Chapter 1
Equivalence Relations and Ellipsoidal Tight Frames

We find the closest and respectively the nearest tight frame to a given frame .Our main tool in the infinite dimensional case is a result we have proven which concerns the decomposition of a positive invertible operator into a strongly converging sum of (not necessarily mutually orthogonal) self-adjoint projections. This decomposition result implies the existence of tight frames in the ellipsoidal surface determined by the positive operator. In the real or complex finite dimensional case, this provides an alternate (but not algorithmic) proof that every such surface contains tight frames with every prescribed length at least as large as $\dim \mathcal{H}$. A corollary in both finite and infinite dimensions is that every positive invertible operator is the frame operator for a spherical frame.

Section(1.1): Distances Between Hilbert Frames

Suppose H is an infinite dimensional separable Hilbert space. A theorem due to Paley-Wiener [198] states the following: let $\{e_i\}_{i\in N}$ be an orthonormal basis of H and let $\{f_i\}_{i\in N}$ be a family of vectors in H. If there exists a constant $\lambda \in [0,1)$ such that

$$\left\|\sum_{i=1}^{n} c_i(e_i - f_i)\right\| \le \lambda \left\|\sum_{i=1}^{n} c_i e_i\right\| = \lambda\left(\sum_{i=1}^{n} |c_i|^2\right)^{1/2} \qquad (1)$$

for all $n, c_1, c_2, \ldots, c_n$, then $\{f_i\}_{i\in N}$is a Riesz basis in H and a frame with bounds $(1-\lambda)^2, (1+\lambda)^2$. An extension of this theorem was given by Christensen in [193] to Hilbert frames and by Christensen and Heil in [194] to Banach frames. Duffin and Eachus ([75]) proposed a converse of the above result by proving that every Riesz basis, after a proper scaling, is close to an orthonormal basis in the sense of (1). We are going to extend this result to Hilbert frames and show some results about quadratic closeness and distance between two frames.

Let I be a countable index set. A family of vectors $\mathcal{F} = \{f_i\}_{i\in I}$ in H is called a(Hilbert) frame if there exist two real numbers $0 < A \le B < \infty$ such that for any $x \in H$ we have:

$$A\ \|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2. \qquad (2)$$

If $A = B$ we call the frame tight. The largest constant A and respectively the smallest constant B that satisfy (2) are called the (optimal) frame bounds.

To a frame $\mathcal{F}$ we associate several objects. Consider the operator

$$T:\ H \to l^2(I),\ T(x) = (\langle x, f_i\rangle)_{i\in I,}$$

called the analysis operator associated to $\mathcal{F}$ (see [199] for terminology). From (2) we get that it is a bounded operator with norm $\|T\| = \sqrt{B}$ and its range is closed. The adjoint of T is given by

$$T^*:\ l^2(I) \to H, \qquad T^*c = \sum_{i\in I} c_i f_i\ ,$$

and is called the synthesis operator. With these two operators we construct the frame operator by $S: H \to H, S = T^*T$ or $S(x) = \sum_{i\in I}\langle x, f_i\rangle f_i$. The condition (2) can then be read as $A.1 \le S \le B.1$ and therefore the frame bounds are $B = \|S\|, A = \|S^{-1}\|^{-1}$d .

To every frame $\mathcal{F}$ one can associate two special frames: one is called the (stan-dard) dual frame and the other (less frequently used) is called the associated tight frame [191]. The (standard) dual frame is defined by

$$\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}, \tilde{f}_i = S^{-1} f_i \qquad (3)$$

and has a lot of useful properties. A few of them are the following:

(a) $\tilde{\mathcal{F}}$ is a frame with frame bounds $\frac{1}{B}, \frac{1}{A}$.

(b) If $\tilde{T}$is the analysis operator associated to $\tilde{\mathcal{F}}$, then $\tilde{T} = TS^{-1}$ and the following resolutions of identity (or reconstruction formulae) hold:

$$1 = \tilde{T}^*T = T^*\tilde{T} \text{ or } x = \sum_{i\in I}\langle x, f_i\rangle \tilde{f}_i = \sum_{i\in I}\langle x, \tilde{f}_i\rangle f_i.$$

(c) In $l^2(I), T$ and $\tilde{T}$ have the same range ($E = Ran\,T = Ran\,\tilde{T}$) and $P = T\tilde{T}^* = \tilde{T}T^*$ is the orthogonal projector onto E.
(d) For any $c \in l^2(I)$ we can consider the set of sequences $d \in l^2(I)$ with the same image as $c, i.e., T^*c = T^*d$; the minimum l^2-norm in this set is achieved by the sequence $c^* = Pc \in E$. The associated tight frame is defined by

$$\mathcal{F}^{\#} = \{f_i^{\#}\}_{i\in I}, f_i^{\#} = S^{-1/2} f_i. \qquad (4)$$

A few properties of the associated tight frame that can be simply checked are the following:
(i) The associated tight frame is a tight frame with frame bound 1.
(ii) If $T^{\#}$ is the analysis operator associated to $\mathcal{F}^{\#}$, then $T^{\#} = TS^{-1/2}$; its range coincides with $E = Ran\,T$, and the orthogonal projector onto E, P, is also equal to $T^{\#}(T^{\#})^*$. We shall come back to this associated tight frame in this section .

So far, we have just listed properties of one frame and some derived frames.

We shall discuss mainly the relations between two frames. Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be two frames in H. We define the following notions:
(a) If Q is an invertible bounded operator $Q : H \to H$ and if $g_i = Qf_i$, then we say that $\mathcal{F}$ and $\mathcal{G}$ are Q-equivalent.
(b)We say they are unitarily equivalent if they are Q-equivalent for a unitary operator Q.
(c)If Q is a bounded operator $Q : H \to H$ (not necessarily invertible) and $g_i = Qf_i$, then we say $\mathcal{F}$ is Q-partial equivalent with $\mathcal{G}$.
(d)We say $\mathcal{F}$ is partial isometric equivalent with $\mathcal{G}$ if there exists a partial isometry $J : H \to H$ such that $g_i = Jf_i$ (then J should satisfy $JJ^* = 1$ since $g_i \in Ran\,J$ and $\mathcal{G}$ is a complete set in H).

The last two relations (Q-partial equivalent and partial isometric equivalent) are not equivalency relations, because they are not symmetric.

We say that a frame $\mathcal{G} = \{g_i\}_{i\in I}$ is (quadratically) close to a frame $\mathcal{F} = \{f_i\}_{i\in I}$ if there exists a positive number $\lambda \geq 0$ such that

$$\|\sum_{i\in I} c_i(g_i - f_i)\| \leq \lambda \|\sum_{i\in I} c_i f_i\| \qquad (5)$$

for any $c = (c_i)_{i\in I} \in l^2(I)$ (see [201]). The infimum of such $\lambda's$ for which (5) holds for any $c \in l^2(I)$ will be called the closeness bound of the frame $\mathcal{G}$ to the frame $\mathcal{F}$ and denoted by $c(\mathcal{G}, \mathcal{F})$.

The closeness relation is not an equivalency relation (it is transitive, but not reflexive, in general). However, if $\mathcal{G}$ is quadratically close to $\mathcal{F}$ with a closeness bound less than 1, then $\mathcal{F}$ is also quadratically close to $\mathcal{G}$ but the closeness bound is different, in general. Indeed, from (5) it follows that $\left\|\sum_{i\in I} c_i \left(g_{i-f_i}\right)\right\| \leq \frac{\lambda}{1-\lambda} \|\sum_{i\in I} c_i\, g_i\|$.

The closeness bound can be related to a relative operator bound used in perturbation theory (see [197]). More specifically, if T^g , T^f denote the analysis operators associated, respectively, to the frames $\mathcal{G}$ and $\mathcal{F}$, then $c(\mathcal{G}, \mathcal{F})$ is the $(T^f)^*$ -bound of $(T^g)^* - (T^f)^*$ (in the terminology of Kato).
The next step is to correct the nonreflexivity of the closeness relation. We say that two frames $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ are near if $\mathcal{F}$ is close to $\mathcal{G}$ and $\mathcal{G}$ is close to $\mathcal{F}$. It is fairly easy to check that this is an equivalency relation. In this case we define the predistance between $\mathcal{F}$ and $\mathcal{G}$, denoted $d^0(\mathcal{F}, \mathcal{G})$ as the maximum between the two closeness bounds:

$$d^0(\mathcal{F}, \mathcal{G}) = max\big(c(\mathcal{F}, \mathcal{G}),\ c(\mathcal{G}, \mathcal{F})\big). \qquad (6)$$

It is easy to prove that d^0 is positive and symmetric, but does not satisfy the triangle inequality. This inconvenience can be removed if we define the (quadratic) distance between $\mathcal{F}$ and $\mathcal{G}$ by

$$d(\mathcal{F}, \mathcal{G}) = log(d^0(\mathcal{F}, \mathcal{G}) + 1). \qquad (7)$$

Then, as we shall see later (Theorem(1.1.7), this is a veritable distance (a metric) on the set of frames which are near to one another.

Since the nearness relation is an equivalency relation, we can partition the set of all frames on H, denoted $\mathcal{F}(H)$, into disjoint equivalent classes, indexed by an index set A:

$$\mathcal{F}(H) = \cup_{\alpha\in A}\, \varepsilon_\alpha \qquad (8)$$

with the following properties:

$$\varepsilon_\alpha \cap \varepsilon_\beta = \phi, \text{ for } \alpha \neq \beta$$

$$\forall \mathcal{F},\mathcal{G} \in \varepsilon_\alpha\ , d(\mathcal{F},\mathcal{G}) < \infty \text{ and } \forall \mathcal{F} \in \varepsilon_\alpha, \mathcal{G} \in \varepsilon_\beta \text{ with } \alpha \neq \beta\ , d(\mathcal{F},\mathcal{G}) = \infty.$$

Let π denote the index projection $\pi: \mathcal{F}(H) \to A$, with $\mathcal{F} \to \pi(\mathcal{F}) = \alpha$ if $\mathcal{F} \in \varepsilon_\alpha$. We shall prove that the partition (8) corresponds to the nondisjoint partition of l^2(I) into closed infinite dimensional subspaces. Moreover, the two equivalency relations introduced before are identical (i.e., two frames are near if and only if they are Q-equivalent) as we shall prove later.

For a frame $\mathcal{G}$ we denote by T^1 the set of tight frames which are quadratically close to $\mathcal{G}$ and by T^2 the set of tight frames such that $\mathcal{G}$ is close to them:

$$\mathcal{T}^1 = \{\mathcal{F} = \{f_i\}_{i\in I} | \mathcal{F} \text{ is a tight frame and } c(\mathcal{G},\mathcal{F}) < +\infty\}, \qquad (9)$$

$$\mathcal{T}^2 = \{\mathcal{F} = \{f_i\}_{i\in I} | \mathcal{F} \text{ is a tight frame and } c(\mathcal{F},\mathcal{G}) < +\infty\}. \qquad (10)$$

Let $d^1 : T^1 \to R_+, d^2 : T^2 \to R_+$ denote the map from each $\mathcal{F}$ to the associated closeness bound, i.e., $d^1(\mathcal{F}) = c(\mathcal{G},\mathcal{F})$ and $d^2(\mathcal{F}) = c(\mathcal{F},\mathcal{G})$. If $\mathcal{G}$ is a tight frame itself, then $\mathcal{G} \in \mathcal{T}^1 \cap \mathcal{T}^2$ and $\min d^1 = \min d^2 = 0$.

Consider now the intersection between these two sets

$$T = \mathcal{T}^1 \cap \mathcal{T}^2 = \{\mathcal{F} = \{f_i\}_{i\in I} | \mathcal{F} \text{ is a tight frame and } d(\mathcal{F},\mathcal{G}) < +\infty\} \subset \varepsilon_{\pi(\mathcal{G})}. \quad (11)$$

In this section we will be looking for the minima of the functions d^1, d^2 and $d|_T$. And we are mainly concerned with the relations introduced previously. We shall prove that Q-equivalence is the same as nearness (in other words, two frames are Q-equivalent if and only if they are near). The following lemmas are fundamental for all constructions and results in this section.

Lemma (1.1.1)[62]: Consider $\mathcal{F}_1 = \{f_i^1\}_{i\in I}$ and $\mathcal{F}_2 = \{f_i^2\}_{i\in I}$ two tight frames in H with frame bounds 1. Denote by T_1 and T_2 respectively their analysis operators. Then:

a) $Ran\, T_2 \subset Ran\, T_1$ if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are partial isometric equivalent;moreover,if J is the corresponding partial isometry,then $ker J \simeq Ran T_1 / Ran T_2$; more specifically $Ker\, J = T_1^*(Ran\, T_1 \cap (Ran\, T_2)^\perp)$;

b) $Ran T_1 = Ran\, T_2$ if and only if $\mathcal{F}_1$and $\mathcal{F}_2$ are unitarily equivalent.

Proof. 1. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are partial isometric equivalent. Then $f_i^2 = Jf_i^2$ and $T_2 = T_1 J^*$ for some partial isometry J. Obviously, $Ran\, T_2 \subset Ran\, T_1$. Now, recall that T_1 and T_2 are isometries from H onto their ranges (since $\mathcal{F}_1$ and $\mathcal{F}_2$ are tight frames with bound 1). Therefore they preserve the scalar product and linear independence. Thus,

$$Ran\, T_1 = T_1(Ran\, J^* \oplus Ker\, J) = T_1 J^*(H) \oplus T_1(Ker\, J) = Ran\, T_2 \oplus T_1(Ker\, J)$$

and $T_1(Ker\, J)$ is the orthogonal complement of Ran T_2into Ran T_1. On the other hand, $T_1^*|_{Ran\, T_1}$ is the inverse of $T_1 : H \to Ran\, T_1$; thus,

$$Ker\, J = T_1^*(Ran\, T_1 \cap (Ran\, T_2)^\perp)$$

fixing canonically the isometric isomorphism $Ker\, J \backsimeq Ran\, T_1/Ran\, T_2$. Conversely, suppose $Ran\, T_2 \subset Ran\, T_1$. Then, the two projectors are $P_1 = T_1 T_1^*$ onto $Ran\, T_1$ and $P_2 = T_2 T_2^*$ onto Ran T_2 and we have $P_1 T_2 = T_2$ .Now, consider $J: H \to H, J = T_2^* T_1$ which acts in the following way:

$$J(x) = \sum_{i\in I} \langle x, f_i^1 \rangle f_i^2.$$

We have

$$JJ^* = T_2^*T_1\,T_1^*T_2 = T_2^*P_1\,T_2 = T_2^*\,T_2 = 1.$$
We want to prove now that $f_j^2 = Jf_j^1$ for all j. We have, for fixed j,
$$Jf_j^1 - f_j^2 = \sum_{i\in I}\left(\langle f_j^1, f_i^1\rangle - \langle f_j^2, f_i^2\rangle\right) f_i^2 = T_2^* c$$
Where $c = \{c_i\}_{i\in I}, ci = \langle f_j^1, f_i^1\rangle - \langle f_j^2, f_i^2\rangle$. On the other hand,
$$0 = f_j^1 - \sum_{i\in I}\langle f_j^1, f_i^1\rangle f_i^1 = \sum_{i\in I}\left(\delta_{ij} - \langle f_j^1, f_i^1\rangle\right) f_i^1 = T_1^* a^j$$
where $a^j = \{a_i^j\}_{i\in I}, a_i^j = \delta_{ij} - \langle f_j^1, f_i^1\rangle$ and δ_{ij} is the Kronecker symbol. Similarly, $0 = T_2^* b^j$ with $b^j = \{b_i^j\}_{i\in I}$, $b_i^j = \delta_{ij} - \langle f_j^2, f_i^2\rangle$. Thus, $a^j \in Ker\,T_1^*$ and $b^j \in KerT_2^*$. But $Ker\,T_1^* = (Ran\,T_1)^{\perp} \subset (Ran\,T_2)^{\perp} = Ker\,T_2^*$.Therefore $a^j \in KerT_2^*$ and then $c^j = a^j - b^j \in KerT_2^*$ which means $T_2^*\,c^j = 0$ or $f_j^2 = Jf_j^1$. Moreover, $T_2 = T_1 J^*$ and, as we have proved before, $Ker\,J = T_1^*\,(Ran\,T_1 \cap (Ran\,T_2)^{\perp})$.
2. The conclusion comes from point 1: the partial isometry will have a zero kernel ($Ker\,J = \{0\}$) and therefore it is a unitary operator (recall that the range of J should be H). This ends the proof of the lemma.
Lemma (1.1.2)[62]: Consider $\mathcal{F}_1 = \{f_i^1\}_{i\in I}$ and $\mathcal{F}_2 = \{f_i^2\}_{i\in I}$ two frames in H. Let us denote by T_1 and T_2respectively, their analysis operators. Then:
a) $Ran\,T_2 \subset Ran\,T_1$ if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$are Q-partial equivalent for some bounded operator Q; furthermore, $KerQ = T_1^*\,(Ran\,T_1 \cap (Ran\,T_2)^{\perp})$.
b) $Ran\,T_1 = Ran\,T_2$ if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are Q-equivalent, for some invertible operator Q.
Proof. Let us denote by $S_1 = T_1^*\,T_1, S_2 = T_2^*T_2$ the frame operators.
a. Suppose $Ran\,T_2 \subset Ran\,T_1$. We have that $\mathcal{F}_1$ is Q-equivalent with $\mathcal{F}_1^{\#}$ $\left((f_i^1)^{\#} = S_1^{-1/2} f_i^1\right)$; $\mathcal{F}_1^{\#}$ is J-partial equivalent with $\mathcal{F}_2^{\#}$ from Lemma (1.1.1), where $J = (T_2^{\#})^* T_1^{\#}$ is a partial isometry, and $\mathcal{F}_2^{\#}$ is Q-equivalent with $\mathcal{F}_2$ with $Q = S_2^{1/2}$ ($f_i^2 = S_2^{\frac{1}{2}}(f_i^2)^{\#}$). By composing , we get $\mathcal{F}_1$is Q-partial equivalent with $\mathcal{F}_2$ *via* $Q = S_2^{1/2} J S_1^{-1/2}$. Furthermore, since S_1 and S_2 are invertible, $KerQ = S_1^{1/2}\,Ker\,J = T_1^*\,(Ran\,T_1 \cap (Ran\,T_2)^{\perp})$.

Conversely, if $\mathcal{F}_1$ is Q-partial equivalent with $\mathcal{F}_2$ and Q is the bounded operator relating $\mathcal{F}_1$ to $\mathcal{F}_2$, then $T_2 = T_1 Q^*$ and obviously $Ran\,T_2 \subset Ran\,T_1$. On the other hand, since $T_1^*\,T_1 = S_1$ is invertible, $Q = T_2^*\,T_1 S_1^{-1}$ and then $\mathcal{F}_1{}^{\#}$ is J-partial equivalent with $\mathcal{F}_2^{\#}$ with $J = S_2^{-1/2}\,Q S_1^{1/2}$. We have
$$JJ^* = S_2^{-1/2} Q S_1^{1/2} S_1^{1/2} Q^* S_2^{-1/2} = S_2^{-1/2} T_2^* P_1 T_2 S_2^{-1/2}$$
where $P_1 = T_1 S_1^{-1}\,T_1^*$ is the orthogonal projection onto Ran T_1. But $Ran\,T_2 \subset Ran\,T_1$; hence, $P_1 T_2 = T_2$. Thus, $JJ^* = S_2^{-1/2} T_2^* T_2 S_2^{-1/2} = 1$,proving that J is a partial isometry.Now we apply the conclusion of Lemma (1.1.1) and obtain that $Ker\,J = (T_1^{\#})^*(Ran\,T_1 \cap (Ran\,T_2)^{\perp})$. Substituting this into $KerQ = S_1^{1/2}\,Ker\,J$ we obtain the result.
b. The statement is obtained from (1) by observing that $KerQ = \{0\}$; since we also know that $RanQ = H, Q$ is therefore invertible with bounded inverse.
Lemma (1.1.3)[62]: Consider $\mathcal{F}_1 = \{f_i^1\}_{i\in I}$ and $\mathcal{F}_2 = \{f_i^2\}_{i\in I}$ two frames in H. Let us denote by $T_1$and T_2, respectively, their analysis operators. Then, $\mathcal{F}_1$ is close to $\mathcal{F}_2$ (i.e., $c(\mathcal{F}_1, \mathcal{F}_2) < \infty$) if and only if $\mathcal{F}_2$ is Q-partial equivalent with $\mathcal{F}_1$ for some bounded operator Q and therefore $Ran\,T_2 \subset Ran\,T_1$. Moreover, $c(\mathcal{F}_1, \mathcal{F}_2) = \|Q - 1\|$.

Proof. $\Longrightarrow$ Suppose $\mathcal{F}_1$ is close to $\mathcal{F}_2$.Then $\|\sum_{i\in I} c_i(f_i^1 - f_i^2)\| \le \lambda \|\sum_{i\in I} c_i f_i^2\|$ for $\lambda = c(\mathcal{F}_1, \mathcal{F}_2). If\ c = \{c_i\}_{i\in I} \in Ker\ T_2^*$,
then necessarily $c \in KerT_1^*$ Therefore, $Ker\ T_2^* \subset Ker\ T_1^*\ or\ Ran\ T_1 = (Ker\ T_1^*)^\perp \subset (KerT_2^*)^\perp = Ran\ T_2$. Now, applying Lemma (1.1.2) we get that $\mathcal{F}_2$ is Q-partial equivalent with $\mathcal{F}_1$. Then, $f_i^1 = Qf_i^2$ and if we denote $v = \sum_{i\in I} c_i f_i^2$ we have

$$\|(Q - 1)v\| \le \lambda \|v\| .$$

The smallest $\lambda \ge 0$ that satisfies the above inequality for any $v \in H$ is $\|Q - 1\|$.
Therefore $c(\mathcal{F}_1, \mathcal{F}_2) = \|Q - 1\|$.
$\Longleftarrow$Suppose $\mathcal{F}_2$ is Q-partial equivalent with $\mathcal{F}_1$. Then, it is easy to check that $c(\mathcal{F}_1, \mathcal{F}_2). = \|Q - 1\|$ and then $\mathcal{F}_1$ is close to $\mathcal{F}_2$. As a consequence of this lemma, we obtain the following result:

Theorem (1.1.4)[62]: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two frames. Then, they are near if and only if they are Q-equivalent for some invertible operator Q. Moreover,

$$d^0(\mathcal{F}_1, \mathcal{F}_2) = max(\|Q - 1\|, \|1 - Q^{-1}\|).$$

Applying this theorem to the set T defined in (11) we obtain the following corollary:

Corollary (1.1.5)[62]: Consider a frame $\mathcal{G} = \{g_i\}_{i\in I}$ in H and consider also the set T defined by (11). Then T is parametrized in the following way:
$T = \{\mathcal{F} = \{f_i\}_{i\in I} \,|f_i = \alpha U g_i^\#$ where $\alpha > 0$ and U is unitary$\}$.

Proof. Indeed, let $\alpha > 0$ and U be unitary. Then, by computing its frame operator one can easily check that $\mathcal{F} = \{f_i\}_{i\in I}, f_i = \alpha U g_i^\#$ is a tight frame with bound α^2. Conversely, suppose $\mathcal{F} = \{f_i\}_{i\in I} \in T$. Then, from Theorem (1.1.4) we obtain $f_i = Qg_i^\#$ for some invertible Q. We compute its frame operator:

$$S^{\mathcal{F}} = \sum_{i\in I}\langle ., f_i\rangle f_i = Q\big(\sum_{i\in I}\langle ., g_i^\#\rangle g_i^\#\big)Q^* = QQ^*$$

Therefore, $QQ^* = A\,.1$ which means that $\frac{1}{\sqrt{A}}Q$ is unitary. Thus $Q = \sqrt{A}\,U$ for some unitary U.

The following result makes a connection between the extension of the Paley and Wiener theorem given by Christensen in [193] and the relations introduced so far.

Theorem (1.1.6)[62]: Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame in H and $\mathcal{G} = \{g_i\}_{i\in I}$ be a set of vectors in H. Suppose there exists $\lambda \in [0, 1)$ such that

$$\left\|\sum\nolimits_{i\in I} c_i(g_i - f_i)\right\| \le \lambda \left\|\sum\nolimits_{i\in I} c_i f_i\right\|$$

for any $n \in N$ and $c_1, c_2, ...$ in C. Then $\mathcal{G}$ is a frame in H and
(a) $\mathcal{G}$ is Q-equivalent with $\mathcal{F}$;
(b) if T^f and T^g are the analysis operators associated respectively to $\mathcal{F}$ and $\mathcal{G}$, then $Ran\ T^f = Ran\ T^g$;
(c) $c(\mathcal{G}, \mathcal{F}) \le \lambda < 1$ and $d^0(\mathcal{G}, \mathcal{F}) < \infty$.

Proof. The conclusion that $\mathcal{G}$ is a frame follows from a stability result proved by Christensen in [193]. As we have checked before, from $c(\mathcal{G}, \mathcal{F}) < 1$ we get $c(\mathcal{F}, \mathcal{G}) \le \frac{\lambda}{1-\lambda} < \infty$. Therefore, $\mathcal{F}$ and $\mathcal{G}$ are near and we can apply Theorem (1.1.4) and complete the proof. Theorem (1.1.4) allows us to partition the set of all frames on H, denoted $\mathcal{F}(H)$, into equivalent classes, as follows: $\mathcal{F}(H) = \bigcup_{\alpha\in A} \varepsilon_\alpha$ where $\varepsilon_\alpha \subset \mathcal{F}(H)$ is a set of frames such that any $\mathcal{F}, \mathcal{G} \in \varepsilon_\alpha, \mathcal{F}$ is Q-equivalent with $\mathcal{G}$ or, equivalent, $\mathcal{F}$ is near to $\mathcal{G}$. Therefore, for each index $\alpha \in A$, the function $d^0 : \varepsilon_\alpha \times \varepsilon_\alpha \to R_+$ is well-defined and finite. We want to show now that the function $d : \varepsilon_\alpha \times \varepsilon_\alpha \to R_+ , d(\mathcal{F}, \mathcal{G}) = log(1 + d^0(\mathcal{F}, \mathcal{G}))$ is a distance on each class ε_α.

Theorem (1.1.7)[62]: The function d defined above is a distance on ε_α. Moreover, for any $\mathcal{F} \in \varepsilon_\alpha$ and $\mathcal{G} \in \mathcal{F}(H)$, if $d\,(\mathcal{F}, \mathcal{G}) < \infty$, then $\mathcal{G} \in \varepsilon_\alpha$.

Proof. The second part of the statement is immediate: if $d(\mathcal{F},\mathcal{G})$is finite so is $d^0(\mathcal{F},\mathcal{G})$; hence, $\mathcal{F}$ is close to $\mathcal{G}$ and therefore they belong to the same class.
To show that d is a distance we need to check only the triangle inequality. Let $\mathcal{F},\mathcal{G},\mathcal{H} \in \varepsilon_\alpha$ then, there exist Q and R invertible bounded operators on H such that $g_i = Qf_i, h_i = Rg_i$ and therefore $h_i = RQf_i$. We have

$$d(\mathcal{F},\mathcal{G}) = log(1 + max(\|Q - 1\|, \|Q^{-1} - 1\|))$$
$$d(\mathcal{G},\mathcal{H}) = log(1 + max(\|R - 1\|, \|R^{-1} - 1\|))$$
$$d(\mathcal{F},\mathcal{H}) = log(1 + max(\|RQ - 1\|, \|Q^{-1}R^{-1} - 1\|))$$

and

$$\begin{aligned}\|RQ - 1\| &= \|(R - 1)(Q - 1) + R + Q - 2\| \\ &\leq \|R - 1\| . \|Q - 1\| + \|R - 1\| + \|Q - 1\| \\ &= (\|R - 1\| + 1)(\|Q - 1\| + 1) - 1.\end{aligned}$$

Hence,

$$log(\|RQ - 1\| + 1) \leq log(\|R - 1\| + 1) + log(\|Q - 1\| + 1).$$

Similarly for $\|Q^{-1}R^{-1} - 1\|$ and therefore $d(\mathcal{F},\mathcal{H}) \leq d(\mathcal{F},\mathcal{G}) + d(\mathcal{G},\mathcal{H})$.

The next step is to relate the partition (8)with the set of infinite dimensional closed subspaces of l^2(I). We suppose H is infinite dimensional and I is countable.Otherwise, the following result still holds providing we replace "infinite dimensional closed subspaces of dimension equal to the dimension of H".

Let us denote by $S(l^2(I))$ the set of all infinite dimensional closed subspaces of $l^2(I)$. Then Lemma (1.1.2) and Theorem (1.1.4) assert that $\mathcal{F}(H)$ is mapped into $S(l^2(I))$ by

$$i : \mathcal{F}(\mathcal{H}) \to S\big(l^2(I)\big), i(\varepsilon_\alpha) = Ran\, T \qquad (12)$$

where T is the analysis operator associated to any frame $\mathcal{F} \in \varepsilon_\alpha$. The natural question that can be asked is whether i is surjective, i.e., if for any closed infinite dimensional subspace of $l^2(I)$ we can find a corresponding frame in $\mathcal{F}(H)$. The answer is yes as the following theorem proves (see Christensen [192], Aldroubi [88] or Holub [196] for this type of argument).

Theorem (1.1.8)[62]: For any infinite dimensional closed subspace E of $l^2(I)$ there exists a frame $\mathcal{F} \in \mathcal{F}(H)$ (and therefore a class ε_α) such that $i(\mathcal{F}) = E$ (in other words, $Ran\, T = E$ with T the analysis operator associated to $\mathcal{F}$). Therefore, i, considered from the set of classes $\varepsilon_\alpha$into $S(l^2(I))$, is a bijective mapping.

Proof. Let $E \subset l^2(I)$ be an infinite dimensional closed subspace. Choose an orthonormal basis $\{d_i\}_{i\in I}$ in E and a basis $\{e_i\}_{i\in I}$ in H (recall H is infinite dimensional and I countable). Let $pi : l^2(I) \to C$ be the canonical projection, $p_i(c) = c_i$, where $c = \{c_j\}_{j\in I}$, let $i \in I$ and $P : l^2(I) \to C$ be the canonical projection onto E.

Let us denote by $\{\delta_i\}_{i\in I}$ the canonical basis in $l^2(I)$, i.e., $\delta_i = \{\delta_{ij}\}_{j\in I}$.

Then, it is known (see [196]) that $\{P\delta_i\}_{i\in I}$ is a tight frame with bound 1 in E (and any tight frame indexed by I with bound 1 in E is of this form, i.e., the orthogonal projection of some orthonormal basis of $l^2(I)$, since

$$\sum_{i\in I}\langle c, P\delta_i\rangle P\delta_i = P\sum_{i\in I}\langle Pc, \delta_i\rangle \delta_i = Pc = c, \quad \forall c \in E$$

We define a tight frame with bound 1 in H in the following way:

$$f_i = \sum_{j\in I}\langle P\delta_i, d_j\rangle e_j = \sum_{j\in I}\langle \delta_i, d_j\rangle e_j = \sum_{ji} p_i\left(d_j\right)e_j .$$

It is easy to show that $f_i's$ are well defined, since $\|f_i\|^2 = \sum_{j\in I}\left|\langle P\delta_i, d_j\rangle\right|^2 = \|P\delta_i\|^2 < \infty$.
Let T be the analysis operator associated to $\{f_j\}_{i\in I}$ and $x \in H$ be arbitrary. Then

$$\langle x, f_i\rangle = \sum_{j\in I} p_i\big(d_j\big)\langle x, e_j\rangle = p_i\big(\sum_{j\in I}\langle x, e_j\rangle d_j\big), \quad \forall i \in I$$

Thus, $T(x) = \{\langle x, f_i\rangle\}_{i\in I} = \sum_{j\in I}\langle x, e_j\rangle d_j$ and obvious Ran $T = E$. It is simple to check that $Tf_i = P\delta_i$ and therefore, $\{f_i\}_{i\in I}$is a tight frame with bound 1.

We are concerned here with the closeness and distance functions d^1, d^2 and $d|_T$ introduced earlier. In fact, we would like to characterize the minima of these functions. Here is the main result:

Theorem (1.1.9)[62]: Consider $\mathcal{G} = \{g_i\}_{i\in I}$ a frame in H with optimal frame bounds $A; B$ and consider the sets $\mathcal{T}^1$, $\mathrm{T}^2$and T introduced in (9), (10) and (11). Let us denote by $\theta = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$ and $\rho = \frac{1}{4}(logB - logA)$. Then the following conclusions hold:

(a) The values of the minima of d^1, d^2 and $d|_\mathrm{T}$ are given by

$$min\, d^1 = \ min d^2 \ = \theta\ min\, d|_\mathrm{T} = \rho.$$

(b) These values are achieved by the following scalings of the associated tight frames of $\mathcal{G}$:

$$\mathcal{F}^1 = \{f_i^1\}_{i\in I}\, , f_i^1 = \frac{\sqrt{A}+\sqrt{B}}{2} g_i^\# \qquad (13)$$

$$\mathcal{F}^2 = \{f_i^2\}_{i\in I}\, , f_i^2 = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} g_i^\#, \qquad (14)$$

$$\mathcal{F}^0 = \{f_i^0\}_{i\in I}\, , f_i^0 = \sqrt[4]{AB}\, g_i^\# \qquad (15)$$

Hence, $d^1(\mathcal{F}^1) \ = d^2\ (\mathcal{F}^2) = \theta\ and\ d(\mathcal{F}^0) \ = \rho.$

(c) Any tight frame that achieves the minimum of one of the three functions d^1, d^2 or d is unitarily equivalent with the corresponding solution (13), (14) or (15) in the following way:

$$(d^1)^{-1}(\theta) = \{K = \{k_i\}_{i\in I} | k_i = Uf_i^1,$$
$$U \text{ unitary and} \left\| U - \frac{2}{\sqrt{A}+\sqrt{B}} S^{1/2} \right\| = \theta\} \qquad (16)$$

$$(d^2)^{-1}(\theta) = \{K = \{k_i\}_{i\in I} | k_i = Uf_i^2$$
$$U \text{ unitary and} \left\| U - \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} S^{-1/2} \right\| = \theta\} \qquad (17)$$

$$d^{-1}(\rho) == \{K = \{k_i\}_{i\in I} | k_i = Uf_i^0,$$
$$U \text{ unitary and} \| U - \sqrt[4]{AB} S^{-1/2} \| = \left\| U - \frac{1}{\sqrt[4]{AB}} S^{1/2} \right\| = \rho\} \qquad (18)$$

where S is the frame operator associated to $\mathcal{G}$. Moreover, any unitary operator that parametrizes $(d^1)^{-1}(\theta), (d^2)^{-1}(\theta)\ or\ d^{-1}(\rho)$ as above, has the value 1 in its spectrum.

Proof. If $\mathcal{G}$ is a tight frame, then $\mathcal{F}^1 = \mathcal{F}^2 = \mathcal{F}^0 \ = \ \mathcal{G}$ and $\theta = \rho = \ 0$ and the problem is solved. Therefore, we may suppose that $A \ < \ B$.

We show this in the following way: In the first step we check that $d^1(\mathcal{F}^1) = d^2(\mathcal{F}^2) = \theta$ and $d(\mathcal{F}^0) \ = \rho$. Then, since $\theta < 1$, it follows that the infimum of $d^1$and d^2 are less than 1. Now, using Corollary (1.1.5) and Theorem (1.1.4) we can reduce our problem to an infimum of an operator norm. In the third step we will prove two lemmas, one to be applied to d^1 and d^2, and the other to d, and this will end, the proof.

(a) Let us check that (13), (14), (15) achieve the desired values for d^1, d^2 and d, respectively. For $f_i^1 = Qg_i$ with $Q = \frac{\sqrt{A}+\sqrt{B}}{2} S^{-1/2}$ we have $d^1(\mathcal{F}^1) = c(\mathcal{G}, \mathcal{F}^1) = \|1 - Q^{-1}\|$. Now,$\sqrt{A} \leq S^{1/2} \leq \sqrt{B}$ where the inequalities cannot be improved. Therefore,

$$-\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} \leq 1 - Q^{-1} \leq \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$$

which means that $\|1 - Q^{-1}\| = \theta$. Similarly, for $f_i^2 = Lg_i$ with $L = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} S^{-1/2}$ we have $d^2(\mathcal{F}^2) = c(\mathcal{F}^2, \mathcal{G}) \ = \|L - 1\|$ and a similar calculus shows that $d^2(\mathcal{F}^2) = \theta$. For F^0 we have $f_i^0 = Rg_i$ with $R = \sqrt[4]{AB} S^{-1/2}$ and therefore

$$d(\mathcal{F}^0) = log\big(1 + max(\ \|R \ - \ 1\|\, , \|1 \ - \ R^{-1}\|)\big).$$

Now, an easy calculation shows that

$$\|R \ - \ 1\| = \|1 \ - \ R^{-1}\| = max\left(\sqrt[4]{\frac{B}{A}} - 1, 1 - \sqrt[4]{\frac{A}{B}}\right) = \sqrt[4]{\frac{B}{A}} - 1\ .$$

Therefore, $d(\mathcal{F}^0) = log\sqrt[4]{\frac{B}{A}} = \rho.$

(b) Since we are looking for the infimum of the functions d^1, d^2 and since $\theta < 1$ we may then restrict our attention to only the tight frames $\mathcal{F} \in T^1$ (or to T^2) such that $d^1(\mathcal{F}) < 1$ (respectively, $d^2(\mathcal{F}) < 1$). But this implies also that $d^2(\mathcal{F}) < \infty$ (respectively, $d^1(\mathcal{F}) < \infty$). Therefore, we may restrict our attention only to tight frames in $T^1 \cap T^2 = T$.
Corollary (1.1.5) tells us that these frames must have the form $\mathcal{F} = \{f_i\}_{i \in I}$ and $f_i = \sqrt{C}Ug_i^{\#} = \sqrt{C}US^{-1/2}g_i$ for some $C > 0$ and U unitary. Hence

$$d^1(\mathcal{F}) = \left\|1 - \frac{1}{\sqrt{C}}S^{\frac{1}{2}}U^{-1}\right\| = \left\|\frac{1}{\sqrt{C}}S^{\frac{1}{2}} - U\right\|, \qquad (19)$$

$$d^2(\mathcal{F}) = \left\|\sqrt{C}US^{\frac{-1}{2}} - 1\right\| = \left\|\sqrt{C}S^{\frac{1}{2}} - U\right\|, \qquad (20)$$

$$d^0(\mathcal{F}) = max\left(\left\|\frac{1}{\sqrt{C}}S^{\frac{1}{2}} - U\right\|, \left\|\sqrt{C}S^{-\frac{1}{2}} - U\right\|\right). \qquad (21)$$

To minimize d is equivalent to minimizing d^0; since d^0 has a simpler expression, we prefer to work with d^0 from now on. Thus, our problem is reduced to find minima of the operator norms (19), (20), (21) subject to $C > 0$ and U unitary.

(c) The next step is to solve these norm problems. For d^1 and d^2 we apply the following lemma to be proved later:

Lemma (1.1.10)[62]: Consider R a selfadjoint operator on H with $a = \|R^{-1}\|^{-1}$ and $b = \|R\|$. Then, the solution of the following inf-problem

$$\mu = \inf_{\substack{\alpha > 0 \\ u\ unitary}} \|\alpha R - U\| \qquad (22)$$

is given by $\mu = \frac{b-a}{b+a}$ and $\alpha = \frac{2}{a+b}$. This infimum is achieved by the identity operator; any other unitary U that achieves the infimum must have 1 in its spectrum.

If we apply this lemma with $R = S^{\frac{1}{2}}, \alpha = \frac{1}{\sqrt{C}}$ and $a = \sqrt{A}, b = \sqrt{B}$, then we get $\mu = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} \equiv \theta$ and $\alpha = \frac{2}{\sqrt{B}+\sqrt{A}}$, hence the parametrization (16) of the solutions. This shows (19). For (20) we apply the lemma with $R = S^{-1/2}, \alpha = \sqrt{C}$ and $a = \frac{1}{\sqrt{B}}, b = \frac{1}{\sqrt{A}}$. We get $\mu = \theta$ and $\alpha = \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}}$, hence the parametrization (17) of the solutions.

Proof. Let $\delta = \alpha - \frac{2}{a+b}$. We denote by $\sigma(X)$ the spectrum of the operator X. Thus, $a, b \in \sigma(R)$. Now, by Weyl 's criterion (see for instance, [200]), there are two sequences of normed vectors in H, $(v_n)_{n \in \mathbf{N}}$ and $((w_n)_{n \in \mathbf{N}}$ such that

$$\|v_n\| = \|w_n\| = 1 \text{ and } lim_n \ \|(R - a)v_n\| = 0, lim_n \ \|(R - b)w_n\| = 0.$$

Consider $\delta > 0$. Let $\varepsilon = \frac{\delta}{2}b$. Then there exists an index N such that for any $n > N, \|Rw_n - bw_n\| \leq \frac{\varepsilon}{\alpha}$. We get $\|\alpha Rw_n\| \geq \alpha b - \varepsilon > 1$ and

$$\|(\alpha R - U)w_n\| \geq \|\alpha Rw_n\| - \|Uw_n\| = \|\alpha Rw_n\| - 1 \geq \alpha b - \varepsilon - 1 = \frac{b-a}{b+a} + \varepsilon.$$

Therefore,

$$\|\alpha R - U\| \geq \frac{b-a}{b+a} + \varepsilon > \frac{b-a}{b+a} = \mu. \qquad (23)$$

Consider now $\delta < 0$. Let $\varepsilon = \frac{\delta}{2}a > 0$. Then, there exists an N such that for any $n > N$, $\|Rv_n - av_n\| \leq \frac{\varepsilon}{\alpha}$. We get $\|\alpha Rv_n\| \leq \alpha a + \varepsilon < 1$ and

$$\|(\alpha R - U)v_n\| \geq |\|\alpha Rv_n\| - \|Uv_n\|| = 1 - \|\alpha Rv_n\| \geq 1 - \alpha a - \varepsilon = \frac{b-a}{b+a} + \varepsilon.$$

Therefore,

$$\|\alpha R - U\| \geq \frac{b-a}{b+a} + \varepsilon > \frac{b-a}{b+a} = \mu. \qquad (24)$$

From (23) and (24) we observe that the infimum of $\|\alpha R - U\|$ has the value $\frac{b-a}{b+a}$ and may be achieved only if $\delta = 0$, i.e., $\alpha = \frac{2}{a+b}$. Thus, the first part of the lemma has been showed.

The set of all unitary U that achieves the infimum is then given by

$$\{U : H \to H | U \text{ unitary and } \left\| \frac{2}{a+b} R - U \right\| = \frac{b-a}{b+a}\}. \qquad (25)$$

We still have to prove that the set (25) contains the identity and 1 is in spectrum of any unitary operator from this set. From $a \le R \le b$ we get $-\frac{b-a}{b+a} \le \frac{2}{a+b} R - 1 \le \frac{b-a}{b+a}$ .Therefore, $\left\| \frac{2}{a+b} R - 1 \right\| \le \frac{b-a}{b+a}$ .But, us we have showed, $\frac{b-a}{b+a}$ is the minimum that can be achieved. Therefore, $\left\| \frac{2}{a+b} R - 1 \right\| = \frac{b-a}{b+a} = \mu$ and thus, 1 is in the set (25).

Now recall the sequence $(v_n)_n$ and the inequality (23) which is realized on $(v_n)_n$. For U in the set (25) we have $\left\| \left(\frac{2}{a+b} R - U \right) v_n \right\| \to \mu$. But

$$\left\| \left(\frac{2}{a+b} R - U \right) v_n \right\|^2 = \frac{4}{(a+b)^2} \langle v_n, R^2 v_n \rangle - \frac{2}{a+b} \langle v_n, (RU + U^* R) v_n \rangle + 1.$$

From $(R - a)v_n \to 0$ we get $\langle v_n, R^2 v_n \rangle \to a^2$. Therefore,

$$\lim_n \langle v_n, (RU + U^* R) v_n \rangle = \frac{a+b}{2} \left(\frac{4a^2}{(a+b)^2} + 1 - \sigma^2 \right) = 2a$$

Now:

$$RU + U^* R = (R - a)U + U^*(R - a) + a(U + U^*)$$

and the previous limit gives $\lim_n \langle v_n, (U + U^*) v_n \rangle = 2$.

Therefore,

$$\|(U - 1)v_n\|^2 = \langle v_n (2 - (U + U^*) v_n \rangle \to 0$$

or $\lim_n \|(U - 1)v_n\| = 0$ which proves $1 \in \sigma(U)$.

Lemma (1.1.11)[62]: Consider R a bounded invertible selfadjoint operator on H with $a = \|R^{-1}\|^{-1}$ and $b = \|R\|$. Then, the solution of the following optimization problem:

$$\mu = \inf_{\substack{a>0 \\ \text{U unitary}}} \max \left(\|\alpha R - U\|, \left\| \frac{1}{\alpha} R^{-1} - U \right\| \right) \qquad (26)$$

is given by $\mu = \sqrt{\frac{b}{a}} - 1$, $\alpha = \frac{1}{\sqrt{ab}}$ and U in the set (27)

$$\{U: H \to H | U \ unitary \text{ and} \left\| \frac{1}{\sqrt{ab}} R - U \right\| = \|\sqrt{ab} R^{-1} - U\| = \sqrt{\frac{b}{a}} - 1\}.$$

Moreover, the set (25) contains the identity and therefore, is not empty and the spectrum of any U contains 1. The solution for d^0 is now straightforward: we apply this lemma to (21) with $R = S^{1/2}$, $\alpha = \frac{1}{\sqrt{C}}$ and $a = \sqrt{A}$, $b = \sqrt{B}$. We get $\mu = mind^0 = \sqrt[4]{\frac{B}{A}} - 1$ and $\alpha = \frac{1}{\sqrt[4]{AB}}$, hence the parametrization (18) of the solution and the proof of theorem is complete.

Proof. First, let us solve the following scalar problem:

$$\bar{\mu} = \inf_{\alpha > 0} max \left(\max_{a \le x \le b} |\alpha x - 1|, \max_{a \le x \le b} \left| \frac{1}{\alpha x} - 1 \right| \right) \qquad (28)$$

Because of monotonicity,

$$\max_{a \le x \le b} |\alpha x - 1| = max(|\alpha a - 1|, |\alpha b - 1|),$$

$$\max_{a \le x \le b} \left| \frac{1}{\alpha x} - 1 \right| = max \left(\left| \frac{1}{\alpha a} - 1 \right|, \left| \frac{1}{\alpha b} - 1 \right| \right)$$

Therefore, $\bar{\mu} = \inf_{\alpha > 0} f(\alpha)$ where

$$f(\alpha) = max \left(|\alpha a - 1|, |\alpha b - 1|, \left| \frac{1}{\alpha a} - 1 \right|, \left| \frac{1}{\alpha b} - 1 \right| \right)$$

It is now simple to check that the infimum may be achieved only when at least two moduli are equal. This condition is fulfilled at the following points:

$$\alpha_1 = \frac{2}{a+b}; \alpha_2 = \frac{1}{a}; \alpha_3 = \frac{1}{a} \pm \frac{1}{a}\sqrt{1-\frac{a}{b}}; \alpha_4 = \frac{1}{\sqrt{ab}}; \alpha_5 = \frac{1}{b}; \alpha_6 = \frac{a+b}{2ab}$$

We evaluate $f(\alpha)$ at these points and we get

$$f(\alpha_1) = \frac{b-a}{2a}\ ; f(\alpha_2) = \frac{b-a}{a}; f(\alpha_3) = \frac{\sqrt{b-a}}{a}\left(\sqrt{b}-\sqrt{b-a}\right),$$

$$f(\alpha_4) = \sqrt{\frac{b}{a}} - 1\ ; f(\alpha_5) = \frac{b-a}{a}; f(\alpha_6) = \frac{b-a}{2a}$$

It is obvious now that $f(\alpha_4) \le f(\alpha_1) = f(\alpha_6) \le f(\alpha_2) = f(\alpha_5) \le f(\alpha_3)$ and therefore, $\bar{\mu} = f(\alpha_4) = \sqrt{\frac{b}{a}} - 1$ and $\alpha_{optim} = \alpha_4 = \frac{1}{\sqrt{ab}}$. Observe also that for $\alpha = \alpha_4$ we have $\max_{a\le x\le b}|\alpha_4 x - 1| = \max\limits_{a\le x\le b}\left|\frac{1}{\alpha_4 x} - 1\right|$.

Let us now return to the norm problem (26). Our claim is that the infimum is achieved for $\alpha = \frac{1}{\sqrt{ab}} = \alpha_4$ and $U = 1$ (the identity) and the value of the infimum is $\mu = \sqrt{\frac{b}{a}} - 1 = \bar{\mu}$. The solution of the scalar problem (28) proves also that the set (27) contains the identity. We are now going to prove that $\mu = \bar{\mu}$ is the optimum and $\alpha = \alpha_4$. As in the previous lemma, consider $(v_n)_{n\ge1}$ and $(w_n)_{n\ge1}$ two sequences of normed vectors in H ($\|v_n\| = \|w_n\| = 1$) such that $\lim_n\|(R-a)v_n\| = 0, \lim\limits_n\|(R-b)w_n\| = 0$.

It is simple to check that $\lim\limits_n\left\|\left(R^{-1}-\frac{1}{a}\right)v_n\right\| = 0$ and $\lim\limits_n\left\|\left(R^{-1}-\frac{1}{b}\right)w_n\right\| = 0$ hold also. Now, consider some $\alpha > 0, \alpha \ne \alpha_4 = \frac{1}{\sqrt{ab}}$. Then, as the scalar problem proved, we have

either $\max\limits_{a\le x\le b}|\alpha x - 1| > \bar{\mu}$ *or* $\max\limits_{a\le x\le b}\left|\frac{1}{\alpha x} - 1\right| > \bar{\mu}$ (29)

Suppose the first inequality holds. Now, either $|\alpha a - 1| > \bar{\mu}$ or $|\alpha b - 1| > \bar{\mu}$. In the former case we use the sequence $(v_n)_n$ as follows: Let $\varepsilon = \frac{1}{2}(|\alpha a - 1| - \bar{\mu}) > 0$ and *let* N_ε be such that$\|(R-a)v_n\| \le \frac{\varepsilon}{\alpha}$ for any $n \ge N_\varepsilon$. Then

$\|(\alpha R - U)v_n\| \ge |\|\alpha R v_n\| - \|U v_n\||=|\alpha\|\alpha v_n + (R-a)v_n\| - 1| \ge |\alpha a - 1| - \alpha\|(R-a)v_n\| > \bar{\mu} + \varepsilon$ which implies $\|\alpha R - U\| > \bar{\mu} + \varepsilon$Similarly, in the later case($|ab - 1>\mu-$ *we take* $\varepsilon=12\ ab-1-\mu>0$ *and* $N\varepsilon$ *such that*

$$\|(R-b)w_n\| \le \frac{\varepsilon}{\alpha} \text{ for any } n \ge N_\varepsilon.\text{Therefore,}$$

$$\|(\alpha R - U)w_n\| \ge |\|\alpha R w_n\| - \|U w_n\||=|\alpha\|b w_n + (R-b)w_n\| - 1|$$
$$\ge |\alpha b - 1| - \alpha\|(R-b)w_n\| > \bar{\mu} + \varepsilon.$$

Thus, in both cases we obtain$\|\alpha R - U\| > \bar{\mu}$. If the second inequality in (29) holds, a similar argument can be used to prove that, for $\alpha \ne \alpha_4$ we have

$$\left\|\frac{1}{\alpha}R^{-1} - U\right\| > \bar{\mu}$$

Therefore, the optimum in (26) is achieved for $\alpha = \frac{1}{\sqrt{ab}}$ and the value of it is $\mu = \sqrt{\frac{b}{a}} - 1$. It is obvious now that the set of unitary operators that achieve the optimum is given by (27) and also that the identity operator is in that set. The only problem that still remains to be proved is that all these unitary operators have 1 in their spectra.

The previous argument shows the following conclusion fix $\delta_0 >$ small enough and let U be in the set (27). Then, for any $0 < \delta \le \delta_0$ the following inequality holds:

$$\bar{\mu} \leq \left\| \left(\delta R + \frac{1}{\sqrt{ab}} R - U \right) w_n \right\|$$

for $n \geq N_\delta$ here N_δ is an integer depending on δ. Then, $\bar{\mu} \leq \left\| \left(\delta R + \frac{1}{\sqrt{ab}} R - U \right) w_n \right\| < \delta \|R\| + \bar{\mu}$ for $n \geq N_\delta$, and it is fairly easy to prove now that $\left\| \left(\frac{1}{\sqrt{ab}} R - U \right) w_n \right\| \to \bar{\mu}$ when $n \to \infty$. Now, by repeating the argument given in the previous lemma we obtain $\lim_n \|(U - 1) w_n\| = 0$ which proves $1 \in \sigma(U)$ and the lemma is showed. (or In this section we introduced and studied a distance between Hilbert frames having the same index set I. This distance partitions the set of frames into equivalency classes characterized (and indexed) by closed subspaces of the space of coefficients $l^2(I)$. Thus, two frames are at a finite distance if and only if their analysis operators have the same (closed) range in $l^2(I)$ and this happens if and only if there exists a bounded and invertible operator on the Hilbert space that maps one frame set into the other.)Next we determined the closest, respectively nearest, tight frame to a given frame. It turns out that these tight frames are scaled versions of the associated tight frame.We point out that the entire theory can be carried out on the set of Hilbert frames over different Hilbert spaces, but indexed by the same index set. All the results are similar, the changes being straightforward.

As a final remark we acknowledge that Lemmas (1.1.1) and (1.1.2) have also been independently obtained by D. Han and D. R. Larson in a recent paper ([79]).

Section (1.2)[45]: Projection Decompositions of Operators

Frames were first introduced by Dufflin and Schaeffer [75] in 1952 as a component in the development of non-harmonic Fourier series, and a paper by Daubechies, Grossmann, and Meyer [94] in 1986 initiated the use of frame theory in signal processing. A frame on a separable Hilbert space $\mathcal{H}$ is defined to be a complete collection of vectors$\{ x_i \} \subset \mathcal{H}$ for which there exist constants $0 < A \leq B$ such that for any $x \in \mathcal{H}$, $A\|x\|^2 \leq \sum_i |\langle x, x_i \rangle|^2 \leq B\|x\|^2$.

The constants A and B are known as the frame bounds. The collection is called a tight frame if $A = B$, and a Parseval frame if $A = B = 1$. (In some of the existing literature, Parseval frames have been called normalized tight frames; however it should be noted that other authors have used the term normalized to describe a frame consisting only of unit vectors.) The length of a frame is the number of vectors it contains, which cannot be less than the Hilbert space dimension. References in the study of frames include [184], [79], and [185].

Hilbert space frames are used in a variety of signal processing applications, often demanding additional structure. Tight frames may be constructed having specified length, components having a predetermined sequence of norms, or with properties making them resilient to erasures. For examples, see [36], [47], and [49]. One area of rapidly advancing research lies in describing tight frames in which all the vectors are of equal norm, and thus are elements of a sphere, [36]. Since frame theory is geometric in nature, it is natural to ask which other surfaces in a finite or infinite dimensional Hilbert space contain tight frames.

By an ellipsoidal surface we mean the image of the unit sphere $S_1 = \{x : \|x\| = 1\}$ under a bounded invertible operator $T \in B(\mathcal{H})$. Let $\mathcal{E}_T$ denote the ellipsoidal surface $\mathcal{E}_T = TS_1$. A frame contained in $\mathcal{E}_T$ is called an ellipsoidal frame, and if it is tight it is called an ellipsoidal tight frame (ETF) for that surface. We say that a frame bound K is attainable for $\mathcal{E}_T$ if there is an ET for $\mathcal{E}_T$ with frame bound K. If an ellipsoid $\mathcal{E}$ is a sphere we will call a frame in $\mathcal{E}$ spherical.

Given an ellipsoid $\mathcal{E}$, we can assume $\mathcal{E} = \mathcal{E}_T$, where T is a positive invertible operator. Given A an invertible operator, let $A^* = U|A^*|$ be the polar decomposition where $|A^*| = (AA^*)^{1/2}$. Then $A = |A^*|U^*$. By taking $T = |A^*|$ we see that $TS_1 = AS_1$. Moreover it is easily seen that the positive operator T for which $\mathcal{E} = \mathcal{E}_T$ is unique.

Throughout the section, $\mathcal{H}$ will be a separable real or complex Hilbert space and for $x;\ y;\ u\ \in \mathcal{H}$, we will use the notation $x \otimes\ y$ to denote the rank-one operator $u \mapsto \langle u, y\rangle x$. Note that $\|x\| = 1$ implies that $x \otimes x$ is a rank-1 projection. There are three theorems in this section. The first gives an elementary construction of $ETF's$ when $\mathcal{H} = \mathbb{R}^n$, and is proved in this Section .

We note that, in the non-degenerate case, the definition of an ellipsoidal surface ε given in Theorem(1.2.2) is equivalent to the definition given in the introduction, specifying that the Hilbert space be $\mathbb{R}^n$. Indeed, if $a_i > 1$ for all $i = 1, \dots, n$ and if $D = diag(a_{1,}\ a_2, \dots, a_n)$, then $\sum_{i=1}^{n} a_i x_i^2 = 1\ iff\ \langle Dx, x\rangle = 1$ if and only if $\left\|D^{\frac{1}{2}}x\right\| 1\ \ iff\ \ D^{\frac{1}{2}}x \in S_1(\mathbb{R}^n)$ if and only if $x \in D^{-1/2}S_1(\mathbb{R}^n)$. So $\varepsilon = \varepsilon_T\ for\ T\ = D^{-1/2}$, and thus ε has the requisite form. To reverse this argument for a non-diagonal positive operator T, first diagonalize it by an orthogonal transformation given by rotations. Reversing the steps will then show that ε_T is equivalent to ε for some choice of positive constants $\{a_{1,}, \dots, a_n\}$. The second theorem is used to prove Theorem(1.2.3) in the infinite dimensional case. It has independent interest in operator theory, and to our knowledge is a new result. The proof, as well as the corresponding result infinite dimensions (Proposition (1.2.6)), is contained in this Section . Some preliminaries are required before we state Theorem(1.2.2).

It is well-known (see [187]) that a separably acting positive operator A decomposes as the direct sum of a positive operator $A_1$with nonatomic spectral measure and a positive operator A_2 with purely atomic spectral measure (i.e., a diagonalizable operator). For $B \in B(\mathcal{H})$, the essential norm of B is $\|B\|_{ess} := \ inf\ \{\|B - K\| : K$ is a compact operator in $B(\mathcal{H})\}$.

In the proof of Proposition (1.2. 11), we have the special case where A is a diagonal operator, $A = diag(a_{1,}\ a_2, \dots)$, with respect to some orthonormal basis. In this case, it is clear that $\|A\|_{ess} =\ sup\ \{\alpha\ > 0 : |a_i|\ \geq \alpha$ for infinitely many $i\}$.

For a positive operator A with spectrum $\sigma(A)$, we have $\|A\| =\ \sup\{\lambda : \lambda \in \sigma(A)\}$ and if A is invertible, then$\|A^{-1}\|^{-1} =\ \inf\{\lambda : \lambda \in \sigma(A)\}$. Similarly, $\|A\|_{ess} = \sup\{\lambda : \lambda \in \sigma_{ess}(A)$ and $\|A^{-1}\|_{ess}^{-1} =\ \inf\{\lambda : \lambda \in \sigma_{ess}(A)\}$.In particular, $\|A^{-1}\|^{-1} \leq\ \|A^{-1}\|_{ess}^{-1} \leq \|A\|_{\ ess} \leq \|A\|$.

For A a positive operator, we say that A has a projection decomposition if A can be expressed as the sum of a finite or infinite sequence of (not necessarily mutually orthogonal) self-adjoint projections, with convergence in the strong operator topology.

Note that in this theorem A need not be invertible. There are theorems in the literature (e.g., [188]) expressing operators as linear combinations of projections and as sums of idempotents (non self-adjoint projections). The decomposition in Theorem(1.2.9) is different in that each term is a self-adjoint projection rather than a scalar multiple of a projection.

The next theorem states that every ellipsoidal surface contains a tight frame. We also include some detailed information about the nature of the set of attainable frame bounds.

Lemma(1.2. 1)[45]: Let $n \in \mathbb{N}$, let $a_{1,}, \dots, a_n\ \geq\ 0$.be such that $\sum_1^n a_j\ = n$ and let

$$\varepsilon = \left\{x = (x_1, \dots, x_n)^t \in \mathbb{R}^n \middle| \sum_1^n a_j x_j^2\ = 1\right\}.$$

Then there is an orthonormal basis $v_{1,}, \dots, v_n$ for $\mathbb{R}^n$ consisting of vectors $v_j\ \in \varepsilon$.

Proof: Proceed by induction on n. The case $n = 1$ is trivial. Assume $n \geq 2$ and without loss of generality suppose $a_{1,} \geq 1$ and $a_{2,} \leq 1$. Let θ be such that $a_{1,}(cos\,\theta)^2 + a_2\ (sin\,\theta)^2 = 1$ and let $b_2 = a_1(sin\,\theta)^2 + a_2(cos\,\theta)^2$. Consider the rotation matrix

$$R = \begin{pmatrix} cos\theta & sin\theta & & \\ -sin\theta & cos\theta & & \\ & & 1 & \\ & & & \ddots 1 \end{pmatrix}$$

Then

$$R^{-1}\varepsilon = \{(y_1, \dots, y_n)^t \in \mathbb{R}^n | \\ y_1^n + 2(a_1 - a_2)y_1y_2cos\theta sin\theta + b_2y_2^2 + \sum_3^n a_j y_j^2 = 1\}.$$

We have $b_2 + \sum_3^n a_j = n - 1$. Let v be the subspace of $\mathbb{R}^n$ consisting of all vectors of the form $(0, x_2, \dots, x_n)^t$. By the induction hypothesis, there is n orthonormal basis $u_2, \dots, u_n$ for v consisting of vectors $u_j \in R^{-1}\varepsilon$. Let $u_1 = (1,0, \dots ,0)^t \in \mathbb{R}^n$, and let $v_j = Ru_j$. Then $v_1, \dots, v_n$ is an orthonormal basis for $\mathbb{R}^n$ consisting of vectors $v_j \in \varepsilon$.

In the case of a general ellipsoid, where $\sum_{j=1}^n a_j = r > 0$, the lemma gives a constant multiple of an orthonormal basis on the ellipsoid.

Theorem(1.2.2)[45]: Let $n;\ k \in \mathbb{N}$ with $n \le k$, let $a_1, \dots, a_n \ge 0$ be such that $r := \sum_1^n a_j > 0$ and consider the (possibly degenerate) ellipsoid

$$\varepsilon = \left\{ x = (x_1, \dots, x_n)^t \in \mathbb{R}^n \left| \sum_1^n a_j x_j^2 = 1 \right. \right\}.$$

Then there is a tight frame for $\mathbb{R}^n$ consisting of k vectors $u_1, \dots, u_k \in \varepsilon$.

This result is valid for degenerate ellipsoids (in which some of the major axes are infinitely long). Our method of proof provides geometric insight to the problem, but does not extend to infinite dimensions.

Proof. Consider the isometry $W: \mathbb{R}^n \to \mathbb{R}^k$ and the projection $P = W^*: \mathbb{R}^k \to \mathbb{R}^n$ given by

$$W(x_1, \dots, x_n)^t = (x_1, \dots, x_n, 0, \dots, 0)^t,$$
$$P(x_1, \dots, x_k)^t = ((x_1, \dots, x_n)^t.$$

Let $a_j = 0\ for\ n + 1 \le j \le k$ and let

$$\varepsilon' = \left\{ y = (y_1, \dots, y_k)^t \in \mathbb{R}^k \left| \sum_1^k a_j y_j^2 = 1 \right. \right\}.$$

By Lemma(1.2. 1), there is a multiple of an orthonormal basis $v_1, \dots, v_k$ for $\mathbb{R}^k$ consisting of vectors $v_1 \in \varepsilon'$. Let $u_j = Pv_j$. Then $u_j \in \varepsilon$. Moreover, $u_1, \dots, u_k$ is a tight frame for $\mathbb{R}^n$, because if $x \in \mathbb{R}^n$, then

$$\sum_{j=1}^k \left|\langle x, u_j\rangle\right|^2 = \sum_{j=1}^k \left|\langle Wx, v_j\rangle\right|^2 = \frac{k}{r}\, \|Wx\|^2 = \frac{k}{r}\|x\|^2.$$

Proposition(1.2.3)[45]: Let $A \in B(\mathcal{H})$ be a finite rank positive operator with integer trace k. If $k \in rank(A)$, then A is the sum of k projections of rank one.

Proof. We will construct unit vectors x1; $x_1, x_2, \dots, x_k$ so that A is the sum of the projections $x_i \otimes x_i$. The proof uses induction on k. Let $n = rank(A)$ and write $\mathcal{H}_n = range(A)$. If $k = 1$, then A must itself be a rank-1 projection.

Assume $k \ge 2$. Select an orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathcal{H}_n$ such that A can be written on $\mathcal{H}_n$ as a diagonal matrix with positive entries $a_1 \ge a_2 \dots a_n > 0$.

Case 1: $k > n$. In this case, we have $a_1 > 1$, so we can take $x_k = e_1$. The remainder on $\mathcal{H}_n$,

$$A - (x_k \otimes x_k) = diag(a_1 - 1, a_2, \dots, a_n),$$

has positive diagonal entries, still has rank n, and now has trace $k - 1 \ge n$. By the inductive hypothesis, the result holds.

Case 2: $k = n$. We now have that $a_1 \ge 1$ and $a_n \le 1$. Given any finite rank, self-adjoint $R \in B(\mathcal{H})$, let $\mu_n(R)$ denote the n-th largest eigenvalue of R counting multiplicity. Note

that $\mu_n(A-(e_1 \otimes e_1)) \geq 0, \mu_n(A-(e_n \otimes e_n)) \leq 0$, and $\mu_n(A-(x \otimes x))$ is a continuous function of $x \in \mathcal{H}_n$. Hence, there exists $y \in \mathcal{H}_n$ such that
$\mu_n(A-(y \otimes y))=0$. Choose $x_k=y$. Note the remainder $(A-(x_k \otimes x_k)) \geq 0$ and Trace $\left(A-(x_k \otimes x_k)\right)=n-1$, Rank $\left(A-(x_k \otimes x_k)\right)=n-1=k-1$.
Again, by the inductive hypothesis, the result holds. _

Lemma (1.2.4)[45]: Let $P_1, P_2, \ldots, P_n$ be mutually orthogonal projections on a Hilbert space $\mathcal{H}$, all of the same nonzero rank k, where k can be finite or Infinite. Let $r_1, r_2, \ldots, r_n$ be nonnegative real numbers, and let $r=\sum_1^n r_i$. Difine the operator

$$A = r_1 P_1 + r_2 P_2 + \cdots + r_n P_n.$$

If the sum r is an integer and $r \geq n$, then there exist rank-k projections $Q_1, \ldots, Q_r$ such that

$$A = Q_1 + Q_2 + \cdots + Q_r.$$

Proof. If $k=1$, then $r=trace(A)$ and we have rank $(A) \leq n \leq r$, so the result follows from Proposition (1.3.3). If $k>1$, each projection P_i can be written as a sum of k mutually orthogonal rank-1 projections:

$$P_i = P_{i1} + P_{i2} + \cdots + P_{ik}.$$

(Here and elsewhere in this proof, sums with indices running from 1 to k should be interpreted as infinite sums in the case where $k=\infty$.) All rank-1 projections P_{ij} are thus mutually orthogonal. Define operators $A_1, \ldots, A_k$ by $A_j = r_1 P_{1j} + r_2 P_{2j} + \cdots + r_n P_{nj}$.
Now, $A=A_1+\cdots+A_k$ and each A_j has rank n and trace r. By Proposition(1.2.3), each A_j can be written as a sum of r rank-1 projections:

$$A_j = T_{j1} + T_{j2} + \ldots + T_{jr}.$$

Note that projections T_{jl} and T_{mp} are orthogonal if $j \neq m$. De_ne the rank-k projections $Q_1, \ldots, Q_r$ by

$$Q_l = T_{1l} + T_{2l} + \cdots + T_{kl}.$$

This gives $A=Q_l+Q_2+\ldots+Q_r$.

Lemma(1.2.5)[45]: Let A be a positive operator with finite spectrum contained in the rationals $\mathbb{Q}$, such that all spectral projections are infinite dimensional, and also such that $\|A\|>1$. Then A is a finite sum of self-adjoint projections.

Proof. By hypothesis, there are mutually orthogonal infinite-rank projections $P_1, \ldots, P_n$ and positive rational numbers $r_1 \geq r_2 \geq \cdots \geq r_n$ such that

$$A = r_1 P_1 + \cdots + r_n P_n.$$

By hypothesis $\|A\|>1$, hence $r_1>1$.
Write $r_i = s_i/t_i$ with s_i and t_i positive integers, and let $s=\sum_{i=1}^n s_i$, $t=\sum_{i=1}^n t_i$. We may assume $s \geq t$, for otherwise we can choose $m \in \mathbb{N}$ such that

$$m s_1 + s_2 + \cdots + s_n \geq m t_1 + t_2 + \cdots + t_n$$

and replace s_1 with $m s_1$ and t with $m t_1$.
Each P_i can be written as a sum of t_i mutually orthogonal infinite rank projections P_{ij} , $j= 1, \ldots, t_i$ which then allows us to write

$$A = \sum_{i=1}^{n} \sum_{j=1}^{t_i} r_i P_{ij}.$$

The operator is now a linear combination of $\sum t_i = t$ mutually orthogonal Projections of infinite rank, and the sum of the coefficients is now an integer $\sum t_i r_i = \sum s_i = s$. Since $s \geq t$, Lemma (1.2.4) implies that A can be written as a sum of s projections.

Lemma (1.2.6)[45]: Let A be a positive operator which has a projection-decomposition. Then either A is a projection or $\|A\|>1$.

Proof. Suppose, to obtain a contradiction, that $\|A\| \leq 1$ and that A is not a projection. By assumption, $\mathrm{A}=\sum \mathrm{P_i}$ with the series converging strongly.

Thus $\mathrm{A}-\mathrm{P_i} \geq 0$ for all i. Then $\mathrm{P_i}(\mathrm{A}-\mathrm{P_i})\mathrm{P_i} \geq 0$, so $\mathrm{P_i A P_i} \geq \mathrm{P_i}$.

Let $K_i = P_i\mathcal{H}$ and $B = P_iA|_{K_i}$. Then B_i is positive and $B_i \geq I_k$ (the identity operator on K_i). Since $\|B_i\| \leq 1$, this implies $B_i = I_k$, and thus

$$P_iAP_i = P_i.$$

Now, $P_i = P_i(\sum_j P_j)P_i = P_i + \sum_{j\neq i} P_iP_jP_i$,so $\sum_{j\neq i} P_iP_jP_i = 0$. Since each $P_iP_jP_i \geq 0$, this implies $P_iP_jP_i = 0$. Thus, $(P_jP_i)^*(P_jP_i) = 0$, so $P_jP_i = 0$. Since this is true for arbitrary i, j with $i \neq j$, this shows that A is the sum of mutually orthogonal projections, and hence is itself a projection.

The contradiction shows the result.

Proposition(1.2.7)[45]: Let A be a positive operator in $B(\mathcal{H})$ with the property that all nonzero spectral projections for A are of infinite rank. If $\|A\| > 1$, then A admits a projection decomposition as a sum of infinite rank projections.

Proof. We will show that A can be written as a sum $A = \sum_{i=1}^{\infty} A_i$ of positive operators, each satisfying the hypotheses of Lemma(1.2.5), where the sum converges in the strong operator topology. We can then decompose each of the operators A_i as a finite sum of projections A_{ij} and then re-enumerate with a single index to obtain a sequence Q_i of projections which sum to A in SOT. Indeed, the partial sums of $\sum Q_i$ are dominated by A, hence$\sum Q_i$ converges strongly to some operator C, and since the partial sums of $\sum A_i$ are also partial sums of $\sum Q_i$, the sequence of partial sums of $\sum Q_i$ has a subsequence which converges to A, and hence $C = A$.

By hypothesis, we have $\|A\| > 1$. We may choose a positive rational number $\alpha > 1$ and a nonzero spectral projection G for A such that $A \geq \alpha G$.

Let $B = A - \alpha G$, so that $B \geq 0$. Using a standard argument, we can write $B = \sum_{i=1}^{\infty} B_i$, where each B_i is a positive rational multiple of a spectral projection for A, with convergence in the SOT. We can write $G = \sum G_i$, G_i as an infinite direct sum of nonzero infinite rank projections, with the requirement that G_i be a subprojection of G which commutes with all the spectral projections for A. (This can clearly be done when the spectral projections for A are all of infinite rank.) Now, let $A_i = \alpha G_i + B_i$. We have

$$\|A_i\| \geq \alpha > 1.$$

By Lemma(1.2.5), it follows that A_i is a finite sum of projections. By the construction, we have the requisite form $A = \sum A_i$.

Proposition(1.2.8)[45]: Let A be a positive operator in $B(\mathcal{H})$ which is diagonal with respect to some orthonormal basis $\{e_i\}$ for the Hilbert space $\mathcal{H}$. Suppose $\|A\|_{ess} > 1$. Then there is a sequence of rank-1 projections $\{P_i\}_{i=1}^{\infty} = 1$ such that $A = \sum P_i$, where the sum converges in the strong operator topology.

Proof. Write A as diag $(a_0, a_1, \ldots)$ and let $E_n = e_n \otimes e_n$. Since $\|A\|_{ess} > 1$, there is a constant $\alpha > 1$ such that $a_i \geq \alpha$ for infinitely many i. Let $k \geq 2$ be an integer such that $1 + 2/(k-1) \leq \alpha$. Permuting if necessary, we can without loss of generality assume that the indices n for which $a_n < \alpha$ are all multiples of k.

Let $B_0 = a_0E_0 + \cdots + a_{k-1}E_{k-1}$. Therefore, we have rank $(B_0) \leq k$ and Trace $(B_0) =$

$$\sum_0^{k-1} a_i \geq a_0 + (k-1)\alpha \geq a_0 + (k-1)\left(1 + \frac{2}{k-1}\right) = a_0 + k + 1.$$

Let L_0 be the greatest integer less than trace (B_0). Then $L_0 \geq k + 1$. Define $\acute{a}_{k-1}$ to be the real number $0 \leq \acute{a}_{k-1} \leq a_{k-1}$ such that if

$$\grave{B}_0 = a_0E_0 + \cdots + a_{k-2}E_{k-2} + \grave{a}_{k-1}E_{k-1},$$

then

$$trace(\grave{B}_0) = L_0 \geq k + 1 > rank(\grave{B}_0).$$

By Proposition(1.2.3), $\grave{B}_0$ can be written as a sum of L_0 rank-1 projections.

In the next step, let $a''_{k-1} = a_{k-1} - \acute{a}_{k-1}$ and let

$$B_1 = a''_{k-1}E_{k-1} + a_kE_k + a_{k+1}E_{k+1} + \cdots + a_{2k-1}E_{2k-1.}$$

Thus rank $(B_1) \leq k + 1$ and

Trace $(B_1) = a''_{k-1} + a_k + (a_{k+1} + \cdots + a_{2k-1}) \geq a''_{k-1} + a_k + (k-1)\alpha$

$$\geq a''_{k-1} + a_k + (k-1)\left(1 + \frac{2}{k-1}\right) = a''_{k-1} + a_k + k + 1 \geq rank(B_1).$$

Construct B'_1 in a similar manner, so that its trace is an integer greater than or equal to its rank. Then B'_1 can be written as a sum of rank-1 projections using Proposition(1.2.3). Proceeding recursively in a like manner, we may write $A = \sum_{j=1}^{\infty} B'_j$ converging in SOT, where each B'_j is a positive operator supported in $E_{jk-1} + \cdots + E_{(j+1)k-1}$ and with trace (B'_j) an integer that is greater than or equal to rank(B'_j). Invoking Proposition(1.2.3) again to write each B'_j as a sum of rank-1 projections, the proposition is showed.

Theorem(1.2.9)[45]: Let A be a positive operator in $B(\mathcal{H})$ for $\mathcal{H}$ a real or complex Hilbert space with infinite dimension, and suppose $\|A\|_{ess} > 1$. Then A has a projection decomposition

Proof . Write $A = A_1 + A_2$, where A_1 and A_2 respectively denote the nonatomic and purely atomic parts of A. Then $\|A_1\|_{ess} = \|A_1\|$, and $\|A\|_{ess} = \max\{\|A_1\|, \|A_2\|_{ess}\}$. So $\|A\|_{ess} > 1$ implies $\|A_1\| > 1$ or $\|A_2\|_{ess} > 1$. Suppose first that $\|A_1\| > 1$. Then there is a nonzero spectral projection P for A_1 and a constant $\alpha > 1$ such that $A_1 P \geq \alpha P$. Let Q be a nonzero spectral projection for A_1 dominated by P such that $P - Q \neq 0$.

Then $A_1 - \alpha Q$ satisfies the hypotheses of Proposition (1.2.7), so is projection decomposable. Also, $QA_2 = A_2 Q = 0, so\ A_2 + \alpha Q$ is a diagonal operator with essential norm greater than or equal to α, and so it is projection decomposable by Proposition (1.2.8). The result follows by decomposing $A_1 - \alpha Q$ and $A_2 - \alpha Q$ as sums of projections and combining the series.

For the case $\|A_1\| \leq 1$ and $\|A_2\|_{ess} > 1$, we use a similar argument. There is a constant $\alpha > 1$ and an infinite rank spectral projection P for A_2 such that $A_2 - \alpha P \geq 0$. Then P dominates a projection Q that commutes with A_2 such that both Q and $P - Q$ are of infinite rank. Then $A_2 - \alpha Q$ satisfies Proposition (1.2.8) and hence has a projection decomposition. The operator $A_1 + \alpha Q$ has norm greater than or equal to α and all of its nonzero spectral projections have infinite rank, so it satisfies the hypotheses of Proposition(1.2.7). Thus, $A_1 + \alpha Q$ has a projection decomposition, and we combine this decomposition with the decomposition of $A_2 + \alpha Q$ to get a projection decomposition for A.

Let $\mathcal{H}$ be a finite or countably infinite dimensional Hilbert space. Let $\{x_j\}_{j \in \mathbb{J}}$ be a frame for $\mathcal{H}$, where $\mathbb{J}$ is some index set. Consider the standard frame operator defined by

$$Sw = \sum_{j \in \mathbb{J}} \langle w, x_j \rangle x_j = \sum_{j \in \mathbb{J}} (x_j \otimes x_j) w.$$

Thus, $S = \sum_{\mathbb{J}} x_j \otimes x_j$, where this series of positive rank-1 operators converges in the strong operator topology (i.e., the topology of pointwise convergence).

In the special case where each $\|x_j\| = 1, S$ is the sum of the rank-1 projections $P_j = x_j \otimes x_j$. If we let $y_j = S^{-1/2} x_j$, then it is well-known that $\{y_j\}_{j \in \mathbb{J}}$ is a Parseval frame (i.e., tight with frame bound 1). If each $\|x_j\| = 1$, then $\{y_j\}_{j \in \mathbb{J}}$ is an ellipsoidal tight frame for the ellipsoidal surface $\varepsilon_{S^{-1/2}} = S^{-1/2} S_1$.

Moreover, it is well-known (see [79]) that a sequence $\{x_j\}_{j \in \mathbb{J}} \subseteq \mathcal{H}$ is a tight frame for $\mathcal{H}$ if and only if the frame operator S is a positive scalar multiple of the identity, i.e., $S = KI$, and in this case K is the frame bound.

The link between Theorem(1.2.9) and Theorem(1.2.11) is the following:

Proposition(1.3.10)[45]: Let T be a positive invertible operator in $B(\mathcal{H})$, and let

$K > 0$ be a positive constant. The ellipsoidal surface $\mathcal{E}_T = TS_1$ contains a tight frame $\{y_i\}$ with frame bound K if and only if the operator $R = KT^{-2}$ admits a projection decomposition. In this case, R is the frame operator for the spherical frame $\{T^{-1}y_i\}$.

Proof. We present the proof in the infinite dimensional setting, and note that the calculations in the finite dimensional case are identical but do not require discussion of convergence. Let $\mathbb{J}$ be a finite or infinite index set.

Assume ε_T contains a tight frame $\{y_j\}_{j\in\mathbb{J}}$ with frame bound K. Then $\sum_{j\in\mathbb{J}} y_j \otimes y_j = KI$, with the series converging in the strong operator topology.

Let $x_j := T^{-1}y_j \in S_1$, so $x_j \otimes x_j$ are projections. We can then compute:

$$R = KT^{-2} = T^{-1}\left(\sum_{j\in\mathbb{J}} y_j \otimes y_j\right)T^{-1} = \sum_{j\in\mathbb{J}} T^{-1}y_j \otimes T^{-1}y_j = \sum_{j\in\mathbb{J}} x_j \otimes x_j.$$

This shows that R can be decomposed as required. Conversely, suppose R admits a projection decomposition $R = \sum P_j$, where $\{P_j\}$ are self-adjoint projections and convergence is in the strong operator topology. We can assume that the P_j have rank-1, for otherwise we can decompose each P_j as a strongly convergent sum of rank-1 projections, and re-enumerate appropriately. Since $P_j \geq 0$, the convergence is independent of the enumeration used. Write $P_j = x_j \otimes x_j$ for some unit vector x_j. Letting $y_j = Tx_j$, we have $y_j \in \varepsilon_T$, and we also have

$$KI = TRT = T\left(\sum_{j\in\mathbb{J}} x_j \otimes x_j\right)T = \sum_{j\in\mathbb{J}} Tx_j \otimes Tx_j = \sum_{j\in\mathbb{J}} y_j \otimes y_j.$$

This shows that $\sum y_j \otimes y_j$ converges in the strong operator topology to KI.

Thus, $\{y_j\}_{j\in\mathbb{J}}$ is a tight frame on ε_T, as required.

Theorem(1.2. 11)[45]: Let T be a bounded invertible operator on a real or complex Hilbert space. Then the ellipsoidal surface ε_T contains a tight frame. If $\mathcal{H}$ is finite dimensional with $n = dim\mathcal{H}$, then for any integer $k \geq n, \varepsilon_T$ contains a tight frame of length k, and every ETF on ε_T of length k has frame bound $K = k[\,trace(T^{-2}]^{-1}$. If $dim\mathcal{H} = \infty$ then for any constant $K > \|T^{-2}\|_{ess}^{-1}$ $\varepsilon_T$contains a tight frame with frame bound K.

We begin by showing that every ellipsoid can be scaled to contain an orthonormal basis.

Proof. Let ε be an ellipsoid. Then $\varepsilon = \varepsilon_T = TS_1$ for some positive invertible $T \in B(\mathcal{H})$. Let K be a positive constant, and let $R = KT^{-2}$.

The condition $K > \|T^{-2}\|_{ess}^{-1}$ implies $\|R\|_{ess} > 1$. So, by Theorem (1.2.9), R admits a projection decomposition, and thus Proposition(1.2.10) implies that E contains a tight frame with frame bound K.

In the finite dimensional case, let $n = dim\mathcal{H}$. Proposition (1.2.10) states that $\mathcal{E}$ will contain a tight frame with frame bound D if and only if KT^{-2} admits a projection decomposition, and by Proposition(1.2.3) this happens if and only if trace(KT^{-2}) is an integer $k \geq n$, and in this case k is the length of the frame. Thus, we have $K = k[trace(T^{-2})]^{-1}$. Therefore, every ellipsoid $\mathcal{E} = \mathcal{E}_T$ contains a tight frame of every length $k \geq n$, and every such tight frame has frame bound $k[trace(T^{-2})]^{-1}$.

Corollary(1.2.12)[45]: Every positive invertible operator S on a separable Hilbert space $\mathcal{H}$ is the frame operator for a spherical frame. If $\mathcal{H}$ has finite dimension n, then for every integer $k \geq n, S$ is the frame operator for a spherical frame of length k, and the radius of the sphere is $\sqrt{trace(S)/k}$. If $\mathcal{H}$ is infinite dimensional, the radius of the sphere can be taken to be any positive number

$$r < \|S\|_{ess}^{1/2}.$$

Proof. In the finite dimensional case, let $c = k/trace(S)$ and $A = cS$, so that trace $(A) = k$. Then,by Proposition(1.2.3), A has a projection decomposition into k rank-1 projections, making A the frame operator for the frame of unit vectors $\|x_i\|_{i=1}^{k}$. Thus, S is the frame operator for $\{x_i/\sqrt{c}\}_{i=1}^{k}$.

When $\mathcal{H}$ has infinite dimension, let c be any constant greater than $\|S\|_{ess}^{-1}$ and let $A = cS$. The hypotheses of Theorem (1.2.9) are satisfied, so A admits a projection decomposition. Then A is the frame operator for a frame $\{x_i\}$ of unit vectors, so S is the frame operator for the spherical frame $\{x_i/\sqrt{c}\}$.

Chapter 2
Parseval Frames and Prescribed Norms

We further investigate several of Parseval frame properties. Finally, we apply the algorithm to several numerical examples . Let $\mathcal{H}$ be a finite dimensional (real or complex) Hilbert space and let $\{a_i\}_{i=1}^{\infty}$ be a non-increasing sequence of positive numbers. Given a finite sequence of vectors $\mathcal{F} = \{f_i\}_{i=1}^{p}$ in $\mathcal{H}$ we find necessary and sufficient conditions for the existence of $r \in N \cup \{\infty\}$ and a Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^{r}$ in $\mathcal{H}$ such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame for $\mathcal{H}$ and $\|g_i\|^2 = a_i$ for $1 \leq i \leq r$.

Section(2.1)[71]: A Generalization Of Gram–Schmidt Orthogonalization

Let $\mathcal{H}$be a finite–*dimensional* Hilbert space. A sequence $(f_i)_{i=1}^{n} \subset \mathcal{H}$ forms a frame, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|g\|^2 \leq \sum_{i=1}^{n} |\langle f_i, g\rangle|^2 \leq B\|g\|^2 \quad \text{for all } g \in \mathcal{H}. \qquad (1)$$

Frames have turned out to be an essential tool for many applications such as, for example, data transmission, due to their robustness not only against noise but also against losses and due to their freedom in design [74, 47]. Their main advantage lies in the fact that a frame can be designed to be redundant while still providing a reconstruction formula. Since the frame operator $Sg = \sum_{i=1}^{n} \langle g, f_i\rangle f_i$ is invertible, each vector $g \in \mathcal{H}$ can be always reconstructed from the values $(g, f_i)_{i=1}^{n}$ via

$$g = SS^{-1}g = \sum_{i=1}^{n} \langle g, f_i\rangle S^{-1} f_i.$$

However, the inverse frame operator is usually very complicated to compute. This difficulty can be avoided by choosing a frame whose frame operator equals the identity. This is one reason why Parseval frames, i.e., frames for which $S = Id$ or equivalently for which A and B in (1) can be chosen as $A = B = 1$, enjoy rapidly increasing attention. Another reason is that quite recently it was shown by Benedetto and Fickus [36] that in $\mathbb{R}^d$ as well as in $\mathbb{C}^d$ finite equal norm Parseval frames, i.e., finite Parseval frames whose elements all have the same norm, are exactly those sequences which are in equilibrium under the so–called frame force, which parallels a Coulomb potential law in electrostatics. In fact, they demonstrate that in this setting both orthonormal sets and finite equal norm Parseval frames arise from the same optimization problem. Thus, in general, Parseval frames are perceived as the most natural generalization of orthogonal bases [183],[74].

Our algorithm is designed to be iterative in the sense that one vector is added each time to an already modified set of vectors and then the new set is adjusted again. In each iteration it not only computes a Parseval frame for the span of the sequence of vectors already dealt with at this point, but also preserves redundancy in an exact way. Moreover, it reduces to Gram–Schmidt orthogonalization if applied to a sequence of linearly independent vectors and each time a linearly dependent vector is added, the algorithm computes the Parseval frame which is closest in l^2 –norm to the already modified sequence of vectors.

The section is organized as follows. In this Section we first state the algorithm and show that it in fact generates a special Parseval frame in each iteration. Additional properties of the algorithm such as, for example, the preservation of redundancy, are treated in this Section. Finally, in this Section we first compare the complexity of our algorithm with the complexity of the Gram–Schmidt orthogonalization and then study the different steps of the algorithm applied to several numerical examples. Throughout this section let $\mathcal{H}$ denote a finite–dimensional Hilbert space. We start by describing our iterative algorithm. On input $n \in \mathbb{N}$ and $f = (f_i)_{i=1}^{n} \subset \mathcal{H}$ the procedure $GGSP$ (Generalized Gram–Schmidt orthogonalization to compute Parseval frames) outputs a Parseval frame $g = (g_i)_{i=1}^{n} \subset \mathcal{H}$ for span$\{(f_i)_{i=1}^{n}\}$ with special properties (see Theorem 2.2). procedure $GGSP(n, f;\ g)$

for $k := 1$ to n do
begin
if $f_k = 0$ then
$g_k := 0;$
else
begin
$g_k := f_k - \sum_{j=1}^{k-1} \langle f_k, g_j \rangle\, g_j;$
if $g_k \neq 0$ then
$g_k := \frac{1}{\|g_k\|} g_k;$
else
begin
for $i := 1\ to\ k-1\ do\ g_i := g_i + \frac{1}{\|f_k\|^2}\left(\frac{1}{\sqrt{1+\|f_k\|^2}} - 1\right)\langle g_i, f_k \rangle f_k;$
$g_k := \frac{1}{\sqrt{1+\|f_k\|^2}} f_k;$
end;
end;
end;
end.

In the remainder of this section the following notation will be used.

Let Φ denote the mapping$(f_i)_{i=1}^n \longmapsto (g_i)_{i=1}^n$ of a sequence of vectors in $\mathcal{H}$ to another sequence of vectors in $\mathcal{H}$ given by the procedure $GGSP$. We will also use the notation $((f_i)_{i=1}^n, g) \coloneqq (f_1, \ldots, f_n, g)$ for $(f_i)_{i=1}^n \subset \mathcal{H}$ and $g \in \mathcal{H}$.

The following result shows that the algorithm not only produces a Parseval frame for span$\{(f_i)_{i=1}^k\}$, but even in each iteration also produces a special Parseval frame for span$\{(f_i)_{i=1}^k\}, k = 1, \ldots, n$.

It is well–known that applying $S^{-\frac{1}{2}}$ to a sequence of vectors $(f_i)_{i=1}^k$ in $\mathcal{H}$ yields a Parseval frame, where S denotes the frame operator for this sequence (see [182]). Moreover, Theorem(2.1. 3) will show that the Parseval frame $(S^{-\frac{1}{2}} f_i)_{i=1}^n$ is the closest in l^2–norm to the sequence $(f_i)_{i=1}^n$. However, in general the computation of the operator $S^{-\frac{1}{2}}$ is not very efficient. In fact, in our algorithm we do not compute $S^{-\frac{1}{2}}((f_i)_{i=1}^n)$. Instead in each iteration when adding a vector, which is linearly dependent to the already modified vectors, we apply $S^{-\frac{1}{2}}$ to those vectors and the added one, where here S denotes the frame operator for this new set of vectors. This eases the computation in a significant manner, since the set of computed vectors already forms a Parseval frame, and nevertheless we compute the closest Parseval frame in each iteration. When we add a linearly independent vector, we orthogonalize this one vector by using a Gram–Schmidt step. Thus this algorithm is also a generalization of Gram–Schmidt orthogonalization.

Theorem(2.1.1)[71]: Let $n \in \mathbb{N}$ and $(f_i)_{i=1}^n \subset \mathcal{H}$. Then, for each $k \in \{1, \ldots, n\}$, the sequence of vectors $\Phi((f_i)_{i=1}^k)$ is a Parseval frame for span$\{(f_i)_{i=1}^k = span\{\Phi((f_i)_{i=1}^k)\}$.
In particular, for each $k \in \{1, \ldots, n\}$, the following conditions hold.

(i) If $f_k \in span\{((f_i)_{i=1}^k)\}$, then

$$\Phi((f_i)_{i=1}^k) = (S^{-\frac{1}{2}}(\Phi((f_i)_{i=1}^{k-1}), f_k)),$$

where S is the frame operator for $(\Phi\left((f_i)_{i=1}^{k-1}), f_k\right)$.

(ii) If $f_k \notin span\{(f_i)_{i=1}^{k-1}\}$, then

$$\Phi\left((f_i)_{i=1}^k\,\right) = \left(\Phi((f_i)_{i=1}^{k-1}), g_k\right), \quad g_k \in \mathcal{H}, \|g_k\| = 1$$

and

$$g_k \perp \Phi\left(\left(f_i\right)_{i=1}^{k-1}\right).$$

Proof. We will prove the first claim by induction and meanwhile in each step we show that, in particular, the claims in (i) and (ii) hold. For this, let l denote the smallest number in $\{1,\dots,n\}$ with $f_l \neq 0$. Obviously, for each $k \in \{1,\dots,l-1\}$, the generated set of vectors g_k (see line 3 of $GGSP$) forms a Parseval frame for $\operatorname{span}\{(f_i)_{i=1}^{k}\} = \{0\}$ and also (i) is fulfilled. The hypothesis in (ii) does not apply here. Next notice that in the case $k = l$ we have $g_k := \frac{1}{\|f_k\|} f_k$ ($line$ 8), which certainly is a Parseval frame for $\operatorname{span}\{(f_i)_{i=1}^{k}\} = span\{f_k\}$. It is also easy to see that (i) and (ii) are satisfied.

Now fix some $k \in \{l+1,\dots,n\}$ and assume that the sequence $(\bar{g}_i)_{i=1}^{k-1} := \Phi((f_i)_{i=1}^{k-1})$ is a Parseval frame for $\operatorname{span}\{(f_i)_{i=1}^{k-1}\} = span\{(\bar{g}_i)_{i=1}^{k-1}\}$. We have to study two cases.

Case 1: The vector $g_k := f_k - \sum_{j=1}^{k-1}\langle f_k, \bar{g}_j\rangle \bar{g}_j$ computed in line 6 is trivial. This implies that

$$span\{(f_i)_{i=1}^{k-1}\} = span\{(\bar{g}_i)_{i=1}^{k-1}\} = span\{(f_i)_{i=1}^{k}\}, \quad (2)$$

since otherwise the Gram–Schmidt orthogonalization step would yield a non–trivial vector. In particular, only the hypothesis in (i) applies.

Now let P denote the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}\{f_k\}$. In order to compute $S^{-\frac{1}{2}}$, where S denotes the frame operator for $((\bar{g}_i)_{i=1}^{k-1}, f_k)$, we first show that each $(I-P)\bar{g}_i, i = 1,\dots,k-1$ is an eigenvector for S with respect to the eigenvalue 1 or the zero vector. This claim follows immediately from

$$\begin{aligned} S(I-P)\bar{g}_i &= \sum_{j=1}^{k-1}\langle (I-P)\bar{g}_i, \bar{g}_j\rangle \bar{g}_j + \langle (I-P)\bar{g}_i, f_k\rangle f_k \\ &= \sum_{j=1}^{k-1}\langle (I-P)\bar{g}_i, \bar{g}_j\rangle \bar{g}_j \\ &= (I-P)\bar{g}_i \end{aligned}$$

since $(\bar{g}_i)_{i=1}^{k-1}$ is a Parseval frame for $\operatorname{span}\{(\bar{g}_i)_{i=1}^{k-1}\}$. Also f_k is an eigenvector for S, but with respect to the eigenvalue $1 + \|f_k\|^2$, which is proven by the following calculation:

$$Sf_k = \sum_{j=1}^{k-1}\langle f_k, \bar{g}_j\rangle \bar{g}_j + \langle f_k, f_k\rangle f_k = (1 + \|f_k\|^2 f_k.$$

Using f_k as an eigenbasis for $P\left(span\{(\bar{g}_i)_{i=1}^{k-1}\}\right)$ and an arbitrary eigenbasis for $(I-P)\left(span\{(\bar{g}_i)_{i=1}^{k-1}\}\right)$, we can diagonalize S to compute $S^{-\frac{1}{2}}$. This together with the fact that $(I-P)\bar{g}_i, i = 1,\dots,k-1$ is an eigenvector for S with respect to the eigenvalue 1 and that $S(I-P)f_k = 0$ yields

$S^{-\frac{1}{2}}\bar{g}_i = \frac{1}{\sqrt{1+\|f_k\|^2}} p\bar{g}_i + (I-P)\bar{g}_i$ for $1 \le i \le k-1$

and

$$S^{-\frac{1}{2}} f_k = 1 - \frac{1}{\sqrt{1+\|f_k\|^2}} f_k .$$

Comparing these equalities with line 11 and 12 of $GGSP$ shows that in fact $\Phi\left(f_i\right)_{i=1}^{k-1}) = \left(S^{-\frac{1}{2}}\left((\bar{g}_i)_{i=1}^{k-1}, f_k\right)\right)$, which is (i). By [183] and (14), this immediately implies that the sequence $\Phi((f_i)_{i=1}^{k})$ is a Parseval frame for $\operatorname{span}\{\Phi((f_i)_{i=1}^{k})\} = span\{(f_i)_{i=1}^{k})\}$.

Case2: The condition in line 7 applies, i.e., we have $g_k := (f_k - \sum_{j=1}^{k-1}\langle f_k, \bar{g}_j\rangle \bar{g}_j) / \left(\left\|f_k - \sum_{j=1}^{k-1}\langle f_k, \bar{g}_j\rangle \bar{g}_j\right\|\right) \neq 0$. Then we set $g_i := \bar{g}_i$ for all $i = 1,\dots,k-1$. Obviously, $\|g_k\| = 1$. Moreover, since by induction hypothesis $(\bar{g}_i)_{i=1}^{k-1}$ forms a Parseval frame, for each $i = 1,\dots,k-1$, we have

$$\langle g_i, f_k - \sum_{j=1}^{K-1} \langle f_k, \bar{g}_j \rangle \bar{g}_j \rangle = \langle g_i, f_k \rangle - \langle g_i, f_k \rangle = 0.$$

Thus g_k is normalized vector, which is orthogonal to $g_1, \ldots, g_{k-1}$. Hence (ii) is satisfied and, for all $h \in span\{(g_i)_{i=1}^k\}$, we obtain

$$\sum_{i=1}^{k} |\langle h, g_i \rangle|^2 = \sum_{i=1}^{k-1} |\langle (I-P)\rangle h, g_i|^2 + |\langle Ph, g_k \rangle|^2 = \|(I-P)h\|^2 + \|Ph\|^2 = \|h\|^2,$$

where P denotes the orthogonal projection of $\mathcal{H}$ onto span$\{g_k\}$. This proves that $\left(g_i)_{i=1}^k\right) = \Phi(f_i)_{i=1}^k)$ is a Parseval frame for span$\{\Phi(f_i)_{i=1}^k)\}$. Moreover, we have span$\{\Phi((f_i)_{i=1}^k)\} = span\{(f_i)_{i=1}^{k-1}, f_k - \sum_{j=1}^{k-1} \langle f_k, g_j \rangle g_j$ = span$\{(f_i)_{i=1}^k\}$. This finishes the proof, since the hypothesis in (i) does not apply in this case.

The algorithm can be seen as a "Gram–Schmidt procedure backwards" in the sense that in each iteration, if the added vector is linearly dependent to the already computed vectors, not only this vector is modified, but also all the other vectors are rearranged with respect to the new vector so that the collection forms a Parseval frame. This way of computation will be demonstrated by several examples in Subsection .

In this section we first determine in general which Parseval frame is the closest to the initial sequence and study which properties of our algorithm this result implies.

Then we investigate several additional properties of the procedure $GGSP$, in particular we characterize those sequences, which lead to orthonormal bases, and we show that Φ regarded as a map from finite sequences to Parseval frames is "almost" bijective. At last, we examine the redundancy of the generated Parseval frame.

Given a sequence $(f_i)_{i=1}^n$ with frame operator S, by [183], the sequence $(S^{-\frac{1}{2}} f_i)_{i=1}^n$ always forms a Parseval frame. The following result shows that this sequence can in fact be characterized as the very same Parseval frame, which is the closest to $(f_i)_{i=1}^n$ in l^2-norm.

Theorem(2.1.2)[71]: If $(f_i)_{i=1}^n \subset \mathcal{H}, n \in \mathbb{N}$ is any frame for $\mathcal{H}$ with frame operator S, then

$$\sum_{i=1}^{n} \left\| f_i - S^{-1/2} f_i \right\|^2 = inf \left\{ \sum_{i=1}^{n} \| f_i - g_i \|^2 : (g_i)_{i=1}^n \text{is a Parseval frame for} \mathcal{H} \right\}.$$

Moreover, $(S^{-1/2} f_i)_{i=1}^n$ is the unique minimizer.

Proof. Let $(e_j)_{j=1}^d, d := dim\mathcal{H}$, be an orthonormal eigenvector basis for $\mathcal{H}$ with respect to S and respective eigenvalues $(\lambda_j)_{j=1}^d$. Then we can rewrite the left–hand side of the claimed inequality in the following way:

$$\sum_{i=1}^{n} \left\| f_i - S^{-1/2} f_i \right\|^2 = \sum_{i=1}^{n} \left\| \sum_{j=1}^{d} \langle f_i, e_j \rangle e_j - \frac{1}{\sqrt{\lambda_j}} \langle f_i, e_j \rangle e_j \right\|^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{d} \left| \langle f_i, e_j \rangle \right|^2 \left| 1 - \frac{1}{\sqrt{\lambda_j}} \right|^2$$

$$= \sum_{j=1}^{d} \left| 1 - \frac{1}{\sqrt{\lambda_j}} \right|^2 \sum_{i=1}^{n} \left| \langle f_i, e_j \rangle \right|^2$$

$$= \sum_{j=1}^{d} \left| 1 - \frac{1}{\sqrt{\lambda_j}} \right|^2 \lambda_j$$

$$= \sum_{j=1}^{d} \left(\lambda_j - 2\sqrt{\lambda_j} + 1 \right).$$

Now let $(g_i)_{i=1}^n$ be an arbitrary Parseval frame for $\mathcal{H}$. Using again the eigenbasis and its eigenvalues, we obtain

$$\sum_{i=1}^{n}\|f_i - g_i\|^2 = \sum_{i=1}^{n}\left\|\sum_{j=1}^{d}\langle f_i, e_j\rangle e_j - \langle g_i, e_j\rangle e_j\right\|^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{d}\left|\langle f_i, e_j\rangle - \langle g_i, e_j\rangle\right|^2$$

$$= \sum_{j=1}^{d}\sum_{i=1}^{n}\left(\left|\langle f_i, e_j\rangle\right|^2 + \left|\langle g_i, e_j\rangle\right|^2 - 2Re\left[\langle f_i, e_j\rangle\overline{\langle g_i, e_j\rangle}\right]\right)$$

$$= \sum_{j=1}^{d}\left(\sum_{i=1}^{n}\left|\langle f_i, e_j\rangle\right|^2 + \sum_{i=1}^{n}\left|\langle g_i, e_j\rangle\right|^2 - 2Re\left[\sum_{i=1}^{n}\langle f_i, e_j\rangle\overline{\langle g_i, e_j\rangle}\right]\right)$$

$$= \sum_{j=1}^{d}\left(\lambda_j + 1 - 2Re\left[\sum_{i=1}^{n}\langle f_i, e_j\rangle\overline{\langle g_i, e_j\rangle}\right]\right).$$

Moreover, we have

$$\sum_{j=1}^{d} Re\left[\sum_{i=1}^{n}\langle f_i, e_j\rangle\overline{\langle g_i, e_j\rangle}\right] \le \sum_{j=1}^{d}\sum_{i=1}^{n}\left|\langle f_i, e_j\rangle\right|\left|\langle g_i, e_j\rangle\right|$$

$$\le \sum_{j=1}^{d}\sqrt{\sum_{i=1}^{n}\left|\langle f_i, e_j\rangle\right|^2}\sqrt{\sum_{i=1}^{n}\left|\langle g_i, e_j\rangle\right|^2}$$

$$= \sum_{j=1}^{d}\sqrt{\lambda_j}.$$

Combining this estimate with the computations above yields

$$\sum_{i=1}^{n}\|f_i - g_i\|^2 \ge \sum_{j=1}^{d}\left(\lambda_j - 2\sqrt{\lambda_j} + 1\right) = \sum_{i=1}^{n}\left\|f_i - S^{-1/2}f_i\right\|^2.$$

Since $(S^{-1/2}f_i)_{i=1}^n$ is a Parseval frame for $\mathcal{H}$, the first claim follows.

