Chapter One

Lie Groups

1.1 Topological Manifolds

Definition 1.1.1.

A manifold M [1]of dimension n ,or n-manifold is a topological space with the following properties **:**

1. M is Hausdorff

2. M is locally Euclidean of dimension n

3. M has a countable basis of open sets

As a matter of notation $dim\ M$ is used for the dimension of M; when $dim\ M = 0$, then M is a countable space with the discrete topology. It follows from the homeomorphism of U and U' that locally Euclidean is equivalent to the requirement that each point p have a neighborhood U homeomorphic to an n-ball in R^n. Thus a manifold of dimension 2 is locally homeomorphic to an open disk, and so on ……

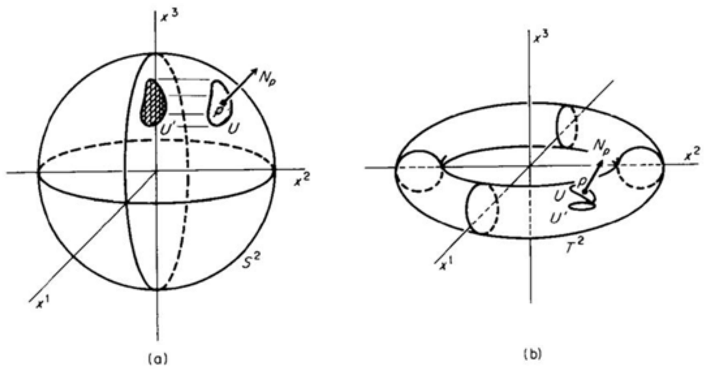
Example 1.1.1.

The simplest examples of manifolds not homeomorphic to open subsets of Euclidean space are the circle S^1 and the 2-sphere S^2, which may be defined to be all points of E^2, or of E^3, respectively, which are a unit distance from a fixed point 0.

There are to be the subspace topology so that (**1**) and (**2**) are immediate. To see that they are locally Euclidean we introduce coordinate axes with 0 as origin in the corresponding ambient Euclidean space. Thus in the case of S^2 we identify R^3 and E^3, and S^2 becomes the unit sphere centered at the origin. At each point p of S^2 we have a tangent plane and a unit normal vector N_p . there will be a coordinate axis which is not perpendicular to N_p

[1] William M.Boothby – "An Introduction to Differential Manifolds and Riemannian Geometry" – Second Edition – Washington University- ST.Lous.Missori- 2003.

and some neighborhood U of p on S^2 will then project in a continuous and one-to-one fashion onto an open set U' of the coordinate plane perpendicular to that axis. In **fig (1.1)** . N_p is not perpendicular to the x_2-axis so for $q \in U$, the projection is given quite explicitly by $\varphi(q) = \left(x^1(q), 0, x^3(q)\right)$, where $\left(x^1(q), x^2(q), x^3(q)\right)$ are the coordinate of q in E^3. Similar considerations show that S^1 is locally Euclidean.



(a) The spherical surface S^2 as a manifold. (b) The torus as a manifold.

Fig(1.1)

Proposition 1.1.1.

The differential of the differentiable mapping h is not (maximum) rank 2 every where[2] .

Proof:

The following Jacobians matrix

$$\begin{pmatrix} \partial_\lambda^x & \partial_\phi^x \\ \partial_\lambda^y & \partial_\phi^y \\ \partial_\lambda^z & \partial_\phi^z \end{pmatrix} = \begin{pmatrix} -\sin\lambda\ \cos\phi & -\cos\lambda\ \sin\phi \\ -\sin\lambda \sin\phi & \cos\lambda\ \cos\phi \\ \cos\lambda & 0 \end{pmatrix}$$

Is of rank 2 on D except at points $(\frac{\pi}{2}, \phi)$ and $(-\frac{\pi}{2}, \phi)$ where it's of rank 1. At every other point there's a reversible 2×2 matrix and the determines are respectively :

[2] "Differential Geometry With Applications to Mechanics and Physics"- Lecture'1'- page '38'- Yves Tapaert- Ouagadougou University - Burkina Faso .

$$\begin{vmatrix} \partial_\lambda^x & \partial_\phi^x \\ \partial_\lambda^y & \partial_\phi^y \end{vmatrix} = -\sin\lambda\ \cos\lambda \qquad \text{(zero if } \lambda = 0)$$

$$-\cos^2\lambda\ \cos\phi \qquad \text{(zero if } \phi = \tfrac{\pi}{2}\ , \tfrac{3\pi}{2})$$

$$\cos^2\lambda\ \sin\phi \qquad \text{(zero if } \phi = 0)$$

Definition 1.1.2 (Chart)

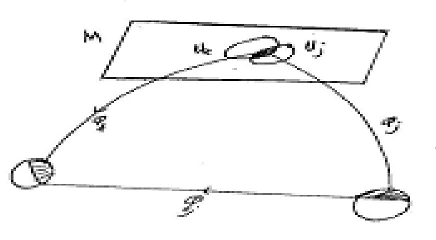
A local chart on M is the pair (U_i, ϕ)consisting of :

1. An open U_i of M

2. A homeomorphic ϕ of U_i onto an open subset $\phi(U_i)$

The open U_i is called domain of the chart .

An arbitrary point of M can belong to two distinct opens, for instance U_i and U_k . the corresponding distinct charts are (U_k, k) and (U_k, ϕ_k). The homeomorphisms ϕ_j and ϕ_k being different we link the opens $\phi_j(U_j)$ and $\phi_k(U_k)$ of F by introducing the following definition. Let us denote the restriction of ϕ_j^{-1} to the open $\phi_k(U_k \cap U_j)$ of F .



Fig(1.2)

Afterwards, the space F will be only R^n . So to each point M is associated a chart (U, ϕ) such that $\varphi(u)$ ia an open of R^n .

Definition 1.1.3

Two (U_j, ϕ_j) and (U_k, ϕ_k) on M, such that $U_j \cap U_j \neq \emptyset$ are called C^q- compatible $(q \geq 1)$ if the over lap mapping $\phi_{kj} = {\phi_k \circ \phi_j^{-1}}/_{U_j \cap U_k}$ is a C^q- defeomorphisms between the open $\phi_j(U_j \cap U_k)$ and $\phi_k(U_j \cap U_k)$ of R^n .

Definition 1.1.4. (Local Coordinates)

The local coordinates x^i of a point P belong to the domain U of a chart (U, ϕ) of M are the coordinates of points $\varphi(P)$ of R^n .

We denoted by $(x^1, \dots \dots \dots, x^n)$, the order n-tuple of real numbers linked to point P.

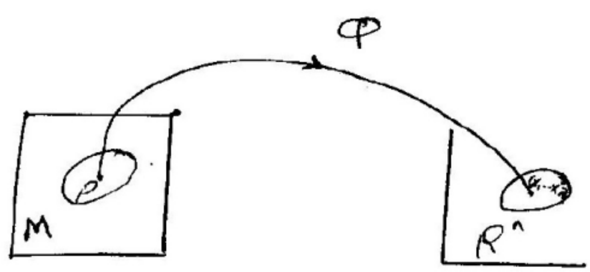


Fig (1.3)

Definition 1.1.5. (Atlas)

An atlas of class C^q on M is a family of charts (U_i, ϕ_i) such that :

1.The domains U_i of charts make up a covering of M.

2. Any charts $(U_i, \phi_i), (U_j, \phi_j)$ of A, with $U_i \cap U_j \neq \phi$, is C^q - compatible.

1.2 Differential Manifold structure

A differential Manifold structure is defined from an atlas representative of it's equivalence class (all the equivalent atlas defining the same differentiable manifold structure).

Definition 1.2.1.

A differentiable manifold structure requires that :

1. The opens of local charts cover M.

2. Two any chartsU_i , ϕ_i , U_j , ϕ_j such that $U_i \cap U_j \neq \phi$, is C^q - compatible

Definition 1.2.2. (Differential Manifolds)

A differentiable or C^∞(or smooth) structure on a topological manifold M is a family $\mathcal{U} = [U_\alpha, \varphi_\alpha]$ of coordinate neighborhoods such that[3]:

1. The U_α cover .

2. For any α, β the neighborhood U_α, φ_α and U_β, φ_β are C^∞-compatible .

3. Any coordinate neighborhood V, ψ compatible with every $U_\alpha, \varphi_\alpha \in \mathcal{U}$ is itself in $\mathcal{U}$.

A C^∞ manifold is a topological manifold together with C^∞-differentiable structure.

Definition 1.2.3.

A differentiable manifold is a pair of Haussdorff space with countable basis and atlas and also it's a manifold if for every point of space there exist an admissible local chart (U, ϕ) such that $(U, \phi) \subset R^n$.

Example 1.2.1.

Sphere S^n , in R^{n+1} let us consider the n- Sphere [4]

$$S^n = \{x = (x^n, \dots \dots, x^{n+1}) | \sum_{i=1}^{n+1} (x^i) = 1\}$$

[3] William M.Boothby – "An Introduction to Differential Manifolds and Riemannian Geometry" – Second Edition – Washington University- ST.Lous.Missori- 2003.

[4] "Differential Geometry With Applications to Mechanics and Physics"- Lecture'1'- page '27'- Yves Tapaert- Ouagadougou University - Burkina Faso .

Answer .

To provide S^n with a differentiable manifold structure we define an atlas consisting of $2n+2$ charts $(1 \leq i \leq n+1)$:

$U_i^+ = \{x \in \ S^n | x^i > 0\}$

$U_i^- = \{x \in \ S^n | x^i < 0\}$

The sphere S^n is really covered with such charts.

Now we must construct transformations between charts (changes of charts) which are C^∞ diffeomorphisms . Let us consider

$\varphi_1^+ : U_i^+ \longrightarrow R^n : x = (x^n, \dots \dots ., x^{n+1}) \longrightarrow x = \left(x^n, \dots \dots, x^{\wedge i}, \dots \dots, x^{n+1}\right)$

Where the symbol ^ means ith coordinates are removed. It is a way the orthogonal projection of the "positive hemisphere" onto the corresponding equatorial "plane". That is really a bicontinuous bijection. Analogically we define

$\varphi_1^- : U_i^- \longrightarrow R^n : x = x \longrightarrow \left(x^n, \dots \dots, x^{\wedge i}, \dots \dots, x^{n+1}\right)$

For instance, let us consider any point x of $U_i^+ \cap U_j^+$ such that the ith and jth coordinates are positive. The following mapping between opens of R^n :

$$\varphi_j^+ \circ (\varphi_i^+)^{-1} : \varphi_i^+\left(U_i^+ \cap U_j^+\right) \longrightarrow \varphi_j^+\left(U_i^+ \cap U_j^+\right) : \left(x^n, \dots \dots, x^{\wedge i}, \dots \dots, x^{n+1}\right) \longrightarrow$$
$$\left(x^1, \dots \dots, x^{j-1}, \sqrt{1 - \sum_{\begin{cases} k=1 \\ k \neq i \end{cases}}^{n+1} (x^k)^2}, \dots \dots ., x^{\wedge j}, \dots \dots, x^{n+1}\right)$$

Is actually a diffeomorphism. A difficulty could have occurred because of the square root but the expression under the radical sign is always positive.

1.3 Submanifold

Definition 1.3.1. A subset N of a C^∞ manifold M is said to have the n- submanifold property if each $p \in N$ has a coordinate neighborhood U, φ on M with local coordinates $x^1, \dots \dots, x^m$ such that :

$\varphi(U) = C_F^m(0)$

$\varphi(U \cap N) = \{x \in C_F^m(0) | x^{n+1} = \cdots = \cdots x^{m=0}\}$

If N has property coordinate neighborhoods of this type[5] are called preferred coordinates (relative to N).

Definition 1.3.2.

A regular submanifold of a C^∞ manifold M is any subspace N with the submanifold property and with the C^∞ structure that the corresponding preferred

coordinate neighborhoods determine on it.

Example 1.3.1.

If $U =]-\frac{\pi}{2}, -\frac{\pi}{2}\ [\times]\ 0{,}2\pi$

$F: U(\subset R^n) \longrightarrow R^3(\lambda, \phi) \longrightarrow (x = \cos\lambda\cos\phi\ , y = \cos\lambda\sin\phi\ , z = \sin\lambda)$

Then $f(U)$ [6]is a two-dimensional submanifold of R^3.

