## 1.1 MATRICES AND VECTORS

### A. Linear Independence and Dependence

Before proceeding, we state without proof the following two theorems from algebra.

**THEOREM A**

A system of n homogeneous linear algebraic equations in n unknowns has a nontrivial solution if and only if the determinant of coefficients of the system is equal to zero.

**THEOREM B**

A system of n linear algebraic equations in n unknowns has a unique solution if and only if the determinant of coefficients of the system is unequal to zero,

**DEFINTION:**

A set of m constant vectors $v_1, v_2, \dots, v_m$ is linearly dependent if there exist a set of m numbers $c_1, c_2, \dots, c_m$ not all of which are zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0$$

**Example 1.1.**

The set of three constant vectors

$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \; , v_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad , v_3 = \begin{pmatrix} 7 \\ 3 \\ 8 \end{pmatrix}$$

Is linearly dependent, since there exists the set of three numbers 2, 3, and —1, none of which are zero, such that

$$2v_1 + 3v_2 + (-1)v_3 = 0$$

**DEFINITION:**

A set of m constant vectors is linearly independent if and only if the set is not linearly dependent. That is, a set of m constant vectors $v_1, v_2, \dots, v_m$ is linearly independent if the relation

$$c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0c_1 = c_2 = \cdots = c_m = 0$$

**Example 1.2.**

The set of three constant vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \quad , \ and \ v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Is liner independent. For where

$$c_1 \ v_1 + c_2 \ v_2 + c_3 \ v_3 = 0, \qquad (1.1)$$

that is,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is linearly independent. For we have

$$c_1 - c_2 = 0, \ \ c_1 + 2c_2 + 2c_3 = 0, \qquad c_1 + c_3 = 0. \qquad (1.2)$$

of three homogeneous linear algebraic equations in the three unknowns $c_1, c_2, c_3$. The determinant of coefficients of this system is

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Thus by Theorem A, with n = 3, the system (1.2) has only the trivial solution $c_1 = c_2 = c_3 = 0$. Thus for the three given constant vectors, the relation (1.1) implies $c_1 = c_2 = c_3 = 0$ ; and so these three vectors are indeed linearly independent.

**DEFINITION:**

The set of m vector functions $\phi_1, \phi_2, \dots, \phi_m$ is linearly dependent on an interval a ≤ t ≤ b if there exists a set of m numbers $c_1, c_2, \dots, c_m$ numbers not all zero, such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_m\phi_m(t) = 0$$

for all t ∈ [a, b].

**Example 1.3.**

Consider the set of three vector functions $\phi_1, \phi_2$ and $\phi_3$ defined for all t by

$$\phi_1(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 5e^{2t} \end{pmatrix}, \qquad \phi_2(t) = \begin{pmatrix} e^{2t} \\ 4e^{2t} \\ 11e^{2t} \end{pmatrix} \ and \ \phi_3(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Respectively. This set of vector functions is linearly dependent on any interval a ≤ t ≤ b. To see this, note that

$$3\begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 5e^{2t} \end{pmatrix} + (-1)\begin{pmatrix} e^{2t} \\ 4e^{2t} \\ 11e^{2t} \end{pmatrix} + (-2)\begin{pmatrix} e^{2t} \\ e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and hence there exists the set of three numbers 3, $-1$, and $-2$, none of which are zero, such that

$$3\phi_1(t) + (-1)\phi_2(t) + (-2)\phi_3(t) = 0$$

for all t ∈ [a, b]

**DEFI NIT1ON:**

A set of m vector functions is linearly independent on an interval if and only if the set is not linearly dependent on that interval. That is, a set of m vector functions $\phi_1, \phi_2, \dots, \phi_m$ is linearly independent on an interval a ≤ t ≤ b if the relation

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_m\phi_m(t) = 0$$

For all t ∈ [a, b] implies that

$$c_1 = c_2 = \cdots = c_m = 0$$

**Example 1.4.**

Consider the set of two vector functions $\phi_1(t)$, and $\phi_2(t)$. Defined for all t

$$\phi_1(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} \ and \ \phi_2(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Respectively. We shall show that $\phi_1$ and $\phi_2$ are linearly independent on any interval $a \le t \le b$. To do this, we assume the contrary; that is, we assume that $\phi_1 \ and \phi_2$ are linear dependent on [a, b]. Then there exist numbers $c_1$ and $c_2$, not both zero, such that

$$c_1\phi_1(t) + c_2\phi_2(t) = 0$$

for all t ∈ [a, b]. Then

$$c_1e^t + c_2e^{2t} = 0,$$

$$c_1e^t + 2c_2e^{2t} = 0,$$

and multiplying each equation through by $e^{-t}$, we have

$$c_1 + c_2e^t = 0,$$

$$c_1 + 2c_2e^t = 0,$$

for all t ∈ [a, b]. This implies that $c_1e^t + c_2e^{2t} = c_1e^t + 2c_2e^{2t}$ and hence 1 = 2, which is an obvious contradiction. Thus the assumption that $\phi_1$ and $\phi_2$ are linearly dependent on [a, b] is false, and so these two vector functions are linearly independent on that interval.

**Note:** If a set of m vector functions $\phi_1, \phi_2, \dots, \phi_m$ is linearly dependent on an interval a ≤ t ≤ b, then it readily follows that for each fixed $t_0$ ∈ [a, b], the corresponding set of m constant vectors $\phi_1(t_0), \phi_2(t_0), \dots, \phi_m(t_0)$ is linearly independent.

Indeed the corresponding set of constant vectors $\phi_1(t_0), \phi_2(t_0), \dots, \phi_m(t_0)$ may be linearly dependent for each $t_0$ ∈ [a, b]. See Exercise 6 at the end of this section.

## B. Characteristic Values and Characteristic Vectors

Let A be a given n x n square matrix of real numbers, and let S denote the set of all n x 1 column vectors of numbers. Now consider the equation

$$Ax = \lambda x \tag{1.3}$$

In the unknown vector $x \in S$, where $\lambda$ is a number. Clearly the zero vector 0 is a solution of this equation for every number $\lambda$. We investigate the possibility of finding nonzero vectors $x \in S$ which are solutions of (1.3) for some choice of the number $\lambda$. In other words, we seek numbers $\lambda$. Corresponding to which there exist nonzero vectors x which satisfy (1.3). These desired values of $\lambda$ and the corresponding desired nonzero vectors are designated in the following.

**DEFINITION:**

A characteristic value (or eigenvalue) of the matrix A is a number 2 for which the equation Ax = $\lambda x$ has a nonzero vector solution x.

A characteristic vector (or eigenvector) of A is a nonzero vector x such that Ax = $\lambda$x for some number $\lambda$.

We proceed to solve this problem. Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is the given n x n square matrix of real numbers, and let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Then Equation (1.3) may be written

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and hence, multiplying the indicated entities,

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

Equating corresponding components of these two equal vectors, we have

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2 \\ \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_n \end{gathered}$$

and rewriting this, we obtain

$$\begin{gathered} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n = 0 \end{gathered} \qquad (1.4)$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n = 0$$

Thus we see that (1.3) holds if and only if (1.4) does. Now we are seeking nonzero vectors x that satisfy (1.3). Thus a nonzero vector x satisfies (1.3) if and only if its set of components $x_1, x_2, \dots, x_n$ is a nontrivial solution of (1.4). By Theorem A of Section (1.1)B, the system (1.4) has nontrivial solutions if and only if its determinant of coefficients is equal to zero, that is, if and only if

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0 \qquad (1.5)$$

It is easy to see that (1.5) is a polynomial equation of the nth degree in the unknown 2. In matrix notation it is written

$$|A - \lambda I| = 0$$

where I is the n x n identity matrix (see Section 1.1A). Thus Equation (1.3) has a nonzero vector solution x for a certain value of $\lambda$ if and only if $\lambda$ satisfies the nth-degree polynomial equation (1.5). That is, the number $\lambda$ is a characteristic value of the matrix A if and only if it satisfies this polynomial equation. We now designate this equation and also state the alternative definition of characteristic value that we have thus obtained.

**DEFINITION:**

Let A = $(a_{ij})$ be an n x n square matrix of real numbers. The characteristic equation of A is the nth-degree polynomial equation

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0 \qquad (1.5)$$

in the unknown $\lambda$; and the characteristic values of A are the roots of this equation.

Since the characteristic equation (1.5) of A is a polynomial equation of the nth degree, it has n roots. These roots may be real or complex, but of course they may or may not all be distinct. Then we say that that root has multiplicity m. If we count each no repeated root once and each repeated root according to its multiplicity, Then we can say that the n x n matrix A has precisely n characteristic values, say $\lambda_1, \lambda_2, \dots, \lambda_n$.

Corresponding to each characteristic value $\lambda_k$ of A there is a characteristic vector $x_k$ (k = 1, 2,..., n). Further, if $x_k$ is a characteristic vector of A corresponding to characteristic value $\lambda_k$, then so is $cx_k$, for any nonzero number c. We shall be concerned with the linear independence of the various characteristic vectors of A. Concerning this, we state the following two results without proof.

Result C. Each of the n characteristic values $\lambda_1, \lambda_2, \dots, \lambda_n$. of the n x n square matrix A is distinct (that is, nonrepeated); and let $x_1, x_2, \dots, x_n$ be a set of n respective corresponding characteristic vectors of A. Then the set of these n characteristic vectors is linearly independent.

Result D. Suppose the n x n square matrix A has a characteristic value of multiplicity m, where $1 < m \leq n$. Then this repeated characteristic value having multiplicity m has p linearly independent characteristic vectors corresponding to it, where $1 \leq p \leq n$.

Now suppose A has at least one characteristic value of multiplicity m, where $1 < m \leq n$; and further suppose that for this repeated characteristic value, the number p of Result D is strictly less than m; that is, p is such that $1 \leq p \leq m$. Then corresponding to this characteristic value of multiplicity m, there are less than m linearly independent characteristic vectors. It follows at once that the matrix A must then have less than n linearly independent characteristic vectors. Thus we are led to the following result:

Result E. If the n x n matrix A has one or more repeated characteristic values, then there may exist less than n linearly independent characteristic vectors of A.

Before giving an example of finding the characteristic values and corresponding characteristic vectors of a matrix, we introduce a very special class of matrices whose characteristic values and vectors have some interesting special properties. This is the class of so-called real symmetric matrices, which we shall now define below. First, however we give a preliminary definition.

**DIFINITION:**

A square matrix A of real numbers is called a real symmetric matrix if$A^T = A$.

For example, the 3x3 square matrix

$$A = \begin{pmatrix} 2 & -1 & 4 \\ -1 & 0 & 3 \\ 4 & 3 & 1 \end{pmatrix}$$

Is a real symmetric matrix since $A^T = A$

Concerning real symmetric matrices, we state without proof the following interesting results:

Result F. All of the characteristic values of a real symmetric matrix are real numbers.
Result G. If A is an n x n real symmetric square matrix, then there exist n linearly independent characteristic vectors of A, whether the n characteristic values of A are all distinct or whether one or more of these characteristic values is repeated.

**Example 1.5.**

Find the characteristic values and characteristic vectors of the matrix

$$A = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}$$

Solution. The characteristic equation of A is

$$\begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$$

Evaluating the determinant in the left member, we find that this equation may be written in the form

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

Or

$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0.$$

Thus the characteristic values of A are

$$\lambda = 2 \ , \qquad \lambda = 3 \ , \qquad and \ \lambda = 5$$

The characteristic vectors corresponding to $\lambda = 2$ are the nonzero vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Such that

$$\begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Thus $x_1, x_2, x_3$ must be non trivial solution of the system

$$7x_1 - x_2 + 6x_3 = 2x_1$$

$$-10x_1 + 4x_2 - 12x_3 = 2x_2$$

$$-2x_1 + x_2 - x_3 = 2x_3$$

That is

$$5x_1 - x_2 + 6x_3 = 0$$

$$-10x_1 + 2x_2 - 12x_3 = 0$$

$$-2x_1 + x_2 - 3x_3 = 0$$

Note that the second of these three equations is merely a constant multiple of the first thus we seek nonzero numbers $x_1, x_2, x_3$ which satisfy the first and third of these equations. Writing these two as equations in the unknowns $x_2$ and $x_3$, we have

$$-x_2 + 6x_3 = -5x_1,$$

$$x_2 - 3x_3 = 2x_1.$$

Solving for $x_2$ and $x_3$, we find

$$x_2 = -x_1 \quad and \quad x_3 = -x_1$$

We see at once that $x_1 =$ k, $x_2 = -$k, $x_3 = -$k is a solution of this for every real k. Hence the characteristic vectors corresponding to the characteristic value $\lambda$ = 2 are the vectors

$$x = \begin{pmatrix} k \\ -k \\ -k \end{pmatrix}$$

where k is an arbitrary nonzero number. In particular, letting k = 1, we obtain the particular characteristic vector

$$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

corresponding to the characteristic value $\lambda$ = 2.

Proceeding in like manner, one can find the characteristic vectors corresponding to $\lambda$ = 3 and those corresponding to $\lambda$ = 5. We find that the components $x_1, x_2, x_3$ of the characteristic vectors corresponding to $\lambda$ = 3 must be a nontrivial solution of the system

$$4x_1 - x_2 + 6x_3 = 0,$$

$$-10x_1 + x_2 - 12x_3 = 0,$$

$$-2x_1 + x_2 - 4x_3 = 0.$$

From these we find that

$$x_2 = -2x_1 \quad and \quad x_3 = -x_1$$

and hence $x_1$ = k, $x_2$ = − 2k, $x_3 =$ −k is a solution for every real k. Hence the characteristic vectors corresponding to the characteristic value $\lambda = 3$ are the vectors

$$x = \begin{pmatrix} k \\ -2k \\ -k \end{pmatrix}$$

where k is an arbitrary nonzero number. In particular, letting k = 1, we obtain the particular characteristic vector

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

corresponding to the characteristic value $\lambda$ = 3.

Finally, we proceed to find the characteristic vectors corresponding to $\lambda$ = 5. We find that the components $x_1, x_2, x_3$ of these vectors must be a nontrivial solution of the system

$$2x_1 - x_2 + 6x_3 = 0,$$
$$-10x_1 - x_2 - 12x_3 = 0,$$
$$-2x_1 + x_2 - 4x_3 = 0.$$

From these we find that

$$x_2 = -2x_1 and\ 3x_3 = -2x_1.$$

We find that $x_1 = 3k, x_2 = -6k, x_3 = -2k$ satisfies this for every real k. Hence the characteristic vectors corresponding to the characteristic value $\lambda = 5$ are the vectors

$$x = \begin{pmatrix} 3k \\ -6k \\ -2k \end{pmatrix}$$

where k is an arbitrary nonzero number. In particular, letting k = 1, we obtain the particular characteristic vector

$$\begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$$

corresponding to the characteristic value $\lambda = 5$.

## 2.1 DIFFERENTIAL OPERATORS AND AN OPERATOR METHOD

### A. Types of Linear Systems

We shall consider the general linear system of two first-order differential equations in two unknown functions x and y is of the form

$$a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)x + a_4(t)y = F_1(t).$$

$$b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)x + b_4(t)y = F_2(t). \quad (2.1)$$

An example of such a system which have constant coefficients is

$$2\frac{dx}{dt} + 3\frac{dy}{dt} - 2x + y = t^2 \quad , \quad \frac{dx}{dt} - 2\frac{dy}{dt} + 3x + 4y = e^t .$$

We shall say that a solution of system (2.1) is an ordered pair of real functions (f,g)

such that $x = f(t), y = g(t)$ simultaneously satisfy both equations of the

system (2.1) on some real interval $a \le t \le b$.

The general linear system of three first-order differential equations in three unknown functions x, y and z and of the form

$$a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)\frac{dz}{dt} + a_4(t)x + a_5(t)y + a_6(t)z = F_1(t).$$

$$b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)\frac{dz}{dt} + b_4(t)x + b_5(t)y + b_6(t)z = F_2(t). \quad (2.2)$$

$$c_1(t)\frac{dx}{dt} + c_2(t)\frac{dy}{dt} + c_3(t)\frac{dz}{dt} + c_4(t)x + c_5(t)y + c_6(t)z = F_3(t).$$

As in the case of system of the form

$$\frac{dx}{dt} + \frac{dy}{dt} - 2\frac{dz}{dt} + 2x - 3y + z = t ,$$

$$2\frac{dx}{dt} - \frac{dy}{dt} + 3\frac{dz}{dt} + x + 4y - 5z = sint ,$$

$$\frac{dx}{dt} + 2\frac{dy}{dt} + \frac{dz}{dt} - 3x + 2y - z = cost .$$

A solution of this system is an ordered triple of real functions (f, g, h) such that $x = f(t)$, $y = g(t)$, $z = h(t)$ simultaneously satisfy all three equations of the system (2.2) on some real interval $a \leq t \leq b$.

System of the form (2.1) and (2.2) contained only first derivatives, and we not consider the basic linear system involving higher derivatives. This is the general linear system of two second-order differential equations in two unknown functions $x$ and $y$ and is a system of the form

$$a_1(t)\frac{d^2x}{dt^2} + a_2(t)\frac{d^2y}{dt^2} + a_3(t)\frac{dx}{dt} + a_4(t)\frac{dy}{dt} + a_5(t)\, x + a_6(t)\, y = F_1(t).$$

$$b_1(t)\frac{d^2x}{dt^2} + b_2(t)\frac{d^2y}{dt^2} + b_3(t)\frac{dx}{dt} + b_4(t)\frac{dy}{dt} + b_5(t)\, x + b_6(t)\, y = F_2(t). \qquad (2.3)$$

We shall be concerned with systems having constant coefficients in this case also an example is provided by

$$2\frac{d^2x}{dt^2} + 5\frac{d^2y}{dt^2} + 7\frac{dx}{dt} + 3\frac{dy}{dt} + 2y = 3t + 1.$$

$$3\frac{d^2x}{dt^2} + 2\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 4x + y = 0.$$

For given fixed positive integers m and n, we could proceed, in like manner, to exhibit other general linear systems of n nth-order differential equations in n unknown functions.

We consider special type of linear system (2.1) which is of the form

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + F_1(t)$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + F_2(t) \qquad (2.4)$$

This is the so-called normal form in the case of two linear differential equations in two unknown functions. The characteristic feature of such a system is apparent from the manner in which the derivatives appear in it. An example of such a system with variable coefficients is

$$\frac{dx}{dt} = t^2x + (t+1)y + t^3,$$

$$\frac{dy}{dt} = \mathrm{t}e^t x + t^2 y - e^t,$$

While one with constant coefficients is

$$\frac{dx}{dt} = 5x + 7y + t^2,$$

$$\frac{dy}{dt} = 2x - 3y + 2t.$$

The normal form in the case of a linear system of three differential equations in three unknown functions x, y, and z is

$$\frac{dx}{dt} = a_{11}(\mathrm{t})\, x + a_{12}(\mathrm{t})\, y + a_{13}(\mathrm{t})\, z + F_1(\mathrm{t}),$$

$$\frac{dy}{dt} = a_{21}(\mathrm{t})\, x + a_{22}(\mathrm{t})\, y + a_{23}(\mathrm{t})\, z + F_2(\mathrm{t}),$$

$$\frac{dz}{dt} = a_{31}(\mathrm{t})\, x + a_{32}(\mathrm{t})\, y + a_{33}(\mathrm{t})\, z + F_3(\mathrm{t}).$$

An example of such a system is the constant coefficient system

$$\frac{dx}{dt} = 3x + 2y + z + t,$$

$$\frac{dy}{dt} = 2x - 4y + 5z - t^2$$

$$\frac{dz}{dt} = 4x + y - 3z + 2t + 1.$$

The normal form in the general case of a linear system of n differential equations in n unknown functions$x_1, x_2, \cdots, x_n$ is

$$\frac{dx_1}{dt} = a_{11}(\mathrm{t})\, x_1 + a_{12}(\mathrm{t})\, x_2 + \cdots + a_{1n}(\mathrm{t})\, x_n + F_1(\mathrm{t})$$

$$\frac{dx_2}{dt} = a_{21}(\mathrm{t})\, x_1 + a_{22}(\mathrm{t})\, x_2 + \cdots + a_{2n}(\mathrm{t})\, x_n + F_2(\mathrm{t}) \quad (2.5)$$

$$\vdots$$

$$\frac{dx_1}{dt} = a_{n1}(\mathrm{t})\, x_1 + a_{n2}(\mathrm{t})\, x_2 + \cdots + a_{nn}(\mathrm{t})\, x_n + F_n(\mathrm{t})$$

An important fundamental property of a normal linear system (2.5) is its relationship to a single nth-order linear differential equation in one unknown function. Specifically, consider the so-called normalized (meaning, the coefficient of the highest derivative is one) nth-order linear differential equation

$$\frac{d^n x}{dt^n} + a_1(\mathrm{t}) \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(\mathrm{t})\frac{dx}{dt} + a_n(\mathrm{t})\, x = \mathrm{F(t)} \qquad (2.6)$$

in the one unknown function x. Let

$$x_1 = x,\ x_2 = \frac{dx}{dt},\ x_3 = \frac{d^2x}{dt^2},\ \cdots,\ x_{n-1} = \frac{d^{n-2}x}{dt^{n-2}},\ x_n = \frac{d^{n-1}x}{dt^{n-1}} \qquad (2.7)$$

From (2.7), we have

$$\frac{dx}{dt} = \frac{dx_1}{dt},\ \frac{d^2x}{dt^2} = \frac{dx_2}{dt},\ \cdots,\ \frac{d^{n-1}x}{dt^{n-1}} = \frac{dx_{n-1}}{dt},\ \frac{d^n x}{dt^n} = \frac{dx_n}{dt}. \qquad (2.8)$$

Then using both (2.7) and (2.8), the single nth-order equation (2.6) can be transformed into

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
&\vdots \\
\frac{dx_{n-1}}{dt} &= x_n \\
\frac{dx_n}{dt} &= -a_n(t)\, x_1 - a_{n-1}(t)\, x_2 - \cdots - a_1(t)\, x_n + F(t)
\end{aligned} \qquad (2.9)$$

Which is a special case of the normal linear system (2.5) of n equations in n unknown functions. Thus we see that a single nth-order linear differential equation of form (2.6) is one unknown function is indeed intimately related to a normal linear system (2.5) of n first-order differential equation in n unknown functions.

