بسم الله الرحمن الرحيم

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Permutation Groups with Graph Theory and Polya's Theory of Counting with Applications

زُمر التباديل مع نظرية الأشكال ونظرية بوليا للعد وتطبيقاتها

A thesis Submitted for Fulfillment for the Requirement of the Degree of Doctor of Philosophy in Mathematics

Prepared by:

MOHAMED ELNEAM AHMED ELNAIR

Supervisor:

Ass. Prof . Adam Abdalla Abkar Hassan

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

الاستهلال

قال تعالى :

(قَالُوا سُبْحَانَكَ لَا عِلْمَ لَنَا إِلَّا مَا عَلَّمْتَنَا إِنَّكَ أَنْتَ الْعَلِيمُ الْحَكِيمُ)

سورة البقرة، الآية (٣٢)

DEDICATION

Dedicated to my family who have spared no efforts to support me throughout the study.

I also dedicate this study to my lecturers who have provided me with full support and follow-up to accomplish this study.

To all persons who helped me through writing this thsis.

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ABSTRACT

This study is carried out in order to investigate permutation groups, Graph theory and Polya's theory of counting with applications together with the relations between them. The key to this relationship is the celebrated Burnside lemma. Aimed to explain the aspects of group theory which are related to them. Moreover numearous groups of permutations and the cyclic structures of their elements together with the orbits of those elements are then used methods and scientific means to enumerate all the possible ways of colourings of a set. This is then used to prove polya’s enumeration theorem (PET). The most important results of this study to obtain some applications of permutation groups, Graph theory and Polya's theory of counting .

ملخص البحث

أجريت هذه الدراسة بغرض التحقق من زمر التباديل مع نظرية الأشكال ونظرية بوليا للعد وتطبيقاتها والعلاقة بينهما. ومفتاح هذه العلاقة هو نظرية برنسايد الشهيرة.وتهدف لدراسة الزمر بانواعها المختلفة.بالاضافة الى الزمر المتناؤبه وتركيب عناصرها وعدد مداراتها.والتى استخدمت الطرق والوسائل العلمية فى حساب عدد كل الطرق الممكنه لتلوين اى مجموعة وبذلك نحصل على الاساس لاثبات نظرية بوليا للعد.من اهم النتائج التى توصلت اليها الدراسة الحصول على بعض التطبيقات على زُمر التبديلات، نظرية الأشكال ونظرية بوليا للعد.

Table of Contents

Contents	Page

Chapter One

Historical Background And Previous Studies

1.1 Historical Background

As noted earlier, Lagrange's work of 1770 initiated the study of permutations in connection with the study of the solution of equations. It was probably the first clear instance of implicit group-theoretic thinking in mathematics. It led directly to the works of Ruffini, Abel, and Galois during the first third of the nineteenth century, and to the concept of a permutation group. Ruffini and Abel proved the unsolvability of the quintic by building on the ideas of Lagrange concerning resolvents. Lagrange showed that a necessary condition for the solvability of the general polynomial equation of degree n is the existence of a resolvent of degree less than n. Ruffini and Abel showed that such resolvents do not exist for $n > 4$. In the process they developed elements of permutation theory. It was Galois, however, who made the fundamental conceptual advances, and who is considered by many as the founder of (permutation) group theory Galios group. He was familiar with the works of Lagrange, Abel, and Gauss on the solution of polynomial equations. But his aim went well beyond finding a method for solvability of equations. He was concerned with gaining insight into general principles, dissatisfied as he was with the methods of his predecessors: also of this century," he wrote, computational procedures have become so complicated that any progress by those means has become impossible [45]

Let an equation be given, whose m roots are *a, b, c, ...* There will always be a group of permutations of the letters *a, b, c, . . .* which has the following property: (1) that every function of the roots, invariant under the substitutions of that group, is rationally known [i.e., is a rational function of the coefficients and any adjoined quantities]; (2) conversely, that every function of the roots, which can be expressed rationally, is invariant under these substitutions. The definition says essentially that the group of an equation consists of those permutations of the roots of the equation which leave invariant all relations among the roots over the field of coefficients of the equation—basically the definition we would give today. Of course the definition does not guarantee the existence of such a group, and so Galois proceeded to demonstrate it. He next investigated how the group changes when new elements are adjoined to the "ground field." His treatment was close to the standard treatment of this matter in a modern algebra text [45].

The Polya enumeration theorem (PET) also known as red field – Polya's theorem, is a theorem in combinatorics, generalizing Burnside's lemma about number of orbits. This theorem was first discovered and published by John Howared Red field in 1927 but its importance was over looked and Red field's publication was not noticed by most of the mathematical community. In dependently the result was proved in 1937 by George polya, who also demonstrated a number of its applications, in particular to enumeration of chemical compounds. The (PET) gave rise to symbolic operators and symbolic methods in enumerative combinatorics and was generatized to the fundamental theorem of combinatorial enumeration [6,10].

Graphs are mathematical structures used to model pair-wise relations between objects from a certain collection. Graph can be defined a set V of vertices and set of edges. Where, V is collection of |V| = n abstract data types. Vertices can be any abstract data types and can be presented with the points in the plane. These abstract data types are also called nodes. A line (line segment) connecting these nodes is called an edge. Again, more Abstractly saying, edge can be an abstract data type that shows relation between the nodes (which again can be an abstract data types). In this document, we would briefly go over through how and what led to the development of the graph theory which revolutionized the way many complicated problems were looked at and were solved. Leonhard Paul Euler (1707- 1783) was a pioneering Swiss mathematician, who spent most of his life in Russia and Germany. Euler (pronounced as OILER) solved the first problem using graph theory and thereby led the foundation of very vast and important field of graph theory. He created first graph to simulate a real time place and situation to solve a problem which was then considered one of the toughest problems.

In, 1736 Euler came out with the solution in terms of graph theory. He proved that it was not possible to walk through the seven bridges exactly one time. In coming to this conclusion, Euler formulated the problem in terms of graph theory. He abstracted the case of Königsberg beliminating all unnecessary features. He drew a picture consistingof "dots" that represented the landmasses and the line-segments representing the bridges that connected those land masses. The resulting picture might have looked somewhat similar to the figure shown below [42].

1.2 The Polya Theory And Permutation Groups in (2009)

This study presents a thorough exposition of the Polya Theory in its enumerative applications to permutations groups. The discussion includes the notion of the power group, the Burnside's Lemma along with the notions on groups, stabilizer, orbits and other group theoretic terminologies which are so fundamentally used for a good introduction to the Polya Theory. These in turn, involve
the introductory concepts on weights, patterns, figure and configuration counting series along with the extensive discussion of the cycle indexes associated with the permutation group at hand. In order to realize the applications of the Polya Theory, the paper shows that the special figure series c(x) =1+ x is useful to enumerate the number of G-orbits of r -subsets of any arbitrary set X. Further-more, the paper also shows that this special figure series simplifies the counting of the number of orbits determined by any permutation group which consequently
determines whether or not the said permutation group is transitive.

Many people have difficulty in doing some counting problems probably because sometimes, a situation comes wherein distinct objects are often considered equal. If a teacher for instance is interested in knowing the number of families represented by her class, then she will consider two children to be equal" if and only if they are siblings. Suppose next we consider the problem of counting nonequivalent bracelets with two beads of three dire rent colors; red (r), blue (b) and green (g). By simple combinational analysis, there will be exactly $3^2 = 9$ possible faces of the bracelets with the above specified colors.

Let us now divide these nine bracelets into groups of bracelets by considering two bracelets similar if one can be obtained from the other by rotation. Then we see that b_2 is rotationally equivalent to b_4 , hence both should belong to the same group. Likewise, b_3 and b_7 are equivalent; b_5 and b_6 are also equivalent. On the other hand, we see that b_1 belongs to a group that contains itself and so does b_8 .Thus, in the sense of grouping these bracelets, we are led to have classified six die rent bracelets that are non-equivalent under rotation.

In this paper, certain enumerative techniques like the one illustrated above will be developed and used for the solutions of some counting problems specifically those that call the notion of permutation groups. A thorough exposition of a powerful tool in the said enumeration will be the central feature of study from a point of view first developed by George Polya in 1938.

1.2.1 The Power Group

Consider two permutation groups G_1 , of order m acting on $X = \{x_1, x_2, \dots , x_d\}$ and another permutation group G_2 of order n acting on Y

$= \{y_1, y_2, \dots, y_e\}$. Here, we refer to the sets X and Y as the object set of G_1 and G_2 respectively.

1.2.2 The Cycle Index Polynomial

Let us begin this section by considering the disjoint cycles of a particular length on every permutation $\pi \in G$.

1.2.3 The Burnside's Lemma

It makes sense to begin this section by studying what it means to say two objects are the same.

Theorem (1.2.1)

If G is a permutation group, then » defines an equivalence relation.

Lemma (1.2.1)

Orb(x) = Orb(y) if and only *if x , y .*

The problem of determining the number of equivalent objects in *X* reduces to the problem of counting the number of distinct G-orbits established by on *X* induced by *G*. One way of doing this is simply to count, that is to compute all *G*-orbits and enumerate them. However, this method seems impractical and even more tedious for complex situations. Fortunately, the Burnside's Lemma which we are going to develop next gives an analytical formula for such counting of *G*-orbits. The Burnside's Lemma is a powerful technique in the counting of G-orbits induced by a permutation group. This technique which is particularly ancient when the order of the group is small is considered one of the essential parts in the development of the Polya Theory.

Theorem (1.2.2) (Burnside's Lemma)

Let G be a permutation group acting on the set X and suppose » is an equivalence relation on X induced by G. If μ is the number of G orbits in X, then,

$$\theta = \frac{1}{|G|}\left(\sum_{\pi \in G} \phi(\pi)\right) \qquad (1.1)$$

1.2.4 The Polya Counting Theory

In many instances, the direct application of the Burnside's Lemma is not practically efficient to permit us to enumerate the distinct G orbits induced by a permutation group. The difficulty perfectly stems from the computation of the number of invariance's for a large ordered group. The Polya's Theorem provides a tool necessary to facilitate this computation.

To formulate and prove Polya's Theorem in an abstract and more concise manner, it is somehow convenient to require the notion of functions and patterns as its enumerations are basically performed over sets whose elements are functions. In the rest of the discussion, we consider X be a set of elements called places ; and let Y be a set of elements called figures . Also, we consider the usual permutation group G acting on X, which we call the configuration group. Moreover, an element f in $Y X$ will be called configuration .

Theorem (1.٢.3) (Polya's Theorem)

The configuration counting series *C(x)* is obtained by substituting the figure counting series c(xk) for each indeterminate sk into the cycle index *Z(G)* of the configuration group. In symbols, *C(x) = Z(G; c(x))* [25].

In combinatorics, there are very few formulas that apply comprehensively to all cases of a given problem. Polya's Counting Theory is a spectacular tool that allows us to count the number of distinct items given a certain number of colors or other characteristics. Basic questions we might ask are, \How many distinct squares can be made with blue or yellow vertices?" or \How many necklaces with n beads can we create with clear and solid beads?" We will count two objects as 'the same' if they can be rotated or ipped to produce the same configuration. While these questions may seem uncomplicated, there is a lot of mathematical machinery behind them. Thus, in addition to counting all possible positions for each weight, we must be sure to not recount the configuration again if it is actually the same as another .We can use Burnside's Lemma to enumerate the number of distinct objects. However, sometimes we will also want to know more information about the characteristics of these distinct objects. Polya's Counting Theory is uniquely useful because it will act as a picture function - actually producing a polynomial that demonstrates what the different configurations are, and how many of each exist. As such, it has numerous applications. Some that will be explored here include chemical isomer enumeration, graph theory and music theory .This paper will first work through proving and understanding Polya's theory, and then move towards surveying applications. Throughout the paper we will work heavily with examples to illuminate the simplicity of the theorem beyond its notation.

Definition (1.3.1):

We will first clarify some basic notation. Let S be a finite set. Then $|S|$ denotes the number of its elements. If G is a group, then $|G|$ represents the number of elements in G and is called the order of the group. Finally, if we have a group of permutations of a set S, then $|G|$ is the degree of the permutation group

1.3.2 The Orbit-Stabilizer Theorem

Also proves the following two theorems.

Theorem . Lagrange's Theorem If G is a Finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left cosets of H in G is $\left|\frac{G}{H}\right|$.

Theorem (1.3.1) Orbit-Stabilizer Theorem

Let G be a Finite group of permutations of a set S.Then, for any i from S,

$$|G| = |orb_G(i)||stab_G(i)|.$$

1.3.3 Burnside's Lemma

provides the following theorem.

Theorem (1.3.2) Burnside's Theorem

If G is a Finite group of permutations on a set S, then the number of orbits of G on S is

$$\frac{1}{G}\sum_{\phi\in G}|fix(\phi)| \qquad (1.2)$$

Burnside's Lemma can be described as Finding the number of distinct orbits by taking the average size of the fixed sets.

Let n denote the number of pairs (ϕ, i), with $\phi \in G$, $i \in S$, and $\phi(i) = i$. We begin by counting these pairs in two ways. First, for each particular ϕ in G, the number of such pairs is exactly $|fix(\phi)|$, as i runs over S. So,

$$n = \sum_{\phi\in G}|fix(\phi)| = \sum_{\phi\in S}|stab_G(i)| \qquad (1.3)$$

We know that if s and t are in the same orbit of G, then $orb_G(s) = orb_G(t)$ and $|stab_G(s)| = |stabG(t)|$. So if we choose $t \in S$, sum over $orb_G(s)$, we have

$$\sum_{t\in orb_G(s)}|stab_G(t)| = |orb_G(s)||stab_G(s)| = |G| \qquad (1.4)$$

Finally, by summing over all the elements of G, one orbit at a time, it follows

$$\sum_{\phi\in\cup G}|fir(\phi)| = \sum_{i\in S}|stab_G(i)| = |G|.(numberoforbits) \qquad (1.5)$$

1.3.4 The cycle index

Note that if S is a Finite set, a permutation of S is a one-to-one mapping of S onto itself. If a permutation π is given, then we can split S into cycles, which are subsets of S that are cyclically permuted by π. If L is the length of a cycle, and s is any element of that cycle, then the cycle consists of

$$s\ ,\ \pi s\ ,\ \pi^2 s,\ \ldots\ ,\ \pi^{L-1}s,$$

Definition (1.3.1)

Let G be a group whose elements are the permutations of S, where $|S|$ = m. We define the polynomial in m variables $x_1, x_2, \ldots x_m$, with nonnegative coefficients, where for each $\phi \in G$ we form the product $x_1^{b2} x_2^{b2} \ldots x_m^{bm}, if \{b_1, b_2, b_3, \ldots.\}$ is the type of ϕ. Then the polynomial

$$P_G(x_1, x_2, \ldots, x_m) = \frac{1}{|G|} \sum_{\phi \in G} x_1^{b1} x_2^{b2} \ldots x_m^{bm} \qquad (1.6)$$

is called the cycle index of G.

This formula closely resembles Burnside's Lemma. The key difference is that now we differentiate between cycles of ifferent lengths, and specify how many of each cycle there are. Later, this will allow us to not only count the number of different objects we seek, but also have an idea of what the appearance of each different object is like.

Consider the simple example when G consists of only the identity permutation. Then the identity permutation is of type $\{m, 0, 0, \ldots\}$ and thus $P_G = x_1^m$.

Theorem (1.3.1) Polya's Enumeration Formula

Let S be a set of elements and G be a group of permutations of S that acts to induce an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using colors $c_1, c_2, \ldots, c_m$ is given by the generating function

$$PG = \left(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, \ldots, \sum_{j=1}^{m} c_j^k \right) \qquad (1.7)$$

where k corresponds to the largest cycle length.

So the inventory of colorings of *S* using three colors, *A, B*, and C.

Theorem . Polya's Fundamental Theorem:

Let *D* and *R* be finite sets and *G* be a permutation group of *D*. The elements of *R* have weights *w(r).* The functions $f \in R^D$ and the patterns *F* have weights *W(f)* and *W(F)*, respectively. Then the pattern inventory is

$$\sum_{F} W(F) = P_G\left\{\sum_{r\in R} w(r), \sum_{r\in R}\left[w(r)\right]^2, \sum_{r\in R}\left[w(r)\right]^3, ...,\right\} \qquad (1.8)$$

where P_G is the cycle index. In particular, if all weights are chosen to be equal to unity, then we obtain the number of patterns = $P_G(|R|, |R|, |R|, ...,)$ [31].

1.4 An Approach for Counting the Number of Specialized Mechanisms Subject to- Non Adjacency Constraints in (2012)

This study presents an improved approach to count the number of specialized mechanisms subject to non-adjacency constraints from a candidate kinematic chain. First, the permutation group of the candidate kinematic chain is found. Next, an inventory polynomial named kinematic king polynomial (KKP) to count the specialized mechanisms is modified from the traditional king polynomial related to the count of moves of a king on a chess. Then, an algorithm to calculate the KKP is presented by operations on labeled joint adjacency matrix (LJAM). Finally, two examples are illustrated to verify the approach. A systematized method of creative mechanism design in type synthesis, the separation of function and structure allows derivation of the necessary topological structures between links and joints and thus deduction of possible topological structures by combination. In general, mechanisms are type synthesized using the following three-step procedure: identifying the appropriate mechanism type (e.g. the numbers or link and joint types and necessary design constraints), enumerating the basic kinematic chains and their required numbers of links and joints, and the specialization of mechanisms. In this latter, each kinematic chain is specialized through the assignation of link and joint types to obtain all possible mechanism configurations. Mechanism specialization has been the subject of many studies, which have variously based the structural synthesis of kinematic chains on graph theory, matrices and combinatorial theory. In some such investigations, generation of mechanisms during the synthesis and specialization process requires complex procedures for detection (or computer-aided assignation) and deletion of isomorphic mechanisms. For checking the correction of the results of specialized mechanism without isomorphism, the Polya's theory is applied to count the number of mechanisms. The basic concepts of Polya's theory of enumeration with application to the structural classification of mechanism were proposed at the first by Freudenstein in 1967. Then Bushsbaun and Freudenstein synthesized the kinematic structure of geared kinematic chains and other mechanisms by using previous work and combinatorial mathematics. In 1991, Yan and

Hwang proposed a methodology for enumerating nonisomorphic specialized mechanisms from a specified kinematic chain using the cycle index of permutation groups to calculate the number of the synthesized mechanisms. This method has also been successfully applied in the number synthesis of general simple joint and multiple joints kinematic chains. Hwang and in applied the concept of generating function and permutation group to generate general specialized mechanisms. Yan and Hung then provided a procedure for generating nonisomorphic specialized mechanisms to identify and count the number of mechanism from kinematic chains subject to design constraints. They applied Polya's theory and Burnside's theorem to count the number of mechanisms with a pair of non-adjacent specialized joints. However, the method can not be applied to count the number of mechanisms with arbitrary number of non-adjacent specialized joints. For instance, the Example 3to count the number of nonisomorphic identified mechanisms, with two nonadjacent prismatic joints, from the Stephenson-III mechanism.

1.4.1 Definition of Permutation groups

A permutation p is a bijection (one-to-one and onto) of a finite set S into itself. For example, the sequence (a_2,a_3,a_1,a_4) is a permutation of the set $s = (a_1,a_2,a_3,a_4)$ in which a1 !(is transformed into) a_2, $a_2 \rightarrow a_3$, $a_3 \rightarrow a_1$ and $a_4 \rightarrow a_4$. In this permutation, $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_1$ forms a cycle, denoted by $[a_1,a_2,a_3]$, with a length of three, while $a_4 \rightarrow a_4$ forms another cycle [a_4] with a length of one. The cyclic representation of this permutation p is denoted by [a_1 a_2 a_3][a_4].

1.4.2 Cycle index of Specialized Mechanisms :

If G is a permutation group of set S, then, because each permutation p in G can be written uniquely as a product of disjoint cycles, the cycle structure representation of a permutation is $x_1^{b1} x_2^{b2} \ldots x_k^{bk} \ldots .$, where x_k is a dummy variable for cycles with a length of k and b_k is their number. For example, the permutation

p = [1][2 6][3 5][4] has the cycle structure representation $x_1^2 x_2^2$.

The cycle index [13] of a permutation group denoted by Ci, is the summation of the cycle structure representations of all the permutations that make up the group's elements divided by the number of permutations (n):

$$C_i(x_1, x_2, x_3, \ldots) = \frac{1}{n} \sum_{p \in PG} x_1^{b1} x_2^{b2} x_3^{b3}. \qquad (1.9)$$

Thus, the first consists of six cycles with a length of one; the second, of two cycles with a length of one and two cycles with a length of two; and the third and fourth, of three cycles with a length of two.

$$C_1(x_1, x_2) = \frac{1}{4}(x_1^6 + x_1^2 + x_2^2 + 2x_2^3) \qquad (1.10)$$

1.4.3 Polya's theory of Specialized Mechanisms :

In the specialization process, once the cycle index has been calculated as shown in Polya's theory can be used to calculate the number of results where T is the allotting type of kinematic pair. For example, if fixed link (F) and link (L) are assigned to the kinematic chain, then $x_1 = F+L$ and $x_2 = F^2+L^2$. Substituting x_1 and x_2 produces the results two possible allotments for a fixed link $2L^5F$, six for two fixed links $6L^4F^2$, and six for three fixed links $6L^3F^3$.

$$I_P = C(T, T^2, T^3), \tag{1.11}$$

$$I_P = \frac{1}{4}(F+L)^6 + (F+L)^2(F^2+L^2)^2 + s(F^2+L^2)^3)$$
$$= L^6 + 2L^5F + 6L^4F^2 + 6L^3F^3 + 6L^2F^4 + 2LF^5 + F^6. \tag{1.12}$$

1.5 Combinatorica: A System for Exploring Combinatorics and Graph Theory in Mathematica

Combinatorica is an extension to the computer algebra system Mathematica that provides over 450 functions for discrete mathematics. It is distributed as a standard package with every copy of Mathematica. Combinatorica facilitates the counting, enumeration, visualization, andmanipulation of permutations, combinations, integer and set partitions, Young tableaux, partiallyordered sets, trees, and (most importantly) graphs. Combinatorica users include mathematicians,computer scientists, physicists, economists, biologists, anthropologists, lawyers, and high school students .Combinatorica has been widely used for teaching and research in discrete mathematics since its initial release in 1990 . The original Combinatorica contained 230 functions, using only 2500 lines of code. Its value lay in the ease with which one could conduct a large variety of experiments on discrete mathematical objects and visualize the results. It was never intended to be a high-performance algorithms library such as LEDA , but more as a mathematical research tool and a prototyping environment for effective technology transfer" of discrete mathematics and algorithms to a diverse applications community. Combinatorica received a 1991 EDUCOM award for distinguished mathematics software. We have recently completed the rest sign cant revision of Combinatorica since its initial release over ten years ago . The new package is essentially a complete rewrite of Combinatorica. Over 80% of the functions have been rewritten and the package has more than doubled in size to 450 functions and 6700 lines of code. Feedback from users, advances in graph theory and combinatorics, faster and more versatile hardware, better versions of Mathematica, and easier access to color graphics were some of factors that motivated this rewrite. In this study, we present an overview of the new Combinatorica along with a summary of lessons learned along the way. To presents an overview of Combinatorica, including representative graphics generated by the package, a description of new features of the revised version. Combinatorica in Action We begin our introduction to Combinatorica with a brief discussion of its design philosophy. We encourage the reader to visit www.combinatorica.com for more information on Combinatorica and related resources such as algorithm animations, graph database, and Java-based graph editor. Pemmaraju and Skiena is the de_nitive guide to Combinatorica.

1.5.1 Applications of Combinatorica

Combinatorica has been widely used for both teaching and research. The research applications typically fall into one of three types:

(1) mathematical research into discrete structures through Combinatorica experiments.

(2) employing Combinatorica to perform discrete simulation modeling, typically by people outside the computer science community.

(3) systematic extensions to Combinatorica for particular applications. Improved graph data structure. The original Combinatorica used the adjacency matrix data structure for graphs, for several reasons which were sound at that time. However, with improvements in technology this eventually became a bottleneck in performance. The new Combinatorica uses an edge list data structure for graphs, partly motivated by increased exigency and partly motivated by the need to store drawing information associated with the graph. Edge lists are linear in the size of the graph, and this makes a huge deference to most graph related functions. The improvement is most dramatic in fast graph algorithms |those that run in linear or near linear time, such as graph traversal, topological sort, and ending connected biconnected components. The implications of this change is felt throughout the package; in running time improvements, memory savings, increased functionality, and better graph drawings. The package can now work with graphs that are about 50-100 times larger than graphs that the old package can deal with [37].

Chapter Two

Permutation Groups

In all the systems of algebra studied so far in the privies chapter, namely groups, we have always used one of two operations addition or multiplication.

This chapter includes some additional groups with a different type of operations, i.e. composition of mappings. They are the symmetric group, alternating group together with Cayley's theorem.

2.1 The Symmetric Group

Definition (2.1):

Let X be any finite set

(1)Let α , $\beta : X \to X$. The composition of α and β denoted by $\alpha\,\beta$ is a mapping of X into X defined by

$$\alpha\,\beta(x) = \alpha\,(\beta(x))\ ,\ \forall x \in X.$$

(2)A one-to-one mapping of a set X onto itself is called a permutation of the set X [1,6,8].

Theorem (2.1):

The set S of all permutations of a set X is a group under composition of mappings.

Proof:

Let α and β be permutations of X, i.e. $\alpha, \beta \in S$. Since α and β are mappings of X onto X, therefore

$$\alpha\,(X) = X,\ \beta\,(X) = X$$

By composition of mappings this implies that:

$$\alpha\,\beta\,(X) = \alpha(\beta(X)) = \alpha(X) = X$$

Therefore $\alpha\,\beta$ is mapping of X onto X.

