Chapter 3

Conservation Laws of Physics

3.1 Introduction

The conservation laws allow us to treat physical objects as point particles, when we are concerned about their kinematics properties. They assist us in solving, in full generality, complicated problems that would otherwise be unattainable. Perhaps more importantly, they give us a first glance into the fundamental properties of our Universe and the symmetries that govern the laws of physics with it.

It was considered natural to obtain ' conservation principles' from ' laws of nature '. Subsequently, with the work of Emmily Noether, emphasis shifted to a reversal of this process, by first examining the symmetry considerations that have conservation principles associated with them , and then deduction of the "laws of nature" from the underlying connections between symmetry and conservation laws. This latter approach has now assumed a fundamental role in the scientific method aimed at examining ' laws of nature', to both test them and / or to discover new laws.

3.2 Concept of a conservation Law

Let us consider an ordinary differential equation[1]

$$F(t, q, \dot{q}, \ddot{q}) = 0 \qquad (3.1)$$

Describing a motion of a dynamical system. Here t is time , $q = (q^1, \dots\dots, q^s)$ are the position coordinates, $q = q(t)$, and $v = \dot{q} \equiv \frac{dq}{dt}$ is the velocity, $\ddot{q} = \frac{d^2q}{dt^2}$.

Definition 3.2.1.

A function $C = C(t, q, v)$ is called a conserved quantity for eq (3.1) if

$$\frac{dC}{dt} = 0 \qquad (3.2)$$

On every solution of Eq(**3.1**)

In other words, the conserved quantity $C = C(t, q, v)$ is constant on each trajectory $q = q(t)$ and therefore is called a constant of motion.

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[1]Raisa Khamitova – "Symmetries and conservation laws. Thesis for the degree of Doctor of Philosophy"- Växjö University Press , Sweden 2009

In classical mechanics Eq(**3.1**) has the form

$$m\ddot{x} = 0 \qquad (3.2)$$

And describes a free motion of a particle with the mass m and a position vector

$\mathbf{x} = (x^1, x^2, x^3)$. The equation has several conserved quantities, e.g. the energy $E = \frac{1}{2}m\mathbf{v}^2$ and the linear momentum $= m$v .

Let us now consider a partial differential equation of p- th order

$$F = \left(x, u, u_{(1)}, u_{(2)}, \dots \dots, u_{(p)}\right) = 0 \qquad (3.4)$$

Where the function F depends on n independent variables x , $x = (x^1, \dots \dots, x^n)$, m dependent variables u, $u = (u^1, \dots \dots, u^m)$, and the first , second ,….., p- th order derivatives of u with respect to x denoted as

$u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \dots \dots, u_{(p)} = \left\{u_{i_1 i_2 \dots i_p}^\alpha\right\}$ respectively, $\alpha = 1, \dots \dots, m$ and other indices change from 1 to .

Definition 3.2.2.

A vector $C = (C^1, \dots \dots, C^n)$ where $C^i = C^i\left(x, u, u_{(1)}, u_{(2)}, \dots \dots, u_{(p)}\right)$, $i = 1, \dots., n$

Is called a conserved vector for Eq (**3.4**) if

$$\text{div}\, C = 0 \qquad (3.5)$$

on every solution of Eq (**3.4**). We can also say that Eq (**3.5**) is a conservation law of Eq(**3.4**) .

a conservation law for a system of partial differential equations can be defined similarly .

instead of dealing with functions $u^\alpha = u^\alpha(x)$ and their derivatives, which are also functions of x, one can treat all variables, x, u and derivatives of u, as independent variables, called differential variables. Variables with the same set of subscripts will be symmetric, for example $u_{ij} = u_{ji}$ and so on. Using the idea of differential variables one can reformulate the definition of a conservation law by introducing the operator of total differentiation with respect to x^i :

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots + u_{ij_1 \dots j_k}^\alpha \frac{\partial}{\partial u_{j_1 \dots j_k}^\alpha} + \cdots \qquad (3.6)$$

Where the usual convention of summation over repeated upper and lower indices is used. Hence

$$\operatorname{div} C|_{(2.4)} \equiv D_i\left(C^i\right)\Big|_{(2.4)} = 0 \qquad (3.7)$$

Where the notation $|_{(2.4)}$ means that the relation holds on any solution of Eq (**3.4**). if one of the variables, for example x^1 , is time t then the component C^1 is called the density of the conservation law.

Remark 3.2.1.

In practical calculations the conservation law (**3.7**) can be rewritten to an equivalent form. If

$$C^1|_{(2.4)} = \tilde{C}^1 + D_2(h^2) + \cdots + D_n(h^n)$$

Then one obtains the following conservation law**:** $D_t\left(\tilde{C}^1\right) + D_2\left(\tilde{C}^2\right) + \cdots + D_n\left(\tilde{C}^n\right) = 0$

Where $\tilde{C}^2 = C^2 + D_t(h^2), \dots \dots, \tilde{C}^n = C^n + D_t(h^n)$, Because $D_t D_i\left(h^i\right) = D_i D_t\left(h^i\right)$.

We can rewrite Eq(3.2) in the following form :

$$\frac{dC}{dt}\Big|_{(2.1)} \equiv D_t(C)|_{(2.1)} = 0 \qquad (3.8)$$

3.3 Hamilton's principle and the Euler-Lagrange equations

Consider again a motion of a dynamical system with a kinetic energy $T(t, q, \dot{q})$ and a potential energy $U(t, q)$. The function

$$\mathcal{L}(t, q, v) = T(t, q, \dot{q}) - U(t, q)$$

is called the Lagrangian of the system.

Hamilton's principle or the principle of least action, states that the true motion of the system between two chosen times t_1 and t_2 is described by the fact that the trajectories of the particles provide an extremum of the action functional

$$\int_{t_1}^{t_2} \mathcal{L}(t, q, v)\, dt \qquad (3.9)$$

This requirement is equivalent to the statement that the Euler-Lagrange equations **:**

$$\frac{\partial \mathcal{L}}{\partial q^\alpha} - D_t\left(\frac{\partial \mathcal{L}}{\partial v^\alpha}\right) = 0\,, \qquad \alpha = 1, \dots \dots, s \qquad (3.10)$$

In the case of several independent variables $x = (x^1, \dots\dots, x^n)$ and dependent variables $u = (u^1, \dots\dots, u^m)$ an action integral has the form

$$\int_V \mathcal{L}\big(x, u, u_{(1)}, u_{(2)}, \dots\dots, u_{(p)}\big)\ dx \tag{3.11}$$

Where V is an arbitrary n-dimensional volume in the space of the variables x and the Lagrangian $\mathcal{L}$ is a function depending on a finite number of differential variables. The corresponding Euler-Lagrange equations have the form :

$$\frac{\partial \mathcal{L}}{\partial u^\alpha} = 0\,, \qquad \alpha = 1, \dots\dots, s \tag{3.12}$$

Where

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha} + \cdots + (-1)^s D_{i_1} D_{i_2} \dots\dots D_{i_s} \ \frac{\partial}{\partial u^\alpha_{j_1 \dots j_k}} + \cdots$$

Is the variational derivative.

3.4 The Action Principle and Derivation of the Euler – Lagrange Equation

To find the equations of motion of a system with n degree of freedom, stated in terms of first-order differential equations rather than the second-order differential equations Newton's laws yield. We define n generalized coordinates $q_1, \dots\dots q_n$. Let the Lagrange, L be defined as

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, \dot{q}_i, t) \tag{3.13}$$

Where T is the kinetic energy of the system and V is the potential energy of the system. Define the action, S of the system to be

$$S = \int L(q_i, \dot{q}_i, t)\ dt \tag{3.14}$$

In order to find the path which nature would "choose" for the system, it turns out that we need to find the local extreme of the action. We therefore need to solve for the stationary points of S, where $\frac{\delta S}{\delta t} = 0$, which means that $\delta S = 0$. So we need to solve for L where

$$\delta S = 0 = \int_{-\infty}^{\infty} dt\ [L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i) - L(q_i, \dot{q}_i)] \tag{3.15}$$

But Taylor expansions tell us that for small changes $(x + \delta x\,, y + \delta y) - f(x, y) = f_x \delta_x + f_y \delta_y$. So expression (**3.15**) is equivalent to

$$\int_{-\infty}^{\infty} dt\ \sum_i \left[\frac{\partial L}{\partial q_i}\ \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\ \delta \dot{q}_i\right] \tag{3.16}$$

Since $\delta\dot{q}_i = \frac{d}{dt}\delta\dot{q}_i$, expression (**3.16**) is equivalent to

$$\int_{-\infty}^{\infty} dt \sum_i \left[\frac{\partial L}{\partial q_i}\ \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{d}{dt}\ \delta \dot{q}_i\right)\right] \tag{3.17}$$

Notice that $\int_{-\infty}^{\infty} dt\ \left[\frac{d}{dt}(fg)\right] = (fg)\big|_{-\infty}^{\infty} = 0$ if $(\pm\infty) = 0$. this implies that, for such functions f and g, $\int_{-\infty}^{\infty} dt\ [\dot{f}g + \dot{g}f] = 0$ by the chain rule, and so

$$\int_{-\infty}^{\infty} dt\ [\dot{f}g] = -\int_{-\infty}^{\infty} dt[\dot{g}f].$$

Therefore, assuming that $\delta q_i = 0$ at $\pm\infty$, expression (**3.17**) is equivalent to

$$\int_{-\infty}^{\infty} dt \sum_i \left[\frac{\partial L}{\partial q_i}\ \delta q_i - \left(\frac{d}{dt}\ \frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right] \tag{3.18}$$

$$\int_{-\infty}^{\infty} dt \sum_i \left[\frac{\partial L}{\partial q_i}\ - \frac{d}{dt}\ \frac{\partial L}{\partial \dot{q}_i}\ \delta q_i\right] = \delta S = 0 \tag{3.19}$$

Each coordinate q_i is independent of the others, and so is each variation δq_i, so each δq_i can have any value. Therefore, the only way to make this integral always equal to 0 is to demand that the bracketed part be equal to 0 for all i.

$$\frac{\partial L}{\partial q_i}\ - \frac{d}{dt}\ \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{3.20}$$

It turns out, that nature would always "choose" the motion of a system to be such that the Lagrangian L obeys this equation .

3.5 Hamilton's Equations

There is yet another way of formulating the equations of motion of a system, equivalent to the Lagrange formulation.

Define $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ and the Hamiltonian = $H \equiv \sum_i p_i \dot{q}_i - L$. This means that

$$dH = \sum_i \left(p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) - \frac{\partial L}{\partial t}\ dt \tag{3.21}$$

But $\frac{\partial L}{\partial \dot{q}_i} = p_i$, so

$$dH = \sum_i \left(p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - p_i d\dot{q}_i\right) - \frac{\partial L}{\partial t}\ dt$$

$$= \sum_i \left(\dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i\right) - \frac{\partial L}{\partial t} dt \tag{3.22}$$

Since H is a Legendre transformation,

$$dH = \sum_i \left(\frac{\partial H}{\partial p_i}\, dp_i + \frac{\partial H}{\partial q_i}\, dq_i\right) + \frac{\partial H}{\partial t}\, dt \qquad (3.23)$$

By comparing equations (**3.20**) and (**3.21**), we get **:**

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \ , \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial \dot{q}_i} \qquad (3.24)$$

This last expression is equal to $-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$ by Euler-Lagrange, which equals $-\dot{p}_i$. So we have :

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \ , \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i \qquad (3.25)$$

Put together, these are Hamilton's equations, another formulation of mechanics equivalent to the Euler-Lagrange equation.

3.5.1 Alternative Derivation

We can also derive Hamilton's equations directly from the action principle. To do this note that $H = \sum_i p_i\, \dot{q}_i - L$ implies that $L = \sum_i p_i\, \dot{q}_i - H$. So $S = \int_{-\infty}^{\infty} dt\, [\sum_i p_i\, \dot{q}_i - H]$ and we want to make $\delta S = 0$ under independent variations of $\dot{q}_i$ and $\dot{p}_i$ variables. So we have

$$\delta S = \int_{-\infty}^{\infty} dt\ \sum_i \left[\dot{q}_i \delta p_i + p_i \frac{\partial}{\partial t}\delta q_i - \frac{\partial H}{\partial p_i}\ \delta p_i - \frac{\partial H}{\partial q_i}\ \delta q_i\right]$$

$$= \int_{-\infty}^{\infty} dt\ \sum_i \left[\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) - \delta q_i \left(\dot{p}_i - \frac{\partial H}{\partial q_i}\right)\right] = 0 \qquad (3.26)$$

So $\frac{\partial H}{\partial p_i} = \dot{q}_i \ , \quad -\frac{\partial H}{\partial q_i} = \dot{p}_i$

3.5.2 Significance of the Hamiltonian:

We see the significance of the Hamiltonian most clearly by looking at its time derivative.

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i\right) + \frac{\partial H}{\partial t} = \sum_i \left(\dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_{i_i}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \qquad (3.27)$$

So $\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, from the definition of H.

So we see that if L is not an explicit function of t, then $\frac{dH}{dt} = 0$, and H is a constant of the motion. Therefore, H can give us a first-order differential equation for the motion, which we can solve with relative case.

Definition 3.5.1.

A conservation law is called a trivial conservation law if , $D_i(C^i) \equiv 0$. Or C^i are smooth functions of $\frac{\delta}{\delta u^\alpha}$, $D_i \frac{\delta}{\delta u^\alpha}$, ... two conservation laws which only differ by a trivial conservation law are regarded as equivalent.

3.6 Symmetry and Conservation Law

Conservation laws and symmetries have always been of considerable interest in science. They are important in the formulation and investigation of many mathematical models. There are several ideas for constructing conservation laws, one of them is conservation laws for differential equations obtained from a variational principle could appear from their symmetries, from works of Jacobi, Klein and Noether. Her we will considered about the work of Noether .

Symmetry in Physics generally means the system must be invariance under any kind of transformation, it become one of the most powerful tools of theoretical Physics. We will see that in Noether's Theorem below that will led to group theory being one of the areas of mathematics most studied by physicists.

3.6.1 Noether's Theorem

Consider a Lagrangian $L(q_i, \dot{q}_i, t)$, with equation of motion $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$.

Let $q_i(t) \longmapsto q_i^{'}(t) = q_i(t) + \varepsilon\delta q_i(t)$ be a (continuous) transformation of the generalized coordinates q_i that leaves the equation of motion unchanged or we can say we whatever have a continuous symmetry of Lagrange, there is an associated conservation law. by continuous symmetry we mean a symmetry with continuous constant parameter, typically infinitesimal "ε" that we can dial, and that measures how far from the identity the transformation is bringing us. In a sense ε measures the size of the transformation.

The condition that the equation of motions are unchanged is equivalent to requiring that the action $S = \int L\, dt$ be invariant, or more generally be unchanged by no more than additive constant term (as the equations of motion are derived from $\delta S = 0$ such a term will vanish).

This means we can allow the Lagrangian to vary by no more than an overall total time derivative, $L \longmapsto L^{'} = L + \alpha \frac{d}{dt} J$. This is because the overall time derivative will integrate out immediately in the action, leaving just an additive constant, and so does not affect the equations of motion **:**

$$S = \int L'\, dt = S = \int \left(L + \alpha \frac{d}{dt} J\right) dt = \int L\ dt + \alpha J(t_2) - \alpha J(t_1) \Rightarrow \delta S = \delta \int L\, dt$$

We could then formally state the theorem as follows:

Theorem (Noether)

Let $q_i(t) \longmapsto q_i^{'}(t) = q_i(t) + \varepsilon_\alpha \delta q_i(t)$ be an infinitesimal transformation of the generalized coordinates, parameterized by the (infinitesimal) quantities ε_α such that under this transformation $L \longmapsto L^{'} = L + \varepsilon_\alpha \frac{d}{dt} J$, then the quantities j_α given by

$j_\alpha \varepsilon_\alpha = \frac{\partial L}{\partial \dot{q}_i}\ \delta q_i \varepsilon_\alpha - J\varepsilon_\alpha$, Are conserved .

Proof:

Consider the variation in the lagrangian caused by the change in the coordinates and their velocities:

$$\delta L = \frac{\partial L}{\partial q_i}\ \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\ \delta \dot{q}_i$$

$$= \frac{\partial L}{\partial q_i}\ \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\ \frac{d}{dt} \delta q_i$$

$$= \frac{\partial L}{\partial q_i}\ \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\ \delta q_i\right) - \left(\frac{d}{dt}\ \frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i \qquad (3.28)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\ \delta q_i\right)$$

Where we have used the equations of motion to eliminate two of the terms. Now for each α, this variation multiplied by ε_α must be equal to the corresponding change $\varepsilon_\alpha \frac{d}{dt} J$ in the Lagrangian, so

$$\varepsilon_\alpha \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\ \delta q_i\right) = \varepsilon_\alpha \frac{d}{dt} J \qquad (3.29)$$

And this implies , $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\ \delta q_i - J\right) = 0$,

Hence

$$j_\alpha = \frac{\partial L}{\partial \dot{q}_i}\ \delta q_i - J \qquad (3.30)$$

Is conserved, or taking into account that the index α refers to numerous transformations we should write [2]

$$j_\alpha \varepsilon_\alpha = \frac{\partial L}{\partial \dot{q}_i}\ \delta q_i \varepsilon_\alpha - J\varepsilon_\alpha$$

3.6.2 Continuous Symmetries and Conservation Laws

Consider a simple mechanical system[3] with a generic action

$$\mathcal{A} = \int_{t_a}^{t_b} dt\ \ L(q(t), \dot{q}(t), t) \qquad (3.31)$$

Suppose $\mathcal{A}$ is invariant under a continuous set of transformations of the dynamical variables :

$$q(t) \longrightarrow q^{'}(t) = f\big(q(t), \dot{q}(t)\big) \qquad (3.32)$$

Where $f\big(q(t), \dot{q}(t)\big)$ is some functional of $q(t)$. Such transformations are called symmetry transformation. Thereby it is important that the equations of motion are not used when establishing the invariance of the action under (**3.32**).

