Chapter Two

Lie algebras

2.1 Introduction

The tangent vectors form Lie algebra on any manifold.

Definition 2.1.1.

A real Lie algebra$\mathfrak{g}$[1]is a vector space over $\mathbb{R}$ with a bilinear map (called the Lie bracket)

$[.,.]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$,

$(x, y) \longmapsto [x, y]$, Such that for all $, y, z \in \mathfrak{g}$,

1. $[x, y] = -[x, y]$

2. $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

A **homomorphism of Lie algebras**[2] is a k-linear map $\alpha: \mathfrak{g} \longrightarrow \mathfrak{g}'$ such that

$\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in \mathfrak{g}$

Condition (**2**) is called **Jacobi identity** .and condition (**1**) applied to $[x + y, x + y]$ shows that the Lie bracket is **skew-symmetric**.

Remark 2.1.1.

Let the commutator $[.,.]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be defined by :

$\left(\exp(x) \exp(y) = exp\left(x + y + \frac{1}{2}[x, y] + \dots \dots ..\right)\right)$ it's Taylor series bilinear skew-symmetry .Then it satisfies the following identity ,called Jacobi identity :

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

This identity can also be written in any of the following equivalent forms:

$$[x, [y, z]] + [[x, y], z] + [y, [x, z]] = 0$$

$$ad\, x. [y, z] = [ad\, x. y, z] + [y, ad\, x. z]$$

---

[1]Notes on Lie Groups – Eugene Lerman – Februaury 15,2012.

[2] Introduction to Lie Groups and Lie algebras – Alexander Kirillov, Jr- department of Math, Suny at Stony Brook, NY 11794, USA.

$$ad[x,y] = ad\ x\ ad\ y - ad\ y\ ad\ x$$

Definition 2.1.2. (Sub algebras)

Let $\mathfrak{g}$ be a Lie algebra . a subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a Lie subalgebra if it is closed under commutator ,for any $x, y \in \mathfrak{h}$, we have $[x,y] \in \mathfrak{h}$. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an ideal if for any $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, we have $[x,y] \in \mathfrak{h}$.

If $\mathfrak{h}$ is an ideal , then ${}^{\mathfrak{g}}/_{\mathfrak{h}}$ has a canonical structure of a Lie algebra .

Definition 2.1.3.

A Lie algebra $\mathfrak{g}$ is said to be commutative (or abelian)[3] if $[x,y] = 0$ for all $x, y \in \mathfrak{g}$. Thus, to give a commutative Lie algebra amounts to giving a finite-dimensional vector space .

An injective homomorphism is sometimes called an embedding , and a surjective homomorphism is sometimes called a Quotient map .

We shall be mainly concerned with finite-dimensional Lie algebras. Suppose that $\mathfrak{g}$ has a basis $\{e_1, e_2, \dots \dots ., e_n\}$, and write

$$[e_i, e_j] = \sum_{l=1}^{n} a_{ij}^{l}\ e_l \ , \quad a_{ij}^{l} \in k \ , \qquad 1 \le i\ , j \le n\ .$$

The a_{ij}^{l} , $1 \le i\ , j, l \le n$, are called the structure constants of $\mathfrak{g}$ relative to the given basis . they determine the bracket on $\mathfrak{g}$.

Definition 2.1.4.

An ideal in a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{a}$ such that $[x, a] \in \mathfrak{a}$ for all $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$ (such that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$) .

Notice that ; because of the skew-symmetry of the bracket

$$[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a} \Leftrightarrow [\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a} \Leftrightarrow [\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a} \text{ and } [\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$$

All left (or right) ideals are two-sided ideals .

Example 2.1.1

Here we have some type of Lie subalgebras of $\mathfrak{gl}_n$:

$\mathfrak{sl}_n = \{A \in M_n(k) | trace\ (A) = 0\}$

[3] Lie Algebras, Algebraic Groups, and Lie Groups J.S. Milne – may 5, 2013.

$\mathfrak{o}_n = \{A \in M_n(k) | A \ is \ skew \ symmetry, \ A + A^t = 0\}$

$\mathfrak{sp}_n = \left\{A \in M_n(k) | \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A + A^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = 0\right\}$

$\mathfrak{b}_n = \{(c_{ij}) | c_{ij} = 0 \ if \ i > j\}$ (upper triangle matrices)

$\mathfrak{n}_n = \{(c_{ij}) | c_{ij} = 0 \ if \ i \geq j\}$ (strictly upper triangular matrices)

$\mathfrak{d}_n = \{(c_{ij}) | c_{ij} = 0 \ if \ i \neq j\}$ (diagonal matrices)

Definition 2.1.5.

Take a fixed element h. Multiplication by h defines the left translation

$$L_h : \begin{matrix} G \longrightarrow G \\ g \longmapsto L_h g = hg \end{matrix}$$

In coordinates, this is expressed as follows:

Assume that $(g) = \alpha^a$. Then left translation induces a motion $L_h : \alpha^a \longmapsto \beta^a(\alpha)$, such that $\phi(hg) = \beta$. Of course there is also the right translation, but that doesnot give different results up to some ordering switches.

Left translation is a bijection of G to itself . it also acts on functions on the manifold : to a function f it associates a new function $L_h f$ which is simply the old function moved along the manifold, i.e.

$(L_h f)(hg) = f(g)$

Also induces a map on tangent vectors, the differential map $dL_h : T_g G \longrightarrow T_{hg} G$

Which similarly maps the vector X at point g to the vector $dL_h \ . \ X$ at point hg defined by

$$(dL_h \cdot X)[f(hg)] = X[f(g)] \qquad (2.1)$$

Remark 2.1.2.

This is sometimes written with a $*$ subscript as $dL_h = L_{h*}$. for maps from $\mathbb{K}^d$ to $\mathbb{K}^m$, this is the familiar Jacobian (the matrix of derivatives $\partial f^a / \partial x^b$) . the differential map allows us to single out a particular kind fields, namely those that are invariant under the differential maps of all left translations.

Definition 2.1.6.

A vector field is called left-invariant if $X|_{hg} = dL_h \cdot X|_g$ for all $g, h \in G$,such that $X|_g$ notation means "the vector field X at point g" in coordinates the components of the vector are evaluated at that point , and it acts on functions defined at g .

also the left hand side of (Eq **2.1**) is again at the same field at the point hg .

$$X|_{hg}[f(hg)] = X|_g[f(g)]$$

Hence this is a restriction of the g- dependence of the vector field X – it does not apply to a vector at a given point. In coordinates this is written as

$$dL_h \cdot X|_g = \mathrm{X}^{\mathrm{a}}(hg)\frac{\partial}{\partial x^a(gh)} = \mathrm{X}^{\mathrm{a}}(g)\frac{\partial}{\partial x^a(g)} = \mathrm{X}^{\mathrm{a}}(g)\frac{\partial x^b(hg)}{\partial x^{a\,(g)}}\frac{\partial}{\partial x^a(gh)}$$
$$= \mathrm{X}^{\mathrm{a}}(g)(dL_h)^b_a\frac{\partial}{\partial x^b(gh)}$$

Definition 2.1.7.

The Lie algebra g of a group G is the space of left-invariant vector fields with the Lie bracket as product.

The Lie algebra is generically denoted by the name of the group in lower case fracture letters, e.g. the Lie algebra of $SU(n)$ is (n) .

If one in particular choose $= e$, left-invariant implies that : $X|_h = d\mathrm{L}_h\, X|_{\mathrm{e}}$.

2.2 One parameter and Local One-Parameter Groups Action on Manifold :

Definition 2.2.1.

Let G be a group and X a set [4]. Then G is said to act on X (on the left) if there is a mapping $\theta: G \times X \longrightarrow X$ satisfying two conditions:

If e is the identity element of , Then **:**

1. $\theta(e, x) = x$ for all $x \in X$

2. If $g_1, g_2 \in G$, then**:** $\theta(g_1, (g_2, x)) = \theta(g_1, g_2, x)$ for all $x \in X$

[4] An introduction to Differentiable Manifols & Riemannian Geometry- William M.Boothby –Washington University- ST. Louis Missouri- 2003

Definition 2.2.2.

If we let $\theta: R \times M \longrightarrow M$ specializied to an action θ of R on , and θ be a mapping which satisfies the two conditions:

1. $\theta_0(P) = P$ for all $P \in M$

2. $\theta_t \circ \theta_s(P) = \theta_{t+s}(P) = \theta_s \circ \theta_t(P)$ for all $P \in M$ and for all $s, t \in R$

Example 2.2.1.

Suppose that $M = R^3$ and $a = (a^1, a^2, a^3)$ is fixed and assumed different from 0. Then $\theta_t(x) = (x^1 + a^1 t, x^2 + a^2 t, x^3 + a^3 t)$ defines a C^∞ action of R on M .To each $t \in R$ we have thus assigned the translation $\theta_t: R^3 \longrightarrow R^3$. taking the point x to the point $x + ta$. This is a free action and the orbits consist of straight lines parallel to the vector a . A particulary simple special case is given by $a = (1,0,0)$ so that $\theta_t(x) = (x^1 + t, x^2, x^3)$.

Suppose that $\theta: R \times M \longrightarrow M$ is any such C^∞ action . Then it defines on M a C^∞-vector field X, which we shall call infinitesimal generator of θ . according to the following prescripition ; for each $P \in M$ we define $X_P: C^\infty(P) \longrightarrow R$ by

$$X_P f = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\left(f\theta_{\Delta t}(P) \right) - f(P) \right] \qquad (2.2)$$

Directly from Eq(2.2) that X_P is a vector at P in the sence of definition and then verify that $P \longrightarrow X_P$ defines a vector field, or we may proceed as follows . Let U, φ be a coordinate neighborhood of $P \in M$ and let $I_\delta \times V$ be an open subset of $(0, P)$ in $\times M$, where $I_\delta = \{t \in R | -\delta < t < \delta\}$, and V , $\delta > 0$ are so chosen that $\theta(I_\delta \times V) \subset U$. In particular, $V = \theta_0(V)$ is contained in U and contains . Restricted to the open set $I_\delta \times V$, we may write θ in local coordinates

$y^1 = h^1(t, x^1, \dots\dots\dots\dots\dots, x^n)$

.

.

$y^n = h^n(t, x^1, \dots\dots\dots\dots\dots, x^n)$

Or $y = h(t, x)$, where $, x = (x^1, \dots\dots\dots, x^n)$ are the coordinates of $q \in V$ and $y = (y^1, \dots\dots\dots, y^n)$ of $\theta_t(q)$,It's image . The h^i are defined and C^∞ on $I_\delta \times \varphi(V)$ and the range of $h(t, x)$ is in $\varphi(U)$. The fact that θ_0 is the identity and $\theta_{t_{1+t_2}} = \theta_{t_1} \circ \theta_{t_2}$ is reflected in the conditions :

$$h^i(0,x) = x^i \quad \text{and} \qquad h^i(t_1+t_2,x) = h^i(t_1 + h(t_2,x))$$

For $1, \dots \dots, n$. Now if $\hat{f} = (x^1, \dots \dots \dots \dots \dots, x^n)$ is the local expression for $f \in C^\infty(P)$.

Then

$$\frac{1}{\Delta t}\left[f\big(\theta_{\Delta t}(P)\big) - f(P)\right] = \frac{1}{\Delta t}\left[\hat{f}\big(h(\Delta t, x)\big) - \hat{f}(x)\right]$$

And

$$X_P f = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\hat{f}\big(h(\Delta t,x)\big) - \hat{f}(x)\right] = \sum_{i=1}^{n} h^i(0,x)\left(\frac{\partial \hat{f}}{\partial x^i}\right)_{\varphi(P)}$$

Where we have used a dot to indicate differentiation with respect to t . This formula is valid for every $P \in V$ and implies that on V, $X_P = \sum h^i(0,x)\, E_{iP}$ with $E_i = \varphi_*^{-1}\left(\partial/\partial x^i\right)$ and $x = \varphi(P)$, which shows that X is a C^∞-vector field over V. Since every point of M lies in such a neighborhood . X is a C^∞ on M .Note that definition of X at $P \in M$ involves only the values of θ on $I_\delta \times V$. That is, like derivatives in general, it is defined locally and involves only values of t near $\theta = 0$.

Definition 2.2.3

If $\theta: G \times M \longrightarrow M$ is the action of a group G on a manifold M . then a vector field X on M is said to be invariant under the action of G or G -invariant if X is invariant under each of the diffeomorphisms θ_g of M to itself .

In brief if $\theta_{g*}(X) = X$ (as in Def **2.2.2**) .

Theorem 2.2.1.

If $\theta: G \times M \longrightarrow M$ is a C^∞ action of R on . then the infinitesimal generator X is invariant under this action. That is $\theta_{t*}(X_P) = X_{\theta_{t(P)}}$ for all $t \in R$.

Proof:

Let $f \in C^\infty(\theta_t\,(P))$ for some $(t,P) \in R \times M$ and compute $\theta_{t*}(X_P)f$:

$\theta_{t*}(X_P)f = X_P(f \circ \theta_t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[f \circ \theta_t\big(\theta_{\Delta t}\,(P)\big) - f \circ \theta_t(P)\right]$.

However, R is Abelian and we have $\theta_t \circ \theta_{\Delta t} = \theta_{t+\Delta t} = \theta_{\Delta t} \circ \theta_t$, So

$\theta_{t*}(X_P)f = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[f \circ \theta_{\Delta t}\big(\theta_t\,(P)\big) - f \circ \theta_t(P)\right] = X_{\theta_t\,(P)} f$.

Since this holds for all , the result follows .

Corollary 2.2.1.

If $X_P = 0$, then for each q in the orbit of P we have $X_q = 0$. that is , at the points of an orbit the associated vector field vanishes identically or is never zero .

Proof:

The orbit of P consists of all $q = \theta_t(P)$ for some $\in R$; thus by the theorem $X_q = \theta_{t*} X_P$. Since θ_t is an isomorphism of $T_P(M)$ onto $T_q(M)$ so that $X_q = 0$ if and only if $X_P = 0$.

Definition 2.2.4.

A local one-parameter group action or flow on a manifold M is a C^∞ map $\theta: W \longrightarrow M$ which satisfies the following two conditions:

$$\theta_0\ (P) = P \qquad\qquad \text{for all } P \in M$$

If $(s, P) \in W$, then $\big(\theta_t(P)\big) = \alpha(P) - s, \beta\big(\theta_t(P)\big) = \beta(p) - s$, and moreover for any t such that $\alpha(p) - s < t < \beta(p) - s$, $\theta_{t-s}(P)$ is defined and $\theta_t \circ \theta_s(P) = \theta_{t+s}(P)$.

Remark 2.2.1.

For local one-parameter actions we may show as in the global case that :

$\theta_{t*}(X_P) = X_{\theta_t\,(P)}$ if $\in V_t$. As before , $F(t) = \theta_t\ (P)$ defined for $\alpha(p) < t < \beta(p)$ is a C^∞-integral curve of X , which is an immersion of $I(P)$ in M provided that $X_P \neq 0$ and is a single point if $X_P = 0$. We shall continue to refer to these curves as orbits of the local one-parameter group. Just as in the global case. It is a consequence of our definitions curves (and points) partition M into a union of mutually disjoint sets . The proof is a same , essentially , as in the globle case .

Definition 2.2.5.

A vector field X on M is said to be complete if it generates a (global) action of R on M, that is , if $W = R \times M$.

Theorem 2.2.2.

Let X be a C^∞ -vector field on a manifold M and $F: M \longrightarrow M$ a diffeomorphism . Let $\theta(t, P)$ dente the C^∞ map $\theta: W \longrightarrow M$ defined by X. Then X is invariant under F if and only if $F\ (\theta(t, P)) = \theta\big(t, F(P)\big)$ whenever both sides are defined .

Proof:

Suppose that X is invariant under a F . if $\theta_P: I(P) \longrightarrow M$ is the integral curve of X with $\theta_P(0) = P$, then the diffeomorphism F takes it to an integral curve $F(\theta_P(t))$ of the vector field $F_*(X)$. Since $F_*(X) = X$ and $F(\theta_P(0)) = F(p)$, from uniqueness of integral curves we conclude that $F(\theta_P(t)) = \theta(t, F(P))$. this proves the "only I" part of theorem .

Now suppose that $F\ (\theta(t, P)) = \theta(t, F(P))$ and prove that $F_*(X) = X_{F(P)}$. This could be done directly from expression for the infinitesimal generator . but we shall proceed in a slightly different way .Let $\theta_t(t) = \theta(t, P)$ and let d/dt be the natural basis of $T_0(R)$. The tangent space to R at $t = 0$,then by definition . $X_P = \theta_P(0) = \theta_{P*}(d/dt)$ and applying the isomorphism $F_*: T_P(M) \longrightarrow T_{F(P)}(M)$ to this definition we have

$$F_*(X_P) = F_* \circ \theta_{P*}(d/dt) = \theta_{F(P)*}(d/dt) = X_{F(P)}.$$

The second equality is the chain rule for the composition of mappings applied to $\theta_P: R \longrightarrow M$ and $: M \longrightarrow M$. the third equality uses the hypothesis that $F \circ \theta_P(t) = \theta_{F(P)}(t)$.

Definition 2.2.6.

Let R be the additive group of real numbers , considered as a Lie group , and let G be an arbitrary Lie group . A one –parameter subgroup H of G is the hoemomorphic image $H = F(R)$ of a homomorphism $: R \longrightarrow G$. It is called trivial if $H = \{e\}$.

Example 2.2.2.

Let G be the group $Gl(3, R)$. We consider two one –parameter subgroups . that is , two homomorphisms F_1, F_2 into G defined as follows ($a, b, c \in R$ are constants):

$$F_1(t) = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{at} \end{pmatrix} \quad \text{and} \quad F_2(t) = \begin{pmatrix} 1 & at & bt + \frac{1}{2}\,act^2 \\ 0 & 1 & ct \\ 0 & 0 & 1 \end{pmatrix}$$

Answer.

Now $Gl(3, R)$ acts naturally on R^3 and hence each F_1 defines an action on R^3 . In the case of

F_1 we have $\theta(t\,, x^1, x^2\,, x^3) = (e^{at}\ x^1, e^{at}\ x^2\,, e^{at}\ x^3)$. Therefore the infinitesimal generator X is given at $x \in R^3$ by **:** $X_x = \dot{\theta}(0, x) = ax^1 \frac{\partial}{\partial x^1} + ax^2 \frac{\partial}{\partial x^2} + \ ax^3 \frac{\partial}{\partial x^3}$

And the integral curves ,or orbits are the lines through the origin (see **Fig 2.1**) .

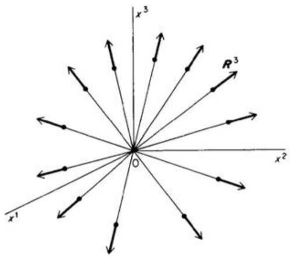


Fig (2.1)

The group $Gl(n, R)$ also acts on $P^{n-1}(R)$, since it preserves the equivalence relation (proportionality) of n-tuples which defines $P^{n-1}(R)$. In particular $Gl(3, R)$ acts on two-dimensional projective space $P^2(R)$. In this case F_1 defines a trivial action $(t, P) \equiv P$.

$$F_1(t) = \begin{pmatrix} \cos at & \sin at & 0 \\ -\sin at & \cos at & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The $SO(3)$ is homomorphism and acts on the unit sphere S^2 in a standard manner . the action is just the usual rotation of the sphere . and F defines a one –parameter group of rotations holding the x^3 axis fixed :

$$\theta(t, x^1, x^2, x^3) = (x^1 \cos at + x^2 \sin at , -x^1 \sin at + x^2 \cos at , x^3)$$

The orbits are the lines of latitude and the generator X is tangent to them and orthogonal to the x^3-axis , $X = 0$ at the north and south poles $(0,0, \pm 1)$. (See **Fig 2.2**)

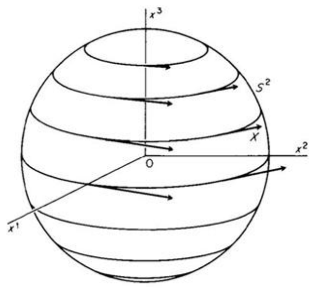


Fig (2.2)

Example 2.2.3.