Let $a,\ b\ \in X$ such that:

$$(\beta\alpha)a = (\beta\alpha)b$$

By composition of mappings the above equation can be written as:

$$\beta(\alpha(a)) = \beta\,(\alpha(b))$$

Since β is a one-to-one mapping, we conclude that

$$\alpha(a) = \alpha(b).$$

Since α is a one-to-one mapping, therefore

$$a = b,$$

This implies that $\alpha\,\beta$ is also a one-to-one mapping. Therefore $\alpha\,\beta \in S$ and the set S is closed with respect to the operation of composition of mappings. Moreover composition of mappings is associative, since by definition

$$((\alpha\beta)\ \gamma)\ a = (\alpha\beta)\ (\gamma\ a) = \alpha(\beta\ \ (\gamma\ a))$$

$$= \alpha(\beta\ \gamma)a) = (\alpha(\beta\ \gamma)a\ \ ,\ a \in X.$$

This implies that $(\alpha\beta)\ \gamma) = \alpha(\beta\ \gamma)$

Consider the mapping $\in: X \rightarrow X$ define by

$$\in(a) = a \qquad , \quad a \in X$$

By definition $\in$ is one-to-one and onto. Therefore $\in \in S$.

Moreover for each $\alpha \in S$ and $a \in X$,

$$\alpha \in (a) = \alpha\,(\in (a)) = \alpha\,(a) = \alpha\,a$$

and

$$(\in \alpha) a = \in (\alpha\, a) = \alpha\, a$$

This implies that:

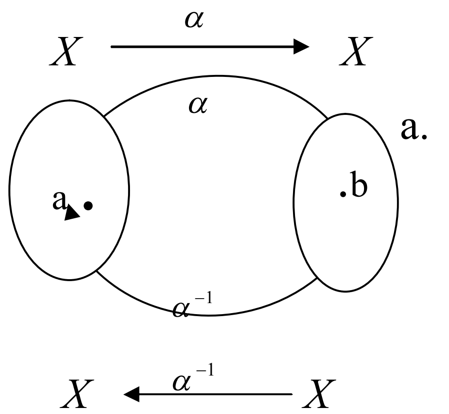
$$\in \alpha = \alpha \in = \alpha$$

Therefore $\in$ is the identity of S.

To prove that each element of S has an inverse let $\alpha \in S$. Since α is a one-to-one mapping of X onto X then the mapping $\alpha^{-1}: X \to X$ defined by

$$\alpha^{-1} a = b \text{ iff } \alpha\, b = a \qquad (2.1)$$

for each $a \in X$, is one-to-one and onto, see figure:



Thus $\alpha^{-1} \in S$. Moreover by equation (2.1)

$$\alpha\alpha^{-1}(a) = \alpha(\alpha^{-1} a) = \alpha\, b = a = \in (a), \ \forall a \in X$$

and

$$\alpha^{-1}\alpha\,(b) = \alpha^{-1}(\alpha\, b) = \alpha^{-1}(a) = b = \in (b), \ \forall b \in X.$$

Therefore

$$\alpha\alpha^{-1} = \alpha^{-1}\alpha = \in.$$

Hence α^{-1} is the inverse of α.

By definition of a group S is a group under composition of mappings [8,13].

Definition (2.2):

(1) Let X be any set. The set S_x of all permutations of X is a group called the symmetric group on X. Moreover every subgroup of S_x is called a group of permutations of X.

(2)If X is finite and has, say, n elements, then we can represent X by $\{1, 2, \ldots, n\}$, and we accordingly denote S_x by S_n. In this case S_n is called the symmetric group of degree n.

Let $f \in S_n$, then f shuffles the elements 1, 2,..., n, and we can represent f explicitly by writing

$$f = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ f(1) & f(2) & f(3) & \ldots & f(n) \end{pmatrix}$$

Hence if $f(1) = i_1, f(2) = i_2, \ldots f_{(n)} = i_n$, therefore

$$\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$$

and

$$f = \begin{pmatrix} 1 & 2 \ldots & n \\ i_1 & i_2 \ldots & i_n \end{pmatrix}$$

Using S_n we shall first study permutations of the particular type described in the following definition [8,1]

Definition (2.3):

An element α of S_n is said to be a cycle of length k if there exist distinct elements $a_1, a_2, \ldots, a_k$, $1 \le k \le n$, of X such that:

$$\alpha(a_1) = a_2,\ \alpha(a_2) = a_3,\ \ldots,\ \alpha(a_{k-1}) = a_k,\ \alpha(a_k) = a_1$$

and $\alpha(a_i) = a_i$ for each element a_i of X other then $a_1, a_2, \ldots, a_k$, i.e. for all a_i, $k < i \le n$, where

$$X = \{a_1, a_2, \ldots, a_n\}$$

Using the above notation for a permutation then

$$\alpha = \begin{pmatrix} a_1 & a_2 \ldots & a_k & a_{k+1} & \ldots a_n \\ a_2 & a_3 \ldots & a_1 & a_{k+1} & \ldots a_n \end{pmatrix}, [7,8].$$

Remark (2.1):

In writing a cycle usually elements fixed by it are omitted. Moreover if the cycle

$$\alpha = \begin{pmatrix} a_1 & a_2 \ldots & a_k & a_{k+1} & \ldots a_n \\ a_2 & a_3 \ldots & a_1 & a_{k+1} & \ldots a_n \end{pmatrix},$$

then after omitting elements fixed by it α is designated by

$$\alpha = (a_1\ a_2\ \ldots a_k)$$

This designation of α is not unique. Using definition (2.3) and remark (2.1) therefore if $a_i \in \{a_1, a_2, \ldots, a_k\}$, $1 \le i \le k$, then the cycles $(a_i\ a_{i+1} \ldots a_k a_1 \ldots a_{i-1})$ and $(a_1\ a_2\ \ldots a_k)$ are equal permutations of X. Hence α can be written as a cyle starting with any a_i, $1 \le i \le k$ .Therefore

$$\alpha = (a_1\ a_2\ \ldots a_k)$$

$$\alpha = (a_2\ a_3\ \ldots a_k\ a_1)$$

$$\vdots$$

$$\alpha = (a_k\ a_1\ a_2\ \ldots a_{k-1}),[8].$$

Example (2.1):

If the cycle $\alpha \in S_6$ is equal to

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 1 & 6 & 2 \end{pmatrix},$$

then by definition(2.3) and remark (2.1) above α can be written as

$$\alpha = (1\ \ 3\ \ 5\ \ 6\ \ 2\ \ 4)$$

Moreover starting with any member of {1, 2, …, 6} e.g. 2 or 3 α can be written in the following respective forms

$$\alpha = (2\ \ 4\ \ 1\ \ 3\ \ 5\ \ 6)$$

or

$$\alpha = (3\ \ 5\ \ 6\ \ 2\ \ 4\ \ 1),[8].$$

2.2 Decomposition of a Permutation Into Cycles And Transpositions:

We start this section with an example.

Example (2.2):

If $S = \{1, 2, 3, 4, 5\}$, then

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

is the permutation such that $\pi(1)=3$, $\pi(2)=5$, $\pi(3)=4$, $\pi(4)=1$, $\pi(5)=2$.

Using this we deduce that if we start with any element $x \in S$ and apply π repeatedly we get $\pi(x)$, $\pi(\pi(x))$, $\pi(\pi(\pi(x)))$, and so on. Since S is finite therefore we must return to x, and there are no repetitions along the way because π is 1:1. We can also denote this by

$$x \to \pi(x) \to \pi(\pi(x)) \to \dots \to x.$$

Using this in the above example we get

$$1 \to 3 \to 4 \to 1\ ,\ 2 \to 5 \to 2\ .$$

Using this and definition of cycle above together with composition of mappings we can write π in the following simple cyclic form, i.e. as a product of cycles.

$$\pi = (1, 3, 4)(2,5).$$

Where the cycle (1, 3, 4) is the permutation of S that is mapping 1 to 3, 3 to 4 and 4 to 1, i.e.

$$1 \to 3 \to 4 \to 1,$$

and similarly the cycle (2, 5) maps 2 to 5 and 5 to 2, i.e.

$$2 \to 5 \to 2$$

the same as π [8,13]

Example (2.3):

Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 4 & 2 & 8 & 1 & 6 \end{pmatrix}$$

In S_8. As in example (2.2) above, we rewrite it as a product of cycles and on omitting 1-cycles, i.e. elements fixed by f, we have

$$f = (1,3,7)\begin{pmatrix} 2 & 4 & 5 & 6 & 8 \\ 5 & 4 & 2 & 8 & 6 \end{pmatrix}$$
$$= (1,3,7)(2,5)\begin{pmatrix} 4 & 6 & 8 \\ 4 & 8 & 6 \end{pmatrix}$$
$$= (1,3,7)(2,5)(6,8)$$

Let S_x be the symmetric group on the set X and σ, $\pi \in S_x$. Using composition of mappings.

$$(\sigma\pi)x = \sigma\,\pi\,(x) = \sigma(\pi\,(x)),\ \forall x \in X \qquad (2.2)$$

We demonstrate in the following example how to evaluate the product $\sigma\,\pi$ [5,8]

Example (2.4):

Consider

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

and

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

in S_4 . Using composition of mappings and by equation (2.2) above

$$(\sigma\,\pi)\,x = \sigma\,(\pi(x)),\ \forall x \in \{1, 2, 3, 4\}.$$

To find the image of $x=1$, using equation (2.2) above, we first find the image $\pi(1)=3$ and using this image we find its image under σ, i.e. $\sigma(3)=1$. Rewriting this using composition of mappings we have

$$(\sigma\,\pi)1 = \sigma\,(\pi(1)) = \sigma(3)=1$$

Alternatively we can rewrite this in the following form:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}(1)$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}1\right]$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}(3) = 1.$$

Hence

$$\sigma\,\pi(1) = \sigma(\pi(1)) = \sigma(3) = 1,$$
$$i.e. \sigma\pi : 1 \to 1$$

Similarly for x=2

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}(2)$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}2\right]$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}(2) = 4,$$

$$i.e. \sigma\pi : 2 \to 4.$$

Continuing in this manner for x= 3, 4 we get

$$\sigma\,\pi : 3 \to 3,\ \sigma\,\pi : 4 \to 2$$

Combining the above results and as in example (2.2) we get

$$\sigma\pi : 1 \to 1,\ 2 \to 4 \to 2,\ 3 \to 3,$$
$$i.e. \sigma\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Alternatively we can rewrite this in the following form by substituting for σ and π from above. Hence

$$\sigma\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Observe that the notation for the product

$$\sigma\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

is rather uneconomical. Hence rewriting it as a product of cycles and omitting 1-cycles, i.e. the elements fixed by $\sigma\pi$, we get

$$\sigma\,\pi = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} = (2 \quad 4).$$ [8].

Definition (2.4):

Two cycles $(a_1 \quad a_2 \ \ldots a_k)$ *and* $(b_1 \quad b_2 \ \ldots b_l)$ of S_n are said to be disjoint if the sets $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_l\}$ have no elements in common [8].

Theorem (2.2):

Every permutation of S_n is a product of disjoint cycles.

Proof:

Since the identity permutation is a product of 1-cycles (of length 1), we assume that γ is not the identity permutation.

Hence start with any symbol a_1 such that $\gamma(a_1) \neq a_1$, and suppose that $\gamma(a_1) = a_2$, $\gamma(a_2) = a_3$, $\gamma(a_3) = a_4$ and so on. Since n is finite we come to the point where, say, $\gamma(a_k)$ equals some one of the symbols $a_1, a_2, \ldots, a_{k-1}$ already used. Since γ is one to one and every other one of these symbols is already known to be the image of some symbol of them under the mapping γ except a_1, therefore we must have $\gamma(a_k) = a_1$. Thus γ has the same permutation effect on the symbols $a_1, a_2, \ldots, a_k$ as the cycle ($a_1, a_2, \ldots, a_k$). If b_1 is a symbol other than $a_1, a_2, \ldots, a_k$ and $\gamma(b_1) \neq b_1$, we proceed as above and obtain a cycle ($b_1, b_2, \ldots, b_l$) which is disjoint from ($a_1, a_2, \ldots, a_k$) since γ is one to one .This implies that γ has the same permutation effect on $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l\}$ as $(a_1, a_2, \ldots, a_k)(b_1, b_2, \ldots, b_l)$. Continuing like this and since n is finite therefore after a finite number of steps we get γ as a product of a finite number of disjoint cycles. After omitting the 1-cycles, therefore

$$\gamma = (a_1, a_2, \ldots, a_k)(b_1, b_2, \ldots, b_l) \ldots (c_1, c_2, \ldots, c_m) \text{ [8,13]}$$

Lemma 2.1:

Disjoint cycles commute.

Proof:

Let σ and π be disjoint cycles of S_n, and suppose that

$$\sigma = (a_1, a_2, \ldots, a_k)$$

and

$$\pi = (b_1, b_2, \ldots, b_l)$$

Let

$$n = \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l, c_1, c_2, \ldots, c_m\}$$

$$=\{1, 2, \ldots, n\}$$

and i be an integer s.t: $1 \leq i \leq n$

To prove that σ and π commute we have three cases:

i\ $i \in \{\sigma\} = \{a_1\ a_2\ ...a_k\}$, *i.e.* $i = a_r,\ 1 \le r \le k$.

Therefore

$$(\sigma\ \pi)i = (a_1\ a_2\ ...a_k)\ (b_1\ b_2\ ...b_l)\ i$$

$$= (a_1\,a_2\,...a_k)[(b_1\,b_2...b_l)a_r]$$

$$= (a_1\,a_2\,...a_k)a_r$$

$$= \begin{cases} a_{r+1}, r \neq k \\ a_1, r = k \end{cases},$$

since $a_r \notin \{\pi\}$.

Similarly

$$(\pi\sigma)i = (b_1\,b_2...b_l)\lfloor(a_1\,a_2\,...a_k)a_r\rfloor = (b_1\,b_2\,...b_l) \begin{cases} a_{r+1}, r \neq k \\ a_1, r = k \end{cases}$$

$$= \begin{cases} a_{r+1}, r \neq k \\ a_1, r = k \end{cases},$$

since $\{\sigma\} \cap \{\pi\} = \phi$.

This implies that $\sigma\ \pi = \pi\ \sigma$.

ii\ $i \in \{\pi\} = \{b_1\,b_2...b_l\}$ *i.e.* $i = b_s, 1 \le s \le l$.

Therefore

$$(\sigma\pi)i = (a_1\, a_2 \dots a_k)\left[(b_1\, b_2 \dots b_l)b_s\right]$$

$$= (a_1\ \ a_2 \dots a_k) \quad \begin{cases} b_{s+1}, s \neq l \\ \\ b_1, s = l \end{cases} \quad = \quad \begin{cases} b_{s+1}, s \neq l \\ \\ b_1, s = l \end{cases}$$

since $b_s \notin \{\sigma\}$.

Similarly

$$(\pi\sigma)i = (b_1\, b_2 \dots b_l)(a_1\, a_2 \dots a_k)b_s$$

$$= (b_1 b_2 \dots b_l)\left[(a_1\, a_2 \dots a_k)b_s\right] = (b_1 b_2 \dots b_l)b_s.$$

Since $b_s \notin \{\sigma\}$, therefore

$$\pi\sigma(i) = \quad \begin{cases} b_{s+1}, s \neq l \\ \\ b_1, s = l \end{cases}$$

Therefore σ π = π σ.

iii\ $i \notin \{\sigma \cup \pi\}$

This implies that $\sigma i = i,\ \pi i = i$. Therefore

$$(\sigma\pi)i = (a_1\, a_2 \dots a_k)(b_1\, b_2 \dots b_l)i$$

$$= (a_1\, a_2 \dots a_k)\left[(b_1\, b_2 \dots b_l)i\right]$$

$$= (a_1\, a_2 \dots a_k)i = i$$

$$(\pi\sigma)i = (b_1\, b_2 \dots b_l)(a_1\, a_2 \dots a_k)i$$

$$= (b_1\, b_2 \dots b_l)\left[(a_1\, a_2 \dots a_k)i\right]$$

$$= (b_1\, b_2 \dots b_l)i = i$$

This implies that $\sigma\, \pi = \pi\, \sigma$. Therefore $\sigma\, \pi = \pi\, \sigma$ for all $1 \leq i \leq n$, i.e. disjoint cycles commute [8,10]

Note:

Let $f \in S_n$. By theorem (2.2)

$$f = f_1 f_2 \cdots f_m,$$

where $f_{1,} \cdots, f_m$ are disjoint cycles. By the above lemma (2.1) therefore we can also write

$$f = f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{i_m},$$

where $\{i_1, \ldots, i_m\} = \{1, 2, \ldots, m\}$

Definition (2.5):

A cycle of length 2 is called a transposition [8].

Lemma 2.2:

Every cycle can be written as a product of transpositions.

Proof:

Let $(a_1 \; a_2 \ldots, a_k)$ be any cycle of $S_{n..}$ Consider the permutation

$$(a_k \;\; a_{k-1})(a_k \;\; a_{k-2}) \ldots (a_k a_2)(a_k a_1).$$

We have two cases to consider

1- Let $a_i \in \{a_1 \;\; a_2 \ldots a_k\}$, therefore

$$(a_1 \;\; a_2 \ldots a_k) a_i = \begin{cases} a_{i+1}, i \neq k \\ a_1, i = k \end{cases}$$

Similarly

$$(a_k \;\; a_{k-1})(a_k \;\; a_{k-2}) \ldots (a_k a_2)(a_k a_1)\, a_i = (a_k \;\; a_{k-1})(a_k \;\; a_{k-2}) \ldots (a_k a_{i+1})(a_k a_i)\, a_i$$
$$= (a_k \;\; a_{k-1}) \ldots (a_k a_{i+1}) a_k$$
$$= (a_k \;\; a_{k-1}) \ldots (a_k a_{i+2}) a_{i+1}$$

$$= \begin{cases} a_{i+1}, i \neq k \\ a_1, i = k \end{cases}$$

This implies that

$$(a_1 a_2 ... a_k) a_i = (a_k \ a_{k-1})(a_k \ a_{k-2})...(a_k a_1) a_i$$

2- Let $b \in \{1, 2, ..., n\}$, such that $b \notin \{a_1 \ a_2 ... a_k\}$ .Therefore such that

$$(a_1 \ a_2 ... a_k) b = b$$

$$(a_k \ a_{k-1})...(a_k a_1) b = b$$

This implies that

$$(a_1 a_2 ... a_k) b = (a_k \ a_{k-1})...(a_k a_1) b = b$$

Using the above two cases therefore

$$(a_1 \ a_2 ... a_k) = (a_k \ a_{k-1})(a_k \ a_{k-2})...(a_k a_2)(a_k a_1) \qquad (2.3)$$

In view of this lemma and theorem (2.2) it follows immediately that every permutation can be expressed as a product of transpositions. Thus we have [8].

Theorem (2.3):

Every permutation of S_n is a product of transpositions.

Example (2.5):

Using equation (2.3) in proof of lemma above, we have:

$$(1 \ 2 \ 3 \ 4) = (4 \ 3)(4 \ 2)(4 \ 1), \ k = 4.$$

A transposition *(ij)* merely interchanges the symbols i and j and leaves the other symbols unchanged. Since $(i \ j)(i \ j) = \in$, the identity permutation, it follows that a transposition is its own inverse.

Moreover this implies that we can insert as many such pairs of identical transpositions as we wish in any decomposition of a permutation into transpositions without changing it. Clearly, then, a permutation can be expressed as a product of transpositions in many different ways [8].

Definition (2.6):

i- A permutation is called an even permutation if it can be expressed as a product of an even number of transpositions.

ii- A permutation is called an odd permutation if it can be expressed as a product of an odd number of transpositions [8,10].

Lemma 2.3:

1/ If α is a product of K transpositions and β is a product of L transpositions then $\alpha\beta$ is a product of $K+L$ transpositions.

2/ The product of two even or two odd permutations is even whereas the product of an even and odd permutations is odd.

3/ α^{-1} is an even (odd) permutation iff α is an even (odd) permutation.

Proof:

1/ If the permutation α can be expressed as a product of K transpositions and the permutation β can be expressed as a product of L transpositions, it is obvious that $\alpha\beta$ is a product of $K+L$ transpositions.

2/ It follows form (1) above that the product of two even or of two odd permutation is an even permutation, whereas the product of an odd permutations and an even permutation is an odd permutation.

3/ Suppose that α is a product of K transpositions, say

$$\alpha = \alpha_1\ \alpha_2 \dots \alpha_k .$$

Then, since S_n is a group and a transposition is it own inverse, it is easy to see that:

$$\alpha^{-1} = (\alpha_1\ \alpha_2 \dots \alpha_k)^{-1} = \alpha_k^{-1}\ \alpha_{k-1}^{-1} \dots \alpha_2^{-1} \alpha_1^{-1} \tag{2.4}$$

Since α_i is a transposition therefore

$$\alpha_i^{-1} = \alpha_i$$

Substituting in equation (2.4) above therefore

$$\alpha^{-1} = \alpha_k\ \alpha_{k-1} \dots \alpha_2 \alpha_1$$

This implies that α^{-1} is an even permutation if and only if α is an even permutation [8].

Lemma 2.4:

The number of transpositions whose product is a given permutation of a finite set is either always even or always odd.

Proof:

Let

$$S = \{1,2,\dots,n\},\ n \geq 2$$

Hence transpositions exist. Our first case is the identity permutation $\in$. Of course $\in$ can be expressed as a product of an even number of transposition e.g. $\in$ =(1 2)(2 1). We show that if:

$$\in = T_1 T_2 ... T_k \qquad (2.5)$$

Where each T_i , $1 \le i \le k$ is a transposition, then k must be even. Choose any integer m, $1 \le m \le n$, which appears in one of the transpositions. Counting from left to right let T_j be the first such transposition. This implies that T_i fixes m for all T_i , $1 \le i \le j$. Since T_i , $i < j$, fixes m therefore $j \le k$.Moreover if $j = k$ then $T_k = (mx)$ for some $x \in S_n$.

Since T_i fixes m for all $1 \le i < j = k$,therefore

$$x = \in (x) = T_1 T_2 ... T_k (x) = m,$$

contradiction since $x \neq m$. Therefore $j < k$. Moreover by choice of m this leads to the following cases.

1. If T_{j+1} contains m then either

i/ The second elements of the transposition T_j, T_{j+1} are equal i.e.

$$T_j\, T_{j+1} = (m, x)(m, x) = \in$$

ii/ or the second elements of T_j, T_{j+1} are not equal i.e.

$$T_j\, T_{j+1} = (m, x)(m, y) = (yxm) = (myx) = (x, y)(m, x),$$

by remark (2.1) and proof of lemma (2.2).

2. If T_{j+1} does not contain m then

i/ Either T_j, T_{j+1} have a common element, i.e.

$$T_j\, T_{j+1} = (m, x)(x, y) = (ymx) = (mxy) = (y, x)(y, m)$$

by remark (2.1) and proof of lemma (2.2).

ii/ or T_j, T_{j+1} have no common element, i.e.

$$T_j\, T_{j+1} = (m, x)(y, z) = (y, z)(m, x)$$

by proof of lemma (2.1).

3. Using the above results therefore $T_j T_{j+1}$ is equal to one of the following forms.

$$(i)\ T_j\, T_{j+1} = \in,\ \textit{the identity}$$
$$(ii)\ T_j\, T_{j+1} = (x,y)(m,x)$$
$$(iii)\ T_j\, T_{j+1} = (y,x)(y,m)$$
$$(iv)\ T_j\, T_{j+1} = (y,z)(m,x)$$

Using equation 3(i) above and substituting for $T_j T_{j+1} = \in$ reduce the number of transpositions in equation (2.5) above by two.

On the other hand using any of the equations 3(ii, iii, iv) and substituting for $T_j T_{j+1}$ will shift the first occurrence of m in equation (2.5)one step to the right, i.e. to T_{j+1}. Then as above and by considering $T_{j+1} T_{j+2}$we get similar results to those in 3(i-iv) after replacing $T_j T_{j+1}$ by $T_{j+1} T_{j+2}$, i.e.either $T_{j+1} T_{j+2} = \in$ or m is shifted to T_{j+2}. By contradiction suppose that

$$T_{s-1}\, T_s \neq \in, \text{ for all } j+3 \leq s \leq k. \qquad (2.6)$$

Then as before this implies that for each $s, j+3 \leq s \leq k$, m is shifted one step to the right. If $s = k$ then as assumed by equation (2.6)

$$T_{k-1}\, T_k = T_{s-1}\, T_s \neq \in$$

Therefore as in equation 3(i, ii, iii) m is shifted to T_k, contradiction as proved above. Therefore for some $j \leq s \leq k, T_{s-1} T_s = \in$. As a result the transpositions in (2.5) are reduced by two transpositions. Since n is finite and m is arbitrary this implies that for each m, $1 \le m \le n$, $\exists\, j$, $1 \le j \le k$ such that $T_j\, T_{j+1} = \in$. Therefore

$$\in = \in \in \ldots \in,$$

where each $\in$ on the right represents a reduction of k by two. This implies that k is even.

More generally suppose that the permutations α and β are equal and that

$$\alpha = T_1 T_2 \ldots T_k$$

$$\beta = T_1^{/} T_2^{/} \ldots T_L^{/}$$

By proof of lemma (2.3) therefore

$$\beta^{-1} = T_L^{/} \ldots T_2^{/} T_1^{/}$$

since every transposition is its own inverse. Moreover

$$\in = \alpha\beta^{-1} = T_1 T_2 \ldots T_k\, T_L^{/} \ldots T_2^{/} T_1^{/}$$

Our special case of $\in$ above shows that $K + L$ is an even number of transpositions. This implies that $K \text{ and } L$ are either both even or both odd.

