Chapter 1

Lyapunov Inequalities

In this chapter some special cases of our results contain the classical Lyapunov inequalities for differential equations as well as only recently developed Lyapunov inequalities for difference equations. We present some Lyapunov type inequalities for discrete linear scalar Hamiltonian systems when the coefficient $c(t)$ is not necessarily nonnegative valued and when the end-points are not necessarily usual zeros, but rather, generalized zeros. The purpose of this work is to establish the time scale version of Lyapunov's inequality as follows: Let $x(t)$ be a nontrivial solution of

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)x^{\sigma}(t) = 0 \quad \text{on } [a,b]$$

Satisfying $x(a) = x(b) = 0$. Then, under suitable conditions on $p,\ r,\ a$ and b, we have

$$\int_a^b p_+(t)\Delta t \geq \begin{cases} \dfrac{r(a)}{r(b)}\dfrac{b-a}{f(d)}, & \text{if } r \text{ is increasing,} \\ \dfrac{r(b)}{r(a)}\dfrac{b-a}{f(d)}, & \text{if } r \text{ is decreasing,} \end{cases}$$

where $p_+(t) = \max\{p(t), 0\}$, $f(t) = (t-a)(b-t)$ and $d \in \mathbb{T}$ satisfies

$$\left|\frac{a+b}{2} - d\right| = \min\left\{\frac{a+b}{2} - s\middle| s \in [a,b] \cap \mathbb{T}\right\}$$

If $\frac{a+b}{2} \in \mathbb{T}$. Here $\mathbb{T}$ is a time scale (see below).

Section (1.1): Time Scales:

Lyapunov inequalities have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications in the theory of differential and difference equations. A nice summary of continuous and discrete Lyapunov inequalities and their applications can be found in the survey [2] by Chen. In this section we present several versions of Lyapunov inequalities that are valid on so-called time scales. The calculus of time scales has been introduced by Hilger [3] in order to unify discrete and continuous analysis. Hence our results presented cover (among other cases) both the continuous (see [2] and also [4]) and discrete (see [2, and also [5, 6, 7]) Lyapunov inequalities. For convenience we now recall the following easiest versions of Lyapunov's inequality.

Theorem (1.1.1)[1]: (Continuous Lyapunov Inequality) Let $p:[a,b]\to\mathbb{R}_+$ be positive-valued and continuous. If the Sturm-Liouville (differential) equation

$$\ddot{x}+p(t)x=0$$

has a nontrivial solution satisfying $x(a)=x(b)=0$, then the Lyapunov inequality

$$\int_a^b p(t)dt>\frac{4}{b-a}$$

holds.

Theorem (1.1.2)[1]: (Discrete Lyapunov Inequality) Let $\{p_k\}_{0\le k\le N}\subset\mathbb{R}_+$ be positive-valued. If the Sturm-Liouville difference equation

$$\Delta^2 x_k+p_k x_{k+1}=0$$

has a nontrivial solution satisfying $x_0=x_N=0$, then the Lyapunov inequality

$$\sum_{k=0}^{N-1}p_k\ge\begin{cases}(2/m+1) & if\ N=2(m+1)\\ \big((2m+1)/m(m+1)\big) & if\ N=2m+1\end{cases}$$

holds.

In this section we prove a Lyapunov inequality that contains both Theorems (1.1.1) and (1.1.2) as special cases. It is valid for an arbitrary time scale, and it reads as follows.

To see how Theorems (1.1.1) and (1.1.2) follow as special cases from Theorem (1.1.5) below, it is at this point only important to know that

- $\mathbb{T}=\mathbb{R}$ corresponds to the continuous case, and $x^\sigma=x, x^\Delta=x, \int_a^b f(t)\Delta t=\int_a^b f(t)dt$, and an rd-continuous function is the same as a continuous function in this case;
- $\mathbb{T}=\mathbb{Z}$ Corresponds to the discrete case, and $x^\sigma(t)=x(t+1), x^\Delta=x^\sigma-x$, $\int_a^b f(t)\Delta t=\sum_{t=a}^{b-a}f(t)$, and any function is rd-continuous in this case.

A short introduction to the time scales calculus is given. We prove the above Theorem (1.1.5) below, and for the proof several lemmas on quadratic functionals connected to the Sturm-Liouville dynamic equation (2) are needed. In the time scales calculus, the concept of a zero of a function is replaced by a so-called generalized zero, and (as in the classical case), a Lyapunov inequality leads immediately to disconjugacy criteria as presented. Two extensions which we have not considered in this Section are the cases when p is not necessarily positive-valued and when the endpoints are not necessarily zeros but generalized zeros. Finally, we extend the theory to linear Hamiltonian dynamic systems of the form

$$x^{\Delta} = A(t)x^{\sigma} + B(t)u, \qquad u^{\Delta} = -C(t)x^{\sigma} - A^{*}(t)u, \tag{1}$$

where A, B and C are square-matrix-valued functions satisfying the properties as given. Such Hamiltonian systems contain in particular Sturm-Liouville equations of higher order, and in particular also equations (2) as presented. Several lemmas concerning certain quadratic functionals connected to the system (1) are needed, and a Lyapunov inequality for Hamiltonian systems (1) is presented, as well as a disconjugacy criterion as an immediate application of the inequality. We also consider so-called right-focal boundary conditions and offer a Lyapunov inequality for this case, too.