We recall also that a Lie group G acts on itself (on the right) by right translations . Thus if we are given a homomorphism : $R \longrightarrow G$, we may define an action θ of R on $M = G$ by

$\theta(t,g) = g_{F(t)}(g) = gF(t)$. we have used R_a to denote right translation : $g_a(g) = ga$. this is a composition of C^∞ maps , F and right translation . it is an action since F is a homomorphism and multiplication is associative :

$$\theta(0,g) = gF(0) = g$$

$$\theta(t+s,g) = gF(t+s) = g\big(F(t)F(s)\big)$$

$$= \Big(g\big(F(t)F(s)\big)\Big) = \theta\big(s,\theta(t,g)\big)$$

Thus the examples above furnish further examples of one –parameter group action . namely on $M = Gl(3,R)$ and $M = O(3)$, respectively .

A left –invariant vector field on G is uniquely determined by it's value at the identity , we may use these ideas to characterized one-parameter subgroups of a Lie group .

Theorem 2.2.3.

Let F:R⟶ G be a one-parameter subgroup of the Lie group G and X the left –invariant vector field on G defined by $X = F(0)$. Then $\theta(t,g) = R_{F(t)}g$ defines on action $\theta: R \times G \longrightarrow G$ of R on G (as a manifold) having X as infinitesimal generator .Conversely, let X be a left-invariant vector field and $\theta: R \times G \longrightarrow G$ the corresponding action . then $F(t) = \theta(t,e)$ ia a one –parameter subgroup of G and $\theta(t,g) = R_{F(t)}g$.

Proof:

Given the $F: R \longrightarrow G$ .then $\theta: R \times G \longrightarrow G$ .defined by $\theta(t,g) = R_{F(t)}g = gF(t)$ is, as we have just seen, an action of R on G. If $a \in G$,then :

$$L_a\theta(t,g) = a\big(gF(t)\big) = (ag)F(t) = \theta(t, L_a(g)) .$$

By (Theorem **2.2.2**) it follows that the generator X of θ is L_a -invariant .for any $\in G$. however $\theta(t,e) = F(t)$, and so $X_e = \theta(0,e) = F(0)$, which proves the first half of the theorem .

For the converse X, being left-invariant , is both C^∞ and complete and it generates an action θ of R on G. By (Theorem 2.2.2) for any left translation L_h we have $L_h\theta(t,g) = \theta\big(t, L_h(g)\big)$ or equivalently, $h\theta(t,g) = \theta(t,hg)$. let $F(t) = \theta(t,e)$ and $h = F(s)$. then this relation implies .

$$F(s)F(t) = F(s)\theta(t,e) = \theta\big(t,\theta(s,e)\big) = \theta(t+s,e) = F(s+t)$$

Thus $t \longrightarrow F(t)$ is a C^∞ homomorphism . But $\dot{F}(0) = \dot{\theta}(0, e) = X_P$ and since X is left-invariant , we see by uniqueness of the action generated by X that $\theta(t, g) = R_{F(t)}(g)$.

Corollary 2.2.2.

There is a one-to-one correspondence between the elements of $T_e(G)$ and one-parameter subgroups of t . For $Z \in T_e(G)$,let $t \longrightarrow F(t, Z)$ denote the (unique) corresponding one-parameter subgroup . Then $F: R \times T_e(G) \longrightarrow G$ is C^∞ and sarisfies $F(t, sZ) = F(st, Z)$.

Proof:

According to (Theorem **2.2.3**) each $Z \in T_e(G)$ determines a unique homomorphism $t \longrightarrow F(t, Z)$ of R onto G such that $\dot{F}(0, Z) = Z$. By extension of the existence theorem , we see that F is C^∞ simultaneously in t and Z [identifying $T_e(G)$ with R^n by some choice of basis]. Using the rule of change of parameter in a curve on a manifold , we have

$$\left[\frac{d}{dt} F(ts, Z)\right]_{t=0} = s \left[\frac{d}{dt} F(t, Z)\right]_{t=0} = sZ .$$

One the other hand $t \longrightarrow F(ts, Z)$ is a homomorphism . therefore , by uniqueness ,

$t \longrightarrow F(st, Z) = t \longrightarrow F(t, sZ)$.

2.3 The Lie Algebra of Vector Fields On a Manifold :

Definition 2.3.1.

A vector space $\mathcal{L}$ over R is a (real) [5]Lie algebra if in addition to it's vector space structure it possesses a product , that is , a map $\mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ taking the pair (X, Y) to the element $[X, Y]$ of $\mathcal{L}$, which has the following properties :

1. It is bilinear over **:**

$$[\alpha_1 X_2 + \alpha_2 X_2 \, , Y] = \alpha_1 [X_1, Y] + \alpha_2 [X_2 \, , Y]$$

$$[X, \alpha_1 Y_1 + \alpha_2 Y_2 \,] = \alpha_1 [X, Y_1] + \alpha_2 [X \, , Y_2]$$

2. It is skew commutative **:** $[X, Y] = -[X, Y]$

3. It satisfies the Jacobi identity **:** $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$

[5] An introduction to Differentiable Manifols & Riemannian Geometry- William M.Boothby – Washington University- ST. Louis Missouri- 2003

Example 2.3.1.

A vector space V^3 of dimension 3 over R with the usual vector product of vector calculus is a Lie algebra .

Example 2.3.2.

Let $\mathcal{M}_n(R)$ denote the algebra of $n \times n$ matrices over R with $X, Y \in \mathfrak{X}(M)$. Then in general the operator $f \longrightarrow X_P(Yf)$ defined on $C^\infty(p) \longrightarrow f$ being a C^∞ function on a neighborhood of P — does not define a vector at P . Determine a C^∞ - vector field ,however ,oddly enough .

$XY - YX$ does it define a vector field $Z \in \mathfrak{X}(M)$ according to the prescription

$Z_P f = (XY - YX)_P\, f = X_P(Yf) - Y_P(Xf)$.

For if $f, g \in C^\infty(p)$, then Xf and Yf are C^∞ on a neighborhood of P ,and this prescription determines a linear map of $C^\infty(p) \longrightarrow R$ .Therefore, if the Leibniz rule holds for Z_P . then Z_P is an element of $T_P(M)$ at each $P \in M$ .Consider $f, g \in C^\infty(p)$. Then $f, g \in C^\infty(U)$ for some open set U containing P. Using the notation $(Xf)_P$ for $X_P f$, the value of $\tilde{X} f$ at P .we have relations :

$(XY - YX)_P\,(fg) = X_P(Yfg) - Y_P(Xfg)$

$= X_P(Yfg + gYf) - Y_P(Xfg + gXf)$

$= (X_P f)(Yg)_P + f(P)X_P(Yg) + (X_P g)(Yf)_P + g(P)X_P(Yf)$

$-(Y_P f)(Xg)_P - f(P)Y_P(Xg) - (Y_P g)(Xf)_P - g(P)(Y_P Xf)$

So that

$Z_P(fg) = (XY - YX)_P\,(fg) = f(P)(XY - YX)_P\, g + g(P)(XY - YX)_P\, f$

$= f(P)Z_P g + g(P)Z_P f$.

Finally, if f is C^∞ on any open set $U \subset M$, then so is $(XY - YX)f$, and therefore Z is a C^∞- vector field on M as claimed .We may define a product on $\mathfrak{X}(M)$ using this fact; namely define the product of X and Y by $[X, Y] = XY - YX$.

Theorem 2.3.1.

$\mathfrak{X}(M)$ with the product $[X, Y]$ is a Lie algebra .

Proof:

If $\alpha, \beta \in R$ [6]and X_1, X_2, Y are C^∞-vector fields, then it is straight-forward to verify that

$[\alpha X_1 + \beta X_2, Y]f = \alpha[X_1, Y]f + \beta[X_2, Y]f$

Thus $[X, Y]$ is linear in the first variable .Since the skew commutativity $[X, Y] = -[Y, X]$ is immediate from the definition, we see that linearity in the first variable implies linearity in the second . Therefore, $[X, Y]$ is bilinear and skew commutativite . There remains the Jacobi identity which follows immediately if we evaluate $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big]$ applied to a C^∞-function f .

Using the definition , we obtain

$\big[X, [Y, Z]\big]f = X\big(([Y, Z])f\big) - [Y, Z](Xf)$

$= X\big(Y(Zf)\big) - X\big(Z(Yf)\big) - Y\big(Z(Xf)\big)$

Permuting cyclically and adding establishes the identity.

Theorem 2.3.2. ($T_eG \cong \mathcal{L}(G)$ as Vector Spaces)

We have two Lie algebra associated with [7]: the tangent space at the identity, T_eG , with the bracket induced by ad , and the left invariant vector fields ,mathcal $L(G)$, with the Lie bracket .In this section we will demonstrate that they are isomorphic as vector spaces .

Define a map $v: T_eG \longrightarrow \mathfrak{X}(G)$ by $v_\xi(g) = T_eL_g(\xi)$

For all $\xi \in T_eG$ and $g \in G$ .Because tangent maps are linear, so is v . For all $\xi \in T_eG$ and $g, h \in G$ we have

$$(T_hL_g)\big(v_\xi(h)\big) = (T_hL_g)\big(T_e\, L_h(\xi)\big) = T_e\big(L_g \circ L_h\big)(\xi) = T_e\, L_{gh}(\xi) = v_\xi(gh)$$

$$= \big(v_\xi \circ L_g\big)(h)\ .$$

Therefore v_ξ is left invariant ,so v really is a map $T_eG \longrightarrow \mathcal{L}(G)$. It's inverse (immediately)

Given by the map **:** $\mathcal{L}(G) \longrightarrow T_eG$, $X \longmapsto X(e) \in T_eG$.

[6] An introduction to Differentiable Manifols & Riemannian Geometry- William M.Boothby –Washington University- ST. Louis Missouri- 2003
[7] About Lie Groups – timothy e. Goldberg –October 6,2005

Theorem 2.3.3. **($T_eG \cong \mathcal{L}(G)$ as Lie algebras)**

To show that T_eG and $\mathcal{L}(G)$ [8]are isomorphic as Lie algebras as well vector fields , we must show that the map **:** $v: T_eG \longrightarrow \mathcal{L}(G)$, Preserves the brackets , i.e. $v_{ad\,(\xi)\eta} = [v_\xi, v_\eta]$

For all $,\eta \in T_eG$. Since the Lie bracket of vector fields can be described easily in terms of flows , it might be helpful to know what the flows of these vector fields look like .

Claim 2.3.1.

Let $\xi \in T_eG$. and $g \in G$. Then the flows of v_ξ through g is the curve $c: \mathbb{R} \longrightarrow G$ given by

$c(t) = L_g \circ \exp(t\xi)$

Proof:

Note that $c(0) = L_g \circ \exp(\vec{0}) = L_g(e) = g$. Let $t \in \mathbb{R}$. Then

$$\dot{c}(0) = \frac{d}{ds}\Big|_{s=t} c(t) = \frac{d}{ds}\Big|_{s=0} c(s+t)$$

$$= \frac{d}{ds}\Big|_{s=0} L_g \circ \exp\big((s+t)\xi\big) = \frac{d}{ds}\Big|_{s=0} L_g \circ \exp\big((t+s)\xi\big)$$

$$= \frac{d}{ds}\Big|_{s=0} L_g(\exp(t\xi)\cdot \exp(s\xi)) = \frac{d}{ds}\Big|_{s=0} L_g \circ L_{\exp(t\xi)}(\exp(s\xi))$$

$$= \frac{d}{ds}\Big|_{s=0} L_{g\exp(t\xi)}(\exp(s\xi))$$

$$= \big(T_eL_{g\exp(t\xi)}\big)(\xi) = v_\xi(g\exp(t\xi)) = v_\xi\big(c(t)\big) .$$

Theorem 2.3.4.

Let $,\eta \in T_eG$, Then **:** $v_{ad\,(\xi)_\eta} = [v_\xi , v_\eta]$

Proof:

Recall that the flow of v_ξ at time $t \in \mathbb{R}$ is the map $G \longrightarrow G$ given by $R_{\exp(t\xi)}$. Let $g \in G$

Then using the definition of Ψ [9]$, Ad,$ and , the linearity of tangent maps, we calculate

[8] About Lie Groups – timothy e. Goldberg –October 6,2005
[9] It is a conguagation map

$$[v_\xi , v_\eta](g) = \frac{d}{ds}\Big|_{t=0} \left(\left(R_{\exp(t\xi)}\right)v_{eta}\right)(g)$$

$$= \frac{d}{ds}\Big|_{t=0} TR_{\exp\ (t\xi)} \circ v_{eta} \circ R^{-1}{}_{\exp\ (t\xi)}\,(g)$$

$$= \frac{d}{ds}\Big|_{t=0} TR_{\exp\ (t\xi)} \circ v_{eta}\ \left(g\ exp(-t\xi)\right)$$

$$= \frac{d}{ds}\Big|_{t=0} TR_{\exp\ (t\xi)} \circ TL_{g\ \exp\ (-t\xi)}(\eta)$$

$$= \frac{d}{ds}\Big|_{t=0} T\left(R_{\exp\ (t\xi)} \circ L_{g\ \exp\ (-t\xi)}\right)(\eta)$$

$$= \frac{d}{ds}\Big|_{t=0} T\left(R_{\exp\ (t\xi)} \circ L_g \circ L_{\exp\ (-t\xi)}\right)(\eta)$$

$$= \frac{d}{ds}\Big|_{t=0} \left(TL_g\right) \circ \left(T\Psi_{\exp(t\xi)}\right)$$

$$= \left(TL_g\right)\left(\frac{d}{ds}\Big|_{t=0} Ad(\exp t\xi)\eta\right)$$

$$= \left(TL_g\right)[\xi, \eta]$$

$$= v_{[\xi,\eta]}(g)\ .$$

2.4 Lie Derivative

Unlike Euclidean spaces [10],the manifold notion doesn't let us simply introduce the derivative notion. Indeed , how shall we compare, for example vectors at various points and how shall we define the derivative of a vector field at a point? A first answer is supplied with the notion of orbits of a one-parameter group .

i. Lie derivative of a Function

Let g be a differentiable function on M .and the tangent vector at point x_0 ,to the orbit of diffeomomrphisms ϕ_t is

$$X_0 = \frac{d}{dt}\, x(t)\Big|_{t=0} = \frac{d}{dt}\, \phi_t\, x_0\Big|_{t=0}$$

We recall that the derivative of (germ) g in X tangency direction, at x_0 , is the real

$$X_0 g = \frac{d}{dt}\, (g \circ \phi_t)\, (x_0)\Big|_{t=0}$$

[10] Differential Geometry with Applications to Mechanics and Physics

In a chart , if the $n\ x^t(t)$ designate local coordinates of $\phi_t\ x_0 = x$ and x_0^t the ones of x_0 , then we know that :

$$X_0 g = X^t \frac{\partial}{\partial x^t}\ g\Big|_{x_0^t} = \frac{\partial g}{\partial x^t}\Big|_{x_0^t} \frac{\partial x^t}{\partial t}\Big|_0$$

Definition 2.4.1.

The Lie derivative of a function g with respect to X ,at point x_0 , is the derivative of g in the direction X :

$$L_{X_0} g = X_0 g = \lim_{t=0} \frac{g(\phi_t x_0) - g(x_0)}{t}$$

More precisely, we compare at x_0 ,the value $g^*\,(x_0) = g(\phi_t x_0)$ of g obtained at point $\phi_t(x_0)$ with the value $g(x_0)$. next we divide by the variation of parameter t and take the limit $t \longrightarrow 0$ .we go back to x_0 along the orbit .

The Lie derivative of a function g with respect to X is the function $L_X g = M \longmapsto L_X g(x)$

Such that **:** $L_X g(x) = X_x g(x)$

In short omitting the bracket **:** $L_X g = Xg = \frac{d}{dt}\ (g \circ \phi_t) = \frac{d}{dt}\ (\phi_t^* g)$.

In local coordinates , the Lie derivative of g with respect to X is expressed by

$$L_X g = X^i \partial_i\ g = \partial_i\ g dx^i\ \left(X^j \partial_j\right) = dg(X)$$

Denoted in short form **:** $L_X g = dg\ X$ (2.3)

Remark 2.4.1.

The gradient of g denoted dg and such that :

$$\langle dg, X\rangle = L_X g$$

Proposition 2.4.1.

Let $f\colon M \longrightarrow N$ be a diffeomorphism , X be a vector field on M , L_X be a differentiable Lie operator on $C^\infty(M)$. then the Lie operator L_X is :

i. Natural with respect to pull-back by f ,that is the following diagram is commutative

$$\begin{array}{ccc} C^\infty(N) & \overset{f^*}{\longrightarrow} & C^\infty(M) \\ L_{dfX}\downarrow & & \downarrow L_x \\ C^\infty(N) & \underset{f^*}{\longrightarrow} & C^\infty(M) \end{array}$$

ii. Natural with respect to restrictions ;that is the following diagram is commutative (for any open U of M) .

$$\begin{array}{ccc} C^\infty(M) & \overset{|U}{\longrightarrow} & C^\infty(U) \\ L_X\downarrow & & \downarrow L_{x|U} \\ C^\infty(M) & \underset{|U}{\longrightarrow} & C^\infty(U) \end{array}$$

Proof:

i. The image of X under f is the vector field dfX on N such that $\forall h \in C^\infty(N)$:

$dfX(h) = X(f^*h) \circ f^{-1}$

That implies **:** $L_X(f^*h) = X(f^*h) = dfX(h) \circ f = f^*L_{dfX}\ h$

ii. The second assertion $L_{X|U}(h|U) = L_X h|U$

Is obvious because $d(h|U) = (dh)|U$

ii. Lie derivative of vector field :

Let $\phi_t: M \longrightarrow M$ be [11]diffeomorphisms, and Let X be the (generating) field of tangent vectors to the ormit of a group of diffeomorphisms ϕ_t passing through x_0 .

Let Y be a vector field associated to a diffeomorphism ψ_t and $Y_{\phi_t x_0}$ be the tangent vector at point $x_t = \phi_t x_0$.

We use the image of this vector under the diffeomorphism $\phi_t^{-1} = \phi_{-t}$ is $d\phi_t^{-1} Y_{\phi_t x_0}$

Where $d\phi_t^{-1}: T_{\phi_t x_0}M \longrightarrow T_{x_0}M$.

"Going backwards " to x_0 along an orbit and comparing the previous image with vector T_{x_0} ,

We define **:** The Lie derivative of vector field Y with respect to X ,at x_0 , is

[11] "Differential Geometry With Applications to Mechanics and Physics"- Yves Tapaert- Ouagadougou University - Burkina Faso .

$$T_{x_0}Y = \lim_{t=0} \frac{1}{t} \left(d\phi_t^{-1} Y_{\phi_t x_0} - T_{x_0} \right)$$

$$= \frac{d}{dt} d\phi_t^{-1} Y \Big|_{t=0}$$

Remark 2.4.2. (Zero Lie derivative)

From the previous definition , if $d\phi_t^{-1} Y_{\phi_t x_0} = T_{x_0}$ then the Lie derivative is zero .