Lemma 2.5:

Let $x_1, x_2, \ldots, x_n$ be independent symbols and p be the polynomial with integral coefficients defined by

$$p = \prod_{i<j} (x_i - x_j), \quad i = 1, \ldots, n \;, \quad j = 1, \ldots, n$$

Let $\alpha \in S_n$ and define

$$\alpha(p) = \prod_{i<j} (x_{\alpha(i)} - x_{\alpha(j)}) \;, i = 1, \ldots, n, \; j = 1, \ldots, n$$

If $\delta = (k,l)$ is a transposition, then

$$\delta(p) = -p$$

Proof:

Consider the product $(x_i - x_k)(x_i - x_L)$ in p. Now $k \neq l$ and without loss of generality let $k < l$. Hence, one of the factors in p .is $x_k - x_l$ and in $\delta(p)$ the corresponding factor is $x_l - x_k$, that is, this factor is just changed in sign under the mapping δ of the subscripts. By definition of p all other factors of p containing x_k and x_l can be paired to form a product of the form $\pm(x_i - x_k)(x_i - x_l)$, with the sign determined by the relative magnitude of *i* to *k* and *l*. But since effect of δ is just to interchange x_k and x_l it follows that the sign of any such product is unchanged. Hence, the only effect of δ is to change the sign of p by. changing only the sign of $x_k - x_l$ and the lemma is established [8,13]

Theorem (2.4):

Every permutation α can be expressed as a product of transpositions. Moreover, if α can be expressed as a product of r transpositions and also as a product of *s* transpositions, then either *r* and *s* are both even or they are both odd.

Proof:

Suppose that α is a permutation of the set

$$A = \{1,2,\ldots,n\}$$

From above every permutation is a product of disjoint cycles by theorem (2.2) and every cycle is a product of transpositions by lemma (2.2). Therefore suppose that

$$\alpha = \beta_1\beta_2...\beta_r = \gamma_1\gamma_2...\gamma_s$$

Where each β_i and γ_j $i = 1, ..., r,\ j = 1,2,...,s$ is a transposition. We need to prove that *r* and *s* are both even or that they are both odd. Let $x_1, x_2,..., x_n$ be independent symbols and let *p* denote the polynomial with integral coefficients defined as above by:

$$p = \prod_{i<j}(x_i - x_j), \quad i = 1,..., n \ , \quad j = 1,..., n$$

Similarly and for each $\alpha \in S_n$ define

$$\alpha(p) = \prod_{i<j}(x_{\alpha(i)} - x_{\alpha(j)}) \ , i = 1,..., n, \ j = 1,..., n$$

By lemma (2.5) and since α is a product of transpositions and in general it is fairly clear that

$$\alpha(p) = \pm p$$

with the sign depending in some way on the permutation α. If

$$\alpha = \beta_1, \beta_2,..., \beta_r,$$

then $\alpha\, p$ can be computed by performing in turn the r transpositions $\beta_1, \beta_2,..., \beta_r$. By lemma above each of these transpositions merely changes the sign of *p*. This implies that:

$$\alpha\,(p) = (-1)^r\, p \qquad (2.7)$$

Similarly using the fact that:

$$\alpha = \gamma_1\,\gamma_2\ ...\gamma_s$$

therefore

$$\alpha(p) = (-1)^s p \tag{2.8}$$

Hence using equations (1) and (2) above we must have:

$$(-1)^r p = (-1)^s p,$$

from which it follows that:

$$(-1)^r = (-1)^s$$

This implies that *r* and *s* are either both even or they are both odd [8,13]

2.3 Alternating Group:

Theorem 2.5:

If $n \geq 2$, then the collection A_n of all even permutations of a finite set of *n* elements form a subgroup of order $\frac{n!}{2}$ of the symmetric group S_n .

Proof:

By theorem(1.1) and lemma(2.3) parts(2,3) A_n *is a* subgroup of S_n.

Let B_n be the set of odd permutations of S_n for , $n \geq 2$. Let *T* be any fixed transposition in S_n. Suppose that *T=(1, 2)*. Define:

$$\lambda_T : A_n \rightarrow B_n$$

by

$$\lambda_T(\alpha) = \alpha T, \quad \forall \alpha \in A_n$$

That is $\alpha \in A_n$ is mapped into $\alpha(1,2)$ by λ_T. Since α is even, the permutation $\alpha(1,2)$ is a product of an odd number of transpositions. So $\alpha(1,2)$ is indeed in B_n.

Suppose $\alpha, \beta \in A_n$ such that:

$$\lambda_T(\alpha) = \lambda_T(\beta)$$

Then

$$\alpha(1,2) = \beta(1,2)$$

Multiplying both sides by $T^{-1} = (1,2)$, therefore we have $\alpha = \beta$. Thus λ_T is one-to-one.

Finally

$$T = (1,2) = T^{-1}$$

If $p \in B_n$ then

$$pT^{-1} \in A_n$$

and

$$\lambda_T(pT^{-1}) = pT^{-1}T = p$$

Thus λ_T is onto B_n. Hence the number of elements in A_n is the same as the number in B_n, since there is a one-to-one correspondence between the elements of the sets. Since

$$S_n = A_n \cup B_n$$

and

$$A_n \cap B_n = \phi\,,$$

This implies that

$$|A_n|, |B_n| = \tfrac{1}{2}|S_n| = \tfrac{1}{2}n! \text{ [1,8]}$$

Definition (2.7):

The set A_n of all even permutations of S_n is a subgroup of S_n. It is called the alternating group on *n* symbols. Moreover $[S_n : A_n] = 2$ [3,8]

2.4 Theorem (Cayley's Theorem):

Every group G is isomorphic to a group of permutations. i.e. isomorphic to a subgroup of S_G.

Proof:

For each $a \in G$ define

$$\theta_a : G \to G \text{ by}$$

$$\theta_a(x) = ax \in G, \ \forall x \in G$$

To prove that $\theta_a \in G$ we prove that it is 1:1 and onto.

Let $x, y \in G$ such that

$$\theta_a(x) = \theta_a(y)$$

Therefore

$$ax = ay$$

This implies that

$$a^{-1}ax = a^{-1}ay$$

i.e. $ex = ey$

i.e. $x = y$

Therefore θ_a is 1:1

let $b \in G$, since

$$a a^{-1}b = b,$$

Let

$$x = a^{-1}b$$

Therefore

$$\theta_a(x) = b$$

This implies that θ_a is onto. Therefore $\theta_a \in S_G$

Let $H = \{\theta_a : a \in G\}$.

Let $\theta_a, \theta_b \in H$, and $x \in G$. By composition of mappings.

$$\theta_a \theta_b(x) = \theta_a(\theta_b(x)) = \theta_a(bx) = a(bx) = ab(x) = \theta_{ab}(x)$$

Since this true for all $x \in G$, therefore

$$\theta_a \theta_b = \theta_{ab} \in H$$

Therefore H is closed under composition of mappings. Since $e \in G$, therefore $\theta_e \in H$. Moreover

$$\theta_e \theta_a = \theta_{ea} = \theta_a \in H$$

Similarly

$$\theta_a \theta_e = \theta_{ae} = \theta_a, \quad \forall \theta_a \in H$$

This implies that θ_e is identity of *H*.

Note that

$$\theta_e(x) = ex = x,$$

For all $x \in G$. Hence θ_e is the identity mapping of *G*, i.e. identity of S_G.

For each $a \in G$, $a^{-1} \in G$ and hence $\theta_a, \theta_{a^{-1}} \in H$ such that

$$\theta_a \theta_{a^{-1}} = \theta_{aa^{-1}} = \theta_e, \text{ identity of } H$$

and

$$\theta_{a^{-1}} \theta_a = \theta_{a^{-1}a} = \theta_e, \text{ identity of } H$$

Therefore $\theta_{a^{-1}}$ is the inverse of θ_a, i.e.

$$(\theta_a)^{-1} = \theta_{a^{-1}} \in H.$$

Let $\theta_a, \theta_b, \theta_c \in H \subseteq S_G$, this implies that

$$\theta_a(\theta_b\theta_c) = \theta_a(\theta_{bc}) = \theta_{a(bc)} = \theta_{(ab)c} = (\theta_a\theta_b)\theta_c,$$

since S_G is associative. Therefore H is a group. Since S_G is a group, This implies that H is a subgroup of S_G.

Define $\phi : G \to H$ by

$$\phi(a) = \theta_a,$$

For each $a \in G$.

Therefore

$$\phi(ab) = \theta_{ab} = \theta_a\theta_b = \phi(a)\ \phi(b).$$

Therefore ϕ is a homomorphism of G into H. By definition of H, ϕ is onto.

To prove that ϕ is 1:1, let $a,b \in G$ such that

$$\phi(a) = \phi(b)$$

This implies that

$$\theta_a = \theta_b$$

By definitions of θ_a and θ_b we have

$$\theta_a(x) = \theta_b(x), \text{ for all } x \in G.$$

i.e.

$$ax = bx, \text{ for all } x \in G$$

Therefore

$$a=b$$

This implies that ϕ is 1:1 and onto. Therefore G is isomorphic to H [3,8].

2.5 Orbits:

Definition (2.7):

Let X be any set and $G \le S_X$, be a group of permutations of X. If $x \in X$, then the orbit of x

$$O_x = \{y \in X : \exists g \in G\, such that\, g(x) = y\}$$

or equivalently

$$O_x = \{g(x) : \forall g \in G\}$$ [1,3,8].

Lemma (2.6):

If $G \le S_X$, then the orbits of G in X partition X.

Proof:

Let I_X be the identity permutation of X. For each $x \in X$, $x = I_X(x)$ and so $x \in O_x$. This implies that

$$X = \bigcup_{x \in X} O_x .$$

Let $x, y \in X$. Suppose that

$$O_x \cap O_y \ne \phi$$

by definition(2.7) above this implies that $\exists g_1, g_2 \in G$ such that $g_1(x) = g_2(y)$. This implies that $x = g_1^{-1} g_2(y)$. Therefore for any $g \in G$

$$g(x) = (g(g_1^{-1} g_2))y \in O_Y .$$

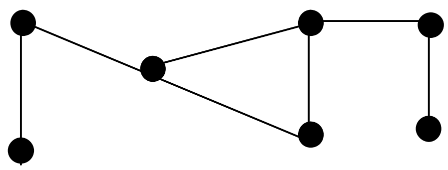
Therefore $O_x \subseteq O_y$. Similarly $O_y \subseteq O_x$. Thus $O_x = O_y$ whenever, $O_x \cap O_y \ne \phi$. Therefore lemma is true [1,3,8].

Chapter Three

Graph Theory

3.1 The Basics:

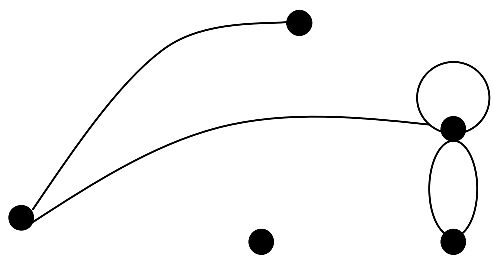
A graph *G* consists of a pair *(V, E)*, where *V* is the set of vertices and E the set of edges. We write *V (G)* for the vertices of *G* and *E(G)* for the edges of *G* when necessary to avoid ambiguity, as when more than one graph is under discussion. If no two edges have the same endpoints we say there are no multiple edges, and if no edge has a single vertex as both endpoints we say there are no loops. A graph with no loops and no multiple edges is a simple graph. The edges of a simple graph can be represented as a set of two element sets; for example, $\{v_1, \ldots, v_7\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$, $\{v_4, v_5\}$, $\{v_5; v_6\}$, $\{v_6, v_7\}$ is a graph that can be pictured as in figure below this graph is also a connected graph: each pair of vertices *v, w* is connected by a path $v = v_1, v_2, \ldots, v_k = w$, where each pair of vertices v_i and v_{i+1} are adjacent [36] .



3.2 Definitions and Fundamental Concepts:

Conceptually, a graph is formed by vertices and edges connecting the vertices.

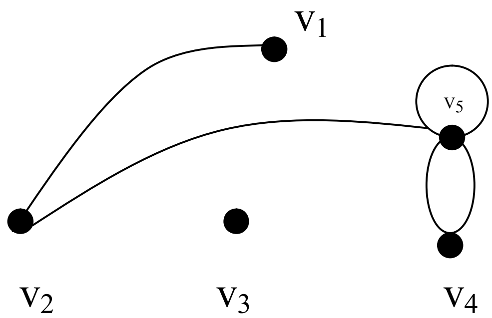
Example(3.1):



Formally, a graph is a pair of sets *(V,E)*, where V is the set of vertices and *E* is the set of edges, formed by pairs of vertices. *E* is a multiuse , in other words, its elements can occur more than once so that every element has a multiplicity. Often, we label the vertices with letters (for example: *a, b, c, . . . or* $v_1, v_2, \ldots$) or numbers *1, 2, . . .* Throughout this lecture material, we will label the elements of *V* in this way [36,37].

Example(3.2):

(Continuing from the previous example) We label the vertices as follows:



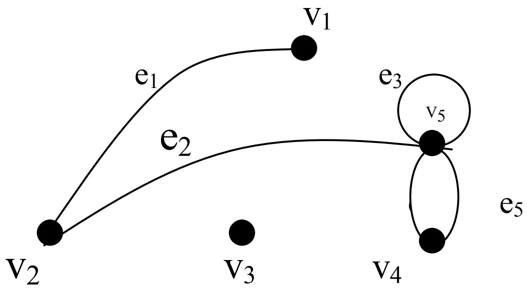
We have $V = \{v_1, . . . , v_5\}$ for the vertices and $E = \{(v_1, v_2), (v_2, v_5), (v_5, v_5), (v_5, v_4), (v_5, v_4)\}$ for the edges. Similarly, we often label the edges with letters (for example: $a, b, c,. . .$ or $e_1, e_2,. . .$) or numbers $1, 2, . . .$ for simplicity.

Remark(3.1):

The two edges (u, v) and (v, u) are the same. In other words, the pair is not ordered [36,37].

Example(3.3):

(Continuing from the previous example) We label the edges as follows:



So $E = \{e_1, . . . , e_5\}$.We have the following menologies:

1. The two vertices u and v are end vertices of the edge (u, v).
2. Edges that have the same end vertices are parallel.
3. An edge of the form (v, v) is a loop.
4. A graph is simple if it has no parallel edges or loops.
5. A graph with no edges (i.e. E is empty) is empty.
6. A graph with no vertices (i.e. V and E are empty) is a null graph.
7. A graph with only one vertex is trivial.
8. Edges are adjacent if they share a common end vertex.

9. Two vertices u and *v* are adjacent if they are connected by an edge, in other words, *(u, v)* is an edge.

10. The degree of the vertex *v*, written as *d(v),* is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately [36,37].

11. A pendant vertex is a vertex whose degree is 1.

12. An edge that has a pendant vertex as an end vertex is a pendant edge.

13. An isolated vertex is a vertex whose degree is 0.

Example(3.4):

(Continuing from the previous example)

- v_4 and v_5 are end vertices of e_5.
- e_4 and e_5 are parallel.
- e_3 is a loop.
- The graph is not simple.
- e_1 and e_2 are adjacent.
- v_1 and v_2 are adjacent.
- The degree of v_1 is 1 so it is a pendant vertex.
- e_1 is a pendant edge.
- The degree of v_5 is 5.
- The degree of v_4 is 2.
- The degree of v_3 is 0 so it is an isolated vertex [36,37].

In the future, we will label graphs with letters, for example:

$G = (V,E)$. The minimum degree of the vertices in a graph G is denoted $\delta(G) = 0$ if there is an isolated vertex in (G). Similarly, we write (G) as the maximum degree of vertices in G [36,38]..

Example(3.5):

(Continuing from the previous example) $\delta(G) = 0$ and (G) = 5.

Remark(3.2):

In this course, we only consider finite graphs, i.e. V and E are finite sets.

Since every edge has two end vertices, we get

Theorem(3.1):

The graph $G = (V,E)$, where $V = \{v_1, . . . , v_n\}$ and $E = \{e_1, . . . , e_m\}$, satisfies

$$\sum_{i=1}^{m} d(v_i) = 2m$$

Corollary:

Every graph has an even number of vertices of odd degree.

Proof

If the vertices $v_1, . . . , v_k$ have odd degrees and the vertices $v_{k+1}, . . . , v_n$ have even degrees, then Theorem

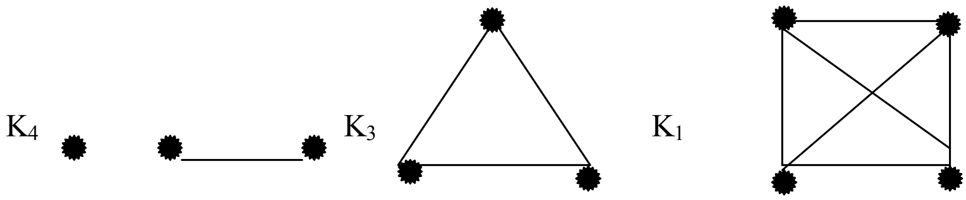
$$d(v_1) + \cdots + d(v_k) = 2n - d(v_{k+1}) - \cdots - d(v_n)$$

is even. Therefore, k is even [36,37]..

Example(3.6):

(Continuing from the previous example) Now the sum of the degrees is 1 + 2 + 0 +2 + 5 = 10 = 2.5 , There are two vertices of odd degree, namely v_1 and v_5 [36,37].

A simple graph that contains every possible edge between all the vertices is called a complete graph. A complete graph with *n* vertices is denoted as *Kn*. The first four complete graphs are given as examples:



The graph $G_1 = (V_1,E_1)$ is a subgraph of $G_2 = (V_2,E_2)$ if

1. $V_1 \subseteq V_2$ and
2. Every edge of G_1 is also an edge of G_2.

3.3 Graph Operations:

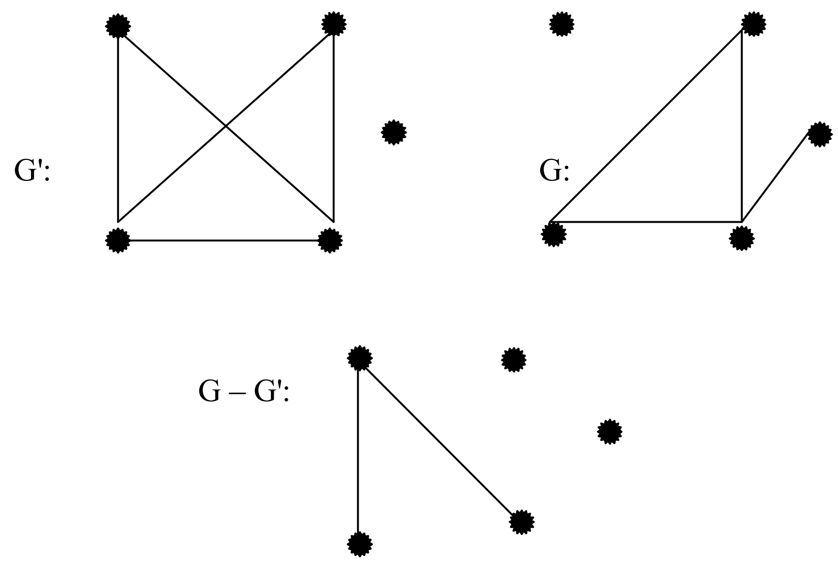
The complement of the simple graph $G = (V,E)$ is the simple graph $G = (V,E)$, where the edges in E are exactly the edges not in G [36,37].

Example(3.7):

The complement of the complete graph *Kn* is the empty graph with *n* vertices. Obviously, $G = G$. If the graphs $G = (V,E)$ and $G' = (V',E')$ are simple and $V' \subseteq V$, then the difference graph is $G - G' = (V,E''),$

where E'' contains those edges from G that are not in G' (simple graph) [36,37].

Example(3.8):



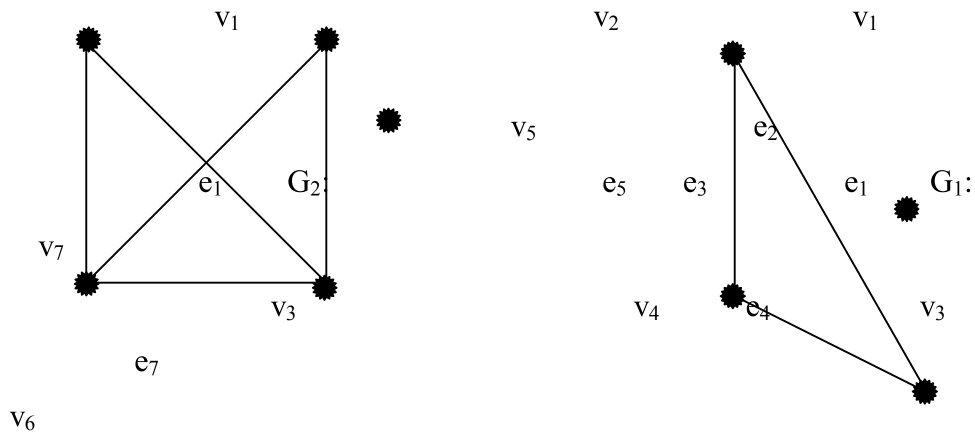
Here are some binary operations between two simple graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$:

- The union is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ (simple graph).
- The intersection is $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ (simple graph).
- The ring sum $G_1 \oplus G_2$ is the subgraph of $G_1 \cup G_2$ induced by the edge set $E_1 \oplus E_2$ (simple graph).

Note: The set operation $\oplus$ is the symmetric difference, i.e. $E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1)$. Since the ring sum is a subgraph induced by an edge set, there are no isolated vertices. All three operations are commutative and associative [36,37].

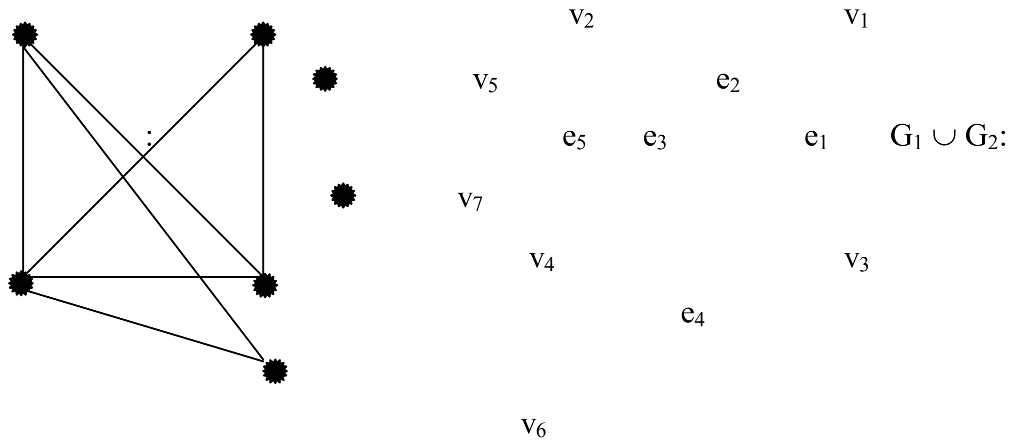
Example(3.9):

For the graphs

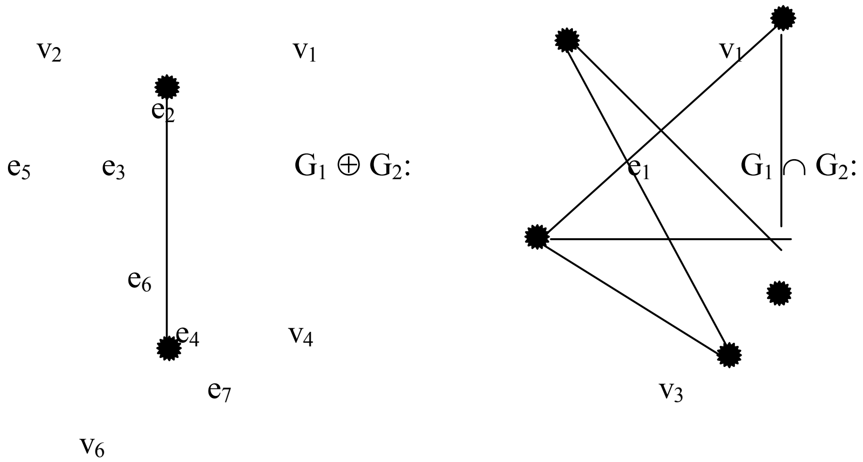


v_1
v_2
v_1
v_5
e_2
e_1
G_2:
e_5
e_3
e_1
G_1:
v_7
v_3
v_4
e_4
v_3
e_7
v_6

We have



v_2
v_1
v_5
e_2
e_5
e_3
e_1
$G_1 \cup G_2$:
v_7
v_4
v_3
e_4
v_6



v_2
v_1
v_1
e_2
e_5
e_3
$G_1 \oplus G_2$:
e_1
$G_1 \cap G_2$:
e_6
e_4
v_4
e_7
v_3
v_6

Remark(3.3):

The operations $\cup$, $\cap$ and $\oplus$ can also be defined for more general graphs other than simple graphs. Naturally, we have to "keep track" of the multiplicity of the edges:

$\cup$: The multiplicity of an edge in $G_1 \cup G_2$ is the larger of its multiplicities in G_1 and G_2.

$\cap$: The multiplicity of an edge in $G_1 \cap G_2$ is the smaller of its multiplicities in G_1 and G_2.

$\oplus$: The multiplicity of an edge in $G_1 \oplus G_2$ is $|m_1 - m_2|$, where m_1 is its multiplicity in G_1 and m_2 is its multiplicity in G_2. (We assume zero multiplicity for the absence of an edge.) In addition, we can generalize the difference operation for all kinds of graphs if we take account of the multiplicity. The multiplicity of the edge e in the difference $G - G'$ is $m_1 \dot{-} m_2 = (m_1 - m_2$, if $m_1 \geq m_2$, if $m_1 < m_2$ (also known as the proper difference), where m_1 and m_2 are the multiplicities of e in G_1 and G_2, respectively. If v is a vertex of the graph $G = (V,E)$, then $G - v$ is the sub graph of G induced by the vertex set $V - \{v\}$. We call this operation the removal of a vertex [36,37].