If the action is subjected successively to two symmetry transformations, the result is again a symmetry transformation. Thus, symmetry transformations form a group called the symmetry group of the system which we will talk about it in the next chapter. For infinitesimal symmetry transformations (**3.32**), the difference

$$\delta_s q(t) \equiv q^{'}(t) - q(t) \qquad (3.33)$$

Which will be a symmetry variation. It has the general form

$$\delta_s q(t) = \epsilon\Delta\big(q(t), \dot{q}(t)\big) \qquad (3.32)$$

Symmetry variations must not be confused with ordinary variations $\delta q(t)$ that used to derive the Euler-Lagrange equations (**3.20**). while the ordinary variations $\delta q(t)$ vanish at initial and final times, $\delta q(t_b) = \delta q(t_a) = 0$, the symmetry variations $\delta_s q(t)$ are usually nonzero at the ends.

[2] Note the important thing to note is that there is a separate conserved quantity for each ε_α – where α is used to index the different transformations. Note also that this formulation of the theorem does not really take into account transforming time, though we can sort handle this – see the example after it. Note also what the theorem essentially means is that for every continuous symmetry there corresponds a conserved quantity, which is a really cool result.

[3] http://users.physik.fu-berlin.de/~kleinert/b6/psfiles/Chapter-7-conslaw.pdf

Let us calculate the change of the action under a symmetry variation (**3.32**). using the chain rule of differentiation and a an integration by parts, we obtain

$$\delta_s \mathcal{A} = \int_{t_a}^{t_b} dt \left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot{q}(t)}\right] \delta_s q(t) + \frac{\partial L}{\partial \dot{q}(t)} \delta_s q(t) \Big|_{t_a}^{t_b} \qquad (3.33)$$

For orbits $q(t)$ that satisfy the Euler-Lagrange equations (**3.20**), only boundary terms survive , and we are left with

$$\delta_s \mathcal{A} = \epsilon \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) \Big|_{t_a}^{t_b} \qquad (3.34)$$

Under the symmetry assumption, $\delta_s \mathcal{A}$ vanishes for any orbit $q(t)$, implying that the quantity

$$Q(t) \equiv \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) \qquad (3.35)$$

Is the same at times $t = t_a$ and $t = t_b$. Since t_b is arbitrary, $Q(t)$ is independent of the time t, i.e., it satisfies

$$Q(t) \equiv Q \qquad (3.36)$$

It is a conserved quantity, a constant of motion. The expression on the right-hand side of (**3.35**) is called Noether charge.

The statement can be generalized to transformations $\delta_s q(t)$ for which the action is not directly invariant but its symmetry variation is equal to an arbitrary boundary term :

$$\delta_s \mathcal{A} = \epsilon \, \Lambda(q, \dot{q}, t) \Big|_{t_a}^{t_b} \qquad (3.37)$$

In this case,

$$Q(t) \equiv \frac{\partial L}{\partial \dot{q}} \Delta(q, \dot{q}, t) - \Lambda(q, \dot{q}, t) \qquad (3.38)$$

Is a conserved Noether charge .

It is also possible to derive the constant of motion (**3.38**) without invoking the action, but starting from the Lagrange . for it we evaluate the symmetry variation as follows :

$$\delta_s L \equiv L(q + \delta_s , \dot{q} + \delta_s \dot{q}) - L(q, \dot{q}) = \left[\frac{\partial L}{\partial q(t)} - \partial_t \frac{\partial L}{\partial \dot{q}(t)}\right] \delta_s q(t) + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}(t)} \delta_s q(t)\right] \quad (3.39)$$

On account of Euler-Lagrange equations (**3.20**), the first term on the right-hand side vanishes as before, and only the last term survives. The assumption of invariance of the action up to a possible surface term in Eq.(**3.37**) is equivalent to assuming that the symmetry variation of the Lagrangian is a total time derivative of some function $\Lambda(q,\dot{q},t)$:

$$\delta_s L(q,\dot{q},t) = \epsilon\ \frac{d}{dt}\Lambda(q,\dot{q},t) \tag{3.40}$$

Inserting this into the left-hand side of (**3.39**), we find

$$\epsilon\ \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\Delta(q,\dot{q},t) - \Lambda(q,\dot{q},t)\right] = 0 \tag{3.41}$$

Thus recovering again the conserved Noether charge (**3.36**). The existence of a conserved quantity for every continuous symmetry is the content of Noether's theorem.

3.7 Alternative Derivation

Let us do the substantial variation in Eq(**3.33**) explicitly, and change a classical orbit $q_c(t)$, that extremizes the action, by an arbitrary variation $\delta_a q(t)$. If this does not vanish at the boundaries, the action changes by a pure boundary term that follows directly from Eq(**3.33**) :

$$\delta_s \mathcal{A} = \frac{\partial L}{\partial \dot{q}}\delta_s q\ \Big|_{t_a}^{t_b} \tag{3.42}$$

From this equation we can derive Noether's theorem in yet another way. Suppose we subject a classical orbit to a new type of symmetry variation, to be called local symmetry transformations, which generalizes the previous symmetry variations Eq(**3.32**) by making the parameter ϵ time- dependent :

$$\delta_s^t q(t) = \epsilon(t)\Delta(q(t),\dot{q}(t),t) \tag{3.43}$$

The superscript t of $\delta_s^t q(t)$ indicates the new time dependence in the parameter $\epsilon(t)$. These variations may be considered as a special set of the general variations $\delta_a q(t)$ introduced above. Thus also $\delta_s^t \mathcal{A}$ must be a pure boundary term of the type Eq(**3.42**). For the subsequent discussion it is to introduce the infinitesimally transformed orbit

$$q^\epsilon(t) \equiv q(t) + \delta_s^t q(t) = q(t) + \epsilon(t)\Delta(q(t),\dot{q}(t),t) \tag{3.44}$$

$$L^\epsilon \equiv L(q^\epsilon(t),\dot{q}^\epsilon(t)) \tag{3.45}$$

Using the time-dependent parameter $\epsilon(t)$,the local symmetry variation of the action can be written as

$$\delta_s^t \mathcal{A} = \int_{t_a}^{t_b} dt\ \left[\frac{\partial L^\epsilon}{\partial \epsilon(t)} - \frac{d}{dt}\ \frac{\partial L^\epsilon}{\partial \dot{\epsilon}(t)}\right]\ \epsilon(t) + \frac{d}{dt}\ \left[\frac{\partial L^\epsilon}{\partial \dot{\epsilon}(t)}\right]\ \epsilon(t)\ \Big|_{t_a}^{t_b} \tag{3.46}$$

Along the classical orbits, the action and satisfies the equation

$$\frac{\delta \mathcal{A}}{\delta \epsilon(t)} = 0 \qquad (3.47)$$

Which translates for a local action to an Euler-Lagrange type of equation **:**

$$\frac{\partial L^{\epsilon}}{\partial \epsilon(t)} - \frac{d}{dt}\frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}(t)} = 0 \qquad (3.48)$$

This can also be checked explicitly by differentiating (**3.45**) according to the chain rule of differentiation **:**

$$\frac{\partial L^{\epsilon}}{\partial \epsilon(t)} = \frac{\partial L^{\epsilon}}{\partial q(t)}\,\Delta(q,\dot{q},t) + \frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}(t)}\dot{\Delta}(q,\dot{q},t) \qquad (3.49)$$

$$\frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}(t)} = \frac{\partial L^{\epsilon}}{\partial \dot{q}(t)}\Delta(q,\dot{q},t) \qquad (3.50)$$

And inserting on the right-hand side the ordinary Euler-Lagrange equations .

We now invoke the symmetry assumption that the action is a pure surface term under the time-independent transformations (**3.43**). this implies that

$$\frac{\partial L^{\epsilon}}{\partial \epsilon} = \frac{d}{dt}\,\Lambda \qquad (3.51)$$

Combining this with (**3.48**), we find that this is the same charge as that derived by the previous method.

3.8 Displacement and Energy Conservation

As a simple but physically important example consider the case that the Lagrangian does not depend exiplicity on time, i.e,that $L(q,\dot{q},t) \equiv L(q,\dot{q})$. Let us perform a time translation on the coordinate frame :

$$t' = t - \epsilon \qquad (3.52)$$

In the new coordinate frame , the same orbit has the new description

$$\dot{q}(t) = q(t) \qquad (3.53)$$

i.e, the orbit $\dot{q}(t)$ at the translated time t' is precisely the same as the orbit $q(t)$ at the original time t. If we replace the argument of $\dot{q}(t)$ in (**3.53**) by t', we describe a time-translated orbit in terms of the original coordinates. This implies the symmetry variation of the form (**3.32**) :

$$\delta_s q(t) = q'(t) - q(t) = q(t' + \epsilon) - q(t)$$

$$= q(t') + \epsilon\dot{q}(t) - q(t) = \epsilon\dot{q}(t) \tag{3.54}$$

The symmetry variation of the Lagrangian is in general

$$\delta_s L = L(q'(t), \dot{q}'(t) - L\big(q(t), \dot{q}(t)\big) = \frac{\partial L}{\partial q}\delta_s q(t) + \frac{\partial L}{\partial \dot{q}}\delta_s \dot{q}(t) \tag{3.55}$$

Inserting $\delta_s q(t)$ from Eq(**3.45**) we find, without using Euler-Lagrange equation,

$$\delta_s L = \epsilon\left(\frac{\partial L}{\partial \dot{q}}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}\right) = \epsilon\frac{d}{dt}L \tag{3.56}$$

This has precisely the form of Eq.(3.39), with $\Lambda = L$ as expected, since time translations are symmetry transformations. Here the function Λ in Eq (**3.39**) happens to coincide with the Lagrangian. According to Eq.(**3.37**), we find the Noether charge

$$Q = \frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q}) \tag{3.57}$$

To be a constant of motion. This is recognized as the Legendre transform of the Lagrangian which is, of course, the Hamiltonian of the system.

Let us briefly check how this Noether charge is obtained from the alternative formula Eq(**3.37**). The time-dependent symmetry variation is here

$$\delta_s^t q(t) = \epsilon(t)\,\dot{q}(t) \tag{3.58}$$

Under which the Lagrangian is changed by

$$\delta_s^t L = \frac{\partial L}{\partial q}\epsilon\dot{q} + \frac{\partial L}{\partial \dot{q}}(\dot{\epsilon}\dot{q} + \epsilon\ddot{q}) = \frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}}\epsilon + \frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}}\dot{\epsilon} \tag{3.59}$$

With $$\frac{\partial L^{\epsilon}}{\partial \dot{\epsilon}} = \frac{\partial L}{\partial \dot{q}}\dot{q} \tag{3.60}$$

And $$\frac{\partial L^{\epsilon}}{\partial \epsilon} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\epsilon\ddot{q} = \frac{d}{dt}L \tag{3.61}$$

This shows that time translations fulfill the symmetry condition Eq(**3.51**), and that the Noether charge Eq(**3.53**) coincides with the Hamiltonian found in Eq.(**3.37**).

3.9 Continuous Symmetry Implies Conserved Charges

Consider a particle moving in two dimensions under the influence of an external potential $U(r)$. The potential is a function only of the magnitude of the vector r. The Lagrangian is then

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) \tag{3.62}$$

Where we have chosen generalized coordinates (r, ϕ). The momentum conjugate to ϕ is $p_\phi = mr^2\dot{\phi}$. The generalized force F_ϕ clearly vanishes[4], since L doesnot depend on the coordinate ϕ. (one says that L is 'cyclic' in ϕ) Thus , although $r = r(t)$ and $\phi = \phi(t)$ will in general be time-dependent, the combination $p_\phi = mr^2\dot{\phi}$ is constant. This is the conserved angular momentum about the $\tilde{z}$ axis .

If instead the particle moved in a potential $U(y)$, independent of x, then writing

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(y) \tag{3.63}$$

We have that the momentum $p_x = \partial L/\partial \dot{x} = m\dot{x}$ is conserved, because the generalized force $F_x = \partial L/\partial x = 0$ vanishes. This situation pertains in a uniform gravitational field, with $U(x, y) = mgy$, independent of x. The horizontal component of momentum is conserved.

In general, whenever the system exihibits a continuous symmetry, there is an associated conserved charge. (The terminology 'charge' is from field theory). Indeed, this is a rigorous result, known as Noether's Theorem. Consider a one-parameter family of transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q, \zeta) \tag{3.64}$$

Where ζ is the continuous parameter. Suppose further (without loss of generality) that at $\zeta = 0$ this transformation is the identity, i.e, $\tilde{q}_\sigma(q, \zeta) = q_\sigma$. The transformation may be nonlinear in the generalized coordinates. Suppose further that the Lagrangian L is invariant under the replacement $q \longrightarrow \tilde{q}$. Then we must have

$$0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} L\left(\tilde{q}, \dot{\tilde{q}}, t\right) = \frac{\partial L}{\partial q_\sigma}\frac{d\tilde{q}_\sigma}{d\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d\dot{\tilde{q}}_\sigma}{d\zeta}\bigg|_{\zeta=0}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\frac{d\tilde{q}_\sigma}{d\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d}{dt}\left(\frac{d\tilde{q}_\sigma}{d\zeta}\right)_{\zeta=0}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\frac{d\tilde{q}_\sigma}{d\zeta}\right)_{\zeta=0} \tag{3.65}$$

Thus, there is an associated conserved charge

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d\dot{\tilde{q}}_\sigma}{d\zeta}\bigg|_{\zeta=0} \tag{3.63}$$

[4] Chapter 7- Noether Theorem - http://users.physik.fu-berlin.de/~kleinert/b6/psfiles/Chapter-7-conslaw.pdf

Example 3.9.1 (one –parameter families of transformations)

Consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U\left(\sqrt{x^2 + y^2}\right) \tag{3.64}$$

In two-dimensional polar coordinates, we have **:** $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$

And we may now , $\tilde{r}(\zeta) = r$

$$\tilde{\phi}(\zeta) = \phi + \zeta \tag{3.65}$$

Note that $\tilde{r}(\zeta) = r$ and $\tilde{\phi}(\zeta) = \phi$, i.e. the transformation is the identity when $\zeta = 0$. We now have $\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{d\tilde{\dot{q}}_\sigma}{d\zeta}\Big|_{\zeta=0} = \frac{\partial L}{\partial \dot{r}} \frac{d\tilde{r}}{d\zeta}\Big|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\tilde{\phi}}{d\zeta}\Big|_{\zeta=0} = mr^2\dot{\phi}$

Another way to derive the same result which is somewhat instructive is to work out the transformation in Cartesian coordinates. We then have

$$\tilde{x}(\zeta) = x\cos\zeta - y\sin\zeta$$

$$\tilde{y}(\zeta) = x\sin\zeta + y\cos\zeta$$

Thus,

$$\frac{\partial \tilde{x}}{\partial \zeta} = -\tilde{y}\ , \ \frac{\partial \tilde{y}}{\partial \zeta} = \tilde{x} \tag{3.66}$$

And

$$\Lambda = \frac{\partial L}{\partial \dot{x}} \frac{d\tilde{x}}{d\zeta}\Big|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}} \frac{d\tilde{y}}{d\zeta}\Big|_{\zeta=0} = m(x\dot{y} - y\dot{x}) \tag{3.67}$$

But

$$m(x\dot{y} - y\dot{x}) = m\tilde{z}\ \cdot \boldsymbol{r} \times \dot{\boldsymbol{r}} = mr^2\dot{\phi} \tag{3.68}$$

As another example, consider the potential

$$U(\rho, \phi, z) = V(\rho, a\phi + z) \tag{3.69}$$

Where (ρ, ϕ, z) are cylindrical coordinates for a particle of mass m, and where a is a constant with dimensions of length. The Lagrangian is

$$\frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - V(\rho, a\phi + z) \tag{3.70}$$

This model possesses a helical symmetry, with a one-parameter family

$$\tilde{\rho}(\zeta) = \rho$$

$$\tilde{\phi}(\zeta) = \phi + \zeta$$

$$\tilde{z}(\zeta) = z - \zeta a$$

Note that **:** $a\tilde{\phi} + \tilde{z} = a\phi + z$

So the potential energy, and the Lagrangian as well, is invariant under this one-parameter family of transformations. The conserved charge for this symmetry is

$$\Lambda = \frac{\partial L}{\partial \dot{\rho}} \frac{d\tilde{\rho}}{d\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}} \frac{d\tilde{\phi}}{d\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{z}} \frac{d\tilde{z}}{d\zeta}\bigg|_{\zeta=0} = m\rho^2\dot{\phi} - ma\dot{z} \tag{3.71}$$

We can check explicitly that Λ is conserved, using the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = \frac{\partial L}{\partial \phi} = -a\frac{\partial V}{\partial z}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z}$$

Thus, $\dot{\Lambda} = \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) - a\frac{d}{dt}(m\dot{z}) = 0$ (3.72)

3.10 Conservation of Linear Angular Momentum

Suppose that the Lagrangian of a mechanical system is invariant under a uniform translation of all particles in the $\hat{\boldsymbol{n}}$ direction. Then our one-parameter family of transformations is given by

$$\tilde{\boldsymbol{x}}_a = \boldsymbol{x}_a + \zeta\hat{\boldsymbol{n}} \tag{3.73}$$

And the associated conserved Noether charge is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{P} \tag{3.74}$$

Where $\boldsymbol{P} = \sum_a \boldsymbol{P}_a$ is the total momentum of the system.

