# Chapter 5

## Applications of Lie algebras

### 5.1 Symmetry of differential equation

### Introduction

Several real – world models are formulated in terms of differential equations (PDE's). quite often these models involve parameters, arbitrary elements or functions which may not be straightforwardly determined through experiment. The symmetry principle which presumes that nature prefers maximally symmetric models has proven to be a powerful tool in determining these unknown parameters appearing in physical models. For instance Newton's inverse square law can be obtained solely from the symmetry principle. Once the unknown parameters of a model are determined the next logical step is to find its solutions given a set of initial or boundary condition.

In this chapter we will use Lie algebraic techniques to solve the PDE, by answering these questions:

How do we find the symmetry Lie algebras ?

How do we use the symmetry Lie algebras to find the solution to the D.E's ?

**Definition 5.1.1. (Symmetry of a System of Differential Equations )**

A $k$th-order $(k \geq 1)$ system $E$ of $s$ differential equations is defined by [1]

$$E^{\sigma}\left(x, u, u_{(1)}, \ldots\ldots, u_{(k)}\right) = 0 \ , \qquad \sigma = 1, \ldots., s \qquad (5.1)$$

where $u \equiv (u^1, u^2, \ldots\ldots, u^m)$ is dependent vector, $x \equiv (x^1, x^2, \ldots\ldots, x^n)$ is the independent vector and $u_{(1)}, \ldots\ldots, u_{(k)}$ are respectively the collection of all first, second, up to $k$th-order partial derivatives. In expanded form

$$u_{(1)} = \{u_i^{\alpha}\}, u_{(2)} = \{u_{ij}^{\alpha}\}, \ldots\ldots., \{u_{i_1,\ldots\ldots i_k}^{\alpha}\}$$

Where $\alpha = 1, \ldots\ldots, m; \quad i, j, i_1, \ldots\ldots, i_k = 1, \ldots\ldots, n.$

[1] Group Classification Of Coupled Partial Differential Equations With Applications to Flow in a Collapsible Channel and Diffusion Processes - Motlatsi Molat - A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy - Johannesburg 2010

The equations appearing in Eq(**5.1**) are of maximal order $k$. In many applications $s = m$, i.e., the number of equations is equal to the number of unknowns.

Note that the system Eq(**5.1**) becomes a system of ODE's if $n = 1$. Otherwise it is a system of PDF's

**Definition 5.1.2.**

A symmetry transformation of the system Eq(**5.1**) is an invertible transformation of the variables $x$ and $u$, namely

$$\bar{x}^i = f^i(x,u), \bar{u}^\alpha = \phi^\alpha(x,u),\ i = 1, \dots, n;\ \alpha = 1, \dots, m \qquad (5.2)$$

That leaves (1.1) form-invariant in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E^\sigma\left(\bar{x}, \bar{u}, \bar{u}_{(1)}, \dots, \bar{u}_{(k)}\right) = 0\ , \qquad \sigma = 1, \dots, s \qquad (5.3)$$

Whenever Eq(**5.1**) is satisfied.

**Example 5.1.1.**

The heat equation $u_t - u_{xx} = 0$ . is invariant under the transformations $\bar{t} = t,\ \bar{x} = x,$ $\bar{u} = au$ for $a \in \mathbb{R}^+$.

The examples of transformations illustrated above are easy to deduce from the corresponding equations, but in general the calculation of symmetry transformations Eq(**5.2**) admitted by the system Eq(**5.1**) leads to non linear equations which are not easy to solve.

However, if we consider symmetries that depend on a small parameter and that form a one – parameter group of transformations, we can 'linearize' these equations and easily solve them : this is an important discovery made by Sophus Lie . The transformation groups can be either local or global, they can be of continuous, discontinuous and mixed type.

### 5.1.3. Canonical Coordinates

If we take simplest first order differential equation to deal with has the form [2]

$$\frac{dy}{dx} = g(x) \qquad \dots\dots\dots\dots (5.4)$$

The solution of this equation is trivial

[2] Lie Groups and Differential Equations -

$$y = G(x) = \int g(x)dx(+\text{additive constant}) = G(x) + c \qquad \dots\dots\dots (5.5)$$

If we can write the solution on form $y - G(x) = 0$,the surface $y + c - G(x) = 0$ is also solution of Eq.(**5.4**).There is a one-parameter group of displacements that maps one solution into another. These displacements can be represented by the Taylor series displacement operator $e^{c\frac{\partial}{\partial y}}$ , for

$$e^{c\frac{\partial}{\partial y}} = [y - G(x) = 0] = y + c - G(x) = 0 \quad \dots\dots.(5.6)$$

We can express the derivative $\frac{dy}{dx}$ as a coordinate $p$. The first O.D.E can be written as form $F(x, y, p) = 0$, where $F(x, y, p) = p - g(x)$. There are two relations among the three variables $x, y, p$. They are given by the surface equation and the constraint equation **:**

**i. Surface equation :** $F(x, y, p) = 0$

**ii. Constraint equation :** $p = dy/dx$ when $F(x, y, p) = 0$ .

These two relations are summarized as follows:

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial p \end{bmatrix} = [p - g(x)] = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix} \quad , \quad \partial/\partial y \begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \dots\dots..(5.7)$$

These two equations will be generalized to determining equation of infinitesimal generator of invariance group and the determining equations for the canonical coordinates.

### 5.1.4. Determining equation

The surface equation must be unchanged under the one-parameter group of transformations, so that

$$F(x, y, p) = 0 \to F\big(\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{p}(\epsilon)\big) \xrightarrow{\epsilon \text{ small}} F(x + \epsilon\xi + y + \epsilon\eta + p + \epsilon\zeta) =$$

$$F(x, y, p) + \epsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p}\right)F(x, y, p) + h.o.t \qquad (5.8)$$

These are leading two term of Taylor series expansion

$$F\big(\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{p}(\epsilon)\big) = e^{\epsilon X}F(x, y, p) \dots\dots\dots (5.9)$$

Where the generator of infinitesimal displacements for the one parameter group that leaves the surface equation invariant is

$$X = \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p}\right) \qquad (5.10)$$

The first two terms in Eq.(**5.8**) and Eq(**5.9**) are

$$F(x, y, p) = 0 \quad , XF(x, y, p) = 0 \qquad (5.11)$$

There are called the determining equations which generalized from Eq.(**5.7**) Specifically, these equations are used to determine the functions $\xi(x, y)$, $\eta(x, y)$and $\zeta(x, y, p)$ that defined the infinitesimal generator $X$.

If an infinitesimal generator $X$ can be constructed from the determining equations, then it is possible to determine a new system of coordinates $R, S, T$ which "straightens out" the surface equation. This is done by solving the determining equations for canonical coordinates. These are a set of P.D.E's that are analogues to the equations to the right hand side of the Eq(**5.7**). for convenience, we summarize the determining equation for the infinitesimal generator and for the canonical coordinates, analogs of the two equations on Eq.(**5.7**), as follows :

$$XF = 0 , \qquad X \begin{bmatrix} R(x, y) \\ S(x, y) \\ T(x, y, p) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (5.12)$$

The three linear partial differential equations on the right determine the new canonical coordinates: the independent variable $R(x, y)$, the dependent variable $S(x, y)$, and the new constraint $T(x, y, p)$ between $R$ and $S$.