3.4 Trees and Forests:

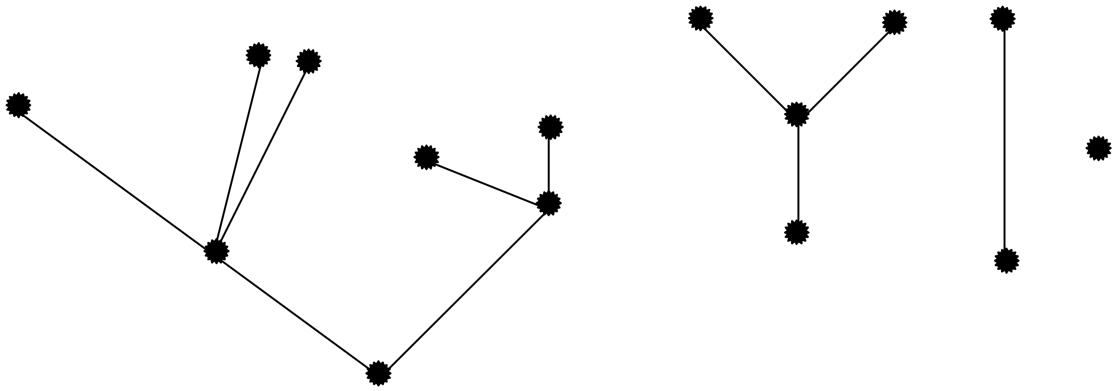
A forest is a circuit less graph. A tree is a connected forest. A sub forest is a sub graph of a forest. A connected sub graph of a tree is a sub tree. Generally speaking, a sub forest (respectively sub tree) of a graph is its sub graph, which is also a forest (respectively tree).

Definition(3.1):

A connected graph G is a tree if it is acyclic, that is, it has no cycles. More generally, an acyclic graph is called a forest.

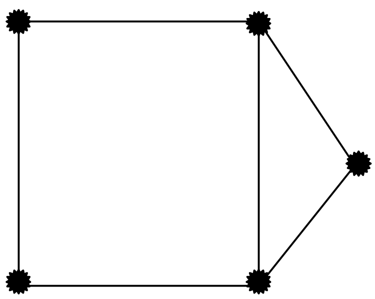
Example(3.10):

Four trees which together form a forest:

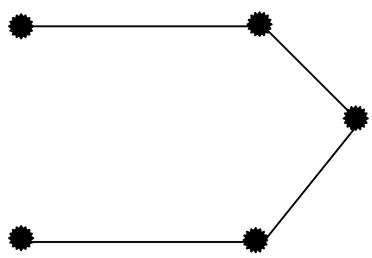


A spanning tree of a connected graph is a sub tree that includes all the vertices of that graph. If T is a spanning tree of the graph G [37].

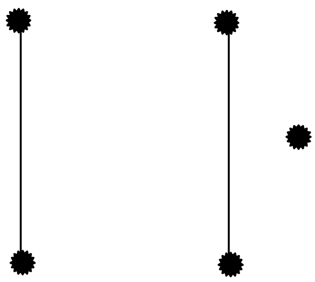
Example(3.11):



Spanning tree:



G:



Co spanning tree

The edges of a spanning tree are called branches and the edges of the corresponding co spanning tree are called links or chords.

Theorem(3.2):

If the graph G has n vertices and m edges, then the following statements are equivalent:

(i) G is a tree.

(ii) There is exactly one path between any two vertices in G and G has no loops.

(iii) G is connected and $m = n - 1$.

(iv) G is circuit less and $m = n - 1$.

(v) G is circuit less and if we add any new edge to G, then we will get one and only one circuit [37].

Remark(3.4):

We can get a spanning tree of a connected graph by starting from an arbitrary subforest M (as we did previously). Since there is no circuit whose edges are all in M, we can remove those edges from the circuit which are not in M. By the sub graph G_1 of G with n vertices is a spanning tree of G (thus G is connected) if any three of the following four conditions hold:

1. G_1 has n vertices.
2. G_1 is connected.
3. G_1 has n − 1 edges.
4. G_1 is circuitless.

Actually, conditions 3 and 4 are enough to guarantee that G_1 is a spanning tree. If conditions 3 and 4 hold but G_1 is not connected, then the components of G_1 are trees and the number of edges in G_1 would be number of vertices − number of components < $n - 1$ (p) [37].

3.5 Directed Trees:

A directed graph is quasi-strongly connected if one of the following conditions holds for every pair of vertices u and v:

(i) $u = v$ or

(ii) there is a directed u–v path in the digraph or

(iii) there is a directed v–u path in the digraph or

(iv) there is a vertex w so that there is a directed w–u path and a directed w–v path.

Example(3.12):

(Continuing from the previous example) The digraph G is quasi-strongly connected. Quasi-strongly connected digraphs are connected but not necessarily strongly connected. The vertex v of the digraph G is a root if there is a directed path from v to every other vertex of G [37].

Theorem(3.3):

For the digraph G with $n > 1$ vertices, the following are equivalent:

(i) G is a directed tree.

(ii) G is a tree with a vertex from which there is exactly one directed path to every other vertexof G.

(iii) G is quasi-strongly connected but $G - e$ is not quasi-trongly connected for any arc e in G.

(iv) G is quasi-strongly connected and every vertex of G has an in-degree of 1 except one vertex whose in-degree is zero.

(v) There are no circuits in G (i.e. not in G_u) and every vertex of G has an in-degree of 1 except one vertex whose in-degree is zero.

(vi) G is quasi-strongly connected and there are no circuits in G (i.e. not in G_u) [37].

3.6 Counting Graphs:

We count the graphs G on m vertices with q edges. Let G denote the set of graphs G on the vertices $M = \{1, 2, . . . ,m\}$. Such a G is a function from the set X of unordered pairs {i, j} of distinct elements of M to the set $Y = \{0, 1\}$, where G ({i, j}) is 1 or 0, according as {i, j} is an edge or a non edge of the graph G.

3.7 Acyclic Directed Graphs:

A directed graph with at least one directed circuit is said to be cyclic. A directed graph is acyclic otherwise. Obviously, directed trees are acyclic but the reverse implication is not true [38].

Theorem(3.4):

We can sort the vertices of a digraph topologically if and only if the graph is acyclic.

Proof.

If the digraph is cyclic, then obviously we can not sort the vertices topologically.

If the digraph G is acyclic, then we can sort the vertices in the following manner

1. We choose a vertex v which is a sink. It exists

$$\alpha(v) \leftarrow \text{n},\ G \leftarrow G - v \text{ and } n \leftarrow n - 1.$$

2. If there is just one vertex v in G, set $\alpha(v) \leftarrow 1$. Otherwise, go back to step 1 [38].

3.8 Graph Coloring:

Let's return now to the subject of assigning frequencies to radio stations so that they don't interfere. The first thing that we will need to do is to turn the map of radio stations into a suitable graph, which should be pretty natural at this juncture. We define a graph $G = (V, E)$ in which V is the set of radio stations and $xy \in E$ if and only if radio station x and radio station y are within 200 miles of each other. With this as our model, then we need to assign different frequencies to two stations if their corresponding vertices are joined by an edge. This leads us to our next topic, coloring graphs [38].

When $G = (V, E)$ is a graph and C is a set of elements called colors, a proper coloring of G is a function $f : V \rightarrow C$ such that if $f(x) \neq f(y)$

whenever xy is an edge in *G*. The least t for which *G* has a proper coloring using a set *C* of t colors is called the chromatic number of *G* and is denoted *c(G)*. In Figure 5.14, we show a proper coloring of a graph using 5 colors. Now we can see that our radio frequency assignment problem is the much-studied question of finding the chromatic number of an appropriate graph [38].

Chapter Four

Polya's Theory of Counting

The Polya enumeration theorem (PET) also known as red field – Polya's theorem, is a theorem in combinatorics, generalizing Burnside's lemma about number of orbits. This theorem was first discovered and published by John Howared Red field in 1927 but its importance was over looked and Red field's publication was not noticed by most of the mathematical community. Independently the result was proved in 1937 by George polya, who also demonstrated a number of its applications, in particular to enumeration of chemical compounds.

The (PET) gave rise to symbolic operators and symbolic methods in enumerative combinatorics and was generatized to the fundamental theorem of combinatorial enumeration [6.10].

4.1 Polya's Theory of Counting

We start our discussion of the theory by some examples which make easy the following of the development of the theory.

Example (4.1):

A disc lies in a plane. It's center is fixed but it is free to rotate. It has been divided into n sectors of angle $\frac{2\pi}{n}$.

Each sector is to be colored Red or Blue.

How many different colorings are there?

One could argue for 2^n. Hence if n=4 then the number of colourings is $2^4 = 16$ [11,15]

On the other hand, what if we only distinguish colourings which can not be obtained from one another by a rotation for example if n=4 and the sectors are numbered 0,1,2,3 in clockwise order around the disc, then there are only 6 ways of coloring the disc: 4R, 4B, 3R1B, 1R3B and RBRB. Hence we have two different answers for the number of colourings of a disc if $n=4$, namely 16 and 6 [11,14].

Now consider an $n \times n$ "chessboard" where $n \geq 2$. Here we colour the squares Red and Blue and two colorings are different only if one can not be obtained from another by a rotation or a reflection.

For n = 2 there are 6 colourings as follows:

Red	Red
Red	Red

Red	Red
Red	Blue

Red	Blue
Red	Blue

Blue	Red
Red	Blue

Red	Blue
Blue	Blue

Blue	Blue
Blue	Blue

on the other hand the chessboard has $n=4$ squares which can be coloured in $2^{n^2} = 2^{2^2} = 16$ different ways which are not necessarily distinct [11,14].

Referring to these examples and in order to determine the exact number of colourings justifies Polya's theory of counting.

The general scenario of the above examples is as follows:

Suppose we have a set X which stands for the set of all colourings of the set of sectors D. To investigate how Polya's theory handles these situations suppose G is a group of permutations of X. G will have a group structure as follows:

Given two members $g_1, g_2 \in G$ and as in chapter (2) above the composition $g_1 \circ g_2$ is defined by

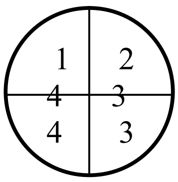
$$g_1 \circ g_2(x) = g_1(g_2(x)), \quad \forall x \in X \; .$$

Then as above (G, o) is a group of permutations of X.

Using the above examples for a disc of four sectors 1, 2, 3, 4 ,where $n=4$,and a chessboard of four squares 1,2,3,4, where $n=2$, we have the following remark [11,15].

Remark (4.1)

(i) Let $D = \{1,2,3,4\}$ be represented by the ordered disc

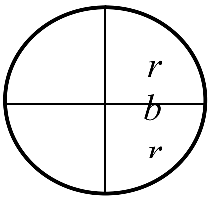


or ordered chessboard

1	2
4	3

where both the sectors of the disc and squares of the chessboard are numbered 1,2,3,4 in clockwise order starting at the upper left of each of the figures above [11,23].

(ii) Suppose there are two colours Blue (b) and Red (r). Let X be the set of all colourings of D, as represented above by a disc or chessboard, by the two colours Blue (b) and Red (r).Using the order of D in figures above, then each element x of X will be represented by a sequence of four elements of the set $\{b, r\}$ written from left to right such that the element on the left of x is the colour of sector or square numbered 1 in figures above. Similarly the following element of x is the colour of sector or square number 2 in figures above and hence forth from left to right of x and in clockwise direction of figures above and conversely, e.g. if $x = rbbr \in X$ then the corresponding coloured disc and chessboard are in figures below



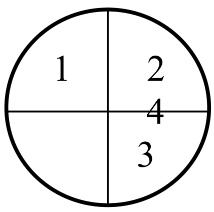
and

r	b
r	b

and conversely [11,25].

Example (4.2):

Suppose D is a disc and the number of sectors of $D = 4$. Moreover suppose we number the sectors of D 1, 2, 3, 4 in clockwise order starting at the upper left as in the figure below:



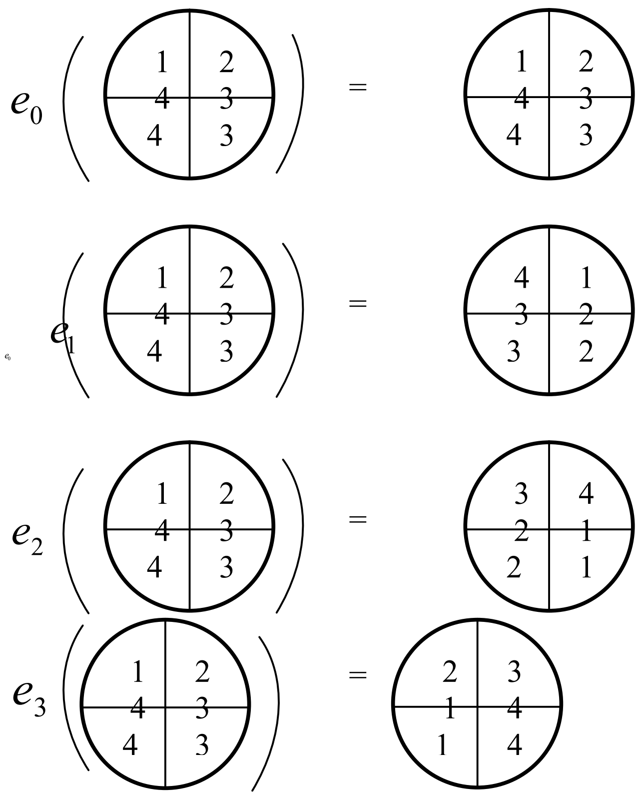
Hence

$$D = \{1,2,3,4\}$$

Suppose

$$G_1 = \{e_0, e_1, e_2, e_3\},$$

where e_0, e_1, e_2, e_3 represent a clockwise rotation of D through 0, 90, 180, 270 degrees respectively. Using the above figures and by definition of e_0, e_1, e_2, e_3 we get



Using the above figures then e_0, e_1, e_2, e_3 have the following cyclic structures

$$e_0 = (1)(2)(3)(4),\ \ e_1 = (1\ \ 2\ \ 3\ \ 4),\ e_2 = (1\ 3)(2\ \ 4),\ \ e_3 = (1\ \ 4\ \ 3\ \ 2)$$

Using the cyclic structures of e_0, e_1, e_2, e_3 , and as in chapter two,we get

$$e_0 e_0 = (1)(2)(3)(4)\ \ (1)(2)(3)(4) = (1)(2)(3)(4) = e_0$$

$$e_0 e_1 = (1)(2)(3)(4)\ \ (1\ \ 2\ \ 3\ \ 4) = (1\ \ 2\ \ 3\ \ 4) = e_1$$

$$e_0e_2 = (1)(2)(3)(4)\ \ (1\ \ 3)(2\ \ 4) = \ (1\ \ 3)(2\ \ 4) = \ e_2$$

$$e_0e_3 = (1)(2)(3)(4)\ \ (1\ \ 4\ \ 3\ \ 2) \quad = (1\ \ 4\ \ 3\ \ 2) = \ e_3$$

$$e_1e_0 = (1\ \ 2\ \ 3\ \ 4)\ (1)(2)(3)(4) \quad = (1\ \ 2\ \ 3\ \ 4) = \ e_1$$

$$e_1e_1 = \ \ (1\ \ 2\ \ 3\ \ 4)\ (1\ \ 2\ \ 3\ \ 4) \quad = (1\ \ 3)(2\ \ 4) = \ e_2$$

$$e_1e_2 = \ \ (1\ \ 2\ \ 3\ \ 4)(1\ \ 3)(2\ \ 4) \quad = (1\ \ 4\ \ 3\ \ 2) \ = \ e_3$$

$$e_1e_3 = \ \ (1\ \ 2\ \ 3\ \ 4)\ (1\ \ 4\ \ 3\ \ 2) = (1)\ \ (2)\ \ (3)\ \ (4) = e_0$$

$$e_2e_0 = (1\ \ 3)(2\ \ 4)(1)(2)(3)(4) \quad = \ (1\ \ 3)(2\ \ 4) = \ e_2$$

$$e_2e_1 = (1\ \ 3)(2\ \ 4)\ \ (1\ \ 2\ \ 3\ \ 4) \quad = (1\ \ 4\ \ 3\ \ 2) \ = \ e_3$$

$$e_2e_2 = (1\ \ 3)(2\ \ 4)\ \ (1\ \ 3)(2\ \ 4) \ = (1)(2)(3)\ (4) \ = \ e_0$$

$$e_2e_3 = \ (1\ \ 3)(2\ \ 4)\ (1\ \ 4\ \ 3\ \ 2) \quad = \quad (1\ \ 2\ \ 3\ \ 4) \ = \ e_1$$

$$e_3e_0 = (1\ \ 4\ \ 3\ \ 2)(1)(2)(3)(4) \quad = \quad (1\ \ 4\ \ 3\ \ 2) = \ e_3$$

$$e_3e_1 = \quad (1\ \ 4\ \ 3\ \ 2)\ (1\ \ 2\ \ 3\ \ 4) = (1)\ \ (2)\ \ (3)\ \ (4) = e_0$$

$$e_3e_2 = (1\ \ 4\ \ 3\ \ 2)\ (1\ \ 3)(2\ \ 4) \quad = \quad (1\ \ 2\ \ 3\ \ 4) \ = \ e_1$$

$$e_3e_2 = (1\ \ 4\ \ 3\ \ 2)(1\ \ 4\ \ 3\ \ 2) \quad = \quad (1\ \ 3)(2\ \ 4) \ = \ e_2$$

Using the above results we get the following composition of mappings table of G_1.

o	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	e_2	e_3	e_0
e_2	e_2	e_3	e_0	e_1
e_3	e_3	e_0	e_1	e_2

Using this table we get

Lemma (4.1):

(G_1, o) is a permutation group of X.

Poof:

By definition each element of G_1 is a permutation of X. moreover by table above:

1- G_1 is closed under composition of mappings.

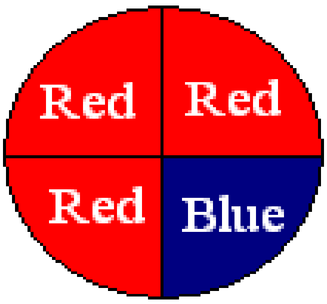
2- e_0 is identity of G_1.

3- Each element has an inverse.

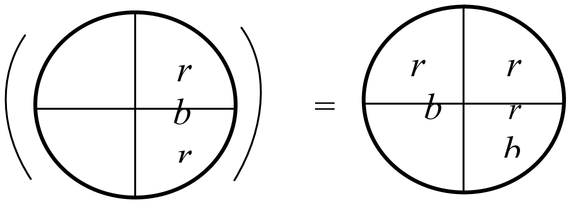
By composition of mappings G_1 is associative.

There fore G_1 is a permutation group of X [11].

Moreover represent elements of X as elements of a sequence from $\{r,b\}^4$, where for example the element $rrbr \in X$ is the element where the colour of $1,2,4$ is Red and the colour of 3 is Blue see figure:



Using the above remark (4.1) and the definitions of members of G_1 as rotations we construct the following table where the first column represents the elements of X and the first row represents the elements of G_1. Moreover each other element is the image of the corresponding elements of G_1 and X which are respectively in the same column and row with it, e.g. for *rbrr*$\in X$ and $e_2 \in G_1$ and by definition of e_2 as a rotation we get the element

e_2 (*rbrr*)= e_2  = *rrrb*

as in the table below

We observe that X has $2^4 = 16$ elements as shown in first column of table below [11,27]

x	e_0	e_1	e_2	e_3
rrrr	*rrrr*	*rrrr*	*rrrr*	*rrrr*
brrr	*brrr*	*rbrr*	*rrbr*	*rrrb*
rbrr	*rbrr*	*rrbr*	*rrrb*	*brrr*
rrbr	*rrbr*	*rrrb*	*brrr*	*rbrr*
rrrb	*rrrb*	*brrr*	*rbrr*	*rrbr*
bbrr	*bbrr*	*rbbr*	*rrbb*	*brrb*
rbbr	*rbbr*	*rrbb*	*brrb*	*bbrr*
rrbb	*rrbb*	*brrb*	*bbrr*	*rbbr*
brrb	*brrb*	*bbrr*	*rbbr*	*rrbb*
rbrb	*rbrb*	*brbr*	*rbrb*	*brbr*
brbr	*brbr*	*rbrb*	*brbr*	*rbrb*
bbbr	*bbbr*	*rbbb*	*brbb*	*bbrb*
bbrb	*bbrb*	*bbbr*	*rbbb*	*brbb*
brbb	*brbb*	*bbrb*	*bbbr*	*rbbb*
rbbb	*rbbb*	*brbb*	*bbrb*	*bbbr*

bbbb	*bbbb*	*bbbb*	*bbbb*	*bbbb*

Table (4.1)

We generalize this example in the following theorem [11,27].

Theorem (4.1):

Let D be a disc divided into n sectors. Denote

$$D = \{0,1,2,\ldots,n\text{-}1\}.$$

If we have two colours Red and Blue to colour D, and if X is the set of all colourings of D without transformation, then

$$|X| = 2^{D}.$$

Moreover the set of permutations of X

$$G = \{e_0, e_1, \ldots, e_{n-1}\},$$

where

$$e_j(x) = x + j \bmod n, x \in X.$$

Stands for a clockwise rotation of x by $2j\pi/n$, is a group [11,26].

Proof:

Obviously $|X| = 2^{|D|} = 2^{n-1}$

Let $e_i, e_j \in G$ then by definition of O

$$(e_i \circ e_j)x = e_i(e_j)(x) = e_i(x + j \bmod n)$$

$$= (x + j \bmod n) + i \bmod n = y + i \bmod n,$$

where $y = x + j \bmod n$. By definition of $j \bmod n$, y is a rotation of x by an angle of $\frac{2\pi j}{n}$.Similarly $z = y + j \bmod n$ is a rotation of y by an angle of $\frac{2\pi i}{n}$. Therefore, z is a rotation of x by

$$\frac{2\pi j}{n} + \frac{2\pi i}{n} = \frac{2\pi}{n}(i + j) = \frac{2\pi}{n}(i + j) \bmod n = e_{i+j}$$

Therefore

$$(e_i \circ e_j)(x) = e_{i+j}(x), \forall_x \in X$$

This implies that

$$e_i \circ e_j = e_{i+j}$$

Hence G_1 is closed under composition of mapping. Moreover composition of mappings is associative. Furthermore e_0 is identity of G since

$$e_\circ \circ e_j = e_{o+j} = e_j = e_{j+o} = e_j o e_o, \forall e_j \in G.$$

Let $e_j \in G, e_j \neq e_0$. This implies that $0 \neq n - j < n$. Since

$$e_j \circ e_{n-j} = e_n = e_0 = e_{n-j} . e_j$$

This implies that

$$(e_j)^{-1} = e_{n-j} \in G.$$

Hence each element has an inverse. Therefore G is a group.

Note: we observe that G isomorphic to Z_n the group of integers modulo n under addition [11].

Example (4.3):

Suppose D is a chessboard where $n=2$. If X is the set of all colourings of D with two colours and without transformation then

$$|D| = n^2 = 4, \qquad |X| = 2^{n^2} = 16.$$

Moreover we number the squares of D 1,2,3,4 in clockwise order starting at the upper left, see figure below

1	2
4	3

Hence

$$D = \{1,2,3,4\}$$

Suppose

$$G_2 = \{e,a,b,c,p,q,r,s\}$$

where e, a, b, c represent a clockwise rotation through 0, 90, 180, 270 degrees respectively. Using the above figure and by definition of e_0, e_1, e_2, e_3 in example (4.2) therefore

$$e = e_0\,,\ a = e_1\,,\ b = e_2\,,\ c = e_3$$

Moreover let p, q represent reflections along one of the vertical and horizontal sides respectively, and r, s represent reflections in the diagonals 1,3 and 2,4 respectively. Using the above figure and by definition of e, a, b, c, p, q, r, s we get the cyclic structures of these elements. Since [11].

$$e\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array},$$

hence $e = (1)\ (2)\ (3)\ (4)$ is identity permutation.

Since

$$a \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline 4 & 1 \\ \hline 3 & 2 \\ \hline \end{array},$$

hence $a = (1\ \ 2\ \ 3\ \ 4)$.

Similarly

$$b\ \ b \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array},$$

and hence $b = (1\ \ 3)\ (2\ \ 4)$.

$$c\ \ c \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array},$$

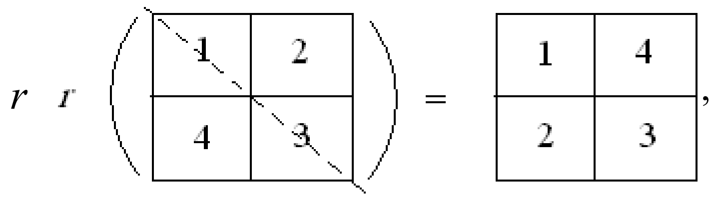
and hence $c = (1\ \ 4\ \ 3\ \ 2)$.

$$p\ \ p \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & 4 \\ \hline \end{array},$$

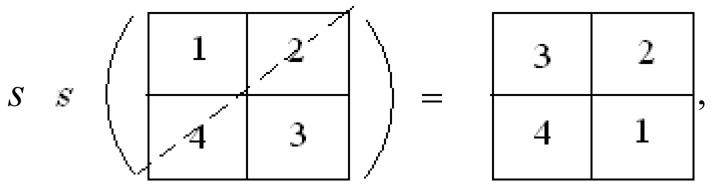
and hence $p = (1\ \ 2)\ (3\ \ 4)$.

$$q\ \ q \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 1 & 2 \\ \hline \end{array},$$

and hence $q = (1\ \ 4)\ (2\ \ 3)$.



and hence $r = (1)\ (2\ \ 4)\ (3)$.



and hence $s = (1\ \ 3)\ (2)\ (4)$.

From example (4.2) above

$$e_0 = e, \qquad e_1 = a, \qquad e_2 = b, \qquad e_3 = c$$

Moreover $G_1 = \{e_0, e_1, e_2, e_3\}$.