A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$, and the (forward and backward) jump operators $\sigma, \rho: \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T}: s < t\}$$

(Supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$), while the graininess $\mu: \mathbb{T} \to \mathbb{R}_{+}$ is given by

$$\mu(t) = \sigma(t) - t.$$

For a function $f: \mathbb{T} \to \mathbb{R}$ we define the derivative f^{Δ} as follows: Let $t \in \mathbb{T}$. If there exists a number a such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$\left|f\big(\sigma(t)\big) - f(s) - \alpha(\sigma(t) - s)\right| \leq \varepsilon|\sigma(t) - s| \quad \text{ for all } s \in U,$$

then f is said to be (delta) differentiable at t, and we call α the derivative of f at t and denote it by $f^{\Delta}(t)$. Moreover, we denote $f^{\sigma} = f \circ \sigma$. The following formulas are useful:

- $f^{\sigma}: f + \mu f^{\Delta}$;

- $(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta}$ ("Product Rule");
- $(f/g)^{\Delta} = (f^{\Delta}g - fg^{\Delta})/(gg^{\sigma})$ ("Quotient Rule").

A function F with $F^{\Delta} = f$ is called an antiderivative of f, and then we define

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $a, b \in \mathbb{T}$. If a function is rd-continuous (i.e., continuous in points t with $\sigma(t) = t$ and left-hand limit exists in points t with $p(t) = t$), then it possesses an antiderivative (see [8]). We have that (see e.g., [8])

$$f(t) \geq 0, \qquad a \leq t < b \text{ implies } \int_a^b f(t)\Delta t \geq 0.$$

Throughout this section we assume $a, b \in \mathbb{T}$ with $a < b$. The two most popular cases of time scales are $\mathbb{T} = \mathbb{R}$, where $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$ and $\mathbb{T} = \mathbb{Z}$, where $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$. Other examples of time scales (to which our inequalities apply as well) are e.g.

$$h\mathbb{Z} = \{hk : k \in \mathbb{Z}\} \quad \text{for some } h > 0,$$
$$q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\} \quad \text{for some } q > 1$$

(which produces so-called q-difference equations),

$$\mathbb{N}^2 = \{k^2 : k \in \mathbb{N}\}, \qquad \left\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\right\}, \qquad \bigcup_{k \in \mathbb{Z}} [2k, 2k+1],$$

or the Cantor set.

We let $\mathbb{T} \subset \mathbb{R}$ be any time scale, $p: \mathbb{T} \to \mathbb{R}$ be rd-continuous with $p(t) > 0$ for all $t \in \mathbb{T}$, and consider the Sturm-Liouville dynamic equation (2) together with the quadratic functional

$$\mathcal{F}(x) = \int_a^b \{(x^{\Delta})^2 - p(x^{\sigma})^2\}(t)\Delta t.$$

Our first auxiliary result reads as follows.

Lemma (1.1.3)[1]: If x solves (1) and if $\mathcal{F}(y)$ is defined, then

$$\mathcal{F}(y) - \mathcal{F}(x) = \mathcal{F}(y - x) + 2(y - x)(b)x^{\Delta}(b) - 2(y - x)(a)x^{\Delta}(a).$$

Proof. Under the above assumptions we find

$$
\begin{aligned}
&\mathcal{F}(y)-\mathcal{F}(x)-\mathcal{F}(y-x)\\
&\quad=\int_a^b\{(y^\Delta)^2-p(y^\sigma)^2-(x^\Delta)^2+p(x^\sigma)^2\\
&\qquad\quad-(y^\Delta-x^\Delta)^2+p(y^\sigma-x^\Delta)^2\}(t)\Delta t\\
&\quad=\int_a^b\{(y^\Delta)^2-p(y^\sigma)^2-(x^\Delta)^2+p(x^\sigma)^2-(y^\Delta)^2+2y^\Delta x^\Delta-(x^\Delta)^2\\
&\qquad\quad+p(y^\sigma)^2-2py^\sigma x^\sigma+p(x^\sigma)^2\}(t)\Delta t\\
&\quad=2\int_a^b\{y^\Delta x^\Delta-py^\sigma x^\sigma+p(x^\sigma)^2-(x^\Delta)^2\}(t)\Delta\, t\\
&\quad=2\int_a^b\left\{y^\Delta x^\Delta+y^\sigma x^{\Delta^2}-(x^\Delta)^2\right\}(t)\Delta t\\
&\quad=2\int_a^b\{yx^\Delta-xx^\Delta\}^\Delta\Delta t=2\int_a^b\{(y-x)x^\Delta\}^\Delta\Delta t\\
&\quad=2\big(y(b)-x(b)\big)x^\Delta(b)-22\big(y(a)-x(a)\big)x^\Delta(a),
\end{aligned}
$$

where we have used the product rule.