### 5.1.5. Dependent Coordinate

The dependent coordinate $S$ is determined from the differential equation $X(x, y, p)S(x, y) = 1$. We require $S$ to be independent of $p$, so the condition defining $S$ reduces to

$$\left(\xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}\right)S(x, y) = 1 \qquad (5.13)$$

The solution is not unique: Any function of $x$ and $y$ that is annihilated by $X$ can be added to the solution. Further, it is not important that $XS = +1$: we could just as well choose a solution satisfying $XS = -1$ or, for that matter, $XS = k \neq 0$, where $k$ is some constant.

### 5.1.6 Invariant Coordinates

**i. Independent Variable**

The two invariant coordinates $R$ and $T$ are unchanged under the one- parameter transformation group. These functions obey $XR = 0$ and $XT = 0$, which are explicitly

$$\left(\xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}\right)R(x, y) = 0 \qquad (5.14)$$

$$\left(\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}+\zeta(x,y)\frac{\partial}{\partial p}\right)T(x,y,p)=0 \qquad (5.15)$$

The solutions are most simply found by the method of characteristics. They obey the differential relations

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)}=\frac{dp}{\zeta(x,y,p)} \qquad (5.16)$$

The first equation is used to construct $R(x,y)$.

**ii. Constraint Variable**

The second equation in Eq(**5.16**) is used to construct $T(x,y,p)$. It is often possible to construct $T$ so that it is a function of p to the first power. When this is possible, it is the preferred form of the non unique expression for the invariant coordinate $T$.

**Definition 5.1.7. ( Invariants)**

A point $(x,u)\in\mathbb{R}^{n+m}$ [3]is an invariant point if it remains unchanged by every transformation of a group $G$, i.e., $(\bar{x},\bar{u})=(x,u)$ , $\forall a\in\mathcal{D}'\subset\mathcal{D}$ .

**Theorem 5.1.1.**

A point $(x,u)\in\mathbb{R}^{n+m}$ is an invariant point of a group $G$ with generator

$$X=\xi^i(x,u)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

if and only if $\xi^i(x,u)=\eta^\alpha(x,u)=0$.

**i. Invariant Function**

**Definition 5.1.8.**

A function $F(x,u)$ is an invariant of a group $G$ if and only if :$F(\bar{x},\bar{u})=F(x,u)$ $\forall x,u,a\in\mathcal{D}'\subset\mathcal{D}$ .

**Theorem 5.1.2.**

A function $F(x,u)$ is an invariant of a group $G$ with the generator $X$ if and only if

$$X(F)=0 \qquad (5.17)$$

[3] Group Classification Of Coupled Partial Differential Equations With Applications to Flow in a Collapsible Channel and Diffusion Processes - Motlatsi Molat - A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy - Johannesburg 2010

The characteristic system for Eq.(**5.17**) is given by

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)} .$$

Thus an arbitrary invariant $F(x,u)$ of the group $G$ is $F = \Lambda\big(I_1(x,u), \ldots, I_{m+n-1}(x,u)\big)$ ,

Where $I_1(x,u), \ldots, I_{m+n-1}(x,u)$ is called a basis of invariants of $G$ (i.e., group $G$ has exactly $m+n-1$ functionally independent invariants). The basis is not unique. One can take, as basic invariants, the left hand side of $m+n-1$ first integrals $I_1(x,u), \ldots, I_{m+n-1}(x,u) = c_{m+n-1}$ . .

**Example 5.1.2.**

Consider the rotation group with generator $X$ in the $(x,y)$ plane given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} .$$

Canonical variables fot rotation along the $v$ axis are obtained from the system $(u) = 0$ , $X(v) = 1$ , i.e., $-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$ , $-y\frac{\partial v}{\partial x} + x\frac{\partial v}{\partial y} = 1$ .

Solving the first system, we write the corresponding characteristic system **:**

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{du}{0} .$$

Therefore the basis invariant is $x^2 + y^2 = c_1$ , the arbitrary invariant is $u = c_2$ and the general invariant is $= f(x^2 + y^2)$ .

**ii. Prolongation of infinitesimal generator ( vector field)**

As with the group of transformation themselves, we can also define the prolongation of the corresponding of infinitesimal generator. Indeed, these will just be the infinitesimal generators of prolonged group action. [4]

**Definition 5.1.9.**

Let $M \subset X \times U$ be an open and suppose $v$ is a vector field on $M$, with corresponding (local) one-parameter group $\exp(\epsilon v)$. Then $n$-th prolongation of $v$, denoted $\mathrm{Pr}^{(n)}v$, will be a vector

[4] Applications of Lie Groups to Differential Equations – by Peter.J.Olver – School of Mathematics-university of Mannesota- Minneapolis, MN55455, USA - 1986

filed on the $n$-th jet space $M^{(n)}$, and it's defined to be infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{Pr}^{(n)}[\exp(\epsilon v)]$. In other word,

$\mathrm{Pr}^{(n)} v|_{(x,u^{(n)})} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{Pr}^{(n)}[\exp(\epsilon v)](x,u^{(n)})$ , For any $(x,u^{(n)}) \in M^{(n)}$.

Note that since the coordinates $(x,u^{(n)})$ on $M^{(n)}$ consist of the independent variables $(x^1, \dots, x^p)$ and all derivatives $u_J^\alpha$ of dependent variables up to order $n$, a vector space on $M^{(n)}$ will in general take the form

$$v^* = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{p} \sum_{J} \phi_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

Where Jet space are :

$x = (x^1, \dots, x^p)$ – independent variables

$u = (u^1, \dots, u^q)$ – dependent variables

$u_J^\alpha = \frac{\partial^k u^\alpha}{\partial x^{j_1} \dots \dots \partial x^k}$ – partial derivatives

$(x,u^{(n)}) = (\dots . x^i \dots . u^\alpha \dots u_J^\alpha) \in J^n$ – jet coordinates

$J^n = p + q^{(n)} = p + q \binom{p+n}{n}$

**iii. Invariant differential**

**Definition 5.1.10.**

A differential function, $F(x,u,u_{(1)}, \dots, u_{(p)})$ for $p \geq 0$[5], is a $p$th-order differential invariant of a group $G$ if : $F(x,u,u_{(1)}, \dots, u_{(p)}) = F(\bar{x}, \bar{u}, \bar{u}_{(1)}, \dots, \bar{u}_{(p)})$

i.e, $F$ is invariant under the prolonged group $G^{[p]}$, where for $p = 0$, $u_{(0)} \equiv u$ and $G^{[p]} \equiv G$.