Answer.

From the (Prop1.1.1) ,we have showed the Jacobian matrix of f is

$$\begin{pmatrix} -\sin\lambda\ \cos\phi & -\cos\lambda\ \sin\phi \\ -\sin\lambda\sin\phi & \cos\lambda\ \cos\phi \\ \cos\lambda & 0 \end{pmatrix}$$

And it's of (maximum) rank 2. So ,f is an immersion. And the mapping f is injective because

$\forall(\lambda, \phi), (\lambda', \phi') \in U: f(\lambda, \phi) = f(\lambda', \phi') \Longrightarrow (\lambda, \phi) = (\lambda', \phi')$

[5] William M.Boothby – "An Introduction to Differential Manifolds and Riemannian Geometry" – Second Edition – Washington University- ST.Lous.Missori- 2003.

[6] "Differential Geometry With Applications to Mechanics and Physics"- Lecture'1'- page '27'- Yves Tapaert- Ouagadougou University - Burkina Faso .

Indeed, from $\sin\lambda = \sin\lambda'$ we deduce $\lambda = \lambda'$, from the equalities

$\cos\lambda\ \cos\phi = \cos\lambda\ \cos\phi'$ and $\cos\lambda\ \sin\phi = \cos\lambda\ \sin\phi'$ we deduce $\cos\phi = \cos\phi'$ and $\sin\phi = \sin\phi'$ (because $\cos\lambda \neq 0$) and thus $\phi = \phi'$. the mapping $f^{-1}: f(U) \longrightarrow U$ is contiuous since $\lambda = \sin^{-1} z$ an $\phi = arc\ tg\frac{x}{y}$. Consequently, $f(U)$ is a submanifold of R^3 .

1.4 Lie Groups

1.4.1 Preface

Before we start our topic we will take a quick a glance on Group Theory because it's very important to understand the concepts of Lie Group which it's what we want talk about on these chapter.

Group Theory was first developed by the likes of Karl Friedrich Gauss (1777-1855) and Evariste Galois (1811-1832) as a means of studying the abstract objects called groups.

1.4.2 Basic Principles

i. Group

A group denoted G is quite simply defined as a collection (set) of objects (called group elements) that can be combined by a binary operation (called group product and denoted by $\odot$) that satisfy some basic properties:

Closure: $if\ a, b \in G \Rightarrow a \odot b = c \in G$

Identity: $\exists e \in G\ such\ that\ g \odot e = e \odot g = g\ , \forall g \in G$

Inverse: $\exists g^{-1} \in G\ s.t.\ g \odot g^{-1} = g^{-1} \odot g = e$

Associativity: $a \odot (b \odot c) = (a \odot b) \odot c$

Note that we require associativity but not commutativity. If commutativity is obeyed, the group thus formed is called "abelian" .

ii. Sub group

A subgroup denoted as H is merely a subset of a group that satisfies the group rules above

$G \supset H \Leftrightarrow \forall h \in H \text{ , } h \in G$ and H is group

iii. Direct Product

A group G is said to be the direct product of it's subgroups $H_1, H_2, H_3, \dots \dots \dots, H_n$ if :

The elements of different subgroups commute,

$h_i \in H_i$, $h_j \in H_j$ and $i \neq j \Rightarrow h_i \odot h_j = h_j \odot h_i$

Every element $g \in G$ can be uniquely expressed as

$g = h_1 \odot h_2 \odot h_3 \dots ..$ where $h_i \in H_i \ \forall i$

This 'direct product structure' of a group denoted as:

$G = H_1 \times H_2 \times H_3 \times \dots \dots \dots \times H_n$

iv. Continuous groups

There are groups where in the elements cannot be enumerated. Rather , the elements are generated via the continuous variation of one or more parameters (which is really define the group) .by example the set of all rotations in two dimensions. It can be shown that this set obeys the group rules in eq.1. this is then just the group of all possible rotations in the plane, that leave the length of the rotated vector invariant.

From (**Fig.1.1**). we can deduce the relationship between the components of the initial vector $\vec{x}_2$.

$x_2 = x_1 \cos\theta - y_1 \sin\theta$

$y_2 = x_1 \sin\theta + y_1 \cos\theta$

Expressing eq. in matrix form, we get

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

So all the elements of this group are of the form**:** $U(\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Two properties of this general transformation are immediately deducible. All matrices $U(\theta)$ are orthogonal i.e. $U^T U = 1$ and have unit determinant i.e.$|U(\theta)| = 1$.

The latter property enforces the invariance of the length of the vector. The group is therefore called 'Special Orthogonal group in two dimension'- SO(2). As it turn out, this group is just one of a whole class of continuous groups called Lie groups.

Definition 1.4.3. (Topological groups)

A topological group G is a topological space[7] which is a group and has the properties that the group operations are contiuous .

Lemma 1.4.1.

Let G be a connected topological group . suppose H is an abstract open subgroup of G . then $= G$.

Proof:

For any $a \in G, L_a: G \longrightarrow G$ given by $g \mapsto ag$ is homeomorphism. Thus , for each $a \in G$, $aH \subseteq G$ is open . since the cosets partition G , and G is connected , we must have $|G/H| = 1$.

Lemma 1.4.2.

Let G be a connected topological group, $U \subseteq G$ a neighborhood of 1. Then U is generates G.

Proof:

For a subset $W \subseteq G$, write $W^{-1} = \{g^{-1} \in G | g \in W\}$. Also, if k is a positive integer , we set $W^k = \{a_1, \dots, a_k | a_i \in W\}$. Let U be as above, and $= U \cap U^{-1}$. then, V is open and $v \in V$ implies that $v^{-1} \in V$. let $= \bigcup_{n=1}^{\infty} V^n$. then H is subgroup and we claim that H is

[7] Notes on Lie Groups – Eugene Lerman – February 15, 2012

open. Notice, that H is precisely the subgroup generated by U, so unless H is open , then $H = G$ and Lemma is proved.

If V^k is open , then $V^{k+1} = \bigcup_{a\in V}(aV^k)$ is open since left multiplication is a homeomorphism.

Definition 1.4.4. (Lie groups)

A Lie group G is a C^∞ manifold with a group structure so that the group operations are smooth. More precisely,the maps

$m: G \times G \to G$ (**multiplication**)

$inv: (g_1, g_2) \mapsto g_1 g_2^{-1}$ (**inversion**)

Are C^∞ maps of manifolds . The dimension of the group G is the dimension of the manifold.

Recall that a d-dimensional manifold is a generalisation of the notion of a surface. We will encounter them in two types: The more intuitive concept is a submanifold, which is a subset of R^m specified by some constraint equations. (One can also define complex manifolds by replacing R with C and "differentiable" with "holomorphic" in the following. However, we will only consider real manifolds, i.e. groups with real parameters.

Otherwise we will make no distinction between real or complex functions, matrices etc.

Example 1.4.1.

$\mathbb{R}$ and $\mathbb{C}$ are evidently Lie groups under addition. (more generally any finite dimensional real or complex vector space is a Lie group under addition).

Example 1.4.2.

$\mathbb{R}/\{0\}$, $\mathbb{R} > 0$, and $\mathbb{C}/\{0\}$ are all Lie groups under multiplication. Also,

$U(1) = \{z \in \mathbb{C}: |z| = 1\}$ is a Lie group under multiplication.

Example 1.4.3.

If G and H is Lie groups then the product $G \times H$ is a Lie group with the evident product structures. In view of (1.4.1) and (1.4.2) we conclude that for $n \in \mathbb{N}$ the torus $\mathbb{T}^n = U(1)^n$ is a Lie group. More generally, for $m, n \in \mathbb{N}$ we have a Lie group $\mathbb{R}^m \times \mathbb{T}^n$.

Example 1.4.4.

The fundamental example of a Lie group is the group $GL(n,\mathbb{R})$,Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices over $\mathbb{R}$ [8]. define

$$GL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) |\ detA \neq 0\}$$

Then, $GL(n,\mathbb{R})$ is a group under the operations $m(A,B) = AB$ and $inv(A) = A^{-1} = \frac{adjA}{\det A}$ where $adj\ A$ denotes the adjugate of A .As these operations are smooth on $GL(n,\mathbb{R})$ considered as a submanifold of $\mathbb{R}^{n^2}$, $GL(n,\mathbb{R})$ is a Lie group called the real general linear group. Completely analogously, we have the Lie group

$$GL(n,\mathbb{C}) = \{A = M_n(\mathbb{C}) | detA \neq 0\}$$

The complex general linear group.

And also,The orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = 1\}$ is a Lie group as a subgroup and submanifold of $GL(n,\mathbb{R})$.

Example 1.4.5.

The following are examples [9]of Lie groups

1. $\mathbb{R}^n$ with the group operation given by addition

2. $(\mathbb{R}^n, \times)$ and $(\mathbb{R}^+, \times)$

3. $S^1 = \{z \in \mathbb{C} : \ |z| = 1\}$,$\times$

[8] Notes on Lie Groups – Eugene Lerman –February 15,2012.

[9] Introduction to Lie Groups and Lie Algebras – Alexander Kirillov,Jr. – Departement of mathematics, Suny At stony Brook,NY 11794, USA.

4. $GL(n,\mathbb{R}) \subset \mathbb{R}^{n^2}$. Many of the groups we will consider will be subgroups of $GL(n,\mathbb{R})$or $(n,\mathbb{C})$.

5. $SU(2) = \{A \in GL(2,\mathbb{C}) |\, A\bar{A}^t = 1, \det A = 1\}$. indeed, one can easily see that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Writing $= x_1 + ix_2$, $\beta = x_3 + ix_4$, $x_i \in \mathbb{R}$, we see that $SU(2)$ is diffeomorfic to $S^3 = \{x_1^2 + \cdots \ldots \ldots + x_4^2\}) \subset \mathbb{R}^4$.

Recall that a space is simply connected if every closed curve (a loop)[10] can be contracted to a point. Clearly , this is not true for a curve that wraps a round S^1.

A general (topological) space is compact if each open cover contains a finite cover. This is a rather abstract (through important) notion. Luckily , for subsets of $\mathbb{R}^n$, there is a simpler criterion: they are if and only if they are closed and bounded.

Clearly, $SO(2)$ and $SU(2)$ are compact (note that we didn't need to introduce parameters for $SU(2)$ to see this). A non-compact example would be $SO(1,1)$, the Lorentz group in two dimensions. It is defined as the group of linear transformations of $\mathbb{R}^2$ which leave the indefinite inner product

$$\langle \vec{v}, \vec{u} \rangle = v_1 u_1 - v_2 u_2$$

Invariant, and have determinant one. It can be written similarly to $SO(2)$ as

$$\Lambda = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a^2 - b^2 = 1$$

And parameterised by $\chi \in \mathbb{R}$ as

$$\Lambda(\chi) = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix}$$

Hence, as a manifold, $SO(1,1) \cong \mathbb{R}$. Actually, since $\Lambda(\chi)\Lambda(\xi) = \Lambda(\chi + \xi)$, this isomorphism hold for the groups as well.

[10] Group Theory (for Physicists)- Christoph L¨udeling - August, 16, 2010.

Theorem 1.4.1 (close subgroup)

Let G be a Lie group and $H < G$ a closed subgroup of [11]. Then, H is a Lie group in the induce topology as an embedded submanifold of G. As a direct consequence we have

Corollary 1.4.1.

If G and $G^{'}$ are Lie groups and $\phi: G \longrightarrow G'$ is a continuous homomorphism, then ϕ is smooth . From the Closed sub group Theorem we can generate quite a few more examples of Lie groups.

Example 1.4.6.

- The real special linear group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | detA = 1\}$
- The complex special linear group $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) | detA = 1\}$
- The special orthogonal group $SO(n, \mathbb{R}) = SL(n, \mathbb{R}) \cap O(n)$
- The unitary group $U(n = \{A \in GL(n, \mathbb{C}) | AA^* = 1\}$

  (where A^* denotes the Hermitian transpose of A)

- The special unitary group $SU(n) = U(n) \cap SL(n, \mathbb{C})$

Example 1.4.7.

Define the Euclidean group of rigid motions, $Euc(n)$.

