Chapter 1
Selfadjoint Analytic Operator and Spectral Functions

The, properties of the local spectral function of a selfadjoint analytic operator function $A(z)$ on Δ_0 and a so-called inner linearization of the operator function $A(z)$ in the subspace $\mathcal{H}(\Delta_0)$ are established. Given two possibly unbounded selfadjoint operators A and G such that the resolvent sets of AG and GA are non-empty, it is shown that the operator AG has a spectral function on $\mathbb{R}$ with singularities if there exists a polynomial $p \neq 0$ such that the symmetric operator $Gp(AG)$ is non-negative. This result generalizes a wellknown theorem for definitizable operators in Krein spaces.

Section (1.1) : Local Spectral Function and Inner Linearization

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$, and denote by $\mathcal{L}(\mathcal{H})$ the set of all bounded linear operators in $\mathcal{H}$. We consider a bounded simply connected domain $\mathcal{D} \subset \mathbb{C}$, that is symmetric with respect to the real axis $\mathbb{R}$, and an $\mathcal{L}(\mathcal{H})$ –valued function $A(z)$ on D which is analytic and selfadjoint, i.e. $A(\bar{z}) = A(z)^*, z \in \mathcal{D}$; in particular, $A(\lambda) = A(\lambda)^*, \lambda \in D \cap \mathbb{R}$. The spectrum $\sigma(A)$, the point spectrum $\sigma_p(A)$, and the resolvent set $\sigma(A)$ of the operator function $A(z)$ are defined in the usual way (see [208], [210]). A real point $\lambda_0 \in \sigma(A)$ is said to be a spectral point of positive type of the operator function $A(z)$, if for each sequence (x_n), satisfying $\|x_n\| = 1$ and $\|A(\lambda_0)x_n\| \to 0$ $if\ n \to \infty$, we have

$$lim\ inf_{n\to\infty} (A'(\lambda_0)x_n, x_n) > 0;$$

(see [208]). The set of all spectral points of positive type of $A(z)$ is denoted by $\sigma_+(A)$.

We fix some real interval $\Delta_0 = [\alpha_0, \beta_0] \subset \mathcal{D}$ and suppose that

$$\Delta_0 \cap \sigma(A) \subset \sigma_+(A), \tag{1}$$

and that $\alpha_0, \beta_0 \in \sigma(A)$. Because of (1) we can choose a complex neighborhood $\mathcal{U}(\subset \mathcal{D})$ of Δ_0 such that $\mathcal{U} \setminus \Delta_0 \subset \sigma(A)$ (see [207]). According to [207], [208] (see also [205]), $A(z)$ admits a linearization Λ in a Krein space $\mathcal{F}$. Here

$$\mathcal{F} = L^2_+(\gamma_0, \mathcal{H})/A(z)L^2_+(\gamma_0, \mathcal{H}) \tag{2}$$

where $\gamma_0(\subset \mathcal{U})$ is a sufficiently smooth simple positive oriented curve which surrounds Δ_0 and passes through the points α_0, β_0. The inner product in $\mathcal{F}$ is defined by the relation

$$\langle f, g\rangle := \frac{1}{2\pi i}\oint_{\gamma_0} A(t)^{-1}f(t), g(\bar{t})dt, f, g \in L^2_+(\gamma_0, \mathcal{H}), \tag{3}$$

followed by the factorization (2). The condition (1) implies that the Krein space $\mathcal{F}$ with the inner product induced by (3) is even a Hilbert space.

In the space $L^2_+(\gamma_0, \mathcal{H})$ we consider the operator Λ_0 of the multiplication by the independent variable. The corresponding operator Λ in the factor space $\mathcal{F}$ is called linearization of the operator function $A(z)$; this operator Λ is selfadjoint in the Hilbert space $\mathcal{F}$.

Let P denote the mapping of the space $\mathcal{H}$ into the space $\mathcal{F}$ which associates with the element $g \in \mathcal{H}$ the equivalence class in $\mathcal{F}$ which contains the vector function $u(t) \equiv g \in L^2_+(\gamma_0, \mathcal{H})$, and let P^* be the adjoint operator from $\mathcal{F}$ into $\mathcal{H}$. The basic relation which connects $A(z)$ and Λ is

$$A(z)^{-1} - B(z) = -P^*(\Lambda - z)^{-1}P, z \in \mathcal{U} \setminus \sigma(A), \tag{4}$$

where $B(z)$ is an analytic in $\mathcal{U}$ operator function which is uniquely defined by the condition that the expression on the left hand side is the principal part of the operator function $A(z)^{-1}$ with respect to γ_0. The linearization is minimal in the sense that (see [207])

$$\mathcal{F} = \overline{span}(\Lambda - z)^{-1}P\mathcal{H} : z \in \mathcal{U} \cap \rho(\Lambda), \tag{5}$$

where $\mathcal{U} \cap \rho(\Lambda)$ can be replaced by any of its nonempty open subsets, and the spectrum of the operator Λ coincides with $\sigma(A) \cap \Delta_0$ (see [207]). The relation

$$P^*(\Lambda - z)^{-1}(\Lambda - \bar{\zeta})^{-1}P = -\frac{1}{2\pi i}P^*\oint_{\gamma_0}\frac{(\Lambda - t)^{-1}dt}{(t - z)(t - \bar{\zeta})}I P = -\frac{1}{2\pi i}\oint_{\gamma_0}\frac{A(t)^{-1}dt}{(t - z)(t - \bar{\zeta})},$$

$z, \zeta \in \mathcal{U}$ and outside of γ_0, implies for these z, ζ the formula

$$\langle(\Lambda - z)^{-1}Px, (\Lambda - \zeta)^{-1}Py\rangle = -\frac{1}{2\pi i}\oint_{\gamma_0} \frac{(A(t)^{-1}x, y)\, dt}{(t - z)(t - \bar{\zeta}),}$$

for the inner product in $\mathcal{F}$. Because of (5), in the Krein space situation (without assumption (1)) this implies that the linearization Λ is uniquely determined up to a weak isomorphism (see [207]), in our situation (with assumption (1)) this implies the uniqueness of the linearization Λ in the Hilbert space $\mathcal{F}$ up to unitary equivalence. It is worth to mention that the quintuple $\{\Lambda, P, P^*; \mathcal{F}, \mathcal{H}\}$ is a spectral node in the sense of [206].

Since the spectrum of the selfadjoint operator Λ is contained in Δ_0, the operator Λ has a spectral function E which is supported on Δ_0 and is defined for all Borel subsets Γ of $\mathbb{R}$. In [208] the $\mathcal{L}(\mathcal{H})$-valued function

$$Q(\Gamma) := P^*E(\Gamma)P, \Gamma\, Borel\, set\, in\, \Delta_0,$$

was called the local spectral function of the operator function $A(z)$ on Δ_0 (in fact, in [208] this notion was used for the function $Q_t := Q([\alpha_0, t]), t \in \Delta_0$); in the following we call the range $ranQ(\Gamma)$ the spectral subspace of $A(z)$ corresponding to Γ. Clearly, the values $Q(\Gamma)$ of the local spectral function are nonnegative operators in $\mathcal{H}$ but in general not projections.

Under the general assumption (1) the local spectral function does not have some of the properties that one usually associates with the term 'spectral function', e.g. its ranges on disjoint intervals can have nonempty intersection. This is excluded if instead of (1) on Δ_0 the Virozub–Matsaev condition (VM) is imposed:

$$(VM) \exists_{\varepsilon}, \delta > 0 : \lambda \in \Delta_0, f \in \mathcal{H}, \|f\| = 1, |(A(\lambda)f, f)| < \varepsilon \Longrightarrow (A'(\lambda)f, f) > \delta.$$

In [208] it was shown that (VM) is a natural condition for a comprehensive spectral theory of the selfadjoint operator function $A(z)$. We also assume that $A(\alpha_0), A(\beta_0)$ are boundedly invertible. It was shown in [208] that under the condition (VM) the operator $Q(\Delta_0)$ has closed range and hence it has the remarkable property that it is uniformly positive on its range. In particular, the spectral subspace $ranQ(\Delta_0) =: \mathcal{H}(\Delta_0)\, of\, A(z)$ corresponding to the interval Δ_0 is a closed subspace of $\mathcal{H}$, which admits the decomposition (see [208])

$$\mathcal{H} = ran\, A(\beta_0) - \dot{+}\ \mathcal{H}(\Delta_0)\ \dot{+}\ ranA(\alpha_0)_+.$$

We mention that the condition (VM) in the case of a finite dimensional space $\mathcal{H}$ reduces to the simpler condition (vm) (see [331]):

$$(vm)\ \lambda_0 \in \Delta_0, (A(\lambda_0)f, f) = 0\, for\, some\, f \in \mathcal{H}, f \neq 0 \Longrightarrow (A'(\lambda_0)f, f) > 0.$$

The basic result of the present note is that if the condition (VM) is satisfied on Δ_0 and A(z) is boundedly invertible in the endpoints of Δ_0, then always

$$ran\, P = \mathcal{F}, and\, P^*\, is\, a\, bijection\, from\, \mathcal{F}\, onto\, \mathcal{H}(\Delta_0). \quad (6)$$

This implies that the spectral subspaces $ranQ(\Gamma)\ of\ A(z)$ for any Borel set $\Gamma \subset \Delta_0$ are closed. It also allows us to show Theorem 7.11in [208] and the following results of [208] in a more compact way and sometimes in a more general form. Moreover, we construct a so-called inner linearization of the analytic operator function $A(z)$, which acts in the subspace $\mathcal{H}(\Delta_0)$of the originally given Hilbert space $\mathcal{H}$.

We have already mentioned that the space $\mathcal{F}$ and the linearization Λ were intoduced in [206] without the restriction (1), i.e. for an arbitrary selfadoint analytic operator function $A(z)$ such that its spectrum is a compact subset of the domain $\mathcal{D}$. In general the space $\mathcal{F}$ is a Krein space and it is 'much larger' than the given space $\mathcal{H}$ (with respect to the mapping P). E.g. for a selfadjoint monic selfadjoint operator polynomial $A(z)$ of degree n and $\mathcal{D} = \mathbb{C}$, the space $\mathcal{F}$ can be chosen to be $\mathcal{H}^n$ (and Λ can be choosen to be the companion operator of $A(z)$). By enlarging the space $\mathcal{H}$ we gain that the linear operator Λ in $\mathcal{F}$ represents in $\mathcal{D}$ the spectral properties of the analytic operator function $A(z)$. However, under the operator function $A(z)$ satisfies the condition (VM) on the interval $[\alpha_0, \beta_0]$ and that the operators $A(\alpha_0)$ and $A(\beta_0)$ are boundedly invertible, the subspace $\mathcal{H}(\Delta_0)$ of $\mathcal{H}$, which is mapped by P bijectively onto $\mathcal{F}$, can be even smaller than $\mathcal{H}$, that is, also $\mathcal{F}$ can be 'smaller' than $\mathcal{H}$; in this situation P will have a nontrivial kernel. In the particular case that $A(\alpha_0) \ll 0, A(\beta_0) \gg 0$, the space $\mathcal{F}$ is of the 'same size' as $\mathcal{H}$, that is, the mapping P is a bijection between $\mathcal{H}$ and $\mathcal{F}$. Clearly, ifmathcal $\mathcal{H}$ is finite–dimensional, $\mathcal{F}$ 'smaller' than $\mathcal{H}$ means $dim\mathcal{F} \leq dim\mathcal{H}$.

The claims (6) are showed in this Section . And we establish some properties of the local spectral function and of the spectral subspaces of $A(z)$. Finally, we introduce in $\mathcal{H}(\Delta_0)$ the inner linearization S of the operator function $A(z)$ corresponding to the interval Δ_0, which is just an isomorphic copy of the linearization $\Lambda\, in\, \mathcal{F}$. As an application, in the case $\mathcal{H}(\Delta_0) = \mathcal{H}$, which is equivalent to $A(\alpha_0) \ll 0\, and\, A(\beta_0) \gg 0$, we give a simple proof of a factorization result of Virozub-Matsaev from [211] and explicit expressions for the factors in terms of the local spectral function of $A(z)$.

Theorem (1.1.4) below is the crucial result of this section. In its proof we need some lemmata which we show first. Recall that the condition (VM) is supposed to hold on the interval Δ_0 and that the endpoints of Δ_0 are regular points for the operator function $A(z)$. For an interval $\Delta \subset \Delta_0$ we introduce the following subspaces of $\mathcal{H}$:

$$\mathcal{H}_\Delta := \overline{P^*E(\Delta)\mathcal{F}}, \mathcal{H}(\Delta) := \overline{P^*E(\Delta)P\mathcal{H}}. \tag{7}$$

If $\Delta = \Delta_0$ the closure in the last relation is superfluous (see [208]); later we will see that all closures are superfluous. From the definition in (7) it follows immediately that

$$\mathcal{H}(\Delta) \subset \mathcal{H}_\Delta; \tag{8}$$

below (Corollary (1.2.5)) it will be shown that in (8) always equality holds.

Lemma (1.1.1)[204] If $\Delta \subset \Delta_0$, then

$$\mathcal{H}_\Delta = \overline{span}\, \{P^*E(\Delta')P\mathcal{H} : \Delta' \subset \Delta_0\}. \tag{9}$$

Proof. Clearly,

$$P^*E(\Delta')P\mathcal{H} \subset P^*E(\Delta')\mathcal{F} \subset P^*E(\Delta')\mathcal{F},$$

and hence for the two sets in (9) the inclusion $\supset$ follows.

Conversely, from the minimality (5) of the linearization Λ we have

$$P^*E(\Delta')\mathcal{F} = \overline{span}\, \{P^*E(\Delta)\mathcal{F}(\Lambda - z)^{-1}P\mathcal{H} : z \in \mathcal{U} \cap \rho(\Lambda)\}.$$

Observing that

$$P^*E(\Delta)\mathcal{F}(\Lambda - z)^{-1}P = \int_\Delta (\Lambda - z)^{-1}P^* \, dE(\lambda)P$$

and approximating the integral by finite Riemann-Stieltjes sums, we see that each element $x \in E(\Delta)\mathcal{F}$ belongs to the set on the right hand side of (9), and the inclusion $\subset$ for the two sets in (9) follows.

The following lemma is a partial extension of [208], where it was showed for $x \in \mathcal{H}(\Delta)$.

Lemma (1.1.2)[204] Let $\Delta = [\alpha, \beta] \subset \Delta_0$. If $x \in \mathcal{H}_\Delta$, then

$$(A(\alpha)x, x) \leq 0, (A(\beta)x, x) \geq 0. \tag{10}$$

Proof. Consider two intervals $\Delta_j = [\alpha_j, \beta_j], j = 1, 2$, such that $\alpha \leq \alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \beta$, and an element

$$x = P^*E(\Delta_1)Px'_1 + P^*E(\Delta_2)Px'_2 =: x_1 + x_2, x'_1, x'_2 \in \mathcal{H}.$$

By [208] we have $(A(\beta_1)x_1, x_1) \geq 0$, and hence, by [330] $(A(\alpha_2)x_1, x_1) \geq 0$. According to [208] this implies

$$(A(\alpha_2)x_1 + x_1), x_1 + x_1) \geq 0,$$

and, again by [208], the relation $(A(\beta)(x_1 + x_1), x_1 + x_1) \geq 0$ follows.

By induction, we obtain the second inequality in (10) for all $x \in \mathcal{H}$ of the form

$$x = \sum_{j=1}^{n} P^*E(\Delta_j)Px'_j, x'_j \in \mathcal{H}, j = 1, 2, \ldots, n, \tag{11}$$

where $\Delta_j = [\alpha_j, \beta_j], \alpha \leq \alpha_1, \beta_j \leq \alpha_{j+1}, j = 1, \ldots, n-1, \beta_n \leq \beta$. If an element x of the form (11) with arbitrary closed intervals Δ_j is given, we can choose a new decomposition of the set $\bigcup_{j=1}^{n} \Delta_j$ into closed intervals such that not any two of them have common inner points, and obtain a representation of x as required in the line after (11). By Lemma (1.1.1) the set of elements x of the form (11) with arbitrary closed intervals Δ_j is dense in $\mathcal{H}_\Delta$, and the second inequality in (10) follows. The proof of the first inequality in (10) is analogous.

Lemma (1.1.3) [204]. Let $\Delta =: [\alpha, \beta] \subset \Delta_0$, denote by P_Δ the orthogonal projection onto $\mathcal{H}_\Delta$ and set

$$A_\Delta(\lambda) := P_\Delta A(\lambda)|_{\mathcal{H}_\Delta}. \quad (12)$$

If $\mathcal{H}$ is separable, then $\sigma_p(A_\Delta)$ is at most countable.

Proof. We can suppose that Δ lies strictly inside Δ_0, i.e.

$$\alpha_0 < \alpha, \beta_0 > \beta. \quad (13)$$

Indeeed, if e.g. $\alpha_0 = \alpha$, we replace the point α_0 by a point α'_0 such that $\alpha'_0 < \alpha_0, [\alpha'_0, \alpha_0] \subset \rho(A)$ and (VM) holds on $[\alpha'_0, \alpha_0]$.

By Lemma 1.1.2, $A_\Delta(\alpha) \leq 0$, $A_\Delta(\beta_0) \geq 0$, and then from (13) and [208] it follows that $A_\Delta(\alpha_0) \ll 0$, $A_\Delta(\beta) \gg 0$. Since (VM) holds for $A_\Delta(\lambda)$ on Δ_0 and $\alpha_0, \beta_0 \in \rho(A_\Delta)$, the operator function $A_\Delta(\lambda)$ on Δ_0 has a linearization Λ_Δ which is a selfadjoint operator in a Krein space $\mathcal{F}_\Delta$ (see [207]), and since the whole spectrum of Λ_Δ is of positive type (see [207]), the space $\mathcal{F}_\Delta$ is uniformly positive, i.e. it is in fact a Hilbert space (see [13]). The separability of $\mathcal{H}_\Delta$ implies the separability of the space $\mathcal{F}_\Delta$, and hence the point spectrum of the selfadjoint operator Λ_Δ is at most countable. By [207], $\rho(A_\Delta) = \rho(\Lambda_\Delta)$.

Theorem (1.1.4)[204]. If the condition (VM) holds on the interval Δ_0 and the endpoints of Δ_0 are regular points for the operator function $A(z)$, then the operators $P : \mathcal{H} \mapsto \mathcal{F}$ and $P^* : \mathcal{F} \mapsto \mathcal{H}$ have the properties

$$ran\, P = \mathcal{F}, ran\, P^* = \mathcal{H}_{\Delta_0},$$

and P^* is a bijection between $\mathcal{F}$ and $\mathcal{H}_{\Delta_0}$.

Proof. Since ran P^* is closed (see [208]) and since $E(\Delta_0) = I$, from the definition of $\mathcal{H}_{\Delta_0}$ we have ran $P^* = \mathcal{H}_{\Delta_0}$. If we show that $ran\ P = \mathcal{F}$, then it follows that P^* is injective and the Theorem is showed.

According to the definition of $\mathcal{F}$ and P, ran $P = \mathcal{F}$ means that for any vector function $f(t) \in L^2_+(\gamma_0, \mathcal{H})$ there exists an element $g \in \mathcal{H}$ such that

$$f(t) - g \in A(t)L^2_+(\gamma_0, \mathcal{H}). \quad (14)$$

In the first step of the proof we show that without loss of generality we can suppose that the space $\mathcal{H}$ is separable. To this end we choose a dense countable subset $T = \{t_j : j = 1, 2, \ldots\}$ of $\mathcal{D}$ and, with the given function $f(t)$ in (14), we consider the closure $\widetilde{\mathcal{H}}$ of the linear span of all elements

$$A(t_{j1})A(t_{j2}) \cdots A(t_{jn})f(t_j), t_j, t_{j1}, t_{j2}, \ldots, t_{jn} \in \mathcal{T}, j, n \in \mathbb{N}.$$

It is easy to check that $f(z) \in \widetilde{\mathcal{H}}$ for all $z \in \mathcal{D}$ and that $\widetilde{\mathcal{H}}$ is an invariant subspace of the operators $A(z), z \in \mathcal{D}$. So the restriction $\tilde{A}(z)\ of\ A(z)$ to the separable Hilbert space $\widetilde{\mathcal{H}}$ is an operator function $\tilde{A}(z)$ in $\widetilde{\mathcal{H}}$ with the same properties as $A(z)$ in $\mathcal{H}$, and the Hilbert space $L^2_+(\gamma_0, \mathcal{H})$ can be considered in a natural way as a subspace of $L^2_+(\gamma_0, \mathcal{H})$. Therefore, if we find an element bg in $\widetilde{\mathcal{H}}$ such that

$$f(t) - \tilde{g} \in A(t)L^2_+(\gamma_0, \mathcal{H}),$$

then for the element g in (14) we can choose $g = \tilde{g}$. So we suppose in the rest of this proof that the space $\mathcal{H}$ is separable.

The relation ran $P = \mathcal{F}$ will be proved if we show that ker $P^* = \{0\}$. Assume $\tilde{x}_0 \in \mathcal{F}, P^*\tilde{x}_0 = 0$. Choose a point $\lambda_0 \notin \sigma_p(A_{\Delta_0})$ that is close to the point $\frac{\alpha_0+\beta_0}{2}$, and set $\Delta_{1,1} := [\alpha_0, \lambda_0]\ and\ \Delta_{1,2} = [\lambda_0, \beta_0]$. Denote $x_1 := P^*E(\Delta_{1,1})\tilde{x}_0$.

The relation $P^*\tilde{x}_0 = 0$ implies

$$P^*E(\Delta_{1,2})\tilde{x}_0 = -P^*E(\Delta_{1,1})\tilde{x}_0,$$

hence $x_1 \cap \mathcal{H}_{\Delta_{1,1}} \cap \mathcal{H}_{\Delta_{1,2}}$. By Lemma (1.1.2), $(A(\lambda_0)x_1, x_1) \geq 0$ and $(A(\lambda_0)x_1, x_1) \leq 0$, i.e.

$$(A(\lambda_0)x_1, x_1 = 0. \quad (15)$$

Since, according to Lemma (1.1.2), $A_{\Delta_{1,1}}(\lambda_0) \geq 0$, relation (15) implies that

$$A_{\Delta_{1,1}}(\lambda_0)x_1 = 0.$$

Using $x_1 \in \mathcal{H}_{\Delta_{1,1}}$, we obtain by precisely the same argument that $A_{\Delta_{1,2}}(\lambda_0)x_1 = 0$. So, the vector $A(\lambda_0)x_1$ is orthogonal to both $\mathcal{H}_{\Delta_{1,j}}, j = 1, 2$, and therefore it is orthogonal to $\mathcal{H}_{\Delta_0}$. Hence

$$A_{\Delta_0}(\lambda_0)x_1 = 0,$$
and the condition $x_1 \in \mathcal{H}_{\Delta_{1,2}}$ implies that $x_1 = 0$. This means that
$$P^*E(\Delta_{1,1})\tilde{x}_0 = P^*E(\Delta_{1,2})\tilde{x}_0 = 0.$$
Now choose points λ'_1, λ'_2 which are close to the middle points of $\Delta_{1,1}$ and $\Delta_{1,2}$ respectively, and such that $\lambda'_j \notin \sigma_p(A_{\Delta_{1,j}}), j = 1, 2$. We denote the corresponding subintervals of Δ_0 by $\Delta_{2,j}, j = 1, 2, 3, 4$. By the same arguments as above we find $P^*E(\Delta_{2,j})\tilde{x}_0 = 0, j = 1, 2, 3, 4$. Continuing this procedure we obtain a sequence of partitions $\{\Delta_{n,j}, j = 1, 2, \ldots, 2^n\}$ of $\Delta_{n,j} = 1, 2, \ldots$, such that
$$P^*E(\Delta_{n,j})\tilde{x}_0 = 0, j = 1, 2, \ldots, 2^n, n = 1, 2, \ldots.$$
Hence with $t_{n,j} \in \Delta_{n,j}$ and $z \in \mathbb{C} \setminus \Delta_0$ we find
$$\langle \tilde{x}_0, \sum_{j=1}^{2^n} (t_{n,j} - z)^{-1} E(\Delta_{n,j}) P\mathcal{H} =\rangle \{0\}.$$
Passing to the $limit\ n \to \infty$ we obtain
$$\langle \tilde{x}_0, (\Lambda - z)^{-1} P\mathcal{H} \rangle = \{0\},$$
and the minimality of the linearization (see (5)) implies that $\tilde{x}_0 = 0$. Thus, ker $P^* = \{0\}$, and hence ran $P = \mathcal{F}$.

Corollary (1.1.5)[204]. For all intervals $\Delta \subset \Delta_0$,
$$P^*E(\Delta)\mathcal{F} = P^*E(\Delta)P\mathcal{H}, \qquad (16)$$
and these sets are closed; in particular $\mathcal{H}_\Delta = \mathcal{H}(\Delta)$.

Proof. The equality (16) follows from the relation ran $P = \mathcal{F}$. Since P^* is an isomorphism and the spectral subspace $E(\Delta)\mathcal{F}$ is closed, the subspace on the left hand side in (16) is closed as well.

Recall that
$$Q(\Gamma) = P^* E(\Gamma) P, \Gamma\ Borel\ set\ of\ \Delta_0, \qquad (17)$$
is the local spectral function of the analytic operator function $A(z)$, the range $ranQ(\Gamma)$ is the spectral subspace of the analytic operator function $A(z)$ corresponding to Γ. Clearly, $Q(\Gamma)$ is a nonnegative operator in $\mathcal{H}$ which, because of Corollary (1.1.5), is uniformly positive on its range. If $\Delta \subset \Delta_0$ is an interval such that the endpoints $\alpha, \beta\ of \Delta$ are not eigenvalues of the operator function $A(z)$, then the operator $Q(\Delta)$ can be expressed directly through $A(z)$ as follows:
$$Q(\Delta) = \frac{1}{2\pi i} \int'_{\gamma(\Delta)} A(z)^{-1}\, dz, \qquad (18)$$
where $\gamma(\Delta)$ is a smooth contour in $\mathcal{U}$ which surrounds Δ and crosses the real axis in α and β orthogonally, the prime at the integral denotes the Cauchy principal value at α and β. The relation (18) follows from the representation of the spectral function of the linearization Λ by means of the resolvent of Λ and the relation (20) (see also [208]).

The properties of the operator valued set function $Q(\Gamma)$ are summarized in the following theorem.

Theorem (1.1.6) [204]. Let $\Gamma, \Gamma_1, \Gamma_2, \ldots \subset \Delta_0$ be Borel sets. Then:

(a) $Q(\Gamma) = P^*E(\Gamma)\mathcal{F}$, and $ranQ(\Gamma)$ is closed.

(b) If $\Gamma_1 \subset \Gamma_2$, then $ranQ(\Gamma_1) \subset ranQ(\Gamma_2)$.

(c) If $Q(\Gamma_1 \cap \Gamma_2) = 0$, then
$$ranQ(\Gamma_1) \cap ranQ(\Gamma_2) = \{0\}, ranQ(\Gamma_1) \dotplus ranQ(\Gamma_2) = ranQ(\Gamma_1 \cup \Gamma_2).$$
(d) If $(\Gamma_j)_1^\infty$ is an infinite sequence such that $Q(\Gamma_j \cap \Gamma_k) = 0$ for all $j \neq k, j, k = 1, 2, \ldots$, then
$$Q\left(\bigcup_{j=1}^{\infty} \Gamma_j\right) = \sum_{j=1}^{\infty} Q(\Gamma_j),$$
where the sum on the right hand side converges strongly and unconditionally.

(e) The point $\lambda_0 \in \Delta_0$ is a regular point of the operator function $A(z)$ if and only if there exists a neighbourhood Γ of λ_0 such that $Q(\Gamma) = 0$.

(f) The point $\lambda_0 \in \Delta_0$ is an eigenvalue of the operator function $A(z)$ if and only if $Q(\{\lambda_0\}) \neq 0$; in this case

$$kerA(\lambda_0) = ranQ(\{\lambda_0\}). \qquad (19)$$

(g) For an open interval Δ the following two statements are equivalent:

(i) dim $ranQ(\Delta) = n$;

(ii) Δ contains only a finite number of points of $\sigma(A)$, all of them are eigenvalues and the sum of their multiplicities is n.

In this case

$$ranQ(\Delta) = span\,\{kerA(\lambda_j) : \lambda_j \in \Delta \cap \sigma(A)\}.$$

(h) The eigenvectors of the operator function $A(z)$, corresponding to different eigenvalues in Δ_0, are linearly independent. If there is an infinite number of such eigenvalues, then the corresponding eigenvectors form a Riesz basis in their closed linear span.

(l) If Δ is a subinterval of Δ_0 and A_Δ is the operator function in $\mathcal{H}(\Delta)$ $(= \mathcal{H}_\Delta)$ defined by (12), then

$$\sigma(A_\Delta) \cap \Delta_0 \subset \overline{\Delta}.$$

Proof. The first equality in (a) is a consequence of (16). All the other statements in (a)–(l) follow from the fact that ran $P = \mathcal{F}$ and that P^* is an isomorphic embedding of $\mathcal{F}$ into $\mathcal{H}$, see Theorem (1.1.4), and the corresponding properties of the spectral function E of the selfadjoint operator Λ in $\mathcal{F}$.

For the proof of (l) we observe that with $\overline{\Delta} =: [\alpha, \beta]$ by Lemma (1.1.2) we have $A_\Delta(\alpha) \leq 0, A_\Delta(\beta) \geq 0$.

Then, by [208], $A_\Delta(\alpha') \ll 0$ for all $\alpha' \in [\alpha_0, \alpha)$ and $A_\Delta(\beta') \gg 0$ for all $\beta' \in (\beta, \beta_0]$, therefore the intervals $[\alpha_0, \alpha)$ and $(\beta, \beta_0]$, belong to $\rho(A_\Delta)$.

Indeed, multiply (20) by $z_0 - z$ and let $z \to z_0$. Then the left hand side converges strongly to $Q(\{z_0\})$, and we can also apply it to elements $y(z)$ which converge strongly for $z \to z_0$. Now for any x we find

$$P^*E(\{\lambda_0\})Px = lim_{z \to z_0}(z - z_0)A(z)^{-1}x,$$

and the limit on the right hand side exists; it is easy to check that this element is in $kerA(z_0)$. The converse inclusion follows if in the above reasoning we choose $y(z) = \frac{A(z)x_0}{z - z_0}$ with x_0 such that $A(z_0)x_0 = 0$.

In this section it is convenient to consider P^* as an operator which acts from $\mathcal{F}$ to $\mathcal{H}(\Delta_0)$. We will use for this operator the special notation P_0^*. By Theorem (1.1.4), the operator P_0^* is boundedly invertible, and it is easy to see that

$$(P_0^*)^*x = Px, \qquad x \in \mathcal{H}(\Delta_0). \qquad (20)$$

Furthermore, in this section also the operators $Q(\Gamma)$ will be considered as operators in $\mathcal{H}(\Delta_0)$. Then $Q(\Delta_0)$ is a positive and boundedly invertible operator.

Under the bijection P_0^* from $\mathcal{F}$ onto $\mathcal{H}(\Delta_0)$ the linearization Λ in $\mathcal{F}$ of the operator function $A(z)$ becomes an operator S which is selfadjoint in $\mathcal{H}(\Delta_0)$ with respect to a suitable Hilbert inner product $(\cdot,\cdot)_0$. More exactly, we equip $\mathcal{H}(\Delta_0)$ with the positive definite inner product

$$(x, y)_0 := (Q(\Delta_0)^{-1}x, y), \qquad x, y \in \mathcal{H}(\Delta_0), \qquad (21)$$

and define in $\mathcal{H}(\Delta_0)$ the operator

$$S := P_0^*\,\Lambda(P_0^*)^{-1}. \qquad (22)$$

Since S acts in a subspace of the originally given space $\mathcal{H}$ (equipped with the inner product (21)) we call S the inner linearization of $A(z)$.

Theorem (1.1.7) [204]. The operator S in (22) is selfadjoint in $\mathcal{H}(\Delta_0)$ with respect to the inner product $(\cdot,\cdot)_0$ from (21),and, if E_S denotes the spectral function of S, for each Borel subset $\Gamma of\ \Delta_0$ we have

$$E_S(\Gamma) = Q(\Gamma)Q(\Delta_0)^{-1}, \qquad ranE_S(\Gamma) = ranQ(\Gamma). \qquad (23)$$

Proof. Under the mapping P^* the (positive definite) inner product $\langle .,. \rangle$ on $\mathcal{F}$ becomes a positive definite inner product $(\cdot,\cdot)_0$ on $\mathcal{H}(\Delta_0)$. In fact, if $\tilde{x}, \tilde{y} \in \mathcal{F}$, $P^*\tilde{x} = x, P^*\tilde{y} = y$ with $x, y \in \mathcal{H}(\Delta_0)$, then, observing (20), we find

$$(x, y)_0 = \langle \tilde{x}, \tilde{y} \rangle = \langle (P_0^*)^{-1}x, (P_0^*)^{-1}y \rangle = ((P_0^*)^{-*}(P_0^*)^{-1}x, y)$$

$$= ((P_0^* P)^{-1}x, y) = Q(\Delta_0)^{-1}x, y).$$

Further, $Q(\Delta_0) = P^*P$ implies $(P_0^*)^{-1} = PQ(\Delta_0)^{-1}$, and we obtain

$$S = P^*\Lambda(P_0^*)^{-1} = \int_{\Delta_0} \lambda dQ(\lambda)Q(\Delta_0)^{-1}. \tag{24}$$

Clearly, S is selfadjoint in the inner product $(\cdot,\cdot)_0$ and for any Borel subset $\Gamma \subset \Delta_0$ the spectral projection $E_S(\Gamma)$ of S in $\mathcal{H}(\Delta_0)$ is given by

$$E_S(\Gamma) = P^*\Lambda(P_0^*)^{-1}PQ(\Delta_0)^{-1} = Q(\Gamma)Q(\Delta_0)^{-1},$$

and also the second equality in (23) follows.

The relation (23) yields also the following corollary.

Corollary (1.1.8)[204]. If Γ_1, Γ_2 are Borel subsets of Δ_0 *and* $Q(\Gamma_1 \cap \Gamma_2) = 0$, then the spectral subspaces $ranQ(\Gamma_1)$ and $ranQ(\Gamma_2)$ are orthogonal in the inner product $(\cdot,\cdot)_0$.

Finally we show that a factorization result from [211] can be obtained in a simple way from the above considerations, and we obtain an explicit form of the factors in terms of the spectral function of $A(z)$.

Theorem (1.1.9)[204]. Suppose that, additionally to the above assumptions, $A(\alpha_0) \ll 0$ and $A(\beta_0) \gg 0$. Then, in a neighbourhood of Δ_0, the operator function $A(z)$ admits the unique factorization

$$A(z) = A_1(z)(S - z). \tag{25}$$

Here the operator $S \in \mathcal{L}(\mathcal{H})$ is selfadjoint with respect to the inner product $(\cdot,\cdot)_0$ on $\mathcal{H}$ and such that

$$\sigma(S) = \sigma(A) \cap \Delta_0,$$

the operator function $A_1(z)$ is boundedly invertible in a neighbourhood of Δ_0, and such that $A_1(z)$ and its inverse $A_1(z)^{-1}$ are analytic there. The operator S and the operator function $A_1(z)$ admit the representations

$$S = \int_{\Delta_0} \lambda dQ(\lambda)Q(\Delta_0)^{-1}, A_1(z) = \int_{\Delta_0} \frac{A(\lambda) - A(z)}{\lambda - z} dQ(\lambda)Q(\Delta_0)^{-1}.$$

Proof. According to [208], $\mathcal{H}(\Delta_0) = \mathcal{H}$. Hence P is an invertible operator from $\mathcal{H}$ to $\mathcal{F}, P^*$ is an invertible operator from $\mathcal{F}$ to $\mathcal{H}$, and $Q(\Delta_0)$ is an invertible operator in $\mathcal{H}$. Consider the operator $S = P^*\Lambda P^{-*}$ from (24) . The relation (4) and the invertibility of P^* imply

$$A(z)^{-1} = -(S - z)^{-1}P^*P + B(z). \tag{26}$$

for all $z \in \mathcal{U} \setminus \Delta_0$, where $\mathcal{U}$ is a neighbourhood of Δ_0. Multiplying (26) from the left by $A(z)$ gives

$$-A(z)(S - z)^{-1}P^*P = I - A(z)B(z). \tag{27}$$

On the other hand, multiplying (26) from the left by $S - z$ implies

$$(S - z)A(z)^{-1} = -P^*P + (S - z)B(z). \tag{28}$$

Denote $A_1(z) := A(z)(S - z)^{-1}$. Taking into account that P^*P is an invertible operator we see from (27) and (28) that $A_1(z)$ is invertible in a neighbourhood of Δ_0 and that $A_1(z)^{-1}$ is analytic there. Moreover,

$$A_1(z) = A(z)(S - z)^1 = A(z) \int_{\Delta_0} (\lambda - z)^{-1} dQ(\lambda)Q(\Delta_0)^{-1}$$

$$= \int_{\Delta_0} \frac{A(z) - A(\lambda)}{\lambda - z} dQ(\lambda)Q(\Delta_0)^{-1}.$$

To show the uniqueness of the factorization we consider another factorization $A(z) = \tilde{A}_1(z)(\tilde{S} - z)$ with the same properties as in (25). It follows that

$$(\tilde{S} - z)(S - z)^{-1} = \tilde{A}_1(z)^{-1} A_1(z).$$

The operator function on the left hand side is analytic outside Δ_0, the function on the right hand side is analytic in a neighbourhood of Δ_0. Thus both operator functions can be extended by analytic continuation to the whole complex plane.

If $z \to \infty$ the left hand side tends to the identity operator, and the claim $S = \tilde{S}$, $A_1(z) = b\,\widehat{A_1}(z)$ follows from Liouville's theorem.

Section (1.2): Products of Selfadjoint Operators

Let A and G be two selfadjoint operators in a Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$ such that either A or G is bounded and boundedly invertible. Then the product AG is selfadjoint in a Krein space. Indeed, if $G(A)$ is bounded and boundedly invertible, then AG is selfadjoint in the Krein space $(\mathcal{H}, [\cdot\,,\cdot]_G)$ $((\mathcal{H}, [\cdot\,,\cdot]_{A^{-1}})$, respectively), where

$$[x, y]_G = (Gx, y), [x, y]_{A^{-1}} = (A^{-1}x, y), x, y \in \mathcal{H}.$$

Conversely, a selfadjoint operator in a Krein space can be written as a product of two selfadjoint operators in a Hilbert space one of which is bounded and boundedly invertible.

The spectrum of a selfadjoint operator in a Krein space is symmetric with respect to the real axis. But even simple examples show that the spectrum of such operators can be empty or cover the entire complex plane. However, some classes of selfadjoint operators in Krein spaces are well-understood. Among those are the definitizable operators. A selfadjoint operator T in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$ is called definitizable if its resolvent set $\rho(T)$ is non-empty and if there exists a polynomial $p \neq 0$ with real coefficients such that

$$[p(T)x, x] \geq 0 \quad for\ all \quad x \in dom\, p(T).$$

This definition goes back to $\mathcal{H}$. Langer who showed that the spectrum of a definitizable operator T – with the possible exception of a finite number of non-real eigenvalues which are poles of the resolvent of T – is real and that T possesses a spectral function on $\mathbb{R}$ with a finite number of singularities, see [33]. Definitizable operators appear in many applications including differential operators with indefinite weights (see, [184], [185], [187], [198], [199], [201]), selfadjoint operator polynomials (see, [189], [32]) and Sturm-Liouville equations with floating singularity (see, [196], [197], [202]).

We extend the spectral theory of definitizable operators from selfadjoint operators in Krein spaces to products $T = AG$ of selfadjoint operators A and G in a Hilbert space which are both allowed to be unbounded and non-invertible. Instead of $\rho(T) \neq \emptyset$ as in the above definition of definitizability we will have to assume that both resolvent sets $\rho(AG)$ and $\rho(GA)$ are non-empty. If this holds, we say that the ordered pair (A, G) of selfadjoint operators A and G is definitizable if there exists a polynomial p with real coefficients such that

$$(p(AG)x, Gx) \geq 0 \qquad for\ all\ x \in dom(AG)^{max\{1, deg(p)\}}.$$

In the Krein space case the condition that $\rho(AG)$ and $\rho(GA)$ be non-empty is equivalent to $\rho(T) \neq \emptyset$ since in this situation the operators $AG\ and\ GA$ are similar. For example, we have $GA = G(AG)G^{-1}$ if G is bounded and boundedly invertible. Moreover, in this case the ordered pair (A, G) is definitizable according to our definition if and only if $T = AG$ is a definitizable operator in the Krein space $(\mathcal{H}, (G\,\cdot, \cdot))$.

In our first main theorem Theorem (1.2.20) we show that the non-real spectrum of a definitizable pair (A, G) of selfadjoint operators $A\ and\ G$ consists of a finite number of poles of the resolvent of AG and that its real spectral points can be classified into the so-called spectral points of positive and negative type and a finite set of critical points. This classification is then used in the proof of our second main result Theorem (1.2.22) to shown the existence of a spectral function for the operator AG. This spectral function behaves similarly as a spectral measure in Dunford's sense (see,[26]) but might have a finite set of singularities which is a subset of the above-mentioned critical points. Our Theorems show that the class of definitizable operators is in fact a subclass of a much larger class of operators with the same spectral properties. We mention that in the special case when both $A\ and\ G$ are bounded, $A \geq 0$ and $0 \notin \sigma p\ (A)$ the existence of a spectral function of AG was already showed in [188].

The techniques used in our proof of the existence of the spectral function are different to those in [33] where an analogue of Stone's formula for selfadjoint operators in Hilbert spaces was used to define the spectral function. Here, we make use of the concept of the spectral points of positive and negative type of symmetric operators in inner product spaces which was introduced by Langer, Markus and Matsaev in [13], see also [2], [22], [11] for the Krein space case. As mentioned above, in Theorem (1.2.20) we show that if the pair (A, G) is definitizable, then there exists a finite number of real points which divide the real line into intervals which are either of positive or negative type with respect to AG. Due to a theorem in [13] this implies the existence of local spectral functions of AG on these intervals. In the proof of Theorem (1.2.22) we "connect" those local spectral functions and thus obtain a spectral function of AG on $\mathbb{R}$ with a finite number of singularities.

In the preliminaries section following this introduction we introduce the spectral points of positive and negative type of a symmetric operator in a (possibly indefinite) inner product space and show that such an operator has a local spectral function on intervals of positive or negative type. We consider products AG of selfadjoint operators A *and* G in a Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$ such that both $\rho(AG)$ *and* $\rho(GA)$ are non-empty. The operator AG is then symmetric with respect to the inner product $(G_0\ \cdot,\cdot)$, where G_0 is the bounded selfadjoint operator given by

$$G_0 := G(AG - \lambda_0)^{-1}(AG - \overline{\lambda_0})^{-1}, \lambda_0 \in \rho(AG)\backslash\mathbb{R},$$

and we analyze the spectra of positive and negative type of AG (corresponding to the inner product $(G_0\ \cdot,\cdot)$). For example, it turns out that these spectra do not depend on the choice of λ_0 . In this Section we particularly make use of the results to show the main Theorems on definitizable pairs of selfadjoint operators, and we apply our results to Sturm-Liouville problems.

Let S be a linear operator in a Banach space X. If S is bounded and everywhere defined, we write $S \in L(X)$. By the resolvent *set* $\rho(S)$ *of* S we understand the set of all $\lambda \in \mathbb{C}$ for which ran$(S - \lambda) = X, ker(S - \lambda) = \{0\}$ and $(S - \lambda)^{-1} \in L(X)$. With this definition of $\rho(S)$, the operator S is closed if $\rho(S)$ is non-empty. The operator S is called boundedly invertible if $0 \in \rho(S)$. The set $\sigma(S) := C\backslash\rho(S)$ is called the spectrum of S. The approximate point spectrum $\sigma_{ap}(S)$ of S is defined as the set of all $\lambda \in \mathbb{C}$ for which there exists a sequence $(x_n) \subset dom\, S$ with $\|x_n\| = 1$ and $(S - \lambda)x_n \to 0$ *as* $n \to \infty$. A point $\lambda \in \mathbb{C}$ does not belong to σ_{ap} (S) if and only if there exists $c > 0$ and an open neighborhood U *of* λ in C such that $\|(S - \mu)x\| \geq c\|x\|$ holds for all $x \in dom\, S$ and all $\mu \in U$.

Throughout this section $(\mathcal{H}, (\cdot\,,\cdot))$ denotes a Hilbert space and G_0 a bounded selfadjoint operator in $\mathcal{H}$. The operator G_0 induces a new inner product $[\cdot\,,\cdot]$ on $\mathcal{H}$ via

$$[x, y] := (G_0x, y), \qquad x, y \in \mathcal{H}.$$

The pair $(\mathcal{H}, [\cdot\,,\cdot])$ is often referred to as a G_0-space. If G_0 is boundedly invertible, $(\mathcal{H}, [\cdot\,,\cdot])$ is called a Krein space. A subspace $\mathcal{L}$ of $\mathcal{H}$ is called uniformly positive (uniformly negative) if there exists $\delta > 0$ such that

$$[x, x] \geq \delta\|x\|^2([x, x] \leq -\delta\|x\|^2, respectively)$$

holds for all $x \in \mathcal{L}. If\ \mathcal{L}$ is closed, then $\mathcal{L}$ is uniformly positive (uniformly negative) if and only if $(\mathcal{L}, [\cdot\,,\cdot])$ $((\mathcal{L}, -[\cdot\,,\cdot])$, respectively) is a Hilbert space. The orthogonal companion of a subspace $\mathcal{L}$ is defined by

$$\mathcal{L}^{[\perp]} := \{x \in \mathcal{H} : [x, \ell] = 0 \quad for\ all \quad \ell \in \mathcal{L}\}.$$

The subspace $\mathcal{L}$ is called ortho-complemented if $\mathcal{H} = \mathcal{L} + \mathcal{L}^{[\perp]}$. If the sum is direct, we write $\mathcal{H} = \mathcal{L}[\dotplus]\mathcal{L}^{[\perp]}$. The symbol $[\dotplus]$ thus denotes the direct $[\cdot\,,\cdot]$-orthogonal sum. The following Lemma will be used frequently, cf. [3].

Lemma(1.2.1)[183]. Let $\mathcal{L} \subset \mathcal{H}$ be a closed subspace. If $(\mathcal{L}, [\cdot\,,\cdot])$ is a Krein space, then $\mathcal{L}$ is ortho-complemented. More precisely, we have

$$\mathcal{H} = \mathcal{L}[\dotplus]\mathcal{L}^{[\perp]}$$

A closed and densely defined linear operator T in $\mathcal{H}$ will be called G_0-symmetric (or $[\cdot\,,\cdot]$-symmetric) if

$$[Tx, y] = [x, Ty] \text{ holds for all } x, y \in dom\, T.$$

This is equivalent to the symmetry of the operator G_0T in the Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$, i.e., $G_0T \subset (G_0T)^*$.

The Riesz-Dunford spectral projection of a closed linear operator T in $\mathcal{H}$ with respect to a spectral set σ of T will be denoted by $E(T;\ \sigma)$. $If\ \sigma\ =\ \{\lambda\}$, we write $E(T;\ \lambda)$ instead of $E(T;\ \{\lambda\})$.

Lemma(1.2.2)[183]. Let T be G_0-symmetric. If $\lambda\ \in\ \mathbb{C}$and $\bar{\lambda}$ are isolated points of the spectrum of T, we have

$$[E(T;\ \lambda)x,y] = \left[x,E(T;\bar{\lambda})\right] \quad for\ all\ x,y\ \in\ \mathcal{H}.$$

Proof. Let $\varepsilon\ >\ 0$ be a number such that the deleted discs $\{\mu\ \in\ \mathbb{C}:\ |\mu\ -\ \lambda|\ \le\ \varepsilon\}\backslash\{\lambda\}$ and $\mu\ \in\ \mathbb{C}:\ \left|\mu\ -\ \bar{\lambda}\right|\le\ \varepsilon\backslash\lambda$ are contained in $\rho(T)$. Define the curves
$\gamma,\psi:\ [0,2\pi]\ \to\mathbb{C}$ by

$$\gamma(t):=\ \lambda\ +\ \varepsilon e^{it}\quad and\quad \psi(t):=\ \bar{\lambda}\ +\ \varepsilon e^{it},\qquad t\in\ [0,2\pi].$$

Then for $x,y\ \in\ \mathcal{H}$ we have

$$[E(T;\ \lambda)x,y] =\ -\frac{1}{2\pi i}\int_{\gamma}[(T-\mu)^{-1}x,y]d\mu$$

$$=\frac{1}{2\pi i}\int_{\gamma^{-1}}[x,(T\ -\ \bar{\mu})^{-1}y]d\mu$$

$$=\frac{1}{2\pi i}\int_{0}^{2\pi}\left[x,(T\ -\ \bar{\lambda}\ -\ \varepsilon e^{it}\ )^{-1}y\right](-i)\varepsilon e^{-it}\,dt = \left[x,-\ \frac{1}{2\pi i}\int_{0}^{2\pi} i\varepsilon e^{it}\,(T\ -\ \psi(t))^{-1}y\,dt\right]$$

$$=\left[x,E(T;\ \bar{\lambda})y\right],$$

where $\gamma^{-1}\,(t)\ :=\ \lambda\ +\ \varepsilon e^{-it}\,,t\ \in\ [0,2\pi]$.

In [13] the spectral points of positive and negative type of a bounded G_0-symmetric operator were introduced. In the following definition these notions are extended to unbounded operators.

Definition(1.2.3)[183]. Let T be a G_0-symmetric operator in $\mathcal{H}$. A point $\lambda\ \in\ \sigma_{ap}\,(T)$ is called a spectral point of positive (negative) type of T if for every sequence $(x_n\,)\ \subset\ dom\,T$ with $\|x_n\|\ =\ 1$ and $(T\ -\lambda)x_n\ \to\ 0\ as\ n\to\infty$ we have

$$\liminf_{n\to\infty}\ [x_n,x_n\,]>\ 0\ \left(\limsup_{n\to\infty}[x_n,x_n\,]<\ 0, respectively\right).$$

The set of all spectral points of positive (negative) type of T will be denoted by $\sigma_+(T)$ ($\sigma_-(T)$, respectively). A set $\Delta\ \subset\ \mathbb{C}$ is said to be of positive (negative) type with respect to T if

$$\Delta\ \cap\ \sigma_{ap}\,(T)\subset\ \sigma_+(T)\big(\Delta\cap\ \sigma_{ap}\,(T)\subset\ \sigma_-(T), respectively\big).$$

The following statements were showed in [13] for bounded operators. However, the proofs can be adopted without difficulties in the unbounded case.

Proposition(1.2.4)[183]. The spectral points of positive and negative type of a G_0-symmetric operator T are real. Moreover, $\sigma_+(T)$ and $\sigma_-(T)$ are open in σ_{ap} (T). In particular, if Δ is a compact interval which is of positive (negative) type with respect to T, then there exists a $\mathbb{C}$ -open neighborhood $\mathcal{U}$ of Δ such that $(\mathcal{U}\backslash\mathbb{R})\cap\sigma_{ap}\,(T)\ =\ \emptyset$ and $\mathcal{U}\ \cap\mathbb{R}$ is of positive type (negative type, respectively) with respect to T. Moreover, there exists $C\ >\ 0$ such that for all $\lambda\ \in\ \mathcal{U}$ we have

$$\|(T\ -\ \lambda)x\|\ \ge\ C|\,Im\,\lambda|\|x\|,\qquad x\in\ dom\,T.$$

Definition(1.2.5)[183]. Let $J\ \subset\ \mathbb{R}$ be a bounded or unbounded open interval and let $s\ \subset\ J$ be a finite set. The system consisting of all bounded Borel subsets Δ of J with $\bar{\Delta}\ \subset\ J$ the boundary points of which are not contained in s will be denoted by $\mathcal{R}_s\,(J)$. $If\ s\ =\ \emptyset$, we simply write $\mathcal{R}(J)$. Let S be a closed and densely defined linear operator in the Banach space X. A set function E mapping from $\mathcal{R}_s\,(J)$ into the set of bounded projections in X is called a local spectral function of S on J (with the set of critical points $s\ =\ s(E)$) if the following conditions are satisfied for all $\Delta,\Delta_1\,,\Delta_2\ \in\ \mathcal{R}_s\,(J)$:

(i) $E(\Delta_1\cap\ \Delta_2)\ =\ E(\Delta_1\,)E(\Delta_2\,)$.

(ii) If $\Delta_1\cap\ \Delta_2\ =\ \emptyset$, then $E(\Delta_1\ \cup\ \Delta_2)\ =\ E(\Delta_1)\ +\ E(\Delta_2\,)$.

(iii) $E(\Delta)$ commutes with every operator $B\ \in\ L(\mathcal{H})$ for which $BS\ \subset\ SB$.

(iv) $\sigma(S|E(\Delta)\mathcal{H})\ \subset\ \overline{\sigma(S)\ \cap\ \Delta}$.

(v) $\sigma(S|(I - E(\Delta))\mathcal{H}) \subset \overline{\sigma(S) \setminus \Delta}$.

The points $\lambda \in s(E)$ for which the strong limits

$$s - lim_{t\to 0}E([\lambda - \varepsilon, \lambda - t]) \; and \; s - lim_{t\to 0}E([\lambda + t, \lambda + \varepsilon])$$

do not exist for sufficiently small $\varepsilon > 0$ are called the singularities *of* E.

Let S be a closed operator in a Banach space and let Δ be a compact set in C. A closed subspace $\mathcal{L}_\Delta \subset$ dom S is called the maximal spectral subspace of S corresponding to Δ if the following holds:

(a) $S\mathcal{L}_\Delta \subset \mathcal{L}_\Delta$.

(b) $\sigma(S|\mathcal{L}_\Delta) \subset \sigma(S) \cap \Delta$.

(c) If $\mathcal{L} \subset$ dom S is a closed subspace such that (a) and (b) hold with $\mathcal{L}_\Delta$ replaced by $\mathcal{L}$then $\mathcal{L} \subset \mathcal{L}_\Delta$.

By $\mathbb{C}^+$ ($\mathbb{C}^-$) we denote the open upper (lower, respectively) halfplane. The following Theorem has been shown for bounded G_0-symmetric operators in [13].

Theorem(1.2.6)[183]. Let J be a bounded or unbounded open interval in $\mathbb{R}$ which is of positive (negative) type with respect to the G_0-symmetric operator T. If each of the sets $\mathbb{C}^+ \cap \rho(T)$ and $\mathbb{C}^- \cap \rho(T)$ has an accumulation point in J, then T has a local spectral function E without critical points on J with the following properties

($\Delta \in \mathcal{R}(J)$):

(i) The subspace $E(\Delta)\mathcal{H}$ is uniformly positive (uniformly negative, respectively).

(ii) The operator $E(\Delta)$ is G_0-symmetric.

(iii) If Δ is compact, then $E(\Delta)\mathcal{H}$ is the maximal spectral subspace of T corresponding to Δ.

Proof. Let J be of positive type with respect to T. As a consequence of the uniqueness of a local spectral function (see [310]) it is sufficient to show that the operator T has a local spectral function on each compact subinterval of J. Let J' be such an interval. By assumption, it is no restriction to assume that J' contains accumulation points of both $\mathbb{C}^+ \cap \rho(T) \; and \; \mathbb{C}^- \cap \rho(T)$. Due to Proposition (1.2.4) there exists an open neighborhood $\mathcal{U} \; of \; J'$ in $\mathbb{C}$ such that $(U\setminus\mathbb{R}) \cap \sigma_{ap}(T) = \emptyset$. Hence, for each $\lambda \in \mathcal{U}\setminus\mathbb{R}$ the operator $T - \lambda$ is semi-Fredholm with $ker(T - \lambda) = \{0\}$. Since the sets $\mathcal{U} \cap \mathbb{C}^+ \cap \rho(T) \; and \; \mathcal{U} \cap \mathbb{C}^- \cap \rho(T)$ both are non-empty, it follows from [200] that in fact $\mathcal{U}\setminus\mathbb{R} \subset \rho(T)$. Moreover, by Proposition (1.2.4) there exists $C > 0$ such that

$$\|(T - \lambda)\|^{-1} \le \frac{C}{|Im\,\lambda|} \tag{29}$$

holds for all $\lambda \in \mathcal{U}\setminus\mathbb{R}$. By [34] the maximal spectral subspace $\mathcal{L}$ of T corresponding to J' exists and $T|\mathcal{L}$ is bounded. As $T|\mathcal{L}$ is also $[\cdot,\cdot]$-symmetric and $\sigma(T|\mathcal{L}) = \sigma+(T|\mathcal{L})$ it follows from [34] that $(\mathcal{L}, [\cdot,\cdot])$ is a Hilbert space. Denote by $E_\mathcal{L}$ the spectral measure of the selfadjoint operator $T|\mathcal{L}$ in $(\mathcal{L}, [\cdot,\cdot])$ and by $P_\mathcal{L}$ the projection onto $\mathcal{L}$ with $ker\, P_\mathcal{L} = \mathcal{L}^{[\perp]}$ which exists due to Lemma (1.2.1). Then $E(\cdot) := E_\mathcal{L}(\cdot)P_\mathcal{L}$ defines a local spectral function of T on J'.

Throughout this section let $A \; and \; G$ be selfadjoint operators in the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$. Each of the statements in the following proposition follows from or is an easy consequence of [192] and [192], see also [193], [203].

Proposition(1.2.7)[183]. Let A and G be selfadjoint operators in $\mathcal{H}$. *If*

$$\rho(AG) \neq \emptyset \quad and \quad \rho(GA) \neq \emptyset, \tag{30}$$

then both operators AG and GA are closed and densely defined and

$$(AG)^* = GA. \tag{31}$$

Moreover,

$$\sigma(AG)\setminus\{0\} = \sigma(GA)\setminus\{0\}. \tag{32}$$

In addition, for $\lambda \in \rho(AG)\setminus\{0\}$ the following relations hold:

$$A(GA - \lambda)^{-1} = \overline{(AG - \lambda)^{-1}A},$$
$$G(AG - \lambda)^{-1} = \overline{(GA - \lambda)^{-1}G}.$$

In our main results (Theorems (1.2.20) and (1.2.22) below) we require that (30) is satisfied. Since in applications this condition might be hard to verify, the following sufficient conditions for (30) may be helpful.

Lemma(1.2.8)[183]. The following conditions are sufficient for (30) to hold:

(a) G is bounded and $\rho(GA) \neq \emptyset$.
(b) G is boundedly invertible and $\rho(AG) \neq \emptyset$.
(c) $(AG)^* = GA$ and $\rho(AG) \neq \emptyset$.
(d) $\rho(AG) \neq \emptyset$, GA is closed and for some $\lambda \in \rho(AG)\backslash\{0\}$ the operator $G(AG - \lambda)^{-1}A$ is bounded on dom A.

Proof. If (c) holds, then $\sigma(GA) = \sigma((AG)^*) = \{\bar{\lambda} : \lambda \in \sigma(AG)\}$ and hence $\rho(AG) \neq \emptyset$. Hence, (c) implies (30). If (b) holds, then AG and GA are closed and $(AG)^* = GA$. If (a) holds, then AG and GA are closed and $(GA)^* = AG$. Therefore, (b) implies (c) and (a) implies (c) with A and G interchanged. Consequently, both (a) and (b) imply (30). Assume now that (d) holds. Then the operator $GA - \lambda$ is injective. Moreover (see also [307]), for $x \in domA$ we have $G(AG - \lambda)^{-1}Ax - x \in dom(GA)$ and

$$(GA - \lambda)(G(AG - \lambda)^{-1}Ax - x) = \lambda x.$$

This shows that $domA \subset ran(GA - \lambda)$ and that $(GA - \lambda)^{-1} \mid domA$ is bounded. As the closure of $(GA - \lambda)^{-1} \mid domA$ coincides with $(GA - \lambda)^{-1}$ (on ran $(GA - \lambda)$), it follows that $(GA - \lambda)^{-1} \in L(\mathcal{H})$.

Indeed, if $AG \in L(\mathcal{H})$, then $domG = \mathcal{H}$ yields $G \in L(\mathcal{H})$. Suppose that (30) holds. Then, according to Proposition (1.2.7), we have $GA = (AG)^* \in L(\mathcal{H})$ and thus $A \in L(\mathcal{H})$.

Proposition(1.2.9)[183]. *If* (30) is satisfied, then the following conditions are equivalent.
(a) AG is boundedly invertible.
(b) $ran(AG) = \mathcal{H}$.
(c) GA is boundedly invertible.
(d) $ran(GA) = \mathcal{H}$.
(e) A and G are boundedly invertible.
In particular, $\sigma(AG) = \sigma(GA)$.

Proof. Clearly, (a) implies (b). Assume that (b) holds. Then $ran\, A = \mathcal{H}$ (which implies $kerA = \{0\}$) and $domA = A^{-1}(AGH) \subset ranG$ which implies $rG = (ranG)^{\perp} \subset (domA)^{\perp} = \{0\}$. Hence, $ker(AG) = \{0\}$ and (a) follows. By interchanging the roles of A and G it is seen that (c) and (d) are equivalent. The equivalence (a) ⇔ (c) is a consequence of (31). Since (a) implies that A is boundedly invertible, (c) implies that G is boundedly invertible and (e) implies both (a) and (c), the proposition is showed.

Corollary(1.2.10)[183] Assume that (30) holds. Then for each $\lambda \in \mathbb{C}$ the following statements hold.
(i) $\lambda \in \sigma(AG) \Leftrightarrow \bar{\lambda} \in \sigma(AG)$.
(ii) $\lambda \in \sigma(AG)\backslash\sigma_{ap}(AG) \Rightarrow \bar{\lambda} \in \sigma_p(AG)$.

Proof. From Propositions (1.2.7) and (1.2.9) it follows that $\lambda \in \rho(AG)$ implies

$$\bar{\lambda} \in \rho((AG)^*) = \rho(GA) = \rho(AG).$$

This showes (i). Let us show (ii) for $\lambda \neq 0$. If $\lambda \in \sigma(AG)\backslash\sigma_{ap}(AG), \lambda \neq 0$, then it is well-known that $\lambda \in \sigma_p((AG)^*) = \sigma_p(GA)$. Hence, there exists $x \in dom(GA)\backslash\{0\}$ such that $GAx = \lambda x$. Therefore, $GAx \in domA$ and $(AG - \lambda)Ax = A(GA - \lambda)x = 0$. Since $Ax \neq 0$ (otherwise, $GAx = 0$ and thus $x = 0$), we conclude that $\lambda \in \sigma_p(AG)$. But (ii) also holds for $\lambda = 0$ as in this case the left-hand side of the implication (ii) is never true. To see this, note that $0 \notin \sigma_{ap}(AG)$ implies that there is a neighborhood $\mathcal{U}$ of zero such that $\mathcal{U} \cap \sigma_{ap}(AG) = \emptyset$. Now, from (ii) for $\lambda \neq 0$ it follows that $\mathcal{U}\backslash\{0\} \subset \rho(AG)$. Hence, the Fredholm index of $AG - \lambda$ for $\lambda \in \mathcal{U}$ is constantly zero. And as $ker(AG) = \{0\}$, it follows that also $0 \in \rho(AG)$.

If (30) is satisfied, by Corollary (1.2.10) there exists $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ such that $\lambda_0, \overline{\lambda_0} \in \rho(AG)$, and thus, the operator

$$G_0 := G(AG - \lambda_0)^{-1}\left(AG - \overline{\lambda_0}\right)^{-1} \tag{33}$$

is bounded. Moreover, due to Proposition (1.2.7) we have

$$\begin{aligned} G_0^* &= (GA - \lambda_0)^{-1}(G(AG - \lambda_0)^{-1})^* \\ &= (GA - \lambda_0)^{-1}((GA - \lambda_0)^{-1}G)^* \end{aligned}$$

$$= (GA - \lambda_0)^{-1}G(AG - \overline{\lambda_0})^{-1}$$
$$= G(AG - \lambda_0)^{-1}(AG - \overline{\lambda_0})^{-1} = G_0$$

and

$$G_0AG = G\,(GA - \lambda_0)^{-1}\,(GA - \overline{\lambda_0})^{-1}AG$$
$$\subset GAG(GA - \lambda_0)^{-1}(GA - \overline{\lambda_0})^{-1}$$
$$= GAG_0 = (G_0AG)^*.$$

This shows that G_0 is selfadjoint and that AG is G_0-symmetric. Equivalently, AG is symmetric with respect to the inner product

$$[x,y] := (G_0x,y), x,y \in \mathcal{H}. \quad (34)$$

Note that the inner product $[\cdot,\cdot]$ is in general not a Krein space inner product. It might even be degenerate.

For the rest of this section we assume that (30) holds and fix $\lambda_0 \in \rho(AG)\backslash\mathbb{R}$, the operator G_0 in (33) and the inner product $[\cdot,\cdot]$ in (34). The spectra of positive and negative type of AG are connected with the inner product $[\cdot,\cdot]$ which itself depends on $\lambda_0 \in \rho(AG)\backslash\mathbb{R}$. The following Lemma shows that $\sigma_+(AG)$ and $\sigma_-(AG)$ are in fact independent of λ_0 .

Lemma(1.2.11)[183]. Let $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma_+(AG)$ ($\lambda \in \sigma_-(AG)$) if and only if for each sequence $(x_n) \subset$ dom AG with $\|x_n\| = 1$ and $(AG - \lambda)x_n \to 0$ as $n \to \infty$we have

$$\liminf_{n\to\infty} (Gx_n, x_n) > 0 \quad \limsup_{n\to\infty} (Gx_n, x_n) < 0, \qquad respectively.$$

Proof. Assume that the condition in the lemma on the approximate eigensequences of AG holds and let $(x_n) \subset dom\,AG$ with $\|x_n\| = 1$ and
$(AG - \lambda)x_n \to 0 \quad as\, n \to \infty$. Set

$$y_n := (\lambda - \lambda_0)(AG - \lambda_0)^{-1}x_n .$$

Then we have

$$(AG - \lambda)y_n = (\lambda - \lambda_0)(x_n + (\lambda_0 - \lambda)(AG - \lambda_0)^{-1}x_n) = (\lambda - \lambda_0)(x_n - y_n).$$

On the other hand,

$$(AG - \lambda)y_n = (\lambda - \lambda_0)(AG - \lambda_0)^{-1}(AG - \lambda)x_n \to 0$$

as $n \to \infty$. Hence $\|y_n\| \to 1$ and since

$$[x_n, x_n] = G(AG - \lambda_0)^{-1}x_n\,, (AG - \lambda_0)^{-1}x_n = \frac{1}{|\lambda - \lambda_0|^2}\,(Gy_n, y_n)$$

we conclude

$$\liminf_{n\to\infty} [x_n, x_n] = \frac{1}{|\lambda - \lambda_0|^2} \liminf_{n\to\infty} (Gy_n, y_n) > 0.$$

Conversely, let $\lambda \in \sigma_+(AG)$ and let $(x_n) \subset domAG\ with\ \|x_n\| = 1\ and\ (AG - \lambda)x_n \to 0\ as\ n \to \infty$. Since $(Gx_n, x_n) = [(AG - \lambda_0)x_n, (AG - \lambda_0)x_n]$,
we obtain from $(AG - \lambda)x_n \to 0\ as\, n \to \infty$:

$$\liminf_{n\to\infty} (Gx_n, x_n) = |\lambda - \lambda 0|^2 \liminf_{n\to\infty} [x_n, x_n] > 0,$$

which shows the assertion.

Corollary(1.2.12)[183] Assume that (30) holds and that $0 \in \sigma_+(AG) \cup \sigma_-(AG)$. Then G is boundedly invertible.

Proof. Suppose that, e.g., $0 \in \sigma_+(AG)$ and that there exists a sequence
$(x_n) \subset domG\ with\ \|x_n\| = 1$ for $n \in N\ and\ Gx_n \to 0\ as\, n \to \infty$. Define

$$y_n := -\lambda_0 (AG - \lambda_0)^{-1}x_n \in dom(GAG)$$

as in the proof of Lemma (1.2.11) (with $\lambda = 0$). Then $AGy_n = \lambda_0(y_n - x_n)$ and $AGy_n = -\lambda_0 A(GA - \lambda_0)^{-1}Gx_n \to 0\ as\, n \to \infty\ as\ A(GA - \lambda_0)^{-1}$ is bounded. Therefore, $\|y_n\| \to 1$ and since $0 \in \sigma_+(AG)$, from Lemma (1.2.11) we conclude $\liminf_{n\to\infty}(Gy_n, y_n) > 0$. But this contradicts $Gy_n = -\lambda_0 (GA - \lambda_0)^{-1}Gx_n \to 0\ as\, n \to \infty$.

Lemma(1.2.13)[183]. Assume that (30) is satisfied. Let $\mathcal{L} \subset$ domAG be a closed subspace such that AG$\mathcal{L} \subset \mathcal{L}$ and $0 \in \rho(AG|\mathcal{L})$. If $\mathcal{H} = \mathcal{L} + \mathcal{L}^{[\perp]}$, then $(\mathcal{L}, [\cdot,\cdot])$ is a Krein space.

Proof. Let $P_{\mathcal{L}}$ be the orthogonal projection (with respect to $(\cdot,\cdot)$) onto $\mathcal{L}\, in \mathcal{H}$. Then, with $G_{\mathcal{L}} := P_{\mathcal{L}}(G_0|\mathcal{L}) \in L(\mathcal{L})$ we have

$$[\ell_1,\ell_2] = (G_{\mathcal{L}}\ell_1,\ell_2)\, for\, \ell_1,\ell_2 \in \mathcal{L}.$$

Hence, $(\mathcal{L},[\cdot,\cdot])$ is a Krein space if and only if $G_{\mathcal{L}}$ is boundedly invertible. Let $\ell \in \ker G_{\mathcal{L}}$. By assumption, for any $x \in \mathcal{H}$ we find $x_1 \in \mathcal{L}\, and\, x_2 \in \mathcal{L}^{[\perp]}$ such that $x = x_1 + x_2$. It follows that

$$(G_0\,\ell,x) = [\ell,x_1 + x_2] = [\ell,x_1] = (G_{\mathcal{L}}\ell,x_1) = 0,$$

and thus $G_0\,\ell = 0$. From

$$0 = G(AG - \lambda_0)^{-1}(AG - \overline{\lambda_0})^{-1}\,\ell = (GA - \lambda_0)^{-1}(AG - \overline{\lambda_0})^{-1}G\ell$$

we conclude $G\ell = 0$ and hence $AG\ell = 0$ which implies $\ell = 0\, as\, 0 \in \rho(AG|\mathcal{L})$. Therefore we have $\mathcal{H} = \mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]}$ (since $ker G_{\mathcal{L}} = \mathcal{L}\cap\mathcal{L}^{[\perp]}$).
Now, suppose that there exists a sequence $(\ell_n) \subset \mathcal{L}$ with $\|\ell_n\| = 1$ and $\|G_{\mathcal{L}}\ell_n\| \to 0\, as\, n\to\infty$. If by P we denote the (G_0-symmetric) projection onto L with $r\,P = \mathcal{L}^{[\perp]}$, we obtain

$$\begin{aligned}\|G_{\mathcal{L}}\ell_n\|^2 &= (G_0\ell_n,PG_0\ell_n) + (G_0\ell_n,(I - P)G_0\ell_n) \\ &= (P_{\mathcal{L}}G_0\ell_n,PG_0\ell_n) + [\ell_n,(I - P)G_0\ell_n] \\ &= (G_{\mathcal{L}}\ell_n,PG_0\ell_n) \\ &\le \|G_{\mathcal{L}}\ell_n\|\cdot\|P\|\cdot\|G_0\|\end{aligned}$$

Hence, $G_0\ell_n \to 0\, as\, n \to \infty$. It is easy to see that $\mathcal{L}^{[\perp]}$ is AG-invarant. Hence, $\mathcal{L}$ is $(AG - \lambda_0)^{-1}$ -invariant. And since $AG|\mathcal{L}$ is bounded, we conclude

$$\begin{aligned}\|G_0\ell_n\| &\le \|(AG - \lambda_0)|\mathcal{L}\|\cdot\|(AG - \lambda_0)^{-1}AG_0\ell_n\| \\ &= \|(AG - \lambda_0)|\mathcal{L}\|\cdot\|A(GA - \lambda_0)^{-1}G_0\ell_n\| \\ &\le \|(AG - \lambda_0)|\mathcal{L}\|\cdot\|A(GA - \lambda_0)^{-1}\|\cdot\|G_0\ell_n\|.\end{aligned}$$

Thus, we have $(AG - \lambda_0)^{-1}(AG - \overline{\lambda_0})^{-1}AG\ell_n = AG_0\ell_n \to 0$, which implies $AG\ell_n \to 0\, as\, n\to\infty$, which is a contradiction to $0 \in \rho(AG|\mathcal{L})$. The Lemma is showed.

Proposition(1.2.14)[183]. Assume that (30) is satisfied. Then for each $\lambda \in \mathbb{C}$ the following statements hold.

(i) If $\lambda \neq 0$ is an isolated point of the spectrum of AG, then the inner product space $\left(E\left(AG;\{\lambda,\bar{\lambda}\}\right)\mathcal{H},[\cdot,\cdot]\right)$ is a Krein space.

(ii) If λ is a pole of the resolvent of AG of order ν then $\bar{\lambda}$ is a pole of the resolvent of AG of order ν.

Proof. For the proof of (i) set $E := E\left(AG;\{\lambda,\bar{\lambda}\}\right)$. As E is $[\cdot,\cdot]$-symmetric by Lemma (1.2.2), it follows that $(I - E)\mathcal{H} \subset (E\mathcal{H})^{[\perp]}$. And since $\mathcal{H} = E\mathcal{H} \dot{+} (I - E)\mathcal{H}$, Lemma (1.2.13) yields the assertion.

By [305] the fact that $\lambda \notin \mathbb{R}$ (the statement for $\lambda \in \mathbb{R}$ is trivial) is a pole of the resolvent of AG of order ν is equivalent to

$$(AG - \lambda)^{\nu}E(AG;\lambda) = 0\, and\, (AG - \lambda)^{\nu-1}E(AG;\lambda) \neq 0.$$

Let $x,v \in E\left(AG;\bar{\lambda}\right)\mathcal{H}$ be arbitrary. From Lemma 1.2.2 we obtain

$$\left[\left(AG - \bar{\lambda}\right)^{\nu}x,v\right] = \left[E\left(AG;\bar{\lambda}\right)\left(AG - \bar{\lambda}\right)^{\nu}x,v\right] = \left(AG - \bar{\lambda}\right)^{\nu}x,E((AG;\lambda)v) = 0.$$

Furthermore, for $u \in E(AG;\lambda)\mathcal{H}$ we have

$$\left[\left(AG - \bar{\lambda}\right)^{\nu}x,u\right] = [x,(AG - \lambda)^{\nu}u] = 0.$$

Hence, $\left[\left(AG - \bar{\lambda}\right)^{\nu}x,y\right] = 0$ for all $y \in E\left(\left(AG;\{\lambda,\bar{\lambda}\}\right)\mathcal{H}\right)$. And as $\left(E\left(AG;\{\lambda,\bar{\lambda}\}\right)\mathcal{H},[\cdot,\cdot]\right)$ is a Krein space by (i), we obtain $\left(AG - \bar{\lambda}\right)^{\nu}x = 0$.

Proposition(1.2.15)[183] Let A_0 be a bounded selfadjoint operator in $\mathcal{H}$ and assume that $G_0A_0G_0 \ge 0$. Then the following statements hold for the bounded G_0-symmetric operator A_0G_0:

(i) $\sigma(A_0G_0) \subset R$,

(ii) $(0,\infty) \cap \sigma(A_0G_0) \subset \sigma_+(A_0G_0)$,

(iii) $(-\infty,0) \cap \sigma(A_0G_0) \subset \sigma_-(A_0G_0)$.

Proof. Let $\lambda \in \sigma_{ap}(A_0G_0)\backslash\{0\}$ and let $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1, n \in \mathbb{N}$, and $(A_0G_0 - \lambda)x_n \to 0\, as\, n\to\infty$. We claim that it is not possible that $lim_{n\to\infty}(G_0A_0G_0x_n,x_n) = 0$. Suppose the contrary. Then, from the Cauchy-Bunyakowski inequality we obtain

$$\|G_0A_0G_0x_n\|^2 \le (G_0A_0G_0x_n, x_n)((G_0A_0G_0)^2x_n, G_0A_0G_0x_n),$$
and hence $G_0A_0G_0x_n \to 0 \; as\; n \to \infty$. As $(A_0G_0 - \lambda)x_n \to 0$, this implies $G_0x_n \to 0$ and hence $A_0G_0x_n \to 0 \; as \; n \to \infty$. A contradiction.

Assume that there exists $\lambda \in \sigma_{ap}(A_0G_0)\backslash\mathbb{R}$. Then there exists $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1$ and $(A_0G_0 - \lambda)x_n \to 0 \; as\; n \to \infty$. Since $[A_0G_0x_n, x_n] - \lambda[x_n, x_n]$ tends to zero as $n \to \infty$ and $[A_0G_0x_n, x_n]$ and $[x_n, x_n]$ both are real for each n, it follows from $\lambda \notin \mathbb{R}$ that $[A_0G_0x_n, x_n]$ tends to zero which contradicts the statement showed above. Hence $\sigma_{ap}(A_0G_0) \setminus \mathbb{R} = \emptyset$, and from Corollary (1.2.10) (ii) we obtain $\sigma(A_0G_0) \subset \mathbb{R}$.

Let $\lambda \in \sigma(A_0G_0), \lambda > 0$. Then $\lambda \in \sigma_{ap}(A_0G_0)$ by Corollary (1.2.10) (ii). Let $(x_n) \subset \mathcal{H} \; with \; \|x_n\| = 1$ and $(A_0G_0 - \lambda)x_n \to 0 \; as \; n \to \infty$. Suppose $\liminf_{n \to \infty} [x_n, x_n] \le 0$. Then from
$$\lambda \liminf_{n \to \infty} [x_n, x_n] = \liminf_{n \to \infty} [(\lambda - A_0G_0)x_n, x_n] + (G_0A_0G_0x_n, x_n) \ge 0$$
it is seen that there exists a subsequence (x_{nk}) such that $G_0A_0G_0x_{nk}, x_{nk}$ tends to zero as $k \to \infty$. But this is a contradiction to the statement showed above, and it follows that
$$\liminf_{n \to \infty} [x_n, x_n] > 0.$$
This shows (ii), and (iii) can be shown similarly.

As a corollary of Proposition (1.2.17) we give another proof of a Theorem of Radjavi and Rosenthal (see [16]). Recall that a closed subspace is hyperinvariant for $T \in L(X), X$ a Banach space, if it is invariant for any operator in $L(X)$ which commutes with T.

Corollary(1.2.16)[183]. Let $S, T \in L(\mathcal{H})$ be selfadjoint such that $STS \ge 0$. If TS is not a constant multiple of the identity, then TS has a non-trivial hyperinvariant subspace.

Proof. If $\sigma(TS) \ne \{0\}$, then the assertion follows from Proposition (1.2.15) and Theorem (1.2.6) (see. [23]. Hence, suppose that $\sigma(TS) = \{0\}$. It is no restriction to assume that $S \; and \; T$ are injective. Otherwise, $ker(TS) \; or \; \overline{ranTS} = ker(ST)^{\perp}$ is hyperinvariant for $TS \; or \; TS = 0$. Hence, T is a non-negative operator and Proposition (1.2.7) yields $\left(\sigma T^{1/2}ST^{1/2}\right) = \{0\}$. But $T^{1/2}ST^{1/2}$ is selfadjoint and thus coincides with the zero operator. This yields $T = S = 0$, a contradiction.

In the following we extend the notion of definitizability of selfadjoint operators in Krein spaces to products (or pairs) of selfadjoint operators in a Hilbert space. As in the previous section let $A \; and \; G$ be selfadjoint operators in the Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$. Again, if (30) is satisfied for $A \; and \; G$ we fix $\lambda_0 \in \rho(AG)$, define the bounded selfadjoint operator G_0 as in (33) and set $[\cdot,\cdot] := (G_0 \; \cdot,\cdot)$.

Definition(1.2.17)[183]. An ordered pair (A, G) of selfadjoint operators is called definitizable if the resolvent sets of $AG \; and \; GA$ are non-empty and if there exists a polynomial $p \ne 0$ with real coefficients such that
$$(p(AG)x, Gx) \ge 0 \quad for \; all \quad x \in dom(AG)^{max\{1,d\}},$$
where $d := deg(p)$. The polynomial p is called definitizing for (A, G).

If G is bounded and boundedly invertible, then AG is selfadjoint in the Krein space $(\mathcal{H}, (G \,\cdot,\cdot))$ and Definition (1.2.17) coincides with the definition of definitizability of the operator AG in this Krein space. The next Lemma shows that the definitizability of (A, G) can also be expressed by means of the inner product $[\cdot\,,\cdot]$.

Lemma(1.2.18)[183]. Assume that (30) is satisfied. Let $p \ne 0$ be a polynomial with real coefficients. Then the following statements are equivalent.

(i) (A, G) is definitizable with definitizing polynomial p.

(ii) $[p(AG)x, x] \ge 0$ *holds for all* $x \in dom\, p(AG)$.

Proof. Let d be the degree of p. If (i) holds and $\in dom(AG)^d$, then with $x := (AG - \lambda_0)^{-1}y \in dom(AG)^{d+1}$ we have
$$\begin{aligned}[p(AG)y, y] &= (p(AG)(AG - \lambda_0)x, G_0(AG - \lambda_0)x) \\ &= ((AG - \lambda_0)p(AG)x, (GA - \overline{\lambda_0})^{-1}Gx) \\ &= (p(AG)x, Gx) \ge 0.\end{aligned}$$

Conversely, assume that (ii) holds and let $x \in dom(AG)^{d+1}$. Then with $y := \left(AG - \lambda_0\right)x \in dom(AG)^d$ the following holds:

$$(p(AG)x, Gx) = (p(AG)(AG - \lambda_0)^{-1}y, G(AG - \lambda_0)^{-1}y)$$
$$= \left(p(AG)y, \left(GA - \overline{\lambda_0}\right)^{-1} G(AG - \lambda_0)^{-1}y\right)$$
$$= (p(AG)y, G_0 y) = [p(AG)y, y] \geq 0.$$

Hence, the proof is finished if $d = 0$. Let $d > 0$ and $x \in dom(AG)^d$. As $\rho(AG) \neq \emptyset$, there exists a sequence $(x_n) \subset dom(AG)^{d+1}$ such that for $k = 0,1,\dots,d$ we have $(AG)^k x_n \to (AG)^k x \ as\ n \to \infty$. Moreover, due to $domAG \subset domG$ and the closedness of $AG\ and\ G$ there exists $c > 0$ such that

$$\|Gu\| \leq c\|u\| + \|AGu\| \quad for\ all\ u \in dom\, AG.$$

Therefore, from $x_n \to x$ and $AGx_n \to AGx$ we conclude $Gx_n \to Gx\ as\ n \to \infty$. This gives $(p(AG)x, Gx) = lim_{n\to\infty} (p(AG)x_n, Gx_n) \geq 0$. The Lemma is showed.

The proof of the following Lemma is similar to that of Lemma (1.2.18) and is therefore omitted.

Lemma(1.2.19)[183]. Let $p \neq 0$ be a polynomial with real coefficients and degree d. Then the following holds:

(a) If (A, G) is definitizable with definitizing polynomial p, then (G, A) is definitizable with definitizing polynomial $\lambda p(\lambda)$.

(b) If G is boundedly invertible, then (A, G) is definitizable with definitizing polynomial p if and only if the relation $(p(GA)x, G^{-1}x) \geq 0$ holds for all $x \in dom(GA)^{max\{1,d\}}$.

It is well-known (see [33]) that the spectrum of a definitizable operator T in a Krein space is real – with the possible exception of a finite number of non-real poles of the resolvent of T – and that T has a spectral function on $\mathbb{R}$ with a finite number of singularities. The following two theorems generalize this result to definitizable pairs of selfadjoint operators.

Theorem(1.2.20)[183]. If (A, G) is definitizable, then the following statements hold.

(a) The non-real spectrum of AG consists of a finite number of points which are poles of the resolvent of AG. Each such point is a zero of every definitizing polynomial for (A, G).

(b) If $\lambda \in \sigma(AG) \cap (\mathbb{R}\backslash\{0\})$ and $p(\lambda) > 0$ for some definitizing polynomial p for (A, G), then $\lambda \in \sigma_+(AG)$.

(c) If $\lambda \in \sigma(AG) \cap (\mathbb{R}\backslash\{0\})$ and $p(\lambda) < 0$ for some definitizing polynomial p for (A, G), then $\lambda \in \sigma_-(AG)$.

Proof. Let p be a definitizing polynomial for (A, G) and set $m := deg(p) + 1$. Let $z_0 \in \mathbb{C}\backslash\mathbb{R}$ such that $p(z_0) \neq 0$. First of all let us show that there exists some $\lambda_1 \in \rho(AG)$ such that

$$z_0^2\, p(z_0)(z_0 - \lambda_1)^{-m-1}\left(z_0 - \overline{\lambda_1}\right)^{-m-1} \notin \mathbb{R}.$$

To see this, choose two open intervals $J_1\ and J_2$ such that $0 \notin J_2$, $z_0 \notin J_1 \times J_2$ and $J_1 \times J_2 \subset \rho(AG)$. With $\lambda = x + iy \in J_1 \times J_2\ and\ z_0 = \alpha_0 + i\beta_0$ we have

$$(z_0 - \lambda)\left(z_0 - \overline{\lambda}\right) = (\alpha_0 - x)^2 - \beta_0^2 0 + y^2 + 2i\beta_0 (\alpha_0 - x) =: f(x, y).$$

The function $f : J_1 \times J_2 \to \mathbb{R}^2$ has the derivative

$$f'(x, y) = \begin{pmatrix} -2(\alpha_0 - x) & 2y \\ -2\beta_0 & 0 \end{pmatrix}.$$

Its determinant equals $4\beta_0 y$ and does therefore not vanish as $0 \notin J_2$ and $z_0 \notin \mathbb{R}$. Hence, $f(J_1 \times J_2)$ is an open set in $\mathbb{C} \setminus \{0\}$, and thus also

$$\left\{z_0^2\, p(z_0)(z_0 - \lambda_1)^{-m-1}\left(z_0 - \overline{\lambda_1}\right)^{-m-1} : \lambda \in J_1 \times J_2\right\} = \{z_0^2\, p(z_0) z^{-m-1} : z \in f(J_1 \times J_2)\}$$

is open. By Lemma (1.2.18) it is no restriction to assume $\lambda_0 = \lambda_1 \ (\neq z_0)$.

$For\ k = 1, 2$ define the rational functions

$$r_k(\lambda) := \lambda^2 p(\lambda)(\lambda - \lambda_0)^{-m-k}\left(\lambda - \overline{\lambda_0}\right)^{-m-k}. \qquad (35)$$

Then $r_1(z_0) \notin \mathbb{R}$. Define the bounded operator

$$A_0 := AGAp(GA)(GA - \lambda_0)^{-m}\left(GA - \overline{\lambda_0}\right)^{-m}. \qquad (36)$$

It is not difficult to see that A_0 is selfadjoint. Moreover, we observe that

$$Gr_2(AG) = GAGAGp(AG)(GA - \lambda_0)^{-m-2}\left(GA - \overline{\lambda_0}\right)^{-m-2}$$
$$= G_0AGp(AG)AG(GA - \lambda_0)^{-m-1}\left(GA - \overline{\lambda_0}\right)^{-m-1}$$
$$= G_0A_0G_0 .$$

Similarly, one showes that

$$r_1(AG) = A_0G_0 .$$

In addition, $G_0A_0G_0 \geq 0$ holds as for $x \in \mathcal{H}$ we have

$$y := AG(AG - \lambda_0)^{-m-1}x \in dom\, p(AG)$$

and

$$(G_0A_0G_0x, x) = \left(GAGAGp(AG)(AG - \lambda_0)^{-m-2}\left(AG - \overline{\lambda_0}\right)^{-m-2}x, x\right)$$
$$= \left(GAG\left(AG - \overline{\lambda_0}\right)^{-m-2}(AG - \lambda_0)^{-1}p(AG)y, x\right)$$
$$= \left(GA\left(AG - \overline{\lambda_0}\right)^{-m-1}G_0p(AG)y, x\right)$$
$$= (G_0p(AG)y, y) = [p(AG)y, y] \geq 0.$$

By virtue of Proposition (1.2.15) we obtain $\sigma(r_1(AG)) = \sigma(A_0G_0) \subset \mathbb{R}$. And since $r_1(\cdot)$ is analytic in a neighborhood of $\sigma(AG) \cup \{\infty\}$, it is a consequence of the spectral mapping theorem [191] that $r_1(\sigma(AG)) \subset \mathbb{R}$ and thus $z_0 \in \rho(AG)$. To complete the proof of (a) it remains to show that each $\lambda \in \sigma(AG)\backslash\mathbb{R}$ is a pole of the resolvent of AG. To this end we show that

$$p(AG)E\left(AG; \{\lambda, \bar{\lambda}\} = 0\right). \tag{37}$$

From this it follows that also $p(AG)E(AG;\, \lambda) = 0$. And since the spectrum of $AG|E(AG;\, \lambda)\mathcal{H}$ coincides with $\{\lambda\}$, we have $(AG - \lambda)^{\alpha}E(AG;\, \lambda) = 0$, where α is the order of λ as a zero of p. This and [191] imply the assertion. So, let us show (37). Let $y \in E(AG;\, \lambda)\mathcal{H}$ and $z \in E\left(AG;\, \bar{\lambda}\right)\mathcal{H}$ be arbitrary. By Lemma (1.2.2) we have $[p(AG)y, y] = [E(AG;\, \lambda)p(AG)y, y] = \left[p(AG)y, EAG;\, \bar{\lambda}y\right] = 0$, $[p(AG)z, z] = 0$ and thus

$$[p(AG)y, z] + [p(AG)z, y] = [p(AG)y, y + z] + [p(AG)z, y + z]$$
$$= [p(AG)(y + z), y + z] \geq 0.$$

But at the same time,

$$-[p(AG)y, z] - [p(AG)z, y] = [p(AG)y, -z] + [p(AG)(-z), y]$$
$$= [p(AG)(y - z), y - z] \geq 0.$$

Hence, $[p(AG)(y + z), y + z] = 0$ and thus $[p(AG)x, x] = 0$ holds for all $x \in E\left(AG; \{\lambda, \bar{\lambda}\}\right)\mathcal{H}$ By polarization we obtain $[p(AG)x, y] = 0$ for all $x, y \in E\left(AG; \{\lambda, \bar{\lambda}\}\right)\mathcal{H}$. But $\left(E\left(AG; \{\lambda, \bar{\lambda}\}\right)\mathcal{H}, [\cdot,\cdot]\right)$ is a Krein space by Proposition (1.2.14) (i), and $p(AG)x = 0$ for all $x \in E\left(AG; \{\lambda, \bar{\lambda}\}\right)\mathcal{H}$ follows. Hence, (a) is showed.

For the proof of (b) we observe that by (a) there exists a definitizing polynomial p for (A, G) such that $p(\lambda_0) \neq 0$. Define the rational function r_1 as in (35). Let $\lambda_1 \in \mathbb{R}\backslash\{0\}$ such that $p(\lambda_1) > 0$. Then also $r_1(\lambda_1) > 0$, and there exists a function g which is analytic on $\mathcal{U} := \bar{\mathbb{C}}\backslash\left\{\lambda_0, \overline{\lambda_0}\right\}$ such that

$$r_1(\lambda) - r_1(\lambda_1) = g(\lambda)(\lambda - \lambda_1), \lambda \in \mathcal{U}.$$

It is obvious that g is a rational function with the poles λ_0 and $\overline{\lambda_0}$, both of order $m + 1$. Therefore, there exists a polynomial q with $q(\lambda_0) \neq 0$) such that

$$g(\lambda) = q(\lambda)(\lambda - \lambda_0)^{-m-1}\left(\lambda - \overline{\lambda_0}\right)^{-m-1}.$$

From the identity

$$\lambda^2p(\lambda) - r_1(\lambda_1)(\lambda - \lambda_0)^{m+1}\left(\lambda - \overline{\lambda_0}\right)^{m+1} = q(\lambda)(\lambda - \lambda_1)$$

we see that $deg(q) = 2m + 1$. Hence, the operator $g(AG)$ is bounded. Let $(x_n) \subset domAG$ be a sequence with $\|x_n\| = 1$ and $(AG - \lambda_1)x_n \to 0$ as $n \to \infty$. With the operator A_0 from (36) we have

$$(A_0G_0 - r_1(\lambda_1))x_n = (r_1(AG) - r_1(\lambda_1))x_n = g(AG)(AG - \lambda_1)x_n \to 0$$

as $n \to \infty$. And since $G_0A_0G_0 \geq 0$ it follows from $r_1(\lambda_1) > 0$ and Proposition (1.2.15) that

$$\liminf_{n \to \infty}[x_n, x_n] > 0.$$

This shows that $\lambda_1 \in \sigma_+(AG)$. The assertion (c) is showed similarly.