**Theorem 5.1.3.**

A differential function, $F(x,u,u_{(1)}, \dots, u_{(p)})$ for $p \geq 0$, is a $p$th-order differential invariant of a group $G$ if

[5] Group Classification Of Coupled Partial Differential Equations With Applications to Flow in a Collapsible Channel and Diffusion Processes - Motlatsi Molat - A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy - Johannesburg 2010

$$X^{[p]} = 0 \qquad (5.18)$$

Where $X^{[p]}$ is the $p$th prolongation of $X$ and for $p = 0, X^{[0]} \equiv X$.

The differential invariants can be obtained by solving the charactrestic equations for Eq.(**5.18**)

**Example 5.1.3.**

The prolongation of the operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

For a second-order ODE with $y$ depending on $x$ is given by

$$X^{[0]} = X + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''}$$

The variables $\zeta_1$, defined are

$$\zeta_1 = D_x(\eta) - y' D_x(\xi)$$

$$= \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y \qquad (5.19)$$

$$\zeta_2 = D_x(\zeta_1) - y'' D_x(\xi)$$

$$= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} + y''(\eta_y - 2\xi_x - 3y'\xi_y) \qquad (5.20)$$

**Definition 5.1.11. Criterion for a Symmetry of D.E**

An invertible transformation acting on the space $(x, u)$ of $E$ is a point symmetry of $E$ provided every solution $h$ of $E$ is mapped onto another solution $\bar{h}$ of $E$.

**Theorem 5.1.4.**

Let $G$ be a group of transformations Eq(**5.4**), admitted by the system $E$. Performing the first-order Taylor expansions of Eq(**5.3**) around $a = 0$, we arrive at the fact that

$$X^{[k]}\left(E^{\sigma}(x, u, u_{(1)}, \dots\dots, u_{(k)})\right) = 0, \quad \sigma = 1, \dots, s. \qquad (5.21)$$

whenever Eq(**5.1**) is satisfied for every group operator $X$ of $G$. Then $G$ consists of symmetries of the system $E$. It can be shown that the converse is also true.

The symmetry condition Eq(**5.21**) can be written compactly as

$$X^{[k]}\left(E^{\sigma}\left(x,u,u_{(1)},\ldots\ldots,u_{(k)}\right)\right)\Big|_{(1.1)}=0\ ,\ \sigma=1,\ldots.,s. \qquad (5.22\ )$$

where$|_{(1.1)}$ means evaluated on the surface.

Equations Eq(**5.22**) are the so-called determining equations. In general the determining equations comprise an over-determined system of linear homogeneous PDE's for the unknown coordinates $\xi^{i}$ and $\eta^{\alpha}$ of the symmetry generator $X$. The solution of the determining system form a vector space, that is , any finite linear combination of symmetries is again a symmetry. This steam from the fact the determining equations are linear.

**5.1.12. Lie's algorithm :**

Below we give a layout of the steps involved in the execution of the procedure for calculating symmetries of :

**1.** Write $E$ such that all the terms are on the left hand side .

**2.**Write the generator of symmetry

$$X=\xi^{i}(x,u)\frac{\partial}{\partial x^{i}}+\eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}$$

**3.**Prolong the symmetry generator $X$ to the order which is the same as that of $E$, i.e.,

$$X^{[k]}=X+\zeta_{i}^{\alpha}\left(x,u,u_{(1)}\right)\frac{\partial}{\partial u_{j}^{\alpha}}+\cdots+\zeta_{i_1\ldots.i_k}^{\alpha}\left(x,u,\ldots.,u_{(k)}\right)\frac{\partial}{\partial u_{i_1\ldots.i_k}^{\alpha}}$$

where the variables $\zeta_{i_1\ldots.i_k}^{\alpha}=D_{ik}\left(\zeta_{i_1\ldots.i_{k-1}}^{\alpha}\right)-u_{i_1\ldots.i_{k-1}}^{\alpha}D_{ik}\left(\xi^{l}\right)$[6].

**4.**Apply the prolonged generator $X^{[k]}$ on $E$ evaluated on the surface Eq(**5.1**) yielding the symmetry conditions

$$X^{[k]}E^{\sigma}\left(x,u,u_{(1)},\ldots\ldots,u_{(k)}\right)\Big|_{(1.1)}=0\quad,\quad\alpha=1,\ldots\ldots,s\ .$$

**5.**Substitute the $\zeta_{i}^{\alpha}$ upon expansion of the symmetry conditions and replace the derivatives which are to be eliminated.

**6.**Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting in an over-determined system of linear homogeneous PDE's in terms of $\xi^{i}$ and $\eta^{\alpha}$.

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[6] Refere to same previous reference – Chapetr One (Lie-Point Symmetries of Differential Equations) – sec [1.3]- page 19.

**7.**Solve the over-determined system for the infinitesimals $\xi^i$ and $\eta^\alpha$ .

**8.**Construct one-parameter groups using (Theorem **5.1.4**).

**5.1.13. Symmetry calculations and Use of Symmetry :**

**i. Symmetry Calculation :**

The following examples illustrate Lie's algorithm for calculating symmetries of DE's.

**Example 5.1.4.**

Consider the equation

$$y'' = \frac{\alpha}{y^3}, \quad \alpha \neq 0 \qquad (5.23)$$

which is a special case of the Ermakov-Pinney equation.

According to Lie's algorithm the vector field

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y \qquad (5.24)$$

is a symmetry generator of Eq(**5.23**) if and only if we have the symmetry condition

$$X^{[2]}\left(y'' - \frac{\alpha}{y^3}\right)\Big|_{(1.32)} = 0$$

where

$$X^{[2]} = X + \zeta_1\partial_{y'} + \zeta_2\partial_{y''}$$

The variables $\zeta_1$ and $\zeta_2$ are given by Eq(**5.19**) and Eq(**5.20**) respectively. However, the turn involving $\zeta_1$ does not contribute in calculations because there is no $y'$ appearing in the equation under consideration.

Thus we have the determining equation

$$\left(\frac{3\alpha}{y^4}\eta + \zeta_2\right)\Big|_{(1.32)} = 0 \qquad (5.25)$$

Upon expansion of Eq(**4.25**) we have

$$\frac{3\alpha}{y^4}\eta + \eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - (y')^3\xi_{yy} + \frac{\alpha}{y^3}(\eta_y - 2\xi_x) - \frac{3\alpha}{y^3}y'\xi_x = 0$$

The separation of the above expression with respect to powers of $y'$ yields the equations

$$(y')^3 : \ \xi_{yy} = 0$$

$$(y')^2 : \eta_{yy} - 2\xi_{xy} = 0$$

$$y' = 2\eta_{xy} - \xi_{xx} - \frac{3\alpha}{y^3}\xi_y = 0$$

$$(y')^0 : \ \frac{3\alpha}{y^4}\eta + \eta_{xx} + \frac{\alpha}{y^3}(\eta_y - 2\xi_x) = 0$$

The general solution of the above system is

$\xi = c_0x^2 + 2xc_1 + c_2$ , $\eta = (c_0x + c_1)y.$

Thus, the symmetry Lie algebra of Eq(5.23) is generated by the operators

$X_1 = \partial_x$ , $X_2 = 2x\partial_x + y\partial_y$ , $X_3 = x^2\partial_x + xy\partial_y$

Hence the symmetry Lie algebra is three-dimensional. The Lie Bracket of the symmetry generators is

$[X_1, X_2] = 2X_1$ , $[X_1, X_3] = X_2$ , $[X_2, X_3] = 2X_3$

**ii. Use of Symmetry :**

Symmetries can be used to reduce the order of a DE. In fact when the equation admits solvable Lie algebra, the ideals of the algebra can be used to perform the reduction.