This particular case is illustrated as follows:

In general way, there is no reason for such an equality apart from when the image of orbit (with tangent vector X) passing through x_0 under ψ_t is the orbit corresponding to X passing through $\psi_t x_0$.

Proposition 2.4.2.

The Lie derivative of vector field Y with respect to X is the Lie bracket of X and .

Proof:

First : Let us point out the following remark .

Let $g: I \times U \longrightarrow R$ be a function defined on $\times U \subset R \times M$. there is a function $h: I \times U \longrightarrow R$

Of class C^t such that **:** $g(t,x) = g(0,x) + th(t,x)$ and $\partial_t g(0,x) = h(0,x)$

The function h such that **:** $h(t,x) = \int_0^t \partial_t g(tu,x)du$

fits the requirements

Indeed , from the change of variable $v = tu$, we deduce :

$$h(t,x) = \frac{1}{t} \int_0^t \partial_v g(v,x)dv = \frac{1}{t} \left(g(t,x) - g(0,x) \right)$$

$$g(t,x) = g(0,x) + th(t,x)$$

And also $h(0,x) = \frac{1}{t} \int_0^t \partial_t g(0,x)du = \partial_t g(0,x)$.

Second : let us prove the proposition

The following comparison between vectors $d\phi_t^{-1} Y - Y$

Leads to

$L_X = \lim_{t=0} \frac{1}{t} \, (d\phi_t^{-1}Y - \; Y)(g) = \lim_{t=0} \frac{1}{t} \, d\phi_t^{-1}(Y - d\phi_t Y)(g)$

$= \lim_{t=0} \frac{1}{t} \, (Y - d\phi_t Y)(g)$

$= \; \lim_{t=0} \frac{1}{t} \, (Y(g) - Y(g \circ \phi_t) \circ \phi_t^{-1})(g)$.

It is from the definition of the image of vector field under ϕ_t ,namely :

$(d\phi_t Y)(g) = g(x) + th(t, x)$

Where

$h(0, x) = \frac{\partial (g \circ \phi_t)}{\partial t} \, (0, x) = Xg$,

the last equality following from the definition of the directional derivation of g along X ,

then , by using the expression of $L_X Y(g)$, we have :

$L_X Y(g) = \lim_{t=0} \frac{1}{t} \, (Y(g) - Y(g) \circ \phi_t^{-1} - tY(h) \circ \phi_t^{-1})$

$= \lim_{t=0} \left(\frac{1}{t} \, (y(g) \circ \phi_t - Y(g)) \circ \phi_t^{-1} - Y(h) \circ \phi_t^{-1}\right)$

$= \frac{d}{dt} \, (Y(g) \circ \phi_t)(0) - \lim_{t=0} Y(h)$, because $\lim_{t=0} \phi_t^{-1} = id$

From the directional derivative of a function $Y(g)$ along **:**

$X(Y(g)) = \frac{d}{dt} \, (Y(g) \circ \phi_t)(x)|_{t=0}$

And since $h(0, x) = Xg$

we deduce **:** $L_X Y(g) = X\big(Y(g)\big) - Y\big(X(g)\big) = [X, Y]g$.

Properties ……

1. ***R*-** bilinearity: $\forall \; X, Y, Z \in \mathfrak{X}(M)$, $\forall a, b \in \boldsymbol{R}$:

$$L_{X+Y}Z = L_X Z + L_Y Z$$

$$L_X(Y + Z) = \; L_X Y + L_X Z$$

$$L_{aX} \, bY = ab L_X Y$$

2. Anticommutative property , $\forall \; X, Y \in \mathfrak{X}(M)$ **:** $L_X Y = -L_Y X$.

Jacobi identity , $\forall\, X, Y, Z \in \mathfrak{X}(M)$:

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] = -\big[Z,[X,Y]\big]$$

$$\Leftrightarrow \qquad L_{[X+Y]}Z = L_X L_Y Z + L_Y L_X Z$$

$$\Leftrightarrow \qquad L_X[Y,Z] = [L_X Y, Z] + [Y, L_X Z]$$

So the properties (**2**) and (**3**) show the algebra is a Lie algebra . The equality proves L_X is a Lie bracket derivation .

Definition 2.4.2. (Lie derivative of differential form)

The Lie derivative[12] of differential form ω with respect to X ,at x_0 ,is

$$(L_X\omega)_{x_0} = \lim_{t=0} \frac{1}{t}\left(\phi_t^* \omega_{\phi_t^* x_0} - \omega_{x_0}\right) \tag{2.4}$$

$$= \frac{d}{dt}\, \phi_t^* \omega\Big|_{t=0}$$

Remark 2.4.3.

$$\frac{d}{dt}\, \phi_t^*\omega = \frac{d}{dt}\, \phi_{t+s}^*\omega\Big|_{s=0} = \frac{d}{ds}\, \phi_{t+s}^*\omega\Big|_{s=0} = \phi_t^*\, \frac{d}{dt}\, \phi_s^*\omega\Big|_{t=0} = \phi_t^*(L_X\, \omega)$$

Remark 2.4.4.

The definition (2.4.2) leads again to the formula (2.3) in the context of real-valued functions :

$$(L_X\, g)_{x_0} = \lim_{t=0} \frac{1}{t}\left(\phi_t^* g(\phi_t x_0) - g(x_0)\right) = \lim_{t=0} \frac{1}{t}\left(g(\phi_t x_0) - g(x_0)\right) .$$

Proposition 2.4.3.

The operator d is natural with respect to L_X ; that is the following diagram is commutative :

$$\begin{array}{ccc} \Omega^P(M) & \xrightarrow{L_X} & \Omega^P(M) \\ d\downarrow & & \downarrow d \\ \Omega^P(M) & \xrightarrow[L_X]{} & \Omega^{P+1}(M) \end{array}$$

In other words , $\forall\, \omega \in \Omega^P(M)$: $dL_X\, \omega = L_X\, d\omega$.

[12] "Differential Geometry With Applications to Mechanics and Physics - Yves Tapaert- Ouagadougou University - Burkina Faso .

Proof:

$$dL_X\,\omega = d\frac{d}{dt}\,\phi_t^*\omega\Big|_{t=0} = \frac{d}{dt}\,\phi_t^*\omega\Big|_{t=0} = \frac{d}{dt}\,d\phi_t^*\omega\Big|_{t=0} = L_X\,d\omega\ .$$

Example 2.4.1.

With the help of diffeomorphisms give an interpretation of the Lie derivative of a vector field X with respect to vector field Y on a manifold .

Given one-parameter groups of diffeomorphisms ϕ_t and ψ_t of which X and Y are the respective generating fields, show that the curve : $t \longmapsto \left(\phi_{-\sqrt{t}} \circ \psi_{-\sqrt{t}} \circ \phi_{\sqrt{t}} \circ \psi_{\sqrt{t}}\right)x$

Is differentiable at $t = 0$ and admits $[X, Y] = L_X Y$ as a corresponding tangent vector .

Answer:

Let x_0 be a point of M $(t = 0)$

2.5 Matrix groups

i. The Lie algebra of matrix Groups

Let us consider $(n, \mathbb{K})$ [13], as a coordinates we choose the entries of the matrices, so that a matrix g is parameterized by $g = g_j^i$. in particular, the identity is $e = \delta_j^i$. then the left translation, as multiplication acts as **:** $L_h g = hg = h_k^i g_j^k$

Its differential is $(dL_h)_j^i{}_k^l = \frac{\partial (hg)_j^i}{\partial g_l^k} = h_k^i \delta_j^l$

The left-invariant vector fields can be obtained from the tangent vectors at the identity . denoted such a vector by $V = V_j^i \left.\frac{\partial}{\partial g_j^i}\right|_{g=e}$

The vector field X_V corresponding to V is given by acting on V with differential ,

$$X_V|_h = dL_h V = (dL_h)_j^i{}_k^l\, V_l^k\, \frac{\partial}{\partial h_j^i} = h_k^i \delta_j{}^l\, V_l^k\, \frac{\partial}{\partial h_j^i} = (hV)_l^k\, \frac{\partial}{\partial h_j^i}$$

The component of X_V at the point V is just hV, interpretend as a matrix product. This gives us a very important formula for the Lie bracket:

[13] Group Theory (for Physicists) - Christoph L¨udeling - August, 16, 2010

Let X_V and X_w be two vector fields obtained from tangent vectors V and W as a bove . the Lie bracket is a new vector field, which at point h is given by

$$[X_V, X_W]|_h = \left((X_V|_h)^i_j \frac{\partial}{\partial h^i_j} (X_W|_h)^k_l - (X_W|_h)^i_j \frac{\partial}{\partial h^i_j} (X_V|_h)^k_l \right) \frac{\partial}{\partial h^k_l}$$

$$= \left(h^i_m V^m_j \frac{\partial}{\partial h^i_j} h^k_n W^n_l - h^i_m W^m_j \frac{\partial}{\partial h^i_j} h^k_n V^n_l \right) \frac{\partial}{\partial h^k_l}$$

$$= h^k_m \left(V^m_j W^j_l - W^m_j V^j_l \right) \frac{\partial}{\partial h^k_l} = h\, [V, W] \frac{\partial}{\partial h}$$

Remark 2.5.1.

In the last line the square brackets indicate not the Lie bracket of vector fields, but the matrix commutator, (it means that we can identify the Lie algebra of $GL(n, \mathbb{C})$ with the components V^i_j of tangent vectors and use the usual matrix commutator as the product which is huge simplification).

Definition 2.5.1. (The Exponential Map)

We define a diffeomorphism of G onto itself as follows :

The points on an integral curve of a left-invariant vector field X (through , at $t = 0$) are defined by **:** $g_X(t): x \longmapsto \exp(tX)x$

Definition 2.5.2.

The mapping $g_X(t) = \exp(tX)$ is called exponential mapping generated by the vector [14]field X. This mapping has the property of a one-parameter subgroup of :

$$g_X(s+t) = exp\big((s+t)X\big) = exp(sX)exp(tX)$$

$$= g_X(s)g_X(t) \ .$$

Therefore, for every $X \in T_eG$, the integral curve of X passing through e at $t = 0$ is

$$g_X: R \longrightarrow G: t \longmapsto \exp(tX)$$

this mapping g_X exists for any real t (the flow is complete) . This smooth $(i.e\ C^\infty)$ homomorphism is a one-parameter subgroup of G.

[14] Mechanics in Differential Geometry - Yves Tapaert – copy right 2006 , koninklije Brill NV,Leiden – the Netherland .

Definition 2.5.3.

The exponential mapping of the Lie algebra of G into G is

$exp_G: T_e G \longrightarrow G: X \longmapsto g_X(1) = \exp X$.

Example 2.5.1.

The group of $(n \times n)$ real nonsingular matrices is called general linear group and denoted $L(n, R)$.

If the space of $(n \times n)$ real matrices is identified with R^{n^2}, then the general linear group is identified with the open submanifold of R^{n^2} defined by a nonzero determinant .

Let $A = \left(a_i^j\right)$ be some element of $GL(n, R)$ with $det\left(a_i^j\right) \neq 0$.

There is an open subset of R^{n^2} , $GL(n, R)$ is provided with a differentiable manifold structure

A neighborhood of A is composed of matrices $B = det\left(b_i^j\right)$ such that $\left|b_i^j - a_i^j\right| < \varepsilon$ where the real ε is chosen small enough so that $det\left(b_i^j\right) \neq 0$. In this neighborhood, the coordinates are defined by the n^2 reals $x_i^j = b_i^j - a_i^j$.

$GL(n, R)$ is a group with the multiplication law of class C^∞ :

$GL(n, R) \times GL(n, R) \longrightarrow GL(n, R): (A, B) \longmapsto AB$

Where $AB = \left(a_i^j b_k^l\right)$.

Besides , $GL(n, R) \longrightarrow GL(n, R): A \longmapsto A^{-1}$ is of class C^∞ , so G is a lie group .

Since R^{n^2} is identical to the tangent space at any of it's points, the tangent space at identity point e of $GL(n, R)$ is naturally identified with R^{n^2} : any tangent vector is a $(n \times n)$ real matrix .

Consider a one-parameter subgroup generated by any matrix $A \in L(R^n, R^n)$ that is an integral curve of left-invariant vector field passing through e (at $t = 0$) and represented by the matrix $g_A(t) = \left((g_A)_i^j\right)$ with $A = \left(\frac{d(g_A)_i^j}{dt}\right)_0$.

Since $g_A(t + \Delta t) = g_A(t) g_A(\Delta t)$,

We easily obtain **:**

$\frac{dg_A}{dt}\ (t) = \ g_A(t)A$

$\Longrightarrow g_A(t) = \exp(tA)$

So, $g_A: R \longrightarrow GL(n,R): t \longmapsto \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$

Is a one-parameter subgroup with $g_A(0) = I$. The exponential mapping is

$$exp: L(R^n, R^n) \longrightarrow GL(n,R): A \longmapsto g_A(1) = \sum_{n=0}^{\infty} \frac{A^n}{n!}\ .$$

Finally ,we find the Lie algebra of $GL(n,R)$ as follows .

For every $\in GL(n,R)$, the left-invariant vector fields on $GL(n,R)$ are defined by

$$X_A: GL(n,R) \longrightarrow L(R^n, R^n): Y \longmapsto YA$$

Here, the Lie bracket is defined by **:** $[A,B] = [X_A, X_B](I)$

If we consider the one-parameter subgroups or integral curves of left-invariant vector fields g_A and g_B , and if we refer to the (Example 2.5.1) giving an interpretation of Lie bracket with the help of diffeomorphisms , we immediately have :

$$[X_A, X_B] = \lim_{t=0} \frac{1}{t^2} \left(g_A(t)\ g_B(t) - g_B(t)\ g_A(t)\right)$$

$$= \lim_{t=0} \frac{1}{t^2} \left((1 + tA + \cdots)(1 + tB + \cdots) - (1 + tB + \cdots)(1 + tA + \cdots)\right)$$

So, at point *e* we obtain : $[A,B] = AB - BA$

And the Lie bracket of any two left-invariant vector fields at e is usual commutator of the two matrices "generating" the fields . the left-invariant vector field generated by this commutator belongs to the Lie algebra of $GL(n,R)$.

$L(R^n, R^n)$, is the lie algebra of $L(n,R)$, the Lie bracket being the matrix commutator . we remark that , given $C \in GL(n,R)$, the mapping

$$Cg_AC^{-1}: R \longrightarrow GL(n,R): t \longmapsto Cg_A(t)C^{-1}$$

Is an integral curve of vector field $X_{CAC^{-1}}$ passing through I. Indeed, we have **:**

$Cg_A(0)C^{-1} = I$

and

$\frac{d}{dt}\ (Cg_A C^{-1})(t) = C\frac{dg_A}{dt}\ (t)C^{-1} = Cg_A(t)AC^{-1}$

$$= (Cg_A C^{-1})(t)CAC^{-1} .$$

In addition, we see that **:** $g_{CAC^{-1}} = Cg_A C^{-1}$ Or $exp(CAC^{-1}) = CexpAC^{-1}$.

Proposition 2.5.1.

If $h: G \longrightarrow H$ is a C^∞ homomorphism of Lie group G into Lie group H, then

$dh_e: L(G) \longrightarrow L(H)$ is a Lie algebra homomorphism .

Proof:

We have $\forall \xi, \eta \in T_e G$:

$$dh_e[\xi,\eta] = dh_e\big([X_\xi, X_\eta]\big)(e) = \big[dh_e X_\xi, dh_e X_\xi\big](e_H)$$

$$= \big[X_{dh_e\xi}, X_{dh_e\eta}\big](e_H) \qquad \text{(homomorphism)}$$

$$= [dh_e\xi, dh_e\eta] .$$

Proposition 2.5.2.

If $h: G \longrightarrow H$ is a C^∞ homomorphism of Lie groups , then $\forall \xi \in L(G)$:

$h(exp_G\xi) = exp_H(dh_e\xi)$.

proof:

The mapping $g: R \longrightarrow H: t \longmapsto h(exp_G t\xi)$

is a one-parameter subgroup of H. Thus, we have $dh_e\xi = \frac{d}{dt}\ g(t)\Big|_{t=0} = \eta$

And $(t) = exp_H t\eta$.

That implies **:** $h(exp_G\xi) = g(1) = exp_H\,\eta = \ exp_H(dh_e\xi)$

ii. The adjoint transformation:

For example, the matrix group $GL(n, R)$ has been considered through a faithfull representation of a matrix transformation of n-dimensional vector space (a representation is termed faithful if it is one-to-one). Besides such a type of representation there is the adjoint representation that we are going to introduce .

First let us consider the inner automophism of G associated with $\in G$, that is

$I_g: G \longrightarrow G: h \longmapsto ghg^{-1} = R_{g^{-1}} L_g h$.

This mapping of G into itself is C^∞ and is a homomorphism because $\forall h, l \in G$:

$I_g(hl) = ghlg^{-1} = ghg^{-1}glg^{-1} = I_g(h)I_g(l)$

In parcitcular , the identity e is mapping by any I_g into e . So, each I_g induces a mapping of T_eG into it self.

Definition 2.5.4.

The adjoint transformation associated with $g \in G$ is the mapping $T_eG \longrightarrow T_eG$ defined by

$Ad_g = \left(dI_g\right)_e = d\left(R_{g^{-1}}L_g\right)(e)$

Remark 2.5.2.

From (Prop **2.5.2**) lets write $\forall g \in G, \forall \xi \in T_eG$:

$exp\left(Ad_g\xi\right) = exp\left(\left(dI_g\right)_e \xi\right) = I_g(exp\xi) = gexp\, \xi g^{-1}$.

Remark 2.5.3.

Considering a one-parameter subgroup of G defined by $t \longmapsto \exp(tX)$, let h and l be any two points of this integral curve of X passing through e at $t = 0$.

The respective images under I_g of the previous points ,that are $h^1 = ghg^{-1}$, $l^1 = glg^{-1}$ and $e = geg^{-1}$,define another curve passing through e such that the tangent vector field is Ad_gX (this $I_g(h\, l) = I_g(h)I_g(l)$.

Thus we denote **:** $I_g\left(exp(tX)\right) = exp\left(t\, Ad_gX\right)$

2.6 Representations of Lie algebras :

We will study representations of the simplest possible Lie algebra, $\mathfrak{sl}(2, \mathbb{C})$ [15]. Recall that this Lie algebra has a basis e, f, h with commutation relations:

$[e, f] = h$, $\quad [h, e] = 2e$, $\quad [h, f] = -2f$

As we proved earlier This Lie algebra is simple .

[15] Introduction to Lie Groups and Lie Algebras – Alexander Kirillov,Jr. – Departement of mathematics, Suny At stony Brook,NY 11794, USA.

The main idea of the study of representation of $\mathfrak{sl}(2,\mathbb{C})$. Is to start by diagonalizing the operator h .

Definition 2.6.1.

Let V be a representation of of $\mathfrak{sl}(2,\mathbb{C})$. A vector $v \in V$ is called vector of weight λ, $\lambda \in \mathbb{C}$ if it is an eigenvector for h with eigenvalue λ **:** $hv = \lambda v$

we denoted by $V[\lambda] \subset V$ the subspace of vectors of weight λ . The following **Lemma** play a key role in the study of representations of $\mathfrak{sl}(2,\mathbb{C})$.

Lemma 2.6.1.

$eV[\lambda] \subset V[\lambda + 2]$

$fV[\lambda] \subset V[\lambda - 2]$

Proof:

Let $v \in V[\lambda]$. Then **:** $hev = [h,e]v + ehv = 2ev + \lambda ev = (\lambda + 2)ev$, So $ev \in V[\lambda + 2]$. the proof for f is similar .

Theorem 2.6.1.