The following example shows that the condition (30) is essential for Theorem (1.2.20) to be valid.

Example(1.2.21)[183]. Let T be a closed and densely defined symmetric operator in the Hilbert space $\mathcal{H}$ which is uniformly positive but not selfadjoint. Then T has a uniformly positive selfadjoint extension A (e.g., the Friedrichs extention). Since for $x \in dom(T^*T)$ we have $(T^*Tx, x) = \| Tx\|^2 \geq \delta\| x\|^2$ with some $\delta > 0$, the selfadjoint operator $|T| := (T^*T)^{1/2}$ is boundedly invertible. We set $G := |T|^{-1}$. Then $AG = T|T|^{-1}$ and hence $(AGx, Gx) \geq 0 \, for \, x \in domAG$. But since AG is bounded while A is unbounded, it follows from Remark (1.2.10) that (30) is not satisfied. Let us now see that the statements (a)–(c) of Theorem (1.2.20) do not apply. For this we note that for $x \in dom\,|T| \, and \, y \in \mathcal{H}$ we have

$$(T|T|^{-1}x, T|T|^{-1}y) = \left((T^*T)^{1/2}x, (T^*T)^{-1/2}y\right) = (x, y)$$

which shows that the operator AG is an isometry with $domAG = \mathcal{H}$ and $ranAG = ranT \neq \mathcal{H}$. The spectrum of AG therefore coincides with the closed unit disk.

Assume that (A, G) is definitizable. Theorem (1.2.20) shows that there is only a finite number of real points which are not contained in $\rho(AG) \cup \sigma_+(AG) \cup \sigma_-(AG)$. In analogy to definitizable operators in Krein spaces these exceptional points will be called the *critical points* of (A, G). By Theorem (1.2.20) each non-zero critical point of (A, G) is a zero of every definitizing polynomial for (A, G). Moreover, if G is not boundedly invertible, then due to Propsition (1.2.9) and Corollary (1.2.12) zero is a critical point of (A, G). The set of the critical points of (A, G) is denoted by $c(A, G)$.

Theorem(1.2.22)[193]. Assume that (A, G) is definitizable. Then the operator AG possesses a spectral function on $\mathbb{R}$ with the set of critical points $s := c(A, G)$.

Proof. The proof is divided into several steps. In step 1 we define the spectral projection $E(\Delta)$ for sets Δ which have a positive distance to s. In step 2, $E(\Delta)$ is defined for compact intervals. This will be used in step 3 to define $E(\Delta)$ for all $\Delta \in \mathcal{R}_s(\mathbb{R})$.

(a) By $\mathcal{R}_{s,0}(\mathbb{R})$ we denote the system of all sets Δ in $\mathcal{R}_s(\mathbb{R})$ with $\Delta \cap s = \emptyset$. In this first step of the proof we define $E(\Delta)$ for $\Delta \in \mathcal{R}_{s,0}(\mathbb{R})$ and show that the set function E on $\mathcal{R}_{s,0}(\mathbb{R})$ satisfies (i)–(v) in Definition (1.2.5). Let p be a definitizing polynomial for (A, G) and let Z be the set of zeros of p. By Theorem (1.2.20) the points in Z divide the real line into intervals which are of either positive or negative type with respect to AG. The set Z contains the critical points of (A, G), but there might be spectral points of AG in Z which are not critical. However, a slight modification of the set Z leads to a finite set Z' of real points which divide $\mathbb{R}$ into intervals $J_1, \ldots, J_n$ of positive or negative type with respect to AG, respectively, such that $Z' \cap \sigma(AG) = s$. By Theorem (1.2.6), on each interval J_k the operator AG has a local spectral function E_k. For $\Delta \in \mathcal{R}_{s,0}(\mathbb{R})$ we set $\Delta_k := \Delta \cap J_k \cap \sigma(AG), k = 1, \ldots, n$, and

$$E(\Delta) := \sum_{k=1}^{n} E_k(\Delta_k).$$

As $\Delta_k \in \mathcal{R}(J_k) \, for \, k = 1, \ldots, n$, this is a section definition. Each of the subspaces $\mathcal{L}_k := E_k(\Delta_k), k = 0, \ldots, n$, is contained in $domAG$ and is AG-invariant. In the following we shall show that $\mathcal{L}_k \cap \mathcal{L}_j = \{0\}$ for $k \neq j$. Let $\lambda \in \mathbb{C}$ be arbitrary. Then $\lambda \notin \overline{\Delta_k} \, or \, \lambda \notin \overline{\Delta_j}$. Assume $\lambda \notin \overline{\Delta_j}$. Then $\lambda \in \rho\left(AG|\mathcal{L}_j\right)$ and thus $ker\left(AG|\mathcal{L}_k \cap \mathcal{L}_j - \lambda\right) = \{0\}. Let \, y \in \mathcal{L}_k \cap \mathcal{L}_j$.
$hen, as \, y \in \mathcal{L}_j$, the vector

$$x := (AG|\mathcal{L}_j - \lambda)^{-1}y = \lim_{\eta \downarrow 0}(AG - (\lambda + i\eta))^{-1}y$$

exists and is contained in both $\mathcal{L}_j$ and $\mathcal{L}_k$. Hence, we have $\lambda \in \rho(AG|\mathcal{L}_k \cap \mathcal{L}_j)$. As this is similarly showed for $\lambda \notin \overline{\Delta_k}$, it follows that $\sigma(AG|\mathcal{L}_k \cap \mathcal{L}_j) = \emptyset$ and hence $\mathcal{L}_k \cap \mathcal{L}_j = \{0\}$. Therefore, as $E_k(\Delta_k)$ and $E_j(\Delta_j)$ commute, we obtain

$$E_k(\Delta_k)E_j(\Delta_j) = E_j(\Delta_j)E_k(\Delta_k) = 0.$$

This shows that $\mathcal{L}_k \dotplus \mathcal{L}_j$ is a subspace and that $\mathcal{L}_k \subset \mathcal{L}_j^{[\perp]}$. In fact, we have shown that

$$E(\Delta)\mathcal{H} = E_0(\Delta_0)\mathcal{H}[\dot{+}] \cdots [\dot{+}]E_n(\Delta_n)\mathcal{H}.$$

With the help of this decomposition it is easily seen that the function E, defined on $R_{s,0}(\mathbb{R})$, satisfies (i)–(v) in Definition (1.2.5).

(b) In this step we define the spectral projection $E([a,b])$ for a compact interval $[a,b] \in \mathcal{R}_s(\mathbb{R})$. To this end choose a', b'; $with\ a < a' < b' < b$ such that there is no critical point of AG in $[a,a'] \cup [b',b]$. We set

$$\Delta_0 := [a,a']\ and\ \Delta_1 := [b',b].$$

Define the spectral subspaces $\mathcal{L}_j := E(\Delta_j)\mathcal{H}, j = 0,1.$ As these are both uniformly definite, on account of Lemma (2.1.1) we have

$$\mathcal{H} = \mathcal{L}_0 [\dot{+}]\mathcal{L}_1 [\dot{+}] \widetilde{\mathcal{H}}, \tag{38}$$

where $\widetilde{\mathcal{H}} = (\mathcal{L}_0 [\dot{+}]\mathcal{L}_1)^{[\perp]} = (I - E(\Delta_0 \cup \Delta_1))\mathcal{H}$. We set $T_j := AG|\mathcal{L}_j, j = 0,1,$ and $\tilde{T} := AG|\widetilde{\mathcal{H}}$. With respect to the decomposition (38) the operator AG decomposes as $AG = T_0 [\dot{+}] T_1 [\dot{+}] \tilde{T}$. As a consequence of the results in step 1 we have

$$\sigma(\tilde{T}) \subset \overline{\sigma(AG)\,(\Delta_0 \cup \Delta_1)}. \tag{39}$$

This implies $(a,a') \cup (b',b) \subset \rho(\tilde{T}). Set\ \Delta := [a',b']$ and denote by $\tilde{E}_\Delta$ the Riesz-Dunford spectral projection of $\tilde{T}$ $(in\ \widetilde{\mathcal{H}})$ corresponding to Δ. Similarly as in the proof of Lemma (1.2.2) it is seen that $\tilde{E}_\Delta\ is\ [\cdot,\cdot]$-symmetric. With respect to the decomposition (38) we now define

$$E([a,b]) := I_{\mathcal{L}_0} [\dot{+}]\, I_{\mathcal{L}_1} [\dot{+}]\, \tilde{E}_\Delta.$$

This is obviously a $[\cdot,\cdot]$-symmetric projection in $\mathcal{H}$ which commutes with the resolvent of AG. Moreover, $\sigma(AG|E([a,b])\mathcal{H}) \subset [a,b]$.

In the following we show that the above definition of $E([a,b])$ is independent of the choice of a' and b'. To this end we show the following claims.

(a) The subspace $E([a,b])\mathcal{H}$ is the maximal spectral subspace of AG corresponding to $[a,b]$.

(b) $E([a,b])$ commutes with every bounded operator which commutes with the resolvent of AG.

For the proof of (a) let $K \subset domAG$ be an AG-invariant (closed) subspace such that $\sigma(AG|K) \subset [a,b]$. By Theorem (1.2.6) the maximal spectral subspaces $\mathcal{K}_j$ of $AG|\mathcal{K}$ corresponding to Δ_j exist, $j = 0,1$. These are uniformly definite with respect to the inner product $[\cdot,\cdot]$. Hence,

$$\mathcal{K} = \mathcal{K}_0 [\dot{+}]\mathcal{K}_1 [\dot{+}] \widetilde{\mathcal{K}},$$

where $\widetilde{\mathcal{K}} = (\mathcal{K}_0 [\dot{+}]\mathcal{K}_1 [\dot{+}])^{[\perp]} \cap \mathcal{K}$ and $\sigma AG|\widetilde{\mathcal{K}} \subset [a',b']$. From $\sigma(AG|\mathcal{K}_j) \subset \Delta_j$ and the maximality of $\mathcal{L}_j$ we conclude $\mathcal{K}_j \subset \mathcal{L}_j, j = 0,1$, and set

$$\mathcal{M} := (\mathcal{L}_0 [\dot{+}]\mathcal{L}_1) + \widetilde{\mathcal{K}}.$$

This sum is direct (and hence $\sigma(AG|\mathcal{M}) \subset [a,b]$): Set $\mathcal{L} := \mathcal{L}_0 [\dot{+}]\mathcal{L}_1$. By [16], $\sigma AG|\mathcal{L} \cap \widetilde{\mathcal{K}} \subset (\Delta_0 \cup \Delta_1) \cap [a',b'] = \{a',b'\}$. From the maximality of $\mathcal{K}_0$ and $\mathcal{K}_1$ it follows that $a',b' \notin \sigma_p AG|\mathcal{L} \cap \widetilde{\mathcal{K}}$. And as the resolvent of $AG|L \cap \widetilde{\mathcal{K}}$ satisfies a growth condition (29) in neighborhoods of Δ_0 and Δ_1, we conclude $\mathcal{L} \cap \widetilde{\mathcal{K}} = \{0\}$.

Now, with $\widetilde{\mathcal{M}} := (\mathcal{L}_0 [\dot{+}]\mathcal{L}_1)^{[\perp]} \cap \mathcal{M}$ we have

$$\mathcal{M} = \mathcal{L}_0 [\dot{+}]\mathcal{L}_1 [\dot{+}] \widetilde{\mathcal{M}}.$$

As $\mathcal{L}_0$ and $\mathcal{L}_1$ are maximal, the spectrum of $AG|\widetilde{\mathcal{M}}$ is contained in $[a',b']$. Since $\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{H}}$ and $\tilde{E}_\Delta \widetilde{\mathcal{H}}$ is the maximal spectral subspace of $AG|\widetilde{\mathcal{H}}$ corresponding to $[a',b']$, this implies $\widetilde{\mathcal{M}} \subset \tilde{E}_\Delta \widetilde{\mathcal{H}}$ and hence $\mathcal{K} \subset \mathcal{M} \subset E([a,b])\mathcal{H}$. (a) is showed.

Let B be a bounded operator in $\mathcal{H}$ which commutes with the resolvent of AG. Then $BAG \subset AGB$ and hence $E(\Delta_j)B = BE(\Delta_j), j = 0,1$, see (iii) in Definition (1. 2.5). Hence, $\mathcal{L}_0$ and $\mathcal{L}_1$ and also their orthogonal companions $\mathcal{L}_0^{[\perp]}$ and $\mathcal{L}_1^{[\perp]}$ are B-invariant. And as $\widetilde{\mathcal{H}} = \mathcal{L}_0^{[\perp]} \cap L\mathcal{L}_1^{[\perp]}$, it follows that with respect to the decomposition (38) the operator B decomposes as $B = B_0 [\dot{+}]B_1 [\dot{+}] \tilde{B}$. Hence, $\tilde{B}\tilde{T} \subset \tilde{T}\tilde{B}$ which implies that $\tilde{B}$ commutes with $\tilde{E}_\Delta$. Finally, we conclude that B commutes with $E([a, b])$, and (b) is showed.

Now, let $a'',b'' \in \mathbb{R}\ with\ a < a'' < b'' < b$ such that $[a, a'']$ and $[b'',b]$ do not contain any point from s and construct a spectral projection of AG corresponding to $[a,b]$ as in step 1 with a'

and b' replaced by a'' and b''. Denote this projection by P. As the maximal spectral subspace of AG corresponding to $[a,b]$ is unique, we have $P\mathcal{H} = E([a,b])\mathcal{H}$ by (a). Therefore, $PE([a,b]) = E([a,b])\ and\ E([a,b])P = P$. But (b) yields that P and $E([a,b])$ commute. Therefore, $P = E([a,b])P = PE([a,b]) = E([a,b])$.

Above, it was shown that $E([a,b])$ commutes with any bounded operator in $\mathcal{H}$ which commutes with the resolvent of AG and that $\sigma(AG|E([a,b])\mathcal{H}) \subset \sigma(AG) \cap [a,b]$ holds. Hence, the projection $E([a,b])$ has the properties (iii) and (v) in Definition (1.2.5). It also satisfies (iv) as due to $(a,a') \cup (b',b) \subset \rho(\tilde{T})$ and (39) we have

$$\sigma(AG|(I - E([a,b]))\mathcal{H}) = \sigma T|I - \tilde{E}_{\Delta}\mathcal{H} = \sigma T\backslash(a,b)$$
$$\subset \sigma(AG)\backslash(\Delta_0 \cup \Delta_1)\backslash(a,b) = \sigma(AG)\backslash[a,b].$$

Moreover, similarly as the proof of $E_k(\Delta_k)E_j(\Delta_j) = 0$ in step 1, it is showed that $E([a,b])E([c,d]) = 0$ for compact intervals $[a,b],[c,d] \in \mathcal{R}_s(\mathbb{R})$ with $[a,b] \cap [c,d] = \emptyset$.

(c) In this last step of the proof we define the spectral projection $E(\Delta)$ for every $\Delta \in \mathcal{R}_s(\mathbb{R})$ and show that the function E, defined on $\mathcal{R}_s(\mathbb{R})$, has the properties (i)–(iv) in Definition (1.2.5). Let $\Delta \in \mathcal{R}_s(\mathbb{R})$. Then each $\alpha \in \Delta \cap s$ is contained in the interior $\Delta_i\ of\ \Delta$. Hence, there exists a compact interval $\Delta_\alpha \subset \Delta$ such that $\Delta_\alpha^i \cap s = \{\alpha\}$. Choose these intervals such that $\Delta_\alpha \cap \Delta_\beta = \emptyset\ for\ \alpha,\beta \in \Delta \cap s, \alpha \neq \beta$, and define the projection $E(\Delta)\ by$

$$E(\Delta) := \sum_{\alpha\in\Delta\cap s} E(\Delta_\alpha) + E\left(\Delta \backslash \bigcup_{\alpha\in\Delta\cap s} \Delta_\alpha\right). \tag{40}$$

Let $\alpha \in s$ and let $[a,b] \in \mathcal{R}_s(\mathbb{R})$such that $(a,b) \cap s = \{\alpha\}$. Furthermore, let $a',b' \in (a,b)$ such that $a' < \alpha < b'$.
From the construction of $E([a,b]), E([a',b])$ and $E([a,b'])$ in step 2 it is seen that

$$E([a,a')) + E([a,b]) = E([a,b']) + E((b',b]) = E([a,b]).$$

With the help of this property it is shown that $E(\Delta)$ in (40) is well-defined.

It remains to verify that E satisfies the conditions (i)–(v) in Definition (1.2.5). Let $\Delta_1, \Delta_2 \in \mathcal{R}_s(\mathbb{R})$. Then $\Delta_j = \Delta_j^1 \cup \Delta_j^2$, where $\Delta_j^1 \cup \Delta_j^2 = \emptyset, \Delta_j^2 \in \mathcal{R}_{s,0}(\mathbb{R})$ and

$$\Delta_j^1 = \bigcup_{\alpha\in\Delta_j\cap s} \Delta_\alpha^j$$

with compact intervals Δ_α^j as above, $j = 1,2$. We may choose the intervals Δ_α^j such that the following holds:

$$(a)\ \Delta_1^2 \cap \Delta_1^2 = \Delta_1^1 \cap \Delta_2^2 = \emptyset,$$
$$(b) \Delta_\alpha^1 = \Delta_\alpha^2 \quad for\ \alpha \in \Delta_1 \cap \Delta_2 \cap s,$$
$$(c)\ \Delta_\alpha^1 \cap \Delta_\beta^2 = \emptyset \quad if\ \alpha \neq \beta.$$

Then we have

$$E(\Delta_1 \cap \Delta_2) = E\big((\Delta_1^1 \cup \Delta_1^2) \cap (\Delta_2^1 \cup \Delta_2^2)\big)$$
$$= E\left((\Delta_1^1 \cup \Delta_2^1) \cup (\Delta_1^2 \cap \Delta_2^2)\right)$$
$$= \sum_{\alpha\in\Delta_1\cap\Delta_2\cap s} E(\Delta_\alpha^1) + E(\Delta_1^2)E(\Delta_2^2).$$

On the other hand,

$$E(\Delta_1)E(\Delta_2) = \sum_{\alpha\in\Delta_1\cap s}\sum_{\beta\in\Delta_2\cap s} E(\Delta_\alpha^1)E\big(\Delta_\beta^2\big) + E(\Delta_1^2)E(\Delta_2^2).$$

And as $E(\Delta_\alpha^1)E\big(\Delta_\beta^2\big) = \delta_{\alpha\beta}E(\Delta_\alpha^1)$, where $\delta_{\alpha\beta}$ is the Kronecker delta, (i) follows.

The proof of (ii) is straightforward and (iii) follows from the facts showed in steps 1 and 2. For the proofs of (iv) and (iv) let $\Delta \in \mathcal{R}_s(\mathbb{R})$. Then $\Delta = \Delta_1 \cup \Delta_2$ where $\Delta_1 \cap \Delta_2 = \emptyset, \Delta_2 \in \mathcal{R}_{s,0}(\mathbb{R})$, and Δ_1 is the union of mutually disjoint compact intervals $\Delta_{\alpha j} \in \mathcal{R}_s(\mathbb{R}), j = 1,\ldots,r,\ with\, \Delta_{\alpha j} \cap s = \{\alpha_j\}$. Due to the definition of $E(\Delta)$ we have

$$E(\Delta)\mathcal{H} = E(\Delta_{\alpha 1})\mathcal{H}[\dot{+}] \cdots [\dot{+}]E(\Delta_{\alpha r})\mathcal{H}[\dot{+}]E(\Delta_2)\mathcal{H}.$$

Hence,

$$\sigma(AG|E(\Delta)\mathcal{H}) \subset (\sigma(AG) \cap \Delta_{\alpha 1}) \cup \cdots \cup (\sigma(AG) \cap \Delta_{\alpha 1}) \cup \overline{\sigma(AG) \cap \Delta_2}$$
$$= (\sigma(AG) \cap \Delta_1) \cup \overline{\sigma(AG) \cap \Delta_2} \subset \overline{\sigma(AG) \cap \Delta}.$$

From $(I - E(\Delta))\mathcal{H} \subset \left(I - E\left(\Delta_{\alpha j}\right)\right)\mathcal{H}\ for\, j = 1,\ldots,r$ and $(I - E(\Delta))\mathcal{H} \subset (I - E(\Delta_2))\mathcal{H}$ we conclude

$$\sigma(AG|(I - E(\Delta))\mathcal{H}) \subset \sigma(AG|(I - E(\Delta_{\alpha j}))\mathcal{H}),$$
$$\sigma(AG|(I - E(\Delta))\mathcal{H}) \subset \sigma(AG|(I - E(\Delta_2))\mathcal{H}),$$

and therefore

$$\sigma(AG|(I - E(\Delta))\mathcal{H}) \subset \overline{\sigma(AG) \setminus \Delta_{\alpha 1}} \cap \ldots \cap \overline{\sigma(AG) \setminus \Delta_{\alpha r}} \cap \overline{\sigma(AG) \setminus \Delta_2}$$
$$\subset \overline{\sigma(AG) \setminus \Delta_1} \cap \overline{\sigma(AG) \setminus \Delta_2}$$
$$\subset \overline{\sigma(AG) \setminus \Delta} \cup \partial\Delta_1,$$

where $\partial\Delta_1$ is the real boundary of Δ_1. This is a finite set which depends on the choice of the $\Delta_{\alpha j}$'s. Hence, the theorem is showed.

Let $w, p\ and\ q$ be real-valued functions on a bounded or unbounded open interval (a, b) such that $w, p^{-1}, q \in L^1_{loc}(a, b)$ and $w > 0$ almost everywhere. The differential expression

$$\tau(f) := \frac{1}{w}(-(pf')' + qf')$$

is then called a Sturm-Liouville differential expression. Usually, the differential operators associated with τ are considered in the weighted L^2-space $L^2_w(a, b)$ which consists of all measurable functions $f : (a, b) \to \mathbb{C}$ for which $f^2 w \in L^1(a, b)$. If

$$\underset{x \in (a,b)}{ess\ inf} w(x) > 0\ and \underset{x \in (a,b)}{ess\ sup} w(x) < \infty,$$

then the topologies of $L^2_w(a, b)$ and $L^2(a, b)$ coincide, and the selfadjoint realizations of τ in $L^2_w(a, b)$ are similar to selfadjoint operators in $L^2(a, b)$. In the following we use the abstract results from the previous section to show that also in more general cases it can make sense to consider differential operators associated with τ in $L^2(a, b)$.

By A denote the operator of multiplication with the function w^{-1} in the Hilbert space $L^2(a, b)$. The operator A is selfadjoint and non-negative (in $L^2(a, b)$). In addition, define the operator $Gmax\ in\ L^2(a, b) by\ G_{max} f := -(pf')' + qf, f \in dom\ G_{max}$, where

$$domG_{max} := \{f \in L^2(a, b) : f, pf' \in AC_{loc} - (a, b), -(pf')' + qf \in L^2(a, b)\}.$$

The selfadjoint realizations of the differential expression

$$\tau_0(f) := -(pf')' + qf$$

in $L^2(a, b)$ are well-known to be restrictions of G_{max}. In what follows let G be a selfadjoint realization of τ_0 in $L^2(a, b)$.

Proposition(1.2.23)[183]. If $w \in L^\infty(a, b)$ and G is boundedly invertible, then the spectrum of the operator AG is real, and AG has a spectral function without singularities on $\mathbb{R}$.

Proof. From $w \in L^\infty(a, b)$ it follows that the operator $A = w^{-1}$ is boundedly invertible in $L^2(a, b)$. Hence, $0 \in \rho(A) \cap \rho(G)$ which implies that both AG and GA are boundedly invertible. Therefore, (30) is satisfied for the selfadjoint operators $A\ and\ G$. Furthermore, for $f \in domAG$ we have $(AGf, Gf) \geq 0$ as A is non-negative. Hence, the pair (A, G) is definitizable with definitizing polynomial $p(\lambda) = \lambda$, and the assertions follow directly from Theorems (1.2.20) and (1.2.22).

Chapter 2
Finite Rank Perturbation and Lipschitz Functions

We obtain more general results about the behavior of double operator integrals of the form $Q = \iint (f(x) - f(y))(x - y)^{-1} dE_1(x) T dE_2(y)$, where E_1 and E_2 are spectral measures. We show that if $T \in S_1$, then $Q \in S$ and if $rank\,T = 1$, then $Q \in S_{1,\infty}$. Finally, if T belongs to the Matsaev ideal S_ω, then Q is a compact operator. It is the aim of this note to show a more general variant of this perturbation result where the assumption on $\rho(B)$ is dropped. As an application a class of singular ordinary differential operators with indefinite weight functions is studied.

Section (2.1): Definitizable Operators

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space, i.e., $\mathcal{K}$ can be written as the direct $[\cdot,\cdot]$-orthogonal sum $\mathcal{K}_+ [\dotplus] \mathcal{K}_-$ of Hilbert spaces $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$, and let A be an operator in $\mathcal{K}$ which coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. In general such selfadjoint operators may have unpleasant spectral properties, e.g., the spectrum may cover the whole complex plane. we consider the special class of definitizable operators. A selfadjoint operator $A\, in\, \mathcal{K}$ is called definitizable if the resolvent set of A is nonempty and there exists a polynomial $p \neq 0$ such that $p(A)$ is a nonnegative operator in the Krein space $\mathcal{K}$, cf. [176,177]. Definitizable operators arise in various applications and have been studied extensively in the last decades, see, [159,160,161,162,163,170,171,172,173,175,176,177,178,179,180]. In connection with spectral problems for Sturm–Liouville operators with indefinite weights definitizable operators were studied in [157,159,160,162,171,172]. In these applications the particular operator of interest can be regarded as a perturbation of a definitizable operator $A_+ \times A_-\, in\, \mathcal{K}$, where A_+ and A_- are selfadjoint operators in $\mathcal{K}_+$ and $\mathcal{K}_-$, respectively. Therefore general perturbation results for definitizable operators are very useful and of great importance.

A classical well-known result on finite rank perturbations of definitizable operators was showed by P. Jonas and $\mathcal{H}$. Langer in [169]. Assume that A is a definitizable selfadjoint operator in the Krein space $\mathcal{K}$, let B be a selfadjoint operator in $\mathcal{K}$ with nonempty resolvent set $\rho(B)$ and suppose that

$$dim\, ran(B - \lambda)^{-1} - ((A - \lambda)^{-1}) < \infty$$

holds for some, and hence for all, $\lambda \in \rho(A) \cap \rho(B)$. Then it was shown in [169] that also the perturbed operator B is definitizable. However, in applications it is often difficult to verify the condition on $\rho(B)$, e.g., for ordinary differential operators with indefinite weights, cf. [162], so that there is a strong desire to have a perturbation result of the above type available without any assumptions on the resolvent set of B. It is the aim of Theorem (2.1.2) in the present note to fill this gap. Instead of a finite rank perturbation in resolvent sense we suppose that the symmetric operator $S = A \cap B$ is of finite defect, i.e., the (graphs) of $A\, and\, B$ by finitely many dimensions. Under this assumption we show the following equivalence for two selfadjoint operators $A\, and\, B$ in a Krein space: A is definitizable if and only if B is definitizable.

In this Section new variant of the perturbation result from [169] is applied to ordinary differential operators with an indefinite weight function. We consider singular differential expressions of order $2n\, on\, \mathbb{R}$ and generalize some of the results in [162].

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let A be a linear operator in $\mathcal{K}$. The symbols $domA, kerA$, and $ranA$ stand for the domain, kernel and range of A, respectively. Suppose that A is a selfadjoint operator in $\mathcal{K}$, i.e., A coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. Then A is said to be definitizable if its resolvent set $\rho(A)$ is nonempty and there exists a real polynomial $p, p \neq 0$, such that

$$[p(A)x, x] \geq 0 \qquad for\ all\ x \in domp(A).$$

It was shown by $\mathcal{H}$. Langer that a definitizable operator A has a spectral function which is defined on all real intervals with boundary points which are not critical points of A, see [176,177]. Moreover, for a definitizable operator A the nonreal spectrum $\sigma(A) \cap (\mathbb{C} \setminus \mathbb{R})$ consists of at most finitely many pairs of eigenvalues which are symmetric with respect to the real line. Note that a selfadjoint operator A with $\rho(A) \neq \emptyset$ and the property that the hermitian form $[A\ \cdot,\cdot]$ defined on $domA$ has finitely many negative squares is definitizable, cf. [177].

Definitizability of selfadjoint operators in Krein spaces can also be characterized in a different form, see Theorem (2.1.1) below. Recall that for a selfadjoint operator $A\ in\ \mathcal{K}$ a point λ from the approximative point spectrum is said to be a spectral point of positive type (negative type) of A if for each sequence $(x_n) \subset domA\ with\ \|x_n\| = 1, n = 1,2,\ldots, and\ \|(A-\lambda)x_n\| \to 0\ for\ n \to \infty$,

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \qquad \left(\limsup_{n\to\infty} [x_n, x_n] < 0, respectively\right)$$

holds, cf. [168,178]. The selfadjointness of A implies that the spectral points of positive and negative type are real. An open set $\Delta \subset \mathbb{R}$ is said to be of positive type (negative type) with respect to $A\ if\ \Delta \cap \sigma(A)$ consists of spectral points of positive type (negative type, respectively). We say that an open set $\Delta \subset \mathbb{R}$ is of definite type with respect to $A\ if\ \Delta$ is either of positive or negative type with respect to A.

The next Theorem follows from [167] and [168] where the concept of local definitizability of selfadjoint operators in Krein spaces is investigated in details.We shall use the equivalent characterization of definitizable operators from Theorem (2.1.1) in the proof of Theorem (2.1.4). The one-point compactifications of the real line and the complex plane are denoted by $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$, respectively.

Theorem(2.1.1)[156]. Let A be a selfadjoint operator in the Krein space $\mathcal{K}$. Then A is definitizable if and only if the following holds:

(i) Every point $\mu \in \mathbb{R}$ has an open connected neighborhood $\mathcal{U}_\mu$ in $\overline{\mathbb{R}}$such that both intervals $\mathcal{U}_\mu \setminus \{\mu\}$ are of definite type with respect to A;

(ii) $\sigma(A) \cap (\mathbb{C}\setminus\mathbb{R})$ consists of at most finitely many isolated points which are poles of the resolvent of A;

(iii) there exist $m \geq 1, M > 0$ and an open neighborhood $\mathcal{O}$ of $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$ such that

$$\|(A-\lambda)^{-1}\| \leq M(1+|\lambda|)^{2m-2}|Im\,\lambda|^{-m} \qquad for\ all\ \lambda \in \mathcal{O}\setminus\overline{\mathbb{R}}.$$

In this section a classical result from [169] on finite rank perturbations of definitizable operators is generalized, see Theorem (2.1.2) below. Roughly speaking we drop the assumption from [169] that the perturbed operator has a nonempty resolvent set. In order to formulate our variant of the perturbation result we remind the reader that a (possibly nondensely defined) operator S in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ is called symmetric if $[Sx, x]$ is real for all $x \in domS$. Recall also that a closed symmetric operator $S\ in\ \mathcal{K}$ is said to be of defect $m \in \mathbb{N}_0$if there exists a selfadjoint extension A of $S\ in\ \mathcal{K}$ such that $dim(graph(A)/\,graph(S)) = m$. Note that m is independent of the choice of the selfadjoint extension $A\ of\ S$.

Theorem(2.1.2)[156]. Let A and B be selfadjoint operators in the Krein space $\mathcal{K}$ and assume that $A \cap B$ is of finite defect. Then A is definitizable if and only if B is definitizable.

Proof. Assume that A is definitizable and let $S := A \cap B$, i.e.,

$$domS = \{f \in domA \cap domB : Af = Bf\},$$
$$Sf = Af = Bf, f \in domS. \tag{1}$$

We will show in the following that $\rho(B)$ is nonempty. Then the assumption that the defect of S is finite implies that

$$dim(dom(A-\lambda)^{-1}/dom(S-\lambda)^{-1}) = dim(dom(B-\lambda)^{-1}/dom(S-\lambda)^{-1})$$

is finite for all $\lambda \in \rho(A) \cap \rho(B)$ and $(A-\lambda)^{-1}f = (B-\lambda)^{-1}f,\ \ f \in dom(S-\lambda)^{-1}$, yields

$$dim\,ran\,(B-\lambda)^{-1} - (A-\lambda)^{-1} < \infty \qquad for\ all\ \lambda \in \rho(A) \cap \rho(B). \tag{2}$$

Therefore the statement of Theorem (2.1.2) follows from [169].

Let $p \neq 0$ be a definitizing real polynomial for the selfadjoint operator A, that is, $p(A)$ is a nonnegative operator in $\mathcal{K}$ and with the exception of at most finitely many points the set $\mathbb{C}\setminus\mathbb{R}$ belongs to $\rho(A)$. It is clear that $p(A)$ is symmetric in the Krein space $\mathcal{K}$ and it follows from $\sigma(p(A)) = p(\sigma(A))$ (see, [166]) that $\rho(p(A)) \cap (\mathbb{C}\setminus\mathbb{R})$ is nonempty. Therefore $p(A)$ is a selfadjoint operator in $\mathcal{K}$ and as $p(A)$ is nonnegative we have

$$\mathbb{C}\setminus\mathbb{R} \subset \rho(p(A)). \tag{3}$$

Observe that dom$S$ in (1) is in general not a dense subspace in $\mathcal{K}$ and therefore the adjoint of S has to be defined in the sense of linear relations, i.e., S^+ is the linear subspace

$$S^+ := \{\{f, f'\} \in K2: [Sg, f] = [g, f'] \quad for\ all\ g \in domS\}$$

of $\mathcal{K} \times \mathcal{K}$ cf., e.g., [164]. Here and in the following the elements of a linear relation are written in curly brackets. Operators are regarded as linear relations via their graphs. Note that the definition of S^+ extends the usual definition of the adjoint of a densely defined operator and, moreover, S^+ is an operator if and only if $domS$ is dense in $\mathcal{K}$.

We claim that for each $\lambda \in \mathbb{C}\backslash \mathbb{R}$ the linear relation $p(S^+)$ (see, e.g., [165,189]), can be decomposed in the form

$$p(S^+) = p(A) \dotplus \{h, \lambda h\}: h \in ker(p(S^+) - \lambda)\}, \tag{4}$$

Where $\dotplus$ denotes the direct sum of subspaces. In fact, $S \subset A\ and\ A = A^+$ implies $A \subset S^+$, and hence also $p(A) \subset p(S^+)$. Therefore the inclusion

$$p(A) \dotplus \{h, \lambda h\}: h \in ker(p(S^+) - \lambda)\} \subset p(S^+)$$

holds and the sum is direct since by (3) we have $ker(p(A) - \lambda) = \{0\}$ for any $\lambda \in \mathbb{C}\backslash \mathbb{R}$. In order to verify the reverse inclusion let $\{f, f'\} \in p(S^+)$. By (3) we have $ran(p(A) - \lambda) = \mathcal{K}, \lambda \in \mathbb{C}\backslash \mathbb{R}$, and hence there exists $\{g, g'\} \in p(A)$ such that $f' - \lambda f = g' - \lambda g$. This, together with $\{f, f' - \lambda f\} \in p(S^+) - \lambda\ and\ \{g, g' - \lambda g\} \in (p(A) - \lambda) \subset (p(S^+) - \lambda)$ implies

$$\{f - g, 0\} = \{f, f' - \lambda f\} - \{g, g' - \lambda g\} \in p(S^+) - \lambda,$$

i.e., $f - g \in ker(p(S^+) - \lambda)$. Thus $\{f, f'\} = \{g, g'\} + \{f - g, \lambda(f - g)\}$ is decomposed as in (4).

Next it will be shown that $p(S^+)$ is a finite dimensional extension of $p(A)$. According to (4) it is sufficient to check that $ker(p(S^+) - \lambda_0)$ is finite dimensional for some $\lambda_0 \in \mathbb{C}\backslash \mathbb{R}$. Observe first that the polynomial $q(\mu) := p(\mu) - \lambda_0$ has no real zeros since p is a real polynomial and $\lambda_0 \in \mathbb{C}\backslash \mathbb{R}$. Hence there exist $m \in \mathbb{N}, k_1, \ldots, k_m \in \mathbb{N}, \beta_1, \ldots, \beta_m \in \mathbb{C}\backslash \mathbb{R}$ and $\alpha \in C \setminus \{0\}$ such that

$$q(\mu) = \alpha \prod_{i=1}^{m} (\mu - \beta_i) k_i\ .$$

Furthermore we can assume that $\lambda_0 \in \mathbb{C}\backslash \mathbb{R}$ was chosen in such a way that none of the nonreal eigenvalues of the definitizable operator A is a zero of q. According to [189]

$$ker\ q(S^+) = ker(p(S^+) - \lambda_0) = \sum_{i=1}^{m} ker(S^+ - \beta_i) k_i \tag{5}$$

holds. As the defect of S is finite, S^+ is a finite dimensional extension of A and from the fact that each β_i belongs to $\rho(A)$ we conclude from

$$S^+ = A \dotplus \{\{g, \beta_i g\}: g \in ker(S^+ - \beta_i)\}$$

that the dimension of $ker(S^+ - \beta_i), i = 1, \ldots, m$, is also finite. In a similar way as for operators one then verifies

$$dim(ker(S^+ - \beta_i) k_i) < \infty$$

and thus (4) and (5) imply

$$n := dimp(S^+/p(A)) = dim\big(kerp(S^+ - \lambda_0)\big) < \infty. \tag{6}$$

Hence, $p(S^+)$ is a finite dimensional extension of $p(A)$. From (6) we conclude that the closed symmetric operator $(p(S^+))^+\ in\ \mathcal{K}$ has finite defect n and $(p(S^+))^+ \subset p(A)$ implies that $(p(S^+))^+$ is nonnegative. Since p is a real polynomial it follows that $p(B)$ is a symmetric operator in $\mathcal{K}$. From $B = B^+$ and $S \subset B$ we obtain $B \subset S^+$, hence $p(B)$ is a restriction of $p(S^+)$ and an extension of $p(S), p(S) \subset p(B) \subset p(S^+)$. As $(p(S^+))^+$ and $p(S)$ have finite defect and $p(B)$ is a symmetric operator, it follows that $p(B)$ admits selfadjoint extensions in $\mathcal{K}$ which are operators. Then it follows in the same way as in the proof of [162] that such a selfadjoint (operator) extension $T\ of\ p(B)$ has a nonempty resolvent set. In fact, by (3) we have $ran(p(A) - \lambda) = \mathcal{K}$ for all $\lambda \in \mathbb{C}\backslash \mathbb{R}$, hence the ranges of $p(S) - \lambda$ are closed and the same holds for the ranges of the finite dimensional extensions $p(B) - \lambda\ and\ T - \lambda, \lambda \in \mathbb{C}\backslash \mathbb{R}$. Suppose now $\rho(T) = \emptyset$. Then it follows that in at least one of the halfplanes there are infinitely many points belonging to $\sigma_p(T). Let\ f_1, \ldots, f_{n+1}$ be eigenvectors corresponding to $n+1$ different eigenvalues of T in that halfplane. Choose vectors $g_1,\ldots,g_{n+1}$ in the dense subspace $domT$ such that $[Tf_i,g_j] = \delta_{ij}, i,j = 1,\ldots,n+1$, holds, cf. [162]. Then the Krein space

$$L := (spanf_1,\ldots,f_{n+1},g_1,\ldots,g_{n+1}\}, [T\ \cdot,\cdot])$$

contains an *(n + 1)*-dimensional neutral subspace. Hence L contains also an $(n+1)$-dimensional negative subspace, which contradicts the fact that T is an n-dimensional extension of the nonnegative operator $p(S)$. Therefore

$$\rho(T) \neq \emptyset.$$

Since $[T\ \cdot\ ,\cdot]$ has finitely many negative squares and $\rho(T) \neq \emptyset$ it follows that T is a definitizable operator, cf. [177]. In particular, the set $\mathbb{C}\setminus\mathbb{R}$ with the possible exception of at most finitely many points belongs to $\rho(T)$. Therefore, up to a finite set each $\lambda \in \mathbb{C}\setminus\mathbb{R}$ is a point of regular type of the finite dimensional restriction $p(B)$ *of* T, that is, $ker(p(B)-\lambda) = \{0\}$ *and* $ran(p(B)-\lambda)$ is closed. This together with $\sigma_p(p(B)) = p(\sigma_p(B))$ and the fact that the range of $B-\lambda$ is closed for all $\lambda \in \rho(A)$ implies that there exists a pair $\{\mu,\bar{\mu}\}, \mu \in \mathbb{C}\setminus\mathbb{R}$, of points of regular type of B, i.e., $ran(B-\mu)$ *and* $ran(B-\overline{\mu})$ are closed and $ker(B-\mu) = ker(B-\overline{\mu}) = \{0\}$. But this is possible only if $ran(B-\mu) = ran(B-\overline{\mu}) = \mathcal{K}$, therefore $\{\mu,\overline{\mu}\} \in \rho(B)$. Thus $\rho(B) \neq \emptyset$ and the statement of Theorem (2.1.2) follows from (2) and [169]. We note for the sake completeness that $\rho(B) \neq \emptyset$ implies $\rho(p(B)) \neq \emptyset$ and hence $p(B)$ is selfadjoint, coincides with T and is an extension of $(p(S^+))^+$. (See [158,168,162].

Corollary(2.1.3)[156].. Let S be a closed symmetric operator of finite defect in the Krein space $\mathcal{K}$ and assume that there exists a selfadjoint extension of S in $\mathcal{K}$ which is definitizable. Then the following holds:

(i) every selfadjoint extension of S in $\mathcal{K}$ which is an operator has a nonempty resolvent set and is definitizable;

(ii) if S is densely defined, then every selfadjoint extension of S in $\mathcal{K}$ has a nonempty resolvent set and is definitizable.

We consider the formal differential expression of order $2n$ *on* $\mathbb{R}$ given by

$$(f) = \frac{1}{r}\Big((-1)^n\big(p_0f^{(n)}\big)^{(n)} + (-1)^{n-1}\big(p_1f^{(n-1)}\big)^{(n-1)} +\cdots+p_nf\Big), \tag{7}$$

where $r, p_0^{-1}, p_1,\ldots,p_n \in L^1_{loc}(\mathbb{R})$ are assumed to be real functions such that $r \neq 0$ *and* $p_0 > 0$ a.e. on $\mathbb{R}$. With the help of the quasi-derivatives

$$f^{[0]} := f, f^{[k]} := \frac{d^kf}{dx^k}, \qquad k = 1,2,\ldots,n-1,$$

$$f^{[n]} := p_0\frac{d^nf}{dx^n}, f^{[n+k]} := p_k\frac{d^{n-k}f}{dx^{n-k}} - \frac{d}{dx}f^{[n+k-1]}, \qquad k = 1,2,\ldots,n,$$

cf. [174,181], the formal expression (7) can be written as

$$(f) = \frac{1}{r}f^{[2n]}. \tag{8}$$

Following the lines of [157,162] we show that under suitable assumptions definitizable selfadjoint operators in a Krein space can be associated to the differential expression ℓ .

For the weight function r the following condition (I) is supposed to hold (cf. [157] and [162]):

(I) There exist $a,b \in \mathbb{R}, a < b$, such that the restrictions $r_+ := r_-\ (b,\infty)$ and $r_- := r \upharpoonright (-\infty,a)$ satisfy $r_+ > 0$ a.e. on (b,∞) *and* $r_- < 0$ a.e. on $(-\infty,a)$.

In the following we agree to choose $a,b \in \mathbb{R}$ in such a way that the sets $\{x \in (a,b): r(x) > 0\}$ and $\{x \in (a,b): r(x) < 0\}$ have positive Lebesgue measure. This is no restriction. We note that the case $r_+ < 0$ *and* $r_- > 0$ can be treated analogously. We do not consider the case that r_+ *and* r_- have the same signs. Under suitable assumptions these cases are contained in the considerations in [162], cf. Remark (2.1.5). below.

Let $L^2_{|r|}(\mathbb{R})$ be the Hilbert space of all equivalence classes of measurable functions f defined on $\mathbb{R}$ for which $\int_{\mathbb{R}} |f(x)|^2|r(x)|dx$ is finite. We equip $L^2_{|r|}(\mathbb{R})$ with the indefinite inner product

$$[f,g]:=\int_{\mathbb{R}} f(x)\overline{g(x)}r(x)dx, \qquad f,g \in L^2_{|r|}(\mathbb{R}), \tag{9}$$

and denote the corresponding Krein space $(L^2_{|r|}(\mathbb{R}),[\cdot,\cdot])$ by $L^2_{|r|}(\mathbb{R})$. The maximal operator $S_{max}f = \ell(f)$ associated to (8) is defined on the dense subspace $\mathcal{D}_{max}$ consisting of all functions $f \in L^2_r(\mathbb{R})$ which have absolutely continuous quasi derivatives $f^{[0]}, f^{[1]},\dots, f^{[2n-1]}$ such that $f \in L^2_r(\mathbb{R})$. The restriction S^0_{min} of S_{max} to functions with compact support is a densely defined symmetric operator in the Krein space $L^2_r(\mathbb{R})$. The minimal operator S_{min} is the closure of S^0_{min} . It is a symmetric operator in $L^2_r(\mathbb{R})$ of defect $m, 0 \le m \le 2n$, and $S^+_{min} = S_{max}$ holds, cf. [162,181]. In particular, the selfadjoint realizations of ℓ in $L^2_r(\mathbb{R})$ are finite dimensional extensions of S_{min} in $L^2_r(\mathbb{R})$.

Denote by ℓ_-, ℓ_{ab} and ℓ_+ the differential expressions on the intervals $(-\infty,a),(a,b)$and (b,∞), respectively, which are defined in the same way as ℓ, except that the functions $r, p_0, p_1,\dots,p_n$ in (7) are replaced by their restrictions onto $(-\infty,a),(a,b)$ and (b,∞), respectively. By condition (I) the inner product (9) is positive definite on functions with support in (b,∞) and negative definite on functions with support in $(-\infty,a)$. Furthermore, (9) is indefinite on functions with support in (a,b). Therefore

$$L^2_{r_+}((b,\infty)) := \left(L^2_{|r_+|}((b,\infty)),[\cdot,\cdot]\right)$$

is a Hilbert space,

$$L^2_{r_-}((-\infty,a)) := \left(L^2_{|r_-|}((-\infty,a)),[\cdot,\cdot]\right)$$

is an anti-Hilbert space, i.e., $\left(L^2_{|r_-|}((-\infty,a)),[\cdot,\cdot]\right)$ is a Hilbert space, and

$$L^2_{r_{ab}}\big((a,b)\big) := \left(L^2_{|r_{ab}|}\big((a,b)\big),[\cdot,\cdot]\right), \qquad r_{ab} := r \upharpoonright (a,b),$$

is a Krein space with infinite positive and negative index. Since a and b are regular endpoints, the minimal closed symmetric operators $S_{min,+}$ and $S_{min,-}$ associated to ℓ_+ and ℓ_- have defect $m, n \le m \le 2n$, cf. [181], and the selfadjoint realizations of ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$ are finite dimensional extensions of $S_{min,+}$ and $S_{min,-}$, respectively.

Theorem(2.1.4)[156]. Suppose that the weight function r satisfies condition (I) and assume that A_+ and A_- are selfadjoint brealizations of ℓ_+ and ℓ_- in the spaces $\mathrm{L}^2_{\mathrm{r}_+}((\mathrm{b},\infty))$ and $\mathrm{L}^2_{\mathrm{r}_-}((-\infty,\mathrm{a}))$, respectively, such that the following holds:

(i) A_+ is semibounded from below and A_- is semibounded from above;

(ii) the set $\mathrm{e} := \sigma(\mathrm{A}_+) \cap \sigma(\mathrm{A}_-)$ is finite;

(iii) there exist disjoint open intervals $\mathcal{J}_1,\dots,\mathcal{J}_{\mathrm{n}0} \subset \mathbb{R}$ and some $\mathrm{j}_0 \in \{1,\dots,\mathrm{n}_0+1\}$ such that

$$\sigma(\mathrm{A}_+)\setminus\{\mathrm{e}\} \subset \bigcup_{\mathrm{k}=1}^{\mathrm{j}_0} \mathcal{J}_{\mathrm{k}} \quad \text{and}\ \sigma(\mathrm{A}_-)\setminus\{\mathrm{e}\} \subset \bigcup_{\mathrm{k}=\mathrm{j}_0+1}^{\mathrm{n}_0+1} \mathcal{J}_{\mathrm{k}}.$$

Then every selfadjoint realization of the differential expression ℓ in the Krein space $\mathrm{L}^2_\mathrm{r}(\mathbb{R})$ has a nonempty resolvent set and is a definitizable operator.

Proof. Denote the minimal closed symmetric operator associated to ℓ_{ab} in the Krein space $L^2_{r_{ab}}((a,b))$ by $S_{min,ab}$. The defect of $S_{min,ab}$ is $2n$, cf. [181]. Let A_{ab} be a selfadjoint extension of $S_{min,ab}$ in the Krein space $L^2_{r_{ab}}((a,b))$. Then according to [258] the spectrum $\sigma(A_{ab})$ is discrete, $\rho(A_{ab})$ is nonempty, the hermitian sesquilinear form $[A_{ab}\ \cdot,\cdot]$ defined on $dom\ A_{ab}$ has finitely many negative squares and A_{ab} is definitizable. Let A_+ and A_- be selfadjoint realizations of ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, such that (i)–(iii) hold. We claim that the direct sum

$$A = A_- \times A_{ab} \times A_+, \qquad domA = domA_- \times domA_{ab} \times domA_+ \tag{10}$$

is a definitizable operator in the Krein space

$$L^2_{r_-}((-\infty,a)) \times L^2_{r_{ab}}((a,b)) \times L^2_{r_+}((b,\infty)) = L^2_r(\mathbb{R}).$$

This will be verified with the help of Theorem (2.1.1). First of all $A_\pm$ are selfadjoint operators in Hilbert or anti-Hilbert spaces and thus their spectrum $\sigma(A_\pm)$ is real. Therefore $\sigma(A) \cap (\mathbb{C}\setminus\mathbb{R}) = \sigma(A_{ab}) \cap (\mathbb{C}\setminus\mathbb{R})$. As A_{ab} is definitizable, condition (ii) in Theorem (2.1.1) is satisfied. Similarly the definitizability of A_{ab} together with the growth properties ofthe resolvents $(A_{\pm} - \lambda)^{-1}, \lambda \in \mathbb{C}\backslash\mathbb{R}$, in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, implies (iii) in Theorem (2.1.1).

It remains to check that each point $\mu \in \mathbb{R}$ has an open connected neighborhood $\mathcal{U}_\mu$ such that both intervals $\mathcal{U}_\mu \backslash\{\mu\}$ are of definite type with respect to A. Assume first $\mu \in \mathbb{R}. As\, A_+$ (A_-) is a selfadjoint operator in a Hilbert space (anti- Hilbert space, respectively) $\sigma(A_+)$ $(\sigma(A_-))$ consists only of points of positive type (negative type, respectively). Now (ii) and (iii) imply that $\mathcal{U}_\mu$ can be chosen such that both intervals $\mathcal{U}_\mu \backslash\{\mu\}$ are of definite type with respect to $A_- \times A_+$. Since $\sigma(A_{ab})$ is discrete we can assume $\mathcal{U}_\mu \setminus \{\mu\} \subset \rho(A_{ab})$ and hence both intervals $\mathcal{U}_\mu \setminus \{\mu\}$ are also of definite type with respect to A. Let us now consider the case $\mu = \infty$. As the hermitian sesquilinear form $[A_{ab}\ \cdot,\cdot]$ has finitely many negative squares it follows that there exist $\mu_+ \in (0,\infty)$ *and* $\mu_- \in (-\infty,0)$ such that the interval (μ_+,∞) is of positive type with respect to A_{ab} and the interval $(-\infty,\mu_-)$ is of negative type with respect to A_{ab}. Since by (i) A_+ and A_- are semibounded from below and above, respectively, μ_+ and μ_- can be chosen such that $(\mu_+,\infty) \subset \rho(A_-)$ and $(-\infty,\mu_-) \subset \rho(A_+)$. As $\sigma(A_+)$ is of positive type and $\sigma(A_-)$ is of negative type we conclude that (μ_+,∞) is of positive type with respect to A and $(-\infty,\mu_-)$ is of negative type with respect to A. Thus (i) in Theorem (2.1.1) holds and it follows that A is a definitizable operator in the Krein space $L^2_r(\mathbb{R})$.

Since $A_{\pm}$ are selfadjoint extensions of the operators $S_{min,\pm}$ and A_{ab} is a selfadjoint extension of $S_{min,ab}$ it is clear that A is a selfadjoint extension of the closed symmetric operator $S = S_{min,-} \times S_{min,ab} \times S_{min,+}$ in $L^2_r(\mathbb{R})$. Furthermore, $domS$ is dense and S has finite defect $m, 4n \leq m \leq 6n$. Hence by Corollary (2.1.3) (ii) every selfadjoint extension of S is definitizable. Since each selfadjoint realization of ℓ in $L^2_r(\mathbb{R})$ is an extension of the minimal operator S_{min} associated to ℓ and $S \subset S_{min}$ the assertion of Theorem (2.1.4) follows.

Remark(2.1.5)[156]. The case that the weight function r is positive (negative) on $(-\infty,a)$ and (b,∞) is not considered in Theorem (2.1.4). We note that, e.g., the positivity of r_+, r_- and the semiboundedness of A_+ and A_- from below imply that for some $\alpha \in \mathbb{R}$ the selfadjoint operator $A - \alpha$, where $A = A_- \times A_{ab} \times A_+$ is as in (10), has a finite number of negative squares and $\sigma(A) \cap (-\infty,\eta)$ is discrete for some $\eta \in \mathbb{R}$. Then the same is true for all selfadjoint realizations of ℓ in $L^2_r(\mathbb{R})$, cf. [162].

Definitizability of selfadjoint realizations of indefinite Sturm–Liouville differential expressions of the form (7)-(8) was already studied in [162]. In addition, the selfadjoint differential operators arising in [162] have finitely many negative squares. The following two corollaries connect Theorem (2.1.4) with the results in [162].

Corollary(2.1.6)[156]. Suppose that the weight function r satisfies condition (I) and assume that A_+ and A_- are selfadjoint realizations of ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, such that $\sigma(A_+) \cap (-\infty,0)$ and $\sigma(A_-) \cap (0,\infty)$ consist of finitely many eigenvalues.

Then every selfadjoint realization B of the differential expression ℓ in the Krein space $L^2_r(\mathbb{R})$ has a nonempty resolvent set and the form $[B\ \cdot,\cdot]$ has finitely many negative squares.

Proof. The assumption that $\sigma(A_+) \cap (-\infty,0)$ *and* $\sigma(A_-) \cap (0,\infty)$ consist of finitely many eigenvalues implies that conditions (i)–(iii) in Theorem (2.1.4) hold. Hence every selfadjoint realization of in $L^2_r(\mathbb{R})$ has a nonempty resolvent set and is definitizable. Furthermore, it is not difficult to see that the selfadjoint operator $A_+ \times A_-$ in $L^2_{r_-}((-\infty,a)) \times L^2_{r_+}((b,\infty))$ has finitely many negative squares (cf., e.g., [160]) and the same holds for the selfadjoint operator $A = A_- \times A_{ab} \times A_+$ in $L^2_r(\mathbb{R})$, cf. (10). Therefore the symmetric operator $S = S_{min,-} \times S_{min,ab} \times S_{min,+}$ also has finitely many negative squares and hence every selfadjoint realization B of ℓ in $L^2_r(\mathbb{R})$ has finitely many negative squares.

Corollary(2.1.7)[156]. Suppose that the weight function r satisfies condition (I) and let $S_{min,+}$ and $S_{min,-}$ be the minimal closed symmetric operators associated to ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively. Assume that there exist $b' \in (b,\infty)$ and $a' \in (-\infty,a)$ such that

$[\,S_{\min,+}\ \cdot,\cdot]$ and $[\,S_{\min,-}\ \cdot,\cdot]$ are positive on the set of functions from $\operatorname{dom} S_{\min,+}$ and $\operatorname{dom} S_{\min,-}$ which have compact support in (b',∞) and $(-\infty,a')$, respectively.

Then every selfadjoint realization B of the differential expression ℓ in the Krein space $L^2_r(\mathbb{R})$ has a nonempty resolvent set and the form $[B\ \cdot,\cdot]$ has finitely many negative squares.

Proof. As in the proof of [162] one verifies that the inner product $[\,S_{min,+}\ \cdot,\cdot]$ has a finite number of negative squares on $dom\,S_{min,+}$. Hence, if A_+ is an arbitrary selfadjoint extension of $S_{min,+}$ in $L^2_{r_+}((b,\infty))$, then also the form $[A_+\ \cdot,\cdot]$ defined on $domA_+$ has a finite number of negative squares, so that $\sigma(A_+)\cap(-\infty,0)$ consists of finitely many eigenvalues. Analogously it follows that for any selfadjoint extension A_- of $S_{min,-}$ the form $-[A_-\ \cdot,\cdot]$ has finitely many positive squares, hence the positive spectrum of A_- in $L^2_{r_-}((-\infty,a)) = (L^2_{|r_-|}((-\infty,a))),-[\cdot,\cdot])$ consists of at most finitely many eigenvalues. Therefore the statement follows from Corollary (2.1.6).

Section (2.2): Perturbed Operators

In this section we study the behavior of Lipschitz functions of perturbed operators. It is well known that if $f\in Lip$, i.e., f is a Lipschitz function and $A\ and\ B$ are selfadjoint operators with difference in the trace class S_1, then $f(A)-f(B)$ does not have to belong to S_1. The first example of such f, $A\ and\ B$ was constructed in [154]. Later in [85] a necessary condition on f was found under which the condition $f(A)-f(B)\in S_1$ implies that $f(A)-f(B)\in S_1$. That necessary condition also implies that the condition $f\in Lip$ is not sufficient.

On the other hand, Birman and Solomyak showed in [81] that if $A-B$ belongs to the Hilbert–Schmidt class S_2, then $f(A)-f(B)\in S_2$ and $\|f(A)-f(B)\|_{S_2}\le\|f\|_{Lip}\,\|A-B\|_{S_2}$, where $\|f\|_{Lip}\overset{\text{def}}{=} sup_{x\neq y}|f(x)-f(y)|\cdot|x-y|^{-1}$. Moreover, it was shown in [81] that in this case $f(A)-f(B)$ can be expressed in terms of the following double operator integral

$$f(A)-f(B)=\iint\frac{f(x)-f(y)}{x-y}dE_A(x)(A-B)dE_B(y). \tag{11}$$

where E_A and E_B are the spectral measures of $A\ and\ B$. We refer the reader to [79], [80], and [81] for the beautiful theory of double operator integrals. Note that the divided difference $(f(x)-f(y))/(x-y)$ is not defined on the diagonal. Throughout this note we assume that it is zero on the diagonal.

In this section we study properties of the operators $f(A)-f(B)$ for selfadjoint operators $A\ and\ B$ such that $A-B$ has rank one or $A-B\in S_1$. Actually, we consider more general operators of the form

$$\mathcal{J}_{E_1,E_2}(f,T)\overset{\text{def}}{=}\iint\frac{f(x)-f(y)}{x-y}dE_1(x)T\,dE_2(y), \tag{12}$$

where E_1 and E_2 are Borel spectral measures on $\mathbb{R}$ and rank $T=1\ or\ T\in S_1$. Duality arguments also allow us to study double operator integrals (12) in the case when T belongs to the Matsaev ideal S_ω. Recall the definitions of the following operator ideals:

$$S_{1,\infty}\overset{\text{def}}{=}\left\{T:\ \|\mathbb{T}\|_{S_{1,\infty}}\overset{\text{def}}{=}\sup_{j\ge0}s_j(T)(1+j)<\infty\right\},$$

$$S\overset{\text{def}}{=}\left\{T:\ \|\mathbb{T}\|_S\overset{\text{def}}{=}(log(2+n))^{-1}\sum_{j=0}^{n}s_j(T)<\infty\right\},$$

and

$$S_\omega\overset{\text{def}}{=}\left\{T:\ \|\mathbb{T}\|_{S_\omega}\overset{\text{def}}{=}\sum_{j=0}^{n}\frac{s_j(T)}{1+j}<\infty\right\}.$$

It is well known that $S_{1,\infty}$ is not a Banach space and its Banach hull coincides with S_Ω. Also recall that the dual space to S_ω can be identified in a natural way with S_Ω.

In [73] contains results on properties of $f(A)-f(B)$ for f in the H¨older class $\Lambda_\alpha, 0<\alpha<1$, and selfadjoint operators $A\ and\ B\ with\ A-B$ in Schatten–von Neuman classes S_p.

Theorem(2.2.1)[153]. Let $f \in Lip$ and let E_1 *and* E_2 be Borel spectral measures on $\mathbb{R}$. If rank $T = 1$, then $\mathcal{J}_{E_1,E_2}(f,T) \in S_{1,\infty}$ and

$$\left\|\mathcal{J}_{E_1,E_2}(f,T)\right\|_{S_{1,\infty}} \leq const \, \|f\|_{Lip}\|T\|.$$

Theorem (2.2.1) immediately implies the following result.

Theorem(2.2.2)[153]. Let $f \in Lip$ and let E_1 *and* E_2 be Borel spectral measures on $\mathbb{R}$. If $T \in S_1$, then $\mathcal{J}_{E_1,E_2}(f,T) \in S$ and

$$\left\|\mathcal{J}_{E_1,E_2}(f,T)\right\|_{S} \leq const \, \|f\|_{Lip}\|T\|.$$

By duality, we obtain the following theorem.

Theorem(2.2.3)[153]. Let $f \in Lip$ and let E_1 *and* E_2 be Borel spectral measures on $\mathbb{R}$. Then the transformer $T \mapsto \mathcal{J}_{E_1,E_2}(f,T)$ defined on S_2 extends to a bounded linear operator from S_ω to the ideal of all compact operator and,

$$\left\|\mathcal{J}_{E_1,E_2}(f,T)\right\| \leq const \, \|f\|_{Lip}\|T\|_{S_\omega}.$$

Using interpolation arguments, we can easily obtain from Theorem (2.2.2) the following fact.

Theorem(2.2.4)[153]. Let $f \in Lip$ and let E_1 *and* E_2 be Borel spectral measures on $\mathbb{R}$. Suppose that $1 \leq p < \infty$ and $\varepsilon > 0$. If $T \in S_p$, then
$\mathcal{J}_{E_1,E_2}(f,T) \in S_{p+\varepsilon}$.

Birman–Solomyak formula (11) allows us to deduce straightforwardly from Theorems (2.2.1), (2.2.2), and (2. 2.3) the following theorem.

Theorem(2.2.5)[153]. Let A *and* B be selfadjoint operators on Hilbert space and let $f \in Lip$. We have

$$(i) if \; rank(A - B) = 1, then \; f(A) - f(B) \in S_{1,\infty} \; and \; \|f(A) - f(B)\|_{S_{1,\infty}} \leq const \, \|f\|_{Lip}\|A - B\|;$$

(ii) $if \; A - B \in S_1, then \; f(A) - f(B) \in S \; and \|f(A) - f(B)\|_{S_\Omega} \leq const \, \|f\|_{Lip}\|A - B\|_{S_1}$;

(iii) $if \; A - B \in S_\omega, then \; f(A) - f(B) \; is \; compact \; and \; \|f(A) - f(B)\| \leq const \, \|f\|_{Lip}\|A - B\|_{S_\omega}$;

(iv) $if \; 1 \leq p < \infty, \varepsilon > 0, and \; A - B \in S_p, then \; f(A) - f(B) \in S_{p+\varepsilon}$.

It is still unknown whether the assumption $T \in S_1$ implies that $\mathcal{J}_{E_1,E_2}(f,T) \in S_{1,\infty}$. If this is true, then the condition $A - B \in S_p$ would imply that $f(A) - f(B) \in S_p$ for $1 < p < \infty$.

To show Theorem (2.2.1), we obtain a weak type estimate for Schur multipliers.

For a kernel function $k \in L^2(\mu \times v)$, we define the integral operator
$\mathcal{J}_k : L^2(v) \to L^2(\mu)$ by

$$(\mathcal{J}_k g)(x) = \int k(x,y)g(y)dv(y), g \in L^2(v).$$

As in the case of transformers from S_1 to S_1 (see [81]), Theorem (2.2.1) reduces to the following fact.

Theorem(2.2.6)[153]. Let μ and v be finite Borel measures on $\mathbb{R}, \varphi \in L^2(\mu), \psi \in L^2(v)$. Suppose that $f \in Lip$ and the kernel function k is defined by

$$k(x,y) = \varphi(x)\frac{f(x) - f(y)}{x - y}(y), \qquad x,y \in \mathbb{R}.$$

Then the integral operator $\mathcal{J}_k : L^2(v) \to L^2(\mu)$with kernel function k belongs to $S_{1,\infty}$ and

$$\|\mathcal{J}_k\|_{S_{1,\infty}} \leq const\|\varphi\|_{L^2(\mu)}\|\psi\|_{L^2(v)}.$$

Proof. Without loss of generality we may assume that $\|\varphi\|_{L^2(\mu)} = \|\psi\|_{L^2(v)} = 1 \quad and \; \|f\|_{Lip} = 1$. Let us fix a positive integer n.

Given $N > 0$, we denote by P_N multiplication by the characteristic function of $[-N,N]$ (we use the same notation for multiplication on $L^2(\mu)$and on $L^2(v)$. Then for sufficiently large values of N,

$$\|\mathcal{J}_k - P_N\mathcal{J}_k P_N\|_{S_2} < \frac{1}{n^{1/2}}. \qquad (13)$$

Clearly, $P_N \mathcal{I}_k P_N$ is the integral operator with kernel function $k_N, k_N(x,y) = \chi_N(x)k(x,y)\chi_N(y)$, where $\chi_N = \chi[-N,N]$ is the characteristic function of $[-N,N]. We\ fix\ N > 0$, for which (13) holds. Consider now the points $x_j, 1 \le j \le r, and\ y_j, 1 \le j \le s$, at which μ and v have point masses and

$$|\varphi(x_j)|^2\mu\{x_j\} \ge \frac{1}{n}, 1 \le j \le r, \qquad and\ |\psi(y_j)|^2 v\{y_j\} \ge \frac{1}{n}, 1 \le j \le s. \tag{14}$$

Clearly, $r \le n\ and\ s \le n$. We define now the kernel function $k_\#$ by

$$k_\#(x,y) = u(x)k_N(x,y)v(y), \qquad x,y \in \mathbb{R},$$

where

$$u(x) \stackrel{\text{def}}{=} 1 - \chi_{\{x_1,\dots,x_r\}}(x) \qquad and\ v(y) \stackrel{\text{def}}{=} 1 - \chi_{\{y_1,\dots,y_s\}}(x)(y).$$

Obviously, the integral operators $\mathcal{I}_{k_N}$ and $\mathcal{I}_{k_\#}$ coincide on a subspace of codimension at most $r + s \le 2n$.

We can split now the interval $[-N,N]$ into no more than n subintervals $I, I \in \mathfrak{I}$, such that

$$\int_I |\varphi(x)|^2 u(x)\, d\mu(x) + \int_I |\psi(y)|^2 v(y)\, dv(y) \le \frac{4}{n}, \qquad I \in \mathfrak{I}.$$

This is certainly possible because of (14).

We have $\mathcal{I}_{k_\#} = \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \mathcal{I}^{(3)}$, where

$$\left(\mathcal{I}^{(1)} g\right)(x) = \int_{\mathbb{R}} \left(\sum_{I \in \mathfrak{I}} \chi_I(x) k_\#(x,y)\chi_I(y)\right) g(y)\, dv(y),$$

$$\left(\mathcal{I}^{(2)} g\right)(x) = \int_{\mathbb{R}} \left(\sum_{I,J \in \mathfrak{I}, I \ne J, |I| \ge |J|} \chi_I(x) k_\#(x,y)\chi_I(y)\right) g(y)\, dv(y),$$

and

$$\left(\mathcal{I}^{(2)} g\right)(x) = \int_{\mathbb{R}} \left(\sum_{I,J \in \mathfrak{I}, |I| < |J|} \chi_I(x) k_\#(x,y)\chi_I(y)\right) g(y)\, dv(y),$$

(we denote by $|I|$ the length of I). It is easy to see that $\left\|\mathcal{I}^{(1)}\right\|_{S_2} \le 4n^{-1/2}$. Let us estimate $\mathcal{I}^{(2)}$. The integral operator $\mathcal{I}^{(3)}$ can be estimated in the same way.

Suppose that $I, J \in \mathfrak{I}, I \ne J$, and $|I| \ge |J|$. For $x \in I\ and\ y \in J$, we have

$$\frac{1}{x-y} = \frac{1}{x - c(J)} + \frac{y - c(J)}{x - c(J)} \cdot \frac{1}{x-y},$$

where $c(J)$ denotes the center of J.

Suppose that $\perp \bar{\psi}\chi_J\ and\ g \perp \bar{\varphi}\bar{f}\chi_J$. Then $\mathcal{I}_2 g = \mathcal{I}_{k_\flat} g$, where

$$k_\flat(x,y) = \sum_{I,J \in \mathfrak{I}, I \ne J, |I| \ge |J|} u(x)\varphi(x)a_{IJ}(x,y)\psi(y)v(y)$$

and

$$a_{IJ}(x,y) = \chi_I(x)\frac{y - c(J)}{x - c(J)} \cdot \frac{f(x) - f(y)}{x - y}\chi_J(y).$$

Thus $\mathcal{I}^{(2)}$ and $\mathcal{I}_{k_\flat}$ coincide on a subspace of codimension at most $2n$.

To estimate the Hilbert–Schmidt norm of $\mathcal{I}_{k_\flat}$, we observe that

$$\left|a_{IJ}(x,y)\right| \le \frac{|J|}{\left(|J| + dist(I,J)\right)}, \qquad x \in I, y \in J.$$

Thus

$$\|\mathcal{I}_{k_\flat}\|_{S_2}^2 \le \sum_{I,J\in\mathfrak{I},I\neq J,|I|\ge|J|}\left(\int_I |\varphi|^2\, u\, d\mu\right)\left(\int_I |\psi|^2\, v\, dv\right)\|a_{IJ}\|_{L^\infty}^2$$
$$\le \frac{4}{n^2}\sum_{I,J\in\mathfrak{I},I\neq J,|I|\ge|J|}\frac{|J|^2}{\left(|J| + dist(I,J)\right)^2}$$

Let us observe that for a fixed $J \in \mathfrak{I}$,

$$\sum_{I,J\in\mathfrak{I},I\neq J,|I|\ge|J|}\frac{|J|^2}{\left(|J| + dist(I,J)\right)^2} \le const. \tag{15}$$

Indeed, we can enumerate the intervals $J \in \mathfrak{I}$ satisfying $I \neq J$ and $|I| \ge |J|$ so that the resulting intervals Ik satisfy $dist(I_k, J) \le dist(I_{k+1}, J)$. Since the intervals I_k are disjoint, we have

$$dist(I_k, J) \ge \frac{k-3}{2}|J|.$$

This easily implies (15). It follows that

$$\|I_{k_\flat}\|_{S_2}^2 \le C\frac{4}{n^2}\,.n = \frac{4C}{n}.$$

Similarly, $\mathcal{I}^{(3)}$ coincides on a subspace of codimension at most $2n$ with an operator whose Hilbert–Schmidt norm is at most 2 $(C/n)^{1/2}$.

If we summarize the above, we see that I_k coincides on a subspace of codimension at most $6n$ with an operator whose Hilbert–Schmidt norm is at most $Kn^{-1/2}$, where K is a constant. Hence, on a subspace of codimension at most $7n$ the operator I_k coincides with an operator whose norm is at most K/n, i.e.,

$$s_{7n}(I_k) \le \frac{K}{n}, \qquad n \ge 1,$$

Note that in the case of operators on the space $L^2(\mathbb{T})$ with respect to Lebesgue measure on the unit circle $\mathbb{T}$, the following related fact was obtained in [155] (see also [89]): if the derivative of f belongs to the Hardy class H^1, φ and ψ belong to $L^\infty(\mathbb{T})$, and the kernel function k is defined by

$$k(\zeta,\tau) = \varphi(\zeta)\frac{f(\tau) - f(\tau)}{\zeta - \tau}\psi(\tau), \qquad \zeta,\tau \in \mathbb{T},$$

then the integral operator Ik on $L^2(\mathbb{T})$belongs to $S_{1,2,} i.e., \sum_{j\ge0}(s_j(I_k))^2(1 + j) < \infty$.

Chapter 3

A class of J-Selfadjoint Operators

We show that for each selfadjoint operator A in an S-space we find an inner product which turns S into a Krein space and A into a selfadjoint operator therein. As a consequence we get a new simple condition for the existence of invariant subspaces of selfadjoint operators in Krein spaces, which provides a different insight into this well-know and in general unsolved problem. Here an extension has stable C-symmetry if it commutes with a fundamental symmetry and, in turn, this fundamental symmetry commutes with S. Such a situation occurs naturally in many applications, here we discuss the case of indefinite Sturm–Liouville operators and the case of a one-dimensional Dirac operator with point interaction.

Section (3.1) : S-spaces

A complex linear space $\mathcal{H}$ with a Hermitian sesquilinear form $[\,\cdot\,,-]$ is called a Krein space if there exists a fundamental decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \qquad (1)$$

with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\,\cdot\,,-]$ such that $(\mathcal{H}_\pm, \pm[\cdot\,,-])$ are Hilbert spaces. If $\mathcal{H}_- \; or \; \mathcal{H}_+$ is finite dimensional, then $(\mathcal{H}, [\,\cdot\,,-])$ is called a Pontryagin space. To each decomposition (1) there correspond a Hilbert space inner product $(\,\cdot\,,-)$ and a selfadjoint operator J with $JJ^* = I,\; J = J^*$ such that

$$[x, y] = (Jx, y) \quad for\; x, y \in \mathcal{H}, \qquad (2)$$

see, e.g., [98,105,139].

Conversely, every bounded and boundedly invertible selfadjoint operator G in a Hilbert space $(\mathcal{H}, (\,\cdot\,,-))$ defines an inner product via

$$[\,\cdot\,,-] := (G\,\cdot\,,-) \qquad (3)$$

and $(\mathcal{H}, [\,\cdot\,,-])$ becomes a Krein space. In particular, if the spectrum of G consists on the positive (or negative) semiaxis only of finitely many isolated eigenvalues of finite multiplicity, then $(\mathcal{H}, [\,\cdot\,,-])$is a Pontryagin space.

Eq. (3) is the starting point for various generalizations. E.g., if G is a bounded selfadjoint operator in $\mathcal{H}$ such that $\sigma(G) \cap (-\infty, \varepsilon)$ consists of finitely many eigenvalues of G with finite multiplicities for some $\varepsilon > 0$, then $(\mathcal{H}, [\,\cdot\,,-])$,where $[\,\cdot\,,-]$ is defined by (3), is called an Almost Pontryagin space, see [134]. Observe that in this case zero is allowed to be an eigenvalue of G with finite multiplicity. Almost Pontryagin spaces and operators therein were considered in various situations, we mention only [128,134,135,136,137,142,146,151,152]. The more general case that G is a bounded selfadjoint operator in $\mathcal{H}$ such that zero is an isolated eigenvalue of G with finite multiplicity gives rise to Almost Krein spaces, see [129]. Spaces with an inner product given by an arbitrary bounded selfadjoint operator were studied, in [141,147]. For applications we refer to [130,131,133,134,135,136,137,140,142,143,144,145,151,152].

In all the above-mentioned generalizations of (1) the selfadjointness of the operator G in $\mathcal{H}$ is maintained and the bounded invertibility is dropped. Obviously, this is the same as generalizing (2) by dropping $JJ^* = I$ and preserving $J = J^*$. From this point of view, it seems natural to generalize (2) the other way: dropping selfadjointness and preserving unitarity of J. The inner product space $(\mathcal{H}, [\,\cdot\,,-])$, where $[\,\cdot\,,-]$ is defined by (2) with a unitary operator J is called an S-space, cf. [148] and also Definition 3.1.1 below. Moreover, the pair $((\,\cdot\,,-), J)$ is called a Hilbert space realization of the S-space $(\mathcal{H}, [\,\cdot\,,-])$. Evidently, by definition every Krein space is a special case of an S-space.

We continue the study of S-spaces and operators therein started in [148,149]. It is known from [149] that the inner products of two Hilbert space realizations $((\,\cdot\,,-)_1, U_1)$ and $((\,\cdot\,,-)_2, U_2)$ define the same topology. Here, we show in particular that U_1 and U_2 are similar operators with respect to this topology, cf. Proposition (3.1.4). The notion of selfadjoint operators in S-spaces. We show that their spectrum is symmetric with respect to the real axis. As a main result we show that to each selfadjoint operator A in an S-space $(\mathfrak{S}, [\,\cdot\,, -])$ we find an inner product $\langle \cdot\,, - \rangle$ on S such that $(\mathfrak{S}, \langle \cdot\,, - \rangle)$is a Krein space with the same topology as $(\mathfrak{S}, [\,\cdot\,, -])$ and A is a selfadjoint operator in the Krein space $(\mathfrak{S}, \langle \cdot\,, - \rangle)$, cf.Theorem (3.1.15).

Moreover, if $((\,\cdot\,, -), U)$ is a Hilbert space realization, we show in Theorem (3.1.15) below that each spectral subspace of U related to a Borel subset Δ of the unit circle which is symmetric with respect to the origin $(i.e. x \in \Delta \, implies \, -x \in \Delta)$ is invariant under A. Hence, in this section we obtain the rather unexpected result: Each selfadjoint operator in an S-space is a selfadjoint operator in a Krein space with many invariant subspaces, showided the spectrum of the operator U from some Hilbert space realization $((\,\cdot\,, -), U)$ of $(S, [\,\cdot\,, -])$ is sufficiently rich, i.e., if it consists of more than two points.

The following definition is taken from [148].

Definition (3.1.1)[127]. A complex linear space $\mathfrak{S}$ with an inner product $[\,\cdot\,, -]$, that is a mapping from $\mathfrak{S} \times \mathfrak{S}$ into $\mathbb{C}$ which is linear in the first variable and conjugate linear in the other, is said to be an S-space if there is a Hilbert space structure in $\mathfrak{S}$ given by a positive definite inner product $(\,\cdot\,, -)$ and if there is a unitary operator U in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$ such that

$$[f, g] = (Uf, g) \quad for\, all\, f, g \in \mathfrak{S}.$$

We refer to $[\,\cdot\,, -]$ as the inner product of S. The pair $((\,\cdot\,, -), U)$ is called a Hilbert space realization of $(\mathfrak{S}, [\,\cdot\,, -])$.

Note, that the inner product $[\,\cdot\,, -]$ is not Hermitian, in general. An S-space is a Krein space if and only if the operator U in Definition (3.2.1) is in addition selfadjoint in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$. For the theory of operators in Krein spaces we refer to [98,105].

Proposition (3.1.2)[127]. Let $\mathfrak{S}$ be a complex linear space with an inner product $[\,\cdot\,, -]$. Then the pair $(\mathfrak{S}, (\,\cdot\,, -))$ is an S-space if and only if there exist a Hilbert space inner product $(\,\cdot\,, -)$ on $\mathfrak{S}$ and a bounded and boundedly invertible normal operator T in $(\mathfrak{S}, (\,\cdot\,, -))$ such that

$$[f, g] = (Tf, g) \quad for\, all\, f, g \in \mathfrak{S}.$$

Proof. We define the operator $U := T\,(T^*T\,)^{-1/2}$ and the inner product

$$\langle x, y \rangle := \big((T^*T\,)^{1/2}x, y\big), x, y \in \mathfrak{S}.$$

Since T is bijective, this is a Hilbert space inner product on S. From the relation $(T\,(T^*T\,)^{-1/2}T^*)^2 = TT^*$ it follows that

$$(TT^*\,)^{\frac{1}{2}} = (T^*T\,)^{\frac{1}{2}} = T\,(T^*T\,)^{-\frac{1}{2}}T^*. \tag{4}$$

Hence, for $x, y \in \mathfrak{S}$ we obtain

$$\langle Ux, y\,\rangle = \left((T^*T\,)^{1/2}T\,(T^*T\,)^{-\frac{1}{2}}x, y\right) = \left(T\,(T^*T\,)^{-\frac{1}{2}}T^*T(T^*T\,)^{-\frac{1}{2}}x, y\right) = [x, y]$$

and

$$\begin{aligned}\langle Ux, Uy \rangle &= \left((T^*T\,)^{\frac{1}{2}}T\,(T^*T\,)^{-\frac{1}{2}}x, T\,(T^*T\,)^{-\frac{1}{2}}y\right) \\ &= \left(T\,(T^*T\,)^{-\frac{1}{2}}T^*T(T^*T\,)^{-\frac{1}{2}}x, T(T^*T\,)^{-\frac{1}{2}}y\right) \\ &= \left(Tx, T(T^*T\,)^{-\frac{1}{2}}y\right) = \left((T^*T\,)^{-\frac{1}{2}}\,T^*Tx, y\right) \\ &= \left((T^*T\,)^{\frac{1}{2}}x, y\right) = \langle\, x, y \rangle,\end{aligned}$$

which shows that U is unitary in $(\mathfrak{S}, \langle \cdot, \cdot \rangle)$ and $(\mathfrak{S}, [\,\cdot\,, -])$ is an S-space.

Lemma (3.1.3)[127]. Let $(\mathfrak{S},[\,\cdot\,,-])$ be an S-space. Then there exists a uniquely defined linear operator $D:\ \mathfrak{S}\ \to \mathfrak{S}$ such that

$$[x,y] = \overline{[y,Dx]}\ \ for\ all\ x,y\ \in \mathfrak{S}. \tag{5}$$

If $((\,\cdot\,,-),U)$ is a Hilbert space realization of $(\mathfrak{S},[\,\cdot\,,-])$,then $D\ = U^2$.

Proof. Let $((\,\cdot\,,-),U)$ be a Hilbert space realization of $(\mathfrak{S},[\,\cdot\,,-])$. Then it is easily seen that U^2 satisfies the relation (5) (with D replaced by U^2). Let $D:\ \mathfrak{S}\ \to\ \mathfrak{S}$ be a linear operator satisfying (5). Then from $[y,Dx]\ =\ [y,U^2x]$ for all $x,y\ \in\ \mathfrak{S}$ we conclude $(Uy,Dx\ -U^2x)\ =\ 0$ for all $x,y\ \in\ \mathfrak{S}$. And since U is bijective, it follows that $D\ =\ U^2$.

The topology of an S-space $(\mathfrak{S},[\,\cdot\,,-])$ is given by the topology induced by the Hilbert space inner product $(\,\cdot\,,-)$ of some Hilbert space realization of $(\mathfrak{S},[\,\cdot\,,-])$. The following proposition states in particular that it does not depend on the choice of the Hilbert space realization, see also [149].

Proposition (3.1.4)[127]. Let $(\mathfrak{S},[\,\cdot\,,-])$. be an S-space and assume that there are two Hilbert space realizations $((\,\cdot\,,-)_1,)$ and $((\,\cdot\,,-)_2,U_2)$ with

$$[f,g]\ =\ (U_1f,g)_1\ =\ (U_2f,g)_2\quad for\ all\ \ f,g\ \in\ \mathfrak{S}.$$

Then $(\,\cdot\,,-)_1$ and $(\,\cdot\,,-)_2$ are equivalent and the Gram operator S, defined by

$$(f,g)_2\ =\ (Sf,g)_1\quad for\quad f,g\ \in \mathfrak{S},$$

is bounded, boundedly invertible and selfadjoint with respect to $(\,\cdot\,,-)_1$ and with respect to $(\,\cdot\,,-)_2$. Moreover, the following statements hold:

(i) $U_1^2\ =\ U_2^2$.

(ii) The spectral measures of S in $(\mathfrak{S},(\,\cdot\,,-)_1)$ and $(\mathfrak{S},(\,\cdot\,,-)_2)$ coincide and we have

$$S =\ U_1U_2^{-1} = U_1^{-1}\,U_2,\qquad and\quad U_1^{-1}\,SU_1\ =\ S^{-1}\ = U_2^{-1}\,SU_2. \tag{6}$$

Hence, the operator S is unitarily equivalent to its inverse.

(iii) The operators $U_1$and U_2 are similar. We have

$$U_1\ =\ S^{1/2}U_2S^{-1/2}.$$

Hence

$$\sigma(U_1)\ =\ \sigma(U_2).$$

Proof. Denote by $\|\,.\,\|_1$ and $\|\,.\,\|_2$ the norms induced by $(\,\cdot\,,-)_1)$ and $(\,\cdot\,,-)_2$, respectively, and set $B_1 :=\ \{y\ \in\ S\colon \|y\|_1\ =\ 1\}$. Then, for $y\ \in\ B_1$ the linear functional

$$F_y :=\ [\cdot\,,y]\ =\ (U_1\,\cdot\,,y)_1\ =\ (U_2\,\cdot\,,y)_2$$

is continuous on both $(\mathfrak{S},(\,\cdot\,,-)_1)$ and $(\mathfrak{S},(\,\cdot\,,-)_2)$. For its corresponding operator norms $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\,\cdot\,,-)_1),\mathbb{C})}$ and $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\,\cdot\,,-)_2),\mathbb{C})}$, respectively, we obtain $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\,\cdot\,,-)_1),\mathbb{C})} = 1$ and $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\,\cdot\,,-)_2),\mathbb{C})} = \|y\|_2$. For all $x\ \in\ \mathfrak{S}$ we have $sup_{y\in B_1}\,|F_y(x)|\ \le \|y\|_1\ <\ \infty$. Due to the principle of uniform boundedness there exists some $c\ \in\ (0,\infty)$ with

$$sup_{y\in B_1}\|F_y\|_{\mathcal{L}((\mathfrak{S},(\,\cdot\,,-)_2),\mathbb{C})}\ \le\ c.$$

This yields $\|y\|_2 \le\ c\|y\|_1\,for\ all\ y\ \in\ \mathfrak{S}$. By interchanging the roles of $\|\,.\,\|_1$and $\|\,.\,\|_2$ we obtain that these two norms are equivalent. Hence, by the well-known Lax–Milgram Theorem there exists a unique bounded linear operator S, selfadjoint in $(\mathfrak{S},(\,\cdot\,,-)_1)$, such that

$$(f,g)_2\ =\ (Sf,g)_1\qquad for\,f,g\ \in\ \mathfrak{S}.$$

It is boundedly invertible since $\|Sf_n\|_1 \to 0\ \ and\,\|f_n\|_1\ =\ 1$ would imply $\|f_n\|_2^2 =\ (Sf_n,f_n)_1 \to 0$ which contradicts the above showen fact that $\|\,.\,\|_1$and $\|\,.\,\|_2$ are equivalent. For $f,g\ \in\ \mathfrak{S}$ we have

$$(Sf,g)_2\ = (S^2f,g)_1 =\ (Sf,Sg)_1\ =\ (f,Sg)_2.$$

Thus, S is also selfadjoint with respect to $(\cdot,-)_2$. Moreover, as $(\cdot,-)_1$ and $(\cdot,-)_2$ are positive definite, the operator S is uniformly positive.

Now we will show (i)–(iii). Statement (i) follows directly from Lemma (3.2.3). The equality of the spectral measures E_1 *and* E_2 of S in $(\mathfrak{S},(\cdot,-)_1)$ and $(\mathfrak{S},(\cdot,-)_2)$follows from the equivalence of the norms $\|.\|_1$ and $\|.\|_2$and Stone's formula (see, [175]),

$$E_1((a,b)) = lim_{\delta\to 0} lim_{+\epsilon\to 0+} \frac{1}{2\pi i}\int_{a+S}^{b-\delta} (S-(\lambda+\in i))^{-1} - (S-(\lambda-\in i))^{-1} d\lambda = E_2((a,b)), \quad (7)$$

where the limit is taken in the strong operator topology. As

$$(Sf,g)_1 = (f,g)_2 = (U_2U_2^{-1}f,g)_2 = [U_2^{-1}f,g] = (U_1U_2^{-1}f,g)_1,$$

we have $S = U_1U_2^{-1}$ and, with (i), we conclude $S = U_1^{-1}U_1^2U_2^{-1} = U_1^{-1}U_2$. We will denote the adjoint with respect to $(\cdot,-)_1$ by the symbol $*_1$ and the adjoint with respect to $(\cdot,-)_2$ by $*_2$. For $f,g \in \mathfrak{S}$ we have

$$(U_2f,g)_2 = (SU_2f,g)_1 = (U_2f,Sg)_1 = (f,U_2^{*1}Sg)_1 = S^{-1}f, U_2^{*1}Sg2 = (f,S^{-1}U_2^{*1}Sg)_2,$$

thus

$$U_2^{*2} = S^{-1}U_2^{*1}S. \quad (8)$$

This implies

$$S = S^{*1} = (U_1U_2^{-1})^{*_1} = (U_2^{-1})^{*_1}1U_1^{-1} = \left(SU_2^{*_2}S^{-1}\right)^{-1}U_1^{-1} = SU_2S^{-1}U_1^{-1},$$

hence, with $S = U_1U_2^{-1}$ we get $S^{-1} = U_1^{-1}SU_1$. Replacing U_1 *by* U_2 and U_2 *by* U_1 also $S^{-1} = U_2^{-1}SU_2$ holds and formula (6) and (ii) are showed.

By (ii) the square root of S in $(\mathfrak{S},(\cdot,-)_1)$ and in $(\mathfrak{S},(.,-)_2)$coincide.We denote the unique positive square root of the operator S by $S^{1/2}$. Since, by (6),

$(U_1^{-1}S^{-\frac{1}{2}}U_1)^2 = U_1^{-1}S^{-1}U_1 = S,$ we have the relation

$$S^{1/2} = U_1^{-1}S^{-\frac{1}{2}}U_1)^2,$$

which yields

$$S^{-1}U_2S^{-\frac{1}{2}} = S^{\frac{1}{2}}S^{-1}U_1S^{-\frac{1}{2}} = S^{-\frac{1}{2}}U_1S^{-\frac{1}{2}} = U_1$$

and (iii) is showed.

For the rest of this section let $(\mathfrak{S},[\cdot,-])$ be an S-space and let $((\cdot,-),U)$ be a fixed Hilbert space realization of $(\mathfrak{S},[\cdot,-])$. In the following all topological notions are related to the Hilbert space topology given by $(\cdot,-)$. Its topology is independent of the particular choice of a Hilbert space realization (see Proposition (3.1.4)).

Let T be a densely defined operator in a Hilbert space with a Hilbert space inner product $(\cdot,-)$. As usual, we denote by T^* the adjoint of T with respect to $(\cdot,-)$. As T is denselydefined, T^* is unique. If T is, in addition, a closed operator, then $T^*$is densely defined, see, [138].

Definition (3.1.5)[127]. Let A be a closed, densely defined operator in an S-space. An adjoint A with respect to $[\cdot,-]$ is defined via the following relations:

$$dom\, A^{\#} := \{g \in \mathfrak{S}: \exists h \in \mathfrak{S}\, with\, [Af,g] = [f,h]\, for\, all\, f \in domA\},$$

$[Af,g] = [f,A^{\#}g] for\, all\, f \in domA\, and\, g \in domA^{\#}$.

Analogously, we define $A^{\#}$ via

$$dom\, A^{\#} := \{g \in \mathfrak{S}: \exists h \in \mathfrak{S}\, with\, [f,A\,g] = [h,g]\, for\, all\, g \in domA\},$$

$$[f,Ag] = [{}^{\#}Af,g] \quad for\, all\, g \in domA\, and\, f \in dom\, {}^{\#}A.$$

In the following proposition (see [149]) we collect some of the properties of $^{\#}A$ and $A^{\#}$. We showide here a short proof in order to make this exposition self-contained.