**B. Differential Operators**

Let x be an n-times differentiable function of the independent variable t. We denote the operation of differentiation with respect to t by the symbol D and call

D a differential operator. In terms of this differential operator the derivative $\frac{dx}{dt}$ is. Denoted by $Dx$. That is,

$$Dx = \frac{dx}{dt}.$$

In like manner, we denote the second derivative of x with respect to t by $D^2x$. Extending this, we denote the nth derivative of x with respect to t by $D^n x$. That is,

$$D^n x = \frac{d^n x}{dt^n} \; (n = 1, 2, \cdots).$$

Further extending this operator notation, we write

$$(D + c)\, x \text{ to denote } \frac{dx}{dt} + cx$$

and

$$(aD^n + bD^m)\, x \text{ to denote } a\frac{d^n x}{dt^n} + b\frac{d^m x}{dt^m}$$

where a, b, and c are constants.

In this notation the general linear differential expression with constant coefficients

$$a_0, a_1, \cdots, a_{n-1}, a_n \;.$$

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x$$

Is written as

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)\, x$$

The operators $D^n, D^{n-1}, \cdots, D$ are to be carried out upon this function. The expression

$$a_0\, D^n + a_1\, D^{n-1} + \cdots + a_{n-1}\, D + a_n$$

By itself, where $a_0, a_1, \cdots, a_{n-1,} a_n$ are constants, is called a linear differential operator with constant coefficients.

**Example 2.1.**

Consider the linear differential operator.

$$3D^2 + 5D - 2$$

If x is a twice differentiable function of t, then

$$(3D^2 + 5D - 2)\, x \text{ denotes } 3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 2\,x.$$

For example, if $x = t^3$, we have

$$(3D^2 + 5D - 2)\, t^3 \;=\; 3\frac{d^2}{dt^2}(t^3) + 5\frac{d}{dt}(t^3) - 2\,(t^3)$$

$$=18t + 15\, t^2 - 2t^3.$$

We shall now discuss certain useful properties of the linear differential operator with constant coefficients. In order to facilitate our discussion, we shall let L denote this operator. That is,

$$L \;=\; a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n,$$

where $a_0, a_1, \cdots, a_{n-1,} a_n$ are constants. Now suppose that $f_1$ and $f_2$ are both n-times differentiable functions of $t$ and $c_1$ and $c_2$ are constants. Then it can be shown that

$$L[c_1\, f_1 + \; c_2f_2] \;=\; c_1L[f_1] \;+\; c_2L[f_2].$$

For example, if the operator L ≡ $3D^2 + 5D - 2$ is applied to $3t^2$ + *2sint*, then

$$L\,[3t^2 \;+\; 2sint] \;=\; 3L\,[t^2] \;+\; 2L\,[sin\, t]$$

Or

$$(3D^2 + 5D - 2)(\,3t^2 \;+\; 2sint)$$
$$=\; 3(3D^2 + 5D - 2)t^2 \;+\; 2(3D^2 + 5D - 2)sint.$$

Now let

$$L_1 \equiv\; a_0\, D^m + a_1\, D^{m-1} + \cdots + a_{m-1}\, D + a_m$$

And

$$L_2 \equiv b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n$$

Be two linear differential operators with constant

Coefficients $a_0, a_1, \cdots, a_{m-1}, a_m$ and$b_0, b_1, \cdots, b_{n-1}, b_n$ respectively. Let

$$L_1 \equiv a_0 r^m + a_1 r^{m-1} + \cdots + a_{m-1} r + a_m$$

And $$L_2 \equiv b_0 r^n + b_1 r^{n-1} + \cdots + b_{n-1} r + b_n$$

Be the two polynomials in the quantity r obtained from the operators $L_1$ and$L_2$,

Respectively, by formally replacing D by r, $D^2$ by$r^2$,⋯, $D^k$ by $r^k$. Let us denote the product of the polynomials $L_1$ (r) and $L_2$ (r) by L(r) that is,

$$L(r) = L_1(r) L_2(r).$$

Then, if f is a function possessing n + m derivatives, it can be shown that

$$L_1 L_2 f = L_2 L_1 f = Lf$$

Where L is the operator obtained from the "product polynomial" L(r) by formally replacing r by D, $r^2$ by $D^2$,⋯, $r^{m+n}$ by $D^{m+n}$ Equation (2.10) indicates two important properties of linear differential operators with constant coefficients. First, it states the effect of first operating on f by $L_2$ and then operating on the resulting function by $L_1$ is the same as that which results from first operating on f by $L_1$ and then operating on this resulting function by$L_2$. Second Equation (2.10) states that the effect of first operating on f by either $L_1$ or $L_2$ and then operating on the resulting function by the other is the same as that which results from operating on f by the "product operator" L.

We illustrate these important properties in the following example.

Example 2.2

Let $L_1 \equiv D^2 + 1,\ L_2 \equiv 3D + 2,\ f(t) = t^3$. Then

$$L_1 L_2 f = (D^2 + 1)(3D + 2)t^3 = (D^2 + 1)(9t^2 + 2t^3)$$
$$= 9(D^2 + 1)t^2 + 2(D^2 + 1)t^3$$

$$= 9(2 + t^2) + 2(6t + t^3) = 2t^3 + 9t^2 + 12t + 18$$

and

$$L_2 L_1 f = (3D + 2)(D^2 + 1)t^3 = (3D + 2)(6t + t^3)$$

$$6(3D + 2)t + (3D + 2) t^3$$

$$= 6(3 + 2t) + (9t^2 + 2t^3) = 2t^3 + 9t^2 + 12t + 18.$$

Finally, L≡ $3D^3 + 2D^2 + 3D + 2$ and

$$Lf = (3D^3 + 2D^2 + 3D + 2)t^3 = 3(6) + 2(6t) + 3(3t^2) + 2t^3$$

$$= 2t^3 + 9t^2 + 12t + 18.$$

Now let L ≡ $a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ , where $a_0, a_1, \cdots, a_{n-1,} a_n$ are

Constants, and let L(r) ≡ $a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$ be the polynomial in r obtained from L by formally replacing D by r, $D^2$ by $r^2$,⋯, $D^n$ by $r^n$ let $r_1, \cdots, r_2, r_n$ be the roots of the polynomial equation L(r) = 0. Then L(r) may be written in the factored form

$$L(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_{n)}.$$

Now formally replacing r by D in the right member of this identity, we may express the operator L ≡ $a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ in the factored form

$$L = a_0(D - r_1)(D - r_2) \cdots (D - r_{n)}$$

We thus observe that linear differential operators with constant coefficients can be formally multiplied and factored exactly as if they were polynomials in the algebraic quantity.

C. an Operator Method for linear Systems with Constant Coefficients

We consider a linear system of the form

$$L_1 x + L_2 y = f_1(t)$$

$$(2.11)$$

$$L_3 x + L_4 y = f_2(x)$$

Where $L_1, L_2, L_3$ and $L_4$ are linear differential operators with constant coefficients.

That is, $L_1$, $L_2$, $L_3$, and $L_4$ are operators of the forms

$$L_1 \equiv a_0D^m+a_1D^{m-1}+\cdots+a_{m-1}D+a_m$$

$$L_2 \equiv b_0D^n+b_1D^{n-1}+\cdots+b_{n-1}D+b_n$$

$$L_3 \equiv \propto_0 D^p+\propto_1 D^{p-1}+\cdots+\propto_{p-1} D+\propto_p$$

$$L_4 \equiv \beta_0D^q+\beta_1D^{q-1}+\cdots+\beta_{q-1}D+\beta_q$$

Where the a’s, b’s, $\propto$’s, and $\beta$’s are constants.

A simple example of a system which may be expressed in the form (2.11) is provided by

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$

$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2$$

Introducing operator notation this system takes the form

$$(2D - 3)x - 2Dy = t,$$

$$(2D + 3)x + (2D + 8)y = 2.$$

This is clearly of the form (2.11), where $L_1 \equiv 2D - 3$, $L_2 \equiv -2D$, $L_3 \equiv 2D + 3$, and $L_4 \equiv 2D + 8$.

Returning now to the general system (2.11), we apply the operator $L_4$ to the first equation of (2.11) and the operator $L_2$ to the second equation of (2.11), obtaining

$$L_4L_1x + L_4L_2y = L_4f_1$$

$$L_2L_3x + L_2L_4y = L_2f_2$$

We now subtract the second of these equations from the first. Since $L_4L_2y = L_2L_4y$

we obtain

$$L_4L_1x - L_2L_3x = L_4f_1 - L_2f_2$$

Or (2.12)

$$(L_4L_1 - L_2L_3)x = L_4f_1 - L_2f_2$$

The expression $L_1L_4$ - $L_2L_3$ in the left member of this equation is itself a linear differential operator with constant coefficients. We assume that it is neither zero nor a nonzero constant and denote it by $L_5$. If we further assume that the functions $f_1$ and $f_2$ are such that the right member $L_4f_1$ - $L_2f_2$ of (2.12) exists, then this member is some function, say $g_1$, of $t$. Then Equation (2.12) may be written

$$L_5x = g_1. \qquad (2.13)$$

Equation (2.13) is a linear differential equation with constant coefficients in the single dependent variable x. We thus observe that our procedure has eliminated the other dependent variable y. We now solve the differential equation (2.13) for x using the methods developed in Chapter 4[in book 1]. Suppose Equation (2.13) is of order N. Then the general solution of (2.13) is of the form

$$x = c_1u_1 + c_2u_2 + \cdots + c_Nu_N + U_1, \qquad (2.14)$$

Where $u_1, u_2, \cdots, u_N$ are $N$ linearly independent solutions of the homogeneous linear equation $L_5x = 0, c_1, c_2, \cdots, c_r$ are arbitrary constants, and $U_1$ is a particular solution of $L_5x = g_1$.

We again return to the system (2.11) and this time apply the operators $L_3$ and $L_1$ to the first and second equations, respectively, of the system. We obtain

$$L_3L_1x + L_3L_2y = L_3f_1$$

$$L_1L_3x + L_1L_4y = L_1f_2$$

Subtracting the first of these from the second, we obtain

$$(L_1L_4 - L_2L_3)y = L_1f_2 - L_3f_1.$$

Assuming that $f_1$ and $f_2$ are such that the right member $L_1f_2$ - $L_3f_1$ of this equation exists, we may express it as some function, say $g_2$, of $t$. Then this equation may be written

$$L_5y = g_2, \qquad (2.15)$$

Where $L_5$ denotes the operator $L_1L_4$ - $L_2L_3$. Equation (2.15) is a linear differential equation with constant coefficients in the single dependent variable y. This time we have eliminated the dependent variable x. solving the differential equation (2.15) for y, we obtain its general solution in the form

$$y = k_1u_1 + k_2u_2 + \cdots + k_Nu_N + U_2, \qquad (2.16)$$

Where $u_1, u_2, \cdots, u_N$ are the $N$ linearly independent solutions of $L_5 y$ = 0(or $L_5 x$=0) that already appear in (2.14), $k_1, k_2, \cdots, k_N$ are arbitrary constants, and $U_2$ is a particular solution of $L_5 y = g_2$.

We thus see that if x and y satisfy the linear system (2.11), then x satisfies the single linear differential equation (2.13) and y satisfies the single linear differential equation (2.15). Thus if x and y satisfy the system (2.11), then x is of the form (2.14) and y is of the form (2.16). However, the pairs of functions given by (2.14) and (2.16) do not satisfy the given system (2.11) for all choices of the constants$c_1, c_2, \cdots, c_N, k_1, k_2, \cdots, k_N$. That is these pairs (2.14) and (2.16) do not simultaneously satisfy both equations of the given system (2.11) for arbitrary choices of the 2N constants$c_1, c_2, \cdots, c_N, k_1, k_2, \cdots, k_N$.

In other words, in order for x given by (2.14) and y given by (2.16) to satisfy the given system (2.11), the 2N constants$c_1, c_2, \cdots, c_N, k_1, k_2, \cdots, k_N$. Cannot all be independent but rather certain of them must be dependent on the others. It can be shown that the number of independent constants in the so-called general solution of the linear system(2.11) is equal to the order of the operator $L_1 L_4$ - $L_2 L_3$ obtained from the determinant

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix}$$

of the operator "coefficients" of x and y in (2.11), provided that this determinant is not zero. We have assumed that this operator is of order N Thus in order for the pair (2.14) and (2.16) to satisfy the system (2.11) only N of the 2N constants in this pair can be independent. The remaining N constants must depend upon the N that are independent. In order to determine which of these 2N constants may be chosen as independent and how the remaining N then relate to the N so chosen, we must substitute x as given by (2.14) and y as given by (2.16) Into the system (2.11).

This determines the relations that must exist among the constants $c_1, c_2, \cdots, c_N, k_1, k_2, \cdots, k_N$ in order that the pair (2.14) and (2.16) constitute the so-called general solution of (2.11). Once this has been done, appropriate substitutions based on these relations are made in (2.14) and/or (2.16) and then the resulting pair (2.14) and (2.16) contain the required number N of arbitrary constants and so does indeed constitute the so-called general solution of system (2.11).

We now illustrate the above procedure with an example.

**Example 2.3.**

Solve the system

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$

$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2 \qquad (2.17)$$

We introduce operator notation and write this system in the form

$$(2D - 3)x - 2Dy = t,$$

$$(2D + 3)x + (2D + 8)y = 2. \qquad (2.18)$$

We apply the operator *(2D + 8)* to the first equation of (2.18) and the operator 2D to the second equation of (2.18), obtaining

$$(2D + 8)(2D - 3)x - (2D + 8)2Dy = (2D + 8)t,$$

$$2D(2D + 3)x + 2D(2D + 8)y = (2D)2.$$

Adding these two equations, we obtain

$$[(2D + 8)(2D - 3) + 2D(2D + 3)]x = (2D + 8)t + (2D)2$$

$$\text{Or } (8D^2 + 16D - 24)x = 2 + 8t + 0 \qquad (2.19)$$

or, finally $(D^2 + 2D - 3)x = t + \frac{1}{4}$.

The general solution of the differential equation (2.19) is

$$x = c_1e^t + c_2e^{-3t} - \frac{1}{3}t - \frac{1\ 1}{3\ 6}. \qquad (2.20)$$

We now return to the system (2.18) and apply the operator $(2D + 3)$ to the first equation of (2.18) and the operator $(2D — 3)$ to the second equation of (2.18). We obtain

$$(2D + 3)(2D - 3)x - (2D + 3)2Dy = (2D + 3)t,$$

$$(2D - 3)(2D + 3)x + (2D - 3)(2D + 8)y = (2D - 3)2.$$

Subtracting the first of these equations from the second, we have

$$[(2D - 3)(2D + 8) + (2D + 3)2D]y = (2D - 3)2 - (2D + 3)t$$

Or

$$(8D^2 + 16D—24)y = 0 - 6 - 2 - 3t \qquad (2.21)$$

Or, finally

$$(D^2 + 2D - 3)y = -\frac{3}{8}t - 1$$

The general solution of the differential equation (2.21) is

$$y = k_1 e^t + k_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12} \ .$$

Thus if x and y satisfy the system (2.17), then x must be of the form(2.20) and y must be of the form(2.22) for some choice of the constants $c_1, c_2, k_1, k_2$.The determinate of the operator "coefficients" of x and y in (2.18) is

$$\begin{vmatrix} 2D - 3 & -2D \\ 2D + 3 & 2D + 8 \end{vmatrix} = 8D^2 + 16D - 24.$$

Since this of order two, the number of independent contents in the general solution of the system (2.17) must also be two. Thus in order for the pair (2.20) and (2.22) to the satisfy the system (2.17) must also be two of the four constants $c_1, c_2, k_1$ and $k_2$ can be independent. In order to determine the necessary relations which must exist among these constants, we substitute $x$ as given by (2.20) and $y$ as given by (2.22) into the system (2.17). substituting into the first equation of (2.17), we have

$$\left[2c_1e^t - 6c_2e^{-3t} - \frac{2}{3}\right] - \left[2k_1e^t - 6k_2e^{-3t} + \frac{1}{4}\right] - \left[3c_1e^t + 3c_2e^{-3t} - t - \frac{11}{12}\right] = t$$

Or

$$(-c_1 - 2k_1)e^t + (-9c_2 + 6k_2)e^{-3t} = 0.$$

Thus in order that the pair (2.20) and (2.22) satisfy the first equation of the system (2.17) we must have

$$-c_1 - 2k_1 = 0,$$

$$-9c_2 + 2k_2 = 0. \qquad (2.23)$$

Substituting of $x$ and $y$ into the second equation of the system (2.17) will lead to relations equivalent to (2.23). Hence in order for the pair (2.20) and (2.22) to satisfy the system (2.17) , the relations (2.23) must be satisfied. Two of the four constants in (2.23) must be chosen as independent. If we chose $c_1$ and $c_2$ as independent, then we have

$$k_1 = -\frac{1}{2}c_1 \text{ and } k_2 = \frac{3}{2}c_2.$$

Using these values for $k_1$ and $k_2$ in (2.22), the resulting pair (2.20) and (2.22) constitute the general solution of the system (2.17). that is, the general solution of (2.17) is given by

$$x = c_1e^t + c_2e^{-3t} - \frac{1}{3}t - \frac{11}{36},$$

$$y = -\frac{1}{2}c_1e^t + \frac{3}{2}c_2e^{-3t} + \frac{1}{8}t + \frac{5}{12},$$

Where $c_1$ and $c_2$ are arbitrary constants. If we had chosen $k_1$ and $k_2$ as the independent constants in (2.23), then the general solution of the system (2.17) would have been written

$$x = -2k_1e^t + \frac{2}{3}k_2e^{-3t} - \frac{1}{3}t - \frac{11}{36},$$

$$y = k_1e^t + k_2e^{-3t} + \frac{1}{8}t + \frac{5}{12}.$$

**An Alternative Procedure**

Here we present an alternative procedure for solving a linear system of the form

$$L_1x + L_2y = f_1(t),$$

$$L_3x + L_4y = f_2(t), \qquad (2.11)$$

Where $L_1, L_2, L_3$ and $L_4$ are linear differential operators with constant coefficient this alternative procedure beings in exactly the same way as the procedure already described. That is, we first apply the operator $L_4$ to the first equation of (2.11) and the operator $L_2$ to the second equation of (2.11), obtaining

$$L_4L_1x + L_4L_2y = L_4f_1,$$

$$L_2L_3x + L_2L_4y = L_2f_2.$$

We next subtract the second from the first, obtaining

$$(L_1L_4 - L_2L_3)x = L_4f_1 - L_2f_2 \quad (2.12)$$

Which, under the same assumptions as we previously made at this point, may be written

$$L_5x = g_1. \quad (2.13)$$

Then we solve tis single linear differential equation with constant coefficients in the single dependent variable $x$. Assuming its order is , we obtain its general solution in the form

$$x = c_1u_1 + c_2u_2 + \cdots + c_Nu_N + U_1 \quad (2.14)$$

Where $u_1, u_2, \dots, u_N$ are $N$ linearly independent solutions of the homogenous linear equation $L_5x = 0, c_1, c_2, \dots, c_N$ are $N$ arbitrary constants, and $U_1$ is particular solution of $L_5x = g_1$.