If the Lagrangian of a mechanical system is invariant under rotations about an axis $\hat{\boldsymbol{n}}$, then

$$\tilde{\boldsymbol{x}}_a = R(\zeta, \hat{\boldsymbol{n}})\boldsymbol{x}_a$$

$$= \boldsymbol{x}_a + \zeta\hat{\boldsymbol{n}} \times \boldsymbol{x}_a + \mathcal{O}(\zeta^2) \tag{3.75}$$

Where we have expanded the rotation matrix $R(\zeta, \hat{\boldsymbol{n}})$ in powers of ζ. The conserved Noether charge associated with this symmetry is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} \times \boldsymbol{x}_a = \hat{\boldsymbol{n}} \cdot \sum_a \boldsymbol{x}_a \times \boldsymbol{P}_a = \hat{\boldsymbol{n}} \cdot L \qquad (3.76)$$

Where L is the total angular momentum of the system.

3.10.1 Invariance of L vs Invariance of S :

Observant readers might object that demanding invariance of L is too strict. We should instead be demanding invariance of the action [5] . Suppose S is invariant under

$$t \longrightarrow \tilde{t}(q, t, \zeta) \qquad (3.77)$$

$$q_\sigma(t) \longrightarrow \tilde{q}_\sigma(q, t, \zeta)$$

Then invariance of S means

$$S = \int_{t_a}^{t_b} dt\ L(q, \dot{q}, t) = \int_{\tilde{t}_a}^{\tilde{t}_b} dt\ L(\tilde{q}, \dot{\tilde{q}}, t) \qquad (3.78)$$

Note that t is a dummy variable of integration, so it doesn't matter whether we call it t or $\tilde{t}$. The endpoints of the integral, however, do change under the transformation, for which $\delta t = \tilde{t} - t$ and $\delta t = \tilde{q}(\tilde{t}) - q(t)$ are both small. Thus,

$$S = \int_{t_a}^{t_b} dt\ L(q, \dot{q}, t) = \int_{\tilde{t}_a + \delta t_a}^{\tilde{t}_b + \delta t_b} dt\ \left\{L(q, \dot{q}, t) + \frac{\partial L}{\partial q_\sigma}\ \tilde{\delta} q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma}\ \tilde{\delta} \dot{q}_\sigma + \cdots\right\} \qquad (3.79)$$

Where

$$\tilde{\delta} q_\sigma(t) \equiv \tilde{q}_\sigma(t) - q_\sigma(t)$$

$$= \tilde{q}_\sigma(\tilde{t}) - \tilde{q}_\sigma(\tilde{t}) + \tilde{q}_\sigma(t) - q_\sigma(t)$$

$$= \delta q_\sigma - \dot{q}_\sigma \delta t + \mathcal{O}(\delta q \delta t) \qquad (3.80)$$

Substracting Eq.(**3.79**) from Eq(**3.80**), we obtain

$$0 = L_b \delta t_b - L_a \delta t_a + \left.\frac{\partial L}{\partial \dot{q}_\sigma}\right|_b \tilde{\delta} \dot{q}_{\sigma,b} - \left.\frac{\partial L}{\partial \dot{q}_\sigma}\right|_a \tilde{\delta} \dot{q}_{\sigma,a} + \int_{\tilde{t}_a + \delta t_a}^{\tilde{t}_b + \delta t_b} dt \left\{\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\right\} \tilde{\delta} q_\sigma(t)$$

$$= \int_{t_a}^{t_b} dt\ \frac{d}{dt} \left\{\left(L - \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma\right) \delta t + \frac{\partial L}{\partial \dot{q}_\sigma} \delta q_\sigma\right\} \qquad (3.81)$$

[5] Indeed, we should be demanding that S only change by a function of the endpoint values.

Where $L_{a,b}$ is $L(q,\dot{q},t)$ evaluated at $t = t_{a,b}$. Thus, if $\zeta \equiv \delta\zeta$ is infinitesimal ,and

$$\delta t = A(q,t)\delta\zeta$$

$$\delta q_\sigma = B_\sigma(q,t)\delta\zeta \tag{3.82}$$

Then the conserved charge is

$$\Lambda = \left(L - \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right)A(q,t) + \frac{\partial L}{\partial \dot{q}_\sigma}B_\sigma(q,t)$$

$$= -H(q,p,t)A(q,t) + p_\sigma B_\sigma(q,t) \tag{3.83}$$

Thus, when $A = 0$,we recover our earlier results, obtained by assuming invariance of L. Note that conservation of H follows from time translation invariance : $t \longrightarrow t + \zeta$, for which $A = 1$ and $B_\sigma = 0$. Here we have written

$$H = p_\sigma \dot{q}_\sigma - L \tag{3.84}$$

And expressed it in terms of the momenta p_σ, the coordinates q_σ, and time t. H is called Hamiltonian.

3.11 The Hamiltonian

The Lgrangian is a function of generalized coordinates, velocities and time. The canonical momentum conjugate to the generalized coordinate q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \tag{3.85}$$

The Hamiltonian is a function of coordinates, momenta, and time. It is defined as the Legendre transform of :

$$H(q,p,t) = \sum_\sigma p_\sigma \dot{q}_\sigma - L \tag{3.86}$$

Let's examine the differential of :

$$dH = \sum_\sigma \left(\dot{q}_\sigma\, dp_\sigma + p_\sigma\, d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma}\, dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma\right) - \frac{\partial L}{\partial t}\, dt$$

$$= \sum_\sigma \left(\dot{q}_\sigma\, dp_\sigma - \frac{\partial L}{\partial q_\sigma}\, dq_\sigma\right) - \frac{\partial L}{\partial t}\, dt$$

Where we have invoked the definition of p_σ to cancel the coefficients of $d\dot{q}_\sigma$. Since

$\dot{p}_\sigma = \partial L/\partial q_\sigma$, we have Hamilton's equations of motion,

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \qquad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \tag{3.87}$$

Thus, we can write $dH = \sum_\sigma (\dot{q}_\sigma \, dp_\sigma + \dot{p}_\sigma \, dq_\sigma) - \frac{\partial L}{\partial t} \, dt$

Dividing by dt, we obtain

$$\frac{\partial H}{\partial p_\sigma} = -\frac{\partial L}{\partial t} \tag{3.88}$$

Which says that the Hamiltonian is conserved (i.e. it doesn't change with time) whenever there is no explicit time dependence to L.

Example 3.11.1.

For a simple $d = 1$ system with $L = \frac{1}{2} m\dot{x}^2 - U(x)$, we have $p = m\dot{x}$ and

$$H = p\dot{x} - L = \frac{1}{2} m\dot{x}^2 + U(x) = \frac{p^2}{2m} + U(x)$$

Example 3.11.2.

Consider the mass point – wedge system analyzed above, with

$$L = \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \frac{1}{2} m(1+\tan^2\alpha)\dot{x}^2 - mgx\tan\alpha$$

The canonical momenta are

$$P = \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + m\dot{x} \tag{3.89}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{X} + m(1+\tan^2\alpha)\dot{x}$$

The Hamiltonian is given by **:** $H = P\dot{X} + p\dot{x} - L$

$$H = \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \frac{1}{2} m(1+\tan^2\alpha)\dot{x}^2 - mgx\tan\alpha \tag{3.90}$$

However, this is not quite H since $H = H(X, x, P, p, t)$ must be expressed in terms of the coordinates and the momenta and not the coordinates and velocities. So we must eliminate $\dot{X}$ and $\dot{x}$ in favor of P and p. We do this by inverting the relations

$$\begin{pmatrix} P \\ p \end{pmatrix} = \begin{pmatrix} M+m & m \\ m & m(1+\tan^2\alpha) \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix}$$

To obtain

$$\begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix} = \frac{1}{m(M+(M+m)\tan^2\alpha)} \begin{pmatrix} m(1+\tan^2\alpha) & -m \\ m & M+m \end{pmatrix} \begin{pmatrix} P \\ p \end{pmatrix} \quad (3.91)$$

Substituting into Eq(**3.90**), we obtain

$$H = \frac{M+m}{2m}\frac{p^2\ \cos^2\alpha}{M+m\sin^2\alpha} - \frac{Pp\ \cos^2\alpha}{M+\sin^2\alpha} + \frac{p^2}{2(M+\sin^2\alpha)} + mgx\ \tan\alpha$$

Notice that $\dot{P} = 0$ since $\frac{\partial L}{\partial X} = 0$. P is the total horizontal momentum of the system (wedge plus particle) and it is conserved.

Example 3.11.3.

Is $= T + U$?

The most general form of the kinetic energy is

$$T = T_2 + T_1 + T_0 \quad (3.92)$$

$$= \frac{1}{2} T^{(2)}_{\sigma\sigma'}\ (q,t)\dot{q}_\sigma\ \dot{q}_{\sigma'} + T^{(1)}_\sigma\ (q,t)\dot{q}_\sigma + T^{(0)}(q,t)$$

Where $T^{(n)}(q,\dot{q},t)$ is homogeneous of degree n in the velocities [6]. We assume a potential energy of the form

$$U = U_1 + U_0 \quad (3.93)$$

$$= U^{(1)}_\sigma(q,t)\dot{q}_\sigma + U^{(0)}_\sigma(q,t)$$

Which allows for velocity-dependent forces, as we have with charged particles moving in an electromagnetic field. The Lagrangian is then

$$L = T - U == \frac{1}{2} T^{(2)}_{\sigma\sigma'}\ (q,t)\dot{q}_\sigma\ \dot{q}_{\sigma'} + T^{(1)}_\sigma\ (q,t)\dot{q}_\sigma + T^{(0)}(q,t) - U^{(1)}_\sigma(q,t)\dot{q}_\sigma + U^{(0)}_\sigma(q,t)$$

The canonical momentum conjugate to q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T^{(2)}_{\sigma\sigma'}\ (q,t)\dot{q}_{\sigma'} + T^{(1)}_\sigma\ (q,t) - U^{(1)}_\sigma(q,t) \quad (3.94)$$

---

[6] A homogeneous function of degree k satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$. It is then easy to prove Euler's theorem, $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf$.

Which is inverted to give

$$\dot{q}_\sigma = T_{\sigma\sigma'}^{(2)^{-1}} \left(p_{\sigma'} - T_{\sigma'}^{(1)} + U_{\sigma'}^{(1)}\right) \tag{3.95}$$

The Hamiltonian is then

$$H = p_\sigma \dot{q}_\sigma - L$$

$$= T_{\sigma\sigma'}^{(2)^{-1}} \left(p_{\sigma'} - T_\sigma^{(1)} + U_\sigma^{(1)}\right)\left(T_{\sigma\sigma'}^{(2)^{-1}} \left(p_{\sigma'} - T_{\sigma'}^{(1)} + U_{\sigma'}^{(1)}\right)\right) - T_0 + U_0$$

$$= T_2 - T_0 + U_0 \tag{3.96}$$

If T_0, T_1 and U_1 vanish, i.e. if $T(q, \dot{q}, t)$ is a homogeneous function of degree two in the generalized velocities, and $U(q,t)$ is velocity-independent, then $H = T + U$. But if T_0 or T_1 is nonzero, or the potential is velocity-dependent then $H \neq T + U$.

Example 3.11.4

Consider a bead of mass m constrained to move along a hoop of radius a. The hoop is further constrained to rotate with angular velocity $\dot{\phi} = \omega$ about the $\hat{z}$-axis, as shown in **Fig 3.1**

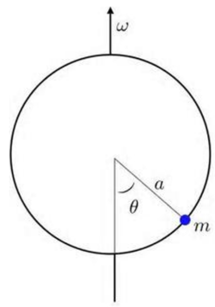

Fig (3.1) : A bead of mass m on a rotating hoop of radius a.

The most convenient set of generalized coordinates is spherical polar (r, θ, ϕ), in which case

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right)$$

$$= \frac{1}{2}ma^2\left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right)$$

Thus, $T_2 = \frac{1}{2}ma^2\, \dot{\theta}^2$ and $T_2 = \frac{1}{2}ma^2\omega^2 \sin^2\theta$. The potential energy is

$$U(\theta) = mga(1 - \cos\theta) \tag{3.97}$$

The momentum conjugate to θ is $p_\theta = ma^2\, \dot{\theta}$, and thus

$H(\theta, p) = T_2 - T_0 + U$

$$= \frac{1}{2} ma^2 \, \dot{\theta}^2 - \frac{1}{2} ma^2 \omega^2 \sin^2 \theta + mga(1 - \cos\theta)$$

$$= \frac{p_\theta^2}{2ma^2} - \frac{1}{2} ma^2 \omega^2 \sin^2 \theta + mga(1 - \cos\theta) \quad (3.98)$$

For this problem, we can define the effective potential

$$U_{eff}(\theta) \equiv U - T_0 = mga(1 - \cos\theta) - \frac{1}{2} ma^2 \omega^2 \sin^2 \theta$$

$$= mga(1 - \cos\theta - \frac{\omega^2}{2\omega_0^2} \sin^2 \theta) \quad (3.99)$$

Where $\omega_0^2 \equiv {}^{g}/_{a}$. The Lagrangian may then be written

$$L = \frac{1}{2} ma^2 \, \dot{\theta}^2 - U_{eff}(\theta) \quad (3.100)$$

And thus the equations of motion are

$$ma^2 \, \ddot{\theta} = -\frac{\partial U_{eff}}{\partial \theta} \quad (3.101)$$

Equilibrium is achieved when $U'_{eff}(\theta) = 0$, which gives

$$\frac{\partial U_{eff}}{\partial \theta} = mga \sin\theta \left\{1 - \frac{\omega^2}{\omega_0^2} \cos\theta\right\} = 0 \quad (3.102)$$

i.e, $\theta^* = 0$, $\theta^* = \pi$, or $\theta^* = \pm \cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right)$, where the last pair of equilibria are present only for $\omega^2 > \omega_0^2$. The stability of these equilibia is assessed by examining the sign of $U''_{eff}(\theta^*)$. We have

$$U''_{eff}(\theta) = mga \, \left\{\cos\theta - \frac{\omega^2}{\omega_0^2} (2\cos^2\theta - 1)\right\} \quad (3.103)$$

Thus,

$$U''_{eff}(\theta^*) = \begin{cases} mga \, \left(1 - \frac{\omega^2}{\omega_0^2}\right) & at \; \theta^* = 0 \\ -mga \, \left(1 + \frac{\omega^2}{\omega_0^2}\right) & at \theta^* = 0 \\ mga \, \left(\frac{\omega^2}{\omega_0^2} - \frac{\omega_0^2}{\omega^2}\right) & at \theta^* = \pm \cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right) \end{cases} \quad (3.104)$$

Thus, $\theta^* = 0$ is stable for $\omega^2 < \omega_0^2$ but becomes unstable when the rotation frequency ω is sufficiently large, i.e. when $\omega^2 > \omega_0^2$. In this regime, there are two new equilibria, at $\theta^* = \pm\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right)$, which are both stable. The equilibrium at $\theta^* = \pi$ is always unstable, independent of the value of ω. The situation is depicted in **Fig.3.2** .