Using lemma (4.1) above and the composition of mappings table of G_1 we deduce that

$$ee = e \quad , \qquad ea = a, \qquad eb = b, \qquad ec = c$$

$$ae = a, \qquad aa = b, \qquad ab = c, \qquad ac = e$$

$$be = b, \qquad ba = c, \qquad bb = e, \qquad bc = a$$

$$ce = c, \qquad ca = e, \qquad cb = a, \qquad cc = b$$

Using the cyclic structure of the elements of G_2 above we get

$$ep = (1)(2)(3)(4)(1\ \ 2)(3\ \ 4) = (1\ \ 2)(3\ \ 4) = p$$

$$eq = (1)(2)(3)(4)(1\ \ 4)(2\ \ 3) = (1\ \ 4)(2\ \ 3) = q$$

$$er = (1)(2)(3)(4)(1)(2\ \ 4)(3) = (1)(2\ \ 4)(3) = r$$

$$es = (1)(2)(3)(4)(1\ \ 3)(2)(4) = (1\ \ 3)(2)(4) = s$$

$$ap = (1\ \ 2\ \ 3\ \ 4)(1\ \ 2)(3\ \ 4) = (1\ \ 3)(2)(4) = s$$

$$aq = (1\ \ 2\ \ 3\ \ 4)(1\ \ 4)(2\ \ 3) = (1)(2\ \ 4)(3) = r$$

$ar = (1\ \ 2\ \ 3\ \ 4)\ (1)(2\ \ 4)(3) = (1\ \ 2)(3\ \ 4) = p$

$as = (1\ \ 2\ \ 3\ \ 4)\ (1\ \ 3)(2)(4) = (1\ \ 4)(2\ \ 3) = q$

$bp = (1\ \ 3)(2\ \ 4)(1\ \ 2)(3\ \ 4) = (1\ \ 4)(2\ \ 3) = q$

$bq = (1\ \ 3)(2\ \ 4)\ (1\ \ 4)(2\ \ 3) = (1\ \ 2)(3\ \ 4) = p$

$br = (1\ \ 3)(2\ \ 4)\ (1)(2\ \ 4)(3) = (1\ \ 3)(2)(4) = s$

$bs = (1\ \ 3)(2\ \ 4)(1\ \ 3)(2)(4) = (1)(2\ \ 4)(3) = r$

$cp = (1\ \ 4\ \ \ 3\ \ 2)(1\ \ 2)(3\ \ 4) = (1)(2\ \ 4)(3) = r$

$cq = (1\ \ 4\ \ \ 3\ \ 2)\ (1\ \ 4)(2\ \ 3) = (1\ \ 3)(2)(4) = s$

$cr = (1\ \ 4\ \ \ 3\ \ 2)\ (1)(2\ \ 4)(3)= (1\ \ 4)(2\ \ 3) = q$

$cs = (1\ \ 4\ \ \ 3\ \ 2)\ (1\ \ 3)(2)(4) = (1\ \ 2)(3\ \ 4) = p$

$pe =(1\ \ 2)(3\ \ 4)(1)(2)(3)(4) = (1\ \ 2)(3\ \ 4) = p$

$pa =(1\ \ 2)(3\ \ 4)(1\ \ 2\ \ 3\ \ 4) = (1)(2\ \ 4)(3) = r$

$pb =(1\ \ 2)(3\ \ 4)(1\ \ 3)(2\ \ 4) = (1\ \ 4)(2\ \ 3) = q$

$pc =(1\ \ 2)(3\ \ 4)(1\ \ 4\ \ \ 3\ \ 2) = (1\ \ 3)(2)(4) = s$

$pp =(1\ \ 2)(3\ \ 4)\ (1\ \ 2)(3\ \ 4) = (1)(2)(3)(4)= e$

$pq =(1\ \ 2)(3\ \ 4)(1\ \ 4)(2\ \ 3) = (1\ \ 3)(2\ \ 4)= b$

$pr =(1\ \ 2)(3\ \ 4)\ (1)(2\ \ 4)(3) = (1\ \ 2\ \ 3\ \ 4)= a$

$ps =(1\ \ 2)(3\ \ 4)\ (1\ \ 3)(2)(4) = (1\ \ 4\ \ \ 3\ \ 2) = c$

$qe =(1\ \ 4)(2\ \ 3)(1)(2)(3)(4) = (1\ \ 4)(2\ \ 3) = q$

$qa =(1\ \ 4)(2\ \ 3)\ \ (1\ \ 2\ \ 3\ \ 4) = (1\ \ 3)(2)(4) = s$

$qb= (1\ \ 4)(2\ \ 3)(1\ \ 3)(2\ \ 4) = (1\ \ 2)(3\ \ 4) = p$

$qc =(1\ \ 4)(2\ \ 3)(1\ \ 4\ \ \ 3\ \ 2) = (1)(2\ \ 4)(3) = r$

$qp =(1\ \ 4)(2\ \ 3)(1\ \ 2)(3\ \ 4) = (1\ \ 3)(2\ \ 4)= b$

$qq =(1\ \ 4)(2\ \ 3)(1\ \ 4)(2\ \ 3) = (1)(2)(3)(4)= e$

$qr = (1\ \ 4)(2\ \ 3)\ (1)(2\ \ 4)(3) = (1\ \ 4\ \ \ 3\ \ 2) = c$

$qs = (1\ \ 4)(2\ \ 3)\ (1\ \ 3)(2)(4) = (1\ \ 2\ \ \ 3\ \ 4) = a$

$re = (1)(2\ \ 4)(3)\ (1)(2)(3)(4) = (1)(2\ \ 4)(3) = r$

$ra = (1)(2\ \ 4)(3)\ (1\ \ 2\ \ \ 3\ \ 4) = (1\ \ 4)(2\ \ 3) = q$

$rb = (1)(2\ \ 4)(3)\ (1\ \ 3)(2\ \ 4) = (1\ \ 3)(2)(4) = s$

$rc = (1)(2\ \ 4)(3)\ (1\ \ 4\ \ \ 3\ \ 2) = (1\ \ 2)(3\ \ 4) = p$

$rp = (1)(2\ \ 4)(3)\ (1\ \ 2)(3\ \ 4) = (1\ \ 4\ \ \ 3\ \ 2) = c$

$rq = (1)(2\ \ 4)(3)(1\ \ 4)(2\ \ 3) = (1\ \ 2\ \ \ 3\ \ 4) = a$

$rr = (1)(2\ \ 4)(3)\ (1)(2\ \ 4)(3) = (1)(2)(3)(4) = e$

$rs = (1)(2\ \ 4)(3)\ (1\ \ 3)(2)(4) = (1\ \ 3)(2\ \ 4) = b$

$se = (1\ \ 3)(2)(4)(1)(2)(3)(4) = (1\ \ 3)(2)(4) = s$

$sa = (1\ \ 3)(2)(4)(1\ \ 2\ \ 3\ \ 4) = (1\ \ 2)(3\ \ 4) = p$

$sb = (1\ \ 3)(2)(4)(1\ \ 3)(2\ \ 4) = (1)(2\ \ 4)(3) = r$

$sc = (1\ \ 3)(2)(4)(1\ \ 4\ \ \ 3\ \ 2) = (1\ \ 4)(2\ \ 3) = q$

$sp = (1\ \ 3)(2)(4)(1\ \ 2)(3\ \ 4) = (1\ \ 2\ \ \ 3\ \ 4) = a$

$sq = (1\ \ 3)(2)(4)(1\ \ 4)(2\ \ 3) = (1\ \ 4\ \ \ 3\ \ 2) = c$

$sr = (1\ \ 3)(2)(4)(1)(2\ \ 4)(3) = (1\ \ 3)(2\ \ 4) = b$

$ss = (1\ \ 3)(2)(4)\ (1\ \ 3)(2)(4) = (1)(2)(3)(4) = e$

We summarize the above results in the following composition of mappings table of G_2.

o	*e*	*a*	*b*	*c*	*p*	*q*	*r*	*s*
e	*e*	*a*	*b*	*c*	*p*	*q*	*r*	*s*
a	*a*	*b*	*c*	*e*	*s*	*r*	*p*	*q*
b	*b*	*c*	*e*	*a*	*q*	*p*	*s*	*r*

c	c	e	a	b	r	s	q	p
p	p	r	q	s	e	b	a	c
q	q	s	p	r	b	e	c	a
r	r	q	s	p	c	a	e	b
s	s	p	r	q	a	c	b	e

Using this table we get

Lemma (4.2):

(G_2, o) is a permutation group of X

Proof:

By definition each element of G_2 is a permutation of X. Moreover by table above

G_2 is closed under composition of mappings.

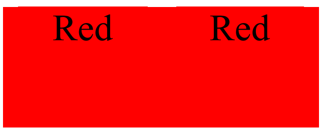
e is identity of G_2.

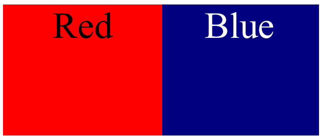
Each element has an inverse.

By composition of mappings G_2 is associative.

Therefore G_2 is a permutation group of X [11,27].

Moreover represent elements of X as elements of a sequence from $\{r,b\}^4$, where for example the element $rrbr \in X$ is the element where the colour of $1,2,4$ is Red and the colour of 3 is Blue see figure





Using the above remark (4.1) and the definitions of members of G_2 as rotations and reflections we construct the following table, where the first column represents the elements of X and the first row represents the elements of G_2 and each other element is the image of the corresponding elements of G_2 and X, which are respectively in the same column and row

with it, e.g. for $rbrr \in X$ and $e_2 \in G_2$ and by definition of e_2 as a rotation we get the element

$$e_2\ (rbrr) = e_2 \left(= \begin{array}{|c|c|}\hline r & b \\ \hline r & r \\ \hline \end{array} \right) = \begin{array}{|c|c|}\hline r & r \\ \hline b & r \\ \hline \end{array}\quad rrrb$$

Similarly by definition of $q \in G_2$

$$q(rbrr) = q \left(\begin{array}{|c|c|}\hline r & b \\ \hline r & r \\ \hline \end{array} \right) = \begin{array}{|c|c|}\hline r & r \\ \hline r & b \\ \hline \end{array} = rrrb$$

as in the table below [11,31].

We observe that X has $2^4 = 16$ elements as shown in first column of table below

x	e	a	b	c	p	Q	r	s
$rrrr$	$rrrr$	$rrrr$	$rrrr$	$rrrr$	$rrrr$	$rrrr$	$rrrr$	$rrrr$
$brrr$	$brrr$	$rbrr$	$rrbr$	$rrrb$	$rbrr$	$rrrb$	$brrr$	$rrbr$
$rbrr$	$rbrr$	$rrbr$	$rrrb$	$brrr$	$brrr$	$rrbr$	$rrrb$	$rbrr$
$rrbr$	$rrbr$	$rrrb$	$brrr$	$rbrr$	$rrrb$	$rbrr$	$rrbr$	$brrr$
$rrrb$	$rrrb$	$brrr$	$rbrr$	$rrbr$	$rrbr$	$brrr$	$rbrr$	$rrrb$
$bbrr$	$bbrr$	$rbbr$	$rrbb$	$brrb$	$bbrr$	$rrbb$	$brrb$	$rbbr$
$rbbr$	$rbbr$	$rrbb$	$brrb$	$bbrr$	$brrb$	$rbbr$	$rrbb$	$bbrr$
$rrbb$	$rrbb$	$brrb$	$bbrr$	$rbbr$	$rrbb$	$bbrr$	$rbbr$	$brrb$
$brrb$	$brrb$	$bbrr$	$rbbr$	$rrbb$	$rbbr$	$brrb$	$bbrr$	$rrbb$
$rbrb$	$rbrb$	$brbr$	$rbrb$	$brbr$	$brbr$	$brbr$	$rbrb$	$rbrb$
$brbr$	$brbr$	$rbrb$	$brbr$	$rbrb$	$rbrb$	$rbrb$	$brbr$	$brbr$
$bbbr$	$bbbr$	$rbbb$	$brbb$	$bbrb$	$bbrb$	$rbbb$	$brbb$	$bbbr$
$bbrb$	$bbrb$	$bbbr$	$rbbb$	$brbb$	$bbbr$	$brbb$	$bbrb$	$rbbb$
$brbb$	$brbb$	$bbrb$	$bbbr$	$rbbb$	$brbb$	$bbrb$	$bbbr$	$brbb$
$rbbb$	$rbbb$	$brbb$	$bbrb$	$bbbr$	$brbb$	$bbbr$	$rbbb$	$bbrb$
$bbbb$	$bbbb$	$bbbb$	$bbbb$	$bbbb$	$bbbb$	$bbbb$	$bbbb$	$bbbb$

Table (4.2)

Lemma (3.3):

If G is a permutation group of a set X, therefore

$$|O_x||S_x| = |G|, \forall x \in X.$$

Proof:

For definitions of O_x and S_x see chapter (2). Moreover *fix* $x \in X$ and define an equivalence relation ~ on G by

$$g_1 \sim g_2 \text{ iff } g_1(x) = g_2(x)$$

We prove that ~ is an equivalence relation [11,41].

(i) Let $g \in G$. Since

$$g(x) = g(x),$$

this implies that g ~ g. Therefore ~ is reflexive.

(ii) Let $g_1 \sim g_2$. This implies that

$$g_1(x) = g_2(x).$$

Therefore

$$g_2(x) = g_1(x)$$

This implies that $g_2 \sim g_1$. Therefore ~ is symmetric.

(iii) Let $g_1 \sim g_2$, $g_2 \sim g_3$. This implies that

$$g_1(x) = g_2(x) and g_2(x) = g_3(x).$$

Therefore

$$g_1(x) = g_3(x)$$

This implies that $g_1 \sim g_3$. Therefore ~ is transitive [11,31].

Therefore ~ is an equivalence relation. Let the equivalence classes be $A_1, A_2, \ldots, A_m$. We first argue that

$$|A_i| = |S_x|, \qquad i = 1,2,\ldots,m$$

Fix i and $g \in A_i$. Then

$$h \in A_i \leftrightarrow g(x) = h(x) \leftrightarrow (g^{-1} \circ h)x = x$$

$$\leftrightarrow (g^{-1} \circ h) \in S_x \leftrightarrow h \in g \circ S_x \qquad (4.1)$$

This implies that $A_i = g \circ S_x$. By definition

$$g \circ S_x = \{g \circ \sigma = \sigma \in S_x\}$$

If $\sigma_1, \sigma_2 \in S_x$ and

$$g \circ \sigma_1 = g \circ \sigma_2$$

Then

$$\sigma_1 = (g^{-1} \circ g)\sigma_1 = g^{-1} \circ (g \circ \sigma_1) = g^{-1} \circ (g \circ \sigma_2) = (g^{-1} \circ g)\sigma_2 = \sigma_2.$$

Thus

$$|g \circ S_x| = |S_x| \qquad (4.2)$$

and therefore

$$|A_i| = |g \circ S_x| \qquad (4.3)$$

Using equations (4.1),(4.2) above therefore

$$|A_i| = |S_x|, \qquad i = 1,2,\dots,m.$$

Finally $m = |O_x|$ since there is a distinct equivalence class for each distinct *g (x)* [11,31].

Using table (4.1) and the lemma (4.1) above we get the following table.

x	O_x	S_x
rrrr	*{rrrr}*	G_1
brrr	*{brrr,rbrr,rrbr,rrrb}*	$\{e_0\}$
rbrr	*{brrr,rbrr,rrbr,rrrb}*	$\{e_0\}$
rrbr	*{brrr,rbrr,rrbr,rrrb}*	$\{e_0\}$
rrrb	*{brrr,rbrr,rrbr,rrrb}*	$\{e_0\}$

bbrr	*{bbrr,rbbr,rrbb,brrb}*	$\{e_0\}$
rbbr	*{bbrr,rbbr,rrbb,brrb}*	$\{e_0\}$
rrbb	*{bbrr,rbbr,rrbb,brrb}*	$\{e_0\}$
brrb	*{bbrr,rbbr,rrbb,brrb}*	$\{e_0\}$
rbrb	*{rbrb,brbr}*	$\{e_0,e_2\}$
brbr	*{rbrb,brbr}*	$\{e_0,e_2\}$
bbbr	*{bbbr,rbbb,brbb,bbrb}*	$\{e_0\}$
bbrb	*{bbbr,rbbb,brbb,bbrb}*	$\{e_0\}$
brbb	*{bbbr,rbbb,brbb,bbrb}*	$\{e_0\}$
rbbb	*{bbbr,rbbb,brbb,bbrb}*	$\{e_0\}$
bbbb	*{bbbb}*	G_1

Table (4.3)

Using table (4.2) and the above lemma (4.2) we get the following table:

x	O_x	S_x
rrrr	*{rrrr}*	G_2
brrr	*{brrr,rbrr,rrbr,rrrb}*	*{e,r}*
rbrr	*{brrr,rbrr,rrbr,rrrb}*	*{e,s}*
rrbr	*{brrr,rbrr,rrbr,rrrb}*	*{e,r}*
rrrb	*{brrr,rbrr,rrbr,rrrb}*	*{e,s}*
bbrr	*{bbrr,rbbr,rrbb,brrb}*	*{e,p}*
rbbr	*{bbrr,rbbr,rrbb,brrb}*	*{e,q}*
rrbb	*{bbrr,rbbr,rrbb,brrb}*	*{e,p}*
brrb	*{bbrr,rbbr,rrbb,brrb}*	*{e,q}*

rbrb	*{rbrb, brbr}*	*{e,b,r,s}*
brbr	*{rbrb, brbr}*	*{e,b,r,s}*
bbbr	*{bbbr,rbbb,brbb,bbrb}*	*{e,s}*
bbrb	*{bbbr,rbbb,brbb,bbrb}*	*{e,r}*
brbb	*{bbbr,rbbb,brbb,bbrb}*	*{e,s}*
rbbb	*{bbbr,rbbb,brbb,bbrb}*	*{e,r}*
bbbb	*{bbbb}*	G_2

Table (4.4)

Theorem (4.2):

Let G be a permutation group of a set X. If $V_{X,G}$ is the number of orbits of G in X, then [11].

$$V_{X,G} = \frac{1}{|G|}\sum_{x\in X}|S_x|$$

Proof:

$$X = \underset{x\in X}{U} O_x$$

See lemma (2.6) above in chapter (2). Suppose

$$X = O_{x_1} U O_{x_2} U \ldots U O_{x_n}$$

where $O_{x_1}, O_{x_2}, \ldots, O_{x_n}$ and the distinct an disjoint orbits of G in X. Therefore

$$V_{X,G} = n$$

By lemma (3.3) above

$$|O_x||S_x| = |G|$$

$$|O_x| = \frac{|G|}{|S_x|}$$

This implies that

$$\frac{1}{|O_x|} = \frac{|S_x|}{|G|}$$

Moreover

$$\sum_{x \in o_{x_i}} \frac{1}{|o_{x_i}|} = \frac{1}{|o_{x_i}|} + \frac{1}{|o_{x_i}|} + \ldots + \frac{1}{|o_{x_i}|} = 1 \qquad (4.4)$$

Since

$$X = \bigcup_{i=1}^{n} O_{x_i},$$

therefore by equation (4.4)

$$\sum_{x \in X} \frac{|S_x|}{|G|} = \sum_{x \in \bigcup_{i=1}^{n} O_{x_i}} \frac{|S_x|}{|G|} = \sum_{x \in O_{x_i}} \frac{|S_x|}{|G|} + \ldots + \sum_{x \in O_{x_n}} \frac{|S_x|}{|G|} = n \quad .$$

Therefore.

$$= \frac{1}{|G|} \sum_{x \in X} |S_x| = \sum_{x \in X} \frac{|S_x|}{|G|} = \quad .$$

Therefore

$$V_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

Example (4.4):

Thus in example (4.2) and using the table (4.1), the number of elements of G_1 is equal to *4*. Similarly and using table (4.1) we find $|S_x|$ for each $x \in X$. Hence each of $rrrr \in X$, and $bbbb \in X$ is fixed by *G*.

Moreover each of the elements *brrr, rbrr, rrbr, rrrb, bbrr, rbbr, rrbb, brrb, brbb, bbrb, bbbr, and rbbb* $\in X$ are fixed by $\{e_0\}$, and each of the elements *rbrb, brbr* $\in X$ is fixed by $\{e_0, e_2\}$. Therefore [11].

$$\sum_{x \in X} |S_x| = 4+1+1+1+1+1+1+1+1+2+2+1+1+1+1+4$$

$$= 24$$

Since $|G_1| = 4$, this implies that

$$V_{X,G_1} = \frac{1}{|G_1|}\sum_{x \in X} |S_x| = \frac{1}{4} \times 24 = 6.$$

Example (4.5):

In example (4.3) and using the table (4.4), the number of elements of G_2 is equal to 8. Similarly and using table (4.4) we find $|S_x|$ for each $x \in X$. Hence each of *rrrr* $\in X$, and *bbbb* $\in X$ is fixed by G. Moreover each of the elements *brrr, rrbr, bbrb, rbbb* $\in X$ are fixed by $\{e,r\}$, and each of the elements *rbrr, rrrb, bbbr,* and *brbb* $\in X$, are fixed by $\{e,s\}$, and each of the elements *bbrr, rrbb* $\in X$ are fixed by $\{e,p\}$, and each of the elements *rbbr, brrb* $\in X$ are fixed by $\{e,q\}$, and each of the elements *rbrb*, *brbr* $\in X$ are fixed by $\{e,b,r,s\}$. Therefore

$$\sum_{x \in X} |S_x| = 8+2+2+2+2+2+2+2+2+4+4+2+2+2+2+8$$

$$= 48$$

Since $|G_2| = 8$, this implies that

$$V_{X,G_2} = \frac{1}{|G_2|}\sum_{x \in X} |S_x| = \frac{1}{8} \times 48 = 6.$$

In what follows we have another look at $V_{X,G}$.

Definition (4.1):

For $g \in G$, define

$$\text{Fix } (g) = \{x \in X : g(x) = x\}$$

Using this definition we prove

Theorem (4.3): (Burnside's Lemma):

Let G be a permutation group of a set X. Then

$$V_{X,G} = \frac{1}{|G|} \sum_{g \in G} | Fix(g) |$$

Proof:

By theorem (4.2) above

$$V_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x| \qquad (4.5)$$

Define a function $A : X \times G \rightarrow \{0,1\}$ by

$$A(x,g) = \begin{cases} 1 & if \quad g(x) = x \\ 0 & if \quad g(x) \neq x \end{cases}$$

By definition of S_x this implies that

$$A(x,g) = 1 \text{ , } iff \ g \in S_x$$

$$A(x,g) = 0 \text{ , } iff \ g \in G - S_x$$

This implies that

$$| S_x | = \sum_{g \in S_x} A(x,g) = \sum_{g \in G} A(x,g)$$

Substituting in equation (4.5) therefore

$$\mathrm{V}_{\mathrm{X,G}} = \frac{1}{|G|}\sum_{x\in G}\sum_{g\in G} A(x,g) = \frac{1}{|G|}\sum_{g\in G}\sum_{x\in G} A(x,g) \qquad (4.6)$$

For a fixed $g \in G$ and by definition (4.1) above

$$\sum_{x\in X} A(x,g) = |Fix(g)|$$

Substituting in equation (4.5) above therefore

$$V_{X,G} = \frac{1}{|G|}\sum_{g\in G} |Fix(g)|$$

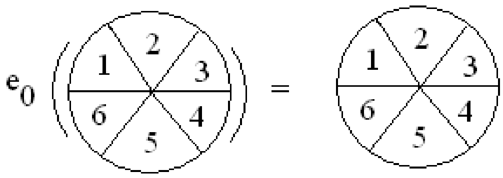
Example (4.6):

Suppose n is a disc with $n = 6$ sectors. Let

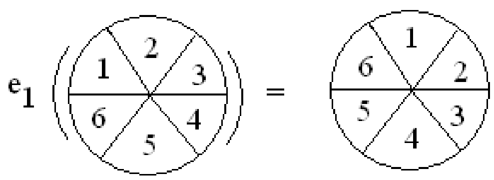
$$G_3 = \{ e_0, e_1, e_2, e_3, e_4, e_5 \}$$

where $e_0, e_1, e_2, e_3, e_4, e_5$ represent a clockwise rotation of D by 0, 60, 120, 180, 240, 300 degrees respectively.

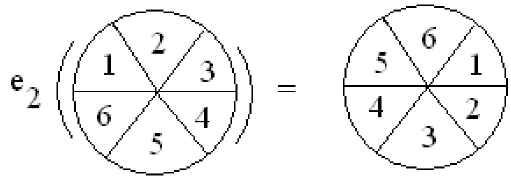
In what follows we find the cyclic structure of the elements of G [11].



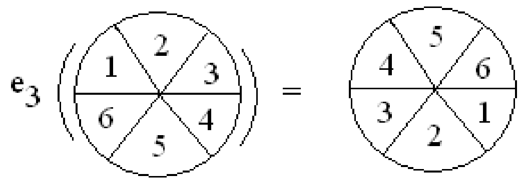
Hence $e_0 = (1)(2)(3)(4)(5)(6)$.



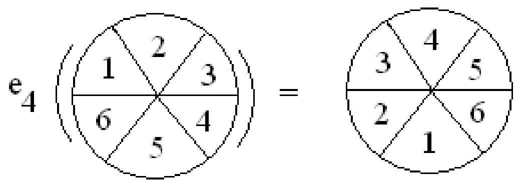
Hence $e_1 = (1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$.



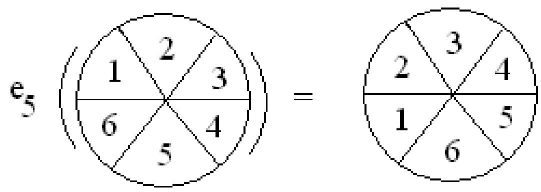
Hence $e_2 = (1\ \ 3\ \ 5)(2\ \ 4\ \ 6)$.



Hence $e_3 = (1\ \ 4)(2\ \ 5)(3\ \ 6)$.