For the moreover part, suppose that $(g_i)_{i=1}^n$is another minimizer. Then, by the above calculation, for each $k \in \{1, \ldots, n\}$, we have

$$Re\langle f_k, e_j\rangle\overline{\langle g_k, e_j\rangle} = \left|\langle f_k, e_j\rangle\right|\left|\langle g_k, e_j\rangle\right| \tag{3}$$

and, for each $j \in \{1, \ldots, d\}$,

$$\sum_{k=1}^{n}\left|\langle f_k, e_j\rangle\right|\left|\langle g_k, e_j\rangle\right| = \sqrt{\sum_{k=1}^{n}\left|\langle f_k, e_j\rangle\right|^2}\sqrt{\sum_{k=1}^{n}\left|\langle g_k, e_j\rangle\right|^2}. \tag{4}$$

Now let $r_{k,j}, s_{k,j} > 0$ and $\theta_{k,j}, \psi_{k,j} \in [0, 2\pi)$ be such that $\langle f_k, e_j\rangle = r_{k,j}e^{i\theta_{k,j}}$ and $\langle g_k, e_j\rangle = s_{k,j}e^{i\psi_{k,j}}$. We compute

$$Re\left[\langle f_k, e_j\rangle\overline{\langle g_k, e_j\rangle}\right] = r_{k,j}s_{k,j}Re\left[e^{i(\theta_{k,j} - \psi_{k,j})}\right] = r_{k,j}s_{k,j}cos\left(\theta_{k,j} - \psi_{k,j}\right).$$

Hence (3) implies that

$$r_{k,j}s_{k,j}\ cos\left(\theta_{k,j} - \psi_{k,j}\right) = r_{k,j}s_{k,j},$$

which in turn yields $\theta_{k,j} = \psi_{k,j}$. Thus $\langle g_k, e_j\rangle = t_{k,j}\langle f_k, e_j\rangle$ for some $t_{k,j} > 0$ for all $k \in \{1, \ldots, n\}, j \in \{1, \ldots, d\}$. By (4), for each $j \in \{1, \ldots, d\}$ there exists some $u_j > 0$ such that

$$u_j\left|\langle f_k, e_j\rangle\right| = \left|\langle g_k, e_j\rangle\right| = t_{k,j}\left|\langle f_k, e_j\rangle\right|.$$

This implies $t_{k,j} = u_j$ for all $k \in \{1, \ldots, n\}$. Hence, for each $k \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$, we obtain the relation

$$\langle g_k, e_j\rangle = u_j\ \langle f_k, e_j\rangle. \tag{5}$$

Since $(g_i)_{i=1}^n$ is a Parseval frame for $\mathcal{H}$, we have

$$1 = \sum_{k=1}^{n}\left|\langle g_k, e_j\rangle\right|^2 = u_j^2\sum_{k=1}^{n}\left|\langle f_k, e_j\rangle\right|^2 = u_j^2\lambda_j.$$

This shows that $u_j = \frac{1}{\sqrt{\lambda_j}}$. Thus, using (5) and the definition of $\left(e_j\right)_{j=1}^d$ and $\left(\lambda_j\right)_{j=1}^d$, it follows that $g_k = S^{-1/2}f_k$ for all $k \in \{1, \ldots, n\}$.

This result together with Theorem(2.1.1) (i) implies the following property of our algorithm.

Corollary(2.1. 3)[71]: In each iteration of $GGSP$, in which a linearly dependent vector is added, the algorithm computes the unique Parseval frame, which is closest to the frame consisting of the already computed vectors and the added one.

Next we characterize those sequences of vectors applied to which the algorithm computes an orthonormal basis. The proof will show that this is exactly the case,when only the steps of the Gram–Schmidt orthogonalization are carried out.

Proposition(2.1.4)[71]: Let $(f_i)_{i=1}^n \subset \mathcal{H}, n \in \mathbb{N}$. The following conditions are equivalent.

(a) The sequence $\Phi((f_i)_{i=1}^n)$ is an orthonormal basis for span$\{(f_i)_{i=1}^n\}$.

(b) The sequence $(f_i)_{i=1}^n$ is linearly independent.

Proof. If (b) holds, only line 6–8 of $GGSP$ will be performed and these steps coincide with Gram–Schmidt orthogonalization, hence produce an orthonormal system. Now suppose that (b) does not hold.This is equivalent to $dim(span\{(f_i)_{i=1}^n\} < n$.

By Theorem(2.1.1), we have $span\{(f_i)_{i=1}^n\} = span\{\Phi((f_i)_{i=1}^n)\}$. This in turn implies $dim(span\{\Phi((f_i)_{i=1}^n\}) < n$. Thus $\Phi((f_i)_{i=1}^n)$ cannot form an orthonormal basis for $span\{(f_i)_{i=1}^n\}$.

The mapping Φ given by the procedure $GGSP$ of a finite sequence in $\mathcal{H}$ to a Parseval frame for a subspace of $\mathcal{H}$ is "almost" bijective in the following sense.

Proposition(2.1.5)[71]: Let Φ be the mapping defined in the previous paragraph. Then Φ satisfies the following conditions.

(a) Φ is surjective.

(b) For each Parseval frame $(g_i)_{i=1}^n \subset \mathcal{H}$, the set $\Phi^{-1}(g_i)_{i=1}^n$equals

$$\left\{ (f_i)_{i=1}^n : f_i = \left\{ \begin{array}{c} \tilde{f}_i \quad if\ span\left\{\left(\tilde{f}_j\right)_{j=1}^{i-1}\right\} = span\left\{\left(\tilde{f}_j\right)_{j=1}^{i}\right\}, \\ \lambda\tilde{f}_i + \varphi, \lambda \in \mathbb{R}^+, \\ \varphi \in span\left\{\left(\tilde{f}_j\right)_{j=1}^{i-1}\right\} \ otherwise \end{array} \right\} \right\} \quad (6)$$

for some $\left(\tilde{f}_j\right)_{j=1}^n \in \Phi^{-1}(\,(g_i)_{i=1}^n)$.

Proof. It is easy to see that each step of the procedure $GGSP$ is reversable which implies (a).

To show (b) we first show that the set (6) is contained in $\Phi^{-1}((g_i)_{i=1}^n)$. For this, let $(f_i)_{i=1}^n$ be an element of the set (6). Notice that, by definition of $(f_i)_{i=1}^n$, we have $span\{(f_i)_{i=1}^k\} = span\{\left(\tilde{f}_i\right)_{i=1}^k\}$ for all $k \in \{1, \dots, n\}$. Since $\left(\tilde{f}_i\right)_{i=1}^n \in \Phi^{-1}((g_i)_{i=1}^n)$, we only have to study the case $span\{(f_i)_{i=1}^{k-1} \neq span\{(f_i)_{i=1}^k\}$ for some $k \in \{1, \dots, n\}$. But then line 8 of $GGSP$ will be performed. Let $\Phi(f_i)_{i=1}^{k-1}$ be denoted by $(\tilde{g}_i)_{i=1}^{k-1}$. By Theorem(2.1.1), the sequence $(\tilde{g}_i)_{i=1}^{k-1}$ forms a Parseval frame for span$\left\{(f_i)_{i=1}^{k-1}\right\}$. Hence

$$\frac{\lambda\tilde{f}_k+\varphi-\sum_{j=1}^{k-1}\langle\lambda\tilde{f}_k+\varphi,\tilde{g}_j\rangle\tilde{g}_j}{\left\|\lambda\tilde{f}_k+\varphi-\sum_{j=1}^{k-1}\langle\lambda\tilde{f}_k+\varphi,\tilde{g}_j\rangle\tilde{g}_j\right\|} = \frac{\lambda\tilde{f}_k-\sum_{j=1}^{k-1}\langle\lambda\tilde{f}_k,\tilde{g}_j\rangle\tilde{g}_j}{\left\|\lambda\tilde{f}_k-\sum_{j=1}^{k-1}\langle\lambda\tilde{f}_k,\tilde{g}_j\rangle\tilde{g}_j\right\|}$$

$$= \frac{\tilde{f}_k-\sum_{j=1}^{k-1}\langle\tilde{f}_k,\tilde{g}_j\rangle\tilde{g}_j}{\left\|\tilde{f}_k-\sum_{j=1}^{k-1}\langle\tilde{f}_k,\tilde{g}_j\rangle\tilde{g}_j\right\|},$$

which proves the first claim.

Secondly, suppose $(f_i)_{i=1}^n \subset \mathcal{H}$ is not an element of (6). We claim that $\Phi((f_i)_{i=1}^n) \neq \Phi\left(\left(\tilde{f}_i\right)_{i=1}^n\right)$, which finishes the proof. Let $k \in \{1, \dots, n\}$ be the largest number such that f_k does not satisfy the conditions in (6). We have to study two cases.

Case 1: Suppose that $f_k \neq \tilde{f}_k$, but $span\left\{(f_i)_{i=1}^{k-1}\right\} = span\{(f_i)_{i=1}^k\}$. Then in the kth iteration line 12 will be performed and we obtain

$$h_k := \frac{1}{\sqrt{1+\|f_k\|^2}} f_k \neq \frac{1}{\sqrt{1+\|\tilde{f}_k\|^2}} \tilde{f}_k =: \tilde{h}_k,$$

since $f_k \neq \tilde{f}_k$. Thus $\Phi\big((f_i)_{i=1}^k\big) \neq \Phi\Big(\big(\tilde{f}_i\big)_{i=1}^k\Big)$

If in the following iterations the condition in line 7 always applies, we are done, since h_k and $\tilde{h}_k$, are not changed anymore. Now suppose that there exists $l \in \{1,\ldots,n\}, l > k$ with span$\{(f_i)_{i=1}^{l-1}\} = pan\{(f_i)_{i=1}^{l}\}$. Then in the lth iteration h_k and $\tilde{h}_k$ are modified in line 11. Since $f_l = \tilde{f}_l$ by choice of k, using a reformulation of line 11, we still have

$$h_k := \frac{1}{\sqrt{1+\|f_l\|^2}} P h_k + (I-P) h_k \neq \frac{1}{\sqrt{1+\|\tilde{f}_l\|^2}} P \tilde{h}_k + (I-P)\tilde{h}_k =: \tilde{h}_k,$$

where P denotes the orthogonal projection onto span$\{f_l\}$.

Case 2: Suppose that $f_k \neq \lambda \tilde{f}_k + \varphi$ for each $\lambda \in \mathbb{R}^+$ and $\varphi \in span\{(f_i)_{i=1}^{k-1}\}$, and also $span\{(f_i)_{i=1}^{k-1}\} \neq span\{(f_i)_{i=1}^{k}\}$. Let $(h_i)_{i=1}^{k-1}$ and $\big(\tilde{h}_i\big)_{i=1}^{k-1}$ denote $\Phi((f_i)_{i=1}^{k-1})$ and $\Phi(\big(\tilde{f}_i\big)_{i=1}^{k-1})$, respectively. If $(h_i)_{i=1}^{k-1} = \big(\tilde{h}_i\big)_{i=1}^{k-1}$, the computation in line 8 in the kth iteration yields

$$h_k := \frac{f_k - \sum_{j=1}^{k-1}\langle f_k, h_j\rangle h_j}{\left\|f_k - \sum_{j=1}^{k-1}\langle f_k, h_j\rangle h_j\right\|} \neq \frac{\tilde{f}_k - \sum_{j=1}^{k-1}\langle \tilde{f}_k, \tilde{h}_j\rangle \tilde{h}_j}{\left\|\tilde{f}_k - \sum_{j=1}^{k-1}\langle \tilde{f}_k, \tilde{h}_j\rangle \tilde{h}_j\right\|} =: \tilde{h}_k.$$

If $(h_i)_{i=1}^{k-1} \neq \big(\tilde{h}_i\big)_{i=1}^{k-1}$ then there exists some $l \in \{1,\ldots,k-1\}$ with $h_l \neq \tilde{h}_l$. In both situations these inequalities remain valid as it was shown in the preceding paragraph.

An important aspect of our algorithm is the redundancy of the computed frame.

Hence it is desirable to know in which way redundancy is preserved throughout the algorithm. For this, we introduce a suitable definition of redundancy for sequences in a finite–dimensional Hilbert space.

Definition (2.1.6)[71]: Let $(f_i)_{i=1}^n \subset \mathcal{H},\ n \in \mathbb{N}$.Then the redundancy $red((f_i)_{i=1}^n)$ of this set is defined by $red((f_i)_{i=1}^n) = \frac{n}{dim\,(span\,\{(f_i)_{i=1}^n\})}$,

where we set $\frac{n}{0} = \infty$.

Indeed in each iteration our algorithm preserves redundancy in an exact way.

Proposition(2.1.7)[71]: Let $(f_i)_{i=1}^n \subset \mathcal{H}, n \in \mathbb{N}$. Then

$$red\big(\Phi((f_i)_{i=1}^n)\big) = red((f_i)_{i=1}^n).$$

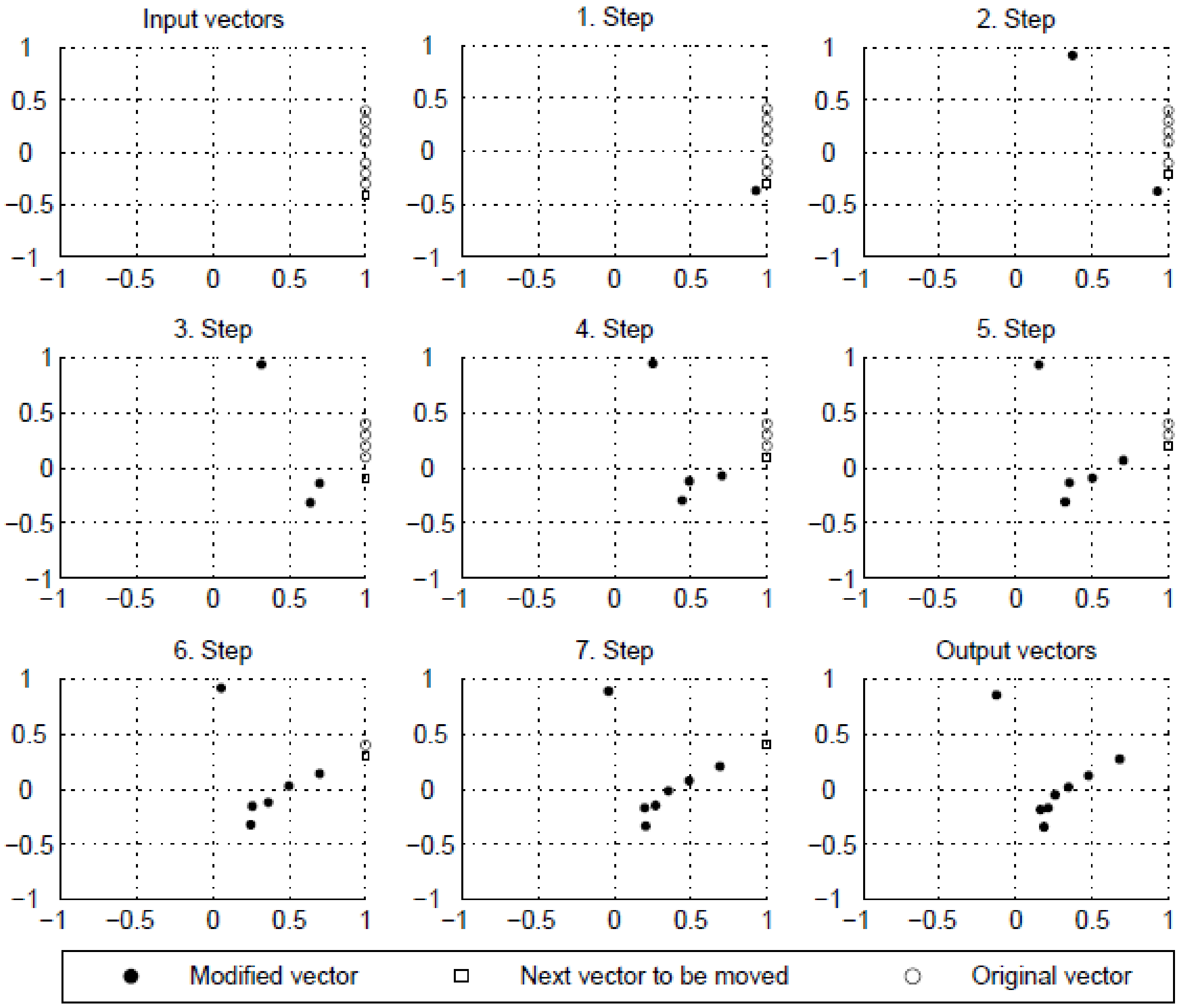
Proof. By Theorem(2.1.1), we have span$\{(f_i)_{i=1}^n\} = span\{\Phi(f_i)_{i=1}^n)\}$. From this, the claim follows immediately.

We will first compare the numerical complexities of the Gram–Schmidt orthogonalization and of GGSP. In a second part the procedure GGSP will be applied to several numerical examples in order to visualize the modifications of the vectors while performing the algorithm.

Only the constants are slightly larger in the new step, which is performed in case of linear dependency. Thus both the Gram–Schmidt orthogonalization and GGSP possess the same numerical complexity of $O(dn^2)$.

In order to give further insight into the algorithm, in this subsection we will study the different steps of GGSP for three examples. The single steps of each example are illustrated by a diagram. In each of these the first image in the uppermost row shows the positions of the vectors of the input sequence. Then in the following images the remaining original vectors and the modified vectors are displayed after each step of the loop in line 0 of GGSP. The original vectors are always marked by a circle and the already computed new vectors are indicated by a filled circle. The vector, which will be dealt with in the next step, is marked by a square.

Recall that, by Theorem(2.1.1), in each step the set of vectors marked with a filled circle forms a Parseval frame for their linear span.



`Figure 1,[71]. GGSP applied to the sequence of vectors ((1, 0.1), (1, 0.2), (1, 0.3),(1, 0.4), (1,−0.1), (1,−0.2), (1,−0.3), (1,−0.4)).

In the first example we consider the sequence of vectors ((1, 0.1), (1, 0.2), (1, 0.3), (1, 0.4), (1,−0.1), (1,−0.2), (1,−0.3), (1,−0.4)). Figure 1 shows the modifications of the vectors while performing the GGSP. The Gram–Schmidt orthogonalization, which is performed in line 6–8 of GGSP, applies twice. In all the following steps the added vector is linearly dependent to the already modified vectors. Therefore we have to go through line 11 and 12, and the vectors already dealt with are newly rearranged in each step.

Figure 2 shows the same example with a different ordering of the vectors. It is no surprise that the generated Parseval frame is completely different from the one obtained in Figure 1, since already the Gram–Schmidt orthogonalization is sensitive to the ordering of the vectors.

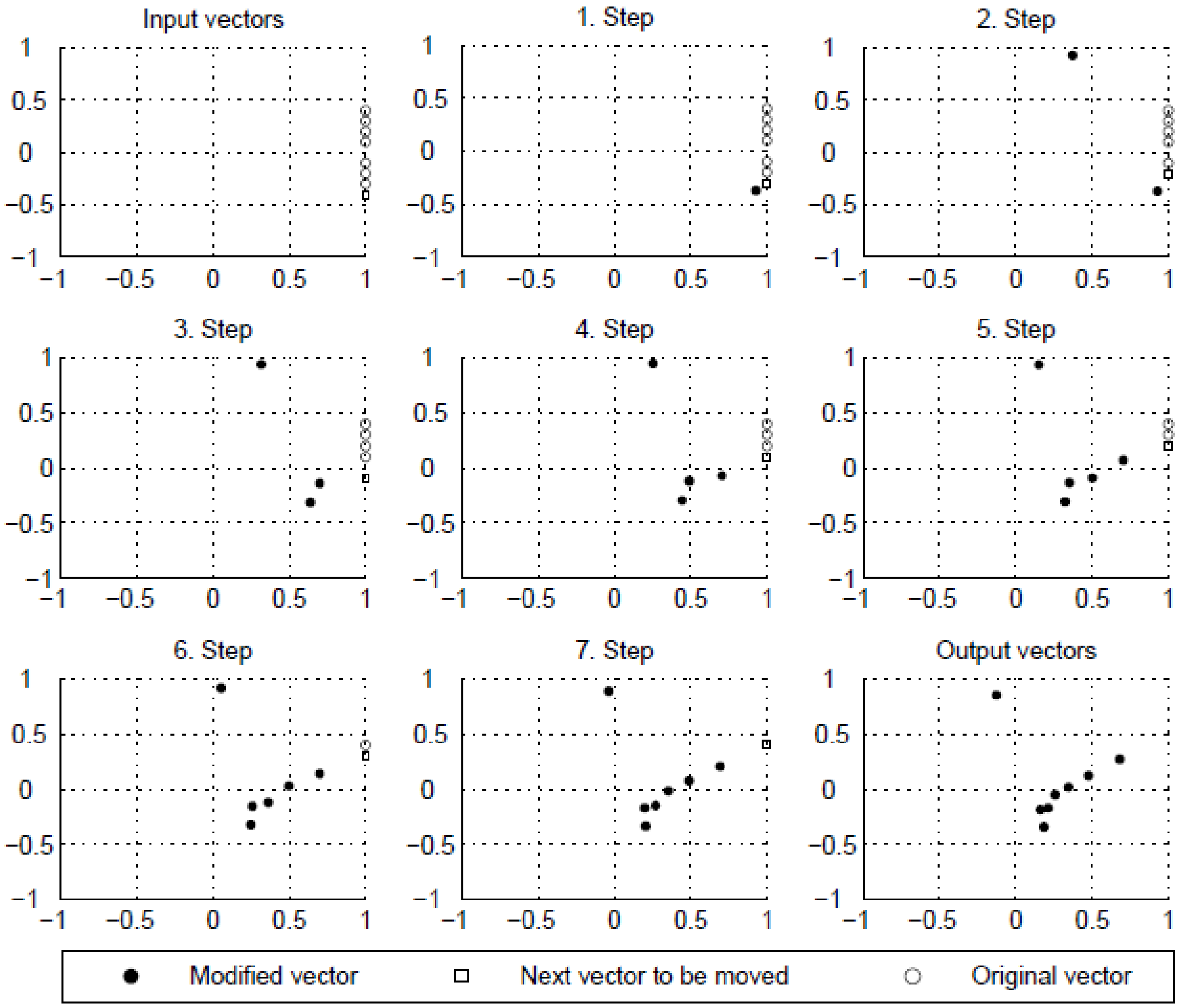


Figure 2,[71]. GGSP applied to the sequence of vectors ((1,−0.4),
.(1,−0.3), (1,−0.2), (1,−0.1), (1, 0.1), (1, 0.2), (1, 0.3), (1, 0.4))

Both generated Parseval frames have in common that the first components of the vectors are almost all positive. Intuitively this is not astonishing, since already all vectors of the input sequence possess a positive first component.

The following example gives further evidence for the claim that the generated Parseval frame inherits the geometry of the input sequence in aparticular way. Here the vectors of the input sequence are located on the unit circle, in particular we consider the sequence of vectors ((1, 0), (√0.5,√0.5), (0, 1), (−√0.5,√0.5), (−1, 0), (−√0.5,−√0.5), (0,−1), (√0.5,−√0.5)). While performing the GGSP the vectors almost keep the geometry of a circle and the final Parseval frame is located on a slightly
deformed circle (see Figure 3). Notice that in the second step of the algorithm the second vector is moved to the position of the third vector (0,1). Hence in all the following computations these two vectors remain indistinguishable.

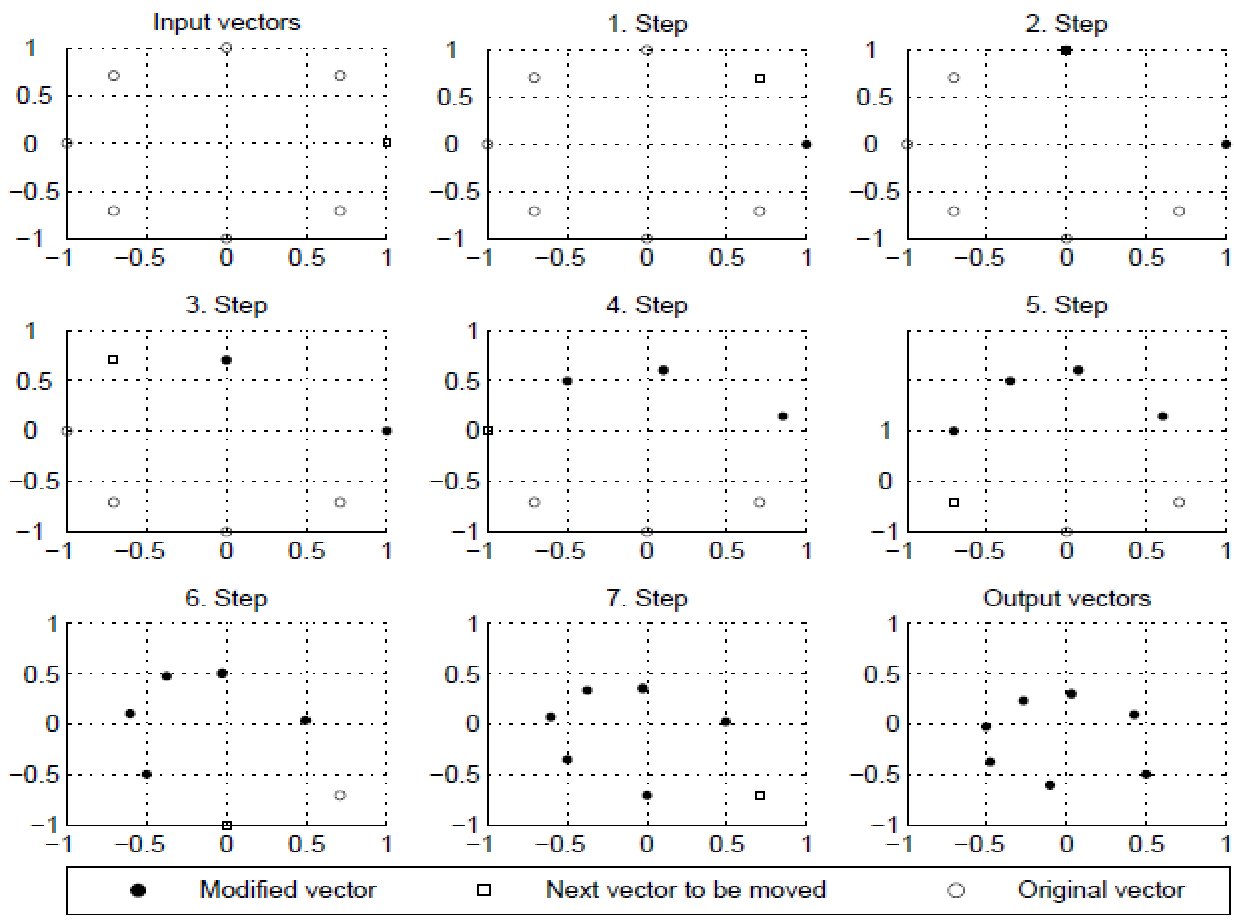


Figure 3,[71]. GGSP applied to the sequence of vectors ((1, 0), ($\sqrt{0.5}$, $\sqrt{0.5}$), (0, 1), ($-\sqrt{0.5}$,$\sqrt{0.5}$), (−1, 0), ($-\sqrt{0.5}$,$-\sqrt{0.5}$), (0,−1), ($\sqrt{0.5}$, $-\sqrt{0.5}$))

The graphical examples seem to indicate that to a certain extent output sequences inherit their geometry from the input sequence. For applications it would be especially important to characterize those input sequences, which generate equal norm Parseval frames or more generally "almost" equal norm Parseval frames (compare [2]).

Section(2.2): Tight Frame Completions

In recent years, the study of frames in finite dimensional Hilbert spaces has been motivated by a large variety of applications, such as signal processing, multiple antenna coding, perfect reconstruction filter banks, and Sampling Theory.

Some particular frames, called tight frames, are of special interest since they allow simple reconstruction formulas. For practical purposes, is often useful to obtain tight frames with some extra "structure", for example with the norms of its elements prescribed (controlled) in advance.

In [180] D. Feng, L. Wang and Y. Wang considered the problem of computing tight completions of a given set of vectors. More explicitly, given a finite sequence $\mathcal{F} = \{f_i\}_{i=1}^{p}$ of vectors in $\mathcal{H}$, how many vectors we have to add in order to obtain a tight frame, and how to find those vectors? [180] provides a complete answer to this question. But when the norms of the additional vectors are required to be one (with the initial set of given vectors of norm one) the authors obtained a lower bound for the number of unit norm vectors we have to add ([180]; but they showed that their lower bound is not sharp in some cases.

Note that this problem may not have a positive solution for a given set of initial vectors and a fixed sequence of "prescribed norms". Therefore we first find conditions for such a tight frame completion to exist. The main tool used here is Theorem (2.2.5), which relates the squared norms of the vectors in a Bessel sequence with the spectrum of its frame operator.

In order to state the main results, we fix some notation used throughout the section. Let $\mathcal{H}$ be a real or complex finite dimensional vector space with dim $\mathcal{H} = n \in N$. Let $\mathcal{F} = \{f_i\}_{i=1}^{p} \subseteq \mathcal{H}$ be a finite sequence with frame operator $S^{\mathcal{F}}$ whose eigenvalues (counted with multiplicity) are $\lambda_1 \geq \ldots \geq \lambda_n$, and let $a = \{a_i\}_{i\in N}$ be a non-increasing sequence of positive real numbers. Finally, let $\alpha = tr(S^{\mathcal{F}})$.

Theorem (2.2.1)[54]: Given $r \in N$, there exists $\mathcal{G} = \{g_i\}_{i=1}^{r} \subseteq \mathcal{H}$ such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame if and only if $\frac{1}{n}\left(\sum_{i=1}^{r}(a_i + \alpha\right) \geq \lambda_1$ and

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq \frac{1}{k}\sum_{i=1}^{r}(a_i + \lambda_{n-i+1}), 1 \leq k \leq min\{n, r\}. \quad (7)$$

On the other hand, there exists an infinite Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame if and only if $\{a_i\}_{i=1}^{\infty} \in \ell^1(N), \frac{1}{n}\left(\sum_{i=1}^{\infty} a_i + \alpha\right) \geq \lambda_1$and

$$\frac{1}{n}\left(\sum_{i=1}^{\infty}(a_i + \alpha\right) \geq \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}), 1 \leq k \leq n. \quad (8)$$

So from Theorem A we get necessary and sufficient conditions for the existence of a sequence $\mathcal{G} = \{g_i\}_{i=1}^{r}$, for some $r \in N \cup \{\infty\}$, with $\|g_i\|^2 = a_i$, and such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame (for some suitable constant). If such a completion exists we say that $\mathcal{F}$ is (a, r) −completable. In case $\mathcal{F}$ is (a, r)-completable, we are then interested in computing the minimum number r_0 of vectors we have to add. In order to state our next result we introduce the following numbers: let $c_0 = \lambda_1$ and for $1 \leq k \leq n$ let

$$c_k = max\left(c_{k-1}, \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1})\right). \quad (9)$$

Theorem (2.2.2)[54]: Assume that $\mathcal{F}$ is (a, r) −completable for some $r \in N \cup \{\infty\}$ and let $r_0 \in N \cup \{\infty\}$ be the minimum such that $\mathcal{F}$ is (a, r_0)-completable.

Then

Case 1: $r_0 < n$ if and only if $c_{r_0} = \frac{1}{n}\sum_{i=1}^{r_0} a_i + \alpha)$.

Case2: $n \leq r_0 < \infty$ if and only if $c_k \neq \frac{1}{n}\left(\sum_{i=1}^{k} a_i + \alpha\right) \forall\, 1 \leq k \leq n-1$ and r_0 *i*s the minimum such that $c_n \leq \frac{1}{n}\sum_{i=1}^{r_0} a_i + \alpha)$.

Case 3: $r_0 = \infty$ if and only if $c_k \neq \frac{1}{n}\left(\sum_{i=1}^{k} a_i + \alpha\right)$ for all $1 \leq k \leq n-1$

and

$$c_n = \frac{1}{n}\left(\sum_{i=1}^{\infty} a_i + \alpha\right)$$

We should remark that although Theorems A and B are of practical interest, they are not efficiently (fast) algorithmic implementable in a computer (see the discussion at the beginning of Section). In this Section we deal with the problem of finding a not so optimal but efficiently algorithmic computable finite tight completion as follows:

Theorem (2.2.3)[54]: Assume that a is a divergent sequence. Let $d \in \mathbb{R}$ be an algorithmic computable upper bound for$\|S^{\mathcal{F}}\|$and let $c = max(d + 1, d + a_1)$. If $r \in N$ is such that

$$\sum_{i=1}^{r-1} a_i < c \cdot n - tr(S^{\mathcal{F}}) \leq \sum_{i=1}^{r} a_i$$

then there exists an algorithmic computable sequence $\mathcal{G} = \{g_i\}_{i=1}^{r}$ such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame and such that $\|g_i\|^2 = a_i$ for $1 \leq i \leq r$. We also consider particular cases of Theorems (2.2.1)and (2.2.2)when $a_i = 1$ for $i \geq 1$.

Throughout the section, $\mathcal{H}$ will be a finite dimensional (real or complex) Hilbert space with dim $\mathcal{H}$ = n ∈ N and $L(\mathcal{H})^+$ will denote the cone of bounded positive semi-definite operators on $\mathcal{H}$. Given $m \in N \cup \{\infty\}$, a sequence $\mathcal{F} = \{f_i\}_{i=1}^{m} \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist numbers $a, b > 0$ such that, for every $f \in \mathcal{H}$,

$$\alpha\|f\|^2 \leq \sum_{i=1}^{m}|\langle f, f_i\rangle|^2 \leq b\|f\|^2 \quad (10)$$

The optimal constants in (10) are called the frame bounds. If the frame bounds a, b coincide, the frame is called a-tight (or simply tight). Finally, tight frames with all its elements having the same norm are called equal norm tight frames.

The sequence $\mathcal{F}$ is Bessel if there exists $b > 0$ such that the upper bound condition in (10) is satisfied. Given a Bessel sequence $\mathcal{F}$, we define its frame operator by

$$S^{\mathcal{F}} f = \sum_{i=1}^{m}\langle f, f_i\rangle f_i. \quad (11)$$

It is easy to see that $S^{\mathcal{F}}$ is a positive semi-definite bounded operator on $\mathcal{H}$. Moreover, $\mathcal{F}$ is a frame if and only if its frame operator $S^{\mathcal{F}}$ is invertible. Indeed, the optimal frame bounds a, b in (10) are respectively λ_{min} ($S^{\mathcal{F}}$) and λ_{max} ($S^{\mathcal{F}}$), the minimum and maximum eigenvalues of $S^{\mathcal{F}}$. In particular, a frame $\mathcal{F}$ is a-tight if and only if $S^{\mathcal{F}} = aI$. For an introduction to the theory of frames and related topics see the books [74, 182].

Given a Bessel sequence $\mathcal{F}$, there is a close relationship between the norms of its elements and the spectrum of $S^{\mathcal{F}}$ that can be expressed in terms of majorization (see [35] for details). First, we introduce some definitions. We say that a sequence $\{a_i\}_{i=1}^{m}$ is summable if $m \in N$, or if $m = \infty$ and $\{a_i\}_{i=1}^{\infty} \in \ell^1(N)$.

Dentition(2.2.4)[54]: Let $a = \{a_i\}_{i=1}^{m}$, $b = \{b_i\}_{i=1}^{s}$ be non-increasing summable sequences of non-negative numbers, with $s, m \in N \cup \{\infty\}$, and let $t = min\{s, m\}$. We say that b majorizes a, noted $b \succ a$, if

$$\sum_{i=1}^{j} b_i \geq \sum_{i=1}^{j} a_i \text{ for } 1 \leq j \leq t \text{ and } \sum_{i=1}^{s} b_i = \sum_{i=1}^{m} \quad (12)$$

If $m = s \in \mathbb{N}$ in Definition(2.2.1) then this notion coincides with the usual vector majorization in $\mathbb{R}^m$ between vectors with non-negative entries which are arranged in non-increasing order (see [181]).

On the other hand, as an immediate consequence of Definition(2.2.1) we see that if $s \in \mathbb{N}$, and then $a \prec b$ if and only if $a \prec (b, 0_n)$ for every $n \in \mathbb{N}$, where
$(b, 0_n) \in \mathbb{R}^{s+n}$, and similarly (a, 0_n) $\prec$ b if $m \in M$.

Now we can state the frame version of the Schur-Horn theorem, which we shall need in the sequel.

Theorem (2.2.5)[54]: Let $a = \{a_i\}_{i=1}^{m}$ be a non-increasing sequence of positive numbers and let $S \in L(\mathcal{H})^+$ with eigenvalues (counted with multiplicity and arranged in non-increasing order) $\Lambda = \{\lambda_j\}_{j=1}^{n}$. Then the following statements are equivalent:

(a) $a \prec \Lambda$.

(b) There exists a Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^{m} \subset \mathcal{H}$ such that $\|g_i\|^2 = a_i$ for $1 \leq i \leq m$ and $S^{\mathcal{G}} = S$.

Definition(2.2.6)[54]: We say that $\mathcal{F}$ is (a, r) $-$completable if there exists $r \in N \cup \{\infty\}$ and a Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^{r} \subset \mathcal{H}$, with $\|g_i\|^2 = a_i$ for $1 \leq i \leq r$,and such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame. We say that $\mathcal{G} = \{g_i\}_{i=1}^{r}$ is an (a, r) -completion of $\mathcal{F}$.

. For the sake of clarity in the exposition, in what follows we consider separately
the cases where $\mathcal{F}$ is (a, r)-completable for some $r \in \mathbb{N}$ and the case $r = \infty$,
although there is no substantial difference in the arguments involved.

Theorem(2.2.7)[54]: Let $r \in \mathbb{N}$. Then $\mathcal{F}$ is (a, r)-completable if and only if

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq \lambda_1 \ and$$

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq$$

$$\frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}), 1 \leq k \leq min\{n, r\}. \quad (13)$$

Proof: Assume that there exists $r \in \mathbb{N}$ and a finite sequence $\mathcal{G} = \{g_i\}_{i=1}^{r}$ such that $S^{\mathcal{F} \cup \mathcal{G}} = S^{\mathcal{F}} + S^{\mathcal{G}} = cI$ and $\|g_i\|^2 = a_i$ for $1 \leq i \leq r$.Then $cI - S^{\mathcal{F}} = S^{\mathcal{G}} \geq 0$; in particular we have $c \geq \|S\| = \lambda_1$. On the other hand, we see that the eigenvalues of

$S^{\mathcal{G}}$ arranged in non-increasing order are $c-\lambda_n \geq \ldots \geq c-\lambda_1 \geq 0$. By Theorem (2.2.3) we have

$$(c-\lambda_n, c-\lambda_{n-1}, \ldots, c-\lambda_1) \succ (a_1, \ldots, a_r). \quad (14)$$

Then, by Definition(2.2.1) we see that $\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq \lambda_1$ and (26) hold, using that $c = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$.

Conversely assume that $\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq \lambda_1$ and (26) hold for $r \in \mathbb{N}$. Set $c = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$ and note that the spectrum of the positive operator $cI - S^{\mathcal{F}}$, $(c-\lambda_n, c-\lambda_{n-1}, \ldots, c-\lambda_1)$, majorizes (in the sense of Definition(2.2.4) $\{a_i\}_{i=1}^{r}$. By Theorem(2.2.5) we conclude that there exists a finite sequence $\mathcal{G} = \{g_i\}_{i=1}^{r}$with $S^{\mathcal{G}} = cI - S^{\mathcal{F}}$ and $\|g_i\|^2 = \alpha_i$for $1 \leq i \leq r$ and we are done.

By inspection of the proof of Theorem(2.2.7), we have the following corollaries.

Corollary (2.2.8)[54]: Using the notations of Theorem(2. 2.7), $\mathcal{F}$ is (a,r)-completable with $r < n$ if and only if, for $1 \leq i \leq n-r$ and $1 \leq k \leq r$,

$$\lambda_i = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right), \text{ and } \lambda_1 \geq \frac{1}{K}\left(\sum_{i=1}^{k} a_i + \lambda_{n-i+1}\right). \quad (15)$$

Corollary(**2. 2.9**)[54] : Let $\mathcal{F}$ be (a,r)- completable for some $r \in \mathbb{N}$. Then

(i)if $r < n$ then $\mathcal{F}$ is not (a,k)-completable for any $k < n$ other than r,

(ii)if $r \geq n$ then $\mathcal{F}$ is (a,k)-completable for every $k \in \mathbb{N}$ with $k \geq r$.

The next result gives different equivalent conditions for a sequence a and vectors $\mathcal{F}$ in order to be (a,r)-completable for some $r \in \mathbb{N}$. First, we define inductively the following numbers: let $c_0 = \lambda_1$ and for $1 \leq k \leq n$ let

$$c_k = max\left(c_{k-1}, \frac{1}{k}\sum_{i=1}^{k} a_i + \lambda_{n-i+1}\right). \quad (16)$$

It is clear from definition that $\lambda_1 \leq c_1 \leq \ldots \leq c_n$.

Proposition(2. 2.10)[54]: Let $r \in \mathbb{N}$. $\mathcal{F}$ is (a,r)-completable if and only if

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) = c_r \text{ for } r < n. \quad (17)$$

Or Moreover, if $c_r = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$some for $r < n$, then $c_r = \lambda_1$.

Poof. A ssume that $\mathcal{F}$ is (a,r)- completable

If $r < n$ note that, by (17) in Corollary (2.2.8), we have $\lambda_1 = c_0 \leq \cdots \leq c_r = \lambda_1$ and $\lambda_1 = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$,

so (17) holds. If $r \geq n$ then min $\{n, r\} = n$ and Theorem (2.2.7) together with the definition of $c_n$imply that

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq c_n.$$

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq c_n \text{ for } r \geq n. \quad (18)$$

So in this case (18) holds. Conversely, if we assume (18), then it is clear $\mathcal{F}$ is (a,r)-completable, by Theorem (2.2.7). Assume now that for some $r < n, \mathrm{c_n} = \frac{1}{\mathrm{n}}\left(\sum_{\mathrm{i}=1}^{\mathrm{r}} a_i + \alpha\right)$. We show that $\mathcal{F}$ is (a,r)-completable; indeed, since $n\mathrm{c_n} = \sum_{\mathrm{i}=1}^{\mathrm{r}} a_i + \alpha$, then

$$rc_r + (n-r)c_r - \sum_{i=1}^{n-r} \lambda_i = \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} \lambda_{n-i+1}$$

So by definition of $\mathrm{c_r}$ we have

$$\sum_{i=1}^{r}(a_i+\lambda_{n-i+1}) \leq rc_r = \sum_{i=1}^{r}(a_i+\lambda_{n-i+1}) - \sum_{i=1}^{n-r}(c_r-\lambda_i) \leq \sum_{i=1}^{r}(a_i+\lambda_{n-i+1}).$$

But then

$\lambda_i = \frac{1}{n}(\sum_{i=1}^{r} a_i + \alpha)$ for $1 \leq i \leq n-r$

and

$$\lambda_1 \geq \max_{1\leq k\leq r} \frac{1}{k}\sum_{i=1}^{k}(a_i+\lambda_{n-i+1}),$$

So $\mathcal{F}$ is (a,r)-completable, by Corollary (2.2.8). The last claim of the proposition is clear from our previous computations.

We are now able to give a formula for the minimum $r \in \mathbb{N}$ such that $\mathcal{F}$ is (a,r)-completable, when such an $r \in \mathbb{N}$ exists.

Theorem(2. 2.11)[54]: Let $\mathcal{F}$ be a (a,r)-completable for some $r \in \mathbb{N}$. Let$r_0 \in \mathbb{N}$ be the minimum such that $\mathcal{F}$ is (a, r_0)-completable. Then

Case 1: $r_0 < n$ if and only if $c_{r_0} = \frac{1}{n}(\sum_{i=1}^{r_0} a_i + \alpha)$

Case2: $r_0 < \geq n$ if and only if $c_k \neq \frac{1}{n}(\sum_{i=1}^{r_0} a_i + \alpha)$ $for\ all$ $1 \leq k \leq n-1$ $and r0\in\mathbb{N}$ is the minimum such that $cn\leq 1ni=1r0ai+\alpha.$

Proof: Note that, by Proposition(2. 2.10), at least one the cases has to be fulfilled by some $r \in \mathbb{N}$. If we assume that case 1 holds for some $r < n$ then,by Proposition (2.2.10) $\mathcal{F}$ is (a,r)-completable. By Corollary(2. 2.9) case 1 does not hold for $k < n$ with $r \neq k$. It is clear that in this case $r_0 = r$.

Assume now that there is no $r < n$ satisfying case 1 above. Then, there exists $r \in \mathbb{N}$ such that$c_n \leq \frac{1}{n}(\sum_{i=1}^{r} a_i + \alpha)$ by Proposition(2. 2.10) we see that $\mathcal{F}$ is (a,r)-completable. It is clear that r_0 is the minimum natural number r satisfying this condition. Finally note that if $r \in \mathbb{N}$ is such that$c_n \leq \frac{1}{n}(\sum_{i=1}^{r} a_i + \alpha)$ then

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i + \alpha\right) \leq c_n \leq \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \Rightarrow \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{r} a_i$$

and $r \geq n$ since for every $i \in \mathbb{N}, a_i > 0$.

The next example shows that it is possible to obtain a set of vectors $\mathcal{F}$ and a sequence a such that $\mathcal{F}$ is (a,r)-completable for only one $r \in \mathbb{N}$ (in virtue ofCorollary (2. 2.9) , $r < n$).

Example(2. 2.12)[54]: Let $\mathcal{F} = \{\sqrt{2}e_1, \sqrt{2}e_2, e_3\}$ in $\mathbb{C}^3$ where $\{e_i\}$ is the canonical orhonormal basis and let $a = \left\{\left(\frac{1}{4}\right)^{i-1}\right\}_{i=1}^{\infty}$. Then, easy computations show that the eigenvalues of $S^{\mathcal{F}}$are $\lambda_1 = 2, \lambda_2 = 2$ and $\lambda_3 = 1$, so $\alpha = trS^{\mathcal{F}} = 5$. By Corollary (2.2.8) $\mathcal{F}$ is $(a,1)$-completable since $\lambda_1 = \frac{1}{3}(a_1 + \alpha)$ and $\lambda_1 \geq a_1 + \lambda_3$ Moreover, it is clear that if we add the vector $e_3$to $\mathcal{F}$ we obtain a 2-tight frame.

On the other hand, it easy to see that $\frac{1}{3}(\sum_{i=1}^{\infty} a_i + \alpha) = \frac{19}{9} < \frac{17}{8} = c_3$ so byProposition (2.2.10) $\mathcal{F}$ is not (a,r)-completable for any $r \geq 3$.

In fact, as the following proposition shows, if $\mathcal{F}$ is (a,r)-completable with $r < n$ the existence of some $r_1 \geq n$ such that $\mathcal{F}$ is (a,r_1)-completable depends only on the tail of the sequence $\{a_i\}_{i=r+1}^{\infty}$.

Proposition(2. 2.13)[54]:Let $\mathcal{F}$ be (a,r)-completable for some $r < n$. There exists $r_1 \in \mathbb{N}$ with $r_1 \geq n$ and such that $\mathcal{F}$ is (a,r_1)-completable if and only if

$$\frac{1}{n}\sum_{i=r+1}^{r_1} a_i \geq \max_{r+1\leq k\leq n}\frac{1}{k}\sum_{i=r+1}^{k} a_i$$

Proof: By Theorem(2. 2.7) $\mathcal{F}$ is (a,r_1)-completable if and only if

$\frac{1}{n}\left(\sum_{i=1}^{r_1} a_i+\alpha\right)\geq\lambda_1$ and $\frac{1}{n}\left(\sum_{i=1}^{r_1} a_i+\alpha\right)\geq\frac{1}{k}\sum_{i=1}^{k}(a_i+\lambda_{n-i+1}), 1\leq k\leq n$

By hypothesis and Corollary(2. 2.8),

$\lambda_i=\frac{1}{n}(\sum_{i=1}^{r} a_i+\alpha), 1\leq i\leq n-r$ and $\lambda_1\geq\frac{1}{k}\sum_{i=1}^{k}(a_i+\lambda_{n-i+1}), 1\leq k\leq r$

Since $\mathcal{F}$ is (a,r)-completable with $r<n$. So $\mathcal{F}$ is (a,r_1)-completable if and only if $\frac{1}{n}\left(\sum_{i=1}^{r_1} a_i+\alpha\right)\geq\frac{1}{k}\sum_{i=1}^{k}(a_i+\lambda_{n-i+1}), r+1\leq k\leq n$

or equivalently, if for every $r+1\leq k\leq n$

$$\sum_{i=1}^{r} a_i+\alpha+\sum_{i=r+1}^{r_1} a_i\geq\frac{n}{k}\left(\sum_{i=1}^{r} a_i+\sum_{i=r+1}^{k} a_i+\alpha-(n-k)\lambda_1\right)$$

$$\sum_{i=1}^{r} a_i+\alpha+\sum_{i=r+1}^{r_1} a_i\geq\frac{n}{k}\left(\sum_{i=1}^{r} a_i+\alpha\right)+\frac{n}{k}\sum_{i=r+1}^{k} a_i-\frac{n-k}{k}\left(\sum_{i=1}^{r} a_i+\alpha\right)$$

$$\sum_{i=1}^{r} a_i+\alpha+\sum_{i=r+1}^{r_1} a_i\geq\sum_{i=1}^{r} a_i+\alpha+\frac{n}{k}\sum_{i=r+1}^{k} a_i$$

$$\sum_{i=r+1}^{r_1} a_i\ \geq\frac{n}{k}\sum_{i=r+1}^{k} a_i,$$

since by hypothesis $\lambda_i=\frac{1}{n}(\sum_{i=1}^{r} a_i+\alpha)$ for $1\leq i\leq n-r$.

In this section we consider some complementary results to those obtained in the previous section and prove Theorems (2.2.1)and(2.2.2).

If $\mathcal{F}=\{f_i\}_{i=1}^{p}$ *and* a are as before, then a necessary condition for $\mathcal{F}$ to be (a,∞)-completable is that $a\in\ell^1(\mathbb{N})$.

Theorem(2. 2.14)[54]: $\mathcal{F}$ is (a,∞)-completable (by a Bessel sequence) if and only if $a\in\ell^1(\mathbb{N}),\frac{1}{n}(\sum_{i=1}^{\infty} a_i+\alpha)\geq\lambda_1$ and

$\frac{1}{n}(\sum_{i=1}^{\infty} a_i+\alpha)\geq\frac{1}{k}\sum_{i=1}^{k}(a_i+\lambda_{n-i+1}), 1\leq k\leq n,$ (19)

or equivalently if $(a\in\ell^1(\mathbb{N})$

$\frac{1}{n}(\sum_{i=1}^{\infty} a_i+\alpha)\geq c_n$. (20)

The proof of Theorem(2. 2.14), which is based on Theorem(2.2.5), is similar to that of Theorem(2. 2.6) and Proposition(2. 2.10).

Proof of Theorem (2.2.1). The first part of the theorem is Theorem(2. 2.6), while the second part is Theorem (2.2.14).

Proof of Theorem (2.2.2). Assume there exists a natural number $r\in\mathbb{N}$ such that $\mathcal{F}$ is (a,r)-completable. Then $r_0\leq r$and in this case the theorem follows from Theorem(2. 2.11). If there is no $r\in\mathbb{N}$ such that $\mathcal{F}$ is (a,r)-completable, then $\mathcal{F}$ is(a,∞)-completable so by Theorem (2.2.14) $a\in\ell^1(\mathbb{N})$ and $\frac{1}{n}(\sum_{i=1}^{\infty} a_i+\alpha)\geq c_n$ if $\frac{1}{n}(\sum_{i=1}^{\infty} a_i+\alpha)>c_n$ then there exists $r\in\mathbb{N}$ such that$\frac{1}{n}(\sum_{i=1}^{r} a_i+\alpha)\geq c_n$. If then, by Proposition(2. 2.11) we get that $\mathcal{F}$ is (a,r)-completable, a contradiction.

We finish with the counter-part of Proposition (2.2.13) for the infinite completion case.

Proposition(2. 2.15)[54]: Let $a\in\ell^1(\mathbb{N})$ and let $\mathcal{F}$ be (a,r)-completable for some $r<n$. Then, $\mathcal{F}$ is (a,∞) $-$completable if and only if $\frac{1}{n}\sum_{i=r+1}^{\infty} a_i\geq\max_{r+1\leq k\leq n}\frac{1}{k}\sum_{i=r+1}^{k} a_i$.

In this section we consider the particular case when $a=\{a_i\}_{i\in N}$ is a constant

sequence, $a_i = 1$ for all $i \in \mathbb{N}$(the general case follows in an analogous way). Note that in this case $\mathcal{F}$ is (a, r)-completable for some $r \in \mathbb{N}$; so we shall compute the minimum natural number r of vectors with norm one we have to add to $\mathcal{F}$ in order
t frame. We keep the notation of the previous section for$\mathcal{F} = \{f_i\}_{i=1}^{p}$,
$^{\mathcal{F}}, \lambda_1 \geq \cdots \geq \lambda_n$ and α.

Theorem(2.2.16)[54]: Let $h := \sum_{i=2}^{n} \lambda_{1-\lambda_i}$, and denote by τ_0 the minimum numbe r of norm one vectors we have to add to $\mathcal{F}$in order to have a tight frame.

Case1:Suppose$h < n$.Then $r_0 = h$ if $h \in N$and $1 + \frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} \leq \lambda_1$in particular, c_h=λ_1). Otherwise, $r_0 = n$.

Case 2:If $h \geq n$, r_0 is the minimum integer greater than or equal to h .

Proof. Assume that $h< n$;then, since $h = n\lambda_1 - \alpha$, we have that $c_n = 1 + \frac{\alpha}{n}$

RIf in addition $h < n$ and $1+\frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} < \lambda_{1,}$,so $c_h = \frac{1}{n}(h + \alpha) = \lambda_{1,}$then $r_0 = h$ byTheorem(2.2.11).Otherwise, $c_k \neq \frac{1}{n}(k + \alpha)$ for all $k < n$ (if $c_k = \frac{1}{n}(k + \alpha)$ for some $k < n$, then by Proposition(2.2.10) $c_k = \lambda_{1,}$and h would be a natural number); since $c_n = 1 + \frac{\alpha}{n}$, the minimum integer greater than or equal to $nc_n - \alpha$ is n so $r_0 = n$ by Theorem (2.2.11).

Finally, $h \geq n$ implies that$c_k \neq \frac{1}{n}(k + \alpha)$ for all $k < n$ and $c_n = \lambda_{1,}$. Therefore, again by Theorem (2.2.11), r_0 is the minimum integer greater than or equal to $n\lambda_{1,} - \alpha = h$.

Example (2.2.17)[54]: This example is taken from [180]. It is interesting because it shows the difference between the cases when we can complete $\mathcal{F}$ to a tight frame with $r < n$ or $r \geq n$ vectors. Let $f_1 = (1.0)$ and $f_2 = (cos\theta, \sin\theta)$ in $\mathbb{R}^2$, and consider $a_i = 1$ $\forall$i.
It easy to see that the eigenvalues of $S^{\mathcal{F}}$ are $1 \pm cos\theta$, hence $h = \lambda_{1-}\lambda_2 = 2|cos\theta|$.
Therefore, by Theorem (2.2.16), the minimum number r_0 of unit vectors we have to add t frame is 2 unless $\theta = \frac{2}{3}\pi$ or $\theta = \frac{4}{3}\pi$ where$r_0 = 1$. Note that When $r_0 = 1$ the tight frame obtained is the well known "Mercedes Benz" (it is–up to rigid rotations, reflections and negation of individual vectors– the only unit norm tight frame on $\mathbb{R}^2$with three elements [47].

A consequence of Theorem (2.2.16) is the characterization of the minimum number of vectors that we have to add in order to get a tight frame, in the particular case when $\mathcal{F}$ is a unit norm tight frame on its linear span.

Proposition (2.2.18)[54]: Let $\mathcal{F}$ =$\{f_i\}_{i=1}^{p}$be a unit norm $\frac{p}{d}$ −tight frame on its span, where $d < n$ is the dimension of span $\mathcal{F}$. Then, the minimum number r_0 of unit norm vectors we have to add to $\mathcal{F}$ in order to obtain a tight frame in$\mathcal{H}$is:

(a) $(n - d)\frac{p}{d}$ if $(n - d)\frac{p}{d} < n$ and $(n - d)\frac{p}{d} \in \mathbb{N}$.

(b) n if $(n - d)\frac{p}{d} < n$ and $(n - d)\frac{p}{d} \notin \mathbb{N}$

(c) the minimum integer greater than or equal to $(n - d)\frac{p}{d}$ if $(n - d)\frac{p}{d} \geq n$.

Proof. Since $\mathcal{F}$is an unit norm tight frame on a subspace of dimension d, the eigenvalues of $S^{\mathcal{F}}$are: $\lambda_i = \frac{p}{d} \geq 1$ for $1 \leq i \leq d$, and $\lambda_i = 0$ for $d + 1 \leq i \leq n$. Therefore, $h = \sum_{i=2}^{n} \lambda_1 - \lambda_i = (n - d)$. Moreover, if $h < n$ and $h \in \mathbb{N}$, then $1+\frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} = \lambda_1$. Indeed,

$$1+\frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} = 1 + \frac{h-(n-d)}{h}\frac{p}{d} = \frac{p}{d} \tag{21}$$

the proposition is then a consequence of Theorem (2.2.16)

Let $\mathcal{F}$ =$\{f_i\}_{i=1}^{p} \subseteq \mathcal{H}$ and assume that a is a divergent sequence. Then, by Remark (2.2.8), $\mathcal{F}$ is (a, r)-completable for some $r \in \mathbb{N}$. From the proof of Theorem(2.2.7) we see that if

$$c = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$$

then equation (26) holds. Therefore, by theorem (2.2.4),theoretically there exists a Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^{p} \subseteq \mathcal{H}$ such that $\|g_i\|^2 = a_i$ for $1 \leq i \leq r$ and $S^{\mathcal{G}} = cI - S^{\mathcal{F}}$. In this case, $\mathcal{G}$ is a (a,r)-completion of $\mathcal{F}$; moreover, if $r \in \mathbb{N}$ is obtained as in Theorem (2.2.11) then $\mathcal{G}$ would be (a, r)-tight completion having the minimum number of vectors for which a tight completion of $\mathcal{F}$ exists. Although constructive, the proof of Theorem (2.2.4) is not practicable; it depends on some matrix decompositions which can not be performed efficiently by a computer for large values of t = $\min_{\{}$n, $r_{\}}$. There are several recent papers related to algorithmic construction of frames with additional properties. In [178] Casazza and Leon considered the problem of constructing frames with prescribed properties from an algorithmic point of view; in particular, they obtained an algorithm for constructing tight frames with pre-scribed norms of its elements, under the admissibility conditions of Theorem(2.2.4). In [180] there is a fast algorithm for constructing tight frames with prescribed norms of its elements based on Householder transformations; in [44] a fast algorithmic proof of some results related to the Schur-Horn theorem is considered and as a consequence a generalized one-sided Bendel-Mickey algorithm (see Theorem (2.2.19) below) is obtained. Still, as far as we know, the problem of constructing a frame for $\mathcal{H}$ with prescribed general (positive definite) frame operator and norms (that are admissible in the sense of Theorem(2.2.4)) using an efficient computable algorithm has not been solved: we remark that for the purposes of this discussion, the diagonalization of a positive semi-definite matrix is considered as not efficiently computable. If such an algorithm is obtained, then optimal tight frame completions can be constructed as described in the first paragraph of this section. In what follows we shall consider a not so optimal tight frame completion of a given set $\mathcal{F} = \{f_i\}_{i=1}^{p}$ but that is efficiently algorithmic computable, based on the generalized one-sided Bendel-Mickey algorithm and the Cholesky's decomposition. Let us begin with the following result from [44]. We remark that our notation is opposite to that in [44] so we translate their result into our terminology.

Theorem (2.2.19) ([44]): Let $a = \{a_i\}_{i=1}^{r}, b = \{b_i\}_{i=1}^{r}$ be two finite and non-increasing sequences of positive numbers such that $a \prec b$. Let X be an $n \times r$ matrix whose squared columns norms are listed by b. Then there is a finite sequence of algorithmic computable plane rotations $U_1, \dots, U_{r-1} \in \mathbb{M}_r(\mathbb{C})$ such that $X(U_1 \dots U_{r-1})$ has squared columns norms listed by a.

Actually, each plane rotation that appears in the theorem above operates non-trivially in the coordinate plane span $\{e_i, e_j\}$ for some $1 \leq i, j \leq r$ (see[44] for details). Note that the initial matrix X and the final matrix $Y = X(U_1 \dots U_{r-1})$ satisfy $XX^* = YY^*$.

Taking into account Theorem (2.2.19), an strategy to construct a frame with pre-scribed frame operator $S \in \mathbb{M}_r(\mathbb{C})$ and norms of its elements listed by a(satisfying the conditions in Theorem(2.2.4)) would be the following: consider a diagonalization $S = U\mathrm{diag}(\lambda_1, \dots, \lambda_n)U^*$ and the factorization $XX^* = S$ with $X = U\mathrm{diag}(\sqrt{\lambda_1}, \dots \sqrt{\lambda_n})$. Note that the squared norms of the columns of X are listed by $(\lambda_1, \dots, \lambda_n)$ so we can apply Theorem (2.2.19) and obtain

$$Y = X(U_1 \dots U_{r-1})$$

with $YY^* = S$ with the squared norms of the columns of Y given by a . Unfortunately, we consider this procedure as not an eciently computable one, so we have to find an alternative approach.

Along this section we prove Theorem C; we begin with an informal discussion of the algorithm. Assume that the non-increasing sequence of positive numbers $\{a_i\}_{i=1}^{\infty}$ forms a divergent series, so that $\mathcal{F}$ is (a, t)-completable for some $t \in \mathbb{N}$. Let $S = S^{\mathcal{F}}$ and let $c >$

$\|S\|$that we shall consider as a variable. We will obtain an algorithmic computable value of c for which the Cholesky's decomposition $cI_n - S = RR^*$ satisfies that the squared norms of the columns of R mayorize$\{a_i\}_{i=1}^{r}$for an integer $r \geq n$. Once we have obtained such c, we apply Theorem (2.2.19) and get a finite sequence $\mathcal{G} = \{g_i\}_{i=1}^{r}$with frame operator $cI_n - S$ and $\|g_i\|^2 = a_i$, for $1 \leq i \leq r$. Let $c > \|S\| + \beta$ so $\lambda_{min}(cI - S) = c - \|S\| \geq \beta$, where $\beta > 0$ is a fixed number controlling the invertibility of $cI - S$. Let $R = R(c)$ be the upper triangular matrix obtained from the Cholesky's decomposition of $cI - S$ (note that the hypothesis on c is made in order that the Cholesky's algorithm becames stable). Then $RR^* = cI - S$ and note that $c - \|S\| = \lambda_{min}(RR^*) = \lambda_{min}(R^*R)$ so, if $C_i(R)$ denotes the i-th column of R then

$$\min_{1 \leq i \leq n} \|C_i(R)\|^2 \geq c - \|S\|,$$

Since $\|C_i(R)\|^2 = (R^*R)_{ii}$ and $(RR^*)_{ii} \geq \lambda_{min}(R^*R)$ for $1 \leq i \leq n$. In particular $\sum_{i=1}^{k} \|C_i(R)\|^2 = \sum_{i=1}^{k} (R^*R)_{ii} \geq k.(c - \|S\|)$.Let $c \geq max(\|S\| + \beta, \|S\| + a_1)$ and note that then

$$c \geq \frac{1}{k} \sum_{i=1}^{k} a_i + \|S\|, \text{for } 1 \leq k \leq n \tag{22}$$

since$\frac{1}{k}\sum_{i=1}^{k} a_i \geq \frac{1}{h}\sum_{i=1}^{h} a_i \; if \; 1 \leq k \leq h \leq n$. Let $r \in \mathbb{N}$ be such that

$$\sum_{i=1}^{r-1} a_i < \sum_{i=1}^{n} \|C_i(R(c))\|^2 = c.n - tr(S) \leq \sum_{i=1}^{r} a_i \tag{23}$$

So $r \geq n$. We define $c' = \frac{1}{n}\sum_{i=1}^{r} a_i + tr(S^{\mathcal{F}})$,where r is defined by (23) so that, if $R(c')$denotes the holesky'Cs decomposition of $c'I - S^{\mathcal{F}}$ then we get

$$(a_i)_{i=1}^{r} \prec (\|C_i(R(c'))\|^2)_{i=1}^{n}.$$

Thus, with this $c' \in \mathbb{R}$ and $r \in \mathbb{N}$we can apply Theorem (2.2.19) to the matrix

$$X = [R(c'), O_{n\times(r-n)}]$$

and get the (efficiently algorithmic computable) $n \times r$ matrix Y such that $YY^* = S$ and$\|C_i(Y)\|^2 = a_i$ for $1 \leq i \leq r$; setting g_i=$C_i(Y)$ we get $\{g_i\}_{i=1}^{r}$ with the desired properties. We briefly resume the previous considerations in the following pseudo-code implementation:

(a)Find an algorithmic computable upper bound d for $\|S\|$.

(b)Compute $c = \max(d + \beta, d + \alpha_1)$ (where $\beta > 0$ is previously fixed) and $r \in \mathbb{N}$ satisfying (23).

(c) Redefine $c = \frac{1}{n}(\sum_{i=1}^{r} a_i + tr(S^{\mathcal{F}}))$.

(d) Compute the Cholesky's decomposition $cI - S = RR^*$.

(e) Apply Theorem(2.2.19) to the $n \times r$ matrix $[R, O_{n\times(r-n)}]$ and get the $n \times r$ matrix Ysuch that $cI - S = YY^*$and $\|C_i(Y)\|^2 = a_i$ for $1 \leq i \leq r$.

(f) Define$g_i = C_i(Y)$ for $1 \leq i \leq r$.

Example (2.2.20)[54]: Assume that$\|f_i\| = 1$ for $1 \leq i \leq p$ and that $\|a_i\| = 1$, so we are looking for unit norm tight completions of a unit norm family of vectors $\mathcal{F}$.

In this case, it is shown in [180] that if $d = [\![\|S^{\mathcal{F}}\| + 1]\!]$, where $[\![h]\!]$ denotes the smallest integer greater than or equal to h, there always exists a unit norm tight completion of $\mathcal{F}$ with $dn - p$ elements. Our arguments above show that there exists an efficiently algorithmic computable unit norm tight completion with $[\![n.(\|S^{\mathcal{F}}\| + 1) - p]\!]$ (assuming that we can compute efficiently$\|S^{\mathcal{F}}\|$ and seting $\beta = 1$). Note that in general we have that

$$n.[\![\|S^{\mathcal{F}}\| + 1]\!] - p \geq [\![n.(\|S^{\mathcal{F}}\| + 1) - p]\!] .$$

Corollary(2.2.21)[202]: Let $a^{m_0} = \{a_i^{m_0}\}_{i,m_0=1}^{m}$ be a non-increasing power sequence of positive numbers and let $S \in L(\mathcal{H})^+$ with eigenvalues (counted with multiplicity and arranged in non-increasing order) $\Lambda = \{\lambda_j^{m_0}\}_{j,m_0=1}^{n}$. Then the following statements are equivalent:

(i) $a^{m_0} \prec \Lambda$.

(ii) There exists a Bessel power sequence $\mathcal{G}^{m_0} = \{g_i^{m_0}\}_{i,m_0=1}^{r} \subseteq \mathcal{H}$ such that $\|g_i^{m_0}\|^2 = a_i^{m_0}$ for $1 \leq i, m_0 \leq m$ and $S^{\mathcal{G}^{m_0}} = S^{m_0}$.

Proof. If we assume that $S^{m_0} > 0$ then the case when $m \in N$ is in [35], while the case when $m = \infty$ is Theorem 4.7 in [35]. If the spectrum of S^{m_0} has zeros (note that this is the case whenever $m < n$) we can reduce to the invertible case, restricting S^{m_0} to the orthogonal complement of ker S^{m_0}.

Corollar(2.2.22)[202]: Let $r \in \mathbb{N}$. Then $\mathcal{F}^{m_0}$ is (a^{m_0}, r)-completable if and only if $\frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right) \geq \lambda_1^{m_0}$ $and \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right) \geq$
$\frac{1}{k}\sum_{m_0=1}^{L}\sum_{i=1}^{k}(a_i^{m_0} + \lambda_{n-i+1}^{m_0})$, $\min(n, r)$. (24)

Proof. Assume that there exists $r \in \mathbb{N}$ and a finite power sequence $\mathcal{G}^{m_0} = \{g_i^{m_0}\}_{i,m_0=1}^{r}$ such that

$S^{\mathcal{F}^{m_0} \cup \mathcal{G}^{m_0}} = S^{\mathcal{F}^{m_0}} + S^{\mathcal{G}^{m_0}} = c^{m_0} I$ and $\|g_i^{m_0}\|^2 = a_i^{m_0}$for $1 \leq i, m_0 \leq r$.Then $c^{m_0} I - S^{\mathcal{F}^{m_0}} = S^{\mathcal{G}^{m_0}} \geq 0$; in particular we have $c^{m_0} \geq \|S\| = \lambda_1^{m_0}$. On the other hand, we see that the eigenvalues of $S^{\mathcal{G}^{m_0}}$ arranged in non-increasing order are $c^{m_0} - \lambda_n^{m_0} \geq \ldots \geq c^{m_0} - \lambda_1^{m_0} \geq 0$. By Theorem (2.2.5) we have

$$\left(c^{m_0} - \lambda_n^{m_0}, c^{m_0} - \lambda_{n-1}^{m_0}, \ldots, c^{m_0} - \lambda_1^{m_0}\right) \succ \left(a_1^{m_0}, \ldots, a_r^{m_0}\right). \qquad (25)$$

Then, by Definition(2.2.1) we see that$\frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0})\right) \geq \lambda_1^{m_0}$ and (7) hold, using that $c^{m_0} = \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right)$.

Conversely assume that $\frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right) \geq \lambda_1^{m_0}$ and (7) hold for $r \in \mathbb{N}$. Set $c^{m_0} = \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right)$ and note that the spectrum of the positive operator $c^{m_0} I - S^{\mathcal{F}^{m_0}}$,$\left(c^{m_0} - \lambda_n^{m_0}, c^{m_0} - \lambda_{n-1}^{m_0}, \ldots, c^{m_0} - \lambda_1^{m_0}\right)$, majorizes (in the sense of Definition (2.2.4)) $\{a_i^{m_0}\}_{i,m_0=1}^{r}$. By Theorem (2.2.5) we conclude that there exists a finite power sequence $\mathcal{G}^{m_0} = \{g_i^{m_0}\}_{i,m_0=1}^{r}$ with $S^{\mathcal{G}^{m_0}} = c^{m_0} I - S^{\mathcal{F}^{m_0}}$ and $\|g_i^{m_0}\|^2 = a_i^{m_0}$for $1 \leq i, m_0 \leq r$ and we are done.

Corollary (2.2.23)[202]: Let $r \in \mathbb{N}$. $\mathcal{F}^{m_0}$ is (a^{m_0}, r)-completable if and only if
$\frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0})\right) = c_r^{m_0}$for $r < n$ (26)
or
$\frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0})\right) \geq c_n^{m_0}$ for $r \geq n$.
(27)

Moreover, if $c_r^{m_0} = \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0})\right)$ for some $r < n$, then $c_r^{m_0} = \lambda_1^{m_0}$.

Poof. A ssume that $\mathcal{F}^{m_0}$ is $(a^{m_0}$,r)- completable. If $r < n$ note that, by (9) in Corollary (2.2.8), we have $\lambda_1^{m_0} = c_0^{m_0} \leq \cdots \leq c_r^{m_0} = \lambda_1^{m_0}$ and $\lambda_1^{m_0} = \frac{1}{n}\left(\sum_{m_0=1}^{r}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right)$,

so (11) holds. If $r \geq n$ then min $\{n, r\} = n$ and Theorem (2.2.7) together with the definition of $c_n$imply that $\frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right) \geq c_n^{m_0}$.

So in this case (12) holds. Conversely, if we assume (27), then it is clear $\mathcal{F}^{m_0}$ is (a^{m_0}, r)-completable, by Theorem (2.2.7). Assume now that for some

$r < n$, $c_n^{m_0} = \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right)$. We show that $\mathcal{F}^{m_0}$is (α^{m_0}, r)-completable; indeed, since $n\, c_n^{m_0} = \left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right)$ then$r\, c_r^{m_0} + (n-r)\, c_r^{m_0} - \sum_{m_0=1}^{L}\sum_{i=1}^{n-r}\lambda_i^{m_0} = \sum_{m_0=1}^{r}\sum_{i=1}^{r} a_i^{m_0}\sum_{i=1}^{L} + \sum_{m_0=1}^{r}\sum_{i=1}^{r}\lambda_{n-i+1}^{m_0}$
So by definition of $c_r^{m_0}$ we have

$\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \lambda_{n-i+1}^{m_0}) \le r\, c_r^{m_0} =$
$\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \lambda_{n-i+1}^{m_0}) - \sum_{m_0=1}^{L}\sum_{i=1}^{n-r}(c_r^{m_0} + \lambda_{n-i+1}^{m_0}) \le \sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} +$
λn−i+1m0).
But then
$\lambda_i^{m_0} = \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right)$ for $1 \le i, m_0 \le n-r$
and

$$\lambda_1^{m_0} \ge \max_{1\le k\le r}\frac{1}{k}\sum_{m_0=1}^{L}\sum_{i=1}^{k}(a_i^{m_0} + \lambda_{n-i+1}^{m_0}).$$

So $\mathcal{F}^{m_0}$ is (α^{m_0}, r)-completable, by Corollary (2.2.8). The last claim of the proposition is clear from the computations.
We give a formula for the minimum $r \in \mathbb{N}$ such that $\mathcal{F}^{m_0}$ is (α^{m_0}, r)-completable, when such an $r \in \mathbb{N}$ exists.

Corollary (2.2.24)[202]: Let$\mathcal{F}^{m_0}$ be a (α^{m_0}, r)-completable for some $r \in \mathbb{N}$. Let $r_0 \in \mathbb{N}$ be the minimum such that $\mathcal{F}^{m_0}$is (α^{m_0}, r_0)-completable. Then
Case1: $r_0 < n$ if and only if $c_{r_0}^{m_0} = \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r_0}(a_i^{m_0} + \alpha^{m_0}\right)$
Case2: $r_0 \ge n$ if and only if $c_k^{m_0} \ne \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{k}(a_i^{m_0} + \alpha^{m_0}\right)$ $for\ all$ $1 \le k \le n-1$ and $r_0 \ge \in \mathbb{N}$ is the minimum such that $c_n^{m_0} \le \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r_0}(a_i^{m_0} + \alpha^{m_0}\right)$.

Proof. Note that, by Proposition (2.2.10), at least one the cases has to be fulfilled by some $r \in \mathbb{N}$. If we assume that case 1 holds for some $r < n$ then,by Proposition (2.2.10), $\mathcal{F}^{m_0}$ is (α^{m_0}, r)-completable. By Corollary (2.2.9) case 1 does not hold for $k < n$ with $r \ne k$. It is clear that in this case $r_0 = r$.
Assume now that there is no $r < n$ satisfying case 1 above. Then, there exists $r \in \mathbb{N}$ such that $c_n^{m_0} \le \frac{1}{n}\left(\sum_{m_0=1}^{L}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\right)$ by Proposition (2.2.10) we see that $\mathcal{F}^{m_0}$ is (α^{m_0}, r)- completable. It is clear that r_0 is the minimum natural number r satisfying this condition. Finally note that if $r \in \mathbb{N}$ is such that $c_n^{m_0} \le \frac{1}{n}\left(\sum_{m_0=1}^{r}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0})\right)$ then

$$\frac{1}{n}\left(\sum_{m_0=1}^{n}\sum_{i=1}^{n}(a_i^{m_0} + \alpha^{m_0})\right) \le c_n^{m_0} \le \frac{1}{n}\left(\sum_{m_0=1}^{r}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0})\right) \Rightarrow \sum_{m_0=1}^{n}\sum_{i=1}^{n} a_i^{m_0} \le \sum_{m_0=1}^{r}\sum_{i=1}^{r}\alpha^{m_0}$$

and $r \ge n$ since for every $i, m_0 \in \mathbb{N}$, $a_i^{m_0} > 0$.
The next example shows that it is possible to obtain a set of vectors $\mathcal{F}^{m_0}$ and a power sequence a^{m_0} such that $\mathcal{F}^{m_0}$ is (α^{m_0}, r)-completable for only one $r \in \mathbb{N}$ (in virtue of Corollary (2.2.9), $r < n$).

Corollary(2.2.25)[202]:
Deduce that

(i) $c_r^{m_0} - c_n^{m_0} = \epsilon$
(ii) $c_3^{m_0} > \frac{19}{9}, n = 3.$

From (12) we can find

$\frac{1}{n}\ (\sum_{m_0=1}^{r}\sum_{i=1}^{r}(a_i^{m_0}\ +\ \alpha^{m_0}\ )) = c_n^{m_0} + \epsilon, \epsilon > 0$

Divide this equation with (11) we have $c_r^{m_0} - c_n^{m_0} = \epsilon$ (28)

In Example (2.2.12), by choosing $n = 3$, we have that $c_3^{m_0} > \frac{19}{9}$.

Corollary (2.2.26)[202]: Let $\mathcal{F}^{m_0}$ be (a^{m_0}, r)-completable for some $r < n$. There exists $r_1 \in \mathbb{N}$ with $r_1 \geq n$ and such that $\mathcal{F}^{m_0}$ is (a^{m_0}, r_1)-completable if and only if

$\frac{1}{n}\ (\sum_{m_0=1}^{r_1}\sum_{i=1}^{r_1}(a_i^{m_0}\ +\ \alpha^{m_0}\ )) \geq \max_{r+1\leq k\leq n} \frac{1}{k}(\sum_{m_0=r+1}^{k}\sum_{i=r+1}^{k} a_i^{m_0}$.

Proof. By Theorem (2.2.7), $\mathcal{F}^{m_0}$ is (a^{m_0}, r_1)-completable if and only if

$$(\frac{1}{k}\sum_{m_0=1}^{k}\sum_{i=1}^{k}(a_i^{m_0}\ +\ \lambda_{n-i+1}^{m_0}\ )) \geq \lambda_1^{m_0} \text{and} \frac{1}{n}\ (\sum_{m_0=1}^{r_1}\sum_{i=1}^{r_1}(a_i^{m_0}\ +\ \alpha^{m_0}\ ))\geq \frac{1}{k}\sum_{m_0=1}^{k}\sum_{i=1}^{k}(a_i^{m_0}\ +\ \lambda_{n-i+1}^{m_0}\ ), 1 \leq k \leq n.$$

By hypothesis and Corollary (2.2.8),

$\lambda_i^{m_0} = \frac{1}{n}\ (\sum_{m_0=1}^{r}\sum_{i=1}^{r}(a_i^{m_0}\ \ \alpha^{m_0}\ )), 1 \leq i, m_0 \leq n-r$ and $\lambda_1^{m_0}\frac{1}{k}\sum_{m_0=1}^{k}\sum_{i=1}^{k}(a_i^{m_0}\ +$ $\lambda n-i+1m0\ ), 1\leq k\leq r$

Since $\mathcal{F}^{m_0}$ is (a^{m_0}, r)-completable with $r < n$. So $\mathcal{F}^{m_0}$ is (a^{m_0}, r_1)-completable if and only if $\frac{1}{n}(\sum_{m_0=1}^{r_1}\sum_{i=1}^{r_1}(a_i^{m_0}\ +\ \alpha^{m_0}\ ) \geq \frac{1}{k}\sum_{m_0=1}^{k}\sum_{i=1}^{k}(a_i^{m_0}\ +\ \lambda_{n-i+1}^{m_0}\ )\ 1 \leq k \leq n$

or equivalently, if for every $r + 1 \leq k \leq n$

$$\sum_{m_0=r+1}^{r}\sum\nolimits_{i=r+1}^{r}(a_i^{m_0}\ +\ \alpha^{m_0}\ ) + \sum_{m_0=r+1}^{r_1}\sum_{i=r+1}^{r_1} a_i^{m_0} \geq \frac{n}{k}\left(\sum_{m_0=1}^{r}\sum_{i=1}^{r} a_i^{m_0} + \sum_{m_0=r+1}^{k}\sum\nolimits_{i=r+1}^{k}(a_i^{m_0}\ +\ \alpha^{m_0} - (n-k)\lambda_1^{m_0})\right)$$

$$\sum_{m_0=1}^{r}\sum\nolimits_{i=1}^{r} a_i^{m_0}\ +\ \alpha^{m_0}\ +\ \sum_{m_0=r+1}^{r_1}\sum_{i=r+1}^{r_1} a_i^{m_0} \geq \frac{n}{k}\left(\sum_{m_0=1}^{r}\sum\nolimits_{i=1}^{r}(a_i^{m_0}\ +\ \alpha^{m_0}\ )\right) + \frac{n}{k}\sum_{m_0=r+1}^{k}\sum_{i=r+1}^{k} a_i^{m_0} - \frac{n-k}{k}(\sum_{m_0=1}^{r}\sum\nolimits_{i=1}^{r}(a_i^{m_0}\ +\ \alpha^{m_0}\ )$$

$$\sum_{m_0=1}^{r}\sum\nolimits_{i=1}^{r} a_i^{m_0}\ +\ \alpha^{m_0}\ +\ \sum_{m_0=r+1}^{r_1}\sum_{i=r+1}^{r_1} a_i^{m_0} \geq \sum_{m_0=1}^{r}\sum\nolimits_{i=1}^{r} a_i^{m_0}\ +\ \alpha^{m_0}\ + \frac{n}{k}\sum_{m_0=r+1}^{k}\sum_{i=r+1}^{k} a_i^{m_0}$$

$$\sum_{m_0=r+1}^{r_1}\sum_{i=r+1}^{r_1} a_i^{m_0} \geq \frac{n}{k}\sum_{m_0=r+1}^{k}\sum_{i=r+1}^{k} a_i^{m_0},$$

since by hypothesis $\lambda_i^{m_0} = \frac{1}{n}\ (\sum_{m_0=1}^{r}\sum_{i=1}^{r}(a_i^{m_0}\ +\ \alpha^{m_0}\ ))$ for $1 \leq i, m_0 \leq n-r$.

Corollary (2.2.27)[202]: $\mathcal{F}^{m_0}$ is (a^{m_0}, ∞)-completable (by a Bessel power sequence) if and only if $a^{m_0} \in \ell^1\ (\mathbb{N}), \frac{1}{n}\ (\sum_{m_0=1}^{\infty}\sum_{i=1}^{\infty}(a_i^{m_0}\ +\ \alpha^{m_0}\ ) \geq \lambda_1^{m_0}$ and

$\frac{1}{n}\ (\sum_{m_0=1}^{\infty}\sum_{i=1}^{\infty}(a_i^{m_0}\ +\ \alpha^{m_0}\ ) \geq \frac{1}{k}\sum_{m_0=1}^{k}\sum_{i=1}^{k}(a_i^{m_0}\ +\ \lambda_{n-i+1}^{m_0}\ ), 1 \leq k \leq n$ (29)

or equivalently if $(a^{m_0} \in \ell^1(\mathbb{N})$

$$\frac{1}{n}\,(\sum_{m_0=1}^{\infty}\sum\nolimits_{i=1}^{\infty} a_i^{m_0} + \alpha^{m_0}\,)\,c_n^{m_0}\,. \qquad (30)$$

The proof of Theorem (2.2.14) which is based on Theorem (2.2.5), is similar to that of Theorem (2.2.11) and Proposition (2.2.10).

Proof of Theorem (A). The first part of the theorem is Theorem (2.2.7), while the second part is Theorem (2.2.14).

Proof of Theorem (B). Assume there exists a natural number $r \in \mathbb{N}$ such that $\mathcal{F}^{m_0}$ is (a^{m_0}, r)-completable. Then $r_0 \le r$ and in this case the theorem follows from Theorem 3.8. If there is no $r \in \mathbb{N}$ such that $\mathcal{F}^{m_0}$ is (a^{m_0}, r)-completable, then $\mathcal{F}^{m_0}$ is (a^{m_0}, ∞)-completable so by Theorem (2.2.14) $a^{m_0} \in \ell^1(\mathbb{N})$ and $\frac{1}{n}\,(\sum_{m_0=1}^{\infty}\sum_{i=1}^{\infty}(a_i^{m_0} + \alpha^{m_0}\,) \ge c_n^{m_0}$. If $\frac{1}{n}\,(\sum_{m_0=1}^{\infty}\sum_{i=1}^{\infty}(a_i^{m_0} + \alpha^{m_0}\,) > c_n^{m_0}$ then there exists $r \in \mathbb{N}$ such that $\frac{1}{n}\,(\sum_{m_0=1}^{r}\sum_{i=1}^{r}(a_i^{m_0} + \alpha^{m_0}\,) \ge c_n^{m_0}$. But then, by Proposition (2.2.11) we get that $\mathcal{F}^{m_0}$ is (a^{m_0}, r)-completable, a contradiction.

Corollary (2.2.28)[202]: Let $h \coloneqq \sum_{m_0=2}^{n}\sum_{i=2}^{n}\lambda_1^{m_0} - \lambda_i^{m_0}$, and denote by r_0 the minimum number of norm one vectors we have to add to $\mathcal{F}^{m_0}$ in order to have a power tight frame.

Case1:Suppose $h < n$.Then $r_0 = h \; if \; h \in N$ and $1 + \frac{1}{h}\sum_{m_0=1}^{h}\sum_{i=1}^{h}\lambda_{n-i+1}^{m_0} \le \lambda_1^{m_0}$(in particular, $c_h^{m_0}$=$\lambda_1^{m_0}$). Otherwise, $r_0 = n$.

Case 2:If $h \ge n$, r_0 is the minimum integer greater than or equal to h .

Proof. Assume that $h< n$;then, since $h = n\lambda_1^{m_0} - \alpha^{m_0}$, we have that $c_n^{m_0} = 1 + \frac{\alpha^{m_0}}{n}$. If in addition $h < n$ and 1+$\frac{1}{h}\sum_{m_0=1}^{h}\sum_{i=1}^{h}\lambda_{n-i+1}^{m_0} < \lambda_1^{m_0}$,so $c_h = \frac{1}{n}(h + \alpha^{m_0}) = \lambda_1^{m_0}$then $r_0 = h$ by Theorem (2.2.11) .Otherwise, $c_k^{m_0} \ne \frac{1}{n}(k + \alpha^{m_0})$ for all $k < n$ (if $c_k^{m_0} = \frac{1}{n}(k + \alpha^{m_0})$
for some $k < n$, then by Proposition (2.2.10) $c_k^{m_0} = \lambda_1^{m_0}$and h would be a natural number); since $c_n^{m_0} = 1 + \frac{\alpha^{m_0}}{n}$, the minimum integer greater than or equal to $nc_n^{m_0} - \alpha^{m_0}$ is n so $r_0 = n$ by Theorem 3.8.

Finally, $h \ge n$ implies that $c_k^{m_0} \ne \frac{1}{n}(k + \alpha^{m_0})$ for all $k < n$ and $c_n^{m_0} = \lambda_1^{m_0}$. Therefore, again by Theorem 3.8, r_0 is the minimum integer greater than or equal to $n\lambda_1^{m_0} - \alpha^{m_0} = h$.

Corollary (2.2.29)[202]: Let $\mathcal{F}^{m_0}$ =$\{f_i^{m_0}{}_i\}_{i,m_0=1}^{p}$be a unit norm $\frac{p}{d} - power$ tight frame on its span, where $d < n$ is the dimension of span $\mathcal{F}^{m_0}$. Then, the minimum number r_0 of unit norm vectors we have to add to $\mathcal{F}^{m_0}$ in order to obtain a power tight frame in $\mathcal{H}$is:

(a) $(n-d)\frac{p}{d}$ if $(n-d)\frac{p}{d} < n$ and $(n-d)\frac{p}{d} \in \mathbb{N}$.

(b) n if $(n-d)\frac{p}{d} < n$ and $(n-d)\frac{p}{d} \notin \mathbb{N}$.

(c) the minimum integer greater than or equal to $(n-d)\frac{p}{d}$ if $(n-d)\frac{p}{d} \ge n$.

Proof. Since $\mathcal{F}^{m_0}$ is an unit norm power tight frame on a subspace of dimension d, the eigenvalues of $S^{\mathcal{F}^{m_0}}$are:$\lambda_i^{m_0} = \frac{p}{d} \ge 1$ for $1 \le i, m_0 \le d$, and$\lambda_i^{m_0} = 0$ for$d + 1 \le i, m_0 \le n$. Therefore, $h = \sum_{m_0=2}^{n}\sum_{i=2}^{n}\lambda_1^{m_0} - \lambda_i^{m_0} = (n-d)\frac{p}{d}$. Moreover, if $h < n$ and $h \in \mathbb{N}$, then 1+$\frac{1}{h}\sum_{m_0=1}^{h}\sum_{i=1}^{h}\lambda_{n-i+1}^{m_0} = \lambda_1^{m_0}$. Indeed,

$$1+\frac{1}{h}\sum_{m_0=1}^{h}\sum_{i=1}^{h}\lambda_{n-i+1}^{m_0} = 1 + \frac{h-(n-d)}{h}\frac{p}{d} = \frac{p}{d} \qquad (31)$$

the proposition is then a consequence of Theorem (2.2.16).

Now we show the following

Corollary(2.2.30)[202]: For $\mathcal{F}^{m_0} = \{f_i^{m_0}\}_{i=1,m_0=1}^{\frac{n(n-\epsilon)}{\epsilon}}$ $\epsilon > 0, n \in N$ be a unit norm $\frac{n}{\epsilon} -$ power tight frame on its span, where $\epsilon > 0$ is the dimension of span $\mathcal{F}^{m_0}$. Then, the minimum number r_0 of unit norm vectors we have to add to $\mathcal{F}^{m_0}$ in order to get a power tight frame in $\mathcal{H}$.

Proof. Given $\mathcal{F}^{m_0}$ is unit norm power tight frame on a subspace of dimension $n-\epsilon$, the eigenvalues of $S^{\mathcal{F}^{m_0}}$ are: $\lambda_i^{m_0} = \frac{n}{\epsilon} \geq 1$, $\Rightarrow 0 \leq \epsilon \leq n$, for $1 \leq i, m_0 \leq n-\epsilon$, and $\lambda_i^{m_0} = 0$ for $n-\epsilon+1 \leq i, m_0 \leq n$. Therefore, $h = \sum_{m_0=2}^{n} \sum_{i=2}^{n} \lambda_1^{m_0} - \lambda_i^{m_0} = \epsilon \frac{n}{\epsilon} = n$. Further, if $h<n$ and $h \in \mathbb{N}$, then $1 + \frac{1}{h} \sum_{m_0=1}^{h} \sum_{i=1}^{h} \lambda_{n-i+1}^{m_0} = \lambda_1^{m_0}$. Therefore,

$$1 + \frac{1}{h} \sum_{m_0=1}^{h} \sum_{i=1}^{h} \lambda_{n-i+1}^{m_0} = \frac{n}{\epsilon}.$$

Chapter 3
The Spectra of Contractions

let $\mathfrak{H}$ be a complex, separable Hilbert space, and let L ($\mathfrak{H}$) denote the set of all bounded, linear operators acting on $\mathfrak{H}$. A contraction T ∈L ($\mathfrak{H}$), i.e., an operator with norm

$\|T\| \leq 1$, belongs to the class

(a) $C_{1.}$, if $\lim_{n\to x} \|T^n h\| \neq 0$, for every $0 \neq h \in \mathfrak{H}$;

(b) $C_{0.}$, if $\lim_{n\to x} \|T^n h\| = 0$, for every $h \in \mathfrak{H}$;

(cl $C_{1.}$, if $T^* \in C_{1.}$;

(d) $C_{0.}$, if $T^* \in C_{0.}$,

Section(3.1): Spectral Classes

let $\mathfrak{H}$ be a complex, separable Hilbert space, and let L ($\mathfrak{H}$) denote the set of all bounded, linear operators acting on $\mathfrak{H}$. A contraction T ∈L ($\mathfrak{H}$), i.e., an operator with norm

$\|T\| \leq 1$, belongs to the class

(a) $C_{1.}$, if $\lim_{n\to x} \|T^n h\| \neq 0$, for every $0 \neq h \in \mathfrak{H}$;

(b) $C_{0.}$, if $\lim_{n\to x} \|T^n h\| = 0$, for every $h \in \mathfrak{H}$;

(cl $C_{1.}$, if $T^* \in C_{1.}$;

(d) $C_{0.}$, if $T^* \in C_0$.

We shall use the terminology and notation of the monograph [29]

1. First of all we recall some facts from the theory of contractions (cf. [29]) which will be needed in the sequel.

Let $T \in \mathscr{L}(\mathfrak{H})$, be a contraction, and let us consider its minimal unitary dilation $U \in \mathscr{L}(\mathfrak{K})$, . It can be proved that the subspace $\mathfrak{L} = ((U - T)\mathfrak{H})$ is wandering for U, and so $M(\mathfrak{L}) = \bigoplus_{n=.\infty}^{\infty} U^n \mathfrak{L}$ reduces U to a bilateral shift. Then the orthogonal complement $\mathfrak{R}_* = \mathfrak{R} \ominus M(\mathfrak{L})$ also reduces U, and the restriction $R_{*,T} := U \backslash \mathfrak{R}_* \in \mathscr{L}(\mathfrak{R}_*)$is called the $*$-residual part of T.

It is known that if T is completely non-unitary (cnu) then U and so $R_{*,T}$ too are absolutely continuous unitary operators. Moreover, if T is of class $C_1,\_$ then T can be injected into $R_{*,T}$: $T <^i R_{*,T}$., i.e., there is an injective operator $X \in \mathscr{L}(\mathfrak{H}, \mathfrak{R}_*)$ which intertwines T and $R_{*,T}$: $XT = R_{*,T}X$.. There is in fact a canonical choice fur X. Namely, the operator $X \in \mathscr{L}(\mathfrak{H}, \mathfrak{R}_*)$ defined by

$$Xh = \lim_{n\to\infty} U^{''} T^{''} h, \qquad h \in \mathfrak{H},$$

will be an injection, intertwining T and $R_{*,T}$ if we assume yet that the point spectrum $\sigma_p(T^*)$ of T^* does not cover the open unit disc $\mathbb{D} = \{\lambda \in \mathbb{C}: |\lambda| < 1\}$, then X will be a quasi-affinity (i.e., an injection with dense range), hence T will be a quasi-affine transform of $R_{*,T}$: $T \prec R_{*,T}$.

In this case, i.e., when $T \in C_1$,. and $\sigma_p(T^*) \not\supset \mathbb{D}$, the $*$-residual part $R_{*,T}$ of T can be characterized as the unitary extension of T. In fact, let us introduce a new scalar product on $\mathfrak{H}$:

$$\langle x, y\rangle_\sim := \lim_{n\to\infty}\langle T^n x, T^n y\rangle \text{ for every } x, y \in \mathfrak{H}.$$

(Since the limit $\lim_{n\to\infty} \|T^n x\|$ clearly exists for every $.x, \in \mathfrak{H}$, the polar identity guarantees the existence of the limit above.) Let $\widetilde{\mathfrak{H}}$ denote the Hilbert space obtained by completing the inner product space $(\mathfrak{H}, (\,.\,,.\,)_{\sim})$. Then the uperator T can be uniquely extended to an isometry $\tilde{T} \in \mathscr{L}(\widetilde{\mathfrak{H}})$. Let us take a point $\lambda \in \mathbb{D}\backslash\sigma_p(T^*)$. Since $ran(T - \lambda I)$ is dense in $\mathfrak{H}$, it follows that $ran(\tilde{T} - RI)$ is dense in $\widetilde{\mathfrak{H}}$. Hence $\tilde{T}$ is a unitary operator, called the unitary extension of T. It can be easily seen that $\tilde{T}$ is unitarily equivalent to

$$R_{*,T}: \tilde{T} \cong R_{*,T}.$$

Finally, we' shall need the following

Definition (3.1. 1)[25]: Let $\mathscr{M}(\mathbb{D})$ denote the system of all nonempty, compact subsets σ of $\mathbb{D}$ such that for every nonempty closed and open subset $\dot{\sigma}$ of σ we have $m(\dot{\sigma} \cap \partial\mathbb{D}) > 0$. (Here and in the sequel m stands for the normalized Lebesgue measure on $\partial\mathbb{D}$.).

Let $\mathscr{M}_0(\mathbb{D})$ denote the subset $\mathscr{M}_0(\mathbb{D}) := \{\alpha \in \mathscr{M}(\mathbb{D}): \alpha \subset \partial\mathbb{D}\}$. It is easy to see that a compact subset α of $\partial\mathbb{D}$ belongs to $,\mathscr{M}_0(\mathbb{D})$if and only if α is regular in the sense that α coincides with the support of the measure $\chi_\alpha dm$. (Here χ_α is the characteristic function of α.) We say that the set $\alpha \in \mathscr{M}_0(\mathbb{D})$is neatly contained in $\sigma \in \mathscr{M}(\mathbb{D})$, denoted by $\alpha \subset^{(n)} \sigma$, if $\alpha \subset \sigma$ and for every closed and open subset $\dot{\sigma}$ of σ we have

$$m(\sigma' \cap \alpha) > 0.$$

Applying the Riesz-Dunford functional calculus we can derive from [29] the following

Theorem(3.1.2)[25]: If T isa cnu C_1,-contraction,then $\sigma(T) \in \mathscr{M}(\mathbb{D}), \sigma(R_{*,T}) \in \mathscr{M}_0(\mathbb{D})$ and $\sigma(R_{*,T})$ is neatly contained in $\sigma(T)$.

Proof: Since $R_{*,T}$,- is absolutely continuous, it follows that $\sigma(R_{*,T}) \in \mathscr{M}_0(\mathbb{D})$. Moreover, applying [29] for T^* and taking into account that $R_{T^*} = (R_{*,T})^*$, where R_{T^*}. is the residual part of T^*, we obtain $\sigma(R_{*,T}) \subset \sigma(T)$.

Let us assume that $\sigma(T)$ is not connected, and let $\dot{\sigma}$ be a non-empty closed and open subset of $\sigma(T)$. Then the Rieszz-Dunford functional calculus (cf. [175]) provides us a subspace $\mathfrak{H}'$, invariant for T such that $\sigma(T\lceil\mathfrak{H}'\rangle) = \dot{\sigma}$. From the preceding part we infer that $\dot{\sigma} \supset \sigma(R_{*,T|\mathfrak{H}'})$, and $m(\sigma(R_{*,T|\mathfrak{H}'})) > O$, since $\mathfrak{H}' \neq \{0\}$. But $R_{*,T|\mathfrak{H}'}$, is unitarily equivalent to $(T|\mathfrak{H}')^{\sim} \cong \tilde{T}|\mathfrak{H}'^{\sim}$, which implies that $\sigma(R_{*,T|\mathfrak{H}'}) \subset \sigma(R_{*,T})$. Therefore $m(\sigma' \cap \sigma(R_{*,T})) > 0$, and the proof is completed.

In [6] it was proved that every set in $.\mathscr{M}(\mathbb{D})$ can serve as the spectrum of a C_{11}-contraction. It is natural to ask whether this is the case in connection with C_{10}-contractions too. First we list some examples:

(a) The simplest C_{10}-contraction is the unilateral shift S of multiplicity

1. Its spectrum is $\sigma(S) = \mathbb{D}$.

(b) More generally, if the defect index d_T, of a contraction $T \in C_{10}$ is finite, then d_{T^*},. must be greater than d_T, and so again $\sigma(T) = \mathbb{D}$.

(c) Gilfeather has shown (cf. [171]) that the spectrum can be thin.

Namely, he provided a weighted bilateral shift $T \in C_{10}$, such that $\sigma(T) = \partial\mathbb{D}$. (Cf. also Eckstein's paper [169].)

(d) The spectrum can be non-circular symmetric. The following example was given by K. Takahashi. Let u be a non-constant function in the Hardy space H^∞' such that $\|u\|_\infty \leq 1$

and $m((\lambda \in \partial\mathbb{D}: |u(\lambda) = 1) > 0$. Then the operator M_u, of multiplication by u in the Hardy space H^2 belongs to C_{10} and its spectrum is $\sigma(M_u) = u(\mathbb{D})$. (In connection with Hardy spaces we refer to [174].)

(e) Beauzamy provided an example for a C_{10}-contraction T such that $\sigma(T) \cap \partial\mathbb{D}$, contains a non-trivial closed arc disjoint from $\sigma(R_{*,T})$ (cf.

[167]).The following theorem together with Theorem(3.1.2) give a complete characterization of the possible spectra of C_{10}-contractions and their *-residual parts.

Propositon(3.1.4)[25]: For every $\alpha \in \mathscr{M}_0(\mathbb{D})$and real number$K > 0$ there exists $a\ C_{10}$-contruction A such that $\sigma(A) = \sigma(R_{*,A}) = \alpha$ and $\|A'\| > K$.

The analogous statement in the case of C_{11}-contractions could be proved easily. In fact a C_{11}-contraction of defect indeces 1 can be found. The C_{10}- case is more difficult, as we saw before the spectrum of C_{10}contractions with finite defect indices is the closed unit disc $\mathbb{D}$ .We are looking for a C_{10}-contraction with properties above among the restrictions of weighted bilateral shifts to their invariant subspaces.

Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis in the Hilbert space $\mathfrak{H}$ where $\mathbb{Z}$ denotes the set of integers. Let $w \in \{w_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $0 < w_n \leq 1$ for every $n \in \mathbb{Z}$.

Throughout this section $T \in \mathscr{L}(\mathfrak{H})$ will denote the weighted bilateral shift with weight sequence w. i.e. $T\ e_n = w_n\ e_{n+1}$ for every $n \in \mathbb{Z}$.

T is clearly a quasi-affine contraction, whose adjoint T^* is also a weighted bilateral shift: $T^*\ e_n = w_n \mid e_n \mid$ for every $n \in \mathbb{Z}$.

An easy computation with weighted shifts (cf. [173, 176]) proves the following

Lemma (3.1.4)[25]: T is of class C_{10} and $\sigma(T) = \partial\mathbb{D}$ lf and only if

(a) $\prod_{n=1}^{\infty} w_n > 0$,

(b) $\prod_{n=\infty}^{-1} w_n = 0$, an

(c) $\lim_{k\to\infty} inf_{n\in I}\,(w_n w_{n+1} \ldots w_{n+k-1})^{1\backslash k} = 1$.

We shall consider special weight sequences.

Definition(3.1.6)[25]: We call $\{r_i\}_{i\in\mathbb{N}}$ a regular ∞-sequenc($\mathbb{N}$ denotes the set of positive integers) if $r_1, = 1, r_i \in \mathbb{N}, r_{i+1} > r_i,, r_{i+2} - r_{i+1} \geq r_{i+1} - r_i$, for every $i \in \mathbb{N}$, and $\{r_i\}_{i\in\mathbb{N}}$ is of density 0, i.e.,

$$\lim_{k\to\infty} \frac{m_k}{k} = 0.$$

Here and always in the sequel m_k denotes the frequency of the sequence

$$\{r_i\}_{i\in\mathbb{N}}, i.e., m_k = \max\{i \in \mathbb{N}: r_i \leq k\}.$$

We call $\{\gamma_i\}_{i\in\mathbb{N}}$ a regular O-sequence if $\gamma_i \in \mathbb{R}, 0 < \gamma_i, < 1, \gamma_i \leqq \gamma_{i+1}$, for every $i \in \mathbb{N}$, and

$$\prod_{i=1}^{\infty} \gamma_i = 0.$$

We say that $w = \{w_n\}_{n\in\mathbb{Z}}$ is a regular $w$eight sequencec orresponding to the regular ∞- and 0-sequences $\{r_i\}_{i\in\mathbb{N}}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$, respectively, if $w_n = \gamma_i$ when n is not of the form $-r_i (i \in \mathbb{N})$ =1 when $n \in \mathbb{Z}$ is not of the above.

Lemma(3.1.5)[25]: If w is a regular weight sequence then $T \in C_{10}$ and $\sigma(T) = \partial\mathbb{D}$.

Proof. We have only to verify property (iii) of Lemma (3.1.3) However, for every $k \in \mathbb{N}$, we have

$$1 \geq \inf_{n\in Z}(w_n \dots w_{n+k-1})^{1/k} = (w_{-1} \dots w_{-k})^{1/k} = \left(\prod_{i=1}^{m_k}\gamma_i\right)^{1/k} \geq \gamma_1^{m_k/k}.$$

Hence $\lim_{k\to\infty}(m_k/k) = 0$ implies that (iii) is fulfilled.

It is wellknown (cf. [173, 176] that the weighted shift T can be considered as a shift operator on a weighted sequence space. In fact, let $\beta = \{\beta_n\}_{n\in z}$ be the sequence defined by

$\beta_n = (w_0 w_1 \dots w_{n\ \ 1})^2$ if $n > 0, = 1$ if $n = 0 = (w_{-1}w_{-2} \dots w_n\ )^{-2}$ if $n < 0$

and let $l^2(\beta)$ denote the L^2-space corresponding to the measure

$$\mu_\beta(\omega) = \sum\nolimits_{n\in\omega}\beta_n \ (\omega \subset \mathbb{Z}):$$

$$l^2(\beta) \coloneqq L^2(\mu_\beta) = \left\{f = \{\hat{f}(n)\}_{n\in\mathbb{Z}}: \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|^2\beta_n < \infty\right\}.$$

Then the shift operator $T_\beta \in \mathscr{L}\,(l^2(\beta))$, defined by

$$T_\beta\chi_{\{n\}} = \chi_{\{n+1\}} \quad \text{for every } n \in \mathbb{Z},$$

is unitarily equivalent to T. In what follows β will always denote the sequence defined above, and we shall also write T instead of T_β,.