Up to this point, we have indeed proceeded just exactly as before. However we now return to system (2.11) and attempt to eliminate from it all terms which involve the derivatives of the other dependent variable $y$. In other words, we attempt to obtain from system (2.11) a relation $R$ which involves the still unknown $y$ but none the derivatives of $y$. This relation $R$ will involve $x$ and/or certain of the derivatives of $x$; but $x$ is given by (2.14) and its derivatives can readily be found from (2.14). Finding these derivatives of $x$ and substituting them and the know $x$ itself the relation $R$, we see that the result is merely a single linear algebraic equation in the one unknown $y$. Solving it, we thus determine $y$ without the need to find (2.15) and (2.16) or to relate the arbitrary constants.

As we shall see, this alternative procedure always applies in an easy straight for wean manner if the operators $L_1, L_2, L_3$ and $L_4$ are all of the first order. However, for system involving one or more higher-order operators, it is generally difficult it eliminate all the derivatives of $y$.

We now give an explicit presentation of the procedure for finding $y$ when $L_1, L_2, L_3,$ and $L_4$ are all first-order operators.

Specifically, suppose

$$L_1 \equiv a_0 D + a_1,$$

$$L_2 \equiv b_0 D + b_1,$$

$$L_3 \equiv \alpha_0 D + \alpha_1,$$

$$L_4 \equiv \beta_0 D + \beta_1,$$

Then (2.11) is

$$(a_0 D + a_1)x + (b_0 D + b_1)y = f_1(t),$$

$$(\alpha_0 D + \alpha_1)x + (\beta_0 D + \beta_1)y = f_2(t)$$

Multiplying the first equation of (2.24) by $\beta_0$ and the second by $-b_0$ and adding, we obtain

$$[(a_0\beta_0 - b_0\alpha_0)D + (a_0\beta_0 - b_0\alpha_1)]x + (b_1\beta_0 - b_0\beta_1)y = \beta_0 f_1(t) - b_0 f_2(t)$$

Note that this involves $y$ but none of the derivatives of $y$. From this, we at once obtain

$$y = \frac{(b_0\alpha_0 - a_0\beta_0)Dx + (b_0\alpha_1 - a_1\beta_0)x + \beta_0 f_1(t) - b_0 f_2(t)}{b_1\beta_0 - b_0\beta_1}, \quad (2.25)$$

Assuming $b_1\beta_0 - b_0\beta_1 \neq 0$. Now $x$ is given by (2.14) and $Dx$ may be found from (2.14)by strightforword differentiation. Then substituting these known expressions for $x$ and $Dx$ into (2.25) , we at once obtain $y$ without the need of obtaining (2.15) and (2.16) and hence without having to determine any relations between constants $c_i$ and $k_i (i = 1,2, \dots, N)$, as in the original procedure.

We illustrate the alternative procedure by applying it to the system of Example 2.3.

**Example 2.4.**

Solve the system

$$2\frac{dx}{dt}-2\frac{dy}{dt}-3x=t,$$

$$2\frac{dx}{dt}+2\frac{dy}{dt}+3x+8y=2. \qquad (2.17)$$

Of example 2.3 by the alternative procedure which we have just described.

Following this alternative procedure, we introduce operator notation and write the system (2.17) in the form

$$(2D-3)x-2Dy=t,$$

$$(2D+3)x+(2D+8)y=2. \qquad (2.18)$$

Now we eliminate $y$, obtain the differential equation

$$D^2+2D-3)x=t+1/4 \qquad (2.19)$$

For $x$ and find its general solution

$$x=c_1e^t+c_2e^{-3t}-\frac{1}{3}t-\frac{11}{36} \qquad (2.20)$$

Exactly as in Example 2.3.

We now proceed using the alternative method. We first obtain from (2.18) a relation which involves the known $y$ but not derivative $Dy$. The system (2.18) of this example is so very simple that we do so by merely adding the equation (2.18). doing so, we at once obtain

$$4Dx+8y=t+2,$$

Which dose indeed involve $y$ but not the derivative $Dy$, as desired. From this, we at once find

$$y=1/8(t+2-4Dx). \qquad (2.26)$$

From (2.20), we find

$$Dx=c_1e^t-3c_2e^{-3t}-\frac{1}{3}.$$

Substituting into (2.26), we get

$$y = \frac{1}{8}\left(t + 2 - 4c_1e^t + 12c_2e^{-3t} + \frac{4}{3}\right)$$

$$= -\frac{1}{2}c_1e^t + \frac{3}{2}c_2e^{-3t} + \frac{1}{8}t + \frac{5}{12}.$$

Thus the general solution of the system may be written

$$x = c_1e^t + c_2e^{-3t} - \frac{1}{3}t - \frac{11}{36},$$

$$y = -\frac{1}{2}c_1e^t + \frac{3}{2}c_2e^{-3} + \frac{1}{8}t + \frac{5}{12},$$

Where $c_1$ and $c_2$ are arbitrary constants.

## 2.2 BASIC THEORY OF LINEAR SYSTEMS IN NORMAI FORM: TWO EQUATIONS IN TWO UNKNOWN FUNCTIONS

### A. Introduction:

We shall begin by considering a basic type of system of two linear differential equations in two unknown functions. This system is of the form,

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + F_1(t),$$
$$\qquad (2.27)$$
$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + F_2(t).$$

We shall assume that the functions$a_{11}$, $a_{12}, F_1$ , $a_{21}$, $a_{22}$, and $F_2$ are all continuous on a real interval a ≤ t ≤ b. If $F_1$(t) and $F_2$(t) are zero for all *t*, then the system (2.27) is called homogeneous; otherwise, the system is said to be non homogeneous.

**Example 2.6.**

The system

$$\frac{dx}{dt} = 2x - y,$$
$$\frac{dy}{dt} = 3x + 6y. \qquad (2.28)$$

Is homogeneous; the system

$$\frac{dx}{dt} = 2x - y - 5t,$$
$$\frac{dy}{dt} = 3x + 6y - 4. \qquad (2.29)$$

Is non homogeneous.

**DEFINITION:**

By a solution of the system (2.27) we shall mean an ordered pair of real functions (*f* , *g*), (2.30)

Each having a continuous derivative on the real interval a ≤ t ≤ b, such that

$$\frac{dF(t)}{dt} = a_{11}(t)F(t) + a_{12}(t)g(t) + F_1(t),$$

$$\frac{dg(t)}{dt} = a_{21}(t)F(t) + a_{22}(t)g(t) + F_2(t),$$

For all *t* such that a ≤ t ≤ b. In other words,

$$x = f\,(t),$$

$$y = g\,(t). \qquad (2.31)$$

Simultaneously satisfy both equations of the system (2.27) identically for

a ≤ t≤b.

Notation. W shall use the notation

$$x = f\,(t),$$

$$y = g\,(t). \qquad (2.31)$$

to denote a solution of the system (2.27) and shall speak of "the solution

$$x = f\,(t),$$

$$y = g\,(t).$$

Whenever we do this, we must remember that the solution thus referred to is really the ordered pair of functions ($f$ , $g$ ) such that (2.31) simultaneously satisfy both equations of the system (2.27) identically on a ≤ t ≤ b.

**Example 2.7.**

The ordered pair of functions defined for all *t* by ($e^{5t}$ , $-3e^{5t}$), which we denote by

$$x = e^{5t},$$

$$y = -3e^{5t}. \qquad (2.32)$$

is a solution of the system (2.24). That is,

$$x = e^{5t},$$

$$y = -3e^{5t}. \tag{2.32}$$

Simultaneously satisfy both equations of the system (2.28). Let us verify this by directly substituting (2.32) into (2.28). We have

$$\frac{d}{dt}(e^{5t}) = 2(e^{5t}) - (-3e^{5t}),$$

$$\frac{d}{dt}(-3e^{5t}) = 3(e^{5t}) + 6(-3e^{5t})$$

Or

$$5e^{5t} = 2e^{5t} + 3e^{5t}$$

$$-15e^{5t} = 3e^{5t} - 18e^{5t}$$

Hence (2.32) is indeed a solution of the system (2.28). Theorem 2.1 is the basic existence theorem dealing with the system (2.27).

**THEOREM (2.1).**

Hypothesis. Let the functions $a_{11}, a_{12}, F_1, a_{21}, a_{22}$ and $F_2$ in the system (2.27) all be continuous on the interval a ≤ t ≤ b. Let $t_0$ be any point of the interval a ≤ t ≤b and let $c_1$ and $c_2$ be two arbitrary constants.

Conclusion. There exists a unique solution

$$x = f(t)$$

$$y = g(t),$$

of the system (2.27) such that

$$f(t_0) = c_1 \text{ and } g(t_0) = c_2 ,$$

and this solution is defined on the entire interval a ≤ t ≤ b.

**Example 2.8.**

Let us consider the system (2.29). The continuity requirements of the hypothesis of Theorem (2.1) are satisfied on every closed interval a ≤ t ≤ b. Hence, given any point $t_0$ and any two constants $c_1$ and $c_2$, there exists a unique solution $x = f(t), y = g(t)$ of the system (2.29) that satisfies the conditions $f(t_0) = c_1, g(t_0) = c_2$. For example, there exists one and only one solution $x = f(t), y = g(t)$ such that $f(2) = 5, g(2) = -7$.

## B. Homogeneous Linear Systems

We shall now assume that $F_1(t)$ and $F_2(t)$ in the system (2.27) are both zero for all and consider the basic theory of the resulting homogeneous lineal system

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y \qquad (2.33)$$

We shall see that this theory is ana1oous to that of the single nth-order homogeneous linear differential equation presented in Section 4.1B[in book b1] Our first result concerning the system (2.33) is the following.

THEOREM (2.2)

Let

$$x = f_1(t) \quad , \quad x = f_2(t) \qquad (2.34)$$

$$y = g_1(t) \quad , \quad y = g_2(t)$$

Be two solutions of the homogeneous linear system (2.33). Let $c_1$ and $c_2$ be two arbitrary constants.

Then

$$x = c_1 f_1(t) + c_2 f_2(t)$$

$$y = c_1 g_1(t) + c_2 g_2(t) \qquad (2.35)$$

is also a solution of the system (2.33).

**DEFINITION:**

The solution (2.35) is called a linear combination of the solutions (2.34). This definition enables us to express Theorem 2.2 in the following alternative form.

THEOREM (2.2) RESTATED

Any linear combination of two solutions of the homogeneous linear system (2.33) is itself a solution of the system (2.33).

**Example 2.9.**

We have already observed that

$$x = e^{5t} \qquad x = e^{3t}$$

and

$$y = -3e^{5t} \qquad y = -e^{3t}$$

Are solutions of the homogeneous linear system (2.28). Theorem 2.2 tells us that

$$x = c_1 e^{5t} + c_2 e^{3t}$$

$$y = -3c_1 e^{5t} - c_2 e^{3t}$$

Where $c_1$ and $c_2$ are arbitrary constants, is also a solution of the system (2.28). For example, if $c_1 = 4$ and $c_2 = -2$ we have the solution

$$x = 4e^{5t} - 2e^{3t},$$

$$y = -12e^{5t} + 2e^{3t}.$$

**DEFINITION:**

Let

$$x = f_1(t) \qquad x = f_2(t)$$

and

$$y = g_1(t) \qquad y = g_2(t)$$

Be two solutions of the homogeneous linear system (2.33). These two solutions are linearly dependent on the interval a ≤ t ≤ b if there exist constants $c_1$ and $c_2$, not both zero, such that

$$c_1 f_1(t) + c_2 f_2(t) = 0$$

$$c_1 g_1(t) + c_2 g_2(t) = 0 \qquad (2.37)$$

For all t such that $a \le t \le b$.

**DEFINITION:**

Let

$$x = f_1(t) \qquad\qquad x = f_2(t)$$

and

$$y = g_1(t) \qquad\qquad y = g_2(t)$$

Be two solutions of the homogeneous linear system (2.33). These two solutions are linearly independent on a ≤ t ≤ b if they are not linearly dependent on a ≤ t ≤b. That is, the solutions $x = f_1(t)$, $y = g_1(t)$ and $x = f_2(t)$, $y = g_2(t)$ are linearly independent on a ≤ t ≤ b

$$c_1 f_1(t) + c_2 f_2(t) = 0$$

$$c_1 g_1(t) + c_2 g_2(t) = 0 \qquad (2.37)$$

For all *t* such that a ≤ t ≤ b implies that

$$c_1 = c_2 = 0.$$

**Example 2.10.**

The solutions

$$x = e^{5t} \qquad\qquad x = 2e^{5t}$$

and

$$y = -3e^{5t} \qquad\qquad y = -6e^{5t}$$

Of the system (2.28) are linearly dependent on every interval a ≤ t ≤ b. For in this case the conditions (2.36) become

$$c_1 e^{5t} + 2c_2 e^{5t}$$

$$-3c_1 e^{5t} - 6\, c_2 e^{5t} = 0 \qquad (2.38)$$

And dearly there exist constants $c_1$ and $c_2$, not both zero, such that the conditions (2.38) hold on a ≤ t ≤ b. For example, let $c_1 = 2$ and $c_2 = -1$.

On the other hand, the solutions

$$x = e^{5t} \qquad\qquad x = e^{3t}$$

and

$$y = -3e^{5t} \qquad\qquad y = e^{3t}$$

Of system (2.28) are linearly independent on a $\leq$ t $\leq$ b. For in this case the conditions (2.37) are

$$c_1 e^{5t} + c_2 e^{3t} = 0$$

$$-3c_1 e^{5t} - c_2 e^{3t} = 0.$$

If these conditions hold for all $t$ such that a $\leq$ t $\leq$ b, then we must have $c_1 = c_2 = 0$.

We now state the following basic theorem concerning sets of linearly independent solutions of the homogeneous linear system (2.33).

**THEOREM (2.3).**

There exist sets of two linearly independent solutions of the homogeneous linear system (2.33).Every solution of the system (2.33) can be written as a linear combination of any two linearly independent solutions of (2.33).

**Example 2.11.**

We have seen that

$$x = e^{5t} \qquad\qquad x = e^{3t}$$

and

$$y = -3e^{5t} \qquad\qquad y = -e^{3t}$$

Constitute a pair of linearly independent solutions of the system (2.28). This illustrates the first part of Theorem (2.3). The second part of the theorem tells us that every solution of the system (2.28) can be written in the form

$$x = c_1 e^{5t} + c_2 e^{3t}$$

$$y = -3c_1 e^{5t} - c_2 e^{3t}$$

Where $c_1$ and $c_2$ are suitably chosen constants.

We now give an analogous definition of general solution for the homogeneous linear system (2.33).

**DEFINITION:**

Let

$$x = f_1(t) \qquad\qquad x = f_2(t)$$

amd

$$y = g_1(t) \qquad\qquad y = g_2(t)$$

Be two linearly independent solutions of the homogeneous linear system (2.33). Let $c_1$ and $c_2$ be two arbitrary constants. Then the solution

$$x = c_1 f_1(t) + c_2 f_2(t)$$

$$y = c_1 g_1(t) + c_2 g_2(t)$$

Is called a general solution of the system (2.33).

**Example 2.12.**

Since

$$x = e^{5t} \qquad\qquad x = e^{3t}$$

and

$$y = -3e^{5t} \qquad\qquad y = -e^{3t}$$

Are linearly independent solutions of the system (2.28), we may write the general solution of (2.28) in the form

$$x = c_1 e^{5t} + c_2 e^{3t}$$

$$y = -3c_1 e^{5t} - c_2 e^{3t}$$

Where $c_1$, and $c_2$ are arbitrary constants.

**THEOREM ( 2.4).**

Two solutions

$$x = f_1(t) \ , x = f_2(t)$$

and

$$y = g_1(t) \ , \ y = g_2(t)$$

Be two solutions of the homogeneous linear system (2.33). A necessary and sufficient condition that these two solutions be linearly independent on a ≤ t≤b.

Is that is determinant

$$\Delta(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix}$$

be different from zero for all *t* such that a ≤ $t$ ≤ b.

Concerning this determinant, we also state the following result.

**THEOREM (2.5).**

The determinant $\Delta(t)$ of theorem (2.4) either is identically zero or vanishes for no *t* on the interval a ≤ t ≤ b.

**Example 2.13.**

Let us employ Theorem 2.4 to verify the linear independence of the solutions

$$x = e^{5t} \qquad\qquad x = e^{3t}$$

and

$$y = -3e^{5t} \qquad\qquad y = -e^{3t}$$

of the system (2.28). We have

$$\Delta(t) = \begin{vmatrix} e^{5t} & e^{3t} \\ -3e^{5t} & -e^{3t} \end{vmatrix} = 2e^{8t} \neq 0.$$

on every closed Interval a ≤ t ≤ b. Thus by Theorem 2.4 the two solutions are indeed linearly independent on a ≤ t ≤ b.

**C. Nonhomogeneous linear Systems**

Let us now return briefly to the nonhomogeneous system (2.27). A theorem and a definition, illustrated by a simple example, will suffice for our purposes here.

**THEOREM (2.6).**

Let

$$x = f_0(t)$$

$$y = g_0(t)$$

be any solution of the nonhomogeneous system (2.27), and let

$$x = f(t),$$

$$y = g(t),$$

be any solution of the corresponding homogeneous system (2.33)

Then

$$x = f(t) + f_0(t)$$

$$y = g(t) + g_0(t)$$

is also a solution of the nonhomogeneous system (2.27).

**DEFINITION:**

Let

$$x = f_0(t)$$

$$y = g_0(t)$$

be any solution of the nonhomogeneous system (2.27), and let

$$x = f_1(t) \ , \ x = f_2(t)$$

and

$$y = g_1(t) \ , \ y = g_2(t)$$

Be two linearly independent solutions of the corresponding homogeneous system (2.33), let $c_1$ and $c_2$ be two arbitrary constants. Then the solution

$$x = c_1 f_1(t) + c_2 f_2(t) + f_0(t)$$

$$y = c_1 g_1(t) + c_2 g_2(t) + g_0(t)$$

will be called a general solution of the nonhomogeneous system (2.27).

**Example 2.14.**

We have the system (2.29),and

$$x = 2t + 1$$

$$y = -t$$

is a solution of the nonhomogeneous system (2.29). The corresponding homogeneous system is the system (2.28), and we have already seen that

$$x = e^{5t} \quad , \quad x = e^{3t}$$

and

$$y = -3e^{5t} \quad , \quad y = -e^{3t}$$

are Linearly independent solutions of this homogeneous system. Theorem 2.6 tells us for example, that

$$x = e^{5t} + 2t + 1$$

$$y = -3e^{5t} - t$$

Is a solution of the nonhomogeneous system (2.29). From the preceding definition we see that the general solution of (2.29) may be written in the form

$$x = c_1 e^{5t} + c_2 e^{3t} + 2t + 1$$

$$y = -3c_1 e^{5t} - c_2 e^{3t} - t$$

Where $c_1$ and $c_2$ are arbitrary constants.

## 2.3 HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS: TWO EQUATIONS IN TWO UNKNOWN FUNCTIONS:

### A. Introduction

In this section we shall be concerned with the homogeneous linear system

$$\frac{dx}{dt} = a_1 x + b_1 y ,$$

$$\frac{dy}{dt} = a_2 x + b_2 y \qquad (2.40)$$

Where the coefficients$a_1$, $b_1$, $a_2$, $b_2$ are real constants. We seek solutions of this system.Remembering the analogy that exists between linear systems and single higher-order linear equations, we might now attempt to find exponential solutions of the system (2.40). Let us therefore attempt to determine a solution of the form

$$x = Ae^{\lambda t}$$

$$y = Be^{\lambda t} \qquad (2.41)$$

where A, B, and $\lambda$ are constants. If we substitute (2.41) into (2.40), we obtain

$$A\lambda e^{\lambda t} = a_1 Ae^{\lambda t} + b_1 Be^{\lambda t}$$

$$B\lambda e^{\lambda t} = a_2 Ae^{\lambda t} + b_2 Be^{\lambda t}$$

These equations lead at once to the system

$$(a_1 - \lambda)A + b_1 B = 0$$

$$a_2 A + (b_2 - \lambda)B = 0 \qquad (2.42)$$

in the unknowns A and B. This system obviously has the trivial solution A = B = 0.