Lemma (1.1.4)[1]: If $\mathcal{F}(y)$ is defined, then for any $r,s\in\mathbb{T}$ with $a\le r<s\le b$

$$\int_r^s\left(y^\Delta(t)\right)^2\Delta t\ge\frac{\big(y(s)-y(r)\big)^2}{s-r}.$$

Proof. Under the above assumptions we define

$$x(t)=\frac{y(s)-y(r)}{s-r}t+\frac{sy(r)-ry(s)}{s-r}.$$

We then have

$$x(r)=y(r),\quad x(s)=y(s),\quad x^\Delta(t)=\frac{y(s)-y(r)}{s-r},\quad \text{and}\quad x^{\Delta^2}(t)=0.$$

Hence x solves the special Sturm-Liouville equation (2) where $p=0$ and therefore we may apply Lemma (1.1.3) to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x) = \int_r^s (x^\Delta)^2(t)\Delta t$$

to find

$$\begin{aligned}
\mathcal{F}_0(y) &= \mathcal{F}_0(x) + \mathcal{F}_0(y-x) + (y-x)(s)x^\Delta(s) - (y-x)(r)x^\Delta(r) \\
&= \mathcal{F}_0(x) + \mathcal{F}_0(y-x) \\
&\geq \mathcal{F}_0(x) \\
&= \int_r^s \left\{\frac{y(s)-y(r)}{s-r}\right\}^2 \Delta t \\
&= \frac{\big(y(s)-y(r)\big)^2}{s-r},
\end{aligned}$$

and this proves our claim.

Using the above Lemma (1.1.4), we now can prove one of the main results of this Section.

Theorem (1.1.5)[1]: (Dynamic Lyapunov Inequality) Suppose $\mathbb{T}$ is a time scale and $a, b \in \mathbb{T}$ with $a < b$. Let $p : \mathbb{T} \to \mathbb{R}_+$ be positive-valued and rd-continuous. If the Sturm-Liouville dynamic equation

$$x^{\Delta^2} + p(t)x^\sigma = 0 \tag{2}$$

has a nontrivial solution x with $x(a) = x(b) = 0$, then the Lyapunov inequality

$$\int_a^b p(t)\Delta t \geq \frac{b-a}{f(d)} \tag{3}$$

holds, where $f: \mathbb{T} \to \mathbb{R}$ is defined by $f(t)\ (t-a)(b-t)$, and where $d \in \mathbb{T}$ is such that

$$\left|\frac{b+a}{2} - d\right| = \min\left\{\left|\frac{b+a}{2} - s\right| : s \in [a,b] \cap \mathbb{T}\right\}.$$

Proof. Suppose x is a solution of (2) with $x(a) = x(b) = 0$. But then we have from Lemma (1.1.3) (with $y = 0$) that

$$\mathcal{F}(x) = \int_a^b \{(x^\Delta)^2 - p(x^\sigma)^2\}(t)\Delta t = 0.$$

Since x is nontrivial, we have that M defined by

$$M = \max\ \{x^2(t) : t \in [a,b] \cap \mathbb{T}\} \tag{4}$$

is positive. We now let $c \in [a,b]$ be such that $x^2(c) = M$. Applying the above as well as Lemma (1.1.4) twice (once with $r = a$ and $s = c$ and a second time with $r = c$ and $s = b$) we have

$$M \int_a^b p(t)\Delta t \geq \int_a^b \{p(x^\sigma)^2\}(t)\Delta t$$

$$= \int_a^b (x^\Delta)^2(t)\Delta t = \int_a^c (x^\Delta)^2(t)\Delta t + \int_c^b (x^\Delta)^2(t)\Delta t$$

$$\geq \frac{\big(x(c) - x(a)\big)^2}{c-a} + \frac{\big(x(b) - x(c)\big)^2}{b-c}$$

$$= x^2(c)\left\{\frac{1}{c-a} + \frac{1}{b-c}\right\}$$

$$= M\frac{b-a}{f(c)} \geq M\frac{b-a}{f(d)},$$

where the last inequality holds because of $f(d) = \max\{f(t): t \in [a,b] \cap \mathbb{T}\}$. Hence, dividing by $M > 0$ yields the desired inequality.

Example (1.1.6)[1]: Here we shortly wish to discuss the two popular cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. We use the notation from the proof of Theorem (1.1.5).

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\min\left\{\left|\frac{a+b}{2} - s\right| : s \in [a,b]\right\} = 0 \quad \text{so that } d = \frac{a+b}{2}.$$

Hence $f(d) = ((b-a)^2/4)$ and the Lyapunov inequality from Theorem (1.1.5) reads

$$\int_a^b p(t)dt \geq \frac{4}{b-a}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we consider two cases. First, if $a + b$ is even, then

$$\min\left\{\left|\frac{a+b}{2} - s\right| : s \in [a,b] \cap \mathbb{Z}\right\} = 0 \quad \text{so that } d = \frac{a+b}{2}.$$

Hence $f(d) = ((b-a)^2/4)$ and the Lyapunov inequality reads

$$\sum_{t=a}^{b-1} p(t) \geq \frac{4}{b-a}.$$

If $a+b$ is odd, then

$$\min\left\{\left|\frac{a+b}{2}-s\right|: s\in[a,b]\cap\mathbb{Z}\right\}=\frac{1}{2}\quad\text{so that } d=\frac{a+b-1}{2}.$$

This time we have $f(d)=((b-a)^2-1/4)$ and the Lyapunov inequality reads

$$\sum_{t=a}^{b-1}p(t)\geq\frac{4}{b-a}\left\{\frac{1}{-(1/(b-a)^2)}\right\}.$$

As an application of the above Theorem (1.1.5) we now prove a sufficient criterion for disconjugacy of (2).