Hence $e_4 = (1\ \ 5\ \ 3)(2\ \ 6\ \ 4)$.



Hence $e_5 = (1\ \ 6\ \ 5\ \ 4\ \ 3\ \ 2)$.

In what follows we prove that G_3 is closed under composition of mappings. We have,

$$e_0e_0 = (1)(2)(3)(4)(5)(6)(1)(2)(3)(4)(5)(6) = (1)(2)(3)(4)(5)(6) = e_0$$

$$e_0e_1 = (1)(2)(3)(4)(5)(6)(1\ 2\ 3\ 4\ 5\ 6) = (1\ 2\ 3\ 4\ 5\ 6) = e_1$$

$$e_0e_2 = (1)(2)(3)(4)(5)(6)(1\ 3\ 5)(2\ 4\ 6) = (1\ 3\ 5)(2\ 4\ 6) = e_2$$

$$e_0e_3 = (1)(2)(3)(4)(5)(6)(1\ 4)(2\ 5)(3\ 6) = (1\ 4)(2\ 5)(3\ 6) = e_3$$

$$e_0e_4 = (1)(2)(3)(4)(5)(6)(1\ 5\ 3)(2\ 6\ 4) = (1\ 5\ 3)(2\ 6\ 4) = e_4$$

$$e_0e_5 = (1)(2)(3)(4)(5)(6)(1\ 6\ 5\ 4\ 3\ 2) = (1\ 6\ 5\ 4\ 3\ 2) = e_5$$

$$e_1 e_0 = (1\ 2\ 3\ 4\ 5\ 6)(1)(2)(3)(4)(5)(6) = (1\ 2\ 3\ 4\ 5\ 6) = e_1$$

$$e_1 e_1 = (1\ 2\ 3\ 4\ 5\ 6)(1\ 2\ 3\ 4\ 5\ 6) = (1\ 3\ 5)(2\ 4\ 6) = e_2$$

$$e_1 e_2 = (1\ 2\ 3\ 4\ 5\ 6)(1\ 3\ 5)(2\ 4\ 6) = (1\ 4)(2\ 5)(3\ 6) = e_3$$

$$e_1 e_3 = (1\ 2\ 3\ 4\ 5\ 6)(1\ 3\ 5)(2\ 4\ 6) = (1\ 5\ 3)(2\ 6\ 4) = e_4$$

$$e_1 e_4 = (1\ 2\ 3\ 4\ 5\ 6)(1\ 5\ 3)(2\ 6\ 4) = (1\ 6\ 5\ 4\ 3\ 2) = e_5$$

$$e_1 e_5 = (1\ 2\ 3\ 4\ 5\ 6)(1\ 6\ 5\ 4\ 3\ 2) = (1)(2)(3)(4)(5)(6) = e_0$$

$$e_2 e_0 = (1\ 3\ 5)(2\ 4\ 6)(1)(2)(3)(4)(5)(6) = (1\ 3\ 5)(2\ 4\ 6) = e_2$$

$$e_2 e_1 = (1\ 3\ 5)(2\ 4\ 6)(1\ 2\ 3\ 4\ 5\ 6) = (1\ 4)(2\ 5)(3\ 6) = e_3$$

$$e_2 e_2 = (1\ 3\ 5)(2\ 4\ 6)(1\ 3\ 5)(2\ 4\ 6) = (1\ 5\ 3)(2\ 6\ 4) = e_4$$

$$e_2 e_3 = (1\ 3\ 5)(2\ 4\ 6)(1\ 4)(2\ 5)(3\ 6) = (1\ 6\ 5\ 4\ 3\ 2) = e_5$$

$$e_2 e_4 = (1\ 3\ 5)(2\ 4\ 6)(1\ 5\ 3)(2\ 6\ 4) = (1)(2)(3)(4)(5)(6) = e_0$$

$$e_2 e_5 = (1\ 3\ 5)(2\ 4\ 6)(1\ 6\ 5\ 4\ 3\ 2) = (1\ 2\ 3\ 4\ 5\ 6) = e_1$$

$$e_3 e_0 = (1\ 4)(2\ 5)(3\ 6)(1)(2)(3)(4)(5)(6) = (1\ 4)(2\ 5)(3\ 6) = e_3$$

$$e_3 e_1 = (1\ 3\ 5)(2\ 4\ 6)(1\ 2\ 3\ 4\ 5\ 6) = (1\ 5\ 3)(2\ 6\ 4) = e_4$$

$$e_3 e_2 = (1\ 4)(2\ 5)(3\ 6)(1\ 3\ 5)(2\ 4\ 6) = (1\ 6\ 5\ 4\ 3\ 2) = e_5$$

$$e_3 e_3 = (1\ 4)(2\ 5)(3\ 6)(1\ 4)(2\ 5)(3\ 6) = (1)(2)(3)(4)(5)(6) = e_0$$

$$e_3 e_4 = (1\ 4)(2\ 5)(3\ 6)(1\ 5\ 3)(2\ 6\ 4) = (1\ 2\ 3\ 4\ 5\ 6) = e_1$$

$$e_3 e_5 = (1\ 4)(2\ 5)(3\ 6)(1\ 6\ 5\ 4\ 3\ 2) = (1\ 3\ 5)(2\ 4\ 6) = e_2$$

$$e_4 e_0 = (1\ 5\ 3)(2\ 6\ 4)(1)(2)(3)(4)(5)(6) = (1\ 5\ 3)(2\ 6\ 4) = e_4$$

$$e_4 e_1 = (1\ 5\ 3)(2\ 6\ 4)(1\ 2\ 3\ 4\ 5\ 6) = (1\ 6\ 5\ 4\ 3\ 2) = e_5$$

$$e_4 e_2 = (1\ 5\ 3)(2\ 6\ 4)(1\ 3\ 5)(2\ 4\ 6) = (1)(2)(3)(4)(5)(6) = e_0$$

$$e_4 e_3 = (1\ 5\ 3)(2\ 6\ 4)(1\ 4)(2\ 5)(3\ 6) = (1\ 2\ 3\ 4\ 5\ 6) = e_1$$

$$e_4 e_4 = (1\ 5\ 3)(2\ 6\ 4)(1\ 5\ 3)(2\ 6\ 4) = (1\ 3\ 5)(2\ 4\ 6) = e_2$$

$e_4 e_5 = (1\ 5\ 3)(2\ 6\ 4)(1\ 6\ 5\ 4\ 3\ 2) = (1\ 4)(2\ 5)(3\ 6) = e_3$

$e_5 e_0 = (1\ 6\ 5\ 4\ 3\ 2)(1)(2)(3)(4)(5)(6) = (1\ 6\ 5\ 4\ 3\ 2) = e_5$

$e_5 e_1 = (1\ 6\ 5\ 4\ 3\ 2)(1\ 2\ 3\ 4\ 5\ 6) = (1)(2)(3)(4)(5)(6) = e_0$

$e_5 e_2 = (1\ 6\ 5\ 4\ 3\ 2)(1\ 3\ 5)(2\ 4\ 6) = (1\ 2\ 3\ 4\ 5\ 6) = e_1$

$e_5 e_3 = (1\ 6\ 5\ 4\ 3\ 2)(1\ 4)(2\ 5)(3\ 6) = (1\ 3\ 5)(2\ 4\ 6) = e_2$

$e_5 e_4 = (1\ 6\ 5\ 4\ 3\ 2)(1\ 5\ 3)(2\ 6\ 4) = (1\ 4)(2\ 5)(3\ 6) = e_3$

$e_5 e_5 = (1\ 6\ 5\ 4\ 3\ 2)(1\ 6\ 5\ 4\ 3\ 2) = (1\ 5\ 3)(2\ 6\ 4) = e_4$

Using the above results we get the following composition of mappings table of G_3.

O	e_0	e_1	e_2	e_3	e_4	e_5
e_0	e_0	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	e_4	e_5	e_0
e_2	e_2	e_3	e_4	e_5	e_0	e_1
e_3	e_3	e_4	e_5	e_0	e_1	e_2
e_4	e_4	e_5	e_0	e_1	e_2	e_3
e_5	e_5	e_0	e_1	e_2	e_3	e_4

Using this table we get

Lemma (4.4):

(G_3,o) is a permutation group of X [11].

Proof:

By definition each element of G_3 is a permutation of X. Moreover by table above

1- G_3 is closed under composition of mappings.

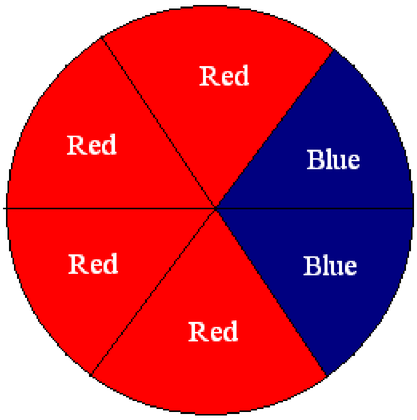
2- e_0 is identity of G_3.

3- Each element has an inverse.

4- By composition of mappings G_3 is associative

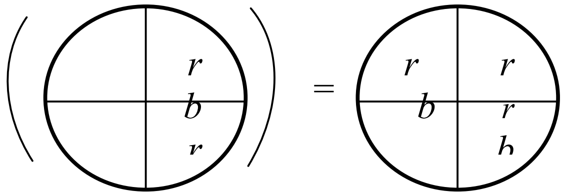
Therefore G_3 is a permutation group of X [11,31].

Moreover, and as in remark (4.1) above, we shall represent elements of X as elements of a sequence from $\{r,b\}$ where for example the element $rrbbrr \in X$ is the element, where the colour of 1, 2, 5, 6 is Red and the colour of 3,4 is Blue, see figure below [11].



Using the above remark and the definition of members of G_3 as rotations we construct the following table where the first column represents the elements of X, and the first row represents the elements of G_3 and each other element is the image of the corresponding elements of $G_3$and X, which are respectively in the same column and row with it, e.g.

for $rbrr \in X$ and $e_2 \in G_3$ and by definition of e_2 as a rotation we get the element

$e_2\ (rbrr) = e_2$ 

$= rrrb$

as in the table below [11].

We observe that X has $2^6 = 64$ elements as shown in first column of table below

x	e_0	e_1	e_2	e_3	e_4	e_5
rrrrrr	*rrrrrr*	*rrrrrr*	*rrrrrr*	*rrrrrr*	*rrrrrr*	*rrrrrr*
brrrrr	*brrrrr*	*rbrrrr*	*rrbrrr*	*rrrbrr*	*rrrrbr*	*rrrrrb*
rbrrrr	*rbrrrr*	*rrbrrr*	*rrrbrr*	*rrrrbr*	*rrrrrb*	*brrrrr*
rrbrrr	*rrbrrr*	*rrrbrr*	*rrrrbr*	*rrrrrb*	*brrrrr*	*rbrrrr*
rrrbrr	*rrrbrr*	*rrrrbr*	*rrrrrb*	*brrrrr*	*rbrrrr*	*rrbrrr*
rrrrbr	*rrrrbr*	*rrrrrb*	*brrrrr*	*rbrrrr*	*rrbrrr*	*rrrbrr*
rrrrrb	*rrrrrb*	*brrrrr*	*rbrrrr*	*rrbrrr*	*rrrbrr*	*rrrrbr*
bbrrrr	*bbrrrr*	*rbbrrr*	*rrbbrr*	*rrrbbr*	*rrrrbb*	*brrrrb*
brbrrr	*brbrrr*	*rbrbrr*	*rrbrbr*	*rrrbrb*	*brrrbr*	*rbrrrb*
brrbrr	*brrbrr*	*rbrrbr*	*rrbrrb*	*brrbrr*	*rbrrbr*	*rrbrrb*
brrrbr	*brrrbr*	*rbrrrb*	*brbrrr*	*rbrbrr*	*rrbrbr*	*rrrbrb*
brrrrb	*brrrrb*	*bbrrrr*	*rbbrrr*	*rrbbrr*	*rrrbbr*	*rrrrbb*
rbbrrr	*rbbrrr*	*rrbbrr*	*rrrbbr*	*rrrrbb*	*brrrrb*	*bbrrrr*
rbrbrr	*rbrbrr*	*rrbrbr*	*rrrbrb*	*brrrbr*	*rbrrrb*	*brbrrr*
rbrrbr	*rbrrbr*	*rrbrrb*	*brrbrr*	*rbrrbr*	*rrbrrb*	*brrbrr*

rbrrrb	*rbrrrb*	*brbrrr*	*rbrbrr*	*rrbrbr*	*rrrbrb*	*brrrbr*
rrbbrr	*rrbbrr*	*rrrbbr*	*rrrrbb*	*brrrrb*	*bbrrrr*	*rbbrrr*
rrbrbr	*rrbrbr*	*rrrbrb*	*brrrbr*	*rbrrrb*	*brbrrr*	*rbrbrr*
rrbrrb	*rrbrrb*	*brrbrr*	*rbrrbr*	*rrbrrb*	*brrbrr*	*rbrrbr*
rrrbbr	*rrrbbr*	*rrrrbb*	*brrrrb*	*bbrrrr*	*rbbrrr*	*rrbbrr*
rrrbrb	*rrrbrb*	*brrrbr*	*rbrrrb*	*brbrrr*	*rbrbrr*	*rrbrbr*
rrrrbb	*rrrrbb*	*brrrrb*	*bbrrrr*	*rbbrrr*	*rrbbrr*	*rrrbbr*
bbbrrr	*bbbrrr*	*rbbbrr*	*rrbbbr*	*rrrbbb*	*brrrbb*	*bbrrrb*
bbrbrr	*bbrbrr*	*rbbrbr*	*rrbbrb*	*brrbbr*	*rbrrbb*	*brbrrb*
bbrrbr	*bbrrbr*	*rbbrrb*	*brbbrr*	*rbrbbr*	*rrbrbb*	*brrbrb*
bbrrrb	*bbrrrb*	*bbbrrr*	*rbbbrr*	*rrbbbr*	*rrrbbb*	*brrrbb*
brbbrr	*brbbrr*	*rbrbbr*	*rrbrbb*	*brrbrb*	*bbrrbr*	*rbbrrb*
brbrbr	*brbrbr*	*rbrbrb*	*brbrbr*	*rbrbrb*	*brbrbr*	*rbrbrb*
brbrrb	*brbrrb*	*bbrbrr*	*rbbrbr*	*rrbbrb*	*brrbbr*	*rbrrbb*
brrbbr	*brrbbr*	*rbrrbb*	*brbrrb*	*bbrbrr*	*rbbrbr*	*rrbbrb*
brrbrb	*brrbrb*	*bbrrbr*	*rbbrrb*	*brbbrr*	*rbrbbr*	*rrbrbb*
brrrbb	*brrrbb*	*bbrrrb*	*bbbrrr*	*rbbbrr*	*rrbbbr*	*rrrbbb*
rbbbrr	*rbbbrr*	*rrbbbr*	*rrrbbb*	*brrrbb*	*bbrrrb*	*bbbrrr*
rbbrbr	*rbbrbr*	*rrbbrb*	*brrbbr*	*rbrrbb*	*brbrrb*	*bbrbrr*
rbbrrb	*rbbrrb*	*brbbrr*	*rbrbbr*	*rrbrbb*	*brrbrb*	*bbrrbr*
rbrbbr	*rbrbbr*	*rrbrbb*	*brrbrb*	*bbrrbr*	*rbbrrb*	*brbbrr*
rbrbrb	*rbrbrb*	*brbrbr*	*rbrbrb*	*brbrbr*	*rbrbrb*	*brbrbr*
rbrrbb	*rbrrbb*	*brbrrb*	*bbrbrr*	*rbbrbr*	*rrbbrb*	*brrbbr*
rrbbbr	*rrbbbr*	*rrrbbb*	*brrrbb*	*bbrrrb*	*bbbrrr*	*rbbbrr*
rrbbrb	*rrbbrb*	*brrbbr*	*rbrrbb*	*brbrrb*	*bbrbrr*	*rbbrbr*
rrbrbb	*rrbrbb*	*brrbrb*	*bbrrbr*	*rbbrrb*	*brbbrr*	*rbrbbr*
rrrbbb	*rrrbbb*	*brrrbb*	*bbrrrb*	*bbbrrr*	*rbbbrr*	*rrbbbr*

bbbbrr	*bbbbrr*	*rbbbbr*	*rrbbbb*	*brrbbb*	*bbrrbb*	*bbbrrb*
bbbrbr	*bbbrbr*	*rbbbrb*	*brbbbr*	*rbrbbb*	*brbrbb*	*bbrbrb*
bbbrrb	*bbbrrb*	*bbbbrr*	*rbbbbr*	*rrbbbb*	*brrbbb*	*bbrrbb*
bbrbbr	*bbrbbr*	*rbbrbb*	*brbbrb*	*bbrbbr*	*rbbrbb*	*brbbrb*
bbrbrb	*bbrbrb*	*bbbrbr*	*rbbbrb*	*brbbbr*	*rbrbbb*	*brbrbb*
bbrrbb	*bbrrbb*	*bbbrrb*	*bbbbrr*	*rbbbbr*	*rrbbbb*	*brrbbb*
brbbbr	*brbbbr*	*rbrbbb*	*brbrbb*	*bbrbrb*	*bbbrbr*	*rbbbrb*
brbbrb	*brbbrb*	*bbrbbr*	*rbbrbb*	*brbbrb*	*bbrbbr*	*rbbrbb*
brbrbb	*brbrbb*	*bbrbrb*	*bbbrbr*	*rbbbrb*	*brbbbr*	*rbrbbb*
brrbbb	*brrbbb*	*bbrrbb*	*bbbrrb*	*bbbbrr*	*rbbbbr*	*rrbbbb*
rbbbbr	*rbbbbr*	*rrbbbb*	*brrbbb*	*bbrrbb*	*bbbrrb*	*bbbbrr*
rbbbrb	*rbbbrb*	*brbbbr*	*rbrbbb*	*brbrbb*	*bbrbrb*	*bbbrbr*
rbbrbb	*rbbrbb*	*brbbrb*	*bbrbbr*	*rbbrbb*	*brbbrb*	*bbrbbr*
rbrbbb	*rbrbbb*	*brbrbb*	*bbrbrb*	*bbbrbr*	*rbbbrb*	*brbbbr*
rrbbbb	*rrbbbb*	*brrbbb*	*bbrrbb*	*bbbrrb*	*bbbbrr*	*rbbbbr*
bbbbbr	*bbbbbr*	*rbbbbb*	*brbbbb*	*bbrbbb*	*bbbrbb*	*bbbbrb*
bbbbrb	*bbbbrb*	*bbbbbr*	*rbbbbb*	*brbbbb*	*bbrbbb*	*bbbrbb*
bbbrbb	*bbbrbb*	*bbbbrb*	*bbbbbr*	*rbbbbb*	*brbbbb*	*bbrbbb*
bbrbbb	*bbrbbb*	*bbbrbb*	*bbbbrb*	*bbbbbr*	*rbbbbb*	*brbbbb*

brbbbb	*brbbbb*	*bbrbbb*	*bbbrbb*	*bbbbrb*	*bbbbbr*	*rbbbbb*
rbbbbb	*rbbbbb*	*brbbbb*	*bbrbbb*	*bbbrbb*	*bbbbrb*	*bbbbbr*
bbbbbb	*bbbbbb*	*bbbbbb*	*bbbbbb*	*bbbbbb*	*bbbbbb*	*bbbbbb*

Table (4.5)

Using table (4.5) and the lemma (4.4) above we get the following table:

x	O_x	*Fix g*
rrrrrr	$\{rrrrrr\}$	G_3
brrrrr	$\{brrrrr, rbrrrr, rrbrrr, rrrbrr, rrrrbr, rrrrrb\}$	$\{e_0\}$
rbrrrr	$\{brrrrr, rbrrrr, rrbrrr, rrrbrr, rrrrbr, rrrrrb\}$	$\{e_0\}$
rrbrrr	$\{brrrrr, rbrrrr, rrbrrr, rrrbrr, rrrrbr, rrrrrb\}$	$\{e_0\}$
rrrbrr	$\{brrrrr, rbrrrr, rrbrrr, rrrbrr, rrrrbr, rrrrrb\}$	$\{e_0\}$
rrrrbr	$\{brrrrr, rbrrrr, rrbrrr, rrrbrr, rrrrbr, rrrrrb\}$	$\{e_0\}$
rrrrrb	$\{brrrrr, rbrrrr, rrbrrr, rrrbrr, rrrrbr, rrrrrb\}$	$\{e_0\}$
bbrrrr	$\{bbrrrr, rbbrrr, rrbbrr, rrrbbr, rrrrbb, brrrrb\}$	$\{e_0\}$
brbrrr	$\{brbrrr, rbrbrr, rrbrbr, rrrbrb, brrrbr, rbrrrb\}$	$\{e_0\}$
brrbrr	$\{brrbrr, rbrrbr, rrbrrb, rrbrrb, rbrrbr\}$	$\{e_0, e_3\}$
brrrbr	$\{brrrbr, rbrrrb, brbrrr, rbrbrr, rrbrbr, rrrbrb\}$	$\{e_0\}$
brrrrb	$\{brrrrb, bbrrrr, rbbrrr, rrbbrr, rrrbbr, rrrrbb\}$	$\{e_0\}$

$rbbrrr$	$\{rbbrrr, rrbbrr, rrrbbr, rrrrbb, brrrrb, bbrrrr\}$	$\{e_0\}$
$rbrbrr$	$\{rbrbrr, rrbrbr, rrrbrb, brrrbr, rbrrrb, brbrrr\}$	$\{e_0\}$
$rbrrbr$	$\{rbrrbr, rrbrrb, brrbrr, rbrrbr, rrbrrb, brrbrr\}$	$\{e_0, e_3\}$
$rbrrrb$	$\{rbrrrb, brbrrr, rbrbrr, rrbrbr, rrrbrb, brrrbr\}$	$\{e_0\}$
$rrbbrr$	$\{rrbbrr, rrrbbr, rrrrbb, brrrrb, bbrrrr, rbbrrr\}$	$\{e_0\}$
$rrbrbr$	$\{rrbrbr, rrrbrb, brrrbr, rbrrrb, brbrrr, rbrbrr\}$	$\{e_0\}$
$rrbrrb$	$\{rrbrrb, brrbrr, rbrrbr, rbrrbr, brrbrr\}$	$\{e_0, e_3\}$
$rrrbbr$	$\{rrrbbr, rrrrbb, brrrrb, bbrrrr, rbbrrr, rrbbrr\}$	$\{e_0\}$
$rrrbrb$	$\{rrrbrb, brrrbr, rbrrrb, brbrrr, rbrbrr, rrbrbr\}$	$\{e_0\}$
$rrrrbb$	$\{rrrrbb, brrrrb, bbrrrr, rbbrrr, rrbbrr, rrrbbr\}$	$\{e_0\}$
$bbbrrr$	$\{bbbrrr, rbbbrr, rrbbbr, rrrbbb, brrrbb, bbrrrb\}$	$\{e_0\}$
$bbrbrr$	$\{bbrbrr, rbbrbr, rrbbrb, brrbbr, rbrrbb, brbrrb\}$	$\{e_0\}$
$bbrrbr$	$\{bbrrbr, rbbrrb, brbbrr, rbrbbr, rrbrbb, brrbrb\}$	$\{e_0\}$
$bbrrrb$	$\{bbrrrb, bbbrrr, rbbbrr, rrbbbr, rrrbbb, brrrbb\}$	$\{e_0\}$
$brbbrr$	$\{brbbrr, rbrbbr, rrbrbb, brrbrb, bbrrbr, rbbrrb\}$	$\{e_0\}$
$brbrbr$	$\{brbrbr, rbrbrb\}$	$\{e_0, e_2, e_4\}$
$brbrrb$	$\{brbrrb, bbrbrr, rbbrbr, rrbbrb, brrbbr, rbrrbb\}$	$\{e_0\}$
$brrbbr$	$\{brrbbr, rbrrbb, brbrrb, bbrbrr, rbbrbr, rrbbrb\}$	$\{e_0\}$
$brrbrb$	$\{brrbrb, bbrrbr, rbbrrb, brbbrr, rbrbbr, rrbrbb\}$	$\{e_0\}$
$brrrbb$	$\{brrrbb, bbrrrb, bbbrrr, rbbbrr, rrbbbr, rrrbbb\}$	$\{e_0\}$
$rbbbrr$	$\{rbbbrr, rrbbbr, brrbbb, brrrbb, bbrrrb, bbbrrr\}$	$\{e_0\}$
$rbbrbr$	$\{rbbrbr, rrbbrb, brrbbr, rbrrbb, brbrrb, bbrbrr\}$	$\{e_0\}$

$rbbrrb$	$\{rbbrrb, brbbrr, rbrbbr, rrbrbb, brrbrb, bbrrbr\}$	$\{e_0\}$
$rbrbbr$	$\{rbrbbr, rrbrbb, brrbrb, bbrrbr, rbbrrb, brbbrr\}$	$\{e_0\}$
$rbrbrb$	$\{rbrbrb, brbrbr\}$	$\{e_0, e_2, e_4\}$
$rbrrbb$	$\{rbrrbb, brbrrb, bbrbrr, rbbrbr, rrbbrb, brrbbr\}$	$\{e_0\}$
$rrbbbr$	$\{rrbbbr, rrrbbb, brrrbb, bbrrrb, bbbrrr, rbbbrr\}$	$\{e_0\}$
$rrbbrb$	$\{rrbbrb, brrbbr, rbrrbb, brbrrb, bbrbrr, rbbrbr\}$	$\{e_0\}$
$rrbrbb$	$\{rrbrbb, brrbrb, bbrrbr, rbbrrb, brbbrr, rbrbbr\}$	$\{e_0\}$
$rrrbbb$	$\{rrrbbb, brrrbb, bbrrrb, bbbrrr, rbbbrr, rrbbbr\}$	$\{e_0\}$
$bbbbrr$	$\{bbbbrr, rbbbbr, rrbbbb, brrbbb, bbrrbb, bbbrrb\}$	$\{e_0\}$
$bbbrbr$	$\{bbbrbr, rbbbrb, brbbbr, rbrbbb, brbrbb, bbrbrb\}$	$\{e_0\}$
$bbbrrb$	$\{bbbrrb, bbbbrr, rbbbbr, rrbbbb, brrbbbb, bbrrbb\}$	$\{e_0\}$
$bbrbbr$	$\{bbrbbr, rbbrbb, brbbrb, brbbrb, rbbrbb\}$	$\{e_0, e_3\}$
$bbrbrb$	$\{bbrbrb, bbbrbr, rbbbrb, brbbbr, rbrbbb, brbrbb\}$	$\{e_0\}$
$bbrrbb$	$\{bbrrbb, bbbrrb, bbbbrr, rbbbbr, rrbbbb, brrbbb\}$	$\{e_0\}$
$brbbbr$	$\{brbbbr, rbrbbb, brbrbb, bbrbrb, bbbrbr, rbbbrb\}$	$\{e_0\}$
$brbbrb$	$\{brbbrb, bbrbbr, rbbrbb, rbbrbb, bbrbbr\}$	$\{e_0, e_3\}$
$brbrbb$	$\{brbrbb, bbrbrb, bbbrbrb, rbbbrb, brbbbr, rbrbbb\}$	$\{e_0\}$
$brrbbb$	$\{brrbbb, bbrrbb, bbbrrb, bbbbrr, rbbbbr, rrbbbb\}$	$\{e_0\}$

$rbbbbr$	$\{rbbbbr, rrbbbb, brrbbb, bbrrbb, bbbrrb, bbbbrr\}$	$\{e_0\}$
$rbbbrb$	$\{rbbbrb, brbbbr, rbrbbb, brbrbb, bbrbrb, bbbrbr\}$	$\{e_0\}$
$rbbrbb$	$\{rbbrbb, brbbrb, bbrbbr, bbrbbr, brbbrb\}$	$\{e_0, e_3\}$
$rbrbbb$	$\{rbrbbb, brbrbb, bbrbrb, bbbrbr, rbbbrb, brbbbr\}$	$\{e_0\}$
$rrbbbb$	$\{rrbbbb, brrbbb, bbrrbb, bbbrrb, bbbbrr, rbbbbr\}$	$\{e_0\}$
$bbbbbr$	$\{bbbbbr, rbbbbb, brbbbb, bbrbbb, bbbrbb, bbbbrb\}$	$\{e_0\}$
$bbbbrb$	$\{bbbbbr, rbbbbb, brbbbb, bbrbbb, bbbrbb, bbbbrb\}$	$\{e_0\}$
$bbbrbb$	$\{bbbbbr, rbbbbb, brbbbb, bbrbbb, bbbrbb, bbbbrb\}$	$\{e_0\}$
$bbrbbb$	$\{bbbbbr, rbbbbb, brbbbb, bbrbbb, bbbrbb, bbbbbr\}$	$\{e_0\}$
$brbbbb$	$\{bbbbbr, rbbbbb, brbbbb, bbrbbb, bbbrbb, bbbbbr\}$	$\{e_0\}$
$rbbbbb$	$\{bbbbbr, rbbbbb, brbbbb, bbrbbb, bbbrbb, bbbbbr\}$	$\{e_0\}$
$bbbbbb$	$\{bbbbbb\}$	G_3

Table (4.6)

Example(4.7):

Using the table (4.6) the number of elements of G_3 is equal to 6. Similarly and using table (4.6) we find $|S_x|$ for each $x \in X$. Hence each of the elements of X is fixed by $\{e_0\}$. Moreover each of the elements *brrbrr, rrbrrb, rbrrbr, bbrbbr, brbbrb,* and *rbbrbb* $\in X$ are fixed by $\{e_0, e_3\}$, and each of the elements *brbrbr,* and *rbrbrb* $\in X$ is fixed by $\{e_0, e_2, e_4\}$. Therefore [11].