To the special sequence $\beta^{(0)} = \{\,\beta_n^{(0)}\}_{n\in\mathbb{Z}}$, where $\beta_n^{(0)} = 1$, for every $n \in \mathbb{Z}$, there corresponds the usual sequence space:

$$l^2 = l^2(\beta^{(0)}) = \left\{f = \{\hat{f}(n)\}_{n\in\mathbb{Z}}: \|f\|^2 = \|f\|_{\beta^{(0)}}^2 = \sum_{n=\infty}^{\infty}\left|\hat{f}(n)\right|^2 < \infty\right\}.$$

Since $\beta_n \geq 1$, for every $n \in \mathbb{Z}$, we infer that

$$\|f\|_\beta \geq \|f\| \ \text{ for every } f \in l^2(\beta),$$

and so

$$l^2(\beta) \subset l^2.$$

As before, let m be the normalized Lebesgue measure on $\partial\mathbb{D}$. The trigonometric system $\{g_n\}_{n\in\mathbb{Z}}$ (where $g_n(\lambda) = \lambda^n$, for every $n \in \mathbb{Z}$) is an orthonormal basis in the Hilbert space $L^2 = L^2(m)$. Hence, the sequence space l^2 can be identified with the function space L^2 via the unitary transformation

$$U: l^2 \to L^2, \quad U: f = \{\hat{f}(n)\}_{n\in\mathbb{Z}} \mapsto \sum_{n=-\infty}^{\infty}\hat{f}(n)g_n.$$

Therefore, every element $f \in l^2$ can be considered as an element of L^2, and conversely.

We note yet that if w is a regular sequence, then

$$\beta_n = 1 \ \text{ if } n \geq 0, = \left(\prod_{i=1}^{m_{\{n\}}}\gamma_i\right)^{-2} \quad \text{if } \ n<0.$$

The following lemma plays an essential role in our construction.

Lemma(3.1.6)[25]: Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions belonging to L^2. There exists a strictly increasing sequence $\{\,r_i^{(0)}\}_{i\in\mathbb{N}}$ of natural numbers such that if $\{r_i\}_{i\in\mathbb{N}}$ is a regular ∞ -sequence satisfying $r_i \geq r_i^{(0)}$, for every $i > i_0$, with some $i_0 \in \mathbb{N}, \{\gamma_i\}_{i\in\mathbb{N}}$ is an arbitrary regular 0-sequence, and w is the regular weight sequence corresponding to $\{r_i\}_{i\in\mathbb{N}}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$, then $f_k \in l^2(\beta)$ for every $k \in \mathbb{N}$.

Proof: Let us choose inductively strictly increasing sequences $\{\,r_{k,i}^{(0)}\}_{i\in\mathbb{N}}$ of natural numbers such tha $r_{k+1,i}^{(0)} \geq r_{k,i}^{(0)}$,

and

$$\prod_{n=-r_{k,i}^{(0)}}^{-\infty} \left|\hat{f}_k(n)\right|^2 < \frac{1}{i!} \quad \text{for every } k, i \in \mathbb{N}.$$

Let us define the sequence $\{r_i^{(0)}\}_{i\in\mathbb{N}}$ by

$$r_i^{(0)} = r_{i,i}^{(0)} \qquad ((i \in \mathbb{N})).$$

Now, let $\{r_i\}_{i\in\mathbb{N}}$ be a regular ∞-sequence such that $r_i \geq r_i^{(0)}$ for every $i > i_0$ with some $i_0 \in \mathbb{N}$, let $\{\gamma_i\}_{i\in\mathbb{N}}$ be an arbitrary regular 0-sequence, and let us consider the regular weight sequence w corresponding to $\{r_i\}_{i\in\mathbb{N}}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$. We shall show that

$$f_k \in l^2(\beta) \qquad \text{for every } k \in \mathbb{N}.$$

In fact, let k be an arbitrary natural number . If $i > \tilde{k} = \max\{k, i_0\}$, then $r_i \geq r_i^{(0)} = r_{i,i}^{(0)} \geq r_{k,i}^{(0)}$. Hence, we obtain

$$\begin{aligned}\|f_k\|_\beta^2 &= \sum_{n=-\infty}^{\infty} \left|\hat{f}_k(n)\right|^2 \beta_n = \sum_{i=1}^{\infty} \sum_{n=-r_i}^{r_{i+1}+1} \left|\hat{f}_k(n)\right|^2 \left(\prod_{i=1}^{m\{n\}} \gamma_i\right)^2 + \sum_{n=0}^{\infty} \left|\hat{f}_k(n)\right|^2, \\ &= \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \gamma_i\right)^{-2} \sum_{n=-r_i}^{-r_{i+1}+1} \left|\hat{f}_k(n)\right|^2 + \sum_{n=0}^{\infty} \left|\hat{f}_k(n)\right|^2 \\ &\leq \sum_{i=\tilde{k}+1}^{\infty} \gamma_1^{-2i}, \frac{1}{i!} + \sum_{i=1}^{\tilde{k}} \gamma_1^{-2i} \sum_{n=-r_i}^{r_{i+1}+1} \left|\hat{f}_k(n)\right|^2 + \sum_{n=0}^{\infty} \left|\hat{f}_k(n)\right|^2 \\ &\leq e^{1/\gamma_1^2} + \sum_{i=1}^{\tilde{k}} \gamma_1^{-2i} \sum_{n=-r_0}^{r_{i+1}+1} \left|\hat{f}_k(n)\right|^2 + \|f_k\|^2 < \infty,\end{aligned}$$

and so $f_k \in l^2(\beta)$. The proof is finished.

Propositon(3.1.7)[25]: For every $\alpha \in \mathscr{M}_0(\mathbb{D})$ and real number $K > 0$ there exists a C_{10}-contruction A such that $\sigma(A) = \sigma(R_{*,A}) = \alpha$ and $\|A'\| > K$.

The analogous statement in the case of C_{11}-contractions could be proved easily. In fact a C_{11}-contraction of defect indeces 1 can be found. The C_{10}- case is more difficult, as we saw before the spectrum of C_{10}contractions with finite defect indices is the closed unit disc $\mathbb{D}$ .We are looking for a C_{10}-contraction with properties above among the restrictions of weighted bilateral shifts to their invariant subspaces.

Proof. Let $\alpha \in \mathscr{M}_0(\mathbb{D})$ be an arbitrary set. Let $\{r_i^{(0)}\}_{i\in\mathbb{N}}$ be the sequence occurring in Lemma (3.1.6), corresponding to the sequence$\{g_{-k}\chi_\alpha\}_{k\in\mathbb{N}}$, where $g_{-k}(\lambda\} = \lambda^{-k}$. Let $\{r_i\}_{i\in\mathbb{N}}$be a regular ∞-sequence such that

$$r_i \geq r_i^{(0)} \text{for every } i > i_0,$$

with some $i_0 \in \mathbb{N}$, and that

$$\sum_{n=1}^{\infty} \frac{m_n}{1+n^2} < \infty$$

holds for the frequences $m_n = max\{ i \in \mathbb{N} : r_i, \leq n \}$. Let $\{\gamma_i\}_{i\in\mathbb{N}}$ be a regular 0-sequence, and let w be the regular weight sequence corresponding to $\{r_i\}_{i\in\mathbb{N}}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$. Let us consider the shift operator $T \in \mathscr{L}(l^2(\beta))$.

Since $g_{-k}\chi_\alpha \in l^2(\beta)$, for every $k \in \mathbb{N}$, and taking into account that $l^2(\beta)$ is a vector space, invariant for the shift, it follows that

$$M_{0,\alpha} \coloneqq \{f\chi_\alpha : f \in M_0\} \subset l^2(\beta)$$

where M_0, denotes the set of trigonometric polynomials. Let $\mathfrak{M}_\alpha(\beta)$ denote the closure of $\mathfrak{M}_{0,\alpha}$ in $l^2(\beta)$. The subspace $\mathfrak{M}_\alpha(\beta)$ is clearly invariant for T, and we define the operator A as the restriction

$$A := T\,|\mathfrak{M}_\alpha(\beta) \in \mathscr{L}\,(\mathfrak{M}_\alpha, (\beta)).$$
In virtue of Lemma(3.1.5) we know that $T \in C_{10}$ and $\sigma(T) = \partial\mathbb{D}$. It immediately follows that $A \in C_{10}$ and A is bounded from below. Taking into consideration that $A\,\mathfrak{M}_{0,\alpha} = \mathfrak{M}_{0,\alpha}$ is dense in $\mathfrak{M}_\alpha(\beta)$ we infer that A is invertible. Since $\partial\sigma(A) \subset \sigma(T) = \partial\mathbb{D}$, it follows that $\sigma(A) \subset \partial\mathbb{D}$. The estimate

$$\|A^{-n}\| = \|T^{-n}|\mathfrak{M}_\alpha(\beta)\| \le \|T^{-n}\| = (w_{-1} \dots w_{-n})^{-1} = \left(\prod_{i=1}^{m_{\{n\}}} \gamma_i\right)^{-1} \le \gamma_1^{m_n},$$

being true for every $n \in \mathbb{N}$, *im*plies that

$$\sum_{n=1}^{\infty} \frac{log\|A^{-n}\|}{1+n^2} \le \sum_{n=1}^{\infty} \frac{m_n log\gamma_1^{-1}}{1+n^2} < \infty.$$

Now, a result of Beauzamy and Rome (see [166]) yields that

$$\sigma(A) = \sigma(A) \cap \partial\mathbb{D} = \sigma(\tilde{A}),$$

where $\tilde{A}$ is the unitary extension of A. However, for every $f \in \mathfrak{M}_\alpha\,(\beta)$ we have

$$\lim_{n\to\infty} \|A^n f\|_\beta = \lim_{n\to\infty} \|T^n f\|_\beta = \|f\|.$$

Hence $\mathfrak{M}_\alpha(\beta)^{\sim} = L^2(\alpha)$ $(:= \{\chi_\alpha, f : f \in L^2\})$ and $\tilde{A} \in \mathscr{L}\,(L^2(\alpha))$ acts as the operator of multiplication by the function $g_1\,(\lambda) = \lambda$. Consequently, we obtain that $\sigma(\tilde{A}) = \alpha$ and so $\sigma(\,A) = \alpha$.

We have to show yet that the norm of A^{-1} can be arbitrarily large. Let us consider the operator A obtained before. Since $A \in C_{10}$ and A is invertible, it follows that the defect numbers of A are equal: $d_A = d_{A^*} = N_0$. Let $\{\Theta(\lambda), \mathfrak{E}, \mathfrak{E}\}$ be a contractive analytic function coinciding with the characteristic function of A. Then Θ is an inner, $*$-outer function. Moreover, $\Theta(\lambda)$is invertible for every $\lambda \in \mathbb{D}$, and the set $s(\Theta) = \{\lambda \in \partial\mathbb{D}: \lambda$ does not lie on an open arc $I \subset \partial\mathbb{D}$ where Θ is analytic unitary valued$\}$ coincides with α. ((see 29].) It is clear that, for every $n \in \mathbb{N}, \{\Theta(\lambda)^n, \mathfrak{E}, \mathfrak{E}\}$ is also an inner, $*$-outer function which is invertible in every point of $\mathbb{D}$ and for which $s(\Theta^n) = \alpha$. We infer that the model operator $A_n = S(\Theta^n)$ is of class C_{10} and $\sigma(A_n\,) = \alpha$. Moreover, we have

$$\|A_n^{-1}\| = \|\Theta(0)^{-n}\|.$$

So the proof will be completed if we show that $\{\,\|\Theta(0)^{-n}\|\}_n$ can be bounded.

Let us suppose that $\mathfrak{E} = \mathfrak{D}_A$, and let $x_0 \in \mathfrak{E}$ be an arbitrary unit vector.

Then $y_0 = -Ax_0 \in \mathfrak{D}A^*$ and $\|y_0\| = q\|x_0\|$, where $0 < q < 1$. Let $Z \in \mathscr{L}\,(\mathfrak{D}_{A^*}, \mathfrak{D}_A)$ be a unitary operator such that $Z\,y_0 = q x_0$. Let us define Θ as the product $\Theta = Z\Theta_A$, where Θ_A, is the characteristic function of A.

Then $\{\Theta(\lambda), \mathfrak{E}, \mathfrak{E}\}$ coincides with Θ_A and

$$\Theta(0)\,x_0 = Z\,\Theta_A\,(0)\,x_0 = Z(-Ax_0) = Zy_0 = q\,x_0.$$

Hence $\Theta(0)^n\,x_0 = q^n x_0$ and so

$$||\Theta(0)^{-n}|| \ge q^{-n} \to \infty \qquad\qquad (n \to \infty)$$

The proof is completed.

In virtue of Proposition (3.1.7) we can prove Theorem(3.1.12) applying the technique based upon the SzNagy-FoiaS functional calculus which was used in [6]. We need some lemmas.

Lemma (3.1.8)[25]: If A is a C_0-contraction and $u \in H^{\alpha}$ is a non-constant function with norm $\|u\|_{\infty} \leq 1$, then the contraction $u(A)$ is also of‘ class C_0.

Proof. Let $w \in H^{\infty}$ be the function $w(\lambda) = (\lambda - \mu_0)/(1 - \tilde{\mu}_0\lambda)$, where $\mu_0 = u(0) \in \mathbb{D}$ and $v := w \circ u \in H^{\infty}$.Then $v(A) = (u(A) - \mu_0 I)(I - \tilde{\mu}_0 u(A))^{-1}$, moreover $u(A) \in C_0$ if and only if $v(A) \in C_0$ (see [29]). Since $v(0) = 0$, v is of the form $v(\lambda) = \lambda v_1(\lambda)$, where $v_1 \in H^{\infty}$ and

$$\|v_1\|_{\infty} = \lim_{r \to 1-0} \sup_{[\lambda]=r} |v_1(\lambda)| = \lim_{r \to 1-0} \sup_{[\lambda]=r} \frac{|v(\lambda)|}{r} = \|v\|_{\infty} \leq 1.$$

Now $v(A) = A\, v_1(A)$ implies that $v(A)^{*n} = v_1(A)^{*n}(A)^{*n}$, for every $n \in \mathbb{N}$. Taking into account $\|v_1(A)^{*n}\| \leq \|v_1(A)\|^n \leq \|v_1\|_{\infty}^n \leq 1$, we infer that

$$\|v(A)^{*n}h\| \leq \|v_1(A)^{*n}\|\|A^{*n}h\| \leq \|A^{*n}h\| \to 0 \qquad (n \to \infty)$$

Hence $v(A) \in C_0$, and so $u(A) \in C_0$.

Lemma (3.1.9)[25]: If A is a C_{10}-contraction and $u \in H^{\infty}$ is a non-constant function such that $\|u\|_{\infty} \leq 1$and $|\, u(\lambda)| = 1$,f or a.e. ,$\lambda \in \partial\mathbb{D} \cap \sigma(A)$ (with respect to the Lebesgue measure), then the contraction $u(A)$ also belongs to C_{10}.

Proof. Since A is of class C_1, it follows that A can be injected into $R_{*,A}: A \prec^{i} R_{*,A}$. On the other hand, A being $cnu, R_{*,A}$ is an absolutely continuous unitary operator. We infer that

$$u(A) \prec^{i} u(R_{*,A}).$$

Taking into consideration that $\sigma(R_{*,A}) \subset (\sigma(A) \cap \partial\mathbb{D})$, cf. Theorem(3.1.2) , the spectral theorem yields that $u(R_{*,A})$ is unitary, and so $u(R_{*,A}) \in C_1$.

Since $u(A)$ can be injected into $u(R_{*,A})$, we obtain that $u(A) \in C_1$.. Now, applying Lemma (3.1.8) we conclude that $u(A)$ is of class C_{10}.

The following statement is the basic tool in our construction.

Proposition (3.1.10)[25]: Let $\Omega \subset \mathbb{D}$ be a simple connected domain whose boundary $\Gamma = \partial\Omega$ is a rectifiable Jordan curve containing a closed arc I of $\partial\mathbb{D}$, $m(I) > 0$. Let us given a .set $\beta \in \mathscr{M}_0(\mathbb{D}), \beta \subset I$, a point $\mu_0 \in \Omega$ and a real number $K > 0$. Then there exists a contraction B such that

(a) $B \in C_{10}$,

(b) $\sigma(B) = \beta$,

(c) $\|(B - \mu_0 I)^{-1}\| > K$, and

(d) l$\|(B - \mu I)^{-1}\| < dist(\mu, \Omega^{-})^{-1}$for every $\mu \in \mathbb{C}\backslash\Omega$.

Proof. The Riemann mapping theorem and Caratheodory’s theorem ensure us a homeomorphism $u: \mathbb{D}^{-} \to \Omega$, which is holomorphic on $\mathbb{D}$. It can be assumed that $u(0) = \mu_0$. Since Γ is a rectifiable Jordan curve, it follows that $\alpha = u^{-1}(\beta) \in \mathscr{M}_0(\mathbb{D})$ [169].

Because of Proposition(3.1.7) we can find a contraction $A \in C_{10}$ such that $\sigma(A) = \alpha$ and $\|A^{-1}\| > 2k$. We define B by the aid of the Sz.-Nagy-Foias functional calculus, namely $B := u(A)$.

Since $u(\sigma(A)) = u(\alpha) = \beta \subset \partial\mathbb{D}$, we infer by Lemma(3.1.9) that $B = u(A) \in C_{10}$. On the other hand, the Foias-Mlak spectral mapping theorem (cf. [170]) yields that $\sigma(B) = u(\alpha) = \beta$.

The relation $u(0) = \mu_0$ implies that $u(\lambda) - \mu_0 = \lambda v(\lambda)$, where $v \in H^{\infty}$ and

$$\|v\|_\infty = \lim_{r\to 1} \sup_{0|\lambda|=r} \left|\frac{u(\lambda)-\mu_0}{\lambda}\right| \le \|u\|_\infty + |\mu_0| \le 2.$$

Since $u(A)-\mu_0 I = A\, v(A)$, it follows $A^{-1} = (u(A)-\mu_0 I)^{-1} v(A)$ and

$$2K < \|A^{-1}\| \le \|(u(A)-\mu_0 I)^{-1}\|\,\|v(A)\| \le \|(B-\mu_0 I)^{-1}\|\|v\|_\infty$$
$$\le 2\|(B-\mu_0 I)^{-1}\|,$$

i.e.,

$$\|(B-\mu_0 I)^{-1}\| > K.$$

While, if $\mu \in \mathbb{C}\backslash\Omega^-$, then $v_\mu(\lambda) = (u(\lambda)-\mu)^{-1} \in H^\infty), \|v_\mu\|_\infty = dist(\mu,\Omega^-)^{-1}$. Hence

$$\|(B-\mu I)^{-1}\| = \|(u(A)-\mu I)^{-1}\| = \|v_\mu(A)\| \le \|v_\mu\|_\infty =, dist(\mu,\Omega^-)^{-1},$$

and the proof is completed.

Lemma (3.1.11)[25]: Let α,σ be sets belonging to $\mathscr{M}_0$ $(\mathbb{D})$ and $,\mathscr{M}(\mathbb{D})$. respectively such that $\alpha \subset^{(n)} \sigma$. Let us given a point $,\mu_0 \in \sigma$ and positive numher. K,ε. There $exists\ a$ contraction T such that

(a) $T \in C_{10}$,
(b) $\sigma(T) \subset \alpha$,
(c) $\|(T-\mu I)^{-1}\| > K$, ifs" $\mu_0 \notin \sigma(T)$and
(d) $\|(T-\mu I)^{-1}\| \le (dist(\mu,\sigma)-\varepsilon)^{-1}$ if dist$(\mu,\sigma) > \varepsilon$.

Proof. Let us consider the open set $(\sigma_\varepsilon = \{\lambda \in \mathbb{C}: dist(\lambda.,\sigma) < \varepsilon\}$, containing σ. Let σ_0 be the component of σ_ε including μ_0. Since $\sigma_\varepsilon\backslash\sigma_0$ is also open, it follows that $(\grave{\sigma} = \sigma_0 \cap \sigma$ is a non-empty closed and open subset of σ. In virtue of $\alpha \subset^{(n)} \sigma$ we get $m(\grave{\sigma} \cap \alpha) > 0$. The set $\sigma_0 \cap \partial\mathbb{D}$ consists of countably many open arcs, hence there exists a closed arc $I \subset (\partial\mathbb{D} \cap \sigma_0)$ such that $m(I \cap \alpha) > 0$. Let β denote the support of the measure $\chi_{I\cap\alpha}\, dm$. Since $I \cap \alpha$ is closed in $\partial\mathbb{D}$, it follows that $\beta \subset I \cap \alpha$, moreover $\beta \in \mathscr{M}_0(\mathbb{D})$. Let us assume first that $\mu_0 \in \mathbb{D}$. There exists a simply-connected domain Ω such that $\Omega \subset \sigma_0 \cap \mathbb{D}$ is a rectifiable Jordan curve, $\partial\Omega \supset I$, and $\mu_0 \in \Omega$. (Note that $\sigma_0 \cap \mathbb{D}$ is also connected.) Now, Proposition(3.1.10) provides us a contraction $T \in C_{10}$ such that $\sigma(T) = \beta(\subset \alpha), \|(T-\mu_0 I)^{-1}\| > K$ and $\|(T-\mu I)^{-1}\| \le dist(\mu,\Omega^{-1})^{-1}$ for every $\mu \in \mathbb{C}\backslash\Omega^{-1}$. Since $\Omega \subset \sigma_0 \subset \sigma_\varepsilon$ it follows that $dist(\mu,\sigma) > \varepsilon$ implies $\mu \notin \Omega^-$. and so $\|(T-\mu I)^{-1}\| \le dist(\mu,\Omega^-)^{-1} \le (dist(\mu,\sigma)-\varepsilon)^{-1}$.

Let us assume now that $\mu_0 \in \partial\mathbb{D}$.. Let us choose a point $\grave{\mu}_0 \in \mathbb{D}$ such that $|\grave{\mu}_0 - \mu_0| < \rho$, where $0 < \rho < \frac{\varepsilon}{2}$. It is clear that $,\grave{\mu}_0 \in \sigma_0$. Now let $T \in C_{10}$ be a contraction corresponding to $\grave{\mu}_0$ and $K+1$ by Proposition(3.1.10). We have only to examine the inverse of $T-\mu_0 I$. Let us suppose that $\mu_0 \notin \sigma(T)$. Since $\|(T-\grave{\mu}_0 I)^{-1}\| > K+1$, there is a unit vector x_0 such that $\|(T-\grave{\mu}_0 I)x_0\| < (K+1)^{-1}$. Hence $\|(T-\mu_0 I)x_0\| \le \|(T-\grave{\mu}_0 I)x_0\| + |\mu_0-\grave{\mu}_0| < (\mathrm{K}+1)^{-1}+\rho$ and so $\|(T-\mu_0 I)^{-1})\| > ((K+1)^{-1}+\rho)^{-1}$. But $((K+1)^{-1}+\rho)^{-1} > K$ if ρ is small enough, and the proof is finished.

Now, we are ready to show Theorem(3.1.12).

Theorem (3.1.12)[25]: If $\alpha \in \mathscr{M}_0(\mathbb{D})$is neatly contained in $\sigma \in \mathscr{M}(\mathbb{D})$ then there exists a C_{10}-contraction T such that $\sigma(T) = \sigma$ and $\sigma(R_{*,T}) = \alpha$.

2. In proving Theorem(3.1.11) first we show that every element of $\mathscr{M}_0(\mathbb{D})$can be the spectrum of a C_{10},-contraction.

Proof. Let us given sets $\alpha \in \mathscr{M}_0(\mathbb{D})$ and $(\sigma \in \mathscr{M}(\mathbb{D})$ such that $\alpha \subset^{(n)} \sigma$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of points of σ, which is dense in σ, and in which every term is repeated infinitely many times. Let $\{K_n\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be sequences of positive numbers such that $\lim_{n\to\infty} K_n = \infty$ and $\lim_{n\to\infty} \varepsilon_n = 0$. For every $n \in \mathbb{N}$, let T_n,, be a C_{10}-contraction corresponding to $\alpha \in \mathscr{M}_0(\mathbb{D}), \sigma \in \mathscr{M}(\mathbb{D})$, $\mu_n \in \sigma, K_n > 0$, and $\varepsilon_n > 0$ by Lemma(3.1.10). Moreover, Proposition(3.1.6) ensures us a contraction T_0, of class C_{10} such that $(\sigma(T_0) = \sigma(R_{*.T_0}) = \alpha$. Let us define T as the orthogonal sum $T = \bigoplus_{n=0}^{\infty} T_n$. It is easy to verify that T belongs to C_{10} and its spectrum $\sigma(T) = \sigma$. Finally, the unitary extension of T being the orthogonal sum of the unitary extensions of T_n's, we conclude that $\sigma(R_{*.1}) = \bigcup_{n=0}^{\infty} \sigma\left(R_{*.T_n}\right) = \alpha$. The proof is completed.

First of all we note that if T belongs to C_{00}, then both its residual and its $*$-residual part: R_T, and $R_{*.T}$, respectively, act on the trivial space $\{0\}$(see[29]).

An important subclass of C_{00}, denoted by C_0 is the system of those cnu contractions T which are annihilated by a non-constant function $u \in H^\infty : u(T) = 0$. The spectrum of a C_0-contraction T can be completely described by the aid of its minimal function $m_T \in H^\infty$ e.g., $\sigma(T) \cap \mathbb{D}$ coincides with the set $\{\lambda \in \mathbb{D}: m_T(\lambda) = 0\}$ [29]. Hence, for a C_0-contraction $T, \sigma(T) \cap \mathbb{D}$ is always countable, moreover the sum $\sum_{\lambda\in\sigma(T)\cap\mathbb{D}} (1 - |\lambda|)$ is finite. The following theorem shows that the spectrum of a countable orthogonal sum of C_0-contractions, and so the spectrum of a C_{00}-contraction, is already an arbitrary compact subset of $\mathbb{D}$.

Theorem (3.1.13)[25]: For every non-empty, compact .suhset K of' $\mathbb{D}$, there exists a contraction $T \in \mathscr{L}(\mathfrak{H})$ such that

(a) T is a countable orthogonal sum of C_0-contractions,

(b) $\sigma(T) = K$, and

(c) T is cyclic, i.e., $V_{n\geq 0} T^n h = \mathfrak{H}$ holds ,fiw a vector $h \in \mathfrak{H}$.

Proof. (a) Let $\{a_n\}_{n\in J}$ be a dense sequence of different points in $K \cap \mathbb{D}$. (Here J is of the form $J = \{n \in \mathbb{N}: n < N)$, where $n \in \mathbb{N}$ or $N = \omega$, the first infinite ordinal. In the case $N = 1$ the set J is empty.) Let μ be the Borel measure on $\mathbb{C}$ defined by

$$\mu(\omega) = \sum_{a_n\in\omega} 2^{-n} \quad \text{for every Bore1 set } \omega \subset \mathbb{C}.$$

Let us consider the Hilbert space $\mathfrak{H}_1 = L^2(\mu)$ and the operator $T_1 \in \mathscr{L}(\mathfrak{H}_1)$ of multiplication by the identity function $f(\lambda) = \lambda. T_1$, is a normal operator and for its spectrum we have

$$K \cap \mathbb{D} \subset \sigma(T_1) = clos(a_n)_{n\in F} \subset K.$$

Moreover, a theorem of Bram (cf. [195]) states that T, has cyclic vectors.

(b) Let v be a finite, positive, Bore1 measure on $\partial\mathbb{D}$ singular with respect to the Lebesgue measure such that $supp\ v = K \cap \partial\mathbb{D}$. (E.g., let $\{b_n\}_{n\in J}$ be a dense sequence in $K \cap \partial\mathbb{D}$ and $v(\omega) = \sum_{b_n\in\omega} 2^{-n}$, for any Bore1 set $\omega \subset \partial\mathbb{D}$) Let $u \in H^\infty$ be the singular inner function deriving from v:

$$u(\lambda) = exp\left[-\int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} dv\right], \qquad \lambda \in \mathbb{D}.$$

Now we define T_2, acting on the space $\mathfrak{H}_2 = H^2 \ominus uH^2$ as the model operator corresponding to u, i.e.,

$$T_2 = S(u) = P_{\mathfrak{H}_2} S|\mathfrak{H}_2,$$
where S denotes the multiplication by $f(\lambda) = \lambda.$ on H^2. Then T_2 will be a C_0-contraction with minimal function u. Moreover, T_2 is cyclic (since S is cyclic), and $\sigma(T_2) = supp\ v = K \cap \partial\mathbb{D}$. [29].

(c) Now T is defined as the orthogonal sum $T = T_1 \oplus T_2$ acting on $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Since T_1 is unitarily equivalent to $\bigoplus_{n\in J} S(m_n)$, where m_n is the Blaschke-factor $m_n(\lambda) = (\frac{\bar{a}_n}{|a_n|})(a_n - \lambda)/(1 - \bar{a}_n\lambda)$ corresponding to a_n, it follows that T is a countable orthogonal sum of C_0-contractions. On the other hand, the spectrum of T is
$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2) = K.$$
(d) Finally, we show that T is cyclic. Let $\zeta_i \in \mathfrak{H}_i$, be a cyclic vector of T_i for $i = 1, 2$. We claim that $\zeta = \zeta_1 \oplus \zeta_2 \in \mathfrak{H}$ will be a cyclic vector for T, i.e., the subspace $\mathfrak{M} = \{p(T)\zeta : p(\lambda)$ 1s a polynomial$\}^-$ coincides with $\mathfrak{H}$.

Taking into account that $u(T)$ can be approximated by the polynomials of T in the strong operator topology, we infer that, for any polynomial $p(\lambda)$, the vector $u(T)\,p(T)\zeta$ belongs to $\mathfrak{M}$. But
$$u(T)p(T)\zeta = u(T_1)p(T_1)\zeta_1 \oplus p(T_2)u(T_2)\zeta_2 = u(T_1)p(T_1)\zeta_1 \oplus 0.$$
If p runs through the set of polynomials, the vectors $p(T_1)\,\zeta_1$ form a dense set, in $\mathfrak{H}_1$. Since $u(T_1)$ is clearly a quasi-affinity, we obtain that
$$\mathfrak{W} \supset \mathfrak{H}_1 \oplus \{0\}.$$
If $\eta \in \mathfrak{H}_2$ and $\varepsilon > 0$ are arbitrarily chosen then, ζ_2 being cyclic for T_2 we can find a polynomial q such that $\|q(T_2)\zeta_2 - \eta\| < \varepsilon$. Then $q(T)\zeta - (q(T_1)\zeta_1 \oplus 0) \in \mathfrak{M}$ and
$$\|(q(T)\zeta - (q)(T_1)\zeta_1 \oplus 0)) - (0 \oplus \eta)\| = \left\|0 \oplus \left(q(T_2)\right)\zeta_2 - \eta\right\| < \varepsilon.$$
Consequently, $\{0\} \oplus \mathfrak{H}_2 \subset \mathfrak{W}$ also holds, and so $\mathfrak{H} = \mathfrak{M}$. The proof is completely finished.

Section(3.2): Asymptotically Nonvanishing Contractions

Let $\mathcal{H}$ be a complex, separable Hilbert space, and let $\mathcal{L}(\mathcal{H})$ stand for the set of all bounded, linear operators acting on $\mathcal{H}$.. An operator $\mathrm{T} \in \mathcal{L}(\mathcal{H})$ is a contraction if $\|\mathrm{T}\| \leq 1$. We say that the contraction T is asymptotically nonvanishing, in notation: $\mathrm{T} \in \mathrm{C}_{*.}$ if there exists a vector $\mathrm{x}_0 \in \mathcal{H}$. such that $\lim_{n\to\infty}\|\mathrm{T}^n\mathrm{x}_0\| > 0$. It is a longstanding open problem whether every asymptotically nonvanishing contraction T, which is not scalar multiple of the identity, has a nontrivial hyperinvariant subspace M. We recall that the (closed) subspace M of' $\mathcal{H}$. is called nontrivial, if $\{0\} \neq \mathrm{M} \neq \mathcal{H}$ and M is hyperinvariant for T, if it is invariant for every operator Q belonging to the commutant $\{\mathrm{T}\}' := \{\mathrm{A} \in \mathcal{L}(\mathcal{H}) : \mathrm{AT} = \mathrm{TA}\}$ of T. The hyperinvariant subspace lattice of T is denoted by Hlat T. It is easy to see that for any contraction $\mathrm{T} \in \mathcal{L}(\mathcal{H})$, the subspace $\mathcal{H}_0(\mathrm{T}) := \{\mathrm{x} \in \mathcal{H} : \lim\limits_{n\to\infty}\|\mathrm{T}^n\mathrm{x}\| = 0\}$ belongs to Hlat T. We write $\mathrm{T} \in \mathrm{C}_0.$ if $\mathrm{H}_0(\mathrm{T}) = \mathcal{H}$ and we write $\mathrm{T} \in \mathrm{C}_1.$ if $\mathcal{H}_0(\mathrm{T}) = \{0\}$. For any $\mathrm{j} = 0, *, 1$, by definition $\mathrm{T} \in \mathrm{C}_{.\mathrm{j}}$ if $\mathrm{T}^* \in \mathrm{C}_{\mathrm{j}}.(\mathcal{H})$ is true for the adjoint operator. Finally, for any choice of $\mathrm{i}, \mathrm{j} = 0, *, 1$, we consider the set $\mathrm{C}_{\mathrm{ij}} := \mathrm{C}_{\mathrm{i}}. \cap \mathrm{C}_{.\mathrm{j}}$. This classification of contractions, according to the asymptotic behaviour of the iterates, was introduced by Bela SzokefalviNagy and Ciprian Foias (see [29]). They showed that if $\mathrm{T} \in \mathrm{C}_{**}$ is nonscalar, then Hlat T is nontrivial, that is $\mathrm{Hlat\,T} \neq \{\{0\}, \mathcal{H}\}$ (see [29]). Therefore, addressing the hyperinvariant subspace problem for asymptotically nonvanishing contractions, we can assume that T is of class C_{10} . In what

follows, we shall consider mainly contractions of class $C_{1.}$. The advantage of the assumption $T \in C_{*.}$ is that a nonzero unitary asymptote $T^{(a)} \in \mathcal{L}(\mathcal{H}_T^{(a)})$ can be associated with T. Namely, if $(.,.)$ stands for the original inner product on $\mathcal{H}$ then a new semi-inner product can be introduced on $\mathcal{H}$ by the formula $[x,y] := \lim_{n\to\infty}\langle T^n x, T^n y\rangle$ $(x, y \in \mathcal{H})$. Forming quotient space and taking completion, we arrive at a Hilbert space $\mathcal{H}_{T,+}^{(a)}$, where T can be continuously extended to an isometry $T_T^{(a)}$). The natural embedding $X_T^+ : \mathcal{H} \to \mathcal{H}_{T,+}^{(a)}, x \mapsto x + \mathcal{H}_0(T)$ is contractive and intertwines T with $T_T^{(a)}$), in notation: $X_T^+ \in T\ (T, T_T^{(a)}) := \{A \in \mathcal{L}(\mathcal{H}, \mathcal{H}_{T,+}^{(a)}) : AT = T_+^{(a)}) A\}$. The unitary asymptote $T^{(a)} \in \mathcal{L}(T_T^{(a)})$ is defined as the minimal unitary extension of the isometry $T_+^{(a)}$), and the canonical intertwining mapping $X_T \in T\ (T, T^{(a)})$ is defined by $X_{T^x} := X_T^+ x$ $(x \in \mathcal{H})$. Clearly,
$\ker X_T = \mathcal{H}_0(T)$ and $(\operatorname{ran} X_T)^- = \mathcal{H}_{T,+}^{(a)}$. Thus, if $T \in C_{*.}(\mathcal{H})$ then the unitary operator $T^{(a)}$ is nonzero (i.e. acts on a nonzero space), and if $T \in C_1(\mathcal{H})$ then the mapping X_T is an injection. For more details in connection with these concepts, we refer to [29], [20], [10] and [157]. It is well-known that any contraction $T \in \mathcal{L}(\mathcal{H})$ can be uniquely decomposed into the orthogonal sum $T = T_1 \oplus T_2 \oplus T_3$ of a completely nonunitary (c.n.u.) contraction T_1, an absolutely continuous (a.c.) unitary operator T_2 and a singular unitary operator T_3 (see [29] and [152]). Applying the Lifting Theorem of Sz.-Nagy and Foias it can be easily verified that the hyperinvariant subspace lattice of T splits into the direct sum $\operatorname{Hlat} T = \operatorname{Hlat}(T_1 \oplus T_2) \oplus \operatorname{Hlat} T_3$ (see [29] and [148]). Thus, we can (and shall) assume in the sequel that the singular unitary component T_3 is zero, that is the contraction T is absolutely continuous. In that case the unitary asymptote $T^{(a)}$ is also a.c. (see [29] and [157], or [20]).
Now, the factorization theorem claims that if $T \in \mathcal{L}(\mathcal{H})$ is an a.c. contraction such that the spectral multiplicity function of the unitary asymptote $T^{(a)}$ dominates the function $n\chi_\omega$ $(n \in \mathbb{N}_\infty)$ and $\omega \in B_1$, then the natural embedding $J_{n,\omega} : H^2(\varepsilon_n) \to \chi_\omega L^2(\varepsilon_n), f \longmapsto \chi_\omega f$ can be factorized into the product $J_{n,\omega} = ZY$, where $Y \in \mathcal{L}(S_n, T)$ and $Z \in \mathcal{L}(T, M_{n,\omega})$, and we have a control on the norms of Y and Z. The hyperinvariant subspace lattices of the operators S_n and $M_{n,\omega}$ are dramatically different. Namely, $\operatorname{Hlat} S_n$ is isomorphic to the lattice of (equivalence classes of) inner functions, while $\operatorname{Hlat} M_{n,\omega}$ is isomorphic to the Boolean lattice of (equivalence classes of) Borel subsets of ω. (See [158].) Now, the question is how the hyperinvariant subspace lattice of the intermediate operator T behaves. We are going to examine under what conditions the properties of T show similarities with those of S_n and when the properties of T are closer to those of $M_{n,\omega}$. The concept of the quasianalytical spectral set π(T) of a $C_{1.}$-contraction T is introduced in this Section . This is a Borel subset of the unit circle, which plays central role in our investigations. The connection of π(T) with the support ρ(T) of the spectral measure of $T^{(a)}$ called the residual set of T) is examined. One of the main results in this section is that Hlat T is nontrivial, if $\pi(T) \neq \rho(T)$. We study the transformation laws concerning these sets in the Sz.-Nagy~Foias functional calculus. As a result, we obtain an abundance of examples for possible pairs of π(T) and ρ(T). It is shown that the contraction T exhibits a 'quasianalytic property' on the quasianalytical spectral set π(T). We devoted to different intertwining relations. The operators in the commutant of T are

studied. It is shown that every nonZero operator in $\{T\}'$ is injective in the quasianalytic, cyclic case. Furthermore, a sufficient condition is given for the existence of an operator $0 \neq Q \in \{T\}'$ with nondense range.

Let H^{∞} denote the Hardy space of bounded, analytic functions, defined on the open unit disc$\mathbb{D}$. We recall that, for any $u \in H^{\infty}$, the radial limit $\lim_{r\to 1} u(rz)$ exists for almost every (a.e.) $z \in \mathbb{T}$ the limit function will be also denoted by u. In connection with the basic properties of H^{∞}, we refer to [9] and [29]. Given any $u, v \in H^{\infty}$, we say that the function u is smaller than v in absolute value, in notation: $u \overset{a}{\prec} v$ if $|u(z)| \leq |v(z)|$ holds, for every $z \in \mathbb{D}$. This relation can be characterized in the following way.

Lemma (3.2.1)[31]: For any functions $u, v \in H^{\infty}$, the following conditions are equivalent:

(a) $u \overset{a}{\prec} v$;

(b) there exists $w \in H^{\infty}$ such that $u = vw$ and $\|w\|_{\infty} \leq 1$

(c) $\|u(z)\| \leq |v(z)|1$ is true for a.e. $z \in \mathbb{T}$, and the inner component of v divides the inner component of u.

In a similar fashion, given any $A, B \in \mathcal{L}(\mathcal{H})$, we say that the operator A is smaller than B in absolute value, in notation: $A \overset{a}{\prec} B$, if $\|Ax\| \leq \|Bx\|$ is true, for every $x \in \mathcal{H}$. This relation can be also easily characterized.

Lemma (3.2.2)[31]: For any operators $A, B \in \mathcal{L}(\mathcal{H})$, the following conditions are equivalent:

(a) $A \overset{a}{\prec} B$;

(b) there exists $C \in \mathcal{L}(\mathcal{H})$ such that $A = CB$, $\|C\| \leq 1$

(c) $A^*A \leq B^*B$.

Clearly, these relations on H^{∞} and $\mathcal{L}(\mathcal{H})$ are reflexive and transitive. We shall consider the following sets of decreasing sequences:

$$D(H^{\infty}) := \{F = \{f_n\}_{n=1}^{\infty} : f_n \in H^{\infty}, f_{n+1} \overset{a}{\prec} f_n \text{ for every } n \in \mathbb{N}\}$$

and

$$D(\mathcal{H}) := \{A = \{A_n\}_{n=1}^{\infty} : A_n \in \mathcal{L}(\mathcal{H}), A_{n+1} \overset{a}{\prec} A_n \text{ for every } n \in \mathbb{N} \} .$$

To any sequence $F = \{f_n\}_{n=1}^{\infty} \in D(H^{\infty})$ we can associate the limit function

$$\varphi_F(z) := \lim_{n\to\infty} |f_n(z)|,$$

defined almost everywhere on the unit circle $\mathbb{T}$. Similarly, to any sequence $A = \{A_n\}_{n=1}^{\infty}$ there corresponds the limit operator

$$\Phi_A := \left(\lim_{n\to\infty} A_n^* A_n\right)^{1/2} .$$

Here the convergence is meant in the strong operator topology; furthermore,
$\|\Phi_A x\| = \lim_{n\to\infty} \|A_n x\|$ is true, for every $x \in \mathcal{H}$.

Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction, and let us consider the Sz.-Nagy-Foias functional calculus for T. We remind the reader that this calculus is a uniquely determined unital algebra-homomorphism $\psi_T : H^{\infty} \to \mathcal{L}(\mathcal{H}), f \mapsto f(T)$, which assigns T to the identical function χ, and which is continuous in the weak-$*$ topologies (see [29] or [3]). It is easy to verify that ψ_T is monotone with respect to the relations introduced before. Indeed, if

$u \overset{a}{\prec} \mathrm{v}(\mathrm{u}, \mathrm{v} \in \mathrm{H}^{\infty})$, then $u = vw$ is true with some $w \in \mathrm{H}^{\infty}, \|w\|_{\infty} \leq 1$, and since$\psi_T$ is contractive, we infer that $\|u(T)x\| = \|w(T)v(T)x\| \leq \|v(T)x\| \; (x \in \mathcal{H})$, that is $u(T) \overset{a}{\prec} \mathrm{v(T)}$. Therefore, given any $F = \{f_n\}_{n=1}^{\infty} \in D(H^{\infty})$, we can form the sequence $F(T) := \{f_n(T)\}_{n=1}^{\infty} \in D(H)$. The subspace

$$\mathcal{H}_0(T, F) := ker\, \Phi_{F(T)} = \{x \in \mathcal{H}: \lim_{n\to\infty} \|f_n(T)x\| = 0\}$$

is clearly hyperinvariant for T. The following propositions show that under specific conditions this subspace is trivial. We note that for a function $\varphi \in L^{\infty} = L^{\infty}(m)$, the notation $\varphi = 0$ or $\varphi > 0$ means that $\varphi(z) = 0$ or $\varphi(z) \in (0, \infty)$, respectively, for a.e. $z \in T$.

Proposition (3.2.3)[31]: Let $T \in \mathcal{L}(\mathcal{H})$be an a.c. contraction, and $F = \{f_n\}_{n=1}^{\infty} \in D(H^{\infty})$. If $\varphi_F = 0$,then $\mathcal{H}_0(T, F) = \mathcal{H}$.

Proof: By Sz.-Nagy's celebrated dilation theorem, there exists a unitary operator U on a larger Hilbert space K, such that $T^n = P_{\mathcal{H}} U^n | \mathcal{H}$ is true for every $n \in \mathbb{N}$, and $K = V_{n=-\infty}^{\infty} U^n \mathcal{H}$. Furthermore, this minimal unitary dilation of T is uniquely determined and absolutely continuous (see [160] and [29]). Let E denote the spectral measure of U, and let E_x be the localization of E at the vector $x \in \mathcal{H}$. Then applying Lebesgue's dominated convergence theorem, we obtain that

$$\left\|\Phi_{F(T)} x\right\| = \lim_{n\to\infty} \|f_n(T)x\| = \lim_{n\to\infty} \| P_{\mathcal{H}} f_n(U)x\|$$
$$\leq \lim_{n\to\infty} \|f_n(U)x\| = \lim_{n\to\infty} \left(\int_{\mathbb{T}} |f_n|^2 dE_x\right)^{1/2} = 0$$

and so $\Phi_{F(T)} = 0$.

Proposition (3.2.4)[31]: Given any a. c. contraction $T \in \mathcal{L}(\mathcal{H})$, the following conditions are equivalent:

(a) $T \in C_{1.}$;

(b) for any $F \in D(\mathrm{H}^{\infty})$, the relation $\varphi_F > 0$ implies that $H_0(T, F) = \{0\}$.

Proof. $(b) \Rightarrow (a)$: This implication is trivial since $\varphi_{F_0} = 1 > 0$ holds for the sequence $F_0 = \{\chi^n\}_{n=1}^{\infty} \in D(H^{\infty})$ and since $H_0(T, F_0) = \mathcal{H}_0(T)$.

$(a) \Rightarrow (b)$: Let us suppose that $T \in C_{1.}$, and let us consider the unitary asymptote $T^{(a)} \in \mathcal{L}(\mathcal{H}_T^{(a)})$ of T and the canonical intertwining mapping $X_T \in \mathcal{L}\,(T, T^{(a)})$.

Let $F = \{f_n\}_{n=1}^{\infty} \in D(H^{\infty})$ be a sequence such that $\varphi_F > 0$. Then, for any vector $x \in \mathcal{H}$. and for any $n \in \mathbb{N}$, we have

$$\|f_n(T)x\| \geq \|X_T\|^{-1} \|X_T f_n(T)x\| = \|X_T\|^{-1} \left\|f_n\left(T^{(a)}\right) X_T x X\right\|$$
$$= \|X_T\|^{-1} \left(\int_{\mathbb{T}} |f_n|^2 \, dE_{X_T x}\right)^{1/2},$$

where E is the spectral measure of the a.c. unitary operator $T^{(a)}$. Lebesgue's Theorem yields that

$$\left\|\Phi_{F(T)} x\right\| \geq \|X_T\|^{-1} \left(\int_{\mathbb{T}} \varphi_F^2 dE_{X_T x}\right)^{1/2} \qquad (x \in \mathcal{H}).$$

Taking into account that X_T is injective and that the measure $E_{X_T x}$ is absolutely continuous, it readily follows that $ker\, \Phi_{F(T)} = \{0\}$.

Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction. We say that T is asymptotically strongly nonvanishing with respect to the Borel set $\alpha \in B_1$, in notation: $T \in C_{1.}(\alpha)$, if $H_0(T,F) = \{0\}$ is fulfilled for every sequence $F \in D(H^\infty)$ satisfying the condition $\chi_\alpha \varphi_F \neq 0$. Since $\chi_\alpha \varphi_{F_0} = \chi_\alpha \neq 0$ is true with $F_0 = \{\chi_n\}_{n=1}^\infty$ we can see that $C_{1.}(\alpha)$ ($\alpha \in B_1$) is a subclass of $C_{1..}$ It also follows immediately from the definition that $C_{1.}(\alpha) \supset C_{1.}(\beta)$ holds, whenever $\alpha \subset \beta (a, \beta \in B_1)$. Furthermore, given any sequence $\{\alpha_n\}_{n=1}^\infty$ of Borel sets in B_1 , we have

$$\cap_{n=1}^\infty C_{1.}(\alpha_n) = C_{1.}(\cup_{n=1}^\infty \alpha_n).$$

Let $\bar{C}_{1.} := \cup_{\alpha \in B_1} \bar{C}_{1.}(\alpha)$ be the set of those a.c. contractions, which are asymptotically strongly nonvanishing with respect to some $\alpha \in B_1$. We make the convention that two Borel subsets α, β of $\mathbb{T}$ are considered equal, if $\chi_\alpha = \chi_\beta$, that is if the symmetric difference $\alpha \,\Delta\, \beta$ is of measure zero.

Proposition (3.2.5)[31]: For any contraction $T \in \tilde{C}_{1.}$, there exists a (unique) largest set $\alpha_T \in B_1$, such that $T \in C_{1.}(\alpha_T)$).

Proof. Setting $\delta := sup\{m(\alpha) : \alpha \in B_1, T \in C_{1.}(\alpha)\}$, let us consider a sequence $\{\alpha_n\}_{n=1}^\infty \subset B_1$ such that $\lim_{n\to\infty} m(\alpha_n) = \delta$, and $T \in C_{1.}(\alpha_n)$, for every $n \in \mathbb{N}$. Since $T \in \cap_{n=1}^\infty C_{1.}(\alpha_n) = C_{1.}(\cup_{n=1}^\infty \alpha_n)$ we can easily see that $\alpha_T = \cup_{n=1}^\infty \alpha_n$ possesses the required properties.

The Borel set $\alpha_T \in B_1$, appearing in the previous proposition, will be called the quasianalytical spectral set of the contraction $T \in \tilde{C}_{1.}$, and it will be denoted by $\pi(T)$. If the a.c. contraction T is of class $C_{1.} \backslash \tilde{C}_{1.}$., then by definition $\pi(T) := \phi$.

Propositon (3.2.6)[31]:

(a) If $S_n \in \mathcal{L}(H^2(\mathcal{E}_n))$ is the unilateral shift of multiplicity $n \in \mathbb{N}_\infty$, then $\pi(S_n) = \mathbb{T}$.

(b) If $U \in \mathcal{L}(\mathcal{H})$ is an a.c. unitary operator, then $\pi(U) = \phi$.

Proof. (a): Let us consider a sequence $F = \{f_k\}_{k=1}^\infty \in D(H^\infty)$ such that $\varphi_F \neq 0$, and a vector $0 \neq x \in H^2(\mathcal{E}_n)$. Taking into account that $x(z) \neq 0$ for a.e. $z \in \mathbb{T}$, we obtain that

$$\left\|\Phi_{F(S_n)} x\right\|^2 = \lim_{k\to\infty} \|f_k(S_n)x\|^2 = \lim_{k\to\infty} \int_{\mathbb{T}} |f_k(z)|^2 \|x(z)\|^2 \mathrm{dm(z)}$$
$$= \int_{\mathbb{T}} \varphi_F(z)^2 \|x(z)\|^2 dm(z) > 0.$$

(b): Let $\chi_\omega dm$ ($\omega \in B_1$) be a scalar spectral measure of the a.c. unitary operator U. Given any $\alpha \in B_1$, we can find $F \in D(H^\infty)$ such that $\chi_\alpha \varphi_F \neq 0$ and the set $\omega_0 := \{z \in \omega : \varphi_F(Z) = 0\}$ is of positive measure. Since

$$ker \Phi_{F(U)} = ker \varphi_F(U) = ran\, \chi_{\omega_0}(U) \neq \{0\},$$

it follows that $U \notin C_{1.}(\alpha)$.

Given any a.c. contraction $T \in \mathcal{L}(\mathcal{H})$ of class $C_{*.}$, there exists a (unique) Borel set $\beta_T \in B_1$ such that χ_{β_T} dm is a scalar spectral measure of the unitary asymptote $T^{(a)}$. This set is called the residual set of the contraction T, and is denoted by $\rho(T)$. We note here that $T^{(a)}$ is unitarily equivalent to the $*$-residual part of the minimal unitary dilation of T (see [157]) , and that, for $c.n.u.$ C_{11}-contractions, $\rho(T)$ is the smallest Borel set on $\mathbb{T}$which is residual for T in the sense used in [29]. We note also that the residual set $\rho(T)$

ιs included in the spectrum $\sigma(T)$ of T; see e.g. [10]. The description of possible spectra of contractions of class $C_{1.}$ was given in [143] and [20].
We proceed with exploring the connection of the sets $\pi(T)$ and $\rho(T)$.
Theorem (3.2.7)[31]: For any a.c. contraction $T \in C_{1.}(\mathcal{H})$ we have $\pi(T) \subset \rho(T)$.
Proof. Let us assume that the set $\alpha := \sigma(T)\backslash\rho(T)$ is of positive measure. Let us choose a strictly decreasing sequence $\{c_n\}_{n=1}^{\infty}$ of real numbers, converging to zero; and, for any $n \in \mathbb{N}$, let $f_n \in H^\infty$ be an outer function such that $|f_n| = \chi_\alpha + c_n \chi_{T\backslash\alpha}$. Now, we consider the sequence $F = \{f_n\}_{n=1}^{\infty} \in D(H^\infty)$, with limit function $\varphi_F = \chi_\alpha$.
In view of the equality $\chi_{\rho(T)}|f_n| = c_n\chi_{\rho(T)}$, we know that $c_n^{-1}f_n(T^{(a)})$ is a unitary operator ($n \in \mathbb{N}$). Thus, given any $0 \neq x_0 \in \mathcal{H}$ Hand $n \in \mathbb{N}$, we have

$$\lim_{j\to\infty}\left\|T^j f_n(T)x_0\right\| = \|X_T f_n(T)x_0\| = \left\|f_n\left(T^{(a)}\right)X_T x_0\right\| = c_n\|X_T x_0\| \le c_n\|x_0\|,$$

and so there exists $j(n) \in \mathbb{N}$ such that $\left\| T^j f_n(T)x_0\right\| \le 2c_n\|x_0\|$ holds, whenever $j \ge j(n)$. Let $k\colon \mathbb{N} \to \mathbb{N}$ be an increasing mapping such that $k(n) \ge j(n)$ is true, for every $n \in \mathbb{N}$; and let us consider the sequence $G = \{g_n\}_{n=1}^{\infty} \in D(H^\infty)$, where $g_n := \chi^{k(n)} f_n$ ($n \in \mathbb{N}$). The relations $\|g_n(T)x_0\| \le 2c_n\|x_0\|$ ($n \in \mathbb{N}$) imply that $0 \neq x_0 \in H_0(T,G)$. On the other hand, the limit function $\varphi_G = \varphi_F = \chi_\alpha$.
Therefore, $\chi_{\pi(T)}\varphi_G = \chi_\alpha \neq 0$ is fulfilled, which yields that $\mathcal{H}_0(T,G) = \{0\}$ must hold, what is a contradiction.
It turns out that if the quasianalytical spectral set and the residual set do not coincide, then we have affirmative answer for the hyperinvariant subspace problem.
Theorem (3.2.8)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{1.}$ $\cdot$ If $\pi(T) \neq \rho(T)$, then T has a nontrivial hyperinvariant subspace.
Proof. Let us suppose that the set $\beta := \rho(T)\backslash\pi(T)$ is of positive measure. We can find a sequence $F = \{f_n\}_{n=1}^{\infty} \in D(H^\infty)$ such that $\chi_\beta\varphi_F \neq 0$ and $\mathcal{H}_0(T,F) \neq 0$. Let us consider the Borel set $\omega_0 := \{z \in \rho(T) : \varphi_F(Z) = 0\}$. Since $X_T f_n(T) = f_n T^{(a)} X_T$ holds for every $n \in \mathbb{N}$, it follows that

$$X_T\mathcal{H}_0(T,F) \subset \mathcal{H}_0^{(a)}\left(T^{(a)},F\right) = ran\chi_{\omega_0}\left(T^{(a)}\right) \neq \mathcal{H}_{\mathrm{T}}^{(a)}.$$

Taking into account that the subspace $\mathcal{H}_0^{(a)}(T^{(a)},F)$ is reducing for $T^{(a)}$, and applying the relation $\bigvee_{n\in\mathbb{N}}(T^{(a)})^{-n}X_T\ \mathcal{H} = \mathcal{H}_T^{(a)}$, we infer that $\mathcal{H}_0(T,F) \neq \mathcal{H}$, and so $\mathcal{H}_0(T,F)$ is a proper hyperinvariant subspace of T.
Now, we examine how the corresponding quasianalytical spectral and residual sets relate if there is some weak similarity connection between two contractions. Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be arbitrary operators. We say that T_1 can be injected into T_2, in notation: $T_1 \overset{i}{\prec} T_2$, if $\mathcal{L}(T_1,T_2)$ contains an injection. The operators T_1 and T_2 are called injection-similar, in notation: $T_1 \underset{\sim}{\mathrm{d}} T_2$, if $T_1 \overset{d}{\prec} T_2$ and $T_2 \overset{d}{\prec} T_1$ hold . We say that T_1 can be densely mapped into T_2, in notation:$T_1 \overset{d}{\prec} T_2$, if (T_1,T_2) contains a transformation with dense range. The operators T_1 and T_2 are densely-similar, in notation: $T_1 \underset{\sim}{\mathrm{d}} T_2$, if $T_1 \overset{d}{\prec} T_2$ and $T_2 \overset{d}{\prec} T_1$ hold simultaneously.
The operator T_1 is called a quasiaffine transform of T_2 , in notation: $T_1 \prec T_2$, if $\mathcal{L}(T_1,T_2)$ contains a quasiaffinity, that is an injection with dense range. Finally, T_1 and T_2 are

quasisimilar, in notation: $T_1 \sim T_2$, if both $T_1 \prec T_2$ and $T_2 \prec T_1$ are fulfilled. These relations played important role in establishing canonical models for special classes of operators; see e.g. [29], [163], [154], [141] or [146].

Proposition (3.2.9)[31]: Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be a.c. contractions of class $C_{1..}$ Then the following statements are true:

(a) $T_1 \overset{i}{\prec} T_2$ implies $\pi(T_1) \supset \pi(T_2)$,
(b) $T_1 \overset{i}{\sim} T_2$implies $\pi(T_1) = \pi(T_2)$,
(c) $T_1 \overset{d}{\prec} T_2$ implies $\rho(T_1) \supset \rho(T_2)$,
(d) $T_1 \,\mathrm{d}\, T_2$ implies $\rho(T_1) = \rho(T_2)$,
(e) $T_1 \overset{\sim}{\prec} T_2$ implies $\pi(T_1) \supset \pi(T_2)$ and $\rho(T_1) \supset \rho(T_2)$,
(f) $T_1 \sim T_2$ implies $\pi(T_1) = \pi(T_2)$ and $\rho(T_1) = \rho(T_2)$.

Proof. (a): Let $Y \in \mathcal{L}(T_1, T_2)$ be an injection. Given any sequence $F = \{f_n\}_{n=1}^{\infty} \in D(H^{\infty})$, the relations $Y f_n(T_1) = f_n(T_2) Y$ $(n \in \mathbb{N})$ yield that $Y(\mathcal{H}_1)_0(T_1, F) \subset (\mathcal{H}_2)_0(T_2, F)$. Thus, if $(\mathcal{H}_2)_0(T_2, F) = \{0\}$ then $(\mathcal{H}_1)_0(T_1, F) = \{0\}$ must be also true; whence the inclusion $\pi(T_2) \subset \pi(T_1)$ readily follows.

(c): Let $Z \in \mathcal{L}(T_1, T_2)$ be a transformation with dense range. Since $X_{T_2} Z \in \mathcal{L}(T_1, T_2^{(a)})$, it follows by the universality property of the pair $(X_{T_1}, T_2^{(a)})$ (see [10]) that there exists a transformation $W \in \mathcal{L}(T_1^{(a)}, T_2^{(a)})$ such that $X_{T_2} Z = W X_{T_1}$. Taking into account that Z has dense range, we infer that

$$\left(W \mathcal{H}_{T_1}^{(a)}\right)^{-} = \bigvee_{n \in \mathbb{N}} \left(T_2^{(a)}\right)^{-n} W \mathcal{H}_{T_1}^{(a)} = \mathcal{H}_{T_2}^{(a)}.$$

We know that the subspace $ker\, W$ is reducing for $T_1^{(a)}$, and that the restriction $T_1^{(a)} | (ker\, W)^{\perp}$ is unitarily equivalent to $T_2^{(a)}$; see [148]. Hence $\rho(T_1^{(a)}) \supset \rho(T_2^{(a)})$ must be true; see e.g. [6].

The remaining statements are immediate consequences of (a) and (c).

We note that if $\omega, \omega' \subset \mathbb{T}$ are Borel sets such that $\phi \neq \omega \neq w', \omega \subset \omega'$, then $S_1 \overset{i}{\prec} M_{1,\omega} \overset{i}{\prec} M_{1,\dot{\omega}}$ and $\rho(S_1) = \mathbb{T}, \rho(M_{1,\omega}) = \omega, \rho(M_{1,\dot{\omega}}) = \omega'$. Therefore, the relation $T_1 \overset{i}{\prec} T_2$ does not yield $\rho(T_1) \subset \rho(T_2)\; nor\; \rho(T_2) \subset \rho(T_1)$.

Now, we are able to extend the validity of Proposition(3.2.6). We recall that the defect operators of the contraction $T \in \mathcal{L}(\mathcal{H})$ are defined by $D_T := (I - T^*T)^{I/2}$ and $D_{T^*} := (I - TT^*)^{1/2}$. The defect spaces of $T$are $D_T := (ran\, D_T)^{-}$ and $D_{T^*} := (ran D_{T^*})$-; and the defect indices of T are given by $d_T := dim D_T$ and $d_{T^*} := dim\, D_{T^*}$. The characteristic function Θ_T of T, introduced by Sz.-Nagy and Foias in [29], is an $\mathcal{L}(D_T, D_{T^*})$-valued, bounded, analytic function, defined on the open unit disc $\mathbb{D}$ by the formula $\Theta_T(z) := (-T + Z D_{T^*}(1 - ZT^*)^{-1} D_T) D_T$.

Corollary (3.2.10)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{1..}$

(a) Let us assume that $T \in C_{10}$. Then $\pi(T) = \mathbb{T}$is true if $d_T < \infty$, or if Dim $kerT^* < \infty$ and there exists a nonzero $\delta \in H^{\infty}$ such that $\Psi \Theta_T = \delta I$ is fulfilled with some $\mathcal{L}(D_{T^*}, D_T)$ -valued, bounded, analytic function Ψ.

(b) If $T \in C_{11}$, then $\pi(T) = \phi$.

Proof. (a): Under both conditions T is injection-similar to a unilateral shift; the case $d_T < \infty$ was discussed in [161], while the other assumption was considered in [31]. Thus, Propositions (3.2.6) and (3.2.9) yield the statement.
(b): If $T \in C_{11}$, then T is quasisimilar to its unitary asymptote $T^{(a)}$; see [29] and [177]. Hence, Propositions(3.2.6) and (3.2.9) can be applied again.
Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{1..}$ Let us assume that $\mathcal{M}$is a proper invariant subspace of T, and let $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the matrix of T in the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. We know that the unitary asymptote $T^{(a)}$ is unitarily equivalent to the orthogonal sum of the corresponding unitary asymptotes of T_1 and T_2: $T^{(a)} \cong T_1^{(a)} \oplus T_2^{(a)}$) ; see [162] and [177]. Hence $\rho(T) = \rho(T_1) \cup \rho(T_2)$ readily follows.
On the other hand, it is clear that the a.c. contraction $T_1 \in \mathcal{L}(\mathcal{H})$ is of class $C_{1.}$, and that $\pi(T_1) \supset \pi(T)$ is fulfilled. Furthermore, it can be easily verified that $\pi(T) = \pi(T_1) \cap \pi(T_2)$ holds, if $T = T_1 \oplus T_2$. As a consequence, we obtain that if the set $\{\pi(T_1), \pi(T_2), \rho(T_1), \rho(T_2)\}$ is not a singleton, and if $T = T_1 \oplus T_2$, then $\pi(T) \neq \rho(T)$, and so T has a nontrivial hyperinvariant subspace by Theorem (3.2.8).
We note that the contraction T_2 , occurring in the previous triangulation, need not be of class $C_{1..}$ Of course, the definition of quasianalytical spectral set can be easily extended for arbitrary a.c. contractions. Namely, given any a.c. contraction $T \in \mathcal{L}(\mathcal{H})$, let us consider the matrix $T = \begin{bmatrix} T_0 & * \\ 0 & T_1 \end{bmatrix}$ of T in the decomposition $\mathcal{H} = \mathcal{H}_0(T) \oplus \mathcal{H}_0(T)^{\perp}$. Then $T_0 \in C_0, T_1 \in C_{1.}$, and the unitary asymptote $T^{(a)}$ can be identified with $T_1^{(a)}$. Hence $\rho(T) = \rho(T_1)$ and it is natural to work with the definition $\pi(T) := \pi(T_1)$ In particular, if $T \in C_0.$, then T_1 acts on the zero space, and then $\rho(T) := \phi$ and $\pi(T) := \phi$. In the special case, when $T = S_1 \in \mathcal{L}(H^2)$ is the simple unilateral shift and $M = \bigvee_{n=1}^{\infty} \chi^n$ we can infer that $\pi(T) = \pi(T_1) = \mathbb{T}$ and $\pi(T_2) = \phi$, thus $\pi(T) \neq \pi(T_1) \cap \pi(T_2)$.
Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction, and let us consider a nonconstant function $u \in H^{\subset}$ with $\|u\|_{\subset} \leq 1$. Taking into account that any a.c. unitary operator is similar to a $c.n.u.$ contraction (see [156]), we infer by [29] that the operator $u(T)$ is also an a.c. contraction. The spectrum of $u(T)$ was characterized in terms of $\sigma(T)$ and u in [151]. Now, we are going to describe the residual and quasianalytical spectral sets of $u(T)$. Let us introduce the measurable set $\Omega(u) := \{z \in \mathbb{T} : |u(z)| = 1\}$. First of all, we note that if T is of class $C_{*.}$ and $\Omega(u) \cap \rho(T) \neq \phi$ (that is $\Omega(u) \cap \rho(T)$ is of positive measure by our convention), then $u(T)$ is of class $C_{*.}$, as well. Indeed, the transformation
$Q \in \mathcal{L}(\mathcal{H}ran\chi_{\omega}), (T^{(a)}))$, defined by $Qx = \chi_{\omega}(T^{(a)})X_{T^x}$ with $\omega := \Omega(u) \cap \rho(T)$, is nonzero and intertwines T with the unitary operator $T^{(a)} \,|ran\chi_{\omega}(T^{(a)})$.
First, we consider the special case, when u is a Mobius transformation.
Proposition (3.2.11)[31]: If $T \in \mathcal{L}(\mathcal{H})$ is an a.c. contraction and if we are given $u(z) = k(z-a)(1-\bar{a}z)^{-1}$ $(k \in \mathbb{T}, a \in \mathbb{D})$, then $\pi(u(T)) = u(\pi(T))$ and $\rho\big(u(T)\big) = u\big(\rho(T)\big)$;furthermore, $u(T)^{(a)}$ is unitarily equivalent to $u(T^{(a)})$.

Proof. In view of [29] we know that T is of class $C_0.$ ($C_{1.}$) if and only if $u(T)$ is of class $C_{0.}$ ($C_{1.}$- respectively). Considering the canonical triangulation of T, the proof can be easily reduced to the case when T and $A := u(T)$ are of class $C_{1.}$
Let us assume that $\pi(T) \neq \phi$ and that $\varphi_F \chi_u(\pi(T)) \neq 0$ is true, for the sequence $F = \{f_n\}_{n=1}^{\infty} \in D(H^\infty)$. Clearly, $\varphi_F \circ u = \varphi_{F\circ u}$ is valid, where $F \circ u := \{f_n \circ u\}_{n=1}^{\infty} \in D(H^\infty)$. Since $\varphi_{F\circ u}\chi_{\pi(T)} \neq 0$ is fulfilled, it follows that $\mathcal{H}_0(T, F\circ u) = \{0\}$.
In virtue of the equations $f_n \circ u)(T) = f_n(A)$ $(n \in \mathbb{N})$, we infer that $\mathcal{H}_0(A, F) = \mathcal{H}_0(T, F\circ u) = \{0\}$. We obtain that $\pi(A) \supset u(\pi(T))$. Taking into account that $T = v(A)$ is true, with the Mobius transformation $v(z) = \bar{k}(z + ka)(1 + \overline{kaz})^{-1}$, the equality $\pi(\,u(T)) = u(\pi(T))$ follows.
As for the residual sets, the equation $X_T T = T^{(a)} X_T$ implies that $X_T u(T) = u(T^{(a)})X_T$. Applying the universality property ofthe pair $(X_A, A^{(a)})$, we infer that there exists a transformation $Y \in \mathcal{L}(\,A^{(a)}, u(T^{(a)}))$ such th;'lt $X_T = YX_A$. As in the proof of Proposition(3.2.9).(iii), we can deduce that $u(T^{(a)}) \overset{i}{\prec} A^{(a)}$. Oonsidering the previous inverse function v of u, yve obtain in a similar way that $v\big(A^{(a)}\big) \overset{i}{\prec} v(A)^{(a)} = T^{(a)}$, whence $A^{(a)} = u(v(A^{(a)})) \overset{i}{\prec} u(T^{(a)})$ follows. Therefore, the unitary operators $u(T^{(a)})$ and $A^{(a)}$ are unitarily equivalent (see [173]), and so $\rho(u(T)) = u(\rho(T))$.
We are able to extend the transformation law of the quasianalytical spectral sets for a larger class of functions. We shall say that the nonconstant function $u \in H^\infty$ is regular, if (i) $\|u\|_\infty = 1, (ii)\, m(\Omega(u)) > 0, (iii)\, u(w)$ is measurable, whenever $\omega \subset \Omega(u)$ is measurable, and (iv) $m(u^{-1}\,(\omega')) > 0$, provided $\omega' \subset u(\Omega(u))$ is measurable and $m(\omega') > 0$. (If $\omega\, \Omega(u)$ is of positive measure, then $u(M_{1,\omega})$ is a nonzero a.c. unitary operator, and so $m(u(\omega)) > 0$.) We note that a function $u \in H^\infty$, with $\|u\|_\infty = 1, |u(0)\,| < 1$ and $m(\Omega(u)) > 0$, is regular, if u is of bounded variation on $\mathbb{T}$; and that this is the case when u is a Riemann map onto a simple Jordan region with rectifiable boundary. (See [144]).
Theorem (3.2.12)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction such that $\pi(T) \neq \phi$. If $u \in H^\infty$ is a regular function and $\Omega(u) \cap \pi(T) \neq \phi$, then $\pi(u(T)) \supset u(\Omega(u) \cap \pi(T))$.
Proof: In view of Proposition (3.2.11), we can assume that $u(0) = 0,$ that is $u = \chi^v$, with $v \in H^\infty, \|v\|_\infty = 1$. Since the quasianalytical spectral set $\pi(T)$ is nonempty, we know that T is of class $C_{*..}$ Let us consider the triangulation $T = \begin{bmatrix} T_0 & * \\ 0 & T_1 \end{bmatrix}$, where T_0 is of class $C_0.$ and T_1 belongs to $C_{1..}$ Then $u(T)$ is of the form $u(T) = \left[u\begin{pmatrix} u(T_0) & * \\ 0 & u(T_1) \end{pmatrix}\right]$. Since $u(T_0) = v(T_0)T_0$, we can see that $u(T_0) \in C_{0..}$ On the other hand, $\Omega(u) \cap \pi(\,T_1) = \Omega(u) \cap \pi(T) \neq \phi$implies that $\varphi_{F_u}\chi_{\pi(T_1)} \neq 0$ is true for the sequence $F_u = \{u^n\}_{n=1}^{\infty} \in D(H^\infty)$, and so $u(T_1)$ is of class $C_{1..}$ Therefore, we may assume that T and $u(T)$ are a.c. contractions belonging to $C_{1..}$ Now, the inclusion $u(\Omega(u) \cap \pi(T)) \subset \pi(u(T))$ can be verified as in the proof of (3.2.11).
We proceed with the description of $\rho(\,u(T)),$ for an arbitrary $u \in H^\infty$. We shall need the following Lemma.

Lemma (3.2.13)[31]: Let $T \in \mathcal{L}(\mathcal{H})$be an a.c. contraction of class $C_{*.}$, and let us assume that the functions $f, g, h \in H^{\infty}$ satisfy the conditions
$\|f\|_{\infty} = \|g\|_{\infty} = 1, |f(0)| < 1, h = fg$ and $\Omega(h) \cap \pi(T) \neq \phi$. Let us consider the a.c. contractions $B = f(T)$ and $C = h(T)$ of class $C_{*..}$ Then, there exists a unique operator $B_C \in \{C^{(a)}\}'$ such that $X_C B = B_C X_C$; furthermore, B_C is an a.c. isometry.

Proof. The existence of a unique $B_C \in \{C^{(a)}\}'$ satisfying the condition $X_C B = B_C X_C$, follows from the universality property of $(X_C, c^{(a)})$; see [10, Section II]. For any $n \in \mathbb{N}$, the relations $h^{n+1} \overset{a}{\prec} h^n f \overset{a}{\prec} h^n$ imply $h^{n+1}(T) \overset{a}{\prec} h^n(T) f(T) \overset{a}{\prec} h^n(T)$.

Tending n to infinity, we obtain that $\|X_C x\| \leq \|X_C B x\| \leq \|X_C x\|$ is true, for every $x \in \mathcal{H}$. Thus, given any vector $y = (C^{(a)})^{-k} X_C x$ $(k \in \mathbb{N}, x \in \mathcal{H})$, we infer that

$$\|B_{Cy}\| = \left\|\left(c^{(a)}\right)^k B_{Cy}\right\| = \|B_C X_C x\| = \|X_C B x\| = \|X_C x\| = \|y\|.$$

Since the set $\cup_{k \in \mathbb{N}} (C^{(a)})^{-k} X_C \mathcal{H}$ is dense in $\mathcal{H}_C^{(a)}$), it follows that B_C is an isometry.

Let us consider the decomposition $\mathcal{H}_C^{(a)} = M_a \oplus M_s$, reducing for B_C, such that $B_C | M_a$ is an a.c. isometry and $B_C | M_s$ is a singular unitary operator. Let $P_s \in \mathcal{L}(\mathcal{H}_C^{(a)})$ denote the orthogonal projection ontoM_s. Since B is an a.c. contraction and $(P_s X_C) B = P_s B_C X_C = B_C (P_s X_C)$, we infer that $P_s X_C = 0$, that is M_a contains the subspace $\mathcal{H}_{C,+}^{(a)}$. Taking into account that M_a is hyperinvariant for B_C, and that the operators $c^{(a)}, (c^{(a)})^{-1}$ commute with B_C, we obtain that $M_a = \mathcal{H}_C^{(a)}$, and so the isometry B_C is a.c.

To formulate the transformation law for the residual sets we introduce some notation. Given a set $\omega \in \mathbb{T}$ of positive measure and a unimodular measurable function $h: \omega \to \mathbb{T}$, the properly essential range of h is defined by $pe - ranh := \{z \in \mathbb{T} : \lim_{r \to 0+} m\ (h^{-1}(D(z,r))) r^{-1} > 0\}$ where $D(z,r) := \{(\zeta \in \mathbb{T} : |\zeta - z| < r\}$. We note that if the Borel measure μ on $\mathbb{T}$, given by $\mu(\omega') := m(h^{-1}(\omega'))$ $(\omega' \subset \mathbb{T})$, is absolutely continuous, then μ is equivalent to the measure $\chi_\omega dm$, where $\omega = pe - ranh$; see [178]. If $u \in H^{\infty}$ is a regular function, then $pe - ran(u|\omega) = u(\omega)$ is true, for any $\omega \in B_1$, included in $\Omega(u)$. (Note that in the latter case both $\chi_{u(\omega)} dm$ and $m \circ (u|\omega)^{-1}$ are scalar spectral measures of the a.c. unitary operator $u\left(M_{1,\omega}\right)$.)

Theorem (3.2.14)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{*.}$, and let $u \in H^{\infty}$ be a function satisfying the conditions $\|u\|_{\infty} = 1, |u(0)| < 1$ and $\Omega(u) \cap \rho(T) \neq \phi$. Then, the unitary asymptote $A^{(a)}$ of the a.c. contraction $A = u(T)$ is unitarily equivalent to the restriction of the normal operator $u(A^{(a)})$ to its hyperinvariant subspace $ran\chi_\omega(T^{(a)})$, where $\omega = \Omega(u) \cap \rho(T)$. Therefore, $\rho\left(u(T)\right) = pe - ran(u|\omega)$;in particular, if u is regular, then $\rho(u(T)) = u(\Omega(u) \cap \rho(T))$.

Proof. In view of Proposition (3.2.11), we may assume that $u(0) = 0$. Hence u is of the form $u = \chi^{v}$, where $v \in H^{\infty}$ and $\|v\|_{\infty} = 1$. Applying Lemma (3.2.13) with $\chi . v, u$ in place of f, g, h, respectively, we obtain that there exists an a.c. isometry $T_A \in \{A^{(a)}\}'$ such that $X_A T = T_A X_A$. In virtue of the equations $A^{(a)} X_A = X_A A = X_A u(T) = U(T_A) X_A$ we infer that the subspace $ker(A^{(a)} - u(T_A))$ − which is reducing for $A^{(a)}$ - contains the subspace $\mathcal{H}_{A,+}^{(a)}$. Thus $ker(A^{(a)} - U(T_A)) = \mathcal{H}_A^{(a)}$, and so $A^{(a)} = u(T_A)$.

Assuming that the isometry T_A is not unitary, the Wold-decomposition of T_A yields that $T_A \cong U \oplus S_n$, where U is an a.c. unitary operator and $n \in \mathbb{N}_{\infty}$. Since $A^{(a)} = u(T_A) \cong$

$u(U) \oplus u(S_n)$ and since $u(S_n)$ is a $c.n.u.$ contraction, we arrive at a contradiction. Therefore, T_A must be an a.c. unitary operator.

In view of the universality of $(X_A, T^{(a)})$, there exists a mapping $Y \in \mathcal{L}\left(T^{(a)}, T_A\right)$such that $Y\,X_T = X_A$. We can easily verify that Y has dense range; see the proof of Proposition (3.2.9).

(iii). Furthermore, the equations $Yu\left(T^{(a)}\right) = u(T_A)Y = A^{(a)}Y$imply that $A^{(a)} \cong u(T^{(a)})\;|ker\;Y)^{\perp}$.

Since the restriction $u(T^{(a)})\;|ker\;Y)^{\perp}$. is unitary, we can see that the subspace $(ker\;Y)^{\perp}$. is contained in the spectral subspace $M_\omega := ran\;\chi_\omega\;(T^{(a)})$, where $\omega := \Omega(u) \cap \rho(T)$. Let $Q\;\in \mathcal{L}(\mathcal{H}_T^{(a)})$ denote the orthogonal projection onto M_ω , and let $R \in\;\mathcal{L}\;\;(T, T^{(a)}|M_\omega\;)$ be defined by $Rx :=\;QX_{T^x}\;(x \in \mathcal{H})$. Taking into account that $u(T^{(a)})|M_\omega$ is unitary and that $R \in \mathcal{L}\;\;(A, u(T^{(a)})|M_\omega)$, we infer by the universality of $(X_A, A^{(a)})$ that there exists a transformation $Z \in \mathcal{L}\;\;\;(A^{(a)}\;, u(T^{(a)})|M_\omega)$ such that $R\;=\;ZX_A$. Clearly,$ran\;Z \supset ran\;R$ and

$$\bigvee_{n=1}^{\infty}\left(T^{(a)}\right)^{-n}R\mathcal{H} = \bigvee_{n=1}^{\infty}Q\left(T^{(a)}\right)^{-n}X_T\mathcal{H} = \left(Q\mathcal{H}_T^{(a)}\right)^{-} = M_\omega$$

Since $(ran\;Z)^{-}$ is reducing for $u(T^{(a)}|M_\omega) = u(T^{(a)})\;|M_\omega$, so it is for $T^{(a)}\;|M_\omega$ as well. Thus, the transformation Z must have dense range, and so we conclude that $u(T^{(a)})|M_\omega\;\;\cong\;\;A^{(a)}|(ker\;Z)^{\perp}$. Now, an application of [153] results in that the operators $A^{(a)}$ and $u(T^{(a)})\;|M_\omega$ are unitarily equivalent.

It is known that the scalar spectral measure of the a.c. unitary operator $u(T^{(a)})\;|M_\omega$ is $\mu \circ (u|\omega)^{-1}$, where $\mu = \chi_\omega\,dm$; see e.g. [6].

Thus, we conclude that $\rho(u(T)) = \rho(u(T^{(a)})|M_\omega) = pe - ran(u|\omega)$.

As an immediate consequence of Theorems (3.2.7),(3.2.12), (3.2.14) and Proposition(3.2.6), we obtain the following

Corollary (3.2.15)[31]: If $T \in \mathcal{L}(\mathcal{H})$ is an a.c. contraction satisfying the condition $\pi(T) = \rho(T) \neq \phi$, then $\pi(u(T)) = \rho(u(T)) =\;u(\Omega(u) \cap \rho(T))$ is true, for any regular function $u \in H^\infty$ such that $\Omega(u) \cap \rho(T) \neq \phi$. In particular, if $T_u\;=\;u(S_1 \in \mathcal{L}(H^2)$ is the analytic Toeplitz operator with symbol u, and $u \in H^\infty$is regular, then $\pi(T_u) = \rho(T_u) = u(\Omega(u))$.

We note here that, in contrast with Proposition(3.2.3), the condition $\varphi_F\chi_\pi(T) =\;0$ does not imply $\mathcal{H}_0(T, F) \neq \{0\}$. Indeed, in view of Corollary(3.2.15), we can find a.c. contractions A, B of class $C_{1.}$-, such that the sets $\pi(A) \cap \pi(B), \pi(A)\backslash\pi(B)$ and $\pi(B)\backslash \pi(A)$ are of positive measure. Let us consider the orthogonal sum $T = A \oplus B$; we know that $\pi(T) = \pi(A)\;\cap \pi(B)$ and $\rho(T) = \rho(A) \cup \rho(B)$. Let $f \in H^\infty$ be an outer function such that If $|f| =\;(1/2)\chi_{\pi(T)} + \chi_{\mathbb{T}}\backslash\pi(T)$, and let us form the sequence $F = \{f^n\}_{n=1}^{\infty} \in D(H^\infty)$. Since $\varphi_F = \chi_{\mathbb{T}}\backslash\pi(T)$, we obtain that $\varphi_F\chi_{\pi(T)} = 0$ and

$$\mathcal{H}_0(T, F) = \mathcal{H}_0(A, F) \oplus \mathcal{H}_0(B, F) = \{0\}.$$

We are going to show that the property of an a.c. contraction T being symptotically strongly nonvanishing on a set $\alpha \in B_1$ is equivalent to a quasianalytic behaviour of T on α . To be more precise, let us introduce some notation. Given an a.c. unitary operator $U \in \mathcal{L}(K)$, we know that $\chi_{\rho(U)}dm$ is a scalar spectral measure of U. Let E denote the spectral measure of U. For any vector $x \in K$, there exists a unique Borel set $\omega(U, x) \subset \mathbb{T}$

such that the measure $\chi_{\rho(U,x)}dm$ is equivalent to the localization E_x of $E\ at\ x$. Clearly, $\omega(U,x) \neq \phi$ exactly when $x \neq 0$. Furthermore, the vector x is cyclic for the commutant of the restriction of U to the spectral subspace $ran\chi_{\omega(U,x)}(U)$, that is $V\{Cx: C\ \{U\}'\} = ran\chi_{\omega(U,x)}\ (U)$.

Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{1.}$,-, and let $\alpha \in B_1$. We say that the contraction T is quasianalytic on the set α, if $\omega(T^{(a)}, X_{T^x}) \supset \alpha$ is fulfilled, for any nonzero vector $x \in \mathcal{H}$.

Theorem (3.2.16)[31]: Given an a.c. contraction $T \in \mathcal{L}(\mathcal{H})$ of class $C_{1.}$ and a Borel set $\alpha \in B_1$, the following conditions are equivalent:

(a) $T \in C_{1.}(\alpha)$,

(b) T is quasianalytic on α.

Proof. $(a) \Longrightarrow (b)$: Let us assume that the contraction T is not quasianalytic on α. Then, there exists a nonzero vector $x_0 \in \mathcal{H}$ such that the set $\omega := \alpha \backslash \omega\left(T^{(a)}, X_T x_0\right)$ is of positive measure. Let us consider a sequence $F = \{f_n\}_{n=1}^{\infty} \in D(H^{\infty})$, with limit function $\varphi_F = \chi_\omega$.Since

$$\lim_{n\to\infty}\|X_T f_n(T)x_0\| = \lim_{n\to\infty}\left\|f_n T^{(a)} X_T x_0\right\| = \left\|\chi_\omega\left(T^{(a)}\right)X_T x_0\right\| = 0$$

we can choose an increasing mapping $k: \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty}\left\|T^{k(n)} f_n(T)x_0\right\| = 0$ see the proof of Theorem(3.2.7). Taking into account that $\varphi_G\chi_\alpha = \chi_\omega \neq 0$ is true for the sequence $G = \{g_n = \chi^{k(n)} f_n\}_{n=1}^{\infty} \in D(H^{\infty})$, we conclude that the contraction T is not of class $C_{1.}(\alpha)$.

$(b) \Longrightarrow (a)$: Assuming that T is quasianalytic on α , let us consider a sequence $F = \{f_n\}_{n=1}^{\infty} \in D(H^{\infty})$, such that $\varphi_F\chi_\alpha \neq 0$. Let E denote the spectral measure of $T^{(a)}$. Given any nonzero vector $x \in \mathcal{H}$, we know that the measure $E_{X_{T^x}}$ is of the form $E_{X_{T^x}} = g_x dm$, where $g_x(z) > 0$ holds for a.e. $z \in \alpha$. Thus, we have

$$\left\|\Phi_{F(T)}x\right\|^2 \geq \|X_T\|^{-2}\int_{\mathbb{T}}|\varphi_F|^2\, g_x dm > 0$$

and so T is asymptotically strongly nonvanishing on α.

In view of this theorem, the a.c. $C_{1.}$-contraction T will be called quasianalytic if $\rho(T) = \pi(T)$.

We note that Theorems ((3.2.7) and(3.2.8) can be also derived from Theorem (3.2.16). For example, if $\pi(T) \neq \rho(T)$, then by Theorem(3.2.16) there exists a nonzero vector $x_0 \in \mathcal{H}$ such that the set $\omega := \rho(T)\backslash\omega(T^{(a)}, X_T x_0)$ is of positive measure. Since the nonzero hyperinvariant subspace $M = \{Cx_0: C \in \{T\}'\}$ - is transformed into the subspace $ran\chi_{\rho(T)\backslash\omega}(T^{(a)})$ by X_T , it follows that M is a proper hyperinvariant subspace of T. We mention that if T is of class C_{11}, , then to every spectral subspace of $T^{(a)}$ there corresponds a hyperinvariant subspace of T; see [29] and [10]. Existence of infinitely many disjoint nontrivial hyperinvariant subspaces of nonquasianalytic type was proved also in [30].

We close this section by posing the following problem.

Question (3.2.17)[31]: Is it true that the unitary asymptote $T^{(a)}$has uniform spectral multiplicity on the quasianalytical spectral set $\pi(T)$, for any a.c. contraction T of class $C_{1.}$? Since T is quasianalytic on $\pi(T)$, we have some evidence to guess that the answer is positive. An affirmative answer to Question(3.2.17) would imply that every a.c.

contraction T of class $C_{1.}$ has a nontrivial hyperinvariant subspace, provided the spectral multiplicity function of the unitary asymptote $T^{(a)}$ is not constant on $\rho(T)$.
Applying Theorem(3.2.16), we are able to prove the following statement.
Theorem (3.2.18)[31]: Let $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$ be a.c. contractions of class $C_{1..}$
(a) If $\mathcal{L}(A, B) \neq \{0\}$, then $\rho(A) \supset \pi(B)$.
(b) If $\pi(A) = \rho(A)$ and $\pi(B)\backslash\pi(A) \neq \phi$, then $\mathcal{L}(A, B) = \{0\}$.
Proof. (a): Let $Y \in \mathcal{L}\ (A, B)$ be a nonzero transformation. By the universality of $(X_A, A^{(a)})$, there exists a transformation $Z \in \mathcal{L}(A^{(a)}, B^{(a)})$ such that $X_B Y = ZX_A$.
Let $x_0 \in \mathcal{H}$ be a vector such that $Yx_0 \neq 0$. Since the contraction B is quasianalytic on the set $\pi(B)$, we know that $\omega(B^{(a)}, X_B Y x_0) \supset \pi(B)$. Taking into account that $X_B Y x_0 = ZX_A x_0$, we infer by [148] that $\rho(A) \supset \omega(B^{(a)}, X_B Y x_0)$.
Statement (b) is an immediate consequence of (a).
As a first application, we prove the following proposition, establishing connection between the commutant of a contraction and its n-th power.
Proposition (3.2.19)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{1.}$, and let us assume that $\pi(T) = \rho(T) = \alpha_n$, where $\alpha_n := \{e^{it} : 0 \leq t \leq 2\pi/n\}, n \in \mathbb{N}$. Then, for the a.c. $C_{1.}$ -contraction T^n, we have $\pi(T^n) = \rho(T^n) = \mathbb{T}$ and $\{T\}' = \{T^n\}'$.
Proof. The equation $\pi(T^n) = \rho(T^n) = \mathbb{T}$ follows from Corollary(3.2.15). Setting $\curlywedge_n := e^{i2\pi/n}$, we know that $\pi(\curlywedge_n^j\, T) = \curlywedge_n^j\, \pi(T) \neq \pi(T)$ is true, for every $0 < j < n, j \in \mathbb{N}$. We infer by Theorem(3.2.18) that $\mathcal{L}\big(T, \curlywedge_n^j\, T\big) = \mathcal{L}\ (\curlywedge_n^j\ T, T) = \{0\}$ holds, whenever $0 < j < n$. Thus, a result of Cowen yields that $\{T^n\}' = \{T\}'$; see [145].
In view of Proposition(3.2.19) and Theorem(3.2.8), the hyperinvariant subspace problem for a.c. $C_{1.}$-contractions, with an arc on $\mathbb{T}$ as a residual set, has an affirmative answer if it has positive answer for a.c. $C_{1.}$ -contractions satisfying the condition $\pi(T) = \rho(T) = \mathbb{T}$. This latter situation seems to be more tractable, since the assumption $\rho(T) = \mathbb{T}$ implies existence in abundance of invariant subspaces, where T is similar to the simple unilateral shift $S_{1.}$; see [28]. Analogous statements of reductive type were proved in [8].
Let us assume now that the a.c. $C_{1.}$-contraction $T \in \mathcal{L}(\mathcal{H})$ is cyclic. We know from [162] that the commutant of T is abelian, that is $\{T\}' = \{T\}''$, and that the adjoint T^* is cyclic, as well. Alternative proofs of these facts can be given in the following way. Since T is cyclic and the transformation $X_T^+ \in \mathcal{L}(T, T_+^{(a)})$ has dense range, it follows that the isometry $T_+^{(a)}$ is cyclic; hence $\{T_+^{(a)}\}'$ is abelian.
By the universality of $(X_T^+, T_+^{(a)})$ (see [10]), there exists an injective algebra-homomorphism $\gamma_+ : \{T\}' \to \{T_+^{(a)}\}'$ such that $X_T^+ C = \gamma_+(C) X_T^+$ is true, for every $C \in \{T\}'$. Thus, the commutativity of $\{T\}'$ is implied by that of $\{T_+^{(a)}\}'$.
On the other hand, if the isometry $T_+^{(a)}$ is cyclic, then so is its adjoint; and since $(T_+^{(a)})^* \prec T^*$, we obtain that T^* is cyclic. For the characterization of cyclic C_{11} -contractions, see [155, Theorem 15].
It is known that commutativity of $\{T\}'$ does not imply cyclicity, in general;
see [147]. In view of Theorem (3.2.18) we can easily provide a large class of noncyclic $C_{1.}$-contractions with abelian commutant.

Example (3.2.20)[31]: Let $A_j \in \mathcal{L}(\mathcal{H}_j), j \in \mathbb{N}$, be a sequence of cyclic a.c. contractions of class $C_{1.}$ such that $\pi(A_j) = \rho(A_j)$ is true for every $j \in \mathbb{N}$, $\pi(A_j)\backslash\pi(A_k) \neq \phi$ whenever $j \neq k$,and $\bigcap_{j\in\mathbb{N}} \rho\left(A_j\right) \neq \phi$. (Corollary(3.2.15) ensures the existence of such a sequence.) Let us form the orthogonal sum $A = \sum_{j=1}^{\infty} \oplus A_j$. In virtue of Theorem(3.2.18), the commutant of A splits into the direct sum of the commutants of the operators $A_j : \{A\}' = \sum_{j=1}^{\infty} \oplus \{A_j\}'$; thus $\{A\}'$ is abelian. On the other hand, the condition $\bigcap_{j\in\mathbb{N}} \rho\left(A_j\right) \neq \phi$ readily implies that the multiplicity of A is infinite.

We say that the a.c. $C_{1.}$-contraction T is a quasiunitary operator, if the canonical intertwining mapping $X_T \in \mathcal{L}(T, T^{(a)})$ has dense range. These operators are characterized in the following proposition. We recall that Θ_T stands for the characteristic function of T, and that $\widetilde{\Theta}_T(z) := \Theta_T(\bar{z})^*$ $(z \in \mathbb{D})$.

Proposition (3.2. 21)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{1..}$. Then the following conditions are equivalent:

(a) T is quasiunitary,

(b) $ker\, \widetilde{\Theta}_T \cap H^{\infty}(D_{T^*}) = \{0\}$,

(c) $\Phi\Theta_T = 0$ implies $\Phi = 0$, for any bounded, analytic, $\mathcal{L}(D_{T^*}, \varepsilon)$-valued function Φ,

(d) $\mathcal{L}(T, S_n) = \{0\}$ is true, for every $n \in \mathbb{N}_\infty$,

(f) $\mathcal{L}(T, S_1 = \{0\}$.

Proof. The implications $(b) \Longrightarrow (c)$ and $(d) \Longrightarrow (f)$ are trivial. For the equivalenceof (a) and (b), see [157]. It was shown in [164] that condition (c) implies (d). Finally, if T is not quasiunitary then $T_+^{(a)}$ is a nonunitary isometry containing S_1 on a reducing subspace, and so (f) implies (a).

We note that the assumption $T \in C_{1.}$ yields that the characteristic function Θ_T is $*$-outer, that $is\, (\widetilde{\Theta}_T H^2(D_{T^*}))^- = H^2(D_T)$. The conditions (b) and (c) express injectivity properties of $\widetilde{\Theta}_T$.

If the a.c. $C_{1.}$-contraction T is not quasiunitary, then it follows by Proposition (3.2.21) that $\mathcal{L}(T, S_1 \neq \{0\}$. Taking into account that the restriction of S_1 to any of its nonzero invariant subspaces is unitarily equivalent to S_1, we obtain that $T \overset{d}{\prec} S_1$. Thus $S_1^* \overset{i}{\prec} T^*$, and so the point spectrum $\sigma_p(T^*)$ of T^* covers the open unit $disc\ \mathbb{D}$.

We conclude that $(ker(T^* - \lambda I))^\perp$ is a nontrivial hyperinvariant subspace of T, for any$\lambda \in \mathbb{D}$. Therefore, the hyperinvariant subspace problem for $C_{*.}$-contractions can be reduced to the case, when T is a quasiunitary operator.

It can be easily seen that the a.c. $C_{1.}$-contraction T is quasiunitary, if the residual set $\rho(T)$ does not cover the unit circle $\mathbb{T}$. Indeed, the condition $\rho(T) \neq \mathbb{T}$ implies that the unitary asymptote $T^{(a)}$ is reductive, that is $LatT^{(a)} = Lat(T^{(a)})^*$, and so the transformation X_T has dense range. It may even happen that such T is of class C_{10}; see [25]. Now, we exhibit an example for a quasiunitary C_{10}-contraction T, with the property $\pi(T) = \rho(T) = \sigma(T) = \mathbb{T}$.

Example (3.2.22)[31]: Let $w: \mathbb{Z} \to [1, \infty)$ be a dissymetric weight, considered by Esterle in [149]; that is w is a decreasing sequence such that $limsup_{n\to-\infty}\, w(n-1)/w(n) < \infty$, $lim_{n\to-\infty}\, w(n)^{1/|n|} = 1$ and $w(n) = 1$, for every $n \geq 0$. Let us assume that w is submultiplicative and quasianalytic, the latter meaning that $\sum_{n=1}^{\infty}(1ogw(-n))\, n^{-2} = \infty$. (For concrete examples, see [149].

Let us consider the Hilbert space

$$L^2(w) := \left\{f \in L^2(\mathbb{T}) : \|f\|_w^2 := \sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|^2 w_n^2 < \infty\right\},$$

and the operator $T_w \in \mathcal{L}(L^2(w))$, defined by $T_w f := \chi f$. The equations$(T_w f)^{\wedge}(n) = \hat{f}(n-1)$ $(n \in \mathbb{Z})$ are valid for the Fourier coefficients, so T_w is a weighted bilateral shift. It is easy to see that T_w is a contraction of class C_{10}. Furthermore, [149] imply that $\sigma(T_w) = \mathbb{T}$ and that $f(z) \neq 0$ is true a.e. on $\mathbb{T}$ for any function $0 \neq f. \in L^2(w)$.

The pair $(X_{T_w}, T_w^{(a)})$ is equivalent to the pair $\left(X_0, T_{w_0}\right)$, where $w_0 \equiv 1$ and $X_0 : L^2(w) \to L^2, f \mapsto f$ is the natural embedding; that is there exists a unitary transformation $Z \in \mathcal{L}\,(T_{w_0}, T_w^{(a)})$ such that $X_{T_w} = ZX_0$. Thus, we infer that T_w is quasianalytic on $\mathbb{T}$, and so $\pi(T_w) = \mathbb{T}$ holds by Theorem (3.2.16). Taking into account that the trigonometric polynomials are contained in $L^2(w)$, we obtain that T_w is a quasiunitary operator.

The invariant subspace problem is open for quasiunitary operators, in general.

On the other hand, it is known from [165] that if the residual set $\rho(T)$ covers the unit circle, then the quasiunitary operator T has disjoint nontrivial invariant subspace.

Let us assume that the a.c. $C_{1.}$-contraction $T \in \mathcal{L}(\mathcal{H})$ is cyclic. Since the commutant $\{T\}'$ is abelian, we know that the subspaces $(ranQ)^-$ and $\ker Q$ belong to $Hlat\, T$, for every operator $Q \in \{T\}'$. The next proposition claims that the nullspaces are all trivial, if the contraction T is quasianalytic. Furthermore, it is sufficient to assume only the cyclicity of $T^{(a)}$, which is a slighter condition than the cyclicity of T; see [163].

Proposition (3.2.23)[31] : Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{1.}$-, and let us assume that $\pi(T) = \rho(T)$ and that the unitary assymptote $T^{(a)}$ is cyclic. Then, every nonzero operator $Q \in \{T\}'$ is injective.

Proof. Let $Q \in \{T\}'$ be an operator with $kerQ \neq \{0\}$. Let $R \in \{T^{(a)}\}'$ be the uniquely determined operator, satisfying the condition $X_T\, Q = RX_T$. Since the unitary operator $T^{(a)}$ is cyclic, there exists a function $\psi \in \chi_{\rho(T)}L^{\infty}$ such that $R = \psi(T^{(a)})$; see [6]. Taking into account that X_T is injective and that $X_T(kerQ) \subset ker\, R$, we infer that the set $\omega := \{z \in \rho(T) : \psi(z) = 0\}$is of positive measure. Thus, $\omega(T^{(a)}, X_T Q x) = \omega(T^{(a)}, \psi(T^{(a)})X_T x) \subset \rho(T)\backslash\omega \neq \rho(T)$ is fulfilled, for every vector $x \in \mathcal{H}$. Since the contraction T is quasianalytic, we conclude that $Q = 0$.

A. Atzmon posed us his conjecture that there exists a nonzero operator $Q \in \{T\}'$ with nondense range, for every cyclic a.c. contraction $T \in \mathcal{L}(\mathcal{H})$ of class $C_{10.}$ Verification of that statement would solve the hyperinvariant subspace problem for cyclic a.c. contractions of class $C_{*..}$ It is natural to start the quest for an appropriate Q in the set of functions of T. It is known that if $v \in H^{\infty}$ is an outer function, then the operator $v(T)$ is a quasiaffinity, see [29].

Hence, we should concentrate on the class of operators $u(T)$, where $u \in H^{\infty}$ is an inner function. For any operator $Q \in \mathcal{L}(\mathcal{H})$, let $\gamma(Q) := inf\{\|Qx\| : x \in \mathcal{H}, \|x\| = 1\}$ denote the lower norm of Q; in the case $\mathcal{H} = \{0\}$, let $\gamma(Q) := 1$. It is known that $\gamma(Q^*) = 0$ is valid exactly when ran $Q \neq \mathcal{H}$. We are going to give a sufficient condition for

$\gamma(u(T)^*) = 0$ in terms of u and the characteristic function Θ_T of T. To that end we introduce the quantity $\eta_*(u,T) := inf\ \{|u(\lambda)| + \gamma(\Theta_T(\lambda)^*): \lambda \in \mathbb{D}\}$.

Lemma (3.2.24)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction of class $C_{*.}$, and let $u \in H^\infty$ be an inner function. If $\eta_*(u,T) = 0$, then $\gamma(u(T)^*) = 0$.

Proof. We shall apply the technique, introduced in [150]. First of all, we may assume that the contraction T is $c.n.u$.. We know from the Sz.-Nagy-Foias model theory of contractions that T is unitarily equivalent to the operator $S(\Theta_T)$ defined on the space $\mathcal{H}(\Theta_T) := K_+ \ominus \{\Theta_T w \oplus \Delta_T w: w \in H^2(D_T)\}$ as the compression $S(\Theta_T) := P_{\mathcal{H}(\Theta_T)} U_+|\mathcal{H}(\Theta_T)$ where $K_+ := H^2(D_{T^*}) \oplus (\Delta_T L^2(D_T))^-, \Delta_T := (I - \Theta_T^*\Theta_T)^{1/2}$ and $U_+ \in \mathcal{L}(K_+)$ is the operator of multiplication by χ. Hence, we can work with $S(\Theta_T)$ instead of T. For short, let us write $\hat{S} = S(\Theta_T), \widehat{\mathcal{H}} = \mathcal{H}(\Theta_T)$, $\ominus = \Theta_T, \Delta = \Delta_T, D = D_T$ and $D_* = D_T$. We note that $T \in C_{*.}$ implies that $D_* \neq \{0\}$.

Given any$\lambda \in \mathbb{D}\}$ and $0 \neq x \in D_*$, let us consider the vector-valued function $E_{\lambda,x} \in H^2(D_*) \subset K_+$, defined by $E_{\lambda,x}(z) := (1 - |\lambda|^2)^{\frac{1}{2}}(1 - \bar{\lambda}Z)^{-1}x\ , z \in \mathbb{D}^-$. Forming power series expansion, we can see that $\|E_{\lambda,x}\|^2 = \|x\|^2$ and $U_+^*E_{\lambda,x} = \bar{\lambda}E_{\lambda,x}$. Let us introduce the projection $F_{\lambda,x} := P_{\widehat{\mathcal{H}}}E_{\lambda,x} \in \widehat{\mathcal{H}}$. A straightforward computation yields that $F_{\lambda,x} = E_{\lambda,x} - G_{\lambda,x}$, where $G_{\lambda,x} = \Theta\Theta(\lambda)^*\ E_{\lambda,x} \oplus \Delta\Theta(\lambda)^* E_{\lambda,x}$. Thus, we have

$$u(\hat{S})^*\ F_{\lambda,x} = P_{\widehat{\mathcal{H}}}u(U_+)^*\ F_{\lambda,x} = P_{\widehat{\mathcal{H}}}\hat{u}(U_+^*)\ E_{\lambda,x} - P_{\widehat{\mathcal{H}}}\hat{u}(U_+^*)G_{\lambda,x}$$
$$= P_{\widehat{\mathcal{H}}}\overline{u(\lambda)}\ E_{\lambda,x} - P_{\widehat{\mathcal{H}}}\hat{u}(U_+^*)G_{\lambda,x} = \overline{u(\lambda)}F_{\lambda,x} - P_{\widehat{\mathcal{H}}}\hat{u}(U_+^*)G_{\lambda,x}$$

whence

$$\|u(\hat{S})^*\ F_{\lambda,x}\| \le |u(\lambda)|\|F_{\lambda,x}\| + \|G_{\lambda,x}\| = |u(\lambda)|\|F_{\lambda,x}\| + \|\Theta(\lambda)^*x\|$$

follows. Taking into account that $\|F_{\lambda,x})\|^2 = \|x\|^2 - \|\Theta(\lambda)^*x\|^2 > 0$, we obtain that

$$\gamma(u(\hat{S})^*) \le |u(\lambda)| + \frac{\|\Theta(\lambda)^*x\|}{(\|x\|^2 - \|\Theta(\lambda)^*x\|^2)^{1/2}}$$

In view of this inequality, we can easily verify that $\eta_*(u,T) = 0$ implies $\gamma(u(T)^*) = \gamma(u(\hat{S})^*) = 0$. The following theorem claims that the range of $u(T)$ is not dense for some inner function u, if the characteristic function Θ_T satisfies some boundary conditions.