But this would only lead to the trivial solution x = 0, y = 0 of the system (2.40). Thus we seek nontrivial solutions of the system (2.42). A necessary and sufficient condition that this system have a nontrivial solution that the determinant

$$\begin{vmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{vmatrix} = 0 \quad (2.43)$$

Expanding this determinant we are led at once to the quadratic equation

$$\lambda^2 - (a_1 + b_2)\,\lambda + (a_1 b_2 - a_2 b_1) = 0 \qquad (2.44)$$

In the unknown $\lambda$. This equation is called the characteristic equation associated with the system (2.40). Its roots $\lambda_1$ and $\lambda_2$ are called the characteristic roots. If the part (2.41) is to be a solution of the system (2.40), then λ in (2.41) must be one of these roots. Suppose

$\lambda = \lambda_1$. Then substituting $\lambda = \lambda_1$ into the algebraic system (2.42), we may obtain a nontrivial solution $A_1$, $B_1$ of this algebraic system. With these values $A_1$, $B_1$ we obtain the nontrivial solution

$$x = A_1 e^{\lambda t}$$

$$y = B_1 e^{\lambda t}$$

of the given system (2.40).

Three cases must now be considered:

1. The roots $\lambda_1$ and $\lambda_2$ are real and distinct.

2. The roots $\lambda_1$ and $\lambda_2$ are real and equal.

3. The roots $\lambda_1$ and $\lambda_2$ are conjugate complex.

**B. Case 1. The Roots of the Characteristic Equations (2.44) are Real and Distinct**

If the roots $\lambda_1$ and $\lambda_2$ of the characteristic equation (2.44) are real and distinct, it appears that we should expect two distinct solutions of the form (2.41), one corresponding to each of the two distinct roots. This is indeed the case. Furthermore, these two distinct solutions are linearly independent. We summarize this case in the following theorem.

**THEOREM 2.7.**

Let the roots $\lambda_1$ and $\lambda_2$ , of the characteristic equation (2.44) associated with the system (2.40) are real and distinct.

Then the system (2.40) has two nontrivial linearly independent solutions of the form

$$x = A_1 e^{\lambda_1 t} \qquad\qquad x = A_2 e^{\lambda_2 t}$$

and

$$y = B_1 e^{\lambda_1 t} \qquad\qquad y = B_2 e^{\lambda_2 t}$$

where $A_1, B_1, A_2$ and $B_2$ are definite constants. The general solution of the system (2.40) may thus be written

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t}$$

$$y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}$$

Where $c_1$ and $c_2$ are arbitrary constants.

**Example 2.15.**

$$\frac{dx}{dt} = 6x - 3y$$

$$\frac{dy}{dt} = 2x + y \qquad (2.45)$$

We assume a solution of the form (2.41):

$$x = Ae^{\lambda t}$$

$$y = Be^{\lambda t} \qquad (2.46)$$

Substituting (2.46) into (2.45) we obtain

$$A\lambda e^{\lambda t} = 6Ae^{\lambda t} - 3Be^{\lambda t}$$

$$B\lambda e^{\lambda t} = 2Ae^{\lambda t} + Be^{\lambda t}$$

and this leads at once to the algebraic system

$$(6 - \lambda)A - 3B = 0$$

$$2A + (1 - \lambda)B = 0 \qquad (2.47)$$

in the unknown $\lambda$. For nontrivial solutions of this system we must have

$$\begin{vmatrix} 6 - \lambda & -3 \\ 2 & 1 - \lambda \end{vmatrix} = 0$$

Expanding this we obtain the characteristic equation

$$\lambda^2 - 7\lambda + 12 = 0$$

Solving this, we find the roots $\lambda_1 = 3$, $\lambda_2 = 4$.

Setting $\lambda_1 = \lambda_2 = 3$ in (2.47), we obtain

$$3A - 3B = 0,$$

$$2A - 2B = 0.$$

A simple nontrivial solution of this system is obviously A = B = 1. With these values of A, B, and $\lambda$ we find the nontrivial solution

$$x = e^{3t}$$

$$y = e^{3t} \tag{2.48}$$

Now setting $\lambda = 4$ in (2.47), we find

$$2A - 3B = 0,$$

$$2A - 3B = 0.$$

A simple nontrivial solution of this system is A = 3, B = 2. Using these values of A, B and $\lambda$ we find the nontrivial solution

$$x = 3e^{4t}, \qquad ,$$

$$y = 2e^{4t}. \tag{2.49}$$

By Theorem 2.7 the solutions (2.48) and (2.49) are linearly independent (one may check this using Theorem 2.4) and the general solution of the system (2.45) may be written

$$x = c_1 e^{3t} + 3e^{4t},$$

$$y = c_1 e^{3t} + 2e^{4t}$$

Where $c_1$and $c_2$are arbitrary constants.

**C. Case 2 The Roots of the Characteristic Equation (2.44) are Real and Equal**

if the two roots of the characteristic equation (2.44) are real and equal, it would appear that we could find only one solution of the form (2.41). Except in the special sub case in which $a_1 = b_2 \neq 0, a_2 = b_1 = 0$. this is indeed true. In general, how shall we then proceed to find a second, linearly independent solution? Recall the analogous situation in which the auxiliary equation corresponding to a single nth-order linear equation has a double root. This would lead us to expect a second solution of the form

$$x = Ate^{\lambda t}$$

$$y = Bte^{\lambda t}$$

However, the situation here is not quite so simple .We must actually seek a second solution of the form

$$x = (A_1 t + A_2)e^{\lambda t}$$

$$y = (B_1 t + B_2)e^{\lambda t}. \qquad (2.50)$$

We shall illustrate this in Example 2.17. We first summarize Case 2 in the following theorem.

**THEOREM 2.9.**

If the roots $\lambda_1$ and $\lambda_2$ of the characteristic equation (2.44) associated with the system (2.40) are real and equal. Let $\lambda$ denote their common value. Further assume that system (2.40) is not such that $a_1 = b_2 \neq 0, a_2 = b_1 = 0$.

Then the system (2.40) has two linearly independent solutions of the form

$x = Ae^{\lambda t}$ $\qquad$ $x = (A_1 t + A_2)e^{\lambda t}$

and

$y = Bte^{\lambda t}$ $\qquad$ $y = (B_1 t + B_2)e^{\lambda t}$

where $, B, A_1$ , $B_1$ , $A_2$ , and $B_2$ are definite constants,$A_1$ and $B_1$ are not both zero, and ${}^{B_1}/_{A_1} = {}^{B}/_{A}$. The general solution of the system (2.40) may thus be written

$$x = c_1 A t e^{\lambda t} + c_2 (A_1 t + A_2) e^{\lambda t}$$

$$y = c_1 B t e^{\lambda t} + c_2 (B_1 t + B_2) e^{\lambda t}$$

**Example 2.16.**

$$\frac{dx}{dt} = 4x - y$$

$$\frac{dy}{dt} = x + 2y \qquad (2.51)$$

We assume a solution of the form (2.41):

$$x = Ae^{\lambda t},$$

$$y = Be^{\lambda t}. \qquad (2.52)$$

Substituting (2.52) into (2.51) we obtain

$$A\lambda e^{\lambda t} = 4Ae^{\lambda t} - Be^{\lambda t},$$

$$B\lambda e^{\lambda t} = Ae^{\lambda t} + 2Be^{\lambda t}.$$

And this leads at once to the algebraic system

$$(4 - \lambda)A - B = 0,$$

$$A + (2 - \lambda)B = 0. \qquad (2.53)$$

In the unknown $\lambda$. For nontrivial solutions of this system we must have

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

Expanding this we obtain the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0,$$

$$(\lambda - 3)^2 = 0. \qquad (2.54)$$

thus the characteristic equation (2.54) has the real and equal roots 3, 3. Setting $\lambda$ = 3 in (2.53), we obtain

$$A - B = 0,$$

$$A - B = 0.$$

A simple nontrivial solution of this system being $A = B = 1$, we obtain the nontrivial solution

$$x = e^{3t}$$

$$y = e^{3t}$$

of the given system (2.50).

Since the roots of the characteristic equation are both equal to 3, we must seek a second solution of the form (2.50), with$\lambda$ = 3. That is, we must determine $A_1$, $A_2$, $B_1$, and $B_2$ (with $A_1$ and $B_1$ not both zero) such that

$$x = (A_1 t + A_2)e^{3t},$$

$$y = (B_1 t + B_2)e^{3t}. \qquad (2.56)$$

is a solution of the system (2.51). Substituting (2.56) into (2.51), we obtain

$$(3A_1 t + 3A_2 + A_1)e^{3t} = 4(A_1 t + A_2)e^{3t} - (B_1 t + B_2)e^{3t}$$

$$(3B_1 t + 3B_2 + B_1)e^{3t} = (A_1 t + A_2)e^{3t} + 2(B_1 t + B_2)e^{3t}$$

These equations reduce at once to,

$$(A_1 - B_1)t + (A_2 - A_1 - B_2) = 0$$

$$(A_1 - B_1)t + (A_2 - B_1 - B_2) = 0$$

In order for these equations to be identities, we must have

$$(A_1 - B_1) = 0, \qquad A_2 - A_1 - B_2 = 0$$

$$(A_1 - B_1) = 0, \qquad A_2 - B_1 - B_2 = 0 \qquad (2.57)$$

Thus in order for (2.56) to be a solution of the system (2.51), the constants $A_1$, $A_2$,

$B_1$, and $B_2$ must be chosen to satisfy the equations (2.57). From the equations $A_1 - B_1 = 0$, we see that $A_1 = B_1$. The other two equations of (2.57) show that $A_2$ and $B_2$ must satisfy

$$A_2 - B_2 = A_1 = B_1. \qquad (2.58)$$

we may choose any convenient nonzero values for $A_1$ and $B_1$. We choose $A_1 = B_1 = 1$ . Then (2.58) reduces to $A_2 - B_2 = 1$, and we can choose any convenient values for $A_2$ and $B_2$ that will satisfy this equation. We choose $A_2 = 1$, $B_2 = 0$. We are thus led to the solution

$$x = (t + 1)e^{3t},$$

$$y = te^{3t}. \qquad (2.59)$$

By Theorem 2.9 the solutions (2.55) and (2.59) are linearly independent. We may thus write the general solution of the system (2.51) in the form

$$x = c_1 e^{3t} + c_2(t + 1)e^{3t},$$

$$y = c_1 e^{3t} + c_2 te^{3t}.$$

where $c_1$ and $c_2$ are arbitrary constants.

**D. Case 3. The Roots of the Characteristic Equation (2.44)are Conjugate Complex**

If the roots $\lambda_1$ and $\lambda_2$ of the characteristic equation (2.44) are the conjugate complex numbers a + bi and a - bi, then we still obtain two distinct solutions

$$x = A^*{}_1\, e^{(a+bi)t} \quad , \quad x = A^*{}_2\, e^{(a-bi)t}$$

$$y = B^*{}_1\, e^{(a+bi)t} \quad , \quad y = B^*{}_2\, e^{(a-bi)t} \qquad (2.60)$$

Of the form (2.41), one is corresponding to each of the complex roots. However, the solutions (2.60) are complex solutions. In order to obtain real solutions in this case we consider the first of the two solutions (2.60) and proceed as follows: We first express the complex constants $A^*{}_1$ and $B^*{}_1$ in this solution in the forms $A^*{}_1 = A_1$ + i$A_2$ and $B^*{}_1$ = $B_1$ + i$B_2$ , where$A_1$, $A_2, B_1$ *and* $B_2$ are real. We then apply Euler's formula $e^{i\theta}$ = $\cos\theta$ + $i\sin\theta$ and express the first solution (2.60) in the form

$$x = (A_1 + iA_2)\, e^{at}(\cos bt + i\sin bt).$$

$$y = (B_1 + iB_2)\, e^{at}(\cos bt + i\sin bt)$$

Rewriting this, we have

$$x = e^{at}[(A_1 \cos bt - A_2 \sin bt) + i(A_2 \cos bt + A_1 \sin bt)],$$

$$y = e^{at}[(B_1 \cos bt - B_2 \sin bt) + i(B_2 \cos bt + B_1 \sin bt)]. \qquad (2.59)$$

It can be shown that a pair [$f_1$(t) + i$f_2$(t), $g_1$(t) + i$g_2$(t)] of complex functions is a solution of the system (2.40) if and only if both the pair [$f_1$(t), $g_1$(t)] consisting of their real parts and the pair [$f_2$(t), $g_2$(t)] consisting of their imaginary parts are solutions of (2.40). Thus both the real part

$$x = e^{at}(A_1 \cos bt - A_2 \sin bt)$$

$$y = e^{at}(B_1 \cos bt - B_2 \sin bt). \qquad (2.62)$$

And the imaginary part

$$x = e^{at}(A_2 \cos bt + A_1 \sin bt),$$

$$y = e^{at}(B_2 \cos bt + B_1 \sin bt). \qquad (2.63)$$

of the solution (2.61) of the system (2.40) are also solutions of (2.40). Furthermore, the solutions (2.62) and (2.63) are linearly independent. We verify this by evaluating the determinant (2.39) for these solutions. We find

$$\Delta(t) = \begin{vmatrix} e^{at}(A_1 \cos bt - A_2 \sin bt) & e^{at}(A_2 \cos bt + A_1 \sin bt) \\ e^{at}(B_1 \cos bt - B_2 \sin bt) & e^{at}(B_2 \cos bt + B_1 \sin bt) \end{vmatrix} \qquad (2.64)$$

$$= e^{2at}\,(A_1 B_1 - A_2 B_2)$$

Now, the constant ${B^*}_1$ is a normal multiple of the constant ${A^*}_1$. If we assume that

$A_1 B_2 - A_2 B_1$ = 0, then it follows that ${B^*}_1$ is a real multiple of ${A^*}_1$. , which contradicts the result stated in the previous sentence. Thus $A_1 B_2 - A_2 B_1 \neq 0$ and ∵ determinant ∆t in (2.64) is unequal to zero. Thus by Theorem 2.4 the solutions (2.62) and (2.63) are indeed linearly independent. Hence a linear combinationof these two real solutions provides the general solution of the system (2.40) in this case. There is no need to consider the second of the two solutions (2.60). We summarize the above results in the following theorem;

**THEOREM 2.9.**

If the roots $\lambda_1$ and $\lambda_2$ of the characteristic equation (2.44) associated with the system (2.40) are the conjugate complex numbers a ± bi.

Then the system (2.40) has two real linearly independent solutions of the form

$$x = e^{at}(A_1 \cos bt - A_2 \sin bt) \ , \quad x = e^{at}(A_2 \cos bt + A_1 \sin bt)$$

And

$$y = e^{at}(B_1 \cos bt - B_2 \sin bt)\ , \qquad y = e^{at}(B_2 \cos bt + B_1 \sin bt)$$

where $A_1$, $A_2, B_1$ $and$ $B_2$ are definite real constants. The general solution of the system (2.40) may thus be written

$$x = e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_2 \cos bt + A_1 \sin bt)]$$

$$y = e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_2 \cos bt + B_1 \sin bt)]$$

where $c_1$ and $c_2$ are arbitrary constanis.

**Example 2.17.**

$$\frac{dx}{dt} = 3x + 2y$$

$$\frac{dy}{dt} = -5x + y \qquad (2.65)$$

We assume a solution of the form (2.41)

$$x = Ae^{\lambda t},$$

$$y = Be^{\lambda t}. \qquad (2.66)$$

Substituting (2.66) into (2.65) we obtain

$$A\lambda e^{\lambda t} = 3Ae^{\lambda t} + 2Be^{\lambda t}$$

$$B\lambda e^{\lambda t} = -5Ae^{\lambda t} + Be^{\lambda t}$$

and this leads at once to the algebraic system

$$(3 - \lambda)A + 2B = 0$$

$$-5A + (1 - \lambda)B = 0 \qquad (2.67)$$

in the unknown $\lambda$.For nontrivial solutions of this system we must have

$$\begin{vmatrix} (3 - \lambda) & 2 \\ -5 & (1 - \lambda) \end{vmatrix} = 0$$

Expanding this, we obtain the characteristic equation

$$\lambda^2 - 4\lambda + 13 = 0.$$

The roots of this equation are the conjugate complex numbers $2 \pm 3i$. Setting $\lambda = 2 + 3i$ in (2.67), we obtain

$$(1-3i)A+2B=0,$$

$$-5A+(-1-3i)B=0.$$

A simple nontrivial solution of this system is $A=2, B=-1+3i$. Using these values we obtain the complex solution

$$x=2e^{(2+3i)t},$$

$$y=(-1+3i)e^{(2+3i)t}.$$

of the given system (2.65). Using Euler's formula this takes the form

$$x=e^{2t}[(2\cos 3t)+i(2\sin 3t)]$$

$$y=e^{2t}[(-\cos 3t-3\sin 3t)+i(3\cos 3t-\sin 3t)].$$

Since both the real and imaginary parts of this solution of system (2.65) are themselves solutions of (2.65), we thus obtain the two real solutions

$$x=2e^{2t}\cos 3t,$$

$$y=-e^{2t}(\cos 3t+3\sin 3t), \qquad (2.68)$$

and

$$x=2e^{2t}\sin 3t,$$

$$y=e^{2t}(3\cos 3t-\sin 3t). \qquad (2.69)$$

Finally, since the two solutions (2.68) and (2.69) are linearly independent we may write the general solution of the system (2.63) in the form

$$x=2e^{2t}[c_1\cos 3t+c_2\sin 3t],$$

$$y=e^{2t}[c_1(-\cos 3t-3\sin 3t)+c_2(3\cos 3t-\sin 3t)].$$

where $c_1$ and $c_2$ are arbitrary constants.

## 2.4 BASIC THEORY OF LINEAR SYSTEMS IN NORMAL FORM: N EQUATIONS IN N UNKNOWN FUNCTIONS.

### A. Introduction

We consider the normal form of the system of n first-order differential equation in n unknown functions $x_1, x_2, \dots, x_n$. As noted in section 2.1 A[in book No.1], this system is of the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + F_1(t),$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + F_2(t), \qquad (2.70)$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + F_n(t).$$

We shall assume that all of the functions defined by $a_{ij}(t), i = 1,2, \dots, n, j = 1,2, \dots, n$ and $F_i(t), i = 1,2, \dots, n$. are continuous on a real interval $a \leq t \leq b$. If all $F_i(t) = 0, i = 1,2, \dots, n$ for all t, then the system(2.70) is called homogeneous. Otherwise, the system is called nonhomogeneous.

**Example 2.18.**

The system

$$\frac{dx_1}{dt} = 7x_1 - x_2 + 6x_3,$$

$$\frac{dx_2}{dt} = -10x_1 + 4x_2 - 12x_3,$$

$$\frac{dx_3}{dt} = -2x_1 + x_2 - x_3 .$$

Is a homogeneous linear system of the type (2.70)with n=3 and having constant coefficient. The system

$$\frac{dx_1}{dt} = 7x_1 - x_2 + 6x_3 - 5t - 6,$$

$$\frac{dx_2}{dt} = -10x_1 + 4x_2 - 12x_3 - 4t + 23,$$

$$\frac{dx_3}{dt} = -2x_1 + x_2 - x_3 + 2.$$

Is a nonhomogeneous linear system of the type (2.70) which n=3, the homogeneous terms being $-5t - 6, -4t + 23$ and 2, respectively we note the system (2.70)can be written more compactly as $\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_i + F_i(t)\,, (i = 1,2, \dots, n)$.

We shall now proceed to express the system in an even more compact manner using vectors and matrices. We introduce the matrix A defined by

$$A(t) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad (2.73)$$

And the vectors F and x defined respectively by

$$F(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix} \qquad and \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad (2.74)$$

Then first by definition of the derivative of vector, and second by multiplication of matrix by a vector followed by addition of vectors, we have respectively

$$\frac{dx}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

And

$$A(t)x + F(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + F_1(t) \\ a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + F_2(t) \\ a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + F_n(t) \end{pmatrix}$$

Comparing the components of $\frac{dx}{dt}$ with the left members of (2.73), we see that system (2.70) can be expressed as the linear vector differential equation

$$\frac{dx}{dt} = A(t)x + F(t) \qquad (2.74)$$

conversely, if A(t) is given by (2.73) and F(t) and x are given by (2.74), then we see that the vector differential equation(2.74) can be expressed as the system (2.73).

Thus, the system (2.73) and the vector differential equation (2.74) both express the same relations and so are equivalent to one another. We refer to (2.74) as the vector differential equation corresponding to the system (2.73), and we shall sometimes call the system (2.73) the sealar from the vector differential equation (2.74). hence forth throughout this section, we shall usually write the system (2.73) as the corresponding vector differential equation(2.74).