Knowing the symmetry of the reduced equation provides further reduction until in some cases the solution is obtained by quadratures.

**Example 5.1.5. (Reduction of order)**

Consider the equation

$$y'' + by' + \frac{2b^2}{9}y + cy^3 = 0 \qquad (5.26)$$

For arbitrary constants $b$ and $c$. The symmetry Lie algebra admitted by the equation is two-dimensional and generated by the operators

$X_1 = \partial_x$ , $X_2 = -\frac{3}{b}\, exp\left(\frac{b}{3}x\right)\partial_x + y\exp\left(\frac{b}{3}x\right)\partial_y$

The Lie Bracket of the operators is

$$[X_1\,, X_2\,] = \frac{3}{b}\,X_2$$

Rewriting $L_2$ in solvable form we use the skew-symmetry property that

$$[X_1\,, X_2\,] = -[X_2\,, X_1\,]$$

Thus, we let $Y_1 = X_2$ and $Y_2 = X_1$ to obtain

$$[Y_1\,, Y_2\,] = -\frac{3}{b}\,Y_1$$

The ideal is spanned by

$$Y_1 = -\frac{3}{b}\,exp\left(\frac{b}{3}x\right)\partial_x + y\exp\left(\frac{b}{3}x\right)\partial_y$$

Therefore we start the reduction with $Y_1$. For a second-order equation we require the invariants of the first prolongation of the operator to reduce the equation. In reducing a third-order equation the second prolongation of the operator is needed and so on……

The first prolongation of $Y_1$ is

$$Y_1^{[1]} = \exp\left(\frac{b}{3}x\right)\left[-\frac{3}{b}\,\partial_x + y\partial_y + \left(2y' + \frac{b}{3}y\right)\partial_{y'}\right]$$

The invariants of the group generated by $Y_1^{[1]}$ are

$$u = y\exp\left(\frac{b}{3}x\right)\quad ,\quad v = \frac{y'}{y^2} + \frac{b}{3y}$$

The second-order equation (5.26) becomes the first-order equation in the new variables, $u$ and $v$, i.e.,

$$\frac{dv}{du} = -\frac{2v^2 + c}{uv} \qquad (5.27)$$

The reduced equation Eq(**5.27**) is variables separable and there is no need for consecutive reduction using $Y_1$. However, Eq. (**5.26**) admits

$$\tilde{Y}_2 = \left(\frac{b}{3}\right)u\partial_u$$

Written in the new coordinates. The solution of Eq. (**5.27**) can be easily found and then written in original coordinates $(x, y)$.

**Example 5.1.6 (Generating solutions )**

The one-parameter group of transformations can be used to generate new solutions from the known ones. The new solutions can comprise nontrivial solutions compared to the known (trivial) solutions from which they were generated.

If $\bar{u} = \bar{h}(\bar{t}, \bar{x})$ is solution of the heat equation, then so is

$\phi(t,x,u,a) = h\big(f^1(t,x,u,a), f^1(t,x,u,a)\big)$

Where the $f^i$ are differentiable functions.

Equivalently in solved form with respect to $u$: $u = H_a(t,x)$ is a one-parameter family of solutions. For instance, if $\bar{u} = \bar{h}(\bar{t}, \bar{x})$ is a solution of the heat equation corresponding to the transformations

$T_{a_1}: \bar{t} = t + a_1$ , $x = \bar{x}$ , $\bar{u} = u$ , Then so is $u = h(t - a_1, x)$.

Also consider the transformations,

$$T_{a_6}: \bar{t} = \frac{t}{1 - 4a_6 t} \;,\; \bar{x} = \frac{x}{1 - 4a_6 t} \;,\; \bar{u} = u\sqrt{1 - 4a_6 t}\,\exp\left[\frac{-a_6 x^2}{1 - 4a_6 t}\right]; \quad a_6 \neq 0$$

Given the constant solution of the heat equation $u = u_0 =$ consatant and expressing $\bar{u}$ in terms of $\bar{t}$ and $\bar{x}$, a new solution (dropping the bars),

$$u = \frac{u_0}{\sqrt{1 + 4a_6 t}} \;\exp\left[\frac{-a_6 x^2}{1 - 4a_6 t}\right]$$

Is generated.

## 5.2 Some applications

Here we use Lie algebraic methods to obtain explicit expressions that approximate the solution of the Cauchy problem defined by [7]

$$\frac{\partial}{\partial t} f(t;x) = A(t;x) f(t;x) \;,\quad f(0;x) = g(x) \qquad \ldots\ldots\ldots\ldots (5.28)$$

Where $x \equiv (x_1, x_2, \ldots\ldots, x_m) \in \mathbb{R}^m$, $g$ is an arbitaray bounded analytic function defined in some open domain in $\mathbb{R}^m$ and

[7] Solution of linear partial differential equations by Lie algebraic methods – Fernando Cases – Departement of Matematiques, Universitat Jaume I, 12071- Castellon, Spain – 10 February 1996.

$$A(t;x) = \sum_{i,j=1}^{m} a_{ij}\,(t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{m} b_{ij}\,(t)x_i\frac{\partial}{\partial x_j} + \sum_{i,j=1}^{m} c_{ij}\,(t)x_i x_j + \sum_{j=1}^{m} d_j\,(t)\frac{\partial}{\partial x_j}$$

$$+ \sum_{j=1}^{m} e_j\,(t)x_j + h(t) \ldots\ldots\ldots\ldots(5.29)$$

The coefficients $a_{ij}\,(t)$,etc. of the differential operator $A(t;x)$ are defined in an open interval of the t-axis containing the origin and are complex-valued bounded analytic functions.

Problem of this type appear frequently in the mathematical physics literature. They include particular cases of the time –dependent linear Fokker-Planck equation, the Schrodinger equation with time-dependent potentials and the Helmholtz equation in the approximation of paraxial wave beams, just to quote a few examples.

One should note that $A(t;x)$ is an element of a Lie algebra $\mathcal{L}$ of finite dimension $n$ under the bracket operation $[B_1, B_2] = B_1 \circ B_2 - B_2 \circ B_1$, where $B_1, B_2 \in \mathcal{L}$ and $\circ$ denotes the operator composition.