Proposition (3.1.6)[127]. The operators $^{\#}A$ *and* $A^{\#}$ are closed, densely defined and satisfy

$$dom\, A^{\#} = U\, domA^{*} = dom\,(A^{*} U^{*}) \quad and\; A^{\#} = UA^{*} U^{*} \tag{9}$$

and

$$dom\ {}^{\#}A = U^{*}\ domA^{*} = dom\,(A^{*} U) \quad and \quad {}^{\#}A = U^{*} A^{*} U \tag{10}$$

Proof. Obviously, we have $f \in dom(A^{*} U^{*})$ if and only if $U^{*}f \in domA^{*}$ which in turn holds if and only if $f \in U\, domA^{*}$. Hence $U\, domA^{*} = dom(A^{*}U^{*})$.

Let $g \in domA^{\#}$. By Definition (3.1.5) we have for all $f \in domA$

$$(f, U^{*}A^{\#}g) = (f, A^{\#}g) = [Af, g] = (Af, U^{*}g).$$

Thus $U^{*}g \in dom\, A^{*}$ $and\; U^{*}A^{\#} \subset A^{*}U^{*}$.

If $g \in dom(A^{*}U^{*})$, then we have for all $f \in domA$

$$[f, UA^{*}U^{*}g] = (f, A^{*}U^{*}g) = (Af, U^{*}g) = [Af, g].$$

Hence $g \in domA^{\#}$ and $A^{\#} \subset UA^{*}U^{*}$. This gives $U^{*}A^{\#} = A^{*}U^{*}$and (9) is showed. The proof of (10) is similar and we omit it here.

Recall that for a densely defined operator T and a bounded operator X in a Hilbert space we have (see [150])

$(XT)^{*} = T^{*}X^{*}$ and, if X is boundedly invertible,

$$(TX)^{*} = X^{*}T^{*}. \tag{11}$$

Proposition (3.1.7)[127]. If $^{\#}A = A^{\#}$ then $AD = DA\; where\; D = U^{2}$.

Proof. If $^{\#}A = A^{\#}$, then from Proposition (3.1.7) and (11) we conclude

$$^{\#}(A)^{\#} = {}^{\#}(UA^{*}U^{*}) = U^{*}(UA^{*}U^{*})^{*}U = A,$$

and hence, with $^{\#}A = A^{\#}$,

$$A = {}^{\#\#}A = U^{*}\left({}^{\#}A\right)^{*}U = U^{*}(U^{*}A^{*}U)^{*}U = (U^{*})^{2}AU^{2} = D^{*}AD.$$

And since D is unitary, the assertion follows.

Corollary (3.1.8)[127]. If $^{\#}A = A^{\#}$ *and* U has no eigenvalues, then A does not have eigenvalues with finite geometric multiplicity.

Proof. By Proposition (3.1.7) we have $AD = DA$. Assume that λ is an eigenvalue of A with finitegeometric multiplicity. From $AD = DA$ it follows that $ker(A - \lambda)$ is invariant under D. Therefore, D (and hence U) has eigenvalues.

Definition (3.1.9)[127]. A densely defined operator A in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$ is called selfadjoint if

$$A = A^{\#}.$$

We have the following characterization for selfadjointness of operators in S-spaces.

Proposition (3.1.10)[127]. For a densely defined operator A in $\mathfrak{S}$ the following assertions are equivalent:

(i) $A = A, i.e., A$ is selfadjoint in $(\mathfrak{S}, [\,\cdot\,, -])$.

(ii) $U^{*}A = A^{*}U^{*}$.

(iii) $UA = A^{*}U$.

(iv) $A = A$.

If one of these equivalent statements holds true we have

$$f \in domA \Leftrightarrow U^{*}f \in domA^{*} \Leftrightarrow Uf \in domA^{*}. \tag{12}$$

Proof. The equivalence of (i) and (ii) follows from (9), the equivalence of (iii) and (iv) follows from (10).

Assume that (ii) holds. For $f \in domA$ we conclude $Uf \in domA^*$. This implies for $f, g \in domA$:

$$(f, UAg) = (A^*U^*f, g) = (U^*Af, g) = (Af, Ug)$$

and we have $Ug \in domA^*$, hence $UA \subset A^*U$. For the other inclusion, we observe by (ii) that $domA^* = U^*\, domA$. For $Ug \in domA^*$ and $f \in domA$ we have $U^*f \in domA^*$ and

$$(U^*f, U^*A^*Ug) = (f, A^*Ug) = (Af, Ug) = (U^*Af, g) = (A^*U^*f, g),$$

thus $g \in dom(A^*)^* = domA$. This gives $U^*A^*Ug = Ag$ and A^*
$U \subset UA$. This showes (iii).

Assume that (iii) holds. For $f \in domA$ we conclude $Uf \in domA^*$. This gives for $f, g \in domA$

$$(U^*Ag, f) = (Ag, Uf) = (g, A^*Uf) = (g, UAf) = (U^*g, Af)$$

and we have $U^*g \in domA^*$, hence $U^*A \subset A^*U^*$. For the other inclusion, we observe by (iii) that $domA^* = U\, domA$. For $U^*g \in domA^*$ and $f \in domA$ we have $Uf \in domA^*$ and

$$(Uf, UA^*U^*g) = (f, A^*U^*g) = (Af, U^*g) = (UAf, g) = (A^*Uf, g),$$

thus $g \in dom(A^*)^* = domA$. This gives $A^*U^*g = U^*Ag$ and $A^*U^* \subset U^*A$. This showes (ii). Moreover, we have shown that (12) holds.

Proposition (3.1.11)[127]. Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\cdot, -])$. Then the spectrumof A is symmetric with respect to the real axis.

Proof. Since $^{\#}A = A^{\#} = UA^*U^*$, cf. Proposition (3.1.6), the operator A is unitarily equivalent to its adjoint. Hence, $\sigma(A) = \sigma(A^*) = \{\bar{\lambda}: \lambda \in \sigma(A)\}$.

Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\cdot, -])$. If $(\mathfrak{S}, [\cdot, -])$ is a Krein space, then U is selfadjoint and thus $\sigma(U) = \sigma_p(U) \subset \{-1, 1\}$. It is well known that the spectrum of A may be rather arbitrary. For example, it can happen that $\sigma(A) = \mathbb{C}$.

Example (3.1.12)[127]. Assume that – in contrast to the Krein space case – $\sigma(U)$ consists of two eigenvalues λ_1, λ_2 with $\lambda_1 \neq -\lambda_2$, e.g., $\sigma(U) = \{1, i\}$. Then $\sigma(U^2) = \{1, -1\}$, and since A commutes with $D = U^2$ by Proposition (3.1.7) the spectral subspaces of D are A-invariant. Since these coincide with the eigenspaces of U corresponding to 1 and i, respectively, we have $A = A_1 \oplus A_i$ and $U = I \oplus iI$ with respect to the decomposition $\mathfrak{S} = ker(U - 1) \oplus ker(U - i)$. From the selfadjointness of A in $(\mathfrak{S}, [\cdot, -])$ we conclude that both A_i and A_i are selfadjoint with respect to the Hilbert space scalar product $(\cdot, -)$ in $ker(U - 1)$ and $ker(U - i)$, respectively. Hence, A is selfadjoint in $(\mathfrak{S}, (\cdot, -))$. In particular its spectrum is real.

This simple example shows that it is not necessarily "better" to know that an operator is selfadjoint in a Krein space than in an S-space. In fact, we will show in the following that every selfadjoint operator in an S-space is also selfadjoint in some Krein space. However, in general (if $\sigma(U) \neq \{e^{it}, -e^{it}\}$ for some $t \in [0, \pi)$) the selfadjointness in the S-space gives us more information about the operator. E.g., we automatically know a whole bunch of invariant subspaces of the operator – namely the spectral subspaces of D.

Definition (3.1.13)[127]. Let G be a bounded selfadjoint operator in the Hilbert space $(\mathfrak{S}, (\cdot, -))$.

A closed and densely defined linear operator T in $\mathfrak{S}$ will be called G-symmetric if $GT \subset (GT)^*$. The operator T is called G-selfadjoint if $GT = (GT)$.

In the following we will deal with the operators

$$G(t) := \frac{1}{2i}\left(e^{it}U - e^{-it}U^*\right), \quad t \in [0, \pi).$$

It is easily seen that all these operators are bounded selfadjoint operators in the Hilbert space $(\mathfrak{S},(\cdot,-))$. We have $G(0) = ImU \ and\ G(\pi/2) = ReU.$ Moreover, the operator $G(t)$ can be factorized in the following way

$$G(t) = \frac{e^{it}}{2i}U^*\left(U^2 - e^{2it}\right) = \frac{e^{it}}{2i}U^*\left(U - e^{-it}\right)\left(U + e^{-it}\right).$$

Therefore, $G(t)$ is boundedly invertible if and only if $e^{-it}, -e^{-it} \in \rho(U)$. In this case $(\mathfrak{S},(G(t)\cdot,-))$ is a Krein space.

Proposition(3.1.14)[127]. Let A be a selfadjoint operator in the S-space $(\mathfrak{S},[\cdot,-])$. Then A is $G(t)$-symmetric for all $t \in [0,\pi)$. If for some $t \in [0,\pi)$ $we\ have\ e^{-it}, -e^{-it} \in \rho(U)$, then the operator A is $G(t)$-selfadjoint.

Proof. Let $t \in [0,\pi)$. Then by Proposition (3.1.10) we have

$$G(t)A = \frac{1}{2i}\left(e^{it}U - e^{-it}U^*A\right) = \frac{1}{2i}\left(e^{it}UA - e^{-it}U^*A\right) = \frac{1}{2i}\left(e^{it}A^*U - e^{-it}A^*U^*\right)$$
$$\subset A^*G(t) = G(t)A^*.$$

This shows that A is $G(t)$-symmetric.

We have by Proposition 3.1.7 $AD = DA$, therefore for each complex number λ

$$(D-\lambda)A \subset A(D-\lambda). \tag{13}$$

We will show that for $\lambda \in \rho(D)$ equality holds,

$$(D-\lambda)A = A(D-\lambda). \tag{14}$$

Let $\lambda \in \rho(D)$. We have to show $dom(A(D-\lambda)) \subset domA$. Consider the Hilbert space $\mathfrak{S}_A := (domA,(\cdot,-)_A)$, where the inner product $(\cdot,-)_A$ is defined by

$$(f,g)A := (f,g) + (Af,Ag), \qquad f,g \in domA.$$

Due to $AD = DA$ the linear manifold domA is D-invariant. Hence, define

$$D_A : \mathfrak{S}_A \to \mathfrak{S}_A, \qquad DAf := Df, \qquad f \in domA.$$

For $f,g \in \mathfrak{S}_A$ we have

$$(D_Af,D_Ag)_A = (Df,Dg) + (ADf,ADg) = (f,g) + (ADf,DAg) = (f,g)_A$$

and DA is an isometric operator in $\mathfrak{S}_A$. Assume that there exists $z \in \mathfrak{S}_A$ with $(D_Af,\ z)_A = 0$ for all $f \in D_A$. That gives

$$-(f,D^*z=)(D_Af,Az) = (Af,D^*Az)$$

for all $f \in \mathfrak{S}_A$ and, hence, $D^*Az \in domA^*$ with $A^*D^*Az = -D^*z.$

By (11) and $AD = DA$ we obtain

$$-D^*z = (DA)^*Az = (AD)^*Az = D^*A^*Az.$$

It follows $A^*Az = -z\ and\ 0 \le (A^*Az,z) = -(z,z) \le 0$. Therefore $z = 0\ and\ D_A$ has a denserange in $\mathfrak{S}_A$. The operator D_A is a unitary operator in $\mathfrak{S}_A$.

For $\lambda \in \rho(D)\setminus\{0\}$, we have

$$ran(D_A-\lambda)^{\perp A} = ker\left(D_A^{-1} - \bar{\lambda}\right) = ker\left(D_A^{-1}\,\bar{\lambda}\left(\overline{\lambda^{-1}} - D_A\right)\right) = \{0\},$$

where $\perp A$ denotes the orthogonal complement in $\mathfrak{S}_A$ with respect to $(\cdot,-)_A$. Hence, for $\lambda \in \rho(D)$, the operator $D_A - \lambda$ has a dense range in $\mathfrak{S}_A$.

In order to show (14) let $f \in dom(A(D-\lambda)).Then\ (D-\lambda)f \in domA$. As $\mathrm{ran}(D_A-\lambda)$ is dense in $\mathfrak{S}_A$, there exists a sequence (f_n) in *dom A* such that

$$\|(D_A-\lambda)f_n - (D-\lambda)f\|_A \to 0 \quad as \quad n\to\infty.$$

From this we conclude

$$f_n \to f \quad and \;\; A(D-\lambda)\, f_n \to A(D-\lambda) f \quad as \quad n \to \infty$$

(in S). But $f_n \in domA$ and from (13) it follows that

$$f_n \to f \; and \; Af_n \to (D-\lambda)^{-1} A(D-\lambda)f \; as \; n \to \infty.$$

Now, it is a consequence of the closedness of A that $f \in domA$ and $(D-\lambda)Af = A(D-\lambda)f$. This shows (14).

The selfadjointness of A in $(\mathfrak{S}, [\,\cdot\,,-])$ is equivalent to $A^*U^* = U^*A$, cf. Proposition (3.1.10). With $\pm e^{-it} \in \rho(U)$ we have $e^{-2it} \in \rho(D)$. This and (14) yield

$$A^*G(t) = \frac{e^{it}}{2i} A^*U^*\left(D - e^{-2it}\right) = \frac{e^{it}}{2i} U^*A\left(D - e^{-2it}\right) = \frac{e^{it}}{2i} U^*\left(D - e^{-2it}\right)A = G(t)A,$$

which is the $G(t)$-selfadjointness of A.

Note that in general the operator A in Proposition (3.1.15) is not $G(t)$-selfadjoint. For example let $U := iI$ and suppose that A is unbounded. Then $G(\pi/2) = 0$ and $G(\pi/2)A$ is the restriction of the zero operator to $domA$, whereas $(G(\pi/2)A)^*$ equals the zero operator on $\mathfrak{S}$. Hence, in this case, A is not $G(\pi/2)$ −selfadjoint.

If $G(t)$ is boundedly invertible, then the space $\mathfrak{S}$ equipped with the inner product $(G(t)\,\cdot\,,-)$ is a Krein space. The following theorem follows immediately from Proposition (3.1.14).

Theorem(3.1.15)[127]. Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,,-])$. If for some $t \in [0,\pi)$ *we have* $e^{-it}, -e^{-it} \in \rho(U)$, then the operator A is selfadjoint in the Krein space$(\mathfrak{S}, (G(t)\,\cdot\,,-))$.

If in the situation of Theorem (3.1.15) the operator U satisfies some additional assumptions, more can be said about the spectrum of A.

Theorem(3.1.16)[127]. Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,,-])$ and assume that there is some $t \in [0,\pi)$ such that $e^{-it}, -e^{-it} \in \rho(U)$. Let $\mathbb{T} = \mathbb{T}_1 \dot\cup \mathbb{T}_2$ be a decomposition of the unit circle, where

$$\mathbb{T}_1 := \{ e^{is} : -t \le s < -t + \pi \} \; and$$
$$\mathbb{T}_2 := \{ e^{is} : -t + \pi \le s < -t + 2\pi \}.$$

If $\mathbb{T}_1 \cap \sigma(U) = \emptyset \; or \; \mathbb{T}_2 \cap \sigma(U) = \emptyset$ then A is selfadjoint in the Hilbert space $(\mathfrak{S}, (G(t)\,\cdot\,,-))$. In particular,

$$\sigma(A) \subset \mathbb{R}.$$

If $\mathbb{T}_1 \cap \sigma(U) \; or \; \mathbb{T}_2 \cap \sigma(U)$ consists of finitely many κ isolated eigenvalues (counted with multiplicity) of U, then the non-real spectrum of A in the open upper half-plane consists of at most κ isolated eigenvalues with finite algebraic multiplicities,

$$\sigma(A) \setminus \mathbb{R} = \{\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}, \dots, \lambda_{\kappa_0}, \overline{\lambda_{\kappa_0}}\} \subset \sigma_p(A),$$

for some κ_0 with $0 \le \kappa_0 \le \kappa$.

Proof. We define

$$\widetilde{U} := e^{it} U.$$

Then $\pm 1 \in \rho(\widetilde{U})$. The operator A is selfadjoint in the S-space $(\mathfrak{S}, [\,\cdot,-]_\sim)$, where $[\,\cdot,-]_\sim$ is given by

$$[f,g]_\sim := (Uf, g) \quad for\, all\, f, g \in \mathfrak{S}.$$

By Theorem (3.1.15), A is selfadjoint in the Krein space $(\mathfrak{S}, (Im\,\widetilde{U}\,\cdot,\cdot))$. If $\mathbb{T}_1 \cap \sigma(U) = \emptyset$ then $Im\,U$ is a uniformly negative operator in the Hilbert space $(\mathfrak{S}, (\,\cdot\,,-))$, and hence A is a selfadjoint operator in the Hilbert space $(S, -(Im\,U\,\cdot,\cdot))$. A similar argument holds for the case $\mathbb{T}_2 \cap \sigma(U) = \emptyset$ and the first assertion of the theorem is shown.

If $\mathbb{T}_1 \cap \sigma(U)$ consists of finitely many isolated eigenvalues of U with finite multiplicity then $Im\,\widetilde{U}$ is a bounded and boundedly invertible selfadjoint operator in the Hilbert space $(\mathfrak{S},(\cdot,-))$. Moreover, the spectral subspace of $Im\,\widetilde{U}$ corresponding to the positive real numbers is finite dimensional. Therefore A is a selfadjoint operator in the Pontryagin space $(\mathfrak{S},(Im\,\widetilde{U}\cdot,\cdot))$ and the second assertion of the Theorem follows from well-known properties of selfadjoint operators in Pontryagin spaces, see, [98,105]. Similar arguments apply if $\mathbb{T}_2 \cap \sigma(U)$ consists of finitely many isolated eigenvalues of U.

The following Theorem is the main result of this section. It shows that the notions of S-space selfadjointness and Krein space selfadjointness coincide.

Theorem(3.1.17)[127]. Let A be a selfadjoint operator in the S-space $(\mathfrak{S},[\cdot,-])$. Then there exists a Krein space inner product $\langle\cdot,-\rangle$ such that A is selfadjoint in the Krein space $(\mathfrak{S},\langle\cdot,-\rangle$ Moreover, if E_U denotes the spectral measure of U and if Δ is a Borel subset of the unit circle T with the property that $\lambda \in \Delta$ implies $-\lambda \in \Delta$, then the spectral subspace $E_U(\Delta)\mathfrak{S}$ is an invariant subspace for A.

Proof. We choose some $\varepsilon \in (0,\pi/2)$ and define

$$\Delta_1 := \{e^{it} : t \in (-\varepsilon,\varepsilon)\} \cup \{-e^{it} : t \in (-\varepsilon,\varepsilon)\}, \Delta_2 := \mathbb{T}\setminus\Delta_1.$$

Let $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ be the spectral subspaces of U corresponding to Δ_1 and Δ_2, respectively, i.e.

$$\mathfrak{S}_1 = E_U(\Delta_1)S \quad and \quad \mathfrak{S}_2 = E_U(\Delta_2)\mathfrak{S}.$$

Then we have

$$\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2.$$

We define the sets

$$\Delta_1^2 := \{e^{it} : t \in (-2\varepsilon,2\varepsilon)\} \quad and \quad \Delta_2^2 := \mathbb{T}\setminus\Delta_1^2 = \{z^2 : z \in \Delta_2\}.$$

If E_{U^2} denotes the spectral measure of U^2 and $h : \mathbb{C} \to \mathbb{C}$ denotes the function given by $h(z) = z^2$, then we deduce from the properties of the functional calculus for unitary operators forj = 1, 2

$$E_{U^2}(\Delta_j^2) = 1_{\Delta_j^2}(U^2) = (1_{\Delta_j^2} \circ h)(U) = 1_{h^{-1}(\Delta_j^2)}(U) = E_U(\Delta_j),$$

where $\mathbf{1}_\Delta$ is the indicator function corresponding to a Borel set Δ and $h^{-1}(\Delta_j^2)$ denotes the pre-image of Δ_j^2 under h. Therefore, the spectral subspace of $D = U^2$ corresponding to Δ_j^2 coincides with $\mathfrak{S}_j, j = 1,2$.

For $\lambda \in \rho(D)$ the operator $(D-\lambda)^{-1}$ commutes with A, cf. (14). With some obvious modifications due to the fact that U is a unitary operator, the projector $E_U(\Delta_j), j = 1,2$, can be written in a similar form as in (7). From this, we conclude

$$E_U(\Delta_j)A \subset AE_U(\Delta_j). \tag{15}$$

Hence, for $x \in domA$ we have $E_U(\Delta_j)x \in domA$ and

$$domA = (\mathfrak{S}_1 \cap domA) \oplus (\mathfrak{S}_2 \cap domA).$$

Moreover, if $x \in \mathfrak{S}_j \cap domA$ then with (15)

$$Ax = E_U(\Delta_j)Ax,$$

which implies that the subspaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are A-invariant. Thus, with respect to the decomposition $\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2$ the operators A *and* U decompose as $A = A_1 \oplus A_2$ *and* $U = U_1 \oplus U_2$, where $A_j = A|\mathfrak{S}_j$ *and* $U_j = U|\mathfrak{S}_j, j = 1,2$. It is easy to see that A_1 is selfadjoint in the S-space $(\mathfrak{S}_1,(U_1\cdot,-))$ and that A_2 is selfadjoint in the S-space $(\mathfrak{S}_2,(U_2\cdot,-))$. Since $i,-i \in \rho(U_1)$ and $1,-1 \in \rho(U_2)$, it follows from Theorem (3.1.15) that there are Krein space inner products $\langle\cdot,-\rangle_1$ and $\langle\cdot,-\rangle_2$ in $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively, such that A_j is selfadjoint in the Krein space $(\mathfrak{S}_j,\langle\cdot,-\rangle_j), j = 1,2$. Hence, A is obviously selfadjoint in the Krein space $(\mathfrak{S}, \langle\cdot,-\rangle)$, where $\langle\cdot,-\rangle$ is given by

$$\langle x , v\rangle := \langle x_1 , y_1\rangle_1 + \langle x_2 , y_2\rangle_2,$$
$$x = x_1 + x_2, y = y_1 + y_2, x_1, x_2 \in \mathfrak{S}_1, y_1, y_2 \in \mathfrak{S}_2.$$

Example(3.1.18)[127]. As an illustration of Theorem (3.1.17) we consider a simple example with 2×2 matrices. Let U be unitary in $\mathbb{C}^2$ and choose an orthonormal basis of $\mathbb{C}^2$ such that the corresponding matrix is diagonal with entries $z_1, z_2 \in \mathbb{T}$. A matrix with entries $a, b, c, d \in \mathbb{C}$ which is selfadjoint in the S-space given by U has to satisfy

$$\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix},$$

cf. Proposition (3.1.10), part (iii).We assume $cb \neq 0$. From this we see that a and d are real, $z_1 = \pm z_2$ *and* $b = \pm\bar{c}$. Hence, either the matrix is selfadjoint (in the case $z_1 = z_2$) or, if $z_1 = -z_2$, we have $b = -\bar{c}$ and the matrix is selfadjoint in the (finite dimensional) Krein space with fundamental symmetry

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Corollary (3.1.19)[212]. Let $\mathfrak{S}_n$ be a complex linear space with an inner product $[\,\cdot\,, -]$. Then the pair $(\mathfrak{S}_n, (\,\cdot\,, -))$ is an S_{n-1}-space if and only if there exist a Hilbert space inner product $(\,\cdot\,, -)$ on $\mathfrak{S}_n$ and a bounded and boundedly invertible normal operators T_{n-1} in $(\mathfrak{S}_n, (\,\cdot\,, -))$ such that

$$\left[\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right] = \left(\sum_{i=1}^{n} T_{i-1} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)$$

for all $\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in \mathfrak{S}_n$.

Proof. We define the operators $U_{n-1} := T_{n-1}\,(\mathrm{T}_{n-1}^* T_{n-1})^{-\frac{1}{2}}$ and the inner product

$$\langle x_n, x_{n+1}\rangle := \left((\mathrm{T}_{n-1}^* T_{n-1})^{\frac{1}{2}} x_n, x_{n+1}\right), x_n, x_{n+1} \in \mathfrak{S}_n.$$

Since T_{n-1} is bijective, this is a Hilbert space inner product on S_{n-1}. From the relation $(T_{n-1}\,(T_{n-1}^* T_{n-1})^{-\frac{1}{2}} T_{n-1}^*)^2 = T_{n-1} T_{n-1}^*$ it follows that

$$(T_{n-1} T_{n-1}^*)^{\frac{1}{2}} = (T_{n-1}^* T_{n-1})^{\frac{1}{2}} = T_{n-1}\,(T_{n-1}^* T_{n-1})^{-\frac{1}{2}} T_{n-1}^*. \tag{16}$$

Hence, for $x_n, x_{n+1} \in \mathfrak{S}_n$ we obtain

$$\langle U_{n-1} x_n, x_{n+1}\rangle = \left((T_{n-1}^* T_{n-1})^{\frac{1}{2}} T_{n-1}\,(T_{n-1}^* T_{n-1})^{-\frac{1}{2}} x_n, x_{n+1}\right)$$
$$= \left(T_{n-1}\,(T_{n-1}^* T_{n-1})^{-\frac{1}{2}} T_{n-1}^* T_{n-1} (T_{n-1}^* T_{n-1})^{-\frac{1}{2}} x_n, x_{n+1}\right) = [x_n, x_{n+1}]$$

and

$$\langle U_{n-1} x_n, U_{n-1} x_{n+1}\rangle = \left((T_{n-1}^* T_{n-1})^{\frac{1}{2}} T_{n-1}\,(T_{n-1}^* T_{n-1})^{-\frac{1}{2}} x_n, T_{n-1}\,(T_{n-1}^* T_{n-1})^{-\frac{1}{2}} x_{n+1}\right)$$
$$= \left(T_{n-1}\,(T_{n-1}^* T_{n-1})^{-\frac{1}{2}} T_{n-1}^* T_{n-1} (T_{n-1}^* T_{n-1})^{-\frac{1}{2}} x_n, T_{n-1} (T_{n-1}^* T_{n-1})^{-\frac{1}{2}} x_{n+1}\right)$$
$$= \left(T_{n-1} x_n, T_{n-1} (\mathrm{T}_{n-1}^* T_{n-1})^{-\frac{1}{2}} x_{n+1}\right) = \left((T_{n-1}^* T_{n-1})^{-\frac{1}{2}}\, T_{n-1}^* T_{n-1} x_n, x_{n+1}\right)$$
$$= \left((T_{n-1}^* T_{n-1})^{\frac{1}{2}} x_n, x_{n+1}\right) = \langle x_n, x_{n+1}\rangle,$$

which shows that U_{n-1} is unitary in $(\mathfrak{S}_n, \langle\cdot,\cdot\rangle)$ and $(\mathfrak{S}_n, [\,\cdot\,, -])$ is an S_{n-1}-space.

Corollary (3.1.20)[212]. Let $(\mathfrak{S}_n, [\,\cdot\,, -])$ be an S_{n-1}-space. Then there exists a uniquely defined linear operator $D_{n-1} : \mathfrak{S}_n \to \mathfrak{S}_n$ such that

$$[x_n, x_{n+1}] = \overline{[x_{n+1}, D_{n-1} x_n]} \text{ for all } x_n, x_{n+1} \in \mathfrak{S}_n. \tag{17}$$

If $((\cdot,-), U_{n-1})$ is a Hilbert space realization of $(\mathfrak{S}_n, [\cdot,-])$,then $D_{n-1} = U_{n-1}^{n+1}$.

Proof. Let $((\cdot,-), U_{n-1})$ be a Hilbert space realization of $(\mathfrak{S}_n, [\cdot,-])$. Then it is easily seen that U_{n-1}^{n+1} satisfies the relation (17) (with D_{n-1} replaced by U_{n-1}^{n+1}). Let $D_{n-1}: \mathfrak{S}_n \to \mathfrak{S}_n$ be a linear operator satisfying (17). Then from $[x_{n+1}, D_{n-1}x_n] = [x_{n+1}, U_{n-1}^{n+1}x_n]$ for all $x_n, x_{n+1} \in \mathfrak{S}_n$ we conclude $(U_{n-1}x_{n+1}, D_{n-1}x_n - U_{n-1}^{n+1}x_n) = 0$, for all $x_n, x_{n+1} \in \mathfrak{S}_n$. And since U_{n-1} is bijective, it follows that $D_{n-1} = U_{n-1}^{n+1}$.

Corollary (3.1.21)[212]. Let $(\mathfrak{S}_n, [\cdot,-])$. be an S_{n-1}-space and assume that there are two Hilbert space realizations $((\cdot,-)_n,)$ *and* $((\cdot,-)_{n+1}, U_{n+1})$ with

$$\left[\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right] = \left(\sum_{i=1}^{n} U_i f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)_r = \left(\sum_{i=1}^{n} U_{i+1} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)_{r+1}$$

$$for\ all\ \sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in \mathfrak{S}_n.$$

Then $(\cdot,-)_n$ and $(\cdot,-)_{n+1}$ are equivalent and the Gram operator S_{n-1}, defined by

$$\left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)_{r+1} = \left(\sum_{i=1}^{n} S_{i-1} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)_r \quad for \sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in \mathfrak{S}_n,$$

is bounded, boundedly invertible and selfadjoint with respect to $(\cdot,-)_n$ and with respect to $(\cdot,-)_{n+1}$. Moreover, the following statements hold:

(i) $U_n^{n+1} = U_{n+1}^{n+1}$.

(ii) The spectral measures of S_{n-1} in $(\mathfrak{S}_n, (\cdot,-)_n)$ and $(\mathfrak{S}_n, (\cdot,-)_{n+1})$ coincide and we have

$$S_{n-1} = U_n U_{n+1}^{n-2} = U_n^{n-2} U_{n+1}, and \quad U_n^{n-2} S_{n-1} U_n = S_{n-1}^{-1} = U_{n+1}^{n-2} S_{n-1} U_{n+1}. \tag{18}$$

Hence, the operator S_{n-1} is unitarily equivalent to its inverse.

(iii) The operators U_n and U_{n+1} are similar. We have

$$U_n = S_{n-1}^{\frac{1}{2}} U_{n+1} S_{n-1}^{-\frac{1}{2}}.$$

Hence

$$\sigma(U_n) = \sigma(U_{n+1}).$$

Proof. Denote by $\|.\|_n$ and $\|.\|_{n+1}$ the norms induced by $(\cdot,-)_n$ and $(\cdot,-)_{n+1}$, respectively, and set $B_n := \{x_{n+1} \in S_{n-1}: \|x_{n+1}\|_n = 1\}$. Then, for $x_{n+1} \in B_n$ the linear functional

$$(F_n)_{x_{n+1}} := [\cdot, x_{n+1}] = (U_n \cdot, x_{n+1})_r = (U_{n+1} \cdot, x_{n+1})_{r+1}$$

is continuous on both $(\mathfrak{S}_n, (\cdot,-)_r)$ *and* $(\mathfrak{S}_n, (\cdot,-)_{r+1})$. For its corresponding operator norms $\|(F_n)_{x_{n+1}}\|_{\mathcal{L}((\mathfrak{S}_n,(\cdot,-)_n),\mathbb{C})}$ and $\|(F_n)_{x_{n+1}}\|_{\mathcal{L}((\mathfrak{S}_n,(\cdot,-)_{n+1}),\mathbb{C})}$, respectively, we obtain $\|(F_n)_{x_{n+1}}\|_{\mathcal{L}((\mathfrak{S}_n,(\cdot,-)_n),\mathbb{C})} = 1$ and $\|(F_n)_{x_{n+1}}\|_{\mathcal{L}((\mathfrak{S}_n,(\cdot,-)_{n+1}),\mathbb{C})} = \|x_{n+1}\|_{n+1}$. For all $x_n \in \mathfrak{S}_n$ we have $sup_{x_{n+1} \in B_n} |(F_n)_{x_{n+1}}(x_n)| \le \|x_{n+1}\|_1 < \infty$. Due to the principle of uniform boundedness there exists some c $\in (0,\infty)$ with

$$sup_{x_{n+1} \in B_n} \|(F_n)_{x_{n+1}}\|_{\mathcal{L}((\mathfrak{S}_n,(\cdot,-)_{n+1}),\mathbb{C})} \le c.$$

This yields $\|x_{n+1}\|_{r+1} \le c\|x_{n+1}\|_r$ for all $x_{n+1} \in \mathfrak{S}_n$. By interchanging the roles of $\|.\|_r$ and $\|.\|_{r+1}$ we obtain that these two norms are equivalent. Hence, by the well-known Lax–Milgram Theorem there exists a unique bounded linear operator S_{n-1}, selfadjoint in $(\mathfrak{S}_n, (\cdot,-)_n)$, such that

$$\left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)_{r+1} = \left(\sum_{i=1}^{n} S_{i-1} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)_r \quad for \sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in \mathfrak{S}_{n-1}.$$

It is boundedly invertible since $\|S_{i-1}f_n\|_r \to 0$ and $\|f_n\|_r = 1$ would imply $\|f_n\|_{n+1}^{n+1} = (S_{i-1}f_n, f_n)_r \to 0$ which contradicts the above showen fact that $\|\,.\,\|_r$ and $\|\,.\,\|_{r+1}$ are equivalent. For $\sum_{i=1}^n f_{i-1}, \sum_{i=1}^n g_{i-1} \in \mathfrak{S}_n$ we have

$$\left(\sum_{i=1}^n S_{i-1}f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_{r+1} = \left(\sum_{i=1}^n S_{i-1}^2 f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_r = \left(\sum_{i=1}^n S_{i-1}f_{i-1}, \sum_{i=1}^n S_{i-1}g_{i-1}\right)_r$$
$$= \left(\sum_{i=1}^n f_{i-1}, \sum_{i=1}^n S_{i-1}g_{i-1}\right)_{r+1}.$$

Thus, S_{n-1} is also selfadjoint with respect to $(\cdot\,,-)_{r+1}$. Moreover, as $(\cdot\,,-)_r$ and $(\cdot\,,-)_{r+1}$ are positive definite, the operator S_{n-1} is uniformly positive.

Now we will show (i)–(iii). Statement (i) follows directly from Lemma (3.1.3). The equality of the spectral measures E_n and E_{n+1} of S_{n-1} in $(\mathfrak{S}_n, (\cdot\,,-)_r)$ and $(\mathfrak{S}_{n-1}, (\cdot\,,-)_{r+1})$follows from the equivalence of the norms $\|.\|_r$ and $\|.\|_{r+1}$ and Stone's formula (see, e.g., [132]),

$$E_n((a, a+\varepsilon)) = lim_{\delta\to 0} lim_{+\in\to 0+} \frac{1}{2\pi i n} \int_{a+S_{n-1}}^{a+\varepsilon-\delta} (S_{n-1} - (\lambda_{n-1} + \in i))^{-1}$$
$$- (S_{n-1} - (\lambda_{n-1} - \in i))^{-1} d\lambda_{n-1} = E_{n+1}((a, a+\varepsilon)), \qquad (19)$$

where the limit is taken in the strong operator topology. As

$$\left(\sum_{i=1}^n S_{i-1}f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_r = \left(\sum_{i=1}^n f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_{r+1} = \left(\sum_{i=1}^n U_{i+1}U_{i+1}^{i-2} f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_{r+1}$$
$$= \left[\sum_{i=1}^n U_{i+1}^{i-2} f_{i-1}, \sum_{i=1}^n g_{i-1}\right] = \left(\sum_{i=1}^n U_i U_{i+1}^{i-2} f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_r,$$

we have $S_{n-1} = U_n U_{n+1}^{n-2}$ and, with (i), we conclude $S_{n-1} = U_n^{n-2} U_n^{n+1} U_2^{n-2} = U_n^{n-2} U_{n+1}$. We will denote the adjoint with respect to $(\cdot\,,-)_r$ by the symbol $*_r$ and the adjoint with respect to $(\cdot\,,-)_{r+1}$ by $*_{r+1}$. For $\sum_{i=1}^n f_{i-1}, \sum_{i=1}^n g_{i-1} \in \mathfrak{S}_n$we have

$$\left(\sum_{i=1}^n U_{i+1}f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_{r+1} = \left(\sum_{i=1}^n S_{i-1}U_{i+1}f_{i-1}, \sum_{i=1}^n g_{i-1}\right)_r = \left(\sum_{i=1}^n U_{i+1}f_{i-1}, \sum_{i=1}^n S_{i-1}g_{i-1}\right)_r$$
$$= \left(\sum_{i=1}^n f_{i-1}, \sum_{i=1}^n U_{i+1}^{*i} S_{i-1}g_{i-1}\right)_r = \left(\sum_{i=1}^n S_{i-1}^{-1}f_{i-1}, \sum_{i=1}^n U_{i+1}^{*i} S_{i-1}g_{i-1}\right)_{r+1}$$
$$= \left(\sum_{i=1}^n f_{i-1}, \sum_{i=1}^n S_{i-1}^{-1} U_{i+1}^{*i} S_{i-1}g_{i-1}\right)_{r+1},$$

thus

$$U_{n+1}^{*n+1} = S_{n-1}^{-1} U_{n+1}^{*n} S_{n-1}. \qquad (20)$$

This implies

$$S_{n-1} = S_{n-1}^{*r} = (U_n U_{n+1}^{n-2})^{*r} = (U_{n+1}^{n-2})^{*r} U_n^{n-2} = \left(S_{n-1} U_{n+1}^{*n+1} S_{n-1}^{-1}\right)^{-1} U_n^{n-2} = S_{n-1} U_{n+1} S_{n-1}^{-1} U_n^{n-2},$$

hence, with $S_{n-1} = U_n U_{n+1}^{n-2}$ we get $S_{n-1}^{-1} = U_n^{n-2} S_{n-1} U_n$. Replacing U_n by U_{n+1} and U_{n+1} by U_n also $S_{n-1}^{-1} = U_{n+1}^{n-2} S_{n-1} U_{n+1}$ holds and formula (18) and (ii) are showed.

By (ii) the square root of S_{n-1} in $(\mathfrak{S}_n(\cdot\,,-)_r)$ and in $(\mathfrak{S}_n, (\,.\,,-)_{r+1})$coincide.We denote the unique positive square root of the operator S_{n-1} by $S_{n-1}^{\frac{1}{2}}$. Since, by (18), $(U_n^{n-2} S_{n-1}^{-\frac{1}{2}} U_n)^2 = U_n^{n-2} S_{n-1}^{-1} U_n = S_{n-1}$, we have the relation

$$S_{n-1}^{\frac{1}{2}} = U_n^{n-2} S_{n-1}^{-\frac{1}{2}} U_n)^2,$$

which yields

$$S_{n-1}^{-1} U_{n+1} S_{n-1}^{-\frac{1}{2}} = S_{n-1}^{\frac{1}{2}} S_{n-1}^{-1} U_n S_{n-1}^{-\frac{1}{2}} = S_{n-1}^{-\frac{1}{2}} U_n S_{n-1}^{-\frac{1}{2}} = U_n$$

and (iii) is shown.

Corollary (3.1.22)[212]. The operators $\mathrm{A}_{\mathrm{n-1}}^{\#}$ and $^{\#}A_{n-1}$ are closed, densely defined and satisfy

$$dom\ \mathrm{A}_{\mathrm{n-1}}^{\#} = U_{n-1}\ domA_{n-1}^{*} = dom\ (A_{n-1}^{*} U_{n-1}^{*}) \ \ and\ \mathrm{A}_{\mathrm{n-1}}^{\#} = U_{n-1} A_{n-1}^{*} U_{n-1}^{*} \qquad (21)$$

and

$$dom\ {}^{\#}A_{n-1} = U_{n-1}^{*}\ domA_{n-1}^{*} = dom\ (A_{n-1}^{*} U_{n-1}) \ \ and\ {}^{\#}A_{n-1} = U_{n-1}^{*} A_{n-1}^{*} U_{n-1} \qquad (22)$$

Proof. Obviously, we have $\sum_{i=1}^{n} f_{i-1} \in dom(A_{n-1}^{*} U_{n-1}^{*})$ if and only if $\sum_{i=1}^{n} U_{i-1}^{*} f_{\ i-1} \in domA_{n-1}^{*}$ which in turn holds if and only if $\sum_{i=1}^{n} f_{i-1} \in U_{n-1}\ domA_{n-1}^{*}$. Hence $U_{n-1}\ domA_{n-1}^{*} = dom(A_{n-1}^{*} U_{n-1}^{*})$.

Let $\sum_{i=1}^{n} g_{i-1} \in dom\mathrm{A}_{\mathrm{n-1}}^{\#}$. By Definition (3.1.5) we have for all $\sum_{i=1}^{n} f_{i-1} \in domA_{n-1}$

$$\left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} U_{i-1}^{*} \mathrm{A}_{i-1}^{\#} g_{i-1}\right) = \left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} \mathrm{A}_{i-1}^{\#} g_{i-1}\right) = \left[\sum_{i=1}^{n} A_{i-1} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right]$$
$$= \left(\sum_{i=1}^{n} A_{i-1} f_{i-1}, \sum_{i=1}^{n} U_{n-1}^{*} g_{i-1}\right).$$

Thus $\sum_{i=1}^{n} U_{n-1}^{*} g_{i-1} \in dom\ A_{n-1}^{*}\ \ and\ U_{n-1}^{*} \mathrm{A}_{\mathrm{n-1}}^{\#} \subset A_{n-1}^{*} U_{n-1}^{*}$.

If $\sum_{i=1}^{n} g_{i-1} \in dom(A_{n-1}^{*} U_{n-1}^{*})$, then we have for all $\sum_{i=1}^{n} g_{i-1} \in domA_{n-1}$

$$\left[\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} U_{i-1} A_{n-1}^{*} U_{i-1}^{*} g_{i-1}\right] = \left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} A_{i-1}^{*} U_{i-1}^{*} g_{i-1}\right) = \left(\sum_{i=1}^{n} A_{i-1} f_{i-1}, \sum_{i=1}^{n} U_{i-1}^{*} g_{i-1}\right)$$
$$= \left[\sum_{i=1}^{n} A_{i-1} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right].$$

Hence $\sum_{i=1}^{n} g_{i-1} \in dom\mathrm{A}_{\mathrm{n-1}}^{\#}\ \ and\ \mathrm{A}_{\mathrm{n-1}}^{\#} \subset U_{n-1} A_{n-1}^{*} U_{n-1}^{*}$. This gives $U_{n-1}^{*} A_{n-1}^{\#} = A_{n-1}^{*} U_{n-1}^{*}$ and (9) is showed. The proof of (22) is similar and we omit it here.

Recall that for a densely defined operator T_{n-1} and a bounded operator X_{n-1} in a Hilbert space we have (see [150])

$(X_{n-1} T_{n-1})^{*} = T_{n-1}^{*} X_{n-1}^{*}$ and, if X_{n-1} is boundedly invertible,

$$(T_{n-1} X_{n-1})^{*} = X_{n-1}^{*} T_{n-1}^{*}. \qquad (23)$$

Corollary (3.1.23)[212]. If $^{\#}A_{n-1} = A_{n-1}^{\#}$ then $A_{n-1} D_{n-1} = D_{n-1} A_{n-1}$ where $D_{n-1} = U_{\mathrm{n-1}}^{\mathrm{n+1}}$.

Proof. If $^{\#}A_{n-1} = A_{n-1}^{\#}$, then from Proposition (3.2.6) and (23) we conclude

$$^{\#}(A^{\#}) = {}^{\#}(U_{n-1} A_{n-1}^{*} U_{n-1}^{*}) = U_{n-1}^{*} (U_{n-1} A_{n-1}^{*} U_{n-1}^{*})^{*} U_{n-1} = A_{n-1},$$

and hence, with $^{\#}A_{n-1} = A_{n-1}^{\#}$,

$$A_{n-1} = {}^{\#\#}A = U_{n-1}^{*} \left({}^{\#}A_{n-1}\right)^{*} U_{n-1} = U_{n-1}^{*} (U_{n-1}^{*} A_{n-1}^{*} U_{n-1})^{*} U_{n-1} = (U_{n-1}^{*})^2 A_{n-1} U^2$$
$$= D_{n-1}^{*} A_{n-1} D_{n-1}.$$

And since D_{n-1} is unitary, the assertion follows.

Corollary (3.1.24)[212]. If $^{\#}A_{n-1} = A_{n-1}^{\#}$ and U_{n-1} has no eigenvalues, then A_{n-1} does not have eigenvalues with finite geometric multiplicity.

Proof. By Proposition (3.1.7) we have $A_{n-1}D_{n-1} = D_{n-1}A_{n-1}$. Assume that λ_{n-1} is an eigenvalue of A_{n-1} with finitegeometric multiplicity. From $A_{n-1}D_{n-1} = D_{n-1}A_{n-1}$ it follows that $\ker(A_{n-1} - \lambda_{n-1})$ is invariant under D_{n-1}. Therefore, D_{n-1} (and hence U_{n-1}) has eigenvalues.

Corollary (3.1.25)[212]. For a densely defined operator A_{n-1} in $\mathfrak{S}_n$ the following assertions are equivalent:

(i) $A_{n-1} = A_{n-1}^{\#}$, i.e., A_{n-1} is selfadjoint in $(\mathfrak{S}_n, [\cdot, -])$.

(ii) $U_{n-1}^{*}A_{n-1} = A_{n-1}^{*}U_{n-1}^{*}$.

(iii) $U_{n-1}A_{n-1} = A_{n-1}^{*}U_{n-1}$.

(iv) $A_{n-1} = A_{n-1}^{\#}$.

If one of these equivalent statements holds true we have

$$\sum_{i=1}^{n} f_{i-1} \in domA_{n-1} \Leftrightarrow \sum_{i=1}^{n} U_{i-1}^{*}f_{i-1} \in domA_{n-1}^{*} \Leftrightarrow \sum_{i=1}^{n} U_{i-1}f_{i-1} \in domA_{n-1}^{*}. \qquad (24)$$

Proof. The equivalence of (i) and (ii) follows from (21), the equivalence of (iii) and (iv) follows from (22).

Assume that (ii) holds. For $\sum_{i=1}^{n} f_{i-1} \in domA_{n-1}$ we conclude $\sum_{i=1}^{n} U_{i-1}^{*}f_{i-1} \in domA_{n-1}^{*}$. This implies for $\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in dom\, A_{n-1}$:

$$\left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} U_{i-1}A_{i-1}g_{i-1}\right) = \left(\sum_{i=1}^{n} A_{i-1}^{*}U_{i-1}^{*}f_{i-1}, \sum_{i-1}^{n} g_{i-1}\right) = \left(\sum_{i-1}^{n} U_{i-1}^{*}A_{i-1}f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)$$
$$= \left(\sum_{i-1}^{n} A_{i-1}f_{i-1}, \sum_{i=1}^{n} U_{i-1}g_{i-1}\right)$$

and we have $\sum_{i=1}^{n} U_{i-1}g_{i-1} \in domA_{n-1}^{*}$, hence $\sum_{i=1}^{n} U_{i-1}A_{i-1} \subset \sum_{i=1}^{n} A_{i-1}^{*}U_{i-1}$. For the other inclusion, we observe by (ii) that $domA_{n-1}^{*} = U_{n-1}^{*}\, domA_{n-1}$. For $\sum_{i=1}^{n} U_{i-1}g_{i-1} \in domA_{n-1}^{*}$ *and* $\sum_{i=1}^{n} f_{i-1} \in domA_{n-1}$ we have $\sum_{i-1}^{n} U_{i-1}^{*}f_{i-1} \in domA_{n-1}^{*}$and

$$\left(\sum_{i=1}^{n} U_{i-1}^{*}f_{i-1}, \sum_{i=1}^{n} U_{i-1}^{*}A_{i-1}^{*}U_{i-1}g_{i-1}\right) = \left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} A_{i-1}^{*}U_{i-1}g_{i-1}\right)$$
$$= \left(\sum_{i-1}^{n} A_{i-1}f_{i-1}, \sum_{i=1}^{n} U_{i-1}g_{i-1}\right) = \left(\sum_{i=1}^{n} U_{i-1}^{*}A_{i-1}f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)$$
$$= \left(\sum_{i=1}^{n} A_{i-1}^{*}U_{i-1}^{*}f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right),$$

thus $\sum_{i=1}^{n} g_{i-1} \in dom(A_{n-1}^{*})^{*} = domA_{n-1}$. This gives $\sum_{i=1}^{n} U_{i-1}^{*}A_{i-1}^{*}U_{i-1}g_{i-1} = \sum_{i=1}^{n} A_{i-1}g_{i-1}$ *and* $\sum_{i=1}^{n} A_{i-1}^{*}\, U_{i-1} \subset \sum_{i=1}^{n} U_{i-1}A_{i-1}$. This showes (iii).

Assume that (iii) holds. For $\sum_{i=1}^{n} f_{i-1} \in domA_{n-1}$ we conclude $\sum_{i=1}^{n} U_{i-1}f_{i-1} \in domA_{i-1}^{*}$. This gives for $\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in domA_{n-1}$

$$\left(\sum_{i=1}^{n} U_{i-1}^{*}A_{i-1}g_{i-1}, \sum_{i=1}^{n} f_{i-1}\right) = \left(\sum_{i=1}^{n} A_{i-1}g_{i-1}, \sum_{i=1}^{n} U_{i-1}f_{i-1}\right) = \left(\sum_{i=1}^{n} g_{i-1}, \sum_{i=1}^{n} A_{i-1}^{*}U_{i-1}f_{i-1}\right)$$
$$= \left(\sum_{i=1}^{n} g_{i-1}, \sum_{i=1}^{n} U_{i-1}A_{i-1}f_{i-1}\right) = \left(\sum_{i=1}^{n} U_{i-1}^{*}g_{i-1}, \sum_{i=1}^{n} A_{i-1}f_{i-1}\right)$$

and we have $\sum_{i=1}^{n} U_{i-1}^{*}g_{i-1} \in domA_{i-1}^{*}$, hence $U_{n-1}^{*}A_{n-1} \subset A_{n-1}^{*}U_{n-1}^{*}$. For the other inclusion, we observe by (iii) that $domA_{n-1}^{*} = U_{n-1}\, domA_{n-1}$. *For* $U_{n-1}^{*}g_{n-1} \in domA_{n-1}^{*}$ and

$\sum_{i=1}^{n} f_{i-1} \in domA_{i-1}$ we have $\sum_{i=1}^{n} U_{i-1}f_{i-1} \in domA_{n-1}^*$ and

$$\left(\sum_{i=1}^{n} U_{i-1}f_{i-1}, \sum_{i=1}^{n} U_{i-1}A_{i-1}^*U_{i-1}^*g_{i-1}\right) = \left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} A_{i-1}^*U_{i-1}^*g_{i-1}\right)$$
$$= \left(\sum_{i=1}^{n} A_{i-1}f_{i-1}, \sum_{i=1}^{n} U_{i-1}^*g_{i-1}\right) = \left(\sum_{i=1}^{n} U_{i-1}A_{i-1}f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)$$
$$= \left(\sum_{i=1}^{n} A_{i-1}^*U_{i-1}f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right),$$

thus $\sum_{i=1}^{n} g_{i-1} \in dom(A_{i-1}^*)^* = domA_{n-1}$. This gives

$$\sum_{i=1}^{n} A_{i-1}^*U_{i-1}^*g_{i-1} = \sum_{i=1}^{n} U_{i-1}^*A_{i-1}g_{i-1}$$

and $\sum_{i=1}^{n} U_{i-1}^*U_{i-1}^* \subset \sum_{i=1}^{n} U_{i-1}^*A_{i-1}$. This showes (ii).Moreover, we have shown that (24) holds.

Corollary (3.1.26)[212]. Let A_{n-1} be a selfadjoint operators in the S_{n-1}-space $(\mathfrak{S}_n, [\cdot, -])$. Then the spectrumof A_{n-1} is symmetric with respect to the real axis.

Proof. Since $A_{n-1} = A_{n-1}^{\#} = U_{n-1}A_{n-1}^*U_{n-1}^*$, cf. Proposition (3.1.6), the operator A is unitarily equivalent to its adjoint. Hence, $\sigma(A_{n-1}) = \sigma(A_{n-1}^*) = \{\bar{\lambda}_{n-1}: \lambda_{n-1} \in \sigma(A_{n-1})\}$.

Let A_{n-1} be a selfadjoint operators in the S_{n-1}-space $(\mathfrak{S}_n, [\cdot, -])$. If $(\mathfrak{S}_n, [\cdot, -])$ is a Krein space, then U_{n-1} is selfadjoint and thus $\sigma(U_{n-1}) = \sigma_p(U_{n-1}) \subset \{-1, 1\}$. It is well known that the spectrum of A_{n-1} may be rather arbitrary. For example, it can happen that $\sigma(A_{n-1}) = \mathbb{C}$.

Corollary (3.1.27)[212]. Let A_{n-1} be a selfadjoint operators in the S_{n-1}-space $(\mathfrak{S}_n, [\cdot, -])$. Then A_{n-1} is $G_{n-1}(t)$-symmetric for all $t \in [0, \pi)$. If for some $t \in [0, \pi)$ we have $e^{-int}, -e^{-int} \in \rho(U_{n-1})$, then the operator A_{n-1} is $G_{n-1}(t)$ $-$selfadjoint.

Proof. Let $t \in [0, \pi)$. Then by Proposition (3.1.10) we have

$$G_{n-1}(t)A_{n-1} = \frac{1}{2in}\left(e^{int}U_{n-1} - e^{-int}U_{n-1}^*A_{n-1}\right) = \frac{1}{2in}\left(e^{int}U_{n-1}A_{n-1} - e^{-int}U_{n-1}^*A_{n-1}\right)$$
$$= \frac{1}{2in}\left(e^{int}A_{n-1}^*U_{n-1} - e^{-int}A_{n-1}^*U_{n-1}^*\right)$$
$$\subset A^*G(t) = G(t)A^*.$$

This shows that A_{n-1} is $G_{n-1}(t)$-symmetric.

We have by Proposition (3.1.7) $A_{n-1}D_{n-1} = D_{n-1}A_{n-1}$, therefore for each complex number λ_{n-1}

$$(D_{n-1} - \lambda_{n-1})A_{n-1} \subset A_{n-1}(D_{n-1} - \lambda_{n-1}). \quad (25)$$

We will show that for $\lambda_{n-1} \in \rho(D_{n-1})$ equality holds,

$$(D_{n-1} - \lambda_{n-1})A_{n-1} = A_{n-1}(D_{n-1} - \lambda_{n-1}). \quad (26)$$

Let $\lambda_{n-1} \in \rho(D_{n-1})$. We have to show $dom(A_{n-1}(D_{n-1} - \lambda_{n-1})) \subset domA_{n-1}$. Consider the Hilbert space $(\mathfrak{S}_n)_{A_{n-1}} := (domA_{n-1}, (\cdot, -)_{A_{n-1}})$, where the inner product $(\cdot, -)_{A_{n-1}}$ is defined by

$$\left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)\sum_{i=1}^{n} A_{i-1} := \left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right) + \left(\sum_{i=1}^{n} A_{i-1}f_{i-1}, \sum_{i=1}^{n} A_{i-1}g_{i-1}\right),$$
$$\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in domA.$$

Due to $A_{n-1}D_{n-1} = D_{n-1}A_{n-1}$ the linear manifold $domA_{n-1}$ is D_{n-1} -invariant. Hence, define

$$(D_{n-1})_{A_{n-1}} : (\mathfrak{S}_n)_{A_{n-1}} \to (\mathfrak{S}_n)_{A_{n-1}}, \sum_{i=1}^{n} D_{i-1} A_{i-1} f_{i-1} := \sum_{i=1}^{n} D_{i-1} f_{i-1} ,$$

$$\sum_{i=1}^{n} f_{i-1} \in domA_{n-1}.$$

For $\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in (\mathfrak{S}_n)_{A_{n-1}}$ we have

$$\left(\sum_{i=1}^{n} (D_{i-1})_{A_{i-1}} f_{i-1}, \sum_{i=1}^{n} (D_{i-1})_{A_{i-1}} g_{i-1}\right)_{A_{n-1}}$$

$$= \left(\sum_{i=1}^{n} D_{i-1}f_{i-1}, \sum_{i=1}^{n} D_{i-1}g_{i-1}\right) + \left(\sum_{i=1}^{n} A_{i-1}D_{i-1}f_{i-1}, \sum_{i=1}^{n} A_{i-1}D_{i-1}g_{i-1}\right)$$

$$= \left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right) + \left(\sum_{i=1}^{n} A_{i-1}D_{i-1}f_{i-1}, \sum_{i=1}^{n} D_{i-1}A_{i-1}g_{i-1}\right) = \left(\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right)_{A_{n-1}}$$

and $A_{n-1}D_{n-1}$ is an isometric operator in $(\mathfrak{S}_n)_{A_{n-1}}$. Assume that there exists $z_n \in (\mathfrak{S}_n)_{A_{n-1}}$ with $((D_{n-1})_{A_{n-1}}f_{n-1}, z_n)_{A_{n-1}} = 0$ for all $f_{n-1} \in (D_{n-1})_{A_{n-1}}$. That gives

$$-\left(\sum_{i=1}^{n} f_{i-1}, D_{i-1}^{*}z_i\right) = \left(\sum_{i=1}^{n} (D_{i-1})_{A_{i-1}}f_{i-1}, \sum_{i=1}^{n} A_{i-1}z_i\right) = \left(\sum_{i=1}^{n} A_{i-1}f_{i-1}, \sum_{i=1}^{n} D_{i-1}^{*}A_{i-1}z_i\right)$$

for all $\sum_{i=1}^{n} f_{i=1} \in (\mathfrak{S}_n)_{A_{n-1}}$ and, hence, $D_{n-1}^{*}A_{n-1}z_n \in domA_{n-1}^{*}$ with $A_{n-1}^{*}D_{n-1}^{*}A_{n-1}z_n = -D_{n-1}^{*}z_n$.

By (23) and $A_{n-1}D_{n-1} = D_{n-1}A_{n-1}$ weobtain

$$-\sum_{i=1}^{n} D_{i-1}^{*}z_i = \left(\sum_{i=1}^{n} D_{i-1}A_{i-1}\right)^{*} \sum_{i=1}^{n} A_{i-1}z_i = \left(\sum_{i=1}^{n} A_{i-1}D_{i-1}\right)^{*} \sum_{i=1}^{n} A_{i-1}z_i = \sum_{i=1}^{n} D_{i-1}^{*}A_{i-1}^{*}A_{i-1}z_i .$$

It follows $A_{n-1}^{*}A_{n-1}z_n = -z_n$ and $0 \le (A_{n-1}^{*}A_{n-1}z_n, z_n) = -(z_n, z_n) \le 0$. Therefore $z_n = 0$ and $(D_{n-1})_{A_{n-1}}$ has a denserange in $(\mathfrak{S}_n)_{A_{n-1}}$. The operator $(D_{n-1})_{A_{n-1}}$ is a unitary operator in $(\mathfrak{S}_n)_{A_{n-1}}$.

For $\lambda_{n-1} \in \rho(D_{n-1}) \setminus \{0\}$, we have

$$ran((D_{n-1})_{A_{n-1}} - \lambda_{n-1})^{\perp A_{n-1}} = ker\big((D_{n-1})_{A_{n-1}}^{-1} - \overline{\lambda_{n-1}}\big)$$

$$= ker\left((D_{n-1})_{A_{n-1}}^{-1} \overline{\lambda_{n-1}}\big(\overline{\lambda_{n-1}^{-1}} - (D_{n-1})_{A_{n-1}}\big)\right) = \{0\},$$

where $\perp A_{n-1}$ denotes the orthogonal complement in $(\mathfrak{S}_n)_{A_{n-1}}$ with respect to $(\cdot, -)_{A_{n-1}}$. Hence, for $\lambda_{n-1} \in \rho(D_{n-1})$, the operator $(D_{n-1})_{A_{n-1}} - \lambda_{n-1}$ has a dense range in $(\mathfrak{S}_n)_{A_{n-1}}$.

In order to show (26) let $\sum_{i=1}^{n} f_{i-1} \in dom(A_{n-1}(D_{n-1} - \lambda_{n-1}))$. *Then* $\sum_{i=1}^{n}(D_{i-1} - \lambda_{i-1})f_{i-1} \in domA_{n-1}$. As ran$((D_{n-1})_{A_{n-1}} - \lambda_{n-1})$ is dense in $(\mathfrak{S}_n)_{A_{n-1}}$, there exists a sequence (f_n) in $domA_{n-1}$ such that

$$\left\|((D_{n-1})_{A_{n-1}} - \lambda_{n-1})f_n - \sum_{i=1}^{n}(D_{i-1} - \lambda_{i-1})f_{i-1}\right\|_{A_{n-1}} \to 0 \quad as \quad n \to \infty.$$

From this we conclude

$$f_n \to \sum_{i=1}^{n} f_{i-1} \quad and \quad A_{n-1}(D_{n-1} - \lambda_{n-1}) f_n \to \sum_{i=1}^{n} A_{i-1}(D_{i-1} - \lambda_{i-1})f_{i-1} \quad as \quad n \to \infty$$

(in S). But $f_n \in domA_{n-1}$ and from (25) it follows that

$$f_n \to \sum_{i=1}^{n} f_{i-1} \quad and\ A_{n-1} f_n \to \sum_{i=1}^{n} (D_{i-1} - \lambda_{i-1})^{-1}\ A_{i-1}(D_{i-1} - \lambda_{i-1}) f_{i-1}\ as\ n \to \infty.$$

Now, it is a consequence of the closedness of A_{n-1} that $\sum_{i=1}^{n} f_{i-1} \in dom A_{n-1}$ and $\sum_{i=1}^{n} (D_{i-1} - \lambda_{i-1}) A_{i-1} f_{i-1} = \sum_{i=1}^{n} A_{i-1}(D_{i-1} - \lambda_{i-1}) f_{i-1}$. This shows (26).

The selfadjointness of A_{n-1} in $(\mathfrak{S}_n, [\cdot, -])$ is equivalent to $A_{n-1}^* U_{n-1}^* = U_{n-1}^* A_{n-1}$, cf. Proposition (3.1.10).

With $\pm e^{-int} \in \rho(U_{n-1})$ we have $e^{-2int} \in \rho(D_{n-1})$. This and (26) yield

$$\begin{aligned} A_{n-1}^* G_{n-1}(t) &= \frac{e^{int}}{2in} A_{n-1}^* U_{n-1}^* \left(D_{n-1} - e^{-2int}\right) = \frac{e^{int}}{2in} U_{n-1}^* A_{n-1} \left(D_{n-1} - e^{-2int}\right) \\ &= \frac{e^{int}}{2in} U_{n-1}^* \left(D_{n-1} - e^{-2int}\right) A_{n-1} = G_{n-1}(t) A_{n-1}, \end{aligned}$$

which is the $G_{n-1}(t)$-selfadjointness of A_{n-1}.

Corollary (3.1.28)[212]. Let A_{n-1} be a selfadjoint operator in the S_{n-1}-space $(\mathfrak{S}_n, [\cdot, -])$ and assume that there is some $t \in [0, \pi)$ such that $e^{-int}, -e^{-int} \in \rho(U_{n-1})$. Let

$\mathbb{T}_{n-1} = \mathbb{T}_n \dot{\cup} \mathbb{T}_{n+1}$ be a decomposition of the unit circle, where

$$\mathbb{T}_n := \left\{ e^{ins} : -t \le s < -t + \pi \right\}\ and$$

$$\mathbb{T}_{n+1} := \left\{ e^{ins} : -t + \pi \le s < -t + 2\pi \right\}.$$

If $\mathbb{T}_n \cap \sigma(U_{n-1}) = \emptyset\ or\ \mathbb{T}_{n+1} \cap \sigma(U_{n-1}) = \emptyset$ then A_{n-1} is selfadjoint in the Hilbert space $(\mathfrak{S}_n, (G_{n-1}(t)\ \cdot, -))$. In particular,

$$\sigma(A_{n-1}) \subset \mathbb{R}.$$

If $\mathbb{T}_n \cap \sigma(U_{n-1})\ or\ \mathbb{T}_{n+1} \cap \sigma(U_{n-1})$ consists of finitely many κ isolated eigenvalues (counted with multiplicity) of U_{n-1}, then the non-real spectrum of A_{n-1} in the open upper half-plane consists of at most κ isolated eigenvalues with finite algebraic multiplicities,

$$\sigma(A_{n-1}) \setminus \mathbb{R} = \{\lambda_n, \overline{\lambda_n}, \lambda_{n+1}, \overline{\lambda_{n+1}}, \dots, (\lambda_{n-1})_{\kappa_0}, \overline{(\lambda_{n-1})_{\kappa_0}}\} \subset \sigma_p(A_{n-1}),$$

for some κ_0 with $0 \le \kappa_0 \le \kappa$.

Proof. We define

$$\widetilde{U_{n-1}} := e^{int} U_{n-1}.$$

Then $\pm 1 \in \rho(\widetilde{U_{n-1}})$. The operator A_{n-1} is selfadjoint in the S_{n-1}-space $(\mathfrak{S}_n, [\cdot, -]_\sim)$, where $[\cdot, -]_\sim$is given by

$$\left[\sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right]_\sim := \left(\sum_{i=1}^{n} U_{i-1} f_{i-1}, \sum_{i=1}^{n} g_{i-1}\right) \quad for\ all \sum_{i=1}^{n} f_{i-1}, \sum_{i=1}^{n} g_{i-1} \in \mathfrak{S}_n.$$

By Theorem (3.1.15), A_{n-1} is selfadjoint in the Krein space $(\mathfrak{S}_n, (Im\,\widetilde{U_{n-1}}\ \cdot, \cdot))$. If $\mathbb{T}_n \cap \sigma(U_{n-1}) = \emptyset$ then $Im\, U_{n-1}$ is a uniformly negative operator in the Hilbert space $(\mathfrak{S}_n, (\cdot, -))$, and hence A_{n-1} is a selfadjoint operator in the Hilbert space $(S_{n-1}, -(Im\, U_{n-1}\ \cdot, \cdot))$. A similar argument holds for the case $\mathbb{T}_{n+1} \cap \sigma(U_{n-1}) = \emptyset$ and the first assertion of the theorem is showed.

If $\mathbb{T}_n \cap \sigma(U_{n-1})$ consists of finitely many isolated eigenvalues of U_{n-1} with finite multiplicity then $Im\,\widetilde{U_{n-1}}$ is a bounded and boundedly invertible selfadjoint operator in the Hilbert space $(\mathfrak{S}_n, (\cdot, -))$. Moreover, the spectral subspace of $Im\,\widetilde{U_{n-1}}$ corresponding to the positive real numbers is finite dimensional. Therefore A_{n-1} is a selfadjoint operator in the Pontryagin space $(\mathfrak{S}_n, (Im\,\widetilde{U_{n-1}}\ \cdot, \cdot))$ and the second assertion of the Theorem follows from well-known properties of selfadjoint operators in Pontryagin spaces, see, [98,105]. Similar arguments apply if $\mathbb{T}_{n+1} \cap \sigma(U_{n-1})$ consists of finitely many isolated eigenvalues of U_{n-1}.

It shows as [128] that the notions of S_{n-1}-space selfadjointness and Krein space selfadjointness coincide.

Corollary (3.1.29)[212]. Let A_{n-1} be a selfadjoint operator in the S_{n-1}-space $(\mathfrak{S}_n, [\cdot, -])$. Then there exists a Krein space inner product $\langle\cdot, -\rangle$such that A_{n-1} is selfadjoint in the Krein space $(\mathfrak{S}_n, \langle\cdot, -\rangle$ Moreover, if $E_{U_{n-1}}$ denotes the spectral measure of U_{n-1} and if Δ_{n-1} is a Borel subset of the unit circle T_{n-1} with the property that $\lambda_{n-1} \in \Delta_{n-1}$ implies $-\lambda_{n-1} \in \Delta_{n-1}$, then the spectral subspace $E_{U_{n-1}}(\Delta_{n-1})\mathfrak{S}_n$ is an invariant subspace for A_{n-1}.

Proof. We choose some $\varepsilon \in (0, \pi/2)$ and define

$$\Delta_n := \{e^{int} : t \in (-\varepsilon, \varepsilon)\} \cup \{-e^{int} : t \in (-\varepsilon, \varepsilon)\}, \Delta_{n+1} := \mathbb{T}_{n-1} \setminus \Delta_n.$$

Let $\mathfrak{S}_{n+1}$ *and* $\mathfrak{S}_{n+2}$ be the spectral subspaces of U_{n-1} corresponding to Δ_n and Δ_{n+1}, respectively, i.e.

$$\mathfrak{S}_{n+1} = E_{U_{n-1}}(\Delta_n)S_{n-1} \quad and \quad \mathfrak{S}_{n+2} = E_{U_{n-1}}(\Delta_{n+1})\mathfrak{S}_n.$$

Then we have

$$\mathfrak{S}_n = \mathfrak{S}_{n+1} \oplus \mathfrak{S}_{n+2}.$$

We define the sets

$$\Delta_n^2 := \{e^{int} : t \in (-2\varepsilon, 2\varepsilon)\} \quad and \quad \Delta_{n+1}^2 := \mathbb{T}_{n-1} \setminus \Delta_n^2 = \{z_n^2 : z_n \in \Delta_{n+1}\}.$$

If $E_{U_{n-1}^{n+1}}$ denotes the spectral measure of U_{n-1}^{n+1} and $h_{n-1} : \mathbb{C} \to \mathbb{C}$ denotes the function given by $h_{n-1}(z_n) = z_n^2$, then we deduce from the properties of the functional calculus for unitary operators for $j = 1, 2$

$$E_{U_{n-1}^{n+1}}((\Delta_{n-1})_j^2) = \mathbf{1}_{(\Delta_{n-1})_j^2}(U_{n-1}^{n+1}) = (\mathbf{1}_{(\Delta_{n-1})_j^2} \circ h_{n-1})(U_{n-1}) = \mathbf{1}_{h_{n-1}^{-1}(\Delta_{n-1})_j^2}(U_{n-1})$$
$$= E_{U_{n-1}}((\Delta_{n-1})_j),$$

where $\mathbf{1}_{\Delta_{n-1}}$ is the indicator function corresponding to a Borel set Δ_{n-1} and $h_{n-1}^{-1}(\Delta_{n-1})_j^2$ denotes the pre-image of $(\Delta_{n-1})_j^2$ under h_{n-1}. Therefore, the spectral subspace of $D_{n-1} = U_{n-1}^{n+1}$ corresponding to $(\Delta_{n-1})_j^2$ coincides with $(\mathfrak{S}_n)_j, j = 1, 2$.

For $\lambda_{n-1} \in \rho(D_{n-1})$ the operator $(D_{n-1} - \lambda_{n-1})^{-1}$ commutes with A_{n-1}, cf. (26). With some obvious modifications due to the fact that U_{n-1} is a unitary operator, the projector $E_{U_{n-1}}((\Delta_{n-1})_j), j = 1, 2$, can be written in a similar form as in (19). From this, we conclude

$$E_{U_{n-1}}((\Delta_{n-1})_j)A_{n-1} \subset A_{n-1}E_{U_{n-1}}((\Delta_{n-1})_j). \tag{27}$$

Hence, for $x_n \in domA_{n-1}$ we have $E_{U_{n-1}}((\Delta_{n-1})_j)x \in domA_{n-1}$ and

$$domA_{n-1} = ((\mathfrak{S}_n)_1 \cap domA_{n-1}) \oplus ((\mathfrak{S}_n)_2 \cap domA_{n-1}).$$

Moreover, if $x_n \in (\mathfrak{S}_n)_j \cap domA$ then with (15)

$$A_{n-1}x_n = E_{U_{n-1}}((\Delta_{n-1})_j)A_{n-1}x_n,$$

which implies that the subspaces $(\mathfrak{S}_n)_1$ and $(\mathfrak{S}_n)_2$ are A_{n-1}-invariant. Thus, with respect to the decomposition $\mathfrak{S}_n = (\mathfrak{S}_n)_1 \oplus (\mathfrak{S}_n)_2$ the operators A_{n-1} *and* U_{n-1} decompose as $A_{n-1} = (A_{n-1})_1 \oplus (A_{n-1})_2$ *and* $U_{n-1} = (U_{n-1})_1 \oplus (U_{n-1})_2$,where $(A_{n-1})_j = A_{n-1}|(\mathfrak{S}_n)_j$ *and* $(U_{n-1})_j = U_{n-1}|(\mathfrak{S}_n)_j$, $j = 1, 2$. It is easy to see that $(A_{n-1})_1$ is selfadjoint in the S_{n-1}-space $((\mathfrak{S}_n)_1, ((U_{n-1})_1 \cdot, -))$ and that $(A_{n-1})_2$ is selfadjoint in the S_{n-1}-space $((\mathfrak{S}_n)_2, ((U_{n-1})_2 \cdot, -))$. Since $i, -i \in \rho((U_{n-1})_n)$ and $1, -1 \in \rho((U_{n-1})_{n+1})$, it follows from Theorem (3.1.15) that there are Krein space inner products $\langle\cdot, -\rangle_n$ and $\langle\cdot, -\rangle_{n+1}$ in $(\mathfrak{S}_n)_1$ and $(\mathfrak{S}_n)_2$, respectively, such that $(A_{n-1})_j$ is selfadjoint in the Krein space $((\mathfrak{S}_n)_j, \langle\cdot, -\rangle_j), j = 1, 2$. Hence, A_{n-1} is obviously selfadjoint in the Krein space $(\mathfrak{S}_n, \langle\cdot, -\rangle)$, where $\langle\cdot, -\rangle$ is given by

$$\langle x_n, x_{n+1}\rangle := \langle x_{n+1}, x_{n+2}\rangle_1 + \langle x_{n+2}, x_{n+3}\rangle_2,$$
$$x_n = x_{n+1} + x_{n+2}, x_{n+1} = x_{n+2} + x_{n+3}, x_{n+1}, x_{n+2} \in (\mathfrak{S}_n)_1, x_{n+2}, x_{n+3} \in (\mathfrak{S}_n)_2.$$

Corollary (3.1.30)[212]. For $A^*_{\sum_{r=1}^m n_r-1}$ be a series of selfadjoint operators in the $S_{\sum_{r=1}^m n_r-1}$-space $\left(\mathfrak{S}_{\sum_{r=1}^m n_r}, [\cdot,-]\right)$. Then there exists a Krein space inner product $\langle\cdot,-\rangle$ such that $A^*_{\sum_{r=1}^m n_r-1}$ is a series ofselfadjoint in the Krein space $(\mathfrak{S}_{\sum_{r=1}^m n_r}, \langle\cdot,-\rangle$. Hence, if $E_{U_{\sum_{r=1}^m n_r-1}}$ denotes a series of the spectral measures of $U_{\sum_{r=1}^m n_r-1}$ and if $\Delta_{\sum_{r=1}^m n_r-1}$ are series of Borel subsets of the unit circle $T_{\sum_{r=1}^m n_r-1}$ with the property that $\lambda_{\sum_{r=1}^m n_r-1} \in \Delta_{\sum_{r=1}^m n_r-1}$ implies $-\lambda_{\sum_{r=1}^m n_r-1} \in \Delta_{\sum_{r=1}^m n_r-1}$, then the series of the spectral subspaces $E_{U_{\sum_{r=1}^m n_r-1}}(\Delta_{\sum_{r=1}^m n_r-1})\mathfrak{S}_{\sum_{r=1}^m n_r}$ are an invariant subspaces for $A^*_{\sum_{r=1}^m n_r-1}$.

Proof. We choose some $\varepsilon \in (0,\pi/2)$ and define

$$\Delta_{\sum_{r=1}^m n_r} := \left\{e^{it\sum_{r=1}^m n_r} : t \in (-\varepsilon,\varepsilon)\right\} \cup \left\{-e^{it\sum_{r=1}^m n_r} : t \in (-\varepsilon,\varepsilon)\right\}, \Delta_{\sum_{r=1}^m n_r+1}: = \mathbb{T}_{\sum_{r=1}^m n_r-1} \backslash \Delta_{\sum_{r=1}^m n_r}.$$

Let $\mathfrak{S}_{\sum_{r=1}^m n_r+1}$ *and* $\mathfrak{S}_{\sum_{r=1}^m n_r+2}$ be the spectral subspaces of $U_{\sum_{r=1}^m n_r-1}$ corresponding to $\Delta_{\sum_{r=1}^m n_r}$ and $\Delta_{\sum_{r=1}^m n_r+1}$, respectively, i.e.

$$\mathfrak{S}_{\sum_{r=1}^m n_r+1} = E_{U_{\sum_{r=1}^m n_r-1}}(\Delta_{\sum_{r=1}^m n_r})S_{\sum_{r=1}^m n_r-1} \quad and \quad \mathfrak{S}_{\sum_{r=1}^m n_r+2} = E_{U_{\sum_{r=1}^m n_r-1}}(\Delta_{\sum_{r=1}^m n_r+1})\mathfrak{S}_{\sum_{r=1}^m n_r}.$$

Then we have

$$\mathfrak{S}_{\sum_{r=1}^m n_r} = \mathfrak{S}_{\sum_{r=1}^m n_r+1} \oplus \mathfrak{S}_{\sum_{r=1}^m n_r+2}.$$

We define the sets

$$\Delta^2_{\sum_{r=1}^m n_r} := \left\{e^{it\sum_{r=1}^m n_r} : t \in (-2\varepsilon, 2\varepsilon)\right\} \ and \ \Delta^2_{\sum_{r=1}^m n_r+1} := \mathbb{T}_{\sum_{r=1}^m n_r-1} \backslash \Delta^2_{\sum_{r=1}^m n_r} = \left\{z^2_{\sum_{r=1}^m n_r} : z_{\sum_{r=1}^m n_r} \in \Delta_{\sum_{r=1}^m n_r+1}\right\}.$$

If $E_{U^{\sum_{r=1}^m n_r+1}_{\sum_{r=1}^m n_r-1}}$ denotes the series of spectral measures of $U^{\sum_{r=1}^m n_r+1}_{\sum_{r=1}^m n_r-1}$ and $h_{\sum_{r=1}^m n_r-1} : \mathbb{C} \to \mathbb{C}$ denotes the function given by $h_{\sum_{r=1}^m n_r-1}(z_{\sum_{r=1}^m n_r}) = z^2_{\sum_{r=1}^m n_r}$, then we deduce from the properties of the functional calculus for unitary operators for j = 1, 2

$$E_{U^{\sum_{r=1}^m n_r+1}_{\sum_{r=1}^m n_r-1}}\left(\left(\Delta_{\sum_{r=1}^m n_r-1}\right)^2_j\right) = \mathbf{1}_{(\sum_{r=1}^m n_r-1)^2_j}\left(U^{\sum_{r=1}^m n_r+1}_{\sum_{r=1}^m n_r-1}\right) = (\mathbf{1}_{\left(\Delta_{\sum_{r=1}^m n_r-1}\right)^2_j} \circ h_{\sum_{r=1}^m n_r-1})(U_{\sum_{r=1}^m n_r-1})$$
$$= \mathbf{1}_{h^{-1}_{\sum_{r=1}^m n_r-1}\left(\Delta_{\sum_{r=1}^m n_r-1}\right)^2_j}(U_{\sum_{r=1}^m n_r-1}) = E_{U_{\sum_{r=1}^m n_r-1}}(\left(\Delta_{\sum_{r=1}^m n_r-1}\right)_j),$$

where $\mathbf{1}_{\Delta_{\sum_{r=1}^m n_r-1}}$ is the indicator function corresponding to a Borel set $\Delta_{\sum_{r=1}^m n_r-1}$ and $h^{-1}_{\sum_{r=1}^m n_r-1}\left(\Delta_{\sum_{r=1}^m n_r-1}\right)^2_j$ denotes the pre-image of $\left(\Delta_{\sum_{r=1}^m n_r-1}\right)^2_j$ under $h_{\sum_{r=1}^m n_r-1}$. Therefore, the series of spectral subspaces of $D_{\sum_{r=1}^m n_r-1} = U^{\sum_{r=1}^m n_r+1}_{\sum_{r=1}^m n_r-1}$ corresponding to $\left(\Delta_{\sum_{r=1}^m n_r-1}\right)^2_j$ coincides with $\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_j$, $j = 1, 2$.

For $\lambda_{\sum_{r=1}^m n_r-1} \in \rho(D_{\sum_{r=1}^m n_r-1})$ the operators $(D_{\sum_{r=1}^m n_r-1} - \lambda_{\sum_{r=1}^m n_r-1})^{-1}$ commutes with $A^*_{\sum_{r=1}^m n_r-1}$, cf. (14). With some obvious modifications due to the fact that $U_{\sum_{r=1}^m n_r-1}$ is a unitary operator, the projector $E_{U_{\sum_{r=1}^m n_r-1}}(\left(\Delta_{\sum_{r=1}^m n_r-1}\right)_j)$, $j = 1, 2$, can be written in a similar form as in (19). From this, we conclude

$$E_{U_{\sum_{r=1}^m n_r-1}}(\left(\Delta_{\sum_{r=1}^m n_r-1}\right)_j)A^*_{\sum_{r=1}^m n_r-1} \subset A^*_{\sum_{r=1}^m n_r-1}E_{U_{\sum_{r=1}^m n_r-1}}(\left(\Delta_{\sum_{r=1}^m n_r-1}\right)_j). \tag{28}$$

Hence, for $x \in domA^*_{\sum_{r=1}^m n_r-1}$ we have $E_{U_{\sum_{r=1}^m n_r-1}}(\left(\Delta_{\sum_{r=1}^m n_r-1}\right)_j)x \in domA^*_{\sum_{r=1}^m n_r-1}$ and

$$domA^*_{\sum_{r=1}^m n_r-1} = (\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_1 \cap domA^*_{\sum_{r=1}^m n_r-1}) \oplus (\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_2 \cap domA^*_{\sum_{r=1}^m n_r-1}).$$

Moreover, if $x_n \in \left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_j \cap domA$ then with (28)

$$A^*_{\sum_{r=1}^m n_r-1}x_n = E_{U_{\sum_{r=1}^m n_r}}(\left(A^*_{\sum_{r=1}^m n_r-1}\right)_j)A^*_{\sum_{r=1}^m n_r-1}x_n,$$

which implies that the subspaces $\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_1$ and $\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_2$ are $A^*_{\sum_{r=1}^m n_r-1}$-invariant. Thus, with respect to the decomposition $\mathfrak{S}_{\sum_{r=1}^m n_r} = \left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_1 \oplus \left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_2$ the operators $A^*_{\sum_{r=1}^m n_r-1}$ *and* $U_{\sum_{r=1}^m n_r}$ decompose as $A^*_{\sum_{r=1}^m n_r-1} = \left(A^*_{\sum_{r=1}^m n_r-1}\right)_1 \oplus \left(A^*_{\sum_{r=1}^m n_r-1}\right)_2$ *and* $U_{n-1} = (U_{n-1})_1 \oplus (U_{n-1})_2$, where $\left(A^*_{\sum_{r=1}^m n_r-1}\right)_j = A^*_{\sum_{r=1}^m n_r-1}\big|(\mathfrak{S}_n)_j$ *and* $(U_{n-1})_j = U_{n-1}\big|(\mathfrak{S}_n)_j$,

$j = 1,2$. It is easy to see that $\left(A^*_{\sum_{r=1}^m n_r-1}\right)_1$ is selfadjoint in the $S_{\sum_{r=1}^m n_r-1}$-space $(\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_1, (\left(U_{\sum_{r=1}^m n_r-1}\right)_1 \cdot,-))$ and that $\left(A^*_{\sum_{r=1}^m n_r-1}\right)_2$ is selfadjoint in the $S_{\sum_{r=1}^m n_r-1}$-space $(\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_2, (\left(U_{\sum_{r=1}^m n_r-1}\right)_2 \cdot,-))$. Since $i,-i \in \rho(\left(U_{\sum_{r=1}^m n_r-1}\right)_1)$ and $1,-1 \in \rho(\left(U_{\sum_{r=1}^m n_r-1}\right)_2)$, it follows from Theorem (3.1.15) that there are Krein space inner products $\langle\cdot,-\rangle_1$ and $\langle\cdot,-\rangle_2$ in $\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_1$ and $\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_2$, respectively, such that $\left(A^*_{\sum_{r=1}^m n_r-1}\right)_j$ is selfadjoint in the Krein space $(\left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_j, \langle\cdot,-\rangle_j), j = 1,2$. Hence, $A^*_{\sum_{r=1}^m n_r-1}$ is obviously selfadjoint in the Krein space $(\mathfrak{S}_{\sum_{r=1}^m n_r}, \langle\cdot,-\rangle)$, where $\langle\cdot,-\rangle$ is given by

$$\langle x_{\sum_{r=1}^m n_r}, x_{\sum_{r=1}^m n_r+1}\rangle := \langle x_{\sum_{r=1}^m n_r+1}, x_{\sum_{r=1}^m n_r+2}\rangle_1 + \langle x_{\sum_{r=1}^m n_r+2}, x_{\sum_{r=1}^m n_r+3}\rangle_2,$$

$$x_{\sum_{r=1}^m n_r} = x_{\sum_{r=1}^m n_r+1} + x_{\sum_{r=1}^m n_r+2}, x_{\sum_{r=1}^m n_r+1} = x_{\sum_{r=1}^m n_r+2} + x_{\sum_{r=1}^m n_r+3}, x_{\sum_{r=1}^m n_r+1}, x_{\sum_{r=1}^m n_r+2} \in \left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_1, x_{\sum_{r=1}^m n_r+2}, x_{\sum_{r=1}^m n_r+3} \in \left(\mathfrak{S}_{\sum_{r=1}^m n_r}\right)_2.$$

Section (3.2) : J -Selfadjoint Operators with Empty Resolvent Set

Let $(\mathfrak{S},[\cdot,\cdot])$ be a Krein space with a non-trivial fundamental symmetry J (i.e., $\mathrm{J}^2 = \mathrm{I}, \mathrm{J} \neq \pm\mathrm{I}$, and $(\mathfrak{S},[\mathrm{J}\ \cdot,\cdot])$ is a Hilbert space) and corresponding fundamental decomposition

$$\mathfrak{S} = \mathfrak{S}_+ \oplus \mathfrak{S}_-, \tag{29}$$

where $\mathfrak{S}_\pm = \frac{1}{2}(\mathrm{I} \pm \mathrm{J})\mathfrak{S}$. Let A be a linear operator in $\mathfrak{S}$ which is J -selfadjoint with respect to the Krein space inner product $[\cdot,\cdot]$. In general, J -selfadjoint operators A are non-selfadjoint in the Hilbert space $(\mathfrak{S},[\mathrm{J}\ \cdot,\cdot])$ and their spectra $\sigma(A)$ are only symmetric with respect to the real axis:

$\mu \in \sigma(A)$ if and only if $\mu \in \sigma(A)$. Moreover, the situation where $\sigma(A) = C$ is also possible.

It is simple to construct infinitely many J -selfadjoint operators with empty resolvent set. For instance, let $\mathcal{K}$ be a Hilbert space and let L be a closed symmetric (non-self-adjoint) operator in $\mathcal{K}$ Consider the operators

$$A := \begin{pmatrix} L & 0 \\ 0 & L^* \end{pmatrix}, J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

in the product Hilbert space $\mathfrak{S} = \mathcal{K} \oplus \mathcal{K}$. Then J is a fundamental symmetry in $\mathfrak{S}$ and A is a J -selfadjoint operator. As $\rho(L) = \emptyset$, it is clear that $\rho(A) = \emptyset$.

This example shows that the property $\rho(A) = \emptyset$ is a consequence of the special structure of A. It is natural to suppose that this relationship can be made more exact for some special types of J -selfadjoint operators.

We investigate a closed symmetric operator S in the Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot) = [J\ \cdot,\cdot]$.

We assume the deficiency indices of S to be $\langle 2,2\rangle$ and we assume that S commutes with the fundamental symmetry J ,

$$S\,J \;=\; J\,S. \tag{30}$$

Hence, S is simultaneously symmetric and J-symmetric.

Our aim is to describe different types of J -selfadjoint extensions of S. For this let Σ_J be the set of all J -selfadjoint extensions of S and let us denote by U the set of all fundamental symmetries which commute with S, by Σ_J^{st} we denote the set of all J -selfadjoint extensions of S which commute with a fundamental symmetry in $\mathfrak{U}$, *by* Υ_J the set of all J -selfadjoint extensions of S which commute with J and by $\Upsilon_{\mathfrak{U}}$, the set of all J -selfadjoint extensions which commute with all operators in $\mathfrak{U}$. By definition we have $J \in \mathfrak{U}$ and

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st}\,. \tag{31}$$

Operators from Σ_J^{st} are said to have the property of stable C-symmetry, see [118]. In particular, they are fundamentally reducible and, hence, similar to selfadjoint operators in Hilbert spaces. Therefore, J -selfadjoint operators with stable C-symmetries admit detailed spectral analysis, see also [96,114], and the set Σ_J^{st} may be useful for the explanation of exceptional points phenomenon in $\mathcal{PT}$ -symmetric quantum mechanics (see [103,112,123,124] and the references therein).

In the case of a simple symmetric operator S, we show in this section that the existence of at least one J -selfadjoint extension of S with empty resolvent set leads to the quite specific structure of the underlying symmetric operator S. Namely, we have in (31) strict inclusions,

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st} \;\; \left(\Upsilon_{\mathfrak{U}} \neq \Upsilon_J \neq \Sigma_J^{st}\right).$$

and it follows from the definition of the classes $\Upsilon_{\mathfrak{U}}$, Υ_J J *and* Σ_J^{st} that we have a rich structure of different extensions of S. Moreover, in Corollary (3.2.16) and Theorem (3.2.17) below we give a full parametrization of the sets $\Upsilon_{\mathfrak{U}}$, Υ_J J *and* Σ_J^{st} in terms of four real parameters.

If, on the other hand, all J -selfadjoint extension of S have non-empty resolvent set, we show (cf. Theorem (3.2.10) below) equality in (31),

$$\Upsilon_{\mathfrak{U}} = \Upsilon_J = \Sigma_J^{st}.$$

Moreover, we have $\mathfrak{U} = \{J\}$. This is in particular the case, if there exists at least one definitizable extension (see Corollary (3.2.11) below).

We show that the existence of at least one J -selfadjoint extension of S with empty resolvent set is equivalent to one of the following statements.

- There exists an additional fundamental symmetry R in $\mathfrak{S}$ such that

$$SR \;=\; RS, J\,R \;=\; -R\,J.$$

- The operator $S_+ := S\restriction_{\mathfrak{S}_+}$ is unitarily equivalent to $S_- := S\restriction_{\mathfrak{S}_-}$, where $\mathfrak{S}_\pm$ are from the fundamental decomposition (29) corresponding to J .
- The characteristic function $s_+$of S_+ (see [126]) is equal to the characteristic function $s_-$of S_-.

If, in addition, the characteristic function of S is not identically equal to zero, we showide a complete description of the set $\mathfrak{U}$ in terms of R *and* J . More precisely (see Theorem (3.2.15) below), $\mathfrak{U}$ consists of all operators C of the form

$$C \;=\; (cosh\chi)\,J \;+\; (sinh\chi)\,J\,R[cos\omega \;+\; i(sin\omega)\,J]$$

with $\chi \in \mathbb{R}$ *and* $\omega \in [0, 2\pi)$.

The operators J *and* R can be interpreted as basis (generating) elements of the complex Clifford algebra $Cl_2(J\,,R) := span\{I, J\,, R, J\,R\}$ and they give rise to a 'rich' family Σ_J^{st} .

The section is structured as follows. And contains auxiliary results related to the Krein space theory and the extension theory of symmetric operators. In the latter case we emphasize the usefulness of the Krein spaces ideology for the description of the set Σ_J of J -selfadjoint extensions of S in terms of unitary 2×2-matrices U and the definition of the characteristic function of S.

We establish a necessary and sufficient condition under which Σ_J contains operators with empty resolventset (Theorem (3.3.7) and Corollary (3.3.9)) and explicitly describe these operators in terms of unitary matrices U (Corollary 3.3.8).

We establish our main result (Theorem (3.2.12)) about the equivalence between the presence of J-selfadjoint extensions of S with empty resolvent set and the commutation of S with a Clifford algebra $Cl_2(J, R)$.This enables one to construct the collection of operators $C\chi, \omega$ realizing the property of stable C-symmetry for extensions $A \in \Sigma_J$ directly in terms of $Cl_2(J, R)$ (Theorem (3.2.15)) and to describe the corresponding subset Σ_J^{st} of extensions $A \in \Sigma_J$ with stable C-symmetry in terms of matrices U (Corollary (3.2.16) and Theorem (3.2.17)).