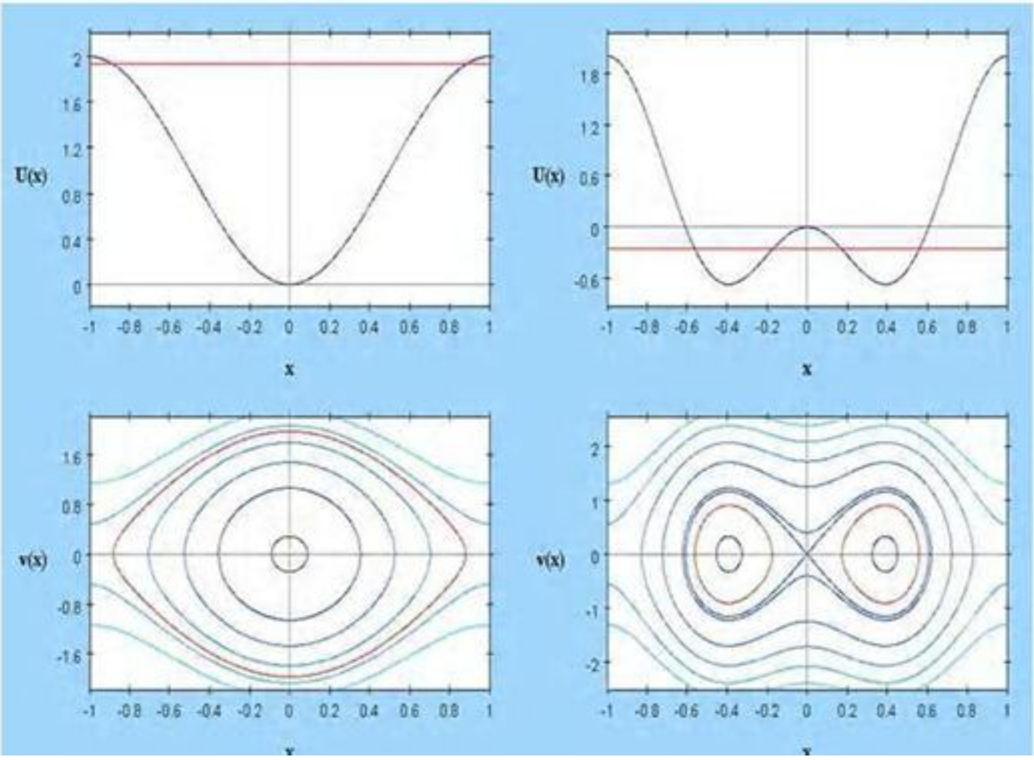

Fig 3.2 : The effective potential $U_{eff}(\theta) = mga\left\{1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\sin^2\theta\right\}$. (The dimensionless potential $\widetilde{U}_{eff} = {U_{eff}}/{mga}$ is shown, where $x = \frac{\theta}{\pi}$). Left panels : $\omega = \frac{1}{2}\sqrt{3}\omega_0$ right panel : $\omega = \sqrt{3}\omega_0$

3.12 Charged particle in a Magnetic Field

Consider next the case of a charged particle moving in the presence of an electromagnetic field. The particle's potential energy is

$$U(\boldsymbol{r}, \dot{\boldsymbol{r}}) = q\phi(\boldsymbol{r}, t) - \frac{q}{c}\boldsymbol{A}(\boldsymbol{r}, t)\cdot\dot{\boldsymbol{r}} \qquad (3.105)$$

Which is velocity-dependent. The kinetic energy is $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2$, as usual. Hence $\phi(\boldsymbol{r})$ is the scalar potential and $\boldsymbol{A}(\boldsymbol{r})$ the vector potential. The electric and magnetic fields are given by

$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$, $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ The canonical momentum is

$$\boldsymbol{p} = \frac{\partial\boldsymbol{L}}{\partial\dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}} + \frac{q}{c}\boldsymbol{A} \qquad (3.106)$$

And hence the Hamiltonian is

$$H(\boldsymbol{r},\boldsymbol{p},t) = \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L$$

$$= m\dot{\boldsymbol{r}}^{\mathbf{2}} + \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} - \frac{1}{2}\,m\dot{\boldsymbol{r}}^{\mathbf{2}} - \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + q\phi$$

$$= \frac{1}{2}\,m\dot{\boldsymbol{r}}^{\mathbf{2}} + q\phi$$

$$= \frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}(\boldsymbol{r},t)\right)^2 + q\phi(\boldsymbol{r},t) \qquad (3.107)$$

If $\boldsymbol{A}$ and ϕ are time-independent, then $H(\boldsymbol{r},\boldsymbol{p})$ is conserved.

Let's work out the equations of motion. We have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}} \qquad (3.108)$$

Which gives

$$m\ddot{\boldsymbol{r}} + \frac{q}{c}\frac{d\boldsymbol{A}}{dt} = -q\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{r}}) \qquad (3.109)$$

Or in component notation

$$m\ddot{x}_i + \frac{q}{c}\frac{\partial A_i}{\partial x_j}\dot{x}_j + \frac{q}{c}\frac{\partial A_i}{\partial t} = -q\frac{\partial \phi}{\partial x_i} + \frac{q}{c}\frac{\partial A_j}{\partial x_i}\dot{x}_j \qquad (3.110)$$

Which is to say

$$m\ddot{x}_i = -q\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j \qquad (3.111)$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk} :

$$B_i = \epsilon_{ijk}\ \frac{\partial A_k}{\partial x_j} \qquad (3.112)$$

And using the result

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km} \qquad (3.113)$$

We have $\epsilon_{ijk}\,B_i = \partial_j A_k - \partial_k A_j$ and

$$m\ddot{x}_i = -q\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}\ \epsilon_{ijk}\ \dot{x}_j\ B_k \qquad (3.114)$$

Or in vector notation, $m\ddot{\boldsymbol{r}} = -q\boldsymbol{\nabla}\phi - \frac{q}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{q}{c}\dot{\boldsymbol{r}} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$

$$= q\boldsymbol{E} + \frac{q}{c}\dot{\boldsymbol{r}} \times (\mathbf{B}) \tag{3.115}$$

Which is of course the Lorentz force law.

3.13 Fast Perturbations: Rapidly Oscillating Fields

Consider a free particle moving under the influence of an oscillating force,

$$m\ddot{q} = F \sin \omega t$$

The motion of the system is then

$$q(t) = q_h(t) = \frac{F \sin \omega t}{m\omega^2} \tag{3.116}$$

Where $q_h(t) = A + Bt$ is the solution to the homogeneous (unforced) equation of motion. Note that the amplitude of the response $q - q_h$ goes as ω^{-2} and is therefore small when ω is large.

Now consider a general $n = 1$ system, with

$$H(q, p, t) = H_0(q, p) + V(q)\sin(\omega t + \delta) \tag{3.117}$$

We assume that ω is much greater than any natural oscillation frequency associated with H_0. We separate the motion $q(t)$ and $p(t)$ into slow and fast components:

$$q(t) = \bar{q}(t) + \zeta(t)$$

$$p(t) = \bar{p}(t) + \pi(t)$$

Where $\zeta(t)$ and $\pi(t)$ oscillate with the driving frequency ω. Since ζ and π will be small , we expand Hamilton's equations in these quantities **:**

$$\dot{\bar{q}} + \dot{\zeta} = \frac{\partial H_0}{\partial \bar{p}} + \frac{\partial^2 H_0}{\partial \bar{p}^2}\pi + \frac{\partial^2 H_0}{\partial \bar{q}\partial \bar{p}}\,\zeta + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}}\,\zeta^2 + \frac{\partial^3 H_0}{\partial \bar{q}\partial \bar{p}^2}\,\zeta\pi + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{p}^3}\,\pi^2 + \cdots$$

$$\dot{\bar{p}} + \dot{\pi} = -\frac{\partial H_0}{\partial \bar{q}} - \frac{\partial^2 H_0}{\partial \bar{q}^2}\zeta - \frac{\partial^2 H_0}{\partial \bar{q}\partial \bar{p}}\,\pi - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^3}\,\zeta^2 - \frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}}\,\zeta\pi - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}\partial \bar{p}^2}\,\pi^2$$
$$- \frac{\partial V}{\partial \bar{q}}\sin(\omega t + \delta) - \frac{\partial^2 V}{\partial \bar{q}^2}\zeta\sin(\omega t + \delta) - \ldots \tag{3.118}$$

We now average over the fast degrees of freedom to obtain an equation of motion for the slow variables $\bar{q}$ and $\bar{p}$ which we here carry to lowest nontrivial order in averages of fluctuating quantities **:**

$$\dot{\bar{q}} = \frac{\partial H_0}{\partial \bar{p}} + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}}\langle \zeta^2 \rangle + \frac{\partial^3 H_0}{\partial \bar{q} \partial \bar{p}^2}\langle \zeta\pi \rangle + \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{p}^3}\langle \pi^2 \rangle \tag{3.119}$$

$$\dot{\bar{p}} = -\frac{\partial H_0}{\partial \bar{q}} - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q}^3}\langle \zeta^2 \rangle - \frac{\partial^3 H_0}{\partial \bar{q}^2 \partial \bar{p}}\langle \zeta\pi \rangle - \frac{1}{2}\frac{\partial^3 H_0}{\partial \bar{q} \partial \bar{p}^2}\langle \pi^2 \rangle - \frac{\partial^2 V}{\partial \bar{q}^2}\langle \zeta \sin(\omega t + \delta) \rangle \tag{3.120}$$

The fast degrees of freedom obey

$$\dot{\zeta} = \frac{\partial^2 H_0}{\partial \bar{q} \partial \bar{p}}\zeta + \frac{\partial^2 H_0}{\partial \bar{p}^2}\pi \tag{3.121}$$

$$\dot{\pi} = -\frac{\partial^2 H_0}{\partial \bar{q}^2}\zeta - \frac{\partial^2 H_0}{\partial \bar{q} \partial \bar{p}}\pi - \frac{\partial V}{\partial \bar{q}}\sin(\omega t + \delta) \tag{3.122}$$

Let us analyze the coupled equations

$$\dot{\zeta} = A\zeta + B\pi$$

$$\dot{\pi} = -C\zeta - A\pi + Fe^{-i\omega t}$$

The solution is of the form

$$\begin{pmatrix} \zeta \\ \pi \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{-i\omega t} \tag{3.123}$$

Plugging in, we find

$$\alpha = \frac{BF}{BC - A^2 - \omega^2} = -\frac{BF}{\omega^2} + \mathcal{O}(\omega^{-4})$$

$$\beta = \frac{(A + i\omega)F}{BC - A^2 - \omega^2} = \frac{iF}{\omega} + \mathcal{O}(\omega^{-3})$$

Taking the real part, and restoring the phase shift δ, we have

$$\zeta(t) = \frac{-BF}{\omega^2}\sin(\omega t + \delta) = \frac{1}{\omega^2}\frac{\partial V}{\partial \bar{q}}\frac{\partial^2 H_0}{\partial \bar{p}^2}\sin(\omega t + \delta)$$

$$\pi(t) = -\frac{F}{\omega}\cos(\omega t + \delta) = \frac{1}{\omega}\frac{\partial V}{\partial \bar{q}}\cos(\omega t + \delta)$$

The desired average, to lowest order are thus

$$\langle \zeta^2 \rangle = \frac{1}{2\omega^2}\left(\frac{\partial V}{\partial \bar{q}}\right)^2 \left(\frac{\partial^2 H_0}{\partial \bar{p}^2}\right)^2$$

$$\langle \pi^2 \rangle = \frac{1}{2\omega^2}\left(\frac{\partial V}{\partial \bar{q}}\right)^2$$

$$\langle \sin(\omega t + \delta) \rangle = \frac{1}{2\omega^2}\frac{\partial V}{\partial \bar{q}}\frac{\partial^2 H_0}{\partial \bar{p}^2}$$

Along with $\langle \zeta\pi \rangle = 0$.

Finally, we substitute the averages into the equations of motion for the slow variables $\bar{q}$ and $\bar{p}$, resulting in the time-independent effective Hamiltonian

$$K(\bar{q},\bar{p}) = H_0(\bar{q},\bar{p}) + \frac{1}{4\omega^2}\frac{\partial^2 H_0}{\partial \bar{p}^2}\left(\frac{\partial V}{\partial \bar{q}}\right)^2 \qquad (124)$$

and the equations of motion $\dot{\bar{q}} = \frac{\partial K}{\partial \bar{p}}$, $\dot{\bar{p}} = -\frac{\partial K}{\partial \bar{q}}$

Example 3.13.1

Consider a pendulum with a vertically oscillating point of support. The coordinates of the pendulum bob are $x = \ell \sin\theta$, $x = a(t) - \ell\cos\theta$

The Lagrangian is easily obtained $L = \frac{1}{2} m\ell^2\dot{\theta}^2 + m\ell\dot{\theta}\sin\theta + mg\ell\cos\theta + \frac{1}{2} m\dot{a}^2 - mg$

$$= \frac{1}{2} m\ell^2\dot{\theta}^2 + m(g + \ddot{a})\ell\cos\theta + \overbrace{\frac{1}{2} m\dot{a}^2 - mga - \frac{d}{dt}(m\ell\dot{a}\sin\theta)}^{these\ may\ be\ dropped} \qquad (3.125)$$

Thus we may take the Lagrangian to be

$$\bar{L} = \frac{1}{2} m\ell^2\dot{\theta}^2 + m(g + \ddot{a})\ell\cos\theta$$

From which we derive the Hamiltonian

$$H(\theta, p_\theta, t) = \frac{p_\theta^2}{2m\ell} - mg\ell\cos\theta - m\ell\ddot{a}\cos\theta$$

$$= H_0(\theta, p_\theta, t) + V_1(\theta)\ \sin\omega t \qquad (3.126)$$

We have assumed $a(t) = a_0 \sin\omega t$, so

$$V_1(\theta) = m\ell a_0 \omega^2 \cos\theta$$

The effective Hamiltonian, per Eq (3.124) is

$$K(\bar\theta, \bar p_\theta) = \frac{\bar p_\theta}{2m\ell^2} - mg\ell\cos\bar\theta + \frac{1}{4} m\, a_0^2\ \omega^2 \sin^2\bar\theta \qquad (3.127)$$

Let's define the dimensionless parameter $\epsilon \equiv \frac{2g\ell}{\omega^2\, a_0^2}$

The slow variable $\bar\theta$ executes motion in the effective potential $V_{eff}\,(\bar\theta) = mg\ell v(\bar\theta)$

$$v(\bar\theta) = -\cos\bar\theta + \frac{1}{2\epsilon}\ \sin^2\bar\theta \qquad (3.128)$$

Differentiating, and dropping the bar on θ, we find that $V_{eff}\,(\theta)$ is stationary when

$$v^{'}(\theta) = 0\ \Longrightarrow \sin\theta\cos\theta = -\sin\theta$$

Thus, $\theta = 0$ and $\theta = \pi$, where $\sin\theta = 0$, are equilibria. When $\epsilon < 1$ (note $\epsilon < 0$ always). There are two new solutions, given by the roots of $\cos\theta = -\epsilon$.

To assess stability of these equilibria, we compute the second derivative

$$v^{''}(\cos^{-1}(-\epsilon)) = \epsilon - \frac{1}{\epsilon} \qquad (3.129)$$

Which is always negative since $\epsilon < 1$ in order for these equilibria to exist. The situation is sketched in **Fig 3.3** , showing $v(\theta)$ for two representative values of the parameter ϵ. For $\epsilon > 1$, the equilibrium at $\theta = \pi$ is unstable, but as ϵ decreases, a subcritical pitchfork bifurcation is encountered at $\epsilon = 1$, and $\theta = \pi$ becomes stable, while the outlying $\theta = \cos^{-1}(-\epsilon)$ solutions are unstable.

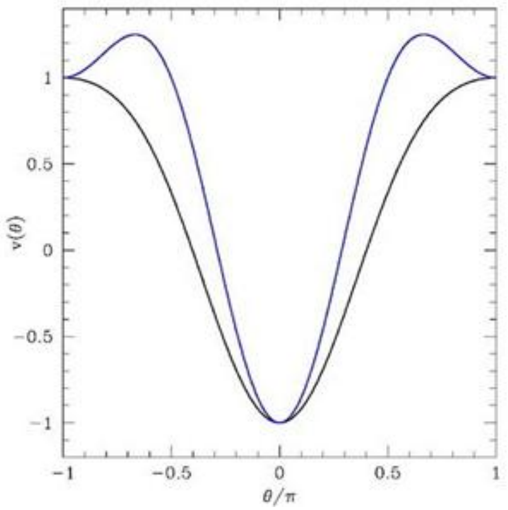

Fig 3.3: Dimensionless potential $v(\theta)$ for $\epsilon = 1.5$ (black curve) and $\epsilon = 0.5$ (blue curve)

3.14 Field Theory

i. Systems with Several Independent Variables

Suppose $\phi_a(\boldsymbol{x})$ depends on several independent variables: $\{x^1, x^2, \dots\dots, x^n\}$.Furthermore, suppose

$$S[\{\phi_a(\boldsymbol{x})\}] = \int_\Omega \mathcal{L}(\phi_a \partial_\mu \phi_a, \boldsymbol{x}) \tag{3.130}$$

i.e. the Lagrangian density $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in R^K. Then the first derivation of S is

$$\delta S = \int_\Omega dx \left\{\frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\frac{\partial\delta\phi_a}{\partial x^\mu}\right\}$$

$$\oint_{\partial\Omega} d\sum n^\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a + \int_\Omega dx \left\{\frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial\mathcal{L}}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right\}\delta\phi_a \tag{3.131}$$

Where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\sum$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ of $\delta\phi_a|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial\mathcal{L}}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) \tag{3.132}$$

As an example, consider the case of a stretched string of linear mass density μ and tension τ. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time t, are the two independent variables. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\,\mu\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 \qquad (3.133)$$

Whence the Euler-Lagrange equations are

$$0 = \frac{\delta S}{\delta y(x,t)} = -\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$$

$$= \tau\frac{\partial^2 y}{\partial x^2} - \mu\frac{\partial^2 y}{\partial t^2}$$

Where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. Thus, $\mu\ddot{y} = \tau y''$, which is the Helmholtz equation. We've assumed boundary conditions where $\delta y(x_a,t) = \delta y(x_b,t) = \delta y(t,x_a) = \delta y(t,x_b) = 0$. The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = \frac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{c}\,j_\mu A^\mu \qquad (3.134)$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \Longrightarrow \partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\,j^\nu \qquad (3.135)$$

Which are Maxwell's equations.

Recall the result of Noether's theorem for mechanical systems :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = 0$$

Where $\tilde{q}_\sigma = \tilde{q}_\sigma(q,\zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize of field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(t,\boldsymbol{x})$$

Where $\{\phi_\sigma(t,\boldsymbol{x})\}$ are a set of fields, which are functions of the independent variables $\{x,y,z,t\}$. we will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct,x,y,z)$. The generalization of $d\Lambda/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)_{\zeta=0} = 0$$

Where there is an implied sum on both μ and a. We can write this as $\partial_{\mu} J^{\mu} = 0$, where

$$J^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_{a})} \frac{\partial \tilde{\phi}_{a}}{\partial \zeta}\bigg|_{\zeta=0}$$

We call $\Lambda = J^{0}/c$ the total charge. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_{\mu} J^{\mu}$ over the spatial region Ω fives

$$\frac{d\Lambda}{dt} = \int_{\Omega} d^{3}\ x\partial_{0} J^{0} = -\int_{\Omega} d^{3}\ x\ \boldsymbol{\nabla}\cdot\boldsymbol{J} = -\oint_{\partial\Omega} d\sum \hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0 \qquad (3.136)$$

Assuming $\boldsymbol{J} = 0$ at the boundary Ω .