Theorem (3.2.25)[31]: Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction of class $C_{10,}$, and let us assume that there exist $z_0 \in \rho(T)$ and $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{D}$ such that $\Theta_T(z_0)$ is a nonunitary isometry, $\lim_{n\to\infty}\lambda_n = z_0$, the sequence $\{\Theta_T(\lambda_n)^*\}_{n=1}^\infty$ converges to $\Theta(z_0)^*$ in the strong operator topology, and $\lim_{n\to\infty}\gamma(\Theta_T(\lambda_n)) = 1$. Then, there exits an inner function $u \in H^\infty$ such that the nonzero operator $u(T)$ has nondense range.

We note that the assumption $T \in C_{10}$ yields for a.e. $z \in \rho(T)$ that $\Theta_T(z)$ is a nonunitary isometry, and that $\{\Theta_T(\lambda_n)^*\}_{n=1}^\infty$ converges strongly to $\Theta(z_0)^*$ whenever λ_n tends to z nontangentially; see [29, Sections $V.2, VI.3$]. Furthermore, if $\Theta_T(z)$ is a nonunitary isometry, then $\gamma(\Theta_T(z)) = 1$ and $\gamma(\Theta(z_0)^*) = 0$.

Proof. Turning to a suitable subsequence, if necessary, we can assume that $\sum_{n=1}^\infty(1 - |\lambda_n|) < \infty$ and that $\lambda_n \neq 0, \gamma(\Theta_T(\lambda_n)) > 1 - 2^{-n}$ are true, for every $n \in \mathbb{N}$. Setting

$$u_n(z) := \frac{\overline{\lambda_n}}{|\lambda_n|}\frac{\lambda_n - z}{1 - \overline{\lambda_n}z} \quad (n \in \mathbb{N}),$$

let us form the Blaschke product $u = \prod_{n=1}^{\infty} u_n$. Let x_0 be a unit vector in $ker\Theta_T(z_0)^*$. Since $\lim_{n\to\infty}\|\Theta_T(\lambda_n)^* x_0\| = \|\Theta_T(z_0)^* x_0\| = 0$ and $u(\lambda_n) = 0 (n \in \mathbb{N})$, we can see that $\eta_*(u, T) = 0$, and so $\gamma(u(T)^*) = 0$ by Lemma (3.2.24). It is clear by [29] that, for any $n \in \mathbb{N}$, we have

$$\gamma\big(u_n(T)\big) = \gamma(T - \lambda_n I)\big(I - \overline{\lambda_n} T\big)^{-1} = \gamma(\Theta_T(\lambda_n)) \geq 1 - 2^{-n}$$

Let us form the partial products $v_N := \prod_{n=1}^{N} u_n$, $N \in \mathbb{N}$. Then

$$\gamma(v_N(T)) \geq \prod_{n=1}^{N} \gamma\,(u_n(T)) \geq \prod_{n=1}^{\infty} (1\, -\, 2^{-n}) =: c > 0$$

holds, for every $N \in \mathbb{N}$. We can select a subsequence $\{u_{N_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} u_{N_k}(z) = u(z)$ is true, for a.e. $z \in \mathbb{T}$ (see [144]). Now, we infer by [29] that the operators $\{u_{N_k}(T)\}_{k=1}^{\infty}$converge to $u(T)$ in the strong operator topology, and so $\gamma(\, u(T)) \geq c > 0$ must be also true.

The relations $\gamma(u(T)) > 0$ and $\gamma(u(T)^*) = 0$ imply that the nonzero operator $u(T)$ has closed range, which is a nontrivial subspace of $\mathcal{H}$.

Chaper 4

Power-Bounded Operator of Class $C_1$and the Hyperinvariant Subspaces

We show that if T is a power-bounded operator of class C_* on a Hilbert space which commutes with a nonzero quasinilpotent operator, then T has a nontrival invariant subspace. We show that if T does not have nontrivial hyperinvariant subspaces for elementary reasons, then T is ampliation quasisimilar to a (BCP)-operator in the classC_{00}: This reduces the hyperinvariant subspace problem for operators in $\ell(\mathcal{H})$ to a very special subcase of itself.

Section(4.1): On Invariant Subspaces

A linear operator T on a Hlibert space $\mathcal{H}$ is called power-bound if $sup_{n\geq 0}\|T^n\| < \infty$. A power-bounded operator T is said to be of class$C_{*.}$ if there exists a nonzero vector $x \in \mathcal{H}$ such that the sequence $\{\|T^n x\|\}_n$ does not converge to 0, and T is of $class\, C_{1.}$. if $\{\|T^n x\|\}_n$ does not converge to 0 for every nonzero vector $x. It$ is still an unsolved problem whether every power-bounded operator of class $C_{*.}$. (in particular, $C_{*.}$.-contraction) has a nontrivial invariant subspace, i.e.,whether there exists a (closed) subspace M of $\mathcal{H}$ such that $\{0\} \neq M \neq \mathcal{H}$ and $TM \subset M$. For partial results on that problem, *see*, e.g., [122] or [134]. In this note we prove the following theorem.

Theorem(4.1. 1)[121]: Assume that T is a power-bounded operator of class $C_{*.}$- on a Hilbert space $\mathcal{H}$, which , commutes with a nonzero quasinilpotent operator. Then T has a nontrivianvariant subspace. This theorem will follow from the following one. We recall that the operator T is called cyclic if it has a cyclic vector, that is, a vector x such that the sequence $\{T^n x\}_{n\geq 0}$ spans the whole space $\mathcal{H}$.

The proof is base on the following construction of the limit isometric operator associated with T (see [10] and [139]).

Given a power-bounded operator T acting on the Hilbert space $\mathcal{H}$, fix a generalized Banach limit glim on $\ell^\infty(N)$ and consider the sesquilinear form $W_T$on $\mathcal{H}$ defined by $W_T(x,y) := g\lim_{n\to\infty}(T^n x, T^n y)\,, x, y \in \mathcal{H}$. Since $\{T^n\}_n$is bounded , it is easy to see that $\text{glim}_{n\to\infty}\|T^n x\| = 0$ if and only if $\inf_{n\geq 0}\|T^n x\| = 0$, and this happens if and only if $\lim_{n\to\infty}\|T^n x\| = 0$. Let $\mathcal{H}_0\,(T)$ be the kernel of w_T, i.e.,

$$\mathcal{H}_0(T) := \{x \in \mathcal{H}: w_T(x,x) = 0\} = \{x \in \mathcal{H}: lim_{n\to\infty}\|T^n x\| = 0\}.$$

Clearly, $\mathcal{H}_0(T)$ is a subspace which is invariant for any operator A in the commutant $\{T\}'$ of the operator T. Furthermore, $\mathcal{H}_0(T) \neq \mathcal{H}$ if and only if T is of class $C_{*.}$, and, $\mathcal{H}_0(T) = \{0\}$ if and only if T is of class $C_{1..}$Thus, Theorem(4.1.1) is an immediate consequence of Theorem (4.1.2).

Let us form the quotient space $\widehat{\mathcal{H}}_T = \mathcal{H}/\mathcal{H}_0(T)$, and let us consider the canonical mapping 1f $\pi_T: \mathcal{H} \to \widehat{\mathcal{H}}_T, \pi_T(x) := x + \mathcal{H}_0(T) =: \hat{x}$.The sesquilinear form $\widehat{w}_T(\hat{x},\hat{y}) := w_T(x\, y)\; (x, y \in \mathcal{H})$ provides an inner product on $\widehat{\mathcal{H}}_T$, so that $\widehat{\mathcal{H}}_T$is a pre-Hilbert space. Let $\hat{T}$ be the operator on $\widehat{\mathcal{H}}_T$ which is defined by $\hat{T}\hat{x} := \widehat{Tx}$. It is easy to see that $\hat{T}$ is an isometry.

Let $\mathcal{H}_T$ be the comlpletion of $\widehat{\mathcal{H}}_T$ and let V_T be the continuous extension of $\hat{T}$, called the isometric asymptote of T in [11]. Any operator $A \in \{T\}'$generates an operatr$\hat{A}$ on $\mathcal{H}_T$ by $\hat{A}\hat{x} := \widehat{Ax}\; (x \in \mathcal{H})$ (and by continuous extension from $\widehat{\mathcal{H}}_T$ to $\mathcal{H}_T$ the mapping $\gamma_T: A \to \hat{A}$ is a contractive algebra-homomorphism from the commutant

$\{T\}'$ of T into the commutant $\{V_T\}'$ of the isometI V_T. Since γ_T is a unital algebra-homomorphism, we obtain the spectral inclusion $\sigma(\hat{A}) \subset \sigma(A)$ $(A \in \{T\}')$. It follows that if A is quasinilpotent then so is $\hat{A}$. It is also clear that $A = 0$ holds if and only if ran $A \in \mathcal{H}_0(T)$.

For a bounded linear operator V on a Hilbert space K, let $\{V\}''$ denote the bicommutant of V. Let $R(V)$ be the set of operators $f(V)$, where f runs through the set of rational functions with poles off the spectrum $\sigma(V)$, and let $A(V)$ be the closure of $R(V)$ in the weak operator topology. We will need the following well-known facts on these algebras.

Lemma(4.1.2)[121]: If V is n isometry on a Hilbert space K,then the abelian Banach algebra $\{V\}''$ ismisimple, and $\{V\}'' = A(V)$.

Proof. For the sake of ompleteness, we sketch the proof. The Hilbert space isometry V splits into tl orthogonal $sum\, V = V_a \oplus U_s$ where V_a is an absolutely continuous isometery and U_s is a singular unitary operator. It is known that $\{V\}'' = \{V_a\}'' \oplus \{U_s\}''$ and $A(V) = A(V_a) \oplus A(U_s)$; see [126] and Rudin's theorem in [8]. Let μ and μ_s denote the normalized Lebesgue measure and the scalar spectral measure of U_s, respectively, on the unit circle T, and let H^∞ be the Hardy subspace of $L^\infty(\mu)$. It can be easily verified that $\{V_a\}'' = \{\varphi(V_a) : \varphi \in H^\infty\}$ if V_a is nonunitary, $\{V_a\}'' = \{\varphi(V_a) : \varphi \in L^\infty(\mu)\}$ if V_a is unitary, and $\{U_s\}'' = \{\psi(U_s) : \psi \in L^\infty(\mu_s)\}$; see [125]. Classical approximation heorems yield that $\{V\}''$ $A(V)$. On the other hand, the previous representation shows that every operator $A \in \{V\}''$ is subnormal, and so $\|A\|$ is equal to the spectral radius $r(A)$, which means that $\{V\}''$does not contain nonzero quasinilpotent operators (or equivalently, the Gelfand transformation,sociated with $\{V\}''$ is injective).

Lemma(4.1. 3)[121]: The isometry V acting on the Hilbert space K is cyclic if and only if its commutant is abelian, that is, $\{V\}' = \{V\}''$.

Proof. Considering the former decomposition $V = V_a \oplus U_s$ we obtain that V is cyclic if and only if both V_a and $U_s$are cyclic. Let us recall that a unitary operator U is cyclic if and only if U is $*$-cyclic, which means that the set $\{U^n x\}_{n=-\infty}^{\infty}$ spans the whole space with a suitable vector x; see [124]. Now, the results in [125] imply the statement.

Theorem(4.1.4)[121]: If T is a power-bounded operator of class $C_{1.}$ on a Hilbert space $\mathcal{H}$ such that T commutes with a nonzero quasinilpotent operator , then T is not cyclic.

Proof. Let us suppose that T has a cyclic vector x. Since $\|\hat{y}\| \le M\|y\|$ holds for every $y \in \mathcal{H}$, where $M = sup\{\|T^n\|\}_{n=0}^{\infty}$ the vector $\hat{x}$ is cyclic for the limit isometry V_T. Let A be the nonzero quasinilpotent operator that commutes with T. Then $\hat{A} = \gamma_T(A)$ commutes with V_T, hence we infer by Lemma (4.1.4) that $\widehat{A} \in \{V_T\}''$. Since $\{V_T\}''$ is semisimple by Lemma(4.1.3), we have $\hat{A} = 0$, and so ran $A \subset \mathcal{H}_0(T) = \{0\}$. Thus $A = 0$, which is a contradiction.

Applying the Riesz-Dunford functional calculus, Theorem(4.1.1) can be easily extended to the following statement.

Corollary (4.1.5)[121]: Let T be a power-bounded operator of class $C_{*.}$ on the Hilbert space $\mathcal{H}$. 1f T commutes with a nonscalar operator A having an isolated spectrum point, then T as a nontrivial invariant subspace. In particular, T has a nontrivial invariant subspace if T commutes with a nonzero, essentially quasinilpotent operator A.The following proposition shows how the statement of lemma (4.1.4) can be transferred to power-bounded operators.

Proposition(4.1. 6)[121]: Let T be a power-bounded operator of class $C_{1.}$ on theHilbert space $\mathcal{H}$, and let us consider the conditions: $(a) T$ is cyclic, $(b) V_T$ is cyclic, (c) $\{T\}' = \{T\}$".Then $(a) \Rightarrow (b) \Rightarrow (c)$, but the reverse implications are false.

Proof : We have already seen that (a) implies (b). If V_T is cyclic then $\{V_T\}'$ is abelian by Lemrr(4.1.3), which implies that $\{T\}'$ is also abelian since the mapping $\gamma_T T$ is one-to-one. In [138], in terms of the Sz.-Nagy-Foias functional model of contractions, examples are given for the case when V_T is cyclic but T is noncyclic.

To show that (c) doe not imply (b), let us consider the simply connected domains $\Omega_+ := \{z \in D : Re\, z > -1/2\}$ and $\Omega_- := \{z \in D: Re\, z < 1/2\}$, where D stands for the 0pen unit disc. Let φ and ψ be conformal mappings of D ontoΩ_+ and ontoΩ_- spectively. Let T_φ and T_ψ be the analytic Toeplitz operators with symbols φ and ψ, respectively, on the Hardy space $H^{2,}$ that is$T_\varphi f := \varphi f, T_\psi f := \psi f$ $(f \in H^2)$, We know by [136] that φ and ψ are (sequential) weak-$*$ generators of the algebra H^∞ and so the operators T_φ and T_ψ have the same invariant subspaces as the operator T_χ where $\chi(z) = z$. Since T_χ . cyclic, it follows that the operators T_φ and T_ψ are cyclic, as well.

It is clear that T_φ and T_ψ are contractions of class $C_{1..}$Furthermore, V_{T_φ} and V_{T_ψ} are unitarily equivalent to the restrictions $M_\alpha := M|\chi_\alpha L^2(\mu)$ and $M_\beta := M\big|\chi_\beta L^2(\mu)$, respectively, where

$$Mf := \chi f \ (f \in L^2(\mu)), a := (\Omega_+)^- \cap T \text{ and } \beta := (\Omega_- \cap T.$$

Let us form the orthogonal sum $T := T_\varphi \oplus T_\psi$. Since V_T is unitarily equivalent to $M_\alpha \oplus M_\beta$ and $\mu(\alpha \cap \beta) > 0$, we obtain that V_T is noncyclic. On the other hand, the conditions $\mu(\beta \setminus \alpha) > 0$ and $\mu(\alpha \setminus \beta) > 0$ imply by [127] that {T}' = $\{T_\varphi\}$'$\oplus$ $\{T_\psi\}$'; see also [134]. Taking into account that T_φ and T_ψ *are* cyclic, we infer that $\{T\}'$ is a semisimple abelian Banach algebra. .

The following examples shows that Lemma(4.1. 2) cannot be generalized to power-bounded operators.

Example(4.1.7)[121]: We recall that the power-bounded operator T is called of *class* C_{11} if both T and its djoint T^* are of class $C_{1..}$ The invariant subspace M is called *quasi-reducing*. if the restriction $T|M$ is of class C_{11}.

Let T be a cyclic, completely non-unitary contraction of class C_{11}. on the Hilbert space $\mathcal{H}$ such that the spectrum of T is the closed unit disc D^-, and V_T is a cyclic bilateral shift The existence of such operators follows from [123]. For a concrete example we refer to [130]. The lattice of the quasi -reducing invariant subspaces of T is isomorphic to the lattice of the spectral subspaces of V_T; see [129] and [29]. Thus, we have an abundance of quasi-reducing subspaces of T. These subspaces are exactly those which can be written in the form $(ran\, A)^-$, where $A \in \{T\}$"; see [129]. Hence, there are many nonzero operators in $\{T\}$" which have nondense range.

On the other hand, since $\sigma(T) = D^-$ and V_T; is a bilateral shift, we infer by Runge's theorem and by [131] that $A(T) = H^\infty(T) := \{u(T): u \in H^\infty\}$. However, for any nonzero function $u \in H^\infty$, the operator u(T) is quasisimilar to $u(V_T)$ (see [29] and [137]), and so u(T) has dense range. Therefore, A(T) is a proper subset of {T}".

Let T be a power-bound operator of class $C_{1.}$ on the Hilbert space $\mathcal{H}$.

Let $A_0(T)$ denote the norm -closure of the set R(T). The norm-continuity of γ_T and the condition $\sigma(T) \supset \sigma(V_T)$ imply that $\gamma_T(A_0(T)) \subset A_0(V_T)$ Since $A_0(V_T) \subset A(V_T) = \{V_T\}''$ and $\{V_T\}''$ is semisimple, we may infer that. $A_0(T)$ is semisimple. This statement was previously showed in [135].

If $\gamma_T(\{T\}'') \subset \{V_T\}''$ holds, then it follows in the same way that $\{T\}''$ is semisimple. However, a look at the operator $T = T_\varphi \oplus T_\psi$ occurring in the proof of Proposition(4.2. 6) shows that the inclusion $\gamma_T(\{T\}'') \subset \{V_T\}''$ does not hold in general. Indeed, the operator $I \oplus 0$ belongs to $\{T\}''$, but $\gamma_T(I \oplus 0) = I \oplus 0$ does not belong to $\{V_T\}''$ Thus, the following problem remains open.

Q**uestion(4.1. 8)[121]:** Is the abelian Banach *algebra* $\{T\}''$ semisimple for every power-bounded Hilbert Space operator T of class $C_{1.}$?

In view of Theorem(4.1. 4) and Proposition (4.1.6), the answer is affirmative if T is cyclic.

As a consequence, we obtain that if the power-bounded operator T of class $C_{1.}$ is of finite multiplicity then the quasinilpotent operators in the commutant of T are nilpotent. So, if T is of finite multiplicity then the problem above can be reduced to the quastion whether every nilpotent operator A in the bicommutant of T is necessarily zero.

The following result on the stability of the semigroup $\{T^n\}_{n\geq 0}$ is related to Theorem (4.1.1) and has an analogous proof.

Theorem(4.2.9)[121]: Suppose that T is a cyclic power-bounded operator on a Hilbert space $\mathcal{H}$. such that T commutes with a quasiinilpotent operator A. Then $\{T^n\}_{n\geq 0}$ is stable on th, range of A, that is, $\lim_{n\to\infty} \|T^n x\| = 0$ holds for every $x \in$ (ran A)-.

In connection with Theorem(4.1. 9), let us also note the following related fact contained in [140].

Theorem(4.1.10)[121]: Let T be a power-bounded operator which commutes with a compact operator K with dense range. Then $\{T^n\}_{n\geq 0}$ is stable 'if and only if T does not have a unimodular eigenvalue.

We note that most of the previous results can be extended without any difficulty to operators T such that the norm-sequence $\{\|T^n\|\}_{n\geq 0}$ is regular in the sense of [132].

Studying these problems in the general Banach space setting, we encounter the obstacle that Lemma(4.1.2) fails, since $\{V\}''$ is not necessarily semisimple if V is an isometry on an arbitary Banach space, see [128].

Section (4,2): Hyperinvariant Subspace Problem

In this section $\mathscr{H}$ will always be a fixed separable, infinite dimensional, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$ .If $\lambda \in \mathbb{C}$ (the complex plane), then the operator $\lambda.1_{\mathscr{H}}$ will be written simply as λ , and the subset of $\mathscr{L}(\mathscr{H})$ consisting of all operators that are not scalar multiples of the identity operator will be denoted by $\mathscr{L}(\mathscr{H})/\mathbb{C}$ If $T \in \mathscr{L}(\mathscr{H})$ then the commutant of T denoted by $\{T\}'$, is the algebra of all operators S in $\mathscr{L}(\mathscr{H})$ such that $ST = TS$.

Recall that a subspace (i.e., closed linear manifold) $\mathscr{M} \subset \mathscr{H}$ is called a nontrivial hyperinvariant subspac $(n.h.s.)$ for T if $(0) \neq \mathscr{M} \neq \mathscr{H}$ and $S\mathscr{M} \subset \mathscr{M}$ for each S in $\{T\}'$. The (presently open) hyperinvariant subs pace problem (for operators on Hilbert

space) is to establish the truth or falsity of the following proposition:
(P_I) Every operator in $\mathscr{L}(\mathscr{H}) \setminus \mathbb{C}$ has *a n.h.s*.

Below we show that (P_I) is equivalent to a very special case of itself, but first we must introduce some additional notation. We denote the spectrum of an operator T in $\mathscr{L}(\mathscr{H})$ by $\sigma(T)$ and the essential (i.e., Calkin) spectrum of T by $\sigma_e(T)$. The sets $\sigma_{le}(T)$ and $\sigma_{re}(T)$ will be, as usual, the left and right essential spectra of T, respectively, and $\sigma_{lre}(\mathrm{T}) := \sigma_{le}(\mathrm{T}) \cap \sigma_{re}(T)$. Moreover we write $\sigma_p(T)$ for the point spectrum of T (i.e., the set of eigenvalues of T) and $r(T)$ for the spectral radius of T. We write also $\mathbb{N}_0$ for the set of nonnegative integers, $\mathbb{D} = \{\xi \in \mathbb{C}: |\xi| < 1\}$,and $\mathbb{T} = \partial\mathbb{D}$. Recall that a subset $\mathcal{D}$ of $\mathbb{D}$ is said to be dominating for $\mathbb{T}$ if almost every point of $\mathbb{T}$ (with respect to Lebesgue arclength measure) is a non tangential limit of a sequence of points from $\mathcal{D}$ Recall also from the theory of dual algebras (cf., e.g., [111]) that a completely non unitary contraction T in $\mathscr{L}(\mathscr{H})$ is called a (BCP)– operator (notation: if $T \in (BCP)$, if $\sigma_e(T) \cap D$ is dominating for $\mathbf{T}$.

The class (BCP) has been studied extensively in the theory of dual algebras (cf. [111]), and, in particular, it is known that (BCP). operators are reflexive [109], which implies that the lattice Lat (T) of invariant subspaces of any (BCP)-operator T is quite large. In fact, it contains a sublattice isomorphic to the lattice of all subspaces of $\mathscr{H}$ [111] and also contains a countably infinite family of cyclic invariant subspaces with the property that any two subspaces from the family have intersection on (0) [108]. Moreover, the (BCP)-operators are, in a sense, "universal dilations", meaning that every direct sum of strict contractions can be realized as a compression to some semi-invariant subspace of an arbitrary (BCP) operator [110]. Recall also from [120] that a completely nonunitary contraction T is said to belong to the class C_{00} if both sequences $\{\{T^n\}_{n=1}^{\infty}$ and $\{(T^*\}_{n=1}^{\infty}$ converge to zero in the strong operator topology and for each $0 \le \theta < 1$. Finally, define

$$\mathbb{A}_0 = \{\xi \in \mathbb{C}: \theta \le |\xi| \le 1\}. \qquad (1)$$

Our principal result in this section is the following:

As an easy corollary of Theorem(4.2.10) , we obtain, as a consequence of the results in this Section below, that proposition (P_I) is equivalent to a(perhaps more amenable, in view of the above remarks about (BCP)-operators), subcase of itself, namely (P_2) Either every (BCP) -operator $T \in C_{00}$ such that $\sigma(T) = \sigma_{le}(T) = \mathbf{D}^-$ has *a n. h. s* . , or there exists $0 < \theta < 1$ such that every (BCP)-operator $T \in C_{00}$ satisfying $\sigma(T) = \sigma_{le}(T) = \mathbf{A}_0$ and $\|T^{-1}\| = 1/\theta$ has *a n.h.s.* In ther words, in this Section we will establish the following:

Theorem(4.2.1)[104]: Proposition (P_I) and (P_2) are quivalent.

We introduce a certain equivalence relation on .$\mathscr{L}(\mathscr{H})$. If n is any cardinal number satisfying $1 \le n \le \aleph_0$, we will write $\mathscr{H}^{(n)}$. for the direct sum of n copies of .$\mathscr{H}$ indexed by the appropriate initial segment of $\mathbf{N}_0$ (i.e., $\mathscr{H}^{(n)} = \oplus_{0 \le k < n} \mathscr{H}_k$ where each $\mathscr{H}_k = \mathscr{H}$), Moreover, for any T in $\mathscr{L}(\mathscr{H})$ we will denote by $T^{(n)}$ the direct sum (ampliation) of n copies of T acting on the space $\mathscr{H}^{(n)}$ in the obvious fashion. The following fact is well known, so no proof is given.

Proposition(4.2.2)[104]: Let T be any operator in in $\mathscr{L}(\mathscr{H}) \setminus \mathbb{C}$. Then T has *a n.h.s.* if and only if for some (every) cardinal number n satisfying $1 < n \le \aleph_0$, $T^{(n)}$ has *a n. h. s.*

Recall next that if S and T are operators in $\mathscr{L}(\mathscr{H})$, then S and T are quasisimilar (notation: $S \sim T$ T) if there exist quasiaffinities X and Y in $\mathscr{L}(\mathscr{H})$ (i.e., $ker\ X = kerX^* = kerY = kerY^* = (0)$) such that $SX = XT$ and $YS = TY$ .The following facts are well-known; for proofs, cf., e.g., [117,119].

Proposition(4.2.3)[104]: Suppose that n is any cardinal number satisfying $1 \le n \le \aleph_0$, and that $\{S_k\}_{0 \le k < n}$ and $\{T_k\}_{0 \le k < n}$, are bounded sequences of operators in $\mathscr{L}(\mathscr{H})$ such that for each $k \in \mathbb{N}_0, S_k \sim T_k$. Then $\hat{S} = \bigoplus_{0 \le k < n} S_k \sim \hat{T} = \bigoplus_{0 \le k < n} \bigoplus_{0 \le k < n} T_k$. Moreover, $\hat{S}$ has *a n. h. s.* if and only if $\hat{T}$ does;

We now introduce a relation on $\mathscr{L}(\mathscr{H})$ that may be new.

Definition(4.2.4)[104]: For T_1 and T_2 in $\mathscr{L}(\mathscr{H})$, we say that T_1 is ampliation quasisimilar to T_2 (notation: $T_1 \overset{a}{\sim} T_2$) if there exist cardinal numbers m and n satisfying $1 \le m, n \le \aleph_0$, such that $T_1^{(m)} \sim T_2^{(n)}$.

Proposition(4.2.5)[104]: Ampliation quasisimilarity is an equivalence relation on $\mathscr{L}(\mathscr{H})$ Furthermore, if $T_1 \overset{a}{\sim} T_2$, then T_1 has *a n.h.s* . if and only if T_2 does. Finally, there are operators T_1 and T_2 in $\mathscr{L}(\mathscr{H})$ such that $T_1 \overset{a}{\sim} T_2$ but $T_1 \nsim T_2$.

Proof: It is clear that he relation $\overset{a}{\sim}$ is reflexive and symmetric. As for transitivity, if $T_1^{(m)} \sim T_2^{(n)}$ and $T_2^{(p)} \sim T_3^{(q)}$, then $T_1^{(mp)} \sim T_2^{(np)} \sim T_3^{(nq)}$ by proposition(4.2.3) so $T_1 \overset{a}{\sim} T_3$.

The fact that if $T_1 \overset{a}{\sim} T_2$, then T_1 has *a n.h.s* . if and only if T_2 does, follows immediately from Propositions(4.2.1) and (4.2.3). Finally, if N is any normal operator in $\mathscr{L}(\mathscr{H})$ of multiplicity one, then $N \overset{a}{\sim} N \oplus N$, but N is not quasisimilar to $N \oplus N$ because, as is well known, two normal operators that are quasisimilar are unitarily equivalent.

In this section, we show Theorems (4.2.10) and (4.2.1). Since (P_1) obviously implies (P_2), to show Theorem (4.2.1) it suffices to show the converse. This follows immediately from Proposition (4.2.6), Theorem (4.2.10), and Proposition (4.2.5), applied in that order, so it is sufficient to prove Theorem (4.2.10), since the following is well-known.

Proposition (4.2.6)[104]: Let $0 \le \theta < 1$ be arbitrarily given. If every T in $\mathscr{L}(\mathscr{H})$ that has the properties (a)-(f) set forth in Theorem (4.2.10) has *a n.h.s.*, then every operator in $\mathscr{L}(\mathscr{H}) \setminus \mathbb{C}$ has a n.h.s..

Proof. If T is a given operator in $\mathscr{L}(\mathscr{H}) \setminus \mathbb{C}$, then T has *a n.h.s* . if and only if some (every) operator $1/\gamma.(T+\lambda)$ where $\gamma \neq 0$ and $\lambda \in \mathbb{C}$ has *a n.h.s* .. The only other property above hat needs a word of explanation is (f) , and if (f) is not satisfied that T has *a n.h.s.* follows from Lomonosov's theorem [118]. (Note that if for some $n \in N, (T-\mu)^n = 0$, then $\sigma_p(T) \neq \phi$).

In order to prove Theorem(4.2.10) , we need some special cases of theorem of Apostol-Herrero-Voiculescu on the closure of similarity orbits of perators [107,106]. (For another exposition, see [112],[113].) The first such result that we will need was proved almost simultaneously and independently in [105,114] For T in $\mathscr{L}(\mathscr{H})$ we write $\mathscr{L}(T)^-$ for the norm closure of the set

$$\{STS^{-1} : S \in \mathscr{L}(\mathscr{H}) \text{ and } 0 \notin \sigma(s)\} .$$

Theorem (4.2.7)[104]: (Apostol- Herrero). Suppose $T \in \mathscr{L}(\mathscr{H})$ is an operator with singleton spectrum $\{\mu\}$ and no (positive, integral) power of $T-\mu$ is *a* compact operator. Then

$\mathscr{L}(T)^-$ consists exactly of all $A \in \mathscr{L}(\mathscr{H})$ such that

(a) $\sigma_e(A) = \sigma_{lre}(A)$,

(b) $\sigma_e(A)$ and $\sigma(A)$ are connected,

(c) $\mu \in \sigma_e(A)$, and

(d) the Fredholm index of $A-\lambda$ is 0 for all λ in $\sigma(A) \setminus \sigma_e(A)$.

The following was proved in [114]. For a different proof, see [116].

Theorem (4.2.8)[104]: (Herrero). Suppose $T \in \mathscr{L}(\mathscr{H})$ and $\sigma(T)$ is a perfect set. Then every normal operator $A \in \mathscr{L}(\mathscr{H})$ such that $\sigma(A) = \sigma(T)$ belongs to $\mathscr{L}(T)^-$.

The last such result that we shall need is also from [114]. See also Theorem 5.8 of [116] for a different proof.

Theorem(4 2.9)[104]: (Herrero). Let T be a normal operator in $\mathscr{L}(\mathscr{H})$ such that $\sigma(T)$ is a perfect set. Then $\mathscr{L}(T)^-$ consists exactly of all A in $\mathscr{L}(\mathscr{H})$ such that

(a) $\sigma(T) \subset \sigma(A)$ and $\sigma(T)$ intersects each component of $\sigma(A)$,

(b) $\sigma_e(T) \subset \sigma_e(A) = \sigma_{lre}(A)$ and $\sigma_e(T)$ intersects each component of $\sigma_e(A)$, and

(c) the Fredholm index of $A-\lambda$ is 0 for all λ . in $\sigma(A) \setminus \sigma_e(A)$.

We can now complete the proof of Theorem (4.2.10).

Theorem(4.2.10)[104]: Let $0 \le \theta < 1$ be arbitrarily given, and let $T \in \mathscr{L}(\mathscr{H})$ have the following properties:

(a) $(1+\theta)/2 \in \sigma(T)$,

(b) the spectral radius $r\left(T-(1+\theta/2)\right)<(1\text{-}\theta)/4$,

(c) $\sigma(T)$ is connected,

(d) $\sigma(T)=\sigma_{lre}(T)$ the point spectrum $\sigma_p(T)$ is empty, and

(f) no (positive, integral) power of $T-\left((1+\theta)/2\right)$ is a compact operator.

Then T is ampliation quasisimilar (see Section 2) to a (BCP) -operator $\widehat{T}$ in the class C_{00} such that $\sigma(\widehat{T})=\sigma_{le}(\widehat{T})=\mathbf{A}_0$ and such that $\|\widehat{T}^{-1}\|=1/\theta$ whenever $\theta>0$. As an easy corollary of Theorem (4.2.10), we obtain, as a consequence of the results see below, that proposition (P_I) is equivalent to a (perhaps more amenable, in view of the above remarks about (BCP)-operators), subcase of itself, namely (P_2) Either every (BCP) -operator $T \in C_{00}$ such that $\sigma(T)=\sigma_{le}(T)=\mathbf{D}^-$ has *a n. h. s* . , or there exists $0<\theta<1$ such that every (BCP)-operator $T \in C_{00}$ satisfying $\sigma(T)=\sigma_{le}(T)$=$\mathbf{A}_0$ and $\|\widehat{T}^{-1}\|=1/\theta$ has *a n.h.s.*

In other words, in Section 3 we will establish the following:

Proof. Let $0 \le \theta < 1$ be arbitrarily given, and let $T \in \mathscr{L}(\mathscr{H}) \setminus \mathbb{C}$ be given that satisfies (a}--(f) of The Theorem (4.2.10). Let $\mathbf{A}_0$ be the annulus (or disc) in (1), and let $\mathscr{D}_0$ in $\mathscr{L}(\mathscr{H}) \setminus \mathbb{C}$ be the disc

$$\mathscr{D}_0 = \left\{\xi \in \mathbb{C} : \left|\xi - \frac{1+\theta}{2}\right| \le \frac{1-\theta}{4}\right\}. \qquad (2)$$

By (a) and (b), we know that $(1+\theta)/2 \in \sigma(T)$ and $\sigma(T) \subset \mathscr{D}_0$. Furthermore, by $(c), \sigma(T)=\sigma_{lre}(T)$ is either the singleton $\{(1+\theta)/2\}$ or is a perfect set. Define the sequences $\{r_n\}$ and $\{s_n\}$ of positive real numbers by

$$r_n = \left((4n+3)\theta+1\right)/(4n+4), \qquad n \in \mathbf{N}_0 \qquad (3)$$

and

$$s_n = (4n+3+\theta)/(4n+4), \qquad n \in \mathbf{N}_0 . \qquad (4)$$

Observe that $\{r_n\}$ is a strictly decreasing sequence satisfying $\theta < r_n \le (1+3\theta)/4$ and $\inf r_n = \theta$, while $\{s_n\}$ is a strictly increasing sequence satisfying $1 > s_n \ge (3+\theta)/4$ and sup $S_n = 1$. Define next the annuli

$$\mathbb{A}_n = \begin{cases} \left\{\xi \in \mathbb{C}: \frac{1+3\theta}{4} \le |\xi| \le s_n\right\}, n\ even \\ \left\{\xi \in \mathbb{C}: r_n \le |\xi| \le \frac{3+\theta}{4}\right\},\ \ n\ \ odd \end{cases} \qquad (5)$$

Let μ denote planar Lebesgue measure on $\mathbb{C}$, and for $n \in \mathbf{N}_0$, let M_z be the normal operator of multiplication by the position function on $\mathscr{H}_n = L_2\left(\mathbf{A}_n, \mu|\mathbf{A}_n\right)$. Let also $N_n \in \mathscr{L}(\mathscr{H})$ be (a norma operator) unitarily equivalent to $M_z \in \mathscr{L}(\mathscr{H}_n)$. One checks next that for each $n \in \mathbf{N}_0, \sigma(N_n) = \sigma_{lre}(N_n) = \mathbf{A}_n$ which is, of course, a perfect set containing $\mathscr{D}_0$. Thus we may apply either Theorem(4. 2.8) (if $\sigma(T) = \{(1+\theta)/2\}$) or Theorems(4. 2.9) and (4.2.10) (otherwise) to conclude that $N_n \in \mathscr{S}(T)^-$ for all $n \in \mathbf{N}_0$. Therefore for each nonegative integer n, there exists an invertible operator S_n in $\mathscr{L}(\mathscr{H})$ such that

$$\left\|S_n T S_n^{-1} - N_n\right\| < (1 - s_n)/2, \qquad n \text{ even} \tag{6}$$

and

$$\left\|S_n T S_n^{-1} - N_n\right\| < (r_n - \theta)/2, \qquad n \text{ odd} \tag{7}$$

Define $\widehat{\mathscr{H}} = \mathscr{H}^{(\aleph_0)}$, and note that since

$$\left\|S_n T S_n^{-1}\right\| - \left\|N_n\right\| \le \left\|S_n T S_n^{-1} - N_n\right\|,$$ and

$$\left\|N_n\right\| = \begin{cases} s_n, & \text{n even} \\ (3+\theta)/4 & n \text{ odd} \end{cases},$$

a short calculation using (3), (4), (6), and (7) shows that

$$\left\|S_n T S_n^{-1}\right\| < 1 \qquad n \in \mathbf{N}_0.$$

Thus, the operator $\widehat{T} = \oplus_{n \in N_0} S_n T S_n^{-1}$ in $\mathscr{L}(\widehat{\mathscr{H}})$ is a completely nonunitary contraction in the class C_{00}, which is obviously ampliation quasisimilar to T, and we begin to study its spectral properties. Note that, for every even n in $\mathbf{N}_0$ and every $x \in \mathscr{H}$, we have from (6) that

$$\begin{aligned} \left\|S_n T S_n^{-1}\right\| &\ge \left\|N_n x\right\| - ((1-s_n)/2)\left\|x\right\| \\ &\ge (\{(1+3\theta)/4\} - (1-s_n)/2)\left\|x\right\|, \\ &\ge\ge ((2s_n + 3\theta - 1)/4)\left\|x\right\| \ge \theta\left\|x\right\|, \end{aligned} \tag{8}$$

which shows, in particular, that if $\theta > 0$, then for n even we have

$$\left\|\left(S_n T S_n^{-1}\right)^{-1}\right\| \le 1/\theta \tag{9}$$

Similarly, for n odd we have from (7) that

$$\left\|S_n T S_n^{-1} x\right\| \ge \left\|N_n x\right\| - ((r_n - \theta)/2)\left\|x\right\|$$

$$\ge (r_n - \{(r_n - \theta)/2\}\left\|x\right\| \ge \theta\left\|x\right\|, \tag{10}$$

which shows that if $\theta > 0$ then (9) is also valid for n odd, and thus, in particular if $\theta > 0$, that $\widehat{T}$ is invertible and

$$\left\|\widehat{T}^{-1}\right\| = \sup\left\|\left(S_n T S_n^{-1}\right)^{-1}\right\| \le 1/\theta \qquad .(11)$$

We also conclude from this that whether or not $\theta > 0$, we have $\sigma_e(\widehat{T}) \subset \sigma(\widehat{T}) \subset \mathbf{A}_0$, as desired.

Next, we will show that $\sigma_{le}(\widehat{T}) = \sigma(\widehat{T}) = \mathbf{A}_0$, and thus that $\widehat{T}$ is a (BCP) -operator.

To this end (whether or not $\theta > 0$), let $\lambda_0 \in \mathbf{A}_\theta^\circ$ (the interior of $\mathbf{A}_0$) be arbitrary. Then, since $inf r_n = \theta$ and $\sup s_n = 1$, there exists a positive integer K_0 such that either

(a) $\lambda_0 \in \mathbf{A}_k$ for all even $k \geq K_0$, or

(b) $\lambda_0 \in \mathbf{A}_k$ for all odd $k \geq K_0$.

Since $\sigma_{le}(N_n) = \mathbf{A}_n$ f(all n in $\mathbf{N}_0$ by construction, we obtain, in the case that (a) is valid, a unit vector $x_n \in \mathscr{H}$ such that $\|(N_n - \lambda_0)x_n\| < 1/n$ for all even $n \geq K_0$, and similarly in case (b) is valid. Now define for each even or odd $n \geq K_0$, depending on whether (a) or (b) is valid, the vector $\widehat{x_n} \in \widehat{\mathscr{H}} = \mathscr{H}^{(\aleph_0)}$ by taking the component of $\widehat{x}_n$ in the nth copy of $\mathscr{H}$ in $\widehat{\mathscr{H}}$ to be x_n and all other components to be zero. It is obvious that the family{$\widehat{x_n} \in \widehat{\mathscr{H}} : n \geq K_0$, n even or odd depending on whether (a) or (b) is valid} is an orthonormal family, and it follows from the inequality

$$\begin{aligned}\|(\widehat{T} - \lambda_0)\widehat{x}_n\| &= \|(S_n T S_n^{-1} - \lambda_0)x_n\| \\ &\leq \|(N_n - \lambda_0)x_0\| + \|S_n T S_n^{-1} - N_n\| \\ &\leq (1/n) + \max\{(1 - s_n)/2(r_n - \theta)/2\}\end{aligned}$$

that $\|(\widehat{T} - \lambda_0)x_n\| \to 0$ and thus that $\lambda_0 \in \sigma_{le}(\widehat{T})$. Since $\sigma_{le}(\widehat{T})$ is closed,

$$\mathbf{A}_0 \subset \sigma_{le}(\widehat{T}) \subset \sigma(\widehat{T}) \subset \mathbf{A}_0 . \tag{12}$$

Finally, we note that if $\theta > 0$, then $\|\widehat{T}^{-1}\| = 1/\theta$ by (11) and [115].

Chapter 5
The Schur-Horn Theorem for Operators and Constructing Finite Frames

Let $\mathcal{H}$ be a Hilbert space. Given a bounded positive definite operator S on $\mathcal{H}$, and a bounded sequence $c = \{c_k\}_{k\in\mathcal{N}}$ of nonnegative real numbers, the pair (S, c) is frame admissible, if there exists a frame $\{f_k\}_{k\in\mathcal{N}}$ on $\mathcal{H}$ with frame operator S, such that $\{f_k\}^2 = c_k, k \in \mathcal{N}$.

The estimate is valid for a fairly general class of frames — requiring that the dimension of the Hilbert space and the number of frame vectors is relatively prime. In addition, we re-phrase our distance estimate to show that certain projection matrices which are nearly constant on the diagonal are close in Hilbert–Schmidt norm to ones which have a constant diagonal.

Indeed, the minimum and maximum eigenvalues of the frame operator are the optimal frame bounds, and the frame is tight precisely when this spectrum is constant. Often, the second-most important design consideration is the lengths of frame vectors: Gabor, wavelet, equiangular and Grassmannian frames are all special cases of equal norm frames, and unit norm tight frame-based encoding is known to be optimally robust against additive noise and erasures. We consider the problem of constructing frames whose frame operator has a given spectrum and whose vectors have prescribed lengths. For a given spectrum and set of lengths, the existence of such frames is characterized by the Schur-Horn Theorem—they exist if and only if the spectrum majorizes the squared lengths—the classical proof of which is nonconstructive. Certain construction methods, such as harmonic frames and spectral tetris, are known in the special case of unit norm tight frames, but even these provide but a few examples from the manifold of all such frames, the dimension of which is known and nontrivial. In this paper, we provide a new method for explicitly constructing any and all frames whose frame operator has a prescribed spectrum and whose vectors have prescribed lengths. The method itself has two parts. In the first part, one chooses eigensteps—a sequence of interlacing spectra—that transform the trivial spectrum into the desired one. The second part is to explicitly compute the frame vectors in terms of these eigensteps; though nontrivial, this process is nevertheless straightforward enough to be implemented by hand, involving only arithmetic, square roots and matrix multiplication.

Section(5.1): Prescribed Norms and Frame Operators

Let $\mathcal{H}$ be a separable Hilbert space and let S be a bounded selfadjoint operator on $\mathcal{H}$. In the first part of this section, we give a complete characterization of the closure in $\ell^\infty(\mathbb{N})$ of the set of possible "diagonals" of S, i.e., the set $C[U_{\mathcal{H}}(S)]$ of real sequences $c = (c_n)_{n\in\mathbb{N}}$ such that

$$\langle Se_n, e_n\rangle = c_n, \qquad n \in \mathbb{N}\,, \tag{1}$$

for some orthonormal basis $B = \{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$.

Note that, if $dim\mathcal{H} = m < \infty$, this can be made in terms of majorization theory. More precisely, the Schur-Horn theorem ensures that $c \in \mathbb{R}^m$ satisfies Eq. (1) for some orthonormal basis if and only if c is majorized by the vector of eigenvalues of S (see Theorem (5.1.2) for a precise formulation). In the general case, we define an analogous form of "the sum of the greatest k eigenvalues" in the following way: given S, a selfadjoint operator on $\mathcal{H}$, and $k \in \mathbb{N}$, we denote

$U_k(S) = sup\{tr\, SP : P \in L(\mathcal{H})$ is an orthogonal projection with $tr\, P = k\}$, and $L_k(S) = -U_k(-S)$.
We prove, based on the results obtained by A.
Neumann in [101], that c belongs to the $\ell^\infty(\mathbb{N})$ -closure of $C[U_{\mathcal{H}}(S)]$ if and only if

$$U_k(c) \le U_k(S) \text{ and } L_k(S) \le L_k(c), k \in \mathbb{N}, \qquad (2)$$

Where

$$U_k(c) = \sup_{|F|=k} \sum_{i\in F} c_i,$$

and

$$L_k(c) = \inf_{|F|=k} \sum_{i\in F} c_i = -U_k(-c).$$

Similarly, if S is a trace class operator, we show that c belongs to the $\ell^1(\mathbb{N})$- closure of $C[U_{\mathcal{H}}(S)]$ if and only if c satisfies formulas (2) and

$$\sum_{n\in\mathbb{N}} c_n = trS.$$

On the other hand, a somewhat technical characterization of the maps U_k and L_k is obtained (see Proposition (5.1.7), which is used to compute these quantities and to prove their basic properties. Related results can be found in R. Kadison [98], [99], and Arveson and Kadison [89] (which appeared during the revision process of this work).
In the second part of this note, these extended Schur-Horn theorems are used to give conditions for the existence of frames with prescribed norms and frame operator. First we recall some basic definitions. Let $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \{1, 2, \dots, m\} := \mathbb{I}_m$, for some $m \in \mathbb{N}$. A sequence $\{f_k\}_{k\in M}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \le \sum_{k\in M} |\langle x, f_k\rangle|^2 \le B\|x\|^2, \qquad for\ every\ x \in \mathcal{H}.$$

For complete descriptions of frame theory and its applications, the reader is referred to [94], [96], [97], [90], or the books by Young [103] and Christensen [74].
Let $\mathcal{F} = \{f_k\}_{k\in M}$, be a frame for $\mathcal{H}$. The operator

$$S : \mathcal{H} \to \mathcal{H}, \text{ given by } S(x) = \sum_{k\in M} \langle x, f_k\rangle f_k, \; x \in \mathcal{H}, \qquad (3)$$

is called the frame operator of $\mathcal{F}$. It is always bounded, positive and invertible (we use the notation $S \in \mathcal{G}l\, (\mathcal{H})^+$).
In recent papers by Casazza and Leon [92], [93], Casazza, Fickus, Leon and
Tremain [91], Dykema, Freeman, Korleson, Larson, Ordower and Weber [45], Kornelson and Larson [100], and Tropp, Dhillon, Heath Jr. and Strohmer [57], the problem of existence and (algorithmic) construction of frames with prescribed norms and frame operator has been considered.
Following [92], [93], we say that the pair $(S, c) \in \mathcal{G}l(\mathcal{H})^+ \times \ell^\infty(\mathbb{M})^+$ is frame admissible if there exists a frame $\mathcal{F} = \{f_k\}_{k\in M}$ on $\mathcal{H}$ such that
(1) $\mathcal{F}$ has frame operator S, and
(2) $\|f_k\|^2 = c_k$ for every $k \in \mathbb{M}$.
In this case, we say that $\mathcal{F}$ is a (S, c)-frame. We denote by $F(S, c)$ the set of all (S, c)-frames on $\mathcal{H}$. Hence the pair (S, c) is frame admissible if $F(S, c) \ne \emptyset$.
It is known (see [92], [57]) that, in the finite dimensional case, there is a connection between frame admissibility and the theory of majorization, in particular, the Schur-Horn theorem. We make this connection explicit both in the finite and infinite dimensional

context. We use the classical Schur-Horn theorem in the finite dimensional case and its extension, developed in the first part of the section, for the infinite dimensional case.

This presentation of the problem allows us to get equivalent conditions for the frame admissibility of a pair $(S,c) \in \mathcal{G}l_n(\mathbb{C})^+ \times \ell^\infty(\mathbb{N})^+$, and necessary conditions for the frame admissibility of a pair $(S,c) \in \mathcal{G}l\,(\mathcal{H})^+ \times \ell^\infty(\mathbb{N})^+$.

We show that, if the pair (S,c) is frame admissible, then $\sum_{k\in\mathbb{N}} c_k = \infty$. and $U_k(c) \leq U_k(S)$ for every $k \in \mathbb{N}$. In particular, $lim\,sup\; c \leq \|S\|_e$, the essential norm of S (see Theorem (5.1.25). Then, by strengthening these conditions we get sufficient conditions for the frame admissibility of pairs $(S,c) \in \mathcal{G}l\,(\mathcal{H})^+ \times \ell^\infty(\mathbb{N})^+$ (Theorem(5.1.28). These conditions are less restrictive than those found by Kornelson and Larson in [100].

We briefly describe the contents of the section. In this Section we fix our notation, and we state the classical Schur-Horn theorem. In the Section we prove the extension of the Schur-Horn theorem for general selfadjoint operators. In this Section we give some reformulations of the notion of frame admissibility which allow us to apply majorization theory to this problem, and we show equivalent conditions for frame admissibility in the finite dimensional case (both for finite or infinite sequences c). In this Section we study the infinite dimensional case, showing separately necessary and sufficient conditions for frame admissibility. In the Section we give several examples for the boundary cases of the conditions studied before. These examples show that, in general, the conditions can not be relaxed further. We also study different types of frames in $F(S,c)$, in terms of their excesses.

Let $\mathcal{H}$ be a separable Hilbert space, and $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. We denote $L_0(\mathcal{H})$ the ideal of compact operators, $\mathcal{G}l\,(\mathcal{H})$ the group of invertible operators, $L(\mathcal{H})_h$ the set of hermitian operators, $L(\mathcal{H})^+$ the set of nonnegative definite operators, $\mathcal{U}(\mathcal{H})$ the group of unitary operators, and $\mathcal{G}l\,(\mathcal{H})^+$ the set of invertible positive definite operators. We denote by $L^1(\mathcal{H})$ the ideal of trace class operators in $L(\mathcal{H})$. We set $L^1(\mathcal{H})_h = L^1(\mathcal{H}) \cap L(\mathcal{H})_h$ and $L^1(\mathcal{H})^+ = L^1(\mathcal{H}) \cap L(\mathcal{H})^+$. We denote by $\ell^1(\mathbb{N})$ the Banach space of complex absolutely summable sequences. By $\ell^1_R(\mathbb{N})$ $(resp.\, \ell^1(\mathbb{N})^+)$ we denote the subsets of real (resp. nonnegative) sequences. Similarly, we use the notations $\ell^\infty(\mathbb{N}), \ell^\infty_R(\mathbb{N})$ and $\ell^\infty(\mathbb{N})^+)$ for bounded sequences. Given an operator $A \in L(\mathcal{H}), R(A)$ denotes the range of A, ker$A$ the nullspace of $A, \sigma(A)$ the spectrum of A, A^* the adjoint of $A, \rho(A)$ the spectral radius of A, and $\|A\|$ the spectral norm of A. We say that A is an isometry (resp. coisometry) if $A^*A = I$ $(resp.\, AA^* = I)$.

We also consider the quotient $A(\mathcal{H}) = L(\mathcal{H})/L_0(\mathcal{H})$, which is a unital C^* −algebra, known as the Calkin algebra. Given $T \in L(\mathcal{H})$, the essential spectrum of T, denoted by $\sigma_e(T)$, is the spectrum of the class $T + L_0(\mathcal{H})$in the algebra $A(\mathcal{H})$. The essential norm $\|T\|_e = inf\{\|T+K\| : K \in L_0(\mathcal{H})\}$ of T is the (quotient) norm of $T + L_0(\mathcal{H})$, also in $A(\mathcal{H})$. Given $S \in L(\mathcal{H})_h$, we define

$$a^+(S) = max\,\sigma_e(S) = \|S\|_e \text{ and } \alpha_-(S) = min\,\sigma_e(S). \quad (4)$$

If $S = \int_{\sigma(S)} t dE(t)$ is the spectral representation of S with respect to the spectral measure E, we shall often consider the following compact operators:

$$S^+ = \int_{[a^+(S),\|S\|]} \big(t - \alpha^+(S)\big) dE(t), \text{ and}$$

$$S_- = \int_{[-\|S\|,\alpha_-(S)]} \big(t - \alpha(S)\big) dE(t) \quad (5)$$

Note that $S_{-} \leq 0 \leq S^{+}$.
Given a subset M of a Banach space $(\chi, \|\cdot\|)$, its closure is denoted by $\overline{M}$ or $cl_{\|\cdot\|}(M)$, and the convex hull of M is denoted by $conv(M)$. Also, given a closed subspace S of $\mathcal{H}$, we denote by P_S the orthogonal (i.e., selfadjoint) projection onto $S. If\ B \in L(\mathcal{H})$ satisfies $P_S B P_S = B$, in some cases we shall use the compression of B to S, (i.e., the restriction of B to S as a linear transformation from S to S), and we say that we consider B as acting on S.
Finally, when $dim\mathcal{H} = n < \infty$, we shall identify $\mathcal{H}$ with $\mathbb{C}^n, L(\mathcal{H})$ with $M_n(\mathbb{C})$, and we use the following notations: $M_n(\mathbb{C})_h$ for $L(\mathcal{H})_h, M_n(\mathbb{C})^+$ for $L(\mathcal{H})^+, U(n)$ for $\mathcal{U}(\mathcal{H})$, and $\mathcal{G}l_n\ (\mathbb{C})$ for $\mathcal{G}l\ (\mathcal{H})$.
Majorization. In this subsection we present some basic aspects of majorization theory. For a more detailed treatment of this notion see [51]. Given $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, denote by $b^{\downarrow} \in \mathbb{R}^n$ the vector obtained by rearranging the coordinates of b in nonincreasing order. If $b, c \in \mathbb{R}^n$ then we say that c is majorized by b, and write $c \prec b$, if

$$\sum_{i=1}^{k} b_i^{\downarrow} \geq \sum_{i=1}^{k} c_i^{\downarrow} \quad k = 1, \ldots, n-1, \text{ and } \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} c_i.$$

Majorization is a preorder relation in $\mathbb{R}^n$ that occurs naturally in matrix analysis.
Definition (5.1.1)[35]: $Let\ \mathbb{M} = \mathbb{N}\ or\ \mathbb{M} = \{1, 2, \ldots, m\} := \mathbb{I}_m$, for some $m \in \mathbb{N}$. Let K be a Hilbert space with $dim K = |\mathbb{M}|$ and let $B = \{e_n\}_{n \in \mathbb{M}}$ be an orthonormal basis of K.
(i) For any $a = (a_n)_{n \in M} \in \ell^{\infty}(\mathbb{M})$, denote by $M_{B,a} \in L(K)$ the diagonal operator given by $M_{B,a} e_n = a_n e_n, n \in \mathbb{M}$. When it is clear which basis we are using, we abbreviate $M_{B,a} = M_a$.
(ii) In particular, for $a \in \mathbb{C}^n$, we denote by $M_a \in M_n(\mathbb{C})$ the diagonal matrix (with respect to the canonical basis of $\mathbb{C}^n$) which has the entries of a on its diagonal.
(iii) The diagonal pinching $C_B : L(K) \to L(K)$ associated to the basis B, is defined by $C_B(T) = M_{B,a}$ where $a = (\langle Te_n, e_n \rangle)_{n \in M}$.
Theorem (5.1.2) (Schur-Horn)[35]: Let $b, c \in \mathbb{R}^n$. Then $c \prec b$ if and only if there exists $U \in \mathcal{U}(n)$ such that

$$C_{\varepsilon}(U^* M_b U) = M_c,$$

where ε is the canonical basis of $\mathbb{C}^n$.
In this section we present a different version of the “infinite dimensional Schur-Horn theorem” given by A. Neumann in [101]. Our approach avoids the somewhat technical distinction between the diagonalizable and nondiagonalizable case. On the other hand, this version can be applied more easily to the problem of frame admissibility in the infinite dimensional case. The main tools we use are the Weyl–von Neumann theorem and the known properties of approximately unitarily equivalent operators.
Given a sequence $a \in \ell_R^{\infty}\ (\mathbb{N})$, Neumann [101] defines

$$U_k(a) = \sup_{|F|=k} \sum_{i \in F} a_i \quad \text{and} \quad L_k(a) = \inf_{|F|=k} \sum_{i \in F} a_i.$$

This generalizes the partial sums which appear in the definition of majorization.
In the first part of this section we shall extend this definition to arbitrary selfadjoint operators on a Hilbert space $\mathcal{H}$. Denote by P_k the set of orthogonal projections onto k-dimensional subspaces of $\mathcal{H}$.
Definition(5.1.3)[35]: Given $S \in L(\mathcal{H})_h$, we define, for any $k \in \mathbb{N}$,

$U_k(S) = \sup_{p\in\mathcal{P}_k} tr(SP)$ and $L_k(S) = \inf_{p\in\mathcal{P}_k} tr(SP) = -U_k(-S)$.

The following result asserts that Definition(5.1.3) extends the natural extrapolation of Neumann's definition for diagonalizable operators.

Proposition(5.1.4)[35]: Let $B = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. If $a \in \ell_R^\infty(\mathbb{N})$, then, for every $k \in \mathbb{N}$,

$$U_k(M_{B,a}) = U_k(a).$$

In order to prove this proposition we need the following technical results.

Lemma(5.1.5)[35]: Let $S \in L_0(\mathcal{H})^+$, and denote by $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq \cdots$ the positive eigenvalues of S, counted with multiplicity (if $dimR(S) < \infty$ we complete this sequence with zeros). Then, for every $k \in \mathbb{N}$,

$$U_k(S) = \sum_{i=1}^{k} \lambda_i.$$

Moreover, if $P \in \mathcal{P}_k$ is the projection onto the subspace spanned by an orthonormal set of eigenvectors of $\lambda_1, \ldots, \lambda_k$, then $U_k(S) = tr(SP)$.

Proof. Fix $k \in \mathbb{N}$. It suffices to show that $tr(SQ) \leq tr(SP) = \sum_{i=1}^{k} \lambda_i$ for every $Q \in \mathcal{P}_k$. This follows from Schur's theorem (the diagonal is majorized by the sequence of eigenvalues), which also holds in this setting (see [102]). _In [101], Neumann proved the following result: if $a \in \ell_R^\infty(\mathbb{N})$,

$a_i^+ = max\{a_i - lim\ sup\ a\ , 0\}, a_i^- min\{a_i - liminf\ a, 0\}, i \in \mathbb{N},$ (6)

then, for every $k \in \mathbb{N}$,

$U_k(a) = U_k(a^+) + k\ limsup\ a$ and $L_k(a) = L_k(a^-) + k\ liminf\ a.$ (7)

The next result extends Eq. (7) to selfadjoint operators. This fact is necessary for the proof of Proposition(5.1.4), but it is also a basic tool in order to deal with the maps U_k and L_k .

Proposition (5.1.6)[35]: Let $S \in L(\mathcal{H})_h$. Then, for every $k \in \mathbb{N}$.,

(i) $U_k(S) = U_k(S^+) + k\ \alpha^+(S)$,

(ii) $L_k(S) = L_k(S_-) + k\ \alpha_-(S)$,

where $\alpha^+(S), \alpha_-(S), S^+, S_-$ are defined in (4) and (5). In particular,

$\lim_{k\to\infty} \frac{U_k(S)}{k} = \alpha^+(S) = \|S\|_e$ and $\lim_{k\to\infty} \frac{L_k(S)}{k} = \alpha_-(S).$ (8)

Proof. Denote $\alpha^+ = \alpha^+(S)$, and

$$P_2 = P_2(S) = E[\,\|S\|_e, \|S\|\,] = E[\,\alpha^+, \|S\|\,], \quad (9)$$

where E is the spectral measure of S. Recall that

$$S^+ = \int_{[\alpha^+,\|S\|]} (t - \alpha^+) dE(t) = (S - \alpha^+)P_2.$$

Then $S - S^+ = S(I - P_2) + \alpha^+ P_2 \leq \alpha^+ I$. Therefore, for every $k \in \mathbb{N}$ and $Q \in \mathcal{P}_k$,

$$tr(SQ) = tr(S^+Q) + tr((S - S^+)Q) \leq U_k(S^+) + k\alpha^+, \quad (10)$$

which shows that $U_k(S) \leq U_k(S^+) + k\alpha^+$ for every $k \in \mathbb{N}$.

To see the converse inequality, suppse first that $tr\,P_2 = +\infty$. Denote by $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq \cdots$ the eigenvalues of S^+, chosen as in Lemma (5.1.5).

Let $Q_k \in \mathcal{P}_k$ be the projection onto the subspace spanned by an orthonormal set of eigenvectors of $\lambda_1 \ldots \lambda_k$. Then $Q_k \leq P_2$. By Lemma(5.1.5) ,

$$tr(SQ_k) = tr(S^+Q_k) + tr\big((S - S^+)Q_k\big) = \sum_{i=1}^{k} \lambda_i + k\alpha^+ = U_k(S^+) + k\alpha^+ .$$

Hence, $U_k(S) = U_k(S^+) + k\alpha^+$. Now, assume that $tr\, P_2 = r < \infty$. If $k \leq r$, the same argument as before shows that $U_k(S) = U_k(S^+) + k\alpha^+$. So, let $k > r$ and take $\varepsilon > 0$. Since$P_\varepsilon = E[\ \alpha^+ - \varepsilon, \alpha^+)$ has infinite rank (otherwise $\|S\|_e \leq \alpha^+ - \varepsilon$), we can take $Q \leq P_\varepsilon$, a projection of rank $k - r. If\ Q_k = Q + P_2$, then

$U_k(S) \geq tr(S\, Q_k) = tr(SP_2) + tr(SQ)$

$= tr(S^+) + r\, \alpha^+ + tr(SP_\varepsilon Q)$

$\geq tr(S^+) + r\, \alpha^+ + (k - r)(\alpha^+ - \varepsilon)$

$= U_k(S^+) + k\, \alpha^+ - \varepsilon(k - r)$.

Since ε is arbitrary, $U_k(S) = U_k(S^+) + k\, \alpha^+$. The formula for $L_k(S)$ follows by applying item 1 to $-S$. Finally, as $S^+ \in L_0(\mathcal{H})^+$, its eigenvalues converge to zero. Hence, by Lemma (5.1.5), we get that

$$\lim_{k\to\infty} \frac{U_k(S^+)}{k} = 0$$

and similarly for $L_k(S_-)$. Therefore, Eq. (8) follows.

Proof of Proposition(5**.**1**.**4) **.** The result follows using Lemma(5.1.5) , Proposition(.5.1.6), Eq. (7) and the following obvious identities: if $S = M_{B,a}$, then

(i) $\alpha^+(S) = lim\ sup\ a$, and $\alpha_-(S) = lim\ inf\ a$,

(ii) $S^+ = M_{B,a^+}\ and\ S_- = M_{B,a^-}$,

where a^+ and a^- are defined as in Eq. (6).

Definition $(\mathbf{5.1.7})[\mathbf{35}]$: Let $\mathcal{H}$ be a Hilbert space, $S \in L(\mathcal{H})$ and B an orthonormal basis of $\mathcal{H}$. Then:

(a) $\mathcal{U}_{\mathcal{H}}(S) = \{U^*SU : U \in \mathcal{U}(\mathcal{H})\}$.

(b) $C[\mathcal{U}_{\mathcal{H}}(S)] = \{c \in \ell^\infty(\mathbb{N}) : M_{B,c} \in C_B(\mathcal{U}_{\mathcal{H}}(S)\}$.

Given a diagonal operator $M_a \in L(\mathcal{H})_h$, Neumann showed that, if $c \in \ell_R^\infty(\mathbb{N})$ the following statements are equivalent [101]):

(i) $c \in \overline{C[\mathcal{U}_{\mathcal{H}}(M_a)]}$.

(ii) $U_k(a) \geq U_k(c)\ and\ L_k(a) \leq L_k(c), k \in \mathbb{N}$.

Now, our objective is to generalize this equivalence to every operator $S \in L(\mathcal{H})_h$ (via a reduction to the diagonalizable case). We need first the following result about approximately unitarily equivalent operators.

Lemma (5.1.8)[35]: Let $S, T \in L(\mathcal{H})_h$. Then $S \in cl_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(T))$ if and only if

$$cl_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(S)) = cl_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(T)) .$$

In this case $U_k(S) = U_k(T)$ and $L_k(S) = L_k(T)$ for every $k \in \mathbb{N}$.

Proof: If $\{V_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{U}(\mathcal{H})$ such that $\|V_nTV_n^* - S\| \underset{n\to\infty}{\to} 0$, then

$$\|V_n^*\, SV_n - T\| = \|V_n^*\,(S - V_nTV_n^*)V_n\| = \|V_nTV_n^* - S\| \underset{n\to\infty}{\to} 0.$$

Hence $cl_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(S)) = l_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(T))$. $U_k(V_nTV_n^*) = U_k(T)$ and $L_k((V_nTV_n^*) = L_k(T)$, for $n, k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and take $P \in \mathcal{P}_k$. Then

$$tr\ SP = \lim_{n\to\infty} tr\, V_nTV_n^*\, P \leq \lim_{n\to\infty} U_k(V_nTV_n^*) = U_k(T).$$

Hence $U_k(S) \leq U_k(T)$. Similarly $L_k(S) \geq L_k(T)$. The reverse inequalities follow from the fact that $V_n^* S V_n \underset{n\to\infty}{\to} T$.

Theorem (5.1.9)[35]: Let $S \in L(\mathcal{H})_h$, and $c \in \ell_R^\infty$ (ℕ). Then the following conditions are equivalent:

(a) $c \in \overline{C[\mathcal{U}_{\mathcal{H}}(S)]}$.

(b) $U_k(S) \geq U_k(c)$ *and* $L_k(S) \leq L_k(c)$ for every $k \in \mathbb{N}$.

If one of these conditions holds, then max $\sigma_e(S) \geq \lim\sup c$ and min $\sigma_e(S) \leq \lim\inf c$.

Proof. The diagonalizable case was proved by Neumann as mentioned before. Note that, in order to deduce our formulation from Neumann's result, we need Proposition(5.1.4). If S is not diagonalizable, there exists a diagonalizable operator $D \in cl_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(S))$. By Lemma (5.1.8), $U_k(D) = U_k(S)$ and $L_k(D) = L_k(S)$ for every $k \in \mathbb{N}$, and $cl_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(D)) = l_{\|\cdot\|}(\mathcal{U}_{\mathcal{H}}(S))$.. This implies that

$$cl_{\|\cdot\|_\infty}(C[\mathcal{U}_{\mathcal{H}}(D)]) = cl_{\|\cdot\|_\infty}(C[\mathcal{U}_{\mathcal{H}}(S)]),$$

because the map $T \mapsto C_B(T)$ is continuous for every orthonormal basis B.

Hence, the general case reduces to the diagonalizable case. The final remark follows from the fact that

$$\lim\sup c = \lim_{k\to\infty} \frac{U_k(c)}{k} \quad \text{and} \quad \lim\inf c = \lim_{k\to\infty} \frac{L_k(c)}{k}, \qquad (11)$$

and Eq. (8).

A similar result can be stated for hermitian operators in $L^1(\mathcal{H})$ and sequences in ℓ_R^1 (ℕ). In this case our result is a slight generalization, using our maps U_k and L_k, of some results due to Neumann.

Definition (5.1.10)[35]: Let $\prod$, be the set of all bijective maps on ℕ and, for any $k \in \mathbb{N}$, denote by $\prod_k \subseteq \prod$, the set of permutations σ such that $\sigma(n) = n$ for every $n > k$. Given $a \in \ell^\infty$(ℕ) and $\sigma \in \prod$, we define:

(a) $\mathrm{a}_\sigma = (a_{\sigma(1)}, a_{\sigma(2)}, \dots)$.

(b) $II.\mathrm{a} = \{\mathrm{a}_\sigma, \sigma \in II\}$, the orbit of a, under the action of II..

(c) $conv(II \cdot \mathrm{a})$, the convex hull of the orbit of a.

(5.1.11)[35]: If b, a are sequences in ℓ_R^1 (ℕ), Neumann [101] proved that the following statements are equivalent:

(a) $b \in cl_{\|\cdot\|}(conv(II \cdot \mathrm{a}))$.

(b) $\sum_{k=1}^\infty b_k = \sum_{k=1}^\infty a_k$ and $U_k(a) \geq U_k(b), L_k(a) \leq L_k(b), k \in \mathbb{N}$.

Proposition(5.1.12)[35]: Let $S \in L^1(\mathcal{H})_h$, and $b \in \ell_R^1$ (ℕ). Then the following statements are equivalent:

(i) $b \in cl_{\|\cdot\|_1}(C[\mathcal{U}_{\mathcal{H}}(S)])$.

(ii) $U_k(S) \geq U_k(b), L_k(S) \leq L_k(b)$ for every $k \in \mathbb{N}$, and $\sum_{k=1}^\infty b_k = trS$.

Proof. $i \Longrightarrow ii$. Note that $cl_{\|\cdot\|_1}(C[\mathcal{U}_{\mathcal{H}}(S)]) \subseteq cl_{\|\cdot\|_\infty}(C[\mathcal{U}_{\mathcal{H}}(S)])$. Hence, by Proposition (5.1.11), $U_k(S) \geq U_k(b)$ and $L_k(S) \leq L_k(b)$ for every $k \in \mathbb{N}$. The equality$\sum_{k=1}^\infty b_k = trS$ clearly holds if $b \in C[\mathcal{U}_{\mathcal{H}}(S)]$. The general case follows from the ℓ^1(ℕ)-continuity of the map $b \longmapsto \sum_{k=1}^\infty b_k$.

$ii \Longrightarrow i$. Let a $\in \ell_R^1(\mathbb{N})$ and $B = \{e_k\}_{k\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$ such that $S = M_{B,a}$. By Lemma (5.1.8) and Proposition (5.1.4), it suffices to show that $cl_{\|\cdot\|_1}\big(conv(II \cdot \text{a})\big) \subseteq cl_{\|\cdot\|_1}(C[\mathcal{U}_{\mathcal{H}}(S)])$.

Claim. $cl_{\|\cdot\|_1}(conv(II \cdot \text{a})) = cl_{\|\cdot\|_1}(conv(\prod_0 \cdot \text{ a})), where\ \prod_0 = \bigcup_{k\in\mathbb{N}} \prod_k$.

Indeed, it is sufficient to prove that $\prod \cdot \text{ a} \subseteq cl_{\|\cdot\|_1}(conv(\prod_0 \cdot \text{ a}))$. Given $\sigma \in \prod, \text{a}_\sigma \in \prod$. a and $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $\sum_{k>N}|a_k| < \varepsilon/2$ and $N_0 \in \mathbb{N}$ such that $\sigma^{-1}(\mathbb{I}_N) \subseteq \mathbb{I}_{N_0}$. There exists $\sigma_0 \in \prod_{N_0}$, such that $\sigma(k) = \sigma_0(k)$ for every $k \in \mathbb{I}_{N_0}$ such that $\sigma(k) \in \mathbb{I}_N$. Therefore,

$$\begin{aligned}\left\|\text{a}_{\sigma} - \text{a}_{\sigma_0}\right\|_1 &= \textstyle\sum_{\sigma(k)\notin\mathbb{I}_N}\left|a_{\sigma(k)} - a_{\sigma_0(k)}\right| \\ &\le \textstyle\sum_{\sigma(k)\notin\mathbb{I}_N}\left|a_{\sigma(k)}\right| + \sum_{\sigma(k)\notin\mathbb{I}_N}\left|a_{\sigma_0(k)}\right| < \varepsilon.\end{aligned}$$

Consider $b \in conv(\prod_0 \cdot \text{ a})$. Then there exists $n \in \mathbb{N}$ such that $b \in conv(\prod_n \text{a})$.

This means that the first n entries of b form a convex combination of permutations of the first n entries of a, and $b_k = a_k$ for every $k > n$. Hence $(b_1, \dots, b_n) \prec (a_1, \dots, a_n)$. Denote $B_n = \{e_k : k \le n\}$ and $\mathcal{H}_n = span\{B_n\}$. Then, by the Schur-Horn Theorem(5.1.2), there exists a unitary $U_0 \in L(\mathcal{H}_n)$ such that

$$M_{B,b}|_{\mathcal{H}_n} = C_{B_n}(U_0^* M_{B,a}|_{\mathcal{H}_n} U_0).$$

Letting

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} \mathcal{H}_n \\ \mathcal{H}_n^{\perp} \end{matrix} \in \mathcal{U}(\mathcal{H})$$

we get that $M_{B,b} = C_B(U^* M_{B,\text{a}} U)$, and $b \in C[\mathcal{U}_{\mathcal{H}}(S)]$. Therefore

$$cl_{\|\cdot\|_1}(conv(II \cdot \text{a})) = cl_{\|\cdot\|_1}\Big(conv\Big(\prod\nolimits_0 \cdot \text{ a}\Big)\Big) \subseteq cl_{\|\cdot\|_1}(C[\mathcal{U}_{\mathcal{H}}(S)]),$$

which completes the proof.

In particular, $cl_{\|\cdot\|_1}(C[\mathcal{U}_{\mathcal{H}}(S)])$ is a convex set. On the other hand, since the maps U_k are convex and the maps L_k are concave for all $k \in \mathbb{N}$, it can be deduced from Theorem (5.1.9) that $cl_{\|\cdot\|_1}(C[\mathcal{U}_{\mathcal{H}}(S)])$ is convex, for every $S \in L(\mathcal{H})_h$. Actually, this fact is known, and can also be deduced from the following results of Neumann [101]:

1. If $S = M_{B,a}$ for some a $\in \ell_R^\infty(\mathbb{N})$ and some orthonormal basis B, then

$$cl_{\|\cdot\|_\infty}\Big(conv\Big(\prod \cdot \text{a}\Big)\Big) = cl_{\|\cdot\|_\infty}(C[u_{\mathcal{H}}(S)]).$$

2. If S is not diagonalizable, then

$$\overline{C[u_{\mathcal{H}}(S)]} = \overline{C[u_{\mathcal{H}}(S^+)]} + [\alpha - (S), \alpha^+(S)]^N + \overline{C[u_{\mathcal{H}}(S_-)]}, \qquad (12)$$

where $a^+(S), a_-(S), S^+, S_-$ are defined in (4) and (5).

Note that formula (12), which holds also for diagonalizable operators, gives another complete characterization of $\overline{C[u_{\mathcal{H}}(S)]}$. It can be used to give an alternative proof of Theorem(5.1.9) , but it can also be deduced from the statement of this theorem, and Proposition(.5.1.5).

Preliminaries on frames. We introduce some basic facts about frames in Hilbert spaces. For a complete description of frame theory and its applications, the reader is referred to

Daubechies, Grossmann and Meyer [94], Aldroubi [88], the review by Heil and Walnut [96] or the books by Young [103] and Christensen [74].

Definition (5.1.13)[35]: Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ a sequence in a Hilbert space $\mathcal{H}$.

(i) $\mathcal{F}$ is called a frame if there exist numbers $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \le B\|f\|^2, \text{ for every } f \in \mathcal{H}. \quad (13)$$

(ii) The optimal constants A, B for Eq. (13) are called the frame bounds for $\mathcal{F}$. The frame $\mathcal{F}$ is called tight if $A = B$, and Parseval if $A = B = 1$. Parseval frames are also called normalized tight frames.

Definition(5.1.14)[35]: Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$be a frame in $\mathcal{H}$. Let K be a separable Hilbert space. Let $B = \{\varphi_n : n \in \mathbb{N}\}$ be an orthonormal basis of K. From Eq. (13), it follows that there exists a unique $T \in L(K, \mathcal{H})$ such that

$$T(\varphi_n) = f_n, \quad n \in \mathbb{N}.$$

We shall say that the triple (T, K, B) is a synthesis (or preframe) operator for $\mathcal{F}$. Another consequence of Eq. (13) is that T is surjective.

Remark (5.1.15)[35]: Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame in $\mathcal{H}$ and (T, K, B) a synthesis operator for $\mathcal{F}$, with $B = \{\varphi_n : n \in \mathbb{N}\}$.

(a) The adjoint $T^* \in L(\mathcal{H}, K)$ of T is given by

$$T^*(x) = \sum_{n\in\mathbb{N}} \langle x, f_n\rangle \varphi_n\,, \;\; x \in \mathcal{H}$$

It is called an analysis operator for $\mathcal{F}$.

(b) By the previous remarks, the operator $S = TT^* \in L(\mathcal{H})^+$, called the frame operator of $\mathcal{F}$, satisfies

$$Sf = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle f_n \text{ , for every } f \in \mathcal{H}. \quad (14)$$

It follows from (13) that $AI \le S \le B1$. So that $S \in \mathcal{G}l(\mathcal{H})^+$. Note that, by formula (14), the frame operator of $\mathcal{F}$ does not depend on the chosen synthesis operator.

Definition(5.1.16)[35]: Let $\mathcal{F} = f_n\}_{n\in\mathbb{N}}$ be a frame in $\mathcal{H}$. The cardinal number

$$e(\mathcal{F}) = dim\left\{(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : \sum_{n\in\mathbb{N}} c_n f_n = 0\right\}$$

is called the excess of the frame. Holub [97] and Balan, Casazza, Heil and Landau [3] proved that, $e(\mathcal{F}) = sup\{\,|I| : I \subseteq \mathbb{N}$ and $\{f_n\}_{n\notin I}$ is still a frame on $\mathcal{H}\}$.

This characterization justifies the name "excess of $\mathcal{F}$". It is easy to see that, for every synthesis operator $(T, K, B)\; of\; \mathcal{F}, e(\mathcal{F}) = dim\, ker\, T$. The frame $\mathcal{F}$ is called a Riesz basis if $e(\mathcal{F}) = 0$, i.e., if the synthesis operators of $\mathcal{F}$ are invertible.

Reformulation of frame admissibility. Recall that, given a sequence $c = (c_k)_{k\in M} \in \ell^\infty(M)^+$ and $S \in \mathcal{G}l\,(\mathcal{H})^+$, we denote by $F(S, c)$ the set of (S, c)-frames, i.e., those frames $\mathcal{F} = \{f_k\}_{k\in M}$ for $\mathcal{H}$, with frame operator S, such that $\|f_k\|^2 = c_k$, for every $k \in \mathbb{M}$, and we say that the pair (S, c) is frame admissible if $F(S, c) \neq \emptyset$. We shall consider the following equivalent formulation of frame admissibility, which makes clear its relationship with the Schur-Horn theorem of majorization theory.

Proposition(5.1.17)[35]: Let $c \in \ell^\infty(\mathbb{M})^+$ and let $S \in \mathcal{G}l(\mathcal{H})^+$. Then the following conditions are equivalent:

(i) The pair (S, c) is frame admissible.

(ii) There exists a sequence of unit vectors $\{y_k\}_{k\in M}$ in $\mathcal{H}$ such that

$$S \sum_{k \in M} c_k y_k \otimes y_k \,,$$

where, if $\mathbb{M} = \mathbb{N}$, the sum converges in the strong operator topology.
(iii) There exists an extension $K = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if we denote

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(K)^+ \qquad (15)$$

Then $c \in C\ [u_K(S_1)]$.
In this case, there exists a frame $\mathcal{F} \in F(S,c)$ with $e(\mathcal{F}) = dim\mathcal{H}_d$

Proof: The equivalence between conditions 1 and 2 is well known (see, for example, [45]). Hence we shall prove 1 ↔3. Assume that $\mathcal{F} = \{f_k\}_{k\in M} \in F(S,c)$. Let (T_0, K_0, B_0) be a synthesis operator for $\mathcal{F}$. Consider the polar decomposition $T_0 = U|T_0|$, where $U : K_0 \to \mathcal{H}$ is a coisometry with initial space $(ker\, T_0)^\perp$ and range $\mathcal{H}$. Note that U^* maps isometrically $\mathcal{H}$ onto ker $T_0^\perp$.
Denote $\mathcal{H}_d = ker\, T_0$, and $K = \mathcal{H} \oplus \mathcal{H}_d$. Let $V: K \to K_0$ be the unitary operator given by

$$V\,(\xi_1, \xi_2) = U^*\xi_1 + \xi_2 \,, for\ (\xi_1, \xi_2) \in \mathcal{H} \oplus \mathcal{H}_d = K.$$

Consider the orthonormal basis $B = V^*(B_0)$ of K, and $T = T_0 V \in L(K, \mathcal{H})$.
Then (T, K, B) is another synthesis operator for $\mathcal{F}$, with ker $T = \mathcal{H}_d$.
Let $T_1 \in L(K)$ given by $T_1 \xi = T\xi \oplus 0_{\mathcal{H}_d}, \xi \in K$. Then $T_1^* T_1 = T^* T_1 = T^* T$, $|T_1| = |T|$, and $T_1 T_1^* = \begin{pmatrix} TT^* & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} = S_1$
If $T_1 = U_1|T_1| = U_1|T|$ is the polar decomposition of T_1, then U_1acts on $\mathcal{H} = (ker\, T_1)^\perp$ as a unitary operator. Hence $W = U_1 + P_{\mathcal{H}_d}\, u(K)$. Since $T_1 = W|T|$,

$$S_1 = T_1 T_1^* = W|T|^2 W^* = W(T^*T)W^* \Rightarrow W^* S_1 W = T^*T.$$

On the other hand, if $B = \{e_k\}_{k\in\mathbb{N}}$, then $\langle T^*Te_k, e_k\rangle = \langle Te_k, Te_k\rangle = \|f_k\|^2 = c_k$ for every $k \in \mathbb{M}$. Therefore,

$$C_B\,(W^*S_1W) = C_B(T^*T) = M_{B,c} \Rightarrow c \in C\ [u_K(S_1)]\,.$$

Conversely, suppose that there exists an extension $K = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ and $V \in u(K)$ such that $M_{B,c} = C_B(V^*S_1V)$, for some orthonormal basis $B = \{e_k\}_{k\in\mathbb{N}}$of K. Let $T = S_1^{1/2}V$. Since S is invertible, we have $R(T) = \mathcal{H}$ and $dim\, ker\, T = dim\mathcal{H}_d$. Thus $\mathcal{F} = \{Te_k\}_{k\in\mathbb{N}}$ is a frame for $\mathcal{H}$, with frame operator $T\,T^*|_{\mathcal{H}} = S_1|_{\mathcal{H}} = S$. Since $T^*T = V^*S_1V$ and $C_B(V^*S_1V) = M_{B,c}$, we have $\|Te_k\|^2 = \langle T\,T^*e_k, e_k\rangle = c_k$, for every $k \in \mathbb{M}$. Hence $\mathcal{F} \in F(S,c)$ with $e(\mathcal{F}) = dim\mathcal{H}_d$.
The finite-dimensional case. In this section we assume that $\mathcal{H}$ is finite dimensional. We shall consider separately the cases of frames of finite or infinite length. Suppose that $S \in M_n(\mathbb{C})^+$ and $|M| = m < \infty$. In this case, the classical Schur-Horn Theorem(5.1.2) gives a complete characterization of frame admissibility for (S,c).

Theorem(5.1.18)[35]: Let $c \in \mathbb{R}^m_{>0}$ and let $S \in \mathcal{G}l_n(\mathbb{C})^+$, with eigenvalues $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Then, the pair (S,c) is frame admissible if and only if

$$\sum_{i=1}^k b_i \geq \sum_{i=1}^k c_i \text{ for } 1 \leq k \leq n-1, \text{ and } \sum_{i=1}^n b_i = \sum_{i=1}^m c_i$$

In other words, if $c \prec (b_1, \ldots, b_n, 0, \ldots, 0) \in \mathbb{R}^m$.
This result was obtained in [92] and [100], from an operator theoretic point of view. Actually the proofs given there can be adapted so as to obtain a proof of the classical

Schur-Horn theorem that is quite conceptual and simpler than those in the literature. Now, we consider frame admissibility for infinite sequences in finite dimensional Hilbert spaces. The case $S = I$ of the next result appeared in [91].

Theorem(5.1.19)[35]: Let $c \in \ell^\infty(\mathbb{N})^+$. Let $S \in \mathcal{G}l_n(\mathbb{C})^+$, with eigenvalues $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Then the following conditions are equivalent:

(a) The pair (S, c) is frame admissible.

(b) $\sum_{i=1}^{k} b_i \geq U_k(c)$, for every $1 \leq k \leq n-1$, and $\sum_{i=1}^{n} b_i = \sum_{i\in\mathbb{N}} c_i$

Proof. Let $b = (b_1, \ldots, b_n, 0, \ldots, 0, \ldots) \in \ell^\infty(\mathbb{N})^+$.

$(b) \Rightarrow (a)$: Let $\mathcal{H}$ be a infinite dimensional Hilbert space, and consider

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in L(\mathbb{C}^n \oplus \mathcal{H}).$$

Then there exists an orthonormal basis $B = \{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{K} = \mathbb{C}^n \oplus \mathcal{H}$ such that $S_1 = M_{B,b}$. Hence, by Proposition (5.1.4),

$$U_k(S_1) = \sum_{i=1}^{k} b_i\, U_k(c), for\ every\ k \in \mathbb{N}.$$

On the other hand, note that $L_k(S_1) = 0 \leq L_k(c)$ for every $k \in \mathbb{N}$ and $\sum_{i=1}^{n} b_i = \sum_{i\in\mathbb{N}} c_i$. Then, by Proposition(5.1.12) , there exists a sequence $\{V_m\}_{m\in\mathbb{N}}$ in $u(K)$ such that

$$C_B\,(V_m^* S_1 V_m) \underset{m\to\infty}{\overset{\|\ \|_1}{\longrightarrow}} M_{c,}$$

where $\|A\|_1 = tr\,|A|$. Therefore, by Proposition(5.1.21), there exists a norm bounded sequence of epimorphisms $T_m : K \to \mathbb{C}^n$ such that $T_m T_m^* = S$ for all $m \in \mathbb{N}$, and $(\|T_m(e_i)\|^2)_{i\in\mathbb{N}} \underset{m\to\infty}{\overset{\ell^1(\mathbb{N})}{\longrightarrow}} c$. Then, by a standard diagonal argument, we can ensure the existence of a subsequence, which we still call $\{T_m\}_{m\in\mathbb{N}}$, such that $T_m(e_i) \underset{m\to\infty}{\longrightarrow} f_i \in \mathbb{C}^n$, with $\|f_i\|^2 = c_i$ for every $i \in \mathbb{N}$.

Let $T_0 : span\,\{B\} \to \mathbb{C}^n$ be the unique (densely defined) operator such that $T_0(e_i) = f_i$ for every $i \in \mathbb{N}$. Note that T_0 is bounded because, if $x = \sum_{i=1}^{r} \alpha_i e_i$ and $C = \sum_{i\in\mathbb{N}} c_i = tr\,S$, then

$$\|T_0(x)\| = \left\|\sum_{i=1}^{r} \alpha_i f_i\right\| \leq \sum_{i=1}^{r} |\alpha_i|\|f_i\|$$
$$\leq (\textstyle\sum_{i=1}^{r} c_i)^{1/2} (\sum_{i=1}^{r} |\alpha_i|^2)^{1/2} \leq C^{1/2}\|x\|$$

The bounded extension of T_0 to K is denoted T.

Claim. $\|T_m - T\| \underset{m\to\infty}{\longrightarrow} 0$

Indeed, let $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ be such that $\sum_{i=i_0}^{\infty} c_i < \varepsilon$. Then there exists $m_i \in \mathbb{N}$ such that

$$\textstyle\sum_{i=i_0}^{\infty} \|T_m(e_i)\|^2 \leq \varepsilon \text{ for every } m \geq m_1 \qquad (16)$$

This is a consequence of the fact that $(\|T_m(e_i)\|^2)_{i=i_0}^{\infty} \underset{m\to\infty}{\overset{\ell^1(\mathbb{N})}{\longrightarrow}} (c_i)_{i=i_0}^{\infty}$. On the other hand, there exists $m_2 \geq m_1$ such that

$$\textstyle\sum_{i=i_0}^{i_0-1} \|T_m(e_i) - f_i\|^2 \leq \text{ for every } m \geq m_2 \qquad (17)$$

Let $m \geq m_2$ and $x = \sum_{i=1}^{r} \alpha_i e_i \in span\,\{B\}$. By equations (16) and (17),

$$\|(T_m - T)(x)\|^2 \leq \left(\sum_{i=1}^{r} |\alpha_i|^2 \left(\sum_{i=1}^{r} \|(T_m - T)(e_i)\|^2\right)\right)$$

$$\leq \|x\|^2 \left(\sum_{i=1}^{i_0-1} \|(T_m - T)(e_i)\|^2 + 2\sum_{i=i_0}^{\infty} \|T_m(e_i)\|^2 + \|T(e_i)\|^2\right) \leq 5\varepsilon \|x\|^2$$

which proves the claim. Therefore

$$TT^* = \lim_{m\to\infty} T_m T_m^* = S.$$

We have proved that the frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}} \in F(S, c)$.

$(a) \Rightarrow (b)$: This follows from Theorem (51.9), applied to S_1 and c, and Proposition(.5.1.17).

Indeed, suppose that $S_1 \neq 0$ (the case $S_1 = 0$ is trivial). Then there exists a sequence $b = (b_1, \ldots, b_m, 0, \ldots, 0, \ldots) \in \ell^1(\mathbb{N})^+$, with $b_m > 0$, and an orthonormal basis $B = \{e_n\}_{n\in\mathbb{N}}$of K such $that\ S_1 = M_{B,b}$. Let $c \in \ell^1(\mathbb{N})^+$. By Proposition(5.1.12), Condition 2 of Theorem(5.1.23) means that c$\in cl_{\|\cdot\|_1}(C[u_K(S_1)])$. But, by Proposition(5.1.12), Condition 1 of Theorem(5.1.23) means that $c \in C[u_K(S_1)]$.

Note that, although $cl_{\|\cdot\|_1}$ (conv($\prod\cdot\ b$)) $= cl_{\|\cdot\|_1}$ $(C[u_K(S_1)]) = C[u_K(S_1)]$, it is not true that conv($\prod\cdot\ b$) is closed, as a subset of $\ell^1(\mathbb{N})^+$. For example, if $b = (1, 0, 0, \ldots)$, then, by Proposition (5.1.12),

$$c = \left(\frac{1}{2^n}\right)_{n\in\mathbb{N}} \in cl_{\|\cdot\|_1}(C[u_K(e_1 \otimes e_1)]) = cl_{\|\cdot\|_1}\text{conv}(\prod\cdot\ b)$$

Nevertheless, $c \notin conv(\prod\cdot\ b)$, because every sequence in conv($\prod\cdot\ b$)has finite nonzero entries. In this case, $c = C_B(x \otimes x) \in C[u_K(e_1 \otimes e_1)]$, where

$$x = \sum_{n\in\mathbb{N}} 2^{-\frac{n}{2}} e_n$$

Throughout this section $\mathcal{H}$ denotes a separable infinite dimensional Hilbert space. The first result gives necessary conditions for frame admissibility:

Theorem(5.1.20)[35]: Let $S \in \mathcal{G}l_n(\mathcal{H})^+$ and $c \in \ell^\infty(\mathbb{N})^+$. If the pair (S, c) is frame admissible, then

$$\sum_{i\in\mathbb{N}} c_i = \infty, \text{ and}$$

$U_k(S) \geq U_k(c)$, for every $k \in \mathbb{N}$. In particular, $lim\ sup\ c \leq \|S\|_e$

Proof. Suppose that there exists a frame $\mathcal{F} \in F(S, c)$. Then, by Proposition (5.1.17), there exists an extension $K = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if we denote

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(K)^+$$

then $c \in C\,[u_K(S_1)]$. Hence, $\sum_{i\in\mathbb{N}} c_i = trM_c = tr\,S_1 = \infty$. On the other hand, by Proposition(5.1.5), $U_k(S) = U_k(S_1)$ for every $k \in \mathbb{N}$. Then, applying Theorem(5.1.9) , the statement follows.

In [100] (see also [91]) there is the following result which gives sufficient conditions for a pair (S, c) in order to be frame admissible:

Theorem (5.1.21)[35]: (Kornelson-Larson). Let $S \in \mathcal{G}l_n(\mathcal{H})^+$ and $c \in l^\infty(\mathbb{N})^+$.

Suppose that $\sum_{i\in\mathbb{N}} c_i = \infty$ and $\|c\|_\infty < \|S\|_e$. Then the pair (S, c) air is frame admissible.

The following result, which generalizes Theorem(5.1.21), strengths slightly the necessary conditions for frame admissibility given by Theorem(5.1.20), to get sufficient conditions. A tight frame version of this result appeared in R. Kadison [98] and [99]. Recall the notation $P_2(S) = E[\|S\|_e, \|S\|\,]$, where E is the spectral measure of $S \in L(\mathcal{H})^+$.

Theorem(5.1.22)[35]: Let $S \in \mathcal{G}l\,(\mathcal{H})^+$and $c \in l^\infty(\mathbb{N})^+$, such that $\sum_{i\in\mathbb{N}} c_i = \infty$. Assume one of the following two conditions:

(i) (a) $tr\, P_2(S) = \infty$,

(b) $U_k(S) \geq U_k(c)$ for every $k \in \mathbb{N}$, and

(c) $\|S\|_e > lim\, sup(c)$.

(ii) (a) $tr\, P_2(S) = r \in \mathbb{N}$,

(b) $U_k(S) > U_k(c) for\ 1 \leq k \leq r,$

(c) $U_k(S) > U_k(c), for\ k > r$, and

(d) $\|S\|_e > lim\, sup(c)$.

Then, the pair (S, c) is frame admissible.

Proof. By Proposition(5.1.13), it suffices to show that there exists a sequence of unit vectors $\{x_k\}_{k\in\mathbb{N}}$ such that $S = \sum_{k\in\mathbb{N}} c_k x_k \otimes x_k$. Assume that the first condition holds. Then, since $\|S\|_e > lim\, sup(c)$, there exist $m_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that

$c_m \leq \|S\|_e - \varepsilon$ for $m \geq m_0$.

Let $\mu_1 \geq \mu_2 \cdots \geq \mu_n \geq \cdots$ be the sequence of eigenvalues of S^+, chosen as in Lemma (5.1.5) Let $\{y_n\}_{n\in\mathbb{N}}$ be an orthonormal system such that $S^+ y_n = \mu_n y_n$. Denote $\lambda_n = \mu_n + \|S\|_e, n \in \mathbb{N}$. Note that $\|S\| \geq \lambda_n \geq \|S\|_e\ and\ \ Sy_n = \lambda_n\, y_n,\ \ n \in \mathbb{N}$. By Proposition(5.1.6), for every $k \in \mathbb{N}$,

$\sum_{i=1}^{k} \lambda_i\, y_i \otimes y_i \leq S$, and $U_k(S) = \sum_{i=1}^{k} \lambda_i$

Let n_0 be the first integer such that $\sum_{i=1}^{n_0} c_i > \sum_{i=1}^{m_0} \lambda_n$

Then $n_0 \geq m_0 + 1$, and $h = \sum_{i=1}^{n_0} c_i - \sum_{i=1}^{m_0} \lambda_i \leq c_{n_0} < \|S\|_e \leq \lambda_{m_0+1}$

Let $c_0 = (c_1, \dots, c_{n_0})$. Since

$$\sum_{i=1}^{k} \lambda_i = U_k(S) \geq U_k(c) \geq U_k(c_0), \qquad 1 \leq k \leq m_0$$

we have c0 $c_0 \prec \left(\lambda_i, \dots, \lambda_{m_0}, h, 0, \dots, 0\right) \in \mathbb{R}^{n_0}$. Denote

$$S_1 = h_{m_0+1} \otimes ym_{0+1} + \sum_{i=1}^{m_0} \lambda_i\, y_i \otimes y_i \leq S$$

and $S_2 = S - S_1$. Then the pair (S_1, c_0), acting on span $\{y_1, \dots, y_{m_0+1}\}$, satisfies the conditions of Theorem (5.1.17). Hence, there exists a set of unit vectors $\{x_1, \dots, x_{n_0}\}$ such that $\sum_{i=1}^{n_0} c_i x_i \otimes x_i = S_1$. Note that $S_2 \geq 0$, $R(S_2)$ is closed (by Fredholm theory), and $\|S_2\|_e = \|S\|_e$. Then we can apply Theorem(5.1.20) to the pair $(S_2, \{c_i\}_{i>n_0})$, acting on $R(S_2)$. So, there exist unit vectors x_k, for $k > n_0$, such that

$$S_2 = \sum_{i=n_0+1}^{\infty} c_i x_i \otimes x_i.$$

Therefore we obtain the rank-one decomposition $S = \sum_{i\in\mathbb{N}} c_i x_i \otimes x_i$.

Assume Condition 2. Note that, by equations (8) and (11), the condition

$\|S\|_e > lim\, sup(c)$ implies that $U_m(S) - U_m(c) \underset{\to\infty}{\longrightarrow} \infty$

(i) Therefore, by item
(c), we can assume that there exists $\delta > 0$ such that
(i) $U_{r+k}(S) - \delta > U_{r+k}(c)$, for every $k \in \mathbb{N}$.
(ii) There exists $m_0 \geq 1$ such that $c_m \leq \|S\|_e - \delta$ for $m \geq m_0$.
Let $m_1 = max\{m_0, r + 1\}$. Let $\mu_1 \geq \cdots \geq \mu_r$ be the greatest eigenvalues of S^+, and let $\{y_1, \ldots, y_r\}$ be an associated orthonormal set of eigenvectors.
Denote $\curlywedge_i = \mu_i + \|S\|_e, \ 1 \leq i \leq r$ and $\curlywedge_i = \|S\|_e - \frac{\delta}{2m_1}, \ r+1 \leq i \leq m_1 + 1$_ Then, by Proposition (5.1.6),
(a) $U_k(S) = \sum_{i=1}^{k} \curlywedge_i$, for $1 \leq k \leq r$, and
(b) $U_k(c) \leq U_k(S) - \delta \leq \sum_{i=1}^{k} \curlywedge_i$, for $r + 1 \leq k \leq m_1 + 1$.
On the other hand, since $Q = E([\|S\|_e - \delta/2m_1, \|S\|_e)$) has infinite rank, there exists an orthonormal set $\{y_{r+1}, \ldots, y_{m_1} + 1\} R(Q)$. Therefore

$$\sum_{i=1}^{m_1+1} \curlywedge_i \, y_i \otimes y_i \leq S$$

Let n_0 be the first integer such that $\sum_{i=1}^{n_0} c_i > \sum_{i=1}^{m_0} \curlywedge_i$. Then $n_0 \geq m_0 + 1$
and

$$h = \sum_{i=1}^{n_0} c_i - \sum_{i=1}^{m_0} \curlywedge_i \leq c_{n_0} \leq \|S\|_e \leq \curlywedge_{m_0} + 1$$

$\sum_{i=1}^{k} \curlywedge_i = U_k(S) \geq U_k(c) \geq U_k(c_0)$, $1 \leq k \leq m_0$, and
$\sum_{i=1}^{k} \curlywedge_i \geq U_k(S) - \delta \geq U_k(c) \geq U_k(c_0)$, $r + 1 \leq k \leq m$,
we have $c_0 \prec (\curlywedge_1, \ldots, \curlywedge_{m_0}, h, 0, \ldots, 0) \in \mathbb{R}^{n_0}$. Denote
$S_{1 = h y_{m_0+1} \otimes} y_{m_0+1} + \sum_{i=1}^{m_0} \curlywedge_i \, y_i \otimes y_i \leq S$ and $S_2 = S - S_1$ then the pair (S_1, c_0) $acting\ on\ span$ $\{y_1 \ldots . y_{m_0+1}\}$ satisfies the conditions of Theorem (5.1.18).Hence there exists a set of unit vectors $\{x_1, \ldots, x_{n_0}\} \subseteq \mathcal{H}$ such that $\sum_{i=1}^{n_0} c_i x_i \otimes x_i = S_1. Note\ that\ S_1 \geq 0, R(S_2)$ is closed (by Fredholm theory),and $\|S_2\|_e = \|S\|_e$.Then we can apply Theorem (5.1.20)to the pair $(S_1, \{c_i\}_{i > n_0}), acting\ on\ R(S_2)$. So there existunit vectors $x_{k,}$for $k > m_0$, Such that $S_2 = \sum_{i=n_0}^{\infty} c_i x_i \otimes x_i$.
Therefore we obtain the rank-one decomposition $S = \sum_{i \in \mathbb{N}} c_i x_i \otimes x_i$
Example (5.1.23)[35]: below shows that the Condition 2 (c) of Theorem (5.1.28) can not be dropped in general.
Corollary(5.1.24)[35]: Let $0 < A \in \mathbb{R}$ and $c \in \ell^{\infty}(\mathbb{N})^+$ be such that $0 < c_i \leq A$, $i \in \mathbb{N}$. Denote $J = \{i \in \mathbb{N} : c_i = A\}$. Assume that $\sum_{i \notin J} c_i = \infty$, and $\limsup_{i \notin J} c_i < A$ (or, equivalently, $\sup_{i \notin J} c_i < A$).
Then the pair (AI, c) is admissible. This means that there exists a tight frame. Some with norms prescribed by c and frame constant A.
In the following example we shall see that

$$U_k(S) > U_k(c), k \in \mathbb{N}, and \ \|S\|_e = lim\,sup(c) \nRightarrow F(S, c) \neq \emptyset .$$

Example(5.1.25) [35]: Let $S = I \in L(\mathcal{H})$ and $a \in (0, 1)$. Let $c \in \ell^{\infty}(\mathbb{N})^+$ be given by $c_1 = p \in (0, 1)$ and

$$c_k = \begin{cases} a^k & if\ k \neq 1\ is\ odd \\ 1 - a^k & if\ k\ \ is\ even \end{cases}$$

Then $0 < c_k < 1$ for $k \in \mathbb{N}, \sum_k c_k = \infty = \sum_k (1 - c_k)$, and $lim\ sup\ c = 1 = \|S\|_e$. Suppose that there exists a frame $\mathcal{F} = \{f_k\}_{k \in \mathbb{N}}\ F(S, c)$. Then

$$\|x\|^2 = \sum_{k \in \mathbb{N}} |\langle x, f_k \rangle|^2, \text{ for every } x \in \mathcal{H}.$$

In particular, we get, for every $j \in \mathbb{N}$,

$$\|f_j\|^2 = \sum_{k \in \mathbb{N}} |\langle f_j, f_k \rangle|^2 = \|f_j\|^4 + \sum_{k \neq j} |\langle f_j, f_k \rangle|^2$$

Thus, if $j \neq 1$, we obtain the inequality

$$|\|f_1, f_j|^2 = |f_j, f_1|^2 \leq \sum_{k \neq j} |\langle f_j, f_k \rangle|^2 = \|f_j\|^2 = \|f_j\|^4 = c_j(1 - c_j)$$

Therefore ,

$$p = \|f_1\|^2 \leq \|f_1\|^4 + \sum_{k \neq 1} c_j(1 - c_j) \qquad (18)$$

Taking $p = \frac{1}{2}$ and $a \in (0, 1)$ such that

$$\frac{a}{1 - a^2} < \frac{1}{4}$$

we get that

$p > p^2 + \frac{a}{1 - a^2}$,

contradicting Eq. (18). Hence, in this case, $F(S, c) = \emptyset$. Note that the pair (S, c) satisfies all of the necessary conditions of Theorem(5.1.20), because $U_k(S) = k = U_k(c)$ for every $k \in \mathbb{N}$.

In the second example we see that, in general,

$$U_k(S) \geq U_k(c), k \in \mathbb{N}\ and\ \|S\|_e > lim\ sup(c) \nRightarrow F(S, c) \neq \emptyset .$$

Example (5.1.26)[35]: Let $S = M_s$ be the diagonal operator, with respect to an orthonormal basis of $\mathcal{H}$, given by $s = \{1 - (i + 1)^{-1}\}_{i \in \mathbb{N}}$, and let $(c_i)_{i \in \mathbb{N}}$ be given by $c_1 = 1$ and $c_i = 1/2$ for every $i \geq 2$. Note that

(i) $1 = \|S\|_e > 1/2 = lim\ sup(c)$,

(ii) $U_1(S) = U_1(c)$, and

(iii) $U_k(S) = k > 1 + (k - 1)/2 = U_k(c)$ for every $k \geq 2$.