In the case of a degenerated Sturm–Liouville expression on a finite interval we describe all J -selfadjoint extensions with an empty resolvent set. Moreover, we consider the case of an indefinite Sturm– Liouville expression on the real line. Imposing an additional boundary conditions at zero, the symmetric operator S is obtained as the orthogonal sum of two symmetric operators related to two differential expressions defined on $\mathbb{R}_+$ *and* $\mathbb{R}_-$, respectively. We are able to show that all J -selfadjoint extensions of S have nonempty resolvent set. This extends some results from [101,102,106,115]. Finally, we consider a one-dimensional impulse and a Dirac operator with point perturbation.

Throughout the section, the symbols $\mathcal{D}(A)$ *and* $\mathcal{R}(A)$ denote the domain and the range of a linear operator A. $A \restriction_{\mathcal{D}}$ is the restriction of A onto a set $\mathcal{D}$. The notation $\sigma(A)$ *and* $\rho(A)$ are used for the spectrum and the resolvent set of A. The sign denotes the end of a proof.

Let $\mathfrak{S}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and with non-trivial fundamental symmetry J $(i.e., J = J*, J^2 = I, \textit{ and } J \neq \pm I)$. The space $\mathfrak{S}$ endowed with the indefinite inner product $[\cdot,\cdot] := (J \cdot,\cdot)$ is called a Krein space $(\mathfrak{S}, [\cdot,\cdot])$. For the basic theory of Krein spaces and operators acting therein we refer to the monographs [98] and [105].

The projectors $P_{\pm} = \frac{1}{2}(I \pm J)$ determine a fundamental decomposition of $\mathfrak{S}$,

$$\mathfrak{S} = \mathfrak{S}_+ \oplus \mathfrak{S}_-, \mathfrak{S}_- = P_-\mathfrak{S}, \mathfrak{S}_+ = P_+\mathfrak{S}, \tag{32}$$

where $(\mathfrak{S}, [\cdot,\cdot])$ and $(\mathfrak{S}_-, -[\cdot,\cdot])$ are Hilbert spaces. With respect to the fundamental decomposition (32), the operator J has the following form

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

A subspace $\mathfrak{L}$ of $\mathfrak{S}$ is called hypermaximal neutral if

$$\mathfrak{L} = \mathfrak{L}^{[\perp]} = x \in \mathfrak{C}: [x, y] = 0, \forall y \in \mathfrak{L}.$$

A subspace $\mathfrak{L} \subset \mathfrak{S}$ is called uniformly positive (uniformly negative) if $[x, x] \geq a^2\|x\|^2$ $(resp. -[x, x] \geq a^2\|x\|^2) a \in \mathbb{R}, a \neq 0, \textit{for all } x \in \mathfrak{L}$. The subspaces $\mathfrak{S}_{\pm}$ in (32) are examples of uniformly positive and uniformly negative subspaces and, moreover, they are maximal, i.e., $\mathfrak{S}_+$ $(\mathfrak{S}_-)$ is not a proper subspace of a uniformly positive (resp. negative) subspace.

Let $\mathfrak{L}_+ (\neq \mathfrak{S}_+)$ be an arbitrary maximal uniformly positive subspace. Then its J -orthogonal complement $\mathfrak{L}_- = \mathfrak{L}_+^{[\perp]}$ is Maximal uniformly negative and the direct J -orthogonal sum

$$\mathfrak{S} = \mathfrak{L}_+ [\dot{+}] \mathfrak{L}_- \tag{33}$$

gives a fundamental decomposition of $\mathfrak{S}$.

With respect to (33) we define an operator C via

$$C = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

We have $C^{2} = I$ and C is a selfadjoint operator in the Hilbert space $(\mathfrak{S}, (\cdot,\cdot)C)$, where the inner product $(\cdot,\cdot)C$ is given by

$$(x, y)C := [Cx, y] = (JCx, y), x, y \in \mathfrak{S}.$$

Note that $(\cdot,\cdot)C$ *and* $(\cdot,\cdot)$ are equivalent, see, [121]. Hence, one can view C as a fundamental symmetry of the Krein space $(\mathfrak{S}, [\cdot,\cdot])$ with an underlying Hilbert space $(\mathfrak{S}, (\cdot,\cdot)C)$.

Summing up, there is a one-to-one correspondence between the set of all decompositions (33) of the Krein space $(\mathcal{H}, [\cdot,\cdot])$ and the set of all bounded operators C such that

$$C2 = I, \quad JC > 0. \tag{34}$$

Definition (3.2.1)[94]. An operator A acting in a Krein space $(\mathfrak{S}, [\cdot,\cdot])$ has the property of C-symmetry if there exists a bounded linear operator C in $\mathfrak{S}$ such that:

(i) $C^{2} = I$;

(ii) $JC > 0$;

(iii) $AC = C A$.

In particular, if A is a J -selfadjoint operator with the property of C-symmetry, then its counterparts

$$A_{\pm} := A \upharpoonright_{\mathfrak{L}_{\pm}}, \mathfrak{L}_{\pm} = 1(I \pm C)\mathfrak{S}$$

are selfadjoint operators in the Hilbert spaces $\mathfrak{L}_{+}$and $\mathfrak{L}_{-}$ endowed with the inner products $[\cdot,\cdot]$ and $-[\cdot,\cdot]$, respectively. This simple observation leads to the following statement, which is a direct consequence of the Phillips theorem [98].

Proposition (3.2.2)[94]. A J -selfadjoint operator A has the property of C-symmetry if and only if A is similar to a selfadjoint operator in $\mathfrak{S}$.

In conclusion, we emphasize that the notion of C-symmetry in Definition (3.2.1) coincides with the notion of fundamentally reducible operator (see, [113]). However, in the context of this section and motivated by [96,97,103,104,111,123,124], we prefer to use the notion of C-symmetry.

Here and in the following we denote by $\mathbb{C}_{+}$ ($\mathbb{C}_{-}$) the open upper (resp. lower) half plane. Let S be a closed symmetric densely defined operator with equal deficiency indices acting in the Hilbert space $(\mathfrak{S}, (\cdot,\cdot))$.

We denote by $\mathfrak{N}_{\mu} = ker(S^{*} - \mu I), \mu \in \mathbb{C} \setminus \mathbb{R}$, the defect subspaces of S and consider the Hilbert space $\mathfrak{M} = \mathfrak{N}_{i} \dotplus \mathfrak{N}_{-i}$

with the inner product

$$(x, y)\mathfrak{M} = 2[(x_{i}, y_{i}) + (x_{-i}, y_{-i})], \tag{35}$$

where $x = x_{i} + x_{-i}$ *and* $y = y_{i} + y_{-i}$ *with* $x_{i}, y_{i} \in \mathfrak{N}_{i}, x_{-i}, y_{-i} \in \mathfrak{N}_{-i}$.

The operator Z which acts as identity operator I on $\mathfrak{N}_{\mathrm{i}}$ and minus identity operator $-I$ on $\mathfrak{N}_{-\mathrm{i}}$ is an example of a fundamental symmetry in $\mathfrak{M}$.

According to the von-Neumann formulas (see, [125,117]) any closed intermediate extension A of S (i.e., $S \subset A \subset S^{*}$) in the Hilbert space $(\mathfrak{S}, (\cdot,\cdot))$ is uniquely determined by the choice of a subspace $M \subset \mathfrak{M}$:

$$A = S^{*} \upharpoonright_{\mathcal{D}(A)}, \mathcal{D}(A) = \mathcal{D}(S) \dotplus M. \tag{36}$$

Let us set $M = \mathfrak{N}_\mu$ $(\mu \in \mathbb{C}_+)$ in (36) and denote by

$$A_\mu = S^* \restriction_{\mathcal{D}(A_\mu)}, \mathcal{D}(A_\mu) = \mathcal{D}(S) \dotplus \mathfrak{N}_\mu, \qquad \forall \mu \in \mathbb{C}_+ \tag{37}$$

the corresponding maximal dissipative extensions of S. The operator-function

$$Sh(\mu) = (A_\mu - i\,I)(A_\mu + i\,I)^{-1} \restriction_{\mathfrak{N}_i} : \mathfrak{N}_i \to \mathfrak{N}_{-i}, \mu \in \mathbb{C}_+ \tag{38}$$

is the characteristic function of S defined by A. Straus, see [168].

The characteristic function $Sh(\cdot)$ is connected with the Weyl function of the symmetric operator S constructed in terms of boundary triplets (see [107], [110]). For instance, if $M(\cdot)$ is the Weyl function of S associated with the boundary triplet $(\mathfrak{N}_i, \Gamma_0, \Gamma_1)$, where

$$\Gamma_0 f = f_i + V f_{-i}, \Gamma_1 f = i f_i - iV f_{-i}, f = u + fi + f_{-i} \in \mathcal{D}(S^*) \tag{39}$$

and $V : \mathfrak{N}_{-i} \to \mathfrak{N}_i$ is an arbitrary unitary mapping, then

$$M(\mu) = i\big(I + V\,Sh(\mu)\big)\big(I - V\,Sh(\mu)\big)^{-1}, \mu \in \mathbb{C}_+. \tag{40}$$

The function $V\,Sh(\cdot)$ in (40) coincides with the characteristic function of S associated with the boundary triplet $(\mathfrak{N}_i, \Gamma_0, \Gamma_1)$, cf. [109].

Another (equivalent) definition of $Sh(\cdot)$ (see [126]) is based on the relation

$$\mathcal{D}(A_\mu) = \mathcal{D}(S) \dotplus \mathfrak{N}\mu = \mathcal{D}(S) \dotplus \big(I - Sh(\mu)\big)\mathfrak{N}_i, \mu \in \mathbb{C}_+, \tag{41}$$

which also allows one to uniquely determine $Sh(\cdot)$.

The characteristic function $Sh(\cdot)$ can be easily interpreted in the Krein space setting. Indeed, according to the von- Neumann formulas, $\mathcal{D}(A_\mu) = \mathcal{D}(S) \dotplus L\mu$, where $L\mu \subset \mathfrak{M}$ is a maximal uniformly positive subspace in the Krein space $(\mathfrak{M}, [\cdot,\cdot]Z)$. Using (41), we conclude that $L\mu = (I - Sh(\mu))\mathfrak{N}_i$ and hence, $-Sh(\mu)$ is the angular operator of $L\mu$ with respect to the maximal uniformly positive subspace $\mathfrak{N}_i$ of the Krein space $(\mathfrak{M}, [\cdot,\cdot]Z)$ (see [140] for the concept of angular operators).

In what follows we assume that S satisfies (30), where J is a fundamental symmetry in $(\mathfrak{M}, (\cdot,\cdot))$.

The condition (30) immediately leads to the special structure of S with respect to the fundamental decomposition (32):

$$S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}, S_+ = S \restriction_{\mathfrak{S}_+}, S_- = S \restriction_{\mathfrak{S}_-}, \tag{42}$$

where $S_\pm$ are closed symmetric densely defined operators in $\mathfrak{S}_\pm$.

Denote by Σ_J the collection of all J -selfadjoint extensions of S and set

$$\Upsilon_J = \{A \in \Sigma_J \mid AJ = JA\}. \tag{43}$$

It is clear that $\Upsilon_J \subset \Sigma_J$ and an arbitrary $A \in \Upsilon_J$ is a simultaneously selfadjoint and J -selfadjoint extension of S. The set Υ_J is non-empty if and only if each symmetric operator $S_\pm$ in (42) has equal deficiency indices. We always suppose that $\Upsilon_J \neq \emptyset$.

Since S satisfies (30) the subspaces $\mathfrak{N}_{\pm i}$ reduce J and the restriction $J \restriction \mathfrak{M}$ gives rise to a fundamental symmetry in the Hilbert space $\mathfrak{M}$. Moreover, according to the properties of Z mentioned above, $JZ = ZJ$ *and* JZ is a fundamental symmetry in $\mathfrak{M}$. Therefore, the sesquilinear form

$$[x,y]_{JZ} = (JZx,y)_{\mathfrak{M}} = 2[(J\,x_i, y_i) - (J\,x_{-i}, y_{-i})] \tag{44}$$

defines an indefinite metric on $\mathfrak{M}$.

It is known (see, [96]) that an arbitrary J -selfadjoint extension A *of* S is uniquely determined by (36), where M is a hypermaximal neutral subspace of the Krein space $(\mathfrak{M}, [\cdot,\cdot]\,J\,Z)$.

In comparison with selfadjoint extensions in the sense of Hilbert spaces, we remark that selfadjoint extensions of S in $(\mathfrak{S}, (\cdot,\cdot))$ are also determined by (36) but then subspaces M are assumed to be hypermaximal neutral in the Krein space $(\mathfrak{M}, [\cdot,\cdot]_Z)$ with the indefinite metric (cf. (44))

$$[x,y]_Z = (Zx,y)_{\mathfrak{M}} = 2[(x_i, y_i) - (x_{-i}, y_{-i})].$$

Denote by $\mathfrak{U}$ the set of all possible C-symmetries of the closed symmetric operator S. By Definition (3.2.1), this means that

$$C \in \mathfrak{U} \Leftrightarrow C^2 = I,\ JC > 0, \qquad SC = C\,S.$$

The next result follows directly from [138]. We repeat principal stages for the reader's convenience.

Lemma (3.2.3)[94]. The set $\mathfrak{U}$ is non-empty and $C \in \mathfrak{U}$ if and only if $C^* \in \mathfrak{U}$.

Proof. It follows from (30) that $J \in \mathfrak{U}$. Therefore, $\mathfrak{U} \neq \emptyset$.

Let $C \in \mathfrak{U}$. The conditions $C^2 = I \text{ and } JC > 0$ are equivalent to the presentation $\mathrm{C} = \mathrm{J}e^{\mathrm{Y}}$, where Y is a bounded selfadjoint operator in $\mathfrak{S}$ such that $J\,Y = -Y\,J$, see [96]. In that case $C^* = Je^{-Y}$ and, obviously, C^* satisfies the relations $C^{*2} = I \text{ and } JC^* > 0.$

Since S commutes with J and C one gets $\mathrm{S}e^{Y} = e^{Y}\,S$. But then $SC^* = Se^{Y}J = e^{Y}J\,S = C^*\,S$. Hence, $\mathrm{C}^* \in \mathfrak{U}$.

Definition (3.2.4)[94]. (See [118].) An operator $A \in \Sigma_J$ has the property of stable C-symmetry if A and S have the property of C-symmetry realized by the same operator C, i.e., there exists $C \in \mathfrak{U}$ with $AC = C\,A$.

Denote

$$\Sigma_J^{st} = \{A \in \Sigma_J \mid \exists C \in \mathfrak{U} \text{ such that } AC = C\,A\}. \tag{45}$$

Due to Definition (3.2.4), Σ_J^{st} consists of J-selfadjoint extensions A of S with the property of stable C-symmetry. It follows from (43) and (45) that $\Sigma_J^{st} \supset \Upsilon_J$. Hence, Σ_J^{st} is non-empty.

Denote

$$\Upsilon_{\mathfrak{U}} = \{A \in \Sigma_J \,|AC = C\,A\ ,\forall C \in \mathfrak{U}\}. \tag{46}$$

It is clear that

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st} \subset \Sigma_J\,. \tag{47}$$

The next Theorem gives a condition for the non-emptiness of the left-hand side of the chain (47).

Theorem (3.2.5)[94]. If the characteristic function $Sh(\cdot)$ of S is boundedly invertible for at least one $\mu \in \mathbb{C}_+$, then $\Upsilon_{\mathfrak{U}} \neq \emptyset$.

Proof. Let $C \in \mathfrak{U}$. Then $S^*C = C\,S^*$ (see the proof of Lemma (3.2.3)) and, hence,

$$C : \mathfrak{N}\mu \to \mathfrak{N}\mu\ ,\forall \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{48}$$

Therefore, $A\mu C = C\,A\mu$ for maximal dissipative extensions $A\mu$ of S (see (37)). This means that the characteristic function $Sh(\cdot)$ defined by (38) commutes with an arbitrary $C \in \mathfrak{U}$, i.e.,

$$Sh(\mu)C = CSh(\mu), \forall \mu \in \mathbb{C}_+, \forall C \in \mathfrak{U}. \tag{49}$$

It follows from Lemma (3.2.3) and (49) that $Sh(\mu) = C^*Sh(\mu)$. Therefore,

$$Sh^*(\mu)C = CSh^*(\mu),\ \forall \mu \in \mathbb{C}_+, \forall C \in \mathfrak{U}. \tag{50}$$

Let $Sh(\mu)$ be boundedly invertible for a certain $\mu \in \mathbb{C}_+$ and let $V : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ be the isometric factor in the polar decomposition of $Sh(\mu)$. Then $V C = CV$ for all $C \in \mathfrak{U}$ (since (49) and (50)). This means that the operator

$$A = S^* \restriction_{\mathcal{D}(A)}, \ \mathcal{D}(A) = \mathcal{D}(S) \dotplus (I + V)\mathfrak{N}_i$$

belongs to $\Upsilon_{\mathfrak{U}}$.

According to (47), an arbitrary $C \in \mathfrak{U}$ determines two operators $C \restriction_{\mathfrak{N}_{\pm i}}$ acting in $\mathfrak{N}_{\pm i}$.

Lemma (3.2.6)[94]. If S is a simple closed symmetric operator, then the correspondence $C \in \mathfrak{U} \to \{C \restriction_{\mathfrak{N}_{+i}}, C \restriction_{\mathfrak{N}_{-i}}\}$ is injective.

Proof. Assume the existence of an operator pair $\{C \restriction_{\mathfrak{N}_{+i}}, C \restriction_{\mathfrak{N}_{-i}}\}$ for two different operators $C, \tilde{C} \in \mathfrak{U}$. Then $(C - \widetilde{C})\mathcal{D}(S^*) \subset \mathcal{D}(S)$. Therefore, $(C - \widetilde{C})\mathfrak{N}_\mu \subset \mathcal{D}(S)$. On the other hand, $\left(C - \widetilde{C}\right)\mathfrak{N}_\mu \subset \mathfrak{N}_\mu$by (47). The obtained relations yield $C f_\mu = \tilde{C} f_\mu$ for any $f_\mu \in \mathfrak{N}_\mu$ *and* $\mu \in \mathbb{C} \setminus \mathbb{R}$. This means that $C = \tilde{C}$.

In what follows we assume that the deficiency indices of the operators $S_\pm$ in (42) are $\langle 1, 1\rangle$. In that case, the defect subspaces $\mathfrak{N}_{\pm i}(S_\pm)$ of $S_\pm$ are one-dimensional and

$$\mathfrak{N}_i(S_+) = (I + Z)(I + J)\mathfrak{M}; \ \mathfrak{N}_{-i}(S_+) = (I - Z)(I + J)\mathfrak{M};$$
$$\mathfrak{N}_i(S_-) = (I + Z)(I - J)\mathfrak{M}; \ \mathfrak{N}_{-i}(S_-) = (I - Z)(I - J)\mathfrak{M}.$$

Hence, $\mathfrak{N}_{\pm i}(S_\pm)$ are orthogonal in the Hilbert space $(\mathfrak{M}, (\cdot,\cdot)_{\mathfrak{M}})$ (see (35)).

Let $\{e_{++}, e_{+-}, e_{-+}, e_{--}\}$ be an orthogonal basis of $\mathfrak{M}$ such that

$$\begin{aligned}
\mathfrak{N}_i(S_+) &= ker(S_+^* - i\,I) = span\{e_{++}\},\\
\mathfrak{N}_i(S_-) &= ker(S_-^* - i\,I) = span\{e_{+-}\},\\
\mathfrak{N}_{-i}(S_+) &= ker(S_+^* + i\,I) = span\{e_{-+}\},\\
\mathfrak{N}_{-i}(S_-) &= ker(S_-^* + i\,I) = span\{e_{--}\},
\end{aligned} \tag{51}$$

and the elements $e_{++}, e_{+-}, e_{-+}, e_{--}$ have equal norms in $\mathfrak{M}$. It follows from the definition of $e_{\pm\pm}$ that

$$Ze_{++} = e_{++}, \ Ze_{+-} = e_{+-}, \ Ze_{-+} = -e_{-+}, \ Ze_{--} = -e_{--},$$
$$Je_{++} = e_{++}, \quad Je_{+-} = -e_{+-}, \quad Je_{-+} = e_{-+}, Je_{--} = -e_{--}. \tag{52}$$

Relations (52) mean that the fundamental decomposition of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$ has the form

$$\mathfrak{M} = \mathfrak{M}_- \oplus \mathfrak{M}_+, \mathfrak{M}_- = span\{e_{+-}, e_{-+}\}, \mathfrak{M}_+ = span\{e_{++}, e_{--}\}. \tag{53}$$

According to the general theory of Krein spaces [98], an arbitrary hypermaximal neutral subspace M of $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$ is uniquely determined by a unitary mapping of $\mathfrak{M}_-$onto $\mathfrak{M}_+$. Since $dim_{\mathfrak{M}_\pm} = 2$ the set of unitary mappings $\mathfrak{M}_- \to \mathfrak{M}_+$ is in one-to-one correspondence with the set of unitary matrices

$$U = e^{i\varphi}\begin{pmatrix} qe^{i\gamma} & re^{i\xi} \\ -re^{-i\xi} & qe^{-i\gamma} \end{pmatrix}, q^2 + r^2 = 1, \ q, r \in \mathbb{R}_+, \varphi, \gamma, \xi \in [0, 2\pi). \tag{54}$$

In other words, formulas (53), (54) allow one to describe a hypermaximal neutral subspace M of $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$ as a linear span

$$M = span\{d_1, d_2\} \tag{55}$$

of elements

$$d_1 = e_{++} + qe^{i(\varphi+\gamma)} e_{+-} + re^{i(\varphi+\xi)} e_{-+},$$

$$d_2 = e_{--} - re^{i(\varphi-\xi)} e_{+-} + qe^{i(\varphi-\gamma)} e_{-+}. \quad (56)$$

This means that (54)–(56) establish a one-to-one correspondence between domains $\mathcal{D}(A) = \mathcal{D}(S) \dot{+} M$ of J -selfadjoint extensions A of S and unitary matrices U. To underline this relationship we will use the notation A_U for the corresponding J -selfadjoint extension A.

It follows from (49) (with $C = J$) that the characteristic function $Sh(\cdot) : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ commutes with J . Combining this fact with the obvious presentations

$$\mathfrak{N}_i = \mathfrak{N}_i(S_+) \oplus \mathfrak{N}_i(S_-) = span\{e_{++}, e_{+-}\},$$

$$\mathfrak{N}_{-i} = \mathfrak{N}_{-i}(S_+) \oplus \mathfrak{N}_{-i}(S_-) = span\{e_{-+}, e_{--}\} \quad (57)$$

and relations (41), (52), we arrive at the conclusion that

$$Sh(\mu)e_{++} = s_+(\mu)e_{-+}, Sh(\mu)e+- = s_-(\mu)e_{--}, \quad (58)$$

where s_j are holomorphic functions in $\mathbb{C}_+$. Moreover, it is easy to see that relations in (58) determine the characteristic functions

$$Sh_+(\mu): \mathfrak{N}_i(S_+) \to \mathfrak{N}_{-i}(S_+), Sh_-(\mu): \mathfrak{N}_i(S_-) \to \mathfrak{N}_{-i}(S_-) \quad (59$$

of the symmetric operators S_+ and S_-, respectively.

We will use the notation

$$S_+ \approx S_-$$

if the identity $e^{i\alpha}S_+(\mu) = S_-(\mu)$ holds for all $\mu \in \mathbb{C}_+$and for a certain choice of a unimodular constant $e^{i\alpha}$, i.e., the sign $\approx$ means equality up to the multiplication by a unimodular constant.

Theorem(3.2.7)[94]. Assume that the deficiency indices of operators $S_\pm$ in the presentation (42) of S are $\langle 1, 1\rangle$. Then J -selfadjoint extensions of S with empty resolvent set exist if and only if $S_+ \approx S_-$.

Proof. It follows from (41) that a J -selfadjoint extension A_U of S with the domain $\mathcal{D}(A_U) = \mathcal{D}(S) \dot{+} M$ has a non-real eigenvalue $\mu \in \mathbb{C}_+$ if and only if U has a non-trivial intersection with the subspace $L_\mu = (I - Sh(\mu))\mathfrak{N}_i$. Therefore,

$$\sigma(A_U) \supset \mathbb{C}_+ \ if\ and\ only\ if\ M \cap L_\mu \neq \{0\}\, \forall \mu \in \mathbb{C}_+.$$

Since A_U is a J -selfadjoint operator, the inclusion $\sigma(A_U) \supset \mathbb{C}_+$ is equivalent to $\sigma(A_U) = \mathbb{C}_+$.

In view of (57) and (58), $L_\mu = (I - Sh(\mu))\mathfrak{N}_i = span\{c_1(\mu), c_2(\mu)\}$, where

$$c_1(\mu) = e_{++} - s_+(\mu)e_{-+}, \qquad c_2(\mu) = e_{+-} - s_-(\mu)e_{--}. \quad (60)$$

Therefore, the relation $M \cap L_\mu \neq \{0\}$ holds if and only if the equation

$$x_1 d_1 + x_2 d_2 = y_1 c_1(\mu) + y_2 c_2(\mu) \quad (61)$$

has a non-trivial solution $x_1, x_2, y_1, y_2 \in C$ $\ for\ all\ \mu \in \mathbb{C}_+$. Substituting (56) and (60) into (61) and combining the corresponding coefficients for $e_{\pm\pm}$ we obtain four relations

$$x_1 = y_1 x_1 q e^{i(\varphi+\gamma)} - x_2 r e^{i(\varphi-\xi)} = y_2,$$

$$x_2 = -y_2 s_-(\mu), x_1 r e^{i(\varphi+\xi)} + x_2 q e^{i(\varphi-\gamma)} = -s_+(\mu) y_1$$

or

$$q e^{i(\varphi+\gamma)} y_1 - \left(1 - r e^{i(\varphi-\xi)} s_-(\mu)\right) y_2 = 0,$$

$$r e^{i(\varphi+\xi)} + s_+(\mu) y_1 - q e^{i(\varphi-\gamma)} s_-(\mu) y_2 = 0.$$

The last system has a non-trivial solution y_1, y_2 for all $\mu \in \mathbb{C}_+$ if and only if its determinant

$$qe^{i(\varphi+\gamma)} \begin{vmatrix} qe^{i(\varphi+\gamma)} & -1+re^{i(\varphi+\xi)}s_-(\mu) \\ re^{i(\varphi+\xi)}+s_+(\mu) & -qe^{i(\varphi-\gamma)} \end{vmatrix} = 0, \forall\mu \in \mathbb{C}_+$$

This is the case if and only if

$$e^{2i\emptyset}s_-(\mu) = re^{i(\emptyset+\xi)} + s_+(\mu) - re^{i(\emptyset-\xi)}s_-(\mu)s_+(\mu), \forall\mu \in \mathbb{C}_+. \tag{62}$$

Further, $Sh(i) = 0$ by the construction (see (38) or (41)). Hence $s_+(i) = s_-(i) = 0$ and relation (62) takes the form $re^{i(\varphi+\xi)} = 0$ $(for\, \mu = i)$ which means that $r = 0$. Therefore, an operator $A_U \in \Sigma_J$ has empty resolvent set if and only

$$e^{2i\varphi}s_-(\mu) = s_+(\mu), \forall\mu \in \mathbb{C}_+. \tag{63}$$

Corollary (3.2.8)[94]. If $s_+ \approx s_-$, then the operators $A_U \in \Sigma_J$ with empty resolvent set are determined by the matrices:

$$U = e^{i\varphi}\begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}, \ \gamma \in [0, 2\pi), \tag{64}$$

where $\varphi \in [0, 2\pi)$ is uniquely determined by (63) if $Sh \not\equiv 0$ and φ is an arbitrary parameter if $Sh \equiv 0$.

Corollary (3.2.9)[94]. Let S be a simple closed symmetric operator. Then Σ_J contains operators with empty resolvent set if and only if the operators $S_\pm$ in (42) are unitarily equivalent.

Proof. Assume that Σ_J contains operators with empty resolvent set and $Sh \not\equiv 0$. Then $s_+ \not\equiv 0$ and (63) holds for a certain $\varphi \in [0, 2\pi)$. Consider unitary mappings $V_\pm : \mathfrak{N}_{-i}(S_\pm) \to \mathfrak{N}_i(S_\pm)$ defined by the relations

$$V_+e_{-+} = e_{++}, \quad V_-e_{--} = e^{2i\varphi}e_{+-}.$$

By virtue of (58) and (59), we get

$$V_+Sh_+(\mu)e_{++} = s_+(\mu)e_{++},$$
$$V_-Sh_-(\mu)e_{+-} = e^{2i\varphi}s_-(\mu)e_{+-} = s_+(\mu)e_{+-}. \tag{65}$$

Then $V_+Sh_+(.)and\ V_-Sh_-(.)$ are the characteristic functions (see, [161]) of $S_\pm$ associated with the boundary triplets $\mathfrak{N}_i(S_\pm), (\Gamma_0, \Gamma_1)\ of\ S_\pm^*$ defined by (39). Identifying the defect subspaces $\mathfrak{N}_i(S_+) = span\{e_{++}\}$ and $\mathfrak{N}_i(S_-) = span\{e_{+-}\}$with $\mathbb{C}$ and using (65) we arrive at the conclusion that the characteristic functions of $S_\pm$ associated with the boundary triplets $(\mathbb{C}, \Gamma_0, \Gamma_1)$ coincide.

The same is true when $Sh \equiv 0$. In that case, $s_+ \equiv s_- \equiv 0$ and the characteristic functions $Sh_\pm\ of\ S_\pm$ are equal to zero.

Since S is a simple symmetric operator, $S_\pm$ are also simple symmetric operators. In that case, the equality of characteristic functions of $S_\pm$ implies the unitary equivalence of $S_\pm$, see, [115,119].

Conversely, if $S_\pm$ are unitarily equivalent then $s_+ = W^{-1}s_-W$, where W is a unitary mapping of $\mathfrak{S}_+$ onto $\mathfrak{S}_-$. Therefore,

$$W : \mathfrak{N}_\mu(s_+) \to \mathfrak{N}_\mu(s_-)\ and\ WSh_+(\mu) = Sh_-(\mu). \tag{66}$$

Assuming $\mu = \pm i$ in the first identity of (66) and using (57), we find $w_1, w_2 \in \mathbb{C}$ with

$$We_{++} = w_1e_{+-}, We_{-+} = w_2e_{--}, |w_1| = |w_2| = 1. \tag{67}$$

It follows from (58) and (67) that

$$WSh_+(\mu)e_{++} = s_+(\mu)We_{\mp} = w_2s_+(\mu)e_{--}$$

and $s_-(\mu)We_{++} = w_1Sh_+(\mu)e_{+-} = w_1s_-(\mu)e_{--}$. Combining the last two identities with the second relation in (66) we obtain $e^{2i\varphi}s_-(\mu) = s_+(\mu)$, where $e^{2i\varphi} = w_1/w_2$. The statement of Corollary (3.2.9) follows now from Theorem (3.2.7).

As above the deficiency indices of operators $S_\pm$ in the presentation (42) of S are supposed to be $\langle 1,1\rangle$. In the following we discuss the different situations which can occur:

• no member of Σ_J has non-empty resolvent set;

• there are members of Σ_J with empty resolvent set. We discuss the cases $Sh(\cdot) \not\equiv 0$ *and* $Sh(\cdot) \equiv 0$

Theorem (3.2.10)[94]. If Σ_J contains no operators with empty resolvent set, then

$$\Upsilon_{\mathfrak{U}} = \Upsilon_J = \Sigma_J^{st}$$

in (47). Moreover, if S is a simple closed symmetric operator, then $\mathfrak{U} = \{J\}$.

Proof. Let $C \in \mathfrak{U}$. It follows from (48) that the operator $C\upharpoonright_{\mathfrak{N}_{\pm i}}$ acts in $\mathfrak{N}_{\pm i}$ and satisfies the relations

$$(C\upharpoonright_{\mathfrak{N}_{\pm i}})^2 = I, \qquad JC\upharpoonright_{\mathfrak{N}_{\pm i}} > 0. \tag{68}$$

Denote by C_1 and C_2 the 2×2-matrix representations of $C\upharpoonright_{\mathfrak{N}_i}$ *and* $C\upharpoonright_{\mathfrak{N}_{-i}}$ with respect to the orthogonal bases e_{++}, e_{+-} *and* e_{-+}, e_{--} of $\mathfrak{N}_i$ *and* $\mathfrak{N}_{-i}$, respectively. Then (68) takes the form

$$C_j^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} C_j > 0,\ j = 1,2\,. \tag{69}$$

(since $C\upharpoonright_{\mathfrak{N}_{\pm i}}$ are determined by (52)). The Hermiticity of the matrix in the second relation of (69) enables one to deduce that a matrix C_j satisfy (69) if and only if

$$C_j = C_{\chi j,\omega j} := \begin{pmatrix} \cosh\chi j & (\sinh\chi j)e^{-i\omega j}\\ -(\sinh\chi j)e^{i\omega j} & -\cosh\chi j\end{pmatrix}, \chi j \in \mathbb{R}, \omega j \in [0,2\pi). \tag{70}$$

Combining (49) with (58) and (70) we get

$$\begin{pmatrix} s_+(\mu) & 0\\ 0 & s_-(\mu)\end{pmatrix}\begin{pmatrix} \cosh\chi_1 & (\sinh\chi_1)e^{-i\omega_1}\\ -(\sinh\chi_1)e^{i\omega_1} & -\cosh\chi_1\end{pmatrix}$$
$$= \begin{pmatrix} \cosh\chi_2 & (\sinh\chi_1)e^{-i\omega_2}\\ -(\sinh\chi_1)e^{i\omega_2} & -\cosh\chi_2\end{pmatrix}\begin{pmatrix} s_+(\mu) & 0\\ 0 & s_-(\mu)\end{pmatrix}. \tag{71}$$

for matrix representations $C_{\chi j,\omega j}$ of the operators $C\upharpoonright_{\mathfrak{N}_{\pm i}}$.

If Σ_J has no operators with empty resolvent set, then $s_+ \not\approx s_-$ (Theorem (3.3.7)). In that case identity (71) holds only in the case $\chi_1 = \chi_2 = 0$, *i.e.*, $C_0, \omega_1 = C_0, \omega_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Therefore, if $s_+ \not\approx s_-$ then

$$C\upharpoonright_{\mathfrak{N}_{\pm i}} = J\upharpoonright_{\mathfrak{N}_{\pm i}}, \forall C \in \mathfrak{U}. \tag{72}$$

Let us consider an arbitrary $A_U \in \Sigma_J^{st}$. Then $A_U C = CA_U$ for some choice of $C \in \mathfrak{U}$. It is known that $A_U C = CA_U$ if and only if $CM = M$, where M is defined by (55) and (56), cf. [138]. This and (72) give $CM = M$ if and only if $JM = M$. Therefore, $A_U J = J A_U$ and $A_U \in \Upsilon_J$. Thus

$$\Upsilon_J = \Sigma_J^{st}.$$

The identity $\Upsilon_U = \Upsilon_J$ is verified in a similar manner.

If S is a simple symmetric operator, then $U = \{J\}$ due to Lemma (3.2.6) and relation (72).

Recall, that a J-selfadjoint operator A in a Krein space $(\mathfrak{S}, [\cdot,\cdot])$ is called definitizable (see [163]) if

$\rho(A) \neq \emptyset$ and there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0, \quad \textit{for all}\ x \in \mathfrak{S}$.

Corollary (3.2.11)[94]. If Σ_J contains at least one definitizable operator, then

$$\Upsilon_U = \Upsilon_J = \Sigma_J^{st}.$$

Proof. If $A \in \Sigma_J$ is definitizable then an arbitrary operator from Σ_J is also definitizable, see [99,100]. Therefore, Σ_J has no operators with empty resolvent sets.

In that case two quite different arrangements for the sets Υ_U, Υ_J, and Σ_J^{st} are possible and they will be discussed in this Section.

We recall that Σ_J contains operators with empty resolvent set if and only if $e^{i\alpha}s_+(\mu) = s_-(\mu), \mu \in \mathbb{C}_+$, for a certain parameter eiα (Theorem (3.2.6)). Here, the functions $s_\pm(\cdot)$ are defined in (58) with the help of the elements $\{e_{\pm\pm}\}$ which are determined up to the multiplication with a unimodular constant. Therefore, without loss of generality, we may assume

$$s_+ = s_-. \tag{73}$$

Theorem (3.2.12)[94]. Let S be a simple closed symmetric operator. Then the set Σ_J contains operators with empty resolvent set if and only if there exists a fundamental symmetry R $(i.e., R^2 = I \textit{ and } R = R^*)$ in $\mathcal{H}$ such that

$$SR = RS, \qquad JR = -RJ. \tag{74}$$

Proof. By virtue of Corollary (3.2.7), the existence of J-selfadjoint extensions of S with empty resolvent set implies that the symmetric operators $S_\pm$ in (42) are unitarily equivalent. Hence, $s_+ = W^{-1} s_- W$, where W is an isometric mapping of $\mathfrak{S}_+$ $onto$ $\mathfrak{S}_-$. It is clear that the operator

$$R = \begin{pmatrix} 0 & W^{-1} \\ W & 0 \end{pmatrix} \tag{75}$$

determined with respect to the fundamental decomposition (30) is a fundamental symmetry in $\mathfrak{S}$ and satisfies (74). Conversely, if (75) hold, then $S_+ = RS_-R$. Therefore $S_\pm$ are unitarily equivalent and Σ_J contains elements with empty resolvent set (Corollary (3.2.8)).

Remark (3.2.13)[94]. If the relations in (74) hold then the existence of J-selfadjoint extensions of S with empty resolvent set can be established without the assumption of simplicity of S in Theorem (3.2.12). Indeed, the operator S is reduced by the decomposition

$$\mathfrak{S} = \mathfrak{S}_0 \oplus \mathfrak{S}_1, \quad \mathfrak{S}_1 = \bigcap_{\forall \mu \in \mathbb{C}\setminus\mathbb{R}} \mathcal{R}(S - \mu I), \tag{76}$$

where $\mathfrak{S}_1$ is the maximal subspace invariant for S on which the operator $S_1 = S \restriction_{\mathfrak{S}_1}$ is selfadjoint; the subspace $\mathfrak{S}_0$ coincides with the closed linear span of all $ker(S^* - \mu I)$ and the restriction $S_0 := S \restriction_{\mathfrak{S}_1}$ is a simple closed symmetric operator in $\mathfrak{S}_0$, see, [151].

If (74) hold, then the restrictions $J_0 := J \restriction_{\mathfrak{S}_1}$ and $R_0 := R \restriction_{\mathfrak{S}_1}$ are fundamental symmetries in $\mathfrak{S}_0$ and they satisfy (74) for S_0. Applying Theorem (3.3.12), we establish the existence of J_0-selfadjoint extensions of S_0 with empty resolvent set. Since an operator $A \in \Sigma_J$ has the decomposition $A = A_0 \oplus S_1$ with respect to (75), where A_0 is a J_0-selfadjoint extension of S_0, the set Σ_J contains J-selfadjoint operators with empty resolvent set.

However, we cannot drop the condition of simplicity of S in Theorem (3.2.11) for the inverse implication. In that case, the existence of a fundamental symmetry R_0 satisfying (74) for S_0 in $\mathfrak{S}_0$ is easily deduced from Theorem (3.2.11) but it is not clear how to extend R_0 to $\mathfrak{S}$ with preservation of the relations in (74).

From (74) one concludes that the four operators I, J , $R, and\, J\, R$ are linearly independent. Hence, the operators $J\, and\, \mathfrak{S}$ can be interpreted as basis (generating) elements of the complex Clifford algebra

$$Cl_2 = span\{I, J\, , R, J\, R\}.$$

Corollary(3.2.14)[94]. Let S satisfy (74) and let $\tilde{J} \in Cl_2$ be a non-trivial fundamental symmetry in $\mathfrak{S}$. Then there exists J -selfadjoint extensions of S with empty resolvent set.

Proof. It is easy to see that an operator $\tilde{J} \in Cl_2$ is a non-trivial fundamental symmetry in $\mathfrak{S}$ $(i.e., \tilde{J}^2 = I\, , J = \tilde{J}^*, \quad and \quad \tilde{J} \neq I\,)$ if and only if

$$\tilde{J} = \alpha_1 J + \alpha_2 R + \alpha_3 i\, JR, \qquad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \qquad \alpha_j \in \mathbb{R}. \tag{77}$$

Denote $\tilde{R} = \beta_1 J + \beta_2 R + \beta_{3i}\, JR,\ where\ \sum \beta_j^2 = 1,\ \beta_j \in \mathbb{R}$. By virtue of (4.10), $\tilde{R}$ is a fundamental symmetry in $\mathfrak{S}$ which commutes with S. Assuming $\alpha_j \beta_j = 0,\ we\ obtain\ \tilde{J}\, \tilde{R} = -\tilde{R}\tilde{J}$. Since J is a fundamental symmetry in $\mathfrak{S}$ which commutes with S, the statement follows from Theorem (3.2.11).

Theorem (3.2.15)[94]. Let S be a simple closed symmetric operator with non-zero characteristic function $Sh(\cdot)$ and let the set Σ_J contains operators with empty resolvent set. Then all operators $C \in \mathfrak{U}$ have the form

$$C := C_{\chi,\omega} = J[(cosh\chi)I + (sinh\chi)R\omega], \tag{78}$$

where R satisfies (74), $R_\omega = Re^{i\omega J} = R[cos\omega + i(sin\omega)\, J\,]$, and $\chi \in \mathbb{R},\ \omega \in [0, 2\pi)$.

Proof. First, we will show $C_{\chi,\omega} \in \mathfrak{U}$. Since Σ_J contains operators with empty resolvent set, there exists a unitary mapping $W : \mathfrak{S}_+ \to \mathfrak{S}_-$ such that $S_+ = W^{-1} S_- W$ (Corollary (3.2.9)). This allows one to determine a fundamental symmetry $R\ in\ \mathfrak{S}$ with the help of formula (75).

By construction, the operator R satisfies (74). Therefore, the subspaces $\mathfrak{N}_{\pm i}$ reduce $R. Let\ \mathcal{R}_1 = (r_{ij}^1)_{i,j=1}^2\ and\ \mathcal{R}_2 = (r_{ij}^1)_{i,j=1}^2$ be the matrix representations of $R \restriction_{\mathfrak{N}_i}$ and $R \restriction_{\mathfrak{N}_{-i}}$ with respect to the bases $e_{++}, e_{+-}\ and\ e_{-+}, e_{--}$ of $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$, respectively. It follows from (67) and (75) that $\mathcal{R}_j = \begin{pmatrix} 0 & w_j^{-1} \\ w_j & 0 \end{pmatrix}$, where $|w_1| = |w_2| = 1$. Moreover, since we assume (73), the parameter φ in the proof of Corollary 3.2.8 is equal to zero and, hence, $w := w_1 = w_2$. The exact value of the unimodular constant w depends on the choice of W. Without loss of generality we may assume that $w = 1$. Then

$$\mathcal{R} := \mathcal{R}_1 = \mathcal{R}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{79}$$

Let us consider the collections of all operators $C_{\chi,\omega}$ determined by (78) It is known that $C_{\chi,\omega} = Je^{\chi R_\omega}$, where $R_\omega = Re^{i\omega} J = R[cos\omega + i(sin\omega)\, J\,]$ is a fundamental symmetry in $\mathfrak{S}$, which anticommutes with J $(i.e., R_\omega\, J = -J\, R_\omega)$, see [138].

Such a representation leads to the conclusion that $C_{\chi,\omega}^2 = I\ and\ JC_{\chi,\omega} > 0$. Moreover $SC_{\chi,\omega} = SC_{\chi,\omega}$ due to (30) and (74). Therefore, an arbitrary $C_{\chi,\omega}$ belongs to $\mathfrak{U}$.

Rewriting (78) as follows

$$C_{\chi,\omega} = (cosh\chi)\, J + (sinh\chi)(cos\omega)\, J\, R - i(sinh\chi)(sin\omega)R$$

and using (79) we obtain that both matrix representations of $C_{\chi,\omega} \restriction_{\mathfrak{N}_i}$ and of $C_{\chi,\omega} \restriction_{\mathfrak{N}_{-i}}$ coincide with

$$C_{\chi,\omega} = \begin{pmatrix} cosh\chi & (sinh\chi)e^{-i\omega} \\ -(sinh\chi)e^{i\omega} & -cosh\chi \end{pmatrix}.$$

Let $C \in \mathfrak{U}$. Then the matrix representations of its restrictions $C \restriction_{\mathfrak{N}_i}$ and $C \restriction_{\mathfrak{N}_{-i}}$ coincide with C_{χ_1,ω_1} *and* C_{χ_2,ω_2} defined by (70). Furthermore, since $Sh(\mu)C = CSh(\mu)$ (see (49)), the identity (71) holds. That is equivalent to the relations $\chi_1 = \chi_2$ *and* $e^{-i\omega_1} = e^{-i\omega_2}$ (since (73) is true and $s_+ \not\equiv 0$).

Setting $\chi = \chi_1 = \chi_2$ and $\omega = \omega_1$, one concludes that the matrix representations C_{χ_j,ω_j} coincides with $C_{\chi,\omega}$. Therefore, $C = C_{\chi,\omega}$due to Lemma 3.2.6. Thus, the collection of operators $\{C_{\chi,\omega}\}$ defined by (78) coincides with $\mathfrak{U}$.

Combining Theorem (3.2.15) with [138], we immediately derive the following statement.

Corollary (3.2.16)[94]. Let S and Σ_J satisfy the condition of Theorem (3.2.15) and let $A_U \in \Sigma_J$ be defined by (54)–(56). Then the strict inclusions

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st}$$

hold and the following relations are true.

(i) A_U belongs to $\Upsilon_{\mathfrak{U}}$ if and only if

$$U = e^{i\frac{\pi}{2}}\begin{pmatrix} 0 & e^{i\xi} \\ -e^{-i\xi} & 0 \end{pmatrix}, \qquad \xi \in [0, 2\pi);$$

(ii) A_U belongs to Υ_J if and only if

$$U = e^{i\varphi}\begin{pmatrix} 0 & e^{i\xi} \\ -e^{-i\xi} & 0 \end{pmatrix}, \varphi, \qquad \xi \in [0, 2\pi);$$

(iii) A_U belongs to $\Sigma_J^{st} \setminus \Upsilon_J$ if and only if

$$U = e^{i\varphi}\begin{pmatrix} qe^{i\gamma} & re^{i\xi} \\ -re^{-i\xi} & qe^{-i\gamma} \end{pmatrix}, \qquad \gamma, \xi \in [0, 2\pi), \qquad q, r > 0, q^2 + r^2 = 1,$$

where $0 < q < |\cos\varphi|$. In that case the operator $A_U$has $C_{\chi,\omega}$ -symmetry, where $\omega = \gamma$ *and* χ is determined by the relation $q = -\tanh\chi \cos\varphi$.

If $Sh \equiv 0$, *then* $s_+(\mu) = s_-(\mu) = 0$ *for all* $\mu \in \mathbb{C}_+$. Therefore, by Theorem (3.2.7), Σ_J contains operators with empty resolvent set and Theorem (3.2.12) and Corollary (3.2.14) hold. However Theorem (3.2.15) is not true due to the fact that the set of all stable C-symmetries $\mathfrak{U}$ is much more greater then the formula (78) provides. That is why the commutation condition (71) is vanished for $s_\pm \equiv 0$ and we cannot establish the relationship between parameters $\chi_1, \omega_1$and χ_2, ω_2 of matrices C_{χ_j,ω_j} (see the proof of Theorem (3.2.15)).

Theorem (3.2.17)[94]. Let S be a simple closed symmetric operator with zero characteristic function and let $A_U \in \Sigma_J$ be defined by (54)–(56).

Then $\Upsilon_{\mathfrak{U}} = \emptyset$ and the strict inclusions

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st}$$

hold.

(i) A_U belongs to Υ_J if and only if

$$U = e^{i\varphi}\begin{pmatrix} 0 & e^{i\xi} \\ -e^{-i\xi} & 0 \end{pmatrix}, \qquad \varphi, \xi \in [0, 2\pi);$$

(ii) A_U belongs to $\Sigma_J^{st} \setminus \Upsilon_J$ if and only if

$$U = e^{i\varphi}\begin{pmatrix} qe^{i\gamma} & re^{i\xi} \\ -re^{-i\xi} & qe^{-i\gamma} \end{pmatrix}, \qquad \varphi, \gamma, \xi \in [0, 2\pi), \qquad q, r > 0, q^2 + r^2 = 1$$

Proof. (i) follows from [138].

In order to show (ii) let $A_U \in \Sigma_J^{st}$. Then $A_U C = C A_U$ for some choice of $C \in \mathfrak{U}$. This is equivalent to the relation $CM = M$,

where $M = span\{d_1, d_2\}$ is defined by (55) and (56) (see the proof of Theorem (3.2.10)). Moreover, it follows from the proof of Theorem (3.2.10) that the operators $C \restriction_{\mathfrak{N}_i}$ and $C \restriction_{\mathfrak{N}_{-i}}$ acts in $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$, respectively and they have the matrix representations C_{χ_1,ω_1} *and* C_{χ_2,ω_2} defined by formula (70).

Combining [160] with Lemma (3.2.6) we conclude that the correspondence

$$C \in \mathfrak{U} \to \{C_{\chi_1,\omega_1}, C_{\chi_2,\omega_2}\}, \qquad \chi_j \in \mathbb{R}, \qquad \omega_j \in [0, 2\pi) \tag{80}$$

is bijective for the case of a zero characteristic function ($Sh \equiv 0$).

It follows from (56) and (70) that

$$\begin{aligned} Cd_1 &= C_{\chi_1,\omega_1}e_{++} + qe^{i(\varphi+\gamma)}C_{\chi_1,\omega_1}e_{+-} + re^{i(\varphi+\xi)}C_{\chi_2,\omega_2}e_{-+} \\ &= k_1e_{++} - [sinh\chi_1e^{i\omega_1} + qe^{i(\varphi+\gamma)} cosh\chi_1]e_{+-} + [re^{i(\varphi+\xi)} cosh\chi_2]e_{-+}\, k_2e_{--}, \end{aligned}$$

where

$$k_1 = cosh\chi_1 + qe^{i(\varphi+\gamma)} sinh\chi_1e^{-i\omega1}, k_2 = -re^{i(\varphi+\xi)} sinh\chi_2e^{i\omega_2}. \tag{81}$$

Taking the definition (56) of d_j into account we conclude that $Cd_1 \in M$ if and only if $Cd_1 = k_1d_1 + k_2d_2$, where k_j are defined by (81). A direct calculation shows that the last identity holds if we set

$$\chi = \chi_1 = \chi_2 = -tanh^{-1} q, \qquad \omega_1 = \frac{\gamma + \varphi}{2}, \qquad \omega_2 = \frac{\gamma - \varphi}{2}. \tag{82}$$

A similar reasoning shows that $Cd_2 \in M$ if we choose parameters χ_j and ω_j according to (82). Note that χ can be defined in (82) only in the case $0 \le q < 1$.

Thus, if $A_U \in \Sigma_J$ is defined by (54)–(56) with $0 \le q < 1$, then choosing parameters χ_j , ω_j due to (82) and using the bijection (70), we establish the existence of $C \in \mathfrak{U}$ such that $A_U C = C A_U$. Therefore $A_U \in \Sigma_J^{st}$. Since $J \in \Upsilon_J$ when $q = 0$ (see (i)) and the spectrum of A_U coincides with $\mathbb{C}$ when $q = 1$, we show (ii).

Let us assume that $A_U \in \Upsilon_{\mathfrak{U}}$. In that case $A_U C = C A_U$ for all $C \in \mathfrak{U}$. Taking (80) into account, we conclude that the element $Cd_1 = C_{\chi_1,\omega_1}e_{++} + qe^{i(\varphi+\gamma)}C_{\chi_1,\omega_1}e_{+-} + re^{i(\varphi+\xi)}C_{\chi_2,\omega_2}e_{-+}$ belongs to M ($i.e., Cd_1 = k_1d_1 + k_2d_2$, where k_j are defined by (81)) for all values of parameters χ_j and ω_j . This is impossible. Hence, $\Upsilon_{\mathfrak{U}} = \emptyset$.

The next statement is a direct consequence of Proposition (3.2.2) and Theorem (3.3.17).

Corollary(3.2.18)[94]. (See [118].) If S is a simple closed symmetric operator with zero characteristic function, then an operator $A_U \in \Sigma_J$ has real spectrum if and only if A_U has stable C-symmetry and, hence, A_U is similar to a selfadjoint operator. Otherwise, the spectrum of A_U coincides with $\mathbb{C}$.

The necessary and sufficient conditions for the Dirichlet eigenvalue problem associated with the Sturm–Liouville equation

$$p\big((x)y'\big)' = \lambda r(x)y, \qquad -\infty < a \le x \le b < \infty \tag{83}$$

to be degenerate were established in [122]. We consider one of the simplest cases where

$$p(x) = r(x) = (sgn\, x)\ and\ [a,b] = [-1,1].$$

Define the closed symmetric operator S associated with the expression $-(sgn\, x)((sgn\, x)y')'$

and boundary conditions $y(-1) = y(1) = 0$ via

$$Sy = -y'' ,$$

with domain

$$\mathcal{D}(S) = \{y \in W_2^2(-1,0) \oplus W_2^2(0,1) y(0\pm) = y(0\pm) = y(\pm 1) = 0\}. \tag{84}$$

Then (83) takes the form $Sy = \lambda y$.

The operator S has deficiency indices $\langle 2,2\rangle$ and it commutes with the fundamental symmetry $J\,y(x) = (sgn\,x)y(x)$ in $\mathfrak{S} = L_2(-1,1)$. The corresponding closed symmetric operators $S_\pm\, y = -y''$ (see (42)) with the domains

$$\mathcal{D}(S_+) = \{y \in W_2^2(0,1) | y(0+) = y'(0+) = y(1) = 0\},$$
$$\mathcal{D}(S_-) = \{y \in W_2^2(-1,0) | y(0-) = y'(0-) = y(-1) = 0\}$$

act in $\mathfrak{S}_+ = L_2(0,1)$ $\quad and \quad$ $\mathfrak{S}_- = L_2(-1,0)$, respectively.

Consider the parity operator $\mathcal{P}y(x) = y(-x)\, and\, set\, R := \mathcal{P}$. It is clear that R is a fundamental symmetry in $L_2(-1,1)$ and it satisfies (74). To describe these operators we observe that solutions $y_\mu^\pm(x)$ of the equations

$$S_\pm^* y - \mu y = -y''(x) - \mu y(x) = 0, \qquad y(\pm 1) = 0, \mu \in \mathbb{C}_+$$

have the form

$$y_\mu^+(x) = \begin{cases} sin\sqrt{\mu}(x-1), & x \in [0,1], \\ 0, & x \in [-1,0], \end{cases}$$

$$y_\mu^-(x) = \begin{cases} 0, & x \in [0,1],]. \\ -sin\sqrt{\mu}(x+1), & x \in [-1,0 \end{cases}$$

Here denotes the branch of the square root defined in $\mathbb{C}$ with a cut along $[0,\infty)$ and fixed by $Im\sqrt{\lambda} > 0 \quad if\, \lambda \notin [0,\infty)$. Moreover, $\sqrt{.}$ is continued to $[0,\infty)$ via $\lambda \mapsto \lambda \geq 0 \quad for\, \lambda \in [0,\infty)$. According to (51), the elements $e_{\pm\pm}$ can be chosen as follows:

$$e_{++} = y_i^+ , \qquad e_{+-} = y_i^- , \qquad e_{-+} = y_{-i}^+ , \qquad e_{--} = y_{-i}^-$$

and the functions $s_\pm(\mu)$ in (58) can be calculated immediately by repeating the arguments in [168]. For completeness we outline the method.

The characteristic function $Sh_+(\mu)\, of\, S_+$ is determined by the first relations in (58) and (59). Employing here (41)

we get

$$y_\mu^+(x) = u(x) + ce_{++} - cs_+(\mu)e_{-+}, u \in \mathcal{D}(S_+), x \in [0,1], \tag{85}$$

where c is a constant which is easily determined by setting $x = 0$ and taking into account the relevant boundary conditions:

$$c = \frac{sin\sqrt{\mu}}{sin\sqrt{i} - s_+(\mu)\, sin\sqrt{-i}} .$$

Differentiating (85) with a subsequent setting $x = 0$ we obtain

$$\sqrt{\mu}\, cos\mu = c\sqrt{i}\, cos\sqrt{i} - cs_+(\mu)\sqrt{-i}\; cos\sqrt{-i}.$$

The last two relations leads to the conclusion:

$$s_+(\mu) = \frac{\sqrt{i}\, sin\, \sqrt{\mu}\, cos\, \sqrt{i}\; - \sqrt{\mu}\, cos\sqrt{\mu}\, sin\sqrt{i}}{\sqrt{-i}\; sin\sqrt{\mu} cos\sqrt{-i}\; - \sqrt{\mu} cos\sqrt{\mu} sin\sqrt{-i}}.$$

Considering the characteristic function $Sh_-\ \ of\ \ S_-$ we obtain the same expression for $s_-(\mu)$. Thus $s_+ = s_- \not\equiv 0$. By Theorem (3.2.7), the set Σ_J of J -selfadjoint extensions of S contains operators with empty resolvent set. Applying Corollary (3.2.8) and taking the explicit form of elements $e_{\pm\pm}$ into account we derive the following description of J -selfadjoint extensions of S with empty resolvent set.

Proposition(3.2.19)[94]. Let S be a symmetric operator in $L_2(-1,1)$ defined by (84) and let $J\, y(x) = (sgn\, x)y(x)\ \ for\ y \in L_2(-1,1)$Then the collection of all possible J -selfadjoint extensions A_γ of S with empty resolvent set is determined by the formulas

$$A_\gamma\, y = y''$$

$$\mathcal{D}(A_\gamma) = \left\{ y \in W_2^2(-1,0) \oplus W_2^2\,(0,1) \,\middle|\, \begin{array}{l} e^{i\gamma}\, y(0+) = y(0-) \\ e^{i\gamma}\, y'(0+) = -y'(0-) \\ y(\pm 1) = 0 \end{array} \right\},$$

where $\gamma \in [0, 2\pi)$ is an arbitrary parameter.

Consider the indefinite Sturm–Liouville differential expression

$$a(y)(x) = (sgn\, x)(y''(x) + q(x)y(x)), x \in \mathbb{R}$$

with a real potential $q \in L^1_{loc}(\mathbb{R})$ and denote by $\mathfrak{S}$ the set of all functions $y \in l_2(\mathbb{R})$ such that y and y' are absolutely continuous and $a(y) \in l_2(\mathbb{R})$. On $\mathfrak{S}$ we define the operator A as follows:

$$Ay = a(y), \qquad \mathcal{D}(A) = \mathfrak{S}. \tag{86}$$

Assume in what follows the limit point case of $a(y)$ at both $-\infty\ and\ +\infty$. Then the operator A is J -selfadjoint in the Krein space $(l_2(\mathbb{R}), [\cdot,\cdot]\, J\,)$, where $J = (sgn\, x)I$, see, [148].

The operator A is a J -selfadjoint extension of the symmetric operator S,

$$S = (sgn\, x)\left(-\frac{d^2}{dx^2} + q\right), \qquad \mathcal{D}(S) = \{y \in \mathfrak{S}\, | y(0) = y'(0) = 0\}. \tag{87}$$

The operator S commutes with J and has deficiency indices $\langle 2,2\rangle$. Its restrictions onto the subspaces $l_2(\mathbb{R}_\pm)$ of the fundamental decomposition $l_2(\mathbb{R}) = l_2(\mathbb{R}_+) \oplus l_2(\mathbb{R}_-)$ coincides with the symmetric operators

$$S_+ = -\frac{d^2}{dx^2} + q_+, \qquad S_- = \frac{d^2}{dx^2} - q_-, \qquad \mathcal{D}(S_\pm) = P_\pm \mathcal{D}(S), q_\pm = q\restriction_{\mathbb{R}_\pm}$$

with deficiency indices $\langle 1,1\rangle$ acting in the Hilbert spaces $\mathfrak{S}_+ = l_2(\mathbb{R}_+)\ and\ \mathfrak{S}_- = l_2(\mathbb{R}_-)$, respectively. Here $P_\pm$ are the orthogonal projectors onto $l_2(\mathbb{R}_\pm)$in $l_2(\mathbb{R})$.

Denote by $c_\mu(\cdot), s_\mu(\cdot)$ the solutions of the equation

$$-f''(x) + q(x)f\,(x) = \mu f\,(x), \qquad x \in \mathbb{R}, \qquad \mu \in \mathbb{C}$$

with boundary conditions

$$c_\mu(0) = s'_\mu(0) = 1, \qquad c'_\mu(0) = s_\mu(0) = 0. \tag{88}$$

Due to the limit point case at $\pm\infty$ there exist unique holomorphic functions $M_\pm(\mu)$ $(\mu \in \mathbb{C}\setminus\mathbb{R})$ such that the functions

$$\psi_\mu^\pm\,(x) = \begin{cases} s_{\pm\mu}(x) - M_\pm(\mu)c_{\pm\mu}(x), & x \in \mathbb{R}_\pm, \\ 0, & x \in \mathbb{R}_\pm \end{cases} \tag{89}$$

belongs to $l_2(\mathbb{R})$. The functions $M_\pm(.)$ are called the Titchmarsh–Weyl coefficients of the differential expression $a(\cdot)$ (see, [158]). They are Nevanlinna functions and they satisfy the following asymptotic behavior

$$M_\pm(\mu) = \pm \frac{i}{\sqrt{\pm\mu}} + O\left(\frac{1}{|\mu|}\right)(\mu \to \infty, 0 < \delta < arg\mu < \pi - \delta) \tag{90}$$

for $\delta \in (0, \pi 2)$, see [150].

The asymptotic behavior (90) was used for justifying the property $\rho(A) \neq \emptyset$ for the concrete J -selfadjoint extension A of S defined by (86), cf. [115]. We extend this result to all operators in Σ_J .

Theorem(3.2.20)[94]. Let the symmetric operator S be defined by (87) and $J = (sgn\, x)I$. Then the set Σ_J of J -selfadjoint extensions of S does not contain operators with empty resolvent set.

Proof. The proof is divided into two steps. In the first one we calculate the characteristic function of S. In the second step we apply Theorem (3.2.7).

Step 1. It follows from the definition of $S_\pm$ and (89) that the defect subspaces $\mathfrak{N}_{\pm i}(S_+)$ coincides with span$\{\psi^+_{\pm i}\}$ and the defect subspaces $\mathfrak{N}_{\pm i}(S_-)$ coincides with span$\{\psi^-_{\mp i}\}$. Therefore, we can choose basis elements $\{e_{\pm\pm}\}$ as follows:

$$e_{++} = \psi^+_i\,, \qquad e_{\mp} = \psi^+_{-i}\,, \qquad e_{+-} = c\psi^-_i\,, \qquad e_{--} = c\psi^-_{-i},$$

where an auxiliary constant $c > 0$ is determined by the condition $\|\psi^+_i\| = \|\psi^-_i\|$

(or, what is equivalent, by the condition $\|\psi^+_{-i}\| = \|\psi^-_{-i}\|$) .This ensures the equality of the norms $\|e_{++}\| = \|e_{+-}\| = \|e_{-+}\| = \|e_{--}\|$.

By virtue of (88) and (89) we have

$$e_{\pm+}(0) = -M_+(\pm i), e'_{\pm+}(0) = 1, e_{\pm-}(0) = -cM_-(\pm i), e'_{\pm-}(0) = c. \tag{91}$$

Using these boundary conditions and, we arrive at the conclusion that the characteristic function $Sh\ of\ S$ is defined by the following functions $s_+(\cdot)\ and\ s_-(\cdot)$ in (58):

$$s_+(\mu) = \frac{M_+(\mu) - M_+(i)}{M_+(\mu) - M_+(-i)}, \qquad s_-(\mu) = \frac{M_-(\mu) - M_-(i)}{M - (\mu) - M_-(-i)}. \tag{92}$$

Step 2. By Theorem (3.2.7) the set Σ_J contains operators with empty resolvent set if and only if $e^{2i\varphi}s_-(\mu) = s_+(\mu),\ \ \mu \in \mathbb{C}_+$, for a certain choice of $\varphi \in [0, 2\pi)$. Tending $\mu \to \infty$ in this identity and taking (90) and (92) into account, we obtain that

$$e^{2i\varphi} = \frac{M_+(i)M - (-i)}{M_+(-i)M_-(i)}. \tag{93}$$

Rewriting $e^{2i\varphi}s_-(\mu) = s_+(\mu)$ with the use of (92) and (93) we get

$$M_+(\mu)M_-(\mu)\left[e^{2i\varphi} - 1\right] + M_+(\mu))M_{-i} - e^{2i\varphi}M_-(i)$$
$$+ M - (\mu)M_+(\mu) - e^{2i\varphi}M_+(-i) = 0. \tag{94}$$

Denote $M_\pm(i) = e^{i\theta} \pm |M_\pm(i)|$, where $\theta_\pm \in (0, \pi)$ (since $Im\, M_\pm(i) > 0$). Then (93) takes the form $e^{2i\varphi} = e^{2i(\theta_+ - \theta_-)}$ and relation (94) can rewriting (after routine transformations) as follows:

$$M_+(\mu)M_-(\mu)sin(\theta_+ - \theta_-) - M_+(\mu)M_-(i)sin\,\theta_+ + M_-(\mu)M_+(i)sin\,\theta_- = 0 \quad \forall\mu \in C+. \tag{95}$$

Since the coefficients $|M_\pm(i)|\ sin\,\theta \pm\ of\ M_\pm(\mu)$ are real, identity (95) cannot be true for the whole $\mathbb{C}_+$ (due to the asymptotic behavior (90)). Therefore, Σ_J does not contain operators with empty resolvent set. (See [115]).

By virtue of Theorems (3.2.10), (3.2.20) the set Σ_J^{st} of J -selfadjoint operators with stable C-symmetry is reduced to the set Υ_J of selfadjoint extensions of S which commute with J in the case of indefinite Sturm–Liouville operators. The set Υ_J consists of all selfadjoint extensions of S with separated boundary conditions on 0, i.e.,

$$A \in \Upsilon_J \Leftrightarrow Ay = a(y), \qquad \mathcal{D}(A) = \{y \in \mathfrak{C} | a_\pm \, f\,(0\pm) - b_\pm \, f\,'(0\pm) = 0\}.$$

Consider the closed symmetric operator

$$S = -i\frac{d}{dx}, \qquad \mathcal{D}(S) = \{y \in W_2^1(\mathbb{R}, \mathbb{C}^2) | y(0) = 0\}$$

in the Hilbert space $L_2(\mathbb{R}, \mathbb{C}^2) := L_2(\mathbb{R}) \otimes \mathbb{C}^2$.

Lemma(3.2.21)[94]. The operator S has deficiency indices $\langle 2, 2\rangle$ and its characteristic function Sh is equal to zero.

Proof. The operator S can be presented as $S = S_1 + S_2$ with respect to the decomposition $L_2(\mathbb{R}, \mathbb{C}^2) = L_2(\mathbb{R}_-, \mathbb{C}^2) \oplus L_2(\mathbb{R}_+, \mathbb{C}^2)$. The restrictions $S_1 = S \restriction_{L_2(\mathbb{R}_-, \mathbb{C}^2)}$ *and* $S_2 = S \restriction_{L_2(\mathbb{R}_+, \mathbb{C}^2)}$ are maximal symmetric operators in the Hilbert spaces $L_2(\mathbb{R}_-, \mathbb{C}^2)$ *and* $L_2(\mathbb{R}_+, \mathbb{C}^2)$, respectively, with deficiency indices $\langle 0, 2\rangle$ *and* $\langle 2, 0\rangle$, respectively. Therefore S has deficiency indices $\langle 2, 2\rangle$ and $\mathfrak{N}_\mu(S) = \mathfrak{N}_\mu(S_2)$ *for all* $\mu \in \mathbb{C}_+$ (since S_2 has deficiency indices $\langle 2, 0\rangle$). An arbitrary $f_\mu \in \mathfrak{N}_\mu(S)$ admits the representation

$$f_\mu = u + f_i, \qquad u \in \mathcal{D}(S_2), \qquad f_i \in \mathfrak{N}_i(S_2).$$

Comparing the obtained formula with (41) we obtain $Sh(\mu) = 0$.

To achieve a non-empty set Σ_J, we have to choose a fundamental symmetry J in such a way that the deficiency indices of $S_\pm$ in (42) are $\langle 1, 1\rangle$. To this end, we write an arbitrary element $y \in L_2(\mathbb{R}, \mathbb{C}^2)$ as follows

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1 \otimes h_+ + y_2 \otimes h_-, \qquad h_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad h_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and consider the fundamental symmetry $J\,y = \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix}$ in $L_2(\mathbb{R}, \mathbb{C}^2)$. In that case, the operators $S_\pm$ in (42) act in the Hilbert spaces $L_2(\mathbb{R}, \mathcal{H}_\pm)$, where $\mathcal{H}_\pm = span\{h_\pm\}$ and they are determined by the formulas

$$S_\pm = -i\frac{d}{dx}, \qquad \mathcal{D}(S_\pm) = \{y \in W_2^1(\mathbb{R}, \mathcal{H}_\pm) | y(0) = 0\}. \tag{96}$$

Obviously, $S_\pm$ have deficiency indices $\langle 1, 1\rangle$. This means that the set Σ_J is non-empty and its elements can be parameterized by unitary matrices U in (54).

In order to describe the subset of J -selfadjoint extensions with empty resolvent set in Σ_J we have to calculate basis elements $\{e_{\pm\pm}\}$ (see (51)) and to apply Corollary (3.2.8).

Denote by

$$y_i(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad y_{-i}(x) = \begin{cases} 0, & x \geq 0, \\ e^{x}, & x < 0, \end{cases}$$

the solutions of the equation $-iy' - \mu y = 0$ ($\mu \in \{i, -i\}$). Using the definition of $S_\pm$ and (51) we obtain

$$e_{++} = yi \otimes h_+, \qquad e_{+-} = y_i \otimes h_-, \qquad e_{-+} = y_{-i} \otimes h_+, e_{--} = y_{-i} \otimes h_-.$$

Corollary (3.2.8) and equalities (55), (56) imply that an arbitrary J -selfadjoint extension A_U with empty resolvent set has the domain $\mathcal{D}(A_U) = \mathcal{D}(S) \dotplus M$, where M is a linear span of elements

$$d_1 = e_{++} + e^{i(\varphi+\gamma)} e_{+-}, \qquad d_2 = e_{--} + e^{i(\varphi-\gamma)} e_{-+}, \qquad \varphi, \gamma \in [0, 2\pi).$$

The obtained expression leads to the following description of J -selfadjoint extensions A_U $(= A_{\varphi\gamma})$ *of* S with empty resolvent set:

$$A_{\varphi\gamma}\, y = -iy', \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{D}(A_{\varphi\gamma}),$$

where $\varphi, \gamma \in [0, 2\pi)$ are arbitrary parameters and

$$\mathcal{D}(A_{\varphi\gamma}) = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W_2^1 \mathbb{R} \setminus \{0\} \otimes \mathbb{C}^2 \middle| \begin{matrix} y_2(0+) = e^{i(\gamma+\varphi)}\, y_1(0+) \\ y_2(0-) = e^{i(\gamma-\varphi)}\, y_1(0-) \end{matrix} \right\}.$$

Let us consider the free Dirac operator D in the space $L_2(\mathbb{R}) \otimes \mathbb{C}^2$:

$$D = -ic\frac{d}{dx} \otimes \sigma_1 + \frac{c_2}{2} \otimes \sigma_3, \qquad \mathcal{D}(D) = W_2^1(\mathbb{R}) \otimes \mathbb{C}^2,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices and $c > 0$.

The closed symmetric Dirac operator

$S = D \restriction \{u \in W_2^1(\mathbb{R}) \otimes \mathbb{C}^2 \,|u(0) = 0\}$

has deficiency indices $\langle 2, 2 \rangle$, see [95], and it commutes with the fundamental symmetry $J = \mathcal{P} \otimes \sigma_3$ *in* $L_2(\mathbb{R}) \otimes \mathbb{C}^2$, where $\mathcal{P}$ is the parity operator $\mathcal{P}y(x) = y(-x)$. In that case, the operators $S_\pm$ in (42) are restrictions of S onto the Hilbert spaces

$$[L_2^{even}(\mathbb{R}) \otimes \mathcal{H}_+] \oplus [L_2^{odd}(\mathbb{R}) \otimes \mathcal{H}_-],$$

$$[Lodd2\,(\mathbb{R}) \otimes \mathcal{H}_+] \oplus [L_2^{even}(\mathbb{R}) \otimes \mathcal{H}_-],$$

respectively, where $\mathcal{H}_\pm$ are as in this section and the closed symmetric operators $S_\pm$ have deficiency indices $\langle 1, 1 \rangle$.

The defect subspaces $\mathfrak{N}_i$ *and* $\mathfrak{N}_{-i}$ *of* S coincide, respectively, with the linear spans of the functions $\{y_{1+}, y_{2+}\}$ and $\{y_{1-}, y_{2-}\}$ where

$$y_{1\pm}(x) = \begin{pmatrix} ie^{\mp it} \\ (sgn\, x) \end{pmatrix} e^{i\tau|x|}, \qquad y_{2\pm}(x) = (sgn\, x) y_{1\pm}(x), \tag{97}$$

$\tau = \frac{i}{c}\sqrt{\frac{c^4}{4} + 1}$, *and* $e^{it} := (\frac{c^2}{2} - i)(\sqrt{\frac{c^4}{4} + 1})^{-1}$, see, e.g., [137].

Using the definition of $S_\pm$ and (51) we obtain

$$e_{++} = y_{1+}, \qquad e_{+-} = y_{2+}, \qquad e_{-+} = y_{1-}, \qquad e_{--} = y_{2-}. \tag{98}$$

The adjoint operator

$$S^* = -ic\frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3$$

is defined on the domain $\mathcal{D}(S^*) = W_2^1(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2$ and an arbitrary J -selfadjoint extension $A_U \in \Sigma_J$ is the restriction of S^* onto $\mathcal{D}(A_U) = \mathcal{D}(S) \dotplus M$, where M is defined by (55) and (56) with $e_{\pm\pm}$ determined by (98).

It is easy to see that the fundamental symmetry $R = (sgn\, x)I$ *in* $L_2(\mathbb{R}) \otimes \mathbb{C}^2$ also commutes with S and $JR = -RJ$. Taking into account Remark (3.2.13) we establish the existence of J -selfadjoint extensions of S with empty resolvent set.

A routine calculation with the use of Corollary (3.2.8) gives that $A_U \in \Sigma_J$ has empty resolvent set if and only if A_U $(= A_\gamma)$ is the restriction of S^* onto the set

$$\mathcal{D}(A_\gamma) = \left\{ y \in W_2^1(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2 \left| \begin{matrix} \Lambda_\gamma\,[y(0+) + y(0-)] = y(0+) - y(0-) \\ y'(0+) + y'(0-) = \Lambda_\gamma\,[y'(0+) - y'(0-)] \end{matrix} \right. \right\}$$

where $\boldsymbol{\Lambda_\gamma} = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}$.

Chapter 4
Functions of Perturbed Normal and Selfadjoint Operators

We also study properties of the operators $f(A) - f(B)$ for $f \in \Lambda_\alpha(\mathbb{R})$ and selfadjoint operators A and B such that $A - B$ belongs to the Schatten–von Neumann class $\boldsymbol{S}p$. We consider the same problem for higher order differences. Similar results also hold for unitary operators and for contractions. We show that if f belongs to the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$,, then it is operator Lipschitz, i.e., $\|f(N_1) - f(N_2)\| \le const \|f\|_{B^1_{\infty,1}} \|N_1 - N_2\|$. We also study properties of $f(N_1) - f(N_2)$ in the case when $f \in \Lambda_\alpha(\mathbb{R}^2)$ and $N_1 - N_2$ belongs to the Schatten–von Neumann class $\boldsymbol{S}_{\boldsymbol{p}}$. In particular, we show that if a function f belongs to the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$, then f is operator Lipschitz and we show that if f satisfies a Hölder condition of order α, then $\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\| \le const max_{1\le j\le n} \|A_j - B_j\|^\alpha$ for all n-tuples of commuting selfadjoint operators $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$. We also consider the case of arbitrary moduli of continuity and the case when the operators $A_j - B_j$ belong to the Schatten–von Neumann class $\boldsymbol{S}_{\boldsymbol{p}}$.

Section(4.1): Functions of Perturbed Operators

It is well known that a Lipschitz function on the real line is not necessarily operator Lipschitz, i.e., the condition,

$$|f(x) - f(y)| \le const\, |x - y|, x, y \in \mathbb{R},$$

does not imply that for selfadjoint operators A and B on Hilbert space,

$$\|f(A) - f(B)\| \le const\, \|A - B\|.$$

The existence of such functions was proved in [82] (see also [90] and [92]). Later in [84] necessary conditions were found for a function f to be operator Lipschitz. Those necessary conditions imply that Lipschitz functions do not have to be operator Lipschitz. It is also well known that a continuously differentiable function does not have to be operator differentiable, see [84] and [85]. Note that the necessary conditions obtained in [84] and [85] are based on the nuclearity criterion for Hankel operators, see [89].

It turns out that the situation dramatically changes if we consider Hölder classes $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$. In this case such functions are necessarily operator Hölder of order α, i.e., the condition:

$$|f(x) - f(y)| const\, |x - y|^\alpha, x, y \in \mathbb{R},$$

implies that for selfadjoint operators A and B on Hilbert space,

$$\|f(A) - f(B)\| \le const\, \|A - B\|^\alpha. \tag{1}$$

Moreover, a similar result holds for the Zygmund class $\Lambda_1(\mathbb{R})$, i.e., the fact that

$$|f(x + t) - 2f(x) + f(x - t)| \le const\, |t|, x, t \in \mathbb{R},$$

and f is continuous implies that f is operator Zygmund, i.e., for selfadjoint operators A and K,

$$\|f(A + K) - 2f(A) + f(A - K)\| \le const\, \|K\|. \tag{2}$$

We also obtain similar results for the whole scale of Hölder–Zygmund classes $\Lambda_\alpha(\mathbb{R})$) for $0 < \alpha < \infty$. Recall that for $\alpha > 1$, the class $\Lambda_\alpha(\mathbb{R})$ consists of continuous functions f such that

$$\left|\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kt)\right| \le const\, |t|^\alpha,\ \ where\ n - 1 \le \alpha \le n.$$

The same problems can be considered for unitary operators and for functions on the unit circle, and for contractions and analytic functions in the unit disk.

To show(1), we use a crucial estimate obtained for trigonometric polynomials and unitary operators in [84] and for entire functions of exponential type and selfadjoint operators in [85]. We state here the result for selfadjoint operators. It can be considered as an analog of Bernstein's inequality.

Let f be an entire function of exponential type σ that is bounded on the real line $\mathbb{R}$. Then for selfadjoint operators A and B with bounded $A - B$ the following inequality holds:

$$\|f(A) - f(B)\| \le const\, \sigma\, \|f\|_{L^\infty(\mathbb{R})} \|A - B\|. \tag{3}$$

Inequality (3) was showed by using double operator integrals and the Birman–Solomyak formula:

$$f(A) - f(B) = \iint \frac{f(x) - f(y)}{x - y} \, dE_A(x)(A - B)dE_B(y),$$

where E_A and E_B are the spectral measures of selfadjoint operators A and B; we refer to [79], [80] and [81] for the theory of double operator integrals. Note that A and B do not have to be bounded, but $A-B$ must be bounded.

To estimate the second difference (2), we use the corresponding analog of Bernstein's inequality which was obtained in [93] with the help of triple operator integrals. To estimate higher order differences, we need multiple operator integrals. We refer the reader to [93] for definitions and basic results on multiple operator integrals.

We also consider in this section the problem of the behavior of functions of operators $f(A)$ under perturbations of A by operators of Schatten–von Neumann class $\boldsymbol{S}_p$ in the case when $f \in \Lambda_\alpha(\mathbb{R})$.

We start with first order differences. We use the notation by $\Lambda_\alpha, 0 < \alpha < \infty$, for the scale of Hölder–Zygmund classes on the unit circle T.

Theorem(4.1.1)[73]. Let $0 < \alpha < 1$. Then there is a constant $c > 0$ such that for every $f \in \Lambda_\alpha$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:

$$\|f(U) - f(V)\| \le c\|f\|^{\Lambda_\alpha} \|U - V\|^\alpha .$$

Theorem(4.1.2)[73]. There exists a constant $c > 0$ such that for every function $f \in \Lambda_1$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:

$$\|f(U) - f(V)\| \le c\|f\|_{\Lambda_1} \left(2 + log_2 \frac{1}{\|U - V\|}\right) \|U - V\| .$$

Note that this result improves an estimate obtained in [82] for Lipschitz functions in the case of bounded selfadjoint operators.

Theorem(4.1.3)[73]. Let n be a positive integer and $0 < \alpha < n$. Then there exists a constant $c > 0$ such that for every $f \in \Lambda_\alpha$ and for an arbitrary unitary operator U and an arbitrary bounded selfadjoint operator A on Hilbert space the following inequality holds:

$$\left|\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(e^{ikA}U)\right| \le c\|f\|_{\Lambda_\alpha} \|A\|^\alpha .$$

Let us consider now a more general problem. Suppose that ω is a modulus of continuity, i.e., ω is a nondecreasing continuous function on $[0,\infty)$ such that $\omega(0) = 0$ and $\omega(x + y) \le \omega(x) + \omega(y), x, \ge 0$. The space Λ_ω consists of functions f on T such that

$$|f(\zeta) - f(\tau)| \le const\, \omega(|\zeta - \tau|), \zeta, \tau \in \mathbb{T}.$$

With a modulus of continuity ω we associate the function ω^* defined by:

$$\omega^*(x) = \int_x^\infty \frac{\omega(t)}{t^2} \, dt, x \ge 0 .$$

Theorem(4.1.4)[73]. Suppose that ω is a modulus of continuity and $f \in \Lambda_\omega$.

If U and V are unitary operators, then

$$\|f(U) - f(V)\| \le const \|f\|_{\Lambda_\omega} \omega^*(\|U - V\|) .$$

In particular, if $\omega^*(x) \le const\, \omega(x)$, then for unitary operators U and V

$$\|f(U) - f(V)\| \le const \|f\|_{\Lambda_\omega} \omega(\|U - V\|) .$$

We have also showed an analog of Theorem (4.1.4) for higher order differences.

We denote here by $(\Lambda_\alpha)_+$ the set of functions $f \in \Lambda_\alpha$, for which the Fourier coefficients $\hat{f}(n)$ vanish for $n < 0$.

Recall that an operator T on Hilbert space is called a contraction if $\|T\| \le 1$. The following result is an analog of Theorem (4.1.3) for contractions.

Theorem(4.1.5)[73]. Let n be a positive integer and $0 < \alpha < n$. Then there exists a constant $c > 0$ such that for every $f \in (\Lambda_\alpha)_+$ and for arbitrary contractions T and R on Hilbert space, the following inequality holds:

$$\left\|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f\left(T+\frac{k}{n}(T-R)\right)\right\| \le c\,\|f\|_{\Lambda_\alpha}\ \|T-R\|^\alpha$$

Note that an analog of Theorem (4.1.4) also holds for contractions.

Theorem(4.1.6)[73]. Let $0<\alpha<1$ and let $f\in\Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A-B$ is bounded. Then $f(A)-f(B)$ is bounded and

$$\|f(A)-f(B)\| \le const\ \|f\|_{\Lambda_\alpha(\mathbb{R})}\ \|A-B\|^\alpha.$$

In this connection we mention the reference [82] where it was showed that for selfadjoint operators A and B with spectra in an interval $[a, b]$ and a function $\varphi\in\Lambda_\alpha(\mathbb{R})$, the following inequality holds:

$$\|\varphi(A)-\varphi(B)\| \le const\ \|\varphi\|_{\Lambda_\alpha(\mathbb{R})}\left(log\left(\frac{b-a}{\|A-B\|}+1\right)+1\right)^2\ \|A-B\|^\alpha$$

(see also [91]).

Theorem(4.1.7)[73]. Suppose that n is a positive integer and $0<\alpha<n$. Let A be a selfadjoint operator and let K be a bounded selfadjoint operator. Then the map,

$$f\mapsto(\Delta_K^n f)(A)\ \underline{\underline{def}}\ \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(A+jK), \tag{4}$$

has a unique extension from $L^\infty\cap\Lambda_\alpha(\mathbb{R})$ to a sequentially continuous operator from $\Lambda_\alpha(\mathbb{R})$ to the space of bounded linear operators on Hilbert space and

$$\|(\Delta_K^n f)(A)\| \le const\ \|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^\alpha.$$

We use the same notation $(\Delta_K^n f)(A)$ for the unique extension of the map (4).

We can also showe an analog of Theorem (4.1.4) for selfadjoint operators.

In this section we consider the behavior of functions of selfadjoint operators under perturbations of Schatten–von Neumann class $\boldsymbol{S_p}$. Similar results also hold for unitary operators and for contractions.

Recall that the spaces $\boldsymbol{S_p}$ and $\boldsymbol{S_{p,\infty}}$ consist of operators T on Hilbert space such that

$$\|T\|_{S_p}\ \underline{\underline{def}}\ \left(\sum_{n\ge 0}s_{\boldsymbol{n}}(T)^p\right)^{\frac{1}{p}}<\infty$$

$$and\ \|T\|_{S_{p,\infty}}\underline{\underline{def}}\ sup_{n\ge 0}\,(1+n)^{\frac{1}{p}}\,s_{\boldsymbol{n}}(T)<\infty.$$

Theorem(4.1.8)[73]. Let $1\le p<\infty, 0<\alpha<1$, and let $f\in\Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A-B\in\boldsymbol{S_p}$. Then

$$f(A)-f(B)\in\boldsymbol{S}_{\frac{p}{\alpha},\infty}$$

$$and\ \|f(A)-f(B)\|_{S_{\frac{p}{\alpha},\infty}} \le\ const\ \|f\|_{\Lambda_\alpha(\mathbb{R})}\ \|A-B\|_{S_p}^{\alpha}.$$

Note that in Theorem (4.1.8). in the case $p>1$ we can replace the condition $A-B\in\boldsymbol{S_p}$ with the condition $A-B\in\boldsymbol{S_{p,\infty}}$.

Using interpolation arguments, we can deduce from Theorem (4.1.8) the following result:

Theorem(4.1.9)[73]. Let $1<p<\infty, 0<\alpha<1$, and let $f\in\Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A-B\in\boldsymbol{S_p}$. Then

$$f(A)-f(B)\in\boldsymbol{S}_{\frac{p}{\alpha},}$$

$$and\ \|f(A)-f(B)\|_{S_{\frac{p}{\alpha}}} \le const\|f\|_{\Lambda_\alpha(\mathbb{R})}\ \|A-B\|_{S_p}^{\alpha}.$$

Theorem(4.1.10)[73]. Suppose that n is a positive integer, α is a positive number such that $n-1\le\alpha<n$, and $n\le p<\infty$.

Let A be a selfadjoint operator and let K be a selfadjoint operator of class $\boldsymbol{S_p}$. Then the operator $(\Delta_K^n f)(A)$ defined in Theorem (4.1.7) belongs to $\mathbf{S}_{\frac{p}{\alpha},\infty}$, and

$$\|(\Delta_K^n f)(A)\|_{S_{\frac{p}{\alpha},\infty}} \le const\, \|f\|_{\Lambda_\alpha(\mathbb{R})}\, \|K\|_{S_p}^{\alpha}.$$

Theorem(4.1.11)[73]. Suppose that n is a positive integer, α is a positive number such that $n - 1 \le \alpha < n, f \in \Lambda_\alpha(\mathbb{R})$, and $n < p < \infty$. Let A be a selfadjoint operator and let K be a selfadjoint operator of class $\boldsymbol{S_p}$. Then the operator $(\Delta_K^n f)(A)$ defined in Theorem (4.1.7) belongs to $\boldsymbol{S}_{\frac{p}{\alpha}}$, and

$$\|(\Delta_K^n f)(A)\|_{S_{\frac{p}{\alpha}}} \le const\, \|f\|_{\Lambda_\alpha(\mathbb{R})}\, \|K\|_{S_p}^{\alpha}.$$

Section (4.2): Perturbed Normal Operators

In this Section we generalize results of the references [85,86,173,74], and [75] to the case of normal operators.

A Lipschitz function f on the real line R (i.e., a function satisfying the inequality $|f(x) - f(y)| \le const\, |x - y|, x, y \in \mathbb{R}$) does not have to be operator Lipschitz, i.e.,

$$\|f(A) - f(B)\| \le const\, \|A - B\|$$

for arbitrary selfadjoint operators $A\ and\ B$ on Hilbert space. The existence of such functions was showed in [82]. Later in [85] and [86] necessary conditions were found for a function f to be operator Lipschitz. In particular, it was shown in [85] that if f is operator Lipschitz, then f belongs locally to the Besov space $B_{11}^1(\mathbb{R})$. This also implies that Lipschitz functions do not have to be operator Lipschitz. Note that in [85] and [86] stronger necessary conditions are also obtained. Note also that the necessary conditions obtained in [85] and [86] are based on the trace class criterion for Hankel operators, see [89].

On the other hand, it was shown in [85] and [86] that if f belongs to the Besov class $B_{\infty 1}^1(\mathbb{R})$, then f is operator Lipschitz. We refer the reader to [84] for information on Besov spaces.

It was shown in [73] and [74] that the situation dramatically changes if we consider Hölder classes $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$. In this case such functions are necessarily operator Hölder of order α, i.e., the condition $|f(x) - f(y)| \le const\, |x - y|^\alpha, x, y \in \mathbb{R}$, implies that for selfadjoint operators $A\ and\ B$ on Hilbert space,

$$\|f(A) - f(B)\| \le const\, \|A - B\|^\alpha.$$

Note that another proof of this result was found in [88].

This result was generalized in [73] and [74] to the case of functions of class $\Lambda_\omega(\mathbb{R})$ for arbitrary moduli of continuity ω. This class consists of functions f on R, for which $|f(x) - f(y)| \le const\omega(|x - y|), x, y \in \mathbb{R}$.

Finally, we mention here that in [75] properties of operators $f(A) - f(B)$ were studied for functions f in $\Lambda_\alpha(\mathbb{R})$ and selfadjoint operators $A\ and\ B$ whose difference $A - B$ belongs to Schatten–von Neumann classes $\mathbf{S}_p$.

We generalize the above results to the case of normal operators. Throughout the section we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$.

Our results are based on the following inequality:

Theorem(4.2.1)[77]. Let f be a bounded function of class $L^\infty(\mathbb{R}^2)$ whose Fourier transform is supported on the disc $\{\zeta \in \mathbb{C}: |\zeta| \le \sigma\}$. Then

$$\|f(N_1) - f(N_2)\| \le const\, \sigma \|N_1 - N_2\|$$

for arbitrary normal operators N_1 and N_2 with bounded difference.

To show Theorem (4.2.1), we obtain a formula for $f(N_1) - f(N_2)$ in terms of double operator integrals. The theory of double operator integrals was developed in [85,87], and [81]. If $E_1\ and\ E_2$ are spectral measures on $\chi_1\ and\ \chi_1$ and Φ is a bounded measurable function on $\chi_1 \times \chi_2$, then the double operator integral

$$\iint_{\chi_1 \times \chi_2} \Phi(s_1, s_2)\, dE_1(s_1) T\, dE_2(s_2)$$

is well defined for all operators T of Hilbert–Schmidt class $\mathbf{S_2}$ and determines an operator of class $\mathbf{S_2}$. For certain functions Φ the transformer $T \mapsto \iint \Phi\, dE_1 T\, dE_2$ maps the trace class $\mathbf{S_1}$ into itself. For such functions Φ one can define by duality double operator integrals for all bounded operators T . Such functions Φ are called Schur multipliers (with respect to the spectral measures E_1 *and* E_2). We refer the reader to [85] for characterizations of Schur multipliers.