Definition (1.1.7)[1]: Equation (2) is called disconjugate on $[a,b]$ if the solution $\tilde{x}$ of (2) with $\tilde{x}(a)=0$ and $\tilde{x}^{\Delta}(a)=1$ satisfies

$$\tilde{x}\tilde{x}^{\sigma}>0\ \text{ on }\ (a,\rho(b)].$$

Lemma (1.1.8)[1]: Equation (2) is disconjugate on $[a,b]$ if and only if

$$\mathcal{F}(x)=\int_a^b\{(x^{\Delta})^2-p(x^{\sigma})^2\}(t)\Delta t>0$$

for all nontrivial x with $x(a)=x(b)=0$.

Proof. This is a special case of [15].

Theorem (1.1.9)[1]: (Sufficient Condition for Disconjugacy of (2) If p satisfies

$$\int_a^b p(t)\Delta t<\frac{b-a}{f(d)},\tag{5}$$

then (2) is disconjugate on $[a,b]$.

Proof. Suppose that (5) holds. For the sake of contradiction we assume that (2) is not disconjugate. But then, by Lemma (1.1.8), there exists a nontrivial x with $x(a)=x(b)=0$ such that $\mathcal{F}(x)\leq 0$. Using this x, we now define M by (4) to find

$$\begin{aligned}M\int_a^b p(t)\Delta t&\geq\int_a^b\{p(x^{\sigma})^2\}(t)\Delta t\\&\geq\int_a^b(x^{\Delta})^2(t)\Delta t\end{aligned}$$

$$\geq \frac{M(b-a)}{f(d)},$$

where the last inequality follows precisely as in the proof of Theorem (1.1.3). Hence, after dividing by $M > 0$, we arrive at

$$\int_a^b p(t)\Delta t \geq \frac{b-a}{f(d)}$$

which contradicts (5) and hence completes the proof.

Remark (1.1.10)[1]: Note that in both conditions (3) and (5) we could

$$\text{replace } \frac{b-a}{f(d)} \text{ by } \frac{4}{b-a},$$

and Theorems (1.1.5) and (1.1.9) would remain true. This is because for $a \leq c \leq b$ we have

$$\frac{1}{c-a} + \frac{1}{b-c} = \frac{(a+b-2c)^2}{(b-a)(c-a)(b-c)} + \frac{4}{b-a} \geq \frac{4}{b-a}$$

We consider the linear Hamiltonian dynamic system (1), where A, B and C are rd-continuous $n \times n$-matrix-valued functions on $\mathbb{T}$ such that $I - \mu(t)A(t)$ is invertible and $B(t)$ and $C(t)$ are positive semidefinite for all $t \in \mathbb{T}$. For the continuous case of this theory we refer to [16] (in particular for Lyapunov inequalities [17]) while [18] is a good reference for the discrete case. A corresponding quadratic functional is given by

$$\mathcal{F}(x,u) = \int_a^b \{u^* B u - (x^\sigma)^* C x^\sigma\}(t)\Delta t\,.$$

A pair (x, u) is called admissible if it satisfies the equation of motion

$$x^\Delta = A(t)x^\sigma + B(t)u.$$

As in the previous section we start with the following auxiliary lemma.

Lemma (1.1.11)[1]: If (x, u) solves (3) and if (y, v) is admissible, then

$$\mathcal{F}(y,v) - \mathcal{F}(x,u) = \mathcal{F}(y-x, v-u)$$
$$+2\, Re[(y\ x)^*(b)u(b) - (y-x)^*(a)u(a)].$$

Proof. Under the above assumptions we calculate

$$\mathcal{F}(y,v) - \mathcal{F}(x,u) - \mathcal{F}(y-x, v-u)$$

$$= \int_a^b \{v^*Bv - (y^\sigma)^* C y^\sigma - u^*Bu - (x^\sigma) C x^\sigma$$

$$-[(v-u)^* B (v-u) - (y^\sigma - x^\sigma)^* C (y^\sigma - x^\sigma)]\}(t)\Delta t$$

$$= \int_a^b \{-2u^*Bu + v^*Bu + u^*Bv$$

$$+2(x^\sigma)^* C x^\sigma - (y^\sigma)^* C x^\sigma - (x^\sigma)^* C y^\sigma\}(t)\Delta t$$

$$= \int_a^b \{-2u^*Bu + 2\mathrm{Re}[u^*Bv] + 2(x^\sigma)^* C x^\sigma - 2\mathrm{Re}[(y^\sigma)^* C x^\sigma]\}(t)\Delta t$$

$$= 2\mathrm{Re}\left(\int_a^b \{u^*(Bv - Bu) + [(x^\sigma)^* - (y^\sigma)^*] C x^\sigma\}(t)\Delta t\right)$$

$$= 2\mathrm{Re}\left(\int_a^b \{u^*(y^\Delta - A y^\sigma - x^\Delta + A x^\Delta)\right.$$

$$\left. +[(x^\sigma)^* - (y^\sigma)^*][-u^\Delta - A^* u]\}\,(t)\Delta t\right)$$

$$= 2\mathrm{Re}\left(\int_a^b \{u^*(y^\Delta - x^\Delta) + (y^\sigma - x^\sigma)^* u^\Delta\right.$$