Every finite-dimensional representation V of $\mathfrak{sl}(2,\mathbb{C})$ can be written in the form $V = \oplus_{\lambda} V[\lambda]$

Where $V[\lambda]$ is defined in (Def **2.6.1**).This decomposition is called weight decomposition of V.

Proof:

Since every representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible, it suffices to prove this for irreducible V. So assume that V is irreducible . Let $V = \sum_{\lambda} V[\lambda]$ be the subspace spanned by eigenvectors of h. By well-known result of linear algebra, eigenvectors with different eigenvalues are linearly independent , so $V' = \oplus V[\lambda]$. By (Lemma 2.6.1) , V' is stable under action of e, f and h. Thus, V' is a subrepresentation. Since we assumed that V is irreducible, and $V' \neq 0$ (h has at least one eigenvector), we see that $V' = V$.

Our main goal will be classification of irreducible finite-dimensional representations. So let V be an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$. Let λ be a weight of V (i.e, $V[\lambda] \neq 0$) which is maximal in the following sense :

$$Re\,\lambda \geq Re\,\lambda' \qquad \text{for every weight } \lambda' \text{ of } V .$$

Such a weight will be called "highest weight of $(V^{''})$ " ,a nd vectors $v \in V[\lambda]$ - highest weight vectors . It is obvious that every finite-dimensional representation has at least one-zero highest weight vector .

Lemma 2.6.2.

Let $\lambda \in \mathbb{C}$. Define M_λ to be the infinite-dimensional vector space with basis $v^0, v^1, \dots ..$

Irreducible representations V_n can also be described more explicitly, as symmetric powers of the usual two-dimnsional representation .

Theorem 2.6.2.

1. For any $n \geq 0$, let V_n be the finite-dimensional vector space basis $v^0, \dots \dots ., v^n$. Define the action of $\mathfrak{sl}(2, \mathbb{C})$ by

$$hv^k = (n - 2k)v^k$$

$$fv^k = (k+1)v^{k+1} \ , \qquad k > n \ ; \qquad fv^n = 0 \qquad (2.5)$$

$$ev^k = (n + 1 - k)v^{k-1} \ , \qquad k > 0 \ ; \qquad ev^0 = 0 \ .$$

Then V_n is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$; we will call it the irreducible representation with highest weight n.

2. For $n \neq m$, representation V_n, V_m are non-isomorphic .

3. Every finite-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ is isomorphic to one of representations V_n .

Proof:

Consider the finite-dimensional representation M_λ. If $\lambda = n$ is a non-negative integer, consider the subspace $M^{'} \subset M_n$ spanned by vectors $v^{n+1}, v^{n+2}, \dots ....$ Then this subspace is actually a subrepresentation. Indeed, it is obviously stable under action of h and f; the only non-trivial relation to check that $v^{n+1} \subset M'$. But $ev^{n+1} = \big(n + 1 - (n + 1)\big)v^n = 0$.

Thus the quotient space M_n/M' is a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. It is obvious that it has basis $v^0, \dots \dots ., v^n$ and that the action of $v^0, \dots \dots ., v^n$ is given by (**2.5**) . irreducibility of this representation is also easy to prove: any subrepresentation must be spanned by some subset of $v, v^1, \dots \dots ., v^n$, but it is easy to see that each of them generates (under the action of $\mathfrak{sl}(2, \mathbb{C})$) the whole representation V_n . therefore , V_N is an irreducible finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. Since $dimV_n = n+1$, it is obvious that V_N are pairwise non-isomorphic .

To prove that every irreducible representation is of this form, let V be an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ and let $v \in V[\lambda]$ be a highest weight vector. By (Lemma 2.6.2.) , V is a quotient of M_λ; in other words , it is spanned by vectors $v^k = \frac{f^k}{k!} v$.

Since v^k have different weights , if they are non-zero, then they must be linearly independent. On the other hand, V is finite-dimensional ; thus, only finitely many of v^i are non-zero . Let n be maximal such that $v^n \neq 0$, so that $v^{n+1} = 0$. Obviously, in this case $v^0, \dots \dots , v^n$ are all non-zero and since they have different weight, they are linearly independent, so they form a basis in V.

Since $v^{n+1} = 0$, we must have $ev^{n+1} = 0$. On the pther hand ,by (2.5) ,we have

$$ev^{n+1} = (\lambda - n)v^n.$$

Since $v^n \neq 0$, this implies that $\lambda = n$ is a non-negative integer . Thus, V is a representation.

2.7 Nilpotent Lie algebras :

Definition 2.7.1.

A Lie algebra $\mathfrak{g}$ is said to be nilpotent[16] if it admits a filtration

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_r = 0 \qquad (2.6)$$

By ideals such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ for $0 \le i \le r-1$. Such a filtration is called a nilpotent series . The condition (**2.6**) to be a nilpotent series is that $\mathfrak{a}_i / \mathfrak{a}_{i+1}$ be in the centre of $\mathfrak{g}/\mathfrak{a}_{i+1}$ for $0 \le i \le r-1$. Thus the nilpotent Lie algebras are exactly those that can be obtained from commutative Lie algebras by successive centeral extensions

$$0 \longrightarrow \mathfrak{a}_1/\mathfrak{a}_2 \longrightarrow \mathfrak{g}/\mathfrak{a}_2 \longrightarrow \mathfrak{g}/\mathfrak{a}_1 \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{a}_2/\mathfrak{a}_3 \longrightarrow \mathfrak{g}/\mathfrak{a}_3 \longrightarrow \mathfrak{g}/\mathfrak{a}_2 \longrightarrow 0$$

…

In another words, the nilpotent Lie algebras from the smallest class containing the commutative Lie algebras and closed under central extensions .

---

[16] Lie Algebras, Algebraic Groups and Lie Groups (chapter one) - J.S. Milne – may 5, 2013.

The lower central series of $\mathfrak{g}$ is

$\mathfrak{g} \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^{i+1} \supset \cdots$

With $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1], \dots \dots, \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i], \dots \dots$

Proposition 2.7.1.

A Lie algebra $\mathfrak{g}$ is nilpotent if and only if it's lower central series terminates with zero .

Proof:

If the lower central series terminates with zero, then it is a nilpotent series. Conversely , if $\mathfrak{g} \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_r = 0$ is nilpotent series, then $\mathfrak{a}_1 \supset \mathfrak{g}^1$ because $\mathfrak{g}/\mathfrak{a}_1$ is commutative , $\mathfrak{a}_2 \supset [\mathfrak{g}, \mathfrak{a}_1] \supset [\mathfrak{g}, \mathfrak{g}^1] = \mathfrak{g}^2$, and so on, until we arrive at $0 = \mathfrak{a}_r \supset \mathfrak{g}^r$.

Let V be a vector space of dimension, and let, $F: V = V_0 \supset V_1 \supset \cdots \supset V_n = 0, \dim V_i = n - i$

Be a maximal flag in . let $n(F)$ be the Lie subalgebra of $\mathfrak{gl}_V$ consisting of the elements x such that $x(V_i) \subset V_{i+1}$ for all i . the lower central series for $n(F)$ has

$$n(F)^j = \{x \in \mathfrak{gl}_V | x(V_i) \subset V_{i+1+j}\}$$

For $j = 1, \dots, n$ .In particular , $n(F)$ is nilpotent . For example ,

$$n_3 = \left\{\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\right\} \supset \left\{\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} \supset \{0\}$$

Is nilpotent series for n_3 .

An extension of nilpotent algebra is solvable , but not necessarily nilpotent . for example, n_3 is nilpotent and $\mathfrak{b}_3/n_3$ is commutative , but $\mathfrak{b}_3$ is not nilpotent when $n \geq 3$.

Proposition 2.7.2.

1. Subalgebras and quotient algebras of nilpotent Lie algebras are nilpotent.

2. A Lie algebra $\mathfrak{g}$ is nilpotent if $\mathfrak{g}/\mathfrak{a}$ is nilpotent for some ideal $\mathfrak{a}$ contained in $z(\mathfrak{g})$.

3. A nonzero nilpotent Lie algebra has nonzero centre .

Proof:

1.The intersection of a nilpotent series for $\mathfrak{g}$ with a Lie subalgebra $\mathfrak{h}$ is nilpotent series for $\mathfrak{h}$, and the image of a nilpotent series for $\mathfrak{g}$ in a quotient algebra $\mathfrak{q}$ is a nilpotent series for $\mathfrak{q}$.

2. For any ideal $\mathfrak{a} \subset z(\mathfrak{g})$, the inverse image of a nilpotent series for $\mathfrak{g}/\mathfrak{a}$ becomes a nilpotent series for $\mathfrak{g}$ when extended by 0 .

3. If $\mathfrak{g}$ is nilpotent , then the last nonzero term $\mathfrak{a}$ in a nilpotent series for $\mathfrak{g}$ is contained in $z(\mathfrak{g})$.

Proposition 2.7.3.

Let $\mathfrak{h}$ be a proper Lie subalgebra of a nilpotent Lie algebra $\mathfrak{g}$; then $\mathfrak{h} \neq n_{\mathfrak{g}}(\mathfrak{h})$.

Proof:

We use induction on the dimension of $\mathfrak{g}$. Because $\mathfrak{g}$ is nilpotent and nonzero , it's centre $z(\mathfrak{g})$ is nonzero . if $z(\mathfrak{g}) \not\subset \mathfrak{h}$,then $n_{\mathfrak{g}}(\mathfrak{h}) \neq \mathfrak{h}$ because $z(\mathfrak{g})$ normalizes $\mathfrak{h}$. if $z(\mathfrak{g}) \subset \mathfrak{h}$, then we can apply induction to the Lie subalgebra ${}^{\mathfrak{h}}/_{z(\mathfrak{g})}$ of ${}^{\mathfrak{g}}/_{z(\mathfrak{g})}$.

2.8 Solvable Lie algebras :

Definition 2.8.1.

A Lie algebra $\mathfrak{g}$ is said to be solvable[17] if it admits a filtration

By ideals such that $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ for $0 \leq i \leq r-1$. Such a filtration is called a solvable series.

The condition (2.6) to be a solvable series is that the quotients $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ commutative for $0 \leq i \leq r-1$. Thus the solvable Lie algebras are exactly those that can be obtained from commutative Lie algebras by successive extensions

$0 \longrightarrow \mathfrak{a}_1/\mathfrak{a}_2 \longrightarrow \mathfrak{g}/\mathfrak{a}_2 \longrightarrow \mathfrak{g}/\mathfrak{a}_1 \longrightarrow 0$

$0 \longrightarrow \mathfrak{a}_2/\mathfrak{a}_3 \longrightarrow \mathfrak{g}/\mathfrak{a}_3 \longrightarrow \mathfrak{g}/\mathfrak{a}_2 \longrightarrow 0$

…

In another words, the solvable Lie algebras from the smallest class containing the commutative Lie algebras and closed under extensions.

The characteristic ideal $[\mathfrak{g}, \mathfrak{g}]$ is called the derived algebra of , and is denoted $\mathcal{D}\mathfrak{g}$ .Clearly $\mathcal{D}\mathfrak{g}$ is contained in every ideal $\mathfrak{a}$ such that $\mathfrak{g}/\mathfrak{a}$ is commutative , and so $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ is the largest commutative quotient of $\mathfrak{g}$. Write $\mathcal{D}^2\mathfrak{g}$ for the second derived algebra $\mathcal{D}(\mathcal{D}\mathfrak{g})$, $\mathcal{D}^3\mathfrak{g}$ for the

[17] Lie Algebras, Algebraic Groups and Lie Groups (chapter one) - J.S. Milne – may 5, 2013.

third derived algebra $\mathcal{D}(\mathcal{D}^2\mathfrak{g})$, and so on. These are characteristic ideals, and the derived series of $\mathfrak{g}$ is the sequence

$$\mathfrak{g} \supset \mathcal{D}\mathfrak{g} \supset \mathcal{D}^2\mathfrak{g} \supset \cdots$$

We sometimes write $\mathfrak{g}'$ for $\mathcal{D}\mathfrak{g}$ and $\mathfrak{g}^{(n)}$ for $\mathcal{D}^n\mathfrak{g}$.

Proposition 2.8.1.

A Lie algebra $\mathfrak{g}$ is solvable if and only if it's derived series terminates with zero .

Proof:

If the derived series terminates with zero, then it is a solvable series. Conversely , if $\mathfrak{g} \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_r = 0$ is a solvable series, then $\mathfrak{a}_1 \supset \mathfrak{g}'$ because $\mathfrak{g}/\mathfrak{a}_1$ is commutative , $\mathfrak{a}_2 \supset \mathfrak{a}_1' \supset \mathfrak{g}''$ because $\mathfrak{a}_1/\mathfrak{a}_2$ is commutative , and so on, until $0 = \mathfrak{a}_r \supset \mathfrak{g}^{(r)}$.

Let V be a vector space of dimension , and let

$$F: V = V_0 \supset V_1 \supset \cdots \supset V_n = 0 \text{ , } \dim V_i = n - i \text{ ,}$$

Be a maximal flag in . Let $\mathfrak{b}(F)$ be the Lie subalgebra of $\mathfrak{gl}_V$ consisting of the elements x such that $x(V_i) \subset V_i$ for all i . then $\mathcal{D}(\mathfrak{b}(F)) = n(F)$ and so $\mathfrak{b}(F)$ is solvable .

For example ,

$$\mathfrak{b}_3 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \supset \{0\}$$

Is nilpotent series for $\mathfrak{b}_3$.

Proposition 2.8.2.

Let k' be a field containing k.A Lie algebra $\mathfrak{g}$ over k is solvable if and only if $\mathfrak{g}k' \stackrel{\text{def}}{=} k' \otimes_k \mathfrak{g}$ is solvable .

Proof:

Obviously, for any subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$, $[\mathfrak{h}, \mathfrak{h}']_{k'} = [\mathfrak{h}_{k'}, \mathfrak{h}'_{k'}]$, and so, under extension of the base field, the derived series of $\mathfrak{g}$ maps to that of $\mathfrak{g}_{k'}$.

Note : we say that an ideal is solvable if it is solvable as a Lie algebra .

Proposition 2.8.3.

1. Subalgebras and quotient algebras of solvable Lie algebras are solvable .

2. A Lie algebra $\mathfrak{g}$ is solvable if it contains an ideal $\mathfrak{n}$ such that both $\mathfrak{n}$ and $\mathfrak{g}/\mathfrak{n}$ are solvable

3. Let $\mathfrak{n}$ be an ideal in a Lie algebra $\mathfrak{g}$, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. if $\mathfrak{n}$ and $\mathfrak{h}$ are solvable, then $\mathfrak{n}+\mathfrak{h}$ is solvable .

Proof:

1.The intersection of a solvable series for $\mathfrak{g}$ with a Lie subalgebra $\mathfrak{h}$ is a solvable series for $\mathfrak{h}$, and the image of a solvable series for $\mathfrak{g}$ in a quotient algebra $\mathfrak{q}$ is a solvable series for $\mathfrak{q}$.

2. Because $\mathfrak{g}/\mathfrak{n}$ is solvable, $\mathfrak{g}^{(m)} \subset \mathfrak{n}$ for some m. Now , $\mathfrak{g}^{(m+n)} \subset \mathfrak{n}^{(n)}$, which is zero for some n .

3. This follows from (2) because $\mathfrak{h}+\mathfrak{n}/\mathfrak{n} \simeq \mathfrak{h}/\mathfrak{h} \cap \mathfrak{n}$ which is solvable by (**1**) .

Corollary 2.8.1.

Every Lie algebra contains a largest solvable ideal .

Proof:

Let $\mathfrak{n}$ be a maximal solvable ideal. If $\mathfrak{h}$ is also a solvable ideal, then $\mathfrak{h}+\mathfrak{n}$ is solvable by (**3**) , and so equals $\mathfrak{n}$; therefore $\mathfrak{h} \subset \mathfrak{n}$.

Definition 2.8.2.

The radical $\mathfrak{r}=r(\mathfrak{g})$ of $\mathfrak{g}$ is the largest solvable ideals in, The radical of $\mathfrak{g}$ is a characteristic ideal .

Definition 2.8.3. (The Cartan's criterion for solvability)

For any $n \times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$, where $Tr(A)=\sum_{i,j} a_{ij}\ b_{ij}=Tr(BA)$

Hence, $Tr_V(x \circ y)=Tr_V(y \circ x)$ for any rndomorphisms x, y of a vector space V, and so

$$\begin{aligned} Tr_V([x,y] \circ z) &= Tr(x \circ y \circ z)-Tr(y \circ x \circ z) \\ &= Tr(x \circ y \circ z)-Tr(x \circ z \circ y) \qquad (2.7) \\ &= Tr(x \circ [y,z]) \end{aligned}$$

Theorem 2.8.2 (Cartan's Criterion)

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}_V$, where V is a finite – dimensional vector space over a field k of characteristic zero. Then $\mathfrak{g}$ is solvable if $Tr_V(x \circ y) = 0$ for all $x, y \in \mathfrak{g}$.

Proof:

We first observe that, if k' is a field containing k, then the theorem is true for $\mathfrak{g} \subset \mathfrak{gl}_V$ if and only if it is true for $\mathfrak{g}_{k'} \subset \mathfrak{gl}_{V_{k'}}$ (because $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_{k'}$ is solvable from (prop **2.8.2**) . therefore, we may assume that the field k is finitely generated over $\mathbb{Q}$ hence embeddable in $\mathbb{C}$, and then that $k = \mathbb{C}$.

We shall show that the condition implies that each $x \in [\mathfrak{g}, \mathfrak{g}]$ defines a nilpotent endomorphism of V. Then Engle's theorem will show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent , in particular, solvable ,and it follows that $\mathfrak{g}$ is solvable because $\mathfrak{g}^{(n)} = (\mathcal{D}\mathfrak{g})^{(n-1)}$.

Let $x \in [\mathfrak{g}, \mathfrak{g}]$, and choose a basis of V for which the matrix of x is upper triangular . Then the matrix of x_s is diagonal, say , diag $(a_1, \dots, a_n)$, and the matrix of x_n is strictly upper triangular . We have to show that $x_s = 0$, and for this it suffices to show that

$$\bar{a}_1 a_1 + \cdots + \bar{a}_n a_n = 0$$

Where $\bar{a}$ is the complex conjugate of a . Note that **:** $Tr_V(\overline{x_s} \circ x) = \bar{a}_1 a_1 + \cdots + \bar{a}_n a_n$,

Because $\overline{x_s}$ has matrix diag $(\bar{a}_1, \dots, \bar{a}_n)$. By assumption , x is a sum of commutators $[y, z]$, and so it suffices to show that **:** $Tr_V(\overline{x_s} \circ [y, z]) = 0$, all $y, z \in \mathfrak{g}$. From the trivial identity (**2.7**) , we see that it suffices to show that **:**

$$Tr_V([\overline{x_s}, y] \circ z) = 0 , \qquad \text{all } y, z \in \mathfrak{g} .$$

This will follow from the hypothisis once we have shown that $[\overline{x_s}, y] \in \mathfrak{g}$. According to

$\overline{x_s} = c_1 x + c_2 x^2 + \cdots + c_r x^r$, for some $c_i \in k$, and so $[\overline{x_s}, \mathfrak{g}] \subset \mathfrak{g}$, Because $[x, \mathfrak{g}] \subset \mathfrak{g}$.

Corollary 2.8.2.

Let V be a finite-dimensional vector space over a field k of characteristic zero , and let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}_V$. if $\mathfrak{g}$ is solvable, then $Tr_V(x \circ y) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$. conversely, if $Tr_V(x \circ y) = 0$ for all $x, y \in [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}$ is solvable .

