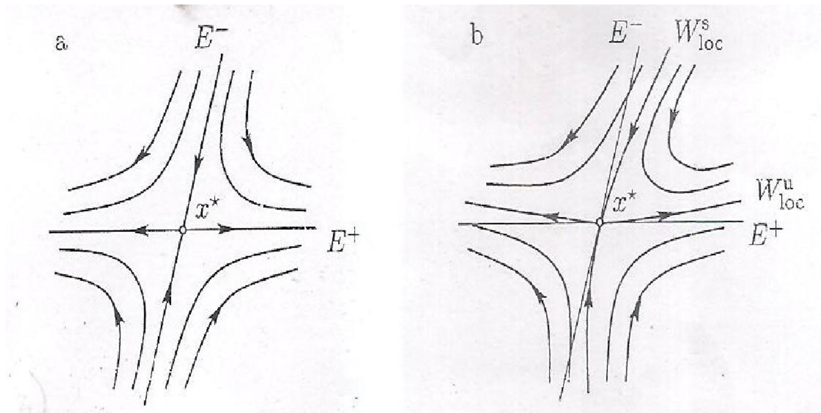


FIGURE (2. 3), Orbit near ahyberbolic fixed point :
(a). Orbit of the linearized system ,

(b). Orbit of the nonlinear system with local stable
and unstable manifold .

We will omit the proof of this result , put we will given in chapter 3 the proof of the center manifold theorem , which relies of similar ideals. Let usnow explain abit more precisely what thisresultmeans . The geometricinterpretation is shown in Fig(2.3)To explain the meanig

of "adifferentable manifold of class C^r
tanget to $E^{\pm}$ and representable as agraph ", let us introduce

acoordinate system along the invariant subspace
of the lineariztion . The vector field near x^* can be written as

$$\begin{aligned} \dot{y}_+ &= A + y_+ + g_+(y_+, y_- \ ) \\ \dot{y}_- &= A - y_- \ + g_-(y_+, y_- \ ), \end{aligned} \qquad (2.29)$$

Where $A_+$is $n_+ n_+$ matric , which has only eigenvalues with positive real parts , and A_- is a $n_- * n_-$ matric , which has only eigenvalues with negative real parts . The terms g_-^+ are non $-$ linear and satisfy $\|g_-^+(y_+, y_- \ )|\|M\|y\|^2$in U, where M is apositive constant . The theorem implies the existence of function of class C^r

$$h^u : u_+ \to R^{n-}, \ \ h^u(0) = 0 \ , \quad \frac{\partial h^u}{\partial y_+}(0) = 0, \qquad (2.30)$$

Where u_+ is neighbour hood of the orgin in R^{n+} , such that the local unstable manifold is give by equation

$$y_- = h^u(y_+) \tag{2.31}$$

Similar, relations hold for the stable manifold in order determine the function h^u let as copute $\dot{y_-}$ in two ways, for given orbit on the unstable manifold

$$\begin{aligned} \dot{y_-} &= A - h^u(y_+) + g_-\big(y_+, h^u(y_+)\big) \\ \dot{y_-} &= \frac{\partial h^u}{\partial y_+}(y_+)\dot{y_+} = \frac{\partial h^u}{\partial y_+}(y_+)[A + y_+ + g_+\big(y_+, h^u(y_+)\big)] \end{aligned} \tag{2.32}$$

Since both expressions must be equal ,we obtain that h^u must be satisfy the partial differential equation

$$\dot{y}_- = A - h^u(y_+) + g_-\big(y_+, h^u(y_+)\big) = \frac{\partial h^u}{\partial y_+}(y_+) \tag{2.33}$$

This equation is difficult to solve in general . since we know by the theorem that $h^u$is of class C^r, we can computeh^u perturbatiely , by inserting its taylor expansion into (2.33) and solving order by order.

Example(2.1.15): Consider , for $n = 2$, the system

$$\begin{aligned} \dot{y}_1 &= y_1 \\ \dot{y}_2 &= -y_2 + {y^2}_1 \end{aligned} \tag{2.34}$$

Then the equation (2.33)reduces to

$$-h^u(y_1) + {y_1}^2 = h^{u'}(y_1)y_1, \tag{2.35}$$

Which adimits the solution

$$h^u(y_1) = \frac{1}{3}{y_1}^2. \tag{2.36}$$

This is confirmed by the explicit solution (2.34),

$$y_1(t) = y_1(0)e^t$$
$$y_2(t) = (y_2(0) - \frac{1}{3}y_1(0)^2)e^{-t} + \frac{1}{3}y_1(0)^2e^{2t} \qquad (2.37)$$

Definition(2.1.16): Let U and W be open set in R^n let $f: U \rightarrow R^n$ and $g: W \rightarrow R^n$ be tow vector fields of class C^r. These vector field are called

- Topologically equivalent in there exists a homeomorphism h:U→ W taking the orbit of $\dot{x} = f(x)$ to the orbits of $\dot{y}=g(y)$ and preserving the since of time ;
- Differentiably equivalent in there exists a diffeomorphism

$h: U \rightarrow W$ taking the orbit of $\dot{x} = f(x)$ to the orbits of $\dot{y}=g(y)$ and preserving the since of time.

in addition , h preserves parametrization of the orbits by time , the vector field are called conjugate . Equivalence means that if αt and Ψt are the flows of the tow systems, then

$$t \circ h = h \circ \alpha_\tau\ (t) \qquad (2.38)$$

On U, where τ is the homeomorphism from R to R. If $\tau(t) = t$ for all t, the system are conjugate

Theorem(2.1.17): (Hartman − Grobman)

Let x^* be ahyberloic equilibirum point of $\dot{x}f(x)$, that is , the matrix

$A = \frac{\partial f}{\partial x}(x^*)$ has no eigen value with zero real part .

Then , in a sufficiently small neighbourhood ofx^*, f is topologically

Conjugate of the linearization $\dot{y}$=Ay.

Note , howevere , that topological equivalence is not avery a strong

property since h − need not be differentaible.

In fact , one can show that all linear system with the same number ofeigenvalues with positive and negative

real part are topolically equivalent (see for instance [HK91]) so for instance , the node and focus in fig(2.2)are topolically equivalent . one the other hand , differentable equivalence is harderto achieve as

shows the following example.

Example $(2.1.18)$: consider the following vector field and its

linearization :

$$\dot{y_1} = 2y_1 + {y_2}^2 \qquad \dot{z_1} = 2z_1$$
$$\dot{y_2} = y_2 \qquad \dot{z_2} = 2z_2 \tag{2.39}$$

The orbit can be found by solving the differential equations

$$\frac{dy_1}{dy_2} = 2\frac{y_1}{y_2} + y_2, \qquad \frac{dz_1}{dz_2} = 2 \tag{2.40}$$

Which admit the soluations

$$y_1 = [c + \log\lfloor y_2 \rfloor \ ]\, {y_2}^2, \qquad z_1 = c{z_2}^2 \tag{2.41}.$$

Because of the logarithm, two flows are C^1_- but not C^2_- conjugat. The theory of normal form allows explain these phenomena, and

the obtain conditions for the existence of such C^r_- conjugcies.

consider the system for $y = x - x^*$,

$$\dot{y} = Ay + g(y) \tag{2.42}$$

We can try to simplify nonlinear term by change coordinates

$y = z + h(z)$, which leads to

$$\dot{z} + \frac{\partial h}{\partial z}(z)\dot{z} = Az + Ah(z) + g\big(z + h(z)\big). \tag{2.43}$$

Assume that h(z)solves the partial differetail equation

$$\frac{\partial h}{\partial z}(z)Az - Ah(z) = g\big(z + h(z)\big). \qquad (2.44)$$

Then we obtain for z the linear equation

$$\dot{z} = Az \qquad (2.45)$$

Unfortuately , we do not khow how to solve the equation (2.44)is genaral.

One can , however , work With Taylor series . To this end , we rewrite the system (2.42) as

$$\dot{y} = Ay + g_2(y) + g_3(y) + \cdots\cdots + g_{r-1}(y) + \mathcal{O}\left(\left|\left|y\right|\right|^r\right). \qquad (2.46)$$

Here the last term is bounded in norm by constant times $\left|\left|y\right|\right|^r$, and the terms $g_k(y)$ are homogeneous polynomial maps of degree k for m

$$R^n \text{ to } R^n \ (g_k(\lambda y) = \lambda^k \ g_k(y) \ \forall\lambda \in R.$$

Let us denote by H_k the set of all such maps . H_k is avector space for the usual addition and multiplication by scalars. For in stance , when $n = 2, H_2$ admits the bases vectors

$$\begin{pmatrix} {y_1}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} {y_2}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ {y_1}^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ {y_2}^2 \end{pmatrix}. \qquad (2.47)$$

We now define alinear map from H_k to itself given by a $d_k : H_k \to H_k$

$$h(y) \longmapsto \frac{\partial h}{\partial y}(y)Ay - Ah. \qquad (2.48)$$

The fundamental result of normal from theorey is the following:

Proposition (2.1.19): For each k, $2 \le k \le r$, choose a complementary space $\mathcal{G}_k$ of the image of a d_k that is , such that $\mathcal{G}_k \oplus a\ d_k\ A(H_k) = H_k$. Then there exists , in aneighbourhood of the orgin , an analytic (polynomial)change of a variables $y = z + h(z)$

transforming (2.46) into

$$\dot{z} = Az + g_2^{res}(z) + g_3^{res}(z) + \cdot \cdots + g_{r-1}^{res}(z) + \mathcal{O}\left(\|z\|^r \right) \quad (2.49)$$

Where $g_k^{res} \in \mathcal{G}_k$, $2 \leq k \leq r$.