$$\left. +2i\ \mathrm{Im}[u^* A x^\sigma + (y^\sigma)^* A^* u]\}\,(t)\Delta t\right)$$

$$= 2\mathrm{Re}\left(\int_a^b \{u^*(y^\Delta - x^\Delta) + (y^\sigma - x^\sigma)^* u^\Delta\}(t)\Delta t\right)$$

$$= 2\mathrm{Re}\left(\int_a^b \{[u^*(y^\Delta - x^\Delta) + (u^\Delta)^*(y^\sigma - x^\sigma)\}(t)\Delta t\right)$$

$$= 2\mathrm{Re}\left(\int_a^b \{[u^*(y-x)]^\Delta\}(t)\Delta t\right)$$

$$= 2\mathrm{Re}\{u^*(b)[y(b) - x(b)] - u^*(a)[y(a) - x(a)]\}$$

$$= 2\mathrm{Re}\{[y-x]^*(b)u(b) - [y-x]^*(a)u(a)\},$$

which is the conclusion we sought.

Notation (1.1.12)[1]: For the remainder of this section we denote by $W(.,r)$ the unique (see [8]) solution of the initial value problem

$$W^{\Delta} = -A^{*}(t)W, \qquad W(r) = I,$$

where $r \in [a,b]$ is given. We also write

$$F(s,r) = \int_{r}^{s} W^{*}(t,r)B(t)W(t,r)\Delta t\,.$$

Observe that $W(t,r) \equiv I$ provided $A(t) \equiv 0$.

Lemma (1.1.13)[1]: Given are W and F as introduced in Notation (1.1.12). If (y,v) is admissible and if $r,s \in \mathbb{T}$ with $a \leq r < s \leq b$ such that $F(s,r)$ is invertible, then

$$\int_{r}^{s} (v^{*}Bv)(t)\Delta t \geq [W^{*}(s,r)y(s) - y(r)]^{*}F^{-1}(s,r)[W^{*}(s,r)y(s) - y(r)].$$

Proof. Under the above assumptions we define

$$x(t) = W^{*^{-1}}(t,r)\{y(r) + F(t,r)F^{-1}(s,r)[W^{*}(s,r)y(s) - y(r)]\}$$

and

$$u(t) = W(t,r)F^{-1}(s,r)[W^{*}(s,r)y(s) - y(r)].$$

Then we have

$$x(r) = y(r), \qquad x(s) = y(s), \qquad u^{\Delta}(t) = -A^{*}(t)u(t),$$

and

$$\begin{aligned} x^{\Delta}(t) &= -\,W^{*^{-1}}(\sigma(t),r)\left(W^{\Delta}(t,r)\right)^{*} x(t) + W^{*^{-1}}(\sigma(t),r)W^{*}(t,r)B(t)u(t) \\ &= W^{*^{-1}}(\sigma(t),r)W^{*}(t,r)A(t)x(t) + W^{*^{-1}}(\sigma(t),r)W^{*}(t,r)B(t)u(t) \\ &= [W(t,r)W^{-1}(\sigma(t),r)]^{*}[A(t)x(t) + B(t)u(t)]. \end{aligned}$$

But

$$\begin{aligned} [W(t,r)W^{-1}(\sigma(t),r)]^{*} &= [W(\sigma(t),r) - \mu(t)W^{\Delta}(t,r)]W^{-1}(\sigma(t),r) \\ &= I + \mu(t)A^{*}(t)W(t,r)W^{-1}(\sigma(t),r) \end{aligned}$$

and therefore $[I - \mu(t)A^{*}(t)]W(t,r)W^{-1}(\sigma(t),r) = I$ so that

$$[I - \mu(t)A(t)]x^{\Delta}(t) = A(t)x(t) + B(t)u(t)$$

and hence

$$x^{\Delta}(t) = A(t)x(t) + \mu(t)A(t)x^{\Delta}(t) + B(t)u(t)$$
$$= A(t)x^{\sigma}(t) + B(t)u(t).$$

Thus (x, u) solves the special Hamiltonian system (1) where $C = 0$ and we may apply Lemma (1.1.11) to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x, u) = \int_r^s (u^* B u)(t)\Delta t$$

to obtain

$$\begin{aligned}
\mathcal{F}_0(y, v) &= \mathcal{F}_0(x, u) + \mathcal{F}_0(y - x, v - u) \\
&\quad +2\mathrm{Re}\{u^*(s)[y(s) - x(s)] - u^*(r)[y(r) - x(r)]\} \\
&= \mathcal{F}_0(x, u) + \mathcal{F}_0(y - x, v - u) \geq \mathcal{F}_0(x, u) \\
&= \int_r^s (u^* B u)(t)\Delta t \\
&= [W^*(s, r)y(s) - y(r)]^* F^{-1}(r, s)[W^*(s, r)y(s) - y(r)].
\end{aligned}$$

which shows our claim.

Remark (1.1.14)[1]: The assumption in Lemma (1.1.13) that $F(s, r)$ is invertible if $r < s$ can be dropped in ease B is positive definite rather than positive semidefinite.

As before, we now may use Lemma (1.1.13) to derive a Lyapunov inequality for Hamiltonian systems.