Proof:

If $\mathfrak{g}$ is solvable, then $Tr_V(x \circ y) = 0$ for $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$. for the converse , note that the condition implies that $[\mathfrak{g}, \mathfrak{g}]$ is solvable by (Theorem 2.8.2). but this implies that $\mathfrak{g}$ is solvable , because $\mathfrak{g}^{(n)} = (\mathcal{D}\mathfrak{g})^{(n-1)}$.

2.9 Simesimple Lie algebra:

Definitions and basic properties 2.9.1.

1. A Lie algebra is called semisimple if it's only commutative ideal is $\{0\}$.

Thus, the Lie algebra $\{0\}$ is semisimple ,but no Lie algebra of dimension 1 or 2 is semisimple. There exists a semisimple Lie algebra of dimension 3, namely, $\mathfrak{sl}_2$.

from (Def **2.8.2**) .

2. A Lie algebra $\mathfrak{g}$ is semisimple if and only if it's radical is zero .

If $r(\mathfrak{g}) = 0$, then every commutative ideal is zero because it is contained in $r(\mathfrak{g})$. Conversely, if $r(\mathfrak{g}) \neq 0$, then the nonzero term of the derived series of $r(\mathfrak{g})$ is a commutative ideal in $\mathfrak{g}$ (it is an ideal in $\mathfrak{g}$ because it is characteristic ideal in $r(\mathfrak{g})$) .

3. A Lie algebra $\mathfrak{g}$ is semisimple if and only if every solvable ideal is zero .

Since $r(\mathfrak{g})$ is the largest solvable ideal, it is zero if and only if every solvable ideal is zero .

4. The quotient $\mathfrak{g}/r(\mathfrak{g})$ of a Lie algebra by it's radical is semisimple .

A non zero commutative ideal in $\mathfrak{g}/r(\mathfrak{g})$ would correspond to a solvable ideal in $\mathfrak{g}$ properly containing $r(\mathfrak{g})$.

5. A product $\mathfrak{g} = \mathfrak{g}_1 \times \dots \dots \times \mathfrak{g}_n$ of semisimple Lie algebras is semisimple .

Let $\mathfrak{a}$ be a commutative ideal in $\mathfrak{g}$; the projection of $\mathfrak{a}$ in $\mathfrak{g}_i$ is zero for each i, and so $\mathfrak{a}$ is zero

Theorem 2.9.1.

If $\mathfrak{g}$ is a semisimple complex Lie algebra[18], then any $x \in \mathfrak{g}$ can be uniquely written in the form

$$x = x_s + x_n$$

[18] Introduction to Lie Groups and Lie Algebras – alexander Kirillov, Jr. department of Mathematics, SUNY at Stony Brook, Stony Brook, NY11794. USA

Where x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Moreover, adx_s = P(ad$x$) for some polynomial $P \in t\mathbb{C}[t]$ depending on x.

Proof:-

Uniqueness immediately follows from uniqueness of Jacobi decomposition for ad x

If $x = x_s + x_n = x_s' + x_n'$, then $(ad\ x)_s = ad\ x_s = ad\ x_s'$, so $ad(x_s - x_s') = 0$. But by definition, a semisimple Lie algebra has zero center, so this implies $x_s - x_s' = 0$. To prove existence, let us write $\mathfrak{g}$ as direct sum of generalized eigenspaces for ad x: $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$, $(\text{ad}\ x - \lambda\ id)^n|_{\mathfrak{g}_\lambda} = 0$ for $\lambda \gg 0$.

Lemma 2.9.1.

$[\mathfrak{g}_\lambda , \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$

Proof

By Jacobi identity, $(\text{ad}\ x - \lambda - \mu)[y, z] = [(\text{ad}\ x - \lambda)\ y, z] + [y, (\text{ad}\ x - \mu)z]$. Thus, if $y \in \mathfrak{g}_\lambda$, $z \in \mathfrak{g}_\mu$, then:

$$(\text{ad}\ x - \lambda - \mu)^n\ [y, z] = \sum \binom{n}{k} [(\text{ad}\ x - \lambda)^k y, (\text{ad}\ x - \mu)^{n-k}\ z] = 0, \text{ for } n \gg 0$$

Definition 2.9.2.

A Lie algebra $\mathfrak{g}$ is called simple if it is not abelian and contains no ideals other than 0 and $\mathfrak{g}$.

The condition that $\mathfrak{g}$ should not be abelian is included rule out one-dimensional Lie algebra : there are many reasons not to include it in the class of simple Lie algebras. One of these reasons is the following lemma .

Lemma 2.9.2.

Any simple Lie algebra is semisimple .

Proof:

If $\mathfrak{g}$ is simple, then it contains no ideals other than 0 and $\mathfrak{g}$. Thus, if $\mathfrak{g}$ contains a nonzero solvable ideal, then it must coincide with $\mathfrak{g}$, so $\mathfrak{g}$ must be solvable . But then $[\mathfrak{g}, \mathfrak{g}]$ is an ideal which is strictly smaller than $\mathfrak{g}$ (because $\mathfrak{g}$ is solvable) and nonzero (because $\mathfrak{g}$ is not abelian). This gives a contradiction .

Definition 2.9.3. (Trace form)

Let $\mathfrak{g}$ be a Lie algebra. A symmetric k-bilinear form $\beta: \mathfrak{g} \times \mathfrak{g} \longrightarrow k$ on $\mathfrak{g}$ is said to be invariant **(or associative)** if **:** $\beta([x,y],z) = \beta(x,[y,z])$ for all $x, y, z \in \mathfrak{g}$

That is , if $\beta([x,y],z) + \beta(y,[x,z]) = 0$ for all $x, y, z \in \mathfrak{g}$

In other words, β is invariant if $\beta(Dy,z) + \beta(y,Dz) = 0$ (2.8)

For all inner derivations D of $\mathfrak{g}$. if (2.8) holds for all derivations, then β is said to be completely invariant .

Lemma 2.9.3.

Let β be an invariant form of $\mathfrak{g}$. and let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. the orthogonal complement $\mathfrak{a}^{\perp}$ of $\mathfrak{a}$ with respect β is again an ideal . If β is nondegenerate , then $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ is commutative .

Proof:

Let $a \in \mathfrak{a}$, $a' \in \mathfrak{a}^{\perp}$, and $x \in \mathfrak{g}$, and consider , $\beta([x,a],a') + \beta(a,[x,a']) = 0$

As $[x,a] \in \mathfrak{a}$, $\beta([x,a],a') = 0$. Therefore $\beta(a,[x,a']) = 0$. As this holds for all $a \in \mathfrak{a}$, we see that $[x,a'] \in \mathfrak{a}^{\perp}$, and so $\mathfrak{a}^{\perp}$ is an ideal .

Now assume that β is nondegenerate. Then $\mathfrak{d} \overset{\text{def}}{=} \mathfrak{a} \cap \mathfrak{a}^{\perp}$ is an ideal in $\mathfrak{g}$ such that $\beta|\mathfrak{d} \times \mathfrak{d} = 0$. For $b, b' \in \mathfrak{d}$ and $x \in \mathfrak{g}$, $\beta([b,b'],x) = \beta(b,[b',x])$, which is zero because $[b',x] \in \mathfrak{d}$.

As this holds for all $x \in \mathfrak{g}$, we see that $[b,b'] = 0$, and so $\mathfrak{d}$ is commutative . The trace form of representation (V,ρ) of $\mathfrak{g}$ is $(x,y) \longmapsto Tr_V\big(\rho(x) \circ \rho(y)\big): \mathfrak{g} \times \mathfrak{g} \longrightarrow k.$

In other words, the trace form $\beta_V: \mathfrak{g} \times \mathfrak{g} \longrightarrow k$ of a $\mathfrak{g}$-module V is $(x,y) \longmapsto Tr_V(x_V \circ y_V)$, $x \in \mathfrak{g}$

Lemma 2.9.4.

The trace form is a symmetric bilinear form on $\mathfrak{g}$, and it is invariant **:**

$\beta_V([x,y],z) = \beta_V(x,[y,z])$, all $x, y, z \in \mathfrak{g}$.

Proof:

It is k-bilinear because ρ is linear, composition of maps is bilinear, and traces are linear . it is symmetry because traces are symmetric . it is invariant because

$$\beta_V([x,y],z) = Tr([x,y]\circ z) \overset{(32)}{\Longrightarrow} Tr(x\circ[y,z]) = \beta_V(x,[y,z]) \qquad \text{for all } x,y,z \in \mathfrak{g}$$

Therefore (Lemma **2.9.3**), the orthogonal complement $\mathfrak{a}^{\perp}$ of an ideal $\mathfrak{a}$ of $\mathfrak{g}$ with respect to a trace form is again an ideal .

Proposition 2.9.1.

If $\mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{gl}_V$ is faithful and $\mathfrak{g}$ is semisimple, then β_V is nondegenerate .

Proof:

We have to show that $\mathfrak{g}^{\perp} = 0$. For this, it suffices to show that $\mathfrak{g}^{\perp}$ is solvable (from **3** def of semisimple) but the pairing

$$(x,y) \longmapsto Tr_V(x_V \circ y_V) \overset{\text{def}}{\cong} \beta_V(x,y)$$

Is zero on $\mathfrak{g}^{\perp}$, and so Cartan's criterion shows that it is solvable .

Definition 2.9.4. (The Cartan's criterion for semisimplicity)

The trace form for the adjoint representation $ad\colon \mathfrak{g} \longrightarrow \mathfrak{gl}_{\mathfrak{g}}$ is called the killing form[19] $k_{\mathfrak{g}}$ on $\mathfrak{g}$. thus ,

$$k_{\mathfrak{g}}(x,y) = Tr_{\mathfrak{g}}(ad(x)\circ ad(y)), \qquad \text{all } x,y \in \mathfrak{g}$$

In orther words , $k_{\mathfrak{g}}(x,y)$ is the trace of the k-linear map **:** $z \longmapsto [x,[y,z]]\colon \mathfrak{g} \longrightarrow \mathfrak{g}$.

Example 2.9.1.

The Lie algebra $\mathfrak{sl}_2$ consists of the 2×2 matrices with trace zero . It has as basis the elements

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

And

$$[h,x] = 2x \quad , \qquad [h,y] = -2y \quad , \qquad [x,y] = h$$

Relative to the basis $\{x,y,h\}$,

$$ad\ x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad\ h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad ad\ y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

[19] Also called the Cartan-killing form . According to Bourbaki (Note Historical to I,II,III), Cartan introduced the "Killing form" in his thesis and proved the two fundamental criteria: a Lie algebra is solvable if its Killing form is trivial; a Lie algebra is semisimple if its Killing form is nondegenerate .

And so the top row $\left(k(x,x),k(x,h),\left(k(x,y)\right)\right)$ of the matrix of k consists of the traces of

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In fact, k has matrix $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$, which has determinant -128 .

Lemma 2.9.5.

Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. The Killing form on $\mathfrak{g}$ restricts to Killing form on $\mathfrak{a}$. i,e.

$k_{\mathfrak{g}}(x,y) = k_{\mathfrak{a}}(x,y)$ all $x,y \in \mathfrak{a}$.

Proof:

If an endomorphism of a vector space V maps V into a subspace W of V, then $Tr_V(\alpha) = Tr_W(\alpha|W)$, because , when we choose a basis for W and extend it to a basis for V, the matrix for α takes the form $\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ with A the matrix of $\alpha|W$. If $x,y \in \mathfrak{a}$, then $ad\ x \circ ad\ y$ is an endomorphism of $\mathfrak{g}$ mapping $\mathfrak{g}$ into $\mathfrak{a}$, and so its trace (on $\mathfrak{g}$) , $k_{\mathfrak{g}}(x,y)$, equals

$Tr_{\mathfrak{a}}(ad\ x \circ ad\ y|_{\ \mathfrak{a}}) = Tr_{\mathfrak{a}}(ad_{\mathfrak{a}}\ x \circ ad_{\mathfrak{a}}\ y) = k_{\mathfrak{a}}(x,y).$

Example 2.9.2

For matrices $X,Y \in \mathfrak{sl}_n$, $k_{\mathfrak{sl}_n}(X,Y) = 2n\,Tr(XY)$

To prove this, it suffices to show that **:** $k_{\mathfrak{gl}_n}(X,Y) = 2n\,Tr(XY)$

For $X,Y \in \mathfrak{sl}_n$. By definition , $k_{\mathfrak{gl}_n}(X,Y)$ is the trace of the map $M_n(k) \longrightarrow M_n(k)$ sending $T \in M_n(k)$ to $XYT - XTY - YTX + TYX$

For any matrix A, the trace of each of the maps $l_A: T \longmapsto AT$ and $r_A: T \longmapsto TA$ is $nTr(A)$, because , as a left or right $M_n(k)$-module , $M_n(k)$ is isomorphic to a direct sum of n-copies of the standard $M_n(k)$-module k^n. Therefore , the traces of the maps $T \longmapsto XYT$ and $T \longmapsto TXY$ are both $nTr(XY)$, while the traces of the maps $T \longmapsto XTY$ and $T \longmapsto YTX$ are both equal to $Tr(l_X \circ r_Y) = n^2 Tr(X)Tr(Y) = 0$

Proposition 2.9.2.

If $k_{\mathfrak{g}}(\mathfrak{g},[\mathfrak{g},\mathfrak{g}]) = 0$, then $\mathfrak{g}$ is solvable; in particular, $\mathfrak{g}$ is solvable if its Killing form is zero .

Proof:

Cartan's criterion for solvability applied to the adjoint representation $ad: \mathfrak{g} \longrightarrow \mathfrak{gl}_{\mathfrak{g}}$ shows that $ad(\mathcal{D}_{\mathfrak{g}})$ is solvable. Hence $\mathcal{D}_{\mathfrak{g}}$ is solvable, and so $\mathfrak{g}$ is solvable.

Theorem 2.9.2 (Cartan's criterion)

A non zero Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate .

Proof:

Because $\mathfrak{g}$ is semisimple , the adjoint representation $ad: \mathfrak{g} \longrightarrow \mathfrak{gl}_{\mathfrak{g}}$ is faithful ,and so this follows from (Prop 2.9.1) . Let $\mathfrak{a}$ be a commutative ideal of $\mathfrak{g}$ — we have to show that $\mathfrak{a} = 0$. For any $a \in \mathfrak{a}$ and $g \in \mathfrak{g}$, we have that **:** $\mathfrak{g} \xrightarrow{ad\ g} \mathfrak{g} \xrightarrow{ad\ a} \mathfrak{a} \xrightarrow{ad\ g} \mathfrak{a} \xrightarrow{ad\ a} 0$,

And so $(ad\ a \circ ad\ g)^2 = 0$. But an endomorphism of a vector space whose square is zero has trace zero (because its minimum polynomial divides X^2) . Therefore

$k_{\mathfrak{g}}\ (a, g) \stackrel{\text{def}}{=} Tr_{\mathfrak{g}}\ (ad\ a \circ ad\ g) = 0$ and $\mathfrak{a} \subset \mathfrak{g}^{\perp} = 0$.

We say that an ideal a Lie algebra is semisimple if it is semisimple as a Lie algebra .

Corollary 2.9.1.

For any semisimple ideal $\mathfrak{a}$ in a Lie algebra $\mathfrak{g}$ and its orthogonal complement $\mathfrak{a}^{\perp}$ with respect to the Killing form **:** $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$

Proof:

Because $k_{\mathfrak{g}}$ is invariant , $\mathfrak{a}^{\perp}$ is an ideal . Now $k_{\mathfrak{g}}|\mathfrak{a} = k_{\mathfrak{g}}$ (Lemma 2.9.3),which is nondegenerate . Hence, $\mathfrak{a} \oplus \mathfrak{a}^{\perp} = 0$.

Corollary 2.9.2.

Let $\mathfrak{g}$ be a Lie algebra over a field k, and let k' be a field containing k . The Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}_{k'}$ is semisimple . The radical $r(\mathfrak{g}_{k'}) \simeq k' \otimes_k r(\mathfrak{g})$.

Proof:

The Killing form of $\mathfrak{g}_{k'}$ is obtained from that of $\mathfrak{g}$ by extension of scalars. The exact sequence

$0 \longrightarrow r(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/r(\mathfrak{g}) \longrightarrow 0$ Gives rise to an exact sequence

$0 \longrightarrow r(\mathfrak{g})_{k'} \longrightarrow \mathfrak{g}_{k'} \longrightarrow (\mathfrak{g}/r(\mathfrak{g}))_{k'} \longrightarrow 0$

As $r(\mathfrak{g})_{k'}$ is solvable and $(\mathfrak{g}/r(\mathfrak{g}))_{k'}$ is semisimple , the sequence shows that $r(\mathfrak{g})_{k'}$ is the largest solvable ideal in $\mathfrak{g}_{k'}$, i.e ., that $r(\mathfrak{g})_{k'} = r(\mathfrak{g}_{k'})$.

Definition 2.9.5. (Cartan Subalgebras)

A Cartan sub-algebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is nilpotent Lie sub-algebra that is equal to its centralizer, such that $\{X \in \mathfrak{g}: [X, \mathfrak{h}] \subset \mathfrak{h}\} = \mathfrak{h}$. For semi-simple Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ being Cartan is equivalent to $\mathfrak{h}$ being a maximal abelian sub-algebra.

Corollary 2.9.3.

In every complex semisimple Lie algebra $\mathfrak{g}$, there exists a Cartan subalgebra.

And any two Cartansubalgebras in $\mathfrak{g}$ have the same dimension. This dimention is called rank of $\mathfrak{g}$ **:** Rank($\mathfrak{g}$) = dim $\mathfrak{h}$.

2.10 Root decomposition and root systems:-

Definition 2.10.1.

A root system is finite **set** of non-zero vectors $\Delta \subseteq \mathbb{E}$ satisfies the following :

i. If $\alpha \in \Delta$, then $\lambda\alpha \in \Delta$ if and only if $\lambda = \pm 1$

ii. If $\alpha, \beta \in \Delta$, then $\sigma_\alpha . \beta \in \Delta$ where $\sigma_\alpha : \mathbb{E} \to \mathbb{E}$ is reflection

Each element of Δ is called a root.

Theorem 2.10.1.

1. We have the following decomposition for $\mathfrak{g}$, called the root decomposition

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where

$\mathfrak{g}_\alpha = \{x \mid [h, x] = \langle \alpha, h \rangle x$ for all $h \in \mathfrak{h}\}$

$R = \{ \alpha \in \mathfrak{h}^* - \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$

The set is called the root system of $\mathfrak{g}$, and sub spaces $\mathfrak{g}_\alpha$ are called the root sub spaces.

2. $[\mathfrak{g}_\alpha , \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (here and below, we let $\mathfrak{g}_0 = \mathfrak{h}$)

3. If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha , \mathfrak{g}_\beta$ are orthogonal with respect to the Killing form K.

4. For any α, the Killing form gives a non-degenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$. in particular, restriction of K to $\mathfrak{h}$ is non-degenerate.

Example 2.10.1.

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{h}$ = diagonal matrices with trace 0. Denote by $e_i : \mathfrak{h} \rightarrow \mathbb{C}$ the functional which computes i^{th} diagonal entry of h **:**

$$e_i : \begin{bmatrix} h_1 & 0 & \ldots. \\ 0 & h_2 & \ldots. \\ 0 & \cdots & h_n \end{bmatrix} \longmapsto h_i$$

Then one easily sees that $\sum e_i = 0$, So $\mathfrak{h}^* = \bigoplus \mathbb{C}\, e_i \,/\, \mathbb{C}(e_1 + \cdots\cdots\cdots + e_n)$. It is easy to see that matrix units E_{ij} are eigen vectors for ad$h$, $h \in \mathfrak{h}$: $[h, E_{ij}] = (h_i - h_j)E_{ij} = (e_i - e_j)(h)E_{ij}$. Thus, the root decompstion is given by

R = $\{e_i - e_j \mid i \neq j\} \subset \bigoplus \mathbb{C}\, e_i \,/\, \mathbb{C}(e_1 + \cdots\cdots\cdots + e_n)$. $\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}$.