Proof: The proof proceeds by induction .Assume that for some k ,

$2 \leq k \leq r \leq r - 1$, we have obtained an equation of the form

$$\dot{y} = Ay + \sum_{j=2}^{k-1} g_j^{res}(y) + g_k(y) + \mathcal{O}\left(\|y\|^{k+1} \right).$$

We decompose the term $g_k(y)$ into resonant and a $-$ resonant part

$$g_k(y) = g_k^{res}(y) + g_k^0(y), \qquad g_k^0(y) \in a\, d_k\, A(H_k), \quad g_k^{res} \in \mathcal{G}_k.$$

There exists $h_k \in H_k$ satisfing

$$a\, d_k\, A\big(h_k(z)\big) := \frac{\partial h_k}{\partial z}(z)Az - Ah_k(z) = g_k^0(z).$$

Observe that $g_j^{res}\big(z + h_k(z)\big) = g_j^{res}(z) + \mathcal{O}\left(\|z\|^{k+1} \right)$ for all j , and asimilar relation holds for g_k. Thus the change of the variables $y = z + h(z)$ yields the equation

$$\begin{aligned} \dot{y} &= \left[\mathbb{I} + \frac{\partial h_k}{\partial z}(z) \right] \dot{z} \\ &= Az + Ah_k(z) + \sum_{j=2}^{k-1} g_j^{res}(z) + g_k(z) + \mathcal{O}\left(\|z\|^{k+1} \right) \\ &= \left[\mathbb{I} + \frac{\partial h_k}{\partial z}(z) \right] Az + \sum_{j=2}^{k} g_j^{res}(z) + \mathcal{O}\left(\|z\|^{k+1} \right), \end{aligned}$$

Where we have used the definition of h to get the second line . Now for sufficiently small z, the matrix [] $+\frac{\partial h_k}{\partial z}$ admits an inverse $\left||z|\right|^{k-1}$. Multiplying the above identity on the left by this inverse , We have proved this induction step. The terms g_j^{res} (z) are called resonant and (2.49) is called the normal from (2.46) the equation

$$a\, d_k\, A\big(h_k(z)\big) = g_k^0\,(z)$$

that has to be satisfied to eliminate the non − resonant terms Of order k is called the homological equation .Whether a terms is resonant or not is a problem of linear algebra , which depends only of the matric A , While it can be difficult to determine coefficients of the resonant terms

in aparticular case, it is in genaral quite easy to find which terms can be

eliminated . In aparticular, the following result holds:

Lemma (2.1.20):Let$(a_1,\cdots\cdots\cdots, a_n)$ be the eigenvalues of A , counting multiplicity. Assume that for each j, $1 \le j \le n$, we have

$$p_1a_1 + \cdots\cdots + p_na_n \ne a_j \qquad (2.50)$$

for all n − tuples non − negative integer $(p_1,\cdots\cdots, p_n)$ satisfying

$$p_1 + \cdots + p_n = k.$$

Then a d_k A is invertible , and is thus there are noresonant terms of

order k.

Proof: We can assume that A is in Jordan canonical from . Consider first the case of a diagonal A .Let $(e_1, \cdots\cdots, e_n)$ be the canonical basis of R^n.for H_k we choose the basis vectors $z_1^{p_1}\cdots z_n^{p_n}e_j$, where

$p_1 + \cdots\cdots\cdots\cdots + p_n = k$ .Then an explicit calculation shows that

$$a\, d_k\, A(z_1^{p_1}\cdots z_n^{p_n}e_j) = (\,p_1a_1 + \cdots\cdots + p_na_n - a_j)z_1^{p_1}\cdots z_n^{p_n}e_j\,.$$

By a assumption , the term in brackets is different from zero. Thus the linear operator a d_k A is diagonal in the chosen basis , with nonzero elements on the diagonal ,which shows that it is invertible . Consider now the case of the a matrix A that is not diagonal, but has elements of the from $a_{jj}+1=1$.then a d_k A applied to a basis vector will contain additional, off- diagonal terms. One of them is proportional to $z_1^{p_1}\cdots z_n^{p_n}e_{j-1}$,While the others are of the from $z_k+1\,\partial_{z_k}(z_1^{p_1}\cdots z_n^{p_n})\,e_j$ .One can show that the basis vector of H_k can be ordered in such a way that a d_k A is represented by a triangular matrix ,with the same diagonal elements at the case of a diagonal A, thus the conclusion is unchanged. The non-resonance condition (2.50) is called a Diophantine condition , since it involves integer coefficients. Thus resonant terms can exist only when the eigenvalues of A satisfy a relation of the from $p_1a_1+\cdots\cdots+p_na_n=a_j$,which is called resonance of order $p_1+\cdots\cdots+p_n$. In example (2.1.18),the relation $2a_2=a_1$ induces a resonance of order 2 ,which makes it impossible to eliminate the term y_2^2 by a polynomial change of coordinates. In order to solve the question of differentiable equivalence, the really difficult problem is to eliminate the reminder $\mathcal{O}\left(\|z\|^r\right)$ in the normal from (2.49) .This problem was solved by Poincarè for sources and sinks , and by Sternberg and Chen for general hyperbolic equilibria (see for instance [Har64]).we state here the main result without proof.

Theorem (2.1.21):(Poincarè-Sternberg-Chen)

Let A be a n×n matrix with no eigenvalues on the imaginary axis .Consider the two equations

$$\begin{aligned}\dot{x}&=Ax+b_1(X)\\ \dot{y}&=Ay+b_2(y)\end{aligned}\qquad(2.51)$$

We assume that b_1 and b_2 are of class C^r , $r\geq 2$,in a neighbourhood of the origin ,that $b_1(x)=\mathcal{O}\left(\|x\|^2\right)$ $b_2(x)=\mathcal{O}\left(\|x\|^2\right)$, and

$b_1(x)$-$b_2(x)$= $\mathcal{O}\left(\|x\|^r \right)$.Then for every k≥ 2,there is an integer N=N(k, n, A) ≥ k such that ,if r≥ N, there exists a map h of class C^k such that the two systems can be transformation x=y+h(y). This result implies that for the system $\dot{y}$=Ay + g(y) to be C^k-conjugate to it is linearization $\dot{z}$=Az ,it must be sufficiently smooth and satisfy non-resonance conditions up to sufficiently high order N, where this order and the conditions depend only on k , n and the eigenvalues of A .In the special case of all eigenvalues of A having real parts with is the same sign (source of sink) , Poincarè showed that N=k .For general hyperbolic equilibria ,N can be much larger than k. Another , even more important consequence , is that vector fields near singular points can be classified by their normal forms ,each normal from being a representative of an equivalence class (with respect to $C^k -$ conjugacy .This property plays an important role in bifurcation theory ,as we shall see in chapter 3. Let us finally remark that in the much more difficult case of non-hyperbolic equilibria, certain results on $C^k -$ conjugacy have been obtained by Siegel, Moser , Takens and other.

Section 2: Periodic Solutions

Definition (2.2.1): Let p≥1 be an integer .A periodic orbit of period p of the map F is a set points $\{x_1^*,\dots,x_p^*\}$ such that

$$F(x_1^*)= x_2^* \text{ , ... } F(x_{P-1}^*) =x_P^* \text{ , } F(x_P^*)= x_1^* \qquad (2.52)$$

Each point of the orbit is called a Periodic point of the period p .Thus a Periodic point x^* of the period p is also a fixed point of F^p .p is called the least period of x^* if $F^j(x^*)\neq x^*$ for 1≤ j ≤ p. To find Periodic orbits of an iterated map, it is thus sufficient to find the fixed points of F^p ,p=1,2,…… . Unfortunately, this becomes usually extremely difficult with increasing p. Moreover ,the number of Periodic orbits of period p often grows very quickly with p .Methods that simplify the search for Periodic orbits are known for special classes of maps. For instance, for two-dimensional conservative maps, there exists a variation method : Periodic orbits of period p correspond to stationary points of some function of R^P TO R. Once a Periodic orbits has been found, the problem of its linear stability is rather easily solved . Indeed, it is sufficient to find the eigenvalues of matrix

$$\frac{\partial F^p}{\partial x}(x_1^*) = \frac{\partial F}{\partial x}\,(F^{p-1}\,(x_1^*))\,\frac{\partial F}{\partial x}(F^{p-2}\,(x_1^*))\,\ldots.\frac{\partial F}{\partial x}(x_1^*)$$

$$=\frac{\partial F}{\partial x}(x_p^*)\,\frac{\partial F}{\partial x}(x_{P-1}^*)\,\ldots..\frac{\partial F}{\partial x}(x_1^*). \qquad (2.53)$$

Note that the result has to be invariant under cyclic permutations of the matrices The dynamics near any point of the Periodic orbit can be inferred from the dynamics near one of them, considered as a fixed points of F^p .thus Periodic orbit can also be classified into sinks, sources ,hyperbolic and elliptic orbits ,and the concepts of nonlinear stability ,invariant manifolds and normal forms can be carried over from fixed points to the Periodic orbits.

Definition (2.2.2): Let f be a vector field, φt its flow and $T > 0$ a constant .A period ic Solutions of period T of f is a function γ(t) such that

$$\dot{\gamma}(t) = f(\gamma(t)\,)\text{ and }\gamma(t+T) = \gamma(t) \quad \forall t. \qquad (2.54)$$

The corresponding closed curve $\Gamma = \{\gamma(t) \;\; :0 \le t \le T\}$ is called a Periodic orbit of period T .Thus each point x of this orbit is a fixed point of φT .T is called the least period of the orbit if $\varphi t(x) \neq x$ for $0< t < T$. Finding Periodic orbits of differential equations is even more difficult than for maps. there exist method which help to find Periodic orbits in a number of particular cases , such as two-dimensional flows ,or systems admitting constants of the motion and small perturbations of them. Let us now assume that we have found a Periodic Solutions γ(t) . we would like to discuss its stability. The difference $y(t) = x(t)\text{-}\,\gamma(t)$ between an arbitrary solution and the Periodic Solution satisfies the equation

$$\dot{y} = f(\gamma(t) + y) - f(\gamma(t)) \quad (2.55)$$

If f is twice continuously differentiable and y is small, we may expand f into Taylor series, which yields

$$\dot{y} = A(t)y + g(y,t), \qquad A(t) = \frac{\partial f}{\partial x}\big(\gamma(t)\big), \quad ||g(y,t)|| \leq M\|y\|^2 \quad (2.56)$$

Let us examine the linearization of this equation, given by

$$\dot{y} = A(t)y \quad (2.57)$$

Note that a similar equation already appeared in the proof of Proposition (2.1.10). This equation admits a unique global solution, which can be represented, because of linearity, as

$$Y(t) = U(t)y(0), \quad (2.58)$$

Where U(t) is an n×n matrix-valued function solving the equation

$$\dot{U}(t) = A(t)U(t), \quad U(0) = \mathbb{I} \quad (2.59)$$

Then function U(t) is called the principal solution of the equation (2.58) .It should be clear that the linear stability of ɼ is related to the asymptotic behavior of the eigenvalues of U(t) .Unfortunately, there is no general method to determine these c. Note, however, that A(t) is periodic in t, and in this case we can say more.