Theorem (1.1.15)[1]: [Lyapunov's Inequality] Assume (1) has a solution (x, u) such that x is nontrivial and satisfies $x(a) = x(b) = 0$. With W and F introduced in Notation (1.1.12), suppose that $F(b, c)$ and $F(c, a)$ are invertible, where $\|x(c)\| = \max_{t \in [a,b] \cap \mathbb{T}} \|x(t)\|$. Let λ be the biggest eigenvalue of

$$F = \int_a^b W^*(t, c)B(t)W(t, c)\Delta t,$$

and let $v(t)$ be the biggest eigenvalue of $C(t)$. Then the Lyapunov inequality

$$\int_a^b v(t)\Delta t \geq \frac{4}{\lambda}$$

holds.

Proof. Suppose we are given a solution (x, u) of (1) such that $x(a) = x(b) = 0$. Lemma (1.1.11) then yields (using $y = v = 0$) that

$$\mathcal{F}(x, u) = \int_a^b \{u^* Bu - (x^\sigma)^* Cx^\sigma\}(t)\Delta t = 0.$$

So we apply Lemma (1.1.13) twice (once with $r = a$ and $s = c$ and a second time with $r = c$ and $s = b$) to obtain

$$\begin{aligned}
\int_a^b [(x^\sigma)^* Cx^\sigma](t)\Delta t &= \int_a^b (u^* Bu)(t)\Delta t \\
&= \int_a^c (u^* Bu)(t)\Delta t + \int_c^b (u^* Bu)(t)\Delta t \\
&\geq x^*(c)W(c, a)F^{-1}(c, a)W^*(c, a)x(c) \\
&\quad + x^*(c)F^{-1}(b, c)x(c) \\
&= x^*(c)[F^{-1}(b, c) - F^{-1}(a, c)]x(c) \\
&\geq 4x^*(c)F^{-1}x(c).
\end{aligned}$$

Here we have used the relation $W(t, r)W(r, s) = W(t, s)$ (see [8 (i)]) as well as the inequality $M^{-1} + N^{-I} \geq 4(M + N)^{-1}$ (see [6] or [19]). Now, by applying the Rayleigh-Ritz theorem [20] we conclude

$$\begin{aligned}
\int_a^b v(t)\Delta t &\geq \int_a^b \frac{\|x^\sigma(t)\|^2}{\|x(c)\|^2}\Delta t \\
&= \frac{1}{\|x(c)\|^2}\int_a^b v(t)\big(x^\sigma(t)\big)^* x^\sigma(t)\Delta t \\
&\geq \frac{1}{\|x(c)\|^2}\int_a^b \big(x^\sigma(t)\big)^* C(t)x^\sigma(t)\Delta t \\
&\geq \frac{1}{\|x(c)\|^2} 4x^*(c)F^{-1}x(c) \\
&\geq \min_{x \neq 0} \frac{x^* F^{-1} x}{x^* x} \\
&= \frac{4}{\lambda},
\end{aligned}$$

and this finishes the proof.

Remark (1.1.16)[1]: If $A \equiv 0$, then $W \equiv I$ and $F = \int_a^b B(t)\Delta t$. If, in addition $B \equiv I$, then $F = b - a$. Note how the Lyapunov inequality $\int_a^b v(t)\Delta t \geq (4/\lambda)$ reduces to $\int_a^b p(t)\Delta t \geq (4/b - a)$ for the scalar case as discussed in Section (1.3).

Theorem (1.1.17)[1]: [Sufficient Condition for Disconjugacy of (1)] the notation from Theorem (1.1.15), if

$$\int_a^b v(t)\Delta t < \frac{4}{\lambda},$$

then (1) is disconjugate on $[a, b]$.

We conclude this section with a result concerning so-called right-focal boundary conditions, i.e., $x(a) = u(b) = 0$.

Theorem (1.1.18)[1]: Assume (1) has a solution (x, u) with x nontrivial and $x(a) = u(b) = 0$. With the notation as in Theorem (1.1.15), the Lyapunov inequality

$$\int_a^b v(t)\Delta t \geq \frac{1}{\lambda}$$

holds.

Proof. Suppose (x, u) is a solution of (1) such that $x(a) = u(b) = 0$ with $a < b$. Choose the point c in $(a, b]$ where $\|x(t)\|$ is maximal. Apply Lemma (1.1.11) with $y = v = 0$ to see that $\mathcal{F}(x, u) = 0$. Therefore,

$$\int_a^b [(x^\sigma)^* C x^\sigma](t)\Delta t = \int_a^b (u^* B u)(t)\Delta t \geq \int_a^c (u^* B u)(t)\Delta t\,.$$

Using Lemma (1.1.13) with $r = a$ and $s = c$, we get

$$\int_a^c (u^* B u)(t)\Delta t \geq [W^*(c, a)x(c) - x(a)]^* F^{-1}(c, a)[W^*(c, a)x(c)x(a)]$$

$$= x^*(c)W(c, a)F^{-1}(c, a)W^*(c, a)x(c)$$

$$= -x^*(c)F^{-1}(a,c)x(c)$$

$$= x^*(c)\left(\int_a^c W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c)$$

$$\geq x^*(c)\left(\int_a^c W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c)$$

$$= x^*(c)F^{-1}x(c).$$

Hence,

$$\int_a^b [(x^\sigma)^* C x^\sigma](t)\Delta t \geq x^*(c)F^{-1}x(c),$$

and the same arguments as in the proof of Theorem (1.1.15) lead us to our final conclusion.