The Killing form on $\mathfrak{h}$ is given by,

$$(h, h') = \sum_{i \neq j} (h_i - h_j)(h'_i - h'_j) = 2\,n \sum_i h_i h'_i = 2\,n\mathrm{tr}\,(hh').$$

From this, it is easy to show that if $\lambda = \sum \lambda_i e_i$, $\mu = \sum \mu_i e_i \in \mathfrak{h}^*$, and λ_i , μ_i are chosen so that $\sum \lambda_i = \sum \mu_i = 0$ (which is always possible), then the corresponding form on $\mathfrak{h}^*$ is given by

$(\alpha, \mu) = \frac{1}{2\,n} \sum_i \lambda_i \mu_i$.

Lemma 2.10.1.

Let $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, then **:** $[e, f] = (e, f)\, H_\alpha$

Proof:-

Let us compute the inner product $([e, f], h)$ for some $h \in \mathfrak{h}$. Since Killing form is invariant, we have

$([e, f], h) = (e, [f, h]) = -\,(e, [h, f]) = \langle h, \alpha \rangle (e, f) = (e, f)\,(h, H_\alpha)$

Since (,) is a non-degenerate form on $\mathfrak{h}$, this implies that $[e, f] = (e, f)\, H_\alpha$.

Lemma 2.10.2.

1. Let $\alpha \in R$, then $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$.

2. Let $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ be such that $(e, f) = \frac{2}{(\alpha, \alpha)}$,and let $h_\alpha = \frac{2\,H_\alpha}{(\alpha, \alpha)}$

Then $\langle h_\alpha, \alpha \rangle = 2$ and the element $e \in \mathfrak{g}_\alpha$ s.t e, f, h_α satisfy the relations of Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We will denote such a sub algebra by $\mathfrak{sl}(2, \mathbb{C})_\alpha \subset \mathfrak{g}$.

Proof:-

Assume that $(\alpha, \alpha) = 0$; then $\langle H_\alpha, \alpha \rangle = 0$. Choose $, f \in \mathfrak{g}_{-\alpha}$ such that $(e, f) \neq 0$ i.e,.

Let $h = [e, f] = (e, f)\ H_\alpha$ and consider the algebra $\mathfrak{a}$ generated by $, f, h$. then we see that $[e, h] = \langle h, \alpha \rangle\ e = 0$, $[h, f] = -\langle h, \alpha \rangle f = 0$, so $\mathfrak{a}$ is solvable Lie algebra . from Lie theorem, we can choose a basis in $\mathfrak{g}$ such that operators ad e, ad$f$, ad$h$ are upper triangular.

Since . $h = [e, f]$,ad$h$ will be strictly upper-triangular and thus nilpotent. But since $h \in \mathfrak{h}$, it is also semisimple. Thus $h = 0$. On the other hand, $h = (e, f)H_\alpha \neq 0$. This contradiction proves the first part of the theorem.

The second part is immediate from definitions and (Lemma **2.10.1**).

Lemma 2.10.3.

Let α be a root , and let $\mathfrak{sl}(2, \mathbb{C})_\alpha$ be the Lie sub algebra generated by $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ and h_α as in the (Lemma **2.10.2**), consider the sub space V = $\mathbb{C}\ h_\alpha \oplus \bigoplus_{k \in \mathbb{Z}, k \neq 0} \mathfrak{g}_{k\alpha} \subset \mathfrak{g}$.

Then V is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$.

Proof:-

Since ad . $\mathfrak{g}_{k\alpha} \subset \mathfrak{g}_{(k+1)\alpha}$, and (Lemma **2.10.2**) , ad $e.\ \mathfrak{g}_{-\alpha} \subset \mathbb{C}\ h_\alpha$, and similarly for f,V is a representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$. Since $\langle h_\alpha, \alpha \rangle = 2$, we see that weight decomposition of V is given by V$[k] = 0$ for odd k and V$[2k] = \mathfrak{g}_{k\alpha}$, V$[0] = \mathbb{C}\ h_\alpha$. in particular, zero weight space V[0] is one-dimensional. Then V is irreducible.

Now we can prove the main theorem about the structure of semi simple Lie algebras.

Theorem 2.10.2.

Let be a complex semi simple Lie algebra with Cartan sub algebra $\mathfrak{h}$ and root decomposition

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$.

1. R spans $\mathfrak{h}^*$ as a vector space, and elements $h_\alpha, \alpha \in R$, span $\mathfrak{h}$ as a vector space

2. For each $\in R$, the root sub space $\mathfrak{g}_\alpha$ is one-dimensional.

3. For any two roots α, β the number $\langle h_\alpha, \alpha \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ is integer .

4. For $\in R$, define the reflection operator $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by, $s_\alpha(\lambda) = \lambda - \langle h_\alpha, \lambda \rangle \alpha = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$

Then for any roots $\alpha, \beta,\ s_\alpha(\beta)$ is also a root. In particular, if $\alpha \in R$, then $-\alpha = s_\alpha(\alpha) \in R$.

5. For any root α , the only multiples of α which are also roots $\pm\alpha$.

6. For roots $\alpha, \beta \neq \pm\alpha$, the sub space V = $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$, is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$.

7. If α, β are roots such that $\alpha + \beta$ is also a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\beta+k\alpha}$.

Proof:-

1. Assume that R does not generate $\mathfrak{h}^*$; then there exists a non-zero $h \in \mathfrak{h}$ such that $\langle h, \alpha \rangle = 0$ for all $\alpha \in R$. But then root decomposition (1) implies that ad$h$ = 0. However, by definition in a semi simple Lie algebra, the center is trivial: $\mathfrak{z}(\mathfrak{g}) = 0$.

The fact that h_α span $\mathfrak{h}$ now immediately follows: using identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ given by the Killing form, elements h_α are identified with non-zero multiples of α.

2. Immediate from (Lemma **2.10.3**) and the fact that in any irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, weight sub spaces are one-dimensional.

3. Consider $\mathfrak{g}$ as a representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$. Then elements of $\mathfrak{g}_\beta$ have weight equal to $\langle h_\alpha, \alpha \rangle$. But from the fact that (V admits a weight decomposition with integer weights: V = $\bigoplus_{n\in\mathbb{Z}}$ V$[n]$) weights of any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ are integer.

4. Assume that $\langle h_\alpha, \alpha \rangle = n \geq 0$. Then elements of $\mathfrak{g}_\beta$ have weight n with respect to action of $\mathfrak{sl}(2,\mathbb{C})_\alpha$. By the same fact above , operator f_α^n is an isomorphism of the space of vectors of weight n with the space of vectors of weight $-n$. In particular, it means that if $v \in \mathfrak{g}_\beta$ is non-zero vector, then$f_\alpha^n\ v \in \mathfrak{g}_{\beta-n\alpha}$ is also non-zero. Thus $\beta - n\alpha = s_\alpha(\beta) \in R$.

5. Assume that $\alpha\ and\ \beta = c\alpha, c \in \mathbb{C}$ are both roots. By part (**3**) ,$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2c$ is integer, so c is a half-integer . same argument shows that $1/c$ is also a half-integer. It is easy to see that this implies that $c = \pm 1, \pm 2, \pm 1/2$. Interchanging the roots if necessary and possibly replacing $\alpha$by $-\alpha$, we have $c = 1$ or $c = 2$.

Now let us consider the sub space V = $\mathbb{C}\, h_\alpha \oplus \bigoplus_{k\in\mathbb{Z}, k\neq 0} \mathfrak{g}_{k\alpha} \subset \mathfrak{g}$.

From (Lemma **2.10.3**) V is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$, and by part (**2**) , V[2] = $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$. Thus, the map ad e_α: $\mathfrak{g}_\alpha \to \mathfrak{g}_{2\alpha}$ is zero. But the results of representation of

$\mathfrak{sl}(2,\mathbb{C})$ show that in an irreducible representation, kernel of e is exactly the highest weight sub space. Thus, we see that V has highest weight 2: V[4] = V[6] = ⋯ = 0. This means that V= $\mathfrak{g}_{-\alpha} \oplus \mathbb{C}\, h_\alpha \oplus \mathfrak{g}_\alpha$, so the only integer multiples of α which are roots are $\pm\alpha$. In particular, 2α is not a root , Combining these two results, we see that if $\alpha, c\alpha$ are both roots, then $c = \pm 1$.

6. Proof is immediate from dim $\mathfrak{g}_{\beta+k\alpha}$ = 1.

7. We already know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\beta+k\alpha}$. since dim$\mathfrak{g}_{\beta+k\alpha}$ = 1, we need to show that for non-zero , $e_\alpha \in \mathfrak{g}_\alpha$, $e_\beta \in \mathfrak{g}_\beta$, we have $[e_\alpha, e_\beta] \neq 0$. This follows from the previous part and the fact that in an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, if $v \in$V$[k]$ is non-zero and V$[k+2] \neq 0$, then $e.v \neq 0$.

Theorem 2.10.3.

i. Let $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ be the real vector space generate by $h_\alpha, \alpha \in R$. Then $\mathfrak{h}= \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$, and the restriction of Killing form to $\mathfrak{h}_\mathbb{R}$ is positive definite.

ii. Let $\mathfrak{h}_\mathbb{R}^* \subset \mathfrak{h}^*$ be the real vector space generated by $\alpha \in R$. Then $\mathfrak{h}^* = \mathfrak{h}_\mathbb{R}^* \oplus i\mathfrak{h}_\mathbb{R}^*$ also, $\mathfrak{h}_\mathbb{R}^* = \{\lambda \in \mathfrak{h}^* | \langle\lambda, h\rangle \in \mathbb{R} \text{ for all } h \in \mathfrak{h}_\mathbb{R}\} = (\mathfrak{h}_\mathbb{R})^*$.

Proof:-

Let us first prove that the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is real and positive definite.

Indeed, (h_α, h_β) = tr (adh_α adh_β) = $\sum_{\gamma\in R}\langle h_\alpha, \gamma\rangle\langle h_\beta, \gamma\rangle$

But by (Theorem **2.10.2**), $\langle h_\alpha, \gamma\rangle\langle h_\beta, \gamma\rangle \in \mathbb{Z}$, so $(h_\alpha, h_\beta) \in \mathbb{Z}$ .Now let $h = \sum c_\alpha h_\alpha \in \mathfrak{h}_\mathbb{R}$. then $\langle h, \gamma\rangle = \sum c_\alpha \langle h_\alpha, \gamma\rangle \in \mathbb{R}$ for any root γ, so (h, h) = tr (ad $h)^2 = \sum_\gamma \langle h, \gamma\rangle^2 \geq 0$

Which proves that the Killing form is positive definite. This shows that $dim_\mathbb{R}\mathfrak{h}_\mathbb{R} \leq \frac{1}{2} dim_\mathbb{R}\mathfrak{h} = r$, where $r = dim_\mathbb{C}\mathfrak{h}$ is the rank of $\mathfrak{g}$. On the other hand, since h_α generate $\mathfrak{h}$ over $\mathbb{C}$, we see that $dim_\mathbb{R}\mathfrak{h}_\mathbb{R} \geq r$. Thus, $dim_\mathbb{R}\mathfrak{h}_\mathbb{R} = r$, so $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$.

Definition 2.10.2. (Abstract root systems)

An abstract root system is a finite set of elements $R \subset E/\{0\}$, where E is a real vector space with a positive definite inner product , such that the following properities hold :

(R1) R generates E as a vector space .

(R2) For any two roots α, β, the number

$n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\beta,\beta)}$ is integer (2.9)

(R3) Let $s_\alpha : E \longrightarrow E$ be definite by $s_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$

Then for any roots $\alpha, \beta, s_\alpha(\beta) \in R$. The number $r = \dim E$ is called the rank of R .

If ,in addition R satisfies the following property

(R4) If $\alpha, c\alpha$ are both roots , then $c = \pm 1$. Then R is called a reduced root system .

Remark 2.10.1.

From proof of (theorem **2.10.2**) it easy to deduce from (**R1**) — (**R3**) that if $\alpha, c\alpha$ are both roots , then $c = \pm 1, \pm 2, \pm \frac{1}{2}$. However , there are indeed examples of non- reduced root systems, which contain α and 2α as roots . Thus, condition (**R4**) does not follow from (**R1**) — (**R3**) .

Note that conditions (**R2**) ,(**R3**) have a very simple geometric meaning. Namely, s_α is reflection around the hyperplane , $L_\alpha = \{\lambda \in E | (\alpha, \lambda) = 0\}$

It can be defined by $s_\alpha(\lambda) = \lambda$ if $(\alpha, \lambda) = 0$ and $s_\alpha(\alpha) = -\alpha$.

Similarly, the number $n_{\alpha\beta}$ also has a simple geometric meaning : if we denote by p_α the operator of orthogonal projection onto the line containing α , then $p_\alpha(\beta) = \frac{n_{\beta\alpha}}{2}\alpha$. Thus, (**R2**) says that the projection of β onto α is a half – integer multiple of α .

Theorem 2.10.4.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra , with root decomposition . then the set of roots $R \subset \mathfrak{h}^*_{\mathbb{R}}/\{0\}$ is a reduced root system .

It's proof coming from (theorem **2.10.3**).

Remark 2.10.2.

We will use it's convenient to introduce, for every root $\alpha \in R$, the corresponding coroot $\langle \alpha^\vee \rangle = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}$.

Note that for the root systems of a semisimple Lie algebra, this coincides with the definition of $h_\alpha \in \mathfrak{h}$ defined by : $\alpha^\vee = h_\alpha$.

Then one easily sees that $\langle \alpha^\vee, \alpha \rangle = 2$ and that :

$$n_{\alpha\beta} = \langle \alpha, \beta^{\vee} \rangle$$

$$s_{\alpha}(\lambda) = \lambda - \langle \lambda, \alpha^{\vee} \rangle \, \alpha$$

If we assume that we have B is non – degenerate on h , so there is an induced isomorphism $B: h \longrightarrow h^*$. by definition, $< s(h), h' >= B(h\,, h')$

Let's calculate

$< sH_\beta, H_\alpha >= B\big(H_\beta, H_\alpha\big) = B(H_\alpha, H_\beta)$ (B Symmetric)

$= B\big(H_\alpha, [X_\beta, Y_\beta]\,\big) = B([H_\alpha, X_\beta], Y_\beta)$ (B invariant)

$= B(X_\beta, Y_\beta) B(H_\alpha)$

$= \frac{1}{2} B([H_\beta, X_\beta], Y_\beta) \beta(H_\alpha)$ ($2X_\beta = [H_\beta, X_\beta]$)

$= \frac{1}{2} B(H_\beta, H_\beta) \beta(H_\alpha)$ (B invariant)

Thus, we have that $s(H_\beta) = \frac{(H_\beta\,, H_\beta)}{2} \beta$, also compute

$(\alpha, \beta) =< \alpha, s^{-1}\beta >= \alpha\left(\frac{2H_\beta}{B(H_\beta, H_\beta)}\right) = \frac{2\alpha(H_\beta)}{B(H_\beta, H_\beta)}$ …………….. (2.10)

Inparticular, letting $\alpha = \beta$, we get $s(H_\beta) = \frac{2\beta}{(\beta,\beta)}$. this is sometimes called the co-root of β , and denoted $\check{\beta}$. then we can use (1) to rewrite this fact

For $\alpha, \beta \in \Delta$, $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in Z$ and $\alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)} \beta \in \Delta \Longrightarrow \Delta \subseteq h^*$ (set of roots)

Now we can define $r_\beta : h^* \longrightarrow h^*$ by $r_\beta(r) = x - \frac{2(\alpha,\beta)}{(\beta,\beta)} \beta$. this is the reflection through the plane orthogonal to β in h^*. The group generated by the r_β for $\beta \in \Delta$ **is a coxeter group** .

Definition 2.10.3.

A root system is irreducible if it cannot be decomposed into the union of two root systems of smaller rank.

Example 2.10.1.

Let us classify all systems of rank 2 which observe that

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \frac{2(\alpha,\beta)}{(\beta,\beta)} = 4cos^2\theta$$

Where θ is the angle between α and , this must be an integer , thus there are not many choices for θ .

$\cos\theta$	0	$\pm\frac{1}{2}$	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{\sqrt{3}}{2}$
θ	$\frac{\pi}{2}$	$\frac{\pi}{3}, \frac{2\pi}{3}$	$\frac{\pi}{4}, \frac{3\pi}{4}$	$\frac{\pi}{6}, \frac{5\pi}{6}$

Choose two vectors with minimal angle between them. If the minimum angle is $\frac{\pi}{2}$, the system is reducible. (notice that α and β can be scaled independently). If the minimal angle is smaller than $\frac{\pi}{2}$, then $r_\beta(\alpha) \neq \alpha$, so the difference $\alpha - r_\beta(\alpha)$ is non-zero integer multiple of β. (in fact, a positive multiple of β since $\theta < \frac{\pi}{2}$).

If we assume $\|\alpha\| \leq \|\beta\|$ we get that $\|\alpha - r_\beta(\alpha)\| < 2\|\alpha\| \leq 2\|\beta\|$.

Example 2.10.2.

Let[20] e_i be the standard basis of $\mathbb{R}^n$, with usual inner product : $(e_i, e_j) = \delta_{ij}$. Let

$E = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n | \sum \lambda_i = 0\}$, and $R = \{e_i - e_j | 1 \leq i, j \leq n, i \neq j\} \subset E$. Then R is a reduced root system . indeed, one easily see that for $\alpha = e_i - e_j$, the corresponding reflection $s_\alpha : E \longrightarrow E$ is transposition of i, j enteries :

$$s_{e_i - e_j}(\dots, \lambda_i, \dots, \lambda_j, \dots) = (\dots, \lambda_i, \dots, \lambda_j, \dots)$$

Clearly, R is stable under such transposition (and, more generally, under all permutations) . thus , condition (**R3**) is satisfied .

Since $(\alpha, \alpha) = 2$ for any $\alpha \in R$, condition (**R2**) is equivalent to $(\alpha, \beta) \in \mathbb{Z}$ for any $\alpha, \beta \in R$ which is immediate .

Finally , condition (**R1**) is obvious. Thus, R is a root system of rank $n - 1$. for historical reasons, this root system is usually referred to as "root system of type A_{n-1} "

Alternatively, one can also define E as a quotient of $\mathbb{R}^n$: $E = \mathbb{R}^n / \mathbb{R}(1, \dots, 1)$

In this description we see that this root system is exactly the root system of Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

[20] Introduction to Lie Groups and Lie Algebras – Alexander Kirillov, Jr. –departement of Mathematics, Suny Brooke, Stony Brooke, NY 11794, USA

Example 2.10.3.