Let $End(V, W)$ denote the vector space of all linear endomorphisms from a vector space V to it self .As a set, we have

$Euc(n) = \{T \in End(\mathbb{R}^n) | \|T_x - T_y\| = \|x - y\| \forall x, y \in \mathbb{R}^n \ \}$

Where $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$. Now, one can check that if $T \in Euc(n)$ and $T(0) = 0$, then $T \in O(n)$. Then, we can write $x \longmapsto T_x - T(0) \in O(n)$ and so $T(x) = \big(T(x) - T(0)\big) + T(0)$. This shows that $T \in \mathbb{R}^n \times O(n)$.

[11] Note on Lie group- Eugen Lerman – February, 15 ,2012

We can think of $Euc(n)$ as a slightly different set. Write

$$Euc(n) = \left\{ \begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix} \middle| A \in O(n), v \in \mathbb{R}^n \right\}$$

If we identify $\mathbb{R}^n$ with the set of all vectors of the form $\begin{bmatrix} w \\ 1 \end{bmatrix}$ with $w \in \mathbb{R}^n$, then we have

$$\begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix} = \begin{bmatrix} Aw + v \\ 1 \end{bmatrix}$$

1.5 Some Differential Geometry

Since Lie groups are analytic manifolds[12], we can apply the apparatus of differential geometry . in particular, it will turn out that almost all information about the Lie group is contained in it's tangent space at the identity, the Lie algebra.

Intuitively, the tangent space is just that : the space of all tangent vectors,i.e. all possible "directions" at a given point. When considering submanifolds, the tangent space can visualized as a plane touching the manifold at the point g, see Fig (1.5) .

Mathematically, the notion of a tangent vector is formalized as a differential opereator , this makes intuitive sense since a tangent vector corresponds to "going" into a particular direction with a certain "speed", i.e. the length of the vector, you notice that you move because things around you change. Hence it is reasonable that tangent vectors measure changes,i.e. they are derivatives.

We introduce a bit of machinery: A curve is a differentiable map $k: \mathbb{R} \supset I \longrightarrow G$,

Where I is some open interval. (note that the map itself is the curve, not just the image).

Definition 1.5.1. (Tangent Vector)

Let $k: (-\varepsilon, \varepsilon) \longrightarrow G$ be a curve with $k(0) = g$. The tangent vector of k in g is the operator that maps each differentiable function $f: G \longrightarrow \mathbb{K}$ to it's directional derivative along , $X: f \longmapsto X[f] = \frac{d}{dt} f(k(t)) \big|_{t=0}$

[12] Group Theory (for Physicists)- Christoph L¨udeling - August, 16, 2010.

The set of all tangent vectors in g is called the tangent space of G in g, $T_g G$. This is naturally a vector space: for two tangent vectors X and Y and a real number λ, define the sum and multiple by

$$(X+Y)[f] = X[f] + Y[f]$$

$$(X\lambda)[f] = \lambda X[f]$$

One can find curves that realise the vectors on the right-hand side, but we only care about the vectors.

Tangent vectors are defined independently of coordinates. Practically, one often needs to calculate a tangent vector in agiven coordinate system,i.e. a particular map ϕ_i. then we have

$$X[f] = \frac{d}{dt}(f \circ k(t)) \Big|_{t=0} = \frac{d}{dt}(f \circ \phi_i^{-1} \circ \phi_i \circ k(t)) \Big|_{t=0}$$
$$= d(f \circ \phi_i^{-1})\big|_g \cdot \tfrac{d}{dt} f(k(t)) \Big|_{t=0}$$

Even more practically: if the elements of V_i,i.e. the coordinates around g, are given by x^a, then it is a common abuse of notation to write the curve as $\phi\big(k(t)\big) = x^a(t)$ and the function $(f \circ \phi_i^{-1})(x^a) = f(\phi_i^{-1}(x))$ as $f(x)$. Thus we get

$$X[f] = \frac{\partial}{\partial x^a} f(x) \cdot \frac{d}{dt} x^a(t)$$

Here we again use the summation convention : an index that appears twice (the a) is summed over. The nice thing about this way of writing the tangent vectoris that we have separated the f-dependent and k-dependent pieces, and we can even write the tangent vector without referring to f as the differential operator

$$X = \tfrac{d}{dt} x^a(t) \Big|_{t=0} \cdot \tfrac{\partial}{\partial x^a} = X^a \partial_a$$

Hence, the partial derivatives along the coordinate directions provide a basis for the tangent space at any given point, called the coordinate basis. Clearly, the dimension of the tangent space is equal to the dimension of the manifold. The X^a are called the components of X. This way of writing a vector comes at the price of introducing a coordinates system, and the components of the vector will depend on the chosen coordinates (as it shouldbe: components depend on the basis). However, so do the partial derivatives, and the vector itself is entirely independent of the coordinates. Hence one often speaks of "the vector X^{a}" .

Definition 1.5.2. (Tangent Space at the identity)

We define the tangent space $T_P(M)$ to M at P to be the set of all mappings $X_P: C^{\infty} \longrightarrow R$ satisfying for all $\alpha, \beta \in R$ and $f, g \in C^{\infty}(P)$ [13] the two conditions:

1. $X_P(\alpha f + \beta g) = \alpha(X_P f) + \beta(X_P g)$ (**Linearity**)

2. $X_P(\alpha f) = (X_P f)g(P) + f(P)(X_P g)$ (**Leibniz rule**)

With the vector space operations in $T_P(M)$ defined by

1. $(X_P + Y_P)f = X_P f + Y_P f$

2. $(\alpha X_P)f = \alpha(X_P f)$

A tangent vector to M at P is any $X_P \in T_P(M)$.

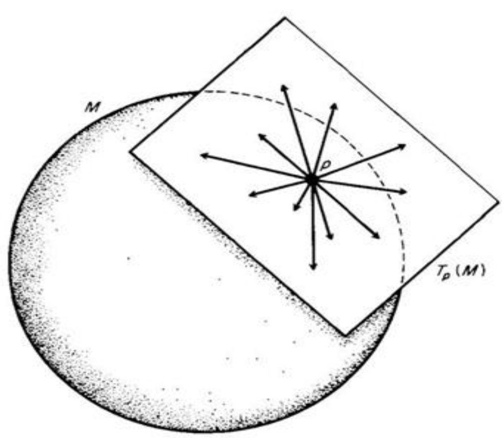


Fig (1.4)

[13] - William M.Boothby – "An Introduction to Differential Manifolds and Riemannian Geometry" – Second Edition – Washington University- ST.Lous.Missori- 2003.

Theorem 1.5.1.

Let $F: M \longrightarrow N$ be a C^{∞} map of manifolds . Then for $P \in M$ the map $F^{*}: C^{\infty}(F(P)) \longrightarrow C^{\infty}(P)$ defined by $F^{*}(f) = f \circ F$ is a homeomorphism of algebras and induces a dual vector space homeomorphism $F_{*}: T_P(M) \longrightarrow T_{F(P)}(N)$. define by $F_{*}(X_P)f = X_P(F^{*}f)$. Which gives $F_{*}(X_P)$ as a map of $C^{\infty}(F(P))$ to . when $F: M \longrightarrow M$ is the identity , both F^{*} and F_{*} are the identity isomorphism . if $H = G \circ F$ is a composition of C^{∞} maps . then $H^{*} = F^{*} \circ G^{*}$ and $H_{*} = G_{*} \circ F_{*}$.

Proof:

The proof consists of routinely checking the statements against definitions . We omit the verification that F^{*} is a homomorphism amd consider F_{*} only . Let $X_P \in T_P(M)$ and

$g \in C^{\infty}(F(P))$; we must prove that the map $F_{*}(X_P): C^{\infty}(F(P)) \longrightarrow R$ is a vector at $F(P)$, that is a linear map satisfying the Leibniz rule . We have

$$F_{*}(X_P)(fg) = X_P F^{*}(fg) = X_P[(f \circ F)(g \circ F)]$$

$$= X_P(f \circ F)g(F(P)) + f(F(P))X_P(g \circ F).$$

And so we obtain

$$F_{*}(X_P)(fg) = (F_{*}(X_P)f)g(F(P)) + f(F(P))F_{*}(X_P)g$$

(linearity is even simpler) .Thus $F_{*}: T_P(M) \longrightarrow T_{F(P)}(N)$. Further , F_{*} is a homomorphism .

$$F_{*}(\alpha X_P + \beta Y_P)f = (\alpha X_P + \beta Y_P)(f \circ F) = \alpha X_P(f \circ F) + \beta Y_P(f \circ F)$$

$$= \alpha F_{*}(X_P)f + \beta F_{*}(Y_P)f$$

$$= [\ \alpha F_{*}(X_P) + \beta F_{*}(Y_P)]f\ .$$

Note: the homomorphism $F_{*}: T_P(M) \longrightarrow T_{F(P)}(M)$ is often called the differential of F.

Corollary 1.5.1.

If $F: M \longrightarrow N$ is a diffeomorphism of M onto an open set $U \subset N$ and $P \in M$, then

$F_{*}: T_P(M) \longrightarrow T_{F(P)}(N)$ is an isomorphism onto.

This follow at once from the last statement of the theorem and the note after (Def **1.5.2**) if we suppose G is inverse to F . Then both $G_* \circ F_*: T_P(M) \longrightarrow T_P(M)$ and

$F_* \circ G_*: T_{F(P)}(N) \longrightarrow T_{F(P)}(N)$ are the identity isomorphism on the corresponding vector space .

Remembering that any open subset of a manifold is a submanifold of the same dimension. We see that if U, φ is a coordinate neighborhood on M, then the coordinate map φ induces an isomorphism $\varphi_*: T_P(M) \longrightarrow T_{\varphi(P)}(R^n)$ of the tandent space at each point $P \in U$ onto

$T_a(R^n) = \varphi(P)$. The map φ_*^{-1} , on the other hand , maps $T_a(R^n)$ isomorphically onto $T_P(M)$

The images $E_{iP} = \varphi_*^{-1}\left(\partial/_{\partial x^i}\right)$, $i = 1, \dots \dots \dots, n$. of the natural basis $\partial/_{\partial x^1}, \dots., \partial/_{\partial x^n}$ at each $a \in \varphi(U) \subset R^n$ determine at $P = \varphi^{-1}(a) \in M$ a basis $E_{1P}, \dots \dots, E_{aP}$ of $T_P(M)$; we call bases the coordinate frames .

Corollary 1.5.2.

To each coordinate neighborhood U on M there corresponds a natural basis $E_{1P}, \dots \dots, E_{nP}$ of $T_P(M)$ for every $P \in U$; in particular, $T_P(M) = \dim M$. Let f be a C^∞ function defined in a neighborhood of P , and $\hat{f} = f \circ \varphi^{-1}$ it's expression in local coordinates relative to $, \varphi$. then $E_{iP} f = \left(\partial y^j / \partial x^i\right)_{\varphi(P)}$. In particular , if $x^i(q)$ is the ith coordinate function , $X_P\, X^i$ is the ith component of X_P in this basis, that is , $X_P = \sum_{i=1}^{n}\left(X_P\, x^i\right) E_{iP}$.

The last statement of the corollary is a restatement of the definition in (Theorem**1.5.1**) for $E_{iP} = \varphi_*^{-1}\left(\partial/_{\partial x^i}\right)$, namely ,

$$E_{iP} f = \left(\varphi_*^{-1}\left(\partial/_{\partial x^i}\right)\right) f = \frac{\partial}{\partial x^i}\left(f \circ \varphi^{-1}\right)\Big|_{v=\varphi(P)}$$

If we take f to be the ith coordinate function. $f(q) = x^i(q)$ and $X_P = \sum \alpha^i E_{iP}$, then

$$X_P\; x^i = \sum_i \alpha^i (E_{iP}\; x^i) = \sum_i \alpha^i \left(\frac{\partial x^i}{\partial x^i}\right)_{\varphi(P)} = \alpha$$

We may use this to derive a standard formula which gives the matrix of the linear map F_* relative to local coordinate systems. Let $F: M \longrightarrow N$ be a smooth map. And let U, φ and V, ψ be coordinate neighborhoods on M and N with $F(U) \subset V$. Suppose that in these local coordinates F is given by $y^i = f^i(x^1, \dots\dots\dots, x^n)$, $\qquad i = 1, \dots\dots\dots, m$

And that P is a point with coordinates $= (a^1, \dots\dots\dots, a^n)$. then $F(P)$ has y coordinates determined by these functions. Further let ${\partial y^i}/{\partial x^i}$ denote ${\partial f^i}/{\partial x^i}$.