Lemma (1.1.19)[219]: If x solves (1) and if $\mathcal{F}(x+\epsilon_1)$ is defined, then

$$\mathcal{F}(x+\epsilon_1) - \mathcal{F}(x) = \mathcal{F}(\epsilon_1) + 2(\epsilon_1)(a+\epsilon)x^\Delta(a+\epsilon) - 2(\epsilon_1)(a)x^\Delta(a).$$

Proof. Under the above assumptions we find

$$\mathcal{F}(x+\epsilon_1) - \mathcal{F}(x) - \mathcal{F}(\epsilon_1)$$

$$= \int_a^{a+\epsilon} \{((x+\epsilon_1)^\Delta)^2 - p((x+\epsilon_1)^\sigma)^2 - (x^\Delta)^2 + p(x^\sigma)^2$$

$$-((x+\epsilon_1)^\Delta - x^\Delta)^2 + p(x+\epsilon_1{}^\sigma - x^\Delta)^2\}(t)\Delta t$$

$$= \int_a^{a+\epsilon} \{((x+\epsilon_1)^\Delta)^2 - p((x+\epsilon_1)^\sigma)^2 - (x^\Delta)^2 + p(x^\sigma)^2 - ((x+\epsilon_1)^\Delta)^2 + 2(x+\epsilon_1)^\Delta x^\Delta - (x^\Delta)^2$$

$$+p((x+\epsilon_1)^\sigma)^2 - 2p(x+\epsilon_1)^\sigma x^\sigma + p(x^\sigma)^2\}(t)\Delta t$$

$$= 2\int_a^{a+\epsilon} \{(x+\epsilon_1)^\Delta x^\Delta - p(x+\epsilon_1)^\sigma x^\sigma + p(x^\sigma)^2 - (x^\Delta)^2\}(t)\Delta t$$

$$= 2\int_a^{a+\epsilon} \left\{(x+\epsilon_1)^\Delta x^\Delta + (x+\epsilon_1)^\sigma x^{\Delta^2} - (x^\Delta)^2\right\}(t)\Delta t$$

$$= 2\int_{a}^{a+\epsilon} \{(x+\epsilon_1)x^{\Delta} - xx^{\Delta}\}^{\Delta}\Delta t = 2\int_{a}^{a+\epsilon} \{(\epsilon_1)x^{\Delta}\}^{\Delta}\Delta t$$

$$= 2\big((x+\epsilon_1)(a+\epsilon) - x(a+\epsilon)\big)x^{\Delta}(a+\epsilon) - 2\big((x+\epsilon_1)(a) - x(a)\big)x^{\Delta}(a),$$

where we have used the product rule.

Lemma (1.1.20)[219]: If $\mathcal{F}(x+\epsilon_1)$ is defined, then for any $r, r+\epsilon_3 \in \mathbb{T}$ with $a \le r < r+\epsilon_3 \le a+\epsilon$

$$\int_{r}^{r+\epsilon_3} \left((x+\epsilon_1)^{\Delta}(t)\right)^2 \Delta t \ge \frac{\big((x+\epsilon_1)(r+\epsilon_3) - (x+\epsilon_1)(r)\big)^2}{\epsilon_3}.$$

Proof. Under the above assumptions we define

$$x(t) = \frac{(x+\epsilon_1)(r+\epsilon_3) - (x+\epsilon_1)(r)}{\epsilon_3}t + \frac{(r+\epsilon_3)(x+\epsilon_1)(r) - r(x+\epsilon_1)(r+\epsilon_3)}{\epsilon_3}.$$

We then have

$$x(r) = (x+\epsilon_1)(r),\quad x(r+\epsilon_3) = (x+\epsilon_1)(r+\epsilon_3),$$

$$x^{\Delta}(t) = \frac{(x+\epsilon_1)(r+\epsilon_3) - (x+\epsilon_1)(r)}{\epsilon_3}, \qquad \text{and } x^{\Delta^2}(t) = 0.$$

Hence x solves the special Sturm-Liouville equation (2) where $p = 0$ and therefore we may apply Lemma (1.1.3) to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x) = \int_{r}^{r+\epsilon_3} (x^{\Delta})^2(t)\Delta t$$

to find

$$\begin{aligned}
\mathcal{F}_0(x+\epsilon_1) &= \mathcal{F}_0(x) + \mathcal{F}_0(\epsilon_1) + (\epsilon_1)(r+\epsilon_3)x^{\Delta}(r+\epsilon_3) - (\epsilon_1)(r)x^{\Delta}(r) \\
&= \mathcal{F}_0(x) + \mathcal{F}_0(\epsilon_1) \\
&\ge \mathcal{F}_0(x) \\
&= \int_{r}^{r+\epsilon_3} \left\{\frac{(x+\epsilon_1)(r+\epsilon_3) - (x+\epsilon_1)(r)}{\epsilon_3}\right\}^2 \Delta t \\
&= \frac{\big((x+\epsilon_1)(r+\epsilon_3) - (x+\epsilon_1)(r)\big)^2}{\epsilon_3},
\end{aligned}$$

and this proves our claim.