In the following theorem E_1 *and* E_2 are the spectral measures of normal operators N_1 *and* N_2. We use the notation

$$\chi_j = Re\, z_j\, , y_j = Im\, z_j\, , A_j = Re\, N_j\, , B_j = Im\, N_j\, , j = 1, 2.$$

Theorem(4.2.2)[77]. Let N_1 *and* N_2 be normal operators such that $N_1 - N_2$ is bounded. Suppose that f is a function in $L^\infty(\mathbb{R}^2)$ such that its Fourier transform $\mathcal{F}$ has compact support. Then the functions

$$(z_1, z_2) \mapsto \frac{f(x_1, y_1) - f(x_1, y_2)}{y_1 - y_2} \; and \; (z_1, z_2) \mapsto \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_2}$$

(are Schur multipliers with respect to E_1 *and* E_2). and

$$f(N_1) - f(N_2) = \iint_{\mathbb{C}^2} \frac{f(x_1, y_1) - f(x_1, y_2)}{y_1 - y_2} dE_1(z_1)(B_1 - B_2)\, dE_2(z_2)$$

$$+ \iint_{\mathbb{C}^2} \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_2} dE_1(z_1)(A_1 - A_2)\, dE_2(z_2). \quad (5)$$

A continuous function f on $\mathbb{R}^2$ is called operator Lipschitz if

$$\|f(N_1) - f(N_2)\| \le const\, \|N_1 - N_2\|$$

for arbitrary normal operators N_1 *and* N_2 whose difference is a bounded operator.

Theorem(4.2.3)[77]. Let f belong to the Besov space $B^1_{\infty 1}(\mathbb{R}^2)$ and let N_1 *and* N_2 be normal operators whose difference is a bounded operator. Then (5) holds and

$$\|f(N_1) - f(N_2)\| \le const\, \|f\|_{B^1_{\infty 1}(\mathbb{R}^2)} \|N_1 - N_2\|\, .$$

In other words, functions in $B^1_{\infty 1}(\mathbb{R}^2)$ must be operator Lipschitz.

As in the case of functions on $\mathbb{R}$, not all Lipschitz functions are operator Lipschitz. In particular, it follows from [85] that if f is an operator Lipschitz function on $\mathbb{R}^2$, then the restriction of f to an arbitrary line belongs locally to the Besov space B^1_{11}.

The next result shows that functions in $B^1_{\infty 1}(\mathbb{R}^2)$ respect trace class perturbations.

Theorem(4.2.4)[77]. Let f belong to the Besov space $B^1_{\infty 1}(\mathbb{R}^2)$ and let N_1 *and* N_2 be normal operators such that $N_1 - N_2 \in \boldsymbol{S_1}$. Then $f(N_1) - f(N_2) \in \boldsymbol{S_1}$ and

$$\|f(N_1) - f(N_2)\|_{\boldsymbol{S_1}} \le const\, \|f\|_{B^1_{\infty 1}(\mathbb{R}^2)} \|N_1 - N_2\|_{\boldsymbol{S_1}}\, .$$

For $\alpha \in (0, 1)$, we consider the class $\Lambda_\alpha(\mathbb{R}^2)$ of Holder functions of order α:

$$\Lambda_\alpha(\mathbb{R}^2) \underline{\underline{def}} \left\{ f : \|f\|_{\Lambda_\alpha(\mathbb{R}^2)} = sup_{z_1 \ne z_2} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} < \infty \right\}.$$

The following result shows that in contrast with the class of Lipschitz functions, a Hölder function of order $\alpha \in (0, 1)$ must be operator Hölder of order α.

Theorem(4.2.5)[77]. There exists a positive number c such that for every $\alpha \in (0, 1)$ and every $f \in \Lambda_\alpha(\mathbb{R}^2)$

$$\|f(N_1) - f(N_2)\| \le c(1 - \alpha)^{-1} \|f\|_{\Lambda_\alpha(\mathbb{R}^2)} \|N_1 - N_2\|^\alpha$$

for arbitrary normal operators N_1 *and* N_2.

Consider now more general classes of functions. Let ω be a modulus of continuity. We define the class $\Lambda_\omega(\mathbb{R}^2)$ by

$$\Lambda_\omega(\mathbb{R}^2) \underline{\underline{def}} \left\{ f : \|f\|_{\Lambda_\omega(\mathbb{R}^2)} = sup_{z_1 \ne z_2} \frac{|f(z_1) - f(z_2)|}{\omega(|z_1 - z_2|)} < \infty \right\}$$

As in the case of functions of one variable (see [73,74]), we define the function ω_* by

$$\omega_*(\chi) \underline{\underline{def}}\, \chi \int_\chi^\infty \frac{\omega(t)}{t^2}\, dt\, , \chi > 0\, .$$

Theorem(4.2.6)[77]. There exists a positive number c such that for every modulus of continuity ω and every $f \in \Lambda_\omega(\mathbb{R}^2)$,

$$\|f(N_1) - f(N_2)\| \leq c\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\omega_*(\|N_1 - N_2\|)$$

for arbitrary normal operators N_1 *and* N_2.

Corollary(4.2.7)[77]. Let ω be a modulus of continuity such that $\omega_*(\chi) \leq const\omega(\chi), \chi > 0$, and let $f \in \Lambda_\omega(\mathbb{R}^2)$. Then

$$\|f(N_1) - f(N_2)\| \leq const\,\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\omega(\|N_1 - N_2\|)$$

for arbitrary normal operators N_1 *and* N_2.

In this section we study properties of $f(N_1) - f(N_2)$ in the case when $f \in \Lambda_\alpha(\mathbb{R}^2), 0 < \alpha < 1$, and N_1 *and* N_2 are normal operators such that $N_1 - N_2$ belongs to the Schatten–von Neumann class $\boldsymbol{S_p}$. The following theorem generalizes (see [75]) to the case of normal operators.

Theorem(4.2.8)[77]. Let $0 < \alpha < 1$ *and* $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$ and for arbitrary normal operators N_1 *and* N_2 with $N_1 - N_2 \in \boldsymbol{S_p}$, the operator $f(N_1) - f(N_2)$ belongs to $\boldsymbol{S_{p/\alpha}}$ and the following inequality holds:

$$\|f(N_1) - f(N_2)\|_{S_{p/\alpha}} \leq c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}\|N_1 - N_2\|_{\boldsymbol{S_p}}^\alpha .$$

For $p = 1$ this is not true even for selfadjoint operators, see [118]. Note that the construction of the counterexample in [75] involves Hankel operators and is based on the criterion of membership of $\boldsymbol{S_p}$ for Hankel operators, see [89].

The following weak version of Theorem (4.2.8) holds:

Theorem(4.2.9)[77]. Let $0 < \alpha < 1$ and let $f \in \Lambda_\alpha(\mathbb{R}^2)$. Suppose that N_1 *and* N_2 are normal operators such that $N_1 - N_2 \in \boldsymbol{S_1}$. Then

$$f(N_1) - f(N_2) \in \boldsymbol{S}_{\frac{\boldsymbol{1}}{\alpha},\infty}, i.e.,$$

$$s_j(f(N_1) - f(N_2)) \leq const\,\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}(1 + j)^{-\alpha}, j \geq 0.$$

Here $s_j(T)$ is the jth singular value of a bounded operator T .

On the other hand, the conclusion of Theorem (4.2.8) remains valid even for $p = 1$ if we impose a slightly stronger assumption on f .

Theorem(4.2.10)[77]. Let $0 < \alpha < 1$ and let f belong to the Besov space $B_{\infty 1}^\alpha(\mathbb{R}^2)$. Suppose that N_1 *and* N_2 are normal operators such that $N_1 - N_2 \in \boldsymbol{S_1}$. Then $f(N_1) - f(N_2) \in \boldsymbol{S}_{\frac{\boldsymbol{1}}{\alpha}}$ and

$$\|f(N_1) - f(N_2)\|_{S_{1/\alpha}} \leq const\|f\|_{B_{\infty 1}^\alpha(\mathbb{R}^2)}\|N_1 - N_2\|_{\boldsymbol{S_1}}^\alpha .$$

We conclude this section with the following improvement of Theorem (4.2.8).

Theorem(4.2.11)[77]. Let $0 < \alpha < 1$ *and* $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$ *every* $l \in \mathbb{Z}_+$, and arbitrary normal operators N_1 *and* N_2 with bounded $N_1 - N_2$, the following inequality holds:

$$\sum_{j=0}^{l}\left(s_j(|f(N_1) - f(N_2)|^{1/\alpha})\right)^p \leq c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}^{p/\alpha}\sum_{j=0}^{l}\left(s_j(N_1 - N_2)\right)^p .$$

Section (4.3): Perturbed Tuples of Selfadjoint Operators

In this section we study the behavior of functions of perturbed tuples of commuting selfadjoint operators. We are going to find sharp estimates for $f(A_1, \dots, A_n) - f(B_1, \dots, B_n)$, where $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-tuples of commuting self-adjoint operators and f is a function on $\mathbb{R}^n$. Our results generalize the results of [85,86,73,74,75,76,77,78] for selfadjoint and normal operators.

Recall that a Lipschitz function f on the real line $\mathbb{R}$ does not have satisfy the inequality

$$\|f(A) - f(B)\| \leq const\|A - B\|$$

for arbitrary selfadjoint operators A and B on Hilbert space, i.e., it does not have to be operator Lipschitz. This was showed in [82]. Later it was shown in [85] and [86] that if f is operator Lipschitz, then f locally belongs to the Besov space $B_{1,1}^1(\mathbb{R})$ (see [84]) which also implies that Lipschitzness is not sufficient for operator Lipschitzness.

On the other hand, it was shown in [85] and [86] that if f belongs to the Besov space $B^1_{\infty,1}(\mathbb{R})$, then f is operator Lipschitz.

The situation changes dramatically if instead of the Lipschitz class, we consider the Hõlder classes $\Lambda_\alpha(\mathbb{R}), 0 < \alpha < 1$, of functions f satisfying the inequality $| f(x) - f(y)| \leq const\,|x - y|^\alpha, x, y \in \mathbb{R}$. It was shown in [73] and [74] that a function f in $\Lambda\alpha(\mathbb{R})$ must be operator Hölder of order α, i.e.

$$\|f(A) - f(B)\| \leq const\|A - B\|^\alpha ,$$

for arbitrary selfadjoint operators A and B. In [73] and [74] also contain sharp estimates of $\|f(A) - f(B)\|$ for functions f of class Λ_ω for arbitrary moduli of continuity ω.

It was also showed in [73] and [75] that if $f \in \Lambda_\alpha, p > 1$, and A and B are selfadjoint operators such that $A - B$ belongs to the Schatten–von Neumann class S_p, then $f(A) - f(B) \in Sp_{/\alpha}$ and

$$\|f(A) - f(B)\|_{Sp_{/\alpha}} \leq const\|A - B\|^\alpha_{S_p}$$

Later in [77] and [78] the above results were generalized to the case of functions of normal operators. Note that the proofs given in [85,86,73,74,75] for selfadjoint operators do not work in the case of normal operators and a new approach was used in [77] and [78].

In this section we consider a more general problem of functions of n-tuples of commuting selfadjoint operators. The case $n = 2$ corresponds to the case of normal operators. It turns out that the techniques used in [78] do not work for $n \geq 3$. We offer in this section a new approach that works for all $n \geq 1$.

We are going to use the technique of double operator integrals developed in [79,80,81]. Double operator integrals are expressions of the form

$$\iint\limits_{\chi_1\times\chi_2} \Phi(s_1, s_2)dE_1(s_1)TdE_2(s_2) \tag{6}$$

where E_1 and E_2 are spectral measures on χ_1 and χ_2, Φ is a bounded measurable function on $\chi_1 \times \chi_2$, and T is an operator on Hilbert space. It was observed in [79,80,81] that the double operator integral (6) is well defined if $T \in s_2$ and determines an operator of class s_2. For certain Φ, the transformer $T \mapsto \iint \Phi dE_1 T\, dE_2$ maps the trace class s_1 into itself. If so, one can define by duality the integral (6) for all bounded operators T . Such functions Φ are called Schur multipliers (with respect to the spectral measures E_1 and E_2). We refer the reader to [85] for characterizations of Schur multipliers.

If χ_1 and χ_2 are Borel subsets of Euclidean spaces, we use the notation $\mathfrak{M}_{\chi_1, \chi_2}$ for the space of Borel functions Φ on $\chi_1 \times \chi_2$ that are Schur multipliers for all Borel spectral measures E_1 *and* E_2 on χ_1 and χ_2.

The proofs of the results of [78] for normal operators were based on the following formula:

$$f(N_1) - f(N_2) = \iint (\mathfrak{D}_\lambda f)(z_1, z_2)dE_1(z_1)(B_1, B_2)dE_2(z_2)$$
$$+ \iint (\mathfrak{D}_x f)(z_1, z_2)dE_1(z_1)(A_1, A_2)dE_2(z_2)$$

Here N_1 and N_2 are normal operators with bounded difference $N_1 - N_2$, $A_j = Re\, N_j , B_j = Im\, N_j , x_j = Re\, z_j , y_j = Im z_j$, f is a bounded function on $\mathbb{R}^2$ whose Fourier transform has compact support,

$$(\mathfrak{D}_x f)(z_1, z_2) = \frac{f(\chi_1, y_2) - f(\chi_2, y_2)}{\chi_1 - \chi_2}, \qquad and\ (\mathfrak{D}_y f)(z_1, z_2) = \frac{f(\chi_1, y_1) - f(\chi_1, y_2)}{y_1 - y_2},$$
$$z_1, z_2 \epsilon\ \mathbb{C}$$

It was shown in [78] that $\mathfrak{D}_x f$ and $\mathfrak{D}_y f$ belong to the space of Schur multipliers $\mathfrak{M}_{\mathbb{R}^2\times\mathbb{R}^2}$.

However, in the case $n \geq 3$ the situation is more complicated. Let (A_1, A_2, A_3) and (B_1, B_2, B_3)

be triples of commuting selfadjoint operators. Suppose that f is a bounded function on $\mathbb{R}^3$ whose Fourier transform has compact support. It can be shown that

$$f(A_1,A_2,A_3)-f(B_1,B_2,B_3)=\iint(\mathfrak{D}_1 f)(x,y)dE_1(x)(A_1-B_1)dE_2(y)$$
$$+\iint(\mathfrak{D}_2 f)(x,y)dE_1(x)(A_2-B_2)dE_2(y)$$
$$+\iint(\mathfrak{D}_3 f)(x,y)dE_1(x)(A_3-B_3)dE_2(y)\ ,$$

whenever the functions $\mathfrak{D}_1 f$, $\mathfrak{D}_2 f$, and $\mathfrak{D}_3 f$ belong to the space of Schur multipliers $\mathfrak{M}_{\mathbb{R}^3\times\mathbb{R}^3}$. Here

$$(\mathfrak{D}_1 f)(x,y)=\frac{f(x_1,x_2,x_3)-f(y_1,x_2,x_3)}{x_1-y_1}\ ,$$
$$(\mathfrak{D}_2 f)(x,y)=\frac{f(y_1,x_2,x_3)-f(y_1,y_2,x_3)}{x_2-y_2}\ ,$$
$$(\mathfrak{D}_3 f)(x,y)=\frac{f(y_1,y_2,x_3)-f(y_1,y_2,y_3)}{x_3-y_3}\ ,$$
$$x=(x_1,x_2,x_3)\ ,y=(y_1,y_2,y_3)$$

The methods of [78] allow us to show that $\mathfrak{D}_1 f$ and $\mathfrak{D}_3 f$ do belong to the space of Schur multipliers $\mathfrak{M}_{\mathbb{R}^3\times\mathbb{R}^3}$. However, as the next result shows, the function $\mathfrak{D}_2 f$ does not have to be in $\mathfrak{M}_{\mathbb{R}^3\times\mathbb{R}^3}$.

Theorem(4.3.1)[72]. Suppose that g is a bounded function on $\mathbb{R}$ such that the Fourier transform of g has compact support and is not a measure. Let f be the function on $\mathbb{R}^3$ defined by

$$f(x_1,x_2,x_3)=g(x_1-x_3).$$

Then $\mathfrak{D}_2 f\notin\mathfrak{M}_{\mathbb{R}^3\times\mathbb{R}^3}$.

Note that it is easy to construct such a function $g, e.g.$,

$$g(x)=\int_0^x t^{-1}sint\,dt$$

.

We show that in the case $n\ \geq\ 3$ it is possible to represent $f\ (A_1,\dots,A_n)\ -\ f\ (B_1,\dots,B_n)$in terms of double operator integrals in a different way. Using such a representation, we obtain analogs of the above results in the case of n-tuples of commuting selfadjoint operators.

The integral representation for $f\ (A_1,\dots,A_n)-\ f\ (B_1,\dots,B_n)$ is based on the following result:

We are going to derive Schur multiplier estimates from the following lemma.

Lemma (4.3.2)[72]. Let $\mathrm{C}\ =\mathrm{Q}\times\mathfrak{R}$ be a cube in $\mathbb{R}^{2n}$ of sidelength L and let Ψ be a C^∞ function on $\frac{3}{2}\mathrm{C}$. Then $\Psi|\mathrm{C}\ \in\ \mathfrak{M}_{\mathrm{Q},\mathcal{R}}$ and

$$\|\Psi\|_{\ \mathfrak{M}_{Q,\mathcal{R}}}\leq const\ max\left[L^{|\alpha|}max_{a\in\frac{3}{2}C}|(D^\alpha\Psi)(a)|:|\alpha|\leq 2n+2\right].$$

The lemma can be showed by expanding Ψ in the Fourier series.

Theorem(4.3.3)[72]. Let $\sigma\ >\ 0$ and let f be a function in $\mathrm{L}^\infty(\mathbb{R}^n)$ whose Fourier transform is supported on $\{\xi\ \in\ \mathbb{R}^n\colon \|\xi\|\ \leq\sigma\}$. Then there exist functions Ψ_j in $\mathfrak{M}_{\mathbb{R}^n\times\mathbb{R}^n}$, $1\ \leq\ j\ \leq n$, such that

$$f(x_1,\dots,x_n)-f(y_1,\dots,y_n)=\sum_{j=1}^{n}(x_j-\ y_j)\Psi_j\ (x_1,\dots,x_n,y_1,\dots,y_n),x_j,y_j\in\mathbb{R}\ , \qquad (7)$$
$$and\ \|\Psi_j\|_{\ \mathfrak{M}_{\mathbb{R}^n\times\mathbb{R}^n}}\leq const\ \sigma\|f\|_{L^\infty(\mathbb{R}^n)}\ .$$

proof. By rescaling, we may assume that $\|f\|_{L^\infty}\leq 1\ and\ \sigma=\ 1.$

We consider the lattice of dyadic cubes in $\mathbb{R}^{2n}=\mathbb{R}^n\times\ \mathbb{R}^n$, i.e., the cubes whose sides are intervals of the form $[\,j^{2k},(j\ +1)^{2k}),j,k\ \in\mathbb{Z}$. We say that a dyadic cube C in $\mathbb{R}^{2n}=\mathbb{R}^n\times\ \mathbb{R}^n$, is admissible if either its sidelength $L(C)$ is equal to 1 or $L(C)\ >\ 1$ and the interior of the cube $2C$, i.e., the cube centered at the center of C with sidelength $2L(C)$, does not intersect the diagonal $\{(x,x)\colon x\ \in\ \mathbb{R}^n\}$. An admissible cube is called maximal if it is not a proper subset of another admissible cube. It is easy to see that the maximal admissible cubes are disjoint and cover $\mathbb{R}^{2n}$. It can also easily be verified that if Q is a dyadic cube in $\mathbb{R}^n$, then there can be at most 6^n dyadic cubes $\mathcal{R}$ in $\mathbb{R}^n$ such that $Q \times \mathcal{R}$ is a maximal admissible cube.

For $l = 2m$, we denote by Dl the set of maximal dyadic cube of sidelength l .

It follows that if Ω is a function on $\mathbb{R}^n \times \mathbb{R}^n$ that is supported on $\cup_{C\in\mathcal{D}_1} C$, then

$$\|\Omega\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}} \le 6^n sup_{C\in\mathcal{D}_1}\|\chi_C\Omega\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}}.$$

We have to define Ψ_j on each maximal admissible cube. Suppose that $C \in \mathcal{D}_1$. We put

$$\Psi_j(x,y) = \int_0^1 (\mathfrak{D}_j f)((1-t)x + ty)dt \ ,(x,y) \ \epsilon\ C = Q \times \mathcal{R} \ ,$$

where $\mathfrak{D}_j f$ is the jth partial derivative of f . It follows from Lemma (4.3.2) that $\|\Psi_j\|_{\mathfrak{M}_{Q,\mathcal{R}}} \le$ const. Suppose now that $l = 2^m > 1$ and $C = Q \times \mathcal{R} \in \mathcal{D}_l$. Let ω be a C^∞ nonnegative even function on $\mathcal{R}$ such that $\omega(t) = 0$ for $t \in [-\frac{1}{2},\frac{1}{2}]$, and $\omega(t) = 1$ for $t \notin [-1,1]$. We put $\Phi_j(x,y) = \omega\left(\frac{x_j - y_j}{l}\right)$, $\Phi = \sum_{j=1}^n \Phi_j$, and define the functions Ξ_j , $1 \le j \le n$, by

$$\Xi_j(x,y) = \begin{cases} \frac{1}{x_j - y_j}\cdot\frac{\Phi_j(x,y)}{\Phi(x,y)}, & x_j \ne y_j \\ 0 , & x_j = y_j \end{cases} .$$

It follows easily from Lemma (4.3.3) that $\|\Xi_j\|_{\mathfrak{M}_{Q,\mathcal{R}}} \le const\ 2^{-m}$. We put now

$$\psi_j(x,y) = (f(x) - f(y))\Xi_j(x,y) \ ,(x,y) \ \epsilon\ C .$$

Clearly, (7) holds for $(x,y) \in C$ and $\|\psi_j\|_{\mathfrak{M}_{Q,\mathcal{R}}} \le const\ 2^{-m}$. The functions ψ_j are now defined on $\mathbb{R}^n \times \mathbb{R}^n$ and $\|\psi_j\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}} \le const\ \sum_{m\ge0} 2^{-m}$. This implies the result.

Theorem(4.3.4)[72]. Let f be a function satisfying the hypotheses of Theorem (4.3.3) and let ψ_j, $1 \le j \le n$, be functions in $\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}$ satisfying (7). Suppose that $(A_1, A_2, \dots, A_n)$ and $(B_1, B_2, \dots, B_n)$are n-tuples of commuting selfadjoint operators such that the operators $A_j - B_j$ are bounded, $1 \le j \le n$. *Then*

$$f(A_1,, \dots, A_n) - f(B_1, \dots, B_n) = \sum_{j=1}^n \iint_{\mathbb{R}^n\times\mathbb{R}^n} \psi_j(x,y)dE_A(x)(A_j - B_j)dE_B(y)$$

and

$$\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\| \le const\ \sigma\|f\|_{L^\infty(\mathbb{R}^n)} max_{1\le j\le n}\|A_j - B_j\|$$

In this section we obtain operator norm estimates for $(A_1, \dots, A_n) - f(B_1, \dots, B_n)$, where $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-tuples of commuting selfadjoint operators.

A function f on $\mathbb{R}^n$ is called operator Lipschitz if $\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\| \le const\ max_{1\le j\le n}\|A_j - B_j\|$ for all n-tuples of commuting selfadjoint operators $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$.

The following theorem can be deduced easily from Theorem (4.3.4):

Theorem(4.3.5)[72]. Let f be a function in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$. Then f is operator Lipschitz.

For $\alpha \in (0,1)$, we define the Hölder class $\Lambda_\alpha(\mathbb{R}^n)$ of functions f on $\mathbb{R}^n$ such that

$$|f\ (x) - f\ (y)| \le const\|\ x - y\|^\alpha_{\mathbb{R}^n} \ ,x,y \in \mathbb{R}^n.$$

For a modulus of continuity ω, the space $\Lambda_\omega(\mathbb{R}^n)$ consists of functions f on $\mathbb{R}^n$ such that

$$|f\ (x) - f\ (y)| \le const\ \omega(\|x - y\|_{\mathbb{R}^n}) \ ,x,y \in \mathbb{R}^n$$

The following results are analogs of the corresponding results of [73] and [74] in the case $n = 1$. The proofs of Theorems (4.3.6) and (4.3.7) are based on Theorem (4.3.4) and use the same methods as in [74].

Theorem(4.3.6)[72]. Let $\alpha \in (0,1)$ and let $f \in \Lambda_\alpha(\mathbb{R}^n)$. Then f is operator Hölder of order α, i.e., $\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\| \le const\ max_{1\le j\le n}\|A_j - B_j\|^\alpha$ for all n-tuples of commuting selfadjoint operators $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$.

Theorem(4.3.7)[72]. Let ω be a modulus of continuity and let $f \in \Lambda_\omega(\mathbb{R}^n)$Then $\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\| \le const\omega_* \left(max_{1\le j\le n}\|A_j - B_j\| \right)$ for all n-tuples of commuting selfadjoint operators $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$, where

$$\omega * (\delta) \underline{\underline{def}} \ \delta \int_\delta^\infty \frac{\omega(t)}{t^2}\, dt \ \ , \delta > 0.$$

In this section we obtain estimates in $\boldsymbol{S_p}$ norms.

Theorem(4.3.8)[72]. Let f be a function in the Besov space $\mathrm{B}^1_{\infty,1}(\mathbb{R}^n)$.. Suppose that $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-tuples of commuting selfadjoint operators such that $A_j - B_j \in \boldsymbol{S_1}$. Then $f(A_1, \dots, A_n) - f(B_1, \dots, B_n) \in \boldsymbol{S_1}$ *and*

$$\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\|_{\boldsymbol{S_1}} \le const\|f\|_{B^1_{\infty,1}(\mathbb{R}^n)} \ max_{1\le j\le n}\|A_j - B_j\|_{\boldsymbol{S_1}} .$$

Theorem(4.3.9)[72]. Let $f \in \Lambda_\alpha(\mathbb{R}^n)$. and let $p > 1$. Suppose that $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-tuples of commuting selfadjoint operators such that $A_j - B_j \in \boldsymbol{S_p}$. Then $f(A_1, \dots, A_n) - f(B_1, \dots, B_n) \in \boldsymbol{S}_{p/\alpha}$ and

$$\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\|_{\boldsymbol{S}_{p/\alpha}} \le const\|f\|_{\Lambda_\alpha(\mathbb{R}^n)} max_{1\le j\le n}\|A_j - B_j\|^\alpha_{\boldsymbol{S_p}}.$$

Note that the conclusion of Theorem (4.3.9) does not hold in the case $p = 1$ even if $n = 1$, see [75].

Theorem(4.3.10)[72]. *Let* f be a function in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$. Suppose that $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-tuples of commuting selfadjoint operators such that $A_j - B_j \in \boldsymbol{S_1}$. Then $f(A_1, \dots, A_n) - f(B_1, \dots, B_n) \in \boldsymbol{S}_{1/\alpha}$ and

$$\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\|_{\boldsymbol{S}_{1/\alpha}} \le const\|f\|_{B^1_{\infty,1}(\mathbb{R}^n)} max_{1\le j\le n}\|A_j - B_j\|^\alpha_{\boldsymbol{S_1}}$$

The proofs of the above theorems are based on Theorem (4.3.4) and use the methods of [75].

In[75] more general results for other operator ideals were obtained in the case $n = 1$. Those results can also be generalized to the case of arbitrary $n \ge 1$. (see [83]) .

Chapter 5

Operators and Functions in Krein Spaces

We study selfadjoint operators in Krein space. Our goal is to show that there is a relationship between the following classes of operators: operators with a compact "corner," definitizable operators, operators of classes (**H**) and **K**(**H**) , and operators of class D_{κ}^{+} . Also, each J-frame induces an indefinite reconstruction formula for the vectors in $\mathcal{H}$, which resembles the one given by a J-orthonormal basis.

Section (5.1): Krein Space and Operators

Let κ be a linear space equipped with an indefinite metric (or, which is the same, a sesquilinear form) $[.,.]$. We assume that κ can be decomposed into the direct orthogonal sum

$$\kappa = \kappa^{+} \dotplus \kappa^{-}, \tag{1}$$

of a positive subspace κ^{+} and a negative subspace κ^{-}. If $\{\kappa^{+}, [.,.]\}$ is a Hilbert space and $\{\kappa^{-}[.,.]\}$ is an anti-Hilbert space (the latter means that $\{\kappa^{-}, -[.,.]\}$ is a Hilbert space), then κ is called a Krein space.

By P^{+} and P^{-} we denote the projection operators on κ^{+} and κ^{-}corresponding to the decomposition (1) and introduce the operator $J = P^{+} - P^{-}$. Then κ is a Hilbert space with respect to the inner product $(.,.) = [J.,.]$. This implies that $[.,.] = (J.,.)$. The Krein space with the inner product thus introduced is called a J-space, and the indefinite metric is called a J-metric. We note that the decomposition (1) is orthogonal both with respect to the J-metric and with respect to the Hilbert inner product.

A subspace $L \subset \kappa$is said to be regular if

$$\kappa = \mathcal{L}[\dotplus]\mathcal{L}^{[\perp]}, \tag{2}$$

where $\mathcal{L}^{[\perp]}$ denotes the J-orthogonal complement of $\mathcal{L}$. In contrast to the Hilbert case, for spaces with indefinite metric, the subspace $\mathcal{L}_0 = \mathcal{L} \cap \mathcal{L}^{[\perp]}$, which is said to be isotropic, is not necessarily trivial, but even if it is trivial, then it may be possible that relation (2) does not hold. We note that $\mathcal{L}$ is regular if and only if it is a Krein space. In particular, the Hilbert and anti-Hilbert subspaces are regular. The latter are also said to be uniformly positive and uniformly negative subspaces, respectively.

It is said that a subspace $\mathcal{L}_{+}$ ($\mathcal{L}_{-}$) belongs to the class $h^{+}(h^{-})$ if $\mathcal{L}_{+}^{0}(\mathcal{L}_{-}^{0})$ is finite-dimensional and the quotient space$\hat{\mathcal{L}}_{+} = \mathcal{L}_{+}/\mathcal{L}_{+}^{0}$, ($\hat{\mathcal{L}}_{-} = \mathcal{L}_{-}/\mathcal{L}_{-}^{0}$) is a Hilbert (anti-Hilbert) space with respect to the induced indefinite metric. Or, equivalently, $\mathcal{L}_{+} \in h^{+}$ ($\mathcal{L}_{-} \in h^{-}$) if it can be decomposed into the sum of a finite-dimensional isotropic subspace and a uniformly positive (uniformly negative) subspace.

We denote the set of maximal nonnegative subspaces and the set of maximal nonpositive subspaces by $\mathfrak{M}^{+}$ and $\mathfrak{M}^{-}$, respectively.

The Krein space with $\kappa = min\{\dim \kappa^{+}, dim\ \kappa^{-}\} < \infty$ is called a Pontryagin space with κ positive or negative squares depending on whether $= dim\ \kappa^{+}$ or $\kappa = \dim \kappa^{-}$, respectively.

One of the main problems of the theory of operators in Krein spaces is the problem of existence of maximal nonnegative and maximal nonpositive invariant subspaces for J-selfadjoint and, which is equivalent, J-unitary operators. Or, more generally, this is the problem of extending a given nonnegative or nonpositive invariant subspace to an invariant subspace which is maximal in its class. Finally, the same problems are posed for operator families. These problems have been solved to a sufficient extent in the case of a Pontryagin space. In the case of a general Krein space, none of these problems has been solved, but several sufficient conditions under which these problems have solutions have been obtained (see [52,63,64], etc.); we refer to the same sources for details concerning the geometry and the theory of operators in spaces with indefinite metric.

We distinguish the following four main classes of operators:

(i) operators with a completely continuous "corner";

(ii) definitizable operators;

(iii) operators of class $K(H)$;

(iv) operators of class D_κ^+ .

Information on classes (i) and (ii) can be found, for example, in [52,63,64]; about the class $K(H)$ and the subclass H, see [64] and [65]; and about operators of class D_κ^+, see [66,67]. In what follows, we assume, unless otherwise specified, that all operators are bounded and defined on the entire space.

Definition (5.1.1)[62]. Suppose that κ is a Krein space, T is a continuous operator, and

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{3}$$

is its matrix representation with respect to the decomposition (1). We shall say that a continuous operator T has a completely continuous corner and write $T \in \mathfrak{S}_{\infty,12}$ if there exists a decomposition of the form (1) such that T_{12} is a completely continuous operator ($T_{12} \in \mathfrak{S}_\infty$).

Definition (5.1.2)[62]. A selfadjoint operator A acting in the Krein space κ is said to be definitizable if there exists a polynomial p such that $p(A)$ is a nonnegative operator, i.e., if $[p(A)x, x] \geq 0$ for all $x \in \kappa$. We denote the set of definitizable operators by the symbol D.

Definition (5.1.3)[62] We shall say that an operator T belongs to the class H, and write $T \in H$, if there exist invariant subspaces $\mathcal{L}^\pm \in \mathfrak{M}^\pm$ for this operator and all such subspaces, respectively, belong to the class $h^\pm$.

Definition (5.1.4)[62]. A set $\mathfrak{U} = \{A\}$ of selfadjoint linear operators acting in a Krein space belongs to the class $K(H)$, $\mathfrak{U} \in K(H)$, if in this space there exists a selfadjoint operator $B \in H$ such that $BA = AB$ for all $A \in \mathfrak{U}$. If $\mathfrak{U}$ consists of a single operator A, then we shall say that A belongs to the class $K(H)$, and we shall write $A \in K(H)$.

Definition (5.1.5)[62]. We shall say that a family $\mathfrak{U} = \{A\}$ of selfadjoint linear operators acting in the Krein space belongs to the class D_κ^+ , $\mathfrak{U} \in D_\kappa^+$, if, for this family, there exists a pair of invariant subspaces $\mathcal{L}^\pm \in \mathfrak{M}^\pm \cap h^\pm$ and $\mathcal{L}^+[\perp]\mathcal{L}^-$ whose dimension is that of the subspace $\mathcal{L}^+ \cap \mathcal{L}^-$ equal to $\kappa, \kappa < \infty$.

Let us find the relationship and distinctions between these operator classes. First, we note that it follows from [107] that $K(H) \subset \mathrm{D}_\kappa^+$. We shall show that a finite family of selfadjoint operators belongs to D_κ^+ if and only if this family belongs to the class $K(H)$.

Let $\mathfrak{U} \in D_\kappa^+$ be a commutative family of selfadjoint operators in a Krein space. It follows from [64] that there exists a decomposition of the form (1) such that all the operators from A have completely continuous corners. In the next section, we present several examples illustrating the difference between these operator classes.

In what follows, in Examples (5.1.6) and (5.1.7), we assume that

$$H = \mathcal{G} \oplus \mathcal{G} \tag{4}$$

is the orthogonal sum of two copies of an infinite-dimensional Hilbert space $\mathcal{G}$ and the J-metric is introduced in $\mathcal{H}$ by using the operator

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{5}$$

Example (5.1.6)[62]. ($\mathfrak{S}_{\infty,12} \not\subset (D \cup K(H))$). Let $A: \mathcal{G} \to \mathcal{G}$ be a nonzero Volterra operator with $\lambda = 0$ in the continuous spectrum. Then the diagonal-with-respect-to-(4) operator $\tilde{A} = diag\{A, A^*\}$ is a Volterra J-selfadjoint operator, $\tilde{A} \notin K(H)$, and $\tilde{A}$ is not definitizable. Indeed, if we had $\tilde{A} \in K(H)$, then $\tilde{A}$ would have a basis system of eigenvectors in the sense of Riesz (see [64]), and if it were definitizable, then the system of its eigenvectors would be complete (see [64]). Either of these properties contradicts the fact that $\tilde{A}$is a Volterra operator.

Example(5.1.7)[62]. ($D \not\subset (K(H) \cup \mathfrak{S}_{\infty,12})$). Now, let $A: \mathcal{G} \to \mathcal{G}$ be a positive completely continuous operator. Then

$$\tilde{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \tag{6}$$

is a J-positive operator with a complete system of eigenvectors (see [64]), but in $\mathcal{H}$ there is no basis composed of these vectors: otherwise, we would have $\tilde{A} \in \mathfrak{S}_\infty$. Hence this operator does not belong to the class (H) . Moreover, (6) gives an example of a definitizable (nonnegative) J-selfadjoint operator for which there does not exist a decomposition of the form (1) such that, with respect to this decomposition, the operator $\tilde{A}$ would have a representation with a completely continuous corner, i.e., $\tilde{A} \notin \mathfrak{S}_{\infty,12}$. The latter follows from the relation $\tilde{A} = A_1 + A_2$, where

$$A_1 = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$

$A_1 \in \mathfrak{S}_\infty$, A_2 is a J-nonnegative operator, and $A_2^2 = 0$. If there were a decomposition of the form (1) such that A had a completely continuous corner, then A_2 would also have a completely continuous corner with respect to the same decomposition. In this case, we see that $A_2^2 = 0$ and the operator A_2 is completely continuous, which contradicts the fact that $\mathcal{G}$ is infinite-dimensional.

Example(5.1.8)[62] $(H \not\subset D)$. The desired example of $\tilde{A}$ can be found rather easily: one can consider a block-operator matrix $\tilde{A} = \{A_1, A_2\}$ diagonal with respect to (1), with completely continuous selfadjoint cyclic operators A_1 and A_2 , whose eigenvalues are nonzero and alternate. However, in this case, $\mathcal{K}$ can be easily represented as a direct orthogonal sum of two $\tilde{A}$-invariant Krein subspaces such that the restrictions of operator $\tilde{A}$ to these subspaces are definitizable operators; moreover, these subspaces can be chosen as Pontryagin spaces. Thus, studying $\tilde{A}$ is reduced to studying definitizable operators or operators in Pontryagin spaces, respectively.

We modify this example and present an operator $\tilde{A}$ for which it is impossible to perform a similar reduction to definitizable operators or, in particular, to operators in a Pontryagin space.

Let $\varepsilon = LS\{e\}$ be a one-dimensional space, and let $\mathcal{G}$ be a separable infinite-dimensional space (LS means the linear span). We form the Krein space $\mathcal{K}$ as the following J-space:

$$\mathcal{K} = \varepsilon \oplus \mathcal{G} \oplus \mathcal{G} \oplus \varepsilon \tag{7}$$

with the operator

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Let $\{f_n\}$ *and* $\{g_n\}$ be orthonormal bases in $\mathcal{G}$. We set

$$f = \sum_{n=1}^{\infty} \frac{1}{2n} f_n \ \ and \ \ g = \sum_{n=1}^{\infty} \frac{1}{2n-1} g_n.$$

The operators

$$A_{22} = \sum_{n=1}^{\infty} \frac{1}{2n} (\cdot, f_n) f_n \ and \ A_{33} = \sum_{n=1}^{\infty} \frac{1}{2n-1} (\cdot, g_n) g_n$$

are completely continuous and selfadjoint, $f \notin ran\ A_{22}$, and $g \notin ran\ A_{33}$. We define the operator $\tilde{A}$ as a block-operator 4 × 4 matrix with respect to (7):

$$\tilde{A} = \begin{bmatrix} 0 & A_{12} & A_{13} & 0 \\ 0 & A_{22} & 0 & A_{12}^* \\ 0 & 0 & A_{33} & -A_{13}^* \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

where $A_{12} = (.,f)e \quad and\, A_{13} = (.,g)e$. The operator $\tilde{A}$ thus defined is completely continuous and J-selfadjoint with respect to J in (8), belongs to the class H, and cannot be decomposed in a finite sum of definitizable operators.

Example(5.1.9)[62]. $(D_\kappa^+ \neq K(H)). Let\, \{e_k\}_{k=-\infty}^{\infty} \cup \{g_0\}$ be an orthonormal basis in the Hilbert space$\mathcal{K}$. We set

$$Je_k = e_k, Je_{-k} = -e_k, \qquad k = 1,2,\ldots, Je_0 = g_0, \qquad Jg_0 = e_0 ,$$
$$A_k x = e_k[x, e_0] + e_0[x, e_k], \qquad where\, x \in \mathcal{H}, \qquad \pm k = 1,2,\ldots.$$

The operator family $\{A_k\}_{\pm k=1}^{\infty}$ belongs to the class D_1^+ , but does not belong to the class $K(H)$. Indeed, all operators from $\{A_k\}_{\pm k=1}^{\infty}$ are J-selfadjoint, the subspace $\mathcal{L}_+ = CLS\{e_k\}_{k=0}^{\infty}$ (from now on, CLS means the closed linear span) is invariant with respect to $\{A_k\}_{\pm k=1}^{\infty}$, and $\mathcal{L}_+ \in \mathfrak{M}^+ \cap h^+$.

Moreover, $dim(\mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]}) = 1$ so that $\{A_k\}_{\pm k=1}^{\infty} \in D_1^+$. Next, the linear hull spanned by the vector e_0 is the common kernel of the family $\{A_k\}_{\pm k=1}^{\infty}$. We assume that a J-selfadjoint operator B commutes with $\{A_k\}_{\pm k=1}^{\infty}$. Then the vector e_0 is an eigenvector for B. Without loss of generality, we can assume that $Be_0 = 0$. In this case, we have $Bg_0 \in CLS\{e_k\}_{k=-\infty}^{\infty}$.

Let us find $e_k\, for\, \pm k = 1,2,\ldots$: we obtain $Be_k = BA_k g_0 = A_k B g_0 = \beta_k e_0$. This implies the following representation of the operator : $Bx = [x,z]e_0 + [x,e_0]z + \xi[x,e_0]e_0$, where $z = \sum_{k=-\infty,k\_=0}^{\infty} \gamma_k e_k,$. But the rank of indefiniteness of the kernel of this operator is infinite, and hence we have $B \notin H$.

In what follows, we show that, for a finite family of commuting operators, the fact that this family belongs to the class D_κ^+ implies that it belongs to (H) .

Theorem(5.1.10)[62]. Let $\mathfrak{U} = \{A_j\}_j^m = 1$ be a finite family of bounded pairwise commuting J-selfadjoint operators in a Krein space. Then the conditions $\mathfrak{U} \in D_\kappa^+ \, and\, \mathfrak{U} \in K(H)$ are equivalent.

Proof. The statement $\mathfrak{U} \in K(H) \Rightarrow \mathfrak{U} \in D_\kappa^+$ follows from [64]. Let us prove that $\in D_\kappa^+ \Rightarrow \mathfrak{U} \in K(H)$. We show that there exists a bounded J-selfadjoint operator X of class H commuting with each of the operators $A_j, j = 1,\ldots,k$. By the definition of the class D_κ^+ , for the family $\mathfrak{U}$, there exist invariant subspaces $\mathcal{L}^\pm \in \mathfrak{M}^\pm$ such that $\mathcal{L}^+[\perp]\mathcal{L}^-$ and

$$\mathcal{L}^\pm = \mathcal{L}^0 \oplus \mathcal{L}_\pm, \tag{10}$$

where $\mathcal{L}^0$ is a finite-dimensional neutral subspace and $\mathcal{L}_\pm$are uniformly definite subspaces. Without loss of generality, we assume that $\mathcal{L}_\pm \subset \mathcal{K}^\pm$, where $\mathcal{K}^\pm$ are components of (1). By $\mathcal{K}$ we denote the angular operator of the subspace $\mathcal{L}^+, i.e., we\, have\, \mathcal{K}: \mathcal{K}^+ \to \mathcal{K}^-$, $\|\mathcal{K}\| \le 1$, and $\mathcal{L}^+ = \{x = x^+ + Kx^+ \mid x+ \in \mathcal{K}^+\}$. $Let\, A_j = \|A_{jik}\|_{\mathrm{i,k=1}}^2$ be the matrix representation of the operators $A_j, j = 1,\ldots,m$, with respect to (1). We shall seek the operator X in the form of a matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{11}$$

whose components satisfy the following conditions: $X_{11} = X_{11}^*, X_{21} = KX_{11}, X_{12} = -X_{21}^* = -X_{11}K^*,\ and\, X_{22} = X_{22}^* = -KX_{11}K^*$. Since we have $\mathcal{L}_\pm \subset \mathcal{K}^\pm$, the operator K is finite-dimensional and partially isometric. Hence it follows from representation (11) that $X \in H$ if and only if $dim\, kerX_{11} < \infty$. The A_j -invariantness of $\mathcal{L}^+$ is equivalent to the condition

$$KA_{j11} + KA_{j12}K - A_{j21} - A_{j22}K = 0, \qquad j = 1,\ldots,m. \tag{12}$$

It follows from (12) that X commutes with A_j if and only if $(A_{j11} + AA_{j12}K)X_{11}$ is a selfadjoint operator in $\mathcal{K}^+$, which, in turn, is equivalent to the selfadjointness of the operator

$$[(P^+ \mid \mathcal{L}^+)^{-1}(A_{j11} + A_{j12}K)(P^+ \mid \mathcal{L}^+)][(P^+ \mid \mathcal{L}^+)^{-1}X_{11}(P^+ \mid \mathcal{L}^+)^{-1*}]. \tag{13}$$

Since, by assumption, $\mathcal{L}^+$ is invariant with respect to A_j and, by construction, invariant with respect to

X, it follows from (13) and the relations

$$A_j \mid \mathcal{L}^+ = (P^+ \mid \mathcal{L}^+) - 1(A_{j11} + A_{j12}K)(P^+ \mid \mathcal{L}^+)$$

that the fact that A_j and X commute is equivalent to the existence of a selfadjoint operator $Y : \mathcal{L}^+ \to \mathcal{L}^+$ with $dim\, kerY < \infty$ such that

$Z_j Y = Y Z_j^*$, $where\ Z_j = A_j |\mathcal{L}^+ , j = 1,\dots,m, (Y = (P^+ \mid \mathcal{L}^+)^{-1} X_{11} (P^+ \mid \mathcal{L}^+)^{-1*})$.

Let $Z_j = \|Z_{jik}\|_{i,k=1}^2$ be the matrix representation of the operator Z_j with respect to the decomposition (10), $j = 1,\dots,m$. Then $Z_{j21} = 0$ and Z_{j22} is a selfadjoint operator. Suppose that $\sigma(Z_{j11}) = \{\lambda_{js}\}_{j\ s=1}^{s}$ is the spectrum of the operator Z_{j11} , P_0 is the orthoprojection from $\mathcal{L}_+\ on\ \mathcal{H}_0 = \cap \{LS\{ker(Z_{j22} - \lambda_{js})\}_{s=1}^{s}\}_{j=1}^{m}$, and $\mathcal{M}$ is the minimal subspace containing $LS\{P_0 Z_{j12}^* \mathcal{L}^0\}_{j=1}^{m}$ and invariant with respect to $Z_{j22}, j = 1,\dots,m$. Hence $dim\mathcal{M} < \infty$, the subspace $\mathcal{H}_0$ is invariant with respect to all operators Z_{j22} , and there exists a decomposition

$$\mathcal{L}_+ = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_m,\ \ Z_{j22}\mathcal{H}_k \subset \mathcal{H}_k, j,\ \ k = 1,\dots,m, \tag{14}$$

such that

$$\sigma\big(Z_{j11} \cap \sigma_p\big(Z_{j22} \,|\mathcal{H}_j\big) = \emptyset, \qquad j = 1,\dots,m, \tag{15}$$

where the symbol σ_p denotes the set of eigenvalues of the corresponding operator. Let $Y = \|Yik\|_{i,k=1}^2$ be the matrix representation of the desired operator . Then $Z_j Y$ is a selfadjoint operator if and only if the following relations hold:

$(i)\ Z_{j11} Y_{11} + Z_{j12} Y_{12}^{*} = Y_{11} Z_{j11}^* + Y_{12} Z_{j12}^*$;

$(ii)\ Y_{12}^* Z_{j11}^* + Y_{22} Z_{j12}^* = Z_{j22} Y_{12}^{*}$;

$(iii)\ Z_{j22} = Y_{22} Z_{j22}$.

This allows us to construct Y so that each of the subspaces in the decomposition (15) is invariant with respect to Y_{22}. We set $Y = \sum_{j=0}^{m} Y_j$, where $Y_j = \|Y_{jik}\|_{i,k=1}^2$ satisfy conditions (i)– (iii) and $Y_{j22}|\mathcal{H}_k = 0\ for\ j \neq k$, j , $k = 1,\dots,m$. If we choose the operators Y_{j22} so that $dim\ ker\ Y_{j22}|\mathcal{H}_j < \infty$, then the operator Y thus constructed is the desired operator.

We define the operators $Y_j , j = 0,1,\dots,m$, as follows: Y_0: $Y_{011} = 0, Y_{012} = 0, and\ Y_{022}$ is the orthoprojection in $\mathcal{H}_0$ on $\mathcal{M}^\perp$; and $Y_j, j = 1,\dots,m. Let\ \widetilde{Z}_j = \prod_{js=1}^{s}(Z_j - \lambda_{js})(Z_j - \bar{\lambda}_{js})$, and let $\widetilde{Z}_j = \|\widetilde{Z}_{jik}\|_{i,k=1}^2$ be its matrix representation with respect to (5.1.10). We set $Y_{j11} = \widetilde{Z}_{j12} \widetilde{Z}_{j12}^*$, $Y_{j12} = \widetilde{Z}_{j12} \widetilde{Z}_{j22}$,$Y_{j21} = Y_{j12}^{*}$,

$$and\ Y_{j22} = \widetilde{Z}_{j22}, \qquad j = 1,\dots,m.$$

Definitizable operators and operators of class $K(H)$ have several similar properties. In particular, if a J-selfadjoint operator belongs to at least one of these classes, then the spectral function of this operator has a finite set of spectral singularities. The character of these singularities is different in the two cases mentioned above. Therefore, it is natural to study the spectral singularities of operators contained simultaneously in both classes.

In what follows, in the statement of Theorem (5.1.13) and in its proof, we shall use the standard terminology concerning the J-spectral function of a definitizable operator and its critical points [68] (see also [64]).

The fact that an operator of class $K(H)$ has a spectral function was first announced as an exercise in [64]. We present the corresponding result in the form used in a detailed proof in [69].

Proposition (5.1.11)[62]. Let the spectrum of a J-selfadjoint operator $A \in D_\kappa^+$ be real,and let the subspace $\mathcal{L}_+$ corresponding to Definition (5.1.5) be invariant with respect to A. Then the operator A has spectral function E_λ with a finite set $\Lambda \subset \mathbb{R}$ of spectral singularities and

(a) $E_\lambda \in AlgA\ for\ any\ \lambda \in \mathbb{R}\setminus \Lambda,\ \sigma\big(A|_{E(\Delta)\mathcal{K}}\big) \subset \bar{\Delta}\ for\ any\ \Delta \in \mathcal{R}_\Lambda, and\ E(\mathbb{R}) = I$;

(b) $if\ \Delta \in \mathcal{R}_\Lambda\ and\ \Delta \cap \Lambda = \emptyset, then\ E(\Delta)\mathcal{K} = \mathcal{K}_\Delta^+ [+]\mathcal{K}_\Delta^-, A\mathcal{K}_\Delta^+ \subset \mathcal{K}_\Delta^+, A\mathcal{K}_\Delta^-\ \mathcal{K}_\Delta^-, \mathcal{K}_\Delta^+\ E(\Delta)\mathcal{L}_+$ is a uniformly positive subspace, and $\mathcal{K}_\Delta^-$ is a uniformly negative subspace(each of the subspaces $\mathcal{K}_\Delta^+$and $\mathcal{K}_\Delta^-$ can degenerate into $\{0\}$);

(c) $if\ \Delta \in \mathcal{R}_\Lambda and\ \Delta \cap \Lambda \neq \emptyset, then\ E(\Delta)(\mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]}) \neq \{0\}$,

where $AlgA$ is the minimal weakly closed algebra generated by A and containing the identity, and $\mathcal{R}_\Lambda$ is the system of all Borel subsets of the real axis for which none of the points from Λ can be a tangency point,although such point can be inner.

Theorem(5.1.12)[62]. For a J-selfadjoint operator $A \in K(H)$ with invariant subspaces $\mathcal{L}^\pm \in \mathfrak{M}^\pm\ and\ \mathcal{L}^+[\perp]\mathcal{L}^-$ to be definitizable,it is necessary and sufficient that there exist a finite set $\{\mu_k\}_{k=1}^m \subset \mathbb{R}$ of points such that the intervals $(-\infty,\mu_1), (\mu_1,\mu_2), \dots, (\mu_m,\infty)$ consist of definite-type points of the J-spectral function E of the operator A.

Proof. The necessity follows from the general theory of definitizable operators [68]. Sufficiency. If a J-selfadjoint operator $A \in K(H)$ has a nonreal spectrum, then the latter consists of finitely many eigenvalues of finite multiplicity [64]. Let $\{\lambda_j, \bar{\lambda}_j\}_{j=1}^s$ be the set of nonreal eigenvalues of the operator A if they exist. We introduce the notation

$$\mathcal{K}_1 = \sum_{j=1}^{s} \mathcal{L}_{\lambda_j}(A) + \mathcal{L}_{\bar{\lambda}_j}(A),$$

where $\mathcal{L}_{\lambda_j}(A)$ and $\mathcal{L}_{\bar{\lambda}_j}(A)$ are the root linear manifolds corresponding to $\lambda_j$and $\bar{\lambda}_j$, respectively, $j = 1,\dots,s, \mathcal{K}_2 = \mathcal{K}_1^{[\perp]}$. Then the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ are invariant with respect to A and $K = \mathcal{K}_1[\dotplus]\mathcal{K}_2$. We write $A_j = A \mid \mathcal{K}_j, j = 1,2$. The operator A_2 has the same properties as the operator A and the additional property that its spectrum is real. This implies the existence of a polynomial p such that its set of zeros contains the sets $\{\lambda_j, \bar{\lambda}_j\}_{j=1}^s\ and\ \{\mu_k\}_{k=1}^m$ and the operator $B := p(A)$ has the property that the negative half-axis contains points of negative type and the positive half-axis contains points of positive type. Since the lengths of Jordan chains of the operator A are uniformly bounded [64], it follows that we can choose p so that $\mathcal{L}_0(B) = kerB$. By assumption, the subspaces $\mathcal{L}^\pm$ are invariant with respect to A and hence with respect to B. We set $\mathcal{L}_0 = \mathcal{L}^+ \cap \mathcal{L}^-$. Then

$$\mathcal{K} = \mathcal{L}_0 \dotplus \mathcal{L}^+ \dotplus \mathcal{L}^- \dotplus J\mathcal{L}_0, \tag{16}$$

where $\mathcal{L}_\pm$ are uniformly definite components of the decomposition of the subspaces $\mathcal{L}^+$. Withrespect to (16), the operator B is represented by the matrix

$$B = \begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ 0 & B_{22} & 0 & B_{24} \\ 0 & 0 & B_{33} & B_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\mathcal{L}_+$ is a uniformly positive subspace and $\mathcal{L}_-$ is a uniformly negative subspace, we have $(B_{22}) \in [0,\infty)\ and\ \sigma(B_{33}) \in (-\infty,0]$. This implies that the operator B is a finite-dimensional perturbation of the J-nonnegative operator

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 \\ 0 & 0 & B_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In view of [70], the operator B, as well as the operator A, is definitizable.

Now, we consider the difference between the spectral function of an operator of class H and the spectral function of an operator of class D_κ^+ . First of all, we note the following assertion.

Proposition(5.1.13)[62]. Let $A \in H$ be a J-selfadjoint operator. Then to all its real eigenvalues, except possibly finite many, there correspond uniformly definite eigensubspaces, and the eigensubspaces corresponding toexceptional eigenvalues are regular or pseudoregular and have a finite rank of indefiniteness and a finite-dimensional isotropic part.

Remark(5.1.14)[62]. Let E_λ be the spectral function of a J-selfadjoint operator $A \in D_\kappa^+$. A function $f(\lambda)$ defined on the set $Supp(E) \setminus \Lambda$ is said to be E-measurable if it is measurable with respect to any Lebesgue–Stieltjes measure μ_σ determined by a function of the form $\sigma(\lambda) = [E_\lambda x, x]$,

$$x \in \widetilde{\mathcal{K}} = CLS_{\Delta \in \mathcal{R}_\Lambda, \Delta \cap \Lambda = \emptyset} E(\Delta)\mathcal{K}, \tag{17}$$

first defined on $Supp(E) \setminus \Lambda$ and then extended continuously to points from Λ. If the space $\mathcal{K}$ is separable, then the measurability of a function with respect to the above set of Lebesgue–Stieltjes measures can be replaced by its measurability with respect to the unique measure μ_σ specified by the function $\sigma(\lambda) = [E_\lambda x, y]$ determined by several specially chosen vectors x *and* y .

Proof. Indeed, in the quotient space $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}} / (\widetilde{\mathcal{K}} \cap \widetilde{\mathcal{K}}^{[\perp]})$, a J-selfadjoint operator $A \in D_\kappa^+$ generates an operator $\hat{A}$, which is selfadjoint with respect to an appropriately chosen canonical inner product, and $\sigma_p(A) \cap \Lambda = \emptyset$. For more details, see, for example, [71]. For our purposes, it is expedient to choose the above vectors in the following special way. We set

$$\widetilde{\mathcal{K}}_+ = CLS_{\Delta \in \mathcal{R}_\Lambda, \Delta \cap \Lambda = \emptyset}\{\mathcal{K}_+(\Delta)\}, \widetilde{\mathcal{K}}_- = CLS_{\Delta \in \mathcal{R}_\Lambda, \Delta \cap \Lambda = \emptyset}\{\mathcal{K}_-(\Delta)\}, \tag{18}$$

where $\mathcal{K}_+(\Delta)$ and $\mathcal{K}_-(\Delta)$ are chosen in accordance with conditions (a)–(c) in Proposition (5.1.11). Then there is a vector $x_+ \in \widetilde{\mathcal{K}}_+$ such that the measurability of the set $X \subset \mathbb{R} \setminus \Lambda$ with respect to the measure $\mu_{\sigma+}$, where $\sigma_+(\lambda) = [E_\lambda x_+, x_+]$, implies the measurability of X with respect to the measure μ_σ , where $\sigma(\lambda) = [E_\lambda y, y]$ for any $y \in \widetilde{\mathcal{K}}_+$; and a similar vector x_- belongs to $\widetilde{\mathcal{K}}_-$. The desired vectors x and y , which completely describe E_λ-measurability, can now be chosen in the form

$$x = x_+ + x_-, \qquad y = x_+ - x_-. \tag{19}$$

It follows from Example (5.1.8) that operators of the class H are, in general, not definitizable, but the next theorem shows that the operators of class H are in some sense similar to definitizable operators.

Theorem(5.1.15)[62]. Suppose that the space $\mathcal{K}$ is separable, $A \in H$ is a J-selfadjoint operator, E_λ is its spectral function with the set of critical points Λ. Then there exists a finite set $\hat{\Lambda} \subset \mathbb{R}, \Lambda \subset \hat{\Lambda}$, and E_λ-measurable sets $X_+, X_- \subset R, X_+ \cap X_- = \emptyset$ *and* $X_+ \cup X_- = \mathbb{R} \setminus \hat{\Lambda}$,such that,for any interval $\Delta \in B_\Lambda \cap \mathbb{R}$,e ach of the subspaces

$$E(\Delta \cap X_+)\mathcal{K} \quad and \quad E(\Delta \cap X_-)\mathcal{K} \tag{20}$$

either degenerates into $\{0\}$ or is, respectively, positive or negative.

Proof. We construct the set $\hat{\Lambda}$ by adding to $\Lambda_{a\,\lambda} \in \sigma_p(A)$ for which the corresponding eigensubspace is indefinite. By Proposition (5.1.13), the set $\hat{\Lambda}$ is finite. To simplify the calculations, we assume that $\Lambda = \hat{\Lambda}$and, under the above assumption, show that the measures generated by the functions

$$\sigma_+(\lambda) = [E_\lambda x_+, x_+] \quad and \quad \sigma_-(\lambda) = -[E_\lambda x_-, x_-]$$

$\big(x_+$ *and* x_- *are the same as in* $(19)\big)$ are mutually singular. First, we note that, according to the above assumption, the atomic components of these measures are mutually singular; hence we can assume that the functions $\sigma_+(\lambda)$ and $\sigma_-(\lambda) =$ are continuous.

In view of Remark (5.1.14) and the decomposition (19), we have $\sigma(\lambda) = \sigma_+(\lambda) + \sigma_-(\lambda)$, and hence $\mu_{\sigma+} \prec \mu_\sigma$ *and* $\mu_{\sigma-} \prec \mu\sigma$ ($\prec$ is the symbol of measure subordination), which implies

$$\sigma_+(\lambda) = \int_{-\infty}^{\lambda} \rho_+(\tau)d\sigma(\tau) \quad , \sigma_-(\lambda) = \int_{-\infty}^{\lambda} \rho_-(\tau)d\sigma(\tau)\ d\sigma(\tau),$$

and $\rho_+(\tau), \rho_-(\tau) \in [0\,;\,1]$ for any $\tau \in \mathbb{R}$. We set $Y = \{\tau : \rho_+(\tau) \neq 0,\ \rho_-(\tau) \neq 0\}$. The statement of the theorem will be showed if we obtain $\mu_\sigma Y = 0$. We assume the contrary, i.e., we assume that $\mu_\sigma Y \neq 0$, and introduce the following two functions:

$$f_+(t) = \begin{cases} \left(\dfrac{\rho_+(\tau) + \rho_-(\tau)}{\rho_+(\tau)}\right)^{1/2} & for\ t \in Y, \\ 0 & for\ t \notin Y, \end{cases}$$

$$f_-(t) = \begin{cases} \left(\dfrac{\rho_+(\tau) + \rho_-(\tau)}{\rho_-(\tau)}\right)^{1/2} & for\ t \in Y, \\ 0 & for\ t \notin Y, \end{cases}$$

Now, we fix an interval $\subset \mathbb{R}, \Delta \in B_\Lambda$, for which $\mu_\sigma(Y \cap \Delta) \neq 0$, and set

$$y_+ = \int_\Delta f_+(t)\, dE_t x_+, \qquad y_- = \int_\Delta f_-(t)\, dE_t x_- .$$

Suppose that $\Delta_1 \subset \Delta$ *and* $z_{\Delta_1} = E(\Delta_1)(y_+ + y_-)$. Then we have

$$[z_{\Delta_1}, z_{\Delta_1}] = \int_{\Delta_1} f_+^2(t)d\sigma_+(t) - \int_{\Delta_1} f_-^2(t)d\sigma_-(t) = \int_{\Delta_1\cap Y} \big(\rho_+(t) + \rho_-(t)\big)\, d\sigma(t)$$

$$- \int_{\Delta_1\cap Y} \big(\rho_+(t) + \rho_-(t)\big)\, d\sigma(t) = 0.$$

We set $\mathcal{L} = CLS_{\Delta\subset\Delta_1}\{z_{\Delta_1}\}$. It is clear that $\mathcal{L}$ is a neutral subspace. And since $\mu_\sigma(Y \cap \Delta) \neq 0$ and the set $Y \cap \Delta$ is free of atomic measure, it follows that L is an infinite-dimensional subspace. So, the operator A has an infinite-dimensional neutral invariant subspace; but this is impossibleby the definition of the class H.

Section (5.2): Krein Spaces and Frames

Frame theory for Hilbert spaces has been thoroughly developed; see, [40,41,42,43]. For a fixed Hilbert space $(\mathcal{H}, \langle\,,\rangle)$, a frame for $\mathcal{H}$ is family of vectors $F = \{f_i\}_{i\epsilon I}$ in $\mathcal{H}$ which satisfies the inequalities

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \ \text{ for every } f \in \mathcal{H}, \tag{21}$$

for positive constants $0 \le A \le B$. The (bounded, linear) operator $S : \mathcal{H} \to \mathcal{H}$ defined by

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i \quad , \qquad f \in \mathcal{H} \tag{22}$$

is known as the frame operator associated to $\mathcal{F}$. The inequalities in (21) imply that S is a (positive) boundedly invertible operator, and it allows to reconstruct each vector $f \in \mathcal{H}$ in terms of the family $\mathcal{F}$ as follows:

$$f = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i \quad (23)$$

The above formula is known as the reconstruction formula associated to $\mathcal{F}$. Notice that if $\mathcal{F}$ is a Parseval frame, i.e. if $S = I$, then the reconstruction formula resembles the Fourier series of f associated to an orthonormal basis $\mathcal{B} = \{b_k\}_{k \in K}$ of $\mathcal{H}$:

$$f = \sum_{k \in K} \langle f, b_k \rangle b_k \ ,$$

but the frame coefficients $\{\langle f, f_i \rangle\}_{i \in I}$ given by $\mathcal{F}$ allow to reconstruct f even when some of these coefficients are missing. Indeed, each vector $f \in \mathcal{H}$ may admit several reconstructions in terms of the frame coefficients as a consequence of the redundancy of $\mathcal{F}$. These are some of the advantages of frames over bases in signal processing applications, when noisy channels are involved; e.g., see [44,45,46].

Given a Krein space $(\mathcal{H}, [\,,])$ with fundamental symmetry J, a J -orthonormalized system is a family $\mathcal{E} = \{e_i\}_{i \in I}$ such that $[e_i, e_j] = \pm\delta_{ij}$, $for\ i, j \in I$. A J-orthonormal basis is a J-orthonormalized system which is also a Schauder basis for $\mathcal{H}$. If $\mathcal{E} = \{e_i\}_{i \in I}$ is a J-orthonormal basis of $\mathcal{H}$ then the vectors in $\mathcal{H}$ can be represented as follows:

$$f = \sum_{i \in I} \sigma_i [f, e_i] e_i \ , f \in \mathcal{H}, \quad (24)$$

where$\sigma_i = [e_i, e_i] = \pm 1$.

J-orthonormalized systems are intimately related to the notion of dual pair. In fact, each J-orthonormalized system generates a dual pair, i.e. a pair $(\mathcal{L}_+, \mathcal{L}_-)$ of subspaces of $\mathcal{H}$ such that $\mathcal{L}_+$ is J-nonnegative, $\mathcal{L}_-$ is J-nonpositive and $\mathcal{L}_+$ is J-orthogonal to $\mathcal{L}_-$, i.e. $[\mathcal{L}_+, \mathcal{L}_-)\ ] = 0$. Moreover, if $\mathcal{E}$ is a J-orthonormal basis of $\mathcal{H}$, the dual pair associated to $\mathcal{E}$ is maximal and the subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ are uniformly J-definite, see [47]. Therefore the dual pair $(\mathcal{L}_+, \mathcal{L}_-)$ is a fundamental decomposition of $\mathcal{H}$. Notice that, considering the Hilbert space structure induced by the above fundamental decomposition, the J-orthonormal basis $\mathcal{E}$ turns out to be an orthonormal basis in the associated Hilbert space. Therefore, each J-orthonormal basis can be realized as an orthonormal basis of $\mathcal{H}$.

Given a pair of maximal uniformly J-definite subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$ of a Krein space $\mathcal{H}$, where $\mathcal{M}_+$ is J-positive and $\mathcal{M}_-$ is J-negative, if $f_\pm = \{f_i\}_{i \in I_\pm}$ is a frame for the Hilbert space $(\mathcal{M}_\pm, \pm[\,,])$, it is easy to see that

$$\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_- \ ,$$

is a frame for $\mathcal{H}$, which produces an indefinite reconstruction formula:

$$f = \sum_{i \in I} \sigma_i [f, g_i] f_i = \sum_{i \in I} \sigma_i [f, g_i] f_i \ , f \in \mathcal{H} \quad (25)$$

where $\sigma_i = sgn[f_i, f_i]$ and $\{g_i\}_{i \in I} i \in I$ is some (equivalent) frame for $\mathcal{H}$ (see Example (5.2.10)).

The aim of this work is to introduce and characterize a particular family of frames for a Krein space $(\mathcal{H}, [\,,])$ – hereafter called J-frames – that are compatible with the indefinite inner product $[\,,]$, in the sense that an indefinite reconstruction formula as in (25) holds (see Proposition (5.2.25)).

Some different approaches to frames for Krein spaces and indefinite reconstruction formulas are developed in [48,49], respectively. As it will be seen along this work, neither of the definitions below is comparable with the J-frame concept introduced here.

In [49], the authors studied when a set of vectors $\{\phi_j\}_{j \epsilon I}$ in a Hilbert spaceH can be scaled to obtain a tight frame $\{\alpha_j \phi_j\}_{j \epsilon I}$, and hence a representation of the form

$$f = \sum_{j \epsilon I} c_j \langle f, \psi_j \rangle \phi_j \quad , f \epsilon \mathcal{H} \quad (26)$$

It turns out that representations as in (26) can exist even when some of the c_j's are negative, and these correspond to what they call ''signed frames''. Indeed, a Bessel family $\{\psi_j\}_{j\epsilon I}$in a Hilbert space $\mathcal{H}$ is called a signed frame with signature $\sigma = (\sigma_j)_{j\epsilon I}, \sigma_j \in \{-1, 1\}$, if there exist $A, B > 0$ with

$$A\|f\|^2 \leq \sum_{j\epsilon I} \sigma_j \left|\langle f, \psi_j\rangle\right|^2 \leq B\|f\|^2 \quad for\ every\ f\epsilon\mathcal{H}$$

Then, each $f\epsilon\mathcal{H}$ can be represented as

$$f = \sum_{j\epsilon I} \sigma_j \langle f, \psi_j\rangle \varphi_j = \sum_{j\epsilon I} \sigma_j \langle f, \varphi_j\rangle \psi_j\ ,$$

where $\{\varphi_j\}_{j\epsilon I}$ is the dual signed frame (see [49]). Observe that this idea can be interpreted as introducing an indefinite inner product (associated to the signature $\sigma = (\sigma_j)_{j\epsilon I}$ in $\ell_2(I)$). But the sampling space $\mathcal{H}$ does not need to be a Krein space.

On the other hand, in [48] the authors consider Krein spaces as sampling spaces. They say that a family $\{f_n\}_{n\in N}$ of vectors in $\mathcal{H}$ is a ''frame for the Krein space $(\mathcal{H}, [\ ,\ ])$'' if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n\epsilon N} |[f, f_n]|^2 \leq B\|f\|_J^2\ ,\ \ for\ every\ f\epsilon\mathcal{H}$$

where $\|\ \|_J$ stands for the norm of the associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. Then, they show that a family $\{f_n\}_{n\in N}$ in $\mathcal{H}$ is a ''frame for the Krein space $(\mathcal{H}, [\ ,\ ])$'' if and only if it is a frame for the Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. This is the major difference between J-frames and this concept, because there are frames for the associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$.
which are not J-frames for the Krein space $(\mathcal{H}, [\ ,\ ])$ (see Example (5.2.9)).

The section is organized as follows: Section contains some preliminaries results both in Krein spaces and in frame theory for Hilbert spaces, and presents the motivation and what is meant by a J-frame. Briefly, a J-frame for the Krein space $(\mathcal{H}, [\ ,\ ])$ is a Bessel family $\mathcal{F} = \{f_i\}_{i\epsilon I}$ with synthesis operator $T : \ell_2(I) \to \mathcal{H}$ such that the ranges of $T_+ := TP_-$ and $T_- := T(1 - P_+)$are maximal uniformly J-positive and maximal uniformly J-negative subspaces, respectively, where $I_+ = \{i \in I: [f_i, f_i] > 0\}$ and P_+ is the orthogonal projection onto $\ell_2(I_+)$,as a subspace of $\ell_2(I)$. It is immediate that J-orthonormal bases are J-frames, because they generate maximal dual pairs [78].

Also, if $\mathcal{F}$ is a J-frame for $\mathcal{H}$, observe that $R(T\ ) = R(T_+) + R(T_-)$ and recall that the sum of a maximal uniformly J-positive and a maximal uniformly J-negative subspace coincides with $\mathcal{H}$ [81]. Therefore, each J-frame is in fact a frame for $\mathcal{H}$ in the Hilbert space sense. Moreover, it is shown that $\mathcal{F}_+ = \{f_i\}_{i\epsilon I_+}$ is a frame for the Hilbert space $(R(T_+), [\ ,\ ])$ and $\mathcal{F}_- = \{f_i\}_{i\epsilon I/I_+}$ is a frame for $(R(T_-), -[\ ,\ ])$, i.e. there exist constants $B_- \leq A_- < 0 < A_+ \leq B_+$ such that

$$A_\pm[f, f] \leq \sum_{i\in I_\pm} |[f, f_i]|^2 \leq B_\pm[f, f] \qquad for\ every\ \ f \in R(T_\pm) \tag{27}$$

The optimal constants satisfying the above inequalities can be characterized in terms of $T_\pm$ and the Gramian operators of their ranges.

This section ends with a geometrical characterization of J-frames, in terms of the (minimal) angles between the uniformly J-definite subspace $R(T_\pm)$ and the cone of neutral vectors of the Krein space.

This Section is devoted to study the synthesis operators associated to J-frames. Given a bounded operator $T : \ell_2(I) \to \mathcal{H}$, it is described under which conditions T is the synthesis operator of a J-frame for the Krein space $\mathcal{H}$.

And the J-frame operator is introduced. Given a J-frame $\mathcal{F} = \{f_i\}_{i\epsilon I}$, the J-frame operator $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$Sf = \sum_{i\in I} \sigma_i [f, f_i] f_i \ , \qquad f \in \mathcal{H} \ ,$$

where $\sigma_i = sgn([f, f_i])$ .This operator resembles the frame operator for frames in Hilbert spaces (see (22)), and it has similar properties, in particular $S = TT^+$ where $T : \ell_2(I) \to \mathcal{H}$ is the synthesis operator of $\mathcal{F}$ and T^+ denotes the J-adjoint of T (see Proposition (5.2.23)). Furthermore, each J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ determines an indefinite reconstruction formula, which depends on the J-frame operator S:

$$f = \sum_{i\in I} \sigma_i \, [f, S^{-1} f_i] f_i = \sum_{i\in I} \sigma_i \, [f, S^{-1} f_i] S^{-1} f_i \, , \qquad for\ every\ f \in \mathcal{H} \ . \tag{28}$$

In this case the family $\{S^{-1} f_i\}_{i\in I}$ turns out to be a J-frame too.

Finally, it will be shown that the J-frame operator of a J-frame $\mathcal{F}$ is intimately related to the projection $Q = P_{R(T_+)//R(T_-)}$ determined by the decomposition $\mathcal{H} = R(T_+) + R(T_-)$, see Theorem (5.2.26).

Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. The groups of linear invertible and unitary operators acting on $\mathcal{H}$ are denoted by $GL(\mathcal{H})$ and $U(\mathcal{H})$, respectively. Also, $L(\mathcal{H})^+$ denotes the cone of positive semidefinite operators acting on $\mathcal{H}$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$.

If $T \in L(\mathcal{H}, \mathcal{K})$ then $T^* \in L(\mathcal{K}, \mathcal{H})$ denotes the adjoint operator of , $R(T)$ stands for its range and $N(T)$ for its nullspace. Also, if $T \in L(\mathcal{H}, \mathcal{K})$ has closed range, $T^\dagger \in L(\mathcal{K}, \mathcal{H})$ denotes the Moore–Penrose inverse of T .

Hereafter, $\mathcal{s} \dotplus \mathfrak{J}$ denotes the direct sum of two (closed) subspaces $\mathcal{s}$ and $\mathfrak{J}$ of $\mathcal{H}$. On the other hand,$\mathcal{s} \oplus \mathfrak{J}$ stands for the (direct) orthogonal sum of them and $\mathcal{s} \ominus \mathfrak{J} := \mathcal{s} \cap (\mathcal{s} \cap \mathfrak{J})^\perp$. The oblique projection onto $\mathcal{s}$ along , denoted by $P_{\mathcal{s}//\mathfrak{J}}$, is the unique projection with range $\mathcal{s}$ and nullspace $\mathfrak{J}$. In particular, $P_{\mathcal{s}} := P_{\mathcal{s}//\mathcal{s}^\perp}$ is the orthogonal projection onto $\mathcal{s}$.

The following result due to Douglas [51], characterizes operator range inclusions. It is quite often used along the Section.

Theorem (5.2.1)[39]. Given Hilbert spaces $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$ and operators $A \in L(\mathcal{K}_1, \mathcal{H})$ and $B \in L(\mathcal{K}_2, \mathcal{H})$, the following conditions are equivalent:

(i) the equation $AX = B$ has a solution in $L(\mathcal{K}_2, \mathcal{K}_1)$;

(ii) $R(B) \subseteq R(A)$;

(iii) there exists $\lambda > 0$ such that $BB^* \leq \lambda AA^*$.

In this case, there exists a unique $D \in L(\mathcal{K}_2, \mathcal{K}_1)$ such that $AD = B$ and $R(D) \subseteq \overline{R(A*)}$; moreover, $N(D) = N(B)$ and $\| D \| = inf\{\lambda > 0 : BB^* \leq \lambda AA^*\}$. The operator D is called the reduced solution of $AX = B$.

Corollary(5.2.2)[39]. Let $T \in L(\mathcal{H})^+ . If\ R(T) = R(T^{1/2})$, then $R(T)$ is closed.

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject see the books by Azizov and Iokhvidov [47] and Bognr [52] and the monographs by Ando [50] and by Dritschel and Rovnyak [53].

Given a Krein space $(\mathcal{H}, [\,,\,])$ with a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dotplus \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\,,\,])$ and $(\mathcal{H}_-, -[\,,\,])$is denoted by $(\mathcal{H}, \langle\,,\,\rangle)$.

Observe that the indefinite metric and the inner product of $\mathcal{H}$ are related by means of a fundamental symmetry, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[x, y] = \langle Jx, y \rangle, x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded respect to the associated Hilbert spaces $(\mathcal{H}, \langle\,,\,\rangle_{\mathcal{H}})$ and $(\mathcal{K}, \langle\,,\,\rangle_{\mathcal{K}})$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the J-adjoint operator of T is defined by $T^+ = J_{\mathcal{H}} T^* J_{\mathcal{K}}$, where $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is J-selfadjoint if $T = T^+$.

A vector $x \in \mathcal{H}$ is J-positive if $[x,x] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is J-positive if every $x \in S, x \neq 0$, is a J-positive vector. A subspace $\mathcal{S}$ of $\mathcal{H}$ is uniformly J-positive if there exists α > 0 such that

$$[x,x] \geq \alpha \| x \|^2, for\ every\ x \in \mathcal{S},$$

where ‖ ‖ stands for the norm of the associated Hilbert space $(\mathcal{H}, \langle , \rangle)$.
J-nonnegative, J-neutral, J-negative, J-nonpositive and uniformly J-negative vectors and subspaces are define analogously.
Remark (5.2.3)[39]. If $\mathcal{S}_+$ is a closed uniformly J-positive subspace of a Krein space$(\mathcal{H}, [\,,])$, observe that $(\mathcal{S}_+, [\,,])$ is a Hilbert space. In fact, the forms, $[\,,]$ and $\langle , \rangle$ are equivalent inner products on $\mathcal{S}_+$, because

$$\alpha \| f \|^2 \leq [f,f] \leq \| f \|^2, for\ every\ f \in \mathcal{S}_+.$$

Analogously, if $\mathcal{S}_-$is a closed uniformly J-negative subspace of $(\mathcal{H}, [\,,])$, $(\mathcal{S}_-, -[\,,])$ is a Hilbert space.
Proposition (5.3.4)[39]. ([47]). Let $\mathcal{H}$ be a Krein space with fundamental symmetry J and $\mathcal{S}$ a J-nonnegative closed subspace of $\mathcal{H}$. Then, $\mathcal{S}$ is the range of a J-selfadjoint projection if and only if $\mathcal{S}$ is uniformly J-positive.
Recall that, given a closed subspace $\mathcal{M}$ of a Krein space $\mathcal{H}$, the Gramian operator of $\mathcal{M}$ is defined by: $G_{\mathcal{M}} = P_{\mathcal{M}} J P_{\mathcal{M}}$, where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$ and J is the fundamental symmetry of $\mathcal{H}$. If $\mathcal{M}$ is J-semidefinite, then $\mathcal{M} \cap \mathcal{M}^{[\perp]}$ coincides with $\mathcal{N} := \{f \in \mathcal{M} : [f,f] = 0\}$. Therefore, it is easy to see that

$$G_{\mathcal{M}} = G_{\mathcal{M} \ominus \mathcal{N}}.$$

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the J-orthogonal companion to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [x,s] = 0\ for\ every\ s \in \mathcal{S}\}.$$

A subspace S of $\mathcal{H}$ is J-non-degenerated if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. Notice that if $\mathcal{S}$ is a J-definite subspace of $\mathcal{H}$ then it is J-non degenerated.

Given closed subspaces $\mathcal{S}$ and $\mathfrak{J}$ of a Hilbert space $\mathcal{H}$, the cosine of the Friedrichs angle between $\mathcal{S}$ and $\mathfrak{J}$ is defined by

$$c(\mathcal{S}, \mathfrak{J}) = sup\{|\langle x,y \rangle| : x \in \mathcal{S} \ominus \mathfrak{J}, \| x \| = 1, y \in \mathfrak{J} \ominus \mathcal{S}, \| y \| = 1\}.$$

It is well known that

$$c(\mathcal{S}, \mathfrak{J}) < 1 \Leftrightarrow \mathcal{S} + \mathfrak{J} \text{ is closed} \Leftrightarrow c(\mathcal{S}^\perp, \mathfrak{J}^\perp) < 1.$$

Furthermore, if $P_{\mathcal{S}}$ and $P_{\mathfrak{J}}$ are the orthogonal projections onto $\mathcal{S}$ and , respectively, then $c(\mathcal{S}, \mathfrak{J}) < 1$ if and only if $(I - P_{\mathcal{S}})P_{\mathfrak{J}}$ has closed range. See [54] for further details.
The next definition is due to Kato, see [55].
Definition (5.2.5)[39]. The reduced minimum modulus $\gamma(T)$ of an operator $T \in L(\mathcal{H}, \mathcal{K})$ is defined by

$$\gamma(T) = inf\{\| Tx \| : x \in N(T)^\perp, \| x \| = 1\}.$$

Observe that $\gamma(T) = sup\{C \geq 0 : C \| x \| \leq \| Tx \|\ for\ every\ x \in N(T)^\perp, \| x \| = 1\}$. It is well known that $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2}$. Also, it can be shown that an operator $T \neq 0$ has closed range if and only if $\gamma(T) > 0$. In this case, $\gamma(T) = \|T^\dagger\|^{-1}$.

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$, respectively, and $T \in L(\mathcal{H}, \mathcal{K})$ then

$$\gamma(T^+) = \gamma(J_{\mathcal{H}} T^* J_{\mathcal{K}}) = \gamma(T^*) = \gamma(T),$$

because $J_{\mathcal{H}}$ (resp. $J_{\mathcal{K}}$) is a unitary operator on $\mathcal{H}$ (resp. $\mathcal{K}$).
Remark (5.2.6)[39]. If $\mathcal{M}_+$ is a closed J-nonnegative subspace of a Krein space $\mathcal{H}$ then

$$\gamma(G_{\mathcal{M}_+}) = \alpha^+, \quad (29)$$

where $\alpha^+ \in [0,1]$ is the supremum among the constants $\alpha \in [0,1]$ such that $\alpha \| f \|^2 \leq [f,f]\ for\ every\ f \in \mathcal{M}_+$. From now on, the constant α^+ is called the definiteness bound of $\mathcal{M}_+$. Notice that α+ is in fact a maximum for the above set and $\mathcal{M}_+$ is uniformly J-positive if and only if $\alpha^+ > 0$.

Analogously, if $\mathcal{M}_-$ is a J-nonpositive subspace then $\gamma(G_{\mathcal{M}_-}) = \alpha^-$, where α^- is the definiteness bound of $\mathcal{M}_-$, i.e.

$$\alpha^- = max\{\alpha \in [0,1] : [f,f] \leq -\alpha \| f \|^2 \ for\ every\ f \in \mathcal{M}_-\}.$$

The following is the standard notation and some basic results on frames for Hilbert spaces, see [40,41,43].

A frame for a Hilbert space $\mathcal{H}$ is a family of vectors $\mathcal{F} = \{f_i\}_{i\in I} \subset \mathcal{H}$ for which there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \qquad for\ every\ f \in \mathcal{H}. \tag{30}$$

The optimal constants (maximal for A and minimal for B) are known, respectively, as the upper and lower frame bounds.

If a family of vectors $\mathcal{F} = \{f_i\}_{i\in I}$ satisfies the upper bound condition in (30), then $\mathcal{F}$ is a Bessel family. For a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$, the synthesis operator $T \in L(\ell_2(I), \mathcal{H})$ is defined by

$$Tx = \sum_{i\in I} \langle x, e_i\rangle f_i,$$

where $\{e_i\}_{i\in I}$ is the standard orthonormal basis of $\ell_2(I)$. It holds that $\mathcal{F}$ is a frame for $\mathcal{H}$ if and only if T is surjective. In this case, the operator $S = TT^* \in L(\mathcal{H})$ is invertible and is called the frame operator. It can be easily verified that

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i \quad , for\ every\ f \in \mathcal{H}. \tag{31}$$

This implies that the frame bounds can be computed as: $A = \| S^{-1} \|^{-1}$ and $B = \| S \|$. From (31), it is also easy to obtain the canonical reconstruction formula for the vectors in $\mathcal{H}$:

$$f = \sum_{i\in I} \langle f, S^{-1} f_i\rangle f_i = \sum_{i\in I} \langle f, f_i\rangle S^{-1} f_i \quad , for\ every\ f \in \mathcal{H},$$

and the frame $\{S^{-1} f_i\}_{i\in I}$ is called the canonical dual frame of $\mathcal{F}$. More generally, if a frame $\mathcal{G} = \{g_i\}_{i\in I}$ satisfies

$$f = \sum_{i\in I} \langle f, g_i\rangle f_i = \sum_{i\in I} \langle f, f_i\rangle g_i \quad , for\ every\ f \in \mathcal{H}, \tag{32}$$

then G is called a dual frame of $\mathcal{F}$.

Let $(\mathcal{H}, \langle\, ,\rangle)$. be a separable Hilbert space that models a signal space. A common task in signal processing applications is to take samples of the signals $x \in \mathcal{H}$, for instance to save or to transmit them. Mathematically, taking samples of a signal can be represented as follows: given a frame $\mathcal{G} = \{g_i\}_{i\in K}$ that spans a closed subspace $\mathcal{S}$ (called the sampling subspace), the samples of $x \in \mathcal{H}$ are given by the family of coefficients $\{\langle x, g_i\rangle\}_{i\in K}$, see [56] and the references therein.

Assume that the signals carrying the desired information are those containing only high frequencies or only low frequencies. In order to clarify the idea, suppose that $x \in \mathcal{H}$ is a piece of music and it is intended to discriminate those fragments where high frequencies are predominant (a trumpet) from those fragments where low frequencies are predominant (a bass).

It turns out that some filters for the signals can be modeled as orthogonal projections acting on $\mathcal{H}$. Hence, consider an ideal low pass filter, i.e. an orthogonal projection $P \in L(\mathcal{H})$, and the complementary filter $I - P$. Therefore, the signals with the same energy at high and low band frequencies $\{x \in \mathcal{H} : \| Px \| = \| (I - P)x \|\}$ are considered disturbances, see, [57,58].

For this particular application, given an arbitrary signal $x \in \mathcal{H}$, the filtered signals Px and $(I - P)x$ are sampled and x is discarded in case that the modulus of the difference $\| Px \|^2 - \| (I - P)x \|^2$ is small enough. Also, notice that sampling both filtered signals $y_1 = Px\ and\ y_2 = (I - P)x$

with frames $\mathcal{g}_1 = \{g_i\}_{i\in I_1}$ *and* $\mathcal{g}_2 = \{h_i\}_{i\in I_2}$, which span $R(P)$ and $N(P)$ respectively, is equivalent to sampling $y = y_1 + y_2 \in \mathcal{H}$ with the frame

$$\mathcal{F} = \{f_i\}_{i\in I} = \{g_i\}_{i\in I_1} \cup \{h_i\}_{i\in I_2}, for\ \mathcal{H} \quad .$$

The space $\mathcal{H}$ can be endowed with an indefinite inner product in order to characterize the set of disturbances as the cone of J-neutral vectors $\mathcal{C}$ of $\mathcal{H}$. Indeed, $J = P - (I - P) = 2P - I$ is a fundamental symmetry which turns $\mathcal{H}$ into a Krein space. Furthermore, a signal is a disturbance if and only if it is J-neutral with respect to the indefinite inner product given by

$$[\, y, z\,] = \langle Py, Pz \rangle - \langle (I - P)y, (I - P)z \rangle ,$$

where $y, z \in \mathcal{H}$ are arbitrary signals.

Observe that the vectors of the frame $\mathcal{F}$ are away from the disturbances set $\mathcal{C}$, i.e. the sampling vectors are not highly correlated with the disturbances (see Remark (5.2.18)). However, once that the cone of disturbances is determined, the following questions naturally arise: Are there other frames whose sampling vectors are not highly correlated with the disturbances? Given an arbitrary frame $\mathcal{F}' = \{f'_i\}_{i\in I}$ $for\ \mathcal{H}\times\mathcal{H}$ is $\mathcal{F}'$ good for this sampling scheme?. How correlated are the sampling vectors in $\mathcal{F}'$ and the cone of disturbances $\mathcal{C}$?

The above discussion motivates the following definition. Let $\mathcal{H}$ be a Krein space with fundamental symmetry J. Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ in $\mathcal{H}$ consider the synthesis operator $T \in L(\ell_2(I), \mathcal{H}). If\ I_+ = \{i \in I : [\, f_i, f_i\,] \geq 0\} and\ I_- = \{i \in I : [\, f_i, f_i] < 0\}$, consider the orthogonal decomposition of $\ell_2(I)$ given by

$$\ell_2(I) = \ell_2(I_+) \oplus \ell_2(I_-), \tag{33}$$

and denote by $P_\pm$ the orthogonal projection onto $\ell_2(I_\pm)$. Also, let $T_\pm = TP_\pm\}. If\ \mathcal{M}_\pm = \overline{span\{f_i : i \in I_\pm\}}$, notice that $span\{f_i : i \in I_\pm\} \subseteq R(T_\pm\}) \subseteq \mathcal{M}_\pm\}$ and $R(T) = R(T_+) + R(T_-)$.

Definition (5.2.7)[39]. The Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-frame for $\mathcal{H}$ if $R(T_+)$ is a maximal uniformly J-positive subspace of $\mathcal{H}$ and $R(T_-)$ is a maximal uniformly J-negative subspace of $\mathcal{H}$.

Notice that, in particular, every J-orthogonalized basis of a Krein space $\mathcal{H}$ is a J-frame for $\mathcal{H}$, because it generates a maximal dual pair, see [47].

If $\mathcal{F}$ is a J-frame, as a consequence of its maximality, $R(T_\pm)$ is closed. So, $R(T_\pm) = \mathcal{M}_\pm$ and, by [50], $\mathcal{M}_+ + \mathcal{M}_- = \mathcal{H}$. Then, it follows that $\mathcal{F}$ is a frame for the associated Hilbert space $(\mathcal{H}, \langle\, ,\rangle)$ because

$$R(T) = R(T_+) + R(T_-) = \mathcal{M}_+ + \mathcal{M}_- = \mathcal{H}.$$

Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$, consider the subspaces $R(T_+)$ and $R(T_-)$ as above. If $K_\pm : \mathcal{D}_\pm \to \mathcal{H}_\pm$ is the angular operator associated to $R(T_\pm)$, the operator of transition associated to the Bessel family $\mathcal{F}$ is defined by

$$\mathcal{F} = K_+P + K_-(I - P) : \mathcal{D}_+ + \mathcal{D}_- \to \mathcal{H},$$

where $P = \frac{1}{2}(I + J)$ is the J-selfadjoint projection onto $\mathcal{H}_+$and $\mathcal{D}_\pm$ is a subspace of $\mathcal{H}_\pm$ (the domain of $K_\pm$, see [59].

Proposition (5.2.8)[39]. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a Bessel family in $\mathcal{H}$. Then, $\mathcal{F}$ is a J-frame if and only if $\mathcal{F}$ is everywhere defined (i.e. $\mathcal{D}_+ + \mathcal{D}_- = \mathcal{H}$) and $\| F \| < 1$.

Proof. See [59].