Theorem (2.2.3): (Floquet)

Let $A(t) = A(t+T)$ for all t .Then principal solution of $\dot{y} = A(t)y$ can be written as

$$U(t) = P(t)e^{Bt}, \quad (2.60)$$

Where $P(t+T) = P(t)$ for all t, $P(0) = \mathbb{I}$, and B is a constant matrix

Proof: The matrix $V(t) = U(t+T)$ satisfies the equation

$$\dot{V}(t) = \dot{U}(t+T) = A(t+T)U(t+T) = A(t)V(t),$$

Which in the same as (2.60), except for the initial value $V(0) = U(T)$. we already saw in the Proposition (2.1.10) the det $U(t) \neq 0$ for all t. Thus the matrix $V(t)U(T)^{-1}$ exists and satisfies (2.60), including the initial condition .By uniqueness of the solution, it must be equal to $U(t)$:

$$V(t)U(T)^{-1} = U(t) \quad \Rightarrow \quad U(t+T) = U(t)U(T).$$

We claim that there exists a matrix B such that $U(T) = e^{BT}$ .To see this, let $\lambda_i \neq 0$ and m_i , i=1,……,m be the eigenvalues of U(T) and their algebraic multiplicities. Let $U(T) = \sum_{i=1}^{m}(\lambda_i P_i + N_i)$ be the decomposition of U(T) into the semi simple and nilpotent parts. Then ,using lemma(2.1.2)it is easy to check that

$$B = \frac{1}{T}\sum_{i=1}^{m}(\log(\lambda_i)P_i - \sum_{j=1}^{m_i}\frac{(-N_i)^j}{j\lambda_i{}^j})$$

Satisfies $e^{BT} = U(T)$. here the some over j is simply the Taylor expansion of $\log(\mathbb{I}+N_i/\lambda_i)$. B is unique up to determination of the logarithums . we now define

$$B = U(t)e^{-Bt}.$$

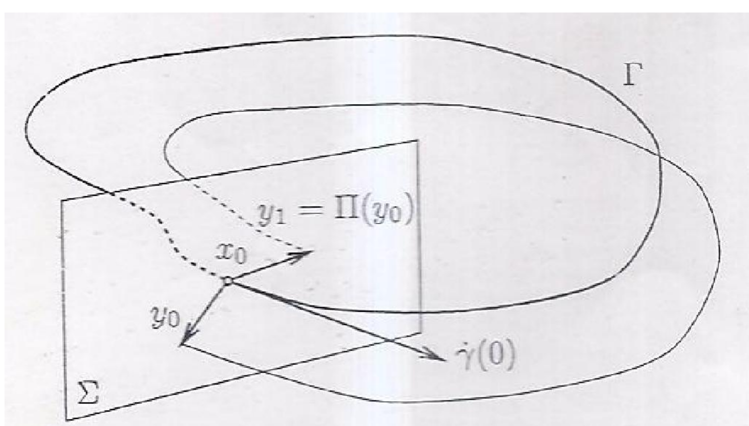


FIGURE (2.4). Definition of the Poincare map associated with the periodic orbit $\gamma(t)$ .Then we have for all t

$$P(t+T) = U(t+T)e^{-B(t+T)} = U(t)e^{BT}e^{-B(t+T)} = P(t).$$

Finally, $P(0)=U(0)=[]$,which completes the proof.

Floquet's theorem shows that the solution of (2.57) can be written as

$$y(t) = P(t)e^{Bt}y(0) \qquad (2.61)$$

Since P(t) is periodic, the long-time behavior depends only on B .the eigenvalues of B are called the characteristic exponents of the equation . e^{Bt} is called the monodrama matrix .and eigenvalues ,called the characteristic exponents time of T. Computing the characteristic exponents is difficult in general, but the existence of the representation (2.61) is already useful to classify the possible behaviors' near a periodic orbit. One we have determined the linear stability of the periodic orbit, we could proceed in a similar way as in the case of a stationary point. in order the determine the nonlinear stability, the existence of invariant manifolds. and similar properties .however, Poincarè invented a remarkable method, which allows to shortcut all these steps by reducing the problem to a simpler one, which has already been studied .Appropriately enough, this method is called the Poincare section.

Definition (2.2.4): Let $\gamma(t)$ be a periodic solution of period $T, x_0=y_0$, and let Σ be a hyper plane transverse to the orbit at x_0 (see fig 2.4) .By continuity of the flow, there is a neighbourhood U of x_0 in Σ such that for all $x = x_0 + y \in U$, we can define a continuous map$\tau(y), \tau(0) = T$,such that ct(x) retuens for the first time to Σ in a vicinity of x_0 at $t= \tau(y)$. The Poincare map Π associated with the periodic orbit in defined by

$$x_0 + \Pi(y) := \varphi\tau(y)(x_0 + y). \qquad (2.62)$$

Proposition (2.2.5): The Poincare map is as smooth as the vector field in a-neighborhood of the origin. The characteristic multipliers of the periodic orbit are given by 1 and the n-1 eigenvalues of the Jacobean matrix $\frac{\partial \pi}{\partial y}(0)$.

Proof: The smoothness of Π follows directly from the smoothness of the follow and the implicit function theorem. Let us now observe that

$$\frac{d}{dt}\dot{\gamma}(t) = \frac{d}{dt}f\big(\gamma(t)\big) = \frac{\partial f}{\partial x}\big(\gamma(t)\big)\dot{\gamma} = A(t)\dot{\gamma}(t).$$

Thus by Floquet's theorem, we can write

$$\dot{\gamma}(t) = P(t)e^{Bt}\dot{\gamma}(0),$$

And, in particular,

$$\dot{\gamma}(0) = \dot{\gamma}(T) = P(T)e^{BT}\dot{\gamma}(0) = e^{BT}\dot{\gamma}(0).$$

This show that $\dot{\gamma}(0)$ is an eigenvector of e^{BT} with eigenvalue 1 . Let $(e_1, \dots, e_{n-1,}\dot{\gamma}(0)), e_1, \dots, e_{n-1,} \in \Sigma$, be a basis of R^n. In this basis,

$$e^{BT} = \begin{pmatrix} e^{B\Sigma T} & 0 \\ \cdots & 1 \end{pmatrix},$$

Where B_Σ is the restriction of B to Σ, and the dots denote arbitrary entries .Now, if we consider momentarily y as a vector in R^n instead of Σ, linearization of (2.62) gives

$$\frac{\partial \pi}{\partial y}(0) = \frac{\partial}{\partial y}(\varphi\tau(y)(\,x_0+y))|_{y=1} = \frac{\partial \varphi T}{\partial t}(x_0)\frac{\partial \tau}{\partial y}(0) + \frac{\partial \varphi T}{\partial t}$$
$$(x_0) = \dot{\gamma}(0)\frac{\partial \tau}{\partial y}(0) + e^{BT}.$$

The first term is a matrix with zero entries except on the last line, so that $\partial_y \prod(0)$ has the same representation as e^{BT} , save for the entries market by dots.In particular, when y is restricted to Σ ,$\frac{\partial \Pi}{\partial y}(0)$ has the same eigenvalues as $e^{B\Sigma T}$. The consequence of this result is that, by studying the Poincare map, we obtain a complete characterization of the dynamics in a neighborhood of the periodic orbit .in particular, if y=0. Considered as a fixed point of the Poincare map, admits invariant manifolds, they can be interpreted as the intersection of Σ and invariant manifolds of the periodic orbit.

Up to now, we have obtained quite a precise picture of the dynamics near hyperbolic equilibrium solutions .one might wonder whether it is of any interest of examine the case of no hyperbolic a matrix chosen at random will have equilibria, since eigenvaluse on the imaginary axis with probability zero. This argument no longer works, however, if the dynamical system depends on a parameter:

$$\dot{x}=f(x,) \text{ or } x_{k+1}=F(x_k, \boldsymbol{\lambda}) \quad (2.63)$$

Where $\boldsymbol{\lambda} \in R$ (or R^p) .by changing $\boldsymbol{\lambda}$, it is quite possible to encounter no hyperbolic equilibria . An important result in this connection is the implicit function theorem :

Theorem (2.2.6): Let N be a neighborhood of (x^*, y^*) in $R^n \times R^m$. Let f:N→ R^n be of class C^r, r≥ 1, and satisfy

$$f(x^*, y^*) = 0, \quad (2.64)$$

$$\det \frac{\partial f}{\partial x}(x^*, y^*) \neq 0. \quad (2.65)$$

then there exists a neighborhood U of y^* in R^m and unique function $\varphi: U \to R^n$ of class C^r such that

$$\varphi(y^*) = x^* \quad (2.66)$$

$$f(\varphi(y), y) = 0, \text{ for all } y \in U. \quad (2.67)$$

This result tells us under which conditions the equation f(x,y)=0 "can be solved with respect to x". assume x^* is an eguilibrium point of $f(x,\lambda_0)$ and let A be the linearization $\frac{\partial f}{\partial x}(x,\lambda_0)$. then the following situations can occur:

- If A has no eigenvaluse with zero real number, then f will admit equilibrium points $x^*(\boldsymbol{\lambda})$ for all in a neighborhood of λ_0. By continuity of the eigenvaluse of a matrix-valued function, $x^*(\boldsymbol{\lambda})$ will be hyperbolic near λ_0 . the curve $x^*(\boldsymbol{\lambda})$ is usually called an equilibrium branch of f.

- If A has one or several eigenvaluse equal to zero, then the implicit function theorem can no longer be applied, and various interesting phenomena can occur. for instance, the number of equilibrium points of f may change at $\boldsymbol{\lambda}=\lambda_0$ .such a situation is called a bifurcation, and (x^*, λ_0) is called a bifurcation point of f.

Chapter 3
Center manifolds and Bifurcation of Differential Equation

Section(3:1): (center manifolds)

One the most useful methods to study the flow near bifurcation point is the center manifolds theorem no hyperbolic equilibrium points.

Definition (3.1.1): Let $u \subset R^n$ be an open set .let $s \subset R^n$ have the structure of differentiable manifold for $x \epsilon u$, let $(t_1^x, \; t_2^x) \ni 0$ be maximal interval such that $\varphi_{t(x)} \in u$ for all $t \in (t_1^x, \; t_2^x)$.

Theorem (3.1.2): let x^* be asingular point of f , where f is of class c^r,r$\geq$ 2 , in a neighbourhood of x^* let $A = \frac{\partial f}{\partial x}(x^*)$ have , respectively , n_+, n_0 and n_-, and such

- W_{loc}^u is the unique local invariant manifold tangent to E_+ at x^* , and $\varphi_{t(x)} \rightarrow x^*$ as $t \rightarrow -\infty$ for all $x \in W_{loc}^u$.
- W_{loc}^u is the unique local inuariant manifold tangent to

E_- at x^* , and $\varphi_t(x) \rightarrow x^*$ as $t \rightarrow \infty$

for all $x \in W_{loc}^s W_{loc}^u$ is tangent to E_0 but not necessrily unique.

Before giving a (partial) proof this result we shall introduce useful lemma from the theorem of differential inequalities.