Still, we have $F(S, c) = \emptyset$. Indeed, suppose that there exists $\mathcal{F} \in F(S, c)$.

Then, by Proposition (5.1.20) there exists an extension $K = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(K)^+ ,$$

then $c \in C\ [u_K(S_1)]$. Let $V \in u(K)$ be such that, in a orthonormal basis $B = \{e_k\}_{k \in \mathbb{N}}$, $M_c = C_B(V^* S_1 V)$. Take $x = P_{\mathcal{H}} V e_1$. We have that $\|x\| \leq 1$ and $\langle Sx, x \rangle = \langle M_c e_1, e_1 \rangle = c_1 = 1$, while $\|S\| = 1$. Then $Sx = x$, and 1 would be an eigenvalue of S, which is false. In this example, Condition 2 (c) of Theorem (5.1.22) does not hold, because $\|S\| = \|S\|_e$, which implies that $r = tr\ P_2(S) = 0$; but $U_1(S) = 1 = U_1(c)$. Note that $\sum_k c_k = \infty = \sum_k (1 - c_k)$ as in the previous example.

The excess of frames in $F(S, c)$. Let $S \in \mathcal{G}l(\mathcal{H})^+$ and $c = (c_i)_{i \in \mathbb{M}} \in \ell^\infty(\mathbb{M})^+$ be such that the pair (S, c) is frame admissible. Then there can be many different types of frames $\mathcal{F} \in F(S, c)$. We consider the set

$$Null(S, c) = \{ e(\mathcal{F}) : \mathcal{F} \in F(S, c) \}.$$

In the example below, we show that this set can be arbitrarily large. Moreover, this example shows that there exists an admissible pair (S,a), satisfying just the necessary conditions of Theorem(5.1.20), and in this case $U_k(S) = U_k(a)$, $k \in \mathbb{N}$, and $m\ sup\ a = \|S\|_e$.

Example(5.1.33)[35]: Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $B = \{x_n\}_{n\in\mathbb{N}}$. Let

$$\mathrm{a} = \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \dots\right) \in \ell^\infty(\mathbb{N})^+, \text{ and } S = M_{B,\mathrm{a}} \in \mathcal{G}l(\mathcal{H})^+$$

Then the frame (Riesz basis) $\mathcal{F}_0 = \{a_n^{\frac{1}{2}} x_n\}_{n\in\mathbb{N}}$ has frame operator S, so that $\mathcal{F}_0 \in F(S,a)$. On the other hand, let

$$\mathcal{F}_1 = \left\{\frac{1}{\sqrt{2}}x_2, x_4, \frac{1}{\sqrt{2}}x_2, x_6, \frac{1}{\sqrt{2}}x_1, x_8, \frac{1}{\sqrt{2}}x_3, x_{10}, \dots\right\}$$

It is easy to see that also $\mathcal{F}_1 \in F(S,a)$, but $e(\mathcal{F}_1) = 1$. Analogously,

$$\mathcal{F}_2 = \left\{\frac{1}{\sqrt{2}}x_2, x_4, \frac{1}{\sqrt{2}}x_2, x_6, \frac{1}{\sqrt{2}}x_8, x_{10}, \frac{1}{\sqrt{2}}x_8, x_{12}, \frac{1}{\sqrt{2}}x_1, \dots\right\} \in F(\mathrm{S},\mathrm{a})$$

with $e(\mathcal{F}_2) = 2$. In a similar way, we can construct frames $\mathcal{F}_k \in F(\mathrm{S},\mathrm{a})$ with $e(\mathcal{F}_k) = k$, for every $k \in \mathbb{N} \cup \{\infty\}$. Note that

$$\mathcal{F}_\infty = \left\{\frac{1}{\sqrt{2}}x_1, x_4, \frac{1}{\sqrt{2}}x_2, x_8, \frac{1}{\sqrt{2}}x_2, x_{12}, \frac{1}{\sqrt{2}}x_3, x_{16}, \frac{1}{\sqrt{2}}x_6, x_{20} \frac{1}{\sqrt{2}}x_6, \dots\right\}.$$

In other words, $\mathcal{F}_\infty$ is the frame induced by the bounded operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ given by

$$T(c_n) = \begin{cases} x_{4k} & if\ n = 2k\ , \\ \frac{1}{\sqrt{2}}x_{2k-1} & if\ n = 6k-5, \\ \frac{1}{\sqrt{2}}x_{4k} - 2 & if\ n = 6k-3, \\ \frac{1}{\sqrt{2}}x_{4k} - 2 & if\ n = 6k-1. \end{cases}$$

Therefore Null $(S,a) = \mathbb{N} \cup [\ \{0,\infty\}$.

Proposition (5.1.28)[35]: Let $S \in (\mathcal{H})^+$ and $c \in \ell^2(\mathbb{N})^+$. Assume that the pair (S,c) is frame admissible and $lim\ inf\ c < min\ \sigma_e(S)$. Then Null $(S,c) = \{\infty\}$.

Proof: Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}} \in F(S,c)$, with $e(\mathcal{F}) = d$. By Proposition(5.1.16) there exists an extension $K = \mathcal{H} \oplus \mathcal{H}_d$ of $\mathcal{H}$ such that, if we denote

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_d \end{matrix} \in L(K)^+$$

then $c \in C\ [u_K(S_1)]$. By Theorem 3.10, $min\ \sigma_e(S_1) \le lim\ inf\ c$. But, if $dim\mathcal{H}_d = e(\mathcal{F}) < \infty$, then $\sigma_e(S_1) = \sigma_e(S)$, which contradicts the fact that

$$lim\ inf\ c < min\ \sigma_e(S).$$

Example (5.1.29)[35]: Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $B = \{x_i\}_{i\in\mathbb{N}}$. Let $\mathrm{a} = (1,2,1,2,\dots)$, $S = M_{B,a} \in \mathcal{G}l(\mathcal{H})^+$ and $c = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \dots\right)$.

We shall show that also $Null(S,c) = \mathbb{N} \cup \{0,\infty\}$. Note that, in this case,

$$\alpha_-(S) = 1 < lim\ inf\ c = \frac{3}{2} = lim\ sup\ c < 2 = \|S\|_e$$

Indeed, take the Riesz basis $\mathcal{F}_0 = \{f_n\}_{n\in\mathbb{N}}$ given by

$$f_n = \begin{cases} \frac{x_n}{\sqrt{2}} + x_{n+1} & if\ n\ is\ odd \\ \frac{-x_{n-1}}{\sqrt{2}} + x_n & if\ \ n\ is\ even \end{cases}$$

It is easy to see that $\mathcal{F}_0 \in F(S,c)$. Using that

$$\left(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}\right) \prec (2,2,2,0),$$

an arbitrary number of packs of four vectors with norm $\sqrt{3/2}$ associated to packs of three even places of the diagonal of S can be interlaced into the previous construction. Each of these packs adds excess 1 to the whole system.

In this way, frames $\mathcal{F}_k \in F(S,c)$ with $e(\mathcal{F}_k) = k$ can be found for every

$$k \in \mathbb{N} \cup [\ \{\infty\}.$$

Section(5-2): Equal-Norm Parseval Frames

A family of vectors $\{f_j\}_{j\in J}$ is a frame for a Hilbert space $\mathcal{H}$ if it provides a stable embedding of $\mathcal{H}$ in $\ell^2(J)$ when each vector in $\mathcal{H}$ is mapped to the sequence of its inner products with the frame vectors. Frames were defined by Duffin and Schaeffer [75] to address some deep questions in non-harmonic Fourier series. Traditionally, frames were most popular in signal processing [77], but today, frame theory has an abundance of applications in pure mathematics, applied mathematics, engineering, medicine and even quantum communication [67,74,77,82,61,64].

Many of these applications give rise to design problems in frame theory, the construction of frames with certain desired properties. Digital transmissions of analog signals, for example, often rely on frames because of their built-in resilience to data loss [48,47], and it has been shown that encoding with equal-norm Parseval frames has certain optimality properties for this purpose [70] (see also [49,65]). Moreover, the use of frames for compensating quantization errors has relied on equal-norm Parseval frames as well [63,66]. Despite their popularity, we know only a few ways to construct such frames analytically [79,36,68,72], mostly with the help of group actions. Success has been claimed for generating a special type of equal-norm Parseval frames with numerical methods [57], however, the analytic verification of convergence remains wanting. The use of frame potentials [36,69] shows the existence of large numbers of equal-norm Parseval frames, but offers little control over additional properties (see [49,72]). Finally, there is an algorithm due to Holmes and Paulsen [49] for turning a Parseval frame into an equal-norm Parseval frame in finitely many moves. Unfortunately, to the best of the authors' abilities, it cannot be combined with the numerical results to provide the existence of an equal-norm Parseval frame in the close vicinity of a nearly equal-norm and nearly Parseval frame, because it does not include a distance estimate. Here, the metric on the set of frames is induced by the norm on the Hilbert space when frames are viewed as vector-valued, square summable functions (see Section 2 for precise definitions).

The closest Parseval frame to a frame $\{f_j\}_{j\in J}$ is known [62,68,71,80]. Also, the closest equal norm frame to a given frame can be found easily [68]. However, despite a significant amount of effort, so far we knew very little about the closest equal-norm Parseval frame to a given frame. This question is known in the field as the Paulsen problem. The main problem here is that finding a close equal-norm frame to a given

frame involves a geometric condition while finding a close Parseval frame involves an algebraic or spectral condition.
We will present the first method for finding an equal-norm Parseval frame in the vicinity of a given frame which gives quantitative estimates for the distance. The new technique we introduce is a system of vector-valued $ODEs$ which induces a flow on the set of Parseval frames that converges to equal-norm Parseval frames. We then bound the arc length traversed by a frame by an integral of the so-called frame energy. With an exponential bound on the frame energy, we derive a quantitative estimate for the distance between our initial, _ϵ-nearly equal-norm and ϵ-nearly Parseval frame $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ for a d-dimensional real or complex Hilbert space and the equal-norm Parseval frame $\mathcal{G} = \{g_1, g_2, \ldots, g_n\}$ obtained as the limit of the flow governed by the ODE system,

$$\left(\sum_{j=1}^{n} \|f_j - g_j\|^2\right)^{1/2} \leq \frac{29}{8} d^2 n(n-1)^8 \epsilon.$$

We also show that the order of ϵ in this estimate cannot be improved.
For our method to work, we must assume that the dimension d of the Hilbert space and the number n of frame vectors are relatively prime. We will use a tensor product technique to show that if our main goal is to produce equal-norm Parseval frames, this is not a serious restriction.
Finally, we show that the Paulsen problem is equivalent to a fundamental problem in matrix theory, and so we find an answer for the corresponding case of this problem.
We believe that the techniques introduced in this section will have application to other "nearness" questions in frame theory, in particular, to the famous equiangular tight frame problem [49,85]. Finding and classifying such frames, or even the easier problem of finding equiangular lines through the origin in $\mathbb{R}^n$ or $\mathbb{C}^n$, started in 1948 by Haantjes [78,73], still leaves a lot to be done. This type of equal-norm Parseval frames is particularly important because of their applications to signal processing [60,85,65,86,81] and to quantum information theory [87,84,76,64].
In this section, we introduce the notation and terminology used throughout the section .
Definition(5. 2.1)[37]: A family of vectors $\mathcal{F} = \{f_j\}_{j\in J}$ is a frame for a Hilbert space $\mathcal{H}$ if there are constants $0 < A \leq B < \infty$ so that

$$A\|x\|^2 \leq \sum_{j\in J} \left|x, f_j\right|^2 \leq B\|x\|^2 \ for \ all \ x \in \mathcal{H}.$$

We call the largest A and smallest B the lower and upper frame bounds respectively. If we can choose $A = B$ then $\mathcal{F}$ is a tight frame and if $A = B = 1$ it is a Parseval frame. If all the frame vectors have the same norm, it is an equal-norm frame. The analysis operator of the frame is the map $V : \mathcal{H} \to \ell^2(J)$ given by $(Vx)_j = \langle x, f_j\rangle$. Its adjoint is the synthesis operator which maps $a \in \ell^2(J)$ to $V^*(a) = \sum_{j\in J} a_j f_j$. The frame operator is the positive, self-adjoint invertible operator $S = V^*V$ on $\mathcal{H}$ and the Grammian is the matrix G with entries $G_{j,k} = \langle f_j, f_k\rangle$_ so that

$$G_{j,k} = (VV^*)_{k,j}, k, j \in \{1,2, \ldots, n\}.$$

Definition(5. 2.2)[37]: (1) A frame $\{f_j\}_{j=1}^{n}$ for a d-dimensional real or complex Hilbert space $\mathcal{H}$ is ϵ-nearly equal-norm with constant c if

$$(1-\epsilon)c \le \|f_j\| \le (1+\epsilon)c, \quad \text{for all } j \in 1,2,\dots,n\}.$$

(2) The frame is ϵ-nearly Parseval if the frame constants can be chosen as $A = 1-\epsilon$ and $B = 1+\epsilon$ so for all $x \in \mathcal{H}$,

$$(1-\epsilon)\|x\|^2 \le \sum_{j\in J} \left|\langle x, f_j\rangle\right|^2 \le (1+\epsilon)\|x\|^2.$$

If a frame satisfies either of these properties (1) or (2) with $\epsilon = 0$ then we say that it is an equal-norm frame or a Parseval frame, respectively.

Definition(5. 2.3)[37]: The ℓ^2-distance between two frames $\mathcal{F} = \{f_j\}_{j=1}^{n}$ and $\mathcal{G} = \{g_i\}_{j=1}^{n}$ for a Hilbert space $\mathcal{H}$ is defined by

$$\|\mathcal{F}-\mathcal{G}\| = \left(\sum_{j=1}^{n} \|f_j - g_j\|^2\right)^{1/2}$$

Two frames $\mathcal{F}$ and $\mathcal{G}$ are ϵ **-close** if $\|\mathcal{F}-\mathcal{G}\| \le \epsilon$.

We can now state the main problem we address in this section.

Problem(5.2.4)[37]: (V. Paulsen). Let $\mathcal{H}$ be a real or complex Hilbert space of dimension d. Given $\epsilon > 0$ and an integer $n \ge d$, find the largest number $\delta > 0$ so that whenever $\{f_j\}_{j=1}^{n}$ is a δ-nearly equal-norm, δ-nearly Parseval frame for a Hilbert space $\mathcal{H}$, there is an equal-norm Parseval frame $\{g_j\}_{j=1}^{n}$ whose ℓ^2-distance to $\{f_j\}_{j=1}^{n}$ is less than ϵ.

The existence of such a δ is assured by an argument of Don Hadwin.

Proposition(5. 2.5)[37]: (D. Hadwin). Given a real or complex Hilbert space $\mathcal{H}$ of dimension d and an integer $n \ge d$, then for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever a frame $\{f_j\}_{j=1}^{n}$ for $\mathcal{H}$ is δ-nearly equal-norm and δ-nearly Parseval, then $\{f_j\}_{j=1}^{n}$ is ϵ-close to an equal-norm Parseval frame.

Proof. We proceed by way of contradiction. If the assertion is false, then there exists some $\epsilon > 0$ and a sequence $\{\delta_m\}_{m=1}^{\infty}$ converging to zero and a sequence of frames $\{f_j^{(m)} : 1 \le j \le n, m \in \{1,2,\dots\}\}$ so that each $\{f_j^{(m)}\}_{j=1}^{n}$ is δ_m-nearly equal-norm and δm-nearly Parseval but for any equal-norm Parseval frame $\{g_j\}_{j=1}^{n}$ we have

$$\sum_{j=1}^{n} \|f_j^{(m)} - g_j\|^2 \ge \epsilon^2.$$

By compactness and switching to a subsequence we may assume that the sequence of frame vectors $\{f_j^{(m)}\}_{m=1}^{\infty}$ has a limit for each fixed $j \in \{1,2,\dots,n\}$,

$$\lim_{m\to\infty} f_j^{(m)} = f_j.$$

By continuity of the spectrum of V^*V in the frame vectors and of the entries in VV^*, it follows that $\{f_j\}_{j=1}^{n}$ is an equal-norm Parseval frame and that its distance to $\{f_j^{(m)}\}_{j=1}^{n}$ goes to zero as $m \to \infty$ which is in contradiction with the assumption that the distance between each

$\{f_j^{(m)}\}_{j=1}^n$ and any equal-norm Parseval frame was bounded below by $\epsilon > 0$.

The diagonal entries of VV^* and the operator inequalities for V^*V are not affected when the frame vectors are multiplied by unimodular constants, because then VV^* is simply conjugated by a diagonal unitary, and V^*V is invariant. Therefore, we can form equivalence classes of frames which share the same nearly equal-norm and nearly Parseval properties. A similar, coarser equivalence relation has already proven useful in the study of frames for erasure coding [47,49].

Definition (5.2.6)[37]: We define two frames $\mathcal{F} = \{f_j\}_{j=1}^n$ and $\mathcal{G} = \{g_j\}_{j=1}^n$ for a real or complex Hilbert space to be switching equivalent if the frame vectors f_j and g_j are collinear and $\|f_j\| = \|g_j\|$ for each $j \in \{1, 2, \ldots, n\}$. Accordingly, we speak of switching a frame $\mathcal{F}$ to a frame $\mathcal{G}$, also denoted $\mathcal{F}^{(c)}$, if we multiply each frame vector by an unimodular constant, $g_j = c_j f_j$ with $|c_j| = 1$ for $j \in \{1, 2, \ldots, n\}$.

Note that unlike the (nearly) equal-norm or Parseval properties, the ℓ^2-distance between two frames is not preserved when one of them is switched. We now define another distance for frames which does not depend on which particular representative of an equivalence class is chosen.

Definition(5. 2.7)[37]: The Bures distance between two frames $\mathcal{F} = \{f_j\}_{j=1}^n$ and $\mathcal{G} = \{g_j\}_{j=1}^n$ for a real or complex Hilbert space $\mathcal{H}$ is defined by

$$d_B(\mathcal{F}, \mathcal{G}) = \left(\sum_{j=1}^{n} \left(\|f_j\|^2 + \|g_j\|^2 - 2\left|\langle f_j, g_j\rangle\right|\right)\right)^{1/2}$$

Two frames $\mathcal{F}$ and $\mathcal{G}$ are ϵ-close in the Bures distance if $d_B(\mathcal{F}, \mathcal{G}) \le \epsilon$.

The Bures distance is only a pseudo-metric on the set of frames, because $d_B(\mathcal{F}, \mathcal{G}) = 0$ only implies $f_j = c_j\, g_j$ $with$ $|c_j| = 1$ for all $j \in \{1, 2, \ldots, n\}$.We have extended its usual definition for a pair of normalized vectors f and g in a real or complex Hilbert space, which assigns their Bures distance to be $\sqrt{2 - 2|f, g|}$, to the setting of vector-valued functions. This way of extending the Bures distance is natural when it is viewed as the solution of a minimization problem.

Lemma(5. 2.8)[37]: Let $\mathcal{H}$ be a Hilbert space over the field of real or complex numbers, here after denoted by $\mathbb{F}$. The value $d_B(\mathcal{F}, \mathcal{G})$ is the solution of the minimization problem

$$d_B(\mathcal{F}, \mathcal{G}) = \min_{c \in \mathbb{T}^n} \left(\sum_{j=1}^n \|f_j - c_j g_j\|^2\right)^{\frac{1}{2}},$$

where $\mathbb{T}^n = \{c \in \mathbb{F}^n : |c_j| = 1 \text{ for all } 1 \le j \le n\}$.

Proof. The equivalence between these two definitions of d_B is seen from the inequality

$$\|f_j - c_j g_j\|^2 = \|f_j\|^2 + \|g_j\|^2 - 2\,\Re \bar{c}_j \langle f_j, g_j\rangle \ge \|f_j\|^2 + \|g_j\|^2 - 2\left|\langle f_j, g_j\rangle\right|,$$

which is saturated (i.e. gives equality) when each c_j is chosen so that $\Re \bar{c}_j \langle f_j, g_j\rangle = |\langle f_j, g_j\rangle|$. Here, $\bar{c}_j$ denotes the complex conjugate of c_j.

The Bures distance is therefore the quotient metric obtained from the ℓ^2-metric when passing from frames to their equivalence classes. From the fact that equal-norm and Parseval properties are switching-invariant, we get an immediate consequence for the closeness of frames.

Corollary(5. 2.9)[37]: A frame $\mathcal{F} = \{f_j\}_{j=1}^{n}$ is ϵ-close to an equal-norm Parseval frame $\mathcal{G} = \{g_j\}_{j=1}^{n}$ in Bures distance if and only if it is ϵ-close to an equal-norm Parseval frame $\grave{\mathcal{G}} = \{\grave{g}_j\}_{j=1}^{n}$ ℓ^2-distance.

Proof. The "only if" part follows from choosing the ℓ^2-distance minimizing equal-norm Parseval frame $\grave{\mathcal{G}}$ in the equivalence class of $\mathcal{G}$. For this frame,

$\|\mathcal{F} - \grave{\mathcal{G}}\| = d_B(\mathcal{F}, \grave{\mathcal{G}}) = d_B(\mathcal{F}, \mathcal{G}) \leq \epsilon$.

The "if" part is clear from the inequality $d_B(\mathcal{F}, \grave{\mathcal{G}}) \leq \|\mathcal{F} - \grave{\mathcal{G}}\|$.

As a final remark before the main part of the section, we will see in this Section that the Paulsen problem is equivalent to a problem in matrix theory.

Problem (5.2.10)[37]: Let the field $\mathbb{F}$ be either the real or complex numbers, and assume $\mathbb{F}^n$ is equipped with the canonical inner product. Given $\epsilon > 0$, find the largest number $\gamma > 0$ so that whenever P is an orthogonal rank-d projection matrix on $\mathbb{F}^n$ with nearly constant diagonal, meaning there is $c > 0$ such that

$$(1 - \gamma)c \leq P_{j,j} \leq (1 + \gamma)c, \quad for\ all\ j \in \{1, 2, \ldots, n\},$$

then there exists an orthogonal projection Q satisfying

(a) $Q_{j,j} = \frac{d}{n}$ for all $j \in \{1, 2, \ldots, n\}$, and

(b) $\left(\sum_{j,k=1}^{n} |P_{j,k} - Q_{j,k}|^2\right)^{1/2} < \epsilon$.

We begin by first finding the closest Parseval frame to a given nearly equal-norm and nearly Parseval frame.

Proposition(5.2.11)[37]: Let $\{f_j\}_{j=1}^{n}$ be an ϵ-nearly Parseval frame for a d-dimensional Hilbert space $\mathcal{H}$, with frame operator $S = V^*V$, then $\{S^{-1/2}f_j\}_{j=1}^{n}$ is the closest Parseval frame to $\{f_j\}_{j=1}^{n}$ and

$$\sum_{j=1}^{n} \|S^{-1/2}f_j - f_j\|^2 \leq d(2 - \epsilon - 2\sqrt{1 - \epsilon}) \leq d\epsilon^2/4.$$

Proof. It is known that $\{S^{-1/2}f_j\}_{j=1}^{n}$ is the closest Parseval frame to $\{f_j\}_{j=1}^{n}$ [62,68,71,80]. We summarize the derivation of this fact.

The squared ℓ^2-distance between $\{f_j\}_{j=1}^{n}$ and $\{g_j\}_{j=1}^{n}$ can be expressed in terms of their analysis operators V and W as

$$\|\mathcal{F} - \mathcal{G}\|^2 = tr[(V - W)(V - W)^*]$$
$$= tr[VV^*] + tr[WW^*] - 2\,\Re tr[VW^*].$$

Choosing a Parseval frame $\{g_j\}_{j=1}^{n}$ is equivalent to choosing the isometry W. To minimize the distance over all choices of W, consider the polar decomposition $V = UP$, where P is positive and U is an isometry. In fact, $S = V^*V$ implies $P = S^{1/2}$, which means its eigenvalues are bounded away from zero.

Since P is positive and bounded away from zero, the term $[VW^*] = tr[UPW^* = trW^*UP]$] is an inner product between W and U. Its magnitude is bounded by the Cauchy Schwarz inequality, and thus it has a maximal real part if $W = U$ which implies $W^*U = I$. In this case,

$V = WP = WS^{1/2}$, or equivalently $W^* = S^{-1/2}V^*$

and we conclude $gj = S^{-1/2}f_j$ for all $j \in \{1, 2, \ldots, n\}$.

After choosing $W = VS^{-1/2}$, the ℓ^2-distance is expressed in terms of the eigenvalues $\{\lambda_k\}_{k=1}^{d}$ of $S = V^*V$ by

$$\| \mathcal{F} - \mathcal{G}\|^2 = tr[S] + tr[I] - 2tr[S^{1/2}]$$
$$= \sum_{k=1}^{d} \lambda_k + d - 2\sum_{k=1}^{d} \sqrt{\lambda_{k}.}$$

If $1-\epsilon \leq \lambda \leq 1+\epsilon$ for all $j \in \{1,2,\ldots,n\}$, calculus shows that the maximum of $\lambda - 2\sqrt{\lambda}$ is achieved when $\lambda = 1-\epsilon$.

Consequently,

$\| \mathcal{F} - \mathcal{G}\|^2 \leq 2d - d\epsilon - 2d\sqrt{1-\epsilon.}$

Estimating $\sqrt{1-\epsilon}$ by the first three terms in its decreasing power series gives the inequality $\| \mathcal{F} - \mathcal{G}\|^2 \leq d\epsilon^2/4$.

We have an upper bound for the distance between a frame and the closest Parseval frame, and for sufficiently small ϵ, we have control over how much of the "nearly equal-norm" property we lose.

Proposition(5.2.13)[37]: Fix $0 \leq \epsilon \leq 1/2$ and let $\{f_j\}_{j=1}^{n}$ be an ϵ-nearly equal-norm frame with constant c which is also an ϵ-nearly Parseval frame with frame operator $S = V^*V$, then $\{S^{-1/2}f_j\}_{j=1}^{n}$ is a Parseval frame and for all $j \in \{1,2,\ldots,n\}$ we have

$$(1-3\epsilon)c^2 \leq \frac{(1-\epsilon)^2}{1+\epsilon}c^2 \leq \left\|S^{-\frac{1}{2}}f_j\right\|^2 \leq \frac{(1+\epsilon)^2}{1-\epsilon}c^2 \leq (1-7\epsilon)c^2.$$

Proof. Since the frame operator $S = V^*V$ is by assumption bounded by $(1-\epsilon)I \leq S \leq (1+\epsilon)I$

we have via the spectral theorem

$$\frac{1}{\sqrt{1+\epsilon}}I \leq S^{-\frac{1}{2}} \leq \frac{1}{\sqrt{1-\epsilon}}I.$$

This means that the image of any unit vector has norm between $1/\sqrt{1+\epsilon}$ and $1/\sqrt{1-\epsilon}$, and for the frame vectors with norm bounds $(1-\epsilon)c \leq \|f_j\| \leq (1+\epsilon)c$, we get

$$\frac{(1-\epsilon)^2}{1+\epsilon}c^2 \leq \|S^{-1/2}f_j\|^2 \leq \frac{(1+\epsilon)^2}{1-\epsilon}c^2.$$

Further, convexity and elementary estimates give together with the assumption $\epsilon \leq 1/2$ the bounds

$$(1-3\epsilon)c^2 \leq \left\|S^{-\frac{1}{2}}f_j\right\|^2 \leq (1+7\epsilon)c^2.$$

Corollary (5.2.13)[37]: Fix $0 \leq \epsilon \leq 1/2$ and let $\{f_j\}_{j=1}^{n}$ be an ϵ-nearly equal-norm frame with constant c which is also an ϵ -nearly Parseval frame with frame operator $= V^*V$, then the norm of each vector $S^{-1/2}f_j, j \in \{1,2,\ldots,n\}$, is bounded by

$$\frac{(1-\epsilon)^3}{(1+\epsilon)^3}\frac{d}{n} \leq \|S^{-1/2}f_j\|^2 \leq \frac{(1+\epsilon)^3}{(1-\epsilon)^3}\frac{d}{n}.$$

Proof. By summing the square-norms of the frame vectors, and using the fact that the Grammian and the frame operator have the same eigenvalues, except possibly for zero, we obtain

$(1-\epsilon)d \leq \sum_{j=1}^{n}\|f_j\|^2 \leq (1+\epsilon)d.$

The nearly equal-norm condition gives

$$(1-\epsilon)d \leq (1+\epsilon)^2c^2n$$

and

$$(1+\epsilon)d \geq (1-\epsilon)^2 c^2 n.$$

This bounds the value of c by

$\frac{(1-\epsilon)d}{(1+\epsilon)^2 n} \leq c^2 \leq \frac{(1+\epsilon)d}{(1-\epsilon)^2 n}$.

Now we combine this with the preceding proposition to obtain

$$\frac{(1-\epsilon)^3 d}{(1+\epsilon)^3 n} \leq \left\|S^{-1/2} f_j\right\|^2 \leq \frac{(1+\epsilon)^3 d}{(1-\epsilon)^3 n}.$$

In the next section, we turn the resulting nearly equal-norm Parseval frame $\left\{S^{-1/2} f_j\right\}_{j=1}^{n}$ Into an equal-norm Parseval frame while measuring the distance between them.

We begin with a dilation argument.We observe that if $\{f_j\}_{j=1}^{n}$ is a Parseval frame for a real or complex Hilbert space, then the Grammian $G = \{f_j, f_k\}_{j,k=1}^{n}$ is an orthogonal projection matrix and we have the expression $G_{j,k} = \langle Ge_j, Ge_k\rangle = \langle V^* e_j, V^* e_k\rangle$ with the canonical orthonormal basis $\{e_j\}_{j=1}^{n}$ on $\ell^2(\{1,2,\ldots,n\})$ and V^*, the adjoint of the analysis operator of $\{f_j\}_{j=1}^{n}$.

Proposition(5.2.14)[37]: Let $\boldsymbol{G}$ be the Grammian of a Parseval frame for a real or complex Hilbert space $\mathcal{H}$, then the system of $ODEs$

$$\frac{d}{dt} e_j(t) = \sum_{k=1}^{n}\left(\left\|Ge_j(t)\right\|^2 - \|Ge_k(t)\|^2\right) e_k(t), \qquad j \in \{1,2,\ldots,n\}, \quad (19)$$

for the vector-valued functions $\{e_j : \mathbb{R}^+ \to \ell^2(\{1,2,\ldots,n\})\}$ with the canonical basis vectors as initial values $\{e_j(0)\}_{j=1}^{n}$ has a unique, global solution on $\mathbb{R}^+$. Moreover, there exists $t \geq 0$ such that $\grave{e}_j(t) = 0$ for all $j \in \{1,2,\ldots,n\}$ if and only if there is a $c > 0$ such that $\left\|Ge_j(t)\right\| = c$ for all $j \in \{1,2,\ldots,n\}$.

Proof. To simplify terminology in the proof, we write $\mathbb{F}^n$ instead of the Hilbert space $\ell^2(\{1,2,\ldots,n\})$, where $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$, depending on whether the Hilbert space $\mathcal{H}$ is real or complex. Moreover, we identify a family of vectors $\{e_j(t)\}_{j=1}^{n}$ in $\mathbb{F}^n$ with a vector $(e_1(t), e_2(t), \ldots, e_n(t)) \in \bigoplus_{j=1}^{n} \mathbb{F}^n \equiv \mathbb{F}^{n^2}$. With this identification, the system of ODEs for $\left\{e_j(t)\right\}_{j=1}^{n}$ combines to an ODE for a single vector-valued function $\varepsilon: \mathbb{R}^+ \to \mathbb{F}^{n^2}$. Since the velocity vector field of the combined ODE is smooth on any bounded set in $\mathbb{F}^{n^2}$, we have local existence and uniqueness of the solution in a sufficiently small neighborhood of $t = 0$.

We first prove that these local solutions preserve orthonormality of $\left\{e_j(t)\right\}_{j=1}^{n}$ and then conclude the existence of global solutions.

Since $\sum_{j=1}^{n} e_j(0) \otimes e_j^*(0) = I$ we only have to show that

$$\frac{d}{dt}\sum_{j=1}^{n} e_j(t) \otimes e_j^*(t) = 0.$$

Denoting $de_j(t)/dt = \grave{e}_j(t)$ and dropping the argument of the vector-valued functions, we compute

$$\frac{d}{dt}\sum_{j=1}^{n} e_j \otimes e_j^* = \sum_{j=1}^{n}\left(\grave{e}_j \otimes e_j^* + e_j \otimes (\grave{e}_j)^*\right)$$

$$= \sum_{j,k=1}^{n}\left(\|Ge_j\|^2 - \|Ge_k\|^2\right) e_k \otimes e_j^* + \left(\|Ge_j\|^2 - \|Ge_k\|^2\right) e_j \otimes e_k^*) = 0.$$

The last step follows from swapping the summation indices in the second term.

Now we invoke that these local solutions are uniformly bounded, because $\{e_j\ (t)\}_{j=1}^{n}$is orthonormal for each $t \geq 0$. This implies that the local solution stays inside the compact set $S^n = \{(e_1, e_2, \ldots, e_n) : \|e_j\| = 1 \text{ for all } j\} \subset \mathbb{F}^{n^2}$. The existence of a unique global solution now follows from the boundedness of the velocity vector field on the compact set S^n, because otherwise the maximal domain $[0, a)$ for a solution would yield a limiting value at a inside S, which we could again use as initial value to find a local solution in the neighborhood of a, and then by the uniqueness of local solutions extend the domain $[0, a)$ to include a neighborhood of a, contradicting the maximality assumption. For more details on this argument, see [83].

Finally, we observe that $\grave{e}_j\ (t) = 0$ for all $j \in \{1, 2, \ldots, n\}$ implies by orthonormality that $\|Ge_j(t)\|^2 - \|Ge_k(t)\|^2 = 0$ for all j and k and thus the family $\{Ge_j\}_{j=1}^{n}$j=1 is equal-norm. Conversely, it follows directly from the definition of the ODE system that all orthonormal bases which G projects to an equal-norm family are fixed points.

By mapping the evolving orthonormal basis with the synthesis operator of a Parseval frame, we obtain a family of Parseval frames which solves a corresponding ODE system.

Proposition(5.2.15)[37]: Let G be the Grammian of a Parseval frame for a real or complex Hilbert space $\mathcal{H}$, let $V : \mathcal{H} \to \ell^2(\{1, 2, \ldots, n\})$ be the analysis operator of the frame, and consider the solution $\{e_j : \mathbb{R}^+ \to \ell^2(\{1, 2, \ldots, n\})\}_{j=1}^{n}$ of the initial value problem given in the preceding proposition, then $f_j\ (t) = V^* e_j(t)$ defines a family of Parseval frames $\{f_j : \mathbb{R}^+ \to \mathcal{H}\}_{j=1}^{n}$ which satisfies the ODE system

$$\frac{d}{dt} f_j(t) = \sum_{k=1}^{n}\left(\|f_j(t)\|^2 - \|f_k(t)\|^2\right) f_k(t), \quad j \in \{1,2, \ldots, n\}, \qquad (20)$$

and V is the analysis operator of $\{f_j(0)\}_{j=1}^{n}$. Conversely, each solution of this ODE system, with a Parseval frame $\{f_j(0)\}_{j=1}^{n}$ having analysis operator V as initial value, is globally defined and unique, and to each such solution corresponds a unique solution for the ODE (19) starting at the canonical basis of $\ell^2(\{1, 2, \ldots, n\})$ such that $V^* e_j(t) = f_j(t)$ for all $t \geq 0$.

Proof. We use the two facts that (1) the projection of any orthonormal basis $\{e_j\}_{j=1}^{n}$with the Grammian G is a Parseval frame for the range of G and that (2) the analysis operator V of a Parseval frame is an isometry, which implies by orthonormality of $\{e_j(t)\}_{j=1}^{n}$ that for any $t \geq 0, \mathcal{F}(t) = \{V^* e_j(t)\}_{\text{j}=1}^{n}$is a Parseval frame for $\mathcal{H}$. Moreover, from the identity $\|Ge_j(t)\| = \|V^* e_j(t)\| = \|f_j(t)\|$ for all $j \in \{1, 2, \ldots, n\}$ and from applying V^* to both sides of the ODE system (19), we deduce that $\mathcal{F}: \mathbb{R}^+ \to \bigoplus_{j=1}^{n} \mathcal{H}$ defines a family of Parseval frames which solves the ODE system (20). The initial value problem for (20) has a unique solution, which is seen by repeating the argument of the preceding proposition with the vector-valued function $\mathcal{F}: \mathbb{R}^+ \to \bigoplus_{j=1}^{n} \mathcal{H}$ instead of ε and with the sphere $S=\{(f_1,f_2,\ldots,f_n):\sum_{j=1}^{n}\left\|f_j\right\|^2=d\}\subset\bigoplus_{j=1}^{n}\mathcal{H}$ instead of S^n. The set S is preserved under the flow because each $\{f_j(t)\}_{j=1}^{n}$ is a Parseval frame, so the trace of its Grammian is equal to its rank, $\sum_{j=1}^{n}\left\|f_j\right\|^2=d$, independent of the choice of $t\geq 0$. Since the solution of the initial value problem (20) is unique, and $\mathcal{F}(t)=\{V^*e_j(t)\}_{j=1}^{n}$ provides a solution when the orthonormal basis evolves under (19), each solution of (20) can be lifted to a unique solution of (19) which has as its initial value $\{e_j(0)\}_{j=1}^{n}$, the canonical orthonormal basis of $\ell^2(\{1,2,\ldots,n\})$.

The reason for introducing the dilation argument with the ODE system for the basis vectors is that the fixed points of (19) are as desired, whereas the set of fixed points of (20) contains more than all equal-norm Parseval frames, see the example below.

Proposition(5.2.16)[37]: Given a family of n vector-valued functions$\{f_j:\mathbb{R}^+\to\mathcal{H}\}_{j=1}^{n}$, satisfying (20), with $\{f_j:(0)\}_{j=1}^{n}a$ Parseval frame, then $\grave{f}_j(0)=0$ for all $j\in\{1,2,\ldots,n\}$ if and only if the frame is equal-norm or the following zero-summing conditions hold:

$$\sum_{j=1}^{n}f_j(0)=\sum_{j=1}^{n}\left\|f_j(0)\right\|^2f_j(0)=0.$$

Proof. In the proof we again omit the explicit time dependence of the frame vectors. From the ODEs system for the frame vectors, we see that if

$$\frac{d}{dt}f_j=\sum_{k=1}^{n}\left(\left\|f_j\right\|^2-\|f_k\|^2\right)f_k=0,$$

then

$$\left\|f_j\right\|^2\sum_{k=1}^{n}f_k=\sum_{k=1}^{n}\|f_k\|^2f_k.$$

Hence, if

$$\frac{d}{dt}f_j=\frac{d}{dt}f_m=0,$$

For $j\neq m\in\{1,2,\ldots,n\}$, then

$$\left\|f_j\right\|^2\sum_{k=1}^{n}f_k=\|f_m\|^2\sum_{k=1}^{n}f_k.$$

That is,

$$\left\|f_j\right\|=\|f_m\|\quad\text{or}\quad\sum_{k=1}^{n}f_k=0.$$

Consequently, if $\frac{d}{dt}f_j=0$ for all $j\in\{1,2,\ldots,n\}$ then the frame is equal-norm or

$$\sum_{k=1}^{n}f_k=\sum_{k}\|f_k\|^2f_k=0.$$

Conversely, if the zero-summing conditions hold, then $\frac{d}{dt}f_j=0$ follows for all $j\in\{1,2,\ldots,n\}$ directly from the definition of the ODE system (20).

Example(5.2.17)[37]: Given a real or complex Hilbert space $\mathcal{H}$ of dimension d and an orthonormal basis $\{e_1,e_2,\ldots,e_d\}$ for $\mathcal{H}$, we can construct a Parseval frame $\{f_j\}_{j=1}^{2d+1}$ by

$$f_j = \begin{cases} \frac{1}{\sqrt{2}} e_j, & 1 \le j \le d, \\ -\frac{1}{\sqrt{2}} e_{j-d}, & d+1 \le j \le 2d, \\ 0, & j = 2d+1. \end{cases}$$

It is straightforward to check that this frame satisfies the zero-summing conditions in the preceding proposition, and is thus a fixed point for the ODE (20), but it is not an equal-norm Parseval frame.

It has been observed numerically that using an example of this type as initial value and dilating the Parseval frame to an orthonormal basis leads to an oscillating behavior of the basis vectors evolving under the ODE system (19). Therefore, one cannot hope to use these ODEs alone to achieve convergence to equal-norm Parseval frames.

Definition(5.2.18)[37]: We define the frame energy of a frame $\mathcal{F} = \{f_j\}_{j=1}^{n}$ by

$$U(\mathcal{F}) = \sum_{j,k=1}^{n} \left(\|f_j\|^2 - \|f_k\|^2 \right)^2 .$$

We will show below that with an appropriate use of intermittent switching, the energy of Parseval frames obtained from piecewise solutions of the ODE (20) decreases rapidly (in fact, exponentially) in time. Together with the following arc length estimate, this amounts to showing a rate of convergence to an equal-norm Parseval frame.

Definition(5.2.19)[37]: Given a family of differentiable vector-valued functions $\mathcal{F} = \{f_j : \mathbb{R}^+ \to \mathcal{H}\}_{j=1}^{n}$ and $0 \le t_1 \le t_2$, the arc length traversed by the family between time t_1 and t_2 is defined by

$$s = \int_{t_1}^{t_2} \left(\sum_{j=1}^{n} \left\| \dot{f}_j(t) \right\|^2 \right)^{1/2} dt.$$

The arc length traversed by the vector-valued function $\mathcal{F}$ evolving under (20) is bounded by an energy integral.

Theorem(5.2.20)[37]: The arc length traversed by the solution $\mathcal{F} : \mathbb{R}^+ \to \bigoplus_{j=1}^{n} \mathcal{H}$ of the ODE system (20) between time t_1 and t_2 is bounded by the energy integral

$$s \le \int_{t_1}^{t_2} \left(U(\mathcal{F}(t)) \right)^{1/2} dt.$$

Proof. We pass from the solution of (20) to the orthonormal basis $\varepsilon = \{e_j : \mathbb{R}^+ \to \mathbb{F}^{n^2}\}_{j=1}^{n}$ evolving under (19), giving $V^* e_j(t) = f_j(t)$, where V^* is the synthesis operator of $\{f_j(0)\}_{j=1}^{n}$. Denoting by G the Grammian, we have by orthonormality,

$$\left\| \frac{d}{dt} e_j \right\|^2 = \sum_{k=1}^{n} \left(\|G e_j\|^2 - \|G e_k\|^2 \right)^2$$

where we have suppressed the explicit time dependence of the orthonormal basis vectors. Summing over all j gives

$$\sum_{j=1}^{n} \left\| \frac{d}{dt} e_j \right\|^2 = \sum_{j,k=1}^{n} \left(\|G e_j\|^2 - \|G e_k\|^2 \right)^2 = U(\mathcal{F}(t)).$$

Finally, again using the Parseval property, $\|Gx\| = \|V^* x\|$ for each $x \in \ell^2\{1,2,\ldots,n\})$, yields $\left\|\dot{f}_j\right\| = \left\|\frac{d}{dt}V^* e_j\right\| = \left\|\frac{d}{dt}Ge_j\right\| \le \left\|\frac{d}{dt}e_j\right\|$,and we have $\sum_{j=1}^{n}\left\|\dot{f}_j(t)\right\|^2 \le U(\mathcal{F}(t))$. Now the definition of arc length provides the desired estimate.

Proposition (5.2.21)[37]: An alternative expression for the frame energy of a Parseval frame $\mathcal{F} = \{f_j\}_{\mathrm{j}=1}^{n}$,

$$U(\mathcal{F}) = 2n\sum_{j=1}^{n}\left\|f_j\right\|^4 - 2d^2,$$

where d is the dimension of $\mathcal{H}$.

Proof. We use the antisymmetry of $\left\|f_j\right\|^2 - \|f_k\|^2$ in j and k to write

$$U(\mathcal{F}) = 2\sum_{j,k=1}^{n}\left(\left\|f_j\right\|^2 - \|f_k\|^2\right)\left\|f_j\right\|^2.$$

Now we can sum over k. Since $\{f_k\}_{\mathrm{k}=1}^{n}$ is Parseval, the square-norms sum to $d = dim(\mathcal{H})$. The result is

$$U(\mathcal{F}) = 2\sum_{j=1}^{n}\left(n\left\|f_j\right\|^4 - d\left\|f_j\right\|^2\right).$$

Again, we can split the two terms into separate sums and carry out the sum over j for the second term to get d again.

Next, we give a closed expression for the time derivative of the frame energy while the frame $\mathcal{F}$ evolves under (20).

Lemma(5.2.22)[37]: If $\mathcal{F} = \{f_j : \mathbb{R}^+ \to \mathcal{H}\}$ is a solution of (20) with a Parseval frame $\{f_j(0)\}_{\mathrm{k}=1}^{n}$ as initial value, then

$$\frac{d}{dt}U(\mathcal{F}(t)) = 4n\sum_{j,k=1}^{n}\langle f_j(t), f_k(t)\rangle\left(\left\|f_j(t)\right\|^2 - \|f_k(t)\|^2\right)^2$$

Proof. Defining $G_{j,k}(t) = \langle Ge_j(t), e_k(t)\rangle = \langle f_j(t), f_k(t)\rangle$ and proceeding with the lifted ODE

$$\dot{e}_j(t) = \sum_{k=1}^{n}\left(G_{j,j}(t) - G_{k,k}(t)\right)e_k(t)$$

we have

$$\frac{d}{dt}\left(G_{j,j}(t)\right)^2 = 2G_{j,j}(t)\sum_{k=1}^{n}G_{j,j}(t)\left(G_{j,j}(t) - G_{k,k}(t)\right).$$

Summing over j and antisymmetrizing $G_{j,j}(t)$ with $G_{k,k}(t)$ gives

$$\frac{d}{dt}U(\mathcal{F}(t)) = 4n\sum_{j,k=1}^{n}G_{j,k}(t)\left(G_{j,j}(t) - G_{k,k}(t)\right)^2$$

In terms of the frame vectors, this is precisely the claimed expression.

Definition(5.2.23)[37]: We define σ_n to be the uniform probability measure on the n-torus $\mathbb{T}^n = \{c \in \mathbb{F}^n : |c_j| = 1 \text{ for all } j\}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. In the complex case, these are all n-tuples of unimodular complex numbers and in the real case n-tuples of $\pm 1's$.We also denote diagonal unitaries $\{D(c)\}$, parametrized by the diagonal entries $(D(c))_{j,j} = c_j$, $|c_j| = 1$ for all $j \in \{1, 2, \ldots, \mathrm{n}\}$.

For later notational convenience, we define

$$W(\mathcal{F}) = 4n \sum_{j,k=1}^{n} \langle f_j, f_k \rangle \left(\|f_j\|^2 - \|f_k\|^2 \right).^2 \qquad (21)$$

We recall the definition of two frames being switching equivalent, meaning the two families consist of vectors that are pairwise collinear and of the same norm.

We now use the switching dependence of W to our advantage.

Proposition(5.2.24)[37]: Given a Parseval frame $\mathcal{F} = \{f_j\}_{\mathrm{j}=1}^{n}$, then there is a choice $c \in \mathbb{T}^n$ such that

$$W\left(\mathcal{F}^{(c)}\right) \leq 0$$

Proof. Let G denote the Grammian of $\mathcal{F}$. For the switched frame $\mathcal{F}^{(c)}$, we have

$$W\left(\mathcal{F}^{(c)}\right) = 4n \sum_{j,k=1}^{n} c_j c_k^* G_{j,k} \left(G_{j,j} - G_{k,k} \right).^2$$

Integrating over the torus $\mathbb{T}^n$ with respect to the switching-invariant measure σ_n gives

$$\int_{\mathbb{T}^n} c_j^* c_k \, d\sigma_n(c) = \delta_{j,k}.$$

Thus we note, since terms with $j = k$ have a vanishing contribution in $W(\mathcal{F}^{(c)})$,

$$\int_{\mathbb{T}^n} W(\mathcal{F}^{(c)}) d\sigma(c) = 0.$$

Since the average is equal to zero, there must be a choice of c which gives $W(\mathcal{F}^{(c)}) \leq 0$.

Next, we compute a lower bound for the variance of $W\left(\mathcal{F}^{(c)}\right)$.

Proposition(5.2.25)[37]: For a fixed Parseval frame $\mathcal{F}$, the variance of $W\left(\mathcal{F}^{(c)}\right)$ with respect to the probability measure σ on the torus $\{c \in \mathbb{T}^n\}$ is

$$\int_{\mathbb{T}^n} \left(W(\mathcal{F}^{(c)})\right) d\sigma(c) = 16n^2 \sum_{j,k} \left|G_{j,j}\right|^2 \left(G_{j,j} - G_{k,k}\right).^4$$

Proof. Similar to the preceding proposition, with the help of

$$\int_{\mathbb{T}^n} c_j c_k^* c_l c_m^* \, d\sigma(c) = \delta_{j,k} \delta_{l,m} + \delta_{j,m} \delta_{k,l}$$

Let $n, d \in \mathbb{N}$ be relatively prime, and define

$$\eta = \min_{\substack{n_1 < n \\ d_1 < d}} \left| \frac{d}{n} - \frac{d_1}{n_1} \right|$$

then we have a lower bound

$$\eta \geq \frac{1}{n(n-1)}.$$

This follows immediately from the fact that since d, n are relatively prime, $dn_1 - d_1 n$ is a nonzero integer. Since $n_1 < n$ and $d_1 < d$ we have

$$\left|\frac{d}{n} - \frac{d_1}{n_1}\right| = \left|\frac{dn_1 - n\, d_1}{n\, n_1}\right| \geq \frac{1}{n\, n_1} \geq \frac{1}{n(n-1)}.$$

Lemma(5.2.26)[37]: Let $n \geq 2, \eta$ as defined above, and let $\mathcal{F} = \{f_j\}_{\mathrm{j=1}}^{n}$ be a Parseval frame for a d-dimensional Hilbert space, then the variance of the random variable $W: c \to W\left(\mathcal{F}^{(c)}\right)$ on the torus $\mathbb{T}^n$ equipped with the uniform probability measure σ_n is bounded below by

$$\frac{16\eta}{(n-1)^7} (U(\mathcal{F}))^2 \leq \int_{\mathbb{T}^n} W\left(\mathcal{F}^{(c)}\right)^2 d\sigma_n \,.$$

Proof. Without loss of generality we can number the frame vectors so that their norms decrease, $j \in \{1, 2, \ldots, n-1$ If $U(\mathcal{F})$ does not vanish then $\|f_1\| > \|f_n\|$ and there is at least one $j \in \{1, 2, \ldots, n-1\}$ such that

$$\|f_j\|^2 > \|f_{j+1}\|^2 \geq (\|f_1\|^2 - \|f_n\|^2)/(n-1)\_$$

This means, if $\grave{j} \leq j$ and $j'' \geq j + 1$, then also

$$\|f_{\grave{j}}\|^2 - \left\|f_{\ j''}\right\|^2 \geq (\|f_1\|^2 - \|f_n\|^2)/(n-1)$$

Thus we have partitioned the frame vectors into two sets, and the difference of square-norms between any pair of vectors containing one from each of these sets is bounded below by $(\|f_1\|^2 - fn\_2)/(n-1)$.

Therefore, the matrix A containing entries $(\|f_1\|^2 - \|f_n\|^2)\,/(n-1)$ is entry-wise bounded below by a matrix (block notation)

$$\grave{A} = \begin{pmatrix} 0 & \epsilon J \\ \epsilon J^* & 0 \end{pmatrix}$$

where J is a block containing all $1's$ an$\epsilon\epsilon = (\|f_1\|^2 - \|f_n\|^2)^4/(n-1)^4$.

If we form the corresponding blocks in the Grammian

$$G = \begin{pmatrix} G_{11} & G_{22} \\ G_{21} & G_{22} \end{pmatrix}$$

then we know $0 \leq G_{11} \leq I$, meaning the eigenvalues of G_{11} are contained in the closed interval $[0,1]$. Since G is an orthogonal projection, $G_{11} = G_{11}^2 + G_{12}G_{21}$ which means

$$tr[G_{12}G_{21}] = tr[G_{11} - G_{11}^2].$$

But that is exactly the squared Frobenius norm of the block G_{12}. Hence,

$$\textstyle\sum_{j,k} \left|G_{j,k}\right|^2 A_{j,k} \geq 2\epsilon tr[G_{11} - G_{11}^2].$$

The smallest number of non-zero entries in $\grave{A}$ is achieved when J contains only one row. If n and d are relatively prime and the vectors are sufficiently near equal-norm, then the diagonal entries of G_{11} are close to d/n and summing them does not give an integer. Therefore, not all eigenvalues are 0 or 1. In fact, a lower bound for the Hilbert–Schmidt square-norm of G_{12} is $tr[G_{11} - G_{11}^2] \geq \eta/(2n-2)]$. This is because at least one of the eigenvalues has distance $\eta/(n-1)$ from $\{0,1\}$ and the function $x \mapsto x(1-x)$ is bounded below by $x \longmapsto x/2$ on $[0, 1/2]$ and by $x \longmapsto 1/2 - x/2$ on $[1/2, 1]$.

Consequently,

$$\sum_{j,k}\left|G_{j,k}\right|^2 A_{j,k} \geq \frac{\left(\|f_1\|^2-\|f_n\|^2\right)^4\eta}{(n-1)^4(n-1)}.$$

Using the equivalence of norms again,

$$\sum_{j,k=1}^{n}\left(\left\|f_j\right\|^2-\|f_k\|^2\right)^2 \leq n(n-1)(\|f_1\|^2-\|f_n\|^2)^2$$

and then applying Proposition(5.2.28)

$$\frac{16\eta}{(n-1)^7}\left(U(\mathcal{F})\right)^2 \leq 16n^2\sum_{j,k=1}^{n}\left|G_{j,k}\right|^2 A_{j,k}\int_{\mathbb{T}^n}\left(\frac{d}{dt}U\left(\mathcal{F}^{(c)}\right)\right)^2 d\sigma(c).$$

Next we will bound $|W(\mathcal{F}^{(c)})|$ by the frame energy.

Lemma(5.2.27)[37]: For a fixed Parseval frame $\mathcal{F}$, the random variable $W : c \mapsto W(\mathcal{F}^{(c)})$ on the torus $\mathbb{T}^n$ is bounded,

$$\left|W(\mathcal{F}^{(c)})\right| \leq dU(\mathcal{F}).$$

Proof. Let B denote the matrix with entries $B_{j,k} = (\left\|f_j\right\|^2-\|f_k\|^2)^2$, and $G^{(c)} = D(c)GD^*(c)$, then $W(\mathcal{F}^{(c)}) = tr[G^{(c)}B]$. Estimating the inner product between $G^{(c)}$ and B gives

$$\left|W(\mathcal{F}^{(c)})\right| = \left|tr\left[G^{(c)}B\right]\right| \leq \max_P tr[P|B|]$$

where the maximum is over all rank-d orthogonal projections P , and the spectral theorem defines $|B| = \sqrt{B^*B}$. According to Perron–Frobenius, the largest eigenvalue of |B| is bounded by $\max_j \sum_{k=1}^{n} B_{i,j}$. Hence,

$$\left|W\left(\mathcal{F}^{(c)}\right)\right| \leq d\ \max_j\sum_{k=1}^{n} B_{j.k} \leq d\sum_{j,k=1}^{n} B_{j.k}.$$

Finally, we observe that $U(\mathcal{F}) = U(\mathcal{F}^{(c)}) = \sum_{j,k=1}^{n} B_{j,k}$.

To finish the quantitative bound on the distance from our initial Parseval frame to our equal norm Parseval frame, we will find an exponential upper bound on the frame energy. Theorem (5.2.22) will then give the needed quantitative upper bound on the arc length.

Lemma(5.2.28)[37]: Let $W:\Omega \to [-a,a], a>0$ be a bounded random variable on a probability space, which induces a normalized Borel measure m on $[-a,a]$. If the expectation and variance of W are $\mathbb{E}[W] = \int_{-a}^{a} x\, dm(x) = 0$ and $\mathbb{E}[W^2] = \int_{-a}^{a} x^2\ dm(x) = \sigma^2 > 0$, then the support of m contains a point in the set $\{x \in [-a,a]: x \leq -\sigma^2/a\}$.

Proof. We consider the polynomial given by $p(x) = (x-a)(x+b)$, then

$$\mathbb{E}[p(W)] = \int_{-a}^{a}(x^2+(b-a)x-ab)\, dm(x) = \sigma^2 - ab.$$

Choosing $b = \sigma^2/a\ gives\ \mathbb{E}[p(W) = 0$, and so

$$suppm \cap \{x \in [-a,a]: p(x) \geq 0\} \neq \emptyset.$$

The subset of $[-a,a]$ where p is non-negative is $[-a,-b]$.

Now we are able to bound $W(\mathcal{F}^{(c)})$ from above by a strictly negative quantity.

Theorem (5.2.29)[37]: Let $n \geq 2, \eta$ as defined above and let $\mathcal{F} = \{f_j\}_{j=1}^{n}$ be a Parseval frame for a d-dimensional Hilbert space, then there exists $c \in \mathbb{T}^n$ such that

$$W(\mathcal{F}^{(c)}) \leq -\frac{16\eta}{(n-1)^7 d} U(\mathcal{F}).$$

Proof. We have that $a = dU(\mathcal{F})$ bounds the magnitude of $W(\mathcal{F}^{(c)})$ and its variance σ^2 is bounded below by $\sigma^2 \geq \frac{16\eta}{(n-1)^7} (U(\mathcal{F}))^2$. The preceding lemma then establishes that there is a choice for $\{c_j\}_{j=1}^{n}$ such that

$$W(\mathcal{F}^{(c)}) \leq \frac{\sigma^2}{dU(\mathcal{F})} \leq \frac{16\eta}{(n-1)^7 d} U(\mathcal{F}).$$

Theorem(5.2.30)[37]: Let $\mathcal{H}$ be a real or complex Hilbert space of dimension d, and let $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ be an ϵ-nearly equal-norm Parseval frame, with $n \geq 2$ and n and d relatively prime, then there exists an equal-norm Parseval frame $\mathcal{G}$ at ℓ^2-distance

$$\|\mathcal{F} - \mathcal{G}\| \leq U(\mathcal{F})^{\frac{1}{2}} \frac{n(n-1)^8 d}{8}.$$

Proof. We let the frame $\mathcal{F}$ serve as the initial value $\mathcal{F}(0)$ for the ODE system (20). Assuming that for each t, we pick $c(t)$ which yields the desired estimate for W, then naively integrating the differential inequality

$$\frac{d}{dt} U(\mathcal{F}^{(c(t))}(t)) = W(\mathcal{F}^{(c(t))}(t)) \leq -\frac{16\eta}{(n-1)^7 d} U(\mathcal{F}(t))$$

obtained in the preceding theorem gives

$$U(\mathcal{F}^{(c(t))}(t)) \leq U(\mathcal{F}(0)) e^{-16\eta t/(n-1)^7 d}.$$

However, we note that there is no guarantee that c is a measurable function. To achieve this, we relax the constant governing the exponential decay.

Choose $0 < \alpha < 1$.We know that for any Parseval frame there is at least one choice of c which gives

$$\frac{d}{dt} U(\mathcal{F}^{(c)}) \leq -\frac{16\eta}{(n-1)^7 d} U(\mathcal{F}) < -\frac{16\alpha\eta}{(n-1)^7 d} U(\mathcal{F}). \quad (22)$$

By the continuity of U and dU/dt in $\mathcal{F}$, we can cover the space of Parseval frames with open sets for which the strict inequality holds with the choice of a corresponding c. To finish the argument we need to patch together the local flows in each open set.

We define a global flow by the appropriate choice of c in each subset. Upon exiting a set at time t , we choose one of the open sets of which the frame $\mathcal{G}(t)$ is an element and continue with the respective flow given by the corresponding choice of c in this subset. Since the cover is open, c is piecewise constant and right continuous.

In the complex case, we choose a countable number of $c's$ which are dense in the torus. By continuity of U and W, for any frame there is a choice in this countable set of $c's$ such that again the strict differential inequality (22) is satisfied. Moreover, the countable family of open sets corresponding to all $c's$ cover the space of all Parseval frames. By the Heine Borel property of the compact set of Parseval frames, there is a finite sub-cover and we can repeat the argument as in the real case.

We recall that switching affects the ℓ^2-distance. Piecewise integrating the differential inequality, including switching when necessary, gives that the frame energy of $\{\mathcal{F}^{(c(t))}(t)\}_{t\in\mathbb{R}^+}$decays exponentially in time. Then using the inequality between arc length and frame energy in Theorem(5.2.22) , we obtain that the sequence $\{\mathcal{F}^{(c(m))}(m)\}_{m=0}^{\infty}$ is Cauchy in the Bures metric, because the series $\sum_{m=0}^{\infty} dB\,(\mathcal{F}^{(c(m))}(m), \mathcal{F}^{(c(m+1))}(m+1))$ is dominated by a geometric series, and hence summable.

Passing to a subsequence converging to an accumulation point $\mathcal{G}'$ then yields that the equalnorm Parseval frame $\mathcal{G}'$ is within Bures distance

$$d_B(\mathcal{F}(0), \mathcal{G}') \leq \int_0^{\infty} U\left(\mathcal{F}^{(c(t))}(t)\right)^{1/2} dt \leq \int_0^{\infty} U\big(\mathcal{F}(0)\big)^{1/2} e^{-8\eta\alpha t/(n-1)^7 d} dt$$

$$= U\big(\mathcal{F}(0)\big)^{1/2} \frac{(n-1)^7 d}{8\eta\alpha}.$$

However, we recall that we can always choose $\mathcal{G}$ in the equivalence class of $\mathcal{G}'$ which minimizes the ℓ^2-distance to $\mathcal{F}$, and obtain the same result for the ℓ^2-distance.

To finish the proof, we recall $\eta \geq 1/n(n-1)$ and use the fact that the set of equal-norm Parseval frames is closed in the compact set of all Parseval frames. Therefore, choosing a sequence of values for α converging to one, we obtain a sequence of frames with an accumulation point within the desired ℓ^2-distance.

Now, putting together the distances we computed above, and taking into account that in the first step we moved from our nearly equal-norm, nearly Parseval frame to the closest Parseval frame, we can give the distance estimate for the Paulsen problem.

Theorem(5.2.31)[37]: Let $n, d \in \mathbb{N}$ be relatively prime, $n \geq 2$, let $0 < \epsilon < \frac{1}{2}$, and assume $\mathcal{F} = \{f_j\}_{j=1}^{n}$ is an ϵ-nearly equal-norm and ϵ-nearly Parseval frame for a real or complex Hilbert space of dimension d, then there is an equal-norm Parseval frame $\mathcal{G} = \{g_j\}_{j=1}^{n}$ such that

$$\|\mathcal{F} - \mathcal{G}\| \leq \frac{29}{8} d^2 n(n-1)^8 \epsilon.$$

Proof. After passing to the closest Parseval frame to the given frame, denoted by $\mathcal{G}(0) = \{S^{-1/2} f_j\}$, we have by the lower and upper bound for the norms of $\{S^{-1/2} f_j\}$ in Corollary (5.2.13) a bound for the frame energy

$$U\big(\mathcal{G}(0)\big) \leq \frac{d^2(n-1)}{n}\left(\frac{(1+\epsilon)^3}{(1-\epsilon)^3} - \frac{(1-\epsilon)^3}{(1+\epsilon)^3}\right)^2.$$

Using convexity and elementary estimates, we infer for $\epsilon < 1/2$ that

$$U\big(\mathcal{G}(0)\big) < 27^2 d^2 \epsilon^2.$$

Now using the preceding theorem, we obtain that there is an equal-norm Parseval frame $\mathcal{G}$ at distance

$$\|\mathcal{G}(0) - \mathcal{G}\| \leq \frac{27}{8} d^2 n(n-1)^8 \epsilon.$$

To complete the proof, we use the triangle inequality,

$$d(\mathcal{F}, \mathcal{G}) \leq d(\mathcal{F}, \mathcal{G}(0)) + d(\mathcal{G}(0), \mathcal{G}) \leq \frac{\sqrt{d}}{2}\epsilon + \frac{27}{8} d^2 n(n-1)^8 \epsilon,$$

and then combine the two contributions after estimating

$$\sqrt{d/2} \leq d^2/2 \leq d^2 n(n-1)^8/4.$$

We conclude with an observation which allows us to reduce the construction of equal-norm Parseval frames to the special case discussed in the previous section.

Lemma(5.2.32)[37]: Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ over the real or complex numbers and equal-norm Parseval frames $\mathcal{F} = \{f_1, \dots, f_{n1}\}$ and $\mathcal{G} = \{g_1, \dots, g_{n_2}\}$ then the family of vectors

$\mathcal{F} \otimes \mathcal{G} = \{f_i \otimes g_j : 1 \le i \le n_1, 1 \le j \le n_2\}$ is an equal-norm Parseval frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof. The Parseval property of $\mathcal{F} \otimes G$ is equivalent to the identity

$$x = \sum_{i,j} \langle x, f_i \otimes g_j \rangle f_i \otimes g_j$$

for all $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$. From the Parseval property of both frames it is clear that this identity holds for any $x = a \otimes b$ with $a \in \mathcal{H}_1$ and $b \in \mathcal{H}_2$. Linearity then establishes the result for general $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$.

The equal-norm property follows from

$$\|f \otimes g\| = \|f\| \|g\|$$

for any pair $(f, g) \in \mathcal{F} \times \mathcal{G}$ and from the equal-norm property of the individual frames.

Corollary(5.2.33)[37]: The construction of an equal-norm Parseval frame of n vectors in a d-dimensional real or complex Hilbert space $\mathcal{H}$ can be reduced to the case of d and n being relatively prime.

Proof. If their greatest common divisor is not one, say $gcd(n, d) = m$, then we can proceed as follows. Consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $dim(\mathcal{H}_1) = d/m$ and $dim(\mathcal{H}_2) = m$.

Now choose an orthonormal basis $\{e_1, e_2, \dots, e_m\}$ for $\mathcal{H}_2$ and construct an equal-norm Parseval frame of n/m vectors $\{f_1, f_2, \dots, f_{n/m}\}$ for $\mathcal{H}_1$. The resulting family of tensor products $\{f_i \otimes e_j : 1 \le i \le n/m, 1 \le j \le m\}$ is an equal-norm Parseval frame for $\mathcal{H}$.

In this section we will show that the estimate for the special case of the Paulsen problem provides a partial answer for Problem(5.2.11) in matrix theory.

Proposition (5.2.34)[37]: If $\{f_j\}_{j \in I}$, $\{g_j\}_{j \in I}$ are frames for $\mathcal{H}$ with analysis operators V_1, V_2 respectively, then

$$\sum_{j \in I} \left\|V_1 f_j - V_2 g_j\right\|^2 < 2(\|V_1\|^2 + \|V_2\|^2) \sum_{j \in I} \left\|f_j - g_j\right\|^2.$$

Proof. Note that for all $j \in I$,

$$V_1 f_j = \sum_{i \in I} \langle f_j , f_i \rangle e_i$$

and

$$V_2 g_j = \textstyle\sum_{i \in I} \langle g_j , g_i \rangle e_i.$$

Hence,

$$\left\|V_1 f_j - V_2 g_j\right\|^2 = \sum_{i \in I} \left|\langle f_j , f_i \rangle - \langle g_j , g_i \rangle\right|^2 = \sum_{i \in I} \left|\langle f_j , f_i - g_i \rangle + \langle f_j - g_j , g_i \rangle\right|^2$$

$$\le 2 \sum_{i \in I} \left|\langle f_j , f_i - g_i \rangle\right|^2 + 2 \sum_{i \in I} \left|f_j - g_j , g_i\right|.^2$$

Summing over j gives

$$\sum_{i\in I}\left\|V_1 f_j - V_2 g_j\right\|^2 \le 2\sum_{j\in I}\sum_{i\in I}\left|\langle f_j, f_i - g_i\rangle\right|^2 + 2\sum_{j\in I}\sum_{i\in I}\left|\langle f_j - g_j, g_i\rangle\right|^2$$

$$= 2\sum_{i\in I}\sum_{j\in I}\left|\langle f_j, f_i - g_i\rangle\right|^2 + 2\|V_2\|^2\sum_{j\in I}\left\|f_j - g_j\right\|^2$$

$$= 2\|V_1\|^2\sum_{i\in I}\left\|f_i - g_i\right\|^2 + 2\|V_2\|^2\sum_{j\in I}\left\|f_j - g_j\right\|^2$$

$$= 2(\|V_1\|^2 + \|V_2\|^2)\textstyle\sum_{j\in I}\left\|f_j - g_j\right\|^2 .$$

Corollary(5.2.35)[37]: Let $\{f_j\}_{j\in I}, \{g_j\}_{j\in I}$ be Parseval frames for $\mathcal{H}$ with analysis operators V_1, V_2 respectively. If

$$\sum_{j\in I}\left\|f_j - g_j\right\|^2 < \epsilon^2,$$

then

$$\sum_{j\in I}\left\|V_1 f_j - V_2 g_j\right\|^2 < 4\epsilon^2.$$

Proof. The analysis operators V_1 and V_2 are isometries, so the preceding proposition simplifies to the desired estimate.

Corollary (5.2.36)[37]: Let $\mathcal{H}$ be a Hilbert space having two Parseval frames $\mathcal{F} = \{f_j\}_{j=1}^n$ *and* $\mathcal{G} = \{g_j\}_{j=1}^n$ at ℓ^2-distance$\|\mathcal{F} - \mathcal{G}\| \le \epsilon$, then their Grammians G and Q satisfy

$$\|G - Q\|_{HS} \equiv \left(\sum_{j,k=1}^{n}\left|G_{j,k} - Q_{j,k}\right|^2\right)^{1/2} < 2\epsilon.$$

. **Corollary(5.2.37)[37]:** Let $n, d \in \mathbb{N}$ be relatively prime, $n \ge 2$, and let $0 < \epsilon < 1/2$. If G is a rank-d orthogonal $n \times n$ projection matrix over $\mathbb{R}$ or $\mathbb{C}$ and there is $c > 0$ such that the diagonal entries satisfy

$$(1 - \epsilon)^2 c^2 \le G_{j,j} \ (1 + \epsilon)^2 c^2$$

for all $j \in \{1, 2, \dots, n\}$, then there is an orthogonal rank-d projection Q with diagonal $Q_{j,j} = \frac{d}{n}$ and

$$\|G - Q\|_{HS} \le \frac{29}{4} d^2\, n(n-1)^8 \epsilon.$$

Proof. The matrix G is the Grammian of a nearly equal-norm Parseval frame. Using the distance estimate in Theorem (5.2.31) and the preceding corollary, we obtain the desired estimate for the Hilbert–Schmidt distance.

Section(5.3): Given Spectrum and Set of Lengths

Letting $\mathbb{K}$ be either the real or complex field, the synthesis operator of a sequence of vectors $F = \{f_n\}_{n=1}^N$ in an M-dimensional Hilbert space $\mathbb{H}_M$over $\mathbb{K}$ is $F : \mathbb{K}^N \to \mathbb{H}_M, Fg := \sum_{n=1}^N g(n) f_n$. Viewing $\mathbb{H}_M$ as $\mathbb{K}^M$, F is the $M \times N$ matrix whose columns are the f_n's. Note that here and throughout, we make no notational distinction between the vectors themselves and the synthesis operator they induce. The vectors F are said to be a frame for $\mathbb{H}_M$ if there exists frame bounds $0 < A \le B < \infty$ such that $A\|f\|^2 \le \|F^*f\|^2 \le B\|f\|^2$ for all $f \in \mathbb{H}_M$. In this finite-dimensional setting, the optimal frame bounds A and B of an arbitrary F are the least and greatest eigenvalues of the frame operator:

$$FF^* = \sum_{n=1}^{N} f_n f_n^* , \qquad (23)$$

Frames provide numerically stable methods for finding over complete decompositions of vectors, and as such are useful tools in various signal processing applications [52**,** 53**].** Indeed, if F is a frame, then any $f \in \mathbb{H}_M$ can be decomposed as

$$f = F\tilde{F}^* f = \sum_{n=1}^{N} \langle f, \tilde{f}_n \rangle f_n, \qquad (24)$$

where $\tilde{F} = \{ \tilde{f}_n \}_{n=1}^{N}$ is a dual frame of F, meaning it satisfies $F\tilde{F}^* = I$. The most often-used dual frame is the canonical dual, namely the pseudoinverse $\tilde{F} = (FF^*)^{-1}F$. Note that computing a canonical dual involves the inversion of the frame operator. As such, when designing a frame for a given application, it is important to retain control over the spectrum $\{\lambda_m\}_{m=1}^{M}$ of FF^*. Here and throughout, such spectra are arranged in nonincreasing order, with the optimal frame bounds A and B being λ_m and λ_1, respectively.

Of particular interest are tight frames, namely frames for which $A = B$. Note this occurs precisely when $\lambda_m = A$ for all m, meaning $FF^* = AI$. In this case, the canonical dual is given by $\tilde{f}_n = \frac{1}{A} f_n$, and(24)becomes an over complete generalization of an orthonormal basis decomposition. Tight frames are not hard to construct: we simply need the rows of F to be orthogonal and have constant squared norm A. However, this problem becomes significantly more difficult if we further require the f_n's—the columns of F to have prescribed lengths.

In particular, much attention has been paid to the problem of constructing unit norm tight frames ($UNTFs$): tight frames for which $\|f_n\| = 1$ for all n. Here, since $MA = Tr(FF^*) = Tr(F^*F) = N$, we see that A is necessarily $\frac{N}{M}$ $UNTFs$ are known to be optimally robust with respect to additive noise[48] and erasures [41, 49]. Moreover, all unit norm sequences F satisfy the zeroth-order Welch bound $Tr[(FF^*)^2] \ge \frac{N^2}{M}$ $UNTF$ [59, 60]; a physics-inspired interpretation of this fact leading to an optimization-based proof of existence of UNTFs is given in [36]. We further know that such frames are commonplace: when $N \ge M + 1$, the manifold of all $M \times N$ real $UNTFs$, modulo rotations, is known to have dimension $(M-1)(N-M-1)$ [46]. Essentially, when $N = M + 1$, this manifold is zero-dimensional since the only $UNTFs$ are regular simplices [47]; each additional unit norm vector injects $M - 1$ additional degrees of freedom into this manifold, in accordance with the dimension of the unit sphere in $\mathbb{R}^M$. Local parametrizations of this manifold are given in [56]. The Paulsen problem involves projecting a given frame onto this manifold, and differential calculus-based methods for doing so are given in [37, 39].

In light of these facts, it is surprising to note how few explicit constructions of $UNTFs$ are known. Indeed, a constructive characterization of all $UNTFs$ is only known for $M = 2$ [47]. For arbitrary M and N, there are only two known general construction techniques: truncations of discrete Fourier transform matrices known as harmonic frames [47] and a sparse construction method dubbed spectral tetris [40]. To emphasize this point, we note that there are only a small finite number of known constructions of 3 × 5 $UNTFs$, despite the fact that an infinite number of such frames exist even modulo rotations, their manifold being of dimension $(M-1)(N-M-1) = 2$. The reason for this is that in order to construct a $UNTF$, one must solve a large system of quadratic equations in many variables: the columns of F must have unit norm, and the rows of F must be orthogonal with constant norm $\left(\frac{N}{M}\right)^{\frac{1}{2}}$.

In this section, we show how to explicitly construct all $UNTFs$, and moreover, how to explicitly construct every frame whose frame operator has a given arbitrary spectrum and whose vectors are of given arbitrary lengths. To do so, we build on the existing theory of majorization and the Schur-Horn Theorem. To be precise, given two nonnegative nonincreasing sequences $\{\lambda_n\}_{n=1}^{N}$ and $\{\mu_n\}_{n=1}^{N}$ we say that $\{\lambda_n\}_{n=1}^{N}$ majorizes $\{\mu_n\}_{n=1}^{N}$, denoted $\{\lambda_n\}_{n=1}^{N} \geq \{\mu_n\}_{n=1}^{N}$, if

$$\sum_{\acute{n}=1}^{n} \lambda_{\acute{n}} \geq \sum_{\acute{n}=1}^{n} \mu_{\acute{n}} \qquad \forall_n = 1, \dots, N-1,$$
$$\sum_{\acute{n}=1}^{N} \lambda_{\acute{n}} = \sum_{\acute{n}=1}^{N} \mu_{\acute{n}}.$$

Viewed as discrete functions over the axis $\{1, \dots, N\}$, having $\{\lambda_n\}_{n=1}^{N}$majorize $\{\mu_n\}_{n=1}^{N}$means that the total area under both curves is equal, and that the area under $\{\lambda_n\}_{n=1}^{N}$ is distributed more to the left than that of $\{\mu_n\}_{n=1}^{N}$. A classical result of Schur [55] states that the spectrum of a self-adjoint positive semidefinite matrix necessarily majorizes its diagonal entries. A few decades later, Horn gave a nonconstructive proof of a converse result [50], showing that if $\{\lambda_n\}_{n=1}^{N} \geq \{\mu_n\}_{n=1}^{N}$, then there exists a self-adjoint matrix that has $\{\lambda_n\}_{n=1}^{N}$ as its spectrum and $\{\mu_n\}_{n=1}^{N}$as its diagonal.

These two results are collectively known as the Schur-Horn Theorem:

Schur-Horn Theorem. There exists a positive semidefinite self-adjoint matrix with spectrum $\{\lambda_n\}_{n=1}^{N}$and diagonal entries $\{\mu_n\}_{n=1}^{N}$ if and only if $\{\lambda_n\}_{n=1}^{N} \geq \{\mu_n\}_{n=1}^{N}$.

Over the years, several methods for explicitly constructing Horn's matrices have been found; see [44] for a nice overview. Many current methods rely on Givens rotations [42, 44, 58], while others involve optimization [43]. With regards to frame theory, the significance of the Schur-Horn Theorem is that it completely characterizes whether or not there exists a frame whose frame operator has a given spectrum and whose vectors have given lengths; this follows from applying it to the Gram matrix F^*F, whose diagonal entries are the values $\{\|f_n\|^2\}_{n=1}^{N}$ and whose spectrum $\{\lambda_n\}_{n=1}^{N}$ is a zero-padded version of the spectrum $\{\lambda_m\}_{m=1}^{M}$ of the frame operator FF^* Indeed, majorization inequalities arose during the search for tight frames with given lengths [38, 45], and the explicit connection between frames and the Schur-Horn Theorem is noted in [35, 52]. This connection was then exploited to solve various frame theory problems, such as frame completion [54].

In this section, we follow the approach of [51] in which majorization is viewed as the end result of the repeated application of a more basic idea: eigenvalue interlacing. To be precise, a nonnegative nonincreasing sequence $\{\gamma_m\}_{m=1}^{M}$ interlaces on another such sequence $\{\beta_m\}_{m=1}^{M}$, denoted $\{\beta_m\}_{m=1}^{M} \sqsubseteq \{\gamma_m\}_{m=1}^{M}$ provided

$$\beta_M \le \gamma_M \le \beta_{M-1} \le \gamma_{M-1} \le \cdots \le \beta_2 \le \gamma_2 \le \beta_1 \le \gamma_1. \qquad (25)$$

Under the convention $\gamma_{M+1} := 0$, we have that $\{\beta_m\}_{m=1}^{M} \sqsubseteq \{\gamma_m\}_{m=1}^{M}$ if and only if $\gamma_{M+1} \le \beta_M \le \gamma_M$ for all $m = 1, \ldots, M$.

Interlacing arises in the context of frame theory by considering partial sums of the frame operator (23). To be precise, given any sequence of vectors $F = \{f_n\}_{n=1}^{N}$ in $\mathbb{H}_M$, then for every $n = 1, \ldots, N$, we consider the partial sequence of vectors $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^{N}$. Note that $F_N = F$ and the frame operator of F_N is

$$F_n F_n^* = \sum_{\acute{n}=1}^{n} f_{\acute{n}} f_{\acute{n}}^* . \qquad (26)$$

Let $\{\lambda_{n;m}\}_{m=1}^{M}$ denote the spectrum (26). For any $n = 1, \ldots, N-1$, (26) gives that $F_{n+1}F_{n+1}^* = F_N F_n^* + f_{n+1} f_{n+1}^*$ and so a classical result [51] involving the addition of rank-one positive operators gives that $\{\lambda_{n;m}\}_{m=1}^{M} \sqsubseteq \{\lambda_{n+1;m}\}_{m=1}^{M}$. Moreover, if $\|f_n\|^2 = \mu_n$ for all $n = 1, \ldots, N$, then for any such n,

$$\sum_{m=1}^{M} \lambda_{n;m} = Tr(F_n F_n^*) = Tr(F_n^* F_n) = \sum_{\acute{n}=1}^{n} \|f_{\acute{n}}\|^2 \sum_{\acute{n}=1}^{n} \mu_{\acute{n}} \qquad (27)$$

Note that as n increases, the Gram matrix grows in dimension but the frame operator does not since $F_n^* F_n : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ but $F_n F_n^* : \mathbb{H}_M \longrightarrow \mathbb{H}_M$. We call a sequence of interlacing spectra that satisfy (27) a sequence of eigensteps:

Definition (5.3.1)[34]: Given nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$, a sequence of eigensteps is a doubly- indexed sequence of sequences $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ for which:

(i) The initial sequence is trivial:

$$\lambda_{0;m} = 0 \quad \forall m = 1, \ldots, M.$$

(ii) The final sequence is $\{\lambda_{n;m}\}_{m=1}^{M} = \lambda_m \quad \forall m = 1, \ldots, M.$

(iii) The sequences interlace:

$$\{\lambda_{n-1;m}\}_{m=1}^{M} \sqsubseteq= \{\lambda_{n;m}\}_{m=1}^{M} \quad \forall n = 1, \ldots, N.$$

(iv) The trace condition is satisfied:

$$\sum_{m=1}^{M} \lambda_{N;m} = \sum_{\acute{n}=1}^{n} \mu_{\acute{n}} \qquad \forall n = 1, \ldots, N.$$

As we have just discussed, every sequence of vectors whose frame operator has the spectrum $\{\lambda_m\}_{m=1}^{M}$ and whose vectors have squared lengths $\{\mu_n\}_{m=1}^{N}$ generates a sequence of eigensteps. In the next section, we adapt a proof technique of [51] to show the converse is true. Specifically, characterizes and proves the existence of sequences of vectors that generate a given sequence of eigensteps. In this Section, We then use this characterization to provide an algorithm for explicitly constructing all such sequences of vectors; see Theorem (5.3.7).Though nontrivial, this algorithm is nevertheless straightforward enough to be implemented by hand in small-dimensional examples, involving only arithmetic, square roots and matrix multiplication. We will see that once the eigensteps have been chosen, the algorithm gives little freedom in picking the frame vectors themselves. That is, modulo rotations, the eigensteps are the free parameters when designing a frame whose frame operator has a given spectrum and whose vectors have given lengths.

The significance of these methods is that they explicitly construct every possible finite frame of a given spectrum and set of lengths. Computing the Gram matrices of such frames produces every possible matrix that satisfies the Schur-Horn Theorem; previous methods have only constructed a subset of such matrices. Moreover, in the special case where the spectrums and lengths are constant, these methods construct every equal norm tight frame. This helps narrow the search for frames we want for applications: tight Gabor, wavelet, equiangular and Grassmannian frames.

The purpose of this section is to prove the following result:

Conversely, any F constructed by this process has $\{\lambda_m\}_{m=1}^{M}$ as the spectrum of FF^* and $\|f_n\|^2 = \mu_n$ for all n.

Moreover, for any F constructed in this manner, the spectrum of $F_n F_n^*$ is $\{\lambda_{n;m}\}_{m=1}^{M}$ for all $n = 1, \ldots, N$.

We note that as it stands, Theorem(5.3.6) is not an easily-implementable algorithm, as Step A requires one to select a valid sequence of eigensteps—not an obvious feat—while Step B requires one to compute orthonormal eigenbases for each F_n. These concerns will be addressed in the following section. We further note that Theorem(5.3.6) only claims to construct all possible such , sidestepping the issue of whether such an F actually exists for a given $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{m=1}^{N}$. This issue is completely resolved by the Schur-Horn Theorem. Indeed, in the case where $M \le N$, [35] shows that there exists a sequence of vectors $F = \{f_n\}_{n=1}^{n}$ in $\mathbb{H}_M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^{M}$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n if and only if $\{\lambda_m\}_{m=1}^{M} \cup \{0\}_{n=M+1}^{N} \succeq \{\mu_n\}_{n=1}^{n}$. In the case where $M > N$, a similar argument shows that such a sequence of vectors exists if and only if $\{\lambda_m\}_{m=1}^{M} \succeq \{\mu_n\}_{n=1}^{n}$ and $\lambda_m = 0$ for all $m = N+1, \ldots, M$. As Step B of Theorem(5.3.6) can always be completed for any valid sequence of eigensteps, these majorization conditions in fact characterize those values $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{n}$ for which Step A can successfully be performed; we leave a deeper exploration of this fact for future work. In order to prove Theorem(5.3.6), we first obtain some supporting results. The following lemma gives a first taste of the connection between eigensteps and our frame construction problem:

Lemma(5.3.2)[34]: Let $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$ be nonnegative and nonincreasing, and let $\{\{\lambda_m\}_{m=1}^{M}\}_{n=0}^{N}$ be any corresponding sequence of eigensteps as in Definition (5.3.1). If a sequence of vectors $F = \{f_n\}_{n=1}^{N}$ has the property that the spectrum of the frame operator $F_n F_n^*$ of $F_n = \{f_n\}_{n=1}^{n}$ is $\{\lambda_m\}_{m=1}^{M}$ for all $n = 1, \ldots, N$, then the spectrum of FF^* is $\{\lambda_m\}_{m=1}^{M}$ and $\|f_n\|^2 = \mu_n$ for all $n = 1, \ldots, N$.

Proof. Definition(5.3.1) (ii) immediately gives that the spectrum of

$$FF^* = F_N F_N^*$$

N is indeed $\{\lambda_n\}_{m=1}^{M} = \{\lambda_{N;m}\}_{m=1}^{M}$, as claimed.

Moreover, for any $= 1, \ldots, N.$, Definition (5.3.1) (iv) gives

$$\sum_{\acute{n}=1}^{n} \|f_{\acute{n}}\|^2 = Tr(F_n^* F_n) = Tr(F_n F_n^*) = \sum_{m=1}^{M} \lambda_{n,m} = \sum_{\acute{n}=1}^{n} \mu_{\acute{n}} \quad (28)$$

Letting $n = 1$ in (28) gives $\|f_1\|^2 = \mu_1$, while for $n = 2, \dots, N$, considering (28) at both n and $n - 1$ gives

$$\|f_n\|^2 = \sum_{\acute{n}=1}^{n} \|f_{\acute{n}}\|^2 - \sum_{\acute{n}=1}^{n-1} \|f_{\acute{n}}\|^2 = \sum_{\acute{n}=1}^{n} \mu_{\acute{n}} - \sum_{\acute{n}=1}^{n-1} \mu_{\acute{n}} = \mu_n$$

The next result gives conditions that a vector must satisfy in order for it to perturb the spectrum of a given frame operator in a desired way(see [51]).

Theorem(5.3.3)[34]: Let $F_n = \{f_{\acute{n}}\}_{\acute{n}-1}^{n}$ be an arbitrary sequence of vectors in $\mathbb{H}_M$ and let $\{\lambda_{n,m}\}_{m=1}^{M}$ denote the eigenvalues of the corresponding frame operator $F_n F_n^*$. For any choice of f_{n+1} in $\mathbb{H}_M$, let $F_{n+1} = \{f_{\acute{n}}\}_{\acute{n}=1}^{n+1}$. Then for any $\lambda \in \{\lambda_{n,m}\}_{m=1}^{M}$ the norm of the projection of f_{n+1} onto the eigenspace $N(\lambda I - F_n F_n^*)$ is given by

$$\left\|P_{n;\lambda} f_{n+1}\right\|^2 = -\lim_{x \to \lambda} (x - \lambda) \frac{P_{n+1}(x)}{P_n(x)}$$

where $p_n(x)$ and $p_{n+1}(x)$ denote the characteristic polynomials of $F_n F_n^*$ and $F_{n+1} F_{n+1}^*$, respectively.

Proof. For the sake of notational simplicity, let $F_n = F, f_{n+1} = f, f_{n+1} = G, P_{n;\lambda} = P_\lambda, p_n(x) = p(x), p_{n+1}(x) = q(x)$,nd let $\lambda_{n;m} = \beta_m$ for all $m = 1, \dots, M$. We will also use I to denote the identity matrix, and its dimension will be apparent from context. To obtain the result, we will express the characteristic polynomial $\tilde{q}(x)$ of the $(n + 1) \times (n + 1)$Gram matrix $G^* G$ in terms of the characteristic polynomial $\tilde{p}(x)$of the $n \times n$ Gram matrix $F^* F$. Written in terms of their standard matrix representations, we have $G = [F \ \ f]$

$$G^* G = \begin{bmatrix} F^* \\ f^* \end{bmatrix} [F \quad f] = \begin{bmatrix} F^* F & F^* f \\ f^* F^* & \|f\|^2 \end{bmatrix} \quad (29)$$

To compute the determinant of $x1 - G^* G$, it is helpful to compute the singular value decomposition $F = U \sum V^*$ and note that for any x not in the diagonal of $\Sigma^* \Sigma_*$ the following matrix W has unimodular determinant:

$$W := \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (x1 - \Sigma^* \Sigma)^{-1} V^* F^* f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} V & V(x1 - \Sigma^* \Sigma)^{-1} V^* F^* f \\ 0 & 1 \end{bmatrix}. \quad (30)$$

Subtracting (29) from xI and conjugating by (30) yields

$$W^*(x1 - G^* G) W$$
$$= \begin{bmatrix} V^* & 0 \\ (V(x1 - \Sigma^* \Sigma)^{-1} V^* F^* f) & 1 \end{bmatrix} \begin{bmatrix} x1 - F^* F & -F^* f \\ -f^* F & x - \|f\|^2 \end{bmatrix} \begin{bmatrix} V & V(x1 - \Sigma^* \Sigma)^{-1} V^* F^* f \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} V^* & 0 \\ (f^* F V (x1 - \Sigma^* \Sigma)^{-1} V^*) & 1 \end{bmatrix}$$
$$\begin{bmatrix} (x1 - F^* F) V & (x1 - F^* F) V (x1 - \Sigma^* \Sigma)^{-1} V^* F^* f - F^* f \\ -f^* F V & x - \|f\|^2 - f^* F V (x1 - \Sigma^* \Sigma)^{-1} V^* F^* f \end{bmatrix} \quad (31)$$

Since $F^* F = V \Sigma^* \Sigma V^*$ then $-F^* F = x1 - V \Sigma^* \Sigma V^* = V(x1 - \Sigma^* \Sigma) V^*$. As such,
$(x1 - F^* F)\, V (x1 - \Sigma^* \Sigma)^{-1} V^* F^* f - F^* f = (V(x1 - \Sigma^* \Sigma)\, V^* V (x1 - \Sigma^* \Sigma)^{-1} V^* F^* f - F{*}f{=}F{*}f{-}F{*}f{=}0 \quad (32)$

Substituting (32) into (31) and again noting $V^*(xI - F^*F)V = x1 - \Sigma^*\Sigma$ gives

$$W^*(x1 - G^*G)W = \begin{bmatrix} V^* & 0 \\ f^*FV(x1-\Sigma^*\Sigma)^{-1}V^* & 1 \end{bmatrix}$$

$$\begin{bmatrix} (x1 - F^*F)V & 0 \\ -f^*FV & x - \|f\|^2 - f^*FV(x1-\Sigma^*\Sigma)^{-1}V^*F^*f \end{bmatrix}$$

$$\begin{bmatrix} V^*(x1-F^*F)V & 0 \\ f^*FV(x1-\Sigma^*\Sigma)^{-1}V^*(x1-F^*F)V - f^*FV & x - \|f\|^2 - f^*FV(x1-\Sigma^*\Sigma)^{-1}V^*F^*f \end{bmatrix}$$

$$= \begin{bmatrix} x1-\Sigma^*\Sigma & 0 \\ 0 & x - \|f\|^2 - f^*FV(x1-\Sigma^*\Sigma)^{-1}V^*F^*f \end{bmatrix} \quad (33)$$

Since W has unimodular determinant, (33) implies

$\tilde{q}(x) := det(xI - G^*G) = det\,[W^*(xI - G^*G)W] = det(x1 - \Sigma^*\Sigma)(x - \|f\|^2 - f*FVx1-\Sigma*\Sigma-1V*F*f$ (34)

To simplify (34), note that since V is unitary,

$\tilde{p}(x) := det(xI - F^*F) = \det[V^*(x1 - F^*F)V] = det(x1 - \Sigma^*\Sigma)$(35)

Moreover, letting $(\Sigma^*\Sigma)(\acute{n},\acute{n})$ denote the $\acute{n}$ th diagonal entry of $\Sigma^*\Sigma$ yields

$$f^*FV(x1-\Sigma^*\Sigma)^{-1}V^*F^*f$$
$$= (V^*F^*f)^*(x1-\Sigma^*\Sigma)^{-1}(V^*F^*f) = \sum_{\acute{n}=1}^{n} \frac{|(V^*F^*f)(\acute{n})|^2}{x-(\Sigma^*\Sigma)(\acute{n},\acute{n})} \quad (36)$$

Substituting (35)and (36)into (34)gives

$$\tilde{q}(x) = \tilde{p}(x)(x - \|f\|^2 - \sum_{\acute{n}=1}^{n} \frac{|(V^*F^*f)(\acute{n})|^2}{x-(\Sigma^*\Sigma)(\acute{n},\acute{n})}) \quad (37)$$

To continue simplifying (37), let $\delta_{\acute{n}}$ denote the $\acute{n}$th standard basis element. Then $V^*F^* = \Sigma^*U^*$ implies that for any $\acute{n} = 1,\ldots,n$,

$$(V^*F^*f)(\acute{n}) = \langle V^*F^*f, \delta_{\acute{n}}\rangle = \langle \Sigma^*U^*f, \delta_{\acute{n}}\rangle = \langle f, U\Sigma\delta_{\acute{n}}\rangle = \begin{cases} \sigma_{\acute{n}}\langle f, \mu_{\acute{n}}\rangle, & \acute{n} \le M, \\ 0, & \acute{n} > M \end{cases} \quad (38)$$

where $\{\sigma_{\acute{n}}\}_{\acute{n}=1}^{min\{M,n\}}$ are the singular values of F. Since $\Sigma^*\Sigma(\acute{n};\acute{n}) = \sigma_{\acute{n}}^2$ for any $\acute{n} = 1,\ldots,min\{M,n\}$, (38) implies

$$\sum_{\acute{n}=1}^{n} \frac{|(V^*F^*f)(\acute{n})|^2}{x-(\Sigma^*\Sigma)(\acute{n},\acute{n})} = \sum_{\acute{n}=1}^{min\{M,n\}} \frac{\sigma_{\acute{n}}^2|\langle f,\mu_{\acute{n}}\rangle^2|}{x-(\Sigma^*\Sigma)(\acute{n},\acute{n})} = \sum_{\acute{n}=1}^{min\{M,n\}} \frac{\sigma_{\acute{n}}^2}{x-\sigma_{\acute{n}}^2}|f,u_{\acute{n}}|^2 \quad (39)$$

Making the change of variables $m = \acute{n}$ in (39) and substituting the result into (37) gives

$$\tilde{q}(x) = \tilde{p}(x)(x - \|f\|^2 - \sum_{m=1}^{min\{M,n\}} \frac{\sigma_{\acute{m}}^2}{x-\sigma_{\acute{m}}^2}|f,u_{\acute{n}}|^2 \qquad \forall x \ne \sigma_1^2,\ldots,\sigma_{\min(M,n)}^2, 0. \quad (40)$$

Here, the restriction that $x \ne \sigma_1^2,\ldots,\sigma_{min\{M,n\}}^2, 0$ follows from the previously stated assumption that x is not equal to any diagonal entry of $\Sigma^*\Sigma$; the set of these entries is $\{\sigma_{\acute{n}}^2\}_{\acute{n}=1}^{n}$ if $M \ge n$ and is $\{\sigma_{\acute{n}}^2\}_{\acute{n}=1}^{M} \cup \{0\}_{\acute{n}=1}^{n}$ if $M < n$. Now recall that $p(x)$ and $q(x)$ are the Mth degree characteristic polynomials of FF^* and GG^* respectively, while $\tilde{p}(x)$ is the nth degree characteristic polynomial of F^*F and $\tilde{q}(x)$ is the $(n+1)$st degree characteristic polynomial of G^*G. We now consider these facts along with (40) in two distinct cases: $n < M$ and $M \le n$. In the case where $n < M$, we have that $p(x) = x^{M-n}\,\tilde{p}(x)$ and $q(x) = x^{M-n-1}\,\tilde{q}(x)$. Moreover, in this case the eigenvalues $\{\beta_m\}_{m=1}^{M}$ of $FF^* = U\Sigma\Sigma^*U^*$ are given by $\beta_m = \sigma_m^2$ for all $m = 1,\ldots,n$ and $\beta_m = 0$ for all $m = n+1,\ldots,M$, implying (40) becomes

$$\frac{q(x)}{x^{M-n-1}}=\frac{p(x)}{x^{M-n}}\left(x-\|f\|^2-\sum_{m=1}^{n}\frac{\beta_m}{x-\beta_m}|\langle f,u_m\rangle|^2\right)$$

$$=\frac{p(x)}{x^{M-n}}\left(x-\|f\|^2-\sum_{m=1}^{M}\frac{\beta_m}{x-\beta_m}|\langle f,u_m\rangle|^2\right)\forall x\neq \beta_1,\dots,\beta_M,(41)$$

In the remaining case where $M \leq n$, we have $\tilde{p}(x) = x^{n-M}\, p(x), \tilde{q}(x) = x^{n+1-M} q(x)$ and $\beta_m = \sigma_m^2$ for all $m = 1, \dots M$,implying (42) becomes

$$x^{n+1-M} q(x) = x^{n-M}\, p(x)\left(x-\|f\|^2-\sum_{m=1}^{M}\frac{\beta_m}{x-\beta_m}|\langle f,u_m\rangle|^2\right)\ \forall x \neq,\dots,\beta_M,0.\,(42)$$

We now note that (43)and (44) are equivalent. That is, regardless of the relationship between M and n, we have

$$\frac{q(x)}{p(x)}=\frac{1}{x}\left(x-\|f\|^2-\sum_{m=1}^{M}\frac{\beta_m}{x-\beta_m}|\langle f,u_m\rangle|^2\right),\qquad \forall x\neq \beta_1,\dots,\beta_M,0$$

Writing $\| f\|^2 = \sum_{m=1}^{M}|\langle f,u_m\rangle|^2$ and then grouping the eigenvalues $\Lambda = \{\beta_m\}_{m=1}^{M}$ according to multiplicity gives

$$\frac{q(x)}{p(x)}=\frac{1}{x}\left(x-\sum_{m=1}^{M}|\langle f,u_m\rangle|^2-\sum_{m=1}^{M}\frac{\beta_m}{x-\beta_m}|\langle f,u_m\rangle|^2\right)=1-\sum_{m=1}^{M}\frac{|\langle f,\mu_m\rangle|^2}{x-\beta_m}=1-\sum_{\hat{\lambda}\in A}\frac{\|P_{\hat{\lambda}}f\|^2}{x-\hat{\lambda}},$$

$$\forall x\notin \Lambda\cup\{0\}$$

As such, for any $\lambda \in \Lambda$.

$$\lim_{x\to\lambda}(x-\lambda)\frac{q(x)}{p(x)}$$

$$=\lim_{x\to\lambda}(x-\lambda)\left(1-\sum_{\hat{\lambda}\in\Lambda}\frac{\|P_{\hat{\lambda}}f\|^2}{x-\hat{\lambda}}=\lim_{x\to\lambda}\left[(x-\lambda)-\|P_\lambda f\|^2-\sum_{\hat{\lambda}\in\lambda}\left\|P_{\hat{\lambda}}f\right\|^2\frac{x-\lambda}{x-\hat{\lambda}}\right]\right)-\|P_\lambda f\|^2$$

yielding our claim.

Though technical, the proofs of the next two lemmas are nonetheless elementary, depending only on basic algebra
and calculus. As such, these proofs are given in the appendix[34].

Lemma(5.3.4)[34]: If $\{\beta_m\}_{m=1}^{M}$ and $\{\gamma_m\}_{m=1}^{M}$ are real and nonincreasing, then $\{\beta_m\}_{m=1}^{M} \sqsubseteq \{\gamma_m\}_{m=1}^{M}$ if and only if

$$\lim_{x\to\beta_m}(x-\beta_m)\frac{q(x)}{p(x)}\leq 0\qquad \forall m=1,\dots,M$$

where $p(x) = \prod_{m=1}^{M}(x-\beta_m)\ and\ \ q(x) = \prod_{m=1}^{M}(x-\gamma_m)$

Lemma(5.3.5)[34]: If $\{\beta_m\}_{m=1}^{M}, \{\gamma_m\}_{m=1}^{M}$ and $\{\delta_m\}_{m=1}^{M}$are real and nonincreasing and

$$\lim_{x\to\beta_m}(x-\beta_m)\frac{q(x)}{p(x)}=\lim_{x\to\beta_m}(x-\beta_m)\frac{r(x)}{p(x)}\qquad \forall m=1,\dots,M$$

where $p(x) = \prod_{m=1}^{M}(x-\beta_m)\,, q(x) = \prod_{m=1}^{M}(x-\gamma_m)$ and $r(x) = \prod_{m=1}^{M}(x-\delta_m)$ then $q(x) = r(x)$

With Theorem(5.3.3) and Lemmas(5.3.2),(5.3.4) and(5.3.5) in hand, we are ready to prove the main result .

Theorem (5.3.6)[34]: For any nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$, every sequence of vectors $F = \{f_n\}_{n=1}^{N}$ in $\mathbb{H}_M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^{M}$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n can be constructed by the following process:

A. Pick eigensteps $\left\{\{\lambda_{n,m}\}_{m=1}^{M}\right\}_{n=0}^{N}$ as in Definition (5.3.1).

B. For each $n = 1, \ldots, N$, consider the polynomial:

$$P_n(x) := \prod_{m=1}^{M} \left(x - \lambda_{n,m}\right). \quad (43)$$

Take any $f_1 \in \mathbb{H}_M$ such that $\|f_1\|^2 = \mu_1$. For each $n = 1, \ldots, N-1$, choose any f_{n+1} such that

$$\left\|P_{n;\lambda} f_{n+1}\right\|^2 = -\lim_{x \longrightarrow \lambda} (x - \lambda) \frac{P_{n+1}(x)}{P_n(x)}, \quad (44)$$

for all $\lambda \in \{\lambda_{N;m}\}_{m=1}^{M}$, where $P_{n;\lambda}$ denotes the orthogonal projection operator onto the eigenspace $N(\lambda I - F_n F_n^*)$ of the frame operator $F_n F_n^*$ of $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^{n}$. The limit in (44) exists and is nonpositive.

Proof. $(\Longrightarrow)$ Let $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_m\}_{n=1}^{N}$n=1 be arbitrary nonnegative nonincreasing sequences, and let $F = \{f_n\}_{n=1}^{N}$ be any sequence of vectors such that the spectrum of $FF^*$is $\{\lambda_m\}_{m=1}^{M}$ and $\|f_n\| = \mu_n$ for all $n = 1, \ldots, N$. We claim that this particular F can be constructed by following Steps A and B.

In particular, consider the sequence of sequences $\left\{\left\{\lambda_{n;m}\right\}_{m=1}^{M}\right\}_{n=0}^{N}$ defined by letting $\left\{\lambda_{n;m}\right\}_{m=1}^{M}$ be the spectrum of the frame operator $F_n F_n^*$ of the sequence $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^{n}$ for all $n = 1, \ldots, N$ and letting $\lambda_{0;m} = 0$ for all m. We claim that $\{\{\lambda_m\}_{m=1}^{M}\}_{n=0}^{N}$ satisfies Definition(5.3.1) and therefore is a valid sequence of eigensteps. Note conditions (i) and (ii) of Definition (5.3.1) are immediately satisfied. To see that $\{\{\lambda_m\}_{m=1}^{M}\}_{n=0}^{N}$ satisfies (iii), consider the polynomials $p_n(x)$ defined by(43) for all $n = 1, \ldots, N$. In the special case where $n = 1$, the desired property (iii) that $\{0\}_{m=1}^{M} \sqsubseteq \left\{\lambda_{1;m}\right\}_{m=1}^{M}$ from the fact that the spectrum $\left\{\lambda_{1;m}\right\}_{m=1}^{M}$of the scaled rank-one projection $F_1 F_1^* = f_1 f_1^*$ is the value $\|f\|^2 = \mu_1$ along with $M - 1$ repetitions of 0, the eigenspaces being the span of f_1 and its orthogonal complement, respectively. Meanwhile if n = 2; : : : ; N, Theorem (5.3.3) gives that

$$\lim_{x \longrightarrow \lambda_{n-1;m}} \left(x - \lambda_{n-1;m}\right) \frac{P_n(x)}{P_{n-1(x)}} = -\left\|P_{n-1;\lambda_{n-1;m}} f_n\right\|^2 \leq 0, \ \forall m = 1, \ldots, M$$

implying by Lemma(5.3.4) that $\left\{\lambda_{n-1;m}\right\}_{m=1}^{M} \sqsubseteq \left\{\lambda_{n;m}\right\}_{m=1}^{M}$ as claimed. Finally, (iv) holds since for any $n = 1, \ldots, N$ we have

$$\sum_{m=1}^{M} \lambda_{n;m} = Tr(F_n F_n^*) = Tr(F_n^* F_n) = \sum_{\acute{n}=1}^{n} \|f_{\acute{n}}\|^2 = \sum_{\acute{n}=1}^{n} \mu_{\acute{n}}.$$

Having shown that these particular values of $\left\{\left\{\lambda_{n;m}\right\}_{m=1}^{M}\right\}_{n=0}^{N}$can indeed be chosen in Step A, we next show that our particular F can be constructed according to Step B. As the method of Step B is iterative, we use induction to prove that it can yield F. Indeed, the only restriction that Step B places on $f_1$is that $\|f_1\|^2 = \mu_1$, something our particular $f_1$satisfies by assumption. Now assume that for any $n = 1, \ldots, N-1$ we have already correctly produced $\{f_{\acute{n}}\}_{\acute{n}=1}^{n}$.