If $A$ doesn't depend explicitly of time $t$, then we can write the solution of Eq.(**5.28**) as

$$f(t;x) \equiv U(t)f(0;x) = e^{tA}g(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.30)$$

Where $\exp(tA)$ should be interpreted as an element in the simply connected Lie group associated with $\mathcal{L}$. Thus one can use the properties of the Lie algebra $\mathcal{L}$ to study the operator $\exp(tA)$. More specifically, a suitable basis for $\mathcal{L}$, with constant generators $A_i$, $i = 1, \ldots, n$, is chosen and then the elements $\exp(tA_i)$ are computed. Next, ordering formulas of Baker, Campbell, Hausdorff and Zassenhaus type are used to write the evolution operator $U(T)$ in the factord form

$$U(T) = \exp(f_1(t)A_1)\,\exp(f_2(t)A_2)\ \ldots\ldots\ \exp(f_n(t)A_n)\ \ldots\ldots\ldots\ldots (5.31)$$

Where $f_i(t)$ are $t$- dependent analytic functions ( with the exception of certain isolated points) linked to the constant coefficients of the operator $A(x)$.

Here we present a modified version of the above algorithm, based entirely on Lie algebraic methods, for solving approximately the Cauchy problem Eq(**5.28**) when the coefficients of $A$ are arbitrary functions of time. The method consists of finding a law-dimensional faithful matrix representation $\hat{Q}$ of the Lie algebra $\mathcal{L}$ and then applying Lie algebraic techniques to obtain the solution of the corresponding image of our partial differential equation in $\hat{Q}$. If the associated Lie groups are also isomorphic, one can get in a straightway explicit expressions for the functions $f_i(t)$ appearing in Eq (**5.31**), and thus a closed – form solution for the

Cauchy problem Eq(**5.2**). This algorithm can easily be implemented for computational purposes for any particular example considered.

Conventionally, the solution of Eq (**5.28**) is formally written in the applications as a time – ordered exponential operator .

$$U(T)P = \left[\exp\left(\int_0^t A(s)ds\right)\right] \equiv I + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^t dt_2 \ldots\ldots \int_0^{t_{n-1}} dt_n \, A(t_1) \ldots\ldots A(t_n) \quad (5.32)$$

But this approach presents two main drawbacks in relation to the previous scheme. First, the treatment depends on whether the coefficients in Eq(**5.28**) are constant or not; in the first case the time – ordered exponential reduces to an ordinary one Eq (**5.29**), whereas in the latter one has to construct the formal series Eq(**5.32**) explicitly. Secondly, it si not easy to evaluate the action of the operator $U(T)$ on $f(0;x)$ and to study the influence of the single factors $A_i$ on the time evolution of $f(t;x)$. On the other hand, from Eq(**5.31**) we may gain insight into the properties of $U(T)$ through a knowledge of the spectral properties of the individual operators $A_i$. We can also consider physical situations where this kind of parameterization is of particular value.

Here, we assume that the solution to the Cauchy problem defined by Eq (**5.28**) is uniquely determined at least for $t$ sufficiently small, provided the initial data $g(x)$ is chosen in some appropriate space of functions $\mathcal{X}$. The resulting flow $f(t;x) = (U(t)g)(x)$ will then be on the given function space $\mathcal{X}$. The verification of this hypothesis leads to very difficult problems on existence and uniqueness of solutions that we shall not consider in this work. Here we will obtain results which are of a formal nature, but nevertheless will have direct practical applications.

### 5.2.1 The algebraic Method

Suppose the linear operator $A(t;x)$ can be expressed in the form

$$A(t;x) = \sum_{i=1}^{n} a_i(t)\, A_i(x)\,, \qquad n \text{ finite} \qquad (5.33)$$

Where the $a_i(t)$ $(i = 1, \ldots., n)$ are scalar functions of time, and $A_1, \ldots., A_n$ are time – independent operators that form a basis of the Lie algebra $\mathcal{L}$ under the bracket operation.

Let us suppose we have found a low – dimensional faithful matrix representation $\hat{Q}$ of $\mathcal{L}$. Using this isomorphism we can consider the associated equation

$$\frac{d\vec{f}}{dt} = \hat{A}(t)\vec{f} \qquad (5.34)$$

which will be referred as the image equation of Eq(**5.28**) in the matrix representation. Here $\hat{A}(t)$ is the $s \times s$ matrix in $\hat{Q}$, image of $A(t; x) \in \mathcal{L}$, and $\vec{f}(t) \in \ell^s$. Equivalently, we can consider the linear equation

$$\frac{d\hat{U}(t)}{dt} = \hat{A}(t)\hat{U}(t), \qquad \hat{U}(0) = \hat{I}, \tag{5.35}$$

where $\hat{I}$ is the $s \times s$ identity matrix, $\vec{f}(t) = \hat{U}(t)\vec{f}(0)$ and the matrix $\hat{A}(t)$ can be written as

$$\hat{A}(t) = \sum_{i=1}^{n} a_i\,(t)\hat{A}_i \tag{5.36}$$

With $\hat{A}_i$ the element of the basis of $\hat{Q}$ associated with the operator $A_i(x)$.

Wei and Norman[8] have shown that if $\hat{U}(t)$ is a solution of Eq (5.35), then there exists a neighborhood of $t = 0$ where it can be represented in the form

$$\hat{U}(t) = \exp\left(f_1(t)\hat{A}_1\right) \exp\left(f_2(t)\hat{A}_2\right) \dots\dots\dots \exp\left(f_n(t)\hat{A}_n\right) \tag{5.37}$$

The $f_i(t)$ being scalar functions of time. Moreover, the $f_i(t)$ satisfy a set of differential equations which depend only on the Lie algebra $\mathcal{L}$ and the coefficients $a_i(t)$'s. This representation is global for all solvable Lie algebras, and for any real $2 \times 2$ system of equations.

Now if the Lie groups associated with the Lie algebras $\hat{Q}$ and $\mathcal{L}$ are also isomorphic, it is possible to express the solution of Eq (5.27) locally as $f(t; x) \equiv U(t)f(0; x)$, with

$$U(T) = \exp(f_1(t)A_1) \exp(f_2(t)A_2) \dots\dots\dots \exp(f_n(t)A_n) \tag{5.38}$$

Finally, by computing explicitly the flows

$$\exp(f_i A_i)g(x), \qquad i = 1, \dots., n \tag{5.39}$$

we obtain a formal expression for the solution of the Eq (**5.28**) in a neighborhood of $t = 0$ in terms of the unknown functions $f_i(t)$.

In general case of a time- dependent operator $A(t; x) \in \mathcal{L}$, the set of differential equations that determine the scalar functions $f_i(t)$, or equivalently the system Eq(**5.35**), cannot be solved by quadaratures. Instead, approximate methods of resolution are required.