As an example, consider the case of a complex scalar field, with Lagrangian density[7]

$$\mathcal{L}(\psi,\psi^{*},\partial_{\mu}\psi,\partial_{\mu}\psi^{*}) = \frac{1}{2}\ K(\partial_{\mu}\psi^{*})(\partial^{\mu}\psi) - U(\psi,\psi^{*})$$

This is invariant under the transformation $\longrightarrow e^{i\zeta}\psi$, $\psi^{*} \longrightarrow e^{i\zeta}\psi^{*}$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = ie^{i\zeta}\psi \qquad , \qquad \frac{\partial\tilde{\psi}^{*}}{\partial\zeta} = ie^{i\zeta}\psi^{*}$$

And, summing over both ψ and ψ^{*} fields, we have

$$J^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}\cdot(i\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi^{*})}\cdot(-i\psi^{*})$$

$$= \frac{K}{2i}\ (\psi^{*}\partial^{\mu}\ \psi - \psi\ \partial^{\mu}\ \psi^{*}) \qquad (3.137)$$

The potential, which depends on $|\psi|^{2}$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

ii. Continuous Symmetry and conserved Currents :

A similar relation between continuous symmetries and constants of motion holds in field theory .

Let $\mathcal{A}$ be the action of an arbitrary field $\varphi(x)$,

$$\mathcal{A} = \int d^{4}\ x\ \mathcal{L}(\varphi,\partial_{\varphi},x)$$

---

[7] We raise and lower indices using the Minkowski metric $g_{\mu\nu} = diag\ (+,-,-,-)$

and suppose that a transformation of the field

$$\delta_s \varphi(x) = \epsilon \Delta\big(\varphi, \partial_\varphi, x\big)$$

Changes the Lagrangian density $\mathcal{L}$ merely by a total derivative

$$\delta_s \mathcal{L} = \epsilon \partial_\varphi \ \Lambda^\mu \qquad (3.138)$$

Or equivalently, that it changes the action $\mathcal{A}$ by a surface term

$$\delta_s \mathcal{A} = \epsilon \int d^4 \ x \partial_\mu \ \Lambda^\mu \qquad (3.139)$$

Then $\delta_s \mathcal{L}$ is called a symmetry transformation.

Given such a symmetry transformation, we can file a current four-vector

$$j^\mu = \frac{\partial L}{\partial \partial_\mu \varphi} \Delta - \ \Lambda^\mu \qquad (3.140)$$

That has no four-divergence

$$\partial_\mu j^\mu (x) = 0 \qquad (3.141)$$

The expression on the right-hand side of (3.139) it is a local conservation law.

The proof of Eq(**3.140**) is just as easy as it was for the mathematical action in section (**3.6**). We calculate the symmetry variation of $\mathcal{L}$ under the symmetry transformation in a similar way as in Eq.(**3.38**), and find

$$\delta_s \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) \delta_s \varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \ \delta_s \varphi \right)$$

$$= \epsilon \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) \Delta + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \ \Delta \right)$$

Then we invoke the Euler-Lagrange equation to remove the first term. Equating the second term with Eq(**3.138**) we obtain

$$\partial_\mu j^\mu \equiv \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \ \Delta - \ \Lambda^\mu \right) = 0 \qquad (3.142)$$

The relation between continuous symmetries and conservation is called Noether's Theorem.

Assuming all fields to vanish at spatial infinitely, we can derive from the local law Eq(**3.142**) a global conservation law for the charge that is obtained from the spatial integral over the charge density J^0 :

$$Q(t) = \int d^3\, x\, J^0(x,t)$$

And add on the right-hand side a spatial integral over a total three-divergence, which vanishes because of the boundary conditions, we find

$$\frac{d}{dt}Q(t) = \int d^3\, x \partial_0\, J^0(x,t) = \int d^3\, x\big[\partial_0\, J^0(x,t) + \partial_i\, J^i(x,t)\big] = 0$$

Thus, the charge is conserved:

$$\frac{d}{dt}Q(t) = 0 \qquad (3.143)$$

3.15 Gross-Pitaevskii model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\psi^* \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - g(|\psi|^2 - n_0)^2$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x},t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\delta S[\psi^*,\psi] = S[\psi^* + \delta\, \psi^*, \psi + \delta\psi]$$

$$= \int dt\ \int d^d x\ \Big\{ i\hbar\psi^* \frac{\partial\delta\psi}{\partial t} + i\hbar\delta\psi^*\ \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\delta\psi^* \cdot \boldsymbol{\nabla}\psi - 2g(|\psi|^2 - n_0)(\psi^*\delta\psi + \psi\delta\, \psi^*)\Big\} \qquad (3.144)$$

$$= \int dt\ \int d^d x\ \left\{\left[-i\hbar\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\nabla^{\mathbf{2}}\psi^* - 2g(|\psi|^2 - n_0)\psi^*\right]\delta\psi + \left[i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^{\mathbf{2}}\psi - 2g(|\psi|^2 - n_0)\psi\right]\delta\psi^*\right\}$$

Where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*,\psi]$ therefore results in the nonlinear Schrodinger equation (NLSE),

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\nabla^{\mathbf{2}}\psi - 2g(|\psi|^2 - n_0)\psi \qquad (3.145)$$

As well as its complex conjugate,

$$-i\hbar\frac{\partial\psi^*}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi^*-2g(|\psi|^2-n_0)\psi^* \qquad (3.146)$$

Note that these equations are indeed the Euler-Lagrange equations **:**

$$\frac{\delta S}{\delta\psi}=\frac{\partial\mathcal{L}}{\partial\psi}-\frac{\partial\mathcal{L}}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\right)$$

$$\frac{\delta S}{\delta\psi^*}=\frac{\partial\mathcal{L}}{\partial\psi^*}-\frac{\partial\mathcal{L}}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\right)$$

With $x^\mu = (t,\boldsymbol{x})$ [8] Plugging in

$$\frac{\partial\mathcal{L}}{\partial\psi}=-2g(|\psi|^2-n_0)\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi}=i\hbar\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi}=-\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \qquad (3.147)$$

And

$$\frac{\partial\mathcal{L}}{\partial\psi^*}=i\hbar\psi-2g(|\psi|^2-n_0)\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi^*}=0 \quad , \quad \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi^*}=-\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi \qquad (3.148)$$

We recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a $U(1)$ invariance, under

$$\psi(\boldsymbol{x},t)\longrightarrow\tilde{\psi}(\boldsymbol{x},t)=e^{i\zeta}\psi(\boldsymbol{x},t)\ ,\quad \psi^*(\boldsymbol{x},t)\longrightarrow\tilde{\psi}^*(\boldsymbol{x},t)=e^{i\zeta}\tilde{\psi}^*(\boldsymbol{x},t)$$

Thus, the conserved Noether current is then

$$J^\mu=\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\frac{\partial\tilde{\psi}}{\partial\zeta}\bigg|_{\zeta=0}+\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\tilde{\psi}^*}\frac{\partial\tilde{\psi}^*}{\partial\zeta}\bigg|_{\zeta=0}$$

$$J^0=-\hbar|\psi|^2$$

$$\boldsymbol{J}=-\frac{\hbar^2}{2im}\,(\psi^*\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\psi^*) \qquad (3.149)$$

Dividing out by $\hbar$, taking $J^0=-\hbar\rho$ and $\boldsymbol{J}=-\hbar\boldsymbol{j}$**,** we obtain the continuity equation,

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$$

Where

$$\rho=|\psi|^2\ ,\quad \boldsymbol{j}=\frac{\hbar}{2im}\,(\psi^*\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\psi^*)$$

[8] In the nonrelativistic case, there is utility in defining $x^0=ct$, so we simply define $x^0=t$.

Are the particle density and the particle current, respectively.

Example 3.15.1.

Consider the Lagrangian which gives rise to Schoedinger's equation.

$$\mathcal{L} = \frac{-\hbar^2}{2m}\nabla\psi^*\nabla\psi + \frac{i\hbar}{2}\left(\psi^*\,\dot{\psi} - \dot{\psi}^*\,\psi\right) \qquad (3.150)$$

In this Lagrangian density, ψ and ψ^* are consider independent functions. The action of this Lagrangian density is symmetric under the transformation $\psi \longrightarrow \psi' = \psi + \epsilon$ where : $\epsilon = \epsilon_R + i\epsilon_I$. because this is still a one dimensional symmetry, there is only conserved current. Also, Λ is zero for this system.

Using Noether's theorem,

$$\rho^\mu = \sum_k \left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\psi_k}{dx^\mu}\right)} \; \frac{\partial\epsilon}{\partial x^\mu} \right)$$

$$\rho^\mu = \left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\psi}{dx^\mu}\right)} \; \frac{\partial\epsilon}{\partial x^\mu} \right) + \left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\psi^*}{dx^\mu}\right)} \; \frac{\partial\epsilon}{\partial x^\mu} \right)$$

$$\rho = \left(\left\{i\hbar\psi^*, \frac{\hbar^2}{2m}\,\nabla\psi^*\right\} + \left\{i\hbar\psi, \frac{\hbar^2}{2m}\,\nabla\psi\right\}\right) \qquad (3.151)$$

The continuity equation of this gives

$$\frac{d\rho^0}{dt} + \nabla.\vec{\rho} = \left(i\hbar\,\dot{\psi}^* + \frac{\hbar^2}{2m}\,\nabla^2\,\psi^* + i\hbar\dot{\psi} + \frac{\hbar^2}{2m}\,\nabla^2\,\psi\right) = 0 \qquad (3.152)$$

This gives the Schoedinger equation of a free particle. Using Gauss's Law, we can find that

$$Q = \int \frac{\hbar^2}{2m}\,(\nabla\psi + \nabla\psi^*)\,dx \qquad (3.153)$$

3.16 Momentum and Angular Momentum

While the conservation law of energy follows from the symmetry of the action under time translations, conservation laws of momentum and angular momentum are found if the action is invariant under translations and rotations.

Consider a Lagrangian of a point particle in a Euclidean space

$$L = L\big(x^i(t), \dot{x}^i(t), t\big) \qquad (3.154)$$

In contrast to previous discussion of time translation invariance, which was applicable to systems with arbitrary Lagrange coordinates $q(t)$, we denote the coordinates here by x^i to emphasize that we now consider Cartesian coordinates. If the Lagrangian does depend only on the velocities $\dot{x}^i$ and not on the coordinates x^i themselves, the system is translationally invariant. If it depends, in addition, only on $\dot{\mathbf{x}}^2 = \dot{x}^i \dot{x}^i$, it is also rotationally invariant.

The simplest example is the Lagrangian of a point particle of mass m in Euclidean space **:**

$$L = \frac{m}{2}\dot{\mathbf{x}}^2 \qquad (3.155)$$

It exhibits both invariances, leading to conserved Noether charges of momentum and angular momentum, as we now demonstrate.

3.17 Translation Invariance in Space

Under a spatial translation, the coordinates x^i change [9]to

$$x'^i = x^i + \epsilon^i \qquad (3.156)$$

Where ϵ^i are small numbers. The infinitesimal translations of a particle path are [compare Eq(**3.132**)]

$$\delta_s x^i(t) = \epsilon^i \qquad (3.157)$$

Under these, the Lagrnagian changes by

$$\delta_s L = L\big(x'^i(t), \dot{x}'^i(t), t\big) - L\big(x^i(t), \dot{x}^i(t), t\big)$$

$$= \frac{\partial L}{\partial x^i}\delta_s x^i = \frac{\partial L}{\partial x^i}\epsilon^i = 0 \qquad (3.158)$$

By assumption, the Lagrangian is independent of x^i, so that the right-hand side vanishes. This is to be compared with the symmetry variation of the Lagrangian around the classical orbit, calculated via the chain rule, and using the Euler-Lagrange equation:

$$\delta_s L = \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta_s x^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\delta_s x^i\right]$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\right]\epsilon^i \qquad (3.159)$$

[9] http://users.physik.fu-berlin.de/~kleinert/kleiner_reb9/psfiles/conslaw.pdf

This has the form Eq(**3.134**), from which we extract a conserved Noether charge Eq(**3.135**) for each coordinate x^i :

$$p^i = \frac{\partial L}{\partial \dot{x}^i} \quad (3.160)$$

These are simply the canonical momenta of the system.

3.18 Rotational Invariance

Under rotations, the coordinates x^i change to

$$\dot{x}'^i = R^i_j \, x^j \quad (3.161)$$

Where R^i_j is an orthogonal 3 × 3-matrix. Infinitesimally, this can be written as

$$R^i_j = \delta^i_j - \omega_k \epsilon_{kij} \quad (3.162)$$

Where $\boldsymbol{\omega}$ is an infinitesimal rotation vector. The corresponding rotation of a particle path is

$$\delta_s x^i(t) = x'^i(t) - x^i(t) = -\omega^k \epsilon_{kij} \, x^j \ (\tau) \quad (3.163)$$

It is useful to introduce the antisymmetric infinitesimal rotation tensor

$$\omega_{ij} \equiv \omega_k \epsilon_{kij} \quad (3.164)$$

In terms of which

$$\delta_s x^i = \omega_{ij} \, x^j \quad (3.165)$$

Then we can write the change of the Lagrangian under $\delta_s x^i$,

$$\delta_s L = L\big(x'^i(t), \dot{x}'^i(t), t\big) - L\big(x^i(t), \dot{x}^i(t), t\big)$$

$$= \frac{\partial L}{\partial x^i} \delta_s x^i + \frac{\partial L}{\partial \dot{x}^i} \delta_s \dot{x}^i \quad (3.166)$$

As

$$\delta_s L = -\left(\frac{\partial L}{\partial x^i} \delta_s x^j + \frac{\partial L}{\partial \dot{x}^i} \dot{x}^j\right) \omega_{ij} = 0 \quad (3.167)$$

If the Lagrangian depends only on the rotational invariants $\mathbf{x}^2, \dot{\mathbf{x}}^2, \mathbf{x} \cdot \dot{\mathbf{x}}$, and on powers thereof the right-hand side vanishes on account of the antisymmetry of ω_{ij}. This ensures the rotational symmetry.

We now calculate once more the symmetry variation of the Lagrangian via the chain rule and find, using the Euler-Lagrange equations,

$$\delta_s L = -\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta_s x^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\delta_s x^i\right]$$

$$= -\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^i}\, x^j\right]\omega_{ij} = \frac{1}{2}\frac{d}{dt}\left[x^i\, \frac{\partial L}{\partial \dot{x}^i} - (i \leftrightarrow j)\right]\omega_{ij} \tag{3.168}$$

The right-hand side yields the conserved Noether charges of type Eq(**3.135**), one for each antisymmetric pair $, j$:

$$L^{ij} = x^i \frac{\partial L}{\partial \dot{x}^j} - x^j \frac{\partial L}{\partial \dot{x}^i} \equiv x^i p^j - x^j p^i \tag{3.169}$$

These are the antisymmetric components of angular momentum.

Had we work with original vector form of the rotation angles ω^k,we would have found the angular momentum in the more commom form:

$$L_k = \frac{1}{2}\,\epsilon_{kij}\, L^{ij} = (\mathbf{x} \times \mathbf{P})^k \tag{3.170}$$

The quantum-mechanical operators associated with these, after replacing $p^i \longrightarrow -\partial/\partial x^i$, have the well-known commutation rules

$$\left[\hat{L}_i, \hat{L}_j\right] = i\epsilon_{kij}\,\hat{L}_k \tag{3.171}$$

In the tensor notation Eq(**3.169**), these become

$$\left[\hat{L}_{ij}, \hat{L}_{kl}\right] = -i\left(\delta_{ik}\hat{L}_{jl} - \delta_{il}\hat{L}_{jk} + \delta_{jl}\hat{L}_{ik} - \delta_{jk}\hat{L}_{il}\right) \tag{3.172}$$

3.19 Center of Mass Theorem

Consider the transformations corresponding to a uniform motion of the coordinate system. We shall study the behavior of a set of free massive point particles in Euclidean space described by the Lagrangian

$$L\left(\dot{x}^i\right) = \sum_n \frac{m_n}{2}\,\dot{\mathbf{x}}_n^2 \tag{3.173}$$

Under Galilei transformations, the spatial coordinates and the time are changed to

$$\dot{x}^i(t) = x^i(t) - v^i t$$

$$t' = t \tag{3.174}$$

Where v^i is the relative velocity along the ith axis. The infinitesimal symmetry variations are

$$\delta_s x^i(t) = \dot{x}^i(t) - x^i(t) = -v^i t \quad (3.175)$$

Which change the Lagrangian by

$$\delta_s L = L\left(x^i - v^i t, \dot{x}^i - v^i\right) - L\left(x^i, \dot{x}^i\right) \quad (3.176)$$

Inserting the explicit form Eq(**3.173**), we find

$$\delta_s L = \sum_n \frac{m_n}{2}\left[\left(\dot{x}_n^i - v^i\right)^2 - \left(\dot{x}_n^i\right)^2\right] \quad (3.177)$$

This can be written as a total time derivative :

$$\delta_s L = \frac{d}{dt}\Lambda = \frac{d}{dt}\sum_n m_n\left[-\dot{x}_n^i\, v^i + \frac{v^2}{2}\, t\right] \quad (3.178)$$

Proving that Galilei transformations are symmetry transformations in the Noether sense. By assumption, the velocities v^i in Eq(**3.174**) are infinitesimal, so that the second term can be ignored.