Theorem 1.5.2.

Let $E_{iP} = \varphi_*^{-1}\left({\partial}/{\partial x^i}\right)$ and $\tilde{E}_{F(P)} = \psi_*^{-1}\left({\partial}/{\partial y^i}\right)$, $i = 1, \dots\dots\dots, n$ and $= 1, \dots\dots\dots, m$. be the basis of $T_P(M)$ and $T_{F(P)}(N)$. respectively, determined by the given coordinate neighborhoods. Then

$$F_*(E_{iP}) = \sum_{j=1}^{m} \left(\frac{\partial y^i}{\partial x^i}\right)_a \tilde{E}_{F(P)} \ . \qquad i = 1, \dots\dots\dots, n$$

In terms of components, if $X_P = \sum \alpha^i E_{iP}$ maps to $F_*(X_P) = \sum \beta^j \ \tilde{E}_{F(P)}$. then we have

$$\beta^j = \sum_{j=1}^{m} \alpha^i \left(\frac{\partial y^i}{\partial x^i}\right)_a . \qquad j = 1, \dots\dots\dots, m$$

The partial derivatives in these formulas are evaluated of $P: a = (a^1, \dots\dots\dots, a^n) = \varphi(P)$.

Proof:

We have $F_*(E_{iP}) = F_* \circ \varphi_*^{-1}\left({\partial}/{\partial x^i}\right)_{\varphi(P)}$ and according to (Corollary **1.5.2**) , to compute its components relative to $\tilde{E}_{jF(P)}$, we must apply this vector as an operator on $C^\infty\big(F(P)\big)$ to the coordinate functions y^j

$$F_*(E_{iP})\, y^j = \left(F_* \circ \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)\right) y^j = \frac{\partial}{\partial x^i}\, y^j\, (F \circ \varphi_*^{-1})(x) = \frac{\partial f^j}{\partial x^i} \ .$$

These derivatives being evaluated at the coordinates of P . that is , at $\varphi(P)$; they could be also written $\left({\partial y^j}/{\partial x^i}\right)_{\varphi(P)}$.

Example 1.5.1.

Suppose M to be a two-dimensional submanifold of R^3, that is a surface. Let W be an open subset, say a rectangle in the (u, v)- plane R^2 and $\theta: W \longrightarrow R^3$ a parameterization of a portion of M (Fig 1.6) .

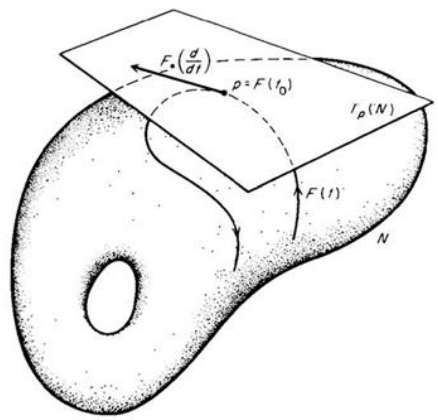


Fig (1.5)

Namely, suppose θ ia an imbedding whose image is an open subset V of ; V, θ^{-1} is a coordinate neighborhood on M . suppose $\theta(u_0, v_0) = (x_0, y_0, z_0)$, where we now use (x, y, z) as the natural coordinates in R^3. We may assume that θ is given by coordinate functions

$$x = f(u, v)\ , \qquad y = g(u, v) \qquad z = h(u, v) \ .$$

Since θ is imbedding , the jacobian matrix $\partial(f, g, h)/\partial(u, v)$ has rank 2 at each point of W. We consider the image of the basis vectors $\partial/_{\partial u}$ and $\partial/_{\partial v}$ at (u_0, v_0) . we denote these by $(X_u)_0$ and $(X_v)_0$. according to the first formula of previous (Theorem**1.5.2**) . they are given by

$$(X_u)_0 = \theta_*\left(\partial/_{\partial u}\right) = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}$$

$$(X_v)_0 = \theta_*\left(\partial/_{\partial v}\right) = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}$$

Where we have written $\partial x/_{\partial u}$, $\partial x/_{\partial v}$ for $\partial f/_{\partial u}$, $\partial f/_{\partial v}$ and so on.. these derivatives being

evaluated at (u_0, v_0) since θ has rank 2, these are linearty independent vectors. And they span a two-dimensional subspace of $T_{(x_0, y_0, z_0)}(R^3)$. this subspace is what we have by our identification then we use the tangent at this point (x_0, y_0, z_0) : it consists of all vectors of the form $\alpha\theta_*\left(\partial/_{\partial u}\right) + \beta\theta_*\left(\partial/_{\partial v}\right) = \alpha(X_u)_0 + \beta(X_v)_0$, $\alpha, \beta \in R$; their initial point of course always is (x_0, y_0, z_0) . it is easily to seen that this subspace is the usual tangent plane to a surface , as we would naturally expect it to be. We use one of standard descriptions of the tangent plane at a point P of surface M in R^3 ; the collection of all tangent vectors at P to curves through P which lie on M. In fact let I an open interval about $t = t_0$ and let us consider a curve on M through (x_0, y_0, z_0) . It is no loss of generality to suppose the curve given by $F: I \longrightarrow W$ composed with $: W \longrightarrow R^3$; thus u, v are functions of t with $u(t_0) = u_0$ and $v(t_0) = v_0$ and the curve is given by

$$\theta\big(f(t)\big) = \Big(x\big(u(t), v(t)\big), y\big(u(t), v(t)\big), z\big(u(t), v(t)\big)\Big)$$

The tangent to the curve at (x_0, y_0, z_0) is given by

$$(\theta \circ F)_*\left(\frac{d}{dt}\right) = \dot{x}(t_0)\frac{\partial}{dx} + \dot{y}(t_0)\frac{\partial}{\partial y} + \dot{z}(t_0)\frac{\partial}{\partial z}$$

Where

$$\dot{x}(t_0) = \left(\frac{dx}{dt}\right)_{t_0} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$$

Evaluated at (x_0, y_0, z_0) and $t = t_0$. Substituting and collecting terms, we have

$$(\theta \circ F)_*\left(\frac{d}{dt}\right) = \frac{du}{dt}\left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}\right)$$

$$= \frac{du}{dt}\,\theta_*\left(\frac{\partial}{\partial u}\right) + \frac{\partial v}{\partial t}\,\theta_*\left(\frac{\partial}{\partial v}\right)$$

$$= \dot{u}(t_0)(X_u)_0 + \dot{v}(t_0)(X_v)_0 \;.$$

If we let $= (t - t_0) + u_0$, $v = v_0$, we obtain just $(X_u)_0 = \theta_*\left(\frac{\partial}{\partial u}\right)$ and analogously $(X_v)_0$ is tangent to the parameter curve $u = u_0$, $v = (t - t_0) + v_0$. the coordinate frame vectors are tangent to the coordinate curves .

Definition 1.5.3. (Vector Fields)

A vector field is a map that associates a vector $X(g) \in T_g G$ to each point $\in G$.

In a given map we can choose the coordinate basis and write the components as functions of the coordinates,i.e.

$$X = X^a(x)\partial_a$$

Clearly, the vector fields form a vector space (over $\mathbb{R}$) themselves[14]. In contrast to the tangent space at a point, which is a d -dimensional vector space, however, the space of vector fields is infinite-dimensional, since every component is a function on$G$.

The vector fields do not only form a vector space, but an algebra. However, one cannot simply act with one vector field on another because the result would not be a first-order differential operator, so the product will be more sophisticated.

Definition 1.5.4.

Given two vector fields X and Y, the Lie bracket is a vector field given by

$$[X,Y][f] = X\big[Y[f]\big] - Y\big[X[f]\big]$$

This is a reflection of the fact that derivatives on manifolds are not directly straight forward. There are tangent vectors, which a priori can only act on (scalar) functions on the manifold, but not on vectors. The Lie bracket allows to extend the action to vector fields. The Lie bracket is thus sometimes called a Lie derivative,

$$\mathcal{L}_X Y = [X,Y]$$

This is not any more truly a directional derivative as it was for functions: it depends not only on X at the point.

To see this, observe that for any function $f: G \longrightarrow \mathbb{K}$, we can define a new vector field $X' = fX$. Assume that$f(g) = 1$, so that $X'|_g = X|_g$. Then one could expect that at g also the derivatives coincide, but actually we have

$\mathcal{L}_{X'} Y = f\mathcal{L}_X Y - Y[f]X$, and the second term doesnot vanish in general.

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[14] Group Theory (for Physicists)- Christoph L¨udeling - August, 16, 2010 .

Definition 1.5.5.

A vector field X of class C^r on M is a function assigning to each point P of M [15]a vector

$X_P \in T_P(M)$ whose components in the frames of any local coordinates U, φ are functions of class C^r on the domain U of the coordinates . Unless otherwise noted we will use vector field to mean C^∞-vector field .

Example 1.5.2.

If we consider $M = R^3 - \{0\}$, then the gravitational field of an object of unit mass at 0 is a C^∞-vector field whose components $\alpha^1, \alpha^2, \alpha^3$ relative to the basis $\partial/\partial x^1 = E_1$, $\partial/\partial x^2 = E_2$, $\partial/\partial x^3 = E_3$ are

$$\alpha^i = -\frac{x^i}{r^3}, \quad i = 1,2,3 \qquad \text{with} \quad r = ((x^1)^2 + (x^2)^2 + (x^2)^3)^2 .$$

Definition 1.5.6.

Let $F: M \longrightarrow N$ a vector field Y on M such that for each $q \in M$ and $P \in F^{-1}(q) \subset N$ we have $F_*(X_P) = Y_q$, then we say that the vector fields X on Y are F-related and we write, briefly, $Y = F_*(X)$.

[We do not require F to be onto: if $F^{-1}(q)$ is empty ,then the condition is vacuously satisfied]

Theorem 1.5.3.

If $F: M \longrightarrow N$ is a diffeomorphism **,** then each vector field X on N is F-related to a uniquely determined vector field Y on M .

Proof:

Since F is diffeomorphism . it has an inverse $G: N \longrightarrow M$, and at each point P we have $F_*: T_P(N) \longrightarrow T_{\varphi(P)}(M)$ is an isomorphism onto G_* an inverse . Thus given a C^∞ -vector field X on N, then at each point q on M. The vector $Y_q = F_*\left(X_{G(q)}\right)$ is uniquely determined . it then remains to check that Y is a C^∞ -vector field . this is immediate if we introduce local coordinates and apply (Theorem **1.5.2**) to the component functions .

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[15] William M.Boothby – "An Introduction to Differential Manifolds and Riemannian Geometry" – Second Edition – Washington University- ST.Lous.Missori- 2003.

Definition 1.5.7.

If $F: M \longrightarrow N$ is a diffeomorphism and X is a C^∞ vector field on M such that $F_*(X) = X$. That is X is F-related to itself, then X is said to be invariant with respect to , or F- invariant .

Theorem 1.5.4.

Let G be a Lie group and $T_e(G)$ the tangent space at the identity , then each $X_e \in T_e(G)$ determine uniquely a C^∞ vector field of X on G which is invariant under left translations .In particular, G is parallelizable .