By following the method of Step B; we show that we can produce the correct f_{n+1} by continuing to follow Step B. To be clear, each iteration of Step B does not produce a unique vector, but rather presents a family of f_{n+1} to choose from, and we show that our particular choice of f_{n+1} lies in this family. Specifically, our choice of f_{n+1}must satisfy (44) for any choice of $\lambda \in \{\lambda_{n;m}\}_{m=1}^{M}$ the fact that it indeed does so follows immediately from Theorem(5.3.3). To summarize, we have shown that by making appropriate choices, we can indeed produce our particular F by following Steps A and, concluding this direction of the proof.

($\Longleftarrow$) Now assume that a sequence of vectors $F = \{f_n\}_{n=1}^{n}$has been produced according to Steps A and B. To be precise, letting $\left\{\{\lambda_{n;m}\}_{m=1}^{M}\right\}_{n=0}^{N}$be the sequence of eigensteps chosen in Step A, we claim that any $F = \{f_n\}_{n=1}^{n}$ constructed according to Step B has the property that the spectrum of the frame operator $F_n F_n^*$ of $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^{n}$ is$\{\lambda_{n;m}\}_{m=1}^{M}$ for all $n = 1, \ldots, N$. Note that by Lemma (5.3.2), proving this claim will yield our stated result that the spectrum of $FF^*$is $\{\lambda_m\}_{m=1}^{M}$ and that $\|f_n\|^2 = \mu_n$ for all $n = 1, \ldots, N$. As the method of Step B is iterative, we prove this claim by induction. Step B begins by taking any f_1 such that $\|f_1\|^2 = \mu_1$. As noted above in the proof of the other direction, the spectrum of $F_1F_1^* = f_1f_1^*$ is the value μ_1 along with $M-1$ repetitions of 0. As claimed, these values match those of $\{\lambda_{1;m}\}_{m=1}^{M}$ to see this, note that Definition(5.3.1) (i) and (iii) give $\{0\}_{m=1}^{M} = \{\lambda_{0;m}\}_{m=1}^{M} \sqsubseteq \{\lambda_{1;m}\}_{m=1}^{M}$and so $\lambda_{1;m} = 0$ for all $m = 2, \ldots, M$, at which point Definition (5.3.1) (iv) implies $\lambda_{1,1} = \mu_1$.

Now assume that for any $n = 1, \ldots, N-1$, the Step B process has already produced $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^{n}$such that the spectrum of $F_nF_n^*$ is $\{\lambda_{n;m}\}_{m=1}^{M}$. We show that by following Step B, we produce an f_{n+1} such that $F_{n+1} = \{f_{\acute{n}}\}_{\acute{n}=1}^{n}$ has the property that $\{\lambda_{N+1;m}\}_{m=1}^{M}$ is the spectrum of $F_{n+1}F_{n+1}^*$. To do this, consider the polynomials $p_n(x)$ and $p_{n+1}(x)$, defined by (43) and pick any f_{n+1} that satisfies(44), namely

$$\lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{p_{n+1}(x)}{p_n(x)} = -\left\|p_{n;\lambda_{n;m}} f_{n+1}\right\|^2 \quad \forall m = 1, \ldots, M. (45)$$

Letting $\{\lambda_{n+1;m}\}_{m=1}^{M}$ denote the spectrum of $F_{n+1}F_{n+1}^*$, our goal is to show that $\{\hat{\lambda}_{n+1;m}\}_{m=1}^{M} = \{\lambda_{n+1;m}\}_{m=1}$. Equivalently, our goal is to show that $p_{n+1}(x) = \hat{p}_{n+1}(x)$ where $\hat{p}_{n+1}(x)$ is the polynomial

$$\hat{p}_{n+1}(x) \coloneqq \prod_{m=1}^{M}(x-\hat{\lambda}_{n+1;m}).$$

Since $p_n(x)$and $\hat{p}_{n+1}$ are the characteristic polynomials of $F_nF_n^*$ and $F_{n+1}F_{n+1}^*$, respectively, Theorem(5.3.3) gives:

$$\lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{\hat{p}_{n+1}(x)}{p_n(x)} = -\left\|p_{n;\lambda_{n;m}} f_{n+1}\right\|^2 \quad \forall m = 1, \ldots, M: \quad (46)$$

Comparing (45)and (46)gives:

$$\lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{p_{n+1}(x)}{p_n(x)} = \lim_{x\to\lambda_{n;m}} (x-\lambda_{n;m})\frac{\hat{p}_{n+1}(x)}{p_n(x)} \qquad \forall m = 1, \ldots, M:$$

implying by Lemma(5.3.5) that $p_{n+1}(x) = \hat{p}_{n+1}(x)$, as desired.

As discussed a two-step process for constructing any and all sequences of vectors $F = \{f_n\}_{n=1}^{N}$ in $\mathbb{H}_M$ whose frame operator possesses a given spectrum $\{\lambda_m\}_{m=1}^{M}$and whose vectors have given lengths $\{\mu_n\}_{n=1}^{N}$. In Step A, we choose a sequence of eigensteps $\left\{\left\{\lambda_{n;m}\right\}_{m=1}^{M}\right\}_{n=0}^{N}$. In the end, the nth sequence $\left\{\lambda_{n;m}\right\}_{m=1}^{M}$ will become the spectrum of the nth partial frame operator $F_n F_n^*$, where $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^{n}$. Due to the complexity of Definition (5.3.1), it is not obvious how to sequentially pick such eigensteps. Looking at simple examples of this problem, such as the one discussed in Example(5.3.8) below, it appears as though the proof techniques needed to address these questions are completely different from those used throughout this section. As such, we leave the problem of parametrizing the eigensteps themselves for future work. In this section, we thus focus on refining Step B.

To be precise, the purpose of Step B is to explicitly construct any and all sequences of vectors whose partial-frame operator spectra match the eigensteps chosen in Step A. The problem with Step B of Theorem(5.3.6) is that it is not very explicit. Indeed for every $n = 1, \dots, N-1$, in order to construct f_{n+1} we must first compute an orthonormal eigenbasis for $F_n F_n^*$, . This problem is readily doable since the eigenvalues $\left\{\lambda_{n;m}\right\}_{m=1}^{M}$ of $F_n F_n^*$ are already known. It is nevertheless a tedious and inelegant process to do by hand, requiring us to, for example, compute QR-factorizations of $\lambda_{n;m} 1 - F_n F_n^*$ for each $m = 1, \dots, M$. This section is devoted to the following result, which is a version of Theorem(5.3.6) equipped with a more explicit Step B; though technical, this new and improved Step B is still simple enough to be performed by hand, a fact which will hopefully permit its future application to both theoretical and numerical problems.

Theorem(5.3.7)[34]: For any nonnegative nonincreasing sequences $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$, every sequence of vectors $F = \{f_n\}_{n=1}^{N}$ in $\mathbb{H}_M$ whose frame operator FF^* has spectrum $\{\lambda_m\}_{m=1}^{M}$ and which satisfies $\|f_n\|^2 = \mu_n$ for all n can be constructed by the following algorithm:

A. Pick eigensteps $\left\{\left\{\lambda_{n;m}\right\}_{m=1}^{M}\right\}_{n=0}^{N}$ as in Definition(5.3.1).

B. Let U_1 be any unitary matrix, $U_1 = \left\{u_{1;m}\right\}_{m=1}^{M}$, and let $f_1 = \sqrt{\mu_1} u_{1,1}$. For each $n = 1, \dots, N-1$:

$B.1$ Let V_n be an $M \times M$ block-diagonal unitary matrix whose blocks correspond to the distinct values of$\left\{\lambda_{n;m}\right\}_{m=1}^{M}$ with the size of each block being the multiplicity of the corresponding eigenvalue.

B.2 Identify those terms which are common to both$\left\{\lambda_{n;m}\right\}_{m=1}^{M}$ and $\left\{\lambda_{n+1;m}\right\}_{m=1}^{M}$. Specifically:

(i) Let $I_n \subseteq \{1, .., M\}$ consist of those indices m such that $\lambda_{n;m} < \lambda_{n;\acute{m}}$ for all $\acute{m} < m$ and such that the multiplicity of $\lambda_{n;m}$as a value in $\left\{\lambda_{\acute{n};\acute{m}}\right\}_{\acute{m}=1}^{M}$ exceeds its multiplicity as a value in $\left\{\lambda_{n+1;\acute{m}}\right\}_{\acute{m}=1}^{M}$

(ii) Let $\boldsymbol{\mathcal{T}_n} \subseteq \{1, \dots, M\}$consist of those indices m such that $\lambda_{n+1;m} < \lambda_{n+1;\acute{m}}$ for all $\acute{m} < m$ and such that the multiplicity of $\lambda_{n;m}$as a value in $\{\lambda_{n+1;\acute{m}}\}_{\acute{m}=1}^{M}$ exceeds its multiplicity as a value in $\{\lambda_{n;\acute{m}}\}_{\acute{m}=1}^{M}$

The sets I_n and $\boldsymbol{\mathcal{T}_n}$ have equal cardinality, which we denote R_n. Next:

(iii) Let π_{I_n} be the unique permutation on $\{1, \dots, M\}$ that is increasing on both I_n and I_n^c and such that

$\pi I_n \in \{1, \dots, I_n\}$ for all $m \in I_n$. Let $\prod_{I_n} be$ the associated permutation matrix $\prod_{I_n} \delta_m = \delta \pi I n m$

(vi) Let $\pi \mathcal{T}_n$ be the unique permutation on $\{1, \dots, M\}$that is increasing on both $\boldsymbol{\mathcal{T}_n}$ and $\boldsymbol{\mathcal{T}}_n^c$ and such that

$\pi_{I_n}(m) \in \{1, \dots, R_n\}$ for all $m \in \mathcal{T}_n$. Let $\prod_{\boldsymbol{\mathcal{T}_n}} be$ the associated permutation matrix $\prod_{\boldsymbol{\mathcal{T}_n}} \delta_m = \delta_{\pi_{I_n}}(m)$

B.3 Let v_n, w_n be the $R_n \times 1$ vectors whose entries are

$$v_n\left(\pi_{I_n}(m)\right) = \left|-\frac{\prod_{\ddot{m} \in \mathcal{T}_n}\left(\lambda_{n;m} - \lambda_{n+1;\acute{m}}\right)}{\prod_{\ddot{m} \in I_n}\left(\lambda_{n;m} - \lambda_{n+1;\acute{m}}\right)}\right|^{1/2}, w_n\left(\pi \mathcal{T}_n(\acute{m})\right) = \left|\frac{\prod_{\ddot{m} \in I_n}\left(\lambda_{n+1;m} - \lambda_{n+1;\acute{m}}\right)}{\prod_{\ddot{m} \in \mathcal{T}_n}\left(\lambda_{n+1;m} - \lambda_{n+1;\acute{m}}\right)}\right|^{1/2}$$

$$\forall m \in I_n m \acute{\in} \boldsymbol{\mathcal{T}_n}.$$

B.4 $f_{n+1} = U_n V_n \prod_{I_n}^T \begin{bmatrix} v_n \\ 0 \end{bmatrix}$, where the $M \times 1$ vector$\begin{bmatrix} v_n \\ 0 \end{bmatrix}$ is v_n padded with $M - R_n$ zeros.

B.5 $U_{n+1} = U_n V_n \prod_{I_n}^T \begin{bmatrix} W_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\boldsymbol{\mathcal{T}_n}}$ where W_n is the $R_n \times R_n$ matrix whose entries are:

$$W_n\left(\pi I_n(m), \pi \mathcal{T}_n(\acute{m})\right) = \frac{1}{\lambda_{n+1;\acute{m}} - \lambda_{n;m}} v_n\left(\pi I_n(m) w_n\left(\pi \mathcal{T}_n(\acute{m})\right)\right)$$

Conversely, any F constructed by this process has $\{\lambda_m\}_{m=1}^M$ as the spectrum of $FF^*$and $\|f_n\|^2 = \mu_n$ for all n.

Moreover, for any F constructed in this manner and any $n = 1, \dots, N$, the spectrum of the frame operator $F_n F_n^*$ arising from the partial sequence $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^n$ is $\{\lambda_{n;m}\}_{m=1}^M$, and the columns of U_n form a corresponding orthonormal eigenbasis for $F_n F_n^*$

Before proving Theorem(5.3.7), we give an example of its implementation, with the hope of conveying the simplicity of the underlying idea, and better explaining the heavy notation used in the statement of the result.

Example(5.3.8)[34]: We now use Theorem (5.3.7) to construct $UNTFs$ consisting of 5 vectors in $\mathbb{R}^3$. Here, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{5}{3}$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \_\mu_5 = 1$. By Step A, our first task is to pick a sequence of eigensteps consistent with definition (5.3.1), that is, pick $\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\}, \{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\}, \{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\}$ and$\{\lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}\}$ that satisfy the interlacing conditions:

$$\{0,0,0\} \sqsubseteq \{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} \sqsubseteq \{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} \sqsubseteq \{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} \sqsubseteq \{\lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}\} \sqsubseteq \left\{\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right\} \quad (47)$$

as well as the trace conditions:

$\lambda_{1;1} + \lambda_{1;2} + \lambda_{1;3}=1,\ \lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3} = 2, \lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3} = 3, \lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}$. (48)

Writing these desired spectra in a table:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	?	?	?	?	$\frac{5}{3}$
$\lambda_{n;2}$	0	?	?	?	?	$\frac{5}{3}$
$\lambda_{n;1}$	0	?	?	?	?	$\frac{5}{3}$

the trace condition (48) means that the sum of the values in the nth column is $\sum_{n'=1}^{n} \mu_{n'} =$ n, while the interlacing condition (47) means that any value $\lambda_{n,m}$ is at least the neighbor to the upper right $\lambda_{n+1,m+1}$ and no more than its neighbor to the right $\lambda_{n+1,m}$. In particular, for $n = 1$, we necessarily have $0 = \lambda_{n,m}$ _1;2 _ _0;1 = 0 and 0 = _0;3 _
_1;3 _ _0;2 = 0 implying that _1;2 = _1;3 = 0. Similarly, for n = 4, interlacing requires that 5
3 = _5;2 _ _4;1 _ _5;1 = 53
and 53
= _5;3 _ _4;2 _ _5;2 = 5
3 implying that _4;1 = _4;2 = 5
3 . That is, we necessarily have:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	?	?	?	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	?	?	?	$\frac{5}{3}$	$\frac{5}{3}$

Applying this same idea again for n = 2 and n = 3 gives 0 = _1;3 _ _2;3 _ _1;2 = 0 and 5
3 = _4;2 _ _3;1 _ _4;1 = 53,
and so we also necessarily have that _2;3 = 0, and _3;1 = 5
3 :

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	0	?	?	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

Moreover, the trace condition (26) at n = 1 gives 1 = _1;1 + _1;2 + _1;3 = _1;1 + 0 + 0 and so _1;1 = 1. Similarly, the
trace condition at n = 4 gives 4 = _4;1 + _4;2 + _4;3 = 5
3 + 53
+ _4;3 and so _4;3 = 2
3 :

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	0	?	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	?	?	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	1	?	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

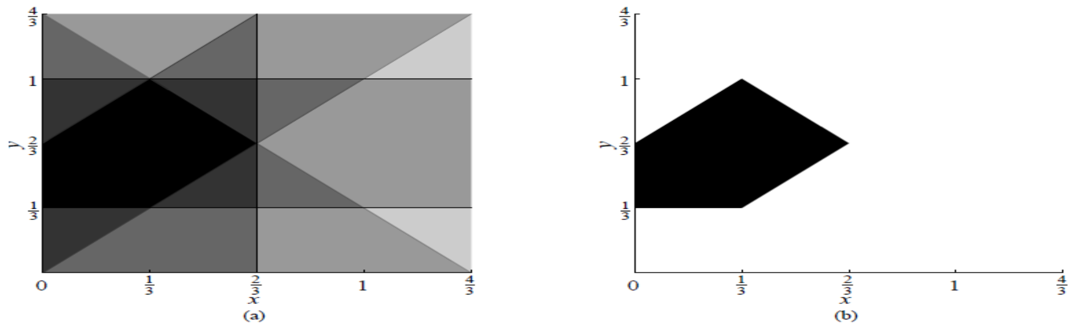


Figure 1: Pairs of parameters (x; y) that generate a valid sequence of eigensteps when substituted into (27). To be precise, in order to satisfy the interlacing requirements of Definition 1, x and y must be chosen so as to satisfy the 11 pairwise inequalities summarized in (28). Each of these inequalities corresponds to a half-plane (a), and the set of (x; y) that satisfy all of them is given by their intersection (b). By Theorem 7, any corresponding sequence of eigensteps (27) generates a 3 _ 5 UNTF and conversely, every 3 _ 5 UNTF is generated in this way. As such, x and y may be viewed as the two essential parameters in the set of all such frames. In particular, for (x; y) that do not lie on the boundary of the set in (b), applying the algorithm of Theorem 7 to (27) and choosing $U_1 = V_1 = V_2 = V_3 = V_4 = \mathrm{I}$ yields the 3 _ 5 UNTF whose elements are given in Table 1.

The remaining entries are not fixed .Inparticular ,welet $\lambda_{3;3}$be come variable x and note that by the trace condition

$3 = \lambda_{3;1} + \lambda_{3;2} + \lambda_{3;3} = x + \lambda_{3;2} + \frac{5}{3}$ and so $\lambda_{3;2} = \frac{4}{3} - x$. Similarly letting $\lambda_{2;2} = y$ gives $\lambda_{2;1} = 2 - y$:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	0	x	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	y	$\frac{4}{3}-x$	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	1	$2-y$	$\frac{5}{3}$	$\frac{2}{3}$	$\frac{5}{3}$

(49)

We take care to note that x and y in (49) are not arbitrary, but instead must be chosen so that the interlacing relations (49) are satisfied. In particular, we have:

$$\{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} \sqsubseteq \{\lambda_{4;1}, \lambda_{4;2}, \lambda_{4;3}\} \Longleftrightarrow x \leq \frac{2}{3} \leq \frac{4}{3} - x \leq \frac{5}{3},$$

$$\{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} \sqsubseteq \{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} \Longleftrightarrow 0 \leq x \leq y \leq \frac{4}{3} - x \leq 2 - y \leq \frac{5}{3},$$

$$\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} \sqsubseteq \{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} \Longleftrightarrow 0 \leq y \leq 1 \leq 2 - y. \qquad (50)$$

By plotting each of the 11 inequalities of (50) as a half-plane (Figure 1(a)), we obtain a 5-sided convex set (Figure 1(b)) of all (x, y) such that (49) is a valid sequence of eigensteps. Specifically, this set is the convex hull of $\left(0, \frac{1}{3}\right)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{2}{3}, \frac{2}{3}\right)$, $\left(\frac{1}{3}, 1\right)$ and $\left(0, \frac{2}{3}\right)$. We note that though this analysis is straightforward in this case, it does not easily generalize to other cases in which M and N are large.

To complete Step A of Theorem (5.3.7), we pick any particular (x, y) from the set depicted in Figure $1(b)$.For example, if we pick $(x, y) = (0, \frac{1}{3})$ then (46) becomes:

n	0	1	2	3	4	5
$\lambda_{n;3}$	0	0	0	0	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;2}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	0	1	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

(51)

We now perform Step B of Theorem 7 for this particular choice of eigensteps. First, we must choose a unitary matrix U_1. Considering the equation for U_{n+1} along with the fact that the columns of $U_N$will form an eigenbasis for F, we see that our choice for U_1merely rotates this eigenbasis, and hence the entire frame F, to our liking. We choose $U_1 = I$ for the sake of simplicity. Thus

$$f_1 = \sqrt{\mu_1} u_{1;1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We now iterate, performing Steps B.1 through B.5 for $n = 1$ to find $f_2$and U_2, then performing Steps B.1 through B.5 for $n = 2$ to find $f_3$and U_3, and so on. Throughout this process, the only remaining choices to be made appear in Step B.1. In particular, for $n = 1$ Step B.1 asks us to pick a block-diagonal unitary matrix V_1whose blocks are sized according to the multiplicities of the eigenvalues $\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} = \{1, 0, 0\}$. That is, consists of a 1×1 unitary block—a unimodular scalar—and a 2×2 unitary block. There are an infinite number of such V_1*'s*, each leading to a distinct frame. For the sake of simplicity, we choose $V_1 = I$. Having completed Step B.1 for $n = 1$, we turn to Step B.2, which requires us to consider the columns of (51) that correspond to $n = 1$ and $n = 2$:

n	1	2
$\lambda_{n;3}$	0	0
$\lambda_{n;2}$	0	$\frac{1}{3}$
$\lambda_{n;1}$	1	$\frac{5}{3}$

(52)

In particular, we compute a set of indices $I_1 \subseteq \{1, 2, 3\}$ that contains the indices m of $\{\lambda_{1;1}, \lambda_{1;2}, \lambda_{1;3}\} = \{1, 0, 0\}$ for which (i) the multiplicity of $\lambda_{1;m}$ as a value of $\{1, 0, 0\}$ exceeds its multiplicity as a value of $\{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} = \{\frac{5}{3}, \frac{1}{3}, 0\}$and (ii) m corresponds to the first occurrence of $\lambda_{1;m}$ as a value of $\{1, 0, 0\}$; by these criteria, we find $I_1 = \{1, 2\}$.

Similarly $\mathcal{J}_1$ if and only if m indicates the first occurrence of a value $\lambda_{2;m}$whose multiplicity as a value of $\{\frac{5}{3}, \frac{1}{3}, 0\}$ exceeds its multiplicity as a value of $\{1, 0, 0\}$, and so$\mathcal{J}_1 = \{1, 2\}$. Equivalently, $I_1$and $\mathcal{J}_1$can be obtained by canceling common terms from (52), working top to bottom; an explicit algorithm for doing so is given in Table 2.

Continuing with Step B.2 for $n = 1$, we now find the unique permutation $\pi I_1: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ that is increasing on both $I_1 = \{1, 2\}$ and its complement $I_1^c = \{3\}$ and takes $I_1$to the first $R_1 = |I_1| = 2$ elements of $\{1, 2, 3\}$. In this particular instance, πI_1happens to be the identity permutation, and so $\Pi_{I_1} = I$. Since $\mathcal{J}_1 = \{1, 2\} = I_1$, we similarly have that $\pi \mathcal{J}_1$ and $\Pi_{\mathcal{J}_1}$are the identity permutation and matrix, respectively.

For the remaining steps, it is useful to isolate the terms in (52) that correspond to $I_1$and $\mathcal{J}_1$:

$$\beta_2 = \lambda_{1;2} = 0, \qquad \gamma_2 = \lambda_{2;2} = \frac{1}{3}$$

$$\beta_1 = \lambda_{1;1} = 1,\ \gamma_1 = \lambda_{2;1} = \tfrac{5}{3}. \quad (53)$$

In particular, in Step B.3, we find the $R_1 \times 1 = 2 \times 1$ vector $v_1$by computing quotients of products of differences of the values in (53):

$$[v_1(1)]^2 = \frac{(\beta_1-\gamma_1)(\beta_1-\gamma_1)}{(\beta_1-\beta_2)} = \frac{\left(1-\frac{5}{3}\right)\left(1-\frac{1}{3}\right)}{(1-0)} = \frac{4}{9} \quad (54)$$

$$[v_1(2)]^2 = \frac{(\beta_2-\gamma_1)(\beta_2-\gamma_2)}{(\beta_2-\beta_1)} = \frac{\left(0-\frac{5}{3}\right)\left(0-\frac{1}{3}\right)}{(0-1)} = \frac{5}{9} \quad (55)$$

yielding $v_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{\sqrt{5}}{2} \end{bmatrix}$. Similary, we compute $w_1 \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ according to the formulas

$$[w_1(1)]^2 = \frac{(\gamma_1-\beta_1)(\gamma_1-\beta_2)}{(\gamma_1-\gamma_2)} = \frac{\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-0\right)}{\left(\frac{5}{3}-\frac{1}{3}\right)} = \frac{5}{6} \quad (56)$$

$$[w_1(2)]^2 = \frac{(\gamma_2-\beta_1)(\gamma_2-\beta_2)}{(\gamma_2-\gamma_1)} = \frac{\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-0\right)}{\left(\frac{1}{3}-\frac{5}{3}\right)} = \frac{1}{6} \quad (57)$$

Next, in Step B.4, we form our second frame element $f_2 = U_1 V_1 \Pi_{I_1}^T \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$

$$f_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{\sqrt{5}}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{\sqrt{5}}{3} \\ 0 \end{bmatrix}$$

As justified in the proof of Theorem(5.3.7), the resulting partial sequence of vectors

$$F_2 = [f_1 \quad f_2] = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} \\ 0 & 0 \end{bmatrix}$$

has a frame operator $F_2 F_2^*$ whose spectrum is $\{\lambda_{2;1}, \lambda_{2;2}, \lambda_{2;3}\} = \{\frac{5}{3}, \frac{1}{3}, 0\}$. Moreover, a corresponding orthonormal eigenbasis for $F_2 F_2^*$is computed in Step B.5; here the first step is to compute the $R_1 \times R_1 = 2 \times 2$ matrix W_1 by computing a pointwise product of a certain 2×2 matrix with the outer product of $v_1$with w_1:

$$w_1 = \begin{bmatrix} \frac{1}{\gamma_1-\beta_1} & \frac{1}{\gamma_2-\beta_1} \\ \frac{1}{\gamma_1-\beta_2} & \frac{1}{\gamma_2-\beta_2} \end{bmatrix} \odot \begin{bmatrix} v_1(1) \\ v_1(2) \end{bmatrix} [w_1(1) \quad w_1(2)] = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{5} & 3 \end{bmatrix} \odot \begin{bmatrix} \frac{2\sqrt{5}}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} \\ \frac{5}{3\sqrt{6}} & \frac{\sqrt{5}}{3\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}$$

Note that $W_1$is a real orthogonal matrix whose diagonal and subdiagonal entries are strictly positive and whose superdiagonal entries are strictly negative; one can easily verify that every W_n has this form. More significantly, the proof of Theorem(5.3.7) guarantees that the columns of

$$U_2 = U_1 V_1 \prod_{I_1}^T \begin{bmatrix} W_1 & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{J}_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & o & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & o & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{6} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & o & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{5}}{6} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

form an orthonormal eigenbasis of $F_2F_2^*$. This completes the $n = 1$ iteration of Step B; we now repeat this process for $n = 2, 3, 4$. For $n = 2$, in Step B.1 we arbitrarily pick some 3×3 diagonal unitary matrix $\square_2$. Note that if we wish our frame to be real, there are only $2^3 = 8$ such choices of V_2. For the sake of simplicity, we choose $V_2 = I$ in this example. Continuing, Step B.2 involves canceling the common terms in

n	1	2
$\lambda_{n;3}$	0	0
$\lambda_{n;2}$	$\frac{1}{3}$	$\frac{4}{3}$
$\lambda_{n;1}$	$\frac{5}{3}$	$\frac{5}{3}$

To find $I_2 = \mathcal{J}_2 = (2)$ and so

$$\prod_{I_1} = \prod_{\mathcal{J}_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In step B.3. we find that $v_2 = w_2 = [1]$. Step B.4 and B.5
Then give that $F_3 = [f_1 \ \ f_2 \ \ f_3]$ and U_3 are

$$F_3 = \begin{vmatrix} 1 & \frac{2}{3} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{\sqrt{6}} \\ 0 & 0 & 0 \end{vmatrix}, \ U_3 = \begin{vmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

The columns of U_3 form an orthonormal eigenbasis for the partial frame operator $F_3F_3^*$ with corresponding eigenvalues $\{\lambda_{3;1}, \lambda_{3;2}, \lambda_{3;3}\} = \{\frac{5}{3}, \frac{4}{3}, 0\}$. For the $n = 3$ iteration, we pick $V_3 = 1$ and cancel the common terms in

n	3	4
$\lambda_{n;3}$	0	$\frac{2}{3}$
$\lambda_{n;2}$	$\frac{4}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	$\frac{5}{3}$	$\frac{5}{3}$

To obtain $I_3 = [2,3]$ and $\mathcal{T}_3 = [1,3]$ implying

In step B.3, we then compute the $R_3 \times 1 = 2 \times 1$ vectors v_3 and w_3 in a manner analogous to
(54), (55),(56) and (57)

$$v_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}, \qquad w_3 = \begin{bmatrix} \frac{\sqrt{5}}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Note that in Step B.4, the role of permutation matrix $\prod_{I_3}^T$ is that it maps the entries of v_3 onto the I_3 indices, meaning that $v_4$lies in the span of the corresponding eigenvectors $\{u_{3;m}\}_{m \in I_3}$

$$f_4 = U_3 V_3 \prod_{I_3}^{T} \begin{bmatrix} v_3 \\ 0 \end{bmatrix} = \begin{vmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} \\ 0 \end{bmatrix}$$

$$= \begin{vmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}$$

In a similar fashion, the purpose of the permutation matrices in Step B.5 is to embed the entries of the 2×2 matrix W_3 into the $I_3 = \{2, 3\}$ rows and $\mathcal{T}_3 = \{1, 3\}$ columns of a 3×3 matrix:

$$U_4 = U_3 V_3 \prod_{I_3}^{T} \begin{bmatrix} w_3 & 0 \\ 0 & 1 \end{bmatrix} \prod \mathcal{T}_3 =$$

$$\begin{vmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} =$$

$$\begin{vmatrix} \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ \frac{\sqrt{5}}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{\sqrt{5}}{\sqrt{6}} \end{vmatrix} = \begin{vmatrix} \frac{-\sqrt{5}}{6} & \frac{\sqrt{5}}{\sqrt{6}} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{\sqrt{6}} & -\frac{\sqrt{5}}{6} \\ \frac{1}{\sqrt{6}} & 0 & \frac{\sqrt{5}}{\sqrt{6}} \end{vmatrix}$$

For the last iteration $n = 4$, we again choose $V_4 = I$ in Step $B.1$. For Step $B.2$, note that since

n	4	5
$\lambda_{n;3}$	$\frac{2}{3}$	$\frac{5}{3}$
$\lambda_{n;2}$	$\frac{5}{3}$	$\frac{5}{3}$
$\lambda_{n;1}$	$\frac{5}{3}$	$\frac{5}{3}$

we have $I_4 = \{3\}$ and $\mathcal{T}_4 = \{1\}$, implying

$$\prod_{I_4} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \prod_{\mathcal{J}_4} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Working through Steps $B.3, B.4$ and $B.5$ yields the $UNTF$:

$$F = F_5 = \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{6} & \frac{1}{6} \\ 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{6} & -\frac{\sqrt{5}}{6} \\ 0 & 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix}, \; U_5 = \begin{bmatrix} \frac{1}{6} & -\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{\sqrt{6}} \\ -\frac{\sqrt{5}}{6} & \frac{5}{6} & \frac{1}{\sqrt{6}} \\ \frac{\sqrt{5}}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \end{bmatrix} \quad (58)$$

We emphasize that the $UNTF\ F$ given in (58) was based on the particular choice of eigensteps given in (51), which arose by choosing $(x, y) = (0, \frac{1}{3})$ in (49). Choosing other pairs (x, y) from the parameter set depicted in Figure 1(b) yields other $UNTFs$. Indeed, since the eigensteps of a given F are equal to those of $U\,F$ for any unitary operator U, we have in fact that each distinct (x, y) yields a $UNTF$ which is not unitarily equivalent to any of the others. For example, by following the algorithm of Theorem(5.3.7) and choosing $U_1 = I\ and\ V_n = 1$ in each iteration, we obtain the following four additional $UNTFs$, each corresponding to a distinct corner point of the parameter set:

$$F = \begin{bmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{\sqrt{5}}{3} & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \\ 0 & 0 & 1 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \text{ for } (x, y) = \left(\frac{1}{3}, \frac{1}{3}\right)$$

$$F = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{8}}{3} & \frac{1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \end{bmatrix} \text{ for } (x, y) = \left(\frac{2}{3}, \frac{2}{3}\right)$$

$$F = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \end{bmatrix} \text{ for } (x, y) = \left(\frac{1}{3}, 1\right)$$

$$F = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{\sqrt{3}} & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{\sqrt{8}}{3} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ 0 & 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \end{bmatrix} \text{ for } (x, y) = \left(0, \frac{2}{3}\right)$$

Notice that, of the four $UNTFs$ above, the second and fourth are actually the same up to a permutation of the frame elements. This is an artifact of our method of construction, namely, that our choices for eigensteps, U_1, and $\{V_n\}_{n=1}^{N-1}$determine the sequence of frame elements. As such, we can recover all permutations of a given frame by modifying these choices.

We emphasize that these four $UNTFs$ along with that of (58) are but five examples from the continuum of all such frames. Indeed, keeping x and y as variables in (49) and applying the algorithm of Theorem (5.3.7)—again choosing

$$f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$f_2 \begin{bmatrix} 1-y \\ \sqrt{y(2-y)} \\ 0 \end{bmatrix}$$

$$f_3 = \begin{vmatrix} \frac{\sqrt{(3_{y-1})(2+3_x-3_y)(2-x-y)}}{\sqrt[6]{1-y}} & -\frac{\sqrt{(5-3y)(4-3x-3y)(y-x)}}{\sqrt[6]{1-y}} \\ \frac{\sqrt{y(3y-1)(2+3x-3y)(2-x-y)}}{\sqrt[6]{(1-y)(2-y)}} & +\frac{\sqrt{(5-3y)(4-3x-3y)(y-x)}}{\sqrt[6]{y(1-y)}} \end{vmatrix}$$

$$f_4 = \begin{bmatrix} -\frac{\sqrt{(4-3x)(3y-1)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)}} - \frac{\sqrt{(4-3x)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)(1-y)}} \\ -\frac{\sqrt{x(3y-1)(y-x)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} + \frac{\sqrt{x(5-3y)(2-x-y)(4-3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} \\ -\frac{\sqrt{(4-3x)y(3x-3y)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)}(2-y)} + \frac{\sqrt{(4-3x)(2-y)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)(1-y)}(2-y)} \\ -\frac{\sqrt{xy(3y-1)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)(2-y)}} - \frac{\sqrt{x(2-y)(5-3y)(2-x-y)}}{4\sqrt{3(2-3x)(1-y)}} \\ \frac{\sqrt{5x(2+3x-3y)(4+3x-3y)}}{6\sqrt{(2-3x)y(2-y)}} + \frac{\sqrt{5(4-3x)(y-x)(2-x-y)}}{2\sqrt{3(2-3x)y(2-y)}} \end{bmatrix}$$

$$f_5 = \begin{bmatrix} -\frac{\sqrt{(4-3x)(3y-1)(2-x-y)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)}} - \frac{\sqrt{(4-3x)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)(1-y)}} \\ -\frac{\sqrt{x(3y-1)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} + \frac{\sqrt{x(5-3y)(2-x-y)(4-3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} \\ -\frac{\sqrt{(4-3x)y(3y-1)(2-x-y)(4-3x-3y)}}{12\sqrt{(2-3x)(1-y)}(2-y)} + \frac{\sqrt{(4-3x)(2-y)(5-3y)(y-x)(2+3x-3y)}}{12\sqrt{(2-3x)(1-y)}} \\ -\frac{\sqrt{xy(3y-1)(y-x)(2+3x-3y)}}{4\sqrt{3(2-3x)(1-y)(2-y)}} - \frac{\sqrt{x(2-y)(5-3y)(2-x-y)(4-3x-3y)}}{4\sqrt{3(2-3x)(1-y)}} \\ \frac{\sqrt{5x(2+3x-3y)(4-3x-3y)}}{6\sqrt{(2-3x)y(2-y)}} + \frac{\sqrt{5(4-3x)(y-x)(2-x-y)}}{2\sqrt{3(2-3x)y(2-y)}} \end{bmatrix}$$

Table 1: A continuum of $UNTFs$. To be precise, for each choice of (x, y) that lies in the interior of the parameter set depicted in Figure 1(b), these five elements form a $UNTF$ for

$\mathbb{R}^3$, meaning that its 3×5 synthesis matrix F has both unit norm columns and orthogonal rows of constant squared norm $\frac{5}{3}$. These frames were produced by applying the algorithm of Theorem 7 to the sequence of eigensteps given in (49), choosing $U_1 = 1$ and $V_n = 1$ for all n. These formulas give an explicit parametrization for a 2-dimensional manifold that lies within the set of all 3×5 $UNTFs$. By Theorem(5.3.7), every such $UNTF$ arises in this manner, with the understanding that (x, y) may indeed be chosen from the boundary of the parameter set and that the initial eigenbasis U_1 and the block-diagonal unitary matrices V_n are not necessarily the identity.

$U_1 = 1$ and $V_n = 1$ in each iteration for the sake of simplicity—yields the frame elements given in Table 1. Here, we restrict (x, y) so as to not lie on the boundary of the parameter set of Figure 1(b). This restriction simplifies the analysis, as it prevents all unnecessary repetitions of values in neighboring columns in (49). Table 1 gives an explicit parametrization for a two-dimensional manifold that lies within the set of all $UNTFs$ consisting of five elements in three-dimensional space. By Theorem(5.3.7), this can be generalized so as to yield all such frames, provided we both (i) further consider (x, y) that lie on each of the five line segments that constitute the boundary of the parameter set and (ii) throughout generalize V_n to an arbitrary block-diagonal unitary matrix, where the sizes of the blocks are chosen in accordance with Step B.1.

Having discussed the utility of Theorem(5.3.7), we turn to its proof.

Proof of Theorem(5.3.7).

($\Leftarrow$) $Let\ \{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$=1 be arbitrary nonnegative nonincreasing sequences and take an arbitrary sequence of eigensteps $\{\{\lambda_{n;m}\}_{m=1}^{M}\}_{n=0}^{N}$ in accordance with Definition(5.3.1). Note here we do not assume that such a sequence of eigensteps actually exists for this particular choice of $\{\lambda_m\}_{m=1}^{M}$ and $\{\mu_n\}_{n=1}^{N}$ if one does not, then this direction of the result is vacuously true.

We claim that any $F = \{f_n\}_{n=1}^{N}$constructed according to Step B has the property that for all $n = 1, \dots, N$, the spectrum of the frame operator $F_n F_n^*$of $F_n = \{f_{\grave{n}}\}_{\grave{n}=1}^{N}$ is $\{\lambda_{n;m}\}_{m=1}^{M}1$, and that the columns of $U_n$form an orthonormal eigenbasis for $F_n F_n^*$. Note that by Lemma(5. 3.3), proving this claim will yield our stated result that the spectrum of FF^* is $\{\{\lambda_m\}_{m=1}^{M}$ and that $\|f_n\|^2 = \mu_n$ for all $n = 1, \dots, N$. Since Step B is an iterative algorithm, we prove this claim by induction on n. To be precise, Step B begins by letting $U_1 = \{u_{1;m}\}_{m=1}^{M}$ and $f_1 = \sqrt{\mu_1} u_{1;1}$.The columns of U_1form an

```
I_n^{(M)} := {1, ..., M}
J_n^{(M)} := {1, ..., M}
For m = M, ..., 1
  If λ_{n;m} ∈ {λ_{n+1;m̀}}_{m̀ ∈ J_n^{(M)}}
    I_n^{(M-1)} := I_n^{(M)} \ {m}
    J_n^{(M-1)} := J_n^{(M)} \ {m̀} Where m' = max{m'' ∈ J_n^{(M)} := λ_{n+1} = λ_{n;m}}
else
I_n^{(M-1)} := I_n^{(m)}
```

09 $\mathcal{T}_n^{(M-1)} \coloneqq \mathcal{T}_n^{(M)}$
10 end if
11 end for
12 $I_n \coloneqq I_n^{(1)}$
13 $\mathcal{T}_n \coloneqq I_n^{(1)}$

Table 2: An explicit algorithm for computing the index sets $I_n$and $\mathcal{T}_n$in Step B.2 of Theorem(5.3.7) orthonormal eigenbasis for $F_1F_1^*$ since $U_1$is unitary by assumption and

$$F_1F_1^*u_{1;m} = \langle u_{1;m,}f_1\rangle\, f_1 = \langle u_{1;m}, \sqrt{\mu_1}u_{1;1}\rangle\sqrt{\mu_1}u_{1;1} = \langle \mu_1 u_{1;m}, u_{1;1}\rangle \begin{cases} \mu_1 u_{1;1} & m = 1 \\ 0 & m \neq 1 \end{cases}$$

for all $m = 1, \dots, M$. As such, the spectrum of $F_1F_1^*$consists of μ_1 and $M-1$ repetitions of 0. To see that this spectrum matches the values of $\{\lambda_{1;m}\}_{m=1}^M$, note that by Definition(5.3.1), we know $\{\lambda_{1;m}\}_{m=1}^M$ {interlaces on the trivial sequence $\{\lambda_{0;m}\}_{m=1}^M = \{0\}_{m=1}^M$ In the sense of (3) implying $\lambda_{1;m} = 0$ for all $m \geq 2$; this in hand, note this definition further gives that $\lambda_{1;1} = \sum_{m=1}^M \lambda_{1;m} = \mu_1$. Thus, our claim indeed holds for $n = 1$.

We now proceed by induction, assuming that for any given $n = 1, \dots, N-1$ the process of Step B has produced $F_n = \{f_{\grave{n}}\}_{\grave{n}=1}^N$ such that the spectrum of $F_nF_n^*$ is $\{\lambda_{n;m}\}_{m=1}^M$and that the columns of U_n form an orthonormal eigenbasis for $F_nF_n^*$. In particular, we have $F_nF_n^*U_n = U_nD_n$where $D_{n+1}$is the diagonal matrix whose diagonal entries are $\{\lambda_{n;m}\}_{m=1}^M$. Defining D_{n+1}analogously from $\{\lambda_{n+1;m}\}_{m=1}^M$, we show that constructing f_{n+1} and U_{n+1} according to Step B implies $F_{n+1}F_{n+1}^*U_{n+1} = U_{n+1}D_{n+1}$where $U_{n+1}$is unitary; doing such proves our claim.

To do so, pick any unitary matrix V_n according to Step B.1. To be precise, let Kn denote the number of distinct values in $\{\lambda_{n;m}\}_{m=1}^M$, and for any $k = 1, \dots, K_n$, let $L_{n;k}$ denote the multiplicity of the kth value. We write the index m as an increasing function of k and l, that is, we write $\{\lambda_{n;m}\}_{m=1}^M$ as $\{\lambda_{n;m(k,l)}\}_{k=1}^{k_n}{}_{l=1}^{l_{m,k}}$ k where $m(k,l) < m(\grave{k},\grave{l})$ if $k < \grave{k}$ or if $k = \grave{k}$ and $l < \grave{l}$. We let V_n be an $M \times M$ block-diagonal unitary matrix consisting of K diagonal blocks, where for any $k = 1, \dots, K$, the kth block is an $L_{n;k} \times L_{n;k}$unitary matrix. In the extreme case where all the values of $\{\lambda_{n;m}\}_{m=1}^M$are distinct, we have that $V_n$is a diagonal unitary matrix, meaning it is a diagonal matrix whose diagonal entries are unimodular. Even in this case, there is some freedom in how to choose V_n; this is the only freedom that the Step B process provides when determining f_{n+1}. In any case, the crucial fact about $V_n$is that its blocks match those corresponding to distinct multiples of the identity that appear along the diagonal of D_n, implying $D_nV_n = V_nD_n$.

Having chosen V_n, we proceed to Step B.2. Here,we produce subsets $I_n$and $\mathcal{T}_n$ of $\{1, \dots, M\}$that are the remnants of the indices of $\{\lambda_{n;m}\}_{m=1}^M$and $\{\lambda_{n+1;m}\}_{m=1}^M$, respectively, obtained by canceling the values that are common to both sequences, working backwards from index M to index 1. An explicit algorithm for doing so is given in Table 2. Note that for each $m = M, \dots, 1$ (Line 03), we either remove a single element from both $I_n^{(m)}$ and $\mathcal{T}_n^{(m)}$ (Lines 04–06) or remove nothing from both (Lines 07–09), meaning that $I_n := I_n^{(1)}$ and $\mathcal{T}_n := \mathcal{T}_n^{(1)}$ have the same cardinality, which we denote R_n. Moreover, since $\{\lambda_{n+1;m}\}_{m=1}^M$ interlaces on $\{\lambda_{n;m}\}_{m=1,}^M$ then for any real scalar λ whose multiplicity as a value of $\{\lambda_{n;m}\}_{m=1}^{M}$ is L, we have that its multiplicity as a value of $\{\lambda_{n+1;m}\}_{m=1}^{M}$ is either $L-, L$ or $L+1$. When these two multiplicities are equal, this algorithm completely removes the corresponding indices from both I_n and J_n. On the other hand, if the new multiplicity is $L-1$ or $L+1$, then the least such index in I_n or $\mathcal{T}_n$ is left behind, respectively, leading to the definitions of I_n or $\mathcal{T}_n$ given in Step B.2. Having these sets, it is trivial to find the corresponding permutations πI_n and $\pi\mathcal{T}_n$ on $\{1,\ldots,M\}$ and to construct the associated projection matrices $\prod_{I_n}$ and $\prod_{\mathcal{T}_n}$.

We now proceed to Step B.3. For the sake of notational simplicity, let $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ denote the values of $\{\lambda_{n;m}\}_{m\in I_n}$, and $\{\lambda_{n+1;m}\}_{m\in\mathcal{T}_n}$ respectively. That is, let $\beta_{\pi I_{n(m)}} = \lambda_{n;m}$ for all $m \in I_n$ and $\gamma_{\pi Jn(m)} = \lambda_{n+1;\mathrm{m}}$ for all $m \in \mathcal{T}_n$.

Note that due to the way in which I_n and J_n and $\mathcal{T}_n$ were defined, we have that the values of $\{\beta_{\pi Jn(m)} = \lambda_{n;\mathrm{m}}$ and $\{\gamma_r\}_{r=1}^{R_n}$ are all distinct, both within each sequence and across the two sequences. Moreover, since $\{\lambda_{n;m}\}_{m\in I_n}$ and $\{\lambda_{n+1;m}\}_{m\in\mathcal{T}_n}$ are nonincreasing while $\pi I_n\ and\ \pi\mathcal{T}_n$ nare increasing on I_n and $\mathcal{T}_n$ respectively, then the values $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ are strictly decreasing. We further claim that $\{\gamma_r\}_{r=1}^{R_n}$ interlaces on $\{\beta_r\}_{r=1}^{R_n}$. To see this, consider the four polynomials:

$$p_n(x) = \prod_{m=1}^{M}(x-\lambda_{n;m}), \quad p_{n+1}(x) = \prod_{m=1}^{M}(x-\lambda_{n+1;m}), b(x) =$$

$$\prod_{r=1}^{R_n}(x-\beta_r), \qquad c(x) = \prod_{r=1}^{R_n}(x-\gamma_r). \tag{59}$$

Since $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ were obtained by canceling the common terms from $\{\lambda_{n;m}\}_{m=1}^{M}$ and $\{\lambda_{n+1;m}\}_{m=1}^{M}$, we have that $p_{n+1}(x)/p_n(x) = c(x)/b(x)$ for all $x \notin \{\lambda_{n;m}\}_{m=1}^{M}$. Writing any $r = 1,\ldots,R_n$ as $r = \pi I_n(m)$ for some $m \in I_n$, we have that since $\{\lambda_{n;m}\}_{m=1}^{M} \sqsubseteq \{\lambda_{n+1;m}\}_{m=1}^{M}$, applying the "only if" direction of Lemma(5.3.5) with "$p(x)$" and "$q(x)$" being $p_n(x)$ and $p_{n+1}(x)$ gives

$$\lim_{x\to\beta_r}(x-\beta_r)\frac{c(x)}{b(x)} = \lim_{x\to\lambda_{n;m}}(x-\lambda_{n;m})\frac{p_{n+1}(x)}{p_n(x)} \leq 0, (60)$$

Since (60) holds for all $r = 1,\ldots,R_n$, applying "if" direction of Lemma(5.3.5) with "$p(x)$" and "$q(x)$" being $b(x)$ and $c(x)$ gives that $\{\gamma_r\}_{r=1}^{R_n}$ indeed interlaces on $\{\beta_r\}_{r=1}^{R_n}$.

Taken together, the facts that $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ are distinct,strictly decreasing and interlacing sequences implies that the $R_n \times 1$ vectors v_n and w_n are well-defined. To be precise, Step B.3 may be rewritten as finding $v_n(r), w_n(\grave{r}) \geq 0$ for all $r, \grave{r} = 1\ldots, R_n$ such that

$$[v_n(r)]^2 = -\frac{\prod_{\grave{r}=1}^{R_n}(\beta_r-\gamma_{\acute{r}})}{\prod_{\substack{\acute{r}=1\\ \acute{r}\neq r}}(\beta_r-\beta_{\acute{r}})}, \quad [w_n(\grave{r})]^2 = \frac{\prod_{\grave{r}=1}^{R_n}(\gamma_{\grave{r}}-\beta_{\acute{r}})}{\prod_{\substack{\acute{r}=1\\ \acute{r}\neq r}}(\gamma_{\grave{r}}-\gamma_{\acute{r}})}, \tag{61}$$

Note the fact that the β_r's and γ_r's are distinct implies that the denominators in (61) are nonzero, and moreover that the quotients themselves are nonzero. In fact, since $\{\beta_r\}_{r=1}^{R_n}$ is

strictly decreasing, then for any fixed r, the values $\{\beta_r - \beta_{\acute{r}}\}\acute{r} \neq r$ can be decomposed into $r-1$ negative values $\{\beta_r - \beta_{\acute{r}}\}_{\acute{r}=1}^{r-1}$ and $R_n - r$ positive values $\{\beta_r - \beta_{\acute{r}}\}_{\acute{r}=r+1}^{R_n}$.
Moreover, since $\{\beta_r\}_{r=1}^{R_n} \sqsubseteq \{\gamma_r\}_{r=1}^{R_n}$, then for any such r, the values $\{\beta_r - \gamma_{\acute{r}}\}_{\acute{r}=1}^{R_n}$ can be broken into r negative values$\{\beta_r - \gamma_{\acute{r}}\}_{\acute{r}=1}^{R_n}$ and $R_n = r$ in (39) is indeed positive values {β$_r$− γ$_r$00}R$_r$00^n=r+1. With the inclusion of an additional negative sign, we see that the quantity defining$[v_n(r)]^2$ in (61) is indeed positive. Meanwhile, the quantity defining $[w_n(\acute{r})]^2$ has exactly $\acute{r}-1$, negative values in both the numerator and denominator, namely $\{\gamma_{\acute{r}} - \beta_{\grave{r}}\}_{\grave{r}=1}^{\acute{r}-1}$ and $\{\gamma_{\acute{r}} - \gamma_{\grave{r}}\}_{\grave{r}=1}^{\acute{r}-1}$respectively .
Having shown that the $v_n$and $w_n$of Step B.3 are well-defined, we now take f_{n+1} and U_{n+1}as defined in Steps B.4 and B.5. Recall that what remains to be shown in this direction of the proof is that $U_{n+1}$is a unitary matrix and that $F_{n+1} = f_{\acute{n}}\}_{\acute{n}=1}^{n+1}$ satisfies $F_{n+1}F_{n+1}^{*}U_{n+1} = U_{n+1}D_{n+1}$. To do so, consider the definition of $U_{n+1}$and recall that U_n is unitary by the inductive hypothesis, V_n is unitary by construction, and that the permutation matrices $\prod_{I_n}$ *and* $\prod_{\mathcal{J}_n}$ *are* orthogonal, that is, unitary and real. As such, to show that U_{n+1} is unitary, it suffices to show that the $R_n \times R_n$ real matrix W_n is orthogonal. To do this, recall that eigenvectors corresponding to distinct eigenvalues of selfadjoint operators are necessarily orthogonal. As such, to show that W_n is orthogonal, it suffices to show that the columns of $W_n$are eigenvectors of a real symmetric operator. To this end, we claim

$$\left(D_{n;I_n} + v_n v_n^T\right)W_n = W_n D_{n+1;\mathcal{J}_n,} W_n^T W_n(r,r) = 1, \forall r = 1,\dots,R_n, \qquad (62)$$

where $D_{n;I_n}$ and $D_{n+1;I_n}$are the $R_n \times R_n$ diagonal matrices whose rth diagonal entries are given by $\beta_r = \lambda_{n;\pi_{I_n}^{-1}(r)}$ and$\gamma_r = \lambda_{n+1;\pi_{\mathcal{J}_n}^{-1}(r)}$ respectively. To prove (62), note that for any ,$\acute{r} = 1,\dots,R_n$,

$$\left[\left(D_{n;I_n} + v_n v_n^T\right)W_n\right](r,\acute{r}) = \left(D_{n;\mathrm{I}_n}W_n\right)(r,\acute{r}) + (v_n v_n^T W_n)(r,\acute{r})\beta_r W_n(r,\grave{r}) + v_n(r)\sum_{\acute{\acute{r}}}^{R_n} v_n\left(\acute{\acute{r}}\right)W_n\left(\acute{\acute{r}},\acute{r}\right) \qquad (63)$$

Rewriting the definition of $W_n$from Step B.5 in terms of $\{\beta_r\}_{r=1}^{R_n}$ and $\{\gamma_r\}_{r=1}^{R_n}$ gives

$$W_n(r,\acute{r}) = \frac{v_n(r)W_n(\acute{r})}{\gamma_{\acute{r}} - \beta_r}. \qquad (64)$$

Substituting (64) into (63) gives

$$\left[\left(D_{n;I_n} + v_n v_n^T\right)W_n\right](\acute{r},r) = \beta_r \frac{v_n(r)w_n(\acute{r})}{\gamma_{\acute{r}} - \beta_r} + v_n(r)\sum_{\acute{\acute{r}}=1}^{R_n} v_n\left(\acute{\acute{r}}\right)\frac{v_n\left(\acute{\acute{r}}\right)w_n}{\gamma_{\acute{r}} - \beta_{\acute{\acute{r}}}}$$

$$= v_n(r)w_n(\acute{r})\left(\frac{\beta_r}{\gamma_{\acute{r}} - \beta_r} + \sum_{\acute{\acute{r}}=1}^{R_n} \frac{\left[v_n\left(\acute{\acute{r}}\right)\right]^2}{\gamma_{\acute{r}} - \beta_{\acute{\acute{r}}}}\right). \qquad (65)$$

Simplifying (65) requires a polynomial identity. Note that the difference $\prod_{\acute{r}=1}^{R_n}(x - \gamma_{\acute{r}}) - \prod_{\acute{r}=1}^{R_n}(x - \beta_{\acute{r}})$ of two monic polynomials is itself a polynomial of degree at most $R_n - 1$, and as such it can be written as the Lagrange interpolating polynomial determined by the $R_n$distinct points $\{\beta_r\}_{r=1}^{R_n}$:

$$\prod_{\acute{\acute{r}}=1}^{R_n}(x-\gamma_{\acute{\acute{r}}})-\prod_{\acute{\acute{r}}=1}^{R_n}(x-\beta_{\acute{\acute{r}}})=\sum_{\acute{\acute{r}}=1}^{R_n}\left(\prod_{r=1}^{R_n}(\beta_{\acute{\acute{r}}-\gamma_r})-0\right)\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}\frac{(x-\beta_r)}{(\beta_{\acute{\acute{r}}-}\beta_r)}$$

$$=\sum_{\acute{\acute{r}}=1}^{R_n}\frac{\prod_{r=1}^{R_n}(\beta_{\acute{\acute{r}}}-\gamma_r)}{\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}}\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}(x-\beta_r), \quad (66)$$

Recalling the expression for $[v_n(r)]^2$given in (61), (66) can be rewritten as

$$\prod_{\acute{\acute{r}}=1}^{R_n}(x-\beta_{\acute{\acute{r}}})-\prod_{\acute{\acute{r}}=1}^{R_n}(x-\gamma_{\acute{\acute{r}}})=\prod_{r=1}^{R_n}\left[v_n(\acute{\acute{r}})\right]^2\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}(x-\beta_r), \quad (67)$$

Dividing both sides of (67) by$\prod_{\acute{\acute{r}}=1}^{R_n}(x-\beta_{\acute{\acute{r}}})$ gives

$$1-\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}\frac{(x-\gamma_{\acute{\acute{r}}})}{(x-\beta_{\acute{\acute{r}}})}=\sum_{\acute{\acute{r}}=1}^{R_n}\frac{\left[v_n(\acute{\acute{r}})\right]^2}{(x-\beta_{\acute{\acute{r}}})}, \forall x\notin\{\beta_r\}_{r=1}^{R_n} (68)$$

For any $\acute{r}=1,\ldots,R_n$, letting $x=\gamma_{\acute{r}}$ in (68) makes the left-hand product vanish, yielding the identity:

$$1=\sum_{\acute{\acute{r}}=1}^{R_n}\frac{\left[v_n(\acute{\acute{r}})\right]^2}{(\gamma_{\acute{\acute{r}}}-\beta_{\acute{\acute{r}}})}, \forall\acute{r}=1,\ldots,R_n \quad (69)$$

Substituting (69) into (65) and then recalling (64) gives

$$\left[\left(D_{n;I_n}+v_nv_n^T\right)W_n\right](\acute{r},r)=v_n(r)w_n(\acute{r})\left(\frac{\beta_r}{\gamma_{\acute{r}}-\beta_r}+1\right)=\gamma_{\acute{r}}\frac{v_n(r)w_n(\acute{r})}{\gamma_{\acute{r}}-\beta_r}=\gamma_{\acute{r}}w_n(r,\acute{r})=\left(w_nD_{n+1;\mathcal{T}_{n},}\right)(r,\acute{r}) \quad (70)$$

As (70) holds for all $r,\acute{r}=1,\ldots,R_n$we have the first half of our claim (62). In particular, we know that the columns of W_n are eigenvectors of the real symmetric operator $D_{n;I_n}+v_nv_n^T$ which correspond to the distinct eigenvalues $\{\gamma_r\}_{r=1}^{R_n}$.

As such, the columns of W_n are orthogonal. To show that W_n is an orthogonal matrix, we must further show that the columns of W_n have unit norm, namely the second half of (62). To prove this, at any $x\notin\{\beta_r\}_{r=1}^{R_n}$ we differentiate both sides of (68) with respect to x to obtain

$$\sum_{\acute{\acute{r}}=1}^{R_n}\left[\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}\frac{(x-\gamma_r)}{(x-\beta_r)}\right]\frac{(\gamma_{\acute{\acute{r}}}-\beta_{\acute{\acute{r}}})}{(x-\beta_{\acute{\acute{r}}})}=\sum_{\acute{\acute{r}}=1}^{R_n}\frac{\left[v_n(\acute{\acute{r}})\right]^2}{(x-\beta_{\acute{\acute{r}}})} \quad \forall x\notin\{\beta_r\}_{r=1}^{R_n} \quad (71)$$

For any $\acute{r}=1,\ldots,R_n$, letting $x=\gamma_{\acute{r}}$ in (71) makes the left-hand summands where $\acute{\acute{r}}\neq\acute{r}$ vanish; by (71), the remaining summand where $\acute{\acute{r}}\neq\acute{r}$can be written as:

$$\frac{1}{w_{n\ (\acute{r})}}=\frac{\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}(\gamma_{\acute{r}}-\gamma_r)}{\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}(\gamma_{\acute{r}}-\beta_r)}=\left[\prod_{\substack{r=1\\r\neq\acute{\acute{r}}}}^{R_n}\frac{(\gamma_{\acute{r}}-\gamma_r)}{(\gamma_{\acute{r}}-\beta_r)}\right]\frac{(\gamma_{\acute{r}}-\beta_r)}{(\gamma_{\acute{r}}-\beta_r)}=\sum_{\acute{\acute{r}}=1}^{R_n}\frac{\left[v_n(\acute{\acute{r}})\right]^2}{(\gamma_{\acute{r}}-\beta_{\acute{\acute{r}}})^2}. \quad (72)$$

We now use this identity to show that the columns of $w_n$have unit norm; for any $\acute{r}=1,\ldots,R_n$, (64) and (72) give

$$(W_n^T W_n)(\acute{r},\acute{r}) = \sum_{\acute{\acute{r}}=1}^{R_n} \left[W_n(\acute{\acute{r}},\acute{r})\right]^2 = \sum_{\acute{\acute{r}}=1}^{R_n} \left(\frac{v_n(\acute{\acute{r}}) w_n(\acute{r})}{\gamma_{\acute{r}} - \beta_{\acute{\acute{r}}}}\right) = [w_n(\acute{r})]^2 \frac{1}{[w_n(\acute{r})]^2} = 1.$$

Having shown that w_n is orthogonal, we have that U_{n+1} is unitary.

For this direction of the proof, all that remains to be shown is that $F_{n+1} + F_{n+1}^* U_{n+1} = U_{n+1} D_{n+1}$. To do this, we write

$F_{n+1} + F_{n+1}^* = F_n + F_n^* + f_{n+1} f_{n+1}^*$ and recall the definition of U_{n+1}:

$$F_{n+1} + F_{n+1}^* U_{n+1} = (F_n + F_n^* + f_{n+1} f_{n+1}^*) U_n V_n \prod_{I_n}^T \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} =$$

$$F_n + F_n^* U_n V_n \textstyle\prod_{I_n}^T \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} = + f_{n+1} f_{n+1}^* U_n V_n \prod_{I_n}^T \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n}. \quad (73)$$

To simplify the first term in (73), recall that the inductive hypothesis gives $F_n + F_n^* U_n = U_n D_n$ and that V_n was constructed to satisfy $D_n V_n = V_n D_n$, implying

$$F_n + F_n^* U_n V_n \textstyle\prod_{I_n}^T \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} = U_n V_n D_n \prod_{I_n}^T \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} = U_n V_n \prod_{I_n}^T (\prod_{I_n}^T D_n \prod_{I_n}^T) \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n}. \quad (74)$$

To continue simplifying (74), note that $\prod_{I_n}^T D_n \prod_{I_n}^T$ *is* itself a diagonal matrix: for any $m, \acute{m} = 1,\ldots,M$, the definition of the permutation matrix $\prod_{I_n}^T$ given in Step B.2 gives

$$\prod_{I_n}^T D_n \prod_{I_n}^T (m,\acute{m}) = \langle D_n \prod_{I_n}^T \delta_{\acute{m}} \prod_{I_n}^T \delta_m \rangle = D_n \delta_{\pi_{I_n(m)}^{-1}} = \begin{cases} \lambda_{n;\pi_{I_n(m)}^{-1}} & m = \acute{m} \\ 0 & m \neq \acute{m}. \end{cases}$$

That is, $\prod_{I_n}^T D_n \prod_{I_n}^T$ *is* the diagonal matrix whose first R_n diagonal entries $\{\beta_r\}_{r=1}^{R_n} = \{\lambda_{n;\pi_{I_n(m)}^{-1}}(r)\}_{r=1}^{R_n}$ match those of the aforementioned $R_n \times R_n$ diagonal matrix $\prod_{I_n}^T$, and whose remaining $M - R_n$ diagonal entries $\{\lambda_{n;\pi_{I_n(m)}^{-1}}\}_{m=R_n+1}^{M}$ the diagonal of an $(M - R_n) \times (M - R_n)$ diagonal matrix $\prod_{I_n}^T$,

$$\prod_{I_n} D_n \prod_{I_n}^T = \begin{bmatrix} D_{n;I_n} & 0 \\ 0 & D_{n;I_n} \end{bmatrix} \quad (75)$$

Substituting(75)into(74)

$$F_n + F_n^* U_n V_n \textstyle\prod_{I_n}^T \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} = U_n V_n \prod_{I_n}^T \begin{bmatrix} D_{n;I_n} & 0 \\ 0 & D_{n;I_n} \end{bmatrix} \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} =$$

$$U_n V_n \textstyle\prod_{I_n}^T \begin{bmatrix} D_{n;I_n} w_n & 0 \\ 0 & D_{n;I_n} \end{bmatrix} \prod_{\mathcal{T}_n}, \quad (76)$$

Meanwhile, to simplify the second term in (73), we recall the definition of f_{n+1} from Step B.4:

$$f_{n+1} f_{n+1}^* U_n V_n \textstyle\prod_{I_n}^T \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} = U_n V_n \prod_{I_n}^T \begin{bmatrix} v_n \\ 0 \end{bmatrix} \begin{bmatrix} v_n^T & 0 \end{bmatrix} \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} = U_n V_n \prod_{I_n}^T \begin{bmatrix} D_{n;I_n} & 0 \\ 0 & D_{n;I_n} \end{bmatrix} \prod_{\mathcal{T}_n}, \quad (77)$$

Substituting (76) and (77) into (73), simplifying the result, and recalling (62) gives

$$F_{n+1} + F_{n+1}^* U_{n+1} \left[\begin{matrix} D_{n;I_n} & 0 \\ 0 & D_{n;I_n} \end{matrix} U_n V_n \prod_{I_n}^T \begin{bmatrix} D_{n;I_n} w_n & 0 \\ 0 & D_{n;I_n} \end{bmatrix} \prod_{\mathcal{T}_n}, \right].$$

By introducing an extra permutation matrix and its inverse and recalling the definition of

U_{n+1}, this simplifies to

$$F_{n+1} + F_{n+1}^* U_{n+1} [v_n^T \quad 0] \begin{bmatrix} w_n & 0 \\ 0 & 1 \end{bmatrix} \prod_{\mathcal{T}_n} = U_n V_n \quad \prod_{I_n}^T \begin{bmatrix} D_{n;I_n} & 0 \\ 0 & D_{n;I_n} \end{bmatrix} \prod_{\mathcal{T}_n}, \quad \Big] \quad (78)$$

We now partition the $\{\lambda_{n+1;m}\}_{m=1}^M$ of D_{n+1} into $\mathcal{T}_n$ and $\mathcal{T}_n^c$ and mimic the derivation (75), writing D_{n+1} in terms of $D_{n+1};\mathcal{T}_n$ and $D_{n+1};\mathcal{T}_n^c$. Note here that by the manner in which I_n and $\mathcal{T}_n$ were constructed, the values of $\{\lambda_{n;m}\}_{m\in I_n^c}$ are equal to those of $\{\lambda_{n+1;m}\}_{\mathcal{T}_n^c}$, as the two sets represent exactly those values which are common to both $\{\lambda_{n;m}\}_{m=1\text{a}}^M and \{\lambda_{n+1;m}\}_{m=1}^M$. As these two sequences are also both in nonincreasing order, we have $D_{n;I_n^c} = D_{n+1;}\mathcal{T}_n^c$ and so

$$\prod_{\mathcal{T}_n} D_{n+1} \prod_{\mathcal{T}_n}^T = \begin{bmatrix} D_{n+1}\mathcal{T}_n & 0 \\ 0 & D_{n+1}\mathcal{T}_n^c \end{bmatrix} = \begin{bmatrix} D_{n+1}\mathcal{T}_n & 0 \\ 0 & D_{n+1}\mathcal{T}_n^c \end{bmatrix}. \quad (79)$$

Substituting (79) into (78) yields $F_{n+1} + F_{n+1}^* U_{n+1} = U_{n+1} D_{n+1}$, completing this direction of the proof.

(⇒) Let $\{\lambda_m\}_{m=1}^M and \{\mu_n\}_{n=1}^N$ be any nonnegative nonincreasing sequences, and let $F = \{f_n\}_{n=1}^N$ be any sequence of vectors whose frame operator $FF^* has \{\lambda_m\}_{m=1}^M$ as its spectrum and has $\|f_n\|^2 = \mu_n$ for all $n = 1, \dots, N$. We will show that this F can be constructed by following Step A and Step B of this result. To see this, for any $n = 1, \dots, N$, let $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^n$ and let $\{\lambda_{n;m}\}_{m=1}^M$ be the spectrum of the corresponding frame operator $F_n F_n^*$. Letting $\lambda_{0;m} := 0$ for all m, the proof of Theorem(5.3.2) demonstrated that the sequence of spectra $\{\{\lambda_{n;m}\}_{m=1}^M\}_{n=0}^N$ necessarily forms a sequence of eigensteps as specified by Definition (5.3.1). This particular set of eigensteps is the one we choose in Step A.

To be precise, let U_1 be any one of the infinite number of unitary matrices whose first column $u_{1:1}$ satisfies $f_1 = \sqrt{u_1} u_{1:1}$

We now proceed by induction, assuming that for any given $n = 1, \dots, N-1$, we have followed. Step B and have made appropriate choices for $\{V_{\acute{n}}\}_{\acute{n}=1}^{n-1}$ so as to correctly produce $F_n = \{f_{\acute{n}}\}_{\acute{n}=1}^n$; we show how the appropriate choice of V_n will correctly produce f_{n+1}. To do so, we again write the nth spectrum $\{\lambda_{n;m}\}_{m=1}^M$ in terms of its multiplicities as $\{\lambda_{n;m(k,l)}\}_{k=1}^{k_n} {}_{l=1}^{L_{n,k}}$. For any $k = 1, \dots, K_n$, Step B of Theorem (5.3.2) gives that the norm of the projection of f_{n+1} onto the kth eigenspace of $F_n F_n^*$ is necessarily given by

$$\left\| P_{n;\lambda_{n;m(k,1)}} f_{n+1} \right\|^2 = -\lim_{x\to\lambda_{n;m(k,1)}} \left(x - \lambda_{n;m(k,1)}\right) \frac{P_{n+1}(x)}{P_n(x)}, \quad (80)$$

where $P_n(x)$ and $P_{n+1}(x)$ are defined by(69).Note that by picking $l = 1, \lambda_{n;m(k,l)}$ represents the first appearance of that particular value in $\{\{\lambda_{n;m}\}_{m=1}^M$. As such, these indices are the only ones that are eligible to be members of the set I_n found in Step B.2. That is, $I_n \subseteq \{m(k,1) : k = 1, \dots, K_n\}$. However, these two sets of indices are not necessarily equal, since I_m only contains m's of the form $m(k,1)$ that satisfy the additional property that the multiplicity of $\lambda_{n;m}$ as a value in $\{\lambda_{n;\acute{m}}\}_{\acute{m}=1}^M$ exceeds its multiplicity as a value in $\{\lambda_{n+1;m}\}_{m=1}^M$. To be precise, for any given $k = 1, \dots, K_n, if \ m(k,1) \in I_n^c$ then $\lambda_{n;m(k,1)}$ appears as a root of $p_{n+1}(x)$ at least as many times as it appears as a root of $p_n(x)$, meaning in this case that the limit in (80) is necessarily zero. If, on the other hand, $m(k,1) \in I_n$, then writing $\pi_{I_n}(m(k,1))$ as some

$r \in \{1,\ldots,R_n\}$ and recalling the definitions of $b(x)$ and $c(x)$ in (69) and $v(r)$ in (71), we can rewrite (80) as

$$\left\|P_{n;\beta_r} f_{n+1}\right\|^2 = \lim_{x\to\beta_r}(x-\beta_r)\frac{P_{n+1}(x)}{P_n(x)} = -\lim_{x\to\beta_r}(x-\beta_r)\frac{c(x)}{b(x)} = -\frac{\prod_{\acute{r}=1}^{R_n}(\beta_r-\gamma_{\acute{r}})}{\prod_{\substack{\acute{r}=1\\ \acute{r}\neq r}}^{R}(\beta_r-\beta_{\acute{r}})} = [v_n(r)]^2 \qquad (81)$$

As such, we can write $f_{n+1}$where each

$$f_{n+1} = \sum_{k=1}^{K_n} P_{n;\lambda_{n;m(k,1)}} f_{n+1} = \sum_{r=1}^{R_n} P_{n;\beta_r} f_{n+1} = \sum_{r=1}^{R_n} v_n(r)\frac{1}{v_n(r)} P_{n;\beta_r} f_{n+1} = \sum_{m\in I_n} v_n\big(\pi I_n(m)\big)\frac{1}{v_n\big(\pi I_n(m)\big)} P_{n;\beta_{\pi I_n(m)}} f_{n+1} \qquad (82)$$

$\frac{1}{v_n(\pi I_n(m))} P_{n;\beta_{\pi I_n(m)}} f_{n+1}$has unit norm by (81). We now pick a new orthonormal eigenbasis $\widehat{U}n := \{\hat{u}_{n;m}\}_{m=1}^{M}$ for $F_n F_n^*$ that has the property that for any $k = 1,\ldots,K_n$, both $\{u_{n;m(k,l)}\}_{l=1}^{L_{n:k}} and \{\hat{u}_{n;m(k,l)}\}_{l=1}^{L_{n:k}}$ span the same eigenspace and, for every $m(k,1) \in I_n$, has the additional property that $\acute{u}_{n;m(k,1)} = \frac{1}{v_n\left(\pi I_k(m(k,1))\right)} P_{n;\beta_{\pi I_n(m(k,1))}} f_{n+1}$.As such becomes

$$f_{n+1} \sum_{m\in I_n} v_n\big(\pi I_n(m)\big)\acute{u}_{n;m} = \widehat{U}_n \sum_{m\in I_n} v_n\big(\pi I_n(m)\big)\delta_m = \widehat{U}_n \sum_{m\in I_n} v_n(r)\delta_{\pi^{-1}_{I_n(r)}} =$$

UnInTr=1Rnvnrδr=UnInTvn0. (83)

Letting Vnbe the unitary matrix $V_n = U_n^*\widehat{U}_n$, the eigenspace spanning condition gives that V_n is block-diagonal whose kth diagonal block is of size $L_{n;k} \times L_{n;k}$. Moreover, with this choice of V_n,(61) becomes

$$f_{n+1} = U_n U_n^* \widehat{U}_n \prod_{I_n}^{T} \begin{bmatrix} v_n \\ 0 \end{bmatrix} = U_n V_n \prod_{I_n}^{T} \begin{bmatrix} v_n \\ 0 \end{bmatrix}$$

meaning that f_{n+1} can indeed be constructed by following Step B.

Chapter 6
Shift-Type and Quasianalytic with Compression of Contractions

The norm estimates in the Factorization Theorem of this paper are sharpened to their best possible form by essential improvements in the proof. As a consequence we obtain that if the residual set of a contraction covers the whole unit circle then those invariant subspaces, where the restriction is similar to the unilateral shift with a similarity constant arbitrarily close to 1, span the whole space. Furthermore, the hyperinvariant subspace problem for asymptotically non-vanishing contractions is reduced to these special circumstances.

In this setting the commutant $\{T\}'$ of T is identified with a quasianalytic subalgebra $\mathcal{F}(T)$ of $L^{\infty}(\mathbb{T})$ containing H^{∞}. Conditions are given for the cases when $\mathcal{F}(T)$ is a Douglas algebra, a pre-Douglas algebra, or a generalized Douglas algebra.

In this chapter it is shown that for every operator $T \in \mathcal{L}_0(\mathcal{H})$ there exists an operator $T_1 \in \mathcal{L}_1(\mathcal{H})$ commuting with T . Thus, the hyperinvariant subspace problems for the two classes are equivalent. The operator T_1 is found as an H^{∞}. Function of T .The existence of an appropriate function, compressing $\pi(T)$ to the whole circle, is proved using potential theoretic tools by constructing a suitable regular compact set on $\mathbb{T}$ with absolutely continuous equilibrium measure.

Section(6.1): Invariant Subspaces of Contractions

One of the greatest achievements of the Sz.-Nagy–Foias theory of Hilbert space contractions is the functional model constructed in the completely non-unitary case. We use this model operator to prove a factorization theorem for asymptotically non-vanishing, absolutely continuous contractions. Namely, it is shown that if the spectral-multiplicity function of the unitary asymptote of the contraction T is at least $n(\in \mathbb{N} \cup \{\aleph 0\})$ on the Borel set $\gamma \subset T$, then the natural embedding: $J\, f \mapsto \chi_\gamma\, f$ of the Hardy space $H^2(\mathfrak{G}_n)$over the n-dimensional Hilbert space $\mathfrak{G}_n$ into the function space $\chi_\gamma\, L^2(\mathfrak{G}_n)$ can be factored into the product $J = ZY$, where Y intertwines the unilateral shift S_n on $H^2(\mathfrak{G}_n)$ with T , and Z intertwines T with the unitary operator M_n, γ of multiplication by the independent variable on $\chi_\gamma\, L^2(\mathfrak{G}_n)$. Furthermore, the norms of the linear transformations Y and Z can be arbitrarily close to 1. This statement is sharpening of the main result in [14], where the norm conditions on Y and Z were weaker. This sharpening requires essential improvements in the proof given in [14].

In this Section we give a brief summary of the unitary asymptotes of contractions, with their representation in the functional model. The Factorization Theorem is formulated in this Section. The first step in its proof is the construction of a vector-sequence in the space $\mathfrak{K}$ of the minimal unitary dilation, which is pointwise orthonormal, and which is transformed by a canonical intertwinerto a sequence which is also pointwise orthonormal. This is carried out in this Section relying on the connection of the defect fields. The results of Section make possible to approximate the previous vectors in $\mathfrak{K}$by vectors in the space $\mathfrak{K}_+$ of the minimal isometric dilation. The proof of the Factorization Theorem is completed in this Section .

It turns out in the Section that the ranges of the possible intertwiners Y span the whole space $\mathfrak{H}$ of the contraction T . In the particular case $n = \aleph 0$ even the ranges of two intertwiners Y and $\widehat{Y}$ span $\mathfrak{H}$. As a consequence we obtain that if the unitary asymptote of

T is of infinite spectralmultiplicity on the whole circle $\mathbb{T}$, then we can find two invariant subspaces of T which span the whole space $\mathfrak{H}$,and where the restrictions of T are similar to the infinite-dimensional unilateral shift; furthermore, the similarity constants can be chosen arbitrarily close to 1. Thus in this case we have a lot of information on the structure of , in particular, T has plenty of invariant subspaces. It can be surprising that the hyperinvariant subspace problem for asymptotically non-vanishing contractions can be reduced to this particular situation, as shown in this Section .

The Banach space of the bounded linear transformations from the Hilbert space $\mathfrak{A}$ to the Hilbert space $\mathfrak{B}$ will be denoted by$\mathcal{L}(\mathfrak{A},\mathfrak{B})$. The C^*-algebra of bounded linear operators acting on $\mathfrak{A}$ is denoted by $\mathcal{L}(\mathfrak{A}) = \mathcal{L}(\mathfrak{A},\mathfrak{A})$.

Let T be a contraction acting on the Hilbert space $\mathfrak{H}$ (that is $T \in \mathcal{L}(\mathfrak{H})$ and $\|T\| \leq 1$). For every $x \in \mathfrak{H}$, the decreasing sequence $\{T^n x\}_n$ is convergent. Hence, by the polar identity the sequence $\{\langle T^n x, T^n y\rangle\}_n$ is also convergent $(x, y \in \mathfrak{H})$. The functional $w_T(x,y) := \lim_n \langle T^n x, T^n y\rangle$ is linear in x, conjugate linear in y, and bounded by 1. Thus, there exists a unique operator A_T on $\mathfrak{H}$ such that $A_T\, x, y = w_T(x,y)$ holds for all $x, y \in \mathfrak{H}$. Since $w_T(x,x) \geq 0\ (x \in \mathfrak{H})$, it follows that $0 \leq A_T \leq I$. Furthermore, the relation $w_T(T\,x, T\,y) = w_T(x,y)$ yields $T^* A_T\, T = A_T$, whence $\|A_T^{1/2}\, T\,x\| = \|A_T^{1/2} x\| (x \in \mathfrak{H})$ follows. Introducing the transformation $X_T^+ : \mathfrak{H} \to \mathfrak{K}_T^+ := (A_T^{\frac{1}{2}}\mathfrak{H})^-, x \mapsto A_T^{\frac{1}{2}} x,$ we obtain that there exists a unique isometry $V_T$on the space $\mathfrak{K}_T^+$ such that $X_T^+\, T = V_T\, X_T^+$. The isometry V_T is called the isometric asymptote of the contraction T . It is clear that the canonical intertwining transformation X_T^+ has dense range. Let W_T denote the minimal unitary extension of $V_T$acting on the Hilbert space $\mathfrak{K}_T$, determined uniquely up to isomorphism (see [19]). The operator W_T is the unitary asymptote of the contraction . The transformation $X_T : \mathfrak{H} \to \mathfrak{K}_T, x \mapsto X_T^+ x$ intertwines T with W_T : $X_T\, T = W_T\, X_T$. Furthermore,

$$\|X_T h\|^2 = \|X_T^+ h\|^2 = \left\|A_T^{1/2} h\right\|^2 = \langle A_T h, h\rangle = \lim_n \|T^n h\|^2$$

is true, for every $h \in \mathfrak{H}$. Thus, the nullspace $\ker X_T$ coincides with the set of vectors whose orbits converge to zero under the action of T . The contraction T is called asymptotically non-vanishing if $\ker X_T \neq \mathfrak{H}$.

The pair (X_T, W_T) has an important universal property. For an operator A acting on a space $\mathfrak{A}$ and an operator B acting on a space $\mathfrak{B}$, the intertwining set $\mathcal{T}(A,B)$ consists of the (bounded linear) transformations $Y : \mathfrak{A} \to \mathfrak{B}$ satisfying the equation $YA = BY$. The commutant of Á is defined by $\{A\}' := \mathcal{T}(A,A)$. Now, it is true that for any unitary operator G acting on a Hilbert space $\mathfrak{G}$, and for any transformation $Y \in \mathcal{T}(T,G)$, there exists a unique transformation $Z \in \mathcal{T}(W_T, G)$ such that $Y = ZX_T$. Furthermore, the commutants and spectra of T and W_T are closely related. For these properties and their extension to larger classes of operators we refer to [13] and [15]. (See also [1] for the study of isometric asymptotes, called isometric extensions there.

The unitary asymptote of the contraction T can be identified with the $*$-residual part of its minimal unitary dilation U acting on the Hilbert space $\mathfrak{K}$. We recall from [19] that the subspace $\mathfrak{L} = ((U - T)\mathfrak{H})^-$is wandering, and the orthogonal sum $M(\mathfrak{L}) := \bigoplus_{k=-\infty}^{\infty} U^k \mathfrak{L}$ is reducing for U. The $*$-residual part $R_* T$ is the restriction of U to its reducing subspace

$\mathcal{R}_*\mathrm{T} := \mathfrak{K} \ominus \mathrm{M}(\mathfrak{L})$. Let us consider the transformation $\widehat{\mathrm{X}}_\mathrm{T} : \mathfrak{H} \to \mathcal{R}_*\mathrm{T}, \mathrm{h} \longmapsto \mathrm{P}_*\mathrm{h}$, where P_* denotes the orthogonal projection in $\mathfrak{K}$ onto the subspace $\mathfrak{K}_{*\mathrm{T}}$. Since $\mathrm{R}_{*\mathrm{T}}\widehat{\mathrm{X}}_\mathrm{T} = \widehat{\mathrm{X}}_\mathrm{T}\,\mathrm{T}$ and $\widehat{\mathrm{X}}_\mathrm{T}\mathrm{h} = \lim_\mathrm{n} \|\mathrm{T}^\mathrm{n}\mathrm{h}\| (\mathrm{h} \in \mathfrak{H})$ hold by [19], it can be easily verified that the pair $(\widehat{\mathrm{X}}_\mathrm{T}, \mathrm{R}_{*\mathrm{T}})$ is equivalent to (X_T, W_T), that is there exists a unitary transformation $Z \in \mathsf{T}\ (W_T, R_{*T})$ such that $\widehat{\mathrm{X}}_\mathrm{T} = \mathrm{Z}X_T$.

The unitary asymptote has a particularly useful representation in the $Sz.-Nagy$–Foias functional model of completely non-unitary $(c.n.u.)$ contractions. We recall the construction of the model operator given in [19]. Let $\mathfrak{E}, \mathfrak{E}_*$ be (separable) Hilbert spaces, and let $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ be a purely contractive analytic function defined on the open unit disc $\mathbb{D}$. It is known that the radial $limit\Theta(\zeta) = lim_{r\to 1-0}\,\Theta(r\zeta)$ exists in the strong operator topology for almost every ζ on the unit circle $\mathbb{T}$. Hence Θ can be considered also as a measurable function defined almost everywhere on $\mathbb{T}$, and taking values in $\mathcal{L}(\mathfrak{E}, \mathfrak{E}_*)$. We can extend Θ to the whole circle $\mathbb{T}$ defining its value by 0 on the exceptional set of measure zero.The defect operator functions associated with Θ are defined b

$\Delta(\zeta) := I - \Theta(\zeta)^*\Theta(\zeta)^{1/2}$ and $\Delta_*(\zeta) := \left(I - \Theta(\zeta)\Theta(\zeta)^*\right)^{\frac{1}{2}} \quad (\zeta \in \mathbb{T})$.

Let us consider the spaces $L^2(\mathfrak{E}), L^2(\mathfrak{E}_*)$ of vector-valued functions, defined with respect to the normalized Lebesgue measure m on $\mathbb{T}$, and the Hardy subspaces $H^2(\mathfrak{E}), H^2(\mathfrak{E}_*)$. Setting

$$\mathfrak{K} := L^2(\mathfrak{E}_*) \oplus (\Delta L^2(\mathfrak{E}))^- \qquad \mathfrak{K}_+ := H^2(\mathfrak{E}_*) \oplus (\Delta L^2(\mathfrak{E}))^-$$
$$\mathfrak{G} := \{\Theta u \oplus \Delta u : u \in L^2(\mathfrak{E})\} \qquad \mathfrak{G}_+ := \{\Theta u \oplus \Delta u : u \in H^2(\mathfrak{E})\}$$

the model space $\mathfrak{H} = \mathfrak{H}\,(\Theta)$ is given by $\mathfrak{H} := \mathfrak{K}_+ \ominus \mathfrak{G}_+$. Let $U^\times$ denote the operator of multiplication by ζ on $L^2(\mathfrak{E}_*) \oplus L^2(\mathfrak{E})$. The subspaces $\mathfrak{K}, \mathfrak{G}$ are reducing for $U^\times$, while $\mathfrak{K}_+\mathfrak{G}_+$ are invariant for $U^\times$.

The model operator $T = S(\Theta)$ is defined by $T := P_+U_+|\mathfrak{H}$, where P_+ denotes the orthogonal projection onto $\mathfrak{H}$ in $\mathfrak{K}_+, U_+, := U^\times|\mathfrak{K}_+$ is the minimal isometric dilation of T , while $U := U^\times|\mathfrak{K}_+$ is the minimal unitary dilation of T .

Let us consider the restriction $\tilde{R}_{*T}$ of $U^\times$ to the reducing subspace $\tilde{\mathfrak{K}}_{+T} := (\Delta_* L^2(\mathfrak{E}_*))^-$ and the transformation $\tilde{X}_T \in \tau(T, \tilde{X}_{*T})$ defined by

$$\tilde{X}_T\,(u \oplus v) := -\Delta_* u + \Theta v\ (u \oplus v \in \mathfrak{H}).$$

Since multiplication by the unitary operator-valued function

$$F(\zeta) = \begin{bmatrix} -\Delta_*(\zeta) & \Theta(\zeta) \\ \Theta(\zeta)^* & \Delta(\zeta) \end{bmatrix} \in \mathcal{L}(\mathfrak{E}_* \oplus \mathfrak{E}) \qquad (\zeta \in \mathbb{T})$$

transfers the subspace $\mathfrak{K}_{+T}$ into $\tilde{\mathfrak{K}}_{+T}$, we obtain that the pair $(\tilde{X}_T, \tilde{R}_T)$ is equivalent to the pair $(\hat{X}_T, R_*T)$, and so to the pair (X_T, W_T). (See [12]) The great advantage of this representation of the unitary asymptote lies in the fact that the intertwining mapping $\tilde{X}_T$ is multiplication by an operator-valued function.

Let T be an absolutely continuous (a.c.) contraction on the Hilbert space $\mathfrak{H}$, that is we assume that the (spectral measure of the) unitary component of T is a.c. with respect to the normalized Lebesgue measure m on the unit circle $\mathbb{T}$. The minimal unitary dilation U of T is a.c. by [19]. It follows that the $*$-residual part R_{*T} is also a.c., and then so is the unitary asymptote W_T of T . The a.c. unitary operator W_T on the (separable) Hilbert space $\mathfrak{K}_T$ is uniquely determined—up to unitary equivalence—by its spectral-multiplicity function (see [4]), which we will call the asymptotic spectral-multiplicity function of the contraction T , and denote by μ_T . We recall that μ_T is a measurable function defined on the unit circle $\mathbb{T}$, and taking values in the set $\mathbb{N} \cup \{0, \aleph_0\}$ of countable cardinals. It is determined by the decreasing sequence of Borel sets:

$$\rho T_{,n} := \{\zeta \in \mathbb{T}: \mu_T(\zeta) \geq n\} \; (n \in \mathbb{N} \cup \{\aleph_0\}).$$

The set $\rho_T := \rho_{T,1}$ is the support of the spectral measure of W_T and is called the residual set of the a.c. contraction T . For its role in the study of T we refer to [17].

The Factorization Theorem establishes intertwining relations between the contraction T and unilateral shifts, exploiting the fine structure of W_T encoded in the spectral-multiplicity function μ_T .

For any cardinal number $1 \leq n \leq \aleph_0$, let $\mathfrak{G}_n$ be a fixed Hilbert space of dimension n. Let us consider the Hilbert space $L^2(\mathfrak{G}_n)$ of vector-valued functions. The σ-algebra of Borel subsets of $\mathbb{T}$ will be denoted by $B_{\mathbb{T}}$. For any $\alpha \in B_{\mathbb{T}}, \widetilde{M}_{n,\alpha}$ is the multiplication by $\chi(\zeta) = \zeta$on the space $L^2(\mathfrak{G}_n, \alpha) := \chi_\alpha L^2(\mathfrak{G}_n)$. Clearly, $\widetilde{M}_{n,\alpha}$ is an a.c. unitary operator with spectral multiplicity function $n\chi_\alpha$. (Here and in the sequel χ_ω stands for the characteristic function of the set ω.

It is known that the Hardy space $H^2(\mathfrak{G}_n)$ of analytic vector-valued functions, defined on $\mathbb{D}$, can be identified with the subspace $L^2_+(\mathfrak{G}_n)$ of $L^2(\mathfrak{G}_n)$, consisting of the functions with zero Fourier coefficients of negative indices (see [19, Section V.1]). Let S_n be the multiplication by $\chi(\zeta) = \zeta \; on \; H^2(\mathfrak{G}_n)$; S_n is clearly a unilateral shift of multiplicity n.For any $\alpha \in B_{\mathbb{T}}$, let us consider the natural embedding $\tilde{J}_{n,\alpha} : H^2(\mathfrak{G}_n) \to L^2(\mathfrak{G}_n, \alpha), \quad f \mapsto \chi_\alpha f$, of $H^2(\mathfrak{G}_n)$ into $L^2(\mathfrak{G}_n, \alpha)$. If $m(\alpha) = 0$ then $L^2(\mathfrak{G}_n, \alpha)$ and $\tilde{J}_{n,\alpha}$ reduce to zero. If $m(\alpha) > 0$ then $\tilde{J}_{n,\alpha}$ is one-to-one, $\|\tilde{J}_{n,\alpha}\| = 1$, and $\tilde{J}_{n,\alpha} S_n = \widetilde{M}_{n,\alpha} \tilde{J}_{n,\alpha}$. (See [9]. For the sake of brevity, we introduce the notation

$$M_{T,n} := \widetilde{M}_{n,\rho T,n} \; and \; J_{T,n} := \tilde{J}_{n,\rho T,n} \qquad (1 \leq n \leq \aleph_0)$$

in connection with the a.c. contraction T . The Factorization Theorem states that $J_{T,n}$can be factored into the product of two mappings intertwining S_n and $M_{T,n}$ with , with a control on the norms of the intertwiners.

The core of the proof of the Factorization Theorem (Theorem(6.1. 1)) is its verification in the functional model. So let us give a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$, and let us consider the model operator $T = S(\Theta) \in \mathcal{L}(\mathfrak{H} = \mathfrak{H}(\Theta))$ constructed at the end of this Section .

Since W_T is unitarily equivalent to $\tilde{R}_{*T}$, we obtain that the asymptotic spectral-multiplicity function

μ_T of T coincides with the function rank $\Delta_*(\zeta)$ ($\zeta \in \mathbb{T}$). Thus

$$\{ \rho_{T,n} = \{\zeta \in \mathbb{T}: rank \; \Delta_*(\zeta) \geq n\}$$

holds for every $1 \leq n \leq \aleph_0$.

First we want to show that there exists a sequence $\{u_i \oplus v_i\}_{0 \leq i < n}$in the dilation space $\mathfrak{K}$, which is pointwise orthonormal on the set $\rho_{T,n,}$and whose transformed sequence $\{-\Delta_* u_i + \Theta v_i\}_{0 \leq i < n}$ in $\widetilde{\mathfrak{R}}_{*T}$ is also pointwise orthonormal on $\rho_{T,n}$. In order to do so we have to make a closer look at the defect functions.

We recall from [19] that the defect operator D_A of a contractive transformation $A \in \mathcal{L}(\mathfrak{G}, \mathfrak{G}_*)$ is the positive contraction defined by $D_A := (I - A^*A)^{1/2} \in \mathcal{L}(\mathfrak{G})$. The closure of its range is the defect space $\mathfrak{D}_A$ of A. Let D_{A^*} and $\mathfrak{D}_{A^*}$ be the analogous objects connected with the adjoint A. It is easy to check that $A^*D_{A^*} = D_A A^*$.

For any $\zeta \in \mathbb{T}$, let $\Delta(\zeta)$ and $\mathfrak{D}(\zeta)$ be the defect operator and the defect space of $\Theta(\zeta)$, respectively. Let $\Delta_*(\zeta)$ and $D_*(\zeta)$ stand for the analogous objects connected with the adjoint transformation $\Theta(\zeta)^*$. All these operator fields and subspace fields are measurable; see [14]. Notice that the direct integrals

$\int_\omega^\oplus (\mathfrak{D}_*(\zeta)) \oplus \mathfrak{D}(\zeta)\, dm(\zeta)$ and $\int_\omega^\oplus \mathfrak{D}_*(\zeta)\, dm(\zeta)$ can be viewed as subspaces in the space $\mathfrak{K}$ of the minimal unitary dilation U, and in R_{*T} , respectively.

The following statement is an improvement of [14]. (Its version formulated in the more general setting considered in [14] can be proved in a similar way. We note that only the pointwise orthogonality of the system $\{k_i\}_i$ below was shown in [14].)

Proposition(6.1.1)[12]: For every cardinal number $1 \le n \le \aleph_0$, there exist sequences $\{u_i\}_{0\le i<n}$ and $\{v_i\}_{0\le i<n}$ of measurable vector fields in $\prod_{\zeta\in\rho T,n} \mathfrak{D}_*(\zeta)$ and $\prod_{\zeta\in\rho T,n} \mathfrak{D}(\zeta)$, respectively, such that

(i) $\{u_i(\zeta) \oplus v_i(\zeta)\}_{0\le i<n}$ forms an orthonormal system in $\mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta)$ for every $\zeta \in \rho_{T,n}$, and

(ii) $\{k_i(\zeta) := -\Delta_*(\zeta)u_i(\zeta) + \Theta(\zeta)v_i(\zeta)\}_{0\le i<n}$ is also an orthonormal system in $\mathfrak{D}_*(\zeta)$ for every $\zeta \in \rho_{T,n}$.

Proof. Let $F : \mathbb{T} \to \mathcal{L}(\mathfrak{E}_* \oplus \mathfrak{C})$ be the unitary operator-valued, measurable function introduced in Section 2. The equation $\Theta(\zeta)^*\Delta_*(\zeta) = \Delta(\zeta)\Theta(\zeta)^*$ yields $F(\zeta)\mathfrak{D}_*(\zeta) \subset \mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta)$ $(\zeta \in \mathbb{T})$. Let us consider the isometry-valued, measurable transformation field $F_0(\zeta): \mathfrak{D}_*(\zeta) \mapsto \mathfrak{D}_*(\zeta) \oplus \mathfrak{D}(\zeta)$ defined by $F_0(\zeta)w := F(\zeta)w = -\Delta_*(\zeta)w \oplus \Theta(\zeta)$ $w(\zeta \in \mathbb{T}, w \in \mathfrak{D}_*(\zeta))$. It is easy to see that the adjoint transformation field $F_0(\zeta)^* : \mathfrak{D}_*(\zeta)\ \mathfrak{D}(\zeta) \to \mathfrak{D}_*(\zeta)$ is defined by $F_0(\zeta)(u \oplus v) = -\Delta_*(\zeta)u + \Theta(\zeta)v$.

Since $dim\mathfrak{D}_*(\zeta) \ge n$ holds for every $\zeta \in \rho_{T,n}$, we can give measurable vector fields $\{k_i\}_{0\le i<n}$ in $\prod_{\zeta\in\rho T,n} \mathfrak{D}(\zeta)$ so that $\{k_i(\zeta)\}_{0\le i<n}$ forms an orthonormal system for each $\zeta \in \rho_{T,n}$ (see [5]). Then the measurable vector fields $\{u_i\}_{0\le i<n}$ and $\{v_i\}_{0\le i<n}$, defined by

$$u_i(\zeta) \oplus v_i(\zeta) := F_0(\zeta)k_i(\zeta)\ (0 \le i < n, \zeta \in \rho_{T,n}),$$

satisfy all the required conditions.

The vector functions (or vector fields) $\{u_i \oplus v_i\}_i$ provided by Proposition (6.1.1) are contained in the space $\mathfrak{K}$ of the minimal unitary dilation. We want to approximate them with functions from the space $\mathfrak{K}_+$ of the minimal isometric dilation U_+. Hence we have to approximate measurable vector-valued functions by analytic ones. For any $u, \acute{u} \in L^2(\mathfrak{E}_*)$ the measurable function $[u, \hat{u}]$ is defined by $[u, \hat{u}](\zeta) := \big(u(\zeta), \hat{u}(\zeta)\big)$ $(\zeta \in \mathbb{T})$. The norm-function of $u \in L^2(\mathfrak{E}_*)$ is denoted by $[u]$, that is $[u](\zeta) := \|u(\zeta)\|$ $(\zeta \in \mathbb{T})$.

We recall from [14] that if $u \in L^2(\mathfrak{E}_*)$ is a unimodular function, that is $[u] \equiv 1$, then for every $0 < \eta < 1$ there exists a function $u^\# \in L^2(\mathfrak{E}_*)$ such that $[u^\#] \equiv 1$ and $[u^\#, u] \ge \eta$

For the approximating purposes, mentioned above, we need the following lemma.

Lemma(6.1.2)[12]: Let $u \oplus v \in L^2(\mathfrak{E}_*) \oplus L^2(\mathfrak{C}) \equiv L^2(\mathfrak{E}_* \oplus \mathfrak{C})$ be a function with the property that its norm-function $[u \oplus v] \equiv 1$. Then, for every $0 < \eta < 1$, there exist a function $u^\# \in H^2(\mathfrak{E}_*)$ and a measurable complex function ψ on $\mathbb{T}$ such that $u^\# \oplus \psi v \equiv 1$ and $|[u^\# \oplus \psi v, u \oplus v]| \ge \eta$.

Proof. Let us give an arbitrary $0 < \eta < 1$, and let us choose positive numbers η_1 and ε satisfying the conditions

$\eta < \eta_1^2 < \eta_1 < 1$ and $\eta_1^2 - 2\varepsilon(1 + 2\varepsilon)^{-1} > \eta$. (1)

Let us consider the decomposition $\mathbb{T} = \beta_1 \cup \beta_2$, where

$$\beta_1 := \{\zeta \in \mathbb{T}: \|u(\zeta)\| \geq \eta_1\} \text{ and } \beta_2 := \mathbb{T} \setminus \beta_1.$$

Given any $e_* \in \mathfrak{E}_*$ with $\|e_*\| = 1$, the function $u_1 \in L^2(\mathfrak{E}_*)$ is defined by

$$u_1(\zeta) := \begin{cases} u(\zeta)/\|u(\zeta)\| & if\ \zeta \in \beta_1 \\ e_* & if\ \zeta \in \beta_2 \end{cases} \quad (2)$$

Since $[u_1] \equiv 1$, by [14] there exists a function $u_1^\# \in L^2(\mathfrak{E}_*)$ such that

$$\eta_1 \leq |[u_1^\#, u_1]| \leq [u_1^\#] \equiv 1. \quad (3)$$

Let us give a positive η_2 so that $\eta_1 < \eta_2 < 1$. Applying [14] for $u \oplus v$, we obtain a function $u_2^\# \oplus v_2^\# \in H^2(\mathfrak{E}_* \oplus \mathfrak{C}) \equiv H^2(\mathfrak{E}_*) \oplus H^2(\mathfrak{C})$ with the properties

$$\eta_2 \leq |[u_2^\# \oplus v_2^\#, u \oplus v]| \leq u_2^\# \oplus v_2^\# \equiv 1. \quad (4)$$

For every $\zeta \in \beta_2$, we have

$\|v(\zeta)\| = (1 - \|u(\zeta)\|^2)^{1/2} \geq (1 - \eta_1^2)^{1/2} > 0$ (5)

Let us consider the decomposition

$v_2^\#(\zeta) = \psi_2(\zeta)v(\zeta) + w(\zeta)$, where $w(\zeta) \perp v(\zeta)$ $(\zeta \in \beta_2)$. (6)

It is clear that the function

$$\psi_2(\zeta) = \|v(\zeta)\|^{-2}\langle v_2^\#(\zeta), v(\zeta)\rangle \ (\zeta \in \beta_2) \quad (7)$$

is measurable. We want to show that the norm of $w(\zeta)$ is as small as we wish if η_2 is sufficiently close to 1.

In view of (4) and applying the Cauchy–Schwarz inequality we infer that.

$\eta_2 \leq |\langle u_2^\#(\zeta) \oplus u_2^\#(\zeta), u(\zeta) \oplus v(\zeta)\rangle| \leq |\langle u_2^\#(\zeta), u(\zeta)\rangle| + [u_2^\#(\zeta), v(\zeta)]$

$\leq \|u_2^\#(\zeta)\| \|u(\zeta)\| + \|u_2^\#(\zeta)\|\|v(\zeta)\| =: \mathrm{k}(\zeta)$

$\leq (\|u_2^\#(\zeta)\|^2 + \|v_2^\#(\zeta)\|^2)^{1/2}(\|u(\zeta)\|^2 + \|v(\zeta)\|^2)^{1/2} = 1$ (8)

holds for every $\zeta \in \mathbb{T}$. Taking the decomposition of the ordered pairs

$$(\|u_2^\#(\zeta)\|, \|v_2^\#(\zeta)\|) = \mathrm{k}(\zeta)(\|u(\zeta)\|, \|v(\zeta)\|) + \big((a)(\zeta), b(\zeta),\big)$$

we obtain that

$$|b(\zeta)|^2 \leq \|a(\zeta), b(\zeta)\|^2 = 1 - k(\zeta)^2 \leq 1 - \eta_2^2,$$

whence

$\|u_2^\#(\zeta)\| = \mathrm{k}(\zeta)\|v(\zeta)\| + b(\zeta) \geq \eta_2\|v(\zeta)\| - (1 - \eta_2^2)^{1/2} \quad (\zeta \in \mathbb{T})$

follows. Applying (5) we conclude that

$$\|v_2^\#(\zeta)\| \geq \eta_2(1 - \eta_1^2)^{1/2} - (1 - \eta_2^2)^{1/2} \quad (9)$$

is true for every $\zeta \in \beta_2$. Let us assume that η_2 is so close to 1 that

$$\eta_2(1 - \eta_{21}^2)^{1/2} - (1 - \eta_2^2)^{1/2} > 0 \quad (10)$$

is fulfilled. One can easily derive from (8) that

$$\|v_2^\#(\zeta)\|\|v(\zeta)\| - |\langle v_2^\#(\zeta), v(\zeta)\rangle| \leq 1 - \eta_2 \quad (\zeta \in \mathbb{T}).$$

It follows by (5)–(7) and (9) that

$$\|\psi_2(\zeta)v(\zeta)\| = |\langle v_2^\#(\zeta), v(\zeta)/\|v(\zeta)\|\rangle| \geq \|v_2^\#(\zeta)\| - (1 - \eta_2)(1 - \eta_1^2)^{-1/2}$$

$\geq \eta_2(1 - \eta_1^2)^{1/2} - (1 - \eta_2^2)^{1/2} - (1 - \eta_2)(1 - \eta_1^2)^{-1/2}$ (11)

holds for every $\zeta \in \beta_2$. Choosing η_2 sufficiently close to 1 we can ensure that

$\eta_2(1 - \eta_1^2)^{1/2} - (1 - \eta_2^2)^{1/2} - (1 - \eta_2)(1 - \eta_1^2)^{-1/2} > 0$ (12)

Applying (6), (11) and (12) we obtain

$$\|v_2^\#(\zeta) - \psi_2(\zeta)v(\zeta)\|^2 = \|v_2^\#(\zeta)\|^2 - \|\psi_2(\zeta)v(\zeta)\|^2$$

$$\leq \|v_2^{\#}(\zeta)\|^2 - (\|v_2^{\#}(\zeta)\| - (1-\eta_2)(1-\eta_1^2)^{-1/2})^2$$
$$\leq \|v_2^{\#}(\zeta)\|(1-\eta_2)(1-\eta_1^2)^{-1/2}$$
$$\leq 2(1-\eta_2)(1-\eta_1^2)^{-1/2} \qquad (\zeta \in \beta_2) \qquad (13)$$

Therefore, assuming that the positive number η_2 satisfies the conditions $\eta_1 < \eta_2 < 1$, (10), (12) and

$$2(1-\eta_2)(1-\eta_1^2)^{-1/2} < \varepsilon^2 \qquad (14)$$

we conclude by (13) that

$$\|v_2^{\#}(\zeta) - \psi_2(\zeta)v(\zeta)\| < \varepsilon \qquad (15)$$

holds for every $\zeta \in \beta_2$.