By calculating $\delta_s L$ once more via the chain rule with the help of the Euler-Lagrange equations, and by equating the result with Eq(**3.178**). we find the conserved Noether charge

$$Q = \sum_n \frac{\partial L}{\partial \dot{x}^i}\, \delta_s x^i - \Lambda$$

$$= \left(-\sum_n m_n\, \dot{x}_n^i\, t + \sum_n m_n\, x_n^i\right) v^i \quad (3.179)$$

Since the direction of the velocity v^i is arbitrary, each component is separately a constant of motion:

$$N^i = -\sum_n m_n\, \dot{x}_n^i\, t + \sum_n m_n\, x_n^i = const \quad (3.180)$$

This is the well-known center of mass theorem. Indeed, introducing the center of mass coordinates

$$x_{CM}^i \equiv \frac{\sum_n m_n x_n^i}{\sum_n m_n} \quad (3.181)$$

And the associated velocities

$$v_{CM}^{i} \equiv \frac{\sum_n m_n \dot{x}_n^i}{\sum_n m_n} \qquad (3.182)$$

The conserved charge Eq(**3.180**) can be written as

$$N^i = \sum_n m_n\left(-v_{CM}^i t + x_{CM}^i\right) \qquad (3.183)$$

The time-independence of N^i implies that the center of mass moves with uniform velocity according to the law

$$x_{CM}^i(t) = x_{0CM}^i + v_{CM}^i\, t \qquad (3.184)$$

Where

$$x_{0CM}^i = \frac{N^i}{\sum_n m_n} \qquad (3.185)$$

Is the position of the center of mass at $t = 0$.

Note that in non-relativistic physics, the center of mass theorem is consequence of momentum conservation since momentum ≡ mass × velocity . In relativistic physics, this is no longer true.

3.20 Conservation Laws resulting from Lorentz Invariance

In relativistic physics, particle orbits are described by functions in space time

$$x^\mu(\tau) \qquad (3.185)$$

Where τ is an arbitrary Lorentz-invariant parameter. The action is an integral over some Lagrangian:

$$\mathcal{A} = \int d\tau\ L(x^\mu(\tau), \dot{x}^\mu(\tau), \tau) \qquad (3.186)$$

Where $\dot{x}^\mu(\tau)$ denotes the derivative with respect to the parameter τ. If the Lagrangian depends only on invariant scalar products $x^\mu x_\mu, x^\mu \dot{x}_\mu, \dot{x}^\mu \dot{x}_\mu$, then it is invariant under Lorentz transformations

$$x^\mu \longrightarrow \dot{x}^\mu = \Lambda^\mu{}_v\, x^v \qquad (3.187)$$

Where $\Lambda^\mu{}_v$ is a 4 × 4 matrix satisfying

$$\Lambda g \Lambda^T = g \qquad (3.188)$$

With the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad (3.189)$$

For a free massive point particle in spacetime, the Lagrangian is

$$L(\dot{x}(\tau)) = -Mc\sqrt{g_{\mu\nu}\ \dot{x}^{\mu}\ \dot{x}^{\nu}} \qquad (3.190)$$

It is reparametrization invariant under $\tau \longrightarrow f(\tau)$, wuth an arbitrary function $f(\tau)$**.** Under translations

$$\delta_{\mu}\ x^{\mu}(\tau) = \ x^{\mu}(\tau) - \ \epsilon^{\mu}(\tau) \qquad (3.200)$$

The Lagrangian is obviously invariant, satisfying $\delta_{\mu}\mathcal{L} = 0$. Calculating this variation once more via the chain rule with the help of the Euler-Lagrange equations, we find

$$0 = \int_{T_{\mu}}^{T_{v}} d\tau \left(\frac{\partial L}{\partial x^{\mu}}\ \delta_{s}x^{\mu} + \frac{\partial L}{\partial \dot{x}^{\mu}}\ \delta_{s}\dot{x}^{\mu}\right)$$

$$= -\ \epsilon^{\mu} \int_{T_{\mu}}^{T_{v}} d\tau \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) \qquad (3.200)$$

From this we obtain the Noether charges

$$p_{\mu} \equiv -\frac{\partial L}{\partial \dot{x}^{\mu}} = Mc\ \frac{\dot{x}_{\mu}(\tau)}{\sqrt{g_{\mu\nu}\ \dot{x}^{\mu}\ \dot{x}^{\nu}}} = Mcu^{\mu} \qquad (3.201)$$

Which satisfy the conservation law

$$\frac{d}{d\tau}p_{\mu}(t) = 0 \qquad (3.202)$$

They are the conserved four-momenta of a free relativistic particle. The quantity

$$u^{\mu} \equiv \frac{\dot{x}_{\mu}(\tau)}{\sqrt{g_{\mu\nu}\ \dot{x}^{\mu}\ \dot{x}^{\nu}}} \qquad (3.203)$$

is the dimensionless relativistic four-velocity of the particle. It has the property $u^{\mu}u_{\mu} = 1$, and it is reparametrization-invariant. By choosing for τ the physical time $t = x^{0}/c$, we can express u^{μ} in terms of the physical velocities $v^{i} = dx^{i}/dt$ as

$$u^{\mu} = \gamma(1, v^i/c)\,, \quad \text{with} \quad \gamma \equiv \sqrt{1 - v^2/c^2} \tag{3.204}$$

Note the minus sign in the definition of Eq(**3.201**) of the canonical momentum with respect to the nonrelativistic case. It is necessary to write Eq(**3.201**) covariantly. The derivative with respect to $\dot{x}^{\mu}$ transforms like a covariant vector with subscript μ, where as the physical momenta are p^{μ}.

$$\Lambda^{\mu}{}_{v} = \delta^{\mu}{}_{v} + \omega^{\mu}{}_{v} \tag{3.205}$$

Where

$$\omega^{\mu}{}_{v} = u^{\mu\lambda}\ \omega_{\lambda v} \tag{3.206}$$

is an arbitrary infinitesimal antisymmetric matrix. An infinitesimal Lorentz transformation of the particle path is

$$\begin{aligned}\delta_s\, x^{\mu}(\tau) &= \dot{x}^{\mu}(\tau) - x^{\mu}(\tau)\\ &= \omega^{\mu}{}_{v}\, x^{\mu}(\tau)\end{aligned} \tag{3.207}$$

Under it the symmetry variation of a Lorentz-invariant Lagrangian vanishes :

$$\delta_s L = \left(\frac{\partial L}{\partial x^{\mu}} x^{v} + \ \frac{\partial L}{\partial \dot{x}^{\mu}} \dot{x}^{v}\right) \omega^{\mu}{}_{v} = 0 \tag{3.208}$$

This is to be compared with the symmetry variation of the Lagrangian calculated via the chain rule with the help of the Euler-Lagrange equation

$$\begin{aligned}\delta_s L &= \left(\frac{\partial L}{\partial x^{\mu}} + \frac{d}{dt}\ \frac{\partial L}{\partial \dot{x}^{\mu}}\right) \delta_s\, x^{\mu} + \frac{d}{d\tau}\ \left[\frac{\partial L}{\partial \dot{x}^{\mu}}\ \delta_s\, x^{\mu}\right]\\ &= \frac{d}{d\tau}\ \left[\frac{\partial L}{\partial \dot{x}^{\mu}}\ \ \dot{x}^{\mu}\right] \omega^{\mu}{}_{v}\\ &= \frac{1}{2}\ \omega^{\mu}{}_{v}\ \frac{d}{d\tau}\ \left(x^{\mu}\ \frac{\partial L}{\partial \dot{x}^{v}} - x^{v}\, \frac{\partial L}{\partial \dot{x}_{\mu}}\right)\end{aligned} \tag{3.209}$$

By equating this with Eq(**3.208**) we obtain the conserved rotational Noether charges [containing again a minus sign as in Eq(**3.201**)]:

$$L^{\mu v} = -x^{\mu}\, \frac{\partial L}{\partial \dot{x}^{v}} + x^{v}\, \frac{\partial L}{\partial \dot{x}_{\mu}} = x^{\mu} p^{v} - x^{v} p^{\mu} \tag{3.210}$$

They are four-dimensional generalizations of the angular momenta Eq(**3.169**). the quantum-mechanical operators

$$\hat{L}^{\mu\nu} \equiv i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) \tag{3.211}$$

Obtained after the replacement $p^{\mu} \longrightarrow i\,\partial/\partial x_{\mu}$ satisfy the four-dimensional spacetime generalization of the commutation relations Eq(**3.209**):

$$\left[\hat{L}^{\mu\nu}, \hat{L}^{\kappa\lambda}\right] = i\left(g^{\mu\kappa}\hat{L}^{\nu\lambda} - g^{\mu\lambda}\hat{L}^{\nu\kappa} + g^{\nu\lambda}\hat{L}^{\mu\kappa} - g^{\nu\kappa}\hat{L}^{\mu\lambda}\right) \tag{3.212}$$

The quantities L^{ij} coincide with the earlier-introduced angular momenta Eq(**3.169**).

The conserved components

$$L^{0i} = x^{0}p^{i} - x^{i}p^{0} \equiv M_{i} \tag{3.213}$$

Yield the relativistic generalization of the center-mass theorem Eq(**3.180**):

$$M_{i} = \text{const.} \tag{3.214}$$

3.21 Generating the Symmetries

The relation between invariances and conservation law has a second aspect. With the help of Poisson brackets, the charges associated with continuous symmetry transformation can be used to generate the symmetry transformation from which they were derived. Explicitly,

$$\delta_{s}\hat{x} = -i\epsilon\left[\hat{Q}, \hat{x}(t)\right] \tag{3.215}$$

The charge Eq (**3.179**) is by definition the Hamiltonian , $Q \equiv H$

Whose operator version generates infinitesimal time displacements by the Heisenberg equation of motion :

$$\delta_{s}\hat{x} = -i\epsilon\left[\hat{H}, \hat{x}(t)\right] \tag{3.216}$$

This equation is obviously the same as Eq(**3.215**).

To quantize the system canonically, we may assume the Lagrangian to have the standard form

$$L(x, \dot{x}) = \frac{M}{2}\,\dot{x}^{2} - V(x) \tag{3.217}$$

So that the Hamiltonian operator becomes, with the canonical momentum $p \equiv \dot{x}$:

$$\hat{H} = \frac{\hat{p}^{2}}{2M} + V(\hat{x}) \tag{3.218}$$

Equation Eq(**3.218**) is then a direct consequence of the canonical equal-time commutation rules

$$[\hat{p}(t), \hat{x}(t)] = -i \,, \quad [\hat{p}(t), \hat{p}(t)] = 0 \quad , \quad [\hat{x}(t), \hat{x}(t)] = 0 \tag{3.219}$$

After quantization, the commutation rule Eq(**3.215**) becomes, with Eq(**3.157**),

$$\epsilon^j = i\, \epsilon^i \left[\hat{p}^i(t), \hat{x}^j(t)\right] \tag{3.220}$$

This coincides with one of the canonical commutation relations (here it appears only for time-independent momenta, since the system is translationally invariant). The relativistic charges Eq(**3.201**) of spacetime generate translations via

$$\delta_s \hat{x}^\mu = \epsilon^\mu = -i\epsilon^\nu [\hat{p}_\nu(t), \hat{x}^\mu(\tau)] \tag{3.221}$$

Similarly we find that the quantized versions of the conserved charges L_i in Eq(**3.170**) generate infinitesimal rotations :

$$\delta_s \hat{x}^j = -\omega^i \epsilon_{ijk}\, \hat{x}^k(t) = i\omega^i \left[\hat{L}_i, \hat{x}^j(t)\right], \tag{3.222}$$

Whereas the quantized conserved charges N^i of Eq.(**3.179**) generate infinitesimal Galilei transformations and that the charges M_i of Eq.(**3.213**) generate pure rotational transformations:

$$\delta_s \hat{x}^j = \epsilon_i \hat{x}^0 = i\epsilon_i \left[M_i, \hat{x}^j\right]$$

$$\delta_s \hat{x}^0 = \epsilon_i \hat{x}^i = i\epsilon_i [M_i, \hat{x}^0] \tag{3.223}$$

Since the quantized charges generate the rotational symmetry transformations, they form a representation of the generators of the symmetry group. They have the same commutation rules with each other as the generator of the symmetry group. The charges Eq(**3.170**) associated with rotations, for example, have the commutation rules

$$\left[\hat{L}_i, \hat{L}_j\right] = i\epsilon_{ijk}\, \hat{L}_j \tag{3.224}$$

Which are the same as those between the 3×3 generators of the three-dimensional rotations $(L_i)_{jk} = -i\epsilon_{ijk}$.

The quantized charges of the generators Eq(**3.210**) of the Lorentz group satisfy the commutation rules Eq(**3.112**) of the 4×4 generators Eq(**3.211**)

$$\left[\hat{L}^{\mu\nu}, \hat{L}^{\mu\lambda}\right] = -ig^{\mu\mu}\, \hat{L}^{\nu\lambda} \tag{3.225}$$

This follows directly from the canonical commutation rules Eq(**3.221**)

3.22 Canonical Energy Momentum Tensor

As an important example for the field theoretic version of the theorem consider the usual case that the Lagrangian density does not depend explicitly on the space time coordinates :

$$\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi) \tag{3.226}$$

We then perform a translation along an arbitrary direction $v = 0,1,2,3$ of space time

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu} \tag{3.227}$$

Under which field $\varphi(x)$ transforms as

$$\varphi'(x') = \varphi(x) \tag{3.228}$$

This equation expresses the fact that the field has the same absolute point in space and time, which in one coordinate system is labeled by the coordinates x^{μ} and in the other by x'^{μ} .

Under an infinitesimal translation of the field configuration coordinate the Lagrangian density undergoes the following symmetry variation

$$\delta_s\mathcal{L} \equiv \mathcal{L}\big(\varphi'(x), \partial\varphi'(x)\big) - \mathcal{L}(\varphi(x), \partial\varphi(x))$$

$$= \frac{\partial\mathcal{L}}{\partial\varphi(x)}\, \delta_s\varphi(x) + \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\varphi}\partial_{\mu}\, \delta_s\varphi(x) \tag{3.229}$$

Where

$$\delta_s\varphi(x) = \varphi'(x) - \varphi(x) \tag{3.230}$$

Is the symmetry variation of the fields. For the particular transformation Eq(**3.228**), the symmetry variation becomes simply

$$\delta_s\varphi(x) = \epsilon^{v}\, \partial_{v}\varphi(x) \tag{3.231}$$

The Lagrangian density Eq(**3.226**) changes by

$$\delta_s\mathcal{L}(x) = \epsilon^{v}\, \partial_{v}\mathcal{L}(x) \tag{3.232}$$

Hence the requirement Eq(**3.139**) is satisfy and $\delta_s\varphi(x)$ is a symmetry transformation. The function Λ happens to coincide with the Lagrangian density

$$\Lambda = \mathcal{L} \tag{3.233}$$

We Can now define a set of currents $j_{\nu}^{\ \mu}$, one for each ϵ^{ν}. In the particular case at hand, the currents $j_{\nu}^{\ \mu}$ are denoted by $\Theta_{\nu}^{\ \mu}$, and read **:**

$$\Theta_{\nu}^{\ \mu} = \frac{\partial \mathcal{L}}{\partial \partial_{\mu} \varphi} \partial_{\nu}\varphi - \delta_{\nu}^{\ \mu} \mathcal{L} \qquad (3.234)$$

They have no four divergence

$$\partial_{\mu} \Theta_{\nu}^{\ \mu}(x) = 0 \qquad (3.235)$$

As a consequence, the total four momentum of the system, defined by

$$P^{\mu} = \int d^3\, x \Theta^{\mu 0}(x) \qquad (3.236)$$

is independent of time .

3.23 Electromagnetism

As an important physical application of the field theoratic Noether theorem, consider the free electromagnetic field with the action

$$\mathcal{L} = -\frac{1}{4c} F_{\lambda\kappa} F^{\lambda\kappa} \qquad (3.237)$$

Where $F_{\lambda\kappa}$ are the components of the field strength $F_{\lambda\kappa} \equiv \partial_{\lambda} A_{\kappa} - \partial_{\kappa} A_{\lambda}$. Under a translation in space and time from x^{μ} to $x^{\mu} - \epsilon\delta_{\nu}^{\mu}$, the vector potential undergoes a similar change as in Eq(**3.128**) :

$$A'^{\mu} = \Lambda^{\mu}(x) \qquad (3.237)$$

As before, this equation expresses the fact that at the same absolute space time point, which in the two coordinate frames is labeled once by x' and once by x, the field components have the same numerical values. The equation transformation law Eq(**3.138**) can be rewritten in an infinitesimal form as

$$\delta_s \mathrm{A}^{\lambda}(x^{\mu}) \equiv A'^{\lambda}(x^{\mu}) - A^{\lambda}(x^{\mu})$$

$$= A'^{\lambda}\left(x'^{\mu} + \epsilon\delta_{\nu}^{\mu}\right) - A^{\lambda}(x^{\mu}) \qquad (3.238)$$

$$= \epsilon\partial_{\nu} A^{\lambda}(x^{\mu}) \qquad (3.239)$$

Under it, the field tensor changes as follows

$$\delta_s F^{\lambda\kappa} = \epsilon\partial_{\nu} F^{\lambda\kappa} \qquad (3.240)$$

So that the Lagrangian density is a total four divergence :

$$\delta_s \mathcal{L} = -\epsilon \frac{1}{2c} F_{\lambda\kappa} \partial_v F^{\lambda\kappa} = \epsilon \partial_v \mathcal{L} \tag{3.241}$$

Thus, the spacetime translations Eq(**3.140**) are symmetry transformations, and the currents

$$\Theta_v^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\lambda} \partial_v A^\lambda - \delta_v^\mu \mathcal{L} \tag{3.242}$$

Are conserved :

$$\partial_\mu \Theta_v^\mu(x) = 0 \tag{3.243}$$

Using $\partial\mathcal{L}/\partial\partial_\mu A^\lambda = -F_\lambda^\mu$, the corrents Eq(**3.242**) become more explicitly

$$\Theta_v^\mu = -\frac{1}{c}\left(F^\mu{}_\lambda \partial_v A^\lambda - \frac{1}{4} \delta_v^\mu F^{\lambda\kappa} F_{\lambda\kappa}\right) \tag{3.244}$$

They form the canonical energy-momentum tensor of the electromagnetic field.

