Dedication

To the my father , my beloved mother, wife who supported me a lot through the years.

Acknowledgments

In the name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the world; and prayers and peace be upon Mohamed his servant and messenger.

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Abstract

We give the structure of the selfadjoint analytic operator functions and in S-spaces. We also give the characterizations of the spectral functions of products of selfadjoint operators and on a class of J-selfadjoint operators with empty resolvent set. We show the Lipschitz functions of perturbed and finite rank perturbations of operators and definitizable operators. Functions of perturbed normal and tuples of selfadjoint and normal operators are studied. Operators, frames, local spectral for definite normal operators in Krein spaces are considered.

الخلاصة

تم إعطاء بناء لدوال المؤثر التحليلي المرافق الذاتي وفي فضاءات – **S**. أيضاً أعطينا تشخيصات الدوال الطيفية لضرب مؤثرات المرافق الذاتي وعلى عائلة مؤثرات المرافق الذاتي – **J** مع الفئة المذابة الخالية. أوضحنا دوال لبشتز لمؤثرات الإرتجاج وإرتجاجات الرتبة المنتهية والمؤثرات القابلة للتعريف. درست الدوال الناظمة الإرتجاج والمرقمات لمؤثرات المرافق الذاتي والناظمة. تم إعتبار المؤثرات والإطارات والطيف الموضعي للمؤثرات الناظمة المحددة في فضاءات كراين.

The Contents

Introduction

We begin continue the study of spectral properties of a selfadjoint analytic operator function $A(z)$ that was started in [5]. It is shown that if $A(z)$ satisfies the Virozub–Matsaev condition on some interval Δ_0 and is boundedly invertible in the endpoints of Δ_0, then the 'embedding' of the original Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{F}$, where the linearization of $A(z)$ acts, is in fact an isomorphism between a subspace $\mathcal{H}(\Delta_0)$ of $\mathcal{H}$ and $\mathcal{F}$. We show that if f is a Lipschitz function on $\mathbb{R}$, A and B are selfadjoint operators such that $rank(A - B) = 1$, then $f(A) - f(B)$ belongs to the weak space $S_{1,\infty}, i.e., s_j(A - B) \leq const(1 + j)^{-1}$. We deduce from this result that if $A - B$ belongs to the trace class S_1 and f is Lipschitz, then $f(A) - f(B) \in S_{\Omega}, i.e., \sum_{j=0}^{n} s_j\left(f(A) - f(B)\right) \leq const\ log(2 + n)$. It was shown by P. Jonas and H. Langer that a selfadjoint definitizable operator A in a Krein space remains definitizable after a finite rank perturbation in resolvent sense if the perturbed operator B is selfadjoint and the resolvent set $\rho(B)$ is nonempty. We study S-spaces and operators therein. An S-space is a Hilbert space $(S, (\cdot, -))$ with an additional inner product given by $[\cdot, -] := (U \cdot, -)$, where U is a unitary operator in $(S, (\cdot, -))$. We investigate spectral properties of selfadjoint operators in S-spaces. We show that their spectrum is symmetric with respect to the real axis. We investigate the set Σ_J of all J-selfadjoint extensions of an operator S which is symmetric in a Hilbert space $\mathfrak{S}$ with deficiency indices $\langle 2, 2\rangle$ and which commutes with a non-trivial fundamental symmetry J of a Krein space $(\mathfrak{S}, [\cdot,\cdot])$,

$$SJ = JS.$$

Our aim is to describe different types of J-selfadjoint extensions of S, which, in general, are non-selfadjoint operators in the Hilbert space $\mathfrak{S}$. One of our main results is the equivalence between the presence of J-selfadjoint extensions of S with empty resolvent set and the commutation of S with a Clifford algebra $Cl_2(J, R)$, where R is an additional fundamental symmetry with $JR = -RJ$. This enables one to parameterize in terms of $Cl_2(J, R)$, the set of all J-selfadjoint extensions of S with stable C-symmetry. We show that if $0 < \alpha < 1$ and f is in the Hölder class $\Lambda_\alpha(\mathbb{R})$, then for arbitrary selfadjoint operators A and B with bounded $A - B$, the operator $f(A) - f(B)$ is bounded and $\|f(A) - f(B)\| \leq const\ \|A - B\|^{\alpha}$. We show a similar result for functions f of the Zygmund class $\Lambda_1(\mathbb{R})$: $\|f(A + K) - 2f(A) + f(A - K)\| \leq const\ \|K\|$, where A and K are selfadjoint operators. Similar results also hold for all Hölder–Zygmund classes $\Lambda_\alpha(\mathbb{R}), \alpha > 0$. Sharp estimates for $f(A) - f(B)$ were obtained for selfadjoint operators A and B and for various classes of functions f on the real line R. We extend those results to the case of functions of normal operators. We show that if f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R}^2), 0 < \alpha < 1$, of functions of two variables, and N_1 and N_2 are normal operators, then $\|f(N_1) - f(N_2)\| \leq const\ \|f\|_{\Lambda_\alpha} \|N_1 - N_2\|^{\alpha}$. We obtain a more general result for functions in the space $\Lambda_\omega(\mathbb{R}^2) = \{f : |f(\zeta_1) - f(\zeta_2)| \leq const\omega(|\zeta_1 - \zeta_2|)\}$ for an arbitrary modulus of continuity ω. We generalize earlier results of Aleksandrov and Peller, Aleksandrov et al. To the case of functions of n-tuples of commuting selfadjoint operators. A definition of frames for Krein spaces is proposed, which extends the notion of J-orthonormal bases of Krein spaces. A J-frame for a Krein space $(\mathcal{H}, [\,,\,])$ is in particular a frame for $\mathcal{H}$ in the Hilbert space sense. But it is also compatible with the indefinite inner product $[\,,\,]$, meaning that it determines a pair of maximal uniformly J-definite subspaces, an analogue to the maximal dual pair associated to a J-orthonormal basis. Sign type spectra are an important tool in the investigation of spectral properties of selfadjoint operators in Krein spaces. We show that also sign type spectra for normal operators in Krein spaces provide insight in the spectral nature of the operator: If the real part and the imaginary part of a normal operator in a Krein space have real spectra only and if the growth of the resolvent of the imaginary part (close to the real axis) is of finite order, then the normal operator possesses a local spectral function defined for Borel subsets of the spectrum which belong to positive (negative) type spectrum. We introduce the spectral points of two-sided positive type of bounded normal operators in Krein spaces. It is shown that a normal operator has a local spectral function on sets which are of two-sided positive type.