**Example 2.19.**

The vector differential equation corresponding to the nonhomogeneous system(2.74) of example 2.26 is

$$\frac{dx}{dt} = A(t)x + F(t)$$

$$A(t) = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, and\ F(t) = \begin{pmatrix} -5t + 6 \\ -4t + 23 \\ 2 \end{pmatrix}$$

Thus we can write this vector deferential equation as

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} -5t + 6 \\ -4t + 23 \\ 2 \end{pmatrix}$$

Where x is the vector with components $x_1, x_2, x_3$ as given above

**DIFINITION:**

By a solution of the vector deferential equation (2.74)we mean as nx1 column vector function

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \qquad (2.73)$$

Whose components $\phi_1, \phi_2, \dots, \phi_n$ each have a continuous derivative on the real interval $a \leq t \leq b$, which is such that

$$\frac{d\phi(t)}{dt} = A(t)\phi(t) + F(t) \qquad (2.74)$$

For all t such that $a \leq t \leq b$. In other words, $x = \phi(t)$ satisfies the vector deferential equation (2.72) identically on $a \leq t \leq b$. That is, the components $\phi_1, \phi_2, \dots, \phi_n$ of $\phi$ are such that

$$\begin{aligned} x_1 &= \phi_1(t), \\ x_2 &= \phi_2(t), \\ &\vdots \\ x_n &= \phi_n(t). \end{aligned} \qquad (2.75)$$

Simultaneously satisfy all n equation of the scalar form (2.70) of the vector differential equation (2.72) for $a \leq t \leq b$. Hence we say that a solution of the system (2.70) is an order of n real function $\phi_1, \phi_2, \dots, \phi_n$, each having continuous on $a \leq t \leq b$, such that

$$\begin{aligned} x_1 &= \phi_1(t), \\ x_2 &= \phi_2(t), \\ &\vdots \\ x_n &= \phi_n(t). \end{aligned} \qquad (2.75)$$

Simultaneously satisfy all n equation of the system (2.70) for $a \leq t \leq b$.

**Example 2.20.**

The vector differential equation corresponding to the homogeneous linear system

$$\frac{dx_1}{dt} = 7x_1 - x_2 + 6x_3,$$
$$\frac{dx_2}{dt} = -10x_1 + 4x_2 - 12x_3, \qquad (2.68)$$
$$\frac{dx_3}{dt} = -2x_1 + x_2 - x_3.$$

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (2.76)$$

The column vector function $\phi$ is defined by

$$\phi(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}$$

Is a solution of the vector differential equation (2.76) on every real interval $a \leq t \leq b$, for $x = \phi(t)$ satisfies (2.76) identically on $a \leq t \leq b$, that is

$$\begin{pmatrix} 3e^{3t} \\ -6e^{3t} \\ -3e^{3t} \end{pmatrix} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}$$

$$x_1 = e^{3t}$$
$$x_2 = -2e^{3t} \qquad (2.77)$$
$$x_3 = -e^{3t}$$

Simultaneously satisfy all three equation of the system (2.71) for $a \leq t \leq b$, and so we call (2.77) a solution of the system.

**THEOREM (2.10).**

Consider the vector differential equation

$$\frac{dx}{dt} = A(t)x + F(t) \qquad (2.72)$$

Corresponding to the linear system (2.70) of n equation in n unknown functions. Let the components $a_{ij}(t)$, $i = 1,2, \dots, n$, $j = 1,2, \dots, n$, of the matrix A(t) and the components $F_i(t)$, $i = 1,2, \dots, n$, of the vector F(t) all be continuous on the real interval $a \leq t \leq b$. Let to be any point of the interval $a \leq t \leq b$, and let

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Be an nx1 column vector of any n numbers $c_1, c_2, \dots, c_n$. Then there exists a unique solution

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}$$

Of the vector differential equation (2.72) such that

$$\phi(t_0) = c; \qquad (2.78)$$

That is

$$\begin{aligned} \phi_1(t_0) &= c_1, \\ \phi_2(t_0) &= c_2, \\ &\vdots \\ \phi_n(t_0) &= c_n. \end{aligned} \qquad (2.79)$$

And this solution is defined on the entire interval $a \leq t \leq b$. Interpreting this theorem in terms of the scalar form of the vector differential equation (2.72), that is, the system (2.70), we state the following : under the stated continuity hypotheses on the functions $a_{ij}$ and $F_i$, given any point to in the interval $a \leq t \leq b$ and any n numbers $c_1, c_2, \dots, c_n$, then there exist a unique solution

$$\begin{aligned} x_1 &= \phi_1(t), \\ x_2 &= \phi_2(t), \\ &\vdots \\ x_n &= \phi_n(t). \end{aligned}$$

Such that

$$\begin{aligned} \phi_1(t_0) &= c_1, \\ \phi_2(t_0) &= c_2, \\ &\vdots \\ \phi_n(t_0) &= c_n. \end{aligned} \qquad (2.79)$$

And this solution is defined for all t such that $a \leq t \leq b$.

## B.HOMOGENEOUS LINEAR SYSTEM

We now assume that all $F_i(t) = 0$, $i = 1,2,\dots,n$, for all t in the linear system (2.67) and consider the resulting homogeneous linear system

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n,\\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n,\\ \vdots\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n.\end{aligned} \qquad (2.80)$$

The corresponding homogeneous equation in equation of the form (2.72) for which F(t) = 0 for all t and hence is

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

Throughout reminder of section 2.6[in book No.1] we shall always make the following assumption whenever we write or refer to the homogeneous vector differential equations (2.81) : we shall assume that (2.81) is the vector differential equation corresponding to the homogeneous linear system (2.80) of n equation in n unknown function and the components $a_{ij}(t), i = 1,2,\dots,n, j = 1,2,\dots,n$, of the n×n matrix A(t) are all continuous on the real interval $a \leq t \leq b$. Our first result corresponding equation (2.81) is an immediate consequence of theorem 2.10

## COROLLARY TO THEOREM (2.10)

Consider the homogeneous vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

Let $t_0$ be any point of $a \leq t \leq b$; and let

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{pmatrix}$$

Be a solution of (2.81) such that $\phi(t_0) = 0$, that is, such that

$$\phi_1(t_0) = \phi_2(t_0) = \cdots = \phi_n(t_0) = 0 \qquad (2.83)$$

Then $\phi(t_0) = 0$ for all t on $a \le t \le b$; that is,

$$\phi_1(t_0) = \phi_2(t_0) = \cdots = \phi_n(t_0) = 0$$

For all t on $a \le t \le b$.

**Proof .**

Obviously $\phi$ defined by $\phi(t_0) = 0$ for all t on $a \le t \le b$ is a solution of the vector differential equation (2.81) which satisfies conditions (2.83). These conditions are of the form (2.79), where $c_1 = c_2 = \cdots = c_n = 0$; and by theorem 2.10, there is a unique solution of the differential equation satisfying such a set of conditions thus $\phi$ such that $\phi(t) = 0$ for all t on $a \le t \le b$ is the only solution of (2.81) such that $\phi(t_0) = 0$.

**THEOREM (2.11)**

A linear combination of m solutions of the homogeneous vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

Is also a solution of (2.81). that is, if the vector functions $\phi_1, \phi_2, \dots, \phi_n$, are solutions of (2.81) and $c_1, c_2, \dots, c_n$, are m numbers, then the vector function

$$\phi = \sum_{k=1}^{m} c_k \phi_k$$

Is also a solution of (2.81).

**Proof.**

We have

$$\frac{d}{dt}\left[\sum_{k=1}^{m} c_k \phi_k(t)\right] = \sum_{k=1}^{m}\left[\frac{d}{dt} c_k \phi_k(t)\right] = \sum_{k=1}^{m} c_k \left[\frac{d\phi_k(t)}{dt}\right]$$

Now since each $\phi_k$ is a solution of (2.81)

$$\frac{d\phi_k(t)}{dt} = A(t)\phi_k(t) \text{ for } k = 1,2,\dots,m.$$

Thus we have

$$\frac{d}{dt}\left[\sum_{k=1}^{m} c_k\phi_k(t)\right] = \sum_{k=1}^{m} c_k A(t)\phi_k(t)$$

We now use results A and B of section 1.1 A. First applying result B to each term in the right member above, and then applying result A(m-1) times , we obtain

$$\sum_{k=1}^{m} c_k A(t)\phi_k(t) = \sum_{k=1}^{m} A(t)\,[c_k\phi_k(t)] = A(t)\sum_{k=1}^{m} c_k\phi_k(t).$$

Thus we have

$$\frac{d}{dt}\left[\sum_{k=1}^{m} c_k\phi_k(t)\right] = A(t)\left[\sum_{k=1}^{m} c_k\phi_k(t)\right];$$

That is,

$$\frac{d\phi(t)}{dt} = A(t)\phi(t)$$

For all t on $a \le t \le b$ thus the linear combination

$$\phi = \sum_{k=1}^{m} c_k\phi_k(t)$$

is a solution of (2.81).

In each of the next four theorem we shall be concerned with n vector functions, and we shall use the following combination for the n vector function for each of the theorems. We let $\phi_1, \phi_2, \dots, \phi_n$, be the n vector functions respectively by

$$\phi_1(t) = \begin{pmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{pmatrix}, \phi_2(t) = \begin{pmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{pmatrix}, \dots, \phi_n(t) = \begin{pmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{pmatrix}. \quad (2.83)$$

Carefully observe the notation scheme. For each vector, the first subscript of a component indicates the row of the component in the vector, where as the second subscript indicates the vector of which the component is an element. For instance, $\phi_{35}$ would be the component occupying the third row of the vector $\phi_5$.

**DEFINITION:**

The n×n determinant

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} \quad (2.84)$$

Is called the Wronskian of the n vector function $\phi_1, \phi_2, \dots, \phi_n$, defined by (2.83). we will denote it by W($\phi_1, \phi_2, \dots, \phi_n$) and its value at t by W($\phi_1, \phi_2, \dots, \phi_n$)(t).

**Theorem (2.12)**

If the n vector functions $\phi_1, \phi_2, \dots, \phi_n$, defined by (2.83) are linearly dependent on $a \le t \le b$, then their Wronskian W($\phi_1, \phi_2, \dots, \phi_n$)(t) equals zero for all t on $a \le t \le b$

**Proof.**

We begin by employing the definition of linear dependence of vector functions on an interval: since $\phi_1, \phi_2, \dots, \phi_n$ are linearly dependent on the

interval $a \le t \le b$, there exist n numbers $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) = 0$$

For all $t \in [a, b]$, now using the definition (2.83) of $\phi_1, \phi_2, \dots, \phi_n$ and writing the proceeding vector relation in the form of the n equivalent relation corresponding components, we have

$$c_1\phi_{11}(t) + c_2\phi_{12}(t) + \cdots + c_n\phi_{1n}(t) = 0,$$

$$c_1\phi_{21}(t) + c_2\phi_{22}(t) + \cdots + c_n\phi_{2n}(t) = 0,$$

$$\vdots$$

$$c_1\phi_{n1}(t) + c_2\phi_{n2}(t) + \cdots + c_n\phi_{nn}(t) = 0.$$

For all $t \in [a, b]$,thus, in particular, these must hold in an arbitrary point $t_0 \in [a, b]$

Thus , letting $t = t_0$ in the preceding n relations, we obtain the homogeneous linear algebraic system

$$\phi_{11}(t_0)c_1 + \phi_{12}(t_0)c_2 + \cdots + \phi_{1n}(t_0)c_n = 0,$$

$$\phi_{21}(t_0)c_1 + \phi_{22}(t_0)c_2 + \cdots + \phi_{2n}(t_0)c_n = 0,$$

$$\vdots$$

$$\phi_{n1}(t_0)c_1 + \phi_{n2}(t_0)c_2 + \cdots + \phi_{nn}(t_0)c_n = 0.$$

In n unknown $c_1, c_2, \dots c_n$, since $c_1, c_2, \dots c_n$ are not all zero, the determinant of coefficients of the preceding system must be zero, by Theorem A of( section 1.1B) the A is, we must have

$$\begin{vmatrix} \phi_{11}(t_0) & \phi_{12}(t_0) & \cdots & \phi_{1n}(t_0) \\ \phi_{21}(t_0) & \phi_{22}(t_0) & \cdots & \phi_{2n}(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1}(t_0) & \phi_{n2}(t_0) & \cdots & \phi_{nn}(t_0) \end{vmatrix} = 0$$

But the left member of this is the Wronskian W $(\phi_1, \phi_2, \dots, \phi_n)(t_0)$ thus we have

$$\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t_0) = 0$$

Since to is an arbitrary point of [a,b], we must have

$$\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$$

For all $t$ on $a \leq t \leq b$.

**Example 2.21.**

In Example 1.3 of section 1.1B we saw that the three vector function $\phi_1, \phi_2 \; and \; \phi_3$ defined respectively by

$$\phi_1(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 5e^{2t} \end{pmatrix}, \qquad \phi_2(t) = \begin{pmatrix} e^{2t} \\ 4e^{2t} \\ 11e^{2t} \end{pmatrix} \; and \; \phi_3(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 2e^{2t} \end{pmatrix}$$

Are linearly dependent on any interval $a \le t \le b$. Therefore, by theorem 2.12, their Wronskian must equal zero for all $t$ on $a \le t \le b$. Indeed, we fined

$$W(\phi_1, \phi_2, \dots, \phi_n)(t) = \begin{vmatrix} e^{2t} & e^{2t} & e^{2t} \\ 2e^{2t} & 4e^{2t} & e^{2t} \\ 5e^{2t} & 11e^{2t} & 2e^{2t} \end{vmatrix} = 0 \text{ for all } t.$$

**Theorem 2.13.**

Let the vector function $\phi_1, \phi_2, \dots, \phi_n$ defined by (2.83) be n solution of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

If the wronskian W $(\phi_1, \phi_2, \dots, \phi_n)(t_0) = 0$ at some $t_0 \in [a, b]$, then

$\phi_1, \phi_2, \dots, \phi_n$ are linearly dependent on $a \le t \le b$.

**Proof.**

Consider the linear algebraic system

$$c_1\phi_{11}(t) + c_2\phi_{12}(t) + \cdots + c_n\phi_{1n}(t) = 0,$$

$$c_1\phi_{21}(t) + c_2\phi_{22}(t) + \cdots + c_n\phi_{2n}(t) = 0,$$

$$\vdots$$

$$c_1\phi_{n1}(t) + c_2\phi_{n2}(t) + \cdots + c_n\phi_{nn}(t) = 0.$$

In the n unknown $c_1, c_2, \dots, c_n$ since the determinant of the coefficients is

W $(\phi_1, \phi_2, \dots, \phi_n)(t_0)$ and W $(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ by hypothesis, this system has a non trivial solution by A of section 1.1 B that is, there exist numbers

$c_1, c_2, \dots, c_n$, not all zero, which satisfy all n equations of system (2.85). these n equations are the n corresponding component relation equivalent to the one vector relation

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + \cdots + c_n\phi_n(t_0) = 0 \qquad (2.86)$$

Thus there exist numbers $c_1, c_2, \dots, c_n$, not all zero, such that (2.86) holds.

Now consider the vector function $\phi$ defined by

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) = 0. \qquad (2.87)$$

For all $t \in [a, b]$.

Since $\phi_1, \phi_2, \dots, \phi_n$ are solution of the differential equation (2.81), by theorem 2.11, the linear combination $\phi$ defined by (2.87) is also a solution of (2.81). Now from (2.86), we see that this solution $\phi$ is such that $\phi(t_0) = 0$ thus by the corollary to theorem 2.10, we must have $\phi(t) = 0$ for all $t \in [a, b]$. That is, using the definition (2.87),

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) = 0$$

For all $t \in [a, b]$, where $c_1, c_2, \dots, c_n$, are not all zero. Thus by definition $\phi_1, \phi_2, \dots, \phi_n$ are linearly dependent on $a \le t \le b$.

**Example 2.22.**

Consider the vector functions $\phi_1, \phi_2\ and\ \phi_3$ defined respectively by

$$\phi_1(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}, \qquad \phi_2(t) = \begin{pmatrix} 2e^{3t} \\ -4e^{3t} \\ -2e^{3t} \end{pmatrix}\ and\ \phi_3(t) = \begin{pmatrix} -3e^{3t} \\ 6e^{3t} \\ 3e^{3t} \end{pmatrix}$$

It is easy to verify that $\phi_1, \phi_2\ and\ \phi_3$ are all solution of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \ \ where\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (2.76)$$

On every real interval $a \le t \le b$. Thus, in particular, $\phi_1, \phi_2\ and\ \phi_3$ are solutions of (2.76) on every interval [a,b] containing $t_0 = 0$. It is easy to see that

$$W(\phi_1, \phi_2, \dots, \phi_n)(0) = \begin{vmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = 0$$

Thus by theorem 2.13, $\phi_1, \phi_2$ *and* $\phi_3$ are linearly dependent on every [a,b] containing 0. Indeed, note that

$$\phi_1(t) + \phi_2(t) + \phi_3(t) = 0$$

For all t on every interval [a,b], and recall the definition of linear dependence.

**Theorem 2.14.**

Let the vector function $\phi_1, \phi_2, \dots, \phi_n$ defined by (2.83) be n solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

On the real interval [a, b]. Then

Either $W(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ for all $t \in [a, b]$,

Or $W(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ for no $t \in [a, b]$.

**Proof.**

Either $W(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ for some $t \in [a, b]$,

Or $W(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ for no $t \in [a, b]$.

If $W(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ for some $t \in [a, b]$, then by theorem 2.13, the solutions $\phi_1, \phi_2, \dots, \phi_n$ are linearly dependent on [a, b]; and then by theorem 2.12 $W(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ for all $t \in [a, b]$. Thus the Wronskian of $\phi_1, \phi_2, \dots, \phi_n$ either equals zero for all $t \in [a, b]$ or equals zero for no $t \in [a, b]$.

**Theorem 2.15.**

Let the vector function $\phi_1, \phi_2, \dots, \phi_n$ defined by (2.83) be n solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

On the real interval [a, b]. These n solutions $\phi_1, \phi_2, \dots, \phi_n$ of (2.81) are linearly independent on [a, b] if and only if

$$\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t) \neq 0$$

For all $t \in [a, b]$.

**Proof .**

By theorem 2.12 and 2.13, the solutions $\phi_1, \phi_2, \dots, \phi_n$ are linearly dependent on [a, b] if and only if $\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t) = 0$ for all $t \in [a, b]$. Hence, $\phi_1, \phi_2, \dots, \phi_n$ are linearly independent on [a, b] if and only if $\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t_0) \neq 0$ for some $t_0 \in [a, b]$. Then by theorem 2.14 $\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t_0) \neq 0$ for some $t_0 \in [a, b]$ if and only if

$\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t) \neq 0$ for all $t \in [a, b]$.

**Example 2.23.**

Consider the vector function $\phi_1, \phi_2$ and $\phi_3$ defined respectively by

$$\phi_1(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}, \qquad \phi_2(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} \ and\ \ \phi_3(t) = \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

It is easy to verify that $\phi_1, \phi_2$ and $\phi_3$ are all solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \ \ where\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (2.76)$$

On every real interval $a \leq t \leq$ b. We calculate

$$W(\phi_1,\phi_2,\phi_3)(t) = \begin{vmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{vmatrix} = -e^{10t} \neq 0$$

For all real t. Thus by theorem 2.15, the solutions $\phi_1, \phi_2$ and $\phi_3$ of (2.76) defined by (2.88) are linearly independent on every real interval [a, b].

**DEFINITION:**

Consider the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x, \qquad (2.81)$$

Where $x$ is an n x1colmun vector.

1. A set of n linearly independent solutions of (2.81) is called a fundamental set of solutions of (2.81).
2. A matrix whose individual columns of fundamental set of solutions of (2.81),is called a fundamental matrix of (2.81), that is, if the vector functions $\phi_1, \phi_2, \ldots, \phi_n$ defined by (2.83) make up a fundamental of solutions of (2.81), then the n×n square matrix

$$\begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}$$

Is a fundamental matrix of (2.81).

**Example 2.24.**

In Example 2.31 we saw that the three vector functions $\phi_1, \phi_2$ and $\phi_3$ defined respectively by

$$\phi_1(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}, \quad \phi_2(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} \text{ and } \phi_3(t) = \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix} \qquad (2.78)$$

Are linearly independent solutions of the differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (2.76)$$

On every real interval [a, b]. Thus these three solutions $\phi_1, \phi_2$ and $\phi_3$ from a fundamental set of differential equation (2.76), and a fundamental matrix of the differential equation is

$$\begin{pmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{pmatrix}$$

We know that the differential equation (2.76) of example 2.31 and 2.32 has the fundamental set of solutions $\phi_1, \phi_2, \phi_3$ defined by (2.88). We now show that every vector differential equation (2.81) has fundamental sets of solutions.