The number of elements fixed by $e_0 = 64$

The number of elements fixed by $e_1 = 2$

The number of elements fixed by $e_2 = 4$

The number of elements fixed by $e_3 = 8$

The number of elements fixed by $e_4 = 4$

The number of elements fixed by $e_5 = 2$

Using the above results we get the following table

g	e_0	e_1	e_2	e_3	e_4	e_5
$\|Fix(g)\|$	64	2	4	8	4	2

By theorem (4.3) above therefore

$$V_{X,G_3} = \frac{1}{|G_3|}\sum_{g\in G_3}|Fix(g)|$$
$$= \frac{1}{6}(64+2+4+8+4+2)$$
$$= \frac{1}{6}\times 84 = 14$$

4.2 Colouring of Cycles of a permutation:

Suppose D is a set of sectors,squares, etc …,G a permutation group of D and X the set of colourings of D. Let $g \in G$ and $x \in X$ such that $x \in Fix\, g.$. We shall investigate in what follows the relation between the colouring of x and the cycles of g. Thus with reference to theorem (4.1) and example (4.6) let

$$G = \{ e_0, e_1, ..., e_{n-1}\}$$

be a permutation group of $D = \{ 1, 2, ..., n\}$, where e_m, $m= 0,..., m-1$ is defined by [11].

$$e_m (i)=(i+m) \bmod n$$

For the sake of brevity we shall write simply $(i+m)$ instead of $(i+m) \bmod n$ in what follows. By definition the cycles of e_0 are $(1), (2), ...,(n)$ since $n \equiv 0 \bmod n$. Next we find the cycles of e_m where $o<m<n$. let $a_m = \gcd(m,n)$ and $K_m = n/a_m$. By definition of e_m the cycle C_i ,$0 \le i \le n$, of e_m containing the element $i \in D$, is $(i, i+m, i+2m, ..., i+(K_m-1)m)$ since n is a divisor of $K_m m$ and not a divisor of $K'm$ for $K' < K_m$. Moreover C_i has K_m elements for each i. Since $a_m K_m=n$, this implies that in total, e_m has a_m cycles $C_0, C_1, \ldots, C_{a_m-1}$. This is because they are disjoint and they also partition D , see theorem (2.2) chapter two,and therefore they together contain n elements [11].

Next observe that if a colouring $x \in X$ is fixed by e_m then elements on the same cycle C_i must be coloured the same. By contraction suppose that for some integer $0 \le b \le K_m - 1$ the colour of $i+bm$ is different from the colour of $i+(b+1)m$ in C_i, say Red versus Blue. Since

$$e_m(i+bm)=i+bm+m=i+(b+1)m,$$

this implies that the colour of $i+(b+1)m$ in $e_m(x)$ will be Red and so $e_m(x) \neq x$, contracting that x is fixed by e_m .

Conversely, if $x \in X$ such that elements of the same cycle of e_m have the same colour in x then $x \in Fix(e_m)$ since e_m fixes its own cycles [11].

Using the above arguments we have the following lemma

Lemma (4.5):

Let G be a permutation group of D. If $g \in G$ and $x \in X$ then $x \in Fixg$ iff x is obtained by giving all members of a cycle of g in D the same colour [11.31] .

Proof:

As above.

Using D,G and X given in section(4.2) above we have following example .

Example (3.8):

Suppose we have two colours to use.Since e_m has a_m cycles therefore

$$|Fix(e_m)| = 2^{a_m} .$$

Since $a_m = \gcd(m,n)$,therefore by applying theorem (4.3) above it follows that

$$V_{X,G} = \frac{1}{n}\sum_{m=0}^{n-1} 2^{\gcd(m,n)} .$$

Remark (4.2):

(1) Using example (3.2), $D=\{1,2,3,4\}$. Moreover as in example (4.2) the sectors of *D* are numbered in clockwise order. If $x \in X$, then as in remark (4.1) above *x* represents a colouring of each member of *D* by the colours r (Red) or b (Blue). Let $C=\{r,b\}$ be the set of colours. Suppose $x \in X$ such that the colour of 1 is *r*, the colour of 2 is *b*, the colour of 3 is *b* and the colour of 4 is *r*. Then using remark (4.1) above

$$x = rbbr.$$

(2) Using this we observe that *x* defines a mapping of *D* into *C*, i.e. $x: D \to C$, given by $x(1) = r$, $x(2) = b$, $x(3) = b$, $x(4) = r$. Similarly and by remark (4.1) this is true for all $x \in X$. This implies that *X*, the set of all colourings of *D* by the colours *r* or *b*, is equal to [11,31].

$$X = \{x : D \to C, x \, is \, a \, colouring \, of \; D\}.$$

(3) Using this and the above lemma encourages us to extend each permutation $g \in D$ to a permutation of *X* such that for every $x \in X$, $g * x \in X$, i.e. $g * x : D \to C$, defined by

$$g * x(d) = x(g^{-1}(d)), \forall d \in D.$$

4.3 Explanation

The colour of $g_* x$ at $d \in D$ is the colour of the element $g^{-1}(d) \in D$ which is mapped to it by *g*, since

$$g(g^{-1}(d)) = (gg^{-1})d = ed = d$$

Consider example (3.1) with $n=4$. Suppose that $g = e_1$, i.e. rotate clockwise by $\pi/2$ and that

$$x(1) = B, x(2) = b, x(3) = r, x(4) = r.$$

Then for example and by definition of $g_* x$ in part (3) or remark (4.1) above

$$g * x(1) = x(g^{-1}(1)) = x(4) = r\,,$$

as before.Now associate a weight W_C with each $c \in C$,where W_C is a symbol which is not necessarily a number.Using this we have the following definition [11,31].

Definition(3.2):

If $x \in X$ then define

$$W(x) = \prod_{d \in D} W_{x(d)}$$

Example (3.9):

Using example (4.3) and table (4.4), where $D = \{1,2,3,4\}$, let $x \in X$ such that $x = bbrr$. Then by remark (4.1) above

$$x(1) = b,\ x(2) = b,\ x(3) = r, x(4) = r\ .$$

Let w_r=R and w_b =B . Substituting for $x(d), d \in D$ in definition of $W(x)$ above therefore

$$W(x) = \prod_{d \in D} W_{x(d)} = \prod_{d \in \{1,2,3,4)} W_{x(d)} = w_{x(1)} \dots w_{x(4)}$$

$$= w_b w_b w_r w_r = B^2 R^2$$

Definition (4.3):

(1) If $S \subseteq X$, define the inventory of S, $W(S)$,to be

$$W(S) = \sum_{x \in S} W(x)$$

(2) If $S^* \subseteq X$ contains one member of each orbit of X under G define the pattern inventory PI to be

$$PI = W(S^*)$$

The problem we discuss now is how to compute the pattern inventory *PI* [11].

Example (3.10):

Using table (4.4) *G* has six orbits in *X*. Since *S** contains one member of each orbit of *X* under *G* let

$$S^* = \{x_1, x_2, ..., x_6\}$$

Moreover $D = \{1,2,3,4\}$ and $x_1 = rrrr$. Using the above remark (4.1) above therefore $x : D \to C$ given by

$$x_1(i) = r \ , \ i = 1,2,3,4\,,$$

Similarly and using table (4.4) if $x_2 = brrr$, therefore

$$x_2(1) = b,\ x_2(i) = r,\ i = 2,3,4\,,$$

As before if $x_3 = bbrr$, therefore

$$x_3(1) = b,\ x_3(2) = b, x_3(3) = r, x_3(4) = r;$$

If $x_4 = rbrb$, therefore

$$x_4(1) = r,\ x_4(2) = b, x_4(3) = r, x_4(4) = b;$$

If $x_5 = bbbr$, therefore

$$x_5(4) = r,\ x_5(i) = b, i = 1,2,3.\,,$$

Finally $x_6 = bbbb$, therefore

$$x_6(i) = b,\ \ i = 1,2,3,4$$

By definitions of *PI* and *W(S)* and *W(x)* above

$$PI = W(S^*) = \sum_{x \in S^*} W(x) = \sum_{x \in S^*} (\prod_{d \in D} W_{x(d)})$$

Substituting for each $W_{x(d)}, d \in D$, as in example (4.9) ,and for each $x \in S^*$ therefore

$$PI = R^4 + R^3B + R^2B^2 + RB^3 + B^4 \qquad (4.7)$$

Using this example and remark (4.1) we conclude the following

Remark (3.3):

Suppose $D = \{1,2,3,4\}$ i.e. D is of size 4, see remark (4.1). Then

(1) Each term of equation (4.7) above represents a colouring x of D except the term $2R^2B^2$.

(2) Since $2R^2B^2=R^2B^2+B^2\ R^2$,i.e. a sum of two terms, hence it represents two colourings of D namely x_3, x_4

(3) The coefficient of each term gives the number of distinct colourings represented by the term, e.g. $2R^2B^2$ means that there are two distinct colourings using 2 Reds and 2 Blues, which are represented by R^2 and B^2 respectively.

(4) The power of each colour in each term represents the number of times the colour is used in that term.

(5) Each term represents an orbit representative $x \in S^*$.

(6) Using parts (1,2,5) above and if we substitute $R=B=1$ in equation (4.7) above we get the number of distinct colourings of D, i.e. $|X|$. [11]

4.4 Polya's Theorem:

Making use of the above examples the problem we discuss below is how to compute more generally the pattern inventory $PI = W(S^*)$ defined above. Hence and using the above remarks the scenario now consists of a set *D* (Domain), a set *C* (colours) and $X = \{x : D \to C\}$ which is the set of all colourings of *D* with the colour set *C*. Suppose *G* is a group of permutations of *D*. Then as in part (3) remark (4.2) above we can extend each permutation $g \in G$ of *D* to a permutation of *X*. Hence if $x \in X$ and $g \in G$ then as in the explanation above

$$g * x(d) = x(g^{-1}(d)) \text{ for all } d \in D$$

Using definitions of *W(x)* and *W(S)* above therefore

$$W(S) = \sum_{x \in S} W(x) = \sum_{x \in S} \prod_{d \in D} W_{x(d)} \tag{4.8}$$

Using this we prove [11,31].

Lemma (3.6):

If *x, y* are in the same orbit of *X* under *G*, then

$$W(x) = W(y)$$

Proof:

Suppose that $g_* x = y$,then by definition of *W(x)* in the explanation

$$W(y) = \prod_{d \in D} W_{y(d)} = \prod_{d \in D} W_{g*x(d)}$$

By definition of $g_* x$ above we get

$$W(y) = \prod_{d \in D} W_{x(g^{-1}(d)} \tag{4.9}$$

Substituting in equation (4.8) above where $S=\{x\}$ and since $g^{-1}D=D$ therefore

$$W(x) = \prod_{d \in D} W_{x(g^{-1}(d))} = \prod_{d \in D} W_{x(d)} \qquad (4.10)$$

Equating equations (4.9) and (4.10) therefore

$$W(y) = W(x)$$

Note that we can go from (4.9) to (4.10) because as d runs over D, $g^{-1}(d)$ also runs over D since $g^{-1}D = D$ [11,31].

Definition (3.4):

Let G be a permutation group of D and $\Delta = |D|$

1- If $g \in G$ has K_i cycles of length i then we define

$$ct(g) = x_1^{K_1} x_2^{K_2} \ldots x_\Delta^{K_\Delta}$$

2- The cycle index polynomial of G, C_G is the defined to be

$$C_G(x_1, x_2, \ldots x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g)$$

Note:

(1) if g has no cycle of length i then $K_i = 0$ and $x_i^{K_i} = 1$. Moreover

$$\sum_{i=1}^{\Delta} ik_i = \Delta$$

(2) x_i represents a variable for any i-cycle of $g \in G$ in D [11,37].

Example (4.11):

Using example (4.3) with $n=2$ we have $D = \{1,2,3,4\}$ and

$$e = (1)(2)(3)(4) \;\;,\; a = (1\;2\;3\;4)$$

$$b = (1\;\;3)(2\;\;4) \;\;,\; c = (1\;4\;3\;2)$$

$$p = (1\;\;2)(3\;\;4) \;\;,\; q = (1\;4)(3\;2)$$

$$r = (1)(3)(2\ \ 4)\ \ ,\ s = (2)(4)(1\ 3)$$

Using this and by definition of $ct(g)$ we get the following table:

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2x_2$	$x_1^2x_2$

By definition of $C_G(x_1,...,x_4)$ therefore

$$C_G(x_1,x_2,x_3,x_4) = \frac{1}{8}\left(x_1^4 + x_4 + x_2^2 + x_4 + x_2^2 + x_2^2 + x_1^2x_2 + x_1^2x_2\right)$$

$$= \frac{1}{8}\left(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2\right)$$

Using the above examples (4.10),(4.11) and remark (4.3) we have [27,38]

Remark (4.4):

(1)Let X be the set of all colourings of the set $D=\{1,2,3,4\}$ and G be a permutation group of D. Suppose $x \in X$ and $g \in G$ such that $g(x) = x$, i.e. $x \in Fix(g)$ .By theorem (4.5) this implies that each cycle of g of length i has one colour Red or Blue say. For each such i-cycle ,and as in part (4) of remark(4.2) above ,this is indicated by R^i or B^i, i=1,2,3,4.Since the possibilities of R^i or B^i are equal, therefore in any calculations we shall replace the variable x_i by $R^i + B^i$, i.e [27,38].

$$x_i = R^i + B^i . \qquad (4.11)$$

This is justified since if we substitute $R=B=1$,as in part(6) of remark (4.2),we get

$$x_i = R^i + B^i = 2$$

which is the exact number of all ways of colouring an i-cycle by one of two colours R or B.

Therefore if we replace x_1 by $R+B$, x_2 by R^2+B^2, x_3 by R^3+B^3, x_4 by R^4+B^4 we get

$$C_G(x_1,\ldots,x_4)=\frac{1}{8}[(R+B)^4+3(R^2+B^2)+2(R+B)^2(R^2+B^2)+2(R^4+B^4)]$$

$$=R^4+R^3B+2R^2B^2+RB^3+B^4=PI \qquad (4.12)$$

as in example (4.10) above [27,37].

(2)More generally for the set of colourings X of any set D and a permutation group G of D and as in the above note in the above definition of *ct(g)* each x_i represents a variable for the colourings of a cycle of $g \in G$ in D of length i. Moreover if we have two colours R and B or more, then by lemma (4.5) any i cycle C_i is fixed by a $g \in G$ iff C_i is a cycle of g and all elements of C_i have one colour. For each cycle of $g \in G$ of length i and as in part (1) above this value of x_i will represent all the ways of colouring an i-cycle of $g \in G$ with one colour R or B.This implies that $\forall x \in Fix\, g$ all the colourings x_i are given by $x_i = R^i + B^i$

(3)Furthermore, equation (4.12) above is the core of Polya's Theory of counting. Using this equation which relates $C_G(x_1,\ldots,x_4)$ with PI and where x_i are the colouring variables of the i-cycles of the elements $g \in G$ in $D, i=1,2,3,4$, together with example (4.9) we shall rewrite x_i as follows:

$$x_i = R^i + B^i = \sum_{c \in C=\{R+B\}} w_c^i, i=1,2,3,4, \qquad (4.13)$$

where $w_R = R$, $w_B = B$ .More generally for any X,D,G as in part (2) and substituting for x_i in definition of $ct(g)$ above implies that

$$ct(g) = x_1^{k_1} x_2^{k_2} \ldots x_\Delta^{k_\Delta} = (\sum_{c\in C} w_c^1)^k (\sum_{c\in C} w_c^2)^{k_2} \ldots (\sum_{c\in C} w_c^\Delta)^{k_\Delta} \qquad (4.14)$$

Moreover and as in part (1)

$$x_i = R^i + B^i \,, \quad \forall x \in Fix(g), i = 1,2,3,\ldots,\Delta,$$

for each i-cycle of g . Using this and examples (4.9),(4.10) and by definitions of $W(S)$ and $W(x)$ above therefore

$$W(Fixg) = \prod_{d\in D} w_{x(d)} = (\sum_{c\in C} w_c^1)^k (\sum_{c\in C} w_c^2)^{k_2} \ldots (\sum_{c\in C} w_c^\Delta)^{k_\Delta}, \qquad (4.15)$$

where k_i is the number of i-cycles in g. We observe that this is also true if C has more than two colours.

Using this remark we prove Polya's Theorem [27,37,38].

Polya's Theorem:

$$PI = C_G\left(\sum_{c\in C} w_c, \sum_{c\in C} w_c^2, \ldots, \sum_{c\in C} w_c^\Delta\right)$$

Proof:

Suppose

$x \sim y$ iff $W(x) = W(y)$.

We prove that ~ is an equivalence relation

(i) Let $x \in X$. Since $W(x) = W(x)$ therefore $x \sim x$, and hence ~ is a reflexive relation.

(ii) If $x \sim y$ therefore $W(x) = W(y)$, this implies that $W(y) = W(x)$. Therefore $y \sim x$, and hence ~ is a symmetric relation.

(iii) If $x \sim y$ and $y \sim z$ therefore $W(x) = W(y)$, and $W(y) = W(z)$. This implies that $W(x) = W(z)$. Therefore $x \sim z$, and hence ~ is a transitive relation. This implies that ~ is an equivalence relation. Let $X = X_1 U X_2 U ... U X_m$ be the equivalence classes of X under the relation $x \sim y$ iff

$$. \; W(x) = W(y), \forall x, y \in X.$$

By above remark (4.1) part(3) g is a permutation of X .Let $g \in G$ .By Lemma (4.6) if $g_* x = y$ therefore

$$W(x) = W(y).$$

By definition of the equivalence classes $X_i, i = 1, ..., m$,this implies that g is a permutation of X_i for each i. Therefore G is permutation group of X_i for each i. This implies that G is a permutation group of X. For each X_i denote the restriction of g to X_i by g^i , i.e. $g^i = g_{X_i}$. Similarly denote the restriction of G to X_i by G^i, i.e. $G^i = G_{X_i}$ .Let G^i have m_i orbits in X_i . If $x \in X_i$ let $W(x) = W_i$, which is a constant for all elements of X_i by definition of X_i. By definition of PI above [11,37].

$$PI = \sum_{x \in S} W(x) \qquad (4.16)$$

Where S^* contains one member of each orbit of X under G. From above G is a permutation group of each $X_i, i = 1, ..., m$ .Moreover and since $G^i = G_{X_i}$, therefore each X_i is a union of m_i orbits of G_{X_i} . This implies that S^* has m_i elements of X_i and for each $i=1,...,m$. Since for each $x \in X_i$, $W(x)=W_i$, therefore substituting in equation (4.16) above

$$PI = \sum_{x \in S} m_i w_i \qquad (4.17)$$

By theorem (4.3) and since m_i is the number of orbit of G (i.e. $G_{X_i} = G^i$) in X_i, therefore $i=1,...,m$

$$m_i = \frac{1}{|G|} \sum_{g \in G} |Fix(g^i)|$$

and where g is denoted by g^i in X_i .substituting in equation (4.17) therefore

$$PI = \sum_{i=1}^{m} w_i (\tfrac{1}{|G|} \sum_{g \in G} \left| Fix(g^i) \right|$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} w_i \left| Fix(g^i) \right| \qquad (4.18)$$

By definition Fix(g) consists of all the elements of X fixed by g. Since X_i are disjoint and $X = X_1 U X_2 U..U X_m$ therefore $Fix(g)$ in X consists of all the elements of Fix(g) in each of X_i. Since g in X_i is denoted by g^i therefore we have,

$$Fix(g) = \bigcup_{i=1}^{m} fix(g^i) \qquad (4.19)$$

Since $X_1,...,X_m$ are pairwise disjoint this implies that

$$\left| Fix(g) \right| = \left| \bigcup_{i=1}^{m} Fix(g^i) \right| = \sum_{i=1}^{m} | Fix(g^i) |$$

where $|Fix(g)|$ and $|Fix(g^i)|$ represents the number of elements in each set. If $S \subseteq X$,then by definition of $W(S)$ above

$$W(S) = \sum_{x \in S} W(x)$$

Applying this to equation (4.19) therefore

$$W(Fix(g)) = W\left(\bigcup_{i=1}^{m} Fix(g^i)\right) \qquad (4.20)$$

By definition

$$W(Fix(g)) = \sum_{x \in Fix(g)} W(x)$$

By equation (4.19) and since $X_1,\dots,X_m$ are disjoint therefore

$$W\left(\bigcup_{i=1}^{m} Fix(g^i)\right) = \sum_{x \in \bigcup_{i-1}^{m} Fix(g^i)} W(x) = \sum_{i=1}^{m}\left(\sum_{x \in Fix(g^i)} W(x)\right) = \sum_{i=1}^{m}\left(\sum_{x \in Fix(g^i)} W_i\right) \qquad (4.21)$$

since $Fix(g^i) \subseteq X_i$ and $W(x) = w_i, \forall x \in X.$ Using this together with equations (4.20) and (4.21) we get. [11,37,38]

$$W(Fix(g)) = \sum_{i=1}^{m} \left|Fix(g^i)\right| w_i$$

Using this equation therefore

$$\frac{1}{|G|}\sum_{g \in G} W(Fix(g)) = \frac{1}{|G|}\sum_{g \in G}\sum_{i=1}^{m} \left|Fix(g_i)\right| w_i \qquad (4.22)$$

By equations (4.18) and (4.22) above therefore

$$PI = \frac{1}{|G|}\sum_{g \in G} W(Fix(g)) \qquad (4.23)$$

By definition of ct(g) and equations (4.21), (4.22) in remark (4.3) above

$$ct(g) = x_1^{k_1} x_2^{k_2} \dots x_\Delta^{k_\Delta} = (\sum w_c^1)^{k_1} (\sum w_c^2)^{k_2} \dots (\sum w_c^\Delta)^{k_\Delta}$$

By equation (4.23) in remark (4.4)

$$W(Fix(g)) = \left(\sum_{c \in C} w^1\right)^{k_1} \left(\sum_{c \in C} w^2\right)^{k_2} \dots \left(\sum_{c \in C} w^\Delta\right)^{k_\Delta} = ct(g) \qquad (4.24)$$

Substituting from equation (4.24) above into equation (4.23) and by definition (4.4) [11,37,38]

$$PI = \frac{1}{|G|}\sum_{g\in G} ct(g) = C_G\left(\sum_{c\in C} w_c, \sum_{c\in C} w_c^2, \ldots \sum_{c\in C} w_c^{\Delta}\right),$$

which Complete the proof.