It follows from the definition that, given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$for the Krein space $\mathcal{H}$, $[\, f_i, f_i\,] \neq 0$ for every i ∈ I, i.e. $I_\pm = \{i \in I : \pm[\, f_i, f_i\,] > 0\}$. This fact allows to endow the coefficients space $\ell_2(I)$ with a Krein space structure. Denote $\sigma_i = sgn([\, f_i, f_i\,]) = \pm 1$ for every $i \in I$. Then, the diagonal operator $J_2 \in L(\ell_2(I))$ defined by

$$J_2\, e_i = \sigma_i\, e_i, for\ every\ i \in I, \tag{34}$$

is a selfadjoint involution on $\ell_2(I)$. Therefore, $\ell_2(I)$with the fundamental symmetry J_2 is a Krein space. Now, if $T \in L(\ell_2(I), \mathcal{H})$ is the synthesis operator of $\mathcal{F}$, the J-adjoints of T , T_+ and T_-can be easily calculated, in fact if $f \in \mathcal{H}$: $T_\pm^\pm f = \pm\sum_{i\in I_\pm}[f, f_i]e_i$,and

$$T^{\pm}f = (T_- + T_-)^+ f = T_+^+ f + T_-^- f = \sum_{i\in I_+}[f,f_i]e_i - \sum_{i\in I_-}[f,f_i]e_i = \sum_{i\in I}\sigma_i[f,f_i]e_i$$

Example(5.2.9)[39]. It is easy to see that not every frame of J-nonneutral vectors is a J-frame: given the Krein space obtained by endowing $\mathbb{C}^3$ with the sesquilinear form $[(x_1,x_2,x_3),(y_1,y_2,y_3)] = x_1\,\overline{y_1} + x_2\,\overline{y_2} - x_3\,\overline{y_3}$, consider $f_1 = \left(1,0,\frac{1}{\sqrt{2}}\right)$, $f_2 = \left(0,1,\frac{1}{\sqrt{2}}\right)$ and $f_3 = (0,0,1)$. Observe that $\mathcal{F} = \{f_1, f_2, f_3\}$ is a frame for $\mathbb{C}^3$ because it is a (linear) basis for the space.

On the other hand, $\mathcal{M}_+ = span\{f_1, f_2\}$ and $\mathcal{M}_- = span\{f_3\}$. If $\left(a, b, \frac{1}{\sqrt{2}}, (a+b)\right)$ is an arbitrary vector in $\mathcal{M}_+$ then

$$[f,f] = |a|^2 + |b|^2 - \tfrac{1}{2}|a+b|^2 = \tfrac{1}{2}|a-b|^2 \geq 0\,,$$

so $\mathcal{M}_+$ is a J-nonnegative subspace of $\mathbb{C}^3$. But $\mathcal{M}_+$ is not uniformly J-positive, because $\left(1,1,\sqrt{2}\right) \in \mathcal{M}_+$ is a (non-trivial) J-neutral vector. Therefore, $\mathcal{F}$ is not a J-frame for $(\mathbb{C}^3, [\,,])$.

The following is a handy way to construct J-frames for a given Krein space. Along this section, it will be shown that every J-frame can be realized in this way.

Example(5.2.10)[39]. Given a Krein space $\mathcal{H}$ with fundamental symmetry J, let $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) be a maximal uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. If $\mathcal{F}_\pm = \{f_i\}_{i\in I_\pm}$ is a frame for the Hilbert space $(\mathcal{M}_\pm, \pm[\,,])$ then $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$ is a J-frame for $\mathcal{H}$.

Indeed, by Remark (5.2.3), $\mathcal{F}_+$ and $\mathcal{F}_-$ are Bessel families in $\mathcal{H}$. Hence, $\mathcal{F}$ is a Bessel family and, if $I = I_+ \cup I_-$ (the disjoint union of I_+ and I_-), the synthesis operator $T \in L(\ell_2(I),\mathcal{H})$ of F is given by

$$Tx = T_+x_+ + T_-x_- \;\; if\; x = x_+ + x_- \in \ell_2(I_+) \oplus \ell_2(I_-) =: \ell_2(I),$$

where $T_\pm : \ell_2(I_\pm) \to \mathcal{M}_\pm$ is the synthesis operator of $\mathcal{F}_\pm$. Then, it is clear that $R(TP_\pm) = \mathcal{M}_\pm$ is a maximal uniformly J-definite subspace of $\mathcal{H}$.

Proposition(5.2.11)[39]. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a J-frame for $\mathcal{H}$. Then, $\mathcal{F}_\pm = \{f_i\}_{i\in I_\pm}$ is a frame for the Hilbert space $(\mathcal{M}_\pm, \pm[\,,])$, i.e. there exist constants $B_- \leq A_- < 0 < A_+ \leq B_+$ such that

$$A_\pm[f,f] \leq \sum_{i\in I_\pm} |[f,f_i]| \leq B_\pm[f,f] \;\; for\; f \in \mathcal{M}_\pm. \tag{35}$$

Proof. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-frame for $\mathcal{H}$, then $R(T_+) = M_+$ is a (maximal) uniformly J-positive subspace of $\mathcal{H}$. So, T_+ is a surjection from $\ell_2(I)$ onto the Hilbert space $(M_+, [\,,])$. Therefore, $\mathcal{F}_+$ is a frame for $(M_+, [\,,])$In particular, there exist constants $0 < A_+ \leq B_+$ such that (35) is satisfied for M_+. The assertion on $\mathcal{F}_-$ follows analogously.

Now, assuming that $\mathcal{F}$ is a J-frame for a Krein space $(\mathcal{H}, [\,,])$, a set of constants $\{B_-, A_-, A_+, B_+\}$ satisfying (35) is going to be computed. They depend only on the definiteness bounds for $R(T_\pm)$, the norm and the reduced minimum modulus of $T_\pm$.

Suppose that $\mathcal{F}$ is a J-frame for a Krein space $(\mathcal{H}, [\,,])$ with synthesis operator $T \in L(\ell_2(I),\mathcal{H})$. Since $R(T_+) = M_+$ is a (maximal) uniformly J-positive subspace of $\mathcal{H}$, there exists $\alpha_+ > 0$ such that $\alpha_+ \|f\|^2 \leq [f,f]$ $for\ every\ f \in M_+$. So,

$$\sum_{i\in I_+} \|f,f_i\|^2 = \|T_+^+ f\|^2 \leq \|T_+^+\|^2\|f\|^2 \leq B_+[f,f] \quad , for\ every\ f \in M_+$$

where $B_+ = \frac{\|T_+^+\|^2}{\alpha_+} = \frac{\|T_+\|^2}{\alpha_+}$. Furthermore, since $N(T_+^+)^\perp = J(M_+)\,, if\ f \in M_+$,

$$\sum_{i\in I_+} \|f,f_i\|^2 = \|T_+^+ f\|^2 \leq \left\|T_+^+ P_{J(M_+)} f\right\|^2 \geq \gamma(T_+^+)^2 \left\|P_{J(M_+)} f\right\|^2 = \gamma(T_+)^2 \left\|P_{M_+} J f\right\|^2$$

$$= \gamma(T_+)^2 \left\|G_{M_+} J f\right\|^2 \geq \gamma(T_+)^2 \gamma\left(G_{M_+}\right)^2 \|f\|^2 \geq A_+[f,f]$$

where $A_+ = \gamma(T_+)^2\gamma(G_{M_+})^2 = \gamma(T_+)^2\alpha_+^2$, see Remark (5.2..6).
Analogously, $A_- = -\gamma(T_-)^2\alpha_-^2$ and $B_- = \frac{\|T_-\|^2}{\alpha_-}$ satisfy Eq. (35) for every $f \in R(T_-) = M_-$,if α_- is the definiteness bound of the (maximal) uniformly J-negative subspace M_-.
Usually, the bounds $A_\pm = \pm\alpha_\pm^2\gamma(T_\pm)^2$ are not optimal for the J-frame $\mathcal{F}$.

Definition(5.2.12)[39]. Let F be a J-frame for the Krein space $\mathcal{H}$. The optimal constants $B_- \leq A_- < 0 < A_+ \leq B_+$ satisfying (35) are called the J-frame bounds of $\mathcal{F}$.

In order to compute the J-frame bounds associated to a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$, consider the uniformly J-definite subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$. Recall that $\mathcal{F}_+ = \{f_i\}_{i\in I}$ is a frame for the Hilbert space $(\mathcal{M}_+, [\,,])$. Then, if $G_+ = G_{\mathcal{M}_+}|_{\mathcal{M}_+} \in GL(\mathcal{M}_+)$ the frame bounds for $\mathcal{F}_+$ are given by $A_+ = \left\|(S_{G_+})^{-1}\right\|_+^{-1}$ *and* $B_+ = \| S_{G_+} \|_+$, where $S_{G_+} = T_+T_+^*G_+$ is the frame operator of $\mathcal{F}_+$and $\| f \|_+ = [f,f]^{1/2} = \| G_+^{1/2}f \|, f \in \mathcal{M}_+$, is the operator norm associated to the inner product $[\,,]$. Therefore,

$$A_+ = \left\|(S_{G_+})^{-1}\right\|_+^{-1} = \| G_+^{1/2}(T_+T_+^*G_+)^{-1} \|^{-1} = \| G_+^{-1/2}(T_+T_+^*)^{-1} \|^{-1},$$

and $B_+ = \| S_{G_+} \|_+ = \left\|G_+^{1/2}T_+T_+^*G_+\right\|$,
Analogously, it follows that $\mathcal{F}_- = \{f_i\}_{i\in I-}$ is a frame for the Hilbert space $(\mathcal{M}_-., -[\,,])$. So, the frame bounds for $\mathcal{F}_-$ are given by

$$A_- = \| G_-^{1/2}(T_-T_-^*G_-)^{-1} \|^{-1} \quad and\; B_- = \left\|G_-^{1/2}T_-T_-^*G_-\right\|$$

where $G_- = G_{\mathcal{M}_-}|_{\mathcal{M}_-} \in GL(\mathcal{M}_-)$. Thus, the J-frame bound associated to $\mathcal{F}$ can be fully characterized in terms of $T_\pm$ and the Gramian operators $G_\pm$ Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ in a Krein space $\mathcal{H}$, the inequalities:

$$A[f,f] \leq \sum_{i\in I} |[f,f_i]|^2 \leq B[f,f] \;\; for\; every\, f \in \mathcal{M} = \overline{span\{f_i : i \in I\}}, \tag{36}$$

with $B \geq A > 0$, ensure thatMis a J-nonnegative subspace of $\mathcal{H}$. However, they do not imply that $\mathcal{M}$ is uniformly J-positive, i.e. $(\mathcal{M}, [\,,])$ is not necessarily a inner product space. See the example below.

Example (5.2.13)[39]. Consider again the Krein space $(\mathbb{C}^3, [\,,])$ as in Example (5.2.9). as it was mentioned before, $\mathcal{M} = span\left\{f_1 = \left(1,0,1/\sqrt{2}\right), f_2 = \left(0,1,1/\sqrt{2}\right)\right\}$ is a J-nonnegative but not uniformly J-positive subspace of $\mathbb{C}^3$.

In this case, the orthogonal basis

$$v_1 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right), \;\; v_2 = \left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right) \;\; and \;\; v_3 = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-1\right),$$

is a basis of eigenvectors of $G_{\mathcal{M}}$, corresponding to the eigenvalues $\lambda_1 = 0, \lambda_2 = 1$ *and* $\lambda_3 = 0$, respectively. Moreover, $\mathcal{M} = span\,\{v_1, v_2\}$. Thus, if $f \in \mathcal{M}$ there exists $\alpha, \beta \in \mathbb{C}$ such that $f = \alpha v_1 + \beta, v_2$ and then, since $G_{\mathcal{M}}v_1 = 0 \in \mathbb{C}^3$, it is easy to see that

$$|[f, f_1]|^2 + |[f, f_2]|^2 = |\beta|^2(|\langle v_2, f_1\rangle|^2 + |\langle v_2, f_2\rangle|^2) = |\beta|^2 = [f,f].$$

Therefore, (36) holds with $A = B = 1$, but $\{ f_1, f_2 \}$ cannot be extended to a J-frame, since $\mathcal{M}$ is not a uniformly J-positive subspace.

The next result gives a complete characterization of the families satisfying (36) for $B \geq A > 0$. It is straightforward to formulate and show analogues of all these assertions for a family satisfying (36) for negative constants $B \leq A < 0$.

Proposition (5.2.14)[39]. Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ in a Krein space $\mathcal{H}$, let $\mathcal{M} = \overline{span\{f_i : i \in I\}}$, and $\mathcal{N} = \mathcal{M} \cup \mathcal{M}^{[\perp]}$ .If there exist constants $0 < A \leq B$ such that

$$A\,[f,f] \leq \sum_{i\in I} |[f,f_i]|^2 \leq B\,[f,f] for\; every\, f \in \mathcal{M}, \tag{37}$$

then $\mathcal{M} \ominus \mathcal{N}$ is a (closed) uniformly J-positive subspace of $\mathcal{M}$. Moreover, if $\mathcal{F}$ is a frame for the Hilbert space $(\mathcal{M}, \langle\,,\rangle)$, the converse holds.

Proof. First, suppose that there exist $0 < A \leq B$ such that (37) holds. So, $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$, or equivalently, $(\mathcal{M}, \langle\,,\rangle)$is a semi-inner product space.

If $T \in L(\ell_2(I),\mathcal{H})$ is the synthesis operator of the Bessel sequence $\mathcal{F}$ and $C = \| T^* \|^2 > 0$, then $TT^* \leq CP_{\mathcal{M}}$. So, using (37) it is easy to see that:

$$A\langle G_{\mathcal{M}}f,f\rangle \leq \|T^+(P_{\mathcal{M}}f)\|^2 = \langle (P_{\mathcal{M}}JTT^*)f,f\rangle\,,\ f\in\mathcal{H}\,. \tag{38}$$

Thus, $0 \leq G_{\mathcal{M}} \leq \frac{C}{A}(G_{\mathcal{M}})^2$. Applying Theorem (5.2.1) it is easy to see that $R((G_{\mathcal{M}})^{1/2}) \subseteq R(G_{\mathcal{M}}) \subseteq R((G_{\mathcal{M}})^{1/2})$.

Moreover, it follows by Corollary (5.2..2) that $R(G_{\mathcal{M}})$ is closed because

$$R(G_{\mathcal{M}}) = R\big((G_{\mathcal{M}})^{1/2}\big).$$

Let $\mathcal{M}' = \mathcal{M} \ominus \mathcal{N}$ and notice that $\mathcal{M}'$ is a closed uniformly J-positive subspace of $\mathcal{H}$. In fact, since $R(G_{\mathcal{M}})$ is closed, there exists $\alpha > 0$ such that

$$[f,f] = \langle G_{\mathcal{M}}f,f\rangle = \| (G_{\mathcal{M}})^{1/2}f \|^2 \geq \alpha \| f \|^2 \ for\ every\ f \in \mathcal{N}(G_{\mathcal{M}}) = \mathcal{M}\ominus\mathcal{N}.$$

Conversely, suppose that $\mathcal{F}$ is a frame for $(\mathcal{M},\langle\,,\rangle)$, i.e. there exist constants $B' \geq A' > 0$ such that $A'P_{\mathcal{M}} \leq TT^* \leq B'P_{\mathcal{M}}$, where $T \in L(\ell_2(I),\mathcal{M})$ is the synthesis operator of $\mathcal{F}$. If $\mathcal{M}' = \mathcal{M}\ominus\mathcal{N}$ is a uniformly J-positive subspace of $\mathcal{H}$, then there exists $\alpha > 0$ such that $\alpha P_{\mathcal{M}'} \leq G_{\mathcal{M}'} \leq P_{\mathcal{M}}$. As a consequence of Theorem (5.2.1), $R((G_{\mathcal{M}'})^{1/2}) = \mathcal{M}' = R(G_{\mathcal{M}'})$. Since $G_{\mathcal{M}} = G_{\mathcal{M}'}$ it is easy to see that

$$A'(G_{\mathcal{M}})^2 = A'(G_{\mathcal{M}'})^2 \leq P_{\mathcal{M}}JTT^*JP_{\mathcal{M}} \leq B'(G_{\mathcal{M}'})^2 = B'(G_{\mathcal{M}})^2.$$

Therefore, $R(P_{\mathcal{M}}JT) = R(G_{\mathcal{M}'}) = R((G_{\mathcal{M}'})^{1/2})$, or equivalently, there exist $B \geq A > 0$ such that

$$AG_{\mathcal{M}} = AG_{\mathcal{M}'} \leq P_{\mathcal{M}}JTT^*JP_{\mathcal{M}} \leq BG_{\mathcal{M}'} = BG_{\mathcal{M}},$$

$$i.e. A[f,f] \leq \sum_{i\in I} |[f,f_i]|^2 \leq B[f,f]\ \ for\ every\ f \in \mathcal{M}.$$

Theorem(5.2.15)[39]. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. If $I_\pm = \{i \in I : \pm[f_i,f_i] \geq 0\}$ and $\mathcal{M}_\pm = \overline{span\{f_i : i \in I_\pm\}}$, then, $\mathcal{F}$ is a J-frame if and only if $\mathcal{M}_\pm \cap \mathcal{M}^{[\perp]} = 0$ and there exist constants $B_- \leq A_- < 0 < A_+ \leq B_+$ such that

$$A_\pm[f,f] \leq \sum_{i\in I_\pm} |[f,f_i]|^2 \leq B_\pm[f,f] \quad for\ every\ f \in \mathcal{M}_\pm. \tag{39}$$

Proof. If $\mathcal{F}$ is a J-frame, the conditions on $\mathcal{M}_\pm$ follow by its definition and by Proposition (5.2.11). Conversely, if $\mathcal{M}_+$ is J-non degenerated and there exist constants $0 < A_+ \leq B_+$ such that

$$A_+[f,f] \leq \sum_{i\in I_\pm} |[f,f_i]|^2 \leq B_+[f,f]\ for\ every\ f \in \mathcal{M}_+$$

then, by Proposition (5.2.14), $\mathcal{M}_+$ is a uniformly J-positive subspace of $\mathcal{H}$. Therefore, there exist constants $0 < A \leq B$ such that

$$A \| P_{\mathcal{M}_+}f \|^2 \leq \| T_+^+ P_{\mathcal{M}_+}f \|^2 \leq B \| P_{\mathcal{M}_+}f \|^2\ for\ every\ f \in \mathcal{H}.$$

But these inequalities can be rewritten as

$$A\,P_{\mathcal{M}_+} \leq P_{\mathcal{M}_+}JT_+T_+^*JP_{\mathcal{M}_+} \leq B\,P_{\mathcal{M}_+}.$$

Then, by Theorem (5.2.1), $R(P_{\mathcal{M}_+}JT_+) = R(P_{\mathcal{M}_+}) = \mathcal{M}_+$. Furthermore, $P_{J(\mathcal{M}_+)}(R(T_+)) = J(\mathcal{M}_+)$ because

$$J(\mathcal{M}_+) = J(R(P_{\mathcal{M}_+}JT_+)) = R((JP_{\mathcal{M}_+}J)T_+) = R(P_{J(\mathcal{M}_+)}T_+) = P_{J(\mathcal{M}_+)}(R(T_+)).$$

Therefore, taking the counterimage of $P_{J(\mathcal{M}_+)}(R(T_+))\ by\ P_{J(\mathcal{M}_+)}$, it follows that

$$\mathcal{H} = R(T_+) \dotplus J(\mathcal{M}_+)^\perp \subseteq \mathcal{M}_+ \dotplus \mathcal{M}_+^{[\perp]} = \mathcal{H}.$$

Thus, $R(T_+) = \mathcal{M}_+$ and $\mathcal{F}_+$ is a frame for $\mathcal{M}_+$. Analogously, $\mathcal{F}_- = \{f_i\}_{i\in I_-}$ is a frame for $\mathcal{M}_-$. Finally, since $\mathcal{F}$ is a frame for $\mathcal{H}$, $\mathcal{H} = R(T) = R(T_+) + R(T_-)$, which shows the maximality of $R(T_\pm)$. Thus, $\mathcal{F}$ is a J-frame for $\mathcal{H}$.

Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a J-frame for $\mathcal{H}$ and consider $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$ the partition of $\mathcal{F}$ into J-positive and J-negative vectors. Moreover, let $\mathcal{M}_\pm$ be the (maximal) uniformly J-definite subspace of

$\mathcal{H}$ generated by $\mathcal{F}_\pm$.

The aim of this section is to show that it is possible to bound the correlation between vectors in $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) and vectors in the cone of neutral vectors $\mathcal{C} = \{n \in \mathcal{H} : [n,n] = 0\}$, in a strong sense:

$$|\langle f, n\rangle| \leq c_\pm \|f\| \|n\|, f \in \mathcal{M}_\pm, n \in \mathcal{C}, \quad (40)$$

for some constants $\frac{\sqrt{2}}{2} \leq c_\pm < 1$. In order to make these ideas precise, consider the notion of minimal angle between a subspace $\mathcal{M}$ and the cone $\mathcal{C}$.

Definition (5.2.16)[39]. Given a closed subspace $\mathcal{M}$ of the Krein space $\mathcal{H}$, consider

$$c_0(\mathcal{M},\mathcal{C}) = sup\{|\langle m,n\rangle| : m \in \mathcal{M}, n \in \mathcal{C}. \|n\| = \|m\| = 1\}. \quad (41)$$

Then, there exists a unique $\theta(\mathcal{M},\mathcal{C}) \in [0,\frac{\pi}{4}]$ such that $cos(\theta(\mathcal{M},\mathcal{C})) = c_0(\mathcal{M},\mathcal{C})$. In this case, $\theta(\mathcal{M},\mathcal{C})$ is the minimal angle between $\mathcal{M}$ and $\mathcal{C}$.

Observe that if the subspace $\mathcal{M}$ contains a non-trivial J-neutral vector (e.g. if $\mathcal{M}$ is J-indefinite or J-semidefinite) then $c_0(\mathcal{M},\mathcal{C}) = 0$, or equivalently, $\theta(\mathcal{M},\mathcal{C}) = 0$. On the other hand, it will be shown that the minimal angle between a uniformly J-positive (resp. uniformly J-negative) subspace $\mathcal{M}$ and $\mathcal{C}$ is always bounded away from 0.

Proposition (5.2.17)[39]. Let $\mathcal{M}$ be a J-semidefinite subspace of $\mathcal{H}$ with definiteness bound α. Then,

$$c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}}\right). \quad (42)$$

In particular, $\mathcal{M}$ is uniformly J-definite if and only if $c_0(\mathcal{M},\mathcal{C}) < 1$.

Proof. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fundamental decomposition of $\mathcal{H}$ and suppose that $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$.

Let $m \in \mathcal{M}\ with\ \|m\| = 1$. Then, there exist (unique) $m_\pm \in \mathcal{H}_\pm$ such that $m = m^+ + m^-$. In this case,

$$\mathbf{1} = \|\boldsymbol{m}\|^2 = \|m^+\|^2 + \|m^-\|^2 \ \boldsymbol{and}\ \boldsymbol{\alpha} \leq [\boldsymbol{m},\boldsymbol{m}] = \|m^+\|^2 - \|m^-\|^2. \quad (43)$$

Claim. For a fixed $m \in \mathcal{M}\ with\ \|m\| = 1, sup\{|\langle m,n\rangle| : n \in \mathcal{C}, \|n\| = 1\} = \frac{1}{\sqrt{2}}(\|m^+\| + \|m^-\|)$.

Indeed, consider $n \in \mathcal{C}\ with\ \|n\| = 1$. Then, there exist (unique) $n_\pm \in \mathcal{H}_\pm$ such that $n = n^+ + n^-$. In this case,

$$0 = [n,n] = \|n^+\|^2 - \|n^-\|^2 \ and\ 1 = \|n\|^2 = \|n^+\|^2 + \|n^-\|^2,$$

which imply that $\|n^+\| = \|n^-\| = \frac{1}{\sqrt{2}}$. Therefore,

$$|\langle m,n\rangle| \leq |\langle m^+,n^+\rangle| + |\langle m^-,n^-\rangle| \leq \frac{1}{\sqrt{2}}(\|m^+\| + \|m^-\|).$$

On the other hand, if $m^- \neq 0$ then let $n_m := \frac{1}{\sqrt{2}}(\frac{m^+}{\|m^+\|} + \frac{m^-}{\|m^-\|})$, otherwise consider $n_m = \frac{1}{\sqrt{2}}(m + z)$, with $z \in \mathcal{H}_-, \|z\| = 1$.

Now, it is easy to see that $n_m \in \mathcal{C}$ and that $|\langle \boldsymbol{m}, \boldsymbol{n_m}\rangle| = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\|\boldsymbol{m}^+\| + \|\boldsymbol{m}^-\|)$ which together with the previous facts show the claim.

Now, let $\mathcal{M}_1 = \{m = m^+ + m^- \in \mathcal{M} : m^\pm \in \mathcal{H}_\pm, \|m\| = 1\}$. Using the claim above it follows that

$$c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}} sup_{m\in\mathcal{M}_1}(\|m^+\| + \|m^-\|). \quad (44)$$

If $\alpha = 1$ then $\mathcal{M}$ is a subspace of $\mathcal{H}_+$. Also, it is easy to see that $c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}}$. Thus, in this particular case,

$$c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}}\right).$$

On the other hand, if $\alpha < 1$, let $k_0 \in \mathbb{N}$ be such that $\frac{1-\alpha}{2} > \frac{1}{2k_0}$. Observe that, by the definition of the definiteness bound, for every integer $k \geq k_0$ there exists $m_k = m_k^+ + m_k^- \in \mathcal{M}_1$ such that $\alpha \leq \|m_k^+\|^2 - \|m_k^-\|^2 < \alpha + \frac{1}{k}$. Then, it follows that

$$\alpha + 1 \leq 2 \| m_k^+ \|^2 < \alpha + 1 + \frac{1}{k},$$

or equivalently, $\sqrt{\frac{1+\alpha}{2}} \leq \| m_k^+ \| < \sqrt{\frac{1+\alpha}{2} + \frac{1}{2k_0}}$. Moreover, $\| m_k^- \| = \sqrt{1 - \| m_k^+ \|^2}$ implies that $\sqrt{\frac{1-\alpha}{2} - \frac{1}{2k}} < \| m_k^- \| \leq \sqrt{\frac{1-\alpha}{2}}$.

Therefore, for every integer $k \geq k_0$ there exists $m_k \in \mathcal{M}_1$ such that

$$\sqrt{\frac{1-\alpha}{2} - \frac{1}{2k}} + \sqrt{\frac{1+\alpha}{2}} < \| m_k^+ \| + \| m_k^- \| < \sqrt{\frac{1+\alpha}{2} + \frac{1}{2k_0}} + \sqrt{\frac{1-\alpha}{2}}.$$

Thus, $c_0(\mathcal{M}, \mathcal{C}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}}\right)$.

Assume now that $\mathcal{M}$ is a J-nonpositive subspace of $(\mathcal{H}, [\,,])$ with definiteness bound α, for $0 \leq \alpha \leq 1$. Then, $\mathcal{M}$ is a J-nonnegative subspace of the antispace $(\mathcal{H}, -[\,,])$, with the same definiteness bound α. Furthermore, the cone of J-neutral vectors for the antispace is the same as for the initial Krein space $(\mathcal{H}, [\,,])$. Therefore, we can apply the previous arguments and conclude that (42) also holds for J-nonpositive subspaces. Finally, the last assertion in the statement follows from the formula in (42).

Let $\mathcal{F}$ be a J-frame for $\mathcal{H}$ as above. Notice that (40) holds for some constant $\frac{\sqrt{2}}{2} \leq c_\pm < 1$ if and only if $c_0(\mathcal{M}_\pm, \mathcal{C}) < 1$,i.e. that the minimal angles $\theta(\mathcal{M}_\pm, \mathcal{C})$ are bounded away from 0. This is intimately related with the fact that the aperture between the subspaces $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) and $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) is bounded away from $\frac{\pi}{4}$, whenever $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$is a fundamental decomposition.

Also, if α is the definiteness bound of $\mathcal{M}$ then $\| K \| = \sqrt{\frac{1-\alpha}{1+\alpha}}$, see [78]. Therefore, $\Phi(\mathcal{M}, \mathcal{H}_+) = \frac{\|K\|}{\sqrt{1+\|K\|^2}} = \sqrt{\frac{1-\alpha}{2}}$. Since $\Phi(\mathcal{M}, \mathcal{H}_+) = \sin\varphi(\mathcal{M}, \mathcal{H}_+)$ for an angle $\varphi(\mathcal{M}, \mathcal{H}_+) \in [0, \frac{\pi}{4}]$ between $\mathcal{M}$ and $\mathcal{H}_+$, it is easyto see that

$$\cos\varphi(\mathcal{M}, \mathcal{H}_+) = \sqrt{1 - \sin^2\varphi(\mathcal{M}, \mathcal{H}_+)} = \sqrt{\frac{1+\alpha}{2}}.$$

Therefore, if $\varphi = \varphi(\mathcal{M}, \mathcal{H}_+)$,

$$\cos\left(\frac{\pi}{4} - \varphi\right) = \frac{\sqrt{2}}{2}(\cos\varphi + \sin\varphi) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{1+\alpha}{2}} + \sqrt{\frac{1-\alpha}{2}}\right) = \cos(\theta(\mathcal{M}, \mathcal{C})),$$

i.e. $\varphi(\mathcal{M}, \mathcal{H}_+) + \theta(\mathcal{M}, \mathcal{C}) = \frac{\pi}{4}$.

Remark (5.2.18)[39]. Regarding the discussion at the beginning of this section, consider any (redundant) J-frame $\mathcal{F} = \{f_i\}_{i \in I}$ for $(\mathcal{H}, [\,,])$. As usual, denote $\mathcal{M}_+$ and $\mathcal{M}_-$ the maximal uniformly J-definite subspaces generated by $\mathcal{F}$. Since $\mathcal{M}_\pm$ is uniformly J-definite, Proposition (5.2.17) shows that $c_0(\mathcal{M}_\pm, \mathcal{C}) < 1$. That is, J-frames showide a class of frames for $\mathcal{H}$ with the desired properties, namely the correlation between the sampling vectors and the cone of disturbances is controlled by $c_0(\mathcal{M}_\pm, \mathcal{C})$ because

$$|\langle f_i, n \rangle| \leq c_0(\mathcal{M}_\pm, \mathcal{C}) \| f_i \| \| n \| \ whenever\ i \in I_\pm\}\ and\ n \in \mathcal{C}. \quad (45)$$

Moreover, later in Proposition (5.2.25), it will be shown that the J-frame $\mathcal{F}$ admits a (canonical) dual J-frame that induces a linear (indefinite) stable and redundant encoding–decoding scheme in which the correlation between both the sampling and reconstructing vectors and the cone of neutral vectors is bounded from above. These remarks showide a quantitative measure of the advantage of considering J-frames with respect to usual frames in this setting.

If $\mathcal{F}$ is a J-frame with synthesis operator , then $QT = T_+ = TP_+$, where $Q = P_{\mathcal{M}_+//\mathcal{M}_-}$. Therefore,

$$Q = QTT^\dagger = TP_+T^\dagger.$$

So, given a surjective operator $T : \ell_2(I) \to \mathcal{H}$, the idempotency of $TP_+T^\dagger$ is a necessary condition for T to be the synthesis operator of a J-frame.

Lemma (5.2.19)[39]. Let $T \in L(\ell_2(I),\mathcal{H})$ be surjective. Suppose that $P_{\mathcal{S}}$ is the orthogonal projection onto a closed subspace $\mathcal{S}$ of $\ell_2(I)$ such that $c(\mathcal{S}, N(T)^{\perp}) < 1$. Then, $TP_{\mathcal{S}}T^{\dagger}$ is a projection if and only if $N(T) = \mathcal{S} \cap N(T) \oplus \mathcal{S}^{\perp} \cap N(T)$.

Proof. Suppose that $Q = TP_{\mathcal{S}}T^{\dagger}$ is a projection. Then, if $P = P_{N(T)^{\perp}}, E = PP_{\mathcal{S}}P$ is an orthogonal projection because it is selfadjoint and

$$E^2 = (PP_{\mathcal{S}}P)^2 = PP_{\mathcal{S}}PP_{\mathcal{S}} = T^{\dagger}(TP_{\mathcal{S}}T^{\dagger})^2T = T^{\dagger}(TP_{\mathcal{S}}T^{\dagger})T = PP_{\mathcal{S}}P = E.$$

Therefore, $(PP_{\mathcal{S}})^k = E^{k-1}P_{\mathcal{S}} = EP_{\mathcal{S}} = (PP_{\mathcal{S}})^2$ *for every* $k \geq 2$. So, by [85], $PP_{\mathcal{S}} = P_{\mathcal{S}} \wedge P = PP_{\mathcal{S}}$. Then, since $P_{\mathcal{S}}$ and P commute, it follows that $N(T) = \mathcal{S} \cap N(T) \oplus \mathcal{S}^{\perp} \cap N(T)$ (see [85]).

Conversely, suppose that $N(T) = \mathcal{S} \cap N(T) \oplus \mathcal{S}^{\perp} \cap N(T)$. Then, $P_{\mathcal{S}}$ and P commute and

$$(TP_{\mathcal{S}}T^{\dagger})^2 = TP_{\mathcal{S}}(TT^{\dagger})PST^{\dagger} = TP_{\mathcal{S}}PP_{\mathcal{S}}T^{\dagger} = TPP_{\mathcal{S}}T^{\dagger} = TP_{\mathcal{S}}T^{\dagger}.$$

Hereafter consider the set of possible decompositions of $\mathcal{H}$ as a (direct) sum of a pair of maximal uniformly definite subspaces, or equivalently, the associated set of projections:

$\mathfrak{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q, R(Q)$ is uniformly J-positive and $N(Q)$ is uniformly J-negative$\}$.

Proposition(5.2.20)[39]. Let $T \in L(\ell_2(I),\mathcal{H})$ be surjective. Then, T is the synthesis operator of a J-frame if and only if there exists $I_+ \subset I$ such that $\ell_2(I_+)$ (as a subspace of $\ell_2(I)$ satisfies $c(N(T)^{\perp}, \ell_2(I_+)) < 1$ and

$$TP_+T^{\dagger} \in Q,$$

where $P_+ \in L(\ell_2(I))$ is the orthogonal projection onto $\ell_2(I_+)$.

Proof. If T is the synthesis operator of a J-frame, the existence of such a subset I_+ has already been discussed before.

Conversely, suppose that there exists such a subset $I_+$of I. Then, since $c(N(T)^{\perp}, \ell_2(I_+)) < 1$ and $= TP_+T^{\dagger} \in Q$, it follows from Lemma (5.2.19) that P_+ and $P = PN(T)^{\perp}$ commute. Therefore,

$$QT = TP_+P = TPP_+ = TP_+,$$

and $(I - Q)T = T(I - P_+). Hence, R(TP_+) = R(Q)$ is (maximal) uniformly J-positive and

$R(T(I - P_+)) = N(Q)$ is (maximal) uniformly J-negative. Therefore $\mathcal{F} = \{Te_i\}_{i\in I}$ is by definition a J-frame for $\mathcal{H}$.

Theorem(5.2.21)[39]. Given a surjective operator $T \in L(\ell_2(I),\mathcal{H})$, the following conditions are equivalent:

(i) There exists $U \in \mathcal{U}(\ell_2(I))$ such that TU is the synthesis operator of a J-frame.

(ii) There exists $Q \in \mathcal{Q}$ such that

$$QTT^*(I - Q)^* = 0. \tag{46}$$

(iii) There exist closed range operators $T_1, T_2 \in L(\ell_2(I),\mathcal{H})$ such that $T = T_1 + T_2, R(T_1)$ is uniformly J-positive, $R(T_2)$is uniformly J-negative and $T_1T_2^* = T_2T_1^* = 0$.

Proof. $(i) \Rightarrow (ii)$: Suppose that there exists $U \in \mathcal{U}(\ell_2(I))$ such that $V = TU$ is the synthesis operator of a J-frame. If $I_{\pm} = \{i \in I : \pm[Ve_i, Ve_i] > 0\}$ *and* $P_{\pm} \in L(\ell_2(I))$ is the orthogonal projection onto $\ell_2(I_{\pm})$, define $V_{\pm} = VP_{\pm}$. Then, $V = V_+ + V_-$ *and* $\mathcal{M}_{\pm} = R(V_{\pm})$ is a maximal uniformly J-definite subspace. So, considering $Q = P_{\mathcal{M}_+//\mathcal{M}_-} \in \mathcal{Q}$, it is easy to see that $QV = V_+, (I - Q)V = V_-$and

$$QTT^*(I - Q)^* = QVV^*(I - Q)^* = V_+V_-^* = VP_+P_-V^* = 0.$$

$(ii) \Rightarrow (iii)$: Suppose that there exists $Q \in \mathcal{Q}$ such that $QTT^*(I - Q)^* = 0$. Defining $T_1 = QT$ and $T_2 = (I - Q)T$, it follows that $T = T_1 + T_2, R(T_1) = R(Q)$ is uniformly J-positive, $R(T_2) = N(Q)$ is uniformly J-negative and

$$T_1T_2^* = T_2T_1^* = 0,$$

because (46) says that $R(T_2^*) = R(T^*(I - Q)^*) \subseteq N(QT) = N(T_1)$.

$(iii) \Rightarrow (i)$: If there exist closed range operators $T_1, T_2 \in L(\ell_2(I),\mathcal{H})$satisfying the conditions of item 3, notice that $T_1T_2^* = 0$ implies that $N(T_2)^{\perp} \subseteq N(T_1)$, or equivalently, $N(T_1)^{\perp} \subseteq N(T_2)$.

Consider the projection $Q = P_{R(T_1)//R(T_2)} \in \mathcal{Q}$ and notice that $QT = T_1$ and $(I - Q)T = T_2$. If $B_1 = \{u_i\}_{i\in I_1}$ is an orthonormal basis of $N(T_1)^{\perp}$, consider the family $\{f_i^+\}_{i\in I_1}$ *in* $\mathcal{H}$ *given by* $f_i^+ =$

Tu_i. $But,\ if\ i\ \in\ I_1,\quad f_i^+ = QTu_i + (I-Q)Tu_i = T_1 u_i R(T_1),$ because $u_i \in N(T_1)^{\perp} \subseteq N(T_2)$. Therefore, $\{f_i^+\}_{i\in I_1} \subseteq R(T_1)$. Since T_1 is an isomorphism between $N(T_1)^{\perp}$ and $R(T_1)$, it follows that $R(T_1) = \overline{span\{f_i^+\}_{i\in I_1}}$.

Analogously, if $\mathcal{B}_2 = \{b_i\}_{i\in I_2}$ is an orthonormal basis of $N(T_1)$the family $\{f_i^-\}_{i\in I_2}$ defined by $f_i^- = Tb_i (i \in I_2)\ lies\ in\ R(T_2)$.
Since T_2 is an isomorphism between $N(T_2)^{\perp}$ and $R(T_2)$, it follows that

$$R(T_2) = T_2(N(T_1)) \subseteq \overline{span\{f_i^-\}_{i\in I_2}} \subseteq R(T_2).$$

Finally, consider $U \in \mathcal{U}(\ell_2(I))$ which turns the standard orthonormal basis $\{e_i\}_{i\in I}\ into\ \mathcal{B}_1 \cup \mathcal{B}_2. Then, if\ V = TU$ and $\mathcal{F} = \{Ve_i\}_{i\in I} = \{f_i^+\}_{i\in I_1} \cup \{f_i^-\}_{i\in I_2}$, it is easy to see that

$$I_+ = \{i \in I : [Ve_i, Ve_i] > 0\} = I_1\ and\ I_- = \{i \in I : [Ve_i, Ve_i] < 0\} = I_2.$$

So, $R(V_+) = R(T_1)$ is maximal uniformly J-positive and $R(V_-) = R(T_2)$ is maximal uniformly J-negative. Therefore, $\mathcal{F}$ is a J-frame for $\mathcal{H}$with synthesis operator $V = TU$.

Definition (5.2.22)[39]. Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$, the J-frame operator $S: \mathcal{H} \to \mathcal{H}$ is defined by

$$Sf = \sum_{i\in I} \sigma_i [f, f_i] f_i,\ \ for\ every\ f \in \mathcal{H},$$

$where\ \sigma_i = sgn([f_i, f_i])$.

The following proposition compiles some basic properties of the J-frame operator.

Proposition(5.2.23)[39]. Let $\mathcal{F} = \{f_i\}_{i\in I}$, be a J-frame with synthesis operator $T \in L(\ell_2(I)), \mathcal{H})$. Then, its J-frame operator $S \in L(\mathcal{H})$ satisfies:

(i) $S = TT^+$;

(ii) $S = S_+ - S_-$, where $S_+ := T_+T_+^+$ and $S_- := -T_-T_-^+$ are J-positive operators;

(iii) S is an invertible J-selfadjoint operator;

(iv) $ind_{\pm}(S) = dim\mathcal{H}_{\pm}$, where $ind_{\pm}(S)$ are the indices of S.

Proof. If $\mathcal{F} = \{f_i\}_{i\in I}$, is a J-frame with synthesis operator $T \in L(\ell_2(I)), \mathcal{H})$., then $T^+f = \sum_{i\in I} \sigma_i [f, f_i] e_i\ for\ f \in \mathcal{H}$. So,

$$TT^+f = T\left(\sum_{i\in I} \sigma_i [f, f_i] e_i\right) = \sum_{i\in I} \sigma_i [f, f_i] f_i = Sf, for\ every\ f \in \mathcal{H}.$$

Furthermore, if $I_{\pm} = \{i \in I : \pm[f_i, f_i] > 0\}$, consider $T_{\pm} = TP_{\pm}$ as usual. Then,

$$TT^+ = (T_+ + T_-)(T_+ + T_-)^+ = T_+T_+^+ + T_-T_-^+ = T_+T_+^+ - (-T_-T_-^+),$$

because $T_+T_-^+ = T_-T_+^+ = 0$. Therefore, $S = S_+ - S_-\ if\ S_{\pm} := \pm T_{\pm}T_{\pm}^+$. Notice that $S_{\pm}$ is a J-positive operator because

$$S_{\pm} = \pm T_{\pm}T_{\pm}^+ = \pm T_{\pm}J_2T_{\pm}^*J = T_{\pm}T_{\pm}^*J$$

To show the invertibility of S observe that, if $Sf = 0$ then $S_+f = S_-f$. But $R(S_+) \cap R(S_-) \subseteq R(T_+) \cap R(T_-) = \{0\}$. Thus, S is injective. On the other hand, $R(S) = S(\mathcal{M}_+^{[\perp]}) + S(\mathcal{M}_-^{[\perp]})$ because$\mathcal{H} = \mathcal{M}_+^{[\perp]} \dot{+} \mathcal{M}_-^{[\perp]}$. But it is easy to see that $\mathcal{M}_{\pm}^{[\perp]} \subseteq N(S_{\pm})$.So, $S(\mathcal{M}_{\pm}^{[\perp]}\}) = S_{\mp}(\mathcal{M}_{\pm}^{[\perp]}\})$ and $R(S) = S_-(\mathcal{M}_{\pm}^{[\perp]}) + S_2(\mathcal{M}_-^{[\perp]}) = R(S_-) + R(S_+) = \mathcal{M}_+ + \mathcal{M}_- = \mathcal{H}$. Therefore, S is invertible.

Finally, the identities $ind_{\pm}(S) = dim\mathcal{H}_{\pm}$ follow from the indices definition. Recall that if $A \in L(\mathcal{H})$ is a J-selfadjoint operator, $ind_+(A)$ is the supremum of all positive integers r such that there exists a positive invertible matrix of the form $([Ax_j, x_k])_{j,k=1,\dots,r}, where\ x_1, \dots, x_r \in \mathcal{H}$ (if no such r exists, $ind_-(A) = 0$). Similarly, $ind_-(A) = ind_+(-A)$ is the supremum of all positive integers m such that there exists a negative invertible matrix of the form $([Ay_j, y_k])_{j,k=1,\dots,m}, where\ y_1, \dots, y_r \in \mathcal{H}$, see [53].

Corollary(5.2.24)[39]. Let $\mathcal{F} = \{f_i\}_{i\in I}$, be a J-frame for $\mathcal{H}$ with J-frame operator $S \in L(\mathcal{H})$. Then, $(S_{\pm}) = \mathcal{M}_{\pm}\ and\ N(S_{\pm}) = \mathcal{M}_+^{[\perp]}$.

Furthermore, if $Q = P_{\mathcal{M}+//M_-}$,

$$S_+ = QSQ + \text{ and } S_- = -(I - Q)S(I - Q)^+. \tag{47}$$

Proof. Recall that $S_+ := T_+T_+^+ = T_+(J_2T_+^*J) = T_+T_+^*J$. Then, $R(S_+) = R(T_+T_+^*J) = R(T_+T_+^*) = R(T_+) = \mathcal{M}_+$ because $R(T_+)$ is closed. Since S_+ is J-selfadjoint, it follows that $N(S_+) = R(S_+)^{[\perp]} = \mathcal{M}_+^{[\perp]}$. Analogously, $R(S_-) = \mathcal{M}_-$ and $N(S_-) = \mathcal{M}_-^{[\perp]}$. Since $S = S_+ - S_-$, if $Q = P_{\mathcal{M}+//M_-}$ then

$$QS = Q(S_+ - S_-) = S_+,$$

by the characterization of the range and nullspace of S_+. Therefore, $SQ_+ = QS = QSQ_+$. Analogously, $S(I-Q)^+ = (I-Q)S = (I - Q)S(I-Q)^+$.

The above corollary states that S is the diagonal block operator matrix

$$S = \begin{pmatrix} S_+ & 0 \\ 0 & -S_- \end{pmatrix}, \tag{48}$$

according to the (oblique) decompositions $\mathcal{H} = \mathcal{M}_-^{[\perp]} \dot{+} \mathcal{M}_+^{[\perp]}$ and $\mathcal{H} = \mathcal{M}_+ \dot{+} \mathcal{M}_-$ of the domain and codomain of S, respectively.

Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ with synthesis operator , there is a duality between $\mathcal{F}$ and the frame $g = \{g_i\}_{i\in I}$ given by $g_i = S^{-1}f_i$: $f_i\ f \in \mathcal{H}$,

$$f = SS^{-1}f = TT^+(S^{-1}f) = T\left(\sum_{i\in I}\sigma_i[S^{-1}f, f_i]e_i\right)$$

$$= \sum_{i\in I}\sigma_i[S^{-1}f, f_i]f_i = \sum_{i\in I}\sigma_i[f, S^{-1}f_i]f_i$$

Analogously,

$$f = S^{-1}Sf = S^{-1}(TT^+f) = S^{-1}\left(\sum_{i\in I}\sigma_i[f, f_i]e_i\right) = \sum_{i\in I}\sigma_i[S^{-1}f, f_i]f_i = \sum_{i\in I}\sigma_i[f, f_i]S^{-1}f_i.$$

Therefore, for every $f \in \mathcal{H}$, there is an indefinite reconstruction formula associated to $\mathcal{F}$:

$$f = \sum_{i\in I}\sigma_i[f, g_i]f_i = \sum_{i\in I}\sigma_i[f, f_i]g_i. \tag{49}$$

The following question arises naturally: is $g = \{S^{-1}f_i\}_{i\in I}$ also a J-frame for $\mathcal{H}$?

Proposition(5.2.25)[39]. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a J-frame for a Krein space $\mathcal{H}$ with J-frame operator S, then $g = \{S^{-1}f_i\}_{i\in I}$ is also a J-frame for $\mathcal{H}$.

Proof. Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $\mathcal{H}$ with J-frame operator S, observe that the synthesis operator of $g = \{S^{-1}f_i\}_{i\in I}$ is $V := S^{-1}T \in L(\ell_2(I), \mathcal{H})$. Furthermore, by Corollary (5.2.24), $S(\mathcal{M}_\mp^{[\perp]}) = \mathcal{M}_\pm$. Then, $S^{-1}(\mathcal{M}_\pm) = \mathcal{M}_\mp^{[\perp]}$ and it follows that $[S^{-1}f_i, S^{-1}f_i] > 0$ if and only if $[f_i, f_i] > 0$.

Thus, $V_\pm = VP_\pm = S^{-1}T_\pm$ and $R(V_+)$ (resp. $R(V_-)$is a maximal uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. So, g is a J-frame for $\mathcal{H}$.

If $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame for a Hilbert space $\mathcal{H}$ with synthesis operator $T \in L(\ell_2(I), \mathcal{H})$., then the family $\{(TT^*)^{-1}f_i\}_{i\in I}$ is called the canonical dual frame because it is a dual frame for $\mathcal{F}$ (see (32)) and it has the following optimal property: Given $f \in \mathcal{H}$,

$$\sum_{i\in I}|\langle f, (TT^*)^{-1}f_i\rangle|^2 \le \sum_{i\in I}|c_i|^2, \text{whenever } f = \sum_{i\in I}c_if_i, \tag{50}$$

for a family $(c_i)_{i\in I} \in \ell_2(I)$. In other words, the above representation has the smallest ℓ_2-norm among the admissible frame coefficients representing f (see [61]).

In a Hilbert space $\mathcal{H}$, it is well known that every positive invertible operator $S \in L(\mathcal{H})$ can be realized as the frame operator of a frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $\mathcal{H}$, see [43]. Indeed, if $B = \{x_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$, consider $T : \ell_2(I) \to \mathcal{H}$ given by $T_{e_i} = S^{1/2}x_i$ for $i \in I$. Then, for every $f \in \mathcal{H}$,

$$TT^*f = \sum_{i\in I}\langle f, S^{1/2}x_i\rangle S^{1/2}x_i = S^{1/2}\left(\sum_{i\in I}\langle S^{1/2}f, x_i\rangle x_i\right) = Sf.$$

Therefore, $\mathcal{F} = \{S^{1/2}x_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and its frame operator is given by S.

Theorem(5.2.26)[39]. Let $S \in GL(\mathcal{H})$ be a J-selfadjoint operator acting on a Krein space $\mathcal{H}$ with fundamental symmetry J. Then, the following conditions are equivalent:

(i) S is a J-frame operator, i.e. there exists a J-frame $\mathcal{F}$ with synthesis operator T such that $S = TT^+$.

(ii) There exists a projection $Q \in \mathcal{Q}$ such that QS is J-positive and $(I - Q)S$ is J-negative.

(iii) There exist J-positive operators $S_1, S_2 \in L(\mathcal{H})$ such that $S = S_1 - S_2$ and $R(S_1)$ $(resp. R(S_2))$ is a uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$.

Proof. (i)→(ii): Follows from Proposition (5.2.23) and Corollary (5.2.24).

(ii)→(iii): If there exists a projection $Q \in \mathcal{Q}$ such that QS is J-positive and $(I - Q)S$ is J-negative, consider the J-positive operators $S_1 = QS$ and $S_2 = -(I - Q)S$. Then, $S = S_1 - S_2$ and, by hypothesis, $R(S_1) = R(Q)$ is uniformly J-positive and $R(S_2) = R(I - Q) = N(Q)$ is uniformly J-negative.

(iii)→(i): Suppose that there exist J-positive operators $S_1, S_2 \in L(\mathcal{H})$ such that $S = S_1 - S_2$ and $R(S_1)$ $(resp. R(S_2))$ is a uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. Denoting $\mathcal{K}_j = R(S_j)\ for\ j = 1, 2$, observe that $A_j = S_jJ|_{\mathcal{K}_j} \in GL(\mathcal{K}_j)^+$.

Therefore, there exists a frame $\mathcal{F}_j = \{f_i\}_{i\in I_j} \subset \mathcal{K}_j$ for $\mathcal{K}_j$ such that $A_j = T_jT_j^*$ j if $T_j \in L(\ell_2(I_1), \mathcal{K}_j)$ is the synthesis operator of $\mathcal{F}_j$, for $j = 1, 2$.

Then, consider $\ell_2(I) := \ell_2(I_1) \oplus \ell_2(I_2)$ and $T \in L(\ell_2(I), \mathcal{H})$ given by

$$Tx = T_1x_1 + T_2x_2,\ if\ x \in \ell_2(I_1), x = x_1 + x_2,\ x_j \in \ell_2(I_j) for\ j = 1, 12.$$

It is easy to see that T is the synthesis operator of the frame $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_2$. Furthermore $\mathcal{F}$ is a J-frame such that $I_+ = I_1$ and $I_- = I_2$.

Finally, endow $\ell_2(I)$ with the indefinite inner product defined by the diagonal operator $J_2 \in L(\ell_2(I))$ given by

$$J_2\, e_i = \sigma_i\, e_i,$$

where $\sigma_i = 1\ if\ i \in I_1$ and $\sigma_i = -1\ if\ i \in I_2$. Notice that $T_1J_2 = T_1$ and $T_2J_2 = -T_2$. Furthermore, $T_1T_2^* = T_2T_1^* = 0$ because $R(T_2^*) = N(T_2)^\perp \subseteq \ell_2(I_1) = \ell_2(I_2) \subseteq N(T_1)$. Thus,

$$TT^+ = TJ_2T^*J = (T_1 + T_2)(T_1^* - T_2^*)J = T_1T_1^*J - T_2T_2^*J = A_1J - A_2J = S_1 - S_2 = S.$$

Given a J-frame $\mathcal{F} = \{f_i\}_{i\in I}$ for $\mathcal{H}$ with J-frame operator $S \in L(\mathcal{H})$, it follows from Corollary (5.2.24) that

$$S(\mathcal{M}_-^{[\perp]}) = \mathcal{M}_+\ and\ S\left(\mathcal{M}_+^{[\perp]}\right) = \mathcal{M}_- \quad (51)$$

i.e. S maps a maximal uniformly J-positive (resp. J-negative) subspace into another maximal uniformly J-positive (resp. J-negative) subspace. The next proposition shows under which hypotheses the converse holds.

Proposition(5.2.27)[39]. Let $S \in GL(\mathcal{H})$ be a J-selfadjoint operator. Then, S is a J-frame operator if and only if the following conditions hold:

(i) there exists a maximal uniformly J-positive subspace T of $\mathcal{H}$ such that $S(T)$ is also maximal uniformly J-positive;

(ii) $[Sf, f] \geq 0$ for every $f \in \mathcal{T}$;

(iii) $[Sg, g] \leq 0$ for every $g \in S(\mathcal{T})^{[\perp]}$.

Proof. If S is a J-frame operator, consider $\mathcal{T} = \mathcal{M}_-^{[\perp]}$ which is a maximal uniformly J-positive subspace $\mathcal{T}$ of $\mathcal{H}$. Then, $S(\mathcal{T}) = \mathcal{M}_+$ is also maximal uniformly J-positive. Furthermore,

$[Sf, f] = [SQ^+f, Q^+f] = [QSQ^+f, f] = [S_+f, f] \geq 0$ for every $f \in \mathcal{T}$, where $Q = P_{\mathcal{M}_+//\mathcal{M}_-}$. Also, $S(\mathcal{T})^{[\perp]} = \mathcal{M}_+^{[\perp]} = N(Q^+) = R((I - Q)^+)$. So,

$$[Sg, g] = [S(I - Q)^+g, (I - Q)^+g] = [(I - Q)S(I - Q)^+g, g] = [-S_-g, g] \leq 0\ for\ every\ g \in S(\mathcal{T})^{[\perp]}.$$

Conversely, suppose that there exists a maximal uniformly J-positive subspace T satisfying the hypotheses. Let $\mathcal{M} = S(\mathcal{T})$, which is maximal uniformly J-positive. Then, consider $= P_{\mathcal{M}//\mathcal{T}^{[\perp]}}$. It is well defined because $\mathcal{T}^{[\perp]}$ is maximal uniformly J-negative, see [91]. Moreover, $Q \in \mathcal{Q}$.
Notice that $R(S(I - Q)^{+}) = S(\mathcal{M}^{[\perp]}) = S(S(\mathcal{T})^{[\perp]}) = S(S^{-1}(\mathcal{T}^{[\perp]})) = \mathcal{T}^{[\perp]}$. Therefore, $QS(I - Q)^{+} = 0$ and

$$QS = QSQ^{+} + QS(I - Q)^{+} = QSQ^{+}.$$

Furthermore, if $[Sf, f] \geq 0$ for every $f \in \mathcal{T}$ then QS is J-positive. Analogously, if $[Sg, g] \leq 0$ for every $g \in S(\mathcal{T})^{[\perp]}$ then $(I - Q)S$ is J-negative. Then, by Theorem (5.2.26), S is a J-frame operator.

As it was proved in Proposition (5.2.23), if an operator $S \in L(\mathcal{H})$ is a J-frame operator then it is an invertible J-selfadjoint operator satisfying $ind_{\pm}(S) = dim(\mathcal{H}_{\pm})$. Unfortunately, the converse is not true.

Example(5.2.28)[39]. Consider the Krein space obtained by endowing $\mathbb{C}^2$ with the sesquilinear form

$$[(x_1, x_2), (y_1, y_2)] = x_1\overline{y_1} - x_2\overline{y_2},$$

and the invertible J-selfadjoint operator S, whose matrix in the standard orthonormal basis is given by

$$S = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then, S satisfies $ind_{\pm}(S) = dim(\mathcal{H}_{\pm})$, but it maps each J-positive vector into a J-negative vector. Then, by Proposition (5.2.27), S cannot be a J-frame operator.

Corollary (5.2.29)[212 Let the sequences $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$ be a Bessel family in $\mathcal{H}$. Then, $\mathcal{F}$ is a sequences of J-frames if and only if $\mathcal{F}$ is everywhere defined (i.e. $\mathcal{D}_{+} + \mathcal{D}_{-} = \mathcal{H}$) and $\| F \| < 1$.

Proof. See [59].

It follows from the definition that, given a sequences of J-frames $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$for the Krein space $\mathcal{H}$, $[(f_r)_i, (f_r)_i] \neq 0$ for every $i \in I_r$, i.e. $(I_r)_{\pm} = \{i \in I_r : \pm[(f_r)_i, (f_r)_i] > 0\}$. This fact allows to endow the coefficients space $\ell_2(I_r)$ with a Krein space structure. Denote $(\sigma_r)_i = sgn([(f_r)_i, (f_r)_i]) = \pm 1$ for every $i \in I_r$. Then, the diagonal operator $J_2 \in L(\ell_2(I_r))$ defined by

$$J_2 (e_r)_i = (\sigma_r)_i (e_r)_i, for\ every\ i \in I_r, \tag{52}$$

is a selfadjoint involution on $\ell_2(I_r)$. Therefore, $\ell_2(I_r)$with the fundamental symmetry J_2 is a Krein space.

Now, if $T \in L(\ell_2(I_r), \mathcal{H})$ is the synthesis operator of $\mathcal{F}$, the J-adjoints of T , T_{+} and T_{-} can be easily calculated, in fact if $f_r \in \mathcal{H}$:

$$\sum_{r \in R} T_{\pm}^{+} f_r = \pm \sum_{i \in (I_r)_{\pm}} \sum_{r \in R} [f_r, (f_r)_i](e_r)_i,$$

And

$$\sum_{r \in \boldsymbol{R}} \boldsymbol{T}^{\pm} f_r = \sum_{r \in \boldsymbol{R}} (\boldsymbol{T}_{+} + \boldsymbol{T}_{-})^{+} f_r = \sum_{r \in \boldsymbol{R}} \boldsymbol{T}_{+}^{+} f_r + \sum_{r \in \boldsymbol{R}} \boldsymbol{T}_{-}^{+} f_r =$$

$$\sum_{i \in (\boldsymbol{I_r})_{+}} \sum_{r \in \boldsymbol{R}_{+}} [f_r, (f_r)_i](\boldsymbol{e}_r)_i - \sum_{i \in (\boldsymbol{I_r})_{-}} \sum_{r \in \boldsymbol{R}_{-}} [f_r, (f_r)_i](\boldsymbol{e}_r)_i = \sum_{i \in \boldsymbol{I_r}} \sum_{r \in \boldsymbol{R}} (\boldsymbol{\sigma}_r)_i [f_r, (f_r)_i](\boldsymbol{e}_r)_i$$

Corollary (5.230)[212]. Let $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$ be a sequences of J-frames for $\mathcal{H}$. Then, $\mathcal{F}_{\pm} = \{(f_r)_i\}_{i \in (I_r)_{\pm}}$ is a sequences of frames for the Hilbert space $(\mathcal{M}_{\pm}, \pm[\,,])$, i.e. there exist constants $(A + \varepsilon)_{-} \leq A_{-} < 0 < A_{+} \leq (A + \varepsilon)_{+}$ such that

$$\sum_{r \in R} A_{\pm}[f_r, f_r] \leq \sum_{i \in (I_r)_{\pm}} \sum_{r \in R_{\pm}} |[f_r, (f_r)_i]| \leq \sum_{r \in R} (A + \varepsilon)_{\pm}[f_r, f_r] \quad for\ f_r \in \mathcal{M}_{\pm}\,. \tag{53}$$

Proof. If $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$ is a sequences of J-frames for $\mathcal{H}$, then $R(T_{+}) = M_{+}$ is a (maximal) uniformly J-positive subspace of $\mathcal{H}$. So, T_{+} is a surjection from $\ell_2(I_r)$ onto the Hilbert space $(M_+, [\,,])$. Therefore, $\mathcal{F}_+$ is a sequences of frames for $(M_+, [\,,])$In particular, there exist constants $0 < A_+ \leq (A+\varepsilon)_+$ such that (53) is satisfied for M_+. The assertion on $\mathcal{F}_-$ follows analogously.

Now, assuming that (see, [23]) $\mathcal{F}$ is a sequences of J-frames for a Krein space $(\mathcal{H}, [\,,])$, a set of constants $\{(A+\varepsilon)_-, A_-, A_+, (A+\varepsilon)_+\}$ satisfying (53) is going to be computed. They depend only on the definiteness bounds for $R(T_\pm)$, the norm and the reduced minimum modulus of $T_\pm$.

Suppose that $\mathcal{F}$ is a sequences of J-frames for a Krein space $(\mathcal{H}, [\,,])$ with synthesis operator $T \in L(\ell_2(I_r), \mathcal{H})$. Since $R(T_+) = M_+$ is a (maximal) uniformly J-positive subspace of $\mathcal{H}$, there exists $(1-\varepsilon) > 0$ such that $\sum_{r\in R}(1-\varepsilon)\parallel f_r \parallel^2 \leq \sum_{r\in R}[f_r, f_r]$ $\quad for\ every\ f_r \in M_+$. So,

$$\sum_{i\in(I_r)_+}\sum_{r\in R}\|f_r, (f_r)_i\|^2 = \sum_{r\in R}\|T_+^+ f_r\|^2 \leq \sum_{r\in R}\|T_+^+\|^2\|f_r\|^2 \leq \sum_{r\in R}(A+\varepsilon)_+[f_r, f_r] \quad ,for\ every\ f_r \in M_+$$

where $(A+\varepsilon)_+ = \frac{\|T_+^+\|^2}{1-\varepsilon} = \frac{\|T_+\|^2}{1-\varepsilon}$. Furthermore, since $N(T_+^+)^\perp = J(M_+)\,, if\ f_r \in M_+$,

$$\sum_{i\in(I_r)_+}\sum_{r\in R_+}\|f_r, (f_r)_i\|^2 = \sum_{r\in R}\|T_+^+ f_r\|^2 \leq \sum_{r\in R}\left\|T_+^+ P_{J(M_+)}^2 f_r\right\|^2 \geq \sum_{r\in R}\gamma(T_+^+)^2\left\|P_{J(M_+)}^2 f_r\right\|^2$$

$$= \sum_{r\in R}\gamma(T_+)^2\left\|P_{M_+}^2 J f_r\right\|^2 = \sum_{r\in R}\gamma(T_+)^2\left\|G_{M_+} J f_r\right\|^2 \geq \sum_{r\in R}\gamma(T_+)^2\gamma\left(G_{M_+}\right)^2\|f_r\|^2$$

$$\geq \sum_{r\in R} A_+[f_r, f_r]$$

where $A_+ = \gamma(T_+)^2\gamma\left(G_{M_+}\right)^2 = \gamma(T_+)^2(1-\varepsilon)^2$.

Analogously, $A_- = -\gamma\,(T_-)^2(\varepsilon-1)^2$ and $(A+\varepsilon)_- = \frac{\|T_-\|^2}{(\varepsilon-1)}$ satisfy Eq. (53) for every $\sum_{r\in R} f_r \in R(T_-) = M_-$,if $(\varepsilon-1)$ is the definiteness bound of the (maximal) uniformly J-negative subspace M_-.

Usually, the bounds $A_+ = +(1-\varepsilon)^2\gamma(T_+)^2$, $A_- = -(\varepsilon-1)^2\gamma(T_-)^2$ and $(A+\varepsilon)_+ = +\frac{\|T_+\|^2}{(1-\varepsilon)}$, $(A+\varepsilon)_- = -\frac{\|T_-\|^2}{(\varepsilon-1)}$ are not optimal for the series of J-frames $\mathcal{F}$.

Corollary(5.2.31)[212]. Given a Bessel family $\mathcal{F} = \{f_i\}_{i\in I}$ in a Krein space $\mathcal{H}$, let $\mathcal{M} = \overline{span\{f_i : i \in I\}}$, and $\mathcal{N} = \mathcal{M} \cup \mathcal{M}^{[\perp]}$ .If there exist constants $0 < A \leq (A+\varepsilon)$ such that

$$\sum_{r\in R} A\,[f_r, f_r] \leq \sum_{i\in I_r}\sum_{r\in R}|[f_r, (f_r)_i]|^2 \leq \sum_{r\in R}(A+\varepsilon)[f_r, f_r]\ ,for\ every\ f_r \in \mathcal{M}, \tag{54}$$

then $\mathcal{M} \ominus \mathcal{N}$ is a (closed) uniformly J-positive subspace of $\mathcal{M}$. Moreover, if $\mathcal{F}$ is a sequences of frames for the Hilbert space $(\mathcal{M}, \langle\,,\rangle)$, the converse holds.

Proof. First, suppose that there exist $\varepsilon, A > 0$ such that (54) holds. So, $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$, or equivalently, $(\mathcal{M}, \langle\,,\rangle)$is a semi-inner product space.

If $T \in L(\ell_2(I_r), \mathcal{H})$ is the synthesis operator of the Bessel sequence $\mathcal{F}$ and $C = \parallel T^* \parallel^2 > 0$, then $TT^* \leq CP_{\mathcal{M}}^2$. So, using (17) it is easy to see that:

$$\sum_{r\in R} A\langle G_{\mathcal{M}} f_r, f_r\rangle \leq \sum_{r\in R}\left\|T^+(P_{\mathcal{M}}^2 f_r)\right\|^2 = \sum_{r\in R}\langle (P_{\mathcal{M}}^2 JTT^*) f_r, f_r\rangle\ , \qquad f_r \in \mathcal{H}. \tag{55}$$

Thus, $0 \leq G_{\mathcal{M}} \leq \frac{C}{A}(G_{\mathcal{M}})^2$. Applying Theorem (5.2.1) it is easy to see that

$$R((G_{\mathcal{M}})^{1/2}) \subseteq R(G_{\mathcal{M}}) \subseteq R((G_{\mathcal{M}})^{1/2}).$$

Moreover, it follows by Corollary (5.2.2) that $R(G_{\mathcal{M}})$ is closed because

$$R(G_{\mathcal{M}}) = R\big((G_{\mathcal{M}})^{1/2}\big).$$

Let $\mathcal{M}' = \mathcal{M} \ominus \mathcal{N}$ and notice that $\mathcal{M}'$ is a closed uniformly J-positive subspace of $\mathcal{H}$. In fact, since $R(G_{\mathcal{M}})$ is closed, there exists $(1-\varepsilon)$ such that

$$\sum_{r\in R}[f_r, f_r] = \sum_{r\in R}\langle G_{\mathcal{M}} f_r, f_r\rangle = \sum_{r\in R} \| (G_{\mathcal{M}})^{1/2} f_r \|^2 \geq \sum_{r\in R} (1-\varepsilon) \| f_r \|^2,$$

$for\ every\ f_r \in \mathcal{N}(G_{\mathcal{M}}) = \mathcal{M} \ominus \mathcal{N}.$

Conversely, suppose that $\mathcal{F}$ is a series of frames for $(\mathcal{M}, \langle\ ,\ \rangle)$, i.e. there exist constants $(A+\varepsilon)' \geq A' > 0$ such that

$$A' P_{\mathcal{M}}^2 \leq TT^* \leq (A+\varepsilon)' P_{\mathcal{M}}^2,$$

where $T \in L(\ell_2(I_r), \mathcal{M})$ is the synthesis operator of $\mathcal{F}$. If $\mathcal{M}' = \mathcal{M} \ominus \mathcal{N}$ is a uniformly J-positive subspace of $\mathcal{H}$, then there exists $(1-\varepsilon)$ such that $(1-\varepsilon)P_{\mathcal{M}'}^2 \leq G_{\mathcal{M}'} \leq P_{\mathcal{M}}^2$. As a consequence of Theorem (5.2.1), $R((G_{\mathcal{M}'})^{1/2}) = \mathcal{M}' = R(G_{\mathcal{M}'})$. Since $G_{\mathcal{M}} = G_{\mathcal{M}'}$ it is easy to see that

$$A'(G_{\mathcal{M}})^2 = A'(G_{\mathcal{M}'})^2 \leq P_{\mathcal{M}}^2 JTT^*JP_{\mathcal{M}}^2 \leq (A+\varepsilon)'(G_{\mathcal{M}'})^2 = (A+\varepsilon)'(G_{\mathcal{M}})^2.$$

Therefore, $R(P_{\mathcal{M}}^2 JT) = R(G_{\mathcal{M}'}) = R((G_{\mathcal{M}'})^{1/2})$, or equivalently, there exist $\varepsilon, A > 0$ such that

$$AG_{\mathcal{M}} = AG_{\mathcal{M}'} \leq P_{\mathcal{M}}^2 JTT^*JP_{\mathcal{M}} \leq (A+\varepsilon)G_{\mathcal{M}'} = (A+\varepsilon)G_{\mathcal{M}},$$

$$i.e. \sum_{r\in R} A\,[f_r, f_r] \leq \sum_{i\in I_r}\sum_{r\in R} |[f_r, (f_r)_i]|^2 \leq \sum_{r\in R}(A+\varepsilon)\,[f_r, f_r]\,, \quad for\ every\ f_r \in \mathcal{M}\,.$$

Corollary(5.2.32)[212]. Let $\mathcal{F} = \{(f_r)_i\}_{i\in I_r}$ be a series of frame for $\mathcal{H}$. If $(I_r)_\pm = \{i \in I_r : \pm[(f_r)_i, (f_r)_i] \geq 0\}$ and $\mathcal{M}_\pm = \overline{span\{(f_r)_i : i \in (I_r)_\pm\}}$, then, $\mathcal{F}$ is a series of J-frame if and only if $\mathcal{M}_\pm \cap \mathcal{M}^{[\perp]} = 0$ and there exist constants $(A+\varepsilon)_- \leq A_- < 0 < A_+ \leq (A+\varepsilon)_+$ such that

$$\sum_{r\in R} A_\pm[f_r, f_r] \leq \sum_{i\in I_\pm}\sum_{r\in R_\pm} |[f_r, f_i]|^2 \leq \sum_{r\in R}(A+\varepsilon)_\pm[f_r, f_r] \quad for\ every\ f_r \in \mathcal{M}_\pm. \tag{56}$$

Proof. If $\mathcal{F}$ is a sequences of J-frames, the conditions on $\mathcal{M}_\pm$ follow by its definition and by Proposition (5.2.11). Conversely, if $\mathcal{M}_+$ is J-non degenerated and there exist constants $0 < A_+ \leq (A+\varepsilon)_+$ such that

$$\sum_{r\in R} A_+[f_r, f_r] \leq \sum_{i\in (I_r)_\pm}\sum_{r\in R_\pm} |[f_r, (f_r)_i]|^2 \leq \sum_{r\in R}(A+\varepsilon)_+[f_r, f_r]\,, \quad for\ every\ f_r \in \mathcal{M}_+\,.$$

then, by Proposition (5.2.14), $\mathcal{M}_+$ is a uniformly J-positive subspace of $\mathcal{H}$. Therefore, there exist constants $\varepsilon, A > 0$ such that

$$\sum_{r\in R} A \| P_{\mathcal{M}_+}^2 f_r \|^2 \leq \sum_{r\in R} \| T_+^+ P_{\mathcal{M}_+}^2 f_r \|^2 \leq \sum_{r\in R}(A+\varepsilon) \| P_{\mathcal{M}_+}^2 f_r \|^2\ , for\ every\ f_r \in \mathcal{H}.$$

But these inequalities can be rewritten as

$$A\,P_{\mathcal{M}_+}^2 \leq P_{\mathcal{M}_+}^2 JT_+T_+^* JP_{\mathcal{M}_+}^2 \leq (A+\varepsilon)\,P_{\mathcal{M}_+}^2.$$

Then, by Theorem (5.2.1), $R(P_{\mathcal{M}_+}^2 JT_+) = R(P_{\mathcal{M}_+}^2) = \mathcal{M}_+$. Furthermore, $P_{J(\mathcal{M}_+)}(R(T_+)) = J(\mathcal{M}_+)$ because

$$J(\mathcal{M}_+) = J(R(P_{\mathcal{M}_+}^2 JT_+)) = R((JP_{\mathcal{M}_+}^2 J)T_+) = R(P_{J(\mathcal{M}_+)}^2 T_+) = P_{J(\mathcal{M}_+)}^2(R(T_+)).$$

Therefore, taking the counter image of $P_{J(\mathcal{M}_+)}^2(R(T_+))\ by\ P_{J(\mathcal{M}_+)}^2$, it follows that

$$\mathcal{H} = R(T_+) \dotplus J(\mathcal{M}_+)^\perp \subseteq \mathcal{M}_+ \dotplus \mathcal{M}_+^{[\perp]} = \mathcal{H}.$$

Thus, $R(T_+) = \mathcal{M}_+$ and $\mathcal{F}_+$ is a frame for $\mathcal{M}_+$. Analogously, $\mathcal{F}_- = \{(f_r)_i\}_{i\in(I_r)_-}$ is a sequences of frames for $\mathcal{M}_-$. Finally, since $\mathcal{F}$ is a sequences of frames for $\mathcal{H}$,

$$\mathcal{H} = R(T) = R(T_+) + R(T_-),$$

which proves the maximality of $R(T_\pm)$. Thus, $\mathcal{F}$ is a sequences of J-frames for $\mathcal{H}$.

Corollary(5.2.33)[212]. Let $\mathcal{M}$ be a J-semidefinite subspace of $\mathcal{H}$ with definiteness bound $(1-\varepsilon)$. Then,

$$c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2-\varepsilon}{2}} + \sqrt{\frac{\varepsilon}{2}}\right). \tag{57}$$

In particular, $\mathcal{M}$ is uniformly J-definite if and only if $c_0(\mathcal{M},\mathcal{C}) < 1$.

Proof. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fundamental decomposition of $\mathcal{H}$ and suppose that $\mathcal{M}$ is a J-nonnegative subspace of $\mathcal{H}$.

Let $m \in \mathcal{M}\ with\ \| m \| = 1$. Then, there exist (unique) $m_\pm \in \mathcal{H}_\pm$ such that $m = m^+ + m^-$. In this case,

$$\mathbf{1} = \| \boldsymbol{m} \|^2 = \| m^+ \|^2 + \| m^- \|^2 \ \boldsymbol{and}\ (1-\varepsilon) \le [\boldsymbol{m},\boldsymbol{m}] = \| m^+ \|^2 - \| m^- \|^2. \tag{58}$$

Claim. For a fixed $m \in \mathcal{M}\ with\ \| m \| = 1, sup\{| \langle m,n \rangle | : n \in \mathcal{C}, \| n \| = 1\} = \frac{1}{\sqrt{2}}(\| m^+ \| + \| m^- \|)$.

Indeed, consider $n \in \mathcal{C}\ with\ \| n \| = 1$. Then, there exist (unique) $n_\pm \in \mathcal{H}_\pm$ such that $n = n^+ + n^-$. In this case,

$$0 = [n,n] = \| n^+ \|^2 - \| n^- \|^2 \ and\ 1 = \| n \|^2 = \| n^+ \|^2 + \| n^- \|^2,$$

which imply that $\| n^+ \| = \| n^- \| = \frac{1}{\sqrt{2}}$. Therefore,

$$| \langle m,n \rangle| \le |\langle m^+,n^+\rangle| + |\langle m^-,n^-\rangle| \le \frac{1}{\sqrt{2}}(\| m^+ \| + \| m^- \|).$$

On the other hand, if $m^- \neq 0$ then let $n_m := \frac{1}{\sqrt{2}}(\frac{m^+}{\|m^+\|} + \frac{m^-}{\|m^-\|})$, otherwise consider $n_m = \frac{1}{\sqrt{2}}(m+z)$, with $z \in \mathcal{H}_-, \| z \| = 1$.

Now, it is easy to see that $n_m \in \mathcal{C}$ and that $|\langle \boldsymbol{m}, \boldsymbol{n_m} \rangle| = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\| \boldsymbol{m}^+ \| + \| \boldsymbol{m}^- \|)$ which together with the previous facts prove the claim.

Now, let $\mathcal{M}_1 = \{m = m^+ + m^- \in \mathcal{M} : m^\pm \in \mathcal{H}_\pm, \| m \| = 1\}$. Using the claim above it follows that

$$c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}} sup_{m\in\mathcal{M}_1}(\| m^+ \| + \| m^- \|). \tag{59}$$

If $\varepsilon = 0$then $\mathcal{M}$ is a subspace of $\mathcal{H}_+$. Also, it is easy to see that $c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}}$. Thus, in this particular case,

$$c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2-\varepsilon}{2}} + \sqrt{\frac{\varepsilon}{2}}\right).$$

On the other hand, if $\varepsilon = 0$, let $k_0 \in \mathbb{N}$ be such that $k_0 > \frac{1}{\varepsilon}$. Observe that, by the definition of the definiteness bound,

for every integer $k \ge k_0$ there exists $m_k = m_k^+ + m_k^- \in \mathcal{M}_1$ such that $(1-\varepsilon) \le \| m_k^+ \|^2 - \| m_k^- \|^2 < (1-\varepsilon) + \frac{1}{k}$. Then, it follows that

$2 - \varepsilon \le 2 \| m_k^+ \|^2 < 2 - \varepsilon + \frac{1}{k}$,

or equivalently, $\sqrt{\frac{2-\varepsilon}{2}} \le \|m_k^+ \| < \sqrt{\frac{2-\varepsilon}{2} + \frac{1}{2k_0}}$. Moreover, $\| m_k^- \| = \sqrt{1 - \| m_k^+ \|^2}$ implies that

$$\sqrt{\frac{\varepsilon}{2} - \frac{1}{2k}} < \| m_k^- \| \le \sqrt{\frac{\varepsilon}{2}}.$$

Therefore, for every integer $k \ge k_0$ there exists $m_k \in \mathcal{M}_1$ such that

$$\sqrt{\frac{\varepsilon}{2} - \frac{1}{2k}} + \sqrt{\frac{2-\varepsilon}{2}} < \| m_k^+ \| + \| m_k^- \| < \sqrt{\frac{2-\varepsilon}{2} + \frac{1}{2k_0}} + \sqrt{\frac{\varepsilon}{2}}.$$

Thus, $c_0(\mathcal{M},\mathcal{C}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2-\varepsilon}{2}} + \sqrt{\frac{\varepsilon}{2}}\right)$.

Assume now that $\mathcal{M}$ is a J-nonpositive subspace of $(\mathcal{H}, [\,,])$ with definiteness bound $(1-\varepsilon)$, for $\varepsilon > 0$. Then, $\mathcal{M}$ is a J-nonnegative subspace of the antispace $(\mathcal{H}, -[\,,])$, with the same definiteness bound α. Furthermore, the cone of J-neutral vectors for the antispace is the same as for the initial Krein space $(\mathcal{H}, [\,,])$. Therefore, we can apply the previous arguments and conclude that (58) also holds for

J-nonpositive subspaces. Finally, the last assertion in the statement follows from the formula in (58).

Let $\mathcal{F}$ be a sequences of J-frames for $\mathcal{H}$ as above. Notice that (56) holds for some constant $\frac{\sqrt{2}}{2} \leq c_{\pm} < 1$ if and only if $c_0(\mathcal{M}_{\pm}, \mathcal{C}) < 1$,i.e. that the minimal angles $\theta(\mathcal{M}_{\pm}, \mathcal{C})$ are bounded away from 0. This is intimately related with the fact that the aperture between the subspaces $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) and $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) is bounded away from $\frac{\pi}{4}$, whenever $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$is a fundamental decomposition.

Corollary(5.2.34)[212]. Let $T \in L(\ell_2(I_r), \mathcal{H})$ be surjective. Suppose that $P_{\mathscr{s}}^2$ is the orthogonal projection onto a closed subspace $\mathscr{s}$ of $\ell_2(I_r)$ such that $c(\mathscr{s}, N(T)^{\perp}) < 1$. Then, $TP_{\mathscr{s}}^2T^{\dagger}$ is a projection if and only if

$$N(T) = \mathscr{s} \cap N(T) \oplus \mathscr{s}^{\perp} \cap N(T).$$

Proof. Suppose that $Q^2 = TP_{\mathscr{s}}^2T^{\dagger}$ is a projection. Then, if $P^2 = P_{N(T)^{\perp}}^2, E = P^2P_{\mathscr{s}}^2P^2$ is an orthogonal projection because it is selfadjoint and

$$E^2 = (P^2P_{\mathscr{s}}^2P^2)^2 = P^2P_{\mathscr{s}}^2P^2P^2P_{\mathscr{s}}^2 = T^{\dagger}(TP_{\mathscr{s}}^2T^{\dagger})^2T = T^{\dagger}(TP_{\mathscr{s}}^2T^{\dagger})T = P^2P_{\mathscr{s}}^2P^2 = E.$$

Therefore, $(P^2P_{\mathscr{s}}^2)^k = E^{k-1}P_{\mathscr{s}}^2 = EP_{\mathscr{s}}^2 = (P^2P_{\mathscr{s}}^2)^2 for\ every\ k \geq 2$. So, by [15],

$$P^2P_{\mathscr{s}}^2 = P_{\mathscr{s}}^2 \wedge P^2 = P^2P_{\mathscr{s}}^2.$$

Then, since $P_{\mathscr{s}}^2$ and P^2 commute, it follows that $N(T) = \mathscr{s} \cap N(T) \oplus \mathscr{s}^{\perp} \cap N(T)$ (see [15]).

Conversely, suppose that $N(T) = \mathscr{s} \cap N(T) \oplus \mathscr{s}^{\perp} \cap N(T)$. Then, $P_{\mathscr{s}}$ and P commute and

$$(TP_{\mathscr{s}}^2T^{\dagger})^2 = TP_{\mathscr{s}}^2(TT^{\dagger})P^2ST^{\dagger} = TP_{\mathscr{s}}^2P^2P_{\mathscr{s}}^2T^{\dagger} = TP^2P_{\mathscr{s}}^2T^{\dagger} = TP_{\mathscr{s}}^2T^{\dagger}.$$

Hereafter consider the set of possible decompositions of $\mathcal{H}$ as a (direct) sum of a pair of maximal uniformly definite subspaces, or equivalently, the associated set of projections:

$\mathfrak{Q} = \{Q^2 \in L(\mathcal{H}) : Q^2 = Q, R(Q^2)$ is uniformly J-positive and $N(Q^2)$ is uniformly J-negative$\}$.

Corollary(5.2.35)[212]. Let $T \in L(\ell_2(I_r), \mathcal{H})$ be surjective. Then, T is the synthesis operator of a J-frame if and only if there exists $(I_r)_+ \subset I_r$ such that $\ell_2((I_r)_+)$ (as a subspace of $\ell_2(I_r)$ satisfies $c(N(T)^{\perp}, \ell_2((I_r)_+)) < 1$ and

$$TP_+^2T^{\dagger} \in Q^2,$$

where $P_+^2 \in L(\ell_2(I_r))$ is the orthogonal projection onto $\ell_2((I_r)_+)$.

Proof. If T is the synthesis operator of a sequences of J-frames, the existence of such a subset $(I_r)_+$ has already been discussed before.

Conversely, suppose that there exists such a subset $(I_r)_+$of (I_r). Then, since $c(N(T)^{\perp}, \ell_2((I_r)_+)) < 1$ and $= TP_+^2T^{\dagger} \in Q^2$, it follows from Lemma (5.2.19) that P_+^2 and $P^2 = P^2N(T)^{\perp}$ commute. Therefore,

$$Q^2T = TP_+^2P^2 = TP^2P_+^2 = TP_+^2,$$

and $(I_r - Q^2)T = T(I_r - P_+^2). Hence, R(TP_+^2) = R(Q^2)$ is (maximal) uniformly J-positive and $R(T(I_r - P_+^2)) = N(Q^2)$ is (maximal) uniformly J-negative. Therefore $\mathcal{F} = \{T(e_r)_i\}_{i\in I_r}$ is by definition a sequences of J-frames for $\mathcal{H}$.

Corollary(5.2.36)[212]. Given a surjective operator $T \in L(\ell_2(I_r), \mathcal{H})$, the following conditions are equivalent:

(i) There exists $U \in \mathcal{U}(\ell_2(I_r))$ such that TU is the synthesis operator of a sequences of J-frames.

(ii) There exists $Q^2 \in \mathcal{Q}$ such that

$$Q^2TT^*(I_r - Q^2)^* = 0. \tag{60}$$

(iii) There exist closed range operators $T_1, T_2 \in L(\ell_2(I_r), \mathcal{H})$ such that $T = T_1 + T_2, R(T_1)$ is uniformly J-positive, $R(T_2)$is uniformly J-negative and $T_1T_2^* = T_2T_1^* = 0$.

Proof. $(i) \Rightarrow (ii)$: Suppose that there exists $U \in \mathcal{U}(\ell_2(I_r))$ such that $V = TU$ is the synthesis operator of a sequences of J-frames. If $(I_r)_{\pm} = \{i \in I_r : \pm[V(e_r)_i, V(e_r)_i] > 0\}\ and\ P_{\pm}^2 \in L(\ell_2(I_r))$ is the orthogonal projection onto $\ell_2((I_r)_{\pm})$, define $V_{\pm} = VP_{\pm}^2$. Then, $V = V_+ + V_-$ $and\ \mathcal{M}_{\pm} = R(V_{\pm})$ is a maximal uniformly J-definite subspace. So, considering $Q = P_{\mathcal{M}_+//\mathcal{M}_-}^2 \in \mathcal{Q}$, it is easy to see that $Q^2V = V_+, (I_r - Q^2)V = V_-$and

$$Q^2TT^*(I_r - Q^2)^* = Q^2VV^*(I_r - Q^2)^* = V_+V_-^* = VP_+^2P_-^2V^* = 0.$$

$(ii) \Rightarrow (iii)$: Suppose that there exists $Q^2 \in \mathcal{Q}$ such that $Q^2TT^*(I_r - Q^2)^* = 0$. Defining $T_1 = Q^2T$ and $T_2 = (I_r - Q^2)T$, it follows that $T = T_1 + T_2, R(T_1) = R(Q^2)$ is uniformly J-positive, $R(T_2) = N(Q^2)$ is uniformly J-negative and

$$T_1T_2^* = T_2T_1^* = 0,$$

because (60) says that $R(T_2^*) = R(T^*(I_r - Q^2)^*) \subseteq N(Q^2T) = N(T_1)$.

$(iii) \Rightarrow (i)$: If there exist closed range operators $T_1, T_2 \in L(\ell_2(I_r), \mathcal{H})$satisfying the conditions of item 3., notice that $T_1T_2^* = 0$ implies that $N(T_2)^\perp \subseteq N(T_1)$, or equivalently, $N(T_1)^\perp \subseteq N(T_2)$.

Consider the projection $Q^2 = P_{R(T_1)//R(T_2)} \in \mathcal{Q}$ and notice that $Q^2T = T_1$ and $(I_r - Q^2)T = T_2$. If $(B_r)_1 = \{(u_r)_i\}_{i\in(I_r)_1}$ is an orthonormal basis of $N(T_1)^\perp$, consider the family $\{(f_r)_i^+\}_{i\in(I_r)_1}$ *in* $\mathcal{H}$ *given by* $(f_r)_i^+ = T(u_r)_i$. *But, if* $i \in (I_r)_1$,

$$(f_r)_i^+ = Q^2T(u_r)_i + (I_r - Q^2)T(u_r)_i = T_1(u_r)_iR(T_1),$$

because $(u_r)_i \in N(T_1)^\perp \subseteq N(T_2)$. Therefore, $\{(f_r)_i^+\}_{i\in(I_r)_1} \subseteq R(T_1)$. Since T_1 is an isomorphism between $N(T_1)^\perp$ and $R(T_1)$, it follows that $R(T_1) = \overline{span\{(f_r)_i^+\}_{i\in(I_r)_1}}$.

Analogously, if $(\mathcal{B}_r)_2 = \{(b_r)_i\}_{i\in(I_r)_2}$ is an orthonormal basis of $N(T_1)$the family $\{(f_r)_i^-\}_{i\in(I_r)_2}$ defined by $(f_r)_i^- = Tb_i(i \in (I_r)_2)$ *lies in* $R(T_2)$.

Since T_2 is an isomorphism between $N(T_2)^\perp$ and $R(T_2)$, it follows that

$$R(T_2) = T_2(N(T_1)) \subseteq \overline{span\{(f_r)_i^-\}_{i\in(I_r)_2}} \subseteq R(T_2).$$

Finally, consider $U \in \mathcal{U}(\ell_2(I_r))$ which turns the standard orthonormal basis $\{(e_r)_i\}_{i\in I_r}$ *into* $(\mathcal{B}_r)_1 \cup (\mathcal{B}_r)_2$. *Then, if* $V = TU$ and $\mathcal{F} = \{V(e_r)_i\}_{i\in I_r} = \{(f_r)_i^+\}_{i\in(I_r)_1} \cup \{(f_r)_i^-\}_{i\in(I_r)_2}$, it is easy to see that

$$(I_r)_+ = \{i \in I_r : [V(e_r)_i, V(e_r)_i] > 0\} = (I_r)_1 \text{ and } (I_r)_- = \{i \in I_r : [V(e_r)_i, V(e_r)_i] < 0\} = (I_r)_2.$$

So, $R(V_+) = R(T_1)$ is maximal uniformly J-positive and $R(V_-) = R(T_2)$ is maximal uniformly J-negative. Therefore, $\mathcal{F}$ is a J-frame for $\mathcal{H}$with synthesis operator $V = TU$.

Corollary(5.2.37)[212]. Let $\mathcal{F} = \{(f_r)_i\}_{i\in I_r}$, be a series of J-frames with synthesis operator $T \in L(\ell_2(I_r)), \mathcal{H})$. Then, its series J-frames operators $S \in L(\mathcal{H})$ satisfies:

(i) $S = TT^+$;

(ii) $S = S_+ - S_-$, where $S_+ := T_+T_+^+$ and $S_- := -T_-T_-^+$ are J-positive operators;

(iii) S is an invertible J-selfadjoint operator;

(iv) $ind_\pm(S) = dim\mathcal{H}_\pm$, where $ind_\pm(S)$ are the indices of S.

Proof. If $\mathcal{F} = \{(f_r)_i\}_{i\in I_r}$, is a series of J-frames with synthesis operator $T \in L(\ell_2(I_r)), \mathcal{H})$., then $\sum_{r\in R} T^+f_r = \sum_{i\in I_r}\sum_{r\in R}(\sigma_r)_i[f_r, (f_r)_i](e_r)_i$ *for* $f_r \in \mathcal{H}$. So,

$$\sum_{r\in R} TT^+f_r = T\left(\sum_{i\in I_r}\sum_{r\in R}(\sigma_r)_i[f_r, (f_r)_i](e_r)_i\right) = \sum_{i\in I_r}\sum_{r\in R}(\sigma_r)_i[f_r, (f_r)_i](f_r)_i = Sf, \text{for every } f_r \in \mathcal{H}.$$

Furthermore, if $(I_r)_\pm = \{i \in I_r : \pm[(f_r)_i, (f_r)_i] > 0\}$, consider $T_\pm = TP_\pm^2$ as usual. Then,

$$TT^+ = (T_+ + T_-)(T_+ + T_-)^+ = T_+T_+^+ + T_-T_-^+ = T_+T_+^+ - (-T_-T_-^+),$$

because $T_+T_-^+ = T_-T_+^+ = 0$. Therefore, $S = S_+ - S_-$ *if* $S_\pm := \pm T_\pm T_\pm^+$. Notice that $S_\pm$ is a J-positive operator because

$$S_\pm = \pm T_\pm T_\pm^+ = \pm T_\pm J_2 T_\pm^* J = T_\pm T_\pm^* J$$

To show the invertibility of S observe that, if $Sf_r = 0$ then $S_+f_r = S_-f_r$. But $R(S_+) \cap R(S_-) \subseteq R(T_+) \cap R(T_-) = \{0\}$. Thus, S is injective. On the other hand, $R(S) = S(\mathcal{M}_+^{[\perp]}) + S(\mathcal{M}_-^{[\perp]})$ because $\mathcal{H} = \mathcal{M}_+^{[\perp]} \dotplus \mathcal{M}_-^{[\perp]}$. But it is easy to see that $\mathcal{M}_\pm^{[\perp]} \subseteq N(S_\pm)$. So, $S(\mathcal{M}_\pm^{[\perp]}\}) = S_\mp(\mathcal{M}_\pm^{[\perp]}\})$ and $R(S) = S_-(\mathcal{M}_\pm^{[\perp]}) + S_2(\mathcal{M}_-^{[\perp]}) = R(S_-) + R(S_+) = \mathcal{M}_+ + \mathcal{M}_- = \mathcal{H}$. Therefore, S is invertible.