Proof:

To each $g \in G$ there corresponds exactly one left translation L_g taking e to g. Therefore if it exists. X is uniquely determined by the formula : $X_g = L_{g*}(X_e)$. Except for differentiability, this formula does define a left invariant vector field for $a \in G$, we have

$$L_{g*}(X_g) = L_{a*} \circ L_{g*}(X_e) = L_{ag*}(X_e) = X_{ag} \quad .$$

We must show that , so determined is C^∞ . Let U, φ be a coordinate neighborhood of e such that $\varphi(e) = (0, \dots \dots \dots ..,0)$ and let V be a neighborhood of e satisfying $V \subset U$. let $g, h \in V$ with coordinates $x = (x^1, \dots \dots .., x^n)$ and $y = (y^1, \dots \dots .., y^n)$. respectively, and let $z = (z^1, \dots \dots .., z^n)$ be the coordinates of the product gh . Then, $z^i = f^i(x, y)$, $i = 1, \dots \dots, n$ are C^∞ functions on $\varphi(V) \times \varphi(V)$. if we write $X_e = \sum_{i=1}^n \gamma^i E_{ie}$, $\gamma^1, \dots \dots \dots, \gamma^n$ real numbers, then according to (Theorem 1.5.2) the formula above for X_g becomes

$$X_g = L_{g*}(X_e) = \sum \gamma^j \left(\frac{\partial f^i}{\partial y^j}\right)_{(x,0)} E_{ig}$$

Since in local coordinates L_g is given by $z^i = f^i(x, y)$, $i = 1, \dots \dots, n$. With the coordinates x of g fixed . It follows that on V the components of X_g in the coordinate frames are C^∞ functions of the local coordinates. However , for any $a \in G$ the open set a V is the diffeomorphic image by L_a of V. Moreover X, as just shown , is L_g-invariant so that for every $g = ah \in aV$ we have $X_g = L_{a*}(X_h)$. It follows that X on aV is L_g-related to X on V and therefore X is C^∞ on aV by (Theorem **1.5.3**) . Since X is C^∞ in a neighborhood of each element of . it is C^∞ on G .

Corollary 1.5.3.

Let G_1 and G_2 be Lie groups and $F: G_1 \longrightarrow G_2$ a homomorphism. Then to each left –invariant vector field X on G_1 there is a uniquely determined left –invariant vector field Y on G_2 which is F-related to .

Proof:

By (Theorem **1.5.4**) , X is determined by X_e , it's value at the identity e_1 of G_1 . Let $e_2 = F(e_1)$ be the identity of G_2 and let Y be the uniquely determined left –invariant vector field on G_2 such that $Y_{e_2} = F_*(X_{e_1})$. That Y should have this value at e_2 is surely a necessary condition for Y to be F-related to X; and it remains only to see whether this vector field Y satisfies $F_*(X_{e_1}) = Y_{F(g)}$ for every $g \in G_1$. If so , Y is indeed F-related (and uniquely determined) . We write the mapping F as a composition $F = L_{F(g)} \circ F \circ L_g$. Using $F(x) = F(g)F(g^{-1}x)$ .and note that since both X on Y are left-invariant by assumption. This gives

$$F(X_e) = L_{F(g)*} \circ F_* \circ L_{(g)} * (X_g)$$

$$F(X_e) = L_{F(g)*} \circ F_*(X_e) = L_{F(g)} Y_e$$

Therefore Y meets all conditions and the corollary is true .

1.6 Action of groups

Important examples of group action are the following actions of G on itself [16] :

Left action: $L_g: G \to G$ is defined by $L_g(h) = gh$

Right action: $R_g: G \to G$ is defined by $R_g(h) = hg^{-1}$

Adjoint action: $Ad_g: G \to G$ is defined by $Ad_g(h) = ghg^{-1}$

Easily sees that left and right actions are transitive; in fact, each of them is simply transitive. It is also easy to see that the left and right actions commute and that $Ad_g = L_g R_g$.

[16]Introduction to Lie Groups and Lie Algebras – Alexander Kirillov,Jr. – Departement of mathematics, Suny At stony Brook,NY 11794, USA.

Each of these actions also defines the action of G on the spaces of functions, vector fields, forms,etc. on G.

Definition 1.6.1. (Translation)

A vector field $v \in Vect(G)$ is left-invariant if $g.v = v$ for every $g \in G$, and right-invariant if $v.g = v$ for every $g \in G$. A vector field is called bi-invariant if it is both left-and-right- invariant.

Let (G, μ, v, e) be a Lie group[17]. For any element $g \in G$ we can consider the left translation $\lambda_g: G \longrightarrow G$ defined by $\lambda_e(h) = gh = \mu(g, h)$. Smoothness of μ immediately implies that λ_g is smooth and $\lambda_g \circ \lambda_{g^{-1}} = \lambda_{g^{-1}} \circ \lambda_g = id_G$. hence $\lambda_g: G \longrightarrow G$ is adiffeomorphism with inverse $\lambda_{g^{-1}}$. Evidently, we have $\lambda_g \circ \lambda_h = \lambda_{gh}$. Similarly, we can consider the right translation by g, which we write as $\rho^g: G \longrightarrow G$.

Again, this is a diffeomorphism with inverse $\rho^{g^{-1}}$, but this time the compatibility with the product reads as $\rho^g \circ \rho^h = \rho^{gh}$. (many basic of group theory can be easily rephrased in terms of the translation mappings).

Example 1.6.1.

The equation $(gh)^{-1} = h^{-1}g^{-1}$ can be interpreted as $v \circ \lambda_g = \rho^{g^{-1}} \circ v$ or as

$v \circ \rho^h = \lambda_{h^{-1}} \circ v$. the definition of the neutral element can be recast as $\lambda_e = \rho^e = id_G$.

Lemma 1.6.1.

1. Let (G, μ, v, e) be a Lie group, for $g, h \in G,\ \xi \in T_g G$ and $\eta \in T_h G$ we have

$$T_{(g,h)}\mu \cdot (\xi, \eta) = T_h \lambda_g \cdot \eta + T_g \rho^h \cdot \xi.$$

2. The inversion map $v: G \to G$ is smooth and for $g \in G$ we have

$$T_g v = -T_e \rho^{g^{-1}} \circ T_g \lambda_{g^{-1}} = -T_g \lambda_{g^{-1}} \circ T_g \rho^{g^{-1}} \quad .$$

[17] Lie Groups – Fall terms 2012/13 - Andreas Cap – Institut fur Mathematik, Universit ¨ at Wien, Nordbergstraße 15, A– ¨ 1090 Wien

In particular, $T_e v = -id$.

Proof:

1. Since $T_{(g,h)}\mu$ is linear, we get $T_{(g,h)}\mu \cdot (\xi, 0) + T_{(g,h)}\mu \cdot (0, \eta)$. Choose a smooth curve $c: (-\epsilon, \epsilon) \to G$ with $c(0) = g$ and $c^{'}(0) = \xi$. Then the curve $t \mapsto (c(t), h)$ represents the tangent vector $(\xi, 0)$ and the compostion of μ with this curve equals $\rho^h \circ c$. Hence, we conclude that $T_{(g,h)}\mu \cdot (\xi, 0) = T_g \rho^h . \xi$.

2. Consider the function $f: G \times G \longrightarrow G \times G$ defined by $f(g, h) = (g, gh)$. From part (1) and the fact $\lambda_e = \rho^e = id_G$ we conclude that for $, \eta \in T_e G$, we get $T_{(e,e)}f \cdot (\xi, \eta) = (\xi, \xi + \eta)$.

Evidently, this is a linear isomorphism $T_e G \times T_e G \longrightarrow T_e G \times T_e G$, so locally around $(e, e), f$ admits a smooth inverse, $\tilde{f}: G \times G \longrightarrow G \times G$. By definition, $\tilde{f}(g, e) = (g, v(g))$ which implies that v is smooth locally around e. Since $v \circ \lambda_{g^{-1}} = \rho^g \circ v$, we conclude that v is smooth locally around any $g \in G$.

By differentiating the equation $e = \mu(g, v(g))$ and using part (**1**) we obtain

$$0 = \mu T_{(g,g^{-1})} \cdot \left(\xi, T_g v \cdot \xi\right) = T_g \rho^{g^{-1}} \cdot \xi + T_{g^{-1}} \lambda_g \cdot T_g v \cdot \xi$$

For any $\xi \in T_g G$. Since $\lambda_{g^{-1}}$ is inverse to λ_g this shows that $T_g v = -T_g \lambda_{g^{-1}} \circ T_g \rho^{g^{-1}}$.

1.7 Left invariant vector fields:

By using left translations to transport around tangent vectors on G. Put $\mathfrak{g} = T_e G$, the tangent space to G at the neutral element $e \in G$. For $X \in \mathfrak{g}$ and $g \in G$ define

$$L_X(g) = T_e \lambda_g \, . \, X \in T_g G$$

Definition 1.7.1.

Let G be a Lie group. A vector field $\xi \in \mathfrak{X}(G)$ is called left invariant if and only if $(\lambda_g) * \xi = \xi$ for all $g \in G$. The space of left invariant vector fields is denoted by $\mathfrak{X}_L(G)$.

Proposition 1.7.1.

Let G be a Lie group and put $\mathfrak{g} = T_e G$. We have **:**

1. The map $G \times \mathfrak{g} \to TG$ defined by $(g, X) \longmapsto L_X(g)$ is a diffeomorphism.

2. For any $X \in \mathfrak{g}$, the map $L_X : G \to TG$ is a vector field on G. The maps $X \longmapsto L_X$ and $\xi \longmapsto \xi(e)$define inverse linear isomorphisms between $\mathfrak{g}$ and $\mathfrak{X}_L(G)$.

Proof:-

1. Consider the map $\varphi: G \times \mathfrak{g} \to \mathrm{TG} \times \mathrm{TG}$ defined by $(g, X) = (0_g, X)$, where 0_g is the zero vector in $T_g G$. Evidently φ is smooth, and by part (1) of lemma the smooth map $T\mu \circ \varphi$ is given by $(g, X) \to L_X(g)$. On other hand, define $\psi: TG \to TG \times TG$ by $\psi(\xi_g) = (0_{g^{-1}}, \xi)$ which is smooth by part (**2**) of lemma.

By part (**1**) of lemma , we see that $T\mu \circ \varphi$ has values in $T_e G = \mathfrak{g}$ and is given by $\xi_g \longmapsto T\lambda_{g^{-1}} \cdot \xi$. This shows that $\xi_g \longmapsto (g, T\lambda_{g^{-1}} \cdot \xi)$ defines a smooth map $TG \to G \times \mathfrak{g}$, which is evidently inverse to $(g, X) \longmapsto L_X(g)$.

2. by definistion $L_X(g) \in T_g G$ and smoothness of L_X follows from (1) ,so $L_X \in \mathfrak{X}(G)$. By definition ,

$$\left((\lambda_g)^* L_X\right)(h) = T_{gh}\lambda_{g^{-1}} L_X(gh) = T_{gh}\lambda_{g^{-1}} \cdot T_e\lambda_{gh} \cdot X$$

And using $T_e\lambda_{gh} = T_h\lambda_g \circ T_e\lambda_h$ we see that equals $T_e\lambda_h \cdot X = L_X(h)$. Since h is arbitrary, $L_X \in \mathfrak{X}_L(G)$ and we have well defined maps in both directions. Of course $L_X(e) = X$, so one composition is the identity. On other hand. If ξ is left invariant and $X = \xi(e)$, then

$$\xi(g) = \left((\lambda_g)^* \xi\right)(g) = T_e\lambda_g \cdot \xi(g^{-1}g) = L_X(g)$$

And thus $= L_X$.

We have used left translations to trivialize the tangent bundle of a Lie group G in (prop**1.6.1**) in same way, one can consider the right trivialization $TG \longrightarrow G \times \mathfrak{g}$ defined by $\xi_g \longmapsto (g, T_g\rho^{g^{-1}} \cdot \xi)$. The inverse of this map is denoted by $(g, X) \longmapsto R_X(g)$, and

R_X is called the right invariant vector field generated by $X \in \mathfrak{g}$. In general, a vector field $\xi \in \mathfrak{x}(G)$ is called right invariant if $(\rho^g)^*\xi = \xi$ for all $g \in G$. The space of right invariant vector fields is denoted by $\mathfrak{x}_R(G)$. As in (prop1) one shows that $\xi = \xi(e)$ and $X \longmapsto R_X$

are inverse bijections between $\mathfrak{g}$ and $\mathfrak{x}_R(G)$.

Proposition 1.7.2.

Let $\boldsymbol{G}$ be a Lie group[18]. Then, the vector space of all left-invariant vector fields on G is isomorphic (as a vector space) to T_1G **.**

Proof:-

Since X is left invariant the following Fig(1.7) commutes

$$\begin{tikzcd} TG \arrow[r, "dL_a"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_a"] & G \arrow[u, "X"'] \end{tikzcd}$$

Fig (1.7)

So that $X(a) = (dL_a)_1\big(X(1)\big)$ for all $a \in G$. We denote that $\Gamma(TG)^G$ the set of all left invariant vector fields on G. Define a map $\phi\colon \Gamma(TG)^G \longrightarrow T_1G$ by $\phi\colon \mathrm{X} \longrightarrow X(1)$. Then , ϕ is linear and injective since if $\mathrm{X}, \mathrm{Y} \in \Gamma(TG)^G$ and $\phi(X) = \phi(Y)$

$X(g) = dL_g(X(1)) = dL_g(X(1)) = Y(g)$, For each $g \in G$.