Lemma (3.13) (Gromwell's inequality) Let φ, α andβ be continuous and real valued

$$\varphi_{(t)} \leq \alpha_{(t)} + \int_a^s \beta_{(s)} \varphi_{(s)} ds \;\; \forall t \in [a, b] \qquad (3.1)$$

Then

$$\varphi_{(t)} \leq \alpha_{(t)} + \int_a^s \beta_{(s)} \varphi_{(s)} \, e^{\int_s^t \beta_{(u)} du} ds \quad \forall t \in [a, b] \qquad (3.2)$$

proof:let

$$R(t) = \int_a^s \beta_{(s)} \varphi_{(s)} ds$$

Then, $\varphi_{(t)} \leq \alpha_{(t)} + R\,(t)$ for all

$t \in [a, b]$ and since $\beta_{(t)} \geq 0$

$$\frac{dR}{dt}(t) = \beta_{(s)} \varphi_{(s)} \leq \beta_{(t)} \varphi_{(t)} + \beta_{(t)} R(t)$$

let $K(s) = \int_a^s \beta_{(u)} du$, the.

$$\frac{dR}{dt}(t) = \beta_{(s)} \varphi_{(s)} \leq \beta_{(t)} \varphi_{(t)} + \beta_{(t)} R(t)$$

let $K(s) = \int_a^s \beta_{(u)} du$, then

$$\frac{d}{dt} e^{-K(s)} R(s) = [R'(s)_{-} \beta_{(s)} R(s)]\, e^{-K(s)} \leq \beta_{(s)} \alpha_{(s)}$$

And thus, integrating from a to t,

$$e^{-K(s)} \leq \int_a^s \beta_{(s)}\, \alpha_{(s)} e^{-K(s)} ds$$

We obtain the conclusion by multiplying expression by $e^{K(s)}$ and the result into (3.1).There exist various generalizations of this result for instance to functions $\beta_{(t)}$ that are only integrable. Let us now proceed to the proof of the center manifold theorem. there exist more or less sophisticated proofs. we will given her a rather straight for word one take from [ca81]. For simple city we consider the case $n_{(+)} = 0$, and we will only prove the existence of a Lipchitz continues center manifold.

To Proof Theorem(3.1.2),we write the system near the equilibrium point as

$$y^{\cdot} = B_{(y)+\, g-(y,z)}$$

$$x^{\cdot} = C_{(z)+g_{(0)}\,(y,z)}$$

Where all eigluseenva of B have strictly negative real parts all eigluseenva of C hone zero real parts, and

$$||g\text{-}(y, z)||, || g_{(0)}\ (y, z)|| \leq M\left(\left||y|\right|^2 + \left||z|\right|^2\right)$$

In a neighborhood of the oregin.we shall prove the existence of local center manifold for a modified equation that agrees with the present equation in small neighborhood of the equilibeium. let

$\psi: R^{n_{(0)}} \longrightarrow [0,1]$ be C^{∞} function such $\psi(x) = 1$,When$||x||\leq 1$ and $\psi(x) = 0$, when $||x||\geq 2$. We introduce the function

$G\text{-}(y,z)=g\text{-}(y,z\ \psi\ (\frac{z}{\varepsilon}))$, Then the system

$$y^{\cdot} = B_{(y)+\ G-(y,z)}$$

$$z^{\cdot} = C_{(z)+G_{(0)}\ (y,z)}$$

Agrees with the original system for

$||z||\leq \epsilon$ we look for a center manifold with eqaution $y = h(z)$, where h is in a will-chosen function space , in which we went to apply Beach's fixed point theorem .let $\rho > 0$ *and* $k > 0$ *be constants*, *and*

let x be set of lipschitz continuous function

$h: R^{n_{(0)}} \longrightarrow R^{n_{(-)}}$ with lipschitz constat k, $\left||h(z)|\right| \leq \rho$ for all $z\epsilon R^{n_{(0)}}$ and $h(0)=0$, is complet space with the supremum norm $|.|$. for $h\epsilon$ X we denote by $\varphi t(.\,, h)$ the flow of the differnetial equation

$$z^{\cdot} = C_{(z)+G_{(0)}\ (h(z),z).}$$

For any solution $(y(s),z(s))t_0 \leq s \leq t$, then relation

$$Y\ (t) = e^{B(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{B(t-s)}G\text{-}(y(s),z(s))ds$$

Is satisfied, as is easily checked by diferntation.among all possible solution, we want to select a class of solutions satisfying

$$Y(t)=\int_{-\infty}^{t} e^{B(t-s)}G\text{-}(y(s),z(s))ds$$

If we set t=0 and require that y(s)= h(z(s)) for alls, we arrive at equality

$$h(z(0))=\int_{-\infty}^{0} e^{B(t-s)}G\text{-}(y(s),z(s))ds$$

Where $z(s) = \varphi t(z(0), h)$ thus we conclude that h is a fixed point of the operator $T{:}x \longrightarrow x$, defind by

$$(Th) = h(z(0))=\int_{-\infty}^{0} e^{Bs}G\text{-}(h(\varphi_{(s)}(z,h)), \varphi_{(s)}\ (z,h))ds,$$

Then h a center manifold of the equation .not that there may be center manifold that do not satisfy this equation ,and thus we will not be proving uniqueness. Now we wont to show that T is a contraction on x for an apposite choice of ε k and p.observe first that since all eigenvalues of B have a strictly negative real part Lemma (2.2.2) implies the existence of positive constants β and Ksuth that

$$||e^{-Bs}y|| \leq Ke^{Bs}\ ||y||\ \forall z \in R^{n-}, \forall s \leq 0$$

Since the eigevaluse of C have zero real parts , the same lemma implies that $e^{Cs}z||$ is a polynomial in s. hence, for every $v > 0$, there exists a constant Q(v) such that

$$||e^{cs}y|| \leq Q(v)e^{v|s|}||z||\ \ \forall\ z \in R^{n0}\ , \forall s \in R$$

It is positive that $Q(v) \longrightarrow\ \infty$ as$v \longrightarrow 0$ We first need to show that TX ⊂x. Assume from now on that $\rho \leq \varepsilon$,so

that $||h(z)|| \leq \varepsilon$. the definition of $G-$ implies that

$$||G\text{-}(h(z)),z)|| \leq M'\varepsilon^2$$

For a constant $M' > 0$ *depending on M Moreover* ,

the derivates of G_0 are of order ε so that thereis constant $M'' > 0$ *depending on M with*

$$||G_0(h(z_1), z_1) - (h(z_2),)|| \leq M''\varepsilon\, [||h(z_1)\text{- }(h(z_2)\, ||+||z_1 \text{ -}z_2\, ||]$$
$$\leq M''\varepsilon\,(1+k)||\, z_1 \text{ -}z_2\, ||.$$

A similar relation hold for G -.The bound on ||G-|| implies that

$$||Th(z)|| \leq \int_{-\infty}^{0} Ke^{\beta s}\, M'\, \varepsilon^2\, ds \leq \varepsilon^2\, \frac{KM'}{\beta},$$

And hence$||Th(z)|| \leq \rho$ proveded $\varepsilon \leq (\beta\,/KM')\,(\rho\,/\,\varepsilon\,)$. Next we want to es timate the lipschitz constant of T .let z_1 ,z_2 $\in$ R^{n0} By definition of the flow $\varphi t(.\,, h)$,

$$\frac{d}{dt}(\varphi t(z_1, h) - \left(\varphi t(z_2, h)\right) = c(\varphi t(z_1, h) - \left(\varphi t(z_2, h)\right) +$$
$$G_0(h(\varphi t((z_1, h), (\varphi t\big((z_1, h)\big) - G_0(h(\varphi t((z_2, h), (\varphi t\big((z_2, h)\big)$$

Taking into account the fact that φ_0=z,we get

$$\varphi t(z_1, h) - \left(\varphi t(z_2, h)\right) = e^{ct(}z_1 - z_2) + \int_0^t e^{c(t-s)}[\, G_0(h(\varphi s((z_1, h),$$
$$(\varphi s\big((z_1, h)\big) - G_0(h(\varphi s((z_2, h), \varphi s\big((z_2, h)\big)]ds$$

For $t \leq 0$, we obtain with the properties of G_0

$$||\, \varphi t(z_1, h) - \big(\varphi t(z_2, h)\big)|| \leq Q(v)\, e^{-vt}||z_1 - z_2|| + \int_0^t Q(v) e^{-v(t-s)}\, M''\varepsilon$$
$$(1+\kappa)||\, \varphi s\,(z_1, h) \text{ - } \varphi s\,(z_2, h)\, ||ds$$

We can now apply gromwell's inequality to $\psi(t) = e^{-v}||\varphi\text{ -}t(z_1, h)$ - $\varphi\text{ -}t(z_2, h)||$, with the result

$$||\, \varphi t(z_1, h) - \big(\varphi t(z_2, h)\big)\, || \leq Q(v)\, e^{-\gamma t}||z_1 - z_2||,$$

Where γ =v+Q(v) $M''\varepsilon(\,1 + \kappa)$ .we can arrange that $\gamma < \beta$,tanking for instance $v = \beta/2$ and ε small enough we thus obtin

$$||Th(z_1)\text{-}Th(z_{2)}|| \leq$$
$$\int_{-\infty}^{0} Ke^{\beta s}\, M''\varepsilon(\,1 + \kappa)||\, \varphi s(z_1, h) - \big(\varphi s(z_2, h)\big)\, ||ds$$

$$\leq \frac{KM''\varepsilon(\,1+\kappa)Q(v)}{\beta-\gamma}||z_1 - z_2||.$$

For any given κ and ν, we can find ε small enough that $||Th(z_1)$-$Th(z_{2)}|| \leq \kappa|| \mathcal{T} - z_2||$.

This complete s the proof that $\mathcal{T}\mathcal{X} \subset \mathcal{X}$

Finally, we want to show that $\mathcal{T}$ is contraction for h_1 , $h_2 \in \mathcal{X}$,

Using$||G_0((\varphi s(h_1(z_1, h),(\varphi s\big((z_1, h)\big) - G_0(h_2(\varphi s((z_2, h),$ $\varphi s\big((z_2, h)\big)|| \leq M''\varepsilon[(1 + \kappa) || \varphi s(z_1, h) - \big(\varphi s(z_2, h)\big) || + |h_1 - h_2|]$,

We obtain similar way by Gromwell's inequality that

$$||\varphi t(z, h_1) - \varphi t(z, h_2)|| \leq 2\frac{Q(v)}{v} M''\varepsilon|h_1 - h_2|$$

(this is a rough estimate , way where we have thrown away some t-depended terms). This leads to the bound

$$||Th_1(z) - Th_2(z)|| \leq \frac{K}{\beta} M''\varepsilon\,[1{+}2M''\varepsilon(1{+}\kappa)\frac{Q(v)}{v}\,]|\,h_2 - h_1|$$

Again , taking ε small enough ,we can achieve that $||h_1(z) - Th_2(z)|| \leq \lambda|h_2 - h_1|$ for some $\lambda < 1$and for all, z∈ $R^{n(0)}$ .this show that T is a contraction , and we have proved the existence of lipschitz contuous center manifold by Bench's fixed point theorem one can proceed in similar way to show T is contraction in space of Lipchitz differentiable function . One can also prove that if f is class C^r ,r≥ 2,then h is also class C^r if f is analytic or C^∞, however, then h will be C^r for all r≥ 1, but in general it will not be analytic ,and not even C^∞.in fact ,the size of the domain in which h is C^r may became smallar and smallar as r goes to infinity ,see [ca81] for examples. As pointed out in the proof the theorem the center manifold is not necessarily unique it is easy to give example of systems admitting a continuous family of center manifolds however , as we shall see these manifolds have to approach each other extremely fast near the equilibrium point and thus the dynamics will be qualitatively the same on all center manifolds.