Finally, the identities $ind_\pm(S) = dim\,\mathcal{H}_\pm$ follow from the indices definition. Recall that if

$A \in L(\mathcal{H})$ is a J-selfadjoint operator, $ind_+(A)$ is the supremum of all positive integers r such that there exists a positive invertible matrix of the form $([Ax_j, x_k])_{j,k=1,\dots,r}$, $where\ x_1, \dots, x_r \in \mathcal{H}$ (if no such r exists, $ind_-(A) = 0$). Similarly, $ind_-(A) = ind_+(-A)$ is the supremum of all positive integers m such that there exists a negative invertible matrix of the form $([Ay_j, y_k])_{j,k=1,\dots,m}$, $where\ y_1, \dots, y_r \in \mathcal{H}$, see [53].

Corollary (5.2.38)[212]. Let $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$, be a sequences of J-frames for $\mathcal{H}$ with sequences of J-frames operators $S \in L(\mathcal{H})$. Then, $(S_\pm) = \mathcal{M}_\pm\ and\ N(S_\pm) = \mathcal{M}_+^{[\perp]}$.

Furthermore, if $Q^2 = P^2_{\mathcal{M}+//M_-}$,

$$S_+ = Q^2 S Q^2 + \ and\ S_- = -(I_r - Q^2) S (I_r - Q^2)^+ \tag{61}$$

Proof. Recall that $S_+ := T_+ T_+^+ = T_+(J_2 T_+^* J) = T_+ T_+^* J$. Then, $R(S_+) = R(T_+ T_+^* J) = R(T_+ T_+^*) = R(T_+) = \mathcal{M}_+$ because $R(T_+)$ is closed. Since S_+ is J-selfadjoint, it follows that $N(S_+) = R(S_+)^{[\perp]} = \mathcal{M}_+^{[\perp]}$. Analogously, $R(S_-) = \mathcal{M}_-\quad and\ N(S_-) = \mathcal{M}_-^{[\perp]}$.

Since $S = S_+ - S_-, if\ Q^2 = P^2_{\mathcal{M}+//M_-}$ then

$$Q^2 S = Q^2(S_+ - S_-) = S_+,$$

by the characterization of the range and nullspace of S_+. Therefore, $SQ_+^2 = Q^2 S = Q^2 S Q_+^2$. Analogously,

$$S(I_r - Q^2)^+ = (I_r - Q^2)S = (I_r - Q^2)S(I_r - Q^2)^+.$$

The above corollary states that S is the diagonal block operator matrix

$$S = \begin{pmatrix} S_+ & 0 \\ 0 & -S_- \end{pmatrix}, \tag{62}$$

according to the (oblique) decompositions $\mathcal{H} = \mathcal{M}_-^{[\perp]} \dotplus \mathcal{M}_+^{[\perp]}$and $\mathcal{H} = \mathcal{M}_+ \dotplus \mathcal{M}_-$ of the domain and codomain of S, respectively.

Corollary(5.2.39)[212]. If $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$ is a sequences of J-frames for a Krein space $\mathcal{H}$ with a sequences of J-frames operator S, then $\mathcal{g}_r = \{S^{-1}(f_r)_i\}_{i \in I_r}$ is also are J-frames for $\mathcal{H}$.

Proof. Given a J-frames $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$ for $\mathcal{H}$ with J-frames operator S, observe that the synthesis operator of $\mathcal{g}_r = \{S^{-1}(f_r)_i\}_{i \in I_r}$ is $V := S^{-1}T \in L(\ell_2(I_r), \mathcal{H})$. Furthermore, by Corollary (5.2.24), $S(\mathcal{M}_\mp^{[\perp]}) = \mathcal{M}_\pm$. Then, $S^{-1}(\mathcal{M}_\pm) = \mathcal{M}_\mp^{[\perp]}$ and it follows that

$$[S^{-1}(f_r)_i, S^{-1}(f_r)_i] > 0\ \textit{if and only if}\ [(f_r)_i, (f_r)_i] > 0.$$

Thus, $V_\pm = VP_\pm^2 = S^{-1}T_\pm$ and $R(V_+)$ (resp. $R(V_-)$is a maximal uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. So, $\mathcal{g}_r$ are J-frames for $\mathcal{H}$.

If $\mathcal{F} = \{(f_r)_i\}_{i \in I_r}$ is a series of frames for a Hilbert space $\mathcal{H}$ with synthesis operator $T \in L(\ell_2(I_r), \mathcal{H})$., then the family $\{(TT^*)^{-1}(f_r)_i\}_{i \in I_r}$ are called the canonical dual frames because it is a dual frames for $\mathcal{F}$ (see (32)) and it has the following optimal property: Given $f_r \in \mathcal{H}$,

$$\sum_{i \in I_r} \sum_{r \in R} |\langle f_r, (TT^*)^{-1}(f_r)_i \rangle|^2 \le \sum_{i \in I_r} |(c_r)_i|^2, whenever \sum_{r \in R} f_r = \sum_{i \in I} (c_r)_i (f_r)_i, \tag{63}$$

for a family $((c_r)_i)_{i \in I_r} \in \ell_2(I_r)$. In other words, the above representation has the smallest ℓ_2-norm among the admissible frame coefficients representing f (see [61]).

Corollary(5.2.40)[212]. Let $S \in GL(\mathcal{H})$ be a J-selfadjoint operator acting on a Krein space $\mathcal{H}$ with fundamental symmetry J. Then, the following conditions are equivalent:

(i) S is a sequences of J-frames operator, i.e. there exists a sequences of J-frames $\mathcal{F}$ with synthesis operator T such that $S = TT^+$.

(ii) There exists a projection $Q^2 \in \mathcal{Q}$ such that QS is J-positive and $(I_r - Q^2)S$ is J-negative.

(iii) There exist J-positive operators $S_1, S_2 \in L(\mathcal{H})$ such that $S = S_1 - S_2$ and $R(S_1)$ $(resp. R(S_2))$ is a uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$.

Proof. (i)→(ii): Follows from Proposition (5.2.23) and Corollary (5.2.22).

(ii)→(iii): If there exists a projection $Q^2 \in \mathcal{Q}$ such that $Q^2 S$ is J-positive and $(I_r - Q^2)S$ is J-negative, consider the J-positive operators $S_1 = Q^2 S$ and $S_2 = -(I_r - Q^2)S$. Then, $S = S_1 - S_2$

and, by hypothesis, $R(S_1) = R(Q^2)$ is uniformly J-positive and $R(S_2) = R(I_r - Q^2) = N(Q^2)$ is uniformly J-negative.

(iii)→(i): Suppose that there exist J-positive operators $S_1, S_2 \in L(\mathcal{H})$ such that $S = S_1 - S_2$ and $R(S_1)$ $(resp. R(S_2))$ is a uniformly J-positive (resp. J-negative) subspace of $\mathcal{H}$. Denoting $\mathcal{K}_j = R(S_j)\ for\ j = 1, 2$, observe that $A_j = S_j J|_{\mathcal{K}_j} \in GL(\mathcal{K}_j)^+$.

Therefore, there exists a sequences of frames $\mathcal{F}_j = \{(f_r)_i\}_{i\in(I_r)_j} \subset \mathcal{K}_j$ for $\mathcal{K}_j$ such that $A_j = T_j T_j^*$ if $T_j \in L(\ell_2((I_r)_1), \mathcal{K}_j)$ is the synthesis operator of $\mathcal{F}_j$, for $j = 1, 2$.

Then, consider $\ell_2(I_r) := \ell_2((I_r)_1) \oplus \ell_2((I_r)_2)$ and $T \in L(\ell_2((I_r)), \mathcal{H})$ given by

$$Tx_r = T_1(x_r)_1 + T_2(x_r)_2,\ if\ x_r \in \ell_2((I_r)_1), x_r = (x_r)_1 + (x_r)_2, \quad (x_r)_j \in \ell_2((I_r)_j)$$
$for\ j = 1, 2$.

It is easy to see that T is the synthesis operator of the frames $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_2$. Furthermore $\mathcal{F}$ is a sequences of J-frames such that $(I_r)_+ = (I_r)_1$ and $(I_r)_- = (I_r)_2$.

Finally, endow $\ell_2(I_r)$ with the indefinite inner product defined by the diagonal operator$J_2 \in L(\ell_2(I))$ given by

$$J_2 (e_r)_i = (\sigma_r)_i (e_r)_i,$$

where $(\sigma_r)_i = 1\ if\ i \in (I_r)_1$ and $(\sigma_r)_i = -1\ if\ i \in (I_r)_2$. Notice that $T_1 J_2 = T_1$ and $T_2 J_2 = -T_2$. Furthermore, $T_1 T_2^* = T_2 T_1^* = 0$ because $R(T_2^*) = N(T_2)^\perp \subseteq \ell_2((I_r)_1) = \ell_2((I_r)_2) \subseteq N(T_1)$. Thus,

$$TT^+ = TJ_2T^*J = (T_1 + T_2)(T_1^* - T_2^*)J = T_1T_1^*J - T_2T_2^*J = A_1J - A_2J = S_1 - S_2 = S.$$

Given a sequences of J-frames $\mathcal{F} = \{(f_r)_i\}_{i\in I_r}$ for $\mathcal{H}$ with sequences of J-frames operator $S \in L(\mathcal{H})$, it follows from Corollary (5.2.24) that

$$S\left(\mathcal{M}_-^{[\perp]}\right) = \mathcal{M}_+\ and\ S\left(\mathcal{M}_+^{[\perp]}\right) = \mathcal{M}_- \tag{64}$$

i.e. S maps a maximal uniformly J-positive (resp. J-negative) subspace into another maximal uniformly J-positive (resp. J-negative) subspace. The next proposition shows under which hypotheses the converse holds.

Corollary(5.2.41)[212]. Let $S \in GL(\mathcal{H})$ be a J-selfadjoint operator. Then, S is a sequences of J-frames operator if and only if the following conditions hold:

(i) there exists a maximal uniformly J-positive subspace T of $\mathcal{H}$ such that $S(T)$ is also maximal uniformly J-positive;

(ii) $\sum_{r\in R}[Sf_r, f_r] \geq 0$ for every $f_r \in \mathcal{T}$;

(iii) $\sum_{r\in R}[Sg_r, g_r] \leq 0$ for every $g_r \in S(\mathcal{T})^{[\perp]}$.

Proof. If S is a sequences of J-frames operator, consider $\mathcal{T} = \mathcal{M}_-^{[\perp]}$ which is a maximal uniformly J-positive subspace $\mathcal{T}$ of $\mathcal{H}$. Then, $S(\mathcal{T}) = \mathcal{M}_+$ is also maximal uniformly J-positive. Furthermore,

$$\sum_{r\in R}[Sf_r, f_r] = \sum_{r\in R}[S(Q^2)^+f_r, (Q^2)^+f_r] = \sum_{r\in R}[Q^2S(Q^2)^+f_r, f_r] = \sum_{r\in R}[S_+f_r, f_r] \geq 0$$

for every $f_r \in \mathcal{T}$,where $Q^2 = P^2_{\mathcal{M}_+//\mathcal{M}_-}$. Also, $S(\mathcal{T})^{[\perp]} = \mathcal{M}_+^{[\perp]} = N((Q^2)^+) = R((I_r - Q^2)^+)$. So,

$$\sum_{r\in R}[Sg_r, g_r] = \sum_{r\in R}[S(I_r - Q^2)^+g_r, (I_r - Q^2)^+g_r] = \sum_{r\in R}[(I_r - Q^2)S(I_r - Q^2)^+g_r, g_r]$$
$$= \sum_{r\in R}[-S_-g_r, g_r] \leq 0\ for\ every\ g_r \in S(\mathcal{T})^{[\perp]}.$$

Conversely, suppose that there exists a maximal uniformly J-positive subspace T satisfying the hypotheses. Let $\mathcal{M} = S(\mathcal{T})$, which is maximal uniformly J-positive. Then, consider $Q^2 = P^2_{\mathcal{M}//\mathcal{T}^{[\perp]}}$. It is well defined because $\mathcal{T}^{[\perp]}$ is maximal uniformly J-negative, see [11]. Moreover, $Q^2 \in \mathcal{Q}$.

Notice that $R(S(I_r - Q^2) = S(\mathcal{M}^{[\perp]}) = S(S(\mathcal{T})^{[\perp]}) = S(S^{-1}(\mathcal{T}^{[\perp]})) = \mathcal{T}^{[\perp]}$. Therefore, $Q^2S(I_r - Q^2)^+ = 0$ and

$$Q^2S = Q^2S(Q^2)^+ + Q^2S(I_r - Q^2)^+ = Q^2S(Q^2)^+.$$

Furthermore, if $\sum_{r\in R}[Sf_r, f_r] \geq 0$ for every $f_r \in \mathcal{T}$ then Q^2S is J-positive. Analogously, if $\sum_{r\in R}[Sg_r, g_r] \leq 0$ for every $g_r \in S(\mathcal{T})^{[\perp]}$ then $(I_r - Q^2)S$ is J-negative. Then, by Theorem (5.2.26) , S is a series of J-frames operator.

As it was showed in Proposition (5.2.23), if the operators $S \in L(\mathcal{H})$ are a J-frames operators then it is an invertible J-selfadjoint operator satisfying $ind_{\pm}(S) = dim(\mathcal{H}_{\pm})$. Unfortunately, the converse is not true.

Chapter 6
Local Spectral Theory and Definite Normal Operators

Moreover, the restriction of the normal operator to the spectral subspace corresponding to such a Borel subset is a normal operator in some Hilbert space. In particular, if the spectrum consists entirely out of positive and negative type spectrum, then the operator is similar to a normal operator in some Hilbert space. We use this result to show the existence of operator roots of a class of quadratic operator polynomials with normal coefficients. In addition, we show that the Riesz–Dunford spectral subspace corresponding to a spectral set which is only of positive type is uniformly positive. The restriction of the operator to this subspace is then normal in a Hilbert space.

Section (6.1): Spectral Theory for Normal Operators in Krein Spaces

Recall that a bounded operator N in a Krein space $(\mathcal{H},[.,.])$ is normal if $N^{+}N = NN^{+}$, where N^{+}denotes the adjoint operator of N with respect to the Krein space (indefinite) inner product $[.,.]$. In contrast to (definitizable) selfadjoint operators in Krein spaces, the knowledge about normal operators is very restricted.

Some results exist for normal operators in Pontryagin spaces. The starting point is a result of Naimark, see [37], which implies that for a normal operator N in a Pontryagin space Π_{κ} there exists a κ-dimensional non-positive common invariant subspace for N and its adjoint N^{+}. In [14], [20] spectral properties of normal operators in Pontryagin spaces were considered and, in the case Π_1 , a classification of the normal operators is given.

There is only a very limited number of results in the study of normal operators in spaces others than Pontryagin spaces. In [9] a definition of definitizable normal operators was given and it was showed that a bounded normal definitizable operator in a Banach space with a regular Hermitian form has a spectral function with finitely many critical points. Let us note that in this case the spectral function is a homomorphism from the Borel sets containing no critical points on their boundaries to a commutative algebra of normal projections, see also [4]. Some advances for Krein spaces without the assumption of definitizability can be found in [5]. We mention that [4] contains some perturbation results for fundamentally reducible normal operators. The case of fundamentally reducible and strongly stable normal operators is considered in [6], [7].

On the other hand, the spectral theory for definitizable (and locally definitizable) selfadjoint operators in Krein spaces is well developed (see, [22], [28], [33] and references therein). One of the main features of definitizable selfadjoint operators in Krein spaces is their property to act locally similarly as a selfadjoint operator in some Hilbert space. More precisely, the spectrum of a definitizable operator consists of spectral points of positive and of negative type, and of finitely many exceptional (i.e., nonreal or critical) points, see [32]. For a real point λ of positive (negative) type of a selfadjoint operator in a Krein space there exists a local spectral function E such that $(E(\delta)\mathcal{H},[.,.])$ (resp. $(E(\delta)\mathcal{H},-[.,.])$) is a Hilbert space for (small) neighbourhoods $\delta\ of\ \lambda$.

In [11], [13] a characterization for spectral points of positive (negative) type was given in terms of normed approximate eigensequences. If all accumulation points of the sequence $([x_n,x_n])$ for each normed approximate eigensequences corresponding to λ are positive (resp. negative) then λ is a spectral point of positive (resp. negative) type. Obviously, the above characterization can be used as a definition for spectral points of positive (negative) type for arbitrary (not necessarily selfadjoint) operators in Krein spaces (as it was done in [2]). It is the main result of this paper that also for a normal operator N in a Krein space $(\mathcal{H},[.,.])$ positive and negative type spectrum implies the existence of a local spectral function for N. However, for this we have to impose some additional assumptions: The spectra of the real and imaginary part of N are real and the growth of the resolvent of the imaginary part (close to the real axis) of N is of finite order. Under these assumptions we are able to show that N has a local spectral function E on each closed rectangle which consists only of spectral points of positive type or of points from the resolvent set of N. The local spectral function E is then defined for all Borel subsets δ of this rectangle and $E(\delta)$ is a selfadjoint projection in the Krein space $(\mathcal{H},[.,.])$.

It has the property that $(E(\delta)\mathcal{H}, [.,.])$ is a Hilbert spaces for all such δ. This implies that the restriction of N to the spectral subspace $E(\delta)\mathcal{H}$ is a normal operator in the Hilbert space $(E(\delta)\mathcal{H}, [.,.])$.

We emphasize that this result showides a simple sufficient condition for the normal operator N to be similar to a normal operator in a Hilbert space: If each spectral point of N is of positive or of negative type and if the spectra of the real and imaginary part of N are real and the growth of the resolvent of the imaginary part is of finite order, then N is similar to a normal operator in a Hilbert space. Actually, in the final section, we use this result to show the existence of an operator root of a quadratic operator pencil with normal coefficients.

In this section we collect some statements on bounded operators in Banach spaces. As usual, by $L(X, Y)$ we denote the set of all bounded linear operators acting between Banach spaces X *and* Y and set $L(X) := L(X, X)$.

In this section a subspace is always a closed linear manifold. The approximate point spectrum $\sigma_{ap}(T)$ of a bounded linear operator T in a Banach space X is the set of all $\lambda \in \mathbb{C}$ for which there exists a sequence $(x_n) \subset X$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ *and* $(T - \lambda)x_n \to 0$ *as* $n \to \infty$. A point in $\sigma_{ap}(T)$ is called an approximate eigenvalue of T. We have

$$\partial\sigma(T) \subset \sigma_{ap}(T) \subset \sigma(T), \tag{1}$$

see [10]. Therefore, $\sigma_{ap}(T) \neq \emptyset$ *if* $X \neq \{0\}$.

The following Lemmas (6.2.1), (6.2.2), (6.2.3) are well known. For their proofs we refer to in [16].

Lemma (6.1.1)[21]. Let S *and* T be two commuting bounded operators in a Banach space X and let p be a polynomial in two variables. Then

$$\sigma(p(S,T)) \subset \{p(\lambda,\mu) : \lambda \in \sigma(S), \mu \in \sigma(T)\}.$$

If, in addition, the operators $S + T$ *and* $i(S - T)$ have real spectra, i.e.,

$$\sigma(S + T) \subset \mathbb{R} \text{ and } \sigma(S - T) \subset i\mathbb{R} \tag{2}$$

then the following identity holds:

$$\sigma(p(S,T)) = \{p(\lambda,\bar{\lambda}) : \lambda \in \sigma(S)\}.$$

In particular, we have

$$\sigma\left(\frac{S+T}{2}\right) = \{Re\,\lambda : \lambda \in \sigma(S)\},$$

$$\sigma\left(\frac{S-T}{2i}\right) = \{Im\,\lambda : \lambda \in \sigma(S)\}.$$

Lemma (6.1.2)[21]. Let T be a bounded operator in a Banach space X and let $\mathcal{L}$ be a subspace of X which is invariant with respect to T. Then

$$\sigma(T|\mathcal{L}) \subset \sigma(T) \cup \rho_b(T),$$

where $\rho_b(T)$ is the union of all bounded connected components of $\rho(T)$. In particular, if $\sigma(T) \subset \mathbb{R}$, we have $\sigma(T|\mathcal{L}) \subset \sigma(T)$.

Lemma (6.1.3)[21]. (Rosenblum's Corollary) Let S and T be bounded operators in the Banach spaces X *and* Y, respectively. If $\sigma(S) \cap \sigma(T) = \emptyset$, then for every $Z \in L(Y, X)$ the operator equation $SX - XT = Z$ has a unique solution $X \in L(Y, X)$. In particular, $SX = XT$ implies $X = 0$.(See [25])

Let T be a bounded operator in a Banach space and let $Q \subset \mathbb{C}$ be a compact set. We say that a subspace $\mathcal{L}_Q$ is the maximal spectral subspace of T corresponding to Q *if* $\mathcal{L}_Q$ is T-invariant, $\sigma(T|\mathcal{L}_Q) \subset \sigma(T) \cap Q$ and if $L \subset \mathcal{L}_Q$ holds for every T-invariant subspace $\mathcal{L}$ with $\sigma(T|\mathcal{L}) \subset Q$. Recall that such a subspace is hyperinvariant with respect to T, i.e., it is invariant with respect to each bounded operator which commutes with T (see [23]).

If the spectrum of the bounded operator T is real, we say that the growth of the resolvent of T is of finite order $n, n \in N \setminus \{0\}$, if for some $c > 0$ there exists an $M > 0$, such that

$$0 < |Im\,\lambda| < c \Rightarrow \|(T - \lambda)\|^{-1} \leq \frac{M}{|Im\,\lambda|^n}. \quad (3)$$

Since the function $\rho \mapsto M/\rho^n, 0 < \rho < 1$, satisfies the Levinson condition (cf. [34]), it is a consequence of (3) and [34] that to each compact interval Δ the maximal spectral subspace $\mathcal{L}_\Delta$ of T corresponding to Δ exists.

By $r(T)$ we denote the spectral radius of a bounded operator T in a Banach space.

Lemma (6.1.4)[21]. Let $T \neq 0$ be a bounded operator in a Banach space with real spectrum such that the growth of its resolvent is of order n. Then for all $k \geq n$ we have

$$\|T^k\| \leq 2^k \|T\|^{k-n} (M + \|T\|^{n-1}) r(T),$$

where $M = sup\{|Im\,\lambda|^n \|(T - \lambda)^{-1}\| : 0 < |Im\,\lambda| < \|T\|\}$.

Proof. For $\rho > 0$ we define the function

$$M(\rho) = sup\{|Im\,\lambda|^n (T - \lambda)^{-1} : 0 < |Im\,\lambda| < \rho\}.$$

It is obvious that this function is non-decreasing and continuous. Therefore, $M(0) := inf_{\rho>0} M(\rho)$ exists. We have $M = M(\|T\|)$.

Let $k \geq n$. Let C be the circle with center 0 and radius $\rho > r(T)$. For $0 < |Im\,\lambda| < \rho$ we have

$$\|(T - \lambda)^{-1}\| \leq \frac{M(\rho)}{|Im\,\lambda|^n}. \quad (4)$$

Observe that for $j \in \mathbb{N}, j \geq 1$, the function $\lambda \mapsto \lambda^{-j} (T - \lambda)^{-1}$ is holomorphic outside of C. Due to $\|(T - \lambda)^{-1}\| = O(|\lambda|^{-1})\ as\ |\lambda| \to \infty$, the Cauchy integral theorem and standard estimates of contour integrals

we obtain

$$\int_C \lambda^{-j} (T - \lambda)^{-1} d\lambda = 0 \quad , j \geq 1.$$

Therefore, the relation

$$\left(\frac{\lambda^2 - \rho^2}{\lambda}\right)^k = \sum_{j=0}^{k} \binom{k}{j} \lambda^{2j-k} (-\rho^2)^{k-j}$$

yields

$$\frac{-1}{2\pi i} \int_C \left(\frac{\lambda^2 - \rho^2}{\lambda}\right)^k (T - \lambda)^{-1} d\lambda = \sum_{j=0}^{k} \binom{k}{j} (-\rho^2)^{k-j} T^{2j-k},$$

where $\lceil k/2 \rceil$ denotes the smallest integer larger than $k/2$. Since $k \geq n$ and

$$\left(\frac{\lambda^2 - \rho^2}{\lambda}\right) = 2|Im\,\lambda|\ for\ \lambda \in C,$$

together with (4) this gives

$$\left\| T^k + \sum_{j=\lceil k/2 \rceil}^{k-1} \binom{k}{j} (-\rho^2)^{k-j} T^{2j-k} \right\| \leq 2^k M(\rho) \rho^{k-n+1},$$

and hence

$$\|T^k\| \leq \left(2^k M(\rho)\rho^{k-n+1} + \sum_{j=\lceil k/2\rceil}^{k-1} \binom{k}{j} \rho^{2(k-j)-1} \|T\|^{2j-k}\right)\rho.$$

Letting $\rho \to r(T)$ we obtain

$$\|T^k\| \leq \left(2^k M(r(T))\|T\|^{k-n} + \sum_{j=\lceil k/2\rceil}^{k-1} \binom{k}{j} \|T\|^{2(k-j)-1} \|T\|^{2j-k}\right) r(T).$$

We have $M(r(T)) \leq M(\|T\|)$, which leads to the desired estimate with $M = M(\|T\|)$.

For a finite interval Δ we denote by $\ell(\Delta)$ the length of Δ.

Corollary (6.1.5)[21]. Let T be as in Lemma (6.1.4). Then there exists $C > 0$ such that for each $k \geq n$, each $\lambda \in \sigma(T)$ and each compact interval Δ with $\lambda \in \Delta \; and \; \ell(\Delta) \leq \|T\|$ we have

$$\|(T|\mathcal{L}_\Delta - \lambda)^k\| \leq 4^k \|T\|^k C \, . \, \ell(\Delta),$$

where $\mathcal{L}_\Delta$ denotes the maximal spectral subspace of T corresponding to Δ.

Proof. We have $\sigma(T_\Delta) \subset \Delta$, where $T_\Delta := T|\mathcal{L}_\Delta$. Clearly, the growth of the resolvent of $T\Delta - \lambda$ is of order n. Since $\|T_\Delta - \lambda\| \leq \|T\| + |\lambda| \leq 2\|T\|$ and $\mathrm{r}(T_\Delta - \lambda) \leq \ell(\Delta)$, Lemma 6.1.4 gives the estimate

$$\|(T_\Delta - \lambda)^k\| \leq 2^k (2\|T\|)^{k-n}\left(\widetilde{M} + 2^{n-1}\|T\|^{n-1}\right)\ell(\Delta)$$

with $\widetilde{M} = sup\{|\, Im\, \mu|^n \; (T_\Delta - \lambda - \mu)^{-1} : 0 < |\, Im\, \mu| < \|T_\Delta - \lambda\|\}$. As λ is real,

$$\begin{aligned} \widetilde{M} &\leq sup\{|\, Im\, \mu|^n \|(T - \lambda - \mu)^{-1}\| : 0 < |\, Im\, \mu| < 2\|T\|\} \\ &\leq sup\{|\, Im\, \mu|^n \|(T - \lambda)^{-1}\| : 0 < |\, Im\, \mu| < 2\|T\|\} \end{aligned}$$

which is independent of $\Delta, k \; and \; \lambda$.

Recall that an inner product space $(\mathcal{H}, [.,.])$ is called a Krein space if there exist subspaces $\mathcal{H}_+ \; and \; \mathcal{H}_-$ such that $(\mathcal{H}_+, [.,.]) \; and \; (\mathcal{H}_-, -[.,.])$ are Hilbert spaces and

$$H = \mathcal{H}_+ \dotplus \mathcal{H}_-, \tag{5}$$

where $\dotplus$ denotes the direct sum of subspaces. We refer to (5) as a fundamental decomposition of the Krein space $(\mathcal{H}, [.,.])$.

An inner product space $(\mathcal{H}, [.,.])$ is called a G-space if $\mathcal{H}$ is a Hilbert space and the inner product $[.,.]$ is continuous with respect to the norm $\|.\|$ on $\mathcal{H}$, that is, there exists $c > 0$ such that

$$|[x, y]| \leq c\|x\|\|y\| \qquad for \; all \; x, y \in \mathcal{H}.$$

Let $(.,.)$ be a Hilbert space inner product on $\mathcal{H}$ inducing $\|.\|$.Then the inner products $(.,.)$ and $[.,.]$ are connected via

$$[x, y] = (Gx, y), x, y \in \mathcal{H},$$

where $G \in L(\mathcal{H})$ is a uniquely determined selfadjoint operator in $(\mathcal{H}, (.,.))$. It is well known that $(\mathcal{H}, [.,.])$ is a Krein space if and only if G is boundedly invertible, see, [3], [8]. A bounded operator A in the G-space $\mathcal{H}$ is said to be [., .]-selfadjoint or G-selfadjoint if

$$[Ax, y] = [x, Ay] \tag{6}$$

holds for all $x, y \in \mathcal{H}$.

Spectral points of definite type, defined below for bounded operators in a G-space, were defined for $[.,.]$-selfadjoint operators in G-spaces in [42] and in [21] for arbitrary operators (and relations) in Krein spaces.

Definition (6.1.6)[21]. For a bounded operator A in the G-space $(\mathcal{H}, [.,.])$ a point $\lambda \in \sigma_{ap}(A)$ is called a spectral point of positive (negative) type of A if for every sequence (x_n) with $\|x_n\| = 1 \; and \; \|(A - \lambda)x_n\| \to 0 \; as \; n \to \infty$, we have

$$lim_{n\to\infty}\ inf\ [xn, xn\ ] > 0(lim_{n\to\infty}\ sup[x_n, x_n\ ] < 0, respectively).$$

We denote the set of all points of positive (negative) type of A by $\sigma_{++}(A)$ $(\sigma_{--}(A), respectively). A\ set\ \Delta \subset \mathbb{C}$ is said to be of positive (negative) type with respect to A if every approximate eigenvalue of A in Δ belongs to $\sigma_{++}(A)$ $(\sigma_{--}(A), respectively)$.

Remark (6.1.7)[21]. If the operator A is [., .]-selfadjoint, then the sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in ℝ (cf. [13]).

The following lemma is well known for selfadjoint operators in Krein spaces and $[.,.]$-selfadjoint operators in G-spaces (see, [22], [13]). The proof for arbitrary bounded operators remains essentially the same. However, for the convenience of the reader we give a short proof here.

Lemma (6.1.8)[21]. Let A be a bounded operator in the G-space $(\mathcal{H}, [.,.])$. Then a compact set $K \subset \mathbb{C}$ is of positive type with respect to A if and only if there exist a neighbourhood U of K in C and numbers $\varepsilon, \delta > 0$ such that for all $x \in \mathcal{H}$ and each $\lambda \in U$ we have

$$\|(A - \lambda)x\| \le \varepsilon\|x\| \Rightarrow [x, x] \ge \delta\|x\|^2 .$$

In this case, the set U is of positive type with respect to A.

Proof. Assume that K is a compact set of positive type with respect to A, i.e., $K \cap \sigma_{ap}(A) \subset \sigma_{++}(A)$. Let $\lambda_0 \in K$. Then it follows from Definition (6.1.6) and the properties of the points of regular type of A that there exist $\varepsilon_0, \delta_0 > 0$ such that for all $x \in \mathcal{H}$ we have

$$\|(A - \lambda_0)x\| \le 2\varepsilon_0\|x\| \Rightarrow [x, x] \ge \delta_0\|x\|^2 .$$

From this we easily conclude that for all $x \in \mathcal{H}$ and all $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \varepsilon_0$ we have

$$\|(A - \lambda)x\| \le \varepsilon_0\|x\| \Rightarrow [x, x] \ge \delta_0\|x\|^2 .$$

Since λ_0 was an arbitrary point in K, the assertion follows from the compactness of K. The converse statement is evident.

One of the main results of [13] is that under a certain condition a $[.,.]$-selfadjoint operator in a G-space has a local spectral function of positive type on intervals which are of positive type with respect to the operator. Let us recall the definition of such a local spectral function and the exact statement for $[.,.]$-selfadjoint operators.

Definition (6.1.9)[21]. Let $(\mathcal{H}, [.,.])$ be a G-space, $A \in L(\mathcal{H})\ and\ S \subset \mathbb{C}$. A set function E mapping from the system $B(S)$ of Borel-measurable subsets of S whose closure is also contained in S to $L(\mathcal{H})$ is called a local spectral function of positive type of the operator A on S if for all $Q, Q_1, Q_2, \ldots \in B(S)$ the following conditions are satisfied:

(i) $(E(Q)\mathcal{H}, [.,.])$ is a Hilbert space and $E(Q)$ is $[.,.]$-selfadjoint.

(ii) $E(Q_1 \cap Q_2) = E(Q_1)E(Q_2)$.

(iii) If $Q_1, Q_2, \ldots \in B(S)$are mutually disjoint, then

$$E\left(\bigcup_{k=1}^{\infty} Q_k\right) = \sum_{k=1}^{\infty} E(Q_k),$$

where the sum converges in the strong operator topology.

(iv) $AB = BA \Rightarrow E(Q)B = BE(Q)\ for\ every\ B \in L(\mathcal{H})$.

(v) $\sigma(A|E(Q)\mathcal{H}) \subset \overline{\sigma(A) \cap Q}$.

(vi) $\sigma(A|(I - E(Q))\mathcal{H}) \subset \overline{\sigma(A) \setminus Q}$.

Note that (ii) implies that $E(Q)$ is a projection for all $Q \in B(S)$ and that from (iii) (or (v)) it follows that $E(\emptyset) = 0. By\ \mathbb{C}^+$ $(\mathbb{C}^-)$ we denote the open upper (lower, respectively) halfplane of the complex plane ℂ.

Theorem (6.1.10)[21]. Let A be a $[.,.]$-selfadjoint operator in the G-space $(\mathcal{H},[.,.])$. If the interval Δ is of positive type with respect to A and if each of the sets $\rho(A) \cap \mathbb{C}^+$ and $\rho(A) \cap \mathbb{C}^-$ accumulates to each point of Δ, respectively, then A has a local spectral function E of positive type on Δ. For each closed interval $\delta \subset \Delta$ the subspace $E(\delta)\mathcal{H}\mathcal{H}$ is the maximal spectral subspace of A corresponding to δ.

For the rest of this section let $(\mathcal{H},[.,.])$ be a Krein space. It is our aim to extend Theorem (6.1.10) to normal operators in Krein spaces. Recall that a bounded operator N in a Krein space $(\mathcal{H},[.,.])$ is called normal if it commutes with its adjoint N^+, i.e.,

$$N^+N = NN^+.$$

By definition the real part of a bounded operator C in a Krein space $(\mathcal{H},[.,.])$ is the operator $(C + C^+)/2$ and the imaginary part is given by $(C - C^+)/2i$. It is clear that both real and imaginary part of an arbitrary bounded operator are $[.,.]$-selfadjoint. Moreover, it is easy to see that a bounded operator in $(\mathcal{H},[.,.])$ is normal if and only if its real part and its imaginary part commute.

Lemma (6.1.11)[21]. Let N be a normal operator in the Krein space $(\mathcal{H},[.,.])$. If ReN and ImN have real spectra only, then $\sigma(N) = \sigma_{ap}(N)$.

Proof. Assume that $\lambda \in \sigma(N) \setminus \sigma_{ap}(N)$. Then $N - \lambda$ has a trivial kernel and ran$(N - \lambda) \neq \mathcal{H}$ is closed. Hence, $\lambda \in \sigma_p(N^+)$. $Set\ \mathcal{L} := kerN^+ - \lambda$. This subspace is N-invariant. By Lemma (6.1.2) the operators $ReN|\mathcal{L}$ and $ImN|\mathcal{L}$ have real spectra. Thus, by Lemma (6.1.1) (with $S = N^+|\mathcal{L}$ and $T = N|\mathcal{L}$) we conclude that $\sigma(N|\mathcal{L}) = \{\mu: \mu \in \sigma(N^+|\mathcal{L})\} = \{\lambda\}$. Hence, $\lambda \in \sigma_{ap}(N|\mathcal{L}) \subset \sigma_{ap}(N)$. A contradiction.

Theorem (6.1.12)[21]. Let N be a normal operator in the Krein space $(\mathcal{H},[.,.])$. If ReN and ImN have real spectra and the growth of the resolvent of ImN is of finite order, then N has a local spectral function of positive type on each closed rectangle $[a,b] \times [c,d]$ which is of positive type with respect to N.

Proof. Let $[a,b] \times [c,d]$ be of positive type with respect to N. Together with Lemma (6.1.11) we have

$$([a,b] \times [c,d]) \cap \sigma(N) \subset \sigma_{++}(N).$$

By Lemma (6.1.8) there exist an open neighbourhood U of $[a,b] \times [c,d]$ in $\mathbb{C}$and numbers $\varepsilon,\delta \in (0,1)$ such that

$$\lambda \in \mathcal{U}, x \in \mathcal{H}, \|(N - \lambda)x\| \leq \varepsilon\|x\| \Rightarrow [x,x] \geq \delta\|x\|^2 . \tag{7}$$

By Corollary (6.1.5), there exists a value $\tau > 0$ such that for each compact interval Δ with length $\ell(\Delta) < \tau$ and any $\lambda \in \Delta \cap \sigma(ImN)$ we have

$$\|(ImN|\mathcal{L}_\Delta - \lambda)^k\| \leq \frac{\delta^{k-1}\varepsilon^k}{2^k} \quad for\ all\ k = k_0, k_0 + 1, \dots, 2k_0 , \tag{8}$$

where k_0 is the order of growth of the resolvent of ImN and $\mathcal{L}_\Delta$ is the maximal spectral subspace of ImN corresponding to the interval Δ.

The proof will be divided into three steps. In the first step we define the spectral subspace corresponding to rectangles $\Delta_1 \times \Delta_2 \subset U$ with $\ell(\Delta_2) < \tau$. In the second step we show some properties of the spectral subspaces defined in step 1. In the third step we define the spectral subspace corresponding to the rectangle $[a,b] \times [c,d]$ and complete the proof.

(1) Let Δ1 and Δ be compact intervals such that $\Delta_1 \times \Delta \subset \mathcal{U}$ and $\ell(\Delta) < \tau$. Note that the inner product space $(\mathcal{L}_\Delta,[.,.])$ is a G-space which is not necessarily a Krein space. Since a maximal spectral subspace is hyperinvariant (see, [23]), the space $\mathcal{L}_\Delta$ is invariant with respect to N, N^+, ReN and ImN. By A_0, B_0, N_0 and $N_{0,+}$ denote the restrictions of , ImN, N and N^+ to $\mathcal{L}_\Delta$,, respectively. Then we have, see Lemma (6.1.2),

$$\sigma(A_0) \subset \sigma(ReN) \subset \mathbb{R}\ and\ \sigma(B_0) \subset \sigma(ImN) \cap \Delta. \tag{9}$$

Moreover, from $N_0 = A_0 + iB_0, N0, + = A_0 - iB_0$, (9) and Lemma (6.1.1) we conclude

$$\sigma(A_0) = \{Re\,\lambda : \lambda \in \sigma(N_0)\}\ and\ \sigma(B_0) = \{Im\,\lambda : \lambda \in \sigma(N_0)\},$$

hence

$$\sigma(N_0) \subset \sigma(A_0) \times \Delta.$$

The operator A_0 is obviously $[.,.]$-selfadjoint. In the following we will show

$$\Delta_1 \cap \sigma(A_0) \subset \sigma_{++}(A_0). \tag{10}$$

To this end set

$$\tilde{\varepsilon} := min\left\{\frac{\varepsilon}{2}, \frac{\delta^{j-2}\varepsilon^j}{2^j\,(\|ImN\| + r(ImN))^{j-1}} : j = 2,\dots,k_0\right\}$$

We may assume that $ImN \neq 0$. Otherwise, the assertion of Theorem (6.1.12). follows directly from Theorem (6.1.10). We will show that for all $\alpha \in \Delta_1 \cap \sigma(A_0)\ and\ for\ all\ x \in \mathcal{L}_\Delta$ we have

$$\|(A_0 - \alpha)x\| \leq \tilde{\varepsilon}\,\|x\| \Rightarrow [x,x] \geq \delta\|x\|^2,$$

which then implies (10), see Lemma (6.1.8). If $\sigma(ImN) \cap \Delta = \emptyset$, then it follows from (9) that $\mathcal{L}_\Delta = \{0\}$, and nothing needs to be shown. Otherwise, there exists $\beta \in \Delta \cap \sigma(ImN)$. Let $\alpha \in \Delta_1 \cap \sigma(A_0)$ and $x \in \mathcal{L}_\Delta$, $\|x\| = 1$, and suppose that $\|(A_0 - \alpha)x\| \leq \tilde{\varepsilon}$. Let us show that for all $j = 1,\dots,2k_0$ we have

$$\left\|(B_0 - \beta)^j x\right\| \leq \frac{\delta^{j-1}\varepsilon^j}{2^j}. \tag{11}$$

For $j = k_0,\dots,2k_0$ this is a direct consequence of (8). Assume now that (11) holds for all $j \in \{k,\dots,k_0\}$ where $k \in \{2,\dots,k_0\}$ but does not hold for $j = k - 1$, i.e.,

$$\|(B_0 - \beta)^{k-1}x\| > \frac{\delta^{k-2}\varepsilon^{k-1}}{2^{k-1}}. \tag{12}$$

Then we have

$$\left\|(N_0 - (\alpha + i\beta))\frac{(B_0 - \beta)^{k-1}x}{\|(B_0 - \beta)^{k-1}x\|}\right\| \leq \frac{\|B_0 - \beta\|^{k-1}\|(A_0 - \alpha)x\| + \|(B_0 - \beta)^k x\|}{\|(B_0 - \beta)^{k-1}x\|}$$

$$\leq \frac{2^{k-1}\,(\|ImN\| + r(ImN))^{k-1}}{\delta^{k-2}\varepsilon^{k-1}}\,\tilde{\varepsilon} + \frac{2^{k-1}}{\delta^{k-2}\delta^{k-1}}\,\frac{\varepsilon^k\varepsilon^{k-1}}{2^k} \leq \frac{\varepsilon}{2} + \delta\,\frac{\varepsilon}{2} \leq \varepsilon.$$

As $\alpha + i\beta \in \Delta_1 \times \Delta \subset U$, it follows from (7) that

$$\delta \leq \left[\frac{(B_0 - \beta)^{k-1}x}{\|(B_0 - \beta)^{k-1}x\|}, \frac{(B_0 - \beta)^{k-1}x}{\|(B_0 - \beta)^{k-1}x\|}\right] \leq \frac{\|(B_0 - \beta)^{2k-2}x\|}{\|(B_0 - \beta)^{k-1}x\|^2}.$$

Owing to $k \leq 2k - 2 \leq 2k_0$, relation (11) holds for $j = 2k - 2$ by assumption, and thus

$$\|(B_0 - \beta)^{k-1}x\| \leq \sqrt{\delta^{-1}\|(B_0 - \beta)^{2k-2}x\|} \leq \sqrt{\frac{\delta^{2k-4}\varepsilon^{2k-2}}{2^{2k-2}}} = \frac{\delta^{k-2}\varepsilon^{k-1}}{2^{k-1}}$$

follows. But this contradicts (12). Hence, (11) holds for $j = k - 1$, and, by induction, for $j = 1$. Hence,

$$\|(N - (\alpha + i\beta))x\| \leq \|(A_0 - \alpha)x\| + \|(B_0 - \beta)x\| \leq \varepsilon.$$

By (7), this yields $[x,x] \geq \delta$ and (10) is showed.

Due to Theorem (6.1.10), the operator $A_0 \in L(\mathcal{L}_\Delta)$ has a local spectral function E_Δ of positive type on Δ_1, and the subspace

$$\mathcal{H}_{\Delta_1} \times \Delta := E_\Delta(\Delta_1)\mathcal{L}_\Delta$$

is the maximal spectral subspace of A_0 corresponding to Δ_1 .Moreover, $\mathcal{H}_{\Delta_1} \times \Delta$ is a Hilbert space with respect to the inner product $[.,.]$. Since $\mathcal{H}_{\Delta_1} \times \Delta$ is invariant with respect to both $N\ and\ N^+$, the $[.,.]$-orthogonal complement

$$\mathcal{H}^{[\perp]}_{\Delta_1 \times \Delta} = \{y \in \mathcal{H} : [y,x] = 0\ for\ all\ x \in \mathcal{H}_{\Delta_1} \times \Delta\}$$

is also $N-\ and\ N^+$-invariant and $\left(\mathcal{H}^{[\perp]}_{\Delta_1 \times \Delta}, [.,.]\right)$ is a Krein space, see, [33]. Moreover, we have

$$(N|\mathcal{H}^{[\perp]}_{\Delta_1 \times \Delta})^+ = N^+|\mathcal{H}^{[\perp]}_{\Delta_1 \times \Delta}.$$

(2) Let $Q := \Delta_1 \times \Delta \subset U$ be a rectangle as in step 1. By Q^i (Δ^i) we denote the complex (real, respectively) interior of the set Q (Δ, respectively). In this step of the proof we shall show that the subspaces $\mathcal{H}_Q\ and\ \mathcal{H}_Q^{[\perp]}$, defined in the first step, have the following properties.

(a) $\sigma\ (N|\mathcal{H}_Q) \subset \sigma(N) \cap Q$.

(b) If $\mathcal{M} \subset \mathcal{H}$ is a subspace which is both $N-\ and\ N^+-$invariant such that

$$\sigma(N|\mathcal{M}) \subset Q,$$

Then $\mathcal{M} \subset \mathcal{H}_Q$.

(c) If $\mathcal{H}_Q = \{0\}\quad then\ Q^i \subset \rho(N)$.

(d) $\sigma\left(N|\mathcal{H}_Q^{[\perp]}\right) \subset \overline{\sigma(N) \setminus Q}$.

(e) If the bounded operator B commutes with N then both $\mathcal{H}_Q$ and $\mathcal{H}_Q^{[\perp]}$ are B-invariant.

(f) $\mathcal{H}_Q$ is the maximal spectral subspace of N corresponding to Q.

By Lemma (6.1.2) and (9) we have

$$\sigma(Im(N|\mathcal{H}_Q)) = \sigma(B_0|\mathcal{H}_Q) \subset \sigma(B_0) \subset \Delta.$$

In addition,

$$\sigma(Re(N|\mathcal{H}_Q)) = \sigma(A_0|\mathcal{H}_Q) \subset \Delta 1 .$$

From this and Lemma (6.1.1) we obtain

$$\sigma(N|\mathcal{H}_Q) \subset Q.$$

Since the spectrum of a normal operator in a Hilbert space coincides with its approximate point spectrum, (a) follows.

Let $\mathcal{M} \subset \mathcal{H}$ be a subspace as in (b). By Lemma (6.1.1) we have

$$\sigma(ImN|\mathcal{M}) \subset \Delta\ and\ \sigma(ReN|\mathcal{M}) \subset \Delta_1 .$$

As $\mathcal{L}_\Delta$ is the maximal spectral subspace of ImN corresponding to Δ, we conclude from the first relation that $\mathcal{M} \subset \mathcal{L}_\Delta$. From the second relation we obtain (b) since $\mathcal{H}_Q$ is the maximal spectral subspace of $ReN|\mathcal{L}_\Delta$ corresponding to the interval Δ_1 , cf. Theorem (6.1.10).

Let us show (c). By definition of $\mathcal{H}_Q$ it follows from $\mathcal{H}_Q = \{0\}$ that $\Delta_1^i \subset \rho(ReN|\mathcal{L}_\Delta)$. Hence, by Lemma (6.1.1) we have

$$\Delta_1^i \times \mathbb{R} \subset \rho(N|\mathcal{L}_\Delta). \tag{13}$$

Let J be a closed interval which contains $\sigma(ImN)$ and let $\delta_1\ and\ \delta_2$ be the two (closed) components of $J \setminus \Delta^i$. By $\mathcal{L}_{\delta_1}$ and $\mathcal{L}_{\delta_2}$ denote the maximal spectral subspaces of ImN corresponding to the intervals $\delta_1\ and\ \delta_2$, respectively.

Set

$$\mathcal{L}_{\Delta^c} := L\mathcal{L}_{\delta_1} \dot{+} \mathcal{L}_{\delta_2} .$$

Obviously, we have

$$\sigma(ImN|\mathcal{L}_{\Delta^c}) \subset \delta_1 \cup \delta_2 . \tag{14}$$

And by [34] we have

$$\mathcal{H} = \mathcal{L}_{\Delta} + \mathcal{L}_{\Delta^c} . \tag{15}$$

It is an immediate consequence of (b) that $ker(N - \lambda) \subset \mathcal{H}_Q = \{0\}$ for $\lambda \in Q^i$. Hence, due to (13) and (15), it remains to show that $Q^i \subset \rho(N|\mathcal{L}_{\Delta^c})$. But this follows directly from (14) and Lemma (6.1.1). Set $\widetilde{N} := N|\mathcal{H}_Q^{[\perp]}$. In order to show (d) we show

$$\mathbb{C}\backslash\overline{(\sigma(N) \setminus Q)} \subset \rho\left(\widetilde{N}\right). \tag{16}$$

Since

$$\mathbb{C}\backslash\left(\overline{\sigma(N)\backslash Q}\right) = \rho(N) \cup Q^i \cup \{\lambda \in \partial Q : \nexists(\lambda_n) \subset \sigma(N)\backslash Q \; with \; lim_{n\to\infty}\lambda_n = \lambda\},$$

and $\rho(N) \subset \rho\left(\widetilde{N}\right)$ by Lemma (6.1.11), it suffices to show

$$Q^i \cup \{\lambda \in \partial Q : \nexists(\lambda_n) \subset \sigma(N) \setminus Q \; with \; lim_{n\to\infty} \lambda_n = \lambda\} \subset \rho\left(\widetilde{N}\right). \tag{17}$$

Let λ be a point contained in the set on the left-hand side of this relation. Then there exists a compact rectangle $R = \Delta'_1 \times \Delta' \subset \mathcal{U}$ with $\lambda \in R^i, \ell(\Delta') < \tau$ and

$$\sigma(N) \cap R \subset Q.$$

Observe that the normal operator $\widetilde{N}$ in the Krein space $\mathcal{H}_{\mathrm{Q}}^{[\perp]}$ satisfies the conditions of Theorem (6.1.12). In particular, relation (7) holds with the same values ε and δ and with N replaced by $\widetilde{N}$. Hence, there exists a subspace $\widetilde{\mathcal{H}}_R$ of $\mathcal{H}_{\mathrm{Q}}^{[\perp]}$ which is N- and N^+ −invariant and has the properties

$$(\tilde{a}) \; \sigma\left(\widetilde{N}|\widetilde{\mathcal{H}}_R\right) \subset R \cap \sigma\left(\widetilde{N}\right),$$
$$(\tilde{c}) \; \widetilde{\mathcal{H}}_R = \{0\} \Rightarrow R^i \subset \sigma\left(\widetilde{N}\right).$$

By virtue of (b) we conclude from ($\tilde{a}$) and Lemma (6.1.11) that $\widetilde{\mathcal{H}}_R \subset \mathcal{H}_Q$. But since $\widetilde{\mathcal{H}}_R$ is also a subspace of $\mathcal{H}_{\mathrm{Q}}^{[\perp]}$, we have $\widetilde{\mathcal{H}}_R = \{0\}$ which by ($\tilde{c}$) implies $R^i \subset \rho\sigma\left(\widetilde{N}\right)$. Hence, $\lambda \in \rho\sigma\left(\widetilde{N}\right)$and therefore (17) holds.

In order to show (e) let $Q_n = \Delta'_n \times \Delta''_n \subset \mathcal{U}$ be closed rectangles such that $(\Delta''_1) < \tau, Q \subset Q_n^i$ for all $n \in \mathbb{N}$ and

$$Q_1 \supset Q_2 \supset \ldots and \; Q = \bigcap_{n=1}^{\infty} Q_n .$$

From (a) and (b) it follows that $\mathcal{H}_Q \subset \bigcap_{n=1}^{\infty} \mathcal{H}_{Q_n}$. Now, it is not difficult to see that $\mathbb{C}\backslash Q \subset \rho\left(N| \bigcap_{n=1}^{\infty} \mathcal{H}_{Q_n}\right)$, and (b) gives

$$\mathcal{H}_Q = \bigcap_{n=1}^{\infty} \mathcal{H}_{Q_n} . \tag{18}$$

Let $E(Q) \; and \; E(Q_n)$ be the $[.,.]$-orthogonal projections onto the Hilbert spaces $\mathcal{H}_Q$ and Q_n , respectively. As these spaces are invariant with respect to both $N \; and \; N^+$, the projections commute with N. Let B be a bounded operator which commutes with N and let $B_Q \in L(\mathcal{H}_Q, \mathcal{H})$ be the restriction of B to $\mathcal{H}_Q$. We obtain

$$\left(N|\mathcal{H}_{Q_n}^{[\perp]}\right)\left[(I - E(Q_n))B_Q\right] = (I - E(Q_n))NB_Q = [(I - E(Q_n))B_Q](N|\mathcal{H}_Q) .$$

The spectra of $N|\mathcal{H}_{Q_n}^{[\perp]}$ and $N|\mathcal{H}_Q$ are disjoint by (a) and (d), and Rosenblum's Corollary (Theorem (6.1.3)) implies $(I - E(Q_n))B_Q$, i.e., $B\mathcal{H}_Q \subset \mathcal{H}_{Q_n}$ for every $n \in \mathbb{N}$. By (18) this yields $B\mathcal{H}_Q \subset \mathcal{H}_Q$. Similarly, one shows that $B\mathcal{H}_{Q_n}^{[\perp]} \subset \mathcal{H}_Q^{[\perp]}$ for all $n \in \mathbb{N}$. From

$$c.l.s.\left\{\mathcal{H}_{Q_n}^{[\perp]} : n \in \mathbb{N}\right\}^{[\perp]} = \bigcap_{n=1}^{\infty} \mathcal{H}_{Q_n}$$

and (18) we deduce

$$\mathcal{H}_Q^{[\perp]} = c.l.s.\left\{\mathcal{H}_{Q_n}^{[\perp]} : n \in \mathbb{N}\right\}.$$

Hence, for $x \in \mathcal{H}_Q^{[\perp]}$ there exists a sequence (x_k) with each x_k in some $\mathcal{H}_{n_k}^{[\perp]}$ such that $x_k \to x$ as $k \to \infty$. Since $B_{x_k} \in \mathcal{H}_Q^{[\perp]}$ and $B_{x_k} \to B_x$ as $k \to \infty$, we conclude $B_x \in \mathcal{H}_Q^{[\perp]}$.

After all which has been showed above, for (f) we only have to show that every N-invariant subspace $\mathcal{M} \subset \mathcal{H}$ with $\sigma(N|\mathcal{M}) \subset Q$ is a subspace of $\mathcal{H}_Q$. Let $\mathcal{M}$ be such a subspace. Then let (Q_n) be a sequence of rectangles as in the proof of (e). From

$$\left(N|\mathcal{H}_{Q_n}^{[\perp]}\right)[(I - E(Q_n))|\mathcal{M}] = [(I - E(Q_n))|\mathcal{M}]\,(N|\mathcal{M})$$

and Rosenblum's Corollary we conclude $(I - E(Q_n))\mathcal{M} = \{0\}$. Therefore, $\mathcal{M} \subset \mathcal{H}_{Q_n}$ for every $n \in \mathbb{N}$ and $\mathcal{M} \subset \mathcal{H}_Q$ follows from (18).

(3) In this step we complete the proof. Let $Q_1 = [a,b] \times \Delta_1 \subset \mathcal{U}$ and $Q_2 = [a,b] \times \Delta_2 \subset \mathcal{U}$ such that $(\Delta_j) < \tau \ for \ j = 1,2$ and assume that Δ_1 and Δ_2 have one common endpoint. Then $Q := Q_1 \cup Q_2 = [a,b] \times (\Delta_1 \cup \Delta_2)$ is also a closed rectangle. Define

$$\mathcal{H}_Q := \mathcal{H}_{Q_1} + \mathcal{H}_{Q_2} = \mathcal{H}_{Q_1}[\dotplus]\left(\mathcal{H}_{Q_1}^{[\perp]} \cap \mathcal{H}_{Q_2}\right).$$

This is obviously a Hilbert space (with respect to [., .]) which is both N- and N^+-invariant. Let us show that the statements (a)–(f) from part 2 of this proof also hold for $\mathcal{H}_Q$. In step 2 the statements (d)–(f) were showed only with the help of (a)–(c). Here, this can be done similarly. Hence, it is sufficient to show only (a)–(c). By (a_j)–(c_j) denote the corresponding properties of $\mathcal{H}_{Q_j}, j = 1,2$. Statement (a) holds since $N|\mathcal{H}_Q$ is a normal operator in the Hilbert space $(\mathcal{H}_Q, [.,.])$ and

$$\sigma(N|\mathcal{H}_Q) = \sigma(N|\mathcal{H}_{Q_1}) \cup \sigma\left(N|\mathcal{H}_{Q_1}^{[\perp]} \cap \mathcal{H}_{Q_2}\right) \subset Q_1 \cup \sigma(N|\mathcal{H}_{Q_2}) \subset Q_1 \cup Q_2.$$

For (b) let $\mathcal{M}$ be a N- and N^+-invariant subspace with $\sigma(N|\mathcal{M}) \subset Q$. Denote by $\mathcal{L}_{\Delta_j}^{\mathcal{M}} \subset \mathcal{M}$ be the maximal spectral subspace of $Im N|\mathcal{M}$ corresponding to $\Delta_j, j = 1,2$. Then, by Lemmas (6.1.1) and (6.1.2),

$$\sigma\left(N|\mathcal{L}_{\Delta_j}^{\mathcal{M}}\right) \subset (\mathbb{R} \times \Delta_j) \cap (\sigma(N|\mathcal{M}) \cup \rho_b(N|\mathcal{M})) \subset (\mathbb{R} \times \Delta_j) \cap Q = Q_j.$$

From (b_j) we obtain $\mathcal{L}_{\Delta_j}^{\mathcal{M}} \subset \mathcal{H}_{Q_j}, j = 1,2$. And since $\mathcal{M} = \mathcal{L}_{\Delta_1}^{\mathcal{M}} + \mathcal{L}_{\Delta_2}^{\mathcal{M}}$ (see [34]) we have $\mathcal{M} \subset \mathcal{H}_Q$.

Suppose that $\mathcal{H}_Q = \{0\}$. Then $\mathcal{H}_{Q_1} = \mathcal{H}_{Q_2} = \{0\}$ and hence $Q_1^i \cup Q_2^i \subset \rho(N)$ by (c_1) and (c_2). Let $R = [a,b] \times [c_1, c_2]$, where c_j is the center of $\Delta_j, j = 1,2$. Then $c_2 - c_1 < \tau$. From $\sigma(N|\mathcal{H}_R) \subset R \subset Q$ and (b) it follows that $\mathcal{H}_R \subset \mathcal{H}_Q = \{0\}$. Hence, $R^i \subset \rho(N)$ which shows (c).

Now it is clear that for $Q = [a,b] \times [c,d]$ we choose a partition $c = t_0 < t_1 < \cdots < t_m = d \ of \ [c,d]$ such that $t_{k+1} - t_k < \tau, \ k = 0,\ldots,m-1$, and define

$$\mathcal{H}_Q := \mathcal{H}_{Q_1} + \cdots + \mathcal{H}_{Q_m},$$

where $Q_k := [a,b] \times [t_{k-1}, t_k], k = 1,\ldots,m$. This subspace is then a Hilbert space with respect to the indefinite inner product $[.,.]$ with the properties (a)–(f). Moreover, $\mathcal{H}_Q$ is both N- and N^+-invariant. Hence, $N|\mathcal{H}_Q$ is a normal operator in the Hilbert space $(\mathcal{H}_Q, [.,.])$ and has therefore a spectral measure E_Q. By $E(Q)$ we denote the $[.,.]$-orthogonal projection onto $\mathcal{H}_Q$. It is now easy to see that

$$E(.) := E_Q(.)E(Q)$$

satisfies conditions (i)–(iii) from Definition (6.1.9). The remaining conditions (iv)–(vi) follow from (e), (a) and (d), respectively. Hence, E is the local spectral function of positive type of N on Q. (See [28]).

In this section we show that Theorem (6.1.12) also holds in the situation when the real part of N is allowed to have nonreal spectrum but the set of definite type with respect to N is a spectral set.

Lemma (6.1.13)[21]. Let N be a normal operator in the Krein space $(\mathcal{H}, [.,.])$ and let σ be a spectral set of N with

$$\sigma \cap \sigma_{ap}(N) \subset \sigma_{++}(N). \tag{19}$$

Then the Riesz-Dunford projection Q of N corresponding to σ is selfadjoint in the Krein space $(\mathcal{H}, [.,.])$ and the corresponding spectral subspace $Q\mathcal{H}$ is invariant with respect to both N and N^+. Moreover, we have

$$\sigma_{ap}(N|Q\mathcal{H}) \subset \sigma_{++}(N|Q\mathcal{H}).$$

Proof. Since N is normal ,Q is also normal, hence it commutes with Q^+. Moreover , Q^+ is the Riesz-Dunford projection corresponding to N^+ and the set $\lambda : \lambda \in U$, so Q^+ also commutes with N. Thus, the projection $Q - Q^+Q$ projects on a subspace $\mathcal{M}$ which is invariant with respect to N. This subspace is neutral. Hence, from (19) it follows that $\sigma_{ap}(N|\mathcal{M}) = \emptyset$. This is only possible if $\mathcal{M} = \{0\}$, and we conclude

$$Q = Q^+Q,$$

that is, Q is a selfadjoint projection. The last statement follows from $\sigma_{ap}(N|Q\mathcal{H}) = \sigma \cap \sigma_{ap}(N)$.

Theorem (6.1.14)[21]. Let N be a normal operator in the Krein space $(\mathcal{H}, [.,.])$. Let σ be a spectral set of N with

$$\sigma \cap \sigma_{ap}(N) \subset \sigma_{++}(N),$$

and let Q be the Riesz-Dunford projection corresponding to σ and N. Assume that

$$\sigma(ImN|Q\mathcal{H}) \subset \mathbb{R}\ (or\ \sigma(ReN|Q\mathcal{H}) \subset \mathbb{R})$$

and that the growth of the resolvent of $ImN|Q\mathcal{H}$ ($ReN|Q\mathcal{H}$, *respectively*) is of finite order. Then the spectral subspace $Q\mathcal{H}$ equipped with the inner product $[.,.]$ is a Hilbert space. Hence, the restriction $N|Q\mathcal{H}$ is a normal operator in the Hilbert space $(Q\mathcal{H}, [.,.])$ and, therefore, possesses a spectral function.

Proof. By Lemma (6.1.13), the space $(Q\mathcal{H}, [.,.])$is a Krein space and $Q\mathcal{H}$ is N^+-invariant. Hence $(N|Q\mathcal{H})^+ = N^+|Q\mathcal{H}\ and\ N|Q\mathcal{H}$ is normal in $Q\mathcal{H}$. Therefore it is no restriction to assume $\sigma_{ap}(N) = \sigma_{++}(N), \sigma(ImN) \subset \mathbb{R}$ and that the resolvent of ImN is of finite order k_0 for some $k_0 \in \mathbb{N}$. For each compact interval Δ denote the maximal spectral subspace corresponding to ImN and Δ (which exists due to [34]) by $\mathcal{L}_\Delta$. It is a consequence of Lemma (6.1.8), that there exist $\varepsilon, \delta > 0\ with\ \delta < 1$ such that for all $\mu \in K$,

$$K := \{\lambda + ib : \lambda \in \sigma(ReN), b \in \sigma(ImN)\},$$

and all $x \in \mathcal{H}$ we have

$$\|(N - \mu)x\| \leq \varepsilon\|x\| \Rightarrow [x, x] \geq \delta\|x\|^2. \tag{20}$$

Let $b \in \sigma(ImN)$. From Corollary (6.1.5) it follows that there exists a compact interval Δ with center b such that

$$(ImN|\mathcal{L}_\Delta - b)^k \leq \frac{\delta^{k-1}\varepsilon^k}{2^k}\ for\ all\ k = k_0, k_0 + 1, \dots, 2k_0\ , \tag{21}$$

where k_0 is the order of growth of the resolvent of ImN.

Since the subspace $\mathcal{L}_\Delta$ is hyperinvariant with respect to ImN, it is ReN-invariant. The operator $ReN|\mathcal{L}_\Delta$ is a bounded operator in $\mathcal{L}_\Delta$ which is $[.,.]$-selfadjoint in the sense that

$$[(ReN)x, y] = [x, (ReN)y] \quad for\ all\ x, y \in \mathcal{L}_\Delta,$$

cf. (6). We define

$$\varepsilon := min \frac{\varepsilon}{2}, \frac{\delta^{j-2}\varepsilon^j}{2^j\,(\|ImN\| + r(ImN))^{j-1}} : j = 2, \ldots, k_0.$$

In a similar way as in step 1 of the proof of Theorem (6.1.12), it is shown here that from $\|(ReN - \lambda)x\| \le \tilde{\varepsilon}\,\|x\|\ for\ x \in \mathcal{L}_\Delta, \|x\| = 1$ and $\lambda \in \sigma(ReN|\mathcal{L}_\Delta)$ it follows that $\|(ImN - b)x\| \le \frac{\varepsilon}{2}\,\|x\|$ and thus

$$\|(N - (\lambda + ib))x\| \le \|(ReN - \lambda)x\| + \|(ImN - b)x\| \le \varepsilon.$$

Thus, with (20), we obtain

$$\sigma_{ap}(ReN|\mathcal{L}_\Delta) \subset \sigma + +(ReN|\mathcal{L}_\Delta).$$

Since $\sigma_{++}(ReN|\mathcal{L}_\Delta) \subset \mathbb{R}$ (see Remark (6.1.7)) we conclude that $\mathbb{C}\setminus \mathbb{R} \subset \mathbb{C} \setminus \sigma_{ap}(ReN|\mathcal{L}_\Delta)$. But as $ReN|\mathcal{L}_\Delta$ is bounded we even have $\mathbb{C}\setminus \mathbb{R} \subset \rho(ReN|\mathcal{L}_\Delta)$ and thus

$$\sigma(ReN|\mathcal{L}_\Delta) = \sigma_{ap}(ReN|\mathcal{L}_\Delta) = \sigma_{++}(ReN|\mathcal{L}_\Delta).$$

It is now a consequence of [37] that $(\mathcal{L}_\Delta, [.,.])$ is a Hilbert space. It is easily seen that also the subspace $\mathcal{L}_\Delta^{[\perp]}$ is invariant with respect to ImN. Consider the operator $:= ImN|\mathcal{L}_\Delta^{[\perp]}$. If Δ_1 is a compact interval which is completely contained in the inner of Δ, then by [34] there exists a spectral subspace $\mathcal{L}_{\Delta_1} \subset \mathcal{L}_\Delta^{[\perp]}$of A such that $(A|\mathcal{L}_{\Delta_1}) \subset \Delta_1$. But as this implies $\sigma(ImN|\mathcal{L}_{\Delta_1}) \subset \Delta$ and $\mathcal{L}_\Delta$ is a maximal spectral subspace, we obtain $\mathcal{L}_{\Delta_1} \subset \mathcal{L}_\Delta$ and thus $\mathcal{L}_{\Delta_1} \subset \mathcal{L}_\Delta \cap \mathcal{L}_\Delta^{[\perp]} = \{0\}$. Hence, $b \in \rho\left(ImN|\mathcal{L}_\Delta^{[\perp]}\right)$ follows.

We are now ready to show $b \in \sigma_{++}(ImN). Let\ (x_n) \subset \mathcal{H}$ be a sequence with $\|x_n\| = 1, n \in \mathbb{N}$, and $(ImN - b)x_n \to 0\ as\ n \to \infty$. Write

$$x_n = u_n + v_n \quad with\ u_n \in \mathcal{L}_\Delta, \quad v_n \in \mathcal{L}_\Delta^{[\perp]}.$$

From $(ImN - b)x_n \to 0$ it follows that also $(ImN - b)v_n \to 0$, and $b \in \rho\left(ImN|\mathcal{L}_\Delta^{[\perp]}\right)$ implies $v_n \to 0\ as\ n \to \infty$. From the fact that $(\mathcal{L}_\Delta, [.,.])$ is a Hilbert space we conclude

$$lim\ sup_{n\to\infty}[x_n, x_n] = lim\ sup_{n\to\infty}[u_n, u_n] + [v_n, v_n]) = lim\ sup_{n\to\infty}[u_n, u_n] > 0.$$

Since $b \in \sigma(ImN)$ was arbitrary, we have $\sigma(ImN) = \sigma_{++}(ImN)$, and it follows from, see, [13] that $(\mathcal{H}, [.,.])$ is a Hilbert space.

In [6] a bounded normal operator N in a Krein space is called strongly stable if there exists a fundamental decomposition (5) such that $\mathcal{H}_+\ and\ \mathcal{H}_-$ are invariant subspaces with respect to N with $\sigma(N|\mathcal{H}_+) \cap \sigma(N|\mathcal{H}_-) = \emptyset$. The following Theorem (6.1.15), showides a new characterization of strongly stable normal operators in Krein spaces. We say that an operator $T \in L(\mathcal{H})$ is similar to a selfadjoint (normal) operator in a Hilbert space if there exists a Hilbert space scalar product (., .) on $\mathcal{H}$ which induces the topology of $(\mathcal{H}, [.,.])$ such that N is selfadjoint (normal, respectively) in the Hilbert space $(\mathcal{H}, (.,.))$.

Theorem (6.1.15)[21]. A normal operator N in the Krein space $(\mathcal{H}, [.,.])$is strongly stable if and only if

$$\sigma(N) = \sigma_{++}(N) \cup \sigma_{--}(N), \tag{22}$$

$$\sigma(ImN) \subset \mathbb{R}\ (or\ \sigma(ReN) \subset \mathbb{R}) \tag{23}$$

and

the growth of the resolvent of ImN $(ReN, respectively)$ is of finite order. (24)

In particular, in this case, N is similar to a normal operator in a Hilbert space.

Proof. Let N be strongly stable. Then (22) follows and (23) and the growth condition follow from the fact that $ImN|\mathcal{H}_\pm$ and $ReN|\mathcal{H}_\pm$ are selfadjoint operators in the Hilbert spaces $(\mathcal{H}_+, [.,.])$ and $(\mathcal{H}_-, -[.,.])$, respectively.

For the converse observe that the sets $\sigma_{++}(N)$ and $\sigma_{--}(N)$ are open in $\sigma(N)$, see Lemma 6.1.8. Therefore, $\sigma_{++}(N)$ and $\sigma_{--}(N)$are spectral sets. Let Q_+ and Q_- be the spectral projections corresponding to these sets, respectively. Then, since $Q_+Q_- = 0$, due to Theorem (6.1.14), the operator $J := Q_+ - Q_-$is a fundamental symmetry in $(\mathcal{H}, [.,.])$ with the desired properties.

In order to show the last statement of Theorem (6.1.15), we denote by N^* the adjoint of N with respect to the Hilbert space inner product $[J.,.]$. Then, from $N^* = N^+$ it follows that N is a normal operator in $(\mathcal{H}, [J.,.])$.

The following theorem shows that (23) and (24) in Theorem (6.1.15), can be replaced by the condition that ReN and ImN have real spectra and that N is similar to a normal operator in a Hilbert space.

Theorem (6.1.16)[21]. Assume that the normal operator N in the Krein space $(\mathcal{H}, [.,.])$ is similar to a normal operator in a Hilbert space and that $\sigma(ReN) \subset R$ and $\sigma(ImN) \subset R$. Then ReN and ImN are similar to selfadjoint operators in a Hilbert space. In particular, their resolvent growths are of first order.

Proof. Let B denote the set of all Borel-measurable subsets of $\mathbb{C}$ and set $Q^* := \lambda : \lambda \in Q$ for $Q \in \mathcal{B}$. Moreover, let $(.,.)$ be a Hilbert space scalar product on $\mathcal{H}$ with respect to which N is normal, let $G \in L(\mathcal{H})$ such that $[.,.] = (G.,.)$ and let E be the spectral measure of the normal operator N in $(\mathcal{H}, (G,.))$. Then E_*, defined by $E_*(Q) := E(Q^*), Q \in \mathcal{B}$, is the spectral measure of N^*. It follows from the properties of E_* that the function E_+ given by $E_+(Q) = G^{-1}E_*(Q)G = E_*(Q)^+ = E(Q^*)^+, Q \in \mathcal{B}$, is a countably additive resolution of the identity for $N^+ = G^{-1}N^*G$ (see [26]), that is, N^+ is a spectral operator in the sense of Dunford [26].

Note that for any compact rectangle $Q \subset \mathbb{C}$ of the type $Q = [a, b] \times [c, d]$ the projection $E(Q)$ (and therefore the projection $E_*(Q)$) commutes with any operator that commutes with N, so the operators $Re(N|E_*(Q)\mathcal{H})$ and $Im(N|E_*Q)\mathcal{H})$ have real spectra (cf. Lemma (6.1.2)). From Lemma (6.1.1) we conclude that $\sigma(N + |E_*(Q)\mathcal{H}) \subset Q$. Since $E_+(Q)\mathcal{H}$ is the maximal spectral subspace of N^+ corresponding to Q (see, e.g., [23]), we have $E_*(Q)\mathcal{H} \subset E_+(Q)\mathcal{H}$ and hence

$$E(Q^*)^+E(Q^*) = E_+(Q)E_*(Q) = E_*(Q) = E(Q^*).$$

Therefore, for all compact rectangles $Q \subset \mathbb{C}$ the projection $E(Q)$ is selfadjoint in the Krein space $(\mathcal{H}, [.,.])$. Since the system of compact rectangles in $\mathbb{C}$ is stable with respect to intersections and generates $\mathcal{B}$, it follows that $E(Q) = E(Q)^+$ for all $Q \in \mathcal{B}$. This implies that for all $Q \in \mathcal{B}$ we have $GE(Q) = GE(Q)^+ = GG^{-1}E(Q)G = E(Q)G$ and thus $GN = NG$. Consequently, $N^+ = G^{-1}N^*G = N^*$, which implies the assertion.

In this section we apply our results to operator pencils. A standard description of damped small oscillations of a continuum or of small oscillations of a pipe, carrying steady-state fluid of ideal incompressible fluid, is done via an equation of the form

$$T\ddot{z} + R\dot{z} + V z = 0, \tag{25}$$

where z is a function with values in a Hilbert space and V and R are (in general) unbounded operators. As a reference (especially for non-selfadjoint coefficients) we mention here only [27], [38] and [29].

The classical approach (see [30], [31]) to such kind of problems is, under some additional assumptions (Vuniformly positive and the closures of the operators $V^{-1/2}\,TV^{-1/2}$ and $V^{-1/2}RV^{-1/2}$ are bounded) to transform the equation in (6.1) via $u = V^{1/2}\,z$ into

$$E\ddot{u} + F\dot{u} + u = 0, \quad (26)$$

with bounded operators E and F. If one is interested in finding solutions of the form

$$u(t) = e^{t\lambda-1}\emptyset_0 ,$$

with a constant vector $\emptyset_0$, then (26) can be written (after multiplication by λ^2) as

$$(\lambda^2 I + \lambda E + F)\emptyset_0 = 0. \quad (27)$$

In the sequel, we will investigate quadratic pencils of the form (27) with $= AC$ and $F = C^2$, where

C is a bounded normal operator in a Hilbert space $\mathcal{H}$ (28)

and

A is a bounded selfadjoint operator in $\mathcal{H}$ which commutes with C. (29)

That is, we investigate the operator pencil L,

$$L(\lambda) := \lambda^2 I + \lambda AC + C^2 . \quad (30)$$

As usual, a value λ for which the equation $L(\lambda)\emptyset = 0$ has a solution $\emptyset \neq 0$ is called an eigenvalue of the operator pencil L and the spectrum $\sigma(L)$ of L is the set of all complex numbers λ for which the operator $L(\lambda)$ is not boundedly invertible. In many cases it turns out (see, [30], [31]) that a successful investigation of the spectral properties of L is achieved by studying the operator roots Z of the quadratic operator equation

$$Z^2 + ACZ + C^2 = 0. \quad (31)$$

If there exists a bounded operator Z_1 which is an operator root, i.e., a solution of (31), then any eigenvalue (eigenvector) of Z_1 is also an eigenvalue (eigenvector, respectively) of the operator pencil L .Moreover $\partial\sigma(Z_1) \subset \sigma(L)$ (see [35]) and the operator pencils L decomposes into linear factors

$$L(\lambda) = \left(\lambda I - \widetilde{Z_1}\right)(\lambda I - Z_1),$$

where $\widetilde{Z_1} = -AC - Z_1$.

The following Theorem on the existence of an operator root of (31) shows how our previous results can be applied.

Theorem (6.1.17)[21]. Assume that the coefficients A and C of the operator pencil $L(\lambda)$ in (30) satisfy (28) and (29). Define on the Hilbert space $\mathcal{H} := H \times H$ an inner product by

$$\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] := (x_1, x_2) - (y_1, y_2)\ for \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in H \times H, \quad (32)$$

where $(.,.)$ denotes the Hilbert space scalar product in H. Then the operator matrix $\mathcal{A}$

$$\mathcal{A} = \begin{bmatrix} 0 & C \\ -C & -AC \end{bmatrix} .$$

is a normal operator in the Krein space $(H \times H, [.,.])$. If the operator $\mathcal{A}$ satisfies the conditions in Theorem (6.1.15), then Equation (31) has an operator root.

Proof. Obviously, $\mathcal{H} = H \times H$ with inner product (32) is a Krein space and the adjoint of $\mathcal{A}$ with respect to $[.,.]$ is given by

$$\mathcal{A}^+ = \begin{bmatrix} 0 & C^* \\ -C^* & -AC^* \end{bmatrix} .$$

From this, we easily conclude $\mathcal{A}\mathcal{A}^+ = \mathcal{A}^+\mathcal{A}$. If the operator $\mathcal{A}$ satisfies the conditions in Theorem (6.1.15), then $\mathcal{A}$ is a strongly stable normal operator in the Krein space $(H \times H, [.,.])$. Hence, there exists a fundamental decomposition (5) such that $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant subspaces with respect to A. Let $K : H \to H$ with $\|K\| < 1$ be the corresponding angular operator, see, [3], such that

$$\mathcal{H}_+ = \left\{\begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix} : x_+ \in H\right\}.$$

Now, $\mathcal{A}\mathcal{H}_+ \subset \mathcal{H}_+$ implies that for every $x_+ \in H$ there exists $y_+ \in H$ with

$$\begin{pmatrix} CKx_+ \\ -Cx_+ - ACKx_+ \end{pmatrix} = \begin{pmatrix} y_+ \\ Ky_+ \end{pmatrix}$$

and we obtain

$$-C - ACK = KCK.$$

Multiplication by C from the right gives

$$(KC)^2 + AC(KC) + C^2 = 0,$$

and the operator KC is an operator root of (6.2.31).

Remark (6.1.18)[21]. If the operator pencil $\hat{L}(\lambda) := \lambda^2 + \lambda AD + D^2$ $with\, D = \frac{1}{2}(C + C^*)$ is hyperbolic, i.e.,

$$(ADx, x)^2 \geq 4(D^2x, x) \quad for\ all \quad x \in H, \|x\| = 1, \tag{33}$$

and if 0 is not an eigenvalue of D, then Equation (31) has an operator root Z_1 which commutes with $\hat{Z}_1$. To see this, we note that (33) implies $(A^2x, x)\|Dx\|^2 = \|Ax\|^2\|Dx\|^2 \geq (Ax, Dx)^2 \geq 4\|Dx\|^2$ and hence $A^2 - 4 \geq 0$.Let $W := (A^2 - 4)^{1/2}$. Now, a simple computation shows that both

$$Z_1 := \frac{1}{2}(W - A)C \quad and \quad Z_2 := -\frac{1}{2}(W + A)C$$

are operator roots of (31), $Z_1Z_2 = Z_2Z_1$ and $L(\lambda) = (\lambda - Z_1)(\lambda - Z_2)$.

Section (6.2): Definite Normal Operators in Krein Spaces

We showed in [19] that for every bounded linear operator A in a Hilbert space $\mathcal{H}$ there exists a Krein space $\mathcal{K}$ and a normal operator B in this Krein space such that $\mathcal{H} \subset \mathcal{K}$ and $B/\mathcal{H} = A$. In other words: every bounded linear operator in a Hilbert space is a "part" of a normal operator in a Krein space. If the Hilbert space $\mathcal{H}$ is finite-dimensional then the Krein space $\mathcal{K}$can even be chosen as $\mathcal{H}$ itself (see [10]).

From this point of view it seems desirable to have a profound spectral theory for bounded normal operators in Krein spaces. But the literature on normal operators in Krein spaces is very limited at present and, in addition, in each of the existing contributions global assumptions on the space or the normal operator are imposed. In [14] the existence of a spectral function for a normal operator in a Pontryagin space was showed and a complete classification of normal operators in a Π_1-space was worked out. In [20] it is stated without proof that there exists a functional calculus for normal operators in Pontryagin spaces. In [9] the concept of definitizability was extended from selfadjoint operators to a class of normal operators in a Krein space the spectrum of which does not have interior points. For such operators the existence of a spectral function with singularities was showed. Another special class of normal operators with a maximal nonnegative invariant subspace was considered in [5,17]. The References [4,6,7,18] deal with bounded and compact perturbations of fundamentally reducible normal operators.

In contrast to the above-quoted References very weak assumptions on the normal operator were imposed in [15] and the notion of the spectrum of positive and negative type for selfadjoint operators in Krein spaces from [11,13] was extended to normal operators. It could be shown that a normal operator has a local spectral function on open subsets of $\mathbb{C}$ which are of positive or negative type. This result is known for arbitrary selfadjoint operators in Krein spaces (see [13]). But due to the global assumptions the result from [15] is not a proper generalization of that in [13]. However, it shows that the spectrum of positive and negative type is also meaningful for normal operators.

We continue the study of the spectral points of positive and negative type for normal operators, but we do not impose any assumptions on the Krein space inner product or the global structure of the operator. As in [15] it is our main objective to tackle the question whether or when a spectral point λ of positive type of the normal operator N in a Krein space has a neighborhood on which there exists a local spectral function for N. We prove that for this it is necessary that $\bar{\lambda}$ is a spectral point of positive type of the Krein space adjoint $N^{+}$of N. This motivates us to introduce the set $\sigma_{++}(N)$ which consists of all $\lambda \in \mathbb{C}$ such that $\lambda \in \sigma_{+}(N)$ and $\lambda \epsilon \sigma_{+}(N^{+})$ and call it the spectrum of two-sided positive type of N. And indeed, we are able to show that a normal operator has a local spectral function on sets which are of two-sided positive type (see Theorem (6.2.17)). Since for a selfadjoint operator A the sets $\sigma_{++}(A)$ *and* $\sigma_{+}(A)$ coincide, Theorem (6.2.17) is a generalization of the above-mentioned result from [13].

At this point and in light of the results in [15] the natural question arises whether the sets $\sigma_{+}(N)$ *and* $\sigma_{++}(N)$ coincide for all normal operators. It is showed in Theorem (6.2.22) that a spectral set which is of positive type is in fact of two-sided positive type. This essentially improves a result from [15] and shows, in particular, that the part of the operator N corresponding to the spectral set is a normal operator in a Hilbert space. But the question whether $\sigma_{+}(N) = \sigma_{++}(N)$ holds in general has to be left open.

Throughout this section, let $(\mathcal{H}, [.,.])$ be a Krein space. For the basic properties of Krein spaces we refer to the monographs [3] and [8]. We fix a Hilbert space norm $\|.\|$ on $\mathcal{H}$ such that

$$|[x,y]| \leq \|x\|\|y\| \quad for\ all\ x,y \in \mathcal{H}$$

Such a norm exists, and all such norms are equivalent, cf. [3,8]. By T^{+} we denote the adjoint of an operator $T \in L(\mathcal{H})$ with respect to the inner product $[.,.]$. The statements of the following lemma will be used frequently without reference, cf. [3].

Lemma (6.2.1)[1]. Let $T \in L(\mathcal{H})$. Then the following statements hold.

(i) $\lambda \in \sigma(T) \iff \bar{\lambda} \in \sigma(T^{+})$

(ii) $\lambda \in \sigma(T) \setminus \sigma_{ap}(T) \Rightarrow \bar{\lambda} \in \sigma_{p}(T^{+}) \subset \sigma_{ap}(T^{+})$

(iii) If $\mathcal{L}$ is a T -invariant subspace, then $\mathcal{L}^{[\perp]}$ is T^{+} -invariant.

Hereby, $\mathcal{L}^{[\perp]}$ denotes the orthogonal companion of $\mathcal{L}$ with respect to the inner product$[.,.]$:

$$\mathcal{L}^{[\perp]} := \{x \in \mathcal{H}: [x,\ell] = 0\ for\ all\ \ell \in \mathcal{L}\}$$

A closed subspace $\mathcal{L} \in \mathcal{H}$ is called uniformly positive (uniformly negative) if there exists $\delta > 0$ such that $[x,x] \geq \delta\|x\|^{2}(-[x,x] \geq \delta\|x\|^{2})$, respectively) holds for all $x \in \mathcal{L}$. Equivalently, the inner product space $(\mathcal{L},[.,.])\big((\mathcal{L},-[.,.]),\text{respectively}\big)$ is a Hilbert space. In this case, we have $\mathcal{H} = \mathcal{L}[\dotplus]\mathcal{L}^{[\perp]}$,where $[\dotplus]$ denotes the direct $[.,.]$ -orthogonal sum.

Let us recall the definition of a local spectral function (of positive type) for a bounded operator,cf. [13].

Definition (6.2.2)[1]. Let $S \subset \mathbb{C}$ be Borel-measurable. By $\mathfrak{B}_{0}(S)$ we denote the system of Borelmeasurable subsets of S whose closure is also contained in S. A mapping E from $\mathfrak{B}_{0}(S)$ into the set of all bounded projections in $(\mathcal{H}, [.,.])$ is called a local spectral function for the operator $T \in L(\mathcal{H})$on S if for all $\Delta, \Delta_{1}, \Delta_{2}, \ldots \in \mathfrak{B}_{0}(S)$ the following conditions are satisfied:

(i) $E(\Delta_{1} \cap \Delta_{2}) = E(\Delta_{1})E(\Delta_{2})$

(ii) If $\Delta_{1}, \Delta_{2}, \ldots \in \mathfrak{B}_{0}(S)$ are mutually disjoint and $\cup_{k=1}^{\infty} \Delta_{k} \in \mathfrak{B}_{0}(S)$, then

$$E\left(\cup_{k=1}^{\infty} \Delta_{k} = \sum_{k=1}^{\infty} E(\Delta_{k})\right)$$

where the sum converges in the strong operator topology.

(iii) $TB = BT \Rightarrow E(\Delta)B = BE(\Delta) for\ every\ B \in L(\mathcal{H})$.

(iv) $(T|E(\Delta)\mathcal{H}) \subset \overline{\sigma(T) \cap \Delta}$.

(v) $\sigma(T|(I - E(\Delta))\mathcal{H}) \subset \overline{\sigma(T) \setminus \Delta}$

A local spectral function E for T on S is said to be of positive (negative) type if for all $\Delta \in \mathfrak{B}_{0}(S)$

(vi) $E(\Delta)\mathcal{H}$ is uniformly positive (uniformly negative, respectively).

For the rest of this section let N be a normal operator in $(\mathcal{H}, [.,.])$, i.e. N commutes with itsadjoint,

$$NN^{+} = N^{+}N$$

The spectral points of positive and negative type defined below were first introduced in [11] for bounded selfadjoint operators.

Definition (6.2.3)[1]. A point $\lambda \epsilon \sigma_{ap}(N)$ is called a spectral point of positive (negative) type of the normal operator N if for every sequence $(x_n) \subset X$ *with* $\|x_n\| = 1$, $n \in \mathbb{N}$, *and* $(T - \lambda)x_n \to 0$ *as* $n \to \infty$. we have

$$\lim_{n\to\infty} inf[x_n, x_n] > 0 \quad \left(\lim_{n\to\infty} sup[x_n, x_n] < 0\,, respectively\right)$$

The set of all spectral points of positive (negative) type of N is denoted by $\sigma_+(N)$ ($\sigma_-(N)$ respectively). A set $\Delta \subset \mathbb{C}$ is said to be of positive (negative) type with respect to N if

$$\Delta \cap \sigma_{ap}(N) \subset \sigma_+(N) \;\; \left(\Delta \cap \sigma_{ap}(N) \subset \sigma_-(N) \;\; respectively\right)$$

A point $\lambda \in \sigma_{ap}(N)$ is called a spectral point of definite type of N if it is either a spectral point of positive type or of negative type of N. Analogously, a set $\Delta \subset \mathbb{C}$ is said to be of definite type with respect to N if it is either of positive or of negative type with respect to N.(See [2 ,12,13,14,15]).

It is immediately seen that, after a slight modification, Definition (6.2.3) can be formulated also for unbounded linear operators or relations. In fact, the spectral points of definite type were introduced and studied in [2] for closed linear relations in Krein spaces. The following lemma is well known (see [2]).

Lemma (6.2.4)[1]. The sets $\sigma_+(N)$ and $\sigma_-(N)$ are open in $\sigma_{ap}(N)$

Lemma (6.2.5)[1]. Let Q be a bounded projection in $\mathcal{H}$ such that

$$B \in L(\mathcal{H}), \qquad NB = BN \Rightarrow \; QB = BQ$$

Then Q is normal. If, in addition, one of the following conditions

(a) $\sigma_{ap}(N|Q\mathcal{H}) \subset \sigma_+(N) \cup \sigma_-(N)$

(b) $Q\mathcal{H}$ is uniformly positive or uniformly negative, holds, then Q is selfadjoint.

Proof. We have (for the second implication apply the adjoint)

$$NN^+ = N^+N \;\Rightarrow QN^+ = N^+Q \;\Rightarrow\; NQ^+Q^+N \;\Rightarrow QQ^+ = Q^+Q.$$

Therefore, Q as well as $P := Q - QQ^+$ are normal projections.Moreover, P commutes with N, and we have $P^+P = 0$ so that the subspace $P\mathcal{H} \subset Q\mathcal{H}$ is neutral. Hence, if (b) holds, then P = 0 follows immediately. If (a) is satisfied, then we have $\sigma_{ap}(N|P\mathcal{H}) = \emptyset$ and thus also P = 0.

The next theorem was shown in [13] in a somewhat more general situation.

Theorem (6.2.6)[1]. Let A be a bounded selfadjoint operator in the Krein space $(\mathcal{H}\,, [.\,,.])$. If the interval Δ is of positive (negative) type with respect to A then A has a local spectral function E of positive type (negative type, respectively) on Δ . If $\delta \epsilon \mathfrak{B}_0(\Delta)$ is compact then $E(\delta)\mathcal{H}$ is the maximal spectral subspace of A corresponding to δ.

Hereby, the maximal spectral subspace of a bounded operator T in a Banach space X corresponding to the compact set $\Delta \in \mathbb{C}$ is a closed T -invariant subspace $\mathcal{L}_\Delta \subset X$ such that $\sigma(T|\mathcal{L}_\Delta) \subset \Delta$ and $\mathcal{L} \subset \mathcal{L}_\Delta$ for any closed T -invariant subspace $\mathcal{L}$ with $\sigma(T|\mathcal{L}_\Delta) \subset \Delta$. If such a subspace $\mathcal{L}_\Delta$ exists, it is obviously unique.

In what follows we will deal with the question whether also a normal operator has a local spectral function of positive type on sets which are of positive type. In the next three lemmas we collect some necessary conditions. The first one is a direct consequence of Lemma (6.2.5). The proof of the second Lemma is straight forward and is left to the reader.

Lemma (6.2.7)[1] If N has a local spectral function E of positive or negative type on the Borel set S, then for each $\Delta \in \mathfrak{B}_0(S)$ the projection $E(\Delta)$ is selfadjoint and commutes with both N and N^+.