Now, ϕ is also surjective, for $v \in T_1G$, define $X_v \in \Gamma(TG)^G$ by $X_v(a) = (dL_a)_1(v)$ for $a \in G$. We claim that X_v is a left invariant vector field . now, $X_v\colon G \longrightarrow TG$ is a C^∞ map of manifolds since if $f \in C^\infty\ G$, then for $a \in G$.

$$(X_v(f))\,(a) = (dL_a(v))f$$

$$= v(f \circ L_a)$$

Now, if $x \in G$ we have

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$$(f \circ L_a)(x) = (f \circ m)(a, x)$$

Which is a smooth map of a, x (here, m is the multiplication map on G).

Thus, $v(f \circ L_a)$ is smooth and hence so is X_v .

We now show X_v is left invariant. For$a, g \in G$, we have

$$(dL_a)(X_v(a)) = dL_a\left((dL_a)_1(v)\right)$$

$$= d\left(L_g \circ L_a\right)(v)$$

$$= d\left(L_{ga}\right)(v)$$

$$= X_v(ga)$$

$$= X_v\left(L_a(a)\right)$$

So, that X_v is left invariant. Therefore ϕ is onto and $\Gamma(TG)^G \cong T_1G$.

We now give T_1G a Lie algebra structure by identifying it with $\Gamma(TG)^G$ with the Lie bracket of vector fields. But, we need to show that $[\,,]$ is in fact a binary operation on $\Gamma(TG)^G$. recall if $f: M \longrightarrow N$ is a smooth map of manifolds and X, Y are f-related if $d\big(X(x)\big) = Y\big(f(x)\big)$ for every $x \in M$. It is a fact from manifold theory that is X, Y and X', Y' are f-related , then so are $[X, Y]$ and $[X', Y']$. but, left invariant vector fields are L_a related for all $a \in G$ by definition.

Proposition 1.7.3.

The Lie bracket of two left vector fields is a left invariant vector field. Thus, we can regard T_1G as a Lie algebra and make the following definition .

Definition 1.7.3.

Let G be a group. The Lie algebra $\mathfrak{g}$ of G is T_1G with the Lie bracket induced by it's identification with $\Gamma(TG)^G$.

Example 1.7.2.

Let $G = (\mathbb{R}^n, +)$. What is ?

Answer:

Notice that for this group, $L_a(x) = a + x$, so that$(dL_a)_0 = id_{T_0\mathbb{R}^n}$. So, $(dL_a)_0(v) = v$ for all $v \in T_0R^n$ and thus $\mathfrak{g} = T_0\mathbb{R}^n \cong \mathbb{R}^n$. So, the Lie algebra contains all constant vector fields, and the Lie bracket is identically 0 .

Example 1.7.3.

Consider the Lie group $GL(n, \mathbb{R})$. We have $T_1GL(n, \mathbb{R}) = M_n(\mathbb{R})$, the set of all $n \times n$ real matrices. For any $A, B \in M_n(\mathbb{R})$, the Lie bracket is the commutator ; that is

$$[A, B] = AB - BA$$

To prove this , we compute X_A , the left invariant vector field associated with the matrix $A \in T_1GL(n, \mathbb{R})$. Now, on $M_n(\mathbb{R})$, we have global coordinate maps given by

$x_{ij}(A) = A_{ij}$, the ijth entry of the matrix B.

So, for $g = GL(n, \mathbb{R})$,

$$\left(X_A(x_{ij})\right)(g) = X_A(I)(x_{ij} \circ L_g) .$$

also, if $h \in GL(n, \mathbb{R})$, then

$$\begin{aligned}(x_{ij} \circ L_g)(h) &= x_{ij}(gh) \\ &= \textstyle\sum_k g_{ik} h_{kj} \\ &= \textstyle\sum_k g_{ik} x_{kj}\ (h)\end{aligned}$$

Which implies that $x_{ij} \circ L_g = \sum_k g_{ik} x_{kj}$.

Now, if $f \in C^\infty(GL(n, \mathbb{R}))$, $X_A(I)f = \frac{d}{dt}\Big|_{t=0} f(I + tA)$, so that

$$X_A(I)x_{ij} = \frac{d}{dt}\Big|_{t=0} x_{ij}(I+tA) = A_{ij} .$$

Putting these remarks together, we see that $X_A\left(x_{ij} \circ L_g\right) = \sum_k g_{ik} A_{kj} = \sum_k x_{ik}(g) A_{kj}$

We are now in a position to calculate the Lie bracket of the left invariant vector fields associated with elements of $M_n(\mathbb{R})$:

$$\left([X_A, X_B]\,(I)\right)_{ij} = [X_A, X_B](I)x_{ij}$$

$$= X_A X_B\left(x_{ij}\right) - X_B X_A\left(x_{ij}\right)$$

$$= \left(X_A\left(\textstyle\sum_k B_{kj}\, x_{ik}\right) - X_B\left(\textstyle\sum_k A_{kj}\, x_{ik}\right)\right)$$

$$= \left(\textstyle\sum_{k,l} B_{kj}\, x_{il}\, A_{lk} - A_{kj}\, x_{il} B_{lk}\right)(I)$$

$$= \textstyle\sum_{k,l} B_{kj}\, \delta_{il}\, A_{lk} - A_{kj}\, \delta_{il} B_{lk}$$

$$= \textstyle\sum_k A_{ik} B_{kj} - \sum_k B_{ik} A_{kj}$$

$$= (AB - BA)_{ij}$$

So, $[A, B] = AB - BA$.

1.8 Lie Group Homomorphism:

Definition 1.8.1.

Let G and H be Lie groups. A map $\rho: G \longrightarrow H$ is a Lie group homomorphism if :

1. ρ is a C^∞ map of manifolds and

2. ρ is a group homomorphism

Furthermore, we say ρ is a lie group isomomrphism if it's a group isomomrphisim and a diffeomorphism .

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras , a Lie algebra homomorphism $\tau: \mathfrak{g} \longrightarrow \mathfrak{h}$ is a map such that :

1. τ is linear

2. $\tau([X,Y]) = [\tau(X), \tau(Y)]$ for all $X, Y \in \mathfrak{g}$.

Now, suppose V is an n-dimensional vector space over $\mathbb{R}$. we define,

$GL(V) = \{A: V \longrightarrow V \mid A$ a linear isomorphism$\}$

Since $V \cong \mathbb{R}^n$, $GL(V) \cong GL(n, \mathbb{R})$.

1.9 The Lie algebras of a Lie group:

For a Lie group G[19], left invariant vector fields $\xi, \eta \in \mathfrak{x}_L(G)$ and an element $g \in G$ we obtain

$$\lambda_g^*[\xi, \eta] = \left[\lambda_g^*\xi\ , \lambda_g^*\eta\right] = [\xi, \eta]$$

So, $[\xi, \eta]$ is left invariant too. Applying this to L_X and L_Y for $X, Y \in \mathfrak{g} = T_e G$, we see that $[L_X, L_Y]$ is left invariant. Defining $[X, Y] \in \mathfrak{g}$ as $[L_X, L_Y](e)$, part 2 of proposition show that$[L_X, L_Y] = L_{[X,Y]}$.

Proposition 1.9.1.

If $X, Y \in \mathfrak{g}$, so is their Lie bracket $[X, Y]$.

Proof:

We need to show that $[X, Y]$ is left-invariant if X and Y are left invariant. We first notice

$Y(f \circ L_a)(b) = Y_b(f \circ L_a) = (dL_a)_b(Y_b)f = Y_{ab}f = (Yf)(L_a b) = (Yf) \circ L_a(b)$

For any smooth function $f \in C^\infty(G)$. Thus

$Y_{ab}f(Yf) = (dL_a)_b(X_b)(Yf) = (X_b)\big((Yf) \circ L_a\big) = X_b Y(f \circ L_a)$

Similarly , $Y_{ab}X\, f = Y_b X(f \circ L_a)$.

[19] Lie Groups – Fall terms 2013/14 - Andreas Cap – Institut fur Mathematik, Universit ¨ at Wien, Nordbergstraße 15, A– ¨ 1090 Wien

Thus,

$$dL_a([X,Y]_b)f = X_bY(f\circ L_a) - Y_bX(f\circ L_a) = X_{ab}(Y\,f) - \; Y_{ab}X\,f = [X,Y]_{ab}(f)\;.$$

Definition 1.9.1.

Let G be a group. The Lie algebras of G is the tangent space $\mathfrak{g} = T_eG$ together with the map $[\,,]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ defined by $[X,Y] = [L_X, L_Y](e)$. [20]

Remark 1.9.1.

From corresponding properties of the Lie bracket of vector fields, it follows immediately that the bracket $[\,,]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ is:

i. Bilinear: $[aX,Y] = a[X,Y]$ and $[X_1 + X_2, Y] = [X_1, Y] + [X_2\,, Y]$

ii. Skew symmetric $([Y,X] - [X,Y])$

iii. Satisfies the Jacobi identity $\big[X,[Y,Z]\big] = \big[[X,Y],Z\big] + \big[Y,[X,Z]\big]$.

In general, one defines a Lie algebra as a real vector space together with a Lie bracket having these three properties.

Example 1.9.1.

Let us consider the fundamental example $G = GL(n,\mathbb{R})$. As a manifold, G is an open subset in the vector space $M_n(\mathbb{R})$, so in particular, $\mathfrak{g} = M_n(\mathbb{R})$ as a vector space. Consider the matrices $A,B,C \in M_n(\mathbb{R})$ we have $A(B+tC) = AB + tAC$, so left translation by A is a linear map. In particular, this implies that for $A \in GL(n,\mathbb{R})$ and $C \in M_n(\mathbb{R}) = T_eGL(n,\mathbb{R})$ we obtain $L_C(A) = AC$. viewed as a function $GL(n,\mathbb{R}) \longrightarrow M_n(\mathbb{R})$, the left invariant vector field L_C is therefore given by right multiplication by C and thus extends to all of $M_n(\mathbb{R})$ .now viewing vector fields on an open subset of $\mathbb{R}^m$ as functions with values in $\mathbb{R}^m$, the Lie bracket is given by $[\xi,\eta](x) = D\eta(x)\big(\xi(x)\big) - D\xi(x)\big(\eta(x)\big)$. Since, right multiplication by a fixed matrix is a linear map, we conclude

[20] - Introduction to Lie Groups and Lie Algebras – Alexander Kirillov,Jr. – Departement of mathematics, Suny At stony Brook,NY 11794, USA.

that $D(L_{C'})(e)(C) = CC'$ for $C, C' \in M_n(\mathbb{R})$. Hence we obtain $[C, C'] = [L_C, L_{C'}](e) = CC' - C'C$, and rhe Lie bracket on $M_n(\mathbb{R})$ is given by the commutator of matrices.

Lemma 1.9.1.

Let $f: M \longrightarrow N$ be smooth map , and let $\xi_i \in \mathfrak{X}(M)$ and $\eta_i \in \mathfrak{X}(N)$ be vector fields for $i = 1,2$. if $\xi_i \sim_f \eta_i$ for $i = 1,2$ then $[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2]$.

Proof:

For a smooth map $\alpha: N \longrightarrow \mathbb{R}$ we have $(Tf \circ \xi) \cdot \alpha = \xi \cdot (\alpha \circ f)$ by definition of the tangent map. Hence $\xi \sim_f \eta$ is equivalent to $\xi \cdot (\alpha \circ f) = (\eta \cdot \alpha) \circ f$ for all $\alpha \in C^\infty(N, \mathbb{R})$. Now , assuming that $\xi_i \sim_f \eta_i$ for $i = 1,2$ we compute

$$\xi_1 \cdot \big(\xi_2 \cdot (\alpha \circ f)\big) = \xi_1 \cdot \big((\eta_2 \cdot \alpha) \circ f\big) = (\eta_1 \cdot (\eta_2 \cdot \alpha)) \circ f$$

From definition of Lie bracket,then $[\xi_1, \xi_2] \cdot (\alpha \circ f) = ([\eta_1, \eta_2] \cdot \alpha) \circ f$

And thus, $[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2]$.