**Theorem 2.16.**

There exist fundamental sets of solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

**Proof.**

We begin by defining a special set of constant vectors $u_1, u_2, \dots, u_n$ defined

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, u_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

That is, in general, for each $i = 1,2, \dots, n$, has ith component one and all other components zero. Now let $\phi_1, \phi_2, \dots, \phi_n$ be the n solution of (2.81) which satisfy the conditions

$$\phi_i(t_0) = u_i \qquad (i = 1,2, \dots, n),$$

That is $\phi_1(t_0) = u_1, \phi_2(t_0) = u_2, \dots, \phi_n(t_0) = u_n$, where $t_0$ is an arbitrary (but fixed) point of [a, b] not that these solutions exist and are unique by theorem 2.10. we now find

$$\mathrm{W}\,(\phi_1, \phi_2, \dots, \phi_n)(t_0) = \mathrm{W}\,(u_1, u_2, \dots, u_n) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0.$$

Then by theorem 2.14 W $(\phi_1, \phi_2, \dots, \phi_n)(t) \neq 0$ for all $t \in [a, b]$, and so by theorem 2.15, solutions $\phi_1, \phi_2, \dots, \phi_n$ from fundamental set of differential equation(2.81).

**Theorem 2.17.**

Let $\phi_1, \phi_2, \dots, \phi_n$ defined by (2.83) be a fundamental set of solutions of the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x \qquad (2.81)$$

And let $\phi$ be an arbitrary solution of (2.81) on the real interval [a, b]. Then $\phi$ can be represented as a suitable linear combination of $\phi_1, \phi_2, \dots, \phi_n$; that is, there exist number $c_1, c_2, \dots, c_n$ such that

$$\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n \qquad on\ [a, b]$$

**Proof.**

Suppose $\phi(t_0) = u_0$ where $t_0 \in [a, b]$ and

$$u_0 = \begin{pmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{n0} \end{pmatrix}$$

A constant vector. Consider the linear algebraic system

$$\begin{aligned} c_1\phi_{11}(t_0) + c_2\phi_{12}(t_0) + \cdots + c_n\phi_{1n}(t_0) &= u_{10}, \\ c_1\phi_{21}(t_0) + c_2\phi_{22}(t_0) + \cdots + c_n\phi_{2n}(t_0) &= u_{20}, \qquad (2.89) \\ &\vdots \\ c_1\phi_{n1}(t_0) + c_2\phi_{n2}(t_0) + \cdots + c_n\phi_{nn}(t_0) &= u_{n0}. \end{aligned}$$

n equation in n unknowns $c_1, c_2, \dots, c_n$ since $\phi_1, \phi_2, \dots, \phi_n$ is a fundamental set of functions on [a, b] and hence by theorem 2.15 W $(\phi_1, \phi_2, \dots, \phi_n)(t_0) \neq 0$.

Now observe that W $(\phi_1, \phi_2, \dots, \phi_n)(t_0)$ is the determinant of coefficients of system (2.89), and so this determinant of coefficients is unequal to zero. Thus by theorem B of section 7.5 B, the system (2.89) has a unique solution for $c_1, c_2, \dots, c_n$. That is, there exists a unique set of numbers $c_1, c_2, \dots, c_n$ such that

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + \cdots + c_n\phi_n(t_0) = u_0,$$

And hence such that

$$\phi(t_0) = u_0 = \sum_{k=1}^{n} c_k\phi_k(t_0).$$

Now consider the vector function $\psi$ defined by

$$\psi(t) = \sum_{k=1}^{n} c_k\phi_k(t_0).$$

By theorem 2.11, the vector function $\psi$ is also a solution of the vector differential equation (2.81). now note that

$$\psi(t_0) = \sum_{k=1}^{n} c_k\phi_k(t).$$

For all $t \in [a, b]$. Thus $\phi$ is expressed as the linear combination

$$\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

Of $\phi_1, \phi_2, \ldots, \phi_n$, where $c_1, c_2, \ldots, c_n$ is the unique solution of system (2.89)

As a result of theorem 2.17, we are led to make the following definition.

**DEFINITION:**

Consider the homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x,$$

Where $x$ is an nx1 column vector. By a general solution of (2.81), we mean solution of the form

$$c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

Where $c_1, c_2, \ldots, c_n$ are n arbitrary number and $\phi_1, \phi_2, \ldots, \phi_n$, is a fundamental set of solutions of (2.81).

**Example 2.25.**

Consider the differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \ \ where\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In example 2.32 we saw that the three vector function $\phi_1, \phi_2$ and $\phi_3$ defined respectively by

$$\phi_1(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}, \qquad \phi_2(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} \ and \ \phi_3(t) = \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

From a fundamental set of differential equation(2.76). Thus by theorem 2.17 if $\phi$ is an arbitrary solution of (2.76), then $\phi$ can be represented as a suitable linear combination of these three linearly independent solutions $\phi_1, \phi_2$ and $\phi_3$ of (2.76).

Further, if $c_1, c_2$ and $c_3$ are arbitrary numbers, we see from the definition that $c_1\phi_1 + c_2\phi_2 + c_3\phi_3$ is a general solution of (2.76) is defined by

$$c_1 \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} + c_3 \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

And can be written as

$$x_1 = c_1 e^{2t} + c_2 e^{3t} + 3c_3 e^{5t},$$

$$x_2 = -c_1 e^{2t} - 2c_2 e^{3t} - 6c_3 e^{5t},$$

$$x_3 = -c_1 e^{2t} - c_2 e^{3t} - 2c_3 e^{5t}\ .$$

Where $c_1, c_2,$ and $c_3$ are arbitrary numbers.

## C. Non homogeneous Linear Systems

We return briefly to the non homogeneous linear vector differential equation

$$\frac{dx}{dt} = A(t)x + F(t) \qquad (2.72)$$

Where A(t) is given by (2.70) and F(t) and $x$ are given by (2.71).

We shall see corresponding homogeneous equation

$$\frac{dx}{dt} = A(t)x. \qquad (2.81)$$

**Theorem 2.18.**

Let $\phi_0$ be any solution of the non homogeneous linear differential equation

$$\frac{dx}{dt} = A(t)x + F(t); \qquad (2.72)$$

Let $\phi_1, \phi_2, \dots, \phi_n$ be a fundamental set of solution of the corresponding homogeneous differential equation

$$\frac{dx}{dt} = A(t)x. \qquad (2.81)$$

And let $c_1, c_2, \dots, c_n$ be n numbers.

Then: (1) the vector function

$$\phi_0 + \sum_{k=1}^{n} c_k \phi_k \qquad (2.91)$$

Is also a solution of the non homogeneous differential equation (2.72) for every choice of $c_1, c_2, \dots, c_n$; and

(2) an arbitrary solution $\phi$ of the non homogeneous differential equation (2.72) of the form (2.91) for suitable choice of $c_1, c_2, \dots, c_n$.

**Proof.**

(1) We show that (2.91) satisfies (2.72) for all choices of $c_1, c_2, \dots, c_n$ we have $\frac{d}{dt}\left[\phi_0(t) + \sum_{k=1}^{n} c_k \phi_k(t)\right] = \frac{d\phi_0(t)}{dt} + \frac{d}{dt}\left[\sum_{k=1}^{n} c_k \phi_k(t)\right]$

Now since $\phi_0$ satisfies (2.72), we have

$$\frac{d\phi_0(t)}{dt} = A(t)\phi_0(t) + F(t);$$

And since by theorem 2.11$\sum_{k=1}^{n} \lambda_k \phi_k$ satisfies (2.81), we also have

$$\frac{d}{dt}\left[\sum_{k=1}^{n} c_k \phi_k\,(t)\right] = A(t)\left[\sum_{k=1}^{n} c_k \phi_k\,(t)\right]$$

Thus

$$\frac{d}{dt}\left[\phi_0(t) + \sum_{k=1}^{n} c_k \phi_k\,(t)\right] = A(t)\phi_0(t) + F(t) + A(t)\left[\sum_{k=1}^{n} c_k \phi_k\,(t)\right]$$
$$= A(t)\left[\phi_0(t) + \sum_{k=1}^{n} c_k \phi_k\,(t)\right] + F(t).$$

That is

$$\frac{d\psi(t)}{dt} = A(t)\psi(t) + F(t)$$

Where

$$\psi = \phi_0 + \sum_{k=1}^{n} c_k \phi_k\,;$$

And so

$$\psi = \phi_0 + \sum_{k=1}^{n} c_k \phi_k\,;$$

Is a solution of (2.72) for every choice of $c_1, c_2, \dots, c_n$.

(2) Now consider an arbitrary solution $\phi$ of (2.72) and evaluate the derivative of the difference $\phi - \phi_0$. We have

$$\frac{d}{dt}[\phi(t) - \phi_0(t)] = \frac{d\phi(t)}{dt} - \frac{d\phi_0(t)}{dt}$$

Since both $\phi$ and $\phi_0$ satisfy (2.72), we have respectively

$$\frac{d\phi(t)}{dt} = A(t)\phi(t) + F(t),$$

$$\frac{d\phi_0(t)}{dt} = A(t)\phi_0(t) + F(t).$$

Thus we obtain

$$\frac{d}{dt}[\phi(t) - \phi_0] = [A(t)\phi(t) + F(t)] - [A(t)\phi_0(t) + F(t)].$$

Which at once reduces to

$$\frac{d}{dt}[\phi(t) - \phi_0(t)] = A(t)[\phi(t) - \phi_0(t)].$$

Thus $\phi - \phi_0$ satisfies the homogeneous equation (2.81). Hence by theorem 2.17, there exist a suitable choice of numbers $c_1, c_2, \dots, c_n$ such that

$$\phi - \phi_0 = \sum_{k=1}^{n} c_k \phi_k$$

Thus the arbitrary solution $\phi$ of (2.72) is of the form

$$\phi = \phi_0 + \sum_{k=1}^{n} c_k \phi_k \qquad (2.91)$$

For a suitable choice of $c_1, c_2, \dots, c_n$.

**DEFINITION:**

Consider the non homogeneous linear vector differential equation (2.72) and the corresponding homogeneous linear differential equation (2.81). by a general solution of (2.72), we mean a solution of the form

$$c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n + \phi_0,$$

Where $c_1, c_2, \dots, c_n$n arbitrary numbers are $\phi_1, \phi_2, \dots, \phi_n$ is a fundamental set of solutions of (2.81), and $\phi_0$ is any solution of (2.72).

**Example 2.26.**

Consider the non homogeneous differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} -5t-6 \\ -4t+23 \\ 2 \end{pmatrix} \qquad (2.72)$$

And the corresponding homogeneous differential equation

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, \;\; where\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (2.71)$$

These were introduced in example 2.26, where they were written out in component form , and (2.71) has been used in example 2.33 and other examples as well.

In example 2.33 we observed that $\phi_1, \phi_2, \phi_3$ defined respectively by

$$\phi_1(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix} , \qquad \phi_2(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} \;\; and \;\; \phi_3(t) = \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

From a fundamental set of the homogeneous differential equation (2.71)[ or (2.76) as it is numbered there]. Now observe that the vector function $\phi_0$ defined by

$$\phi_0(t) = \begin{pmatrix} 2t \\ 3t-2 \\ -t+1 \end{pmatrix}$$

As a solution of the non homogeneous differential equation (2.72) .

Thus a general solution of (2.72) is given by

$$x = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t) + \phi_0(t),$$

That is

$$x = c_1 \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} + c_3 \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix} + \begin{pmatrix} 2t \\ 3t-2 \\ -t+1 \end{pmatrix},$$

Where $c_1, c_2,$ and $c_3$ are arbitrary numbers . Thus a general solution of (2.72) can be written as

$$x_1 = c_1e^{2t} + c_2e^{3t} + 3c_3e^{5t} + 2t,$$

$$x_2 = -c_1 e^{2t} - 2c_2 e^{3t} - 6c_3 e^{5t} + 3t - 2,$$

$$x_3 = -c_1 e^{2t} - c_2 e^{3t} - 2c_3 e^{5t} - t + 1.$$

Where $c_1, c_2,$ and $c_3$ are arbitrary numbers.

## 2.5 HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS: N EQUATIONS IN N UNKNOWN FUNCTIONS

### A. Introduction

We now consider the normal form of homogeneous linear system of n first-order differential equations in n unknown functions $x_1, x_2, \dots, x_n$ where all of the coefficient constants. To be more specific we shall discuss the case in which each coefficient is a real number. Hence the system to be considered is of the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n,$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n, \qquad (2.92) \qquad \frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n.$$

Where all of the $a_{ij}, i = 1,2, \dots, n, j = 1,2, \dots, n$, are real numbers, introducing the $n \times n$ constant matrix of real numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad (2.93)$$

and the vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad (2.94)$$

The system (2.92) can be expressed as the homogeneous linear vector differential equation

$$\frac{dx}{dt} = Ax \qquad (2.95)$$

The real constant matrix A that appears in (2.95) and is defined by (2.93) is called the coefficient matrix of (2.95).

We seek solutions of the system (2.92), that is, of the corresponding vector differential equation (2.95), we shall proceed by analogy with the presentation in Section 2.4 A. Doing this, we seek nontrivial solutions of system (2.92) of the form

$$\begin{aligned} x_1 &= \alpha_1 e^{\lambda t} \\ x_2 &= \alpha_2 e^{\lambda t} \\ &\vdots \\ x_n &= \alpha_n e^{\lambda t} \end{aligned} \qquad (2.96)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\lambda$ are numbers . Letting

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \qquad (2.97)$$

and using (2.94) we see that the vector form of the desired solution (2.96) is

$$x = \alpha e^{\lambda t}$$

Thus we seek solutions of the vector differential equation (7. 1 29) which are of the form

$$x = \alpha e^{\lambda t} \qquad (2.98)$$

where $\alpha$ is a constant vector and $\lambda$ is a number.

Now substituting (2.98) into (2.95), we obtain

$$\lambda \alpha e^{\lambda t} = A\alpha e^{\lambda t}$$

which reduces at once to

$$A\alpha = \lambda\alpha \qquad (2.99)$$

and hence to

$$(A - \lambda I)\alpha = 0$$

Where I is the n x n identity matrix . Written pt in terms of components, this is the system of n homogeneous linear algebraic equations

$$(a_{11} - \lambda)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n = 0,$$

$$a_{21}\alpha_1 + (a_{22} - \lambda)\alpha_2 + \cdots + a_{2n}\alpha_n = 0, \quad (2.100) a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn} - \lambda)\alpha_n = 0.$$

in the n unknowns $\alpha_1, \alpha_2, \dots, \alpha_n$By Theorem A of Section 7.5 B, this system has a nontrivial solution if and only if

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0, \qquad (2.101)$$

that is, in matrix notation,

$$|A - \lambda I| = 0.$$

Looking back at Section 7.5 C, we recognize Equation (2.101) as the characteristic equation of the coefficient matrix A = $(a_{ij})$ of the vector differential equation (2.95). We know that this is an nth-degree polynomial equation in $\lambda$, and we recall that its roots $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic values of A. Substituting each characteristic value $\lambda_i$(i = 1,2, . . . , n), into system (2.100), we obtain the corresponding nontrivial solution

$$\alpha_1 = \alpha_{1i}, \alpha_2 = \alpha_{2i}, \dots, \alpha_n = \alpha_{ni}$$

(i = 1,2,...,n), of system (7. 134). Since (2.100) is merely the component form of (2.99), we recognize that the vector defined by

$$\alpha^{(i)} = \begin{pmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \\ \alpha_{ni} \end{pmatrix} \quad (i = 1,2,\dots,n) \qquad (2.102)$$

is a characteristic vector corresponding to the characteristic value $\lambda_i$(i= 1, 2, ..., n). Thus we see that if the vector differential equation

$$\frac{dx}{dt} = Ax \qquad (2.95)$$

has a solution of the form

$$x = \alpha e^{\lambda t} \qquad (2.98)$$

then the number $\lambda$ must be a characteristic value $\lambda_i$ of the coefficient matrix A and the vector $\alpha$ must be a characteristic vector $\alpha^{(i)}$ corresponding to this characteristic value $\lambda_i$.

**B. Case of n Distinct Characteristic Values**

Suppose that each of the ii characteristic values $\lambda_1, \lambda_2, \dots, \lambda_n$ of the n x n square coefficient matrix A of the vector differential equation is distinct (that is non repeated); and let $\alpha^{(1)}$ , $\alpha^{(2)}$ ,…, $\alpha^{(n)}$ be a set of n respective corresponding characteristic vectors of A. Then the n distinct vector functions $x_1, x_2, \dots, x_n$ defined respectively by

$$x_1(t) = \alpha^{(1)} e^{\lambda_1 t}, x_2(t) = \alpha^{(2)} e^{\lambda_2 t}, \dots, x_n(t) = \alpha^{(n)} e^{\lambda_n t} \qquad (2.103)$$

are solutions of the vector differential equation (2.95) on every real interval [a, b]. This is readily seen as follows: From (2.99), for each i = 1, 2, …, n, we have $\lambda_i \alpha^{(i)} = A\alpha^{(i)}$

and using this and the definition (2.103) of $x_i(t)$, we obtain

$$\frac{dx_i(t)}{dt} = \lambda_i \alpha^{(i)} e^{\lambda_i t} = A\alpha^{(i)} e^{\lambda_i t} = Ax_i(t)$$

which states that $x_i(t)$ satisfies the vector differential equation

$$\frac{dx}{dt} = Ax \qquad (2.95)$$

on $[a, b]$.

Now consider the Wronskian of the *n* solutions $x_1, x_2, \dots, x_n$ defined by (2.103) we find

$$W(x_1, x_2, \dots, x_n)(t) = \begin{vmatrix} \alpha_{11} e^{\lambda_1 t} & \alpha_{12} e^{\lambda_2 t} & \cdots & \alpha_{1n} e^{\lambda_n t} \\ \alpha_{21} e^{\lambda_1 t} & \alpha_{22} e^{\lambda_2 t} & \cdots & \alpha_{2n} e^{\lambda_n t} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} e^{\lambda_1 t} & \alpha_{n2} e^{\lambda_2 t} & \cdots & \alpha_{nn} e^{\lambda_n t} \end{vmatrix}$$

$$= e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix}$$

By Result C of Section 1.1 C, the *n* characteristic vectors $\alpha^{(1)}$ , $\alpha^{(2)}$ ,…, $\alpha^{(n)}$ are linearly Independent. Therefore

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} \neq 0$$

Further, It is clear that

$$e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)} \neq 0.$$

For all t. Thus W $(x_1, x_2, \dots, x_n)(t) \neq 0$ for all t on [a,b]. Hence by Theorem 7. 15, the solutions $x_1, x_2, \dots, x_n$, of vector differential equation (2.95) defined by

(2.103), are linearly independent on [a,b] and so form a fundamental set of solutions of (2.95) on [a,b]. Thus a general solution of (2.95) is given by

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n,$$

Where $c_1, c_2, \ldots, c_n$ are n arbitrary numbers. We summarize the results obtained in the following theorem:

**THEOREM 2.19.**

Consider the vector differential equation

$$\frac{dx}{dt} = Ax \qquad (2.95)$$

Where A is an n×n real constant matrix. Suppose each of the n characteristic value t $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A is distinct; and let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}$ be a set of n respective corresponding characteristic vectors of A. Then on every real interval [a, b] the n functions defined by

$$\alpha^{(1)} e^{\lambda_1 t}, \alpha^{(2)} e^{\lambda_2 t}, \ldots, \alpha^{(n)} e^{\lambda_n t}$$

From a linearly independent set (fundamental set) of solution of (2.95)

$$x = c_1 \alpha^{(1)} e^{\lambda_1 t} + c_2 \alpha^{(2)} e^{\lambda_2 t} + \cdots + c_n \alpha^{(n)} e^{\lambda_n t},$$

Where $c_1, c_2, \ldots, c_n$ are n arbitrary numbers, is a general solution of (2.95) on

[a, b]

**Example 2.27.**

Consider the homogeneous linear system

$$\frac{dx_1}{dt} = 7x_1 - x_2 + 6x_3,$$

$$\frac{dx_2}{dt} = -10x_1 + 4x_2 - 12x_3,$$

$$\frac{dx_n}{dt} = -2x_1 + x_2 - x_3.$$

Or in matrix form

$$\frac{dx}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} x, where\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (2.105)$$

We assume the solution of the form

$$x = \alpha e^{\lambda t},$$

$$x_1 = \alpha_1 e^{\lambda t},$$

$$x_2 = \alpha_2 e^{\lambda t},$$

$$x_3 = \alpha_3 e^{\lambda t}.$$

Substituting (2.106) into (2.104) and dividing through by $e^{\lambda t} \neq 0$, we obtain

$$\alpha_1 \lambda = 7\alpha_1 - \alpha_2 + 6\alpha_3,$$

$$\alpha_2 \lambda = -10\alpha_1 + 4\alpha_2 - 12\alpha_3,$$

$$\alpha_3 \lambda = -2\alpha_1 + \alpha_2 - \alpha_3.$$

$$(7-\lambda)\alpha_1 - \alpha_2 + 6\alpha_3 = 0,$$

$$-10\alpha_1 + (4-\lambda)\alpha_2 - 12\alpha_3 = 0, \quad (2.107)$$

$$-2\alpha_1 + \alpha_2 + (-1-\lambda)\alpha_3 = 0.$$

This homogeneous linear algebraic system in $\alpha_1, \alpha_2, \alpha_3$ has a non trivial solution if and only if the determinant of its coefficients equals zero, that is, if and only if

$$\begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0 \quad (2.108)$$

Clearly this is the characteristic equation of the coefficient matrix

$$A = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} \quad (2.109)$$

Of the given system (2.104) [or (2.105)]. It's a cubic equation in $\lambda$; and its roots that is, the vector differential equation $\lambda_1, \lambda_2, \lambda_3$ are the characteristic values of the matrix A given by (2.109) expanding the determinant involved, we see that the characteristic equation (2.108) of A may be written

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

Or in factored form

$$(\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$$

Thus the roots of the characteristic equation (2.108) are

$$\lambda_1 = 2, \lambda_2 = 3 \; and \; \lambda_3 = 5 \quad (2.110)$$

A characteristic vector corresponding to $\lambda_1 = 2$ is a non zero vector

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad (2.111)$$

Whose component are a non trivial solution $\alpha_1, \alpha_2, \alpha_3$ of algebraic system (2.107) when $\lambda = 2$ equivalent it's a non zero vector given by (2.111) such that

$$\begin{pmatrix} 7 & -1 & 6 \\ 10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Starting in either of this completely equivalent ways, we at once find that $\alpha_1, \alpha_2, \alpha_3$ must be anon trivial solution of the system

$$5\alpha_1 - \alpha_2 + 6\alpha_3 = 0,$$

$$-10\alpha_1 + \mathbf{2}\alpha_2 - 12\alpha_3 = 0,$$

$$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0,$$

We have already solved this algebra problem in example (2.25) (except for the notational difference of having used $\alpha$'s here) looking back at the example, we found that a characteristic vector corresponding to $\alpha_1 = 2$ is given by

$$\alpha^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Likewise reference to example (2.25) shows that characteristic vector corresponding to $\lambda_2 = 3$ and $\lambda_3 = 5$

$$\alpha^{(2)} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \text{ And } \alpha^{(3)} = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$$

Respectively thus around mental set of solutions of (2.104) [or (2.105)] is

$\alpha^{(1)} e^{\lambda_1 t}$ , $\alpha^{(2)} e^{\lambda_2 t}$, $\alpha^{(3)} e^{\lambda_3 t}$ , that is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{3t} and \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} e^{5t}$$

Or rewriting these slightly

$$\begin{pmatrix} e^{2t} \\ -e^{2t} \\ e^{2t} \end{pmatrix}, \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} and \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

Respectively. A general solution of the system may thus be expressed as

$$x_1 = c_1 e^{2t} + c_2 e^{3t} + 3 c_3 e^{5t}$$

$$x_2 = -c_1 e^{2t} - c_2 e^{3t} - 6 c_3 e^{5t}$$

$$x_3 = -c_1 e^{2t} - c_2 e^{3t} - 2 c_3 e^{5t}$$

Where $c_1, c_2 and\ c_3$ are arbitrary numbers.