The approximation scheme we adopt here is to apply the so-called Fer factorization to the matrix equation Eq(**5.35**). Where its properties as a symplectic integration algorithm have

[8]J.Wei and E. Norman, on global representations of the solutions of linear differential equations as a product of exponentials, proc, Amer. Math, soc, 15 (1964) 327-334.

also been established for Hamiltonian systems of ordinary differential equations. In particular, it allows to construct explicit convergent approximations to the slution of the initial value problem Eq(**5.35**) in a neighborhood of $t = 0$, so that, once this solution has been obtained, comparison with Eq (**5.37**) leads to the corresponding expressions for the functions $f_i(t)$.

The general characteristics of the Fer factorization are included in following result :

**Theorem 5.2.1.**

Let $\hat{A}(t)$ and $\hat{U}(t)$ be two bounded linear operators acting on a Euclidean space, with $\|\hat{A}(t)\|$ a continuous function. Then :

(a) The solution of the initial value problem

$$\frac{d\hat{U}(t)}{dt} = \hat{A}(t)\hat{U}(t), \qquad \hat{U}(0) = \hat{I} \tag{5.40}$$

May be expressed in the form

$$\hat{U}(t) = e^{F_1} \ldots\ldots e^{F_n}\,\hat{U}_n\,, \tag{5.41}$$

With

$$\frac{d\hat{U}(t)}{dt} = H_i(t)\hat{U}_i\ ,\ \hat{U}_i(0) = \hat{I}\ ,$$

$$F_{i+1} = \int_0^t H_i\,(t')\,dt'\ , \qquad H_0 \equiv \hat{A}(t) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5.42)}$$

$$H_{i+1} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}\ j}{(j+1)!}\ \underbrace{[F_{i+1}\,,[F_{i+1}\,,\ldots[F_{i+1}\,,H_i]\ldots\ldots]]}_{j\ \ times}$$

$(j > 0)$ and therefore it can be written as an infinite product of exponentials

$$\hat{U}(t) = e^{F_1} \ldots\ldots e^{F_n} \ldots\ldots \tag{5.43}$$

(b) This infinite product is convergent if the operators $H_i(t)$ are bounded and $\|H_i(t)\|(i > 0)$ are continuous functions, only for times $t$ such that

$$\int_0^t \|\hat{A}(t')\|\,dt' < \xi \tag{5.44}$$

Where $\xi$ is the nonzero solution of the equation

$$\xi = \int_0^{\xi} \frac{1-e^{2x}(1-2x)}{2x}\,dx \qquad (\xi \simeq 0.861) \tag{5.45}$$

Here convergence has to be understood as

$\lim_{n\to\infty} \int_0^t \|H_n(s)\| \; ds = 0$

When the functions involved in Eq (5.40) belong to a solvable Lie algebra, then a finite product of exponentials is attained for the linear operator $\hat{U}(t)$ in the term, $n = 1,2, \ldots$ by doing $\hat{U}_n = \hat{I}$. Thus, we obtain an approximate expression for the evolution operator in the form

$$\hat{U}(t) \simeq \hat{V}_n(t) = e^{F_1} \ldots \ldots . e^{F_n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.46)$$

In that case we have the following result concerning the error bounds of the approximation :

**Theorem 5.2.2.**

Let $E_n(t)$ be the difference between the exact and the approximate solution of Eq (**5.40**),

$$E_n(t) = \hat{U}(t) - \hat{V}_n(t) \qquad (5.47)$$

Then

$$\|E_n(t)\| \le K_n(t) \exp(\textstyle\sum_{i=0}^{n} K_i(t)), \qquad n \ge 1 \qquad (5.48)$$

With

$$K_0 \equiv \int_0^t \|\hat{A}(t')\| \, dt'$$

$$K_{n+1} \equiv \int_0^{K_n} \frac{1 - e^{2x}(1 - 2x)}{2x} \, dx$$

If we denote $K_0 = \alpha\xi$, with $0 < \alpha < 1$, then it can be shown that

$$\|E_n(t)\| \le \alpha^{2^n} g_n(\alpha)\xi \qquad (5.49)$$

Where $g_n$ is a function that tends to a constant as $n$ increases. Therefore the rate of convergence of the procedure is very fast.

Therefore, Fer's factorization provides a reliable and computationally well adapted Lie algebraic method to obtain approximate solutions to the linear equation Eq(**5.40**), and consequently, convergent expressions for the characteristic ordering functions $f_i(t)$ of Eq (**5.42**). These expression are valid in a neighborhood of $t = 0$ and involve only quadratures. The method also allows to compute explicitly the region of convergence and the error bound of the approximation.

We can summarise the proposed algebraic method for solving the Cauchy problem Eq(**5.28**) as the following computational algorithm :

**Step (1) :** Identify the algebra involved in the problem and a low – dimensional faithful matrix representation.

**Step (2) :** Apply the Fer factorization to the image equation Eq(**5.37**) in that matrix representation.

**Step (3) :** Obtain the ordering functions $f_i(t)$ by comparison with the corresponding Wei-Norman representation Eq (**5.37**).

**Step (4) :** Compute explicity the flows Eq(**5.38**) and finally the action of the operator $U(t)$ Eq(**5.37**) on the function $g(x)$.

**Example 5.2.1.**

As a first application we take

$$A(t;x) = a(t)\partial^2 + b(t)x\partial + a(t)\partial + h(t) \qquad (5.50)$$

Where the notation $\partial \equiv \partial/\partial x$ has been used. This corresponding to a one – dimensional Fokker-Planck ( or forward Kolmogorov ) equation [9]whose diffusion and drift coefficients are both arbitrary functions of time. It is used in a stochastic treatement of a given macroscopic system. More specifically, the Fokker-Planck equation is an equation of motion for the distribution function $f(t;x)$ of the fluctuating macroscopic variables that describe the system.

If we identify the operators $A_1 = I$, $A_2 = x\partial$, $A_3 = \partial$, $A_4 = \partial^2$ as the basis of the Lie algebra $\mathcal{L}$ in this case, then the basic bracket operators are given by

$$[A_2\,, A_3] = -A_3\,, \quad [A_2\,, A_4] = -2A_4\,, \quad [A_3\,, A_4] = 0 \qquad (\mathbf{5.51})$$

And therefore the sub-algebra $L \equiv \langle A_2\,, A_3\,, A_4 \rangle$ is solvable. It is easy to realize that a matrix representation for these operators is provided by

$$\hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \hat{A}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (\mathbf{5.52})$$

Thus specifying the matrix image of our partial differential equation in the form of Eq (**5.34**), or equaivalently,

[9] H.Risken, the Fokker-Planck equation (springer, Berlin, 2nd ed, 1989 )

$$\frac{d\hat{U}}{dt} = \hat{A}(t)\hat{U}(t) \qquad (5.53)$$

With $\hat{A}(t) = \sum_{i=1}^{4} a_i\,(t)\hat{A}_i$ and $a_1(t) = h(t), a_2(t) = b(t), a_3(t) = d(t), a_4(t) = a(t)$ .