For a set $\Delta \in \mathbb{C}$ we define $\Delta^* := \{\bar{\lambda} : \lambda \in \Delta\}$

Lemma (6.2.8)[1]. If E is a local spectral function of positive (negative) type for N on the Borel set S, then E_+ , defined by

$$E_+(\Delta) := E(\Delta^*), \Delta \in \mathfrak{B}_0(S^*)$$

is a local spectral function of positive type (negative type, respectively) for N^+ on S^*.

By $B_r(\lambda)$ we denote the disk with center $\lambda \in \mathbb{C}$ and radiusr > 0 .

Lemma (6.2.9)[1]. If N has a local spectral function of positive type on the open set S then the following statements hold:

(a) S is of positive type with respect to N.

(b) S^* is of positive type with respect to N^+.
(c)$\sigma_{ap}(N) \cap S = \sigma(N) \cap S$.
(d) $\sigma_{ap}(N^+) \cap S^* = \sigma(N^+) \cap S^*$
(e) The approximate eigensequences for $N - \lambda$ and $N^+ - \bar{\lambda}$ coincide for each $\lambda \in \sigma_{ap}(N) \cap S$.
Proof. In view of Lemma (6.2.8) it suffices to show only (a), (c) and that approximate eigensequences of $N - \lambda$ are also approximate eigensequences of $N^+ - \bar{\lambda}$ for $\lambda \in \sigma_{ap}(N) \cap S$.

Let $\lambda \in S \cap \sigma(N)$ and choose $\varepsilon > 0$ such that $B_o := \overline{B_\varepsilon(\lambda)}$ is contained in S. We set $\mathcal{L}_o := E(B_o)$, where E is the local spectral function of positive type of N on S. As $E(B_o)$is selfadjoint by Lemma (6.2.7), we have $\mathcal{L}_1 := \mathcal{L}_0^{[\perp]} = (I - E(B_o))\mathcal{H}$ and thus $\mathcal{H} = \mathcal{L}_o[\dotplus]\mathcal{L}_1$. The subspace $\mathcal{L}_o$ is N- and N^+ - invariant. Hence, the same holds for $\mathcal{L}_1$. Set $N_j := N|\mathcal{L}_j \,, j = 0,1$.

It follows from (v) that $\lambda \in \rho(N_1)$. And as $(N_o) \cup \sigma(N_1)$, we conclude $\lambda \in \sigma(N_o)$. But N_o is a normal operator in a Hilbert space by (vi) and thus $\lambda \in \sigma_{ap}(N_o) \subset \sigma_{ap}(N)$. This shows (c). Let $((x_n) \subset \mathcal{H}$ be an approximate eigensequence for $N - \lambda$ and let $(x_{j,n}) \subset \mathcal{L}_j, j = 0,1$, such that $x_n = x_{0,n} + x_{1,n} \,, n \in \mathbb{N} \; as \; \lambda \in \rho(N_1)$, we conclude from $(N_1 - \lambda)x_{1,n} \to 0$ that $x_{1,n} \to 0 \; as \; n \to \infty$. Hence, from the uniform positivity of $\mathcal{L}_o$ we obtain

$$\lim_{n\to\infty} inf[x_n, x_n] = \lim_{n\to\infty} inf[x_{0,n}, x_{o,n}] > 0 \,,$$

and (a) is showed. Moreover,

$$\begin{aligned} \|(N^+ - \bar{\lambda})x_n\| &\le \|(N^+ - \bar{\lambda})x_{0,n}\| + \|(N^+ - \bar{\lambda})x_{1,n}\| \\ &\le \delta[(N^+ - \bar{\lambda})x_{0,n}, (N^+ - \bar{\lambda})x_{0,n}] + \|N^+ - \bar{\lambda}\|\|x_{1,n}\| \\ &= \delta[(N - \lambda)x_{o,n} \,, (N - \lambda)x_{0,n}] + \|N^+ - \bar{\lambda}\|\|x_{1,n}\| \end{aligned}$$

with some $\delta > 0$. This tends to zero as $n \to \infty$.

The next Lemma shows that parts of the necessary conditions in Lemma (6.2.9) are always satisfied for an open set which is of positive type with respect to N. By $\mathcal{L}_\lambda(T)$ we denote the root subspace of $T \in L(\mathcal{H})$ corresponding to $\lambda \in \mathbb{C}$.

Lemma (6.2.10)[1]. Let $\lambda \in \sigma_+(N)$. Then the following statements hold.
(i) The approximate eigensequences for $N - \lambda$ are also approximate eigensequences for $N^+ - \bar{\lambda}$.
(ii) $\bar{\lambda} \epsilon \sigma_{ap}(N^+)$
(iii) $ker\,(N - \lambda) \subset (N^+ - \bar{\lambda})$
(iv) $\mathcal{L}_\lambda(T) = ker(N - \lambda)$
Proof. Clearly, (ii) and (iii) follow from (i). So, let us show (i). To this end let (x_n) be an approximate eigensequence for $N - \lambda$. Then

$$(N - \lambda)(N^+ - \bar{\lambda})\, x_n \to 0 \; as \; n \to \infty. \tag{34}$$

Suppose that lim supn $sup_{n\to\infty}\|(N^+ - \bar{\lambda})x_n\| > 0$. Then there exists a subsequence (x_{nk}) of (x_n) and $\delta > 0$ such that $\|(N^+ - \bar{\lambda})x_n\| \to \delta \; as \; k \to \infty$
.But as $\lambda \in \sigma_+(N)$, it follows from (1) that

$$\lim_{k\to\infty} inf\,[(N - \lambda)(N^+ - \bar{\lambda})x_{nk}, x_{nk}] = \lim_{k\to\infty} inf\,[(N - \lambda)(N^+ - \bar{\lambda})x_{nk}] > o \,,$$

which contradicts (34). Therefore, $(N^+ - \bar{\lambda})x_n \to 0 \; as \; n \to \infty$.

It remains to show (iv). Let $x, u \in \mathcal{H}$ such that
$(N - \lambda)x = o \;\; and \; (N - \lambda)x = u$. Then (iii) yields $(N^+ - \bar{\lambda})x = 0$ and thus

$$[x, x] = [(N - \lambda)u, x] = [u, (N^+ - \bar{\lambda})x] = 0$$

which implies $x = 0 \; as \; \lambda \,\epsilon\, \sigma_+(N)$.

It follows from Lemmas (6.2.4) and (6.2.9) that the necessary condition (e) in Lemma (6.2.9) for the existence of a local spectral function of positive type for N in an open neighborhood S of $\lambda \,\epsilon \sigma_+(N)$. is satisfied if the approximate eigensequences for $N^+ - \bar{\lambda}$ are also approximate eigensequences for $N - \lambda$. Obviously, this is equivalent to the following implication:

$$\lambda \,\epsilon \sigma_+(\mathrm{N}) \to \bar{\lambda} \,\epsilon\, \sigma_+ \; (\mathrm{N}^+) \tag{35}$$

We return to this problem show there that (2) is true if $(\mathcal{H}, [.,.])$ is a Pontryagin space.

Motivated by Lemma (6.2.9), we define a new class of spectral points for normal operators.

Definition (6.2.11)[1]. A point $\lambda \in \mathbb{C}$ is called a spectral point of two-sided positive (negative) type of the normal operator N if

$$\lambda \in \sigma_+(N) \ and \ \bar{\lambda} \in \sigma_+(N^+)$$
$$\left(\lambda \in \sigma_-(N) \ and \ \bar{\lambda} \in \sigma_- \ (N^+), respectively\right).$$

The set of all spectral points of two-sided positive (negative) type of N is denoted by $\sigma_{++}(N)$ $\sigma_{--}(N)$ respectively). A set $\Delta \subset \mathbb{C}$ is said to be of two-sided positive (negative) type with respect to N if

$$\Delta \cap \sigma(N) \subset \sigma_{++}(N) \ (\Delta \cap \sigma(N) \subset \sigma_{--}(N)), respectively$$

In the sequel we restrict ourselves to the investigation of the spectrum of two-sided positive type. Similar results hold for spectral points and sets of two-sided negative type.(See [13]).

Remark (6.2.12) and Lemma (6.2.13) below directly follow from Lemma (6.2.10).

Remark (6.2.12)[1]. For a set Δ we have $\Delta \cap \sigma(N) \subset \sigma_{++}(N)$ if and only if

$$\Delta \cap \sigma_{ap}(N) \subset \sigma_+(N) \ and \ \Delta^* \cap \sigma_{ap}(N^+) \subset \sigma_+(N^+).$$

Lemma (6.2.13)[1]. Let $\lambda \in \sigma_{++}(N)$. Then the following holds.

(i) The approximate eigensequences for $(N-\lambda)$ and $\left(N^+ - \bar{\lambda}\right)$ coincide.

(ii) $ker(N-\lambda) = ker\left(N^+ - \bar{\lambda}\right) = \mathcal{L}_\lambda(N) = \mathcal{L}_{\bar{\lambda}}(N^+)$

Note that for each $\lambda \in \mathbb{C}$ the operator

$$A(\lambda) := (N-\lambda)\left(N^+ - \bar{\lambda}\right)$$

is selfadjoint (in the Krein space $(\mathcal{H}, [.,.])$ .The following Lemma shows that the spectrum of two-sided positive type of N is closely related to the sign type behaviour of the zero point with respect to the operators $A(\lambda)$. This correspondence will serve as the starting point for the construction of the local spectral function in this section.

Lemma (6.2.14)[1]. For all $\lambda \in \mathbb{C}$ we have

$$\lambda \in \sigma_{++}(\mathrm{N}) \Leftrightarrow 0 \in \sigma_+(\mathrm{A}(\lambda))$$

Proof. Let $\lambda \in \sigma_{++}(N)$. Then, clearly, $0 \in \sigma_{ap}(\left(N^+ - \bar{\lambda}\right)(N-\lambda))$. Let (x_n) be an approximate eigensequence for $\left(N^+ - \bar{\lambda}\right)(N-\lambda)$. Suppose that there exists a subsequence (x_{nk}) of (x_n) such that $\lim_{k\to\infty}\left\|\left(N^+ - \bar{\lambda}\right)x_{nk}\right\| > 0$. Then from $\bar{\lambda} \in \sigma_+(N^+)$ we obtain a contradiction:

$$0 = \lim_{k\to\infty} inf\left[\left(N^+ - \bar{\lambda}\right)(N-\lambda)x_{nk}, x_{nk}\right] = \lim_{k\to\infty} \inf\left[(N-\lambda)x_{nk}, \left(N^+ - \bar{\lambda}\right)x_{nk}\right] > o$$

Therefore, $(N-\lambda)x_n \to 0 \ as \ n \to \infty$ and thus $\lim inf_{n\to\infty}[x_n, x_n] > 0$. $as \ \lambda \in \sigma_+(N)$Conversely, assume that $0 \in \sigma_+(\left(N^+ - \bar{\lambda}\right)(N-\lambda))$Then $\lambda \in \sigma_{ap}(N)$. or $\bar{\lambda} \in \sigma_{ap}(N^+)$. Assume, e.g., that $\lambda \in \sigma_{ap}(N)$ and let (x_n) be an approximate eigensequence for $N-\lambda$. Then (x_n) is also an approximate eigensequence for $\left(N^+ - \bar{\lambda}\right)(N-\lambda)$ and thus $\lim inf_{n\to\infty}[x_n, x_n] > 0$ follows.

Hence, $\lambda \in \sigma_+(N)$and therefore $\bar{\lambda} \in \sigma_{ap}(N^+)$ by Lemma (6.2.10). A similar reasoning as above shows $\bar{\lambda} \in \sigma_+(N^+)$.

For a compact set $K \in \mathbb{C}$ and $\varepsilon > 0$ we set

$$B_\varepsilon(K) := \cup_{\lambda \in K} B_\varepsilon(\lambda)$$

Lemma (6.2.15)[1]. Let $K \in \mathbb{C}$ be a compact set which is of two-sided positive type with respect to N, i.e.

$$K \cap \sigma(N) \subset \sigma_{++}(N)$$

Then there exist$\varepsilon_0, \delta_0 > 0$ such that for all $\mu \in [0, \varepsilon_0^2]$ all $\lambda \in \overline{B_{\varepsilon_0}(K)}$ and all $x \in \mathcal{H}$ the following implications hold:

(a) $\|(A(\lambda) - \mu)x\| \le \varepsilon_0\|x\| \Rightarrow [x, x] \ge \delta_0\|x\|^2$.

(b) $\|(N-\lambda)x\| \le \varepsilon_0\|x\| or \left\|\left(N^+ - \bar{\lambda}\right)x\right\| \le \varepsilon_0\|x\| \Rightarrow [x, x] \ge \delta_0\|x\|^2$

In particular,

$$\overline{B_{\varepsilon_0}(K)} \cap \sigma(N) \subset \sigma_{++}(N) \tag{36}$$

and for all $\lambda \in \overline{B_{\varepsilon_0}(K)}$ we have

$$[0, \varepsilon_0^2] \cap \sigma(A(\lambda)) \subset \sigma_+(A(\lambda)) \tag{37}$$

Proof. Assume that it is not true that there are $\varepsilon_1, \delta_1 > 0$ such that for all $(\mu, \lambda, x) \epsilon [0, \varepsilon_1^2] \times \overline{B_{\varepsilon_1}(K)} \times \mathcal{H}$ we have

$$\|(A(\lambda) - \mu)x\| \leq \varepsilon_1 \|x\| \Rightarrow [x, x] \geq \delta_1 \|x\|^2$$

Then for each $n \in \mathbb{N}$ there exist $\mu_n \epsilon \left[0, \frac{1}{n^2}\right], \lambda_n \epsilon \overline{B_{\frac{1}{n}}(K)}$ and $x_n \epsilon \mathcal{H}$ with $\|x_n\| = 1$ such that $\|(A(\lambda) - \mu_n)x_n\| \leq \frac{1}{n}$ *and* $[x_n, x_n] < \frac{1}{n}$. As $\overline{B_1(K)}$ is compact and $\lambda_n \epsilon \overline{B_1(K)}$ for all $\epsilon \mathbb{N}$, there exists a subsequence λ_{nk} of λ_n which converges to some $\lambda_0 \epsilon \overline{B_1(K)}$. But $\lambda_{nk} \in \overline{B_{\frac{1}{n_k}}(K)}$ so that $\lambda_0 \epsilon K$. It follows that

$$\|A(\lambda_0)x_{n_k}\| \leq \left\|\left(A(\lambda_0) - A(\lambda_{n_k})\right)x_{n_k}\right\| + \|(A(\lambda_{n_k}) - \mu_{n_k})x_{n_k}\| + |\mu_{n_k}|$$
$$\leq \left\|\left(A(\lambda_0) - A(\lambda_{n_k})\right)x_{n_k}\right\| + \frac{1}{n_k} + \frac{1}{n_k^2}.$$

Since the function $A: \mathbb{C} \to L(\mathcal{H})$ is continuous, it follows that $A(\lambda_0)x_{n_k} \to 0$ *as* $k \to \infty$. This implies $\lambda_0 \in \sigma(N)$ and hence $\lambda_0 \in \sigma_{++}(N)$. By Lemma (6.2.14), $0 \in \sigma_+(A(\lambda_0))$ which is a contradiction to $[x_n, x_n] < \frac{1}{n}$.

In a similar way it can be shown that there exist $\varepsilon_2, \delta_2 > 0$ such that for all $(\lambda, x) \in \overline{B_{\varepsilon_2}(K)} \times \mathcal{H}$ we have

$$\|(N - \lambda)x\| \leq \varepsilon_2 \|x\| \ or \ \left\|\left(N^+ - \bar{\lambda}\right)x\right\| \varepsilon_2 \|x\| \Rightarrow [x, x] \geq \delta_2 \|x\|^2 .$$

With $\varepsilon_0 := min\{\varepsilon_1, \varepsilon_2\}$ *and* $\delta_0 := \{\delta_1, \delta_2\}$ the assertion follows.

Concerning the "in particular"-part, note that (37) holds due to (b) and Lemma 6.2.1(ii). For the proof of (3) let $\lambda \in \overline{B_{\varepsilon_0}(K)} \cap \sigma(N)$. By Lemma (6.2.1) (ii), either $\lambda \epsilon \sigma_{ap}(N)$ *or* $\bar{\lambda} \epsilon \sigma_{ap}(N^+)$. In the first case, we obtain $\lambda \epsilon \sigma_+(N)$ from (b). By Lemma (6.2.10) (ii) this implies $\bar{\lambda} \epsilon \sigma_{ap}(N^+)$, and (b) yields $\bar{\lambda} \epsilon \sigma_+(N^+)$

Hence, $\epsilon \sigma_{++}(N)$. This follows analogously in the case $\bar{\lambda} \epsilon \sigma_{ap}(N^+)$.

Corollary (6.2.16)[1]. The set $\sigma_{++}(N)$ is open in $\sigma(N)$.

As $\sigma_{++}(N)$ is open in $\sigma(N)$, it is sufficient to show Theorem (2.6.17) only for open sets S. For the proof we need two preparatory lemmas.

Lemma (6.2.17)[1]. Let $K \in \mathbb{C}$ be a compact set which is of two-sided positive type with respect to N. Then there exists $\varepsilon_0 > 0$ such that for each disk $B_\varepsilon(\lambda) \subset B_{\varepsilon_0}(K)$ with radius $\varepsilon \epsilon (0, \varepsilon_0]$ there exists a closed subspace $\mathcal{L}_{\lambda,\varepsilon} \subset \mathcal{H}$ with the following properties:

(a) $\left(\mathcal{L}_{\lambda,\varepsilon}, [.,.]\right)$ is a Hilbert space which is both N- and N^+ -invariant.

(b) $\sigma(N|\mathcal{L}_{\lambda,\varepsilon}) \subset \sigma(N) \cap \overline{B_\varepsilon(\lambda)}$

(c) If $\mathcal{M} \subset \mathcal{H}$ is a closed subspace which is both N- and N^+-invariant such that $(\mathcal{M}, [.,.])$ is a Hilbert space with

$$\sigma(N|\mathcal{M}) \subset \overline{B_\varepsilon(\lambda)}$$

Then $\mathcal{M} \subset \mathcal{L}_{\lambda,\varepsilon}$.

(d) If $B \epsilon L(\mathcal{H})$ commutes with both N and N^+, then $\mathcal{L}_{\lambda,\varepsilon}$ and $\mathcal{L}_{\lambda,\varepsilon}^{[\perp]}$ both are B-invariant.

(e) $\sigma(N) \cap B_\varepsilon(\lambda) \neq \emptyset \Rightarrow \mathcal{L}_{\lambda,\varepsilon} \neq \{0\}$

Proof. Choose $\varepsilon_0 > 0$ according to Lemma (6.2.15) and let $\lambda \in \mathbb{C}$ and $\varepsilon \epsilon (0, \varepsilon_0]$ such that $B_\varepsilon(\lambda) \subset B_{\varepsilon_0}(K)$. By Lemma (6.2.15) we have

$$[0, \varepsilon^2] \cap \sigma\left(A(\lambda)\right) \subset \sigma_+\left(A(\lambda)\right) \ and \ \overline{B_\varepsilon(\lambda)} \cap \sigma(N) \subset \sigma_{++}(N) . \quad (38)$$

By Lemma (6.2.4) there exists $\delta > 0$ such that $[-\delta, 0) \cap \sigma\left(A(\lambda)\right) \subset \sigma_+\left(A(\lambda)\right)$. Due to Theorem (6.2.6) the operator $A := A(\lambda)$ has a local spectral function E_A of positive type on $[-\delta, \varepsilon^2]$. Due to (iii) and (vi) the subspace

$$\mathcal{L}_{\lambda,\varepsilon} := E_A([-\delta, \varepsilon^2])\mathcal{H}$$

is uniformly positive as well as N- and N^+-invariant. Therefore, the restriction $N|\mathcal{L}_{\lambda,\varepsilon}$ is a normal operator in the Hilbert space $\left(\mathcal{L}_{\lambda,\varepsilon}, [.,.]\right)$ with the adjoint $N^+|\mathcal{L}_{\lambda,\varepsilon}$ and

$$A\left|\mathcal{L}_{\lambda,\varepsilon} = \left(\left(N|\mathcal{L}_{\lambda,\varepsilon}\right)^+ - \bar{\lambda}\right)\left(\left(N|\mathcal{L}_{\lambda,\varepsilon}\right) - \lambda\right)\right.$$

Hence, $A|\mathcal{L}_{\lambda,\varepsilon}$ is a non-negative selfadjoint operator in a Hilbert space which implies

$$(-\delta, 0) \subset \rho(A)$$

Let$f(z) := (\bar{z} - \bar{\lambda})(z - \lambda) = |z - \lambda|^2$, $z \epsilon \mathbb{C}$. This is a continuous function on $\mathbb{C}$, and we obtain

$$\sigma\left(N|\mathcal{L}_{\lambda,\varepsilon}\right) = \sigma\left(f\left(N|\mathcal{L}_{\lambda,\varepsilon}\right)\right) = f\left(\sigma\left(N|\mathcal{L}_{\lambda,\varepsilon}\right)\right).$$

Therefore, $z \in \left(N|\mathcal{L}_{\lambda,\varepsilon}\right)$ implies $f(z) \in [0, \varepsilon^2]$ and thus $\in \overline{B_\varepsilon(\lambda)}$. Since $\left(N|\mathcal{L}_{\lambda,\varepsilon}\right) = \sigma_{ap}\left(N|\mathcal{L}_{\lambda,\varepsilon}\right) \subset \sigma(N)$, (b) is showed.

A subspace $\mathcal{M}$ as in (c) is obviously A-invariant, and we have

$$\sigma(\mathrm{N}|\mathcal{M}) = \sigma\left(f(N|\mathcal{M})\right) = f\left(\sigma(N|\mathcal{M})\right) \subset f\overline{\left(B_\varepsilon(\lambda)\right)} \subset [0, \varepsilon^2]$$

And since $\mathcal{L}_{\lambda,\varepsilon}$ is the maximal spectral subspace of A corresponding to $[0, \varepsilon^2]$, it follows that $\mathcal{M} \subset \mathcal{L}_{\lambda,\varepsilon}$.

If $B \epsilon L(\mathcal{H})$ as in (d), then BA = AB and (d) follows from (iii).

For the proof of (e) assume that $\mathcal{L}_{\lambda,\varepsilon} = \{0\}$. Then $E_A([-\delta, \varepsilon^2]) = 0$ and (v) implies $(-\delta, \varepsilon^2) \subset \rho(A)$. $If\ z\ \varepsilon\ \sigma\ (N) \cap\ B_\varepsilon(\lambda)\ , then\ z\ \epsilon\ \sigma_{++}(N)$ and there exists an approximate eigensequence (x_n) for both $N - z$ and $N^+ - \bar{z}$. Consequently, $(A|\lambda - z|^2)x_n \to 0\ \ as\ n \to \infty$ which contradicts $(-\delta, \varepsilon^2) \subset \rho(A)$. Therefore, $\sigma\ (N) \cap\ B_\varepsilon(\lambda) = \emptyset$.

Note that the subspaces $\mathcal{L}_{\lambda,\varepsilon}$ in Lemma (6.2.18) are uniquely determined by (a)–(c).

Lemma (6.2.18)[1] Let K andε_0 be as in Lemma (6.2.18). Let $\Delta_1, \Delta_2, \ldots \Delta_m \subset \overline{\left(B_{\varepsilon_0}(\lambda)\right)}$ be closed sets such that for each $j\ \in\ \{1, \ldots, m\}$ there exists a closed subspace $\mathcal{L}_j \subset \mathcal{H}$ with

(a_j) $(\mathcal{L}_{\mathrm{j}}, [.,.])$ is a Hilbert space which is both N- and N^+ -invariant.

(b_j) $\sigma\ (N|\mathcal{L}_j) \subset \sigma(N) \cap \Delta_j$.

(c_j) If $\mathcal{M} \subset \mathcal{H}$ is a subspace which is both N- and N^+ - invariant such that $(\mathcal{M}, [.,.])$ is a Hilbert space with

$$\sigma\ (N|\mathcal{M}) \subset \Delta_j,$$

then $\mathcal{M} \subset \mathcal{H}$.

(d_j)If $B \epsilon L(\mathcal{H})$ commutes with both N- and N^+, then $\mathcal{L}_j$ and $\mathcal{L}_j^{[\perp]}$ both are B-invariant.

Then the subspace $\mathcal{L}_o = \mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_m$ is closed and satisfies $(a_0) - (d_0)$, where $\Delta_0 := \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m$

Proof. We only show Lemma (6.2.19) for $m\ =\ 2$. The general case then follows by induction. For $j\ \in\ \{1, 2\}$ denote by E_j the $[\cdot,\cdot]$-orthogonal projection onto $\mathcal{L}_j$ and define

$$E_0 := E_1 + E_2 - E_1 E_2 = E_1 + (1 - E_1)E_2\,.$$

From (a1) it follows that E_1 commutes with N and N^+. By $(d_2), E_1 E_2 = E_2 E_1$. Hence, E_0 is a selfadjoint projection, and the following relation holds:

$$\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}_1 [\dot{+}] \left(\mathcal{L}_1^{[\perp]} \cap \mathcal{L}_2\right) E_o \mathcal{H}\,.$$

By [12], $(\mathcal{L}_0, [.,.])$ is a Hilbert space. Thus, (a_0) holds, and (b_0)as well as (d_0) are easily verified.

Let $\mathcal{M}$ be a subspace as in (c_0). Then $N|\mathcal{M}$ is a normal operator in the Hilbert space $(\mathcal{M}, [.,.])$. Let F be its spectral measure. Then

$$\mathcal{M} = F(\Delta_1)\mathcal{M}[\dot{+}]\left(I_\mathcal{M} - F(\Delta_1)\right)\mathcal{M}$$

We have $\sigma\ (N|F(\Delta_1)\mathcal{M}) \subset \Delta_1$ and

$$\sigma\ \left(N|\left(I_\mathcal{M} - F(\Delta_1)\right)\mathcal{M}\right) \subset \overline{\sigma(N|\mathcal{M})\backslash\Delta_1} \subset \overline{(\Delta_1 \cup \Delta_2)} \subset \Delta_2.$$

Hence, from (c_1) and (c_2) we conclude $F(\Delta_1)\mathcal{M} \subset \mathcal{L}_1$ and $\left(I_\mathcal{M} - F(\Delta_1)\right)\mathcal{M} \subset \mathcal{L}_2$ and therefore $\mathcal{M} \subset \mathcal{L}_0$.

A proof of the following lemma can be found in [16].

Lemma (6.2.19)[1]. (Rosenblum's Corollary). Let$\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $S \in L(\mathcal{X})$ and$T \in L(\mathcal{Y})$. $If\ \sigma(S) \cap \sigma(T) = \emptyset$ then for every $Z \epsilon L(\mathcal{Y}, \mathcal{X})$ the operator equation

$$SX - XT = Z$$

has a unique solution $X \epsilon L(\mathcal{Y}, \mathcal{X})$. In particular, $SX = XT$ implies $X = 0$.
We are now prepared to show Theorem (6.2.20). By Δ^i we denote the interior of a subset $\Delta \in \mathbb{C}$.
Theorem (6.2.20)[1]. Let N be a bounded normal operator in the Krein space $(\mathcal{H}, [.,.])$ and let $S \epsilon \mathbb{C}$ be a Borel set which is of two-sided positive type with respect to N. Then N has a local spectral function E of positive type on S. If $\Delta \in \mathfrak{B}_0(S)$ is compact, then $E(\Delta)\mathcal{H}$ is the maximal spectral subspace of N corresponding to Δ .
Proof. The proof is divided into three steps. In the first two steps it is shown that Theorem (6.2.20) holds for compact sets $S = K$. More precisely, in step 1 it is shown that there exists a spectral subspace $\mathcal{L}_0$ for N corresponding to a compact set Δ_0 containing K which has the properties $(a_0) - (d_0)$ in Lemma (6.1.19) and (i)–(iii) below. In the second step the local spectral function of N on K is defined via the orthogonal projection onto $\mathcal{L}_0$ and the spectral measure of the normal operator $N|\mathcal{L}_0$ in the Hilbert space $(\mathcal{L}_0, [.,.])$. In the last step we show that Theorem (6.2.20) holds for open sets S.

1. Let K be a compact set of two-sided positive type with respect to N and let $\varepsilon_0 > 0$ be as in Lemmas (6.2.15) and (6.2.17). Then choose some $\varepsilon_1 \in (0, \varepsilon_0)$ *and* $\lambda_1, \ldots, \lambda_m \epsilon K$ such that

$$K \subset \cup_{j=1}^{m} B_{\varepsilon_1}(\lambda_j) \subset \cup_{j=1}^{m} \overline{B_{\varepsilon_1}(\lambda_j)} =: \Delta_0 \subset B_{\varepsilon_0}(K)$$

By Lemma (6.2.17) and Lemma (6.2.18) there exists a closed subspace $\mathcal{L}_0 \subset \mathcal{H}$. satisfying $(a_0) - (d_0)$ in Lemma (6.2.18). We will show that $\mathcal{L}_0$ also has the following properties:
(i)$\sigma\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right) \subset \overline{\sigma(N)\backslash \Delta_0}$.
(ii) If $B \epsilon L(\mathcal{H})$ with $BN = NB$, then $\mathcal{L}_0$ and $\mathcal{L}_0^{[\perp]}$ are B-invariant.
(iii) $\mathcal{L}_0$ is the maximal spectral subspace of N corresponding to Δ_0.
First of all we show

$$\overline{B_{\varepsilon_0}(K)} \cap \sigma\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right) \subset \sigma_{++}\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right). \tag{39}$$

Since $\overline{B_{\varepsilon_0}(K)} \cap \sigma(N) \subset \sigma_{++}(N)$, it suffices to show that

$$\overline{B_{\varepsilon_0}(K)} \cap \sigma\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right) \subset \sigma_{ap}\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right),$$

cf. Lemma (6.2.13) (i). Let $\lambda \epsilon \overline{B_{\varepsilon_0}(K)} \cap \sigma\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right)$ and assume $\lambda \notin \sigma_{++}\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right)$. Then $\bar{\lambda} \in \sigma_p\left(N^+\left|\mathcal{L}_0^{[\perp]}\right.\right)$, from which $\lambda \in \sigma(N)$. follows. But this implies $\lambda \in \sigma_{++}(N)$ and therefore $\lambda \epsilon \sigma_p\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right)$ (see Lemma (6.2.13) (ii)). A contradiction.

Let $\epsilon \mathbb{C}\backslash \overline{\sigma(N)\backslash \Delta_0}$. . Then $\sigma(N)\backslash \Delta_0$ does not accumulate to λ which means that there exists $\varepsilon' > 0$ such that $(N) \cap B_{\varepsilon'}(\lambda) \subset \Delta_0$. Due to (39) and Lemma (6.2.17) there exist $\varepsilon \epsilon (0, \varepsilon')$ and a closed N- and N^+ - invariant subspace $\mathcal{M} \subset \mathcal{L}_0^{[\perp]}$. such that $(\mathcal{M}, [.,.])$. is a Hilbert space and $\sigma(N|\mathcal{M}) \subset \rho\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right) \cap \overline{B_\varepsilon(\lambda)}$. *As* $\sigma\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right) \subset \sigma(N)$, we have $\sigma(N|\mathcal{M}) \subset \Delta_0$. From (c_0) we conclude $\mathcal{M} \subset \mathcal{L}_0$. But $\mathcal{M} \subset \mathcal{L}_0^{[\perp]}$ and thus $\mathcal{M} = \{0\}$ From Lemma (6.2.17) (e) we obtain $B_\varepsilon(\lambda) \subset \rho\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right)$ which shows (i).

For the proofs of (ii) and (iii) let (δ_n) be a sequence of positive numbers such that

$$\varepsilon_0 > \delta_1 > \delta_2 > \cdots > \varepsilon_1 \; and \; \delta_n \downarrow \varepsilon_1 \; as \; n \to \infty$$

Set

$$\delta_n := \cup_{j=1}^{m} \overline{B_{\delta_n}(\lambda_j)} \subset B_{\varepsilon_0}(K)$$

Then (recall that Δ_0 was defined similarly)

$$\Delta_0 = \cap_{n=1}^{\infty} \Delta_n \;\; and \;\; \Delta_0 \subset \Delta_n^i \;\; for\, every\, n \in \mathbb{N} \, .$$

For $n \in \mathbb{N}$ n ∈ N by $\mathcal{L}_n$ we denote the closed subspace which satisfies $(a_n) - (d_n)$ in Lemma (6.2.18). We have

$$\mathcal{L}_{n+1} \subset \mathcal{L}_n \; , \; \mathcal{L}_0 \subset \mathcal{L}_n \;\; for\, all\, n \in \mathbb{N} \backslash \{0\} \;\; and\, \mathcal{L}_0 = \cap_{n=1}^{\infty} \mathcal{L}_n \quad . \tag{40}$$

Indeed, the first two inclusions follow immediately from the properties (c_k) , $k \in \mathbb{N}$, in Lemma (6.2.18). Hence, $\mathcal{L}_0 \subset \cap_{n=1}^{\infty} \mathcal{L}_n =: \mathcal{M}$, and it is not difficult to see that $\sigma(N|\mathcal{M}) \subset \Delta_0$ (consider $\lambda \notin \Delta_0$ and show $\lambda \epsilon \rho(N|\mathcal{M})$. The last relation in (40) follows now from (c_o).

Let $B\epsilon L(\mathcal{H})$ such that $BN = NB$ and set $B_0 := B|\mathcal{L}_0 \,\epsilon\, L(\mathcal{L}_0,\mathcal{H})$. By (E_n) we denote the $[.,.]$ -orthogonal projection in $\mathcal{H}$ onto $\mathcal{L}_n$, $n \in \mathbb{N}$. As $\mathcal{L}_n$ and $\mathcal{L}_n^{[\perp]}$ both are N-invariant, E_n commutes with N. Hence, the following relation holds:

$$\left(N\left|\mathcal{L}_n^{[\perp]}\right.\right)\left((I-E_n)B_0\right) = (I-E_n)NB_0 = \left((I-E_n)B_0\right)N|\mathcal{L}_0 \,.$$

By (i) and (b_0) the spectra of $N\left|\mathcal{L}_n^{[\perp]}\right.$ and $N|\mathcal{L}_0$ are disjoint. Thus, due to Rosenblum's Corollary (Lemma (6.2.19)) it follows that $(I - E_n)B_0 = 0$, or equivalently, $B\mathcal{L}_0 \subset \mathcal{L}_n$ for every $n \in \mathbb{N}$. By virtue of (40), $\mathcal{L}_0$ is B-invariant. Similarly, one shows $B\mathcal{L}_n^{[\perp]} \subset \mathcal{L}_0^{[\perp]}$ for each $n \in \mathbb{N}$. It is easy to see that

$$\text{c.l.s.}\left\{\mathcal{L}_n^{[\perp]} : n \in \mathbb{N}\right\}^{[\perp]} = \cap_{n=1}^{\infty} \mathcal{L}_n = \mathcal{L}_0 \,.$$

Hence, $B\mathcal{L}_n^{[\perp]} \subset \mathcal{L}_0^{[\perp]}$ follows immediately from

$$\text{c.l.s}\left\{.\mathcal{L}_n^{[\perp]} : n \in \mathbb{N}\right\} = \mathcal{L}_0^{[\perp]}.$$

Let $\mathcal{M} \subset \mathcal{H}$ be a closed N-invariant subspace such that $(N|\mathcal{M}) \subset \Delta_0$. Then from Rosenblum's Corollary and the relation

$$\left(N\left|\mathcal{L}_n^{[\perp]}\right.\right)\left((I-E_n)\mathcal{M}\right) = \left((I-E_n)|\mathcal{M}\right)(N|\mathcal{M})$$

we conclude $\mathcal{M} \subset \mathcal{L}_n$ for all $n \in N$, and thus $\mathcal{M} \subset \mathcal{L}_0$, which shows (iii).

2. Let us complete the proof of Theorem (6.2.20) for $S = K$. Let Δ_0 and $\mathcal{L}_0$ be as in step 1. The operator $N_0 := N|\mathcal{L}_0$ is a normal operator in the Hilbert space $(\mathcal{L}_0, [.,.])$ and has therefore a spectral measure E_0 . By Q we denote the $[.,.]$-orthogonal projection onto $\mathcal{L}_0$ and define

$$E(\Delta) := E_0(\Delta)Q \,, \Delta \in \mathfrak{B}_0(K) \,.$$

For each $\Delta \in \mathfrak{B}_0(K)$ the operator $E(\Delta)$ is a selfadjoint projection, and it is easily seen that E has the properties (i)–(iii) and (vi) in Definition (6.2.2). Let
$\Delta \in \mathfrak{B}_0(K)$. Then

$$\sigma\,(N|E(\Delta)\mathcal{H}\,) = \sigma\,(N_0|E_0(\Delta)\mathcal{L}_0\,) \subset \overline{\sigma(N_0) \cap \Delta}$$
$$\subset \overline{\sigma(N) \cap \Delta_0} \cap \Delta = \overline{\sigma(N) \cap \Delta}$$

And since

$$\left((I-E_n)\right)\mathcal{H} = \mathcal{L}_0^{[\perp]}[\dot{+}]\left((E_0(\Delta)\mathcal{L}_0)^{[\perp]} \cap \mathcal{L}_0\right),$$

we have

$$\sigma\left(N\middle|\left(I-E(\Delta)\right)\mathcal{H}\right) = \sigma\left(N\left|\mathcal{L}_0^{[\perp]}\right.\right) \cup \sigma\left(N_0\middle|(E_0(\Delta)\mathcal{L}_0)^{[\perp]} \cap \mathcal{L}_0\right)$$
$$\subset \overline{\sigma(N)\backslash\Delta_0} \cup \overline{\sigma(N_0)\backslash\Delta}$$

Moreover, if $\Delta \subset K$ is closed, then $E(\Delta)\mathcal{H}$ is the maximal spectral subspace of N corresponding to Δ (we say that E has the property (M)): if $\mathcal{M} \subset \mathcal{H}$ is a closed N-invariant subspace such that $\sigma(N|\mathcal{M}) \subset \Delta$, then $\mathcal{M} \subset \mathcal{L}_0$ by (iii) in step 1 and hence, we have $\sigma(N_0|\mathcal{M}) \subset \Delta$. From this and the properties of the spectral measure E_0 of N_0 we obtain $\mathcal{M} \subset E_0(\Delta)\mathcal{L}_0 = E_0(\Delta)Q\mathcal{H} = E(\Delta)\mathcal{H}$. In particular, this shows that the definition of E does not depend on the choice of ε_0 and ε_1. Indeed, if ΔE is another local spectral function of positive type for N on K with the property (M), then $\check{E}(\Delta) = E(\Delta)$ for all closed sets $\Delta \subset K$. And as the system of the closed subsets of K is a generator of the σ -algebra $\mathfrak{B}_0(K)$ which is stable with respect to intersections, $\check{E} = E$ follows.

3. Finally, we show that Theorem (6.2.20) holds for open sets S. Clearly, it is no restriction to assume that S is bounded. For a closed set $K \subset S$ denote by E_K the local spectral function of positive type of N on K (with the property (M)), defined in the previous steps. We set

$$E(\Delta) := E_{\bar{\Delta}}(\Delta) \,, \qquad \Delta \in \mathfrak{B}_0(S) \,.$$

It is evident that E satisfies (iii)–(vi) in Definition (6.2.2) (with T replaced by N). Moreover, if $\Delta \in S$ is closed, then $E(\Delta)\mathcal{H} = E_\Delta(\Delta)\mathcal{H}$ is the maximal spectral subspace of N corresponding to Δ. It remains to show that E satisfies (i) and (ii). To see this, note that for two closed sets $K_1, K_2 \subset S$ with $K_1 \subset K_2$ we have

$$E_{K_2}|\mathfrak{B}_0(K_1) = E_{K_1}$$

since $E_{K_2}|\mathfrak{B}_0(K_1)$ is a local spectral function of positive type for N on K_1 with the property (M) and must therefore coincide with E_{K_1}. Let $\Delta \in \mathfrak{B}_0(S)$, $\epsilon \mathbb{N}$, as in (ii), and set $\Delta := \cup_{k=1}^{\infty} \Delta_k$ .Then

$$E(\Delta) = E_{\bar{\Delta}}(\Delta) = \sum_{k=1}^{\infty} E_{\bar{\Delta}}(\Delta_j) = \sum_{k=1}^{\infty} E_{\overline{\Delta_j}}(\Delta_j) = \sum_{k=1}^{\infty} E(\Delta_j)$$

in the strong operator topology. It is showed similarly that E satisfies (i). The theorem is showed.

The following corollary is a direct consequence of Theorem (6.2.20).

Corollary (6.2.21)[1]. Let $\lambda_0 \epsilon \sigma_{++}(N)$ be an accumulation point of $\rho(N)$. Then there exist $\varepsilon > 0$ and $C > 0$ such that for all $\lambda \epsilon \mathfrak{B}_\varepsilon(\lambda_0) \cap \rho(N)$ we have

$$\|(N-\lambda)^{-1}\| \le \frac{C}{dis(\lambda, \sigma(N))}$$

In particular, an isolated spectral point of N which is of two-sided positive type is a pole of order one of the resolvent of N.

Proof. Choose $\varepsilon > 0$ such that $\mathfrak{B}_0 := \overline{\mathfrak{B}_{2\varepsilon}(\lambda_0)}$ is of two-sided positive type with respect to N. Denoteby E the local spectral function of N on $\mathfrak{B}_0$ and set $\mathcal{L}_0 := E(\mathfrak{B}_0)\mathcal{H}$. Then $\mathfrak{B}_0^i \subset \rho \left\| \left(N \left| \mathcal{L}_0^{[\perp]} \right. \right) \right\|$

$$\left\| \left(\left(N \left| \mathcal{L}_0^{[\perp]} \right. \right) - \lambda \right)^{-1} \right\| \le C_1 \quad for\ all \quad \lambda \in \mathfrak{B}_\varepsilon(\lambda_0).$$

The restriction of N to $\mathcal{L}_0$ is a normal operator in a Hilbert space. Therefore, for any $x \in \mathcal{L}_0$ and $\lambda \epsilon \mathfrak{B}_\varepsilon(\lambda_0) \cap \rho(N)$ we have the well-known inequality

$$[(N-\lambda)x, (N-\lambda)x] \ge dist\left(\lambda, \sigma(N|\mathcal{L}_0)\right)^2 [x, x]$$

As the subspace $\mathcal{L}_0$ is uniformly positive, this implies

$$\|(N-\lambda)x\|^2 \ge dist\big(\lambda, \sigma(N)\big)^2 . \delta \|x\|^2,$$

with some $\delta > 0$, and the assertion follows.

Let σ be a spectral set for N (i.e. a subset of $\sigma(N)$ which is both open and closed in $\sigma(N)$) which is of positive type with respect to N. In [15] it was shown that the Riesz–Dunford spectral subspace of N corresponding to σ is uniformly positive if the spectrum of the imaginary part $ImN = \frac{1}{2i}(N - N^+)$ is real and if there exist $C > 0$ and $m \in \mathbb{N}$ such that

$$\|(ImN - \lambda)^{-1}\| \le \frac{C}{|Im\lambda|^m} \tag{41}$$

holds for all non-real λ in a neighborhood of $\sigma(ImN)$. The same holds if the above conditions are satisfied for the real part $Re\ N = \frac{1}{2}(N + N^+)$ instead for ImN. The following theorem shows that these assumptions are redundant.

Theorem (6.2.22)[1]. Let σ be a spectral set for N, let Q be the Riesz–Dunford projection of N corresponding to σ and assume that

$$\sigma \cap \sigma_{ap}(N) \subset \sigma_+(N) . \tag{42}$$

Then Q is selfadjoint and $Q\mathcal{H}$ is uniformly positive. In particular, $N|Q\mathcal{H}$ is a normal operator in the Hilbert space$Q\mathcal{H}, [.,.]$.

Proof. The projection Q is selfadjoint by Lemma (6.2.5) (see also [15]). This implies that the inner product space $Q\mathcal{H}, [.,.]$ is a Krein space which is invariant with respect to both N and N^+ Moreover, we have

$$(N|Q\mathcal{H})^+ = N^+|Q\mathcal{H}$$

and $\sigma_{ap}(N|Q\mathcal{H}) = \sigma_+(N|Q\mathcal{H})$. It is therefore no restriction to assume $\mathcal{H} = Q\mathcal{H}$ and $\sigma_{ap}(N) = \sigma_+(N)$. In view of Remark (6.2.12) and Theorem (6.2.20) it only remains to show that $\mathbb{C}$ is of positive type with respect to N^+. i.e. $\sigma_{ap}(N^+) \subset \sigma_+(N^+)$. Let $\bar{\lambda} \epsilon \sigma_{ap}(N^+)$ We have to show that the approximate eigensequences for $\left(N - \bar{\lambda}\right)$ are also approximate eigensequences for $N - \lambda$. To this end we introduce the Banach space

$$\breve{\mathcal{H}} := \ell^\infty(\mathcal{H}) \,/\, c_0(\mathcal{H}),$$

where by $\ell^{\infty}(\mathcal{H})$ we denote the space of all bounded sequences (x_n) in $\mathcal{H}$ with norm $\|(x_n)\|_{\ell^{\infty}(\mathcal{H})} = sup_n\|x_n\|$. and $c_0(\mathcal{H})$ is the closed subspace $\ell^{\infty}(\mathcal{H})$ consisting of the sequences (x_n) with lim $\|x_n\| = 0$. It is not difficult to show that the norm of a coset $[(x_n)] \in \breve{\mathcal{H}}$ is given by

$$\|[(x_n)]\|_{\breve{\mathcal{H}}} = lim_{n\to\infty}sup\|x_n\|$$

Consider the operators $\breve{N}$ and $\breve{N^+}$ in $\breve{\mathcal{H}}$, defined by

$$\breve{N}[(x_n)] := [(N(x_n))] \quad and \quad \breve{N^+}[(x_n)] := [(N^+(x_n))]\,, [(x_n)] \in \mathcal{H} \quad .$$

The operators $\breve{N}$ and $\breve{N^+}$are well-defined and $\breve{N}, \breve{N^+} \in L(\mathcal{H})$ holds where $\|\breve{N}\| \leq \|N\|$ and $\|\breve{N^+}\| \leq \|N^+\|$. As $\breve{N}$ and $\breve{N^+}$ commute , also $\breve{N}$ and $\breve{N^+}$ commute.

Observe that if (x_n) is an approximate eigensequence for $N - \lambda$ then $[(x_n)] \in ker(\breve{N} - \lambda)$. Conversely, if $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1$ for $n \in \mathbb{N}$ such that $[(x_n)] \in ker(\breve{N} - \lambda)$, then (x_n) is an approximate eigensequence for $N - \lambda$. An analogue correspondence holds for $(N^+ - \bar{\lambda})$ and $(\breve{N^+} - \bar{\lambda})$. Therefore, we have to show that

$$ker(\breve{N^+} - \bar{\lambda}) \subset ker(\breve{N} - \lambda) \ .$$

To see this, we define the subspace

$$\mathcal{M} := \overline{(\breve{N} - \lambda)ker(\breve{N^+} - \bar{\lambda})}$$

This subspace is $\breve{N}$-invariant. We are done if we can show that $\mathcal{M} = \{0\}$, or equivalently, $\sigma_{ap}(\breve{N}|\mathcal{M}) = \emptyset$ Thus, suppose that there exist a sequence $(\breve{x_m}) \subset \mathcal{M}$ and $\mu \in \mathbb{C}$ such that

$$\|\breve{x_m}\|_{\breve{\mathcal{H}}} = 1 \ and \quad lim_{m\to\infty}\|(\breve{N} - \mu)\breve{x_m}\|_{\breve{\mathcal{H}}} = 0 \ .$$

For each $m \in \mathbb{N}$ there exists a sequence $\left(x_{\mathrm{n}}^{(m)}\right) \in \ell^{\infty}(\mathcal{H})$ such that $\breve{x_m} = \left[\left(x_n^{(m)}\right)\right]$. Let $m \in \mathbb{N}$. As $\left[\left(x_n^{(m)}\right)\right] \in \mathcal{M}$, there exists $\left[\left(u_n^{(m)}\right)\right] \in ker(\breve{N^+} - \bar{\lambda})$ such that $\left[\left(x_n^{(m)}\right)\right] - (\breve{N} - \lambda)\left[\left(u_n^{(m)}\right)\right] \to 0 \ as \ m \to \infty \ in \ \breve{\mathcal{H}}$. Hence, the following holds:

(a)$lim_{m\to\infty}limsup_{n\to\infty}\left\|x_n^{(m)} - (N - \lambda)u_n^{(m)}\right\| = 0$,

(b) $\forall\, m \in \mathbb{N} : lim_{n\to\infty}\left\|(N^+ - \bar{\lambda})u_n^{(m)}\right\| = 0$,

(c) $\forall\, m \in \mathbb{N} : limsup_{n\to\infty}\left\|x_n^{(m)}\right\| = 1$,

(d) $lim_{m\to\infty}limsup_{n\to\infty}\left\|(N - \mu)u_n^{(m)}\right\| = 0$.

It is not difficult to see that from (a)–(d) it follows that for each $k \in N$ there exist $\mathrm{m}_k, n_k \in \mathbb{N}$ such that

(a') $\left\|x_{n_k}^{(m_k)} - (N - \lambda)u_{n_k}^{(m_k)}\right\| < \dfrac{1}{k}$,

(b') $\left\|(N^+ - \bar{\lambda})u_{n_k}^{(m_k)}\right\| < \dfrac{1}{k}$,

(c') $\dfrac{1}{2} \leq \left\|u_{n_k}^{(m_k)}\right\| \leq 2$,

(d') $\left\|(N - \mu)u_{n_k}^{(m_k)}\right\| < \frac{1}{k}$.

Set $x_k := x_{n_k}^{(m_k)} \ and \ u_k := u_{n_k}^{(m_k)}$. From (c') and (d') we conclude $\mu \in \sigma_{ap}(N)$ and hence $\mu \in \sigma_+(N)$. Consequently,

$$lim_{k\to\infty}inf[x_k, x_k] > 0\,.$$

On the other hand, we have

$$\begin{aligned}|[x_k, x_k]| &\leq |[x_k - (N - \lambda)u_k, x_k]| + |[(N - \lambda)\mathrm{u}_k, x_k - (N - \lambda)u_k]| \\ &\quad + |[(N - \lambda)u_k, (N - \lambda)u_k]| \\ &\leq \frac{2}{k} + \frac{1}{k}\|(N - \lambda)u_k\| + \left|[(N^+ - \bar{\lambda})u_k, (N^+ - \bar{\lambda})u_k]\right| \\ &\leq \frac{2}{k} + \frac{1}{k}\left(\frac{1}{k} + \|x_k\|\right) + \frac{1}{k^2} \leq \frac{6}{k}\,,\end{aligned}$$

which is a contradiction.

In what follows we derive some direct consequences of Theorem (6.2.22) (see also Lemma (6.2.13) and Corollary (6.2.21)).

Corollary (6.2.23)[1]. If σ is a spectral set of N which is of positive type with respect to N, then σ is of two-sided positive type with respect to N. In particular, if $\lambda \in \sigma_+(N)$ is an isolated point of $\sigma(N)$, then $\lambda \in \sigma_{++}(N)$, and λ is a pole of order one of the resolvent of N.

Corollary (6.2.24)[1]. If $\dim\mathcal{H} < \infty$, then $\sigma_+(N) = \sigma_{++}(N)$. In particular, if for some $\lambda \in \sigma(N)$ the inner product $[\cdot,\cdot]$ is positive definite on $ker(N - \lambda)$ or on $ker(N^+ - \bar{\lambda})$, then

$$ker(N - \lambda) = ker(N^+ - \bar{\lambda}) = \mathcal{L}_\lambda(N) = \mathcal{L}_{\bar{\lambda}}(N^+)$$

Corollary (6.2.25)[1]. Let $S \subset \mathbb{C}$ be an open set and assume that N has a local spectral function E on S. Then S is of positive type with respect to N if and only if E is a local spectral function of positive type.

Proof. If E is of positive type, then S is of positive type with respect to N by Lemma (6.2.9). Conversely, assume that S is of positive type with respect to N. Let $\Delta \epsilon \mathfrak{B}_0(S)$. Then from Lemma (6.2.5) we conclude that $\Delta \epsilon E(\Delta)$ is selfadjoint. It remains to show that $(Q\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space. As N^+ commutes with Q, it follows that $N|Q\mathcal{H}$ is a normal operator in the Krein space $(Q\mathcal{H}, [\cdot,\cdot])$ with $(N|Q\mathcal{H})^+ = N^+|Q\mathcal{H}$. The assertion is now a consequence of $\sigma_{ap}(N|QH) \subset \sigma_+(N|QH)$ and Theorem (6.2.22).

A bounded operator T in $(\mathcal{H}, [\cdot,\cdot])$ is said to be fundamentally reducible if there exists a fundamental decomposition $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$ of $\mathcal{H}$ such that both $\mathcal{H}_+$ and $\mathcal{H}_-$ are T -invariant. Note that a fundamentally reducible normal operator is always normal in a Hilbert space. A fundamentally reducible operator T is called strongly stable if

$$\sigma(T|\mathcal{H}_+) \cap \sigma(T|\mathcal{H}_-) = \emptyset,$$

cf. [6]. The following corollary was already showed in [15] under the additional assumption that $\sigma(\mathrm{Im}N) \subset \mathbb{R}$ and that a growth condition (41) on the resolvent of $(ImN$ holds near $\mathbb{R}$. Here, it immediately follows from [6] and Theorem (6.2.22).

Corollary (6.2.26)[1] The following statements are equivalent.

(i) N is strongly stable.

(ii) There exists $\delta > 0$ such that every normal operator X with $\|X - N\| < \delta$ is fundamentally reducible.

(iii) $\sigma(N) = \sigma_+(N) \cup \sigma_-(N)$.

In view of Corollary (6.2.23) the question arises whether the sets $\sigma_+(N)$ and $\sigma_{++}(N)$ possibly even coincide. We cannot give a definite answer to this question here. However, the following proposition shows that a possible counterexample can only be found in an infinite-dimensional Krein space which is not a Pontryagin space.

Proposition (6.2.27)[1]. If $(\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space, then $\sigma_+(N) = \sigma_{++}(N)$.

Proof. It is easy to see (see also [14]) that the space $\mathcal{H}$ can be decomposed into a direct orthogonal sum $\mathcal{H} = \mathcal{H}_1[\dot{+}]\mathcal{H}_2$ with closed $N-$ and N^+-invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\dim\mathcal{H}_1 < \infty$ and the operators $ReN|\mathcal{H}_2$ and $\mathrm{Im}N|\mathcal{H}_2$have real spectra. Set $N_j := N|\mathcal{H}_j$, $j = 1, 2$. Owing to the properties of selfadjoint operators in Pontryagin spaces and [15] the operator N_2 has a local spectral function of positive type on neighborhoods of spectral points of positive type of N_2. The assertion now follows from Lemma (6.2.9) and Corollary (6.2.24).

List of symbol

Symbol	Page
inf : Infimum	1
L_+^2 : Hilbert space	1
Vm : Vivozub-Matsaev	2
ran : range	2
vm : Vivozub-Matsaev	2
dim : dimension	3
Ker : Kernel	5
dom : domain	8
max : maximum	8
deg : degree	8
Sup: Supremum	10
Im : Imaginary	11
Min :minimum	27
Lip: Lipschitz	29
const : constant	30
dist : distant	32
L^∞: Lebesgue space	32
$\oplus$: Orthogonal decomposition	33
sgn: sign	66
ι_2: Hilbert space	68
arg : argument	68
$\otimes$: Tensor product	69
S_p: Schatten- vcn Neumunn	74
$B_{\infty,1}^1$: Besov class	75
diag : diagonal	83
Cls : Closed Linear span	85
$\ominus$: Direct difference	93
ind : index	105
ap : approximate point	116
Re : Real	130
ι^∞: Hilbert space	139

References

[1] Friedrich Philipp, Locally definite normal operators in Krein spaces, Journal of Functional Analysis 262 (2012) 4929–4947
[2] T.Ya. Azizov, J. Behrndt, P. Jonas, C. Trunk, Spectral points of definite type and type π for linear operators and relations in Krein spaces, J. Lond. Math. Soc. 83 (2011) 768–788.
[3] T.Ya. Azizov, I.S. Iokhvidov, Linear Operators in Spaces with an Indefinite Metric, John Wiley & Sons, Ltd., Chichester, 1989.
[4] T.Ya. Azizov, P. Jonas, On compact perturbations of normal operators in a Krein space, Ukrain. Mat. Zh. 42 (1990) 1299–1306.
[5] T.Ya. Azizov, V.A. Strauss, Spectral decompositions for special classes of self-adjoint and normal operators on Krein spaces, in: Aurelian Gheondea, et al. (Eds.), Spectral Analysis and Its Applications. Ion Colojoara Anniversary Volume, Theta, 2003, in: Theta Ser. Adv. Math., vol. 2, Theta, 2003, pp. 45–67.
[6] Ts. Bayasgalan, On strongly stable normal operators on Krein space, Acta Math. Hungar. 81 (1998) 275–278.
[7] Ts. Bayasgalan, Fundamental reducibility of normal operators on Krein space, Stud. Sci. Math. Hungar. 35 (1999) 147–150.
[8] J. Bognár, Indefinite Inner Product Spaces, Springer-Verlag, Berlin–Heidelberg–New York, 1974.
[9] A.Yu. Evnin, V.A. Strauss, Normal operators in Banach spaces with a Hermitian form, in: Collection: Applied Problems of Mathematical Analysis, Chelyabinsk Polytechnical Institute, Chelyabinsk, 1986, pp. 101–105 (in Russian).
[10] I. Gohberg, P. Lancaster, L. Rodman, Indefinite Linear Algebra and Applications, Birkhäuser, Basel, 2005.
[11] P. Lancaster, A.S.Markus, V.I.Matsaev, Definitizable operators and quasihyperbolic operator polynomials, J. Funct. Anal. 131 (1995) 1–28.
[12] H. Langer, Spectral functions of definitizable operators in Krein spaces, Lect. Notes Math. 948 (1982) 1–46.
[13] H. Langer, A.S. Markus, V.I. Matsaev, Locally definite operators in indefinite inner product spaces, Math. Ann. 308 (1997) 405–424.
[14] H. Langer, F.H. Szafraniec, Bounded normal operators in Pontryagin spaces, Oper. Theory Adv. Appl. 162 (2006) 231–251.
[15] F. Philipp, V. Strauss, C. Trunk, Local spectral theory for normal operators in Krein spaces, submitted for publication, arXiv:1204.1479v1.
[16] H. Radjavi, P. Rosenthal, Invariant Subspaces, Dover Publications, Inc., Mineola, New York, 2003.
[17] V.A. Strauss, On models of function type for a special class of normal operators in Krein spaces and their polar representation, Methods Funct. Anal. Topology 13 (2007) 67–82.
[18] K. Veseli´c, A spectral theorem for a class of J-normal operators, Glasnik Mat. 5 (25) (1970) 97–102.
[19] J. Wu, Normal extensions of operators to Krein spaces, Chinese Ann. Math. Ser. B 8 (1987) 36–42.
[20] C. Xiaoman, H. Chaocheng, Normal Operators in $\Pi\kappa$ -spaces, Northeast. Math. J. 1 (1985) 247–252.
[21] F. Philipp, V. Strauss, and C. Trunk, Local spectral theory for normal operators in Krein spaces, Math. Nachr. 286, No. 1, 42 – 58 (2013) / DOI 10.1002/mana.201000141.
[22] T. Ya. Azizov, P. Jonas, and C. Trunk, Spectral points of type $\pi+$ and $\pi-$ of selfadjoint operators in Krein spaces, J. Funct. Anal. **226**, 114–137 (2005).
[23] I. Colojoar˘a, C. Foias¸, Theory of Generalized Spectral Operators, Gordon and Breach (Science Publishers, Inc., New York, 1968).
[24] J. B. Conway, A Course in Functional Analysis (Springer, New York, 1985).
[25] Yu. L. Daletskii and M. G. Krein, Stability of solutions of differential equations in Banach space, Am. Math. Soc. (Translated from Russian) (1974).
[26] N. Dunford and J. T. Schwartz, Linear Operators, Part III, Spectral Operators (John Wiley & Sons, New York, 1971).
[27] B. Jacob and C. Trunk, Spectrum and analyticity of semigroups arising in elasticity theory and hydromechanics, Semigroup Forum **79**, 79–100 (2009).
[28] P. Jonas, On locally definite operators in Krein spaces, in: Spectral Theory and Applications, Theta Ser. Adv. Math. **2** (2003), Theta, Bucharest (95–127).
[29] N. D. Kopachevsky and S. G. Krein, Operator Approach to Linear Problems of Hydrodynamics, Self-adjoint Problems for an Ideal Fluid Vol. 1 (Birkh¨auser, Basel, 2001).
[30] M. G. Krein and H. Langer, On some mathematical principles in the linear theory of damped oscillations of continua I, Integral Equations Oper. Theory **1**, 364–399 (1978).
[31] M. G. Krein and H. Langer, On some mathematical principles in the linear theory of damped oscillations of continua II, Integral Equations Oper. Theory **1**, 539–566 (1978).

[32] H. Langer, Spektraltheorie linearer Operatoren in J-R¨aumen und einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 + \lambda B + C$, Habilitationsschrift (Technische Universit¨at Dresden, 1965).
[33] H. Langer, Spectral functions of definitizable operators in Krein spaces, Lect. Notes Math. **948**, 1–46 (1982).
[34] I. Lyubich and V. I. Matsaev, On operators with decomposable spectrum, Mat. Sbornik **56**(98), 433–468 (1962) (Russian); Engl. transl. AMS Transl. (2) **47**, 89–129 (1965).
[35] A. Markus, Introduction to the spectral theory of polynomial operator pencils, AMS Transl. Math. Monographs **71**, Providence Rhode Island (1988).
[36] V. Mueller, Spectral theory of linear operators and spectral systems in Banach algebras (Birkh¨auser, Basel, 2007).
[37] M. A. Naimark, On commuting unitary operators in spaces with indefinite metric, Acta Sci. Math. (Szeged) **24**, 177–189 (1963).
[38] A. A. Shkalikov, Operator pencils arising in elasticity and hydrodynamics the instability index formula, Recent developments in operator theory and its applications (Winnipeg, MB, 1994), Oper. Theory Adv. Appl. **87**, 358–385 (1996).
[39] J.I. Giribet, A. Maestripieri, F. Martinez Peria , P.G. Massey, On frames for Krein spaces, J.I. Giribet et al. J. Math. Anal. Appl. 393 (2012) 122–137
[40] P.G. Casazza, The art of frame theory, Taiwanese J. Math. 4 (2) (2000) 129–201.
[41] O. Christensen, An introduction to frames and Riesz bases, in: Applied and Numerical Harmonic Analysis, Birkhںuser, Boston, 2003.
[42] O. Christensen, in: A.H. Siddiqi, I.S. Duff, O. Christensen (Eds.), Recent Developments in Frame Theory, in: Modern Mathematical Models, Methods and Algorithms for Real World Systems, Anamaya Publishers, New Delhi, India, 2006.
[43] D. Han, D.R. Larson, Frames, bases and group representations, Mem. Amer. Math. Soc. 147 (697) (2000).
[44] B.G. Bodmann, V.I. Paulsen, Frames, graphs and erasures, Linear Algebra Appl. 404 (2005) 118–146.
[45] R.B. Holmes, V.I. Paulsen, Optimal frames for erasures, Linear Algebra Appl. 377 (2004) 31–51.
[46] T. Strohmer, R.W. Heath Jr., Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.
[47] I.S. Iokhvidov, T.Ya. Azizov, Linear Operators in Spaces with an Indefinite Metric, John Wiley and Sons, 1989.
[48] K. Esmeral Garcيa, E. Wagner, Frames in Krein spaces, Preprint.
[49] I. Peng, S. Waldron, Signed frames and Hadamard products of gram matrices, Linear Algebra Appl. 347 (1–3) (2002) 131–157.
[50] T. Ando, Linear Operators on Krein Spaces, Hokkaido University, Sapporo, Japan, 1979.
[51] R.G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, Proc. Amer. Math. Soc. 17 (1966) 413–416.
[52] J. Bognلr, Indefinite Inner Product Spaces, Springer-Verlag, 1974.
[53] M.A. Dritschel, J. Rovnyak, Operators on indefinite inner product spaces, in: Peter Lancaster (Ed.), in: Fields Institute Monographs no. 3, vol. 3, Amer. Math. Soc., 1996, pp. 141–232.
[54] F. Deutsch, The angle between subspaces of a Hilbert space, in: Approximation Theory, Wavelets and Applications (Maratea, 1994), in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 454, Kluwer Acad. Publ., Dordrecht, 1995, pp. 107–130.
[55] T. Kato, Perturbation Theory for Linear Operators, Springer, New York, 1966.
[56] M. Unser, Sampling 50 years after Shannon, Proc. IEEE 88 (2000) 569–587.
[57] A. Buades, B. Coll, J.-M. Morel, Nonlocal image and movie denoising, Int. J. Comput. Vis. 76 (2) (2008) 123–140.
[58] M.A.S. Masoum, S. Jamali, N. Ghaffarzadeh, Detection and classification of power quality disturbances using discrete wavelet transform and wavelet networks, IET Sci. Meas. Technol. 4 (2010) 193–205.
[59] A. Grod, S. Kuzhel, V. Sudilovskaya, On operators of transition in Krein spaces, Opuscula Math. 31 (1) (2010) 49–59.
[60] N.I. Akhiezer, I.M. Glazman, Theory of Linear Operators in Hilbert Space, Dover Publ. Inc., 1993.
[61] R.J. Duffin, A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952) 341–366.
[62] T. Ya. Azizov, L. I. Sukhocheva, and V. A. Shtraus, Operators in Krein Space, Mathematical Notes, vol. 76, no. 3, 2004, pp. 306–314.

[63] T. Ya. Azizov and I. S. Iokhvidov, “Linear operators in spaces with indefinite metric and their applications,” in: Mathematical Analysis [in Russian], vol. 17, Itogi Nauki i Tekhniki [Progress in Science and Technology], Vsesoyuz. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, 1979, pp. 113–205.
[64] T. Ya. Azizov and I. S. Iokhvidov, Foundations of Theory of Linear Operators in Spaces with Indefinite Metric [in Russian], Nauka, Moscow, 1986.
[65] L. I. Sukhocheva, “On properties of operators of Helton’s class,” Vestnik Voronezh Gos. Univ. Ser. Fiz. Matem. (2001), no. 1, 120–123.
[66] V. A. Shtraus, “Structure of operators doubly commuting with an operator of class K(H) ,” Ukrain. Mat. Zh. [Ukrainian Math. J.], 38 (1986), no. 6, 805.
[67] V. A. Shtraus, “Structure of a family of commuting J-self-adjoint operators,” Ukrain. Mat. Zh. [Ukrainian Math. J.], 41 (1989), no. 10, 1431–1433.
[68] H. Langer, “Spectral functions of definitizable operators in Krein spaces,” in: Lecture Notes in Math., vol. 948, Springer-Verlag, Berlin–Heidelberg–New York, 1982, pp. 1–46.
[69] T. Ya. Azizov and V. A. Shtraus (Strauss), “Spectral decompositions for classes of self-adjoint and normal operators on Krein spaces,” in: Spectral Theory and Its Appl. Proceedings. Theta, 2003, pp. 45–67.
[70] P. Jonas and H. Langer, “Compact perturbations of definitizable operators,” J. Operator Theory, 2 (1979), 63–77.
[71] N. I. Akhiezer and I. M. Glazman, Theory of Linear Operators in Hilbert Space [in Russian], vol. 1, Vichsha Shkola, Kharkov, 1977.
[72] Fedor Nazarov a, Vladimir Peller, Functions of perturbed tuples of self-adjoint operators, *F. Nazarov, V.* Peller / C. R. Acad. Sci. Paris, Ser. I 350 (2012) 349–354.
[73] A.B. Aleksandrov, V.V. Peller, Functions of perturbed operators, C. R. Acad. Sci. Paris, Ser. I 347 (2009) 483–488.
[74] A.B. Aleksandrov, V.V. Peller, Operator H○lder–Zygmund functions, Adv. Math. 224 (2010) 910–966.
[75] A.B. Aleksandrov, V.V. Peller, Functions of operators under perturbations of class $\boldsymbol{S}\, p$, J. Funct. Anal. 258 (2010) 3675–3724.
[76] A.B. Aleksandrov, V.V. Peller, Functions of perturbed unbounded self-adjoint operators. Operator Bernstein type inequalities, Indiana Univ. Math. J. 59 (4) (2010) 1451–1490.
[77] A.B. Aleksandrov, V.V. Peller, D. Potapov, F. Sukochev, Functions of perturbed normal operators, C. R. Acad. Sci. Paris, Ser I 348 (2010) 553–558.
[78] A.B. Aleksandrov, V.V. Peller, D. Potapov, F. Sukochev, Functions of normal operators under perturbations, Adv. Math. 226 (2011) 5216–5251.
[79] M.S. Birman, M.Z. Solomyak, Double Stieltjes operator integrals, in: Problems of Math. Phys., vol. 1, Leningrad. Univ., 1966, pp. 33–67 (in Russian);
English transl.: in: Top. Math. Phys., vol. 1, Consultants Bureau Plenum Publishing Corporation, New York, 1967, pp. 25–54.
[80] M.S. Birman, M.Z. Solomyak, Double Stieltjes operator integrals. II, in: Problems of Math. Phys., vol. 2, Leningrad. Univ., 1967, pp. 26–60 (in Russian);
English transl.: in: Top. Math. Phys., vol. 2, Consultants Bureau Plenum Publishing Corporation, New York, 1968, pp. 19–46.
[81] M.S. Birman, M.Z. Solomyak, Double Stieltjes operator integrals. III, in: Problems of Math. Phys., vol. 6, Leningrad. Univ., 1973, pp. 27–53 (in Russian).
[82] Yu.B. Farforovskaya, The connection of the Kantorovich–Rubinshtein metric for spectral resolutions of selfadjoint operators with functions of operators, Vestn. Leningr. Univ. 19 (1968) 94–97 (in Russian).
[83] E. Kissin, V.S. Shulman, Operator smoothness in Schatten norms for functions of several variables: Lipschitz conditions, differentiability and unbounded derivations, in press.
[84] J. Peetre, New Thoughts on Besov Spaces, Duke Univ. Press, Durham, NC, 1976.
[85] V.V. Peller, Hankel operators in the theory of perturbations of unitary and self-adjoint operators, Funktsional. Anal. i Prilozhen. 19 (2) (1985) 37–51 (in Russian); English transl.: Funct. Anal. Appl. 19 (1985) 111–123.
[86] V.V. Peller, Hankel operators in the perturbation theory of unbounded self-adjoint operators, in: Analysis and Partial Differential Equations, in: Lect. Notes Pure Appl. Math., vol. 122, Dekker, New York, 1990, pp. 529–544.
[87] M.S. Birman, M.Z. Solomyak, Double Stieltjes operator integrals. II, Problems of Math. Phys. Leningrad. Univ. 2 (1967) 26–60 (in Russian). English transl.: Topics Math. Physics, vol. 2, Consultants Bureau Plenum Publishing Corporation, New York, 1968, pp. 19–46.

[88] Yu.B. Farforovskaya, L. Nikolskaya, Operator Holderness of Hölder functions, Algebra i Analiz. (2010), in press (in Russian).
[90] Yu.B. Farforovskaya, An estimate of the norm of $|f\ (B)\ -\ f\ (A)|$ for selfadjoint operators A and B, Zap. Nauchn. Sem. Leningrad. Otdel.
Mat. Inst. Steklov. (LOMI) 56 (1976) 143–162 (Russian).
[91] Yu.B. Farforovskaya, L. Nikolskaya, Modulus of continuity of operator functions, Algebra i Analiz 20 (3) (2008) 224–242.
[92] T. Kato, Continuity of the map $S\ \mapsto\ |S|$ for linear operators, Proc. Japan Acad. 49 (1973) 157–160.
[93] V.V. Peller, Multiple operator integrals and higher operator derivatives, J. Funct. Anal. 233 (2006) 515–544.
[94] Sergii Kuzhel , Carsten Trunk, On a class of J -self-adjoint operators with empty resolvent set, S. Kuzhel, C. Trunk / J. Math. Anal. Appl. 379 (2011) 272–289.
[95] S. Albeverio, F. Gesztesy, R. Hőegh-Krohn, H. Holden, Solvable Models, Quantum Mechanics, Springer-Verlag, Berlin, Heidelberg, New York, 1988; second ed. with an Appendix by P. Exner, Amer. Math. Soc., Chelsea Publishing, Providence, 2005.
[96] S. Albeverio, U. Günther, S. Kuzhel, J -Self-adjoint operators with C-symmetries: Extension theory approach, J. Phys. A 42 (2009) 105205–105227.
[97] S. Albeverio, S. Kuzhel, Pseudo-Hermiticity and theory of singular perturbations, Lett. Math. Phys. 67 (2004) 223–238.
[98] T.Ya. Azizov, I.S. Iokhvidov, Linear Operators in Spaces with an Indefinite Metric, John Wiley & Sons, Chichester, 1989.
[99] T.Ya. Azizov, J. Behrndt, F. Philipp, C. Trunk, On domains of powers of linear operators and finite rank perturbations, in: Spectral Theory in Inner
Product Spaces and Applications, in: Oper. Theory Adv. Appl., vol. 188, Birkhںuser, 2009, pp. 31–36.
[100] T.Ya. Azizov, J. Behrndt, C. Trunk, On finite rank perturbations of definitizable operators, J. Math. Anal. Appl. 339 (2008) 1161–1168.
[101] J. Behrndt, On the spectral theory of singular indefinite Sturm–Liouville operators, J. Math. Anal. Appl. 334 (2007) 1439–1449.
[102] J. Behrndt, F. Philipp, Spectral analysis of singular ordinary differential operators with indefinite weights, J. Differential Equations 248 (2010) 2015– 2037.
[103] C.M. Bender, Making sense of non-Hermitian Hamiltonians, Rep. Progr. Phys. 70 (2007) 947–1018.
[104] C.M. Bender, S.P. Klevansky, Nonunique C operator in PT quantum mechanics, Phys. Lett. A 373 (2009) 2670–2674.
[105] J. Bognar, Indefinite Inner Product Spaces, Springer-Verlag, Berlin, 1974.
[106] B. C´ urgus, H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, J. Differential Equations 79 (1989) 31–61.
[107] V.A. Derkach, M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. 95 (1991) 1–95.
[108] W.N. Everitt, On a property of the m-coefficient of a second-order linear differential equation, J. London Math. Soc. 4 (1971/1972) 443–457.
[109] M.L. Gorbachuk, V.I. Gorbachuk, M.G. Krein's Lectures on Entire Operators, Oper. Theory Adv. Appl., vol. 97, Birkhںuser, 1997.
[110] M.L. Gorbachuk, V.I. Gorbachuk, A.N. Kochubei, Theory of extensions of symmetric operators and boundary-value problems for differential equations, Ukrain. Mat. Zh. 41 (1989) 1299–1313.
[111] U. Günther, S. Kuzhel, PT -symmetry, Cartan decompositions, Lie triple systems and Krein space-related Clifford algebras, J. Phys. A 43 (2010) 392002, 10 pp.
[112] U. Günther, I. Rotter, B. Samsonov, Projective Hilbert space structures at exceptional points, J. Phys. A 40 (2007) 8815–8833.
[113] P. Jonas, On a class of selfadjoint operators in Krein spaces and their compact perturbations, Integral Equations Operator Theory 11 (1988) 351–384.
[114] S. Hassi, S. Kuzhel, On J -self-adjoint operators with stable symmetry, arXiv:1101.0046v1 [math-FA], 2010.
[115] I.M. Karabash, M.M. Malamud, Indefinite Sturm–Liouville operators (sgn x(− d2 dx2+q(x))) with finite-zone potentials, Oper. Matrices 1 (2007) 301–368.
[116] I. Karabash, C. Trunk, Spectral properties of singular Sturm–Liouville operators with indefinite weight sgn x, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009) 483–503.

[117] A. Kuzhel, S. Kuzhel, Regular Extensions of Hermitian Operators, VSP, Utrecht, 1998.
[118] S. Kuzhel, O. Shapovalova, L. Vavrykovich, On J -self-adjoint extensions of the Phillips symmetric operator, Methods Funct. Anal. Topology 16 (2010) 333–348.
[119] A.N. Kochubei, On extensions and characteristic functions of symmetric operators, Izv. Akad. Nauk Armen. SSR 15 (1980) 219–232 (in Russian); English translation: Sov. J. Contemp. Math. Anal. 15 (1980).
[120] A.N. Kochubei, About symmetric operators commuting with a family of unitary operators, Funktsional. Anal. i Prilozhen. 13 (1979) 77–78.
[121] H. Langer, Spectral functions of definitizable operators in Krein spaces, in: Functional Analysis Proceedings of a Conference Held at Dubrovnik, Yugoslavia, November 2–14, 1981, in: Lecture Notes in Math., vol. 948, Springer-Verlag, Berlin, Heidelberg, New York, 1982, pp. 1–46.
[122] A.B. Mingarelli, Characterizing degenerate Sturm–Liouville problems, Electron. J. Differential Equations 130 (2004) 1–8.
[123] A. Mostafazadeh, Pseudo-Hermitian quantum mechanics, arXiv:0810.5643 [quant-ph].
[124] A. Mostafazadeh, Krein-space formulation of PT-symmetry, CPT-inner products, and pseudo-hermiticity, Czechoslovak J. Phys. 56 (2006) 919–933.
[125] M.A. Naimark, Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space, Frederick Ungar Publishing Co., New York, 1968.
[126] A.V. Straus, On extensions and characteristic function of symmetric operator, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968) 186–207 (in Russian); English translation: Math. USSR Izv. 2 (1968) 181–204.

[127] Friedrich Philipp a, Franciszek Hugon Szafraniec b, Carsten Trunk, Selfadjoint operators in S-spaces, Journal of Functional Analysis 260 (2011) 1045–1059.

[128] T.Ya. Azizov, About compact operators which are selfadjoint with respect to a degenerated indefinite inner product, in: Mat. Issled., vol. 26, 1972, pp. 237–240, Kishinyov, UII:4 (in Russian).
[129] T.Ya. Azizov, L.I. Soukhotcheva, Linear operators in almost Krein spaces, in: Oper. Theory Adv. Appl., vol. 175, 2007, pp. 1–11.
[130] P. Binding, R. Hryniv, Full and partial range completeness, in: Oper. Theory Adv. Appl., vol. 130, 2002, pp. 121– 133.
[131] V. Bolotnikov, C.-K. Li, P. Meade, C. Mehl, L. Rodman, Shells of matrices in indefinite inner product spaces, Electron. J. Linear Algebra 9 (2002) 67–92.
[132] N. Dunford, J.T. Schwartz, Linear Operators. Part II: Spectral Theory, John Wiley & Sons, 1963.
[133] M.-A. Henn, C. Mehl, C. Trunk, Hyponormal and strongly hyponormal matrices in inner product spaces, LinearAlgebra Appl. 433 (2010) 1055–1076.
[134] M. Kaltenbäck, H. Winkler, H. Woracek, Almost Pontryagin spaces, in: Oper. Theory Adv. Appl., vol. 160, 2005, pp. 253–371.
[135] M. Kaltenbäck, H. Woracek, Selfadjoint extensions of symmetric operators in degenerated inner product spaces, Integral Equations Operator Theory 28 (1997) 289–320.
[136] M. Kaltenbäck, H. Woracek, On extensions of Hermitian functions with a finite number of negative squares, J. Operator Theory 40 (1998) 147–183.
[137] M. Kaltenbäck, H. Woracek, The Krein formula for generalized resolvents in degenerated inner product spaces, Monatsh. Math. 127 (1999) 119–140.
[138] T. Kato, Perturbation Theory for Linear Operators, second edition, Springer, 1976.
[139] M.G. Krein, Introduction to the theory of indefinite J -spaces and to the theory of operators in those spaces, in: Amer. Math. Soc. Transl., vol. 93, 1970, pp. 103–176.
[140] P. Lancaster, A.S. Markus, P. Zizler, The order of neutrality for linear operators on inner product spaces, Linear Algebra Appl. 259 (1997) 25–29.
[141] H. Langer, A.S. Markus, V.I. Matsaev, Locally definite operators in indefinite inner product spaces, Math. Ann. 308 (1997) 405–424.
[142] H. Langer, R. Mennicken, C. Tretter, A self-adjoint linear pencil $Q-\lambda P$ of ordinary differential operators,Methods Funct. Anal. Topology 2 (1996) 38–54.
[143] C. Mehl, A.C.M. Ran, L. Rodman, Semidefinite invariant subspaces: degenerate inner products, in: Oper. Theory Adv. Appl., vol. 149, 2004, pp. 467–486.
[144] C. Mehl, L. Rodman, Symmetric matrices with respect to sesquilinear forms, Linear Algebra Appl. 349 (2002) 55–75.
[145] C. Mehl, C. Trunk, Normal matrices in inner product spaces, in: Oper. Theory Adv. Appl., vol. 175, 2007, pp. 193– 209.

[146] F. Philipp, C. Trunk, G-selfadjoint operators in almost Pontryagin spaces, in: Oper. Theory Adv. Appl., vol. 188, 2008, pp. 207–235.
[147] B.C. Ritsner, The Theory of Linear Relations, No. 846-82, Dep. VINITI, Voronezh, 1982 (in Russian).
[148] F.H. Szafraniec, Two-sided weighted shifts are almost Krein normal, in: Oper. Theory Adv. Appl., vol. 188, 2008, pp. 245–250.
[149] F.H. Szafraniec, Dissymmetrising inner product spaces, manuscript, in preparation.
[150] J. Weidmann, Linear Operators in Hilbert Spaces, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
[151] H.Woracek, An operator theoretic approach to degenerated Nevanlinna–Pick interpolation, Math. Nachr. 176 (1995) 335–350.
[152] H. Woracek, Resolvent matrices in degenerate inner product spaces, Math. Nachr. 213 (2000) 155–175.
[153] F.L. Nazarov, V.V. Peller, Lipschitz functions of perturbed operators, arXiv:0905.4855vl [math. FA] 29 May 2009.
[154] Yu.B. Farforovskaya, An example of a Lipschitzian function of selfadjoint operators that yields a nonnuclear increase under a nuclear perturbation. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 30 (1972), 146–153 (Russian).
[155] V.V.Peller Metric properties of the averaging projection onto the set of Hankel matrices, Dokl. Akad Nauk SSSR 278 (1984), 271-285. English Transl. in Soviet Math. Dokl., 30 (1984), 362-368.
[156] Tomas Ya. Azizov , Jussi Behrndt , Carsten Trunk, On finite rank perturbations of definitizable operators, J. Math. Anal. Appl. 339 (2008) 1161–1168.
[157] J. Behrndt, On the spectral theory of singular indefinite Sturm–Liouville operators, J. Math. Anal. Appl. 334 (2) (2007) 1439–1449.
[158] J. Behrndt, Finite rank perturbations of locally definitizable selfadjoint relations in Krein spaces, J. Operator Theory, in press.
[159] J. Behrndt, C. Trunk, Sturm–Liouville operators with indefinite weight functions and eigenvalue depending boundary conditions, J. Differential Equations 222 (2006) 297–324.
[160] J. Behrndt, C. Trunk, On the negative squares of indefinite Sturm–Liouville operators, J. Differential Equations 238 (2007) 491–519.
[161] P. Binding, R. Hryniv, H. Langer, B. Najman, Elliptic eigenvalue problems with eigenparameter dependent boundary conditions, J. Differential Equations 174 (2001) 30–54.
[162] B. C′ urgus, H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, J. Differential Equations 79 (1989) 31–61.
[163] A. Dijksma, H. Langer, Operator theory and ordinary differential operators, in: Lectures on Operator Theory and Its Applications, in: Fields Inst. Monogr., vol. 3, Amer. Math. Soc., 1996, pp. 79–139.
[164] A. Dijksma, H.S.V. de Snoo, Symmetric and selfadjoint relations in Krein spaces I, Oper. Theory Adv. Appl. 24 (1987) 145–166.
[165] A. Dijksma, H.S.V. de Snoo, Symmetric and selfadjoint relations in Krein spaces II, Ann. Acad. Sci. Fenn. Math. 12 (1987) 199–216.
[166] N. Dunford, J. Schwartz, Linear Operators Part I, General Theory, Interscience Publishers, New York, 1958.
[167] P. Jonas, On a class of selfadjoint operators in Krein space and their compact perturbations, Integral Equations Operator Theory 11 (1988) 351–384.
[168] P. Jonas, On locally definite operators in Krein spaces, in: Spectral Theory and Applications, Ion Colojoarˇa Anniversary Volume, Theta Foundation, 2003, pp. 95–127.
[169] P. Jonas, H. Langer, Compact perturbations of definitizable operators, J. Operator Theory 2 (1979) 63–77.
[170] P. Jonas, C. Trunk, On a class of analytic operator functions and their linearizations, Math. Nachr. 243 (2002) 92–133.
[171] I.M. Karabash, M.M. Malamud, Indefinite Sturm–Liouville operators $\left((sgn\,x)\frac{d^2}{dx^2} + q(x)\right)$ with finite-zone potentials, Oper. Matrices 1 (3) (2007) 301–368.
[172] I.M. Karabash, C. Trunk, Spectral properties of singular Sturm–Liouville operators with indefinite weight sgn x, submitted for publication.
[173] Q. Kong, H. Wu, A. Zettl, M. Möller, Indefinite Sturm–Liouville problems, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 639–652.
[174] M.G. Krein, The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications, I–II, Mat. Sb. (N.S.) (Rec. Math.) 20 (62) (1947) 431–495; Mat. Sb. (N.S.) 21 (63) (1947) 365–404 (in Russian).

[175] P. Lancaster, A. Markus, V. Matsaev, Definitizable operators and quasihyperbolic operator polynomials, J. Funct. Anal. 131 (1995) 104–129.
[176] H. Langer, Spektraltheorie linearer Operatoren in J-Räumen und einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 + \lambda B + C$, Habilitationsschrift, Technische Universität Dresden, 1965.
[177] H. Langer, Spectral functions of definitizable operators in Krein spaces, in: Functional Analysis Proceedings of a Conference, Dubrovnik, Yugoslavia, November 2–14, 1981, in: Lecture Notes in Math., vol. 948, Springer-Verlag, Berlin, 1982, pp. 1–46.
[178] H. Langer, A. Markus, V. Matsaev, Locally definite operators in indefinite inner product spaces, Math. Ann. 308 (1997) 405–424.
[179] H. Langer, R. Mennicken, M. Möller, A second order differential operator depending nonlinearly on the eigenvalue parameter, Oper. Theory Adv. Appl. 48 (1990) 319–332.
[180] H. Langer, B. Najman, C. Tretter, Spectral theory for the Klein–Gordon equation in Krein spaces, Proc. Edinb. Math. Soc., in press.
[181] M.A. Naimark, Linear Differential Operators, Part II: Linear Differential Operators in Hilbert Space, Frederick Ungar Publishing Co., New York, 1968.
[182] A. Sandovici, Some basic properties of polynomials in a linear relation in linear spaces, Oper. Theory Adv. Appl. 175 (2007) 231–240.
[183] T. Ya. Azizov, M. Denisov, and F. Philipp, Spectral functions of products of selfadjoint operators, Math. Nachr., 1 – 18 (2012) / DOI 10.1002/mana.201200169.
[184] T. Ya. Azizov, J. Behrndt, and C. Trunk, On finite rank perturbations of definitizable operators, J. Math. Anal. Appl. 339(2), 1161–1168 (2008).
[185] J. Behrndt and F. Philipp, Spectral analysis of singular ordinary differential operators with indefinite weights, J. Differ. Equations 248, 2015–2037 (2010).
[186] J. B. Conway, A Course in Functional Analysis (Springer, New York, 1985).
[187] B. C´ urgus and H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, J. Differ. Equations 79, 31–61 (1989).
[188] M. Denisov, The spectral function for some products of self-adjoint operators,Matematicheskie Zametki 81(6), 948–951 (2007).
[189] A. Dijksma and H. Langer, Operator theory and ordinary differential operators, in: Lectures on Operator Theory and its Applications, Fields Institute for Research in Mathematical Sciences, edited by B¨ottcher, Providence, RI: American Mathematical Society, Fields Institute Monographs, Albrecht et al. (Toronto, Canada 1994) Vol. 3 (1996), pp. 75–139.
[190] A. Dijksma and H. S. V. de Snoo, Symmetric and selfadjoint relations in Krein spaces I, Oper. Theory Adv. Appl. 24, 145–166 (1987).
[191] N. Dunford and J. T. Schwartz, Linear Operators Part I: General Theory (Wiley, New York, 1958).
[192] V. Hardt, A. Konstantinov, and R. Mennicken, On the spectrum of the product of closed operators, Math. Nachr. 215, 91–102 (2000).
[193] V. Hardt and R.Mennicken, On the spectrum of unbounded off-diagonal 2×2 operator matrices in Banach spaces, Oper. Theory Adv. Appl. 124, 243–266 (2001).
[194] P. Jonas, On the functional calculus and the spectral function for definitizable operators in Krein space, Beitr¨age zur Analysis 16, 121–135 (1981).
[195] P. Jonas and H. Langer, Compact perturbations of definitizable operators, J. Oper. Theory 2, 63–77 (1979).
[196] P. Jonas and C. Trunk, On a class of analytic operator functions and their linearizations, Math. Nachr. 243, 92–133 (2002).
[197] P. Jonas and C. Trunk, A Sturm-Liouville problem depending rationally on the eigenvalue parameter, Math. Nachr. 280(15), 1709–1726 (2007).
[198] I. M. Karabash and M. M. Malamud, Indefinite Sturm-Liouville operators $\left((sgn\, x)\frac{d^2}{dx^2} + q(x)\right)$ with finite-zone potentials, Oper. Matrices 1, 301–368 (2007).
[199] I. M. Karabash and C. Trunk, Spectral properties of singular Sturm-Liouville operators, Proc. R. Soc. Edinb. Sect. A 139, 483–503 (2009).
[200] T. Kato, Perturbation Theory for Linear Operators, second edition (Springer, 1976).
[201] Q. Kong, H. Wu and A. Zettl, Singular left-definite Sturm-Liouville problems, J. Differ. Equations 206, 1–29 (2004).
[202] H. Langer, R. Mennicken, and M. M¨oller, A second order differential operator depending nonlinearly on the eigenvalue parameter, Oper. Theory Adv. Appl. 48, 319–332 (1990).

[203] F. Philipp, A. C. M. Ran, and M. Wojtylak, Local Definitizability of $T^{[*]}T$ and $TT^{[*]}$, Integral Equations Oper. Theory 71, 491–508 (2011).
[204] Heinz Langer, Alexander Markus and Vladimir Matsaev, Self-adjoint Analytic Operator Functions:Local Spectral Function and Inner Linearization, Integr. equ. oper. theory 63 (2009), 533–545.
[205] I. Gohberg, M.A. Kaashoek, D.C. Lay: Equivalence, linearization and decomposition of holomorphic operator functions. J. Funct. Anal. 28 (1978), 102–144.
[206] M.A. Kaashoek, C.V.M. van der Mee, L. Rodman: Analytic operator functions with compact spectrum, I. Spectral nodes, linearization and equivalence. Integral Equations Operator Theory 4 (1981), 504–547.
[207] H. Langer, A. Markus, V. Matsaev: Linearization and compact perturbation of self– adjoint analytic operator functions. Oper. Theory: Adv. Appl. 118 (2000), 255–285.
[208] H. Langer, A. Markus, V. Matsaev: Self-adjoint analytic operator functions and theirlocal spectral function. J. Funct. Anal. 235 (2006), 193–225.
[209] H. Langer, M. Langer, A. Markus, C. Tretter: The Virozub–Matsaev condition and spectrum of definite type for self-adjoint operator functions. Compl. Anal. Oper. Theory 2 (2008), 99–134.
[210] A.S. Markus: Introduction to the Spectral Theory of Polynomial Operator Pencils. AMS Translations of Mathematical Monographs, vol. 71, 1988.
[211] A.I. Virozub, V.I. Matsaev: The spectral properties of a certain class of selfadjoint operator–valued functions. Funct. Anal. Appl. 8 (1974), 1–9.
[212] Showgy Hussen AbdAlla and Yousif Abdelatuif Mohammed, Function of Perturbed Tuples of Selfadjoint Operators and the Locally Normal Operators in Krein Spaces, Sudan University of Scince and Technology, Sudan, (2015).