We return to the vector differential equation,

$$\frac{dx}{dt} = Ax, \qquad (2.95)$$

Where A is an n x n real constant matrix and reconsider the result started in theorem (2.19) in that theorem we stated that if each of the n characteristic values $\lambda_1, \lambda_2, \ldots$ of **A** is distinct; and if $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}$ is a set of n respective corresponding characteristic vectors of **A,** Then the n functions defined by

$$\alpha^{(1)} e^{\lambda_1 t}, \alpha^{(2)} e^{\lambda_2 t}, \ldots, \alpha^{(n)} e^{\lambda_n t}$$

Form a fundamental set of solution of (2.95) note that although we assume that

$\lambda_1, \lambda_2, \ldots, \lambda_n$ Are distinct, we do not require that they be real. Thus complex characteristic values may be present however, since A is a real matrix, any complex characteristic values must suppose $\lambda_1 = a + bi \; and \; \lambda_2 = a - bi$ form such a pair. Te the corresponding solutions are

$$\alpha^{(1)} e^{(a+bi)t} \text{And} \alpha^{(2)} e^{(a-bi)t},$$

And these solutions are complex solutions. Thus if one or more conjugate complex of characteristic values occur, the fundamental set defined by $\alpha^{(i)} e^{\lambda_i t}$, i = 1,2,…,n, contains complex function. However, in such a case, this fundamental maybe replaced by another fundamental set, all of whose members are real functions this is a accomplished exactly as explained in section 2.4 D and illustrated in example 2.18 C.

We again consider the vector differential equation

$$\frac{dx}{dt} = Ax, \qquad (2.95)$$

Where A is an n x n real constant matrix; but here we given a brief introduction the case in which A has a repeated characteristic value to be definite we suppose that A has a real characteristic value $\lambda_1$of multiplicity m, where$1 < m \leq n$, and they are the other characteristic value $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$, (if there are any) are distinct. B result D of section 7.5 C, we know that the repeated characteristic value $\lambda_1$, of multiplicity m has p linearly independent characteristic vectors where$1 \leq p \leq m$, now consider two sub case (1), p = m, and (2) $p < m$. In sub case (1), there are m linearly independent characteristic vector$\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$ corresponding $\lambda_1$ then the n functions defined by

$$\alpha^{(1)} e^{\lambda_1 t}, \alpha^{(2)} e^{\lambda_2 t}, \ldots, \alpha^{(m)} e^{\lambda_1 t},$$

$$\alpha^{(m+1)} e^{\lambda_{m+1} t}, \ldots, \alpha^{(n)} e^{\lambda_n t}$$

Form a linearly independent set of n solutions of deferential equation (2.95), and general solution of (2.95) is a linear combination of these n solutions having arbitrary numbers as the "constant of combination"

**Example 2.28.**

Consider the homogeneous linear system

$$\frac{dx_1}{dt} = 3x_1 + x_2 - x_3,$$

$$\frac{dx_2}{dt} = x_1 + 3x_2 - 3x_3, \qquad (2.14)$$

$$\frac{dx_n}{dt} = 3x_1 + 3x_2 - x_3.$$

Or in matrix form,

$$\frac{dx}{dt} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} x \ , \qquad where\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (2.14)$$

We assume a solution of the form.

$$x = \alpha_1 e^{\lambda t}$$

That is

$$x_1 = \alpha_1 e^{\lambda t},$$
$$x_2 = \alpha_2 e^{\lambda t},$$

$$x_3 = \alpha_3 e^{\lambda t}.$$

Substituting (2.114) into (2.112) and dividing through by $e^{\lambda t} \neq 0$, we obtain

$$\alpha_1 \lambda = 3\alpha_1 + \alpha_2 - \alpha_3,$$

$$\alpha_2 \lambda = \alpha_1 + 3\alpha_2 - \alpha_3,$$

$$\alpha_3 \lambda = 3\alpha_1 + \alpha_2 - \alpha_3,$$

or

$$(3-\lambda)\alpha_1 + \alpha_2 - \alpha_3 = 0,$$

$$\alpha_1 + (3-\lambda)\alpha_2 - \alpha_3 = 0,$$

$$3\alpha_1 + \alpha_2 + (-1-\lambda)\alpha_3 = 0.$$

This homogenous linear algebraic system in $\alpha_1, \alpha_2, \alpha_3$ has anon trivial solution if and only if the determinate of its coefficients equals zero, that is if and only if

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = 0 \qquad (2.116)$$

Of course this is the characteristic equation of the coefficients matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} \qquad (2.117)$$

Of the given system (2.112) [or (2.113)]. It's a cubic equation in $\lambda$; and its roots $\lambda_1, \lambda_2, \lambda_3$ are the characteristic values of the matrix A is given by (2.117) expanding the determinant involved, we see that the characteristic equation (2.116) of A maybe written

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0,$$

Or in factored form

$$(\lambda-1)(\lambda-2)(\lambda-2) = 0 .$$

Thus the roots of the characteristic equation (2.116) are

$$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 2 \qquad (2.118)$$

Note that the real number 1 is a distinct characteristic value of the coefficient matrix (2.117) of the given system (2.112) but the real number 2 is a repeated characteristic value of this coefficient matrix.

We first consider the distinct characteristic value $\lambda_1 = 1$, a characteristic vector corresponding to $\lambda_1 = 1$ is a non zero vector

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \qquad (2.119)$$

Whose components are a non trivial solution $\alpha_{1,}\alpha_2, \alpha_3$ of the algabric system (2.115), when $\lambda = 1$ equivalently it's a non zero vector or given by (2.119) such that

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 1 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Starting in either of these completely equivalent ways, we at once find that $\alpha_{1,}\alpha_2, \alpha_3$ must be non trivial solutions of the system

$$2\alpha_1 + \alpha_2 - \alpha_3 = 0,$$

$$\alpha_1 + \mathbf{2}\alpha_2 - \alpha_3 = 0,$$

$$3\alpha_1 + \mathbf{3}\alpha_2 - 2\alpha_3 = 0,$$

Note that $\alpha_1 = k, \alpha_2 = k, \alpha_3 = 3k$ is a solution of this system for every real k. hear the characteristic vectors corresponding to the characteristic value $\lambda = 1$, are the vectors

$$\alpha = \begin{pmatrix} k \\ k \\ 3k \end{pmatrix}$$

Where k is an arbitrary non zero number. In particular, letting k = 1 , we obtain the particular characteristic vector

$$\alpha^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Corresponding to $\lambda = 1$. Thus the corresponding solution of the form

$$\alpha e^{\lambda_1 t}$$

That is

$$\begin{pmatrix}1\\1\\3\end{pmatrix}e^{t} \qquad (2.120)$$

We now turn to the repeated characteristic value $\lambda_2 = \lambda_3 = 2$. To be more specific this characteristic value 2 has multiplicity $m = 2 < 3 = n$,

Where n of course denotes the common number of rows and column of the coefficient matrix (2.117) of the given system (2.112).

A characteristic vector corresponding to this double characteristic value $\lambda_2 = \lambda_3 = 2$ a non zero vector

$$\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} \qquad (2.121)$$

Whose components are a non trivial solution $\alpha_{1,}\alpha_2, \alpha_3$ of algebraic system (2.14 a) when $\lambda = 2$ equivelantly. It's a non zero vector given by (2.121) such that

$$\begin{pmatrix}3 & 1 & -1\\1 & 3 & -1\\3 & 3 & -1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = 2\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}$$

Starting in either of these completely equivalent ways, we at once find that $\alpha_{1,}\alpha_2, \alpha_3$ must be a non trivial solution of the system

$$\alpha_1 + \alpha_2 - \alpha_3 = 0,$$

$$\alpha_1 + \alpha_2 - \alpha_3 = 0,$$

$$3\alpha_1 + \mathbf{3}\alpha_2 - 3\alpha_3 = 0,$$

Note the each of these three relations is equivalent to both of the other two and only relationship among $\alpha_{1,}\alpha_2, \alpha_3$ is that given most simply by

$$\alpha_1 + \alpha_2 - \alpha_3 = 0,$$

Clear there exist two linearly independent of the form (2.121) whose components satisfy this relation (2.122) for example if $\alpha_1 = 1, \alpha_2 = 1$ *and* $\alpha_3 = 0$

We obtain the vector

$$\alpha^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

And if $\alpha_1 = 1, \alpha_2 = 1 \; and \;\; \alpha_3 = 0$, we obtain the vector

$$\alpha^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

First note that the component of each of these two vectors $\alpha^{(2)} \, and \, \alpha^{(3)}$ do satisfy (2.122) and hence each is characteristic vector corresponding to the double root $\lambda_1 = \lambda_3 = 2$, next note that these vectors $\alpha^{(2)} \, and \, \alpha^{(3)}$ are indeed linear independent (use the definition of linear independence of a set of constant vectors) thus the characteristic value $\lambda = 2$ of multiplicity m = 2 has the p = 2 linearly independent characteristic vectors

$$\alpha^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{and} \quad \alpha^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Corresponding to it hence this is illustration of sub case (1) of the discussion preceding this example those corresponding to the two fold characteristic value $\lambda = 2$, there are two linearly independent solution of the form $\alpha^{(3)} e^{\lambda t}$ of the given system these are

$$\alpha^{(2)} e^{2t} and \; \alpha^{(3)} e^{2t}$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} \; and \; \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad (2.123)$$

Respectively, hence a fundamental set of solutions of the given system (2.112) [or 2.113)] consist of the

Or rewriting these slightly

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} e^{t}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t}, and \; \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix}, \begin{pmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{pmatrix}, and \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$$

$$x_1 = c_1 e^t + (c_2 + c_3)e^{2t}$$

$$x_2 = c_1 e^t - c_2 e^{2t}$$

$$x_3 = 3c_1 e^t + c_3 e^{2t}$$

Where $c_1, c_2$ $and$ $c_3$ are arbitrary numbers.

One type of vector differential equation (2.95) which always leads to sub case one p = m, in the case of repeated characteristic value $\lambda_1$ is that in which n x n coefficient matrix A of (2.95) is real symmetric matrix for then by result G of section (7.5) c there always exist n linearly independent characteristic vectors of A regard less of whether the n characteristic values of A are all distinct or not. We now turn to a very brief consideration of sub case (2), $p < n$ in this case there are less than m linearly independent solutions of differential equation (2.95) of the form $\alpha^{(1)} e^{\lambda_1 t}$ corresponding to $\lambda_1$; and so there is not a fundamental set of solutions of the form $\alpha^{(k)} e^{\lambda_k t}$, where $\lambda_k$ is a characteristic value of A and $\alpha^{(k)}$ is a characteristic vector corresponding to $\lambda_k$ clearly we must seek linearly independent solutions of another form.

To discover what other forms of solution to seek we look back at the analogous situation in section (2.4) C. the result there suggest the following: if $\lambda_1$ is a characteristic value of multiplicity n = 2 and p = 1 $<$ m, then we seek linearly independent solution of the form

$$\alpha e^{\lambda_1 t} \text{ and } \alpha e^{\lambda_1 t} + \beta e^{\lambda_1 t};$$

Where $\alpha$ is a characteristic vector corresponding to $\lambda_1$, that is $\alpha$ satisfies

$$(A - \lambda_1 I)\, \alpha = 0;$$

And $\beta$ is a vector which satisfies the equation

$$(A - \lambda_1 I)\, \beta = \alpha$$

If $\lambda_1$ is a characteristic value of multiplicity m $>$ 2, and p $<$ m then the forms of the m linearly independent solutions corresponding to $\lambda_1$ depend upon whether $p = 1, 2, \ldots, or\ m - 1$

## 3.1 APPLICATIONS

There are many Physical Problems that involve a number of separate elements linked together in some manner. For example electrical networks have this chapter and in other fields. In these and similar Case, the corresponding mathematical problems consists of a system of two or more differential equations, which can always be written as first order equations.

### A.Applications to Mechanics

Systems of linear differential equations originate in the mathematical formulation of numerous problems in mechanics. We consider one such problem in the following example. Another mechanics problem leading to a linear system in given in.

### Example 3.1

On a smooth horizontal plane BC (for example, a smooth table top) an object $A_1$ is connected to a fixed point P by a mass less spring $S_1$ of natural length $L_1$. An object $A_2$ is then connected to $A_1$ by a mass less spring $S_2$ of natural length $L_2$ in such a way that the fixed point P and the centers of gravity $A_1$ and $A_2$ all lie in a straight line (Figure 3.1).

The object $A_1$ is then displaced a distance $a_1$ to the right or left of its equilibrium
position $O_1$, the object $A_2$ is displaced a distance $a_2$ to the right or left of its equilibrium position $O_2$ and at time t = 0 the two objects are released (Figure 3.2). What are the positions of the two objects at any time t > 0.

Formulation. We assume first that the plane BC is so smooth that frictional forces may be neglected. We also assume that no external forces act upon the system. Suppose object $A_1$ has mass $m_1$ and object $A_2$ has mass $m_2$. Further suppose spring $S_1$ has spring constant $k_1$ and spring $S_2$ has spring constant $k_2$. Let $x_1$ denote the displacement of $A_1$ from its equilibrium position $O_2$ at time t $\geq$ 0 and assume that $x_2$ is positive when $A_2$ is to the right of $O_2$(Figure 3.3).Consider the forces acting on $A_1$ at time t > 0. There are two such forces, $F_1$ and $F_2$, where $F_1$ is exerted by spring $S_1$ and $F_2$ is exerted by spring $S_2$. By Hooke's law

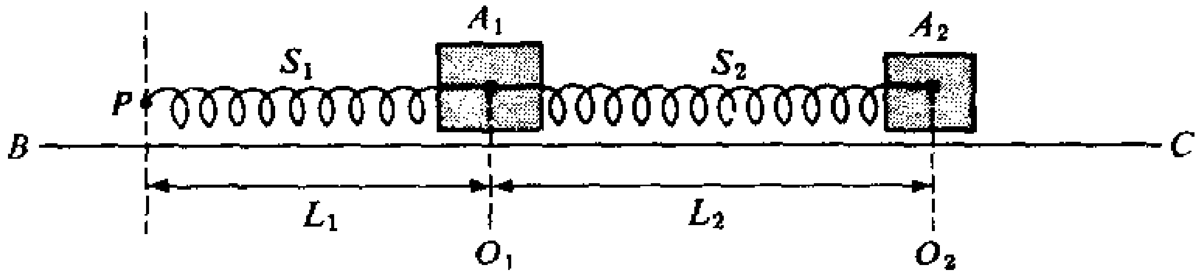


Figure 3.1

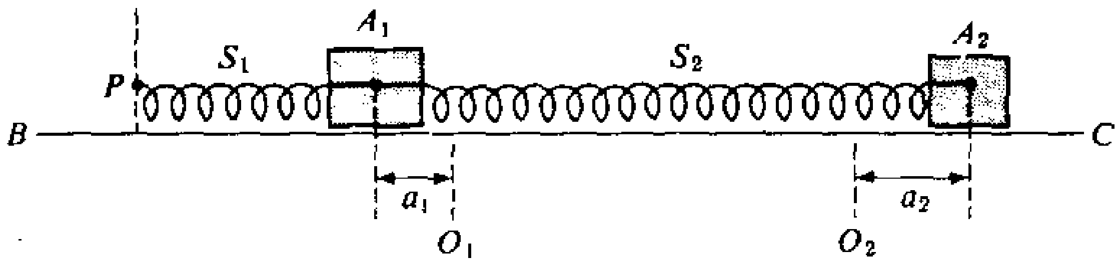


Figure 3.2

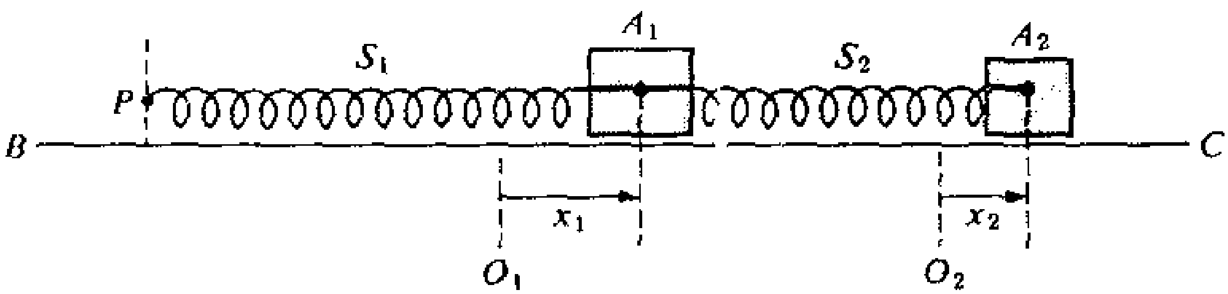


Figure 3.3

(Section 5.1): the force $F_1$ is of magnitude$k_1|x_1|$. Since this force is exerted toward the left when $A_1$ is to the right of $O_1$ and toward the right when $A_1$ is to the left of $O_1$, we have F1 = $-k_1x_1$. Again using Hooke's law, the force $F_2$ is of magnitude $k_2$s, where s is the elongation of $S_2$ at time t. Since s = $|x_2 - x_1|$, we see that the magnitude of $F_2$ is$k_2|x_2 - x_1|$. Further, since this force is exerted toward the left when $x_2 - x_1 < 0$ and toward the right when $x_2 - x_1 > 0$, we see that $F_2 = k_2(x_2 - x_1)$.