Let us consider an example $\mathfrak{su}(3)$ contains the traceless Hermitean matrices

n	θ	$\frac{\vec{\alpha}^2}{\vec{\beta}^2}$
0	$90°$	Arbitrary
1	$60°, 120°$	1
2	$45°, 135°$	$\frac{1}{2}, 2$
3	$30°, 150°$	$\frac{1}{3}, 3$

The possible angle and relative lengths of roots

(in physicist's convention), which is an eight – dimensional space. The customary basis is $T_a = \lambda_a/2$, where the Gell-Mann matrices λ_a are **:**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The first three are an obvious embedding of the Pauli matrices of $\mathfrak{su}(2)$. they are normalized

To $tr\, T_a T_b = \frac{1}{2}\,\delta_{ab}$. this is chosen such that $[T_1, T_2] = iT_3$. to make it consistent with $\langle T_i, T_j \rangle = \delta_{ij}$, we choose the normalization to be $k = 2$. then the Killing matric is $g_{ij} = \delta_{ij}$, and we do not have care about upper and lower indices on the structure constants , i.e , $f_{abc} = f_{ab}^{c}$. the independent nonvanishing structure constants are

$$f^{123} = 2f^{147} = 2f^{246} = 2f^{257} = -2f^{156} = -2f^{367} = \frac{2}{\sqrt{3}}\, f^{458} = \frac{2}{\sqrt{3}}\, f^{678} = 1\,.$$

This algebra has rank two . As Cartan generators one usually chooses $H_1 = T_3$ and $H_2 = T_8$, which are already diagonal, so they commute . to find the roots , we have to diagonalise the adjoint action of the Cartan elements . A straight forward calculation gives

$$E_{\pm(1,0)} = \frac{1}{\sqrt{2}}\,(T^1 \pm iT^2)\,,\ E_{\pm\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{2}}\,(T^4 \pm iT^5)\,,\ E_{\pm\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{2}}\,(T^6 \pm iT^7)\,.$$

So the roots are : $\alpha^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\alpha^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\alpha^3 = (1,0)$

And their negatives. we will use a notation where superscripts label the roots, while subscripts label the vector components . (So the subscripts are actually lower indices, while the superscripts are not proper indices, and they do not take part in the summation convention. This is even more confusing because both sub – and superscripts have the same range).

Of course the T_a not only give the adjoint representation by acting on themselves, but they naturally act on $\mathbb{C}^3$. For any matrix algebra , this is called the defining or vector representation , and it is denoted by it's dimension as 3 . Since they are already diagonal, the eigenvalues of H_1 and H_2 are simply the diagonal elements ,and the eigenvectors are the standard basis of $\mathbb{C}^3$. Hence , the weights are

$$|\omega^1\rangle \equiv \begin{pmatrix}1\\0\\0\end{pmatrix} = \left|\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right)\right\rangle , \quad |\omega^2\rangle \equiv \begin{pmatrix}0\\1\\0\end{pmatrix} = \left|\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)\right\rangle \ , |\omega^3\rangle \equiv \begin{pmatrix}0\\0\\1\end{pmatrix} = \left|\left(0,-\frac{1}{\sqrt{3}}\right)\right\rangle$$

Note that indeed the differences of weights are roots .

There is even a third representation we can construct from the T_a's , which is called thecomplex conjugate representation : Clearly, if the generators $-T_a^*$. This is of course true for any representation. Since, the Cartan generators are diagonal and real, the weights just receive an overall minus sign, in particular, they are different. (This is in contrast to the adjoint representation, which is isomorphic to it's complex conjugate representation). So we have a representation, again three-dimensional ,called $\overline{\mathbf{3}}$, with states

$$\langle v^1| = \langle -\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right)\Big| , \quad \langle v^2| = \langle -\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right)\Big| \ , \quad \langle v^3| = \langle\left(0,\frac{1}{\sqrt{3}}\right)\Big|$$

The weights of the vector representation.

2.11 Automorphisms and Weyl group :

Most important information about the root system is contained in the number $n_{\alpha\beta}$ rather than in inner product themselves . this motivates the following definition :

Definition 2.11.1.

Let $R_1 \subset E_1$, $R_2 \subset E_2$ be two root systems . An isomorphism $\varphi: R_1 \longrightarrow R_2$ is a vector space isomorphism $\varphi: E_1 \longrightarrow E_2$ which also gives a bijection R_1 simeq R_2 and such that $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ for any $\alpha\beta \in R_1$.

Note that : the condition $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ will be automatically satisfied if φ preserves inner product . However, not every isomorphism of root systems preserves inner products . for example : for any $c \in R_+$, the root systems R and $cR = \{c\alpha , \alpha \in R\}$ are isomorphic . The isomorphism is given by $v \longmapsto cv$, which doesnot preserve the inner product .

A special class of automorphisms of a root system R are those generated by reflections s_α .

Definition 2.11.2.

The Weyl group W of a root system R is the subgroup of $GL(E)$ generated by reflections $s_\alpha, \alpha \in R$.

Lemma 2.11.1.

1. The Weyl group W is a finite subgroup in the orthogonal group $O(E)$, and the root system R is invariant under the action of W .

2. For any $w \in W$, we have $s_{w(\alpha)} = w s_\alpha w^{-1}$.

Proof:

Since every reflection s_α is an orthogonal transformation, $W \subset O(E)$. Since $s_\alpha(R) = R$ (by axioms of a root system) , we have $w(R) = R$ for any $w \in W$. Moreover, if some $w \in W$ leaves every root invariant, then $w = id$ (because R generates E). Thus, W is a subgroup of the group $Aut(R)$ of all automorphisms of R . Since R is a finite set, $Aut(R)$ is finite; thus W is also finite . The second identity is obvious : indeed , $w s_\alpha w^{-1}$ acts as identity on the hyperplane $wL_\alpha = L_{w(\alpha)}$, and $w s_\alpha w^{-1}\big(w(\alpha)\big) = -w(\alpha)$, so it is a reflection corresponding to root $w(\alpha)$.

Example 2.11.1.

Let R be the root system of type A_{n-1} (from Example **2.10.2**) . Then W is the group generated by transfositions s_{ij} . it is easy to see that these transpositions generate the symmetric group s_n ; thus, for this root system $W = s_n$.

In particular, for root system A_1 (i.e., root system of $\mathfrak{sl}(2,\mathbb{C})$) ,we have $W = s_2 = \mathbb{Z}_2 = \{1,\sigma\}$ where σ acts on $E \simeq \mathbb{R}$ by $\lambda \longmapsto -\lambda$.

It should be noted, however, that not all automorphisms of a root system are given by elements of Weyl group . for example, for A_n , $n > 2$, the automorphism $\alpha \longmapsto -\alpha$ is not in the Weyl group .

Pair of roots and rank two root systems:

We take R is reduce root system . and also from condition (**R2**),(**R3**) impose very strong restrictions on relative position of two roots .

Theorem 2.11.1.

Let $\alpha, \beta \in R$ roots which are not multiples of one another , with $|\alpha| \geq |\beta|$,and let φ be the angle between them. Then we must have one of the following possibilities :

1. $\varphi = \pi/2$ (i.e., α, β are orthogonal), $n_{\alpha\beta} = n_{\beta\alpha} = 0$

2. $\varphi = 2\pi/3$, $|\alpha| = |\beta|$, $n_{\alpha\beta} = n_{\beta\alpha} = -1$

3. $\varphi = \pi/3$, $|\alpha| = |\beta|$, $n_{\alpha\beta} = n_{\beta\alpha} = 1$

4. $\varphi = 3\pi/4$, $|\alpha| = \sqrt{2}|\beta|$, $n_{\alpha\beta} = -2$, $n_{\beta\alpha} = -1$

5. $\varphi = \pi/4$, $|\alpha| = \sqrt{2}|\beta|$, $n_{\alpha\beta} = 2$, $n_{\beta\alpha} = 1$

6. $\varphi = 5\pi/6$, $|\alpha| = \sqrt{3}|\beta|$, $n_{\alpha\beta} = 3$, $n_{\beta\alpha} = 1$

7. $\varphi = \pi/6$, $|\alpha| = \sqrt{3}|\beta|$, $n_{\alpha\beta} = -3$, $n_{\beta\alpha} = -1$

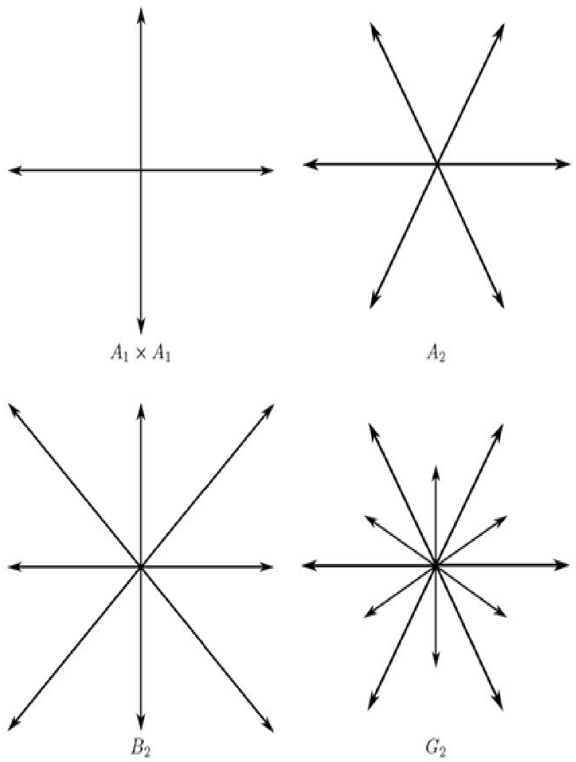
Proof:

Recall $n_{\alpha\beta}$ defined by (**2.9**) . Since $(\alpha, \beta) = |\alpha||\beta| \cos\varphi$, we see that $n_{\alpha\beta} = 2\frac{\sqrt{|\alpha|}}{\sqrt{|\beta|}} \cos\varphi$. thus , $n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2\varphi$. Since $n_{\alpha\beta} n_{\beta\alpha} \in \mathbb{Z}$, this means that $n_{\alpha\beta} n_{\beta\alpha}$ must be one of 0,1,2,3. Analyzing each of these possibilities and using $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{|\alpha|}{|\beta|}$ if $\cos\varphi \neq 0$, we get the statement of the theorem .

Theorem 2.11.2.

1. Let $A_1 \times A_1$, A_2 , B_2 , G_2 be the sets of vectors in $\mathbb{R}^2$ shown in (FIG 2.3) then each of them is a rank two root system .

2. Any rank two reduced root system is isomorphic to one of root systems $A_1 \times A_1$, A_2 , B_2 , G_2



Fig(2.3)

Proof:

Proof of part (**1**) is given by explicit analysis . Since for any pair of vectors in these systems, the angle and ratio of lengths is among one of the possibilities listed in (Theorem **2.11.1**) , condition (**R2**) is satisfied . It is also to see that condition (**R3**) is satisfied .

To prove the second part, assume that R is a reduced rank 2 root system . Let us choose α, β to be two roots such that the angle φ between them is as large as possible and $|\alpha| \geq |\beta|$. Then $\varphi \geq \pi/2$ (otherwise , we could take the pair $\alpha, s_\alpha(\beta)$ and get a larger angle). Thus, we must be in one of situations (**1**) ,(**2**) ,(**3**) ,(**6**) of (Theorem **2.11.1**) .

Consider the example , case (**2**) : $|\alpha| = |\beta|, \varphi \geq 2\pi/3$. by definition of root system , R is stable under reflections s_α , s_β . But successively applying these two reflections to α, β we get exactly the root system of type A_2 generated by α, β .

To show that in this case $R = A_2$, note that if we have another root γ which is not in A_2 , then γ must be between some of the roots of A_2 (since R is reduced) . Thus, the angle between γ and some root δ is less than $\pi/3$, and the angle between γ and $-\delta$ is greater than $2\pi/3$, which is impossible because angle between α, β is the maximal possible . Thus,

$R = A_2$. Similar analysis shows that in cases (**1**) , (**3**) , (**6**) of (Theorem **2.11.1**) , we will get $R = A_1 \times A_1$, B_2 , G_2 respectively .

Result 2.11.1.

Let $(\alpha, \beta) \in R$ be two roots such that $(\alpha, \beta) < 0, \alpha \neq c\beta$. then $\alpha, \beta \in R$.

Definition 2.11.3. (Positive roots and simple roots)

In order to proceed with classification of root systems , we would like to find for each root system some small set of "generating roots" , similar to what was done in the previous section of rank 2 root systems . in general it can be done as follows ;

Let $t \in E$ be such that for any root $\alpha, (t, \alpha) \neq 0$ (such elements t are called regular) .Then we can write

$$R = R_+ \sqcup R_-$$

$$R_+ = \{ \alpha \in R | (\alpha, t) > 0\} , \qquad R_- = \{ \alpha \in R | (\alpha, t) < 0\}$$

Such a decomposition will be called a polarization of R. Note that polarization depends on the choice of t . the roots $\alpha \in R_+$ will be called positive, and the roots $\alpha \in R_-$ will be called negative .

Definition 2.11.4.

A root $\alpha \in R_+$ is called simple if it cannot be written as a sum of two positive roots .

We will denote the set of simple roots by $\prod \subset R_+$.

Lemma 2.11.2.

Every positive root can be written as a sum of simple roots .

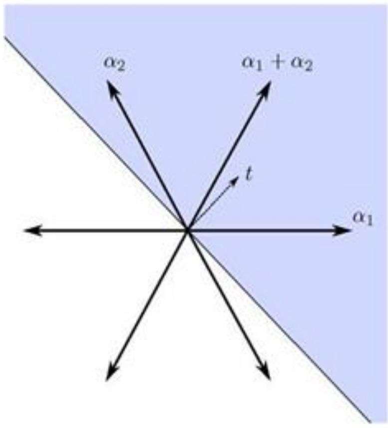
Proof:

If a positive root α is not simple , it can be written in the form $\alpha = \alpha' + \alpha''$, with $\alpha' + \alpha'' \in R_+$ and $(\alpha', t) < (\alpha, t), (\alpha'', t) < (\alpha, t)$. if α', α'' are not simple, we can apply the same argument to them to write them as a sum of positive roots . Since (α, t) can only take finitely many values, the process will terminate after finitely many steps .

Example 2.11.2.

Let us consider the root system A_2 and let t be as shown in **Figure** below . Then there are three positive roots : two of them are denoted by α_1, α_2 and the third one is $\alpha_1 + \alpha_2$.

Thus,one easily sees that α_1, α_2 are simple roots, and $\alpha_1 + \alpha_2$ is not simple .



Positive and simple roots for A_2, Fig (2.4)

Lemma 2.11.3.

If $\alpha, \beta \in R_+$ are simple , then $(\alpha, \beta) \leq 0$.

Proof:

Assume that $(\alpha, \beta) > 0$. Then , applying (Result 2.11.1) to $-\alpha, \beta$, we see that $\beta' = \alpha - \beta \in R$. If $\beta' \in R_+$, then $\beta = \beta' + \alpha$ can not be simple . if $\beta' \in R_-$, then $-\beta' \in R_+$, so $\alpha = -\beta' + \beta$ cannot be simple . this contradiction shows that $(\alpha, \beta) > 0$ is impossible .

Second we can apply the master formula , $\frac{\vec{\beta}.\vec{\alpha}}{\vec{\alpha}.\vec{\alpha}} = -\frac{1}{2}\ (p - q)$

q measures how often we can substract α from β without leaving root space . But we saw that already $\beta - \alpha$ is not a root , so $q = 0$ and

$$\vec{\beta}.\vec{\alpha} = -\frac{1}{2} p \vec{\alpha}.\vec{\alpha} \leq 0$$

By the same argument

$$\vec{\alpha}.\vec{\beta} = -\frac{1}{2} p' \vec{\beta}.\vec{\beta} \leq 0$$

Hence the angle between simple roots and the relative lengths are

$$\cos\theta = -\frac{\sqrt{pp'}}{2} \quad , \qquad \frac{\alpha^2}{\beta^2} = \frac{p'}{p}$$

In particular, the angle is constrained to be $90° \leq \theta < 180°$. the first constraint comes because the cosine is nonpositive, the second because the roots are positive , so they lie in a half-space .

Theorem 2.11.3.

Let $R = R_+ \sqcup R_- \subset E$ be a root system . then the simple roots form a basis of the vector space E.

Proof:

By (Lemma **2.11.2**) every positive root can be written as linear combination of simple roots . Since R spans E , this implies that the set of simple roots spans E .

Simple roots are linearly independent. To see this, consider a linear combination

$$\gamma = \sum_{\alpha} c_\alpha\, \alpha$$

We can find coefficients c_α such that $\gamma = 0$. Since all α are positive, the c_α cannot all have the same sign. Hence we can split γ into strictly positive and negative pieces,

$$\gamma = \sum_{c_\alpha > 0} c_\alpha\, \alpha - \left(- \sum_{c_\alpha > 0} c_\alpha\, \alpha \right) = \mu - v$$

Now consider the norm of γ :

$$\gamma^2 = (\mu - v)^2 = \mu^2 + v^2 - 2\mu \,.\, v$$

Clearly, μ and v cannot vanish, so their norm is positive. However, since μ and v are both positive linear combinations of simple roots, their scalar product is negative. Hence, the norm γ never vanishes, so no linear combination of simple roots can be zero.

Additionally, the simple roots form a basis : if this was not the case , there would be a vector $\vec{\xi}$ which is orthogonal to all simple roots . but it is easy to see that any positive root can be written as a linear combination of simple roots with non-negative integer coefficients,

$\gamma = \sum_{\alpha\ simple} k_\alpha\, \alpha$

This follows by induction: It is obviously true for simple roots themselves. Any other positive root can be written as a sum of positive roots, hence the statement follows. Since the roots are linearly independent, the decomposition is unique , and we can associate to any positive root its level $k = \sum_\alpha k_\alpha$.

Then we have $\vec{\xi}.\vec{\alpha} = 0$ for all roots α , so we see that the operator $\vec{\xi}.\vec{H}$ commutes with all elements of the algebra : $[\vec{\xi}.\vec{H}, H_i] = [\vec{\xi}.\vec{H}, E_\alpha] = 0$

But this means that $\vec{\xi}.\vec{H}$ is in the center of the algebra, which is trivial for a semisimple algbra

Hence, there is no such ξ ,and the simple roots form a basis of $\mathbb{R}^r$. Hence, in particular the number of simple roots is equal to the rank of the algebra .

We can find all positive roots . that is , given the simple roots , we can determine whether a linear combination $\gamma_k = \sum k_\alpha \alpha$ is a root or not .by induction over the level and using the master formula again . the key points is that for the simple roots, i.e. those at level one all $q^i = 0$ since the difference of simple roots is never a root . Hence, from the master formula we can find the p^i , and thus the allowed roots on level two . Now for these roots, we by construction know the q^i , hence we again can find the p^i , and continue this process until we found all the roots, i.e. until at some level all roots have all $p^i = 0$.

Lemma 2.11.4.

Let $v_1, \dots \dots, v_k$ be a collection of non-zero vectors in a Euclidean space E such that for $i \neq j$, $(v_i, v_j) \leq 0$. Then $\{v_1, \dots \dots, v_k\}$ are linear independent .

Corollary 2.11.1.

Every $\alpha \in R$ can be uniquely written as linearly combination of simple roots with integer coefficients **:** $\alpha = \sum_{i=1}^{r} n_i \alpha_i \quad , \quad n_i \in \mathbb{Z}$

Where $\{\alpha_1, \dots \dots, \alpha_r\} = \prod$ is the set of simple roots .if $\alpha \in R_+$, then all $n_i \geq 0$; if $\alpha \in R_-$, then all $n_i \leq 0$. For positive root $\alpha \in R_+$, we define its height by ht $(\sum n_i \alpha_i) = \sum n_i \in \mathbb{Z}_+$

so that $\text{ht}(\alpha_i) = 1$. im many cases, statements about positive roots can be proved by induction in height .

Example 2.11.3.