Let us give $\varphi_1, \varphi_2 \in H^\infty$ with absolute value

$$|\varphi_1| = \chi\beta_1 + \varepsilon\chi\beta_2, \qquad |\varphi_2| = \varepsilon\chi\beta_1 + \chi\beta_2 \qquad (16)$$

and let us introduce the functions

$$\tilde{u} := \varphi_1 u_1^{\#} + \varphi_2 u_2^{\#} \in H^2(\mathfrak{E}_*) \text{and} \qquad \tilde{\psi}(\zeta) := \begin{cases} 0 & for\ \zeta \in \beta_{1,} \\ \varphi_2(\zeta)\psi_2(\zeta) & for\ \zeta \in \beta_{2.} \end{cases}$$

For every $\zeta \in \beta_{1,}$, we have

$$\tilde{u}(\zeta) \oplus \tilde{\psi}(\zeta)v(\zeta) = \left(\varphi_1(\zeta)u_1^{\#}(\zeta) + \varphi_2(\zeta)u_2^{\#}(\zeta)\right) \oplus 0,$$

and so

$$1 - \varepsilon \leq \left\|\tilde{u}(\zeta) \oplus \tilde{\psi}(\zeta)v(\zeta)\right\| \leq 1 + \varepsilon \qquad (\zeta \in \beta_1) \qquad (17)$$

readily follows by (3), (4) and (16). Furthermore,

$$\left|\langle \tilde{u}(\zeta) \oplus \tilde{\psi}(\zeta)v(\zeta), u(\zeta) \oplus v(\zeta)\rangle\right| =$$
$$|\varphi_1(\zeta)\langle u_1^{\#}(\zeta), \|u(\zeta)\|u_1(\zeta)\rangle + \varphi_2(\zeta)\langle u_2^{\#}(\zeta), u(\zeta)\rangle| \geq \eta_1^2 - \varepsilon \qquad (\zeta \in \beta_1)$$
(18)

is clearly true by (2)–(4) and (16). On the other hand, for every $\zeta \in \beta_{2.}$, we have

$$\left(\tilde{u}(\zeta) \oplus \tilde{\psi}(\zeta)v(\zeta) = \varphi_1(\zeta)u_1^{\#}(\zeta) \oplus 0\right) + \varphi_2(\zeta)\left(u_2^{\#}(\zeta) \oplus v_2^{\#}(\zeta)\right)$$
$$+ \left(0 \oplus \varphi_2(\zeta)\left(\psi_2(\zeta)v(\zeta) - u_2^{\#}(\zeta)\right)\right)$$

Hence, applying (15) together with (3), (4) and (16), one can easily verify that

$$1 - 2\varepsilon \leq \left\|\tilde{u}(\zeta) \oplus \tilde{\psi}(\zeta)v(\zeta)\right\| \leq 1 + 2\varepsilon \qquad (\zeta \in \beta 2) \qquad (19)$$

and

$$\left|\langle \tilde{u}(\zeta) \oplus \tilde{\psi}(\zeta)v(\zeta), u(\zeta) \oplus v(\zeta)\rangle\right| \geq \eta_2 - 2\varepsilon \geq \eta_1^2 - 2\varepsilon \quad (\zeta \in \beta 2) \quad (20)$$

Notice that $1 - 2\varepsilon > \eta_1^2 - 2\varepsilon > 0$ by (1). In virtue of (17) and (19) there exists an outer function $\varphi \in H^\infty$ with the property

$$|\varphi| = [\tilde{u} \oplus \tilde{\psi}v].$$

Defining $u^{\#} \in H^2(\mathfrak{E}_*)$ and the measurable function ψ by $u^{\#} := \varphi^{-1}\tilde{u}$ and $\psi := \varphi^{-1}\tilde{\psi}$, the equation $[u^{\#} \oplus \psi v] \equiv 1$ is clearly fulfilled. Finally, the relations (17)–(20) and (1) readily imply that

$$|\langle u^{\#}(\zeta) \oplus \psi(\zeta)v(\zeta), u(\zeta) \oplus v(\zeta)\rangle| \geq (\eta_2 - 2\varepsilon)|\varphi(\zeta)|^{-1}$$
$$\geq (\eta_1^2 - 2\varepsilon)(1 + 2\varepsilon)^{-1} \geq \eta$$

is true for every $\zeta \in \mathbb{T}$. Thus the proof is complete.

Since we shall work with vectors approximating an orthonormal system, we need a statement which describes how perturbation of an isometry on elements of an orthonormal basis affects the norm and the lower bound of the operator. Such a statement is the content of the following lemma taken from [14].

Lemma (6.1.3)[12]: Let $1 \leq n \leq \aleph_0$ be a cardinal number, let $\{g_i\}_{0 \leq i<n}$ be an orthonormal basis in the Hilbert space $\mathfrak{G}_n$, and let $\{f_i\}_{0 \leq i<n}$ be an orthonormal system in a Hilbert space $\mathfrak{J}$. Let us give constants $0 < \delta < c < 1$ and a sequence $\{\delta_i\}_{0 \leq i<n}$ of positive numbers satisfying the condition $\sum_{0 \leq i<n} \delta_i^2 \leq \delta^2$ For any $0 \leq i < n$, let $f_i^{\#} \in \mathfrak{J}$ be a vector of the form $f_i^{\#} = c_i f_i + s_i$, where $c \leq |c_i| \leq 1$ and $\|s_i\| \leq \delta_i$.

Then there exists a uniquely determined transformation $A \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{J})$ such that $Ag_i = f_i^{\#}$ holds, for every $0 \leq i < n$. Furthermore, for this transformation A we have

$$c - \delta \leq \Lambda(A) \leq \|A\| \leq 1 + \delta.$$

Now we are ready to prove Theorem (6.1.1) for the model operator $T = S(\Theta)$.

Proposition(6.1. 4)[12]: The statement of Theorem(6.1.5) is true for the c.n.u. contraction $T = S(\Theta)$.

Proof: Let us fix a cardinal number $1 \leq n \leq \aleph_{0,.}$, and let us assume that $m(\rho T, n) > 0$. Recall that $\rho T, n = \{\zeta \in \mathbb{T}: rank \Delta_*(\zeta) \geq n\}$. For simplicity, we shall use the notation $\gamma := \rho T, n$. Let us give an arbitrary $\varepsilon > 0$.

Let $\{u_i\}_{0 \leq i<n}$ and $\{v_i\}_{0 \leq i<n}$ be measurable vector-valued functions obtained by applying Proposition(6.1.1). We extend these functions to the whole circle T in the following way. Given any orthonormal system $\{e_{*i}\}_{0 \leq i<n}$ in $\mathfrak{G}_*$, let $u_i(\zeta) := e_{*i}$ and $v_i(\zeta) := 0$, for every $\zeta \in \mathbb{T} \setminus \gamma$ and $0 \leq i < n$. It is clear that $u_i \oplus v_i \in \mathfrak{K}$ for every $0 \leq i < n$. Furthermore, $\{u_i(\zeta) \oplus v_i(\zeta)\}_{0 \leq i<n}$ forms an orthonormal system for every $\zeta \in \mathbb{T}$, and

$$\{k_i(\zeta) := -\Delta_*(\zeta) u_i(\zeta) + \Theta(\zeta) v_i(\zeta)\}_{0 \leq i<n} \subset \mathfrak{D}_*(\zeta) \qquad (21)$$

is also an orthonormal system for every $\zeta \in \gamma$.

Let us give constants $0 < \delta < c < 1$ and a sequence $\{\delta_i\}_{0 \leq i<n}$ of positive numbers with the property$\sum_{0 \leq i<n} \delta_i^2 \leq \delta^2$. Applying Lemma (6.1.2) we obtain that, for every $0 \leq i < n$, there exist $u_i^{\#} \in H^2(\mathfrak{E}_*)$ and a measurable complex function ψ_i on $\mathbb{T}$ such that

$$\eta_i \leq \left|\left[u_i^{\#} \oplus \psi_i v_i, u_i \oplus v_i\right]\right| \leq \left[u_i^{\#} \oplus \psi_i v_i\right] \equiv 1 \qquad (22)$$

holds with $\eta_i := max(c, (1 - \delta_i^2)^{1/2})$. Then $u_i^{\#} \oplus \psi_i v_i \in \mathfrak{K}_+$ is clearly true for every $0 \leq i < n$. In view of (22), these functions can be written in the form

$$u_i^{\#}(\zeta) \oplus \psi_i(\zeta) v_i(\zeta) = c_i(\zeta) u_i(\zeta) \oplus v_i(\zeta) + r_i(\zeta) \oplus s_i(\zeta) \qquad (23)$$

where

$$c \leq \eta_i \leq |c_i(\zeta)| \leq (\zeta \in \mathbb{T}, 0 \leq i < n) \qquad (24)$$

and

$$\|r_i(\zeta) \oplus s_i(\zeta)\|^2 = 1 - |c_i(\zeta)|^2 \leq 1 - \eta_i^2 \leq \delta_i^2 \; (\zeta \in \mathbb{T}, 0 \leq i < n). (25)$$

Let us fix an orthonormal basis $\{g_i\}_{0 \leq i<n}$ in $\mathfrak{G}_n$. Given any $\zeta \in \mathbb{T}$, in virte of (23)–(25),Lemma (6.1.3) implies the existence of a uniquely determined transformation $\Phi(\zeta) \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{E}_* \oplus \mathfrak{E})$ satisfying the condition

$$\Phi(\zeta) g_i = u_i^{\#}(\zeta) \oplus \psi_i(\zeta) v_i(\zeta) \text{ for every } 0 \leq i < n;$$

furthermore,

$$c - \delta \leq \Lambda(\Phi(\zeta)) \leq \|\Phi(\zeta)\| \leq 1 + \delta. \qquad (26)$$

We shall write $\chi_{\mathbb{T}} g_i$ for the constant function in $H^2(\mathfrak{G}_n)$ with value g_i . Since $\Phi(\chi_{\mathbb{T}} g_i) = u_i^{\#} \oplus \psi_i v_i$ is a measurable vector-valued function for every $0 \leq i < n$, it follows that the bounded transformation-valued function Φ is measurable (see [5]). The transformation $M(\Phi)$ of multiplication by Φ maps $L^2(\mathfrak{G}_n)$ into $\mathfrak{K}$, and clearly $M(\Phi)H^2(\mathfrak{G}_n) \subset \mathfrak{K}_+$. Let us consider the restriction

$$Y_+ := M(\Phi)\backslash H^2(\mathfrak{G}_n) \in \mathcal{L}(H^2(\mathfrak{G}_n), \mathfrak{K}_+).$$

It is evident that

$$Y_+S_n = U_+Y_+ \ and \ Y_+(\chi_{\mathbb{T}}g_i) = u_i^\# \oplus \psi_i v_i (0 \leq i < n). \qquad (27)$$

Let $\tilde{P}_+ \in \mathcal{L}(\mathfrak{K}_+, \mathfrak{H})$ stand for the transformation defined by $\tilde{P}_+x := P_+x \ (x \in \mathfrak{K}_+)$, where P_+ is the orthogonal projection onto $\mathfrak{H}$ in $\mathfrak{K}_+$. We know from [19] that

$$\tilde{P}_+U_+ = T\,\tilde{P}_+. \qquad (28)$$

Now the transformation $Y \in \mathcal{L}(H^2(\mathfrak{G}_n), \mathfrak{H})$ is defined by

$$Y := \tilde{P}_+Y_+. \qquad (29)$$

The relations (27)–(29) result in that

$$YS_n = T\,Y. \qquad (30)$$

Furthermore, in view of (26) we obtain that

$$\|Y\| \leq \|Y_+\| \leq \|\Phi\|_\infty \leq 1 + \delta. \qquad (31)$$

Let us consider the vector-valued functions

$$h_i := Y(\chi_{\mathbb{T}}g_i) \in \mathfrak{H} \ and \ k_i^\# := \tilde{X}_T\, h_i \in \tilde{\tilde{\mathfrak{K}}}_{*T} \ (0 \leq i < n). \qquad (32)$$

Let $\tilde{\tilde{\mathfrak{K}}}_{*+}$ be the reducing subspace of $\tilde{\tilde{\mathfrak{K}}}_{*T}$ generated by the vectors $\{\chi_\gamma\, k_i^\#\}_{0\leq i<n}$, and let us consider the restriction $\tilde{\tilde{\mathfrak{K}}}_{*+} := \tilde{R}_{*T}\,|\tilde{\tilde{\mathfrak{K}}}_{*+}$. Let $\tilde{Q}_+ \in \mathcal{L}(\tilde{\tilde{\mathfrak{K}}}_{*T}, \tilde{\tilde{\mathfrak{K}}}_{*+})$ stand for the transformation defined by $\tilde{Q}_+x := Q_+x \ (x \in \tilde{\tilde{\mathfrak{K}}}_{*T})$, where Q_+ is the orthogonal projection onto $\tilde{\tilde{\mathfrak{K}}}_{*+}$ in $\tilde{\tilde{\mathfrak{K}}}_{*T}$.

Then clearly

$$\tilde{Q}_+\tilde{\tilde{\mathfrak{K}}}_{*T} = \tilde{R}_{*+}\tilde{Q}_+ \ \text{and} \qquad \tilde{Q}_+k_i^\# = \chi_\gamma\, k_i^\# \ (0 \leq i < n). \qquad (33)$$

Introducing the transformation $\tilde{X}_+ : \mathfrak{K}_+ \to \tilde{\tilde{\mathfrak{K}}}_{*T}$ defined by

$$\tilde{X}_+(u \oplus v) := -\Delta_* u + \Theta v \qquad (u \oplus v \in \mathfrak{K}_+),$$

we can see from the equation $\Theta\Delta = \Delta_*\Theta$ that ker $\tilde{X}_+ \supset \mathfrak{G}_+$. Consequently

$$k_i^\# = \tilde{X}_T\, \tilde{P}_+\big(u_i^\# \oplus \psi_i v_i\big) = \tilde{X}_+\big(u_i^\# \oplus \psi_i v_i\big)$$
$$= -\Delta_* u_i^\# + \Theta(\psi_i v_i) \ (0 \leq i < n). \qquad (34)$$

We infer by (21) and (23) that for any $0 \leq i < n$ the function $k_i^\#$ is of the following form on the set γ :

$k_i^\#(\zeta) = c_i(\zeta)k_i(\zeta) + y_i(\zeta)$, where $yi(\zeta) := -\Delta * (\zeta)ri(\zeta) + \Theta(\zeta)si(\zeta) \ (\zeta \in \gamma)$. (35)

Recalling that the operator $F(\zeta)$ in this Section is an isometry, it follows by (25) that

$$\|y_i(\zeta)\| \leq \|r_i(\zeta) \oplus s_i(\zeta)\| \leq \delta_i \ (\zeta \in \gamma, 0 \leq i < n). \qquad (36)$$

Taking into account that $\{k_i(\zeta)\}_{0\leq i<n}$ is an orthonormal system for $\zeta \in \gamma$, Lemma (6.1.3) yields by (35), (36) and (24) that, for any $\zeta \in \gamma$, there exists a (unique) transformation $\Psi(\zeta) \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{E}_*)$ such that

$$\Psi(\zeta)g_i = k_i^\#(\zeta) \text{ for every } 0 \leq i < n; \qquad (37)$$

furthermore,

$$c - \delta \leq \Lambda(\Psi(\zeta)) \leq \|\Psi(\zeta)\| \leq 1 + \delta. \qquad (38)$$

For any $\zeta \in \mathbb{T} \setminus \gamma$ let us set $\Psi(\zeta) := 0 \in \mathcal{L}(\mathfrak{G}_n, \mathfrak{E}_*)$. It can be easily verified, as before for Φ, that the bounded function Ψ is measurable. The transformation $M(\Psi)$ of multiplication by Ψ maps $L^2(\mathfrak{G}_n, \gamma)$ into $\tilde{\tilde{\mathfrak{K}}}_{*+}$; let us consider the mapping

$$Z_+ := M(\Psi)\backslash L^2(\mathfrak{G}_n, \gamma) \in \mathcal{L}L^2(\mathfrak{G}_n, \gamma), \tilde{\mathfrak{K}}_{*+}$$

It is evident that

$$Z_+(\chi_\gamma\, g_i) = \chi_\gamma\, k_i^\# \quad \text{for every } 0 \le i < n. \qquad (39)$$

We infer by (38) that

$$c - \delta \le \Lambda(Z_+) \le \|Z_+\| \le 1 + \delta. \qquad (40)$$

Since Z_+ has dense range by (39), we obtain that Z_+ is invertible, and so (40) yields

$$\|Z_+^{-1}\| \le (c-\delta)^{-1}. \qquad (41)$$

Taking into account that $M_{T,n} = \widetilde{M}_{n,\gamma}$, we can see that

$$Z_+ M_{T,n} = \tilde{R}_{*+} Z_+. \qquad (42)$$

Now, the transformation $Z \in \mathcal{L}(\mathfrak{H}, L^2(\mathfrak{G}_n, \gamma))$ is defined by

$$Z := Z_+^{-1} \tilde{Q}_+ \tilde{X}_T\ . \qquad (43)$$

The intertwining relations $\tilde{X}_T\, T = \tilde{R}_{*T}\, \tilde{X}_T$, (33) and (42) yield that

$$ZT = M_{T,n} Z. \qquad (44)$$

Taking into account that the mappings $\tilde{Q}_+$ and $\tilde{X}_T$ are contractions, it follows from (41) that

$$\|Z\| \le (c - \delta)^{-1}. \qquad (45)$$

Choosing the constants δ and c sufficiently close to 0 and 1, respectively, it can be achieved that $1 + \delta < 1 + \varepsilon$ and $(c - \delta)^{-1} < 1 + \varepsilon$ hold. Hence, by the inequalities (31) and (45) we conclude that

$$\|Y\| < 1 + \varepsilon \text{ and } \|Z\| < 1 + \varepsilon. \qquad (46)$$

Finally, in view of (32), (33), (39) and (43) we have for any $0 \le i < n$ that

$$ZY(\chi_{\mathbb{T}} g_i) = Zh_i = Z_+^{-1}\, \tilde{Q}_+ X_T\, h_i = Z_+^{-1}\, \tilde{Q}_+ k_i^\# = Z_+^{-1}\, \chi_\gamma k_i^\#$$
$$= \chi_\gamma g_i = \tilde{J}_{n,\gamma}(\chi_{\mathbb{T}} g_i) = J_{T,n}(\chi_{\mathbb{T}} g_i). \qquad (47)$$

Since ZY and $J_{T,n}$ intertwine S_n with $M_{T,n}$, equalities (47) imply

$$ZY = J_{T,n} \qquad (48)$$

The relations (46) and (48) show that the mappings Y and Z possess all the required properties.

Now we complete the proof of the main result.

Theorem(6.1.5)[12]: Let T be an $a.c.$ contraction on the Hilbert space $\mathfrak{H}$. For every cardinal number

$1 \le n \le \aleph_0$, and for every $\varepsilon > 0$, there exist transformations $Y \in \mathcal{T}(S_n, T)$ and $Z \in \mathcal{T}(T, M_{T,n})$ satisfying the conditions:

(i) $ZY = J_{T,n}$, and

(ii) $\|Y\| < 1 + \varepsilon, \|Z\| < 1 + \varepsilon.$

Notice that if $m(\rho_{T,n}) = 0$ then $J_{T,n} = 0$, and so the transformations $Y = 0$ and $Z = 0$ evidently possess the required properties. The statement of the previous theorem becomes nontrivial when $m(\rho_{T,n}) > 0$.

Proof. Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$. Let us give a cardinal number $1 \le n \le \aleph_{0,.}$ and a positive ε.

The contraction T can be decomposed into the orthogonal sum $T = T_u \oplus T_c$, where T_u is an a.c. unitary operator and T_c is a $c.n.u.$ contraction. It is known (see e.g. [4]) that $T_u = W_{T_u}$ is unitarily equivalent to the orthogonal $\oplus_{k\in\mathbb{N}} M_{\alpha_k}$, where $\alpha_k := \rho T_{u,k}$

and$M_{\alpha_k} = \widetilde{M}_{1,\alpha_k}(k \in \mathbb{N})$. Let $Q_u \in T(\oplus_{k\in\mathbb{N}} NM\alpha k, Tu)$ be a unitary transformation; then

$$\tilde{Q} := Q_u \oplus I \in T\left(\left(\bigoplus_{k\in N} M_{\alpha_k}\right) \oplus T_c, T_u \oplus T_c\right)$$

is also unitary. Given an arbitrary $0 < c < 1$, for every $k \in \mathbb{N}, let\ \vartheta_k \in H^\infty$ be an outer function with absolute value $|\vartheta_k| = c\chi\alpha_k + \chi_{\mathbb{T}}\backslash\alpha_k$, and let us consider the $c.n.u.$ contraction $T_k = S(\vartheta_k)$.

By [19] there exists an affinity $Q_k \in T(T_k, M_{\alpha_k})$ satisfying the conditions

$$1 = \|Q_k\| \leq (Q_k^{-1}) \leq c^{-1} \quad (k \in \mathbb{N}).$$

The $c.n.u.$ contraction $\tilde{T} := (\oplus_{k\in\mathbb{N}} T_k) \oplus T_c$ is unitarily equivalent to a model operator T =S(Θ) by [19], let $\grave{Q} \in T(\grave{T}, \tilde{T}, T)$ be a unitary transformation. Then the affinity

has the properties

$$Q\grave{T} = TQ \text{ and } 1 = \|Q\| \leq \|Q^{-1}\| \leq c^{-1}. \quad (49)$$

Clearly, $\mu T_k = \chi\alpha_k$ holds for every $k \in \mathbb{N}$, and so the asymptotic spectral-multiplicity functions of the contractions T and $\grave{T}$ coincide: $\mu_T = \mu_{\grave{T}}$.

Therefore

$$M_{T,n} = M_{\grave{T},n} and\ J_{T,n} = J_{\grave{T},n} \quad (50)$$

Given an arbitrary $0 < \delta < 1$, Proposition(6.1.4) provides us with mappings $\grave{Y} \in T(S_n, \grave{T})$ and $\grave{Z} \in T(\grave{T}, M_{\grave{T},n})$ satisfying the conditions

$$\grave{Z}\grave{Y} = J_{\grave{T},n} \text{ and } \|\grave{Y}\| < 1 + \delta, \|\grave{Z}\| < 1 + \delta. \quad (51)$$

Then $Y := Q\grave{Y} \in T(S_n, T), Z := \grave{Z}Q^{-1} \in T(T, M_{T,n})$, and we conclude by (49)–(51) that

$$ZY = \grave{Z}\grave{Y} = J_{\grave{T},n} = J_{T,n}$$

and

$$\|Y\| < 1 + \delta, \qquad \|Z\| < (1 + \delta)c^{-1}$$

Choosing δ and c sufficiently close to 0 and 1, respectively, we can ensure that $1 + \delta < 1 + \varepsilon$ and $(1 + \delta)c^{-1} < 1 + \varepsilon$.

The proof is complete.

We supplement the statement of Theorem (6.1.5) by showing that every factorization of any embedding $\tilde{J}_{n,\alpha}$ through intertwining mappings with the contraction T is necessarily attached to the set $\rho_{T,n}$.

Proposition (6.1.6)[12]: Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$. Let us give a cardinal number $1 \leq n \leq \aleph_0$ and a Borel set α on the unit circle $\mathbb{T}$. If there exist transformations $Y \in \mathcal{L}(S_n, T)$ and $Z \in \mathcal{L}(T, \widetilde{M}_{n,\alpha})$ with the property $ZY = \tilde{J}_{n,\alpha}$, then α is a.e. contained in $\rho_{T,n}$, that in $m(\alpha \backslash \rho_{T,n}) = 0$.

Proof. By the universal property of (X_T, W_T) there exists a unique transformation

$L \in \mathcal{L}\ (W_T, \widetilde{M}_{n,\alpha})$ such that $Z = LX_T$. Since W_T and $\widetilde{M}_{n,\alpha}$ are unitaries it follows that $L \in \mathcal{L}\ (W_T^*, \widetilde{M}_{n,\alpha}^*)$ is also true. Hence the subspace $\mathfrak{K}_1 := kerL$ is reducing for W_T , that is $W_T = W_0 \oplus W_1$ in the decomposition $\mathfrak{K}_T = \mathfrak{K}_0 \oplus \mathfrak{K}_1$. Taking into account that $L\mathfrak{K}_T \supset Z\mathfrak{H} \supset J_{n,\alpha}H^2(\mathfrak{G}_n)$ and that $(L\mathfrak{K}_T)^-$ reduces $\widetilde{M}_{n,\alpha}$, we infer that L has dense range. Considering the polar decomposition $L = V_L|L|$ of L, one can easily check that the unitary transformation $V_0 := V_L|\mathfrak{K}_0 \in \mathcal{L}(\mathfrak{K}_0, L^2(\mathfrak{G}_n, \alpha))$ intertwines W_0 with $\widetilde{M}_{n,\alpha}$ (see the proof of [19]).

Let us assume that W_1 is unitarily equivalent to the model operator $\bigoplus_{k\in\mathbb{N}} M_{\beta_k}$, where $\{\beta_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of Borel subsets of $\mathbb{T}$. Then it is easy to verify the following unitary equivalence relations:

$$W_T \simeq \widetilde{M}_{n,\alpha} \oplus \left(\bigoplus_{k\in N} M_{\beta_k}\right) \simeq \bigoplus_{k\in N} M_{\beta_k} ,$$

where the decreasing sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset B_{\mathbb{T}}$ is defined by $\alpha_k := \alpha \cup \beta_k$ for all $k \in \mathbb{N}$ if $n = \aleph_0$, while in the case $n < \aleph_0$ we have

$$\alpha_k := \begin{cases} \alpha \cup \beta_k & for\ 1 \le k \le n, \\ \beta_k \cup (\beta_{k-n} \cap \alpha) & for\ n < k. \end{cases}$$

(See e.g. the proof of [19].) We conclude that $\mu_T(\zeta) \ge n$ holds for a.e. $\zeta \in \alpha$, and so $m(\alpha \setminus \rho_{T,n}) = 0$.

If T is an a.c. contraction then, for any $1 \le n \le \aleph_0$ and $\varepsilon > 0$, let $y(T, n, \varepsilon)$ stand for the set of those mappings $Y \in \mathcal{L}\ (S_n, T)$ which satisfy the conditions

$$ZY = J_{T,n}, \qquad \|Y\| < 1 + \varepsilon, \qquad \|Z\| < 1 + \varepsilon$$

with an appropriate $Z \in \mathcal{L}\ (T, M_{T,n})$ (depending on Y). We know that $J_{T,n}$ is one-to-one, and then so is every $Y \in y(T, n, \varepsilon)$, whenever $m(\rho_{T,n}) > 0$. The following proposition states that the ranges of the transformations in $y(T, n, \varepsilon)$ together span the whole space of T . (Though a modified version of this section is contained in [14], we present here a more streamlined discussion for the sake of completeness.)

Proposition (6.1.7)[12]: Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$, and let us assume that $m(\rho_{T,n}) > 0$ holds for a cardinal number $1 \le n \le \aleph_0$.

(a) For every $\varepsilon > 0$ we have

$$\bigvee \{Y H^2(\mathfrak{G}_n): Y \in y(T, n, \varepsilon)\} = \mathfrak{H}$$

(b) If $n = \aleph_0$ then, for every $\varepsilon > 0$, there exist $Y, \hat{Y} \in y(T, n, \varepsilon)$ such that

$$YH^2(\mathfrak{G}_n) \vee \hat{Y}H^2(\mathfrak{G}_n) = \mathfrak{H}.$$

Proof. It is sufficient to verify the statement for the model operator $T = S(\Theta)$. (See the analogous reduction in the proof of Theorem(6.1.1).)

For every $0 \le i < n$, setting an arbitrary vector $e_{*i} \in \mathfrak{E}_*$ with $\|e_{*i}\| \le \sqrt{2}/2$ a vector-valued function $v_i \in (\Delta L^2(\mathfrak{E}))$ with $[v_i] \le \sqrt{2}/2$, a non-negative integer k_i and an integer L_i, let us consider the vectors

$$\hat{u}_i := \chi^{k_i} e_{*i} \in H^2(\mathfrak{E}_*) \text{ and } \hat{v}_i := \chi^{l_i} v_i \in (\Delta L^2(\mathfrak{E}))^-,$$

where $\chi(\zeta) = \zeta\ (\zeta \in \mathbb{T})$. It is clear that $\hat{u}_i \oplus \hat{v}_i \in \mathfrak{K}_+$ and $[\hat{u}_i \oplus \hat{v}_i] \le 1\ (0 \le i < n)$.

Fixing a positive ε, let us give the constants $0 < \delta < c < 1$ and the sequences $\{\delta_i\}_{0\le i<n}$,

$\{\eta_i\}_{0\le i<n}$ as in the proof of Proposition(6.1.4). For every $0 \le i < n,$ one can choose positive numbers $\hat{\eta}_i$ and ε_i so that

$$\eta_i < \hat{\eta}_i < 1 \text{ and } \eta_i < (1 - \varepsilon_i)\hat{\eta}_i - \varepsilon_i .$$

Let $\{u_i^{\#} \oplus \psi_i v_i\}_{0\le i<n} \subset \mathfrak{K}_+$ be a sequence satisfying the condition

$$(\eta_i <)\, \hat{\eta}_i \le \left|\left[u_i^{\#} \oplus \psi_i v_i, u_i \oplus v_i\right]\right| \le \left[u_i^{\#} \oplus \psi_i v_i\right] \equiv 1 \qquad (0 \le i < n)$$

instead of (22). Since $[\,(1 - \varepsilon_i)\left(u_i^{\#} \oplus \psi_i v_i\right) + \varepsilon_i(\hat{u}_i \oplus \hat{v}_i)] \le 1$
and

$$\eta_i < (1-\varepsilon_i)\hat{\eta}_i - \varepsilon_i \le \left|\left[(1 - \varepsilon_i)(u_i^{\#} \oplus \psi_i v_i) + \varepsilon_i(\hat{u}_i \oplus \hat{v}_i), u_i \oplus v_i\right]\right| \le 1$$

we infer that

$$\begin{aligned}(1 - \varepsilon_i)(u_i^{\#}(\zeta) \oplus \psi_i(\zeta)v_i(\zeta)) + \varepsilon_i(\hat{u}_i(\zeta) \oplus \hat{v}_i(\zeta))\\ = \hat{c}_i(\zeta)u_i(\zeta) \oplus v_i(\zeta) + \hat{r}_i(\zeta) \oplus \hat{s}_i(\zeta)\end{aligned}$$

holds for every $\zeta \in \mathbb{T}$ and $0 \le i < n$, where
$c \le \eta_i \le |\hat{c}_i(\zeta)| \le 1$ and

$$\|\hat{r}_i(\zeta) \oplus \hat{s}_i(\zeta)\| \le (1-\eta_i^2)^{1/2} \le \delta_i$$

We note that the relations (22)–(25) are also valid. The procedure described in the proof of Proposition (6.1.4) yields transformations Y and $\hat{Y}$ in $y(T,n,\varepsilon)$ such that

$$Y(\chi_{\mathbb{T}} g_i) = P_+\left(u_i^{\#} \oplus \psi_i v_i\right)$$

and

$$\hat{Y}(\chi_{\mathbb{T}} g_i) = (1- \varepsilon_i)P_+\left(u_i^{\#} \oplus \psi_i v_i\right) + \varepsilon_i P_+(\hat{u}_i \oplus \hat{v}_i)$$

are true for every $0 \le i < n$. Thus

$$\hat{Y}(\chi_{\mathbb{T}} g_i) - (1-\varepsilon_i)Y(\chi_{\mathbb{T}} g_i) = \varepsilon_i P_+(\hat{u}_i \oplus \hat{v}_i) \qquad (0 \le i < n).$$

If $n = \aleph_0$ $t$hen we can choose the sequence $\{\hat{u}_i \oplus \hat{v}_i\}_{0\le i<n}$ to be total in $\mathfrak{K}_+$, and so (b) is obviously fulfilled. If $n < \aleph_0$ then a sequence of finite sequences

$$\left\{\{\hat{u}_i^{(k)} \oplus \hat{v}_i^{(k)}\}_{0\le i<n} : k \in \mathbb{N}\right\}$$

can be chosen to be total in $\mathfrak{K}_+$,. Denoting by $\hat{Y}_k$ $(k \in \mathbb{N})$ the resulting transformations in $y(T,n,\varepsilon)$, we obtain that the subspaces $\{\hat{Y}H^2(\mathfrak{G}_n) \vee \hat{Y}_k H^2(\mathfrak{G}_n)\}_{k\in\mathbb{N}}$ together span the whole space $\mathfrak{H}$, which proves (a).
If $\rho_{T,n}$ coincides with the whole circle $\mathbb{T}$, or more precisely, if $m(\rho_{T,n}) = 1 (= m(\mathbb{T}))$ then the embedding $J_{T,n}$ is an isometry, and so the conditions

$$ZY = J_{T,n}, \qquad \|Y\| < 1 + \varepsilon, \qquad \|Z\| < 1 + \varepsilon$$

imply that

$$\Lambda(Y) > (1+\varepsilon)^{-1}$$

Therefore, the restriction $T\,|YH^2(\mathfrak{G}_n)$ is similar to the unilateral shift S_n, and the intertwining affinity $Y_0 \in \mathcal{L}(S_n, T\,|YH^2(\mathfrak{G}_n))$, defined by $Y_0 g := Yg$, is close to unitary if ε is small. For an a.c. contraction T , for any $1 \le n \le \aleph_0$ and $\varepsilon > 0$, Lat(T,n,ε) stands for the set of those invariant subspaces $\mathfrak{M}$ of T , where the restriction $T\,|\mathfrak{M}$ is similar to S_n, and the similarity can be implemented by an affinity $Q \in \mathcal{L}(S_n, T\,|\mathfrak{M})$ with the properties

$$(1+\varepsilon)^{-1} < \Lambda(Q) \le \|Q\| < 1 + \varepsilon.$$

We have seen that

$$\{\, YH^2(\mathfrak{G}_n) : Y \in y(T,n,\varepsilon)\} \subset Lat(T,n,\varepsilon) \qquad (52)$$

provided $m(\rho_{T,n}) = 1$. In view of Proposition(6.1.7) and (52) we obtain the following statement.

Theorem (6.1.8)[12]: Let T be an a.c. contraction on the Hilbert space $\mathfrak{H}$, and let us assume that $m(\rho_{T,n}) = 1$ holds for a cardinal number $1 \leq n \leq \aleph_0$.

(a) For every $\varepsilon > 0$, the subspaces in Lat(T, n, ε) span the whole space $\mathfrak{H}$:

$$\bigvee Lat(T, n, \varepsilon) = \mathfrak{H}.$$

(b) If n=ℵ0 then, for every $\varepsilon > 0$, there exist two subspaces $\mathfrak{M}, \widehat{\mathfrak{M}} \in Lat(T, n, \varepsilon)$ such that

$$\mathfrak{M} \vee \widehat{\mathfrak{M}} = \mathfrak{H}.$$

We recall that the hyperinvariant subspace lattice Hlat$A$ of an operator $A \in \mathcal{L}(\mathfrak{A})$ consists of those subspaces which are invariant for every operator in the commutant $\{A\}'$ of A. The hyperinvariant subspace problem asks whether every Hilbert space operator $A \in \mathcal{L}(\mathfrak{A})$, which is not scalar multiple of the identity, has a proper hyperinvariant subspace, that is $Hlat A \neq \{\{0\}, \mathfrak{A}\}$ holds. The positive answer is known only under additional assumptions, for example, in the class of normal operators because of the Spectral Theorem, or in the class of compact operators by the celebrated Lomonosov theorem (see e.g. [4]). Existence of proper hyperinvariant subspaces was proved in [16] under an orbit condition for asymptotically non-vanishing operators of regular norm-sequence.

Let T be an arbitrary asymptotically non-vanishing contraction on the Hilbert space $\mathfrak{H}$. It is known that T can be decomposed into the orthogonal sum $T = T_a \oplus U_s$ of an a.c. contraction T_a and a singular unitary operator U_s . Taking into account that the minimal unitary dilation of T_a is a.c., we infer by the Lifting theorem (see [19]) that the intertwining sets $\mathcal{L}\,(T_a, U_s)$ and $\mathcal{L}\,(U_s, T_a)$ consist only of the zero transformation. Hence the commutant of T splits into the direct sum of the commutants of T_a and U_s: $\{T\}' = \{T_a\}' \oplus \{U_s\}'$, and then the same is true for the hyperinvariant subspace lattices too: $Hlat\, T = Hlat\, T_a \oplus Hlat U_s$. Thus, in the quest for proper hyperinvariant subspaces we may assume that the asymptotically non-vanishing contraction T is absolutely continuous.

Let us consider the residual set $\rho_T = \rho_{T,1}$ of T . Since ρ_T is of positive Lebesgue measure, there exists a point $\zeta_0 \in \mathbb{T}$ which is of full density for ρ_T . Replacing T by $\bar{\zeta}_0 T$, we may assume that $\zeta_0 = 1$. (We recall that $lim_{n\to\infty} m(E_n \cap \rho_T)/m(E_n) = 1$ holds whenever the sequence $\{E_n\}_{n=1}^{\infty} \subset B_{\mathbb{T}}$ shrinks to 1 nicely, see [18].) Let us consider the singular inner function $\vartheta \in H^\infty$ defined by

$$\vartheta(\lambda) = exp(\lambda + 1)/(\lambda - 1) \qquad (\lambda \in \mathbb{D}),$$

and let us form the operator $A := \vartheta(T)$. We know from [19] that A is also an a.c. contraction. Furthermore, by [17] the residual set of A is $\rho_A = \vartheta(\rho_T)$. The following lemma ensures us that ρ_A essentially covers the whole circle $\mathbb{T}$.

Lemma (6.1.9)[12]: If the point $1 \in \mathbb{T}$ is of full density for the set $\alpha \in B_{\mathbb{T}}$, then $m(\vartheta(\alpha)) = 1$.

Proof. Notice that ϑ is analytic on $\mathbb{C}\setminus\{1\}$ and

$$\vartheta(e^{it}) = exp[-i\, cot(t/2)], \qquad t \in (0, 2\pi).$$

For any integer $n \in \mathbb{Z}$, let $t_n \in (0, 2\pi)$ be defined by $cot\left(\frac{t_n}{2}\right) = 1 + n.2\pi$. It is clear that $\mathbb{T}\setminus\{1\}$ is the union of the disjoint arcs $\omega_n := \{e^{it} : t_{n+1} < t \leq t_n\}$ $(n \in \mathbb{Z})$, and that $\vartheta_n := \vartheta|\omega_n : \omega_n \to \mathbb{T}$ is a continuous bijection for every $n \in \mathbb{Z}$. So the set $\vartheta(\alpha) = \bigcup_{n\in\mathbb{Z}} \vartheta_n\,(\alpha \cap \omega_n)$ is measurable.
Let us consider the complement $\gamma = T\setminus\vartheta(\alpha)$ and the Borel sets $\beta_n = \vartheta_n^{-1}(\gamma)$ $(n \in \mathbb{N})$. Taking into account that for any $(0 <)t_{n+1} < s_1 < s_2 \leq t_n (< t_0 = \pi/2)$ the inequality

$$cot(s_1/2) - cot(s_2/2) = (sin\, s_*)^{-2}(s_2/2 - s_1/2) \leq 8t_{n+1}^{-2}(s_2 - s_1)$$

is valid, we can easily infer that

$$m(\beta_n) \geq (m(\gamma)/8)t_{n+1}^2 \qquad (n \in \mathbb{N}). \qquad (53)$$

Since the arcs $\tilde{\omega}_n := \bigcup_{k=n}^{\infty} \omega_k = \{e^{it} : 0 < t \leq t_n\}$ $(n \in \mathbb{N})$ shrink to 1 nicely,

$$\lim_{n\to\infty} \frac{m(\tilde{\omega}_n \cap \alpha)}{m(\tilde{\omega}_n)} = 1 \qquad (54)$$

must hold by the assumption. On the other hand, in view of (53) we have

$$\frac{m(\tilde{\omega}_n \cap \alpha)}{m(\tilde{\omega}_n)} = \frac{1}{m(\tilde{\omega}_n)} \sum_{k=n}^{\infty} m(\omega_k \cap \alpha) \leq 1 - \frac{1}{m(\tilde{\omega}_n)} \sum_{k=n}^{\infty} m(\beta_k) \leq$$
$$1 - \frac{m(\gamma)}{2t_n} \sum_{k=n+1}^{\infty} t_k^2 \qquad (n \in \mathbb{N}) . \qquad (55)$$

Starting from the inequalities $1/(2s) \leq cot\, s \leq 2/s$ $(s \in (0, \pi/4))$, one can easily derive that

$$1/(8n) \leq t_n \leq 1/n \qquad (n \in \mathbb{N}) \qquad (56)$$

The relations (55) and (56) together imply

$$\frac{m(\tilde{\omega}_n \cap \alpha)}{m(\tilde{\omega}_n)} \leq 1 - \frac{m(\gamma)}{128} \cdot n \sum_{k=n+1}^{\infty} k^{-2} \leq 1 - \frac{m(\gamma)}{128}\frac{n}{n+1} \qquad (n \in \qquad (57)$$

Tending n to infinity in (57), we conclude that $m(\gamma) \leq 0$, that is $m(\gamma) = 0$. Thus Lemma(6.1.9) yields that $m(\rho_A) = 1$. For every $r \in (0, 1)$, we set $\vartheta_r(\lambda) := \vartheta(r\lambda)$ $(\lambda \in \mathbb{D})$.
Since $\vartheta_r(T)$ is the norm-limit of polynomials of T , and $\vartheta_r(T)$ converges to $\vartheta(T)$ in the strong operator topology as r tends to 1, we obtain that every operator commuting with T will commute with $A = \vartheta(T)$ as well. Therefore $\{T\}' \subset$, w$\{A\}'$ hence

$$Hlat\, T \supset Hlat A$$

follows. Let us form the inflation $B = A^{(\aleph_0)}$ of A acting on the orthogonal sum $\mathfrak{H}^{(\aleph_0)}$ of infinitely many copies of $\mathfrak{H}$. Clearly, B is an a.c. contraction with $m(\rho_B, \aleph_0) = 1$. Furthermore, it can be easily verified that

$$HlatB = \{\mathfrak{M}^{(\aleph_0)} : \mathfrak{M} \in HlatA\}.$$

Thus $Hlat\, T$ contains a sublattice which is isomorphic to $HlatB$, and so we have arrived at the following reduction theorem.
Theorem (6.1.10)[12]: If every absolutely continuous contraction B with $m(\rho_B, \aleph_0) = 1$ has a proper hyperinvariant subspace, then so does every asymptotically non-vanishing (non-scalar) contraction T too.

We note that the subspace $ker\, X_T$ of vectors with vanishing orbits is clearly hyperinvariant for the asymptotically non-vanishing contraction T on $\mathfrak{H}$. Hence we may assume that $kerX_T$, and $kerX_{T^*}$ as well, are trivial subspaces. Since $kerX_T \neq= \mathfrak{H}$ we obtain that $kerX_T = \{0\}$, and so T is a $C_{1.}$-contraction according to the classification in

[19, Section II.4]. If $kerX_{T^*} = \{0\}$ holds true also, that is when T is a C_{11}-contraction, then a subset of Hlat T is isomorphic to $HlatW_T$ by [19]. (For a more complete description of the hyperinvariant subspace lattices of C_{11}-contractions we refer to [19], [10],[13].) Thus we may assume that $kerX_{T^*} = \mathfrak{H}$, and so T is a C_{10}-contraction. Then $A = \vartheta(T)$ is also a C_{10}-contraction (see[11]), and so is $B = A^{(\aleph_0)}$ too. Therefore, we can concentrate on C_{10}-contractions.

Section(6.2): Contractions and Function Algebras

Let $\mathcal{H}$ be an infinite dimensional, separable, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$stand for the algebra of bounded, linear operators acting on $\mathcal{H}$. With an operator $T \in \mathcal{L}(\mathcal{H})$ the following operator algebras can be naturally associated. The set of (analytic) polynomials $p(T)$ of T is denoted by $W_0(T)$, while the set of rational functions $q(T)$ of T with poles off the spectrum $\sigma(T)$ is denoted by $R_0(T)$. The closures of these unital algebras in the weak operator topology (coinciding with the closures in the strong operator topology) are $W(T)$ and $\mathfrak{R}(T)$, respectively. The commutant $\{T\}'$ of T consists of those operators C in $\mathcal{L}(\mathcal{H})$, which commute with T: $TC = CT$. A subspace (i.e., closed linear manifold) $\mathcal{M}$ of $\mathcal{H}$ is invariant for T, if $Tx \in \mathcal{M}$ holds for every $x \in \mathcal{M}$. The set Lat T of all invariant subspaces of T forms a complete lattice. The trivial subspaces $\{0\}$ and $\mathcal{H}$ clearly belong to Lat T. For a non-empty set $A \subset \mathcal{L}(\mathcal{H})$f operators, Lat $A := T\{LatA : A \in \mathcal{A}\}$ is the lattice of common invariant subspaces. Since the operator algebras $W(T), \mathcal{R}(T), \{T\}'$ form an increasing sequence, the corresponding invariant subspace

$$Lat\ T = LatW(T), \mathcal{R}lat\ T := Lat\mathcal{R}(T), Hlat\ T := Lat\{T\}'$$

form a decreasing sequence. The invariant subspace problem (ISP) asks whether Lat T is non-trivial (i.e., different from $\{\{0\}, \mathcal{H}\}$) for every operator $T \in \mathcal{L}(\mathcal{H})$. The hyperinvariant subspace problem (HSP) asks whether $Hlat\ T$ is non-trivial for every operator $T \in \mathcal{L}(\mathcal{H})$ which is not scalar multiple of the identity. These are arguably the most challenging open questions in operator theory. Since multiplication of T by a non-zero scalar does not alter the associated algebras, studying these questions we may assume that the operator T is a contraction: $\|T\| \leq 1$. We recall that contractions can be classified according to the asymptotic behaviour of their iterates and the iterates of their adjoints. Namely, T is of class $C_0\cdot$ if T is stable, that is $lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in \mathcal{H}$.

The contraction T is of class $C_1\cdot$ if, on the contrary, T is asymptotically nonvanishing, that is $lim_{n\to\infty} \|T^n x\| > 0$ for every $x \in \mathcal{H} \setminus \{0\}$. We say that T is of class $C_{\cdot j}$, if the adjoint T^* of T is of class $C_j\cdot$ $(j = 0, 1)$. Finally, T is of class C_{ij} if T simultaneously belongs to the classes $C_i\cdot$ and $C_{\cdot j}$ $(i, j = 0, 1)$.

Every operator T with $\|T\| < 1$ is obviously a C_{00}-contraction. Hence $(ISP)$and (HSP) in the class of C_{00}-contractions are equivalent to the general problems.

On the other hand, (HSP) has been settled affirmatively in the class of C_{11}-contractions (see [15]). Taking into account that the subspace $\{\mathcal{H}_0(T) := x \in \mathcal{H} : lim_{n\to\infty} \|T^n x\| = 0\}$ is hyperinvariant for T (i.e., $\mathcal{H}_0(T)$ belongs to Hlat T), we can reduce (ISP) and (HSP) concerning non-C_{00}- contractions to the class of C_{10}-contractions. Our aim in this note is to get closer to the solution of the (HSP) for C_{10}-contractions in the cyclic case. We recall that the operator $T \in \mathcal{L}(\mathcal{H})$ is cyclic if there exists a vector $x \in \mathcal{H}$ such that its orbit $\{T^n x\}_{n=0}^{\infty}$ spans the whole space $\mathcal{H}$.

The advantage of considering C_{10}-contractions is shown by their connection with unitary operators, described below. If A and B are operators on the Hilbert spaces $\mathcal{H}$ and K, respectively, then the intertwining set $\mathcal{L}(A,B)$ consists of those bounded, linear transformations $Y \in \mathcal{L}(\mathcal{H},\mathrm{K})$ which satisfy the condition $YA = BY$. Setting a contraction $T \in \mathcal{L}(\mathcal{H})$, we say that (X,V) is an intertwining pair for T, if V is a unitary operator on a separable Hilbert space K and $X \in \mathcal{L}\,(T,V)$. The pair (X,V) is called contractive, if $\|X\| \leq 1$. There exists a contractive intertwining pair (X,V) for T, which is universal in the sense that given any contractive intertwining pair (X',V') there exists a unique contractive $Y \in \mathcal{L}\ (V,V')$ such that $X' = YX$. Such a universal contractive pair (X,V) is called a unitary asymptote of T. (Sometimes plainly V is called a unitary asymptote of T, when the existence of X is tacitly meant.) This pair is uniquely determined up to isomorphism. Namely, if (X_1,V_1) and (X_2,V_2) are unitary asymptotes of T, then there exists a unitary transformation $Z \in \mathcal{L}\,(V_1,V_2)$ such that $ZX_1 = X_2$.
Let us assume that (X,V) is a unitary asymptote of T. Then, given any intertwining pair (X',V') for T, there exists a unique mapping $Y \in \mathcal{L}\,(V,V')$ such that $X' = YX$; in addition $\|Y\| = \|X'\|$ Furthermore, for every $C \in \{T\}'$ there exists a unique $D \in \{V\}'$ satisfying the condition $XC = DX$. The transformation $\gamma: \{T\}' \to \{V\}', C \mapsto D$ is a contractive, unital algebra-homomorphism.
We know that $\|Xh\| = lim_{n\to\infty} \|T^n h\|$ holds for every $h \in \mathcal{H}$; hence $kerX = \mathcal{H}_0(T)$. Furthermore, we have $\bigvee_{n\in\mathbb{N}} V^{-n} X\mathcal{H} = K. Thus\ K = \{0\}$ exactly when $X = 0$, and this happens if and only if T is a $C_0\cdot$-contraction. On the other hand, X is injective precisely when T is of class $C_1.$. (For details we refer to [4] and [24].)
The contraction T can be uniquely decomposed into the orthogonal sum $T = T_0 \oplus T_1$, where T_0 is a completely non-unitary $(c.n.u.)$ contraction and T_1 is a unitary operator. We assume in the sequel that the contraction T is absolutely continuous (a.c.), that is, its unitary part $T_1$is an a.c. unitary operator. The latter means that the spectral measure of T_1 is a.c. with respect to the normalized Lebesgue measure m on the unit circle $\mathbb{T}$. Since the unitary asymptote V is unitarily equivalent to the $*$-residual part of the minimal unitary dilation of T, we infer that V is an a.c. unitary operator. (See [29], or [1] for a direct proof.) Therefore V is determined up to unitary equivalence by its spectral multiplicity function $\delta V : \mathbb{T} \to \mathbb{N} \cup \{0,\infty\}$. For any $n \in \mathbb{N}$, we consider the measurable set $\omega(V,n) := \{\zeta \in \mathbb{T} : \delta V\,(\zeta) \geq n\}$. The Borel set $\omega(V) := \omega(V,1)$ supports the spectral measure of V. The residual set of T is defined by $\omega(T) := \omega(V)$, and is determined up to sets of zero Lebesgue measure.
We note that (ISP) is answered affirmatively, actually Lat T has a rich structure with infinitely many invariant subspaces, if $\omega(T) = \mathbb{T}$ (see [13]).
The definition of another characteristic set associated with the a.c. contraction $T \in \mathcal{L}(\mathcal{H})$ relies on the Sz.-Nagy–Foias functional calculus. For any $p \in [1,\infty]$, the Hardy space H^p can be identified with the subspace of functions with vanishing Fourier coefficients of negative indices in $L^p := L^p(m)$ (see [9]). The aforementioned calculus for T is the uniquely determined contractive, unital algebra homomorphism

$$\Phi_T : H^\infty \to \mathcal{L}(\mathcal{H}), f \longmapsto f\,(T),$$

which is continuous in the weak- $*$ topologies, and which transforms the identical function $\chi(\xi) = \xi$ into T (see [34]). We can introduce partial ordering relations on H^∞

and $\mathcal{L}(\mathcal{H})$in the following way. For $f, g \in H^{\infty}$, the relation $f \prec g$ holds if $|f(z)| \leq |g(z)|$ for every z in the open unit disc $\mathbb{D}$. For $A, B \in \mathcal{L}(\mathcal{H})$ the relation $A \prec B$ holds if $\|Ax\| \leq \|Bx\|$ for every $x \in \mathcal{H}$.

It is easy to see that Φ_T is monotone with respect to these relations. (Note that $f \prec g$ yields $f = gh$ with $h \in H^{\infty}$, $\|h\|_{\infty} \leq 1$.) Setting any decreasing sequence $F = \{f_n\}_{n=1}^{\infty}$ in H^{∞}, let us consider the limit function $\varphi_F(\xi) := lim_{n\to\infty} |f_n(\xi)|$ defined for almost every$\zeta \in \mathbb{T}$,and the hyperinvariant subspace$\mathcal{H}_0(T, F) := \{x \in \mathcal{H} : \lim_{n\to\infty}\|f_n(T)x\| = 0$.

We say that the a.c. contraction T is non-vanishing on the Borel set α of $\mathbb{T}$, if $\mathcal{H}_0(T, F) = \{0\}$ whenever the decreasing sequence F is non-vanishing on α, i.e., whenever $\{\zeta \in \alpha: \varphi_F(\xi) > 0\}$ is of positive Lebesgue measure. There exists a largest Borel set with this property, called the quasianalytic spectral set of T, and denoted by $\pi(T)$. Notice that $\pi(T)$ is determined up to sets of measure zero.

For any Borel subsets α, β of $\mathbb{T}$, it is reasonable to use the notation $\alpha \subset \beta, \alpha = \beta$ or $\alpha \neq \beta$ subsequently in the broader sense that $m(\alpha\backslash\beta) = 0, m(\alpha \vartriangle \beta) = 0$ or $m(\alpha \vartriangle \beta) > 0$, respectively (reflecting the corresponding relations between the characteristic functions χ_α and χ_β, as elements of L^{∞}). If $\pi(T) \neq \emptyset$, then considering the decreasing sequence $\{\chi^n\}_{n=1}^{\infty}$ it can be seen that T is a $C_1\cdot$-contraction. It was shown in [25] that $\pi(T) \subset \omega(T)$ always holds; furthermore, $\pi(T) \neq \omega(T)$ implies that $Hlat\, T$ is non-trivial. The a.c. contraction T is called quasianalytic, if $\pi(T) = \omega(T)$. We conclude that (HSP) in C_{10} can be reduced to the quasianalytic case. In this section we study cyclic, quasianalytic C_{10}contractions.

Our work is organized as follows. In this Section spectral mapping theorems for the residual set and for the quasianalytic spectral set are proved, extending and sharpening earlier results in [25]. The question concerning uniform spectral multiplicity on the quasianalytic spectral set, posed in [25], is answered negatively.

Cyclic, quasianalytic C_{10}-contractions are related to the particular ones, where the quasianalytic spectral set covers the whole circle $\mathbb{T}$. This special class is the subject of study in the remaining sections. The commutant $\{T\}'$ is connected with a quasianalytic function algebra $\mathcal{F}(T)$, located between H^{∞} and L^{∞}. The functional calculus Φ_T is extended from H^{∞} to the broader set $\mathcal{F}(T)$. The effect of M¨obius transformation is examined, and spectral relations are proved. Finally, Section deals with characterization of the cases, when $\mathcal{F}(T)$ is a certain kind of generalized Douglas algebra.

Let $\mathcal{B}_{\mathbb{T}}$ denote the σ-algebra of Borel sets on $\mathbb{T}$. Assume that $\omega_0 \in \mathcal{B}_{\mathbb{T}}$ is of positive measure and $h: \omega_0 \to \mathbb{T}$ is a Borel measurable function. Consider the Lebesgue decomposition $\mu_h = \mu_{h,a} + \mu_{h,s}$ of the induced measure $\mu_h(\omega) := m(h^{-1}(\omega))$ $(\omega \in \mathcal{B}_{\mathbb{T}})$. Taking the Radon–Nikodym derivative $g_h = d\mu_{h,a}/dm$ of the a.c. component, the Borel set ω_h is defined by $\omega_h := \{\zeta \in \mathbb{T} : g_h(\zeta) > 0\}$.

The set ω_h is determined up to sets of zero Lebesgue measure, and is called the properly essential range of h. We also use the notation $pe - ranh := \omega_h$.It is known that $lim_{r\to 0+} m(h^{-1}(D(\zeta, r))/r > 0$ for almos every $\zeta \in \omega_h$,and $lim_{r\to 0+} m(h^{-1}(D(\zeta, r))/r = 0$ for almost every $\zeta \in \mathbb{T} \setminus \omega_h$. (Here $D(\zeta, r) := \{\zeta \cdot e^{2\pi i t} : t \in \mathbb{R}, |t| \leq r/2\}$.) Clearly, ω_h is contained in the essential range of h, which is the complement of the largest open set Λ on $\mathbb{T}$ satisfying the condition $\mu_h(\Lambda) = 0$. If $\omega_0 = \emptyset$ (in the broader sense), then pe-ran $h = \emptyset$.

Let V be an a.c. unitary operator acting on the infinite dimensional, separable Hilbert space K. Notice that (I, V) is a unitary asymptote of V, and so the residual set $\omega(V)$ is the support of the spectral measure of V. The Borel set $\omega(V)$ has positive measure, since the space K is non-zero.

The function $h \in H^\infty$ is called partially inner, if $\|h\|_\infty = 1, |h(0)| < 1$, and if the Borel set $\Omega(h) := \{\zeta \in \mathbb{T} : |h(\zeta)| = 1\}$ is of positive measure.

Lemma(6. 2.1)[13]: Setting V and h as above, and assuming $\omega(V) \subset \Omega(h)$, let us consider the operator $W := h(V)$.

(a) W is an a.c. unitary operator with $\omega(W) = pe - ran(h|\omega(V))$.

(b) If $\mathcal{M} \in LatV$ and $\bigvee_{n\in\mathbb{N}} V^{-n}\mathcal{M} = K$, then

$\mathcal{M} \in LatW$ and $\bigvee_{n\in\mathbb{N}} W^{-n}\mathcal{M} = K$.

Proof. (a): Recall that the operator $h(V)$ given by the $Sz.-Nagy-Foias$ calculus coincides with the operator yielded by the spectral measure $E: \mathcal{B}_\mathbb{T} \to \mathcal{P}(K)$ of V: $h(V) = \int_{\omega(V)} hdE$.

(Here $\mathcal{P}(K)$ stands for the set of orthogonal projections acting on K.) Since h is unimodular on $\omega(V)$, it follows that W is unitary. The a.c. unitary operator V is similar to a $c.n.u.$ contraction Q (see [29]). Hence W is similar to $h(Q)$. Since the contraction $h(Q)$ is also $c.n.u.$, we conclude that W is an a.c. unitary operator (see [29]). Setting $h_0 := h|\omega(V)$, the formula $\grave{E}(\omega) := E(h_0^{-1}(\omega))$ $(\omega \in \mathcal{B}_\mathbb{T})$ clearly defines a spectral measure. Since

$$\int_\mathbb{T} \chi\, dE' = \int_\mathbb{T} hdE = W,$$

we infer that E' must be the spectral measure of W. The measure $\chi_\omega(W)$dm is obviously equivalent (mutually $a.c.$) to μ_{h_0} . Hence μ_{h_0} is $a.c.$, and so it is equivalent to $\chi_{\omega h_0} dm$. Therefore $\omega(W) = pe - ranh_0$.

(b): It is evident that $LatV$ is contained in $LatW$. Suppose that the subspace $\mathcal{M}$ is invariant for V, and the smallest reducing subspace of V containing$\mathcal{M}$is K. If $V|\mathcal{M}$ is unitary, then $\mathcal{M}$ is reducing for V, and so $\mathcal{M} = K$. Thus, we may assume that the restriction $V|\mathcal{M}$ is a non-unitary isometry. Let us consider the Wold decomposition $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$, where $V_0 := V|\mathcal{M}_0$ is a unilateral shift and $V_1 := V|\mathcal{M}_1$ is unitary. Since $\mathcal{M}_0$ is non-zero, we infer that $\mathbb{T} = \omega(V) \subset \Omega(h)$, and so h is a non-constant inner function. By the assumption, $K = \bigvee_{n\in\mathbb{N}} V^{-n}\ \mathcal{M} = \widetilde{M}_0 \oplus \mathcal{M}_1, where\ \widetilde{M}_0 = \bigvee_{n\in\mathbb{N}} V^{-n}\mathcal{M}_0$. Taking the corresponding decomposition for W, we obtain that $W|\mathcal{M} = W_0 \oplus W_1$, where $W_0 = h(V_0)$ and $W_1 = h(V_1)$ is unitary; hence $\bigvee_{n\in\mathbb{N}} W^{-n}\mathcal{M} = (\bigvee_{n\in\mathbb{N}} W^{-n}\mathcal{M}_0) \oplus M_1$. We have to show that $\bigvee_{n\in\mathbb{N}} W^{-n}\mathcal{M}_0 = \widetilde{\mathcal{M}}_0$. Considering the functional model of the unilateral shifts, it is enough to verify that $\bigvee_{n\in\mathbb{N}} \bar{h}^n H^n 2 = L^2$. Setting $a := h(0) \in \mathbb{D}$, let us form the inner function $u := b_a \circ h$, where $b_a(z) = (z - a)/(1 - \bar{a}z)$ $(z \in \mathbb{D})$ is the M¨obius function, corresponding to a. Let $\widetilde{M}_u$ and $\widetilde{M}_h$ denote the unitary operators of multiplication by u and h, respectively, on the space L^2. In view of the relations $\widetilde{M}_u = b_a(\widetilde{M}_h)\ and\ \widetilde{M}_h = b_{-a}(\widetilde{M}_u)$ we can see that Lat $\widetilde{M}_h = Lat\ \widetilde{M}_u$. Taking into account that the subspace $L^2 \ominus (\bigvee_{n\in\mathbb{N}} \bar{h}^n H^2)$ is invariant for $\widetilde{M}_h$ and orthogonal to H^2, we can reduce the proof to show that $\bigvee_{n\in\mathbb{N}} \bar{u}^n H^2 = L^2$. However,

u has the form $u = \chi v$, where $v \in H^\infty$ is an inner function. Thus $\bar{u}^n H^2 \supset \bar{\chi}^n \, \bar{v}^n v^n H^2 = \bar{\chi}^n H^2$ holds for every $n \in \mathbb{N}$, and so $\bigvee_{n\in\mathbb{N}} \bar{u}^n \, H^2 = L^2 \supset \bigvee_{n\in\mathbb{N}} \bar{\chi}^n \, H^2 = L^2$.

The previous proof yields that the measure $\mu_h|\Omega(h)$ $is\ a.c$. Thus we obtain the following statement, which can be considered as an extension of the $F.\&M.$ Riesz Theorem (see [9]).

Corollary(6 2.2)[13]: If $h \in H^\infty$ is partially inner, then for every Borel set $\omega \in \mathcal{B}_{\mathbb{T}}$ the condition $m(\omega) = 0$ implies $m(h^{-1}(\omega) \cap \mathbb{T}) = 0$.

We say that the partially inner function $h \in H^\infty$ is regular, if for every Borel subset Ω of $\Omega(h)$ the image $h(\Omega)$ is also a Borel set on $\mathbb{T}$, and for every Borel subset $\dot{\omega}$of $h(\Omega)$ the condition $m(\dot{\omega}) > 0$ implies $m((h|\Omega)^{-1}(\dot{\omega})) > 0$. It is easy to check that in that case $pe - ran(h|\Omega) = h(\Omega)$.

Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction. Let $(X_T \,, V_T\,)$ be a unitary asymptote of T, the a.c. unitary operator $V_T$acting on the Hilbert space K_T . The spectral measure of V_T is denoted by E_T . Given a partially inner function $h \in H^\infty$we set $\omega(T,h) := \omega(T) \cap \Omega(h)$. Taking the spectral subspace $K_{T,h} := E_T\,(\omega(T,h))K_T$, we consider the unitary operator $V_{T,h} := V_T\,|K_{T,h}$ and the intertwining mapping $X_{T,h} \in \mathcal{L}\,(T, V_{T,h})$ defined by $X_{T,h}, \chi := E_T(\omega(T,h))X_T, \chi\ (x \in \mathcal{H})$. The operator $h(T)$ is also an a.c. contraction (see the proof of Lemma (6.2.1).

Theorem(6.2.3)[13]: Under the previous conditions the pair $(X_{T,h}, h(V_{T,h}))$ is a unitary asymptote of $h(T)$, and so $\omega(h(T)) = pe - ran(h|\omega(T,h))$.

Proof. For convenience we introduce the notation $A := h(T). Let$ (X_A, V_A) be a unitary asymptote of the a.c. contraction A, V_A acting on the Hilbert space K_A. Since $T \in \{A\}'$, there exists a unique operator $T_A \in \{V_A\}'$ such that $X_A T = T_A X_A$; in addition: $\|T_A\| \leq \|T\| \leq 1$. The space K_A splits into the orthogonal sum $K_A = K_0 \oplus K_1$ reducing for T_A, where $T_{A,0} := T_A|K_0$ is an a.c. contraction and $T_{A,1} := T_A|K_1$ is a singular unitary operator. Let P_1 denote the orthogonal projection onto K_1 in K_A. Since the mapping $X_{A,1}\,x := P_1 X_A\,x\ (x \in \mathcal{H})$ intertwines the a.c. contraction T with the singular unitary operator $T_{A,1}$, it follows that $X_{A,1} = 0$. Hence $X_A\mathcal{H}$ is contained in K_0. Taking into account that K_0 is hyperinvariant for T_A, we infer that $K_A = \bigvee_{n\in\mathbb{N}} V_A^{-n}\, X_A\mathcal{H}$ is included in K_0. Thus $K_A = K_0$, and so T_A is an a.c. contraction.

The equation $X_A T = T_A X_A$ yields $X_A A = X_A h(T) = h(T_A)X_A$. Since $h(T_A) \in \{V_A\}', it$ follows that $h(T_A) = V_A$. Regarding the decomposition $T_A = T'_A \oplus T''_A$, where T'_A is a c.n.u. contraction and T''_A is an a.c. unitary operator, we obtain that $h(T_A) = h(T'_A) \oplus h(T''_A\,)$, where $h(T'_A\,)$ is a $c.n.u.$ contraction. Taking into account that $h(T_A) = V_A$ is unitary, we conclude that T_A is an a.c. unitary operator.

Since $(X_T\,, V_T\,)$ is a unitary asymptote of T and since the contractive mapping X_A intertwines T with the unitary operator T_A, there exists a unique contraction $Y \in \mathcal{L}\,(V_T\,, T_A)$ such that $X_A = YX_T$. With respect to the decomposition $K_T = K_{T,h}, \oplus K'_{T\,,h}$ the a.c. unitary operator $V_T$has the form $V_T = V_{T,h} \oplus V'_{T\,,h}$. Then $h(V_T) = h(V_{T,h}) \oplus h(V'_{T\,,h})$, where $h(V_{T,h})$ is unitary and $h(V'_{T\,,h})$ is a C0·- contraction. Since $Y|K'_{T\,,h}$ intertwines $h(V'_{T\,,h})$ with the unitary operator $h(T_A)$, it follows that $Y|K'_{T\,,h} = 0$. Clearly

$$Y_h := Y|K'_{T,h} \in \mathcal{L}\,(h(V_{T,h}), h(T_A)).$$

The contractive mapping $X_{T,h}$, intertwines A with $h(V_{T,h})$, and (X_A, V_A) is a unitary asymptote of A, thus there exists a unique contraction $Z \in \mathcal{L}\,(V_A, h(V_{T,h}))$ such that $X_{T,h} = ZX_A$. Since $Y_h Z \in \{V_A\}'$ and

$$X_A = YX_T = YE_T\,(\omega(T,h))X_T = Y_h X_{T,h} = Y_h Z X_A,$$

it follows that $Y_h Z = I$. The transformations Y_h and Z being contractive, the equation $Y_h Z = I$ yields that Z is an isometry.

We know also $X_{T,h} = ZX_A = ZYX_T$, whence $ranX_{T,h} \subset ran\,Z$. The subspace $\mathcal{M} := (X_{T,h}\,\mathcal{H})-$ is invariant for the a.c. unitary operator $V_{T,h}$. The spectral subspace$K_{T,h}$is hyperinvariant for V_T, hence $\widetilde{\mathcal{M}} := \bigvee_{n\in\mathbb{N}} V_A^{-n}\,\mathcal{M}$ is contained in $K_{T,h}$. Since the subspace $\widetilde{\mathcal{M}} \oplus K'_{T,h}$ is reducing for V_T and contains the range of X_T, it follows that $\widetilde{\mathcal{M}} = K_{T,h}.If\ \omega(T,h) = \emptyset$, then $K_{T,h} = \{0\}$ and so $ZK_A = K_{T,h}.If\ \omega(T,h)\,,\emptyset$, then Lemma (6. 2.1) yields that

$$K_{T,h} = \bigvee_{n\in\mathbb{N}} h\,(V_{T,h})^{-n}\mathcal{M}.$$

Since Z intertwines the unitaries V_A and $h(V_{T,h})$, it follows that the subspace ZK_A is reducing for $h(V_{T,h})$. Taking into account that ZK_A contains $\mathcal{M}$, we conclude that $ZK_A = K_{T,h}$. Therefore Z and Y_h are unitary transformations, and so the pair $(X_{T,h}\,, h(V_{T,h}))$—being equivalent to (X_A, V_A) is a unitary asymptote of A. Applying Lemma(6. 2.1) we obtain that

$$\omega(A) = \omega(h(V_{T,h})) = pe - ran(h|\omega(T,h)).$$

Notice that if $\omega(T,h) = \emptyset$, then $K_{T,h} = \{0\}$, and since $V_{T,h}$is a unitary asymptote of $h(T)$, it follows that $h(T)$ is of class $C_{0..}$ In particular, we obtain that $h(T)$ is a $C_0\cdot$-contraction, whenever T is a $C_{0.}$-contraction.

Theorem(6. 2.3) is an improvement of [25], with a streamlined proof, completely identifying the unitary asymptote (X_A, V_A). The following results extend the statements in [25].

Theorem(6.2.4)[13]: Under the conditions of the previous theorem we have

$\pi(h(T)) \supset pe - ran(h|\pi(T,h))$,

where $\pi(T,h) := \pi(T) \cap \Omega(h)$.

Proof: We may assume that $\pi(T,h)\,,\emptyset$; then T must be a $C_{1.}$-contraction.

For convenience let us use the notation $\omega_0 := \pi(T,h), h_0 := h|\omega_0$, and $\grave{\omega}_0 := pe - ranh_0$. Assume that the decreasing sequence $F = \{f_n\}_{n=1}^{\infty}\ of\ H^{\infty}$-functions is non-vanishing on $\grave{\omega}_0$, that is, the $set\ \grave{\omega}_1 := \{\zeta \in \grave{\omega}_0 : \varphi_F\,(\xi) > 0\}$ is of positive measure. By Corollary (6.2.2) the measure $\mu_{h_0}(\omega) = m(h_0^{-1}\,(\omega))\,(\omega \in \mathcal{B}_{\mathbb{T}})$ is a.c., and so it is equivalent to$\chi_{\grave{\omega}_0}$dm. Hence $\omega_1 := h_0^{-1}(\grave{\omega}_0)$ is a subset of $\pi(T)$ with positive measure. The decreasing sequence $F \circ h = \{f_n \circ h\}_{n=1}^{\infty}$ does not vanish on $\pi(T)$, namely $\varphi_{F\circ h}(\zeta) > 0$ for almost every $\zeta \in \omega_1$. We conclude that

$$\mathcal{H}_0(h(T), F) = \mathcal{H}_0(T, F \circ h) = \{0\},$$

and so $\pi(h(T))$ contains $\grave{\omega}_0$.

Combining the previous theorems with the fact that the quasianalytic spectral set is always included in the residual set, we obtain the following result.

Corollary (6.2.5)[13]: Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction, and let $h \in H^{\infty}$ be a partially inner function. If $\omega(T,h) = \pi(T,h)$, then

$$\omega(h(T)) = \pi(h(T)) = pe - ran(h|\omega(T,h)).$$

This statement provides a tool for the construction of quasianalytic contractions, starting with some a.c. contractions with non-vanishing quasianalytic spectral sets. Notice also that if T is quasianalytic and $\mathcal{M}$ is a non-zero invariant subspace of T, then the relations $\pi(T|M) \supset \pi(T) = \omega(T) \supset \omega(T|\mathcal{M})$ yield that the restriction $T|\mathcal{M}$ is also quasianalytic, with the same quasianalytic spectral set as T. Taking cyclic subspaces, we can get examples for cyclic quasianalytic contractions.

We present example for a quasianalytic contraction with non-uniform spectral multiplicity function. (see [25].

Example(6 2.6)[13]: Let $S \in \mathcal{L}(\mathcal{H}^2)$ be the unilateral shift, defined by $Sf := \chi f$.

The $F.\&M.$ Riesz Theorem yields that $\pi(S) = \mathbb{T}$. Let h be a conformal Riemann mapping of the open unit disc $\mathbb{D}$ onto the simply connected open set $G := \left\{re^{it} : 0 < r < 1, 0 < t < \frac{3\pi}{2}\right\}$.

Since the boundary ∂G of G is a Jordan curve, we may infer, by a theorem of Charath´eodory, that h extends to a homeomorphism from the closed unit disc $\mathbb{D}^-$ onto the closure G^- of G (see [21]). Clearly $\Omega(h) = h^{-1}(\alpha)$, where $\alpha = \{e^{it} : 0 \le t \le 3\pi/2\}$. Since ∂G is rectifiable, we obtain that h is a regular partially inner function (see [21]), and so $pe - ran(h|\Omega(h)) = h(\Omega(h)) = \alpha$ By Corollary (6.2.5) the analytic Toeplitz operator $h(S)$ is quasianalytic with $\pi(h(S)) = \alpha$. Considering restriction of $h(S)$ to a cyclic subspace $\mathcal{H}$, we obtain a cyclic a.c. contraction $Q \in \mathcal{L}(\mathcal{H})$ such that $\pi(Q) = \omega(Q) = \alpha$ _. Let (X_Q, V_Q) be a unitary asymptote of Q, where $V_Q \in \mathcal{L}(K_Q)$. Since $\bigvee_{n\in\mathbb{N}} V_Q^{-n} X_Q\mathcal{H} = K_Q$, it follows that the a.c. unitary operator V_Q is also cyclic. Hence V_Q is unitarily equivalent to the operator M_α of multiplication by χ on the space $L^2(\alpha) := \chi_\alpha L^2$.

We know by Corollary(6.2.5) that the a.c. contraction $T := Q^2$ is quasianalytic with $\pi(T) = \mathbb{T}. Let\ (X_T, V_T)$ be a unitary asymptote of T. We conclude by Theorem(6.2.3) that V_T is unitarily equivalent to M_α^2: $V_T \simeq M_\alpha^2$. On the other hand, it is easy to verify that $M_\alpha^2 \simeq M_{\mathbb{T}} \oplus M_{\alpha_1}$, where $\alpha_1 = \{e^{it} : 0 \le t \le \pi\}$.

Therefore $V_T \simeq M_{\mathbb{T}} \oplus M_{\alpha_1}$, and so the spectral multiplicity function $\delta_{V_T} = 1 + \chi_{\alpha_1}$ is not constant on $\pi(T) = \mathbb{T}$.

Let $T \in \mathcal{L}(\mathcal{H})$ be a C_{10}-contraction. Then T is clearly $c.n.u.$, and so it is an a.c. contraction. Let (X, V) be a unitary asymptote of $T, V \in \mathcal{L}(\mathcal{K})$. We say that T belongs to the class $\mathcal{L}_0(\mathcal{H})$ if it is also quasianalytic, and if the unitary operator V is cyclic. Assuming $T \in \mathcal{L}_0(\mathcal{H})$, the cyclicity of V implies that the commutant $\{V\}'$ is abelian. Furthermore, the canonical algebra-homomorphism $\gamma: \{T\}' \to \{V\}'$, defined by $\gamma(C)X = XC$, is one-to-one since X is injective.

Hence the commutant $\{T\}'$ is necessarily abelian too.

We note that V is evidently cyclic when T is. However, V can be cyclic even when T is not cyclic. Indeed, example was given in [30] for a non-cyclic C_{10}- contraction T with defect indices $d_T = 1$ and $d_{T^*} = 2$ (yielding $T \in \mathcal{L}_0(\mathcal{H})$ by Proposition (6.2.8)).

Let $\mathcal{L}_1(\mathcal{H})$ denote the subclass, consisting of those operators $T \in \mathcal{L}_0(\mathcal{H})$ which satisfy the condition $\pi(T) = T$. Recall that the contractions in $\mathcal{L}_1(\mathcal{H})$have many invariant subspaces, while the (ISP) is still open in the class $\mathcal{L}_0(\mathcal{H})$.

The following result shows that the (HSP) in $\mathcal{L}_0(\mathcal{H})$ is strongly related to the (HSP) in $\mathcal{L}_1(\mathcal{H})$.

Theorem(6.2.7) [13]: If $T \in \mathcal{L}_0(\mathcal{H})$ and $\omega(T)$ contains a non-trivial closed arc α, then there exists a contraction $\tilde{T} \in \mathcal{L}_1(\mathcal{H})$ such that $\{T\}' = \{\tilde{T}^{\sim}\}'$, and so $Hlat\,T = Hlat\,\tilde{T}$.

Proof. Let ϑ_0 be a Riemann mapping from $\mathbb{D}$ onto the upper half disc $\mathbb{D}_+ := \{z \in \mathbb{D} : Imz > 0\}$. By Charath´eodory's theorem, ϑ_0 can be extended to a homeomorphism between $\mathbb{D}^-$ and $\mathbb{D}_+^-$. Setting $\lambda_1 = \vartheta_0^{-1}(1), \lambda_2 = \vartheta_0^{-1}(i)$, and $\lambda_3 = \vartheta_0^{-1}(-1)$, let us pick$\zeta_1, \zeta_2, \zeta_3 \in \mathbb{T}$ so that ζ_1, ζ_3 are the two end points of α, ζ_2 is an inner point of α, and the orientation of the triples $(\lambda_1, \lambda_2, \lambda_3)$ and $(\zeta_1, \zeta_2, \zeta_3)$ are the same on $\mathbb{T}$. There exists a unique linear fractional mapping ψ such that $\psi(\xi_j) = \lambda_j$ for $j = 1,2,3$. Clearly $\psi(\mathbb{T}) = \mathbb{T}$ and $\psi(\mathbb{D}) = \mathbb{D}$. The function $\vartheta := \vartheta_0 \circ (\psi|\mathbb{D}^-) \in H^\infty$ is a homeomorphism between $\mathbb{D}^-$and $\mathbb{D}_+^-$ furthermore, $\Omega(\vartheta) = \alpha$.

By Corollary (6.2.5), we have

$$\pi(A) = \omega(A) = \vartheta(\alpha) = \mathbb{T}_+ := \{e^{it} : 0 \le t \le \pi\}$$

for the a.c. contraction $A := \vartheta(T)$. Since $\pi(A) \neq \emptyset$ it follows that A is a $C_{1\cdot}$- contraction. Furthermore, applying Theorem(6. 2.3) for the adjoint of T we obtain that $A^* = \tilde{\vartheta}(T^*)$ is a $C_0{\cdot}$-contraction. (We recall that by definition $\tilde{\vartheta}(z) := \overline{\vartheta(\bar{z})}$.) Hence the contraction A is of class C_{10}. Since the Ces`aro means of the Fourier series of ϑ converge uniformly to ϑ by Fej'er's theorem, by the norm-continuity of the functional calculus Φ_T we infer that $A \in W(T)$. On the other hand, ϑ is univalent and $\vartheta(\mathbb{D}) = \mathbb{D}_+$ is a Charath´eodory domain (i.e., a simply connected bounded open set whose boundary coincides with the boundary of the unbounded component of the complement of its closure). Thus ϑ is a sequential weak-$*$ generator in H^∞ by a result of Sarason (see [14]). Hence there exists a sequence $\{p_n\}_{n=1}^\infty$ of polynomials such that $\{p_n \circ \vartheta\}_{n=1}^\infty$ converges to the identical function χ in the weak-$*$ topology. By the weak-$*$ continuity of Φ_T the operators $p_n(A) = (p_n \circ \vartheta)(T)\ (n \in \mathbb{N})$ weak-$*$ converge to $\chi(T) = T$, and so $T \in W(A)$. Therefore $W(T) = W(A)$, which yields coincidence of the commutants: $\{T\}' = \{A\}'$.

Because of the cyclicity assumption, $M_\omega(T)$ is a unitary asymptote of T (with an appropriate $X \in \mathcal{L}(T, M_\omega(T))$). By Theorem(6 2.3) the operator $\vartheta(M_\alpha)$ will be a unitary asymptote of A. Repeating the preceding argument for M_α in place of T, we obtain $W(M_\alpha) = W(\vartheta(M_\alpha))$. Thus $\vartheta(M_\alpha)$ is cyclic together with M_α. We conclude that $A \in \mathcal{L}_0(\mathcal{H})$with $\pi(A) = \mathbb{T}_+$.

Let us define the C_{10}-contraction $\tilde{T} \in \mathcal{L}(\mathcal{H})$ by $\tilde{T} := A^2$. By Corollary (6.2.5) we have $\pi(\tilde{T}) = \omega(\tilde{T}) = \mathbb{T}$. By virtue of Theorem(6. 2.3) we know that $M_{\mathbb{T}} \simeq M_{\mathbb{T}_+}^2+$ is a unitary asymptote of $\tilde{T}$. Hence $\tilde{T}$is of class $\mathcal{L}_1(\mathcal{H})$. Moreover, we conclude by [25] that $\{\tilde{T}\}' = \{A\}' = \{T\}'$.

It remains open whether the assumption on the existence of an arc in the residual set can be removed in the previous theorem. We note that in [27] the (HSP) for arbitrary C_{10}-contractions was reduced to the case where $\pi(T) = \mathbb{T}$. Now we have concentrated on the cyclic case, establishing exact coincidence for the commutants. Analogous reductions were made also in [6] and [3].

Now we exhibit examples of operators belonging to the class $\mathcal{L}_1(\mathcal{H})$. We recall that the defect operator of a contraction T is $D_T := (I - T^*T)^{1/2}$, the defect subspace of T is $D_T = (D_T\mathcal{H})^-$ and the characteristic function $\Theta_T : \mathbb{D} \to \mathcal{L}(D_T, \mathcal{D}_{T^*})$ of T is defined by

$\Theta_T(z) := [-T + zD_{T^*}(I - zT^*)^{-1}D_T]|\mathcal{D}_T$. The defect index d_T is the dimension of the defect space $\mathcal{D}_T$ (see [29]).

Proposition(6.2.8)[13]: Let $T \in \mathcal{L}(\mathcal{H})$ be a C_{10}-contraction with defect indices satisfying the condition $d_{T^*} = d_T + 1 < \infty$.Then $T \in \mathcal{L}_1(\mathcal{H})$, and for any unitary asymptote (X, V) of T, the restriction $V|(X\mathcal{H})^-$ is a simple unilateral shift.

Proof. Since T is a C_{10}-contraction, Θ_T is an inner and $*$-outer function, that is $\Theta_T(\zeta)$ is an isometry for almost every $\zeta \in \mathbb{T}$ and $(\tilde{\Theta}_T H^2(\mathcal{D}_{T^*}))^- = H^2(\mathcal{D}_T)$. Let us consider the functional model of T. Let U denote the unitary operator of multiplication by the identical function $\chi(\xi) = \xi$ on the space $L^2(\mathcal{D}_{T^*})$ of vector-valued functions. The model operator $S(\Theta_T)$ is defined on the space $\mathcal{H}(\Theta_T) = H^2(\mathcal{D}_{T^*}) \ominus \Theta_T H^2(\mathcal{D}_T)$ by $S(\Theta_T) := P_{\mathcal{H}(\Theta_T)}U|\mathcal{H}(\Theta_T)$, where $P_{\mathcal{H}(\Theta_T)}$ is the orthogonal projection onto $\mathcal{H}(\Theta_T)$ in $H^2(\mathcal{D}_{T^*})$. The contraction T is unitarily equivalent to $S(\Theta_T)$, that is there exists a unitary transformation $W \in \mathcal{L}(T, S(\Theta_T))$. (See [29] for details.) The assumption on the defect numbers yields that

$\Delta_{*T}(\zeta) := I - \Theta_T(\zeta)\Theta_T(\zeta)^*$ is a projection of rank 1 for almost every $\zeta \in \mathbb{T}$. Setting theunitary operator $R_{*,T} := U|\mathcal{R}_{*,T}$ on the space $R_{*,T} = \Delta_{*,T} L^2(\mathcal{D}_{T^*})$, and the transformation $X_{*,T} \in \mathcal{L}(\mathcal{H}(\Theta_T), \mathcal{R}_{*,T})$ defined by $X_{*,T}h := \Delta_{*,T}h$, the pair $(X_{*,T}W, \mathcal{R}_{*,T})$ will be a unitary asymptote of T (see [27]). Since the spectral multiplicity function of $\mathcal{R}_{*,T}$is constant 1 on $\mathbb{T}$, it follows that $\mathcal{R}_{*,T}$ is cyclic and $\omega(T) = \mathbb{T}$.

By a result of $Sz.-Nagy–Foias$, the assumptions imply that T is a quasiaffine transform of the unilateral shift $S \in \mathcal{L}(H^2)$, defined by $Sf = \chi f$ (see [30]).

Thus there exists a quasiaffinity $Y \in \mathcal{L}(T, S)$; we recall that Y is injective and has dense range. Given any decreasing sequence $F = \{f_n\}_{n=1}^{\infty}$ in H^∞, which is nonvanishing on $\mathbb{T}$, the relation $\mathcal{H}_0(S, F) = \{0\}$ readily yields $\mathcal{H}_0(T, F) = \{0\}$ since $f_n(S)Y = Yf_n(T)$ ($n \in \mathbb{N}$). Therefore $\pi(T) = \mathbb{T}$, and so $T \in \mathcal{L}_1(\mathcal{H})$.

Let $\tilde{S} \in \mathcal{L}(L^2)$ stand for the bilateral shift defined by $\tilde{S}f := \chi f$; $\tilde{S}$ is the minimal unitary extension of S. Clearly $\tilde{Y} \in \mathcal{L}(T, \tilde{S})$, where $\tilde{Y}h := Yh$ ($h \in \mathcal{H}$).

If (X, V) is a unitary asymptote of T, then there exists a unique $Z \in \mathcal{L}(V, \tilde{S})$ such that $ZX = \tilde{Y}$. Since $V \simeq \tilde{S}$, we can easily conclude that $V|(X\mathcal{H})^- \simeq S$.

We note that the previous statement can be extended to C_{10}-contractions T with $d_T = \infty$, assuming that $\Delta_{*,T}(\zeta)$ is of rank 1 for almost every $\zeta \in \mathbb{T}$., dim $ker\, T^* < \infty$, and that there exists a non-zero $\delta \in H^\infty$ such that $\Psi\Theta_T = \delta I$ holds with some bounded, analytic function $\Psi : \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{D}_T)$. Under these conditions T is a quasiaffine transform of S by a result of Takahashi in [31].

Further examples for operators in $\mathcal{L}_1(\mathcal{H})$ can be given by taking cyclic subspaces of countable orthogonal sums of operators belonging to $\mathcal{L}_1(\mathcal{H})$.

From now on we assume that $T \in \mathcal{L}_1(\mathcal{H})$. Let (X, V) be a unitary asymptote of T, where $V \in \mathcal{L}(K)$ is a cyclic a.c. unitary operator with $\omega(V) = \mathbb{T}$. The functional calculus for V, resulting from the spectral decomposition, is the uniquely determined isometric, unital $*$-homomorphism $\phi: L^\infty \to \mathcal{L}(K), f \longmapsto f(V)$ which is continuous with respect to the weak-$*$ and weak operator topologies, and which sends the identity function χ into V. The range of ϕ is the abelian commutant $\{V\}'$. (See e.g. [5].) Taking the $*$-isomorphism $\varphi: \{V\}' \to L^\infty, f(V) \longmapsto f$,let us consider the contractive, unital algebra-homomorphism

$$\hat{\gamma} := \varphi o \gamma : \{T\}' \to L^{\infty}$$

satisfying the condition $\hat{\gamma}(\vartheta(T)) = \vartheta$ for every $\vartheta \in H^{\infty}$.

Lemma (6.2.9)[13]: The map $\hat{\gamma}$ is independent of the particular choice of the unitary asymptote (X, V).

Proof. The straightforward proof is left as an exercise.

We use the notation $\hat{\gamma}_{\mathrm{T}}$ for the uniquely determined mapping $\hat{\gamma}$ introduced above.

Notice that $\hat{\gamma}_{\mathrm{T}}$ is injective, since T is of class C_1.. The range $\mathcal{F}(T)$ of $\hat{\gamma}_{\mathrm{T}}$ is a subalgebra of L^{∞}, containing H^{∞}, which we call the functional commutant of the contraction T. We say that a subalgebra A of L^{∞} is a quasianalytic function algebra, if A contains H^{∞}, and if $f(\zeta) \neq 0$ for almost every $\zeta \in \mathbb{T}$, whenever $f \in A\backslash\{0\}$.

Proposition (6.2.10)1[3]: For any $T \in \mathcal{L}_1(\mathcal{H})$, the function algebra $\mathcal{F}(T)$ is quasianalytic.

Proof. Given $f \in \mathcal{F}(T) \setminus \{0\}$, consider the non-zero operator $C \in \{T\}'$ with $\hat{\gamma}_{\mathrm{T}}(C) = f$. Let $v \in \mathcal{H}$ be an arbitrary non-zero vector. Since T is quasianalytic, the vector Xv is cyclic for the algebra $\{V\}'$; that is, the set $\{DXv : D \in \{V\}'\}$ is dense in K (see [25]). Taking into account that V is a cyclic unitary, we infer that Xv must be separating for $\{V\}'$. Thus $XCv = f(V)Xv \neq 0$, whence $Cv \neq 0$ follows. Then $f(V)Xv = XCv$ is also cyclic for $\{V\}'$, and so $f(\zeta) \neq 0$ for almost every $\zeta \in \mathbb{T}$ on account of $\omega(V) = T$.

Taking the inverse of $\hat{\gamma}_{\mathrm{T}}$, we obtain the unital algebra-homomorphism $\widehat{\Phi}_T : \mathcal{F}(T) \to \mathcal{L}(\mathcal{H})$, defined by $\widehat{\Phi}_T(f) = C$ whenever $\hat{\gamma}_{\mathrm{T}}(C) = f$. The mapping $\widehat{\Phi}_T$is an extension of the Sz.-Nagy–Foias calculus Φ_T , with range coinciding with the commutant $\{T\}'$. It is expansive: $\|f(T)\| \geq \|f\|_{\infty}$ for every $f \in \mathcal{F}(T)$, since $\hat{\gamma}_{\mathrm{T}}$ is contractive. Taking into account that Φ_T is contractive, we deduce that $\|\vartheta(T)\| = \|\vartheta\|_{\infty}$ holds for every $\vartheta \in H^{\infty}$.

Let us examine the effect of the M¨obius transformation on the extended functional calculus. Given $a \in \mathbb{D}$, the formula $b_a(z) := (z - a)(1 - \bar{a}z)^{-1}$ defines a regular inner function in H^{∞}, called the M¨obius function corresponding to a. Let us consider the $c.n.u.$ contraction $T_a := b_a(T)$ and the a.c. unitary operator $V_a := b_a(V)$. Since $T = b_{-a}(T_a)$ and $V = b_{-a}(V_a)$, it follows that $W(T) = W(T_a)$ and W(V) = W(V_a). We can see that V_a is also cyclic.

Applying Theorems(6. 2.3) and(6. 2.4) we obtain that (X, V_a) is a unitary asymptote of the contraction T_a belonging to $\mathcal{L}_1(\mathcal{H})$,. It is evident that $\{T\}' = \{Ta\}'$, $\{V\}' = \{Va\}'$, and the mapping γ is the same for T as for T_a. The weak- $*$ continuous $*$-homomorphism $\tau_a : L^{\infty} \to L^{\infty}, f \mapsto f \circ b_a$ transforms the identical function χ into b_a. By the uniqueness of the functional calculus ϕ_a for V_a, we infer that $\phi \circ \tau_a = \phi_a$, whence $\varphi_a = \tau_{-a} \circ \varphi$ follows. Thus $\hat{\gamma}_{\mathrm{T_a}} = \varphi_a \circ \gamma = \tau_{-a} \circ \varphi \circ \gamma = \tau_{-a} \circ \hat{\gamma}_{\mathrm{T}}$, and we arrive at the following statement.

Proposition(6.2.11)[13]: If $T \in \mathcal{L}_1(\mathcal{H})$,then $T_a = b_a(T) \in \mathcal{L}_1(\mathcal{H})(a \in \mathbb{D})$. Furthermore, $\mathcal{F}(T_a) = \{f \circ b_{-a} : f \in \mathcal{F}(T)\}, \hat{\gamma}_{\mathrm{T_a}} = \tau_{-a} \circ \hat{\gamma}_{\mathrm{T}}$ and $\widehat{\Phi}_{\mathrm{T_a}} = \widehat{\Phi}_T \circ \tau_a$; that is, $f(T_a) = (f \circ b_a)(T)$ holds for every $f \in \mathcal{F}(T_a)$.

The following statement relates the spectrum of T to the function algebra $\mathcal{F}(T)$.

Proposition(6.2.12) [13]: Let us assume that $T \in \mathcal{L}_1(\mathcal{H})$.

(a) The point $a \in \mathbb{D}$ is in the spectrum $\sigma(T)$ of T exactly when the conjugate $\overline{b_a}$ of the M¨obius function b_a does not belong to $\mathcal{F}(T)$.

(b) The spectrum $\sigma(T)$ covers the closed unit disc $\mathbb{D}^-$ if and only if $\mathcal{F}(T)$ does not contain the conjugate of any non-constant inner function.

Proof. (a): If $a \in \mathbb{D} \setminus \sigma(T)$, then $b_a(T)$ is invertible and $b_a(T)^{-1} \in \{T\}'$.
Hence $\overline{b_a} = \frac{1}{b_a} = \hat{\gamma}_T(b_a(T)^{-1}) \in \mathcal{F}(T)$. Conversely, if $\overline{b_a} \in \mathcal{F}(T)$, then there exists $C \in \{T\}'$ such that $\hat{\gamma}_T(C) = \overline{b_a}$. Since

$$\hat{\gamma}_T(Cb_a(T)) = \hat{\gamma}_T(C)\,\hat{\gamma}_T(b_a(T)) = \overline{b_a}b_a = 1$$

and $\hat{\gamma}_T$ is injective, it follows that $Cb_a(T) = I$. Hence $b_a(T)$ is invertible, and so $a \in \mathbb{D} \setminus \sigma(T)$.

(b): In view of (a), it is enough to show that if $\bar{\vartheta} \in \mathcal{F}(T)$ holds for some nonconstant inner function $\vartheta \in H^\infty$, then $\overline{b_a} \in \mathcal{F}(T)$ is also true for some $a \in \mathbb{D}$.

Let $C \in \{T\}'$ be the operator with $\hat{\gamma}_T(C) = \bar{\vartheta}$. By Frostman's theorem we can find a non-zero $\lambda \in \mathbb{D}$ such that $|\lambda| < \|C\|^{-1}$ and the composition $b = b_\lambda \circ \vartheta$ is a Blaschke product (see [7]). The function

$$\bar{b} = \overline{b_\lambda \circ \vartheta} = (\bar{\vartheta} - \bar{\lambda})(1 - \lambda\bar{\vartheta})^{-1} = \hat{\gamma}_T(C_{\bar{\lambda}})$$

belongs to $\mathcal{F}(T)$, since $C_{\bar{\lambda}} := b_{\bar{\lambda}}(C)$ is in $\{T\}'$. Clearly, b has the form $b = b_a\eta$ where $a \in \mathbb{D}$ and $\eta \in H^\infty$ is inner. We conclude that $\bar{b}_a = \bar{b}\eta \in \mathcal{F}(T)$.

By the $F.\&M.$ Riesz Theorem, the Hardy space H^∞ itself is a quasianalytic function algebra. Further examples for quasianalytic algebras can be given by the aid of inner functions. We recall that a function $\eta \in H^\infty$ is inner, if $|\eta(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$. Let H_i^∞ stand for the set of all inner functions. Given any non-empty subset $\mathcal{B} \subset H_i^\infty$, the algebra

$$[\overline{\mathcal{B}}, H^\infty]_0 := \left\{\vartheta \prod_{j=1}^{k} \overline{\eta_j} : \vartheta \in H^\infty, \{\eta_j\}_{j=1}^{k} \subset \mathcal{B}, k \in \mathbb{N}\right\},$$

generated by $\overline{\mathcal{B}} \cup H^\infty$, is clearly quasianalytic. Its norm closure
$[\overline{\mathcal{B}}, H^\infty] := ([\overline{\mathcal{B}}, H^\infty]_0)^-$
is called the Douglas algebra, associated with $\mathcal{B}$. By the celebrated Chang–Marshall Theorem, every closed subalgebra A of L^∞ containing H^∞ is a Douglas algebra (see [7]). We shall call $[\overline{\mathcal{B}}, H^\infty]_0$ the pre-Douglas algebra corresponding to $\mathcal{B}$. While all pre-Douglas algebras are quasianalytic, the non-trivial Douglas algebras don't have this property.

Lemma(6.2.13)[13] : The only quasianalytic Douglas algebra is H^∞.

Proof. Let A be a closed subalgebra of L^∞, which properly contains H^∞ : $A \supset H^\infty$, $A \neq H^\infty$. Then $A \supset [\bar{\chi}_{\lrcorner}, H^\infty] = C(\mathbb{T}) + H^\infty$,
where $C(\mathbb{T})$ stands for the space of continuous functions on $\mathbb{T}$ (see [22]). Thus A is not quasianalytic.

The following theorem characterizes the case when the functional commutant is a Douglas algebra. We need some notation. Given a contraction $T \in \mathcal{L}(\mathcal{H})$, let (X, V) be a unitary asymptote of T. The definition of the positive contraction
$A_T := X^*X \in \mathcal{L}(\mathcal{H})$ is independent of the special choice of (X, V); in particular, the equations
$\langle X^*Xh, h\rangle = \|Xh\|^2 = \lim_{n\to\infty}\|T^n h\|^2 = \lim_{n\to\infty}\langle T^*T^n h, h\rangle \quad h \in \mathcal{H}$

show that $A_T = \lim_{n\to\infty} T^{*n}T^n$ in the strong operator topology. (Notice that the sequence $\{T^{*n}T^n\}_{n=1}^{\infty}$ is decreasing.) The mapping $L_T : \{T\}' \to \mathcal{L}(\mathcal{H}), C \mapsto A_T C$ is a bounded, linear transformation. If the contraction T is a.c., then

$$H^\infty(T) := ran\Phi_T \subset W(T).$$

Theorem(6.2.14)[13]: For any contraction $T \in \mathcal{L}_1(\mathcal{H})$, the following conditions are equivalent:

(a) $\mathcal{F}(T)$ is a Douglas algebra,

(b) $\widehat{\Phi}_T$ is bounded,

(c) $\widehat{\Phi}_T$ is an isometry,

(d) L_T is bounded from below,

(e) L_T is an isometry,

(f) $\mathcal{F}(T) = H^\infty$,

(g) $\{T\}' = H^\infty(T)$.

Proof. We know that $\widehat{\gamma}_T : \{T\}' \to L^\infty$ is an injective, bounded, linear transformation, with ran$\widehat{\gamma}_T = \mathcal{F}(T)$. By the Closed Graph Theorem, $\mathcal{F}(T)$ is closed if and only if $\widehat{\Phi}_T$ is bounded.

Let (X, V) be a unitary asymptote of T, and let us consider the algebra homomorphism $\gamma : \{T\}' \to \{V\}'$. For any $F \in \{V\}'$, the operator $B = X^*FX \in \mathcal{L}(\mathcal{H})$ is T-Toeplitz: $T^*BT = B$. Given any $x, y \in \mathcal{H}$ and $n, k \in \mathbb{N}$, we have

$$|\langle FV^{-n}Xx, V^{-n}Xy\rangle| = |\langle Bx, y\rangle| = |\langle BT^k x, T^k y\rangle| \le \|B\| \cdot \|T^k x\| \cdot \|T^k y\|.$$

Running k to infinity, it follows that

$$|\langle FV^{-n}Xx, V^{-n}Xy\rangle| \le \|B\|. \|Xx\| \cdot \|Xy\| = \|B\| \cdot \|V^{-n}Xx\| . \|V^{-n}Xy\|.$$

Hence $\|F\| = \|B\|$, and so the mapping $\Gamma : \{V\}' \to \mathcal{L}(\mathcal{H}), F \mapsto X^*FX$ is a linear isometry. It can be shown that the range of Γ coincides with the set of all T-Toeplitz operators (see [26]). Since $\Gamma \circ \gamma = L_T$, we conclude that $\widehat{\Phi}_T$ is bounded (respectively an isometry) if and only if L_T is bounded from below (respectively an isometry). (We note that the previous discussions can be carried out for any $C_{1.}$ −contractions.)

Taking into account that the Sz.-Nagy–Foias functional calculus, Φ_T is an isometry for a contraction $T \in \mathcal{L}_1(\mathcal{H})$, the remaining implications follow from Lemma (6.2.13).

We obtain that if $\mathcal{F}(T)$ is a Douglas algebra, then $Hlat\, T = Lat\, T$ has an abundant supply of subspaces. The following sufficient condition can be frequently checked easily.

Proposition(6.2.15) [13]: If $T \in \mathcal{L}_1(\mathcal{H})$is a quasiaffine transform of the simple unilateral shift $S \in \mathcal{L}(H^2)$, then $\mathcal{F}(T)$ is a Douglas algebra.

Proof. From the proof of Proposition(6 .2.8) we can see that a unitary asymptote of T can be given in the form $(X, \tilde{S})$, where $\tilde{S} \in \mathcal{L}(L^2)$ is the bilateral shift, $X \in \mathcal{L}\,(T, \tilde{S})$ and $(X\mathcal{H})^- = H^2$. Given any $f \in \mathcal{F}(T)$, let us consider the operator $f(T) \in \{T\}'$. The equation $Xf(T) = f(\tilde{S})X$ implies

$$fH^2 = f(\tilde{S})(X\mathcal{H})^- \subset (Xf\,(T)\mathcal{H})^- \subset (X\mathcal{H})^- = H^2.$$

Hence $f \in H^2$, and so $\mathcal{F}(T) = H^\infty$ is a Douglas algebra.

Proposition (6.2.8) provides contractions satisfying the conditions of the previous proposition. Now we turn to pre-Douglas algebras.

Theorem(6.2.16) [13]: Let $T \in \mathcal{L}_1(\mathcal{H})$. If $\mathcal{F}(T)$ is a pre-Douglas algebra, then $R(T) = \{T\}'$, and so $Rlat\, T = Hlat\, T$.

Proof. We may assume that the pre-Douglas algebra $\mathcal{F}(T)$ is different from H^∞. Setting $\mathcal{B} := \{\eta \in H_i^\infty : \tilde{\eta} \in \mathcal{F}(T)\}$, we have $[\mathcal{B}, H^\infty]_0 = \mathcal{F}(T)$, and $\tilde{\eta}(T) = \eta(T)^{-1} \in \{T\}'$ whenever $\eta \in \mathcal{B}$. We have to show that $\tilde{\eta}(T) \in \mathcal{R}(T)$, for every non-constant $\eta \in \mathcal{B}$.