Example (3.1.4): the system

$$y^{\cdot}=-y \qquad (3.3)$$

$$z^{\cdot}=-z^3$$

Admits a two parameter family of center manifolds

$$Y=h(z,c_1,c_2)=\begin{cases} c_1 e^{-1/2z^2} & \text{for } z>0 \\ 0 & \text{for } z=0 \\ c_2\, e^{-1/2z^2} & \text{for } z<0 \end{cases} \qquad (3.4)$$

The operator T in the proof of theorem (3.1.2) admits a unique fixed point $h(z)\equiv 0$, but there exist other center manifolds which are not fixed points of T note however, that all function $h(z,\ c_1,c_2)$ have identically zero Taylor expansions at =0. Now we will discuss the properties of center manifolds. We assume in this discussing that x^* is a non- hyperbolic equilibrium point of f such that $A=\frac{\partial f}{\partial x}(x^*\ )$ has $n_0 \geq 1$ eigenvalues withzero real parts and $n_- \geq 1$ eigenvalues with negative real parts in appopriates we can write

$$y^{\cdot}=By+g\text{-}(y,z) \qquad (3.5)$$

$$z^{\cdot}=Cz+g_0(y,z)$$

Where all eigevalues of B have strictly negative real parts all eigenvaluse of C have zero real part and g-(y,z), $g_0(y,z)$ are nonliear termes theorem (3.1.2) show the existence of local center manifolds with parametric equation y=h(z).the dynamics on this locally invariant manifolds is governed by the elation

$$u^{\cdot}=c_u+g_0(h(u),u) \qquad (3.6)$$

Equation (3.6) has the advantage to be of lower dimension than (3.5) and thus easier to analyses the following reslut show that (3.6) is good approximation of (3.5).

Theorem (3.1.5): if the origin of (3.6) is stable (asymptotically stable, unstable),then origin of (3.1.5) is asymptotically stable ,unstable). Assume that the origin of (3.1.6) is stable for any solution(y(t),z(t)) of (3.1.5) with (y(0),z(0)) sufficiently small, there exist a solution u(t)of (3.1.6) and a constant $\gamma > 0$ such that

$$Y(t)=h(u(t))+o(e^{-\gamma t})$$

$$Z(t)=u(t)+ o(e^{-\gamma t}) \qquad (3.7)$$

Let us now discuss how to compute center manifolds in the proof the theorem (3.1.2) we used that fact h is a functional operator T while beige useful to prove existence of center manifold T not very helpful for the computation of h but there exists another operator for this purpose replacing y by h(z) in (3.5) we obtain

$$\frac{\partial h}{\partial z}(z)[Cz+g_0(h(z),z)]=Bh(z)+g\text{-}(h(z),z) \qquad (3.8)$$

For functions $\emptyset = R^{n0} \longrightarrow R^{n-}$ which are continuously differentiable in a neighborhood of the origin let us define

$$(L\emptyset)(z)=\frac{\partial \emptyset}{\partial z}(z)[\, Cz+g_0(\emptyset\,(z),z)]\text{-}B\emptyset\,(z)\text{-}g\text{-}(\emptyset\,(z),z) \qquad (3.9)$$

Then (3.8) implies that (Lh)(z)=0 for any center manifold of (3.5) this equation is impossible to solve in general However its solutions can be computed pertubatively and the approximation procedure is justified by the following

Theorem (3.1.6): let u be a neighborhood of origin in R^{n0} and let $\emptyset \in C^1(U, R^{n-})$ satisfy $\emptyset(0) = 0$ and $\frac{\partial \emptyset}{\partial z}(0)=0$if there is a q$>$ 1suth that $(L\emptyset)(z)=O||z||^q$ as z$\longrightarrow$ 0 then$||h(z)\text{-} \emptyset(z)|| = \mathcal{O}||z||^q$ as z$\longrightarrow$ 0 for any center manifold h An important consequence of the result is that if $h_1$and $h_2$are two different center manifold of x^* , then one must have $h_1(z) - h_2(z) = \mathcal{O}||z||^q$ and for all q$>$ 1 i.e,all center manifolds have the same Taylor expansion at z=0

Example (3.1.7): consider the two-dimensional system

$$y^{\cdot}=-y+cz^2 \qquad (3.10)$$

$$z^{\cdot}=yz-z^3$$

Where c is a real parameter we wont to determine the stability of the origin Naively, one might think that since the first equation suggest that y(t) converges to zero as $t\longrightarrow \infty$ the dynamics can be approximimated by projecting on the line y=0 this would lead to the conduction that the origin is asymptotically stable because $z^{\cdot}=-z^3$ when y=0 we will now compute the center manifold in order to find the cornet answer to the equation of stability the operator (3.9) has the form

$$(L\emptyset)(z)=\emptyset'(z)[z\emptyset(z)-z^3]+\emptyset(z)-cz^2 \qquad (3.11)$$

Theorem (3.1.6): allows us solve the equation (lh)(z)=0 perturbatively, by an Ansatz of the from

$$h(z)=h_2 z^2+h_3 z^3+h_4 z^4+\mathcal{O}(z^5) \qquad (3.12)$$

the equation (Lh)(z)=0

$$(Lh)(z)=(h_2-c)z^2+h_3 z^3+[h_{4+2}h_2(h_2-1)z^4+\mathcal{O}(z^5)=0 \qquad (3.13)$$

Which requires $h_2=c$, $h_3=0$and $h_4=-2c(c-1)$. Hence the center manifold has a Taylor expansion of the from

$$h(z)=cz^2-2c(c-1)z^4+\mathcal{O}(z^5) \qquad (3.14)$$

and the motion on the center manifold is governed by the equation

$$u^{\cdot}=uh(u)-u^3=(c-1)u^3-2c(c-1)u^5+\mathcal{O}(z^5) \qquad (3.15)$$

It is easy show (using for instance, u^2 as a Lipunov function)

That the equilibrium point u=0 asymptotically stable if $c<1$and unstable if $c>1$by Theorem (3.1.5) we conclude that the origin of system (3.10) asymptotically stable if $c<1$ and unstable if $c>1$

which contradicts the naïve approach when $c > 1$. The case c=1 is special in the case the function $h(z)= z^2$ is an exact solution of the equation$(Lh)(z)=0$ and the curve $y=z^2$ is the unique center manifold of (3.1.10) which has the particularity to consist only of equilibrium points the origin is stable

Section (3.2) Bifurcation of Differential Equation

We consider in this section parameter-dependent differential equations of the from

$$\dot{x}=f(x,\lambda) \qquad (3.16)$$

Where $x \epsilon D \subset R^n$, $\lambda \in \Lambda \subset R^p$ and f is of class $c^r$for some $r \geq 2$we assume that $(x^*,0)$ is a bifurcation point (3.16) which that

$$f(x^*,0)=0$$

$$\frac{\partial f}{\partial x}(x^*,0)=A, \qquad (3.17)$$

Where the matrix A has $n_0 \geq 1$eigevalues on the imaginary axis when $\lambda = 0$, the equilibrium point x^* admits a center manifolds we would like however to examine the dynamics of (3.16) for all λ in neighbourhood of 0 where the center manifold theorem cannot be applied directly there is however ,an elegant trick to solve this problem in the enlarged phase space $DX\ \Lambda$, consider the system

$$\dot{x}=f(x,\lambda)$$

$$\dot{y}=0 \qquad (3.18)$$

It admits $(x^*,0)$ as a non-hyperbolic equilibrium point the linearization of (3.18)around this point is a matrix of the from

$$\begin{pmatrix} A & \frac{\partial f}{\partial \lambda}(x^*,0) \\ 0 & 0 \end{pmatrix} \qquad (3.19)$$

Which has $n_0 + p$ eigenvalues on the imaginary axis including the

(possibly multiple) eigenvalues zero this matrix can be made black-diagonal by a linear change of variables where one of the black contains all eigenvalues with zero part in variables the system (3.18) becomes

$$y^{\cdot}=By+g_{-}(y,z,\lambda)$$

$$z^{\cdot}=Cz+D\lambda+g_{0}(y,z,\lambda) \quad (3.20)$$

$$y^{\cdot}=0$$

Here the $(n-n_0)x(n-n_0)$ matrix B has only eigenvalues with nonzero eal parts ,the n_0xn_0 matrix C has all eigenvalues on the imaginarynaxis ,D is matrix of size n_0xp and

$$||g_{-}(y,z,\lambda)||,||g_{0}(y,z,\lambda)||\leq ||y||^2+||z||^2+||\lambda||^2 \quad (3.21)$$

In neighborhood of the origin , for some positive constants M we can theorem (3.16) which shows the existence of local inverting center manifold of the from $y=h(z,\lambda)$ the dynamics on the manifold is the dimensional equation

$$u^{\cdot}=cu+D\lambda+g_{0}(h(u,\lambda),u,\lambda). \quad (3.22)$$

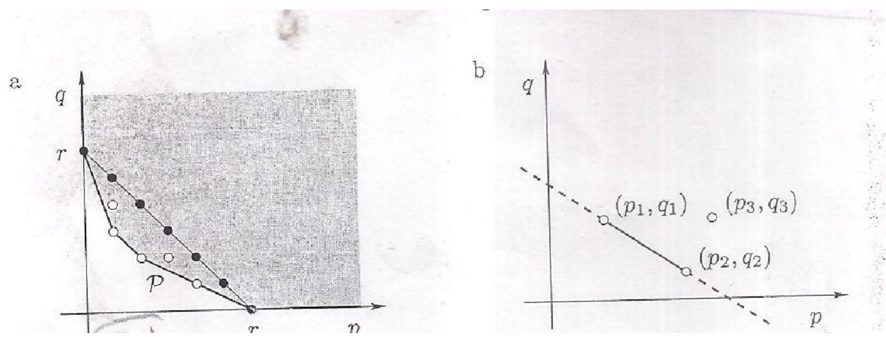
if a has no eigenvalues with positive real part Theorem (3.1.5) show that this equation gives a good approximation to the dynamics of (3.16) for small λ and near $x=x^{*}$ the big advantage is that generically , the number of eigenvalues on the imaginary axis at the bifurcation point is small ,and thus the reduct equation (3.22) is of low dimension with these preliminaries it becomes possible to investigate bifurcations in a systematic way recall that A is a real matrix matrix , and thus its eigenvalues are either real or appear in complex conjugate pairs thus the two simplest bifurcations in,whith we will consider below involve either a single zero eigenvalues or pair of conjugate imaginary more complicated bifurcations correspond to double zero eigenvalues a zero eigenvalues and two conjugate imaginary ones and so on these case are however less ''generic'', and we will not discuss them there .