3.24 Dirac Field

We now turn to the Dirac field which has the well-known action

$$\mathcal{A} = \int d^4 x\, \mathcal{L}(x) = \int d^4 x \bar{\psi}(x)\left(i\gamma^\mu \overleftrightarrow{\partial}_\mu - M\right)\psi(x) \tag{3.245}$$

Where γ^μ are the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{3.246}$$

Here σ^μ , $\tilde{\sigma}^\mu$ are four 2×2 matrices

$$\sigma^\mu \equiv \left(\sigma^0, \sigma^i\right) \cdot \tilde{\sigma}^\mu \equiv \left(\sigma^0, -\sigma^i\right) \tag{3.247}$$

Whose zeroth component is the unit matrix

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.248}$$

And whose spatial components consist of the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.249}$$

On behalf of the algebraic properties of the Pauli matrices

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\ \sigma^k, \tag{3.250}$$

The Dirac matrices Eq(**3.246**) satisfy the anticommutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{3.251}$$

Under spacetime translations

$$x'^\mu = x^\mu - \epsilon^\mu, \tag{3.252}$$

The Dirac field transforms in the same way as the previous scalar and vector fields :

$$\psi'(x') = \psi(x) \tag{3.253}$$

Or infinitesimally :

$$\delta_s\psi(x) = \epsilon^\mu\partial_\mu\psi(x) \tag{3.254}$$

The same is true for the Lagrangian density, where

$$\mathcal{L}'(x') = \mathcal{L}(x) \tag{3.255}$$

and

$$\delta_s\mathcal{L}(x) = \epsilon^\mu\partial_\mu\mathcal{L}(x) \tag{3.256}$$

Thus we obtain the Noether current

$$\Theta_\nu^{\ \mu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^\lambda}\ \partial_\nu\psi^\lambda + c.c. - \delta_\nu^{\ \mu}\mathcal{L} \tag{3.257}$$

With the local conservation law

$$\partial_\mu\Theta_\nu^{\ \mu}(x) = 0 \tag{3.258}$$

From Eq(**3.146**) we see that

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^\lambda} = \frac{1}{2}\ \bar{\psi}\,\gamma^\mu \tag{3.259}$$

So that we obtain the canonical energy-momentum tensor of the Dirac field **:**

$$\Theta_\nu^{\ \mu} = \frac{1}{2}\ \bar{\psi}\,\gamma^\mu\partial_\nu\psi^\lambda + c.c. - \delta_\nu^{\ \mu}\mathcal{L} \tag{3.260}$$

3.25 Angular Momentum

Let us now turn to angular momentum in field theory. Consider first the case of a scalar field $\varphi(\dot{x})$. Under a rotation of the coordinates,

$$x'^{i} = R^{i}{}_{j}\ x^{j} \tag{3.261}$$

The field does not change, if considered at the same space point, i.e.

$$\varphi'\left(x'^{i}\right) = \varphi\left(x^{i}\right) \tag{3.262}$$

The infinitesimal symmetry variation is **:**

$$\delta_s \varphi(x) = \varphi'(x) = \varphi(x) \tag{3.263}$$

Using the infinitesimal form Eq(**3.138**) of Eq(**3.161**),

$$\delta x^{i} = -\omega_{ij}\ x^{j} \tag{3.264}$$

We see that

$$\delta_s \varphi(x) = \varphi'\left(x^0, x'^{i} - \delta x^{i}\right) - \varphi(x)$$

$$= \partial_i \varphi(x)\, x^j\, \omega_{ij} \tag{3.265}$$

Suppose we are dealing with a Lorentz-invariant Lagrangian density that has no explicit x-dependence **:**

$$\mathcal{L} = \mathcal{L}(\varphi(x), \partial\varphi(x)) \tag{3.266}$$

Then the symmetry variation is

$$\delta_s \mathcal{L} = \mathcal{L}(\varphi'(x), \partial\varphi'(x)) - \varphi(\varphi(x), \partial\varphi(x))$$

$$= \frac{\partial\mathcal{L}}{\partial\varphi(x)} \delta_s \varphi(x) + \frac{\partial\mathcal{L}}{\partial\partial\varphi(x)} \partial_\mu \delta_s \varphi(x) \tag{3.267}$$

For a Lorentz-invariant $\mathcal{L}$, the derivative $\partial\mathcal{L}/\partial\partial_\mu\varphi$ is a vector proportional to $\partial_\mu\varphi$. For the Lagrangian density, the rotational symmetry variation Eq.(**3.166**) becomes

$$\delta_s \mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\varphi} \delta_i \varphi x^j + \frac{\partial\mathcal{L}}{\partial_\mu \varphi} \partial_\mu\left(\partial_i \mathcal{L} x^j\right)\right] \omega_{ij}$$

$$= \left[(\partial_i \mathcal{L}) x^j + \frac{\partial\mathcal{L}}{\partial\partial_j \varphi}\ \partial_i \varphi\right] \omega_{ij} = \partial_i\left(\mathcal{L} x^j\, \omega_{ij}\right) \tag{3.268}$$

The right-hand side is a total derivative. In arriving at this result, the antisymmetry of φ_{ij} has been used twice: one in order to drop the second term in the brackets, which is possible since $\partial\mathcal{L}/\partial\partial_i\varphi$ is proportional to $\partial_i\varphi$, as consequence of the assumed rotational invariance of $\mathcal{L}$, and once, in order to pull x^j inside the last parentheses.

Calculating $\delta_s\mathcal{L}$ once more with the help of Euler-Lagrange equations gives

$$\delta_s\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\mathcal{L}}\delta_s\varphi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\,\partial_\mu\delta_s\varphi \qquad (3.269)$$

$$= \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)\delta_s\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta_s\varphi\right)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_i\varphi x^j\right)\omega_{ij}$$

Thus the Noether charges

$$L^{ij,\mu} = \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_i\varphi x^j - \delta_i^{\ \mu}\mathcal{L}\,x^j\right) - (i \leftrightarrow j) \qquad (3.270)$$

have no four-divergence

$$\partial_\mu L^{ij,\mu} = 0 \qquad (3.271)$$

The associated charges

$$L^{ij} = \int d^3\ x\, L^{ij,\mu} \qquad (3.272)$$

Are called the total angular momenta of the field system. In terms of the canonical energy-momentum tensor

$$\Theta_v^{\ \mu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_v\varphi - \delta_v^{\ \mu}\mathcal{L} \qquad (3.273)$$

The current density $L^{ij,\mu}$ can also be written as

$$L^{ij,\mu} = x^i\,\Theta^{j\mu} - x^j\,\Theta^{i\mu} \qquad (3.274)$$

3.25.1 Four-Dimensional Angular Momentum

A similar procedure can be applied to pure Lorentz transformation. An infinitesimal boost to rapidity ζ^i produces a coordinate change

$$x'^{\mu} = \Lambda^{\mu}{}_{v}\, x^{v} = x^{\mu} + \delta^{\mu}{}_{i}\zeta^{i}x^{v} + \delta^{\mu}{}_{0}\zeta^{i}x^{i} \tag{3.275}$$

This can be written as

$$\delta x^{\mu} = \omega^{\mu}{}_{v}\, x^{v} \tag{3.276}$$

$\omega_{ij} = 0$

$$\omega_{0i} = -\omega_{i0} = \zeta^{i} \tag{3.277}$$

With the tensor $\omega^{\mu}{}_{v}$, the restricted Lorentz transformations and the infinitesimal rotations can be treated on the same footing. The rotations have the form Eq(**3.276**) for the particular choice $\omega_{ij} = \epsilon_{ijk}\, \omega^{k}$

$$\omega_{0i} = \omega_{i0} = 0 \tag{3.278}$$

We can now identify the symmetry variations of the field as being

$$\delta_{s}\varphi(x) = \varphi'(x'^{\mu} - \delta x^{\mu}) - \varphi(x)$$

$$= -\partial_{\mu}\varphi(x)x^{v}\omega^{\mu}{}_{v} \tag{3.279}$$

Just as in Eq(**3.268**), the Lagrnagian density transforms as the total derivative

$$\delta_{s}\varphi(x) = -\partial_{\mu}(\mathcal{L}x^{v})\omega^{\mu}{}_{v} \tag{3.280}$$

And we obtain the Noether currents

$$L^{\mu v,\lambda} = -\left(\frac{\partial \mathcal{L}}{\partial \partial_{\lambda}\varphi} x^{\lambda}\varphi\, x^{v} - \delta^{\mu\lambda}\, \mathcal{L}\, x^{v}\right) + (\mu \leftrightarrow v)$$

$$= x^{\mu}\, \Theta^{v\lambda} - x^{v}\, \Theta^{\mu\lambda} \tag{3.281}$$

These currents have no four-divergence

$$\partial_{\lambda} L^{\mu v,\lambda} = 0 \tag{3.282}$$

The associated charges

$$L^{\mu v} \equiv \int d^{3}\, x\, L^{\mu v,0} \tag{3.283}$$

are independent of time.

For the particular form of $\omega_{\mu\nu}$ in Eq(**3.277**), we find time independent components L^{i0}. The components L^{ij} coincide with the previously-derived angular momenta.

The constancy of L^{i0} is the relativistic version of the center of mass theorem (3.52). indeed, since

$$L^{i0} = \int d^3\, x \left(x^i \Theta^{00} - x^0 \Theta^{i0}\right) \qquad (3.284)$$

we can define the relativistic center of mass

$$x^i_{CM} = \frac{\int d^3\, x\; \Theta^{00}\; x^i}{\int d^3\, x\; \Theta^{00}} \qquad (3.285)$$

and the average velocity

$$v^i_{CM} = c\,\frac{\int d^3\, x\; \Theta^{i0}}{\int d^3\, x\; \Theta^{00}} = c\,\frac{P^i}{P^0} \qquad (3.286)$$

Since $\int d^3\, x\; \Theta^{i0} = P^i$ is the constant momentum of the system, also v^i_{CM} is a constant. Thus, the constancy of L^{0i} implies the center of mass to move with the constant velocity

$$x^i_{CM}(t) = x^i_{0CM} + v^i_{0CM} t \qquad (3.287)$$

with $x^i_{0CM} = L^{0i}/P^0$. The quantities $L^{\mu\nu}$ are referred to as four-dimensional orbital angular momenta.

It is important to point out that the vanishing divergence of $L^{\mu\nu,\lambda}$ makes $\Theta^{\nu\mu}$ symmetric :

$$\partial_\lambda L^{\mu\nu,\lambda} = \partial_\lambda\left(x^\mu \Theta^{\nu\lambda} - x^\nu \Theta^{\mu\lambda}\right)$$

$$= \Theta^{\nu\mu} - \Theta^{\nu\mu} = 0 \qquad (3.288)$$

Thus, translationally invariant field theories whose orbital momentum is conserved have always a symmetric canonical energy-momentum tensor.

$$\Theta^{\mu\nu} = \Theta^{\nu\mu} \qquad (3.289)$$

3.25.2 Spin Current

If the field $\varphi(x)$ is no longer a scalar but carries spin degrees of freedom, the derivation of the four-dimensional angular momentum becomes slightly more involved.

3.26 Electromagnetic Fields

Consider first the case of electromagnetism where the relevant field is the four-vector potential $A^{\mu}(x)$. When going to a new coordinate frame

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \tag{3.290}$$

the vector field at the same point remains unchanged in absolute space-time. However, since the components A^{μ} refer to two different basic vectors in the different frames, they must be transformed accordingly. Indeed, since A^{μ} is a vector and transforms like x^{μ}, it must satisfy the relation characterizing a vector filed:

$$A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}\, A^{\nu}(x) \tag{3.291}$$

For an infinitesimal transformation

$$\delta_s x^{\mu} = \omega^{\mu}{}_{\nu}\, x^{\nu} \tag{3.292}$$

This implies a symmetry variation

$$\delta_s A^{\mu}(x) = A'^{\mu}(x) - A^{\mu}(x) = A'^{\mu}(x - \delta x) - A^{\mu}(x)$$

$$= \omega^{\mu}{}_{\nu}\, A^{\nu}(x) - \omega^{\lambda}{}_{\nu}\, x^{\nu}\partial_{\lambda}\, A^{\mu} \tag{3.293}$$

The first term is a spin transformation, the other an orbital transformation. The orbital transformation can also be written in terms of the generators $\hat{L}_{\mu\nu}$ of the Lorentz group defined as

$$\delta_s^{orb}\; A^{\mu}(x) = -i\omega^{\mu\nu}\, \hat{L}_{\mu\nu}\, A(x) \tag{3.294}$$

It is convenient to introduce 4×4 spin transformation matrices $L_{\mu\nu}$ with the matrix elements:

$$\left(L_{\mu\nu}\right)_{\lambda\kappa} \equiv i\left(g_{\mu\lambda}\, g_{\nu\kappa} - g_{\mu\kappa}\, g_{\nu\lambda}\right) \tag{3.295}$$

They satisfy the same commutation relations Eq(**3.212**) as the differential operators $\hat{L}_{\mu\nu}$ defined in Eq.(**3.211**). By adding together the two generators $\hat{L}_{\mu\nu}$ and $L_{\mu\nu}$, we form the operator of total four-dimensional angular momentum

$$\hat{J}_{\mu\nu} \equiv \hat{L}_{\mu\nu} + L_{\mu\nu} \tag{3.296}$$

and can write the symmetry variation Eq(**3.293**) as

$$\delta_s^{orb} \;\; A^\mu(x) = -i\omega^{\mu\nu} \, \hat{J}_{\mu\nu} \; A(x) \tag{3.297}$$

If the Lagrangian density involves only scalar combinations of four-vectors A^μ, and if it has no explicit x-dependence, it changes under Lorentz transformations like a scalar filed :

$$\mathcal{L}^{'}(x^{'}) \equiv \mathcal{L}\big(A^{'}(x^{'}), \partial^{'} A^{'}(x^{'})\big) = \mathcal{L}(A(x), \partial A(x)) \equiv \mathcal{L}(x) \tag{3.298}$$

Infinitesimally, this makes the symmetry variation a pure gradient term:

$$\delta_s \mathcal{L} = -(\partial^\mu \mathcal{L}\, x^\nu) \tag{3.299}$$

Thus, Lorentz transformations in the Noether sense. Following Noether's construction Eq(**3.270**), we calculate the current of total four-dimensional angular momentum :

$$J^{\mu\nu,\lambda} = \frac{\partial \mathcal{L}}{\partial \partial_\lambda A_\mu} \; A^\nu - \left(\frac{\partial \mathcal{L}}{\partial \partial_\lambda \; A^\kappa} \; \partial^\mu A^\kappa x^\nu - \delta^{\mu\lambda} \mathcal{L}\, x^\nu\right) - (\mu \leftrightarrow \nu) \tag{3.300}$$

The last two terms have the same form as the current $L^{\mu\nu,\lambda}$ of the four-dimensional angular momentum of the scalar field. Here they are the currents of the four-dimensional orbital angular momentum :

$$L^{\mu\nu,\lambda} = -\left(\frac{\partial \mathcal{L}}{\partial \partial_\lambda \; A^\kappa} \; \partial^\mu A^\kappa x^\nu - \delta^{\mu\lambda} \mathcal{L}\, x^\nu\right) + (\mu \leftrightarrow \nu) \tag{3.301}$$

Note that this current has the form

$$L^{\mu\nu,\lambda} = -i \frac{\partial \mathcal{L}}{\partial \partial_\lambda \; A^\kappa} \; \hat{L}^{\mu\nu} A^\kappa + \left[\delta^{\mu\lambda} \mathcal{L}\, x^\nu - (\mu \leftrightarrow \nu)\right] \tag{3.302}$$

where $\hat{L}^{\mu\nu}$ are the differential operators of four-dimensional angular momentum in the commutation rules (**3.12**).

just as the scalar case Eq(**3.298**), the currents Eq(**3.302**) can be expressed in terms of the canonical energy-momentum tensor as

$$L^{\mu\nu,\lambda} = x^\mu \Theta^{\nu\lambda} - x^\nu \Theta^{\mu\lambda} \tag{3.303}$$

The first term in Eq(**3.300**),

$$\Sigma^{\mu\nu,\lambda} = \left[\frac{\partial \mathcal{L}}{\partial \partial_\lambda \; A_\nu} \; A^\nu - (\mu \leftrightarrow \nu)\right] \tag{3.304}$$

Is referred to as the spin current. It can be written in terms of the 4×4 generators Eq(**3.295**) of the Lorentz group as

$$\Sigma^{\mu\upsilon,\lambda} = -i\frac{\partial\mathcal{L}}{\partial\partial_\lambda A^\kappa}(L^{\mu\upsilon})_{\kappa\sigma}A^\sigma \qquad (3.305)$$

The two currents together,

$$J^{\mu\upsilon,\lambda}(x) \equiv L^{\mu\upsilon,\lambda}(x) + \Sigma^{\mu\upsilon,\lambda}(x) \qquad (3.306)$$

Are conserved, satisfying $\partial_\lambda J^{\mu\upsilon,\lambda}(x) = 0$. Individually, they are not conserved. Thre total angular momentum is given by the charge

$$J^{\mu\upsilon} = \int d^3\ x\, J^{\mu\upsilon,0}\,(x) \qquad (3.307)$$

It is a constant of motion. Using the conservation law of the energy-momentum tensor we find, just as in Eq(**3.288**), that the orbital angular momentum satisfies

$$\partial_\lambda\ L^{\mu\upsilon,\lambda}(x) = -[\Theta^{\mu\upsilon}(x) - \Theta^{\upsilon\mu}(x)] \qquad (3.308)$$

From this we find the divergence of the spin current

$$\partial_\lambda\ \Sigma^{\mu\upsilon,\lambda}(x) = -[\Theta^{\mu\upsilon}(x) - \Theta^{\upsilon\mu}(x)] \qquad (3.309)$$

For the charges associated with orbital and spin currents

$$L^{\mu\upsilon}(t) \equiv \int d^3\ x\, L^{\mu\upsilon,0}\,(x)\,, \qquad \Sigma^{\mu\upsilon} \equiv \int d^3\ x\, \Sigma^{\mu\upsilon,0}\,(x) \qquad (3.310)$$

this implies the following time dependence :

$$\dot{L}^{\mu\upsilon}(t) = -\int d^3\ x\,[\Theta^{\mu\upsilon}\ (x) - \Theta^{\upsilon\mu}(x)]$$

$$\dot{\Sigma}^{\mu\upsilon}(t) = -\int d^3\ x\,[\Theta^{\mu\upsilon}\ (x) - \Theta^{\upsilon\mu}(x)] \qquad (3.311)$$

Thus fields with a nonzero spin density have always a non-symmetric energy momentum tensor.