Now the Wei-Norman factorization Eq(**5.38**), when applied to the matrix equation Eq(**5.53**), leads to the expression

$$\hat{U}(t) = e^{f_1}\begin{bmatrix} 1 & f_3 & f_4 + \frac{1}{2} f_3^2 \\ 0 & e^{f_2} & e^{f_2} f_3 \\ 0 & 0 & e^{2f_2} \end{bmatrix} \qquad (5.54)$$

With the functions $f_i(t),\ i = 1, \dots, 4$ to be determined.

In this case, by applying Fer's factorization ( Theorem **5.2.1**) to Eq (**5.53**) we obtain the exact solution as

$$\hat{U}(t) = e^{F_1} e^{F_2}\ , \qquad (5.55)$$

Where

$$F_1 = \sum_{i=1}^{4} \alpha_i\,(t)\hat{A}_i\ , \qquad \alpha_i(t) = \int_0^t a_i\,(s)ds \qquad (5.56\ )$$

And

$$F_2 = \alpha_3^{(2)}(t)\hat{A}_3 + \alpha_4^{(2)}(t)\hat{A}_4\ , \qquad \alpha_i^{(2)} = \int_0^t h_i^{(1)}\,(s)ds$$

$$h_3^{(1)}(t) = \frac{1}{\alpha_2^2}\,(-\alpha_2 e^{\alpha_2} + e^{\alpha_2} - 1)(\alpha_3\alpha_2 - \alpha_2\alpha_3)\ , \qquad (5.57)$$

$$h_4^{(1)}(t) = \frac{1}{2\alpha_2^2}\,(-2\alpha_2 e^{2\alpha_2} + e^{2\alpha_2} - 1)(\alpha_4\alpha_2 - \alpha_2\alpha_4)\ .$$

If we evaluate explicitly the exponentials of Eq (**5.55**) and compare the matrix thus obtained with the expression Eq(**5.54**), after some algebra we obtain the exact expressions for the ordering functions $f_i(t)$ in terms of quadratures

$$\left.\begin{aligned} f_1(t) &= \int_0^t h(s)ds \\ f_2(t) &= \int_0^t b(s)ds \\ f_3(t) &= \int_0^t d(s)e^{f_2(s)}ds \\ f_4(t) &= \int_0^t a(s)e^{f_2(s)}ds \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots (5.58)$$

The same expressions can be obtained, of course, by writing down and solving the differential equations satisfied by the functions $f_i(t)$. This is possible here because the Lie algebra involved is solvable.

Finally, by using the easily derivable expressions [**5.38, 5.41**]

$$exp\left[a(t)x\frac{\partial}{\partial x}\right]\ g(x) = g\left(e^{a(t)}x\right)\ , \tag{5.59}$$

$$exp\left[a(t)\frac{\partial}{\partial x}\right]\ g(x) = g(x + a(t))\ , \tag{5.60}$$

$$exp\left[a(t)\frac{\partial^2}{\partial x^2}\right]\ g(x) = \frac{1}{\sqrt{4\pi a(t)}}\int_{-\infty}^{+\infty} \exp\left[-\frac{(y-x)^2}{4a(t)}\right] g(y)dy\ , \tag{5.61}$$

we find for $f(t;x)$,

$$f(t;x) = \frac{e^{f_1(t)}}{\sqrt{4\pi f_4(t)}}\int_{-\infty}^{+\infty} dy\ g(y) \exp\left[-\frac{\left[y-\left(xe^{f_2(t)}+f_3(t)\right)\right]^2}{4f_4(t)}\right], \qquad t > 0 \tag{5.62}$$

A result previously obtained in [**5.38,5.41**] with different algebraic techniques.

**Example 5.2.2.**

Next we consider the operator $A(t;x)$ given by

$$A(t;x) = a(t)\partial^2 + \ b(t)x\partial + c(t)x^2\ , \tag{5.63}$$

Where $a(t), b(t), c(t)$ are complex valued bounded analytic functions. This constitutes a generalization of a linear Fokker-Planck equation. If we denote

$$A_1 = I, \qquad A_2 = \frac{1}{4}\ (1 + 2x\partial), \quad A_3 = \frac{1}{2}\ x^2, \quad A_4 = \frac{1}{4}\ \partial^2, \tag{5.64}$$

Then these operators form a basis of the Lie algebra $\mathcal{L}$, the basic bracket relations are

$$[A_2\ , A_3] = A_3\ , \quad [A_2\ , A_4] = -A_4\ , \quad [A_3\ , A_4] = -A_2 \tag{5.65}$$

And the sub-algebra $\langle A_2\ , A_3\ , A_4\rangle$ can be identified with $SU(1,1)$, which is not solvable. A matrix representation of the $SU(1,1)$ generators is provided by

$$\hat{A}_2 = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \hat{A}_3 = \begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}, \qquad \hat{A}_4 = \frac{1}{2}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \tag{5.66}$$

And the image of the operator $A(t;x)$ under this representation can be written as Eq (**5.36**) with $a_1(t) = -\frac{1}{2}b(t), \quad a_2(t) = 2b(t), \quad a_3(t) = 2c(t), \quad a_4(t) = 4a(t)$.

If we apply the Wei-Norman factorization to the linear Eq (**5.35**), the corresponding solution can be now represented as

$$\hat{U}(t) = e^{f_1}\begin{bmatrix} e^{f_2/2}\left(1-\frac{1}{2}f_3 f_4\right) & -f_3 e^{f_2/2} \\ \frac{1}{2} f_4\, e^{-f_2/2} & e^{-f_2/2} \end{bmatrix} \tag{5.67}$$

In a neighborhood of $t = 0$. In this case the system of differential equations that determine the functions $f_i$ cannot be solved by quadratures for arbitrary coefficients $a_i(t)$. Nevertheless, Fer's factorization provides an iterative procedure for obtaining convergent approximations to the matrix $\hat{U}(t)$ in terms of quadratures. More specifically, by applying (Theorem **5.2.1**) we get up to order $n$

$$\hat{U}(t) \simeq \hat{V}_n(t) = e^{F_1} e^{F_2} \dots \dots . e^{F_n}, \tag{5.68}$$

with

$$F_1 = \sum_{j=1}^{4} \alpha_j^{(1)}(t)\alpha \hat{A}_j\ , \qquad \alpha_j^{(1)}(t) = \int_0^t a_j(s)ds$$

$$F_{i+1} = \sum_{j=2}^{4} \alpha_j^{(i+1)}(t)\hat{A}_j\ , \qquad \alpha_j^{(i+1)}(t) = \int_0^t h_j^{(1)}(s)ds\ , \quad i = 1, \dots\dots, n-1, \tag{5.69}$$

where $h_j^{(1)}(t)$, $j = 2,3,4$, are the coordinates of the matrix $H_{i+1}$ with respect to the basis $\{A_i\}$ Eq (**5.42**), which depend both on the coefficients $h_j^{(i-1)}$ and $\alpha_j^{(1)}$. A simple calculation shows that

$$e^{F_i} = (\cosh \omega_i) I_2 + \frac{\sinh\ \omega_i}{\omega_i}\ B_{(i)}\ , \tag{5.70}$$

where

$$\omega_i \equiv \frac{1}{2}\sqrt{\alpha_2^{(i)2} - 2\alpha_3^{(i)}\alpha_4^{(i)}}, \quad B_{(i)} = \frac{1}{2}\begin{bmatrix} \alpha_2^{(i)} & -2\alpha_3^{(i)} \\ \alpha_4^{(i)} & -\alpha_2^{(i)} \end{bmatrix}, \tag{5.71}$$

and $I_2$ denotes the $2 \times 2$ identity matrix.