Now we discussing the one-Dimensional center manifold. We first discuss bifurcations involving a single eigenvalues equal to zero i.e. $n_0=1$ and c=0 the dynamics on the one–dimensional center manifold is governed by equation of the from

$$u^{\cdot}=f(x,\lambda) \quad u\epsilon R \tag{3.23}$$

We will also assume that $\lambda\ \epsilon R$ the origain (0,0)is a binfurcation point of (3.23) meaning

$$F(0,0)=0, \ \frac{\partial f}{\partial u}(0,0)=0. \tag{3.24}$$



Figure(3.1) (a) Definition on the Newton polgn.white circles coresspondto points (ρ,q) such that $c_{\rho q} \neq 0$,black circles to point $\rho +\ q=r$ the slopes of full lines correspond to possible exponents equilibrium branches (b) the setting of the proof of proposition (3.2.2). In order to understand the dynamics for small u and λ ,we need in particular to determine the singular points of f ,that is we have the equation $F(u,\lambda)=0$ in a neigh bourhood of the origin. Note the second condition in (3.24) implies that we cannot apply the implicit function theorem. Let us start by expanding F in Taylor series,

$$F(u,\lambda)=\sum_{\substack{\rho+\ q\leq r\\ \rho,q\geq 0}} c_{\rho q}u^p\lambda^q+\sum_{\substack{\rho+\ q\leq r\\ \rho,q\geq 0}} u^p\lambda^q\, R_{\rho q}(u,\lambda), \tag{3.25}$$

Where the functions $R_{\rho q}$ are continuous near the origin, $R_{\rho q}(0,0)=0$

$$c_{\rho q} = \frac{\partial^{\rho+q}}{\rho!q!\,\partial u^{p}\,\partial\lambda^{q}}(0,0) \qquad (3.26)$$

The bifurcation conditions (3.2.9) amount to $c_{00}=c_{01}=0$. An elegan way to describe the solution of $F(u,\lambda)=0$ is based on Newton's polygon.

Definition (3.2.1): consider the set

$$A=\{(\rho,q)\in N^2: \rho+q\le r, \text{and } c_{\rho q}\neq 0\} \qquad (3.27)$$

(incase F is analytic , we simple drop the condition $c_{\rho q}\neq 0\le r$) for each $(\rho,q)\in A$,we construct the sector $\{(x,y)\in R^2: x\ge p \text{ and } y\le q\}$. the Newton polygon pof (3.2.10) is the bracken line inR^2 defined by the convex envelope of union of all these sector's see Fig(3.1)

Proposition (3.2.2): Assume for simplicity that $c_{\rho q}\neq 0$ whenever $\rho+q=r$.Assume further that the equation $F(u,\lambda)=0$ admits a solution of from $u=C|\lambda|^{\mu}(1+\rho(\lambda))$ for small $\lambda$where $C\neq 0$ and $\rho(\lambda)\to 0$ continuously $\to 0$ then Newton's polygon must have a segment of slope $-\mu$.

Proof :it is sufficient to consider a function F of the from

$$F(u,\lambda)=\sum_{i=1}^{3} c_i u^{p_i}\lambda^{q_i}$$

Indeed, if the expansion (3.25)contains only two terms, the result is immediate , and if it has more then three terms, one can proceed by induction the hypothesis implies

$$\sum_{i=1}^{3}\sigma_i c_i C^{p_i}|\lambda|^{\mu p_i+q_i}(1+\rho(\lambda))^{p_i}=0$$

Where $\sigma_i=\pm 1$ Assume for definiteness that

$p_1\mu+q_1\le p_2\mu+q_2\le p_3\mu+q_3$ consider

$$\sigma_1 c_1 C^{p_1}(1+\rho(\lambda))^{p_1}+\sum_{i=1}^{3}\sigma_i c_i C^{p_i}|\lambda|^{\mu p_i+q_i-\mu p_i-q}\;(1+\rho(\lambda))^{p_i}=0$$

The exponent of $|\lambda|$ is strictly positive thus tanking the limit $\lambda \to o$ we obtain $C^{p_i} = 0$ a contradiction we conclude that we must have $p_1\mu+q_1 = p_2\mu+q_2$. Graohically, the relation $p_1\mu+q_1 = p_2\mu+q_2 \leq p_3\mu+q_3$ means that $\mu=\frac{q_2-q_1}{p_1-p_2}$ is minus the slop of the segment from (p_1, q_1) to $(p_2, q_2)$and that (p_3, q_3) lies above ,see Fig(3.1).This result does not prove the existence of equilibrium branches $u=u^*(\lambda)$, but it tells us where to look by drawing Newton's polygon ,we obtain the possible values of μ By inserting the

$u^*(\lambda) = C|\lambda|^{\mu}(1 + \rho)$ into the equation $F(u, \lambda) = 0$ we can determine whether of not such a branch exists this is mainly a matter no solution other then (0,0), while the equation $u^2 + \lambda^2=0$ admits no solution other then(0,0), while the equation $u^2 - \lambda^2=0$ does admits solutions $u=\pm\lambda$.

Example (3.2.3): (Saddle-Node Bifurcation)

We now illustrate the procedure of equilibrium branches in generic case r=2 ,$c_{20} \neq 0, c_{01} \neq 0$ then

$$F(u, \lambda)= c_{01}\lambda + c_{20}u^2 + c_{11}u\lambda+c_{02}\lambda^2+\sum_{p+q=2} u^p\lambda^q R_{pq}(u, \lambda), \qquad (3.28)$$

And Newton's polygon has two vertices (0,1),connected by a segment with slope -1/2. Proposition (3.2.2) tells us that if there is an equilibrium branch, then it must be of the from

$$u = |\lambda|^{1/2}(1+\rho(\lambda)),\text{where } \lim_{\lambda\to 0}\rho(\lambda) = 0$$

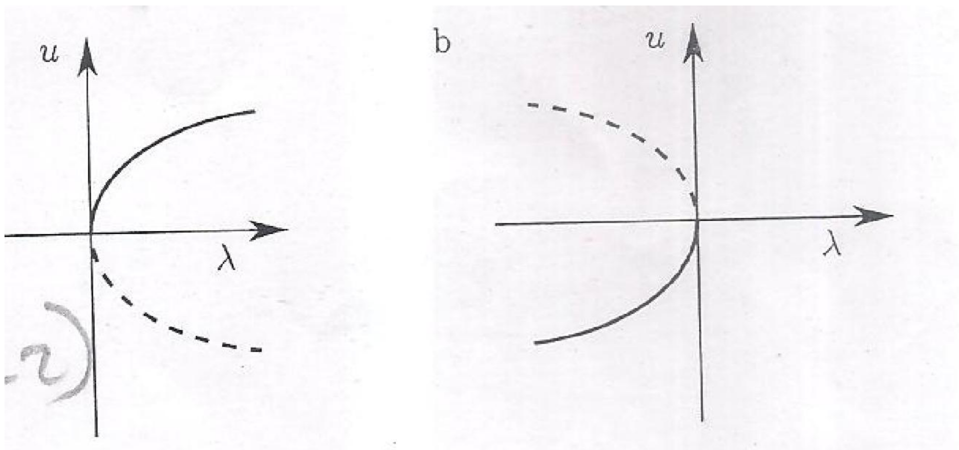
in fact ,it turns out to be easier to express λ as a function of uIn neighborhood of origin ,there exists a continuous function $\rho^-(u)$ with $\rho^-(0)=0$such that $F(u, \lambda)=0$ if and only if

$$\lambda = \frac{c_{20}}{c_{01}}u^2(1+\rho^-(u)) \qquad (3.29)$$

Proof: proposition (3.2.2) indicates that any equilibrium branch must be of the from $\lambda=c^-u^2(1+\rho^-(u))$ .define the function

$G(\rho^{-},u)=\frac{1}{u^2}F(u, c^{-}u^2(1+\rho^{-}))$ using the expansion(3.28) of F,it is easy to see that

$$\lim_{u\to 0}G(0,u)= c_{01}C^{-}+c_{20}$$



FIGURE(3.2)

saddle-node bifurcation, (a) in the case $c_{01} > 0, c_{20} < 0$(direct bifurcation) and (b) in case $c_{01} > 0, c_{20} > 0$(*indirect bifurcation*) .the other case are similar with stable and unstable branches interchanged. full curves indicate stable equilibrium branches ,while dashed curves indicate stable equilibrium branches. And thus G(0,0)=0 if and only if C^{-}=-c_{20}/c_{01}.moreover ,using a Taylor expansion to

$$\lim_{u\to 0}\frac{\partial G}{\partial \rho^{-}}(0,u)= c_{01}C^{-} \neq 0$$

Thus, by the implicit function theorem there exists, for small u,a unique function $\rho^{-}(u)$ suth that $\rho^{-}(0)=0$ and $G(\rho^{-}(u),u)=0$ Expressing u as a function of λ,we find the existence of two equilibrium branches

$$u=u^{*} \pm (\lambda) = \pm\sqrt{-\frac{c_{01}}{c_{20}}\lambda[1+\rho(\lambda)]}, \qquad (3.30)$$

which exist only for sing λ=-sing(c_{01}/c_{20}) their stability can be determined by using the Taylor expansion

$$\frac{\partial F}{\partial u}(u,\lambda)=2c_{20}u+c_{11}\lambda+\sum_{p+q=2} u^p\lambda^q\, R_{pq}(u,\lambda), \tag{3.31}$$

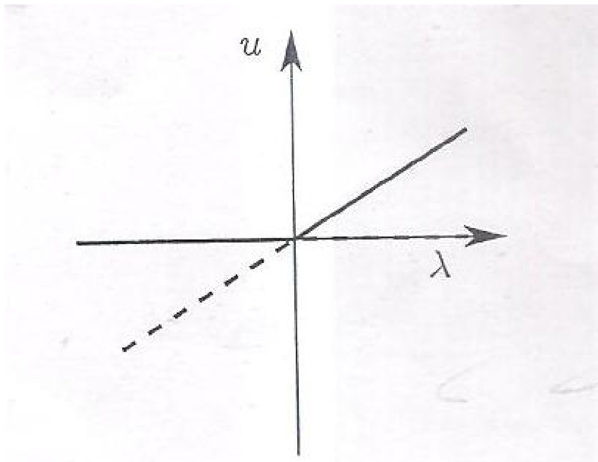
Where R_{pq}^{-} are some continuous functions vanishing at the origin. Inserting (3.30) we get

$$\frac{\partial F}{\partial u}(u^* \pm(\lambda),\lambda)= 2c_{20}\sqrt{-\frac{c_{01}}{c_{20}}\lambda}+\mathcal{O}(\sqrt{|\lambda|}\,) \tag{3.32}$$

We thus obtain following cases, depending on the sings of the coefficients:

1. If $c_{01} > 0$and $c_{20} < 0$,the barnches exist for λ>0,$u^*_+$is stable and $u^*_-$is unstable ;
2. If $c_{01} < 0$and $c_{20} > 0$,the barnches exist for λ>0,$u^*_+$is unstable and $u^*_-$is stable ;
3. If $c_{01} < 0$and $c_{20} < 0$,the barnches exist for λ>0,u^*_+isun stable and $u^*_-$is unstable ;
4. If $c_{01} > 0$and $c_{20} > 0$,the barnches exist for λ>0,$u^*_+$is unstable and $u^*_-$is stable ;

These bifurcations are called saddle –node bifurcation, because when considering the full system instead of its restriction to the center manifold, they involve a saddle and node .if the branches exist for λ>0. The bifurcation is called direct, and if they exist for λ<0 it is called indirect Fig(3.2)



FIGURE(3.3): Tran critical bifurcation, in the case of equation (3.2.21) with $c_{20}<0$ and $c_{11}>0$

It is important to observe that the qualitative depends only on those co-efficient in the Taylor series which correspond to vertices of Newton's polygon thus we could have thrown away all other terms ,to consider only the truncated equation , or normal from,

$$\dot{u}=c_{20}u^2+c_{01}\lambda \qquad (3.33)$$

FIGURE (3.3) Trancitical bifurcation, in the case of equation (3.36) with $c_{20} < 0$ and $c_{11} > 0$

Example(3.2.4) :(Transcritical bifurcation)

let us consider next the slightly less generic case where $c_{01}=0$,but$c_{20} \neq 0$, $c_{11} \neq 0$, $c_{20} \neq 0$ then the taylor expansion of F takes th from

$$F(u,\lambda)= c_{20}u^2+c_{11}u\lambda+c_{02}\lambda^2+\sum_{\rho+q=2} u^p\lambda^q R_{\rho q}(u,\lambda) \qquad (3.34)$$

and Newton's polygonthethere vertices (0,2),(1,1)and (2,0) connected by segments of slope-1 proposion3.2.2 tells us to look for equilibrium branches of the from $u=c\,\lambda(1+p(\lambda))$ proceeding in similar way as lemma (3.2.3) we obtain the conditions

$$c_{20}C^2+c_{11}C=0$$

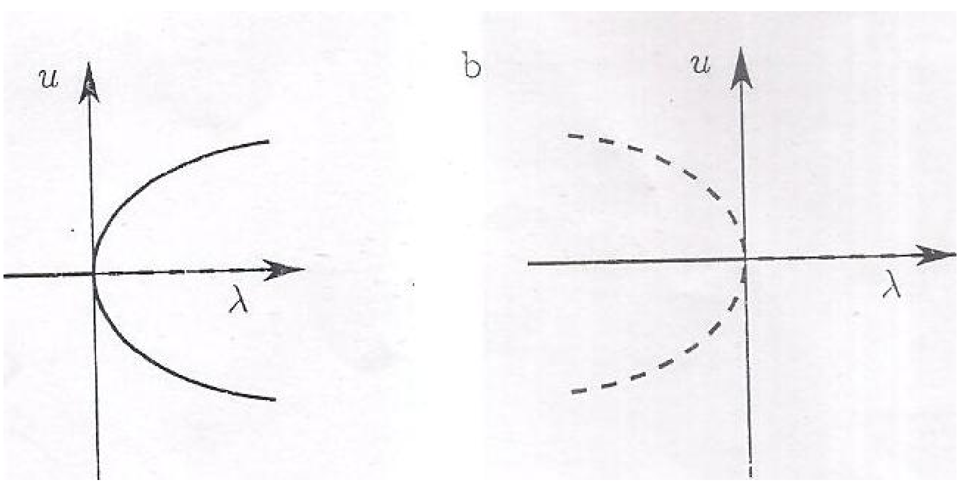
$$2\ c_{20}C^2+c_{11}C\neq 0 \qquad (3.35)$$

For the existence of unique equilibrium branches of this from .we thus conclude that if $c_{11}{}^2$-$4c_{20}c_{02}>0$there are two intersecting equilibrium branches .it is easy to see that the linearization of f around such a branches is $(2\ c_{20}+c_{11})\lambda+\mathcal{O}(\lambda)$,and thus one of the barnches is stable ,the other is unstable ,and they exchange stability at the bifurcation point this bifurcation is called transcritical .

if$c_{11}{}^2$-$4c_{20}c_{02}<0$there are no equilibrium barnches near the origin ,finally ,if $c_{11}{}^2$-$4c_{20}c_{02}=0$there may be several barnches with the same slop through the origin let us point out that the condition $c_{02}\neq 0$is not essential in fact one often carries out a change of variables taking one of the equilibrium branches to the $\lambda-$axis the resulting normal from is

$$u^{\cdot}=c_{20}u^2+c_{01}u\lambda \qquad (3.36)$$

One of the equilibrium branches is $u\equiv 0$thus slope other barnch and the stability depend on the signs $c_{20}$and c_{11}Fig(3.3).



FIGURE(3.4):pitchfork bifurcation,(a) in the case $c_{11}>0, c_{30}<0$(supercritical binfurcation)

Example (3.2.5):(pitchfork Bifurcation)

One can go on that forever, considering case with more coefficients the Taylor series equal to zero, which is especially interesting unless one has to do with a concrete problem However ,sometimes symmetries of the differential equation may cause many terms in the Taylor expansion to vanish consider the case $F \in C^3$ satisfying

$$F(-u, \lambda)=-F(-u, \lambda) \quad (-u, \lambda) \tag{3.37}$$

Then $c_{pq}=0$ for even p in fact , $F(-u, \lambda)/u$ is C^2nearthe origin and thus we can write

$$F(-u, \lambda)=u[c_{11}\lambda+c_{30}u^2+c_{12}\lambda^2+\sum_{\rho+q=2} u^p\lambda^q R_{\rho q}{}^{-}(u, \lambda) \ ] \tag{3.38}$$

Then term in brackets is similar to the expansion for the sadle-nodebinfurcation, and thus we obtain similar equilibrium branches in addition ,there is the equilibrium branches $u \equiv 0$.

Depending on the sigs of the confidents, we have the following case:

(1). If $c_{11} > 0$and $c_{30}<0$ the branch u=0 is stable for $\lambda <0$ and unstable for $\lambda>0$ and two additional stable branches exist for $\lambda>0$;

(2). If $c_{11} < 0$and $c_{30}>0$ the branch u=0 is stable for $\lambda <0$ and unstable for $\lambda>0$ and two additional stable branches exist for $\lambda>0$;

(3). If $c_{11} > 0$and $c_{30}>0$ the branch u=0 is stable for $\lambda <0$ and unstable for $\lambda>0$ and two additional stable branches exist for $\lambda<0$;

(4). If $c_{11} < 0$and $c_{30}<0$ the branch u=0 is stable for $\lambda <0$ and unstable for $\lambda>0$ and two additional stable branches exist for $\lambda>0$;

This situation in called a pitchfork binfurication, which is said to be supercritical if stable equilibrium are created, and subcritical is unstable equilibrium branches are destroyed (Fig .3.4) the normal for of the pitchfork bifurcation is

$$u^{\cdot}=c_{11}\lambda+c_{30}u^3. \qquad (3.39)$$

In physic ,this equation is written in the form

$$u^{\cdot} = \frac{\partial V}{\partial u}(u,\lambda), \qquad V(u,\lambda) \quad =-\frac{c_{11}}{2}\lambda u^2-\frac{c_{30}}{4}u^4 \qquad (3.40)$$

If $c_{30}<0$,the function $V(u,\lambda)$ has one ro two minima and in the latter case it is called a double –well potential. Now we study the two-dimensional center manifold hopf bifurcation and now we will discussing we consider the case $n_0=0$,with c having eigenvaluse$\pm iw_0$, where $w_0 \neq 0$the dynamics on the center manifold is governed by a two –dimensional of the form

$$u^{\cdot} =F(u,\lambda), \qquad (3.41)$$

Where we shall assume that $\lambda \in$ Rand $F\in C^3$.since$\frac{\partial f}{\partial u}(0,0)=C$ is invertible,the implicit function theorem(theorem3.0.1) show the existence near $\lambda=0$,of a unique equilibrium branch $u^*(\lambda)$ with $u(0)$ and $f(u^*(\lambda),\lambda)=0$.by continuity of the eigenvalues of matrix-valued function ,the linearization of F around this branch has eigenvalues $a(\lambda)\ \pm iw(\lambda)$,where $w(0)=w_0$ and $a(0)\ =0$,A translation of $-u^*(\lambda)$ followed by linear change of variables ,puts the system (3.41) into form

$$\begin{pmatrix} u_1{}^* \\ u_2{}^* \end{pmatrix}=\begin{pmatrix} a(\lambda) & -w(\lambda) \\ w(\lambda) & a(\lambda) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}+\begin{pmatrix} g_{1(u_1},u_2,\lambda \\ g_{2(u_1},u_2,\lambda \end{pmatrix}, \qquad (3.42)$$

Where the g_i are nonlinear terms satisfying

$$||g_i g_{1(u_1},u_2,\lambda\ )||\leq M(u_1{}^2+u_2{}^2)$$

for shall $u_1$and $u_2$and some constant M>0. Our strategy is now going to be simplify the nonlinear terms as much as possible following the theory of normal forms devolved in subsection (2.4) it turns out to be useful to introduce the complex variable $z=u_1+iu_2$ (an idea going back to Poincare),which satisfies an equation of the form .

$$\dot{z}=[a(\lambda)+iw(\lambda)]z+g(z,\bar{z}\ \lambda), \qquad (3.43)$$

Where $\bar{z}$ is the complex conjugate of z .this should actually be considered as a two-dimensional system for the independent varibles z and $\bar{z}$:

$$\dot{z}=[\ a(\lambda)+iw(\lambda)]z+g(z,\bar{z}\ \lambda),$$

$$\bar{z}=[\ a(\lambda)-iw(\lambda)]\ \bar{z}+\bar{g}\ (z,\bar{z}\ \lambda), \qquad (3.44)$$

Lemma(2.1.20) shows that monomials of the form $c_p z^{p_1} \bar{z}^{p_1}$ in the nonlinear term g can be eliminated by a nonlinear change of variables, provided the non-resonance condition (2.50) is satisfied .in the present case , this condition has the form

$$(p_{1+}p_2\text{-}1)a(\lambda)+\ (p_{1-}p_2\ \mp 1)iw(\lambda)\neq 0 \qquad (3.45)$$