Definition 1.9.2.

A Lie algebra homomorphism between two Lie algebras A and B (over the same field) is a linear map that preserves the Lie bracket ,i.e. a map

$$f: \begin{cases} A \longrightarrow B \\ a \longmapsto f(a) \end{cases}$$

$f([a, b]) = [f(a), f(b)]$

An invertible Lie algebra homomorphism is a Lie algebra isomorphism.

Proposition 1.9.2.

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

1. If $\varphi: G \longrightarrow H$ is a smooth homomorphism then $\varphi' = T_e\varphi: \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism of Lie algebras, i,e. $\varphi'([X, Y]) = [\varphi'(X), \varphi'(Y)]$ for all $X, Y \in \mathfrak{g}$.

2. If G is commutative, then the Lie bracket on $\mathfrak{g}$ is identically zero.

Proof:

1. The equation $\varphi(gh) = \varphi(g)\varphi(h)$ can be interpreted as $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$. Differentiating this equation in $e \in G$, we obtain $T_g\varphi \circ T_e\lambda_{\varphi(g)} = T_e\lambda_{\varphi(g)} \circ \varphi'$. Inserting $X \in T_eG = \mathfrak{g}$, we get $T_g\varphi \cdot L_X(g) = L_{\varphi'(X)}(\varphi(g))$,and hence the vector fields $L_X \in \mathfrak{X}(G)$ and $L_{\varphi'(X)} \in \mathfrak{X}(H)$ are φ-related for each $X \in \mathfrak{g}$. Form the lemma, we conclude that for $X, Y \in \mathfrak{g}$ we get $T_\varphi \circ [L_X, L_Y] = [L_{\varphi'(X)}, L_{\varphi'(Y)}] \circ \varphi$. Evaluated in $e \in G$ this gives $\varphi'([X,Y]) = [\varphi'(X), \varphi'(Y)]$.

2. If G is commutative, then $(gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}$ so the inversion map $v: G \longrightarrow G$ is a group homomorphism. Hence by part (**1**) , $v': \mathfrak{g} \longrightarrow \mathfrak{g}$ is a Lie algebra homomorphism

By part (**2**) of lemma $v' = id$ and we obtain

$-[X,Y] = v'([X,Y]) = [v'(X), v'(Y)] = [-X, -Y] = [X,Y]$

and thus $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Proposition 1.9.3.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and inversion $v: G \longrightarrow G$, then we have :

1. $R_X = v^*(L_{-X})$ for all $X \in \mathfrak{g}$.

2. For $X, Y \in \mathfrak{g}$, we have $[R_X, R_Y] = R_{-[X,Y]}$.

3. For all $X, Y \in \mathfrak{g}$, we have $[L_X, R_Y] = 0$

Proof:

The equation $(gh)^{-1} = h^{-1}g^{-1}$ can be interpreted as $v \circ \rho^h = \lambda_{h^{-1}} \circ v$.

In particular, if $\xi \in \mathfrak{X}_L(G)$ then

$(\rho^h)^* v^* \xi = (v \circ \rho^h)^* \xi = (\lambda_{h^{-1}} \circ v)^* \xi = v^* {\lambda_{h^{-1}}}^* \xi = v^* \xi$

So $v^*\xi$ is right invariant. Since $v^*\xi(e) = T_e v \cdot \xi(e) = -\xi(e)$.

Using part (**1**) we compute

$[R_X, R_Y] = [v^*L_{-X}, v^*L_{-Y}] = v^*[L_{-X}, L_{-Y}] = v^*L_{[X,Y]} = R_{-[X,Y]}$.

Consider the vector field $(0, L_Y)$ on $G \times G$ whose value in (g, h) is $(0_g, L_X(h))$. by part (**1**) of (Lemma**1.6.1**) $T_{(g,h)}\mu \cdot \left(0_g, L_X(h)\right) = T_h\lambda_g \cdot L_X(h) = L_X(gh)$

Which shows that $(0, L_X)$ is μ- related to L_X . like wise, $(R_Y, 0)$ is μ- related to R_Y , so by (Lemma **1.6.1**) the vector field $0 = [(0, L_X), (R_Y, 0)\ ]$ is μ- related to $[L_X, R_Y]$. since μ is subjective , this is implies that $[L_X, R_Y] = 0$.

1.10 Exponential Map:

Given a Lie group and it's Lie algebra [21], we would like to construct an exponential map from $\mathfrak{g} \longrightarrow G$, which will help to give some information about the structure of $\mathfrak{g}$.

Proposition 1.10.1.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, for each $X \in \mathfrak{g}$, there exists a map $\gamma_X : \mathbb{R} \longrightarrow G$ satisfying : $\gamma_X(0) = 1_G$

$$\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t) = X \ ,$$

and $\gamma_X(s+t) = \gamma_X(s)\gamma_X(t)$

Proof:

Consider the Lie algebra map $\tau : \mathbb{R} \longrightarrow \mathfrak{g}$ defined by $\tau : t \longrightarrow tX$ for all $\in \mathfrak{g}$. Now, $\mathbb{R}$ is connected and simply connected, so by (Theorem **1.5.4**) . There exists a unique Lie group map $\gamma_X : \mathbb{R} \longrightarrow G$ such that $(d\gamma_X)_0 = \tau$; which is to say

$$\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t) = X$$

[21] Notes on Lie Groups – Eugene Lerman – February 15, 2012

This motivates the following definition :

Definition 1.10.1.

Let G be a Lie group with Lie algebra . define the exponential map $exp\colon \mathfrak{g} \longrightarrow G$ by $\exp(X) = \gamma_X(1)$.

Lemma 1.10.1.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\in \mathfrak{g}$. write $\tilde{X}$ for the left invariant vector field on $\mathfrak{g}$ with $\tilde{X}(1) = X$. Then, $\phi_t(a) = a\gamma_X(t)$, is the flow of $\tilde{X}$. in particular; $\tilde{X}$ is complete ; i.e. the flow exists for all $t \in \mathbb{R}$.

Proof:

For $a \in G$, we have

$$\left.\frac{d}{dt}\right|_{t=s} a\,\gamma_X(t) = (dL_a)_{\gamma_X(s)}\left(\left.\frac{d}{dt}\right|_{t=s} \gamma_X(t)\right)$$

$$= (dL_a)_{\gamma_X(s)}\left(\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t+s)\right)$$

$$= (dL_a)_{\gamma_X(s)}\left(\left.\frac{d}{dt}\right|_{t=0} \gamma_X(s)\,\gamma_X(t)\right)$$

$$= (dL_a)_{\gamma_X(s)}\left(\left.\frac{d}{dt}\right|_{t=0} L_{\gamma_X(s)}(\gamma_X(t))\right)$$

$$= \left(dL_{a\gamma_X(s)}\right)_1 \left(\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t)\right)$$

$$= \left(dL_{a\gamma_X(s)}\right)_1 \, (X)$$

$$= \tilde{X}(a\gamma_X(s)) \qquad \text{(since } \tilde{X} \text{ is left-invariant)}$$

So, $a\gamma_X(t)$ is the flow of $\tilde{X}$ and exists for all .

Lemma 1.10.2.

The exponential map is C^{∞} .

Proof:

Consider the vector field V on $G \times \mathfrak{g}$ given by : $V(a, X) = (dL_a(X), 0)$ Then,$V \in C^{\infty}(G, \mathfrak{g})$ and the claim is that the flow of V is given by $\psi_t(g, X) = (g\gamma_X(t), X)$. To prove this claim , consider the following :

$$\frac{d}{dt}\Big|_{t=0} (g\gamma_X(t), X) = \left(dL_{a\gamma_{X(s)}}(X), 0\right) = V(\, g\gamma_X(s), X)$$

From which we can conclude that γ_X depends smoothly on X.

Now, we note that the map $\phi: \mathbb{R} \times G \times \mathfrak{g}$ defined by $\phi(t, a, X) = (a\gamma_X(t), X)$ is smooth .

Thus, if $\pi_1: G \times \mathfrak{g} \longrightarrow G$ is projection on the first factor ,$(\pi_1 \circ)(1_G, X) = \gamma_X(1) = \exp(x)$ is C^{∞} .

Lemma 1.10.3.

For all $X \in \mathfrak{g}$ and for all $\in R$, $\gamma_{tX}(1) = \gamma_X(t)$.

Proof:

The intent is to prove that for all $\in \mathbb{R}$, $\gamma tX(s) = \gamma_X(ts)$. Now, $s \longmapsto \gamma_{tX}(s)$ is the integral curve of the left invariant vector field tX through 1_G . But, $\tilde{X} = t\tilde{X}$, so if we prove that , $\gamma_X(ts)$ is an integral curve of $t\tilde{X}$ through 1_G , by uniqueness the Lemma will be established .

To prove this , first let $\sigma(s) = \gamma_X(ts)$. Then, $\sigma(0) = \gamma_X(0) = 1_G$. we also have

$$\frac{d}{ds}\sigma(s) = \frac{d}{ds}\,\gamma_X(ts)$$

$$= d\,\frac{d}{du}\Big|_{u=ts} \gamma X(u)$$

$$= t\tilde{X}(\,\gamma_X(ts)\,)$$

$$= t\tilde{X}(\sigma(s))$$

So, $\sigma(s)$ is also an integral curve of $t\tilde{X}$ through 1_G . Thus, $\gamma_{tX}(s) = \gamma_X(ts)$, and in particular , when $s = 1$ we have $\gamma_{tX}(1) = \gamma_X(t)$

Now, we will prove the nice fact about exponential map .

Proposition 1.10.2.

Let G be a Lie group and $\mathfrak{g}$ it's Lie algebra . identify both $T_0\mathfrak{g}$ and T_1G with $\mathfrak{g}$. then $(d\ exp)_0 : T_0\mathfrak{g} \longrightarrow T_1G$ is the identity map .

Proof:

By using the previous (Lemma **1.10.3**) we have

$$(d\ \exp)_0(X) = \frac{d}{dt}\Big|_{t=0} \exp(0 + X)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{tX}(1)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_X(t) = \mathrm{X}$$

Corollary 1.10.1.

For all $t_1 , t_2 \in \mathbb{R}$,

1. $(exp(t_1 + t_2)X\ ) = expt_1X + expt_2\,X$.

2. $\exp(-tX) = (\exp(tX))^{-1}$

1.11 Representation of Lie Group :

Definition 1.11.1.

An action of a Lie group G[22] an a manifold M is an assignment to each $g \in G$ a diffeomorphism $\rho(g) \in DiffM$ such that $\rho(1) = \rho(g)\rho(h)$ and such that the map

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$G \times M \to M : (g, m) \longmapsto \rho(g).m$ is smooth map.

Example 1.11.1.

The group $GL(n, \mathbb{R})$ (and thus any it's Lie subgroup) acts on $\mathbb{R}^n$. The group $O(n, \mathbb{R})$ acts on the sphere $S^{n-1} \subset \mathbb{R}^n$. The group $U(n)$acts on the sphere $S^{2n-1} \subset \mathbb{C}^n$.

Definition 1.11.2.

Let G be a Lie group and V a vector space[23]. A representation of a Lie group is a map $\rho: G \longrightarrow GL(V)$ of Lie groups .

For a Lie group G, consider the action of G on itself by conjugation: for each $g \in G$ we have a diffeomorphism $c_g : G \longrightarrow G$ given by $c_g(a) = gag^{-1}$.

Notice that $c_g(1) = 1$, and we have an invertible linear map $\left(dc_g\right)_1 : \mathfrak{g} \longrightarrow \mathfrak{g}$.

Now, $c_{g_1 g_2} = c_{g_1} \circ c_{g_2}$ for all $g_1 g_2 \in G$, and hence $\left(dc_{g_1}\right)_1 \left(dc_{g_1}\right)_1 = \left(d\, c_{g_1 g_2}\right)_1$.

Definition 1.11.3.