Let R be the root system of type A_{n-1} or equivalent, the root system of $\mathfrak{sl}(n, \mathbb{C})$.(from Example 2.10.1 and Example 2.10.2) Choose the polarization as follows :

$$R_+ = \{e_i - e_j \,|\, i < j\}$$

(The corresponding root subspaces E_{ij} , $i < j$,generate the Lie algebra $\mathfrak{n}$ of strictly upper-triangle matrices in $\mathfrak{sl}(n, \mathbb{C})$) . Then it is easy to show that the simple roots are

$$\alpha_1 = e_1 - e_2 \ , \ \alpha_2 = e_2 - e_3 \ , \ \ldots\ldots \ , \ \alpha_{n-1} = e_{n-1} - e_n$$

And indeed, any positive root can be written as a sum of simple roots with non-negative integer coefficients . for example , $e_2 - e_4 = (e_2 - e_3) + (e_3 - e_4) = \alpha_2 + \alpha_3$. the height is given by : $\mathrm{ht}(e_i - e_j) = j - i$.

Definition 2.11.5 (Weight and root lattices)

In study of root systems of simple Lie algebras , we will frequently use the following lattices. (recall that a lattice in real vector space E is an abelian group generated by a basis in E) . by suitable change of basis any lattice $L \subset E$ can be identified with $\mathbb{Z}^n \subset \mathbb{R}^n$.

Every root system $R \subset E$ gives rise to the following lattices :

Q = { abelian group generated by $\alpha \in R$ } $\subset E$

$Q^\vee$ = { abelian group generated by $\alpha^\vee, \alpha \in R$ } $\subset E^*$

Lattice Q is called the root lattice of R, and $Q^\vee$ is the coroot lattice .

To justify the use of the word lattice , we need to show that $Q, Q^\vee$ are indeed generated by a basis in E(respectively E^*) . This can be done as follows:

Fix a polarization of R and let $\prod = \{\alpha_1, \ldots\ldots\ldots, \alpha_r\}$ be the corresponding system of simple roots . Since every root can be written as a linear combination of simple roots with integer coefficients (Corollary 2.11.1) , one has $Q = \bigoplus \mathbb{Z}_{\alpha_i}$ Which shows that Q indeed a lattice, $Q^\vee = \bigoplus \mathbb{Z}_{\alpha_i^\vee}$.

Even more important in the applications to representation theory of semisimple Lie algebras is the weight lattice $P \subset E$ defined as follows :

$P = \{\lambda \in E | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R \} = \{\lambda \in E | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha^\vee \in Q^\vee\}$.

In other words, $P \subset E$ is exactly the dual lattice of $Q^\vee \subset E^*$. Elements of P are frequently called integral weights .

Since $Q^\vee$ is generated by $\alpha_i^\vee$, the weight lattice can also be defined by

$P = \{\lambda \in E | \langle \lambda, \alpha_i^{\vee} \rangle \in \mathbb{Z}$ for all simple roots α_i }

One can easily defined a basis in P. Namely, defined fundamental weights $\omega_i \in E$ by**:** $\langle \omega_i, \alpha_i^{\vee} \rangle = \delta_{ij}$, Then one can easily sees that so defined ω_i form a basis in E and that **:** $P = \oplus_i \left(\mathbb{Z}_{\omega_i} \right)$.

Finally note that by the axioms of a root system, we have $n_{\alpha\beta} = \langle \alpha, \beta^{\vee} \rangle \in \mathbb{Z}$ for any roots α, β . Thus, $R \subset P$ which implies that $Q \subset P$.

However, in general $P \neq Q$, as the examples below show. Since both P, Q are free abelian groups of rank r, general theory of finitely generated abelian groups implies that the quotient group P/Q is a finite abelian group. It is also possible to describe the order $|P/Q|$ in terms of the matrix $\alpha_{ij} = \langle \alpha_i^{\vee}, \alpha_j \rangle$. (from Example **2.10.2**)

Example (2.11.4) :

Consider the root system A_1. It has the unique positive root α , so $Q = \mathbb{Z}_{\alpha}$, $Q^{\vee} = \mathbb{Z}_{\alpha^{\vee}}$. if we define the inner product $(,)$ by $(\alpha, \alpha) = 2$, and use this product to identify $E^* \simeq E$, then under this identification $\alpha^{\vee} \longmapsto \alpha$, $Q^{\vee} \rightrightarrows Q$. Since $\langle \alpha, \alpha^{\vee} \rangle = 2$, we see that the fundamental weight is $\omega = \frac{\alpha}{2}$, and $P = \mathbb{Z}\frac{\alpha}{2}$ .Thus, in this case $P/Q = \mathbb{Z}_2$.

Example (2.11.5):

For the root system A_2 , the root and weight lattices are shown in **Fig (2.5)** . this figure also shows simple roots α_1, α_2 and fundamental weights ω_1, ω_2 .

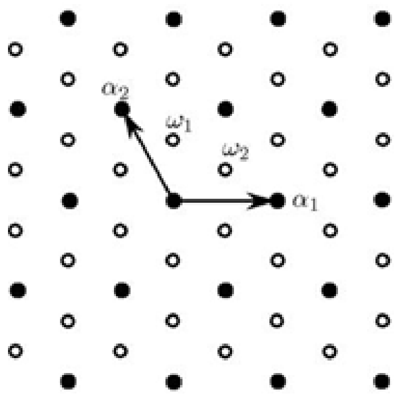


Fig (2.5)

It is easy to see from the figure (and also easy to prove algebraically) that one can take α_1, ω_2 as a basis of P, and that $\alpha_1, 3\omega_2 = \alpha_2 + 2\alpha_1$ is a basis of Q . Thus, $P/Q = \mathbb{Z}_3$.

2.12 Weyl chambers

We recall that a polarization is defined by an element $t \in E$ which does not lie on any of the hyperplanes orthogonal to roots :

$$t \in E \Big/ \bigcup_{\alpha \in R} L_\alpha$$

$$L_\alpha = \{\lambda \in E | (\alpha, \lambda) = 0\}$$

Moreover, the polarization actually depends not on t itself but only on the signs of (t, α); thus, polarization is unchanged if we change t as long as we do not cross any of the hyperplanes . This justifies the following definition .

Definition 2.12.1

A Weyl chamber is a connected component of the complement to the hyperplanes :

$C =$ connected component of $(E / \bigcup_{\alpha \in R} L_\alpha)$

For example , for root system A_2 there are 6 Weyl chambers ; one of them is shaded in the **Fig(2.5)**

Clearly, to specify a Weyl chamber we need to specify, for each hyperplane L_α , on which side of the hyperplane the Weyl chamber lies . Thus, a Weyl chamber is defined by a system of inequalities of the form , $\pm(\alpha, \lambda) > 0$. (one inequality for each pair of roots $\alpha, -\alpha$). Any such system of inequalities defines either an empty set or a Weyl chamber.

Lemma 2.12.1.

The closure $\bar{C}$ of a Weyl chamber C is unbounded convex cone . The boundary $\partial\bar{C}$ is a union of finite number of codimension one faces : $\partial\bar{C} = \bigcup F_i$. Each F_i is a closed convex unbounded subset in one of the hyperplanes L_α, given by a system of inqualities. The hyperplanes containing F_i are called walls of C .

Theorem 2.12.1.

The Weyl group acts transitively on the set of Weyl chambers .

Proof:

Let us say that two Weyl chambers C, C' are adjacent if they have a common codimension one face F (obviousely, they have to be on different sides of F). If L_α is the hyperplane containing this common face F, then we will say that C, C' are adjacent chambers separated by L_α .

Corollary 2.12.1

Every weyl chamber has exactly $r =$ rank (R) walls . Walls of positive Weyl chamber C_+ are L_{α_i} , $\alpha_i \in \prod$.

Proof:

For positive Weyl chamber C_+ , this follows (Lemma **2.11.4**) . Since every Weyl chamber can be written in the form $C = w(C_+)$ for some $w \in W$, all Weyl chambers have the same number of walls .

2.13 Simple reflections

Is it possible to recover R from the set of simple roots $\prod$? the answer is based on the use of Weyl group .

Theorem 2.13.1.

Let R be a reduced root system , with fixed polarization $R = R_+ \sqcup R_-$, Let $\prod = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots . Consider reflections corresponding to simple roots $s_i = s_{\alpha_i}$ (they are called simple reflections).

1. The simple reflections s_i generates W.

2. $W(\prod) = R$: every $\alpha \in R$ can be written in the form $w(\alpha_i)$ for some $w \in W$ and $\alpha_i \in \prod$.

Proof:

We start by proving the following result

Lemma 2.13.2.

Any Weyl chamber can be written as : $C = s_{i_1} \dots s_{i_l} (C_+)$

For some sequence of indices $i_1, \dots, i_l \in \{1, \dots, r\}$

Proof:

By the construction given in the proof (Theorem 2.12.1) , we can connect C_+ , C by a chain of adjacent Weyl chambers $C_0 = C_+, C_1, \dots, C_l = C$. then $C = s_{\beta_1} \dots s_{\beta_l} (C_+)$, where L_{β_i} is the hyperplane separating C_{i-1} and C_i .

Since β_1 separates $C_0 = C_+$ from C_1 , it means that L_{β_1} is one of the walls of C_+ . Since the walls of C_+ are exactly hyperplanes L_{α_i} corresponding to simple roots (Colloray 2.12.1) we see that $\beta_1 = \pm\alpha_{i_1}$ for some index $i_1 \in \{1, \dots \dots, r\}$, so $s_{\beta_1} = s_{i_1}$ and $C_1 = s_{i_1}(C_+)$.

Consider now the hyperplane $L_{\beta_2} = s_{i_1}(L)$ for some hyperplane L which is a wall of C_+ . Thus, we get that $\beta_2 = \pm s_{i_1}(\alpha_{i_2})$ for some index i_2. By (Lemma **2.11.1**), we therefore have $s_{\beta_2} = s_{i_1} s_{i_2}\ s_{i_1}$ and thus , $s_{\beta_2} s_{\beta_1} = s_{i_1} s_{i_2}\ s_{i_1} . s_{i_1} = s_{i_1} s_{i_2}$ and $C_2 = s_{i_1} s_{i_2}(C_+)$

Repeating the same argument , we finally get that $C = s_{i_1} \dots\dots s_{i_l}\ (C_+)$ and the indices i_k are computed inductively, by : $\beta_k = s_{i_1} \dots \dots s_{i_{k-1}}(\alpha_{i_k})$ which completes the proof of the lemma

Now the theorem easily follow. Indeed , every hyperplane L_α is a wall of some Weyl chamber C. Using the lemma, we can write $C = w(C_+)$ for some $w = s_{i_1} \dots\dots s_{i_l}$. Thus, $L_\alpha = w(L_{\alpha_j})$ for some index j, so $\alpha = \pm w(\alpha_j)$ and $s_\alpha = w s_j\ w^{-1}$, which proves both statements of the theorem .

Corollary 2.13.1.

The root system R can be recovered from the set of simple roots $\prod$.

Proof:

Given $\prod$,we can recover W as the group generated by s_i and then recover $R = W(\prod)$.

Let us say that a root hyperplane L_α separates two Weyl chambers C, C' if these two chambers are on different sides of H_α , i.e. $\alpha(C),\ \alpha(C^{'})$ have different signs .

Definition 2.13.1.

Let R be a reduced root system, with set of simple roots $\prod$. Then we define , for an element $w \in W$. Its length by $l(w) =$ number of root hyperplanes hyperplanes separating C_+ and $w(C_+) = |\{\alpha \in R_+ | w(\alpha) \in R_-\}|$. it should be denoted that $l(w)$ depends not only on w itself but also on the choice of polarization $R = R_+ \sqcup R_-$ or equivalently, the set of simple roots .

Example 2.13.1.

Let $w = s_i$ be a simple reflection. Then the Weyl chambers C_+ and $s_i(C_+)$ are separated by exactly one hyperplane, namely L_{α_i} . Therefore, $l(s_i) = 1$, and $\{\alpha \in R_+ | w(\alpha) \in R_-\} = \{\alpha_i\}$

In other words, $s_i(\alpha_i) = -\alpha_i \in R_-$ and s_i permutes elements of $R_+ \backslash \{\alpha_i\}$.

This example is very useful in many arguments involving Weyl group, such as the following lemma .

Lemma 2.13.2

Let $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$, Then: $\langle \rho, \alpha_i^\vee \rangle = \frac{2(\rho\ ,\alpha_i)}{(\alpha_i,\alpha_i)} = 1$

Proof:

Writing $\rho = \left(\alpha_i + \sum_{\alpha \in R_{+\setminus\{\alpha_i\}}} \alpha\right)/2$ and using results of (Example **2.13.1**) , we see that $s_i(\rho) = \rho - \alpha_i$. On the other hand, by definition $s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle\, \alpha_i$.

2.14 Dynkin diagrams and classification of root systems :

There is an obvious construction which allows one to construct larger root systems from smaller ones . Namely, if $R_1 \subset E_1$ and $R_2 \subset E_2$ are two root systems, then we can define $R = R_1 \sqcup R_2 \subset E_1 \oplus E_2$, with the inner product on $E_1 \oplus E_2$ defined so that $E_1 \perp E_2$. so define that R is again root system .

Definition 2.14.1

A root system R is called reducible if it can be written in the form $R = R_1 \sqcup R_2$, with $R_1 \perp R_2$. Otherwise , R is called reducible .

Lemma 2.14.1.

Let R be a reduced root system, with given polarization, and let $\prod$ be the set of simple roots .

If R is reducible : $R = R_1 \sqcup R_2$, then $\prod = \prod_1 \sqcup \prod_2$, where $\prod_i = \prod \cap R_i$ is the set of simple roots for R_i . Conversely, if $\prod = \prod_1 \sqcup \prod_2$, with $\prod_1 \perp \prod_2$, then $R = R_1 \sqcup R_2$, where R_i is the root system generated by $\prod_i$.

Proof:

The first part is obvious . to prove the second part notice that if $\alpha \in \prod_1$, $\beta \in \prod_2$, then $s_\alpha(\beta) = \beta$ and s_α , s_β commute . Thus, if we denote by W_i the group generated by simple reflections $s_\alpha , \alpha \in \prod_i$, then $W = W_1 \times W_2$, and W_1 acts trivially on $\prod_2$, W_2 acts trivially on $\prod_1$ .Thus, $R = W(\prod_1 \sqcup \prod_2) = W_1(\prod_1) \sqcup W_2(\prod_2)$.

It can be shown that every reducible root system can be uniquely written in the form $R_1 \sqcup R_2 \ldots \ldots \sqcup R_n$, where R_i are mutually orthogonal irreducible root systems . Thus, in order to classify all root systems, it suffices to classify all reducible root systems , R is an irreducible root system and $\prod$ is corresponding set of simple roots . We assume that we have chosen an order on the set of simple roots : $\prod = \{\alpha_1, \dots\dots, \alpha_r\}$.

The compact way of describing relative position of roots $\alpha_i \in \prod$ is by writing all inner products between these roots . However, this is not invariant under isomorphisms of root systems . A better way of describing relative position of simple roots is given by cartan matrix

Definition 2.14.2.

The Cartan matrix A of a set of simple roots $\prod \subset$ R is the $r \times r$ matrix with entries

$$\alpha_{ij} = n_{\alpha_j \alpha_i} = \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$$

The following properties of Cartan mtrix immediately follow from definitions and from known properties of simple roots.

Lemma 2.14.2

i. For any $i, \alpha_{ii} = 2$.

ii. For any $i \neq j$, α_{ij} is a non-positive integer : $\alpha_{ij} \in \mathbb{Z}$, $\alpha_{ij} \leq 0$.

iii. For any $i \neq j$, $\alpha_{ij}\, \alpha_{ji} = 4\cos^2 \varphi$, where φ is angle between α_i , α_j . if $\varphi \neq \frac{1}{2}$, then

$$\frac{|\alpha_i|^2}{|\alpha_j|^2} = \frac{\alpha_{ji}}{\alpha_{ij}}$$

Example 2.14.1.

For the root system A_n , the Cartan matrix is

$$A = \begin{bmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & \cdots & & & \\ & -1 & 2 & & & & \\ & \vdots & & & & \vdots & \\ & & & \cdots & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}$$

Because $2\,\frac{(e_i - e_{i+1}\,, e_{i+1} - e_{i+2})}{(e_i - e_{i+1}\,, e_i - e_{i+1})} = -1$, (Entries which are not shown zeroes).

Definition 2.14.3.

Let $\prod$ be a set of simple roots of a root system R. The Dynkin diagram of $\prod$ is the graph constructed as follows :

For each simple root α_i, we construct a vertex v_i of the Dynkin diagram (traditionally, vertices are drawn as small circles rather than as dots) .

For each pair of simple roots $\alpha_i \neq \alpha_j$, we connect the corresponding vertices by n edges, when n depends on the angle φ between α_i, α_j :

For $\varphi = \pi/2$, $n = 0$ (vertices are not connected)

For $\varphi = 2\pi/3$, $n = 1$ (case of A_2 system)

For $\varphi = 3\pi/4$, $n = 2$ (case of B_2 system)

For $\varphi = 5\pi/6$, $n = 3$ (case of G_2 system)

Finally, for every pair of distinct simple roots $\alpha_i \neq \alpha_j$, if $|\alpha_i| \neq |\alpha_j|$ and they are not orthogonal , we orient the corresponding (multiple) edge by putting on it an arrow pointing towards the shorter root .

Example 2.14.2.

The Dynkin diagrams for rank two root systems are shown in (Fig 2.6)

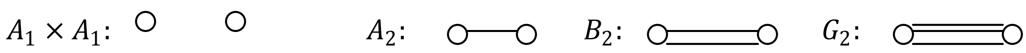


Fig (2.6)

Theorem 2.14.1.

Let $\prod$ be a set of simple roots of a reduced root system R .

1. The Dynkin diagrams is connected if and only if R is irreducible .

2. The Dynkin diagram determines the Cartan matrix A .

3. R is determined by the Dynkin diagram uniquely up to an isomorphism : if R, R' are two reduced root systems with the same Dynkin diagram, then they are isomorphic .

Proof:

1. Assume that R is reducible; by (Lemma **2.14.1**) we have $\prod = P_{i_1} \sqcup \prod_2$, with $\prod_1 \perp \prod_2$. thus ,by construction of Dynkin diagram, it will be a disjoint union of the Dynkin diagram of $\prod_1$ and the Dynkin diagram of $\prod_2$. proof in the opposite direction is similar .

2. Dynkin diagram determines, for each pair of simple roots α_i, α_j , the angle between them and shows which of them is longer . Since all possible configurations of two roots are listed in (Theorem **2.11.1**) , one easily sees that this information, together with $(\alpha_i, \alpha_j) \leq 0$, uniquely determines $n_{\alpha_i \alpha_j}$, $n_{\alpha_j \alpha_i}$.

3. By part (**2**) the Dynkin diagram determines $\prod$ uniquely up to an isomorphism. $\prod$ determines R uniquely up to an isomorphism .

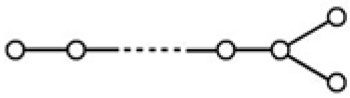
Theorem 2.14.2.

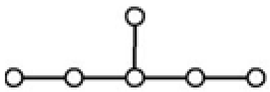
Let R be a reduced irreducible root system . Then its dynkin diagram is isomorphic to one of the diagrams below (in each diagram, the number of vertices is equal to the subscript, so A_n has exactly n vertices) :

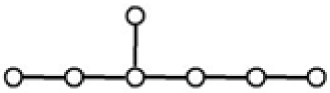
A_n $(n \geq 1)$: 

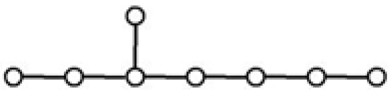
B_n $(n \geq 2)$: 

C_n $(n \geq 2)$: 

D_n $(n \geq 4)$: 

E_6 : 

E_7: 

E_8: 

F_4: 

G_2: 

Fig (2.7)

Conversely, each of these diagrams does appear as a Dynkin diagram of a reduced irreducible root system .