Where the signs $\mp$respectively, to the first and second equation in (3.44) this condition can also be checked directly by carrying out the transformation $z=\xi+h_p\xi^{p_1}\bar{\xi}^{p_2}$ in (3.44) condition (3.45) hardest to satisfy for $\lambda=0$ where it becomes

$$(p_{1-}p_2\ \mp 1)iw_0\ \neq\ 0 \qquad (3.46)$$

Since $w_0\neq 0$ by assumption ,this relation always holds for $p_{1+}p_2=2$ so quadratic terms can always be eliminated the only resonant term of order 3 in g is $z^2\ \bar{z}=|z|^2z$, corresponding to $(p_{1,}p_2)=(2,1)$ (Likewise ,the term $z^2\ \bar{z}=|z|^2z$ is resonant in $\bar{g}$.)we conclude from proposition (2.2.19) that there exist a polynomial change of variable $z=\xi+h(\xi,\bar{\xi})$,transforming (3.43) into

$$\xi^{\cdot}=[a()+iw(\lambda)]\xi \ +c(\lambda)|\xi|^2\,\xi+R(\xi,\xi^{-},\lambda) \qquad (3.47)$$

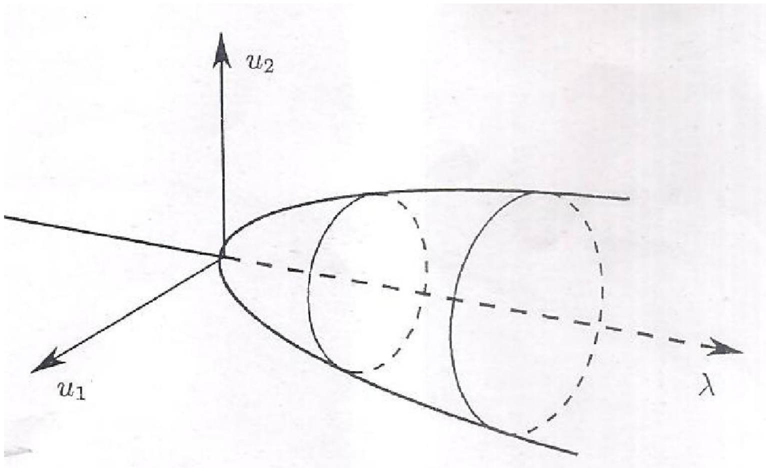


FIGURE (3.5): supercritical Hopf bifurcation .The stationary solution $(u_1,,u_2)=(0,0)$ is stable for $\lambda<0$ and unstable for $\lambda>0$ for positive λ,a stable periodic orbit close to a circle of radius $\sqrt{\lambda}$ appears. Where $c(\lambda)\in$ Cand $R(\xi,\xi^{-},\lambda)=\mathcal{O}(|\xi|^3)$ (meaning that $|\xi|^{-3}$ $R(\xi,\xi^{-},\lambda)=\mathcal{O}$ Equation (3.47) is this normal form of our binfurcation. T_0 analyse it further ,we introducte polar coordinates $\xi=re^{i\varphi}$,in which the system becomes

$$r^{\cdot}=a(\lambda)r+Rec(\lambda)r^3+R_1(r,\varphi,\lambda)$$

$$\varphi^{\cdot}=w(\lambda)+Imc(\lambda)r^2+R_2(r,\varphi,\lambda) \qquad (3.48)$$

Where $R_1=\mathcal{O}(r^3)$ and $R_2=\mathcal{O}(r^2)$we henceforth assume thst $a'(0)\neq 0$ and changing λ into-λif necessary we may assume that $a'(0)>0$ if we discard the remainder R_1,the first eqaution in (3.48) describes a pitchfork bifurcation for r which is superitical if $Rec(0)<0$ and subcritical $Rec(0)>0$ the first case corresponds to the appearance of a stable periodic orbit ,of amplitude $\sqrt{-a(\lambda)/Rec(\lambda)}$ Fig(3.5)the second to the destruction of an unstable periodic orbit the rotation frquenncy on this orbit is $w_0+\mathcal{O}(\lambda)$. It remains to show that this picture is not destroyed by the remainders $R_1$and R_2not that in the present case we cannot apply the Poincare-Chen theorem (theorem (2.1.21)because the linear part is not hyperbolic.

Theorem (3.2.6):(Andronov-Hop)

Assume that the system $x^{\cdot}$=f(x, λ) admits an equilibrium branch $x^*(\lambda)$such that the linearization of at $x^*(\lambda)$ has two eigenvaluse $a(\lambda) \pm iw(\lambda)$ with a(0)=0, a'>0and w(0)≠ 0.and all other eigenvalues have strictly negative real parts if the coefficient c(λ) in the normal form (2.47) staistfies Re c(0) ≠ 0 then

(1). If Re c(0)<0,the system admits a stable isolated periodic orbits for small positive λ close to a circle with radius proportional to $\sqrt{\lambda}$(supercritical case)

(2). If Re c(0)>0,the system admits a unstable isolated periodic orbits for small positive λ close to a circle with radius proportional to $\sqrt{-\lambda}$(subercritical case)

Proof: There exist various proofs of this result. One of them is based on the method of averaging another one of the Poincare – Benison theorem since we did not introduce these methods we will give a straightforward geometrical proof .the main idea is to consider the set $\{(r,\varphi): \varphi = 0, r > 0\}$ as a Poincare section, and to examine the associated Poincare map. consider the case Re c(0)<0.if we assume that o< λ if we assume that o<λ<<1 ,set r=$\sqrt{\lambda}$p

This bifurcation is called Poincare-Andropov-Hopf one should note that it is the first time we prove the existence of periodic orbits in any generality. Bifurcation tions with eigenvalues crossing axis are common. The nature of the bifurcation depends crucially one the sign of Re c(0) can be determined by a straightforward though rather tedious computation and is given for instance in [GH83,p.152].

Section (3.3) :(Bifurcations of Maps)

We turn now to parameter-dependent iterated maps of the form

$$x_{k+1}=F(x_k,\lambda), \tag{3.49}$$

With $x \epsilon D \subset R^n$, $\lambda \epsilon R^p$and $F \in C^r$ for some $r \geq 2$. we assume again that (x^*,0)

$$F(x^*,0)= x^*$$

$$\frac{\partial F}{\partial x}(x^*,0)=A, \tag{3.50}$$

Where A has $n_0 \geq 0$ eigenvalues of module 1 .for simplicity, we assume that all other eigenvalues of A have a module strictly smaller than 1 in appropriate Coordinates we can thus write this system as

$$u_{k+1}=Bu_k+g\text{-}(u_k, z_k, \lambda_k)$$

$$z_{k+1}=Cu_k+D\lambda_k + g_0(y_k, z_k, \lambda_k) \tag{3.51}$$

$$\lambda_{k+1}=\lambda_k$$

Where all eigenvalues of B are inside the unit circle, all eigenvalues of C are on the unit circle, and g-and go are nonlinear terms. One can prove, in much the same way we used for differential equations, the existence of a local invariant center manifold $y=h(z, \lambda)$This manifold has similar properties as in the ODE case; it is Locally attractive, and can be computed solving approximately the equation

$$h(C_z+ D\,\lambda+g_0(h(z,\lambda), z, \lambda))= Bh((z,\lambda)+g_-(h(z,\lambda), z,\lambda)) \tag{3.52}$$

The dynamics on this manifold is governed by the n_0-dimensional map

$$u_{k+1}= Cu_k+D\lambda + g_0(u_k, \lambda), u_k, \lambda) \tag{3.53}$$

Since C is a real matrix with all eigenvalues on the unit circle the most generic case are the following:

(1). One eigenvalues equal t_0 1: $n_0 = 1$ and C=1

(2). One eigenvalues equal t_0 -1: $n_0 = 1$ and C=-1

(3). Tow complex conjugate eigenvalues of module $1n_0 = 2$ and C having eigenvalue $e^{\pm 2\pi\theta 0}$ with $2\theta_0 \ni z$.

The fixed point are obtained by the equation G(u, λ)=0,with behaves exactly as the equation F(u, λ)=0

For the saddle-node bifurcation we have

$$u_{k+1}=u_k+c_{01}\,\lambda+c_{20}u_k^{\,2} \tag{3.54}$$

For the transcitical bifurcation one can reduce the equation to

$$u_{k+1}=u_k+c_{11}\,\lambda u_k+c_{20}u_k^{\,2} \tag{3.55}$$

And for the pitchfork bifurcation the normal form is given by

$$u_{k+1}=u_k+c_{11}\,\lambda u_k+c_{30}u_k^{\,3} \tag{3.56}$$

Example (3.3.1):(Period-Doubling Bifurcation)

We turn now to the case n_0=1,C=-1 and assume that the map is of class $C^3$the map restricted to the center manifold has the form

$$u_{k+1} = u_k+G(u_k, \lambda)$$

$$G(u, \lambda)= c_{01}\,\lambda + c_{02}\lambda^2 + c_{11}u\,\lambda + c_{20}u^2 + c_{30}u^3 + \dots \tag{3.57}$$

We first note that the implication theorem can be applied to the equation $u_{k+1} = u_k$ and yields the existence of a unique equilibrium branch through the origin it has from

$$u=u^*(\lambda)=\frac{c_{01}}{2}+o(\lambda) \tag{3.58}$$

And changes stability as λ passes though 0 if $c_{11} + c_{20}c_{01} \neq 0$ the something must happen to nearby orbits to understand what is going on it is useful to determine the second iterates A straightforward computation gives

$$u_{k+2} = -u_{k+1} + G(u_{k+1}, \lambda)$$

$$= u_k - G(u_k, \lambda) \_ G(-u_k + G(u_k, \lambda), \lambda)$$

$$= u_k + c_{01}(c_{11} + c_{01}c_{20})\lambda^2 - 2(c_{11} + c_{01}c_{20}) u_k \lambda - \ldots \quad (3.59)$$

Now let us consider the equation $u_{k+2} = u_k$ the solutions of which yield orbit of period 2 we find that in addition to the solution $u = u^*(\lambda)$ there exist solution of the from

$$u^2 = \frac{c_{11} + c_{01}c_{20}}{c_{30} + {c_{20}}^2} + o(\lambda) \quad (3.60)$$

Theorem (3.3.2) let F (.,λ):R→ R be a one –parameter family of maps of the form (3.57),I.e. such that F(.,0) admits 0 as a fixed point with linearization -1 assume

$$c_{11} + c_{01}c_{20} \neq 0$$

$$c_{30} + {c_{20}}^2 \neq 0 \quad (3.61)$$

This bifurcations called a period doubling flip or sub harmonic it called supercritical if stable cycle of period 2 is created and subcritical if an unstable orbit of period is destroyed. We end this section by discussing Hop Bifurcation and Invariant Tore we finally consider what happens when two eigenvalues cross the unit circle in complex plane then we have to study the map

$$u_{k+1} = Cu_k + G(u_k, \lambda) \quad (3.62)$$

Where $u \in R^2$ and C has eigenvalues $e^{\pm 2\pi i\theta}$ with $2\theta \notin z$ we shall assume that $\lambda \in R$ and $G \in C^3$ As in the case of differential equation the implicit function theorem show the existence of an equilibrium $u^*(\lambda)$ through the origin.

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