The Adjoint representation of a Lie group G is the representation $Ad: G \longrightarrow GL(\mathfrak{g})$ defined by

$$Ad(g) = \left(dc_g\right)_1$$

The Adjoint representation of a Lie algebra $\mathfrak{g}$ is the representation

$ad: \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) = Hom(\mathfrak{g}, \mathfrak{g})$ by defined by : $ad(X) = (d\, Ad)_1\, (X)$

Proposition 1.11.1.

Suppose G is a Lie group .then, for all $\in \mathbb{R}$,$g \in G$ and $X \in \mathfrak{g}$ we have

1. $g\, exp(tX) g^{-1} = \exp\left(tAd(g)(X)\right)$ and

2. $Ad\, (\exp(tX)) = \exp\left(t\, ad(X)\right)$

[23] Notes on Lie Groups – Eugene Lerman – February 15 ,2012

Proof:

For the first statement , apply naturality of exp to the Fig (1.7)

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}(g)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "c_g"] & G \end{tikzcd}$$

Fig (1.7)

Similarly, to prove, apply the naturality of exp to the Fig(1.8)

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \end{tikzcd}$$

Fig (1.8)

Example 1.11.2

We compute what Ad and ad are as maps when $= GL(n, \mathbb{R})$. recall that for any $A, g \in G$ we have the conjugation map $c_g(A) = gAg^{-1}$. Note that conjugation is linear . thus for $X \in \mathfrak{g}$ we have

$$Ad(g)(X) = \left(dc_g\right)_I (X)$$

$$= \frac{d}{dt}\Big|_{t=0} c_g\left(exp(tX)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} gexp(tX)\, g^{-1}$$

$$= g\left(\frac{d}{dt}\Big|_{t=0} gexp(tX)\right) g^{-1}$$

$$= gXg^{-1}$$

Also, for $X, Y \in \mathfrak{g}$

$$ad\,(X)Y = \frac{d}{dt}\Big|_{t=0}\ Ad\left(exp(tX)\right)Y$$

$$= \frac{d}{dt}\Big|_{t=0}\ exp(tX)\ Y\ exp(-tX)$$

$$= \left(\frac{d}{dt}\Big|_{t=0}\ exp(tX)\ \right) Y \exp(-0X) + \exp(0X) Y \left(\frac{d}{dt}\Big|_{t=0}\ exp(-tX)\ \right)$$

$$= \ XY + Y(-X)$$

$$= [X, Y]$$

The commutator of the matrices $, Y$.

Theorem 1.11.1.

Let G be a Lie group . then, for any $X, Y \in \mathfrak{g}$.

$$ad(X)Y = [X, Y]$$

Proof:

First note that

$$ad(X)Y = \ \frac{d}{dt}\Big|_{t=0} \text{Ad}\,(\exp tx)Y$$

$$= \frac{d}{dt}\Big|_{t=0} d\left(C_{exptx}\right)_1(Y)\ .$$

Also, recall that we have shown:

$$c_g(a) = gag^{-1} = \ \left(R_{g^{-1}} \circ L_g\right)(a)$$

$\left(d\ L_g\right)_1(Y) = \tilde{Y}\ (g)$, where $\tilde{Y}$ is the left invariant vector field with $\tilde{Y}(1) = Y$,

The flow $\phi_t^{\tilde{Y}}$ of $\tilde{Y}$ is given by , $\phi_t^{\tilde{Y}}(a) = a\ (\exp tX) = \ R_{\exp\ tX}\ (a)$

$$[\tilde{X}, \tilde{Y}](a) = \frac{d}{dt}\Big|_{t=0} d\left(\phi_t^{\tilde{X}}\right)\left(\tilde{Y}\left(\phi_t^{\tilde{X}}(a)\right)\right)$$

$$(exptX)^{-1} = \exp(-tX)$$

Then we get

$$ad(X)Y = \frac{d}{dt}\Big|_{t=0} dR_{\exp(-tX)}\left(d\, L_{\exp\, tX}\, Y\right)$$

$$= \frac{d}{dt}\Big|_{t=0} dR_{\exp(-tX)}\left(\tilde{Y}\,(exptX)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} d\left(\phi_t^{\tilde{X}}\right)\left(\tilde{Y}\left(\phi_t^{\tilde{X}}(1)\right)\right)$$

$$= [\tilde{X}, \tilde{Y}]\,(1) \qquad , \quad \text{(by 4)}$$

1.12 Operation on representations:

Definition 1.12.1 (Subrepresentations and Quotients)

Let V be a representation of G(respectively)[24]. A subrepresentation is a vector subspace $W \subset V$ stable under the action: $\rho(g)W \subset W$ for all $g \in G$ (respectively, $\rho(x)W \subset W$ for all $x \in \mathfrak{g}$).

If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then $W \subset V$ is a subrepresentation for G and only if it is a sub representation for $\mathfrak{g}$.

If $W \subset V$ is a subrepresentation, then the quotient space V/W has a canonical strcture of a representation. It will be called factor representation, or the quotient representation.

Lemma 1.12.1. (Direct sum and tensor product)

Let W, V be representations of G (respectively $\mathfrak{g}$). Then there is a canonical structure of a representation on $V^*, V \oplus W, V \otimes W$.

Proof:

Action of G on $V \oplus W$ is given by $\rho(g)(v + w) = \rho(g)v + \rho(g)w$, and similarly for $\mathfrak{g}$

For tensor product, we define $\rho(g)(V \otimes W) = \rho(g)v \otimes \rho(g)w$. However, action of $\mathfrak{g}$ is trickier: indeed naïve definition $\rho(x)(V \otimes W) = \rho(x)v \otimes \rho(x)w$ doesn't define a

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representation (it is not linear in x). Instead, if we write $x = \gamma^{\cdot}(0)$ for some one-parameter family $\gamma(t)$ in a Lie group G with $\gamma(0) = 1$, Then

$$\rho(g)(V \otimes w) = \frac{d}{dt}\Big|_{t=0} (\gamma(t)v \otimes \gamma(t)w) = (\gamma^{\cdot}(0)v \otimes \gamma(t)w) + (\gamma(0)v \otimes \gamma^{\cdot}(t)w)$$

$$= \rho(x)v \otimes w + V \otimes \rho(x)w$$

By Liebnitz rule . Thus, we define $\rho(x)(V \otimes w) = \rho(x)v \otimes w + v\rho(x)w$. it is easy to shows, even without using Lie group G, that so defined action is indeed a representation of $\mathfrak{g}$ on $V \otimes W$. To define action of $G, \mathfrak{g}$ on V^*, we require that the natural pairing $V \otimes V^* \longrightarrow \mathbb{C}$ be a morphism of representations, considering $\mathbb{C}$ as the trivial representation. This gives, for $v \in V$, $v^* \in V^*$, $\langle \rho(g)v, \rho(g)v^* \rangle = \langle v, v^* \rangle$, so action of G in V^* is given by $\rho V^*(g) = \rho(g^{-1})^t$, where for $A: V \longrightarrow V$, we denote by A^t the adjoint operator $V^* \longrightarrow V^*$.

Similarly, for the action of $\mathfrak{g}$ we get $\langle \rho(x)v, v^* \rangle + \langle v, \rho(g)v^* \rangle = 0$, so $\rho V^*(x) = -\big(\rho v(x)\big)^t$

Example 1.12.1.

Let V be a representation of G(respectively $\mathfrak{g}$). Then the space $End(V) \simeq V \otimes V^*$ of linear operators on V is also a representation, with the action given by $g: A \longmapsto \rho v(g) A \rho v(g^{-1})$ maps $Hom(V, W)$ between two representations is also a representation with the action defined by $g: A \longmapsto \rho w(g) A \rho v(g^{-1})$ for $g \in G$ (respectively, $x: A \longmapsto \rho w(x)A - A\rho v(x)$ for $x \in \mathfrak{g}$) .

Similarly, the space of bilinear forms on V is also a representation, with action given by

$$gB(v, w) = B(g^{-1}v, g^{-1}w), \ g \in G$$

$$xB(v, w) = -\big(B(x.v, w) + B(v, x.w)\big) \ , \ x \in \mathfrak{g} .$$

Definition 1.12.2. (Invariants)

Let V be a representation of a Lie group G. A vector $v \in V$ is called invariant if:

$\rho(g)v = v$ for all $g \in G$. The subspace of invariant vectors in V is deonted by V^G.

Similarly, let V be a representation of a Lie algebra $\mathfrak{g}$. A vector $v \in V$ is called invariant if $\rho(x)v = 0$ for all $x \in \mathfrak{g}$. The subspace of invariant vectors in V is denoted by $V^{\mathfrak{g}}$.

If G is a connected Lie group with the Lie algebra $\mathfrak{g}$, then for any representation V of G, we have $V^G = V^{\mathfrak{g}}$.

Example 1.12.2.

Let V, W be representations and $Hom(V, W)$ be the space of linear maps $V \longrightarrow W$, with the action of G defined as in Example (**1.11.2**) . Then $(Hom(V, W))^G = Hom_G(V, W)$ is the space of intertwining operators. In particular, this shows that

$$V^G = (Hom(\mathbb{C}, W))^G = Hom_G(V, W)$$

With $\mathbb{C}$ considered as a trivial representation .

Example 1.12.3.

Let B be a bilinear form on a representation V. Then B is invariant under the action of G defined in (Example **1.11.2**) iff **:** $B(gv, gw) = B(v, w)$ For any $\in G$,$v, w \in V$.

Similarly, B is invariant under the action of $\mathfrak{g}$ iff **:** $B(x.v, w) + B(v, x.w) = 0$, for any $x \in \mathfrak{g}$,$v, w \in V$

Definition 1.12.3. (Irreducible representations)

A non-zero representation V of G or $\mathfrak{g}$ is called **irreducible** or **simple** if it has no subrepresentations other than $0, V$. Otherwise V is called **reducible** .

Example 1.12.4.

Space $\mathbb{C}^n$, considered as a representation of $SL(n, \mathbb{C})$, is irreducible .

If representation V is not irreducible (such representations are called reducible), then it has a non-trivial subrepresentation W and thus, V can be included in a short exact sequence $0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$; thus, in a certain sense it is built out of simple pieces. The natural question is whether this exact sequence splits,i.e, whether we can write $V = W \oplus (V/W)$ as a representation. If so then repeating this process, we can write V as a direct sum of irreducible representations.

Definition 1.12.4.

A representation is called completely reducible (or semisimple) if it is isomorphic to a direct sum of irreducible representations: $\simeq \bigoplus V_i$, V_i irreducible .

In such a case one usually groups together isomorphic summands writing $\simeq \bigoplus n_i V_i$, $n_i \in \mathbb{Z}_+$ where V_i are pairwise non-isomorphic irreducible representations . The numbers n_i are called multiplicities.

However, as the following example shows, not every representation is completely reducible.

Example 1.12.5.

Let $G = \mathbb{R}$,so $\mathfrak{g} = \mathbb{R}$. Then a representation of $\mathfrak{g}$ is the same as a vector space V with a linear map $\mathbb{R} \longrightarrow End(V)$; obviously, every such map is of the form $t \longmapsto tA$ for some $A \in End(V)$ which can be arbitrary. The corresponding representation of the group $\mathbb{R}$ is given by $t \longmapsto \exp(tA)$. Thus, classifying representation of $\mathbb{R}$ is equivalent to classifying linear maps $V \longrightarrow V$ up to a change of basis. Such a classification is known (Jordan normal form) but non-trivial.

If v is an eigenvector of A then the one-dimensional space $\mathbb{C}v \subset V$ is invariant under A and thus a subrepresentation in . Since every operator in a complex vector space has an eigenvector , this shows that every representation of $\mathbb{R}$ is reducible, unless it is one-dimensional. Thus, the only irreducible representations of $\mathbb{R}$ are one-dimensional.

Now, it see that writing a representation given by $t \longmapsto \exp(tA)$ as a direct sum of irreducible ones is equivalent to diagonalizing A. So a representation is completely reducible iff A is diagonalizable . Since not every linear operator is diagonalizable, not every representation is completely reducible .

Thus, more modest goals of the representation theory would be answering the following questions **:**

1. For a given Lie group G, classify all irreducible representations of .

2. For a given representation V of a Lie group G, given that is completely reducible, find explicity the decomposition of V into direct sum of irreducibles .

3. For which Lie groups G all representation are completely reducible?