In general, the current density $J^{\mu\upsilon,\lambda}$ of total angular momentum reads

$$J^{\mu\upsilon,\lambda} = \left(\frac{\partial\delta_s^x\mathcal{L}}{\partial\partial_\lambda\ \omega_{\mu\upsilon}} - \delta^{\mu\lambda}\mathcal{L}\,x^\upsilon\right) - (\mu \leftrightarrow \nu) \qquad (3.312)$$

By the chain rule of differentiation, the derivative with respect to $\partial_\lambda\ \omega_{\mu\upsilon}(x)$ can come only from field derivatives, for a scalar field

$$\frac{\partial \delta_s^x \mathcal{L}}{\partial \partial_\lambda\ \omega_{\mu\nu}(x)} = \frac{\partial \mathcal{L}}{\partial \partial_\lambda \varphi} \frac{\partial \delta_s^x \varphi}{\partial \omega_{\mu\nu}(x)} \tag{3.314}$$

and for a vector field

$$\frac{\partial \delta_s^x \mathcal{L}}{\partial \partial_\lambda\ \omega_{\mu\nu}(x)} = \frac{\partial \mathcal{L}}{\partial \partial_\lambda A^\kappa} \frac{\partial \delta_s^x A^\kappa}{\partial \omega_{\mu\nu}(x)} \tag{3.315}$$

The alternative rule of calculating angular momenta is to introduce spacetime-dependent transformations

$$\delta^x x = \omega^\mu{}_\nu(x) x^\nu \tag{3.316}$$

under which the scalar fields transform as

$$\delta_s \varphi = -\partial_\lambda \varphi \omega^\mu{}_\nu(x) x^\nu \tag{3.317}$$

and the Lagrangian density as

$$\delta_s^x \varphi = -\partial_\lambda \mathcal{L} \omega^\mu{}_\nu(x) x^\nu = -\partial_\lambda (x^\nu \mathcal{L}) \omega^\mu{}_\nu(x) \tag{3.318}$$

By separating spin and orbital transformations of $\delta_s^x A^\kappa$ we find the two contributions $\sigma^{\mu\nu,\lambda}$ and $L^{\mu\nu,\lambda}$ to the current $J^{\mu\nu,\lambda}$ of the total angular momentum, the latter receiving a contribution from the second term in Eq(**3.312**).

3.27 Dirac Field

We now turn to the Dirac field. Under a Lorentz transformation Eq(**3.290**), this transforms according to the law

$$\psi(x') \xrightarrow{A} \psi'_\Lambda(x) = D(\Lambda)\psi(x) \tag{3.319}$$

where $D(\Lambda)$ are the 4×4 spinor representation matrices of the Lorentz group. Their matrix elements can most easily be spacified for infinitesimal transformations. For an infinitesimal Lorentz transformation

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu \tag{3.320}$$

under which the coordinates are changed by

$$\delta_s x^\mu = \omega^\nu{}_\mu\, x^\nu \tag{3.321}$$

The spin components transform under the representation matrix

$$D\left(\delta_{\mu}{}^{\nu} + \omega_{\mu}{}^{\nu}\right) = \left(1 - i\frac{1}{2}\,\omega_{\mu\nu}\sigma^{\mu\nu}\right) \qquad (3.322)$$

Where $\sigma_{\mu\nu}$ are the 4×4 matrices acting on the spinor space

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_{\mu}, \gamma_{\nu}\right] \qquad (3.323)$$

From the anticommutation rules Eq(**3.251**) it is easy to verify that the spin matrices $S_{\mu\nu} \equiv \sigma_{\mu\nu}/2$ satisfy the same commutation rules Eq(**3.212**) as the previous orbital and spin-1 generators $\hat{L}_{\mu\nu\mu}$ and $L_{\mu\nu}$ of Lorentz transformations.

The field has the symmetry variation .

$$\delta_{s}\psi(x) = \psi^{'}(x) - \psi(x) = D\left(\delta_{\mu}{}^{\nu} + \omega_{\mu}{}^{\nu}\right)\psi(x - \delta x) - \psi(x)$$

$$= -i\frac{1}{2}\,\omega_{\mu\nu}\sigma^{\mu\nu}\psi(x) - \omega^{\lambda}{}_{\nu}x^{\nu}\partial_{\lambda}\psi(x)$$

$$= -i\frac{1}{2}\,\omega_{\mu\nu}\left(S^{\mu\nu} + \hat{L}^{\mu\nu}\right)\psi(x) \equiv -i\frac{1}{2}\,\omega_{\mu\nu}\,\hat{J}^{\mu\nu}\psi(x) \qquad (3.324)$$

The last line showing the separation into spin and orbital transformation for a Dirac particle.

Since the Dirac Lagrangian is Lorentz-invariant, it changes under Lorentz transformations like a scalar field:

$$\mathcal{L}^{'}(x^{'}) = \mathcal{L}(x) \qquad (3.325)$$

Infinitesimally, this amounts to

$$\delta_{s}\mathcal{L} = -\left(\partial_{\mu}\mathcal{L}\,x^{\nu}\right)\omega^{\mu}{}_{\nu} \qquad (3.326)$$

With the Lorentz transformations being symmetry transformations in Noether sense, we calculate the current of total four-dimensional angular momentum extending the formulas Eq(**3.281**) and Eq(**3.299**) for scalar field and vector potential. The result is

$$J^{\mu\nu,\lambda} = \left(-i\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\psi}\,\sigma^{\mu\nu}\psi - i\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\psi}\,\hat{L}^{\mu\nu}\psi + c.c.\right) + \left[\sigma^{\mu\lambda}\mathcal{L}x^{\nu} - (\mu \leftrightarrow \nu)\right] \qquad (3.327)$$

As before in Eq(**3.300**) and Eq(**3.281**), the orbital part of Eq(**3.327**) can be expressed in terms of the canonical energy-momentum tensor as

$$L^{\mu\nu,\lambda} = x^{\mu}\Theta^{\nu\lambda} - x^{\nu}\Theta^{\mu\lambda} \qquad (3.338)$$

The first term in Eq(**3.236**) is the spin current

$$\Sigma^{\mu\nu,\lambda} = \frac{1}{2}\left(-i\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\sigma^{\mu\nu}\psi + c.c.\right) \qquad (3.339)$$

Inserting Eq(**3.285**), this becomes explicitly

$$\Sigma^{\mu\nu,\lambda} = -\frac{i}{2}\,\bar{\psi}\gamma^{\lambda}\sigma^{\mu\nu}\psi = \frac{i}{2}\,\bar{\psi}\gamma^{[\mu\ \ \nu\ \ \lambda]}\psi = \frac{1}{2}\,\epsilon^{\mu\nu\lambda\kappa}\,\bar{\psi}\gamma^{\kappa}\psi \qquad (3.340)$$

The spin density is completely antisymmetric in the three indices.

The conservation properties of the three currents are the same as in Eqs.(**3.307-3.311**).

3.28 Internal Symmetries

In quantum field theory, an important role is played by internal symmetries. They do not involve any change in the space time coordinate of the fields, whose symmetry transformations have the simple form

$$\phi^{'}(x) = e^{-i\alpha G}\,\phi(x) \qquad (3.341)$$

Where G are the generators of some Lie group and α the associated transformation parameters. The filed ϕ may have several indices on which the generators G act as a matrix. The symmetry variation associated with Eq(**3.341**) is obviously

$$\delta_s\phi^{'}(x) = -i\alpha G\phi(x) \qquad (3.342)$$

The most important example is that of a complex field ϕ and a generator $G = 1$, where Eq(**3.341**) is simply a multiplication by a constant phase factor. One also speaks of $U(1)$-symmetry.

Other important examples are those of a triple or an octet of fields ϕ_i with G being the generatore of an $SU(2)$ vector representation or an $SU(3)$ octet representation (the adjoint representations of these groups). The first case is associated with charge conservation in electromagnetic interactions, the other two with isospin and $SU(3)$ invariance in strong interactions. The latter symmetries are however, not exact.

3.28.1. $U(1)$- Symmetry and Charge Conservation

Given a Lagrangian density $\mathcal{L}(x) = \mathcal{L}(\phi(x), \partial\phi(x), x)$ depending only on the absolute squares $|\phi|^2, |\partial\phi|^2, |\phi\partial\phi|$. Then $\mathcal{L}(x)$ is invariant under $U(1)$-transformations

$$\delta_s\phi(x) = -i\phi(x) \qquad (3.342)$$

Indeed:

$$\delta_s \mathcal{L} = 0 \tag{3.343}$$

On the other hand, we find by the chain rule of differentiation **:**

$$\delta_s \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu \phi}\right)\delta_s \phi + \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right]\delta_s \phi = 0 \tag{3.344}$$

The Euler-Lagrange equation removes the first part of this, and inserting Eq(**3.342**) we find by comparison with Eq(**3.343**) that

$$j_\mu = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\phi \tag{3.345}$$

is a conserved current .

For a free relativistic complex scalar field with a Lagrangian density

$$\mathcal{L}(x) = \partial_\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi \tag{3.346}$$

We have to add the contributions of real and imaginary parts of the field ϕ in formula Eq(**3.345**). then we obtain the conserved current

$$j_\mu = -\varphi^* \overleftrightarrow{\partial}_\mu \varphi \tag{3.347}$$

Where $\varphi^* \overleftrightarrow{\partial}_\mu \varphi$ denotes the left-minus-right derivative:

$$\varphi^* \overleftrightarrow{\partial}_\mu \varphi \equiv \varphi^* \partial_\mu \varphi - \left(\partial_\mu \varphi^*\right)\varphi \tag{3.348}$$

For a free Dirac field, we find from Eq(**3.345**) the conserved current

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x) \tag{3.349}$$

3.28.2. $\boldsymbol{SU(N)}$ -Symmetry

For more general internal symmetry groups, the symmetry variations have the form

$$\delta_s \varphi = -i\alpha_i G_i \varphi \tag{3.350}$$

And the conserved currents are

$$j_i^\mu = -i\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} G_i \varphi \tag{3.351}$$

3.28.3 Broken Internal Symmetries

The physically important symmetries $SU(2)$ of isospin and $SU(3)$ are not exact. The Lagrnage density is not strictly zero. In this case we remember the alternative derivation of the conservation law. we introduce the space time-dependent parameters $\alpha(x)$ conclude from the extremality property of the action that

$$\partial_\mu \frac{\partial \mathcal{L}^\epsilon}{\partial \partial_\mu \alpha_i(x)} = \frac{\partial \mathcal{L}^\epsilon}{\partial \alpha_i(x)} \qquad (3.352)$$

This implies the divergence law for the above derived current

$$\partial_\mu j_i^\mu(x) = \frac{\partial \mathcal{L}^\epsilon}{\partial \alpha_i} \qquad (3.353)$$

3.30 Generating the Symmetry Transformations on Quantum Fields

As in quantum mechanical systems, the charges associated with the conserved currents of the previous section can be used to generate the transformations of the fields from which they were derived. One merely has to invoke the canonical field commutation rules.

As an important example, consider the currents Eq(**3.351**) of an iternal $U(N)$-symmetry. Their charges

$$Q^i = -i \int d^3 x \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} G_i \varphi \qquad (3.354)$$

can be written as

$$Q^i = -i \int d^3 x \pi G_i \varphi \qquad (3.355)$$

where $\pi(x) \equiv \partial \mathcal{L} / \partial \partial_\mu \varphi(x)$ is the canonical momentum of the field $\varphi(x)$. After quantization, these fields satisfy the canonical commutation rules:

$$[\pi(\mathbf{x}, t), \varphi(\mathbf{x}', t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

$$[\varphi(\mathbf{x}, t), \varphi(\mathbf{x}', t)] = 0$$

$$[\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0 \qquad (3.356)$$

From this we derive directly the commutation rule between the quantized charges Eq(**3.355**) and the field $\varphi(x)$:

$$\left[Q^i, \hat{\varphi}(x)\right] = -\alpha^i G_i \varphi(x) \qquad (3.357)$$

We also find that the commutation rules among the quantized charges are

$$\left[\hat{Q}^i, \hat{Q}^j\right] = \left[G^i, G^j\right] \qquad (3.358)$$

Since these coincide with those of the matrices G_i, the operators Q^i are seen to form a representation of the generators of the symmetry group in the Fock space .

It is important to realize that the commutation relations Eq(**3.357**) and Eq(**3.358**) remain also valid in the presence of symmetry breaking terms, as long as these do not contribute to the canonical momentum of the theory. Such terms are called soft symmetry breaking terms. The charges are no longer conserved, so that we must attach a time argument to the commutation relations Eq(**3.357**) and Eq(**3.358**). All times in these relations must be the same, in order to invoke the equal-time canonical commutation rules.

The most important example is the canonical commutation relation Eq(**3.221**) itself, which holds also in the presence of any potential $V(q)$ in the Hamiltonian. This breakes translation symmetry, but does not contribute to the canonical momentum $p = \partial L / \partial \dot{q}$. In this case, the relation generalizes to

$$\epsilon^i = i\epsilon^i\left[\hat{p}^i(t), \hat{x}^j(t)\right] \qquad (3.359)$$

Which is correct thanks to the validity of the canonical commutation relations Eq(**3.219**) at arbitrary equal times, also in the presence of a potential.

Another important example are the commutation rules of the conserved charges associated with the Lorentz generators Eq(**3.338**):

$$J^{\mu v} \equiv \int d^3x\, J^{\mu v,0}(x) \qquad (3.360)$$

Which are the same as those of the 4×4-matrices Eq(**3.294**), and those of the quantum mechanical generators to be:

$$\left[\hat{J}^{\mu v}, \hat{J}^{\mu\lambda}\right] = -ig^{\mu\mu}\hat{J}^{v\lambda} \qquad (3.361)$$

The generators $J^{\mu v} \equiv \int d^3x\, J^{\mu v,0}(x)$ are sums $J^{\mu v} = L^{\mu v}(t) + \Sigma^{\mu v}(t)$ of charges Eq(**3.309**) associated with orbital and spin rotations. According to Eq(**3.310**), the individual charges are time-dependent. Only their sum is conserved. Nevertheless, they both generate Lorentz transformations : $L^{\mu v}(t)$ on the spacetime argument of the fields, and $\Sigma^{\mu v}(t)$ on the spin indices. As a consequence, they both satisfy the commutation relations Eq(**3.361**) :

$$\left[\hat{L}^{\mu v}, \hat{L}^{\mu\lambda}\right] = -ig^{\mu\mu}\,\hat{L}^{v\lambda}$$

$$\left[\hat{\Sigma}^{\mu v}, \hat{\Sigma}^{\mu\lambda}\right] = -ig^{\mu\mu}\,\hat{\Sigma}^{v\lambda} \qquad (3.362)$$

The commuatators Eq(**3.358**) have played an important role in developing a theory of strong interactions, where they first appeared in the form of a charge algebra of the broken

symmetry $SU(3) \times SU(3)$ of weak and electromagnetic charges. This symmetry will be discuss in the next chapter .