In this way we can write an approximation to the $\hat{U}(t)$ as

$$\hat{V}_n(t) = e^{\alpha_1^{(1)}}\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \tag{5.72}$$

whence, by comparing with Eq (**5.67**), we get

$$f_1(t) = \alpha_1^{(1)} = -\frac{1}{2}\int_0^t b(s)ds\ , \qquad f_2(t) = -2\log u_{22}$$

$$f_3(t) = -\ u_{12}u_{22}\ \ , \qquad f_4(t) = 2u_{21}\ {u_{22}}^{-1} \qquad (5.73)$$

i.e., approximate explicit expressions for the ordering functions $f_i(t)$ in terms of quadratures. This procedure converges to the true solution $\widehat{U}(t)$ as $n \to \infty$, and therefore to the functions $f_i$ , in time intervals $[0, t]$ such that

$$\int_0^t \|\widehat{H}(s)\| ds < \xi\ , \qquad (5.74)$$

with

$$\widehat{H}(t) = \begin{bmatrix} b(t) & -2c(t) \\ 2a(t) & -b(t) \end{bmatrix} \quad , \qquad (5.75)$$

Finally, the solution of Eq (**5.28**), with $A(t;x)$ given by Eq(**5.53**), can be found, by applying Step 4 , as in the preceding example, thus obtaining the expression

$$f(t;x) = \frac{1}{\sqrt{\pi f_4(t)}} \exp\left[f_1(t) + \frac{1}{4}f_2(t) + \frac{1}{2}\ f_3(t)\ x^2\ e^{f_2}\right] \times \int_{-\infty}^{+\infty} dy\ g(y) \exp\left[\frac{-\left(y - xe^{f_2(t)}\right)^2}{f_4(t)}\right] \qquad (5.76)$$

for $t > 0$.

**Example 5.2.3.**

We consider the equation [**5.40,5.42**]

$$\left(\frac{\partial^2}{\partial x \partial y} + b(t)y\frac{\partial}{\partial y} + c(t)xy + \frac{\partial}{\partial t}\right) f(t;x,y) = 0 \qquad (5.77)$$

Subject to the initial condition $f(0;x,y) = \phi(x,y)$. This two-dimensional parabolic PDE is a particular case of an equation introduced and solved by Lambropoulos[10] when the coefficients $b$ and $c$ are constants. Later Wilcox[11] obtained a closed-form solution by normal-ordering exponential operators techniques. In the following we apply the method outlined in the previous section to solve the general case of arbitrary time-dependent coefficients. In Lambropoulos one instance of a physical problem in which a special form of this equation arises is presented.

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[10] P.Lambropoulos, Solution of the differential equation $\left(\frac{\partial^2}{\partial x \partial y} + b(t)y\frac{\partial}{\partial y} + c(t)xy + \frac{\partial}{\partial t}\right)P = 0$, J.Math.Phys. 8(1967) 2167-2169.

[11] R.Wilcox, Closed –form solutionof differential equation$\left(\frac{\partial^2}{\partial x \partial y} + b(t)y\frac{\partial}{\partial y} + c(t)xy + \frac{\partial}{\partial t}\right)P = 0$, by normal –ordering exponential operators ,J.Math. Phys. 11(1970) 1235-1237.

As in the previous examples, if we introduce the operators

$$A_1 = I, \quad A_2 = \frac{1}{2}\left(1 + x\partial_x + y\partial_y\right), \qquad A_3 = \frac{1}{2}xy, \qquad A_4 = \partial_{xy}^2 \ , \tag{5.78}$$

Then Eq (5.76) can be written as Eq (5.27) with $A(t;x,y) = \sum_{i=1}^{4} a_i\,(t)A_i$ and $a_1(t) = b(t)$,

$a_2(t) = -2b(t)$, $a_3(t) = -2c(t)$, $a_4(t) = -1$. Moreover, we have the basic bracket operations (5.64) of the algebra $SU(1,1)$, so the same steps of the (Example **5.2.2**). , when applied to this case, leads to ordering functions

$$f_1(t) = \int_0^t b(s)ds \ , \qquad \varphi(t) \equiv e^{-f_2(t)/2} = u_{22}(t)$$

$$f_3(t) = -u_{12}\,u_{22} \ , \qquad f_4(t) = 2u_{21}\,u_{22}{}^{-1} \ , \tag{5.79}$$

where the coefficients $u_{ij}\,(t)$ are evaluated by means of Fer's factorization. If we denote

$$\delta(t) = -f_1(t) - \frac{1}{2}f_2(t) = \log\varphi(t) - \int_0^t b(s)ds$$

$$\Omega(t) = \frac{1}{\varphi(t)}\,u_{12}(t) \tag{5.80}$$

$$\beta(t) = -2u_{21}(t)\frac{1}{\varphi(t)}$$

then the solution of Eq (**5.77**) is given by

$$f(t;x,y) = exp[-\delta(t) - \Omega(t)xy]R\left(t;\frac{x}{\varphi(t)},\frac{y}{\varphi(t)}\right) \tag{5.81}$$

Under the assumption that

$$R(t;x,y) = exp\left[-\beta(t)\partial_{xy}^2\right]\phi(x,y) \tag{5.82}$$

exists. In particular case of constant coefficients, Fer's expansion leads to the exact solution $\widehat{U}(t) = e^{F_1}$, or equivalently,

$$\varphi(t) = \cosh\omega t + \frac{b}{\omega}\sinh\omega t$$

$$\delta(t) = \log\varphi(t) - bt, \tag{5.83}$$

$$\beta(t) = \frac{\gamma}{c\varphi} \ ,$$

$$\Omega(t) = \frac{\gamma}{\varphi}$$

with

$$\omega \equiv \sqrt{b^2 - c}\,, \qquad \gamma \equiv \frac{c}{\omega}\sinh \omega t \tag{5.84}$$

This is just the solution obtained by Wilcox for Eq (**5.77**) in the time-independent case.

## Conclusion

It's very imazing beautiful thing that we are interesting in groups, because we are interesting is symmetries and also because We are living in locally Minkoviski space, it where we live.

Groups can be powerful tools that we can use it to anderstand what happening around us, We made a glance on ' conservation principles' , Subsequently, with the work of Emily Noether .

We employed the symmetries to construct the invariant solutions wherever applicable. The solution of the optimal system problem allows the classification of all the invariant solutions, i.e. solutions that are left unchanged by sub-algebras of the symmetry Lie algebra. The invariant solutions of a given equation satisfy an equation with a reduced number of the independent variables.

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