If $\eta$is a M¨obius function of the form

$$\eta(z) = \frac{k(z - a)}{1 - \bar{a}z} (k \in \mathbb{T}, a \in \mathbb{D}),$$

then $a \notin \sigma(T)$ by Proposition(6.2.12) . Exploiting the fact that $\widehat{\Phi}_T$ is a unital algebra-homomorphism, we obtain

$$\tilde{\eta}(T) = \eta(T)^{-1} = \bar{k}(T - aI)^{-1}(I - \bar{a}T) \in \mathcal{R}_0(T) \subset \mathcal{R}(T).$$

Observe that $\eta \in B, \eta = \eta_1\eta_2$ $(\eta_1, \eta_2 \in H_i^\infty)$ implies $\bar{\eta_1} = \bar{\eta}\eta_2 \in \mathcal{F}(T)$. Thus $\bar{\eta}(T) \in \mathcal{R}_0(T)$, whenever $\eta \in \mathcal{B}$ is a finite Blaschke product.

Let us assume now that $\eta \in \mathcal{B}$ is an infinite Blaschke product: $\eta = \prod_{n=1}^{\infty} b_{a_n}$, where $\{a_n\}_{n=1}^{\infty} \subset \mathbb{D}$ and $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. Here we use the notation $b_0(z) := z$ and $b_a(z) := -(\bar{a}/|a|)(z - a)/(1 - \bar{a}z)\ for\ a \in \mathbb{D}\setminus\{0\}$. For any $N \in \mathbb{N}$, we set $B_N := \prod_{n=1}^{N} b_{a_n}$. We know that $\overline{B_N} \in \mathcal{F}(T)$ and $\overline{B_N}(T) = B_N(T)^{-1} \in \mathcal{R}_0(T)$. The operator $\eta(T)$ is invertible with $\eta(T)^{-1} = \bar{\eta}(T)$, hence $\delta := inf\{\|\eta(T)x\| : x \in \mathcal{H}, \|x\| = 1\} > 0$. Since the sequence $\{B_N\}_{n=1}^{\infty}$ is bounded and $lim_{N\to\infty} B_N(z) = \eta(z)$ for every $z \in \mathbb{D}$, it follows that $B_N(T)$ converges to $\eta(T)$ in the weak operator topology, as N tends to infinity (see [29]). Given any $x \in \mathcal{H}, \|x\| = 1$, and setting $y = \|\eta(T)x\|^{-1}\eta(T)x$, we infer

$$\begin{aligned} \text{Liminf}_{N\to\infty} \| B_N(T)x\| &\geq \lim_{N\to\infty} |\langle B_N(T)x, y\rangle| \\ &= |\langle \eta(T)x, y\rangle| = \|\eta(T)x\| \geq \delta. \end{aligned}$$

Taking into account that $\| B_{N+1}(T)x\| \leq \|b_{N+1}(T)\| \, \|B_N(T)x\| \leq \|B_N(T)x\|$ $(N \in \mathbb{N})$, we obtain that $\|B_N(T)x\| \geq \delta$ for every $N \in \mathbb{N}$. Thus $\{B_N(T)^{-1}\}_{N=1}^{\infty}$ is a bounded sequence.

There exists a subsequence $\{B_{N_j}\}_{j=1}^{\infty}$ such that $B_{N_j}(T)^{-1}$ converges to an operator $C \in \mathcal{L}(\mathcal{H})$ in the weak operator topology, and $lim_{j\to\infty} B_{N_j}(\zeta) = \eta(\zeta)$ holds for almost every $\zeta \in \mathbb{T}$ (see, e.g., [5]). Then C is necessarily in $\mathcal{R}(T)$. Furthermore, $B_{N_j}(T)$ converges to $\eta(T)$ in the strong operator topology (see [15]), and so the product $I = B_{N_j}(T)^{-1}B_{N_j}(T)$ converges to $C\eta(T)$ in the weak operator topology, as $j \to \infty$. Thus $C\eta(T) = I$, whence $\bar{\eta}(T) = \eta(T)^{-1} = C \in R(T)$ follows.Now let $\eta \in B$ be an arbitrary non-constant inner function. For convenience, we use the notation $A := \eta(T)$. By Frostman's theorem we can find $a \in \mathbb{D}$ so that $0 < |a| < 4^{-1}\|A^{-1}\|^{-1}$, and ba $\circ$ $b_a \circ \eta = b$ is a Blaschke product. (Here $b_a(z) := (z - a)/(1 - \bar{a}z)$.) Since $b(T) = b_a(A) \in \{T\}'$ is invertible, it follows that $b \in \mathcal{B}$, and so $D := b_a(A)^{-1} = \bar{b}(T) \in \mathcal{R}(T)$. The equation $D = (A - aI)^{-1}(I - \bar{a}A)$ readily yields $D + \bar{a}I = A^{-1}(I + aD)$. Taking into account that

$$\|aD\| \leq |a| \| A^{-1}\| \, \|(I - aA^{-1})\|^{-1} \ \|I - \bar{a}A\| < 1,$$

we conclude that $(I + aD)^{-1} = \sum_{n=0}^{\infty} \boldsymbol{a^n} D^n \in \mathcal{R}(T)$, and so we obtain that $\bar{\eta}(T) = A^{-1} = (D + \bar{a}I)(I + aD)^{-1} \in \mathcal{R}(T)$ holds too.

The following type of function algebras were studied by Tolokonnikov in [32].

A subalgebra $\mathcal{A}$ of L^∞, containing H^∞, is called a generalized Douglas algebra, if $f \in \mathcal{A}, \lambda \in \mathbb{C}$ and $|\lambda| > \| f\|_\infty$ imply $1/(f - \lambda) \in \mathcal{A}$. We proceed with a spectral characterization of the case when the functional commutant is of this kind.

Theorem(6.2.17)[13]: Let $T \in \mathcal{L}_1(\mathcal{H})$. Then $\mathcal{F}(T)$ is a generalized Douglas algebra if and only if $\widehat{\Phi}_T$ preserves the spectral radius: $r(f(T)) \; r(f) = \| f\|_\infty$ for every $f \in \mathcal{F}(T)$.

Proof. The commutant $\{T\}'$ is a Banach algebra, and for any $C \in \{T\}'$, the spectrum of C in $\{T\}'$ is the same as the spectrum of C in $\mathcal{L}(\mathcal{H})$. Since $\hat{\gamma}_T: \{T\}' \to L^\infty$ is a unital algebra-homomorphism, we infer that $\sigma(f(T))$ contains $\sigma(f)$, which is the essential range of f, for every $f \in \mathcal{F}(T)$. Hence $r(f(T)) \geq r(f) = \| f\|_\infty$ always holds.

Let us assume that $\mathcal{F}(T)$ is a generalized Douglas algebra. Setting $f \in \mathcal{F}(T)$, $\lambda \in \mathbb{C}, |\lambda| > \| f\|_\infty$, we know that $g = 1/(f - \lambda) \in \mathcal{F}(T)$. Then $g(f - \lambda) = 1$ implies $g(T)(f(T) - \lambda I) = I$, and so λ is in the resolvent set of $f(T)$. Thus $r(f(T)) \leq \| f\|_\infty$.

Let us assume now that $\widehat{\Phi}_T$ preserves the spectral radius. Setting $f \in \mathcal{F}(T)$, $\lambda \in \mathbb{C}, |\lambda| > \| f\|_\infty$, the relation $\| f\|_\infty = r(f) = r(f(T))$ yields that $f(T) - \lambda I$ is invertible. Its inverse C necessarily belongs to $\{T\}'$, and so $C = g(T)$ for some $g \in \mathcal{F}(T)$. Since $\widehat{\Phi}_T$ is injective, the equality $(f(T) - \lambda I)g(T) = I$ implies $(f - \lambda)g = 1$. Hence $1/(f - \lambda) = g \in \mathcal{F}(T)$.

In view of Proposition(6.2.12) and Theorem(6.2.14), the spectrum $\sigma(T)$ is the closed unit disc $\mathbb{D}^-$ when $\widehat{\Phi}_T$ *is* an isometry, or equivalently, when $\mathcal{F}(T)$ is a Douglas algebra. The next theorem describes the spectrum, when $\mathcal{F}(T)$ is a generalized Douglas algebra, but not a Douglas algebra.

Theorem (6.2.18)[13]: Let $T \in \mathcal{L}_1(\mathcal{H})$.. *If* $\mathcal{F}(T)$ is a generalized Douglas algebra, different from H^∞, then $\sigma(T) = \mathbb{T}$.

Proof. For any $a \in \mathbb{D}$, let us consider the operator $T_a = b_a(T)$, where $b_a(z) = (z - a)/(1 - \bar{a}z)$. By Proposition (6.2.11) we know that $T_a \in \mathcal{L}_1(\mathcal{H})$and $\mathcal{F}(b_a) = \{f \circ b_{-a} : f \in \mathcal{F}(T)\}$. We conclude that $\mathcal{F}(T_a)$ is also a generalized Douglas algebra, different from H^∞. We infer by [32] that $\bar{\chi} \in \mathcal{F}(T_a)$, whence $\bar{b}_a = \bar{\chi} \circ b_a \in \mathcal{F}(T)$ follows. Thus a is in the resolvent set of T in view of Proposition (6.2.12), and so $\sigma(T) \subset \mathbb{T}$. On the other hand, $\hat{\gamma}_T$ shrinks the spectrum, as we have seen in the proof of Theorem (6.2.17). Therefore $\sigma(T) \supset \sigma(\chi) = \mathbb{T}$.

Example(6.2.19)[13]: Given $0 < \delta < 1$, let us consider the simply connected domain $G = \{re^{it} : \sqrt{\delta} < r < 1, 0 < t < \pi\}$

Let ϑ_0 be a conformal mapping of $\mathbb{D}$ onto G, satisfying the condition $\vartheta_0(\zeta) = \zeta$ for $\zeta = 1, i, -1$. Then $\vartheta := \vartheta_0^2$ will be a regular partially inner function with $\Omega(\vartheta) = \mathbb{T}_+ := \{e^{it} : 0 \leq t \leq \pi\}$ and $pe-ran(\vartheta|\mathbb{T}_+) = \vartheta(\mathbb{T}_+) = \mathbb{T}$.

The simple unilateral shift $S \in \mathcal{L}(H^2), Sf = \chi f$ belongs to $\mathcal{L}_1(H^2)$. Furthermore, $(J, \bar{S})$ is a unitary asymptote of S, where $J : H^2 \to L^2, f \mapsto f$ is the natural embedding, and $\bar{S} \in \mathcal{L}(L^2), \bar{S}f = \chi f$ is the simple bilateral shift. Clearly $\omega(S, \vartheta) = \mathbb{T}_+$. Let us consider the analytic Toeplitz operator $T := \vartheta(S) = T_\vartheta \in \mathcal{L}(H^2)$. By Theorem (6.2.3) the pair $(J_+, \vartheta(\bar{S}_+))$ is a unitary asymptote of T, where $\bar{S}_+ := \bar{S}|L^2(\mathbb{T}_+)$ and $J_+ : H^2 \to L^2(\mathbb{T}_+), f \mapsto \chi_{\mathbb{T}_+} f$.

In virtue of Theorem(6. 2.4) we can also see that $\pi(T) = \mathbb{T}$.
Set $\vartheta_1 := \vartheta|\mathbb{T}_+$. Since the boundary of G is a rectifiable Jordan curve, the mapping $Z : L^2 \to L^2(\mathbb{T}_+), f \mapsto (f \circ \vartheta_1)|\grave{\vartheta}_1|^{1/2}$ is a unitary transformation, intertwining $\bar{S}$ with $\vartheta(\bar{S}_+)$ (see [5] or [23]). Thus $\vartheta(\bar{S}_+)$ is cyclic, and so $T \in \mathcal{L}_1(H^2)$.. As in the proof of Theorem (6.2.7), we obtain that $\{T\}' = \{S\}' = \{T_h : h \in H^\infty\}$. Therefore, for every $\lambda \in \mathbb{C}\backslash\sigma(T)$, the inverse $(T - \lambda I)^{-1}$ is an analytic Toeplitz operator, whence
$$\sigma(T) = \vartheta(\mathbb{D})^- = \{re^{it} : \delta \leq r \leq 1, 0 \leq t \leq 2\pi\}$$
can be easily derived.
To identify the functional commutant $\mathcal{F}(T)$, let us consider the mapping
$\gamma: \{T\}' \to \{\vartheta(\bar{S}+)\}', C \mapsto D$, where $J_+C = DJ_+$. We know that $\{T\}' = \{S\}'$.
Furthermore, for any $h \in H^\infty$, we have $h(S) \in \{T\}', h(\bar{S}_+) \in \{\vartheta(\tilde{}\bar{S}_+)\}'$ and $J_+h(S) = h(\bar{S}_+)J_+$, hence $\gamma(h(S)) = h(\bar{S}_+)$. Since $\phi: L^\infty \to \{\vartheta(\bar{S}_+)\}', f \mapsto Zf(\bar{S})Z^*$ is the functional calculus for $\vartheta(\bar{S}_+)$, we infer that $g \in \mathcal{F}(T)$ holds exactly when there exists a function $h \in H^\infty$ such that $Zg(\bar{S})Z^* = h(\bar{S}_+)$; and then $\widehat{\Phi}_T(g) = T_h$. For any $f \in L^\infty$ we have $Zg(\bar{S})f = (g \circ \vartheta_1)(f \circ \vartheta_1)|\grave{\vartheta}_1|^{1/2}$ and $h(\bar{S}_+)Zf = (h|\mathbb{T}_+)(f \circ \vartheta_1)|\grave{\vartheta}_1|^{1/2}$. Thus $\mathcal{F}(T) = g \in L^\infty : \{g \circ \vartheta_1 = h|\mathbb{T}_+$ for some $h \in H^\infty\}$
We can get further examples by choosing G to be any simply connected domain in $\mathbb{D}$, whose boundary ∂G is a rectifiable Jordan curve, satisfying the condition $\partial G \cap \mathbb{T} = \mathbb{T}_+$.

Section(6.3): Quasianalytic Spectral Sets of Cyclic Contractions

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded, linear operators acting on $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$ let $\{T\}' = \{C \in \mathcal{L}(\mathcal{H}): CT = TC\}$ denote the commutant of T, and let Hlat $T = Lat\{T\}'$ stand for the hyperinvariant subspace lattice of T. The Invariant Subspace Problem (ISP) asks whether every operator $T\,\mathcal{L}(\mathcal{H})$ has a non-trivial invariant subspace, that is if Lat $T \neq \{\{0\}, \mathcal{H}\}$. In a similar fashion, the Hyperinvariant Subspace Problem (HSP) is whether every operator $T \in \mathcal{L}(\mathcal{H})\backslash\mathbb{C}I$ has a non-trivial hyperinvariant subspace. These problems are arguably the most challenging open questions in operator theory. From the point of view of subspaces one can normalize the operators to have norm at most 1, hence in what follows we shall only consider contractions. In the present work we shall show that for a relatively large class of contractions $\mathcal{L}_0(\mathcal{H})$, see its definition below) the problem (HSP) is equivalent to (HSP) for a special subclass $(\mathcal{L}_1(\mathcal{H}))$, the members of which have rich invariant subspace lattice. The reduction will be achieved by establishing that for every $T \in \mathcal{L}_0(\mathcal{H})$ there is a $T_1 \in \mathcal{L}_1(\mathcal{H})$ which commutes with T .ThisT_1 will be obtained as a function $f(T)$ of T , where f is a special conformal map lying in the disk algebra.
The existence of f will be proven via potential theory.
We define some classes of contractions. These concepts were introduced (in the non-cyclic case too) in [11], where it was shown, among others, that non-quasianalytic contractions (to be defined below) do have proper hyperinvariant subspaces. Thus, in the quest for such subspaces one should concentrate on quasianalytic contractions.
Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction:$\|T\| \leq 1$ We recall that the pair (X, V) is a unitary asymptote of T ,if
(i) V is a unitary operator acting on a Hilbert space κ,

(ii) $X \in \mathcal{L}(\mathcal{H}, \kappa)$ is a contractive mapping intertwining T with V : $\|X\| \leq 1,, XT = VX$ and

(iii) for any similar contractive intertwining pair $(\acute{X}, \acute{V})$ there exists a unique contractive transformation $Y \in \mathcal{L}(\kappa, \acute{\kappa})$ such that $YV = \acute{V} Y$ and $\acute{X} = YX$.

For the existence and uniqueness of unitary asymptotes we refer to [4] (see also [10]). We assume that T is of class C_{10}, which means that

(a) T is asymptotically non-vanishing: $\lim_{n\to\infty}\|T^n\| > 0$ for every $0 \neq x \in \mathcal{H}$ and

(b) the adjoint T^* is stable: $\lim_{n\to\infty}\|(T^*)^n x\| = 0$ for every $x \in \mathcal{H}$.

Then the intertwining mapping X is injective, and the unitary operator V is absolutely continuous. Let us also assume that V is cyclic:

$\bigvee_{n=0}^{\infty} V^n y = \kappa$ for some $y \in \kappa$. Then, for some measurable subset $\alpha \subset \mathbb{T}$ of the unit circle, V is unitarily equivalent to the multiplication operator M_α on the Hilbert space $L^2(\alpha)$ by the identity function $\boldsymbol{\chi}(\zeta) = \zeta$: $M_\alpha f = \boldsymbol{\chi} f, f \in L^2(\alpha)$.

So from now on we may assume $\kappa = L^2(\alpha)$ and $Vf = \boldsymbol{\chi} f, f \in L^2(\alpha)$.The set α is uniquely determined up to sets of zero Lebesgue measure, and is called the residual set of T , denoted by $\omega(T)$.

We say that T is quasianalytic on a measurable subset β of $\mathbb{T}$,if $(Xh)(\zeta) \neq 0$ for a.e. $\zeta \in \beta$ whenever $0 \neq h \in \mathcal{H}$. Taking the union of a sequence of quasianalytic sets, whose measures converge to the supremum (of measures of all quasianalytic sets), we obtain that there exists a largest quasianalytic set for T , denoted by $\pi(T)$.Theset $\pi(T)$ is determined up to sets of zero Lebesgue measure, and is called the quasianalytic spectral set of T . Clearly, $\pi(T)$. is included in $\omega(T)$. The contraction T is quasianalytic, if $\pi(T) = \omega(T)$.

in [13] introduced distinctive classes of quasianalytic contractions. The class $\mathcal{L}_0(\mathcal{H})$ consists of the operators $T \in \mathcal{L}(\mathcal{H})$ satisfying the conditions:

(i) T is a C_{10}-contraction,

(ii) the unitary operator V is cyclic, and

(iii) T is quasianalytic.

The subclass $\mathcal{L}_1(\mathcal{H})$ consists of those operators $T \in \mathcal{L}_0(\mathcal{H})$, which satisfy also the additional condition:

(iv) $\pi(T) = \mathbb{T}$

Every operator $T \in \mathcal{L}_1(\mathcal{H})$ has a rich invariant subspace lattice Lat T ;see[11]. Let us consider also the class $\tilde{\mathcal{L}}(\mathcal{H})$ of those (non-scalar) contractions $T \in \mathcal{L}(\mathcal{H})$, which are non-stable (i.e., $\lim_{n\to\infty}\|\mathrm{T}^n \mathrm{x}\| > 0$ for some $x \in \mathcal{H}$), and where the unitary asymptote V is cyclic. Clearly

$$\mathcal{L}_1(\mathcal{H}) \subset \mathcal{L}_0(\mathcal{H}) \subset \tilde{\mathcal{L}}(\mathcal{H}).$$

We emphasize that from the point of view of invariant subspaces these classes are very natural.

Namely, we know from [11] that the (HSP) in the class $\tilde{\mathcal{L}}(\mathcal{H})$is equivalent to the (HSP) in the class $\mathcal{L}_0(\mathcal{H})$. Furthermore, if the (HSP) has positive answer in $\tilde{\mathcal{L}}(\mathcal{H})$, then the (ISP) has an affirmative answer in the large class of contractions T , where T or T^* is non-stable. As was mentioned earlier, the (ISP) in $\mathcal{L}_1(\mathcal{H})$ is answered affirmatively. Actually, a lot of information is at our disposal on the structure of operators in $\mathcal{L}_1(\mathcal{H})$,

which may be helpful in the study of the (HSP) in this class; see [12]. It was proved in [13] that if $T \in \mathcal{L}_0(\mathcal{H})$ and $\pi(T)$ contains an arc then there exists $T_1 \in \mathcal{L}_1(\mathcal{H})$such that $\{T\}' = \{T_1\}'$and so $Hlat T = Hlat\, T_1$.
Therefore, we obtain the following corollary.

Corollary (6.3.1)[1]: The (HSP) in the class $\mathcal{L}_0(\mathcal{H})$ is equivalent to the (HSP) in the class $\mathcal{L}_1(\mathcal{H})$.

These results are related to those in [7,6,3,12].

We provide an operator T_1 in $\mathcal{L}_1(\mathcal{H}) \cap \{T\}'$ as a function of , using the $Sz.$-Nagy–Foias functional calculus; see [19]. We shall apply the spectral mapping theorem established in [13]. The existence of a function $f \in H^\infty$, satisfying the conditions $f(T) \in \mathcal{L}_0(\mathcal{H})$ and $\pi(f(T)) = f(\pi(T)) = \mathbb{T}$, is based on Theorem(6.3.3) below.

Let m denote the linear Lebesgue measure both on the real line and on the unit circle. A domain $G \in \mathbb{C}$ is called a circular comb domain if it is obtained from the open unit disc $\mathbb{D}$ by deleting countably many radial segments of the form $\{r\,\zeta : \rho < r < 1\}$ with some $0 < \rho < 1$ and $\zeta \in \mathbb{T}$.

Theorems(6.3.7) and(6.3.2) should be compared to [15]. Here the additional absolute continuity of the extremal measure is the key to our results.

In this Section the functional calculus within the class $\mathcal{L}_0\,(\mathcal{H})$ is discussed, and Theorem(6.3.1) is proved relying on Theorem (6.3.7). The proofs of Theorems(6.3.7) and (6.3.2) are given in this Section .

In order to get C_{10} -contractions, we consider functions in the Hardy class H^∞ with specific boundary behavior.

Let M be the σ-algebra of Lebesgue measurable sets on $\mathbb{T}$. For a complex function f defined on the open unit disc $\mathbb{D}$, let $\Omega(f)$ be the set of those points $\zeta \in \mathbb{T}$, where the radial limit $\lim_{r\to 1-0} f(r\zeta) =: f(\zeta)$ exists and is of modulus 1: $|f(\zeta\,)| = 1$. It can be easily seen that if f is continuous on $\mathbb{D}$, then $\Omega(f) \in \boldsymbol{M}$.

For any $f \in H^\infty$ the radial limit exists almost everywhere on $\mathbb{T}$ by Fatou's theorem; see [9].

We recall from [12] that $f \in H^\infty$ is a partially inner function, if

(i) $|f(0)| < 1 = \|f\|_\infty$, and

(ii) $m(\Omega\,(f)) > 0$.

Note that (i) implies $f\,[\mathbb{D}] \subset \mathbb{D}$ by the Maximum Principle. Furthermore, Corollary (6.3.2) of [13] states that $m(f^{-1}[\omega\,]) = 0$ for every $\omega \in M$ with $m(\omega\,) = 0$ (recall also that every set of measure 0 is included in a Borel set of measure zero). Hence, for any $\Omega \in M, \Omega \subset \Omega(f),$ the measure $\mu: M \to [0, 2\pi],\ \mu(\omega) = m(f^{-1}\,[\,\omega] \cap \Omega\,)$ is absolutely continuous with respect to m.

The properly essential range of the restriction $f|_\Omega$ is defined by

$$pe - ran(f|_\Omega\,) := \{\zeta \in \mathbb{T}: (d\mu\,/d\,m)(\,\zeta\,) > \,0\}\,.$$

Note that the Radon–Nikodym derivative $d\mu\,/dm$, and so the Lebesgue measurable set $pe - ran(f|_\Omega\,)$ too, is determined up to sets of measure zero. The spectral mapping theorems in this Section of [13] are formulated in terms of this kind of range.

The properly essential range is just the range of the function under some regularity conditions.

We introduce this regularity property of a partially inner function in a somewhat different (and simpler) manner than in [13]. We say that a function $g: \Omega \to \mathbb{T}$, where $\Omega \subset \mathbb{T}$ is a measurable subset of $\mathbb{T}$, is weakly absolutely continuous, if , $\omega \subset \Omega, m(\omega) = 0$, implies $m(g[\omega]) = 0$. The partially inner function $f \in H^\infty$ is called regular, if $f|_{\Omega(f)}$ is a weakly absolutely continuous function. The following lemma shows that this definition is essentially the same as the one given in [11] and [13], replacing Borel sets occurring there by Lebesgue measurable sets.

Lemma(6.3.2)[1]: Let $f \in H^\infty$ be a partially inner function.

(a) Then f is regular if and only if for every measurable set $\Omega \subset \Omega(f)$ the image set $f\,[\,\Omega]$ is also measurable.

(b) If f is regular and $\Omega \in M, \Omega \subset \Omega(f)$, then $pe-ran(f|_\Omega) = f(\Omega\,)$.

Recall that $pe-ran(f|_\Omega\,)$ is determined only up to measure zero, so the equality $pe-ran(f|_\Omega\,) = f(\Omega\,)$is also understood up to measure zero.

Proof. (a): We sketch the proof of this known equivalence. Suppose that f is regular, and let $\Omega \in M, \Omega \subset \Omega(f)$. Since $f|_\Omega$ is the pointwise limit of a sequence of continuous functions, it follows from Egorov's theorem that $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and $f\,[\,\Omega_1]$ are F_σ-sets and $m(\,\Omega_2) = 0$. Hence, by assumption, $m(f\,[\,\Omega_2\,]) = 0$ and thus $f\,[\Omega\,] \in M$.

Conversely, if f is non-regular, then $m(f\,[\omega\,]) = 0$ fails for some $\omega \subset \Omega(f)$ with $m(\omega\,) = 0$.

There is a non-measurable subset $\acute{\Omega}$ of $f[\omega]$. Thus $\Omega = f^{-1}[\,\acute{\Omega}\,] \cap \omega \in M$, while $f\,[\Omega] = \acute{\Omega} \notin M$.

(b): The sets $\omega_1 = f\,[\Omega]$ and $\omega_2 = pe-ran(f|_\Omega)$ are in M. Let us consider the measure μ occurring in the definition of ω_2, and let $g = d\mu\,/dm$. Since

$$\int_{\omega_2\backslash\omega_1} g dm = \mu(\omega_2\backslash\omega_1) = m((f|_\Omega)^{-1}[\omega_2\backslash\omega_1]) = m\phi = 0$$

and $g(\zeta) > 0$ for $\zeta \in \omega_2\backslash\omega_1$, it follows that $m(\omega_2\backslash\omega_1) = 0$. On the other hand, we have

$$m((f|_\Omega)^{-1}[\omega_1\backslash\omega_2]) = \mu(\omega_1\backslash\omega_2) = \int_{\omega_1\backslash\omega_2} g dm = 0$$

since $g(\zeta) = 0$ for (almost all) $\zeta \in \omega_1\backslash\omega_2$; thus $m(\omega_1\backslash\omega_2) = 0$ by the regularity condition.

Applying the functional calculus, for an operator in $\mathcal{L}_0(\mathcal{H})$ we want to get another operator in $\mathcal{L}_0(\mathcal{H})$, which means that the cyclic property should be preserved. Hence, univalent functions will be considered in the sequel. We recall that $f : \mathbb{D} \to \mathbb{C}$ is called a univalent function (or a conformal map) if it is analytic and injective. The range $G = f\,[\mathbb{D}]$ of f is a simply connected domain, different from $\mathbb{C}$. The boundary ∂G of G is a non-empty closed set. It is known that the geometric properties of ∂G are reflected in the analytic properties of f . For example ∂G is a curve (i.e. a continuous image of the unit circle) exactly when f belongs to the disk algebra A, and then $\partial G = f\,[\mathbb{T}]$ (see [15]). We recall that the disk algebra A consists of those analytic complex functions on $\mathbb{D}$, which can be continuously extended to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. We focus our attention to the class

$$A_1 := \{f \in A: f|_{\mathbb{D}}\ is\ univalent\,\}$$

The following proposition shows that every partially inner function in A_1 has an almost injective unimodular component. The cardinality of a set H is denoted by $|H|$. For distinct points
$\zeta_1, \zeta_2 \in \mathbb{T}$, the open arc determined by ζ_1 and ζ_2 is defined by $\widehat{\zeta_1\zeta_2} = \{e^{it}: t_1 < t < t_2\}$, where $t_1 < t_2 < t_1 + 2\pi$ and $\zeta_1 = e^{it_1}, \zeta_2 = e^{it_2}$.

Proposition(6.3.3)[1]: Let f A_1 be a partially inner function**.**

(a) If $f(\zeta_1) = f(\zeta_2) = w$ holds for distinct points $\zeta_1, \zeta_2 \in \Omega(f)$, then for one of the arcs $I = \widehat{\zeta_1\zeta_2}$ or $I = \widehat{\zeta_2\zeta_1}$ we have $m(I \cap \Omega(f)) = 0$ and $f(\zeta) = w$ for every $\zeta \in I \cap \Omega(f)$.

(b) The set $M = \{w \in \mathbb{T} : |f^{-1}[w]| > 1\}$ of multiple image points on $\mathbb{T}$ is countable.

(c) For any Borel subset Ω of $\Omega(f)$ with $m(\Omega) > 0$ we have $f[\Omega] = pe - ran(f|_\Omega)$ if and only if $f|_\Omega$ is weakly absolutely continuous.

Proof. Statement (b) is an easy consequence of statement (a).

We sketch the proof of (a), which is based on ideas taken from the proof of the related in [15].Let S denote the segment joining ζ_1 with ζ_2. Then $J = f[S]$ is a (closed) Jordan curve in $\mathbb{D} \cup \{w\}$. Let us consider the open sets $G_1 = G \cap int\, J$ and $G_2 = G \cap \text{ext J}$, where $G = f[\mathbb{D}]$. It is easy to check that $D_1 = f^{-1}[G_1], D_2 = f^{-1}[G_2]$ are the connected components of $\mathbb{D} \setminus S$, and $G_1 = f[D_1], G_2 = f[D_2]$. We may assume that $\partial D_1 = S \cup \widehat{\xi_1\xi_2}$; the other case $\partial D_1 = S \cup \widehat{\xi_2\xi_1}$ can be treated similarly. For every $\zeta \in \widehat{\xi_1\xi_2} \cap \Omega(f)$ we have

$f(\zeta) \in G_1 \cap \mathbb{T} = \{w\}$. Since $m(f^{-1}[w]) = 0$, the statement follows.

Turning to the proof of (c) notice first that $\Omega(f)$ is a compact set on $\mathbb{T}$. In view of (b) the system $S = \{\omega; \omega \subset (f), \omega, f(\omega) \text{ are Borel measurable}\}$ is a σ-algebra on $\Omega(f)$ containing compact sets; hence S consists of the Borel subsets of $\Omega(f)$.

Setting $\omega_1 = f[\Omega]$ and $\omega_2 = pe - ran(f|_\Omega)$ we know that $m(\omega_2 \setminus \omega_1) = 0$ always holds, and $m(\omega_1 \setminus \omega_2) = 0$ whenever $f|_\Omega$is weakly absolutely continuous; see the proof of Lemma (6.3.2). Assuming that $f|_\Omega$is not weakly absolutely continuous, there exists a Borel set $\omega \subset \Omega$ such that $m(\omega) = 0$ and $m(\acute{\omega}) > 0$ for $\acute{\omega} = f[\omega]$. Applying (b) again, we can see that $\int_{\acute{\omega}} g dm = \mu(\acute{\omega}) = m((f|_\Omega)^{-1}[\acute{\omega}]) = 0$ holds for $g = d\mu/dm$, and so $m(\omega_2 \cap \acute{\omega}) = 0$, whence $m(\omega_1 \setminus \omega_2) \geq m(\acute{\omega}) > 0$ follows.

The following theorem describes the functional calculus within the class $\mathcal{L}_0(\mathcal{H})$. It plays crucial role in the proof of Theorem (6.3.5).

Theorem(6.3.4)[1]: Setting $T \in \mathcal{L}_0(\mathcal{H})$, let $f \in A_1$ be a regular partially inner function such that $m(\pi(T) \cap \Omega(f)) > 0$. Then $T_0 = f(T) \in \mathcal{L}_0(\mathcal{H})$ and we have $\pi(T_0) = f[\pi(T) \cap \Omega(f)]$.

Proof. By Proposition(6.3.9) the set $M = \{w \in \mathbb{T}: |f^{-1}[w]| > 1\}$ is countable, hence $m(M) = 0$ yields $m(f^{-1}[M]) = 0$. Deleting $f^{-1}[M]$ from the quasianalytic spectral set (which is determined up to sets of measure zero), we may assume that f is injective on the $set\, \alpha = \pi(T)\Omega(f) \in M$. We know also that $\beta = f[\alpha] \in M$, and $m(\alpha) > 0, m(\beta) > 0$. Furthermore,the restriction $\phi = f|_\alpha \to \beta$ is a bijection, and for any $\omega \subset \alpha$ we have $\omega \in M$ if and only if $\phi[\omega] \in M$, and $m(\omega) = 0$ exactly when $m(\phi[\omega]) = 0$. We use the notation $\tilde{\alpha} = \pi(T) = \omega(T)$.

Let $(X, M_{\tilde{\alpha}})$ be a unitary asymptote of , with a properly chosen contractive intertwining mapping $X: XT = M_{\tilde{\alpha}} X$.Since T is a completely non-unitary contraction, it follows that

$T_0 = f(T)$is also a completely non-unitary contraction (see Chapter III in [19]). We know that T_0 is quasianalytic and $\pi(T_0) = \beta$ (see Corollary 5 in [13] and Proposition 6). The condition $m(\pi(T_0)) > 0$ yields $T_0 \in C_1$, and $T \in C_0$ readily implies $T_0 \in C_0$. Furthermore, in [13] the pair $(X_0, \phi(M_\alpha))$ is a unitary asymptote of T_0, where $X_0 v = \chi_\alpha Xv \; (v \in \mathcal{H})$ (here χ_α is the characteristic function of the set α). We know that $\phi(M_\alpha)$ is an absolutely continuous unitary operator because T_0 is an absolutely continuous contraction. It remains to show that $\phi(M_\alpha)$ is cyclic.

Let us introduce the measure v on

$$M(\beta) = \{\omega \in M: \omega \subset \beta\}$$

via

$$v(\omega) = m(\phi^{-1})[\omega].$$

The properties of ϕ imply that v is equivalent to (mutually absolutely continuous with) the Lebesgue measure on β. Let us consider the unitary operator $N_v \mathcal{L}(L^2(v)), N_v g = \chi g$, which is unitarily equivalent to M_β (see [5]). It is easy to verify that $Z : L^2(v), g \longmapsto g \circ \phi$ is a unitary transformation, intertwining N_v with $\phi(M_\alpha) : ZN_v = \phi(M_\alpha)Z$.

Therefore, $\phi(M_\alpha)$is unitarily equivalent to M_β , and so it is cyclic.

Now we proceed with the proof of Theorem (6.3.5) relying on the statement of Theorem (6.3.6).

Theorem (6.3.5)[1]: For every operator $T \in \mathcal{L}_0(\mathcal{H})$ there exists $T_1 \in \mathcal{L}_1(\mathcal{H})$ commuting with $T : TT_1 = T_1 T$.

Since the commutants$\{T\}'$ and$\{T_1\}'$ are abelian (see e.g.this Section in [13]), the relation $TT_1 = T_1 T$ implies$\{T\}' = \{T_1\}'$, and so $Hlat\, T = Hlat\, T_1$.

Proof. Let T be a contraction in the class $\mathcal{L}_0(\mathcal{H})$, and let us consider the quasianalytic spectral set $\Omega = \pi(T)$ of positive measure. By Theorem (6.3.6) there exist a compact set $\tilde{\Omega} \subset \Omega$ and a function $f \in A_1$ such that $f[\mathbb{D}]$ is a circular comb domain, $f^{-1}[\mathbb{T}] = \tilde{\Omega}$, and $f|_{\tilde{\Omega}}$ is weakly absolutely continuous. In other words, f is a regular partially inner function with $\Omega(f) = \tilde{\Omega}$ *and* $f[\tilde{\Omega}] = \mathbb{T}$. Applying Theorem (6.3.6) we conclude that $T_1 = f(T) \in \mathcal{L}_0(\mathcal{H})$ and $\pi(T_1) = f[\pi(T) \cap \Omega(f)] = f[\tilde{\Omega}] = \mathbb{T}$, whence $T_1 \in \mathcal{L}_1(\mathcal{H})$ follows. Being norm-limit of polynomials of , the operator T_1commutes with T .

First we prove Theorem(6.3.6) applying Theorem(6.3.9).

Theorem (6.3.6)[1]: If Ω is a measurable subset of the unit circle $\mathbb{T}$ of positive (linear) measure, then there are a compact set $\tilde{\Omega} \subset \Omega$ and a conformal map f from $\mathbb{D}$ onto a circular comb domain such that f can be extended to a continuous function on the closed unit disc $\overline{\mathbb{D}}, f^{-1}[\mathbb{T}] = \tilde{\Omega}$, and $m(f[\omega]) = 0$ for every Borel subset ω of $\tilde{\Omega}$ of zero measure.

Here, and in what follows, $f[A] := \{f(a): a \in A\}$ is the range of f when restricted to A, and $f^{-1}[B] := \{b : f(b) \in B\}$ is the complete inverse image of the set B under the map f . When $B = \{b\}$ has only one element, then we write $f^{-1}[b]$instead of $f^{-1}[\{b\}]$. Theorem (6.3.6) will be derived from the subsequent Theorem (6.3.9). To formulate it wee need some potential theoretical preliminaries. For all these facts see [16,8] or

[17].Let K be a compact set on $\mathbb{C}$, and let $\mathcal{P}(K)$ be the system of all probability (Borel) measures supported on K .The potential

$$p_v(z) = \int_K log \ |z - w| \, dv \, (w)$$

of a measure $v \in \mathcal{P}(K)$ is a subharmonic function on $\mathbb{C}$, which is harmonic on $\mathbb{C} \setminus K$ .The (logarithmic) capacity of K is defined by cap$(K) = exp(M(K))$, where

$$M(K) = sup \left\{ \int_K p_v dv : v \in \mathcal{P}(K) \right\}$$

If cap $(K) > 0$, then there exists a unique measure $\mu_K \in \mathcal{P}(K)$, called the equilibrium measure of K, which is maximizing the energy integral:

$$\int_K p_{\mu_K} d\mu_K = M(K);$$

we write $p_K = p_{\rho_K}$ for short. By Frostman's theorem there is an F_σ -subset F of K with cap(F)=0 such that $p_K(z) = M(K)$ for all $z = K \setminus F$, and $p_K(z) > M(K)$ for all $z \in F \cup (\mathbb{C} \setminus K)$. The compact set K is called regular, if the potential p_K is continuous on $\mathbb{C}$, or equivalently, if the previous exceptional set F is empty.

Proof. Let $\Omega \subset \mathbb{T}$ be a set of positive Lebesgue measure, and let $\Omega_1 \subset \Omega$ be a compact subset of positive measure. Applying rotation we may assume that 1 is a density point of Ω_1 ; let $\acute{\Omega}_1$ be its reflection onto the real axis. The compact set $\Omega_2 = \Omega_1 \cap \acute{\Omega}_1$ is of positive measure and symmetric with respect to $\mathbb{R}$. Let us consider the bijective Joukovskii map $\varphi: \mathbb{D} \to \bar{\mathbb{C}} \setminus [-1,1]$, defined by $\varphi(z) = (z + 1/z)/2$; the continuous extension to $\bar{\mathbb{D}}$ is also denoted by φ. Then $E = \varphi\,[\,\Omega_2\,]$ is a compact subset of $[-1,1]$ with positive measure, and $\Omega_2 = \varphi^{-1}[\varphi\,[\,\Omega_2]]$ because of the symmetry of Ω_2 .

By Theorem (6.3.9) there is a regular compact subset K of E with an absolutely continuous equilibrium measure μ_K .Let $[a, b]$ be the smallest interval containing K. Consider the analytic function

$$\Phi(z) = \exp(-\int_K log \ (z - t) d\mu_K \, (t) + log \, cap(K))$$

on the upper half plane $\mathbb{H}_+ = \{z \in \mathbb{C}: \Im\, z > 0\}$ with that branch of log which is positive on $(0, \infty)$. It is easy to see that for every $x \in \mathbb{R}$ the function ratio $\Phi(z)/|\Phi(z)|$ converges to $exp[-i\pi\mu_K((x, \infty))]$ as $z \to x$ from the upper half plane. Since $|\Phi(z)| = exp(-P_K(z)) \cdot$ cap(K) and K is regular, it follows that Φ can be continuously extended to the closure of $\mathbb{H}_+$ in $\bar{\mathbb{C}}$; $\Phi(\infty) = 0$. We can see that $\Phi(K)$ coincides with the lower circle $\mathbb{T}_- = \{z \in \mathbb{T}: \Im z \le 0\}$, $\Phi(\bar{\mathbb{R}} \setminus (a, b)) = [-1, 1]$, and each component I of $(a, b) \setminus K$ is mapped by Φ onto a radial segment of the form $\{r\zeta : \rho < r < 1\}$ with some $0 < \rho < 1$ and $\zeta \in \mathbb{T}_-$. It can be shown also that Φ is univalent; see [1]. Since $\Phi(x) = exp[-i\pi\mu_K((x, \infty))]$ for $x \in K$ and μ_K is absolutely continuous, it follows that sets of measure zero on K are mapped by Φ into sets of measure zero.

Let G_+ be the domain $\Phi(H_+)$, and G_- its reflection onto the real axis. Since $\Phi(z)$ is real for $z \in \mathbb{R} \setminus [a, b]$, using the reflection principle we can extend Φ via the definition

$\Phi(z) = \overline{\Phi(\bar{z})}$, $\Im z < 0$ to a conformal map of the domain $\bar{\mathbb{C}}\setminus[a,b]$ onto the circular comb domain $G = G_+ \cup G_- \cup (-1,1)$. Then $f = \Phi \circ \varphi$ is a conformal map from $\mathbb{D}$ onto G, it belongs to the disk algebra, and we have $f[\tilde{\Omega}] = \mathbb{T}, f[\mathbb{T}\setminus\tilde{\Omega}] \subset \mathbb{D}$ for the compact set $\tilde{\Omega} = \varphi^{-1}[K] \subset \Omega$ .If $\omega \subset \tilde{\Omega}$ is of zero linear measure, then $f[\omega]$ is also of zero linear measure. Thus $\tilde{\Omega}$ and f have all the properties set forth in the theorem.
Note also that for compact, symmetric Ω the measure of $\Omega\setminus\tilde{\Omega}$ can be made as small as we wish.
To show Theorem (6.3.9) we need two lemmas.
Lemma(6.3.7)[1]: Let $1 \le \zeta_1 < \alpha_1 < \zeta_2 < \alpha_2 < \cdots < \zeta_1 < \alpha_1$. Then for $x, y \in [-1, 0]$ we have

$$\frac{1}{2} \le \prod_{s=1}^{l}\left(\frac{\zeta_s - x}{\alpha_s - x} \Big/ \frac{\zeta_s - y}{\alpha_s - y}\right) \le 2. \tag{58}$$

In a similar manner, if $1 \le \beta_1 < \zeta_1 < \beta_2 < \cdots < \beta_1 < \zeta_1$ then for $x, y \in [-1,0]$ we have

$$\frac{1}{2} \le \prod_{s=1}^{l}\left(\frac{\zeta_s - x}{\beta_s - x} \Big/ \frac{\zeta_s - y}{\beta_s - y}\right) \le 2. \tag{59}$$

Proof. The inequalities (59) are obtained by taking reciprocal in (58) and switching the role of β_s , ζ_s and ζ_s, α_s. Similarly, in proving (58) we may assume without loss of generality that $y \le x$.
The product in (58) can be written as

$$\prod_{s=1}^{l}\left(\frac{\zeta_s - x}{\beta_s - y} \Big/ \frac{\alpha_s - x}{\alpha_s - y}\right) = \left(\frac{\zeta_1 - x}{\zeta_1 - y} \Big/ \frac{\alpha_1 - x}{\alpha_1 - y}\right) \prod_{s=1}^{l-1}\left(\frac{\zeta_{s+1} - x}{\zeta_{s+1} - y} \Big/ \frac{\alpha_s - x}{\alpha_s - y}\right)$$

($l \ge 2$ can be assumed). Since $(t - x)/(t - y)$ is increasing on $(0, \infty)$, it immediately follows from the left hand side that the product in question is at most 1. On the other hand, by the same token the second factor on the right is at least 1, so the product is at least as large as

$$\frac{\zeta_1 - x}{\zeta_1 - y} \Big/ \frac{\alpha_1 - x}{\alpha_1 - y} \ge \frac{\zeta_1 - x}{\zeta_1 - y} \ge \frac{1}{2}.$$

Let $\beta_1 < \alpha_1 < \cdots < \beta_l < \alpha_l$ be positive integers, and let $\zeta_s \in (\beta_s, \alpha_s)$ for every $1 \le s \le l$.
Taking the geometric mean of the products in (58) and (59) of Lemma (6.3.7) it follows that

$$\frac{1}{2} \le \prod_{s=1}^{l}\left(\frac{|x - \zeta_s|}{\sqrt{|x - \alpha_s||x - \beta_s|}} \Big/ \frac{|y - \zeta_s|}{\sqrt{|y - \alpha_s||y - \beta_s|}}\right) \le 2 \tag{60}$$

for every $x, y \in [-1, 0]$. Multiplying everything by (-1), and changing the notation it follows that (60) holds also, when α_s , β_s are negative integers and $x, y \in [0,1]$.Let $\mathbb{Z}$ denote the set of integers. Via scaling (multiplying everything by 2^{-N} ($N \in \mathbb{N}$) and applying translation), we obtain that (60) is true if α_s , $\beta_s \in 2^{-N}\mathbb{Z}$ for every $1 \le s \le l$ and $x, y \in [\frac{j-1}{2^N}, \frac{j}{2^N}]$ with some $j \in \mathbb{Z}$ satisfying the condition

$$j/2^N < \beta_1 \ or\ (j-1)/2^N > \alpha_l. \tag{61}$$

Given $N \in \mathbb{N}$ let $I_{N,j} = [(j-1)2^{-N}, j2^{-N}]$ for any $j \in \mathbb{Z}$. Setting a non-empty set $S \subset \{k \in \mathbb{N}: k \le 2^N\}$ of non-consecutive indexes, let us consider the compact set $F = \bigcup_{j \in S} I_{N,j}$, which can be written in the form $F = \bigcup_{s=1}^{n}[a_s, b_s]$ with $a_1 < b_1 < a_2 < b_2 < \cdots < b_n$ ($n \ge 2$). The equilibrium measure μ_F of F is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}$, and its density function is given by the formula

$$\psi(t) = (d\mu_F/dm)(t) = \frac{1}{\pi}\frac{\prod_{s=1}^{n-1}|t-\tau_s|}{\prod_{s=1}^{n}\sqrt{|t-\alpha_s||t-b_s|}}dt, \qquad t \in F, \qquad (62)$$

where the numbers $\tau_s \in (b_s, a_{s+1})$ $(1 \le s \le n-1)$ are the unique solution of the system of equations

$$\int_{b_k}^{a_{k+1}} \frac{\prod_{s=1}^{n-1}|t-\tau_s|}{\prod_{s=1}^{n}\sqrt{|t-a_s||t-b_s|}}dt = 0, \qquad 1 \le k \le n-1. \qquad (63)$$

This is a linear system in the coefficients of the polynomial $\prod_{s=1}^{n-1}|t-\tau_s|$.
When $n = 1$ then the product in the numerator (62) is replaced by 1. For all these see [18] and [17].

Lemma (6.3.8)[1]: Let $0 < \eta < 1/2, j \in S$, and H a measurable subset of $I_{N,j}$ $(N, S, F$ and $I_{N,j}$ are as before). If

$$m(H) \ge (1-2\eta)m\left(I_{N,j}\right), \qquad (64)$$

then

$$\mu_F(H) \ge \left(1-2^{29}\eta^{\frac{1}{2}}\right)\mu_F\left(I_{N,j}\right). \qquad (65)$$

Proof. We shall give an estimate of the density function ψ on $I_{N,j}$. Assuming that $I_{N,j} \subseteq [a_r, b_r]$, this estimate depends on the position of $I_{N,j}$ inside $[a_r, b_r]$.

Case I. $a_r, b_r \notin I_{N,j}$,i.e $I_{N,j}$ lies inside (a_r, b_r).For $x, y \in I_{N,j}$ we can write

$$\frac{\psi(x)}{\psi(y)} = \sqrt{\frac{|y-a_1|}{|x-a_1|}\Big/\frac{|x-b_n|}{|y-b_n|}}\cdot\frac{\theta_{1,r-1}(x)}{\theta_{1,r-1}(y)}\cdot\frac{\theta_{1,n-1}(x)}{\theta_{1,n-1}(y)}, \qquad (66)$$

where

$$\theta_{k,l}(x) = \frac{\prod_{s=k}^{l}|x-\tau_s|}{\prod_{s=k}^{l}\sqrt{|x-a_{s+1}||x-b_s|}}$$

$(\theta_{1,0} = \theta_{n,n\ 1} = 1$ by definition). Since each factor in this decomposition (66) of $\psi(x)/\psi(y)$ lies between $1/2$ and 2 by (61), it follows that

$$\frac{1}{8}\psi(y) \le \psi(x) \le 8\psi(y). \qquad (67)$$

Case II. Precisely one of a_r, b_r belongs to $I_{N,j}$. Then either $j2^{-N} = b_r$ or $(j-1)2^{-N} = a_r$, say $j2^{-N} = b_r$. We shall consider only the situation when $1 < r < n$, for the other options (i.e. when $r = 1$ or $r = n$) are simpler. In this case

$$\pi\psi(x) = \frac{|x-\tau_r|}{\sqrt{|x-b_r||x-a_{r+1}|}}\cdot\theta_1(x)\theta_2(x), \qquad (68)$$

where

$$\theta_1(x) = \frac{1}{\sqrt{|x-a_1|}}.\theta_{1.r-1}(x)$$

and

$$\theta_2(x) = \frac{1}{\sqrt{|x-b_n|}}.\theta_{r+1,n-1}(x).$$

Next we prove that here

$$\tau_r - b_r \ge 2^{-8}2^{-N}. \qquad (69)$$

If $\tau_r - b_r \ge 2^{-N}$ then there is nothing to prove, so let us assume that $\tau_r \in [b_r, b_r + 2^{-N}]$.For $t \in [b_r, b_r + 2^{-N}]$ the claim (61) gives the bounds

$$\frac{\theta_i(b_r)}{4} \leq \theta_i(t) \leq 4\theta_i(b_r), \ i = 1,2. \qquad (70)$$

For k = r Eq. (63) can be written as

$$\int_{b_r}^{a_{r+1}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}} \,.\, \theta_1(t)\theta_2(t)dt = 0,$$

So

$$\int_{b_r}^{\tau_r} \frac{\tau_r - t}{\sqrt{(t-b_r)(a_{r+1}-t)}} \theta_1(t)\theta_2(t)dt = \int_{\tau_r}^{a_{r+1}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}} \theta_1(t)\theta_2(t)dt$$

$$\geq \int_{\tau_r}^{b_r+2^{-N}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}} \theta_1(t)\theta_2(t)dt.$$

In view of (70) this gives after division by $\theta_1(b_r)\theta_2(b_r)$ the inequality

$$\int_{b_r}^{\tau_r} \frac{\tau_r - t}{\sqrt{(t-b_r)(a_{r+1}-t)}} 16dt \geq \int_{\tau_r}^{b_r+2^{-N}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}} \frac{1}{16} dt.$$

If we make a linear substitution so that $[b_r, b_r + 2^{-N}]$ becomes $[0,1]$ and make use that for $0 \leq \tau \leq 2^{-8}$ and $l \in \mathbb{N}$ the inequality

$$\int_0^{\tau} \frac{\tau - u}{\sqrt{u(l-u)}} 16du < \int_{\tau}^{1} \frac{u-\tau}{\sqrt{u(l-u)}} \frac{1}{16} du$$

holds, we can conclude (69).

Now (69)immediately gives that for $x, y \in I_{N,j}$

$$\frac{|x-\tau_r|}{|y-\tau_r|} \leq 2^9. \qquad (71)$$

Next note that along with (70) the bounds

$$\frac{\theta_i(y)}{4} \leq \theta_i(x) \leq 4\theta_i(y) \quad (i = 1,2) \qquad (72)$$

are also true for $x, y \in I_{N,j}$ (since $(j-1)2^{-N}$ is not an endpoint of a subinterval of F), so (68), (71) and (72) yield for $x, y \in I_{N,j}$

$$\frac{\psi(x)\sqrt{|x-b_r|}}{\psi(y)\sqrt{|y-b_r|}} \leq 16 \frac{|x-\tau_r|}{|y-\tau_r|} \sqrt{\frac{|y-a_{r+1}|}{|x-a_{r+1}|}} \leq 2^{14}.$$

By reversing the role of x and y and then fixing y to be the center of $I_{N,j}$ we can conclude with $c = \sqrt{|b_r - y|}\,\psi(y)$

$$c2^{-14} \frac{1}{\sqrt{b_r - x}} \leq \psi(x) \leq c2^{14} \frac{1}{\sqrt{b_r - x}}, \qquad x \in I_{N,j}\,. \quad (73)$$

Case III. $a_r, b_r \in I_{N,j}$. Then $I_{N,j} = [a_r, b_r]$. In this case (72) holds only on the right half $I_{N,j}^{+}$ of $I_{N,j}$, so we can conclude (73) (with $y = (a_r + b_r)/2$) only there. However, an analogous argument gives that on the left half $I_{N,j}^{-}$ of $I_{N,j}$ we have

$$c2^{-14} \frac{1}{\sqrt{x-a_r}} \leq \psi(x) \leq c2^{14} \frac{1}{\sqrt{x-a_r}}. \qquad (74)$$

Thus, we have the estimates (68), (73)or(74) for ψ on $I_{N,j}$, depending on the position of the interval $I_{N,j}$ in the set F .

Let now H be a measurable subset of $I_{N,j}$ with measure $m(H) \geq (1-2\eta)m(I_{N,j})$ and let $H_0 = I_{N,j} \setminus H$. Assume that Case III holds for the interval $I_{N,j}$. (In Case II the same argument can be applied, and in Case I the computations based on (67) are actually much simpler, giving a better estimate.) Let $I^+$and I^- denote the right half and the left half of the interval $I_{N,j}$, respectively. Then, using (73) on I^+, we can see that

$$\int_{H_0\cap I^+}\psi(x)dx \le \int_{H_0\cap I^+} c2^{14}\frac{1}{\sqrt{b_r-x}}dx$$
$$\le c2^{14}2m(H_0)^{1/2} \le c2^{15}(2\eta)^{1/2}m\left(I_{N,j}\right)^{1/2}$$
$$\le c2^{15}\eta^{1/2}2m(I^+)^{1/2} = \eta^{1/2}2^{15}c\int_{I^+}\frac{1}{\sqrt{b_r-x}}dx$$
$$= \eta^{1/2}2^{29}\int_{I^+}\frac{c2^{-14}}{\sqrt{b_r-x}}dx \le \eta^{1/2}2^{29}\int_{I^+}\psi(x)\,dx.$$

Since a similar bound can be given for the integral over $H_0 \cap I^-$ using (74), it follows that $\mu_F(H_0 \le)2^{29}\eta^{1/2}\mu_F\left(I_{N,j}\right)$. Then we conclude that $\mu_F(H) \ge (1 - 2^{29}\eta^{1/2}\ \mu_F\left(I_{N,j}\right))$ as was to be proved.

Now we are ready to show Theorem (6.3.9).

Theorem(6.3.9)[1]: Let $E \subset \mathbb{R}$ be a compact set of positive Lebesgue measure. Then for every $\boldsymbol{\varepsilon} > 0$, there is a regular compact set $K \subset E$ such that $m(E \setminus K) < \boldsymbol{\varepsilon}$, and μ_K is absolutely continuous with respect to the Lebesgue measure on the real line $\mathbb{R}$.

Proof. Without loss of generality we may assume that the compact set E of positive Lebesgue measure is contained in $[0,1]$. For an $N \in \mathbb{N}$ and $\delta > 0$ let us consider the finite set

$$S(E,N,\delta) := \left\{j \in \mathbb{N}: m\left(E \cap I_{N,j}\right) \ge (1-\delta)m\left(I_{N,j}\right)\right\},$$

and let

$$E(N,\delta) := \cup\ \left\{I_{N,j}\ :\ j \in S(E,N,\delta)\right\}.$$

By Lebesgue's density theorem almost all $x \in E$ belongs to all $E(N,\delta)$ for sufficiently large N,i.e. to

$$\cup_{M=1}^{\infty}\cap_{N=M}^{\infty}(E \cap E(N,\delta)).$$

Thus

$$\lim_{M\to\infty} m(\cap_{N=M}^{\infty}\left(E \cap E(N,\delta)\right)) = m(E).$$

Whence

$$\lim_{N\to\infty} m\left(E \cap E(N,\delta)\right) = m(E)$$

follows.

Let there be given an $\varepsilon \in (0, m(E)/4)$. Set $\varepsilon_n = \varepsilon/2^n$ for $n \in \mathbb{N}$, and recursively define the positive integers $N_1 < N_2 < \cdots$and the closed sets $E \supset E_1 \supset E_2 \supset \cdots$in the following manner.

Let N_1 be so large that

$$m(E \setminus E(N_1,\varepsilon_1)) < \varepsilon_1,$$

and set $E_1 = E \cap E(N_1,\varepsilon_1)$. In general, if N_n, E_n have already been defined, then select a large $N_{n+1} > N_n$ so that

$$m\left(E_n \setminus E_n(N_{n+1},\varepsilon_{n+1}\right) < \varepsilon_{n+1}/2^{N_n},$$

and let $E_{n+1} = E_n \cap E_n(N_{n+1},\varepsilon_{n+1})$. We obtain the sequences $\{N_n\}_{n=1}^{\infty}$ and$\{E_n\}_{n=1}^{\infty}$ The compact subset K of E is defined by $K = \cap_{n=1}^{\infty} E_n$.

Setting $N_0 = 0$ and $E_0 = E$,we have $m(E_n \setminus E_{n+1}) < \varepsilon_{n+1}1/2^{N_n}$ for every $n \ge 0$, hence

$$m(E \setminus K) < \sum_{n=0}^{\infty} \varepsilon_{n+1} \ /2^{N_n} = \sum_{n=0}^{\infty} \varepsilon/2^{\mathrm{n+1+N_n}} < \varepsilon,$$

in particular $m(K\ ) > 3m(E)/4 > 0$. Furthermore, given $n \in \mathbb{N}$ for every $j \in S(E_{n-1}, N_n, \varepsilon_n)$ we have $E_{n-1} \cap I_{N_n ,j} = E_n \cap I_{N_n ,j}$ and so, by the definition of $S(E_{n-1}, N_n, \varepsilon_n\ )$,we have $m(E_n \cap I_{N_n ,j}) \geq (1 - \varepsilon_n)m(I_{N_n ,j})$. Since for $k \geq 0$

$$m(E_{n+k} \setminus E_{n+k+1}) \leq \varepsilon_{n+k+1} \ /2^{N_{n+k}} \leq \varepsilon_n/2^{N_n+k+1} = \frac{\varepsilon_n}{2^{k+1}} m(I_{N_n ,j})$$

it follows

$$m\left(K \cap I_{N_n ,j}\right) \geq m\left(E_n \cap I_{N_n ,j}\right) - \sum_{k=0}^{\infty} m(E_{n+k} \setminus E_{n+k+1})$$

$$\geq (1 - 2\varepsilon_n)m\left(I_{N_n ,j}\right). \qquad (75)$$

Set $z_0 \in K$, and for any $k \in \mathbb{N}$ let

$$K_k = K \cap \{z \in \mathbb{C}: 2^{-k-1} \leq |z - z_0| \leq 2^{-k}\} .$$

For every $n \in \mathbb{N}$ there is an index $j_n \in S(E_{n-1}, N_n, \varepsilon_n)$ such that $z_0 \in I_{N_n j_n,}$. Since $cap(H\ ) \geq m(H\ )/4$ for any Borel subset of the real line, applying (75) we obtain

$$cap(K_{N_{n+1}}\ ) \geq \frac{m(K_{N_{n+1}})}{4} \geq \frac{1}{4}\left(\frac{1}{4} - 2\varepsilon_n\right) \mathrm{m}(I_{N_n j_n,}) \geq 2^{-\mathrm{N_n}-1}.2^{-4},$$

Whence

$$\frac{\mathrm{N}_n + 1}{log(1/\, cap(K_{N_{n+1}}))} \geq \frac{1}{2}$$

follows (provided $n \geq 3$). Thus

$$\sum_{K=1}^{\infty} \frac{\mathrm{k}}{log\,(1/\,cap\,(K_k))} = \infty$$

and so Wiener's criterion (see [16, Theorem 5.4.1]) yields that the compact set K is regular.

It remains to show that the measure μ_K is absolutely continuous. Let $V \subset K$ be a set of measure zero, and let $U = K \setminus V$ .Forn $n \in \mathbb{N}$, let us consider the set

$$F_n = E_{n-1}(N_n, \varepsilon_n) = \bigcup \{I_{N_n ,j} \ : \ j \in \ S_n\},$$

where $S_n = S(E_{n-1}\ , N_n, \varepsilon_n)$. We know from (75) that

$$m(U \cap I_{N_n ,j}\ ) = m(K \cap I_{N_n ,j}\ ) \geq (1 - 2\varepsilon_n)m(I_{N_n ,j}\ )$$

holds for every $j \in S_n$. Then Lemma(6.3.9) implies

$$\mu_{F_n}\,(U \cap I_{N_n ,j}) \geq \left(1 - 2^{29}\varepsilon_n^{1/2}\right)\mu_{F_n}\left(I_{N_n ,j}\right).$$

Summing up for $j \in S_n$ we get

$$\mu_{F_n}\,(U) \geq 1 - 2^{29}\varepsilon_n^{\frac{1}{2}}$$

Since $K \subset F_n \subset \mathbb{R}$ the measure μ_K is obtained by adding to the restriction $\mu_{F_n}|K$ the so called balayage $\mu_{F_n}|(F_n \setminus K)$ onto K (see [17]). Therefore

$$\mu_K(U) \geq \ \mu_{F_n}\,(U) \geq 1 - 2^{29}\varepsilon_n^{\frac{1}{2}}\,,$$

and so

$$\mu_K\,(V\ ) = 1 - \mu_K(U) \leq 2^{29}\varepsilon_n^{\frac{1}{2}}$$

hold for every $n \in \mathbb{N}$. By letting n tend to infinity we conclude that $\mu_K(V) = 0$.

Corollary(6.3.10)[202]: Let$\sum_{k=1}^{m'} f_k \in H^\infty$ be a partially inner function.

(a)Then $\sum_{k=1}^{m'} f_k$is regular if and only if for every measurable set $\Omega_{k-1} \subset \Omega_{k-1}(\sum_{k=1}^{m'} f_k)$ the image set $\sum_{k=1}^{m} f_k\ [\Omega_{k-1}]$ is also measurable.

(b)If $\sum_{k=1}^{m'} f_k$ is regular and $\Omega_{k-1} \in M, \Omega_{k-1} \subset \Omega_{k-1}(\sum_{k=1}^{m'} f_k)$, then $pe - ran(\sum_{k=1}^{m'} f_k \sum_{k=1}^{m'} f_k\, |_{\Omega_{k-1}}) = f \sum_{k=1}^{m'} f_k\,(\Omega_{k-1})$.

Recall that $pe - ran(\sum_{k=1}^{m'} f_k\, |_{\Omega_{k-1}})$ is determined only up to measure zero, so the equality $pe - ran(\sum_{k=1}^{m'} f_k\, |_{\Omega_{k-1}}) = \sum_{k=1}^{m'} f_k\,(\Omega_{k-1})$ is also understood up to measure zero.

Proof. (a): We sketch the proof of this known equivalence. Suppose that $\sum_{k=1}^{m'} f_k$ is regular, and let $\Omega_{k-1} \in M, \Omega_{k-1} \subset \Omega_{k-1}(\sum_{k=1}^{m'} f_k)$. Since $f \sum_{k=1}^{m'} f_k\, |_{\Omega_{k-1}}$ is the pointwise limit of a sequence of continuous functions, it follows from Egorov's theorem that $\Omega_{k-1} = \Omega_k \cup \Omega_{k+1}$, where Ω_k and $\sum_{k=1}^{m'} f_k\ [\Omega_k]$ are F_σ-sets and $m(\Omega_{k+1}) = 0$. Hence, by assumption, $m(\sum_{k=1}^{m'} f_k\ [\Omega_{k+1}]) = 0$ and thus $\sum_{k=1}^{m'} f_k\ [\Omega_{k-1}] \in M$.

Conversely, if $\sum_{k=1}^{m'} f_k$ is non-regular, then $m(\sum_{k=1}^{m'} f_k\ [\omega]) = 0$ fails for some $\omega \subset \Omega_{k-1}(\sum_{k=1}^{m'} f_k)$ with $m(\omega) = 0$.

There is a non-measurable subset Ω'_{k-1} of $\sum_{k=1}^{m'} f_k\,[\omega]$. Thus $\Omega_{k-1} = \sum_{k=1}^{m'} f_k^{-1}\ [\Omega'_{k-1}] \cap \omega \in M$, while $\sum_{k=1}^{m'} f_k\ [\Omega_{k-1}] = \Omega'_{k-1} \notin M$.

(b): The sets $\omega_k = \sum_{k=1}^{m'} f_k\ [\Omega_{k-1}]$ and $\omega_{k+1} = pe - ran(\sum_{k=1}^{m'} f_k\, |_{\Omega_{k-1}})$ are in M. Let us consider the measure μ occurring in the definition of ω_{k+1}, and let $\sum_{k=1}^{m'} g_k = d\mu/dm$. Since

$$\int_{\omega_{k+1}\backslash\omega_k} \sum_{k=1}^{m'} g_k\, dm = \mu(\omega_{k+1}\backslash\omega_k) = m\left((\sum_{k=1}^{m'} f_k\, |_{\Omega_{k-1}})^{-1}[\omega_{k+1}\backslash\omega_k] \right) = m\phi = 0$$

and $\sum_{k=1}^{m'} g_k(\zeta) > 0$ for $\zeta \in \omega_{k+1}\backslash\omega_k$, it follows that $m(\omega_{k+1}\backslash\omega_k) = 0$. On the other hand,we have

$$m\left(\left(\sum_{k=1}^{m'} f_k\, |_{\Omega_{k-1}}\right)^{-1}[\omega_k\backslash\omega_{k+1}]\right) = \mu(\omega_k\backslash\omega_{k+1}) = \int_{\omega_k\backslash\omega_{k+1}} \sum_{k=1}^{m'} g_k\, dm = 0$$

since $\sum_{k=1}^{m'} g_k(\zeta) = 0$ for (almost all) $\zeta \in \omega_k\backslash\omega_{k+1}$; thus $m(\omega_k\backslash\omega_{k+1}) = 0$ by the regularity condition.

Applying the functional calculus, for the power operator in $\mathcal{L}_{n-1}(\mathcal{H})$ we want to get another power operator in $\mathcal{L}_{n-1}(\mathcal{H})$, which means that the cyclic property should be preserved. Hence, univalent functions will be considered in the sequel. We recall that $\sum_{k=1}^{m'} f_k : \mathbb{D} \to \mathbb{C}$ is called a univalent function (or a conformal map) if it is analytic and injective. The range $G_k = \sum_{k=1}^{m'} f_k\ [\mathbb{D}]$ of $\sum_{k=1}^{m'} f_k$ is a simply connected domain, different from $\mathbb{C}$. The boundary ∂G_k of G_k is a non-empty closed set. It is known that the geometric properties of ∂G_k are reflected in the analytic properties of $\sum_{k=1}^{m'} f_k$. For example ∂G_k is a curve (i.e. a continuous image of the unit circle) exactly when$\sum_{k=1}^{m'} f_k$ belongs to the disk algebra A, and then $\partial G_k = \sum_{k=1}^{m} f_k\ [\mathbb{T}]$ (see [15]). We recall that the disk algebra A consis$G_k$ts of those analytic complex functions on $\mathbb{D}$, which can be continuously extended to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. We focus our attention to the class

$$A_k := \{\sum_{k=1}^{m'} f_k \in A: \sum_{k=1}^{m'} f_k \mid_{\mathbb{D}} \, is\ univalent\ \}.$$

The following proposition (see [1]) shows that every partially inner function in A_k has an almost injective unimodular component. The cardinality of a set H is denoted by $|H|$. For distinct points $\zeta_k, \zeta_{k+1} \in \mathbb{T}$, the open arc determined by $\zeta_k$and $\zeta_{k+1}$is defined by $\widehat{\zeta_k\zeta_{k+1}} = \{e^{it}: t_k < t < t_{k+1}\}$, where $t_k < t_{k+1} < t_k + 2\pi$ and $\zeta_k = e^{it_k}, \zeta_{k+1} = e^{it_{k+1}}$.

Corollary(6.3.11)[202]: Let $\sum_{k=1}^{m'} f_k \in A_k$ be a partially inner function**.**

(a) If $\sum_{k=1}^{m'} f_k(\zeta_1) = \sum_{k=1}^{m'} f_k(\zeta_2) = w$ holds for distinct points $\zeta_k, \zeta_{k+1} \in \Omega_{k-1}\left(\sum_{k=1}^{m'} f_k\right)$, then for one of the arcs $I = \widehat{\zeta_k\zeta_{k+1}}$ or $I = \widehat{\zeta_{k+1}\zeta_k}$we have $m(I \cap \Omega_{k-1}(\sum_{k=1}^{m'} f_k)) = 0$ and $\sum_{k=1}^{m'} f_k(\zeta) = w_k$ for every $\zeta \in I \cap \Omega_{k-1}(\sum_{k=1}^{m'} f_k)$.

(b) The set $M = \{w_k \in \mathbb{T} : |\sum_{k=1}^{m'} f_k^{-1}[w_k]| > 1\}$ of multiple image points on $\mathbb{T}$ is countable.

(c) For any Borel subset Ω_{k-1} of $\Omega_{k-1}(\sum_{k=1}^{m'} f_k)$ with $m(\Omega_{k-1}) > 0$ we have $\sum_{k=1}^{m'} f_k[\Omega_{k-1}] = pe-ran(\sum_{k=1}^{m'} f_k \mid_{\Omega_{k-1}})$ if and only if $\sum_{k=1}^{m'} f_k \mid_{\Omega_{k-1}}$is weakly absolutely continuous.

Proof. Statement (b) is an easy consequence of statement (a).

We sketch the proof of (a), which is based on ideas taken from the proof in [16]. Let S denote the segment joining ζ_1 with ζ_2. Then $J = \sum_{k=1}^{m'} f_k[S]$ is a (closed) Jordan curve in $\mathbb{D} \cup \{w\}$. Let us consider the open sets $G_k = G_{k-1} \cap int\, J$ and $G_{k+1} = G_{k-1} \cap$ ext J, where $G_{k-1} = \sum_{k=1}^{m'} f_k[\mathbb{D}]$. It is easy to check that $D_k = \sum_{k=1}^{m'} f_k^{-1}[G_k]$, $D_{k+1} = \sum_{k=1}^{m'} f_k^{-1}[G_{k+1}]$ are the connected components of $\mathbb{D} \setminus S$, and $G_k = \sum_{k=1}^{m'} f_k[D_k], G_{k+1} = \sum_{k=1}^{m'} f_k[D_{k+1}]$. We may assume that $\partial D_k = S \cup \widehat{\xi_k\xi_{k+1}}$; the other case $\partial D_k = S \cup \widehat{\xi_{k+1}\xi_k}$ can be treated similarly. For every $\zeta \in \widehat{\xi_k\xi_{k+1}} \cap \Omega_{k-1}(\sum_{k=1}^{m} f_k)$ we have $\sum_{k=1}^{m'} f_k(\zeta) \in \overline{G_k} \cap \mathbb{T} = \{w_k\}$. Since $m(\sum_{k=1}^{m'} f_k^{-1}[w_k]) = 0$, the statement follows.

Turning to the proof of (c) notice first that $\Omega_{k-1}(\sum_{k=1}^{m'} f_k)$ is a compact set on $\mathbb{T}$. In view of (b) the system

$S = \{\omega; \omega \subset \left(\sum_{k=1}^{m'} f_k\right), \omega, \sum_{k=1}^{m} f_k(\omega)$ are Borel measurable$\}$

is a σ-algebra on $\Omega_{k-1}(\sum_{k=1}^{m'} f_k)$ containing compact sets; hence S consists of the Borel subsets of $\Omega_{k-1}(\sum_{k=1}^{m'} f_k)$.

Setting $\omega_k = \sum_{k=1}^{m'} f_k[\Omega_{k-1}]$ and $\omega_{k+1} = pe-ran(\sum_{k=1}^{m'} f_k \mid_{\Omega_{k-1}})$ we know that $m(\omega_{k+1} \setminus \omega_k) = 0$ always holds, and $m(\omega_k \setminus \omega_{k+1}) = 0$ whenever $\sum_{k=1}^{m'} f_k \mid_{\Omega_{k-1}}$ is weakly absolutely continuous; see the proof of Lemma (6.3.2). Assuming that $\sum_{k=1}^{m} f_k \mid_{\Omega_{k-1}}$ is not weakly absolutely continuous, there exists a Borel set $\omega \subset \Omega_{k-1}$ such that $m(\omega) = 0$ and $m(\acute{\omega}) > 0$ for $\acute{\omega} = \sum_{k=1}^{m'} f_k[\omega]$. Applying (b) again, we can see that $\sum_{k=1}^{m'} f_k \sum_{k=1}^{m'} f_k f_{\acute{\omega}} \sum_{k=1}^{m'} g_k\, dm = \mu(\acute{\omega}) = m((\sum_{k=1}^{m'} f_k \mid_{\Omega_{k-1}})^{-1}[\acute{\omega}]) = 0$ holds for $\sum_{k=1}^{m'} g_k = d\mu/dm$, and so $m(\omega_{k+1} \cap \acute{\omega}) = 0$, whence $m(\omega_k \setminus \omega_{k+1}) \geq m(\acute{\omega}) > 0$ follows.

The following theorem describes the functional calculus within the class $\mathcal{L}_{n-1}(\mathcal{H})$. It plays crucial role in the proof of Theorem (6.3.5).

Corollary(6.3.12)[202]: Setting $T^{2n-1} \in \mathcal{L}_{n-1}(\mathcal{H})$, let $\sum_{k=1}^{m'} f_k \in A_k$ be a regular partially inner function such that $m(\pi\,(T^{2n-1}) \cap \Omega_{k-1}(\sum_{k=1}^{m'} f_k)) > 0$. Then $T_{n-1}^{2n-1} = \sum_{k=1}^{m'} f_k\,(T^{2n-1}) \in \mathcal{L}_{n-1}(\mathcal{H})$ and we have $\pi(T_{n-1}^{2n-1}) = \sum_{k=1}^{m'} f_k\,[\pi(T^{2n-1}) \cap \Omega_{k-1}\,(\sum_{k=1}^{m'} f_k)]$.

Proof. By Proposition 6 the set $M = \{w_k \in \mathbb{T}: |\sum_{k=1}^{m'} f_k{}^{-1}\,[w_k]| > 1\}$ is countable, hence $m(M) = 0$ yields $m(\sum_{k=1}^{m'} f_k{}^{-1}[M]) = 0$. Deleting $\sum_{k=1}^{m'} f_k{}^{-1}[M]$ from the quasianalytic spectral set (which is determined up to sets of measure zero), we may assume that $\sum_{k=1}^{m'} f_k$ is injective on the $set\ \alpha = \pi\,(T^{2n-1}) \cap \Omega_{k-1}(\sum_{k=1}^{m'} f_k) \in M$. We know also that $\beta = \sum_{k=1}^{m'} f_k\ [\alpha] \in M$, and $m(\alpha) > 0, m(\beta) > 0$. Furthermore, the restriction $\phi = \sum_{k=1}^{m'} f_k\ |_\alpha \to \beta$ is a bijection, and for any $\omega \subset \alpha$ we have $\omega \in M$ if and only if $\phi[\omega] \in M$, and $m(\omega) = 0$ exactly when $m(\phi[\omega]) = 0$. We use the notation $\tilde{\alpha} = \pi\,(T^{2n-1}) = \omega\,(T^{2n-1})$.

Let $(X, M_{\tilde{\alpha}})$ be a unitary asymptote of , with a properly chosen contractive intertwining mapping $X: XT^{2n-1} = M_{\tilde{\alpha}}\,X$.

Since T^{2n-1} is a completely non-unitary power contraction, it follows that $T_{n-1}^{2n-1} = \sum_{k=1}^{m'} f_k\,(T^{2n-1})$ is also a completely non-unitary power contraction (see [19]). We know that T_{n-1}^{2n-1} is quasianalytic and $\pi(T_{n-1}^{2n-1}) = \beta$ (see [13]). The condition $m(\pi(T_{n-1}^{2n-1})) > 0$ yields $T_{n-1}^{2n-1} \in C_k$, and $T^{2n-1} \in C_{k-1}$ readily implies $T_{n-1}^{2n-1} \in C_{k-1}$. Furthermore, by [13] the pair $(X_0, \phi(M_\alpha))$ is a unitary asymptote of T_{n-1}^{2n-1}, where $X_0 v = \chi_\alpha X v\ (v \in \mathcal{H})$ (here χ_α is the characteristic function of the set α). We know that $\phi(M_\alpha)$ is an absolutely continuous unitary power operator because T_{n-1}^{2n-1} is an absolutely continuous power contraction. It remains to show that $\phi(M_\alpha)$ is cyclic.

Let us introduce the measure v on

$$M(\beta) = \{\omega \in M: \omega \subset \beta\}$$

via

$$v(\omega) = m(\phi^{-1})[\omega].$$

The properties of ϕ imply that v is equivalent to (mutually absolutely continuous with) the Lebesgue measure on β. Let us consider the unitary operator $N_v \in \mathcal{L}(L^2(v)), N_v g_k = \chi g_k$, which is unitarily equivalent to M_β (see [4]). It is easy to verify that $Z: L^2(v) \to L^2(\alpha),\ g_k \longmapsto g_k \circ \phi$ is a unitary transformation, intertwining N_v with $\phi(M_\alpha): ZN_v = \phi\,(M_\alpha)Z$. Therefore, $\phi(M_\alpha)$ is unitarily equivalent to M_β [1], and so it is cyclic.

Now we show the following:

Corollary(6.3.13)[202]: For every power operator $T^{2n-1} \in \mathcal{L}_{n-1}(\mathcal{H})$ there exists $T_n^{2n-1} \in \mathcal{L}_n(\mathcal{H})$ commuting with $T^{2n-1}: T^{2n-1}T_n^{2n-1} = T_n^{2n-1}T^{2n-1}$.

Since the commutants $\{T^{2n-1}\}'$ and $\{T_n^{2n-1}\}'$ are abelian (see [13]), the relation $T^{2n-1}T_n^{2n-1} = T_n^{2n-1}T^{2n-1}$ implies $\{T^{2n-1}\}' = \{T_n^{2n-1}\}'$, and so $Hlat\,T^{2n-1} = Hlat\,T_n^{2n-1}$.

Proof. Let T^{2n-1} be a power contraction in the class $\mathcal{L}_{n-1}(\mathcal{H})$, and let us consider the quasianalytic spectral set $\Omega_{k-1} = \pi(T^{2n-1})$ of positive measure. By Theorem (6.3.6) there exist a compact set $\tilde{\Omega}_{k-1} \subset \Omega_{k-1}$ and a function $f_k \in A_k$ such that $\sum_{k=1}^{m'} f_k[\mathbb{D}]$ is a circular comb domain, $\sum_{k=1}^{m'} f_k^{-1}[\mathbb{T}] = \tilde{\Omega}_{k-1}$, and $\sum_{k=1}^{m'} f_k \mid_{\tilde{\Omega}_{k-1}}$ is weakly absolutely continuous. In other words, $\sum_{k=1}^{m} f_k$ is a regular partially inner function with $\Omega_{k-1}(\sum_{k=1}^{m} f_k) = \tilde{\Omega}_{k-1}$ *and* $\sum_{k=1}^{m} f_k[\tilde{\Omega}_{k-1}] = \mathbb{T}$. Applying Theorem (6.3.6) we conclude that $T_n^{2n-1} = \sum_{k=1}^{m'} f_k(T^{2n-1}) \in \mathcal{L}_{n-1}(\mathcal{H})$ and $\pi(T_n^{2n-1}) = \sum_{k=1}^{m} f_k[\pi(T^{2n-1}) \cap \Omega_{k-1}(\sum_{k=1}^{m'} f_k)] = \sum_{k=1}^{m'} f_k[\tilde{\Omega}_{k-1}] = \mathbb{T}$, whence $T_n^{2n-1} \in \mathcal{L}\sum_{k=1}^{m'} f_{k_n}(\mathcal{H})$ follows. Being norm-limit of polynomials of T^{2n-1}, the power operator T_n^{2n-1} commutes with T^{2n-1}.

Corollary(6.3.14)[202]: If Ω_{k-1} is a measurable subset of the unit circle $\mathbb{T}$ of positive (linear) measure, then there are a compact set $\widetilde{\Omega_{k-1}} \subset \Omega_{k-1}$ and a conformal map $\sum_{k=1}^{m'} f_k$ from $\mathbb{D}$ onto a circular comb domain such that $\sum_{k=1}^{m'} f_k$ can be extended to a continuous function on the closed unit disc $\overline{\mathbb{D}}$, $\sum_{k=1}^{m'} f_k^{-1}[\mathbb{T}] = \widetilde{\Omega_{k-1}}$, and $m(\sum_{k=1}^{m'} f_k[\omega]) = 0$ for every Borel subset ω of $\widetilde{\Omega_{k-1}}$ of zero measure.

Here, and in what follows, $\sum_{k=1}^{m'} f_k[A] := \{\sum_{k=1}^{m'} f_k(a) : a \in A\}$ is the range of $\sum_{k=1}^{m'} f_k$ when restricted to A, and $\sum_{k=1}^{m'} f_k^{-1}[B] := \{b : \sum_{k=1}^{m'} f_k(b) \in B\}$ is the complete inverse image of the set B under the map $\sum_{k=1}^{m'} f_k$. When $B = \{b\}$ has only one element, then we write $\sum_{k=1}^{m'} f_k^{-1}[b]$ instead of $\sum_{k=1}^{m'} f_k^{-1}[\{b\}]$.

Proof . Let $\Omega_{k-1} \subset \mathbb{T}$ be a set of positive Lebesgue measure, and let $\Omega_k \subset \Omega_{k-1}$ be a compact subset of positive measure. Applying rotation we may assume that 1 is a density point of Ω_k ; let $\acute{\Omega}_k$ be its reflection onto the real axis. The compact set $\Omega_{k+1} = \Omega_k \cap \acute{\Omega}_k$ is of positive measure and symmetric with respect to $\mathbb{R}$. Let us consider the bijective Joukovskii map $\varphi: \mathbb{D} \to \bar{\mathbb{C}} \setminus [-1,1]$, defined by $\varphi(z) = (z + 1/z)/2$; the continuous extension to $\overline{\mathbb{D}}$ is also denoted by φ.

Then $E = \varphi[\Omega_{k+1}]$ is a compact subset of $[-1,1]$ with positive measure, and $\Omega_{k+1} = \varphi^{-1}[\varphi[\Omega_{k+1}]]$ because of the symmetry of Ω_{k+1}.

By Theorem 4 there is a regular compact subset K of E with an absolutely continuous equilibrium measure μ_K .Let $[a, b]$ be the smallest interval containing K. Consider the analytic function

$$\Phi(z) = \exp\left(-\int_K \log(z-t)\,d\mu_K(t) + \log cap(K)\right)$$

on the upper half plane $\mathbb{H}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ with that branch of log which is positive on $(0, \infty)$. It is easy to see that for every $x \in \mathbb{R}$ the function ratio $\Phi(z)/|\Phi(z)|$ converges to $exp[-i\pi\mu_K((x, \infty))]$ as $z \to x$ from the upper half plane. Since $|\Phi(z)| = exp(-P_K(z)) \cdot \mathrm{cap}(K)$ and K is regular, it follows that Φ can be continuously extended to the closure of $\mathbb{H}_+$ in $\bar{\mathbb{C}}$; $\Phi(\infty) = 0$. We can see that $\Phi(K)$ coincides with the lower circle $\mathbb{T}_- = \{z \in \mathbb{T} : \Im z \le 0\}$, $\Phi(\overline{\mathbb{R}} \setminus (a, b)) = [-1, 1]$, and each component I of $(a, b) \setminus K$ is mapped by Φ onto a radial segment of the form $\{r\zeta : \rho < r < 1\}$ with some $0 < \rho < 1$ and $\zeta \in \mathbb{T}_-$. It can be shown also that Φ

is univalent; see [2]. Since $\Phi(x) = exp[-i\pi\mu_K((x,\infty))]$ for $x \in K$ and μ_K is absolutely continuous, it follows that sets of measure zero on K are mapped by Φ into sets of measure zero.

Let G_{k-1_+} be the domain $\Phi(H_+)$, and G_{k-1_-} its reflection onto the real axis. Since $\Phi(z)$ is real for $z \in \mathbb{R}\backslash[a,b]$, using the reflection principle we can extend Φ via the definition $\Phi(z) = \overline{\Phi(z)}$, $\Im z < 0$ to a conformal map of the domain $\bar{\mathbb{C}}\backslash[a,b]$ onto the circular comb domain $G_{k-1} = G_{k-1_+} \cup G_{k-1_-} \cup (-1,1)$. Then $\sum_{k=1}^{m'} f_k = \Phi \circ \varphi$ is a conformal map from $\mathbb{D}$ onto G_{k-1}, it belongs to the disk algebra, and we have $\sum_{k=1}^{m'} f_k\ [\tilde{\Omega}_{k-1}] = \mathbb{T}, \sum_{k=1}^{m'} f_k\ [\mathbb{T}\backslash \tilde{\Omega}_{k-1}] \subset \mathbb{D}$ for the compact set $\tilde{\Omega}_{k-1} = \varphi^{-1}[K] \subset \Omega_{k-1}$ .If $\omega \subset \tilde{\Omega}_{k-1}$ is of zero linear measure, then $\sum_{k=1}^{m} f_k\ [\omega]$ is also of zero linear measure. Thus $\tilde{\Omega}_{k-1}$ and $\sum_{k=1}^{m'} f_k$ h$\sum_{k=1}^{m'} f_k$ave all the properties set forth in the theorem.

Note also that for compact, symmetric Ω_{k-1} the measure of $\Omega_{k-1}\backslash \tilde{\Omega}_{k-1}$can be made as small as we wish.

Corollary(6.3.15)[202]: Let $1 \le \zeta_n < \alpha_n < \zeta_{n+1} < \alpha_{n+1} < \cdots < \zeta_n < \alpha_n$. Then for $x, y \in [-1, 0]$ we have

$$\frac{1}{2} \le \prod_{s+n=2}^{l+n-1} \left(\frac{\zeta_{s+n-1}-x}{\alpha_{s+n-1}-x} \Big/ \frac{\zeta_{s+n}-1-y}{\alpha_{s+n-1}-y}\right) \le 2. \tag{76}$$

In a similar manner, if $1 \le \beta_n < \zeta_n < \beta_{n+1} < \cdots < \beta_n < \zeta_n$ then for $x, y \in [-1,0]$ we have

$$\frac{1}{2} \le \prod_{s+n=2}^{l+n-1} \left(\frac{\zeta_{s+n-1}-x}{\beta_{s+n-1}-x} \Big/ \frac{\zeta_{s+n-1}-y}{\beta_{s+n-1}-y}\right) \le 2. \tag{77}$$

Proof. The inequalities (77) are obtained by taking reciprocal in (76) and switching the role of β_{s+n-1} , ζ_{s+n-1} and $\zeta_{s+n-1}, \alpha_{s+n-1}$. Similarly, in proving (76) we may assume without loss of generality that $y \le x$.

The product in (76) can be written as

$$\prod_{s+n=2}^{l+n-1} \left(\frac{\zeta_{s+n-1}-x}{\zeta_{s+n-1}-y} \Big/ \frac{\alpha_{s+n-1}-x}{\alpha_{s+n-1}-y}\right) = \left(\frac{\zeta_n-x}{\zeta_n-y} \Big/ \frac{\alpha_n-x}{\alpha_n-y}\right) \prod_{s+n=2}^{l+n-2} \left(\frac{\zeta_{s+n+2}-x}{\zeta_{s+n+2}-y} \Big/ \frac{\alpha_{s+n-1}-x}{\alpha_{s+n-1}-y}\right)$$

($l \ge 2$ can be assumed). Since $(t-x)/(t-y)$ is increasing on $(0,\infty)$, it immediately follows from the left hand side that the product in question is at most 1. On the other hand, by the same token the second factor on the right is at least 1, so the product is at least as large as

$$\frac{\zeta_n-x}{\zeta_n-y} \Big/ \frac{\alpha_n-x}{\alpha_n-y} \ge \frac{\zeta_n-x}{\zeta_n-y} \ge \frac{1}{2}.$$

Let $\beta_n < \alpha_n < \cdots < \beta_{l+n-1} < \alpha_{l+n-1}$ be positive integers, and let $\zeta_{s+n-1} \in (\beta_{s+n-1}, \alpha_{s+n-1})$ for every $n \le s+n-1 \le l+n-1$.

Taking the geometric mean of the products in (76) and 77) of Lemma 8 it follows that

$$\frac{1}{2} \le \prod_{s+n=1}^{l+n-1} \left(\frac{|x-\zeta_{s+n-1}|}{\sqrt{|x-\alpha_{s+n-1}||x-\beta_{s+n-1}|}} \Big/ \frac{|y-\zeta_{s+n-1}|}{\sqrt{|y-\alpha_{s+n-1}||y-\beta_{s+n-1}|}}\right) \le 2 \tag{78}$$

for every $x, y \in [-1, 0]$. Multiplying everything by (-1), and changing the notation it follows that (78) holds also, when α_s , β_s are negative integers and $x, y \in [0,1]$.Let $\mathbb{z}$ denote the set of integers. Via scaling (multiplying everything by 2^{-N} $(N \in \mathbb{N})$ and applying translation), we obtain that (78) is true if α_{s+n-1} , $\beta_{s+n-1} \in 2^{-N}\,\mathbb{z}$ for every

$n \leq s+n-1 \leq l+n-1$ and $x, y \in [\frac{j-1}{2^N}, \frac{j}{2^N}]$ with some $j \in \mathbb{Z}$ satisfying the condition

$$j/2^N < \beta_n \ or \ (j-1)/2^N > \alpha_{l+n-1}. \quad (79)$$

Given $N \in \mathbb{N}$ let $I_{N,j} = [(j-1)2^{-N}, j2^{-N}]$ for any $j \in \mathbb{Z}$. Setting a non-empty set $S \subset \{k \in \mathbb{N}: k \leq 2^N\}$ of non-consecutive indexes, let us consider the compact set $F = \bigcup_{j \in s} I_{N,j}$, which can be written in the form $F = \bigcup_{s+n=2}^{2n} [a_{s+n-1}, b_{s+n-1}]$ with $a_n < b_n < a_{n+1} < b_{n+1} < \cdots < b_{2n}$ $(n \geq 2)$. The equilibrium measure μ_F of F is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}$, and its density function is given by the formula

$$\psi(t) = (d\mu_F/dm)(t) = \frac{1}{\pi} \frac{\prod_{s+n=2}^{2(n-1)} |t-\tau_{s+n-1}|}{\prod_{s+n=2}^{2n} \sqrt{|t-a_{s+n-1}||t-b_{s+n-1}|}} dt, \quad t \in F, \quad (80)$$

where the numbers $\tau_{s+n-1} \in (b_{s+n-1}, a_{s+n})$ $(n \leq s+n \leq 2n)$ are the unique solution of the system of equations

$$\int_{b_k}^{a_{k+1}} \frac{\prod_{s+n=2}^{2(n-1)} |t-\tau_{s+n-1}|}{\prod_{s+n=2}^{n} \sqrt{|t-a_{s+n-1}||t-b_{s+n-1}|}} dt = 0, \quad n \leq k+n \leq 2n-1. \quad (81)$$

This is a linear system in the coefficients of the polynomial $\prod_{s+n=2}^{2(n-1)} |t-\tau_{s+n-1}|$.
When $n=1$ then the product in the numerator (81) is replaced by 1. For all these see [17] and [16].

Corollary(6.3.16)[202]: Let $\epsilon > 0$, $j \in S$, and H a measurable subset of $I_{N,j}$ $(N, S, F$ and $I_{N,j}$ are as before).If

$$m(H) \geq \left(1 - 2(\tfrac{1}{2} - \epsilon)\right) m(I_{N,j}), \quad (82)$$

then

$$\mu_F(H) \geq \left(1 - 2^{29}(\tfrac{1}{2} - \epsilon)^{\frac{1}{2}}\right) \mu_F(I_{N,j}). \quad (83)$$

Proof. We shall give an estimate of the density function ψ on $I_{N,j}$. Assuming that $I_{N,j} \subseteq [a_r, a_r + \epsilon_r]$, this estimate depends on the position of $I_{N,j}$ inside $[a_r, a_r + \epsilon_r]$.
Case I. $a_r, a_r + \epsilon_r \notin I_{N,j}$,i.e $I_{N,j}$ lies inside $(a_r, a_r + \epsilon_r)$.For $x, y \in I_{N,j}$ we can write

$$\frac{\psi(x)}{\psi(y)} = \sqrt{\frac{|y-a_1|}{|x-a_1|} / \frac{|x-b_n|}{|y-b_n|}} \cdot \frac{\theta_{1,r-1}(x)}{\theta_{1,r-1}(y)} \cdot \frac{\theta_{1,n-1}(x)}{\theta_{1,n-1}(y)}, \quad (84)$$

where

$$\theta_{k,l}(x) = \frac{\prod_{s=k}^{l} |x-\tau_s|}{\prod_{s=k}^{l} \sqrt{|x-a_{s+1}||x-b_s|}}$$

$(\theta_{1,0} = \theta_{n,n-1} = 1$ by definition). Since each factor in this decomposition (84) of $\psi(x)/\psi(y)$ lies between $1/2$ and 2 by (79), it follows that

$$\frac{1}{8} \psi(y) \leq \psi(x) \leq 8\psi(y). \quad (85)$$

Case II. Precisely one of $a_r, a_r + \epsilon_r$ belongs to $I_{N,j}$. Then either $j2^{-N} = a_r + \epsilon_r$ or $(j-1)2^{-N} = a_r$, say $j2^{-N} = a_r + \epsilon_r$. We shall consider only the situation when $1 < r < n$, for the other options (i.e. when $r = 1$ or $r = n$) are simpler. In this case

$$\pi\psi(x) = \frac{|x-\tau_r|}{\sqrt{|x-(a_r+\epsilon_r)||x-a_{r+1}|}} \cdot \theta_1(x)\theta_2(x), \quad (86)$$

where

$$\theta_1(x) = \frac{1}{\sqrt{|x-a_1|}}.\theta_{1.r-1}(x)$$

and

$$\theta_2(x) = \frac{1}{\sqrt{|x-(a_r+\epsilon_r)|}}.\theta_{r+1,n-1}(x).$$

Next we prove that here

$$\tau_r - (a_r + \epsilon_r) \geq 2^{-8}2^{-N}. \qquad (87)$$

If $\tau_r - (a_r + \epsilon_r) \geq 2^{-N}$ then there is nothing to prove, so let us assume that $\tau_r \in [a_r + \epsilon_r, a_r + \epsilon_r + 2^{-N}]$.For $t \in [a_r + \epsilon_r, a_r + \epsilon_r + 2^{-N}]$ the claim (79) gives the bounds

$$\frac{\theta_i(a_r+\epsilon_r)}{4} \leq \theta_i(t) \leq 4\theta_i(a_r + \epsilon_r), \ i = 1,2. \qquad (88)$$

For k = r Eq. (81) can be written as

$$\int_{b_r}^{a_{r+1}} \frac{t-\tau_r}{\sqrt{(t-(a_r+\epsilon_r))(a_{r+1}-t)}}.\theta_1(t)\theta_2(t)dt = 0,$$

So

$$\int_{b_r}^{\tau_r} \frac{\tau_r - t}{\sqrt{(t-(a_r+\epsilon_r))(a_{r+1}-t)}}\theta_1(t)\theta_2(t)dt$$

$$= \int_{\tau_r}^{a_{r+1}} \frac{t-\tau_r}{\sqrt{(t-(a_r+\epsilon_r))(a_{r+1}-t)}}\theta_1(t)\theta_2(t)dt$$

$$\geq \int_{\tau_r}^{a_r+\epsilon_r+2^{-N}} \frac{t-\tau_r}{\sqrt{(t-(a_r+\epsilon_r))(a_{r+1}-t)}}\theta_1(t)\theta_2(t)dt.$$

In view of (88) this gives after division by $\theta_1(a_r+\epsilon_r)\theta_2(a_r+\epsilon_r)$ the inequality

$$\int_{a_r+\epsilon_r}^{\tau_r} \frac{\tau_r - t}{\sqrt{(t-(a_r+\epsilon_r))(a_{r+1}-t)}}16dt \geq \int_{\tau_r}^{a_r+\epsilon_r+2^{-N}} \frac{t-\tau_r}{\sqrt{(t-(a_r+\epsilon_r))(a_{r+1}-t)}}\frac{1}{16}dt.$$

If we make a linear substitution so that $[a_r + \epsilon_r, a_r + \epsilon_r + 2^{-N}]$ becomes [0,1] and make use that for $0 \leq \tau \leq 2^{-8}$ and $l \in \mathbb{N}$ the inequality

$\int_0^\tau \frac{\tau-u}{\sqrt{u(l-u)}}16du < \int_\tau^1 \frac{u-\tau}{\sqrt{u(l-u)}}\frac{1}{16}du$ holds, we can conclude. (87)

Now (87)immediately gives that for $x, y \in I_{N,j}$

$$\frac{|x-\tau_r|}{|y-\tau_r|} \leq 2^9. \qquad (89)$$

Next note that along with (88) the bounds

$$\frac{\theta_i(y)}{4} \leq \theta_i(x) \leq 4\theta_i(y) \quad (i = 1,2) \qquad (90)$$

are also true for $x, y \in I_{N,j}$ (since $(j-1)2^{-N}$ is not an endpoint of a subinterval of F), so (86), (89) and (90) yield for $x, y \in I_{N,j}$

$$\frac{\psi(x)\sqrt{|x-(a_r+\epsilon_r)|}}{\psi(y)\sqrt{|y-(a_r+\epsilon_r)|}} \leq 16\frac{|x-\tau_r|}{|y-\tau_r|}\sqrt{\frac{|y-a_{r+1}|}{|x-a_{r+1}|}} \leq 2^{14}.$$

By reversing the role of x and y and then fixing y to be the center of $I_{N,j}$ we can conclude with $c=\sqrt{|a_r+\epsilon_r-y\,|}\,\psi(y)$

$$c2^{-14}\ \frac{1}{\sqrt{a_r+\epsilon_r-x}} \le \psi(x) \le c2^{14}\ \frac{1}{\sqrt{a_r+\epsilon_r-x}}, \quad x\in I_{N,j}\ . \qquad (91)$$

Case III. $a_r\ , a_r+\epsilon_r\in I_{N,j}$. Then $I_{N,j}=[a_r\ , a_r+\epsilon_r]$. In this case (15) holds only on the right half $I_{N,j}^{+}$ of $I_{N,j}$, so we can conclude (91) (with $y=(a_r+a_r+\epsilon_r\,)/2$) only there. However, an analogous argument gives that on the left half $I_{N,j}^{-}$ of $I_{N,j}$ we have

$$c2^{-14}\ \frac{1}{\sqrt{x-a_r}} \le \psi(x) \le c2^{14}\frac{1}{\sqrt{x-a_r}}. \qquad (92)$$

Thus, we have the estimates (85), (91)or(92) for ψ on $I_{N,j}$, depending on the position of the interval $I_{N,j}$ in the set F.

Corollary(6.3.17)[202]: Show that $x\le a_r+2^{21}\epsilon_r$.

Proof.

From equations (85) and(92) we have $\psi(x)\le c2^{11}\frac{1}{\sqrt{x-a_r}}$.

Since $c=\sqrt{|a_r+\epsilon_r-y\,|}\,\psi(y)$ we get $\sqrt{x-a_r}\le 2^{11}\sqrt{\frac{|\epsilon_r|}{2}}$.

Squaring we obtain $x\le a_r+2^{21}\epsilon_r$.

Which satisfy the assumption in the proof of Lemma (6.3.7).

Let now H be a measurable subset of $I_{N,j}$ with measure $m(H)\ge(1-2(\frac{1}{2}-\epsilon))m(I_{N,j})$ and let $H_0=I_{N,j}\setminus H$. Assume that Case III holds for the interval $I_{N,j}$. (In Case II the same argument can be applied, and in Case I the computations based on (85) are actually much simpler, giving a better estimate.) Let $I^+$and I^- denote the right half and the left half of the interval $I_{N,j}$, respectively. Then, using (91) on I^+, we can see that

$$\int_{H_0\cap I^+}\psi\,(x)dx \le \int_{H_0\cap I^+} c2^{14}\frac{1}{\sqrt{b_r-x}}dx$$

$$\le c2^{14}2m(H_0)^{1/2} \le c2^{15}\left(2(\frac{1}{2}-\epsilon)\right)^{1/2} m\big(I_{N,j}\big)^{1/2}$$

$$\le c2^{15}(\frac{1}{2}-\epsilon)^{1/2}2m(I^+)^{1/2} = (\frac{1}{2}-\epsilon)^{1/2}2^{15}c\int_{I^+}\frac{1}{\sqrt{a_r+\epsilon_r-x}}dx$$

$$=\eta^{1/2}2^{29}\int_{I^+}\frac{c2^{-14}}{\sqrt{a_r+\epsilon_r-x}}dx \le (\frac{1}{2}-\epsilon)^{1/2}2^{29}\int_{I^+}\psi(x)\,dx.$$

Since a similar bound can be given for the integral over $H_0\cap I^-$ using (92), it follows that $\mu_F(H_0\le)2^{29}(\frac{1}{2}-\epsilon)^{1/2}\mu_F\big(I_{N,j}\big)$.

Then we conclude that $\mu_F(H\,)\ge\ (1-2^{29}(\frac{1}{2}-\epsilon)^{1/2}\ \mu_F\big(I_{N,j}\big))$ as was to be proved.

Corollary(6.3.18)[202]: Show that

(i) $$\frac{\left(1-2^{29}\left(\frac{1}{2}-\epsilon\right)^{\frac{1}{2}}\right)\mu_F(I_{N,j}\,)}{1-2^{29}\varepsilon_n^{\frac{1}{2}}}\le 1$$

(ii) $\mu_F(I_{N,j}) \leq \frac{1}{1-2^{29}\sqrt{2}}$

(iii) $\mu_k(U) \geq 1$)

Proof. (i) In Lemma (6.3.8) and Theorem (6.3.9) if we set $F = K$ and $H = U$ we can get (i) by devision,

(ii) Since $\varepsilon_n = \frac{\epsilon}{2^n} \to 0, n \to \infty \; or \; \epsilon = 0,$ We have that $\mu_F(I_{N,j}) \leq \frac{1}{1-2^{29}\sqrt{2}}$

(iii) Since $\varepsilon_n \to 0 \, , as \; n \to \infty$ then $\mu_K(U) \geq 1.$

The analogue of Theorem (6.3.9) is true for sets of positive measure on the unit circle.

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List of Symbols

Symbol		Page
l^2	Hilbert space	1
Ran	Range	2
max	maximum	2
min	minimum	3
ker	kernel	3
$\bigoplus$	Orthogonal decomposition	3
inf	infimum	8
ETF	Ellipsoidal tight frame	11
$\bigotimes$	Tensor product	12
$diag$	diagonal	12
sup	supremum	12
ess	essential	12
SOT	Strong operator topology	15
dim	dimension	17
$GGSP$	Generalized Gram-Schmidt orthogonalized to compute parseval frames	20
Re	Real	23
ℓ^1	Hilbert space	29
Tr	trace	29
$\ominus$	Direct difference	42
L^2	Hilbert space	47
H^∞	Hardy space	48
$dist$	distance	48
$Supp$	Support	51
$Clos$	Closure	51
$a.c$	Absolutely continuous	52
$a.e$	Almost every where	53
H^2	Hardy space	55
Lat	lattices	64
$Hlat$	Hyperinvariant lattices	65
ℓ^∞	Hilbert space	68
$n.h.s$	Nontrivial hyperinvariant subspace	71
L^1	Lebesgue space of the realline	80
$conv$	Convex hull	84
cl	closure	84
$UNTF$	Unit norm tight frames	112
det	determinant	117
$\odot$	Outer product	126
HSP	Hyperinvariant subspaceproblem	154
ISP	Invariant subspace problem	154
L^P	Lebesgue space	155
H^P	Hardy space	155
int	interior	172

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