Now applying Newton's law (Section 3.2) to the object $A_1$, we obtain the differential equation

$$m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1) \qquad (3.1)$$

We now turn to the object $A_2$ and consider the forces that act upon it at time t > 0. There is one such force, $F_3$, and this is exerted by magnitude $k_2$s = $k_2|x_2 - x_1|$, since $F_3$ is exerted toward the left when $x_2 - x_1$ > 0 and toward the right when $x_2 - x_1$ < 0, we see that $F_2 = -k_2(x_2 - x_1)$. Applying Newton's second law to the object $A_2$, we obtain the differential equation

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) \qquad (3.2)$$

In addition to the differential equations (3.1) and (3.2), we see from the statement of the problem that the initial conditions are given by

$$x_1(0) = a_1 \ , \ x_1{}'(0) = 0 \, , \quad x_2(0) = a_2 \ , \quad x_2'(0) = 0 \qquad (3.3)$$

The mathematical formulation of the problem thus consists. of the differential equations (3.1) and (3.2) and the initial conditions (3.3). Writing the differential equations in the form

$$m_1 \frac{d^2x_1}{dt^2} + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2 \frac{d^2x_2}{dt^2} - k_2x_1 + k_2x_2 = 0 \qquad (3.4)$$

We see that they form a system of homogeneous linear differential equations with constant coefficients.

Solution of a specific case. Rather than solve the general problem consisting of the system (3.4) and conditions (3.3), we shall carry through the solution in a particular case that was chosen to facilitate the work. Suppose the two objects $A_1$ and $A_2$ are each of unit mass, so that $m_1 = m_2 = 1$. Further, suppose that the springs $S_1$ and $S_2$ have spring constants $k_1 = 3$ and $k_2 = 2$, respectively. Also, we shall take $a_1 = -1$ and $a_2 = 2$. Then the system (3.4) reduces to

$$\frac{d^2x_1}{dt^2} + 5x_1 - 2x_2 = 0$$

$$\frac{d^2x_2}{dt^2} - 2x_1 + 2x_2 = 0 \qquad (3.5)$$

and the initial conditions (3.3) become

$$x_1(0) = -1\ ,\ x_1{}'(0) = 0\ ,\quad x_2(0) = 2\ ,\quad x_2'(0) = 0 \qquad (3.6)$$

Writing the system (3.5) in operator notation, we have

$$(D^2 + 5)x_1 - 2x_2 = 0$$

$$-2x_1 + (D^2 + 2)x_2 = 0 \qquad (3.7)$$

We apply the operator $(D^2 + 2)$ to the first equation of (3.7), multiply the second equation of (3.7) by 2, and add the two equations to obtain

$$\left((D^2 + 2)(D^2 + 5) - 4\right)x_1 = 0$$

or

$$(D^4 + 7D^2 + 6)x_1 = 0. \qquad (3.8)$$

The auxiliary equation corresponding to the fourth-order differential equation (3.8) is

$$m^4 + 7m^2 + 6 = 0 \text{ or } (m^2 + 6)(m^2 + 1) = 0$$

Thus the general solution of the differential equation (3.8) is

$$x_1 = c_1 \sin t + c_2 \cos t + c_3 \sin \sqrt{6}t + c_4 \cos \sqrt{6}t \qquad (3.9)$$

We now multiply the first equation of (3.7) by 2, apply the operator $(D^2 + 5)$ to the second equation of (3.7), and add to obtain the differential equation

$$(D^4 + 7D^2 + 6)x_2 = 0 \qquad (3.10)$$

for $x_2$. The general solution of (3.10) is clearly

$$x_2 = k_1 \sin t + k_2 \cos t + k_3 \sin \sqrt{6}t + k_4 \cos \sqrt{6}t\ . \qquad (3.11)$$

The determinant of the operator "coefficients" in the system (3.7) is

$$\begin{vmatrix} D^2 + 5 & -2 \\ -2 & D^2 + 2 \end{vmatrix} = D^4 + 7D^2 + 6$$

Since this is a fourth-order operator, the general solution of (3.5) must contain four independent constants. We must substitute $x_1$ given by (3.9) and $x_2$ given by(3.11) into the equations of the system (3.5) to determine the relations that must exist among the constants $c_1, c_2, c_3, c_4, k_1, k_2, k_3$ and $k_4$ in order that the pair (3.9) and (3.11). Substituting, we find that

$$k_1 = 2c_1 \ , \ k_2 = 2c_2 \ , \ k_3 = -\tfrac{1}{2}c_3 \ , \ k_4 = -\tfrac{1}{2}c_4.$$

Thus the general solution ofthe system (3.5) is given by

$$x_1 = c_1 \sin t + c_2 \cos t + c_3 \sin\sqrt{6}t + c_4 \cos\sqrt{6}t \qquad (3.12)$$

$$x_2 = 2c_1 \sin t + 2c_2 \cos t - \frac{1}{2}c_3 \sin\sqrt{6}t - \frac{1}{2}c_4 \cos\sqrt{6}t$$

We now apply the initial conditions (3.6). Applying the conditions

$x_1 = -1 , \frac{dx_1}{dt} = 0$ at t = 0 to the first of the pair (3.12), we find

$$-1 = c_2 + c_4, \qquad (3.13)$$

$$0 = c_1 + \sqrt{6}c_3.$$

Applying the conditions $x_2 = 2 , \frac{dx_2}{dt} = 0$ at t = 0 to the second of the pair (3.12), we obtain

$$2 = 2c_2 - \tfrac{1}{2}c_4, \qquad (3.14)$$

$$0 = 2c_1 - \frac{\sqrt{6}}{2}c_3.$$

From Equations (3.13) and (3.14), we find that

$$c_1 = 0 , c_2 = \frac{3}{5} , c_3 = 0 , c_4 = -\frac{8}{5}$$

Thus the particular solution of the specific problem consisting of the system (3.5) and the conditions (3.6) is

$$x_1 = \frac{3}{5}\cos t - \frac{8}{5}\cos\sqrt{6}t,$$

$$x_2 = \frac{6}{5}\cos t + \frac{4}{5}\cos\sqrt{6}t\,.$$

**B. Applications to Electric Circuits**

consider the network shown in Figure 3.4.

This network consists of the three loops ABMNA, BJKMB, and ABJKMNA. Points such as B and M at which two or more circuits join are called junction points or branch points. The direction of current flow has been arbitrarily assigned and indicated by arrows.

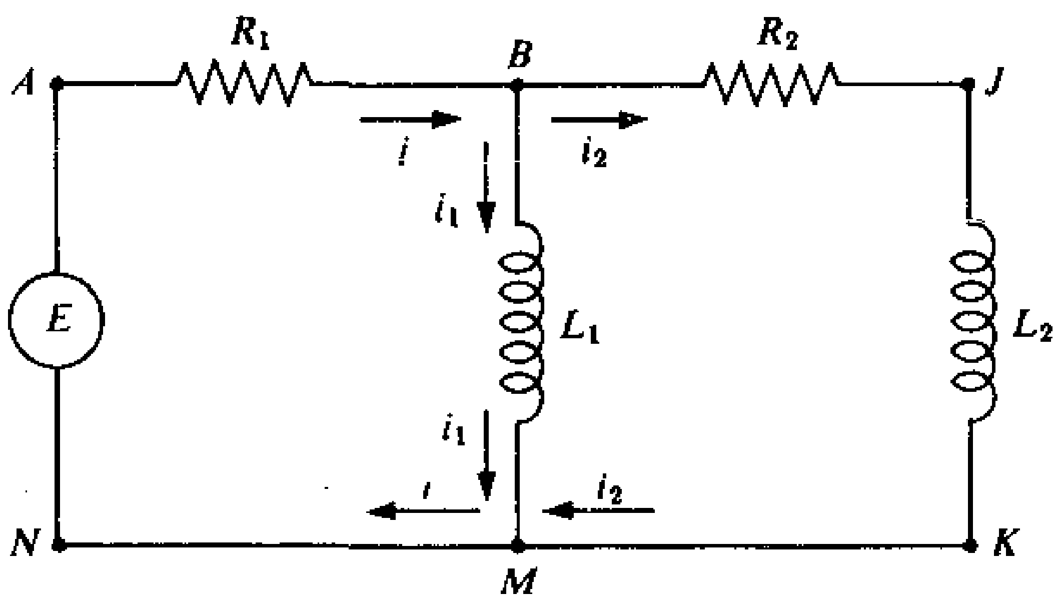


Figure 3.4

in order to solve problems involving multiple loop networks we shall need two fundaniental laws of circuit theory. One of these is Kirchhoff's voltage law, The other basic law that we shall employ is the following:

Kirchhoff's Current Law. In an electrical network the total current flowing into a junction point is equal to the total current flowing away from the junction point.
As an applying of these laws we consider the following problem dealing with the circuit of Figure 3.4.

**Example 3.2**

Determine the currents in the electrical network of Figure 3.4, if E is an electromotive force of 30 V, $R_1$ is a resistor of 10 Ω, $R_2$ is a resistor of 20 Ω, $L_1$ is an inductor of 0.02 H, $L_2$ is an inductor of 0.04 H, and the currents are initially zero.

Formulation. The current flowing in the branch MNAB is denoted by i, that flowing on the branch BM by $i_1$, and that flowing on the branch BJKM by $i_2$. . We now apply Kirchhoff's voltage law to each of the three loops

ABMNA, BJKMB, and ABJKMNA.

For the loop ABMNA the voltage drops are as follows:

1. Across the resistor $R_1$: l0 i.
2. Across the inductor $L_1$ : 0.02 $\frac{di_1}{dt}$.

Thus applying the voltage law to the loop ABMNA, we have the equation $0.02\frac{di_1}{dt} + 10\ i = 30$ (3.15)

For the loop BJKMB, the voltage drops are as follows:

1. Across the resistor $R_2$ : 20 $i_2$.

2.Across the inductor $L_2$ : 0.04 $\frac{di_2}{dt}$.

3. Across the inductor $L_1$ : 0.02 $\frac{di_1}{dt}$.

The minus sign enters into 3 since we traverse the branch MB in the direction opposite to that of the current $i_1$ as we complete the loop BJKMB. Since the loop BJKMB contains no electromotive force, upon applying the voltage law to this loop we obtain the equation

$$-0.02\ \frac{di_1}{dt} + 0.04\frac{di_2}{dt} + 20i_2 \quad (3.16)$$

For the Joop ABJKMNA, the voltage drops are as follows:

1. Across the resistor $R_1$ : 10 i.

2.Across the resistor $R_2$: 20 $i_2$.

3. Across the inductor $L_2$ : 0.04 $\frac{di_2}{dt}$.

Applying the voltage law to this ioop, we obtain the equation

$$10\, i + 0.04\frac{di_2}{dt} + 20i_2 = 30 \quad (3.17)$$

We observe that the three equations (3.15), (3.16), and (3.17) are not all independent. For example, we note that (3.16) may be obtained by subtracting (3.15) from (3.17). Thus we need to retain only the two equations (3.15) and (3.17).

We now apply Kirchhoff's current law to the junction point B. From this we see at once that

$$i = i_1 + i_2 \quad (3.18)$$

In accordance with this we replace i by $i_1 + i_2$ in (3.15) and (3.17) and thus obtain the linear system

$$0.02\frac{di_1}{dt} + 10i_1 + 10i_2 = 30, \quad (3.19)$$

$$10i_1 + 0.04\frac{di_2}{dt} + 30i_2 = 30.$$

Since the currents are initially zero, we have the initial conditions

$$i_1(0) = 0 \quad and \quad i_2(0) = 0 \quad (3.20)$$

Solution.

We introduce operator notation and write the system (3.19)

$$(0.02D + 10)i_1 + 10i_2 = 30, \quad (3.21)$$

$$10i_1 + (0.04D + 30)i_2 = 30.$$

We apply the operator (0.04D + 30) to the first equation of (3.21), multiply the second by 10, and subtract to obtain

$$[(0.04D + 30)(0.02D + 10) - 100]i_1 = (0.04D + 30)30 - 300$$

Or

$$(0.0008D^2 + D + 200)i_1 = 600$$

Or finally

$$(D^2 + 125D + 250000)i_1 = 750000 \qquad (3.22)$$

We now solve the differential equation (3.22) for $i_1$. The auxiliary equation is

$$m^2 + 1250m + 250000 = 0$$

$$(m + 250)(m + 1000) = 0.$$

Thus the complementary function of Equation (3.22) is

$$i_{1\prime c} = c_1 e^{-250t} + c_2 e^{-1000t},$$

And a particular integral is obviously $i_{1\prime p}$ = 3. Hence the general solution of the differential equation (3.22) is

$$i_1 = c_1 e^{-250t} + c_2 e^{-1000t} + 3. \qquad (3.23)$$

Now returning to the system (3.21), we multiply the first equation of the system by 10; apply the operator (0.02 + 10) to the second equation. After

Simplifications we obtain the differential equation

$$(D^2 + 1250D + 250000)i_2 = 0$$

For $i_2$ the general solution of this differential equation is clearly

$$i_2 = k_1 e^{-250t} + k_2 e^{-1000t}. \qquad (3.24)$$

Since the determinant of the operator "coefficients" in the system (3.21) is a second- order operator, the general solution of the system (3.19) must contain two independent constants. We must substitute $i_1$ given by (3.24) and $i_2$ given by (3.24) into the equations of the system (3.19) to determine the relations that must exist among the constants $c_1, c_2, k_1, k_2$ in order that the pair (3.23) and (3.24) represent the general solution of (3.19). Substituting, we find that

$$k_1 = -\frac{1}{2}c_1 \ , \ k_2 = c_2 \qquad (3.25)$$

Thus the general solution of the system (3.19) is given by

$$i_1 = c_1 e^{-250t} + c_2 e^{-100t} + 3, \qquad (3.26)$$

$$i_2 = -\frac{1}{2}c_1 e^{-250t} + c_2 e^{-100t}.$$

Now applying the initial conditions(3.20), we find that $c_1 + c_2 + 3 = 0$ and $-\frac{1}{2}c_1 + c_2 = 0$ and hence $c_1 = -2$ and $c_2 = -1$. Thus the solution of the linear system (3.19) that satisfies the conditions (3.20) is

$$i_1 = -2e^{-250t} - e^{-1000t} + 3,$$

$$i_2 = e^{-250t} - e^{-1000t}.$$

Finally, using (3.18) we find that

$$i = -e^{-250t} - e^{-1000t} + 3.$$

We observe that the current $i_2$ rapidly approaches zero. On the other hand, the currents, $i_1$ and $i = i_1 + i_2$ rapidly approach the value 3.

**C.Application To Mixture Problems:**

**Example 3.3**

Two tanks X and Y are interconnected (see Figure 3.5). Tank X initially contains 100 liters of brine in which there is dissolved 5 kg of salt, and tank Y initially contains 100 liters of brine in which there is dissolved 2 kg of salt. Starting at time t = 0, (1) pure water flows into tank X at the rate of 6 liters/min, (2) brine flows from tank X into tank y at the rate of 8 liters/min, (3) brine is pumped from tank Y back into tank X at the rate of 2 liters/min, and (4) brine flows out of tank Y and away from the system at the rate of 6 liters/min. The mixture in each tank is kept uniform by stirring. How much salt is in each tank at any time t > 0?

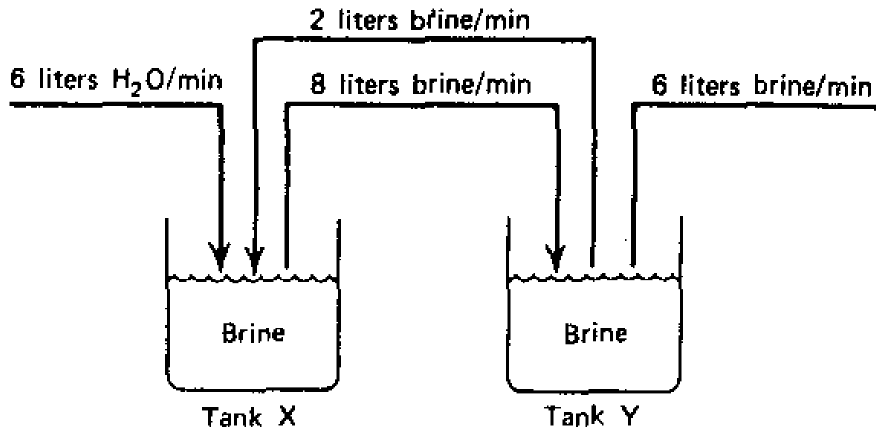


Figure 3.5

Formulation. Let x = the amount of salt in tank X at time t, and let y = the amount of salt in tank Y at time t, each measured in kilograms. Each of these tanks initially contains 100 liters of fluid, and fluid flows both in and out of each tank at the same rate, 8 liters/min, so each tank always contains 100 liters of fluid. Thus the concentration of salt at time t in tank X is x/100 (kg/liter) and that in tank Y is y/l00 (kg/liters).

The only salt entering tank X is in the brine that is pumped from tank Y back into tank X. Since this enters at the rate of 2 liters/min and contains y/100 kg/liter, the rate at which salt enters tank X is 2y/100. Similarly, the only salt leaving tank X is in the brine that flows from tank X into tank Y. Since this leaves at the rate of 8 liters/min and contains x/100 kg/liter, the rate at which salt leaves tank X is 8x/100. Thus we obtain

the differential equation(see section 3.3C in Book No.4)

$$\frac{dx}{dt} = \frac{2y}{100} - \frac{8x}{100} \qquad (3.27)$$

for the amount of salt in tank X at time t. in a similar way, we obtain the differential equation

$$\frac{dy}{dt} = \frac{8x}{100} - \frac{8y}{100} \qquad (3.28)$$

for the amount of salt in tank Y at time t. Since initially there was 5 kg of salt in tank X and 2 kg in tank Y, we have the initial conditions

$$x(0) = 5\ , \qquad y(0) = 2 \tag{3.29}$$

Thus we have the linear system consisting of differential equations (3.27) and (3.28) and initial conditions (3.29).

Solution.

We introduce operator notation and write the differential equations (3.27) and

(3.28) in the forms

$$\left(D + \frac{8}{100}\right)x - \frac{2}{100}y = 0 \tag{3.30}$$

$$-\frac{8}{100}x + \left(D + \frac{8}{100}\right)y = 0$$

We apply the operator $(D + \frac{8}{100})$to the first equation of (3.30), multiply the second equation by $\frac{2}{100}$, and add to obtain

$$\left[\left(D + \frac{8}{100}\right)\left(D + \frac{8}{100}\right) - \frac{16}{(100)^2}\right]x = 0$$

which quickly reduces to

$$\left[D^2 + \frac{16}{100}D + \frac{48}{(100)^2}\right]x = 0 \tag{3.31}$$

We now solve the homogeneous differential equation (3.31) for x. The auxiliary equation is

$$m^2 + \frac{16}{100}m + \frac{48}{(100)^2} = 0,$$

Or

$$\left(m + \frac{4}{100}\right)\left(m + \frac{12}{100}\right) = 0,$$

with real distinct roots ${(-1)}/{25}$ and ,${(-3)}/{25}$. Thus the general solution of equation (3.31) is

$$x = c_1 e^{-(1/25)t} + c_2 e^{-(3/25)t} \qquad (3.32)$$

Now applying the so-called alternative procedure of Section 2.1C(see Book No.4), we obtain from system (3.30) a relation that involves the unknown y but not the derivative Dy. The system (3.30) is so especially simple that the first equation of this system is itself such a relation. Solving this for y, we at once obtain

$$y = 50Dx + 4x \qquad (3.33)$$

From (3.32), we find

$$Dx = -\frac{c_1}{25} e^{-(1/25)t} - \frac{3c_2}{25} e^{-(3/25)t}$$

Substituting into (3.33), we get

$$y = 2c_1 e^{-(1/25)t} - 2c_2 e^{-(3/25)t}$$

Thus the general solution of the system (3.30) is

$$x = c_1 e^{-(1/25)t} + c_2 e^{-(3/25)t} \qquad (3.34)$$

$$y = 2c_1 e^{-(1/25)t} - 2c_2 e^{-(3/25)t}$$

We now apply the initial conditions (3.29). We at once obtain

$$c_1 + c_2 = 5,$$

$$2c_1 - 2c_2 = 2,$$

from which we find

$$c_1 = 3 \quad , \quad c_2 = 2.$$

Thus the solution of the linear system (3.30) that satisfies the initial conditions (3.29) is

$$x = 3e^{-(1/25)t} + 2e^{-(3/25)t}$$

$$y = 6e^{-(1/25)t} - 4e^{-(3/25)t}$$

These expressions give the amount of salt x in tank X, and the amount y in tank

Y, respectively, each measured in kilograms, at any time t (min) > 0. Thus, for example, after 25 min, we find

$$x = 3e^{-1} + 2e^{-3} \approx 1.203\ (kg),$$

$$y = 6e^{-1} - 4e^{-3} \approx 2.008\ (kg).$$

Note that as $t \to \infty$, both x and y → 0. Thus is in accordance with the fact that

no salt at all (but only pure water) flows into the system from outside.

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