Chapter Five

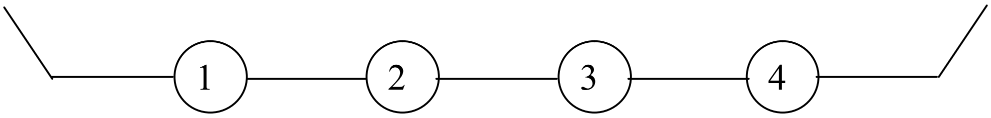
Applications

5.1 Beads Necklaces

Using Polya's theory of counting and the results leading to it we have the following applications.

Example 5.1

As an application of some of the results in Chapter 4 consider $D=\{1,2,3,4\}$, i.e. a 4 beads necklaces as shown below,



subject to the permutation group

$$G=\{\pi_1,\pi_2\}=\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\right\},$$

where π_1 represents a horizontal rotation by 0 degree and π_2 represents a horizontal rotation by 180 degrees.

Using this therefore

- $|G|=2$
- $\pi_1=(1)(2)(3)(4)$, $\pi_2=(1\ \ 4)(2\ \ 3)$
- $|Fix(\pi_1)|=4$, $|Fix(\pi_2)|=0$

By Burinside's lemma Theorem 4.3 in Chapter 4 above

$$V_{X,G}=\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$$

$$=\frac{1}{2}(4+0)=2$$

Using Definition 4.3 part (1) and Example 4.11 in Chapter 4 above we get the following table:

g	π_1	π_2
$ct(g)$	x_1^4	x_2^2

Substituting in Definition 4.3 part (2) we get

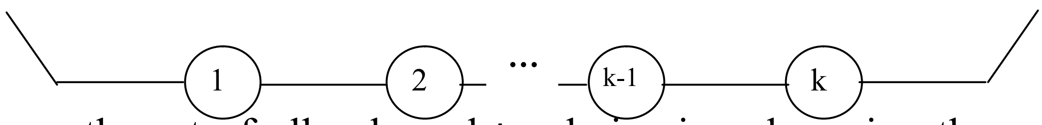
$$PI = C_G(x_1, x_2) = \frac{1}{|G|}\sum_{g\in G} ct(g)$$

Therefore

$$PI = C_G(x_1, x_2) = \frac{1}{2}(x_1^4 + x_2^2).$$

Example 5.2

Suppose we have a necklace with K beads as shown below.



,

Hence the set of all coloured beads is given by using the group G of permutations of the set of K beads $D = \{1,2,3,\ldots,K\}$ given by

$$G = \{\pi_1, \pi_2\} = \left\{ \begin{pmatrix} 1 & 2 & \ldots & K \\ 1 & 2 & \ldots & K \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \ldots \\ K & K-1 & K-2 & \ldots \end{pmatrix} \right\}$$

where π_1 represents a horizontal rotation by 0 degree and π_2 represents a horizontal rotation by 180 degrees.

Using this therefore

- $|G| = 2$
- $\pi_1 = (1)(2)\ldots(K)$, $\pi_2 = (1 \quad K)(2 \quad K-1)(3 \quad K-2)\ldots$
- $|Fix(\pi_1)| = K$, $|Fix(\pi_2)| = 0$, *if K is even*
- $|Fix(\pi_1)| = K$, $|Fix(\pi_2)| = 1$, *if K is odd*

(i) If K is even, by Burinside's lemma Theorem 4.3 in Chapter 4 above,

$$V_{X,G} = \frac{1}{|G|}\sum_{g\in G} |Fix(g)|$$

$$=\frac{1}{2}(K+0)=\frac{K}{2}$$

Using Definition 4.3 part (1) and Example 4.11 in Chapter 4 above we get the following table:

g	π_1	π_2
$ct(g)$	x_1^K	$x_2^{K/2}$

Substituting in Definition 4.3 part (2) we get

$$PI=C_G(x_1,x_2)=\frac{1}{|G|}\sum_{g\in G}ct(g)$$

Therefore

$$PI=C_G(x_1,x_2)=\frac{1}{2}(x_1^K+x_2^{K/2}) \qquad (5.1)$$

(ii) If K is odd, by Burinside's lemma Theorem 4.3 in Chapter 4 above,

$$V_{X,G}=\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$$

$$=\frac{1}{2}(K+1)=\frac{K+1}{2}$$

Using Definition 4.3 part (1) and Example 4.11 in Chapter 4 above we get the following table:

g	π_1	π_2
$ct(g)$	x_1^K	$x_1\,x_2^{K-1/2}$

Substituting in Definition 4.3 part (2) we get

$$PI=C_G(x_1,x_2)=\frac{1}{|G|}\sum_{g\in G}ct(g)$$

Therefore

$$PI = C_G(x_1, x_2) = \frac{1}{2}(x_1^K + x_1 x_2^{K-1/2}) \qquad (5.2)$$

Using equations (5.1) and (5.2) we get

$$PI = C_G(x_1, x_2) = \begin{cases} \frac{1}{2}(x_1^K + x_2^{K/2}) & \text{Iif k is even} \\ & \text{if k is even} \\ \frac{1}{2}(x_1^K + x_1 x_2^{K-1/2}). & \end{cases}$$

Example 5.3

Using Example 5.2 above, where number of beads is K, we have,

Cycle index is $\left(x_i = \sum_{c \in C} w_c^n \right)$

Moreover

$$PI = C_G(x_1, x_2) = \begin{cases} \frac{1}{2}(x_1^K + x_2^{K/2}) & \text{Iif k is even} \\ & \text{if k is even} \\ \frac{1}{2}(x_1^K + x_1 x_2^{K-1/2}). & \end{cases}$$

Using Remark 4.2 in Chapter

(i) If $K=2$ and there is only one colour blue (B) then

$$x_1 = B \quad , \quad x_2 = B^2$$

Therefore and as above

$$PI = C_G(x_1, x_2) = \tfrac{1}{2}\left(B^2 + (B^2)^1\right) = \tfrac{1}{2} \times 2B^2 = B^2$$

As in chapter (4) above, example (4.10), P´olya's enumeration Theorem and by substituting $B = 1$ therefore

$$PI = C_G = 1$$

(ii) If K = 3 and there is only one colour (B) then

$$PI = C_G(x_1, x_2) = \tfrac{1}{2}\left(B^3 + B(B^2)^1\right) = B^3 .$$

Substituting $B = 1$, therefore

$$PI = C_G(x_1, x_2) = 1$$

Using (i) and (ii) therefore for any K and only one colour (B) we have

$$PI = C_G(x_1, x_2) = B^K = 1$$

Example 5.4

Using Example 5.2, where number of beads is K and $G=\{\pi_1,\pi_2\}$, we have,

Cycle index is $\left(x_i = \sum_{c\in C} w_c^n \right)$,

$$PI = C_G(x_1,x_2) = \begin{cases} \frac{1}{2}(x_1^K + x_2^{K/2}) & \text{Iif k is even} \\ \\ \frac{1}{2}(x_1^K + x_1 x_2^{K-1/2}). & \text{if k is even} \end{cases}$$

Using Remark 4.2 in Chapter 4 above and if there are 2 colours (Blue, Red) then P′olya's enumeration Theorem gives

$$x_1 = (B+R) \quad , \quad x_2 = (B^2 + R^2)$$

For $K = 2$ therefore

$$PI = C_G(x_1,x_2) = \tfrac{1}{2}\left((B+R)^2 + (B^2+R^2)^1\right)$$
$$= R^2 + BR + B^2$$

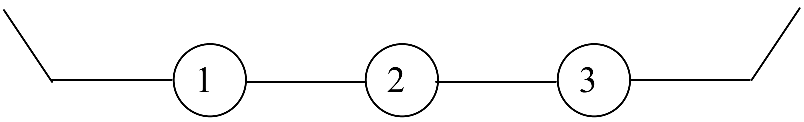
As in Example 5.3 above and substituting $B=R=1$, therefore

$$PI = C_G(x_1,x_2) = 1+1+1 = 3,$$

i.e. There are 3 distinct colourings.

Example 5.5

Suppose we have a necklace with 3 beads and a set of 2 bead colours Blue (b) and Red (r) as shown below,



Hence the set of all coloured beads is given by using the group G of permutations of the set of three beads $D=\{1,2,3\}$ given by

$$G=\{\pi_1,\pi_2\}=\left\{\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix},\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}\right\},$$

where π_1 represents a horizontal rotation by 0 degree and π_2 represents a horizontal rotation by 180 degrees.

Using this therefore

- $|G| = 2$
- $\pi_1 = (1)(2)(3)$, $\pi_2 = (1 \ \ 3)(2)$
- $|Fix(\pi_1)| = 3$, $|Fix(\pi_2)| = 1$

By Burinside's lemma Theorem 4.3 in Chapter 4 above

$$V_{X,G} = \frac{1}{|G|}\sum_{g\in G} |Fix(g)|$$
$$= \frac{1}{2}(3+1) = 2$$

Using Definition 4.3 part (1) and Example 4.11 in Chapter 4 above we have the following table:

g	π_1	π_2
$ct(g)$	x_1^3	$x_1 x_2$

Substituting in Definition 4.3 part (2) we get P´olya's enumeration Theorem gives

$$PI = C_G(x_1, x_2) = \frac{1}{|G|}\sum_{g\in G} ct(g)$$

Using Example 5.2 above for K = 3 have

$$PI = C_G(x_1, x_2) = \tfrac{1}{2}\left((B+R)^3 + (B+R)(B^2+R^2)^1\right)$$
$$= \frac{1}{2}\left((B+R)(B+R)^2 + (B^3+R^3)\right)$$
$$= R^3 + 2B^2R + 2BR^2 + B^3$$

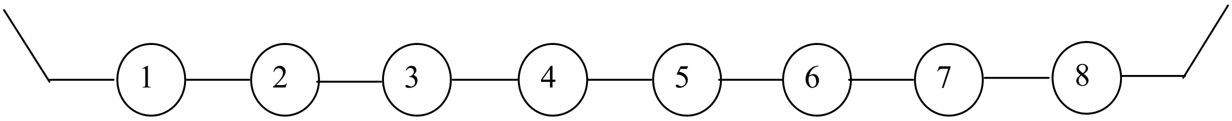
As in Example 5.3 above substituting $B=R=1$, therefore

$$PI = C_G(x_1, x_2) = 1 + 2 + 2 + 1 = 6,$$

i.e. There are 6 distinct colourings.

Example 5.6

Suppose we have a necklace with 8 beads and a set of 2 colours Blue (B) and Red (R) as shown below,



Hence the set of all coloured beads is given by using the group G of permutations of the set of 8 beads $D=\{1,2,\ldots,8\}$ given by

$$G=\{\pi_1,\pi_2\}=\left\{\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&3&4&5&6&7&8\end{pmatrix},\begin{pmatrix}1&2&3&4&5&6&7&8\\8&7&6&5&4&3&2&1\end{pmatrix}\right\},$$

where π_1 represents a horizontal rotation by 0 degree and π_2 represents a horizontal rotation by 180 degrees.

Using this therefore

- $|G|=2$
- $\pi_1=(1)(2)(3)(4)(5)(6)(7)(8)$
- $\pi_2=(1\quad 8)(2\quad 7)(3\quad 6)(4\quad 5)$
- $|Fix(\pi_1)|=8 \quad , \quad |Fix(\pi_2)|=0$

By Burinside's lemma Theorem 4.3 in Chapter 4 above

$$V_{X,G}=\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$$
$$=\frac{1}{2}(8+0)=4$$

Using Definition 4.3 part (1) and Example 4.11 in Chapter 4 above we get the following table:

g	π_1	π_2
$ct(g)$	x_1^8	x_2^4

Substituting in Definition 4.3 part (2) we get P′olya’s enumeration Theorem gives

$$PI = C_G(x_1, x_2) = \frac{1}{|G|}\sum_{g\in G} ct(g)$$

Using equation (5.2) in Example 5.2 above and for $K = 8$ we have

$$PI = C_G(x_1,x_2) = \frac{1}{2}((B+R)^8 + (B^2+R^2)^4) \qquad (5.3)$$

since

$$(B+R)^8 = C_0^8 B^8 R^0 + C_1^8 B^7 R + C_2^8 B^6 R^2 + C_3^8 B^5 R^3 +^0 R^8$$
$$C_4^8 B^4 R^4 + C_5^8 B^3 R^5 + C_6^8 B^2 R^6 + C_7^8 B\ R^7 + C_8^8 B$$

where

$$C_0^8 = C_8^8 = 1 \qquad and \qquad R^0 = B^0 = 1$$
$$C_1^8 = C_7^8 = 8 \qquad and \qquad C_2^8 = C_6^8 = 28$$
$$C_3^8 = C_5^8 = 56 \qquad and \qquad C_4^8 = 70$$

and

$$(B^2+R^2)^2 = B^4 + 2B^2R^2 + R^4$$
$$(B^2+R^2)^4 = (B^2+R^2)^2(B^2+R^2)^2)$$
$$= (B^4 + 2B^2R^2 + R^4)(B^4 + 2B^2R^2 + R^4$$
$$= B^8 + 2B^6R^2 + B^4R^4 + 2B^6R^2 + 2B^6R^2 +$$
$$4B^4R^4 + 2B^2R^6 + B^4R^4 + 2B^2R^6 + R^8$$
$$B^8 + R^8 + 4B^6R^2 + 6B^4R^4 + 4B^2R^6 \qquad (5.4)$$

This implies that

$$(B+R)^8 = B^8 + 8B^7R + 28B^6R^2 + 56B^5R^3 + 70B^4R^4 + 56B^3R^5 +$$
$$28B^2R^6 + 8BR^7 + R^8 \qquad (5.5)$$

in equation (5.3) above we get:

$$PI = C_G(x_1,x_2) = \tfrac{1}{2}(B^8 + 8B^7R + 28B^6R^2 + 56B^5R^3 +$$

$$70B^4R^4 + 56B^3R^5 + \ 28B^2R^6 + 8B\ R^7 + R^8$$
$$+ B^8 + R^8 + 4B^6R^2 + 6B^4R^4 + 4B^2R^6)$$

$$=\tfrac{1}{2}\Big(2B^8+2R^8+8B^7R+32B^6R^2+56B^5R^3$$
$$+76B^4R^4+56B^3R^5+32B^2R^6+8B\ R^7\Big)$$
$$=B^2+R^2+4B^7R+16B^6R^2+28B^5R^3$$
$$+38B^4R^4+28B^3R^5+16B^2R^6+4B\ R^7$$

as in Example 5.3 above and substituting $B=R=1$ therefore

$$PI=C_G(x_1,x_2)=1+1+4+16+28+38+28+16+4=136,$$

i.e. There are 136 distinct colourings.

Example 5.7

Using Example 5.2, where number of beads is K, we have,

Cycle index is $\left(x_i=\sum_{c\in C} w_c^n\right)$,

$$PI=C_G(x_1,x_2)=\begin{cases}\frac{1}{2}(x_1^K+x_2^{K/2}) & \text{Iif k is even}\\ & \text{if k is even}\\ \frac{1}{2}(x_1^K+x_1x_2^{K-1/2}). & \end{cases}$$

Using Remark 4.2 in Cl[illegible] if there are 3 colours (Blue, Red, White) then P´olya's enumeration Theorem gives

$$x_1=(B+R+W)\quad,\quad x_2=(B^2+R^2+W^2)$$

(i) If $K=2$, therefore

$$PI=C_G(x_1,x_2)=\tfrac{1}{2}\Big((B+R+W)^2+(B^2+R^2+W^2)^1\Big)$$
$$=B^2+R^2+W^2+BR+BW+RW$$

As in Example 5.3 above and substituting $B=R=W=1,$ therefore

$$PI=C_G(x_1,x_2)=1+1+1+1+1+1=6,$$

i.e. There are 6 distinct colourings.

(ii) If $K=3$, therefore

$$PI=C_G(x_1,x_2)=\frac{1}{2}((B+R+W)^3+(B+R+W)(B^2+R^2+W^2)) \qquad (5.6)$$

where

$$(B+R+W)^3 = (B+R+W)(B+R+W)(B+R+W)$$

$$= (B+R+W)(B^2+BR+BW+BR+$$

$$R^2+RW+BW+RW+W^2)$$

$$= (B+R+W)(B^2+R^2+W^2+2BR+2BW+2RW)$$

$$= B^3+BR^2+BW^2+2B^2R+2B^2W+2BRW+$$

$$B^2R+R^3+RW^2+2BR^2+2BRW+2R^2W+$$

$$B^2W+R^2W+W^3+2BRW+2BW^2+2RW^2$$

$$= B^3+R^3+W^3+3BR^2+3BW^2+3B^2R+$$

$$3B^2W+6BRW+3R\,W^2+3R^2W$$

and

$$(B+R+W)(B^2+R^2+W^2) = B^3+BR^2+BW^2+B^2R+$$

$$R^3+RW^2+B^2W+R^2W+W^3$$

$$= B^3+R^3+W^3+BR^2+BW^2+$$

$$B^2R+B^2W+RW^2+R^2W$$

Substituting in equation (5.2) above we get:

$$PI = C_G(x_1,x_2) = \tfrac{1}{2}(B^3+R^3+W^3+3BR^2+3BW^2+$$

$$3B^2R+3B^2W+6BRW+3RW^2+3R^2W+B^3+$$

$$R^3+W^3+BR^2+BW^2+B^2R+B^2W+RW^2+R^2W)$$

$$= \tfrac{1}{2}(2B^3+2R^3+2W^3+4BR^2+4BW^2+$$

$$4B^2R+4B^2W+4RW^2+4R^2W+6BRW)$$

$$= B^3+R^3+W^3+2BR^2+2BW^2+2B^2R+.$$

$$2B^2W+2RW^2+2R^2W+3BRW$$

As in Example 5.3 above and substituting *B=R=W=1*, therefore

$$PI = C_G(x_1,x_2) = 1+1+1+2+2+2+2+2+2+3 = 18,$$

i.e. There are 18 distinct colourings.

5.2 Chemical Compounds

In chemistry, P´olya's Enumeration Theorem can be used to find isomers of a given molecule. Two molecules are said to be isomers if they are composed of the same number and types of atoms, but have different structure. Let us illustrate this with C_5H_{12}. Figure below shows two chemical isomers that correspond to the hydrocarbon C_5H_{12}.

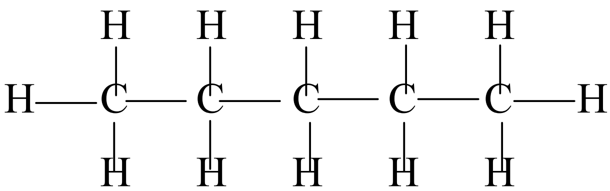


Figure (1.a)

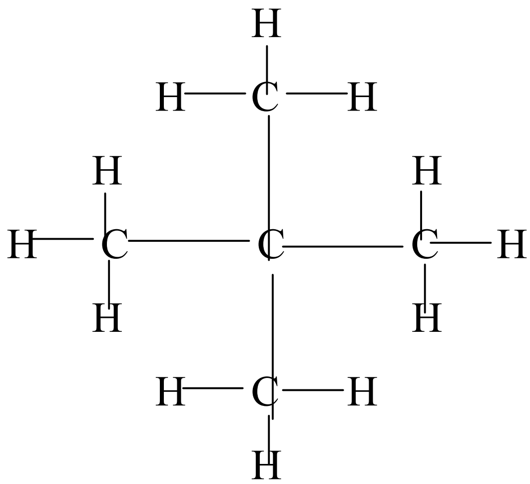


Figure (1.b)

Chemical isomers corresponding to the hydrocarbon C_5H_{12}

Example 5.8:

Cyclobutane is a hydrocarbon constructed of 4 carbon atoms arranged cyclically with 2 hydrogen atoms attached to each carbon, as illustrated in figure below.

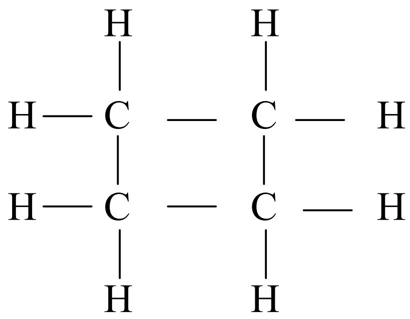


Figure : Cyclobutane structure

How many isomers may be obtained by replacing 2 hydrogen atoms with nitrogen?

Let the 8 bonds to the carbon atoms be our elements in N = {1, 2, 3, 4, 5, 6, 7, 8} and let C = {hydrogen, nitrogen} with the weights !(hydrogen) = H, !(nitrogen) = N. We can graphically visualize Cyclobutane as a cube, where the 4 cyclically arranged carbon atoms are at the center of the cube and each hydrogen atom represents a vertex of the cube, therefore G_V will be used to discount reflectional and rotational symmetry. Then P´olya's Enumeration Theorem
gives

$$Z_{G_V}(H + N, \ldots, H^8 + N^8)$$
$$= H^8 + H^7N + 3H^6N^2 + 3H^5N^3 + 7H^4N^4 + 3H^3N^5 + 3H^2N^6$$
$$+ HN^7 + N^8.$$

Hence there are 3 possible isomers with 6 hydrogens and 2 nitrogens as highlighted above.

Example 5.9:

Continuing with this cyclobutane, how many isomers can be obtained by replacing 2 hydrogens with oxygen and 3 with nitrogen? Now we have three colors C = {hydrogen, nitrogen, oxygen} with weights (hydrogen) = H, (nitrogen) = N, (oxygen) = 0. P´olya's Enumeration Theorem gives

$$Z_{GV}(H + N + O, \ldots, H^8 + N^8 + O^8)$$

$$= \frac{1}{24}\,[(H + N + O)^8 + 8(H + N + O)^2(H^3 + N^3 + O^3)^2 + 9(H^2 + N^2 + O^2)^4 + 6(H^4 + N^4 + O^4)^2]$$

$$= \frac{1}{24}[24H^8 + 24H^7N + 72H^6N^2 + 72H^5N^3 + 168H^4N^4 + 72H^3N^5$$
$$+ 72H^2N^6 + 24HN^7 + 24N^8 + 24H^7O + 72H^6NO + 168H^5N^2O$$
$$+ 312H^4N^3O + 312H^3N^4O + 168H^2N^5O + 72HN^6O + 24N^7O$$
$$+ 72H^6O^2 + 168H^5NO^2 + 528H^4N^2O^2 + 576H^3N^3O^2$$
$$+ 528H2N^4O^2 + 168HN^5O^2 + 72N^6O^2 + 72H^5O^3$$
$$+ 312H^4NO^3 + 576H^3N^2O^3 + 576H^2N^3O^3 + 312HN^4O^3$$
$$+ 72N^5O^3 + 168H^4O^4 + 312H^3NO^4 + 528H^2N^2O^4$$
$$+ 312HN^3O^4 + 168N^4O^4 + 72H^3O^5 + 168H^2NO^5 + 168HN^2O^5$$
$$+ 72N^3O^5 + 72H^2O^6 + 72HNO^6 + 72N^2O^6 + 24HO^7 + 24NO^7$$
$$+ 24O^8]$$

$$= H^8 + H^7N + 3H^6N^2 + 3H^5N^3 + 7H^4N^4 + 3H^3N^5 + 3H^2N^6$$
$$+ HN^7 + N^8 + H^7O + 3H^6NO + 7H^5N^2O + 13H^4N^3O$$
$$+ 13H^3N^4O + 7H^2N^5O + 3HN^6O + N^7O + 3H^6O^2$$
$$+ 7H^5NO^2 + 22H^4N^2O^2 + 24H^3N^3O^2 + 22H^2N^4O^2$$
$$+ 7HN^5O^2 + 3N^6O^2 + 3H^5O^3 + 13H^4NO^3 + 24H^3N^2O^3$$
$$+ 24H^2N^3O^3 + 13HN^4O^3 + 3N^5O^3 + 7H^4O^4 + 13H^3NO^4$$
$$+ 22H^2N^2O^4 + 13HN^3O^4 + 7N^4O^4 + 3H^3O^5 + 7H^2NO^5$$
$$+ 7HN^2O^5 + 3N^3O^5 + 3H^2O^6 + 3HNO^6 + 3N^2O^6$$

$+ HO^7 + NO^7 + O^8.$

Example 5.10:

Let us find the number of isomers in example 6.7 with 3 hydrogens. Let us set the weights as follows: !(hydrogen) = H, !(nitrogen) = 1, !(oxygen) = 1. P´olya's enumeration Theorem gives

$Z_{G_V}(H + 2, \ldots, H^8 + 2)$

$$= \frac{1}{24}\,[(H + 2)^8 + 8(H + 2)^2(H^3 + 2)^2 + 9(H^2 + 2)^4 + 6(H^4 + 2)^2]$$

$$= \frac{1}{24}\,[552 + 1152H + 2112H^2 + 1920H^3 + 1488H^4 + 480H^5$$

$$+ 216H^6 + 48H^7 + 24H^8]$$

$$= 23 + 48H + 88H^2 + 80H^3 + 62H^4 + 20H^5 + 9H^6 + 2H^7 + H^8.$$

Chapter Six

Conclusion And References

Conclusion

-The study link between permutation Groups and Polya's theory of counting .

- The study shows some of the applications on necklace with K beads.
- The study shows some of the applications enumeration of chemical compounds.
- The study uses the colouring after the theory of counting
- The study uses fundamental theorem of combinatorial enumeration.
- The study generalizing Burnside's lemma about number of orbits.
- In this model of theory selection, social learning even with preferential attachment does not generally yield significantly better outcomes than individual learning.
- The one exception occurs when people discount the past here, social learning enables both the identification of successful theories and their rapid spread throughout the population.

-This suggests that discounting the past provide a good balance between retaining what one has learnt through one's own experience while being able to respond rapidly to what other's have learnt.

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