Using the above Lemma (1.1.4), we now can show one of the main results of this section, Theorem (1.1.3)(see [22]) .

Theorem (1.1.21)[219]: Suppose $\mathbb{T}$ is a time scale and $a, a+\epsilon \in \mathbb{T}$ with $\epsilon > 0$. Let $p : \mathbb{T} \to \mathbb{R}_+$ be positive-valued and rd-continuous. If the Sturm-Liouville dynamic equation

$$x^{\Delta^2} + p(t)x^{\sigma} = 0 \tag{6}$$

has a nontrivial solution x with $x(a) = x(a+\epsilon) = 0$, then the Lyapunov inequality

$$\int_a^{a+\epsilon} p(t)\Delta t \geq \frac{\epsilon}{f(d)} \tag{7}$$

holds, where $f: \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = (t-a)(a+\epsilon-t)$, and where $d \in \mathbb{T}$ is such that

$$\left|\frac{2a+\epsilon}{2} - d\right| = \min\left\{\left|\frac{2a+\epsilon}{2} - t + \epsilon_2\right| : (t+\epsilon_2) \in [a, a+\epsilon] \cap \mathbb{T}\right\}.$$

Theorems (1.1.1) and (1.1.2) follows as special cases from Theorem (1.1.3) below [22], it is at this point only important to know that

(i) $\mathbb{T} = \mathbb{R}$ corresponds to the continuous case, and $x^{\sigma} = x, x^{\Delta} = x, \int_a^{a+\epsilon} f(t)\Delta t = \int_a^{a+\epsilon} f(t)dt$, and an rd-continuous function is the same as a continuous function in this case;

(ii) $\mathbb{T} = \mathbb{Z}$ Corresponds to the discrete case, and $x^{\sigma}(t) = x(t+1), x^{\Delta} = x^{\sigma} - x$, $\int_a^{a+\epsilon} f(t)\Delta t = \sum_{t=a}^{a+\epsilon-1} f(t)$, and any function is rd-continuous in this case.

In the time scales calculus, Martin Bohiner , Stephen Clark and Jerry Ridenhour [22] that proof the concept of a zero of a function is replaced by a so-called generalized zero, and a Lyapunov inequality leads immediately to disconjugacy criteria as presented with an extend given done by[22] to linear Hamiltonian dynamic systems of the form

$$x^{\Delta} = A(t)x^{\sigma} + B(t)u, \qquad u^{\Delta} = -C(t)x^{\sigma} - A^*(t)u, \tag{8}$$

Where A, B and C are square-matrix-valued functions for complete description (see [22]).

Theorem (1.1.22)[219]: (Sufficient Condition for Disconjugacy of (2) If p satisfies

$$\int_{a}^{a+\epsilon} p(t)\Delta t < \frac{\epsilon}{f(d)}, \tag{9}$$

then (2) is disconjugate on $[a, a+\epsilon]$.

Proof. Suppose that (9) holds. For the sake of contradiction we assume that (2) is not disconjugate. But then, by Lemma (1.1.8), there exists a nontrivial x with $x(a) = x(a+\epsilon) = 0$ such that $\mathcal{F}(x) \leq 0$. Using this x, we now define M by (4) to find

$$\begin{aligned} M\int_{a}^{a+\epsilon} p(t)\Delta t &\geq \int_{a}^{a+\epsilon} \{p(x^{\sigma})^2\}(t)\Delta t \\ &\geq \int_{a}^{a+\epsilon} (x^{\Delta})^2(t)\Delta t \\ &\geq \frac{M(\epsilon)}{f(d)}, \end{aligned}$$

where the last inequality follows precisely as in the proof of Theorem (1.1.3) Hence, after dividing by $M > 0$, we arrive at

$$\int_{a}^{a+\epsilon} p(t)\Delta t \geq \frac{\epsilon}{f(d)}$$

Which contradicts (9) and hence completes the proof.

Lemma (1.1.23)[219]: If (x, u) solves (3) and if $(x+\epsilon_1, u+\epsilon_4)$ is admissible, then

$$\begin{aligned} \mathcal{F}(x+\epsilon_1, u+\epsilon_4) - \mathcal{F}(x, u) &= \mathcal{F}(\epsilon_1, \epsilon_4) \\ &\quad +2\, Re\big[\big((x+\epsilon_1)x\big)^{*}(a+\epsilon)u(a+\epsilon) - (\epsilon_1)^{*}(a)u(a)\big]. \end{aligned}$$

Proof. Under the above assumptions we calculate

$$\begin{aligned} &\mathcal{F}\big((x+\epsilon_1), u+\epsilon_4\big) - \mathcal{F}(x, u) - \mathcal{F}(\epsilon_1, \epsilon_4) \\ &= \int_{a}^{a+\epsilon} \{u + {\epsilon_4}^{*}Bu + \epsilon_4 - ((x+\epsilon_1)^{\sigma})^{*}C(x+\epsilon_1)^{\sigma} - u^{*}Bu - (x^{\sigma})Cx^{\sigma} \\ &\qquad - [(\epsilon_4)^{*}B(\epsilon_4) - ((x+\epsilon_1)^{\sigma} - x^{\sigma})^{*}C((x+\epsilon_1)^{\sigma} - x^{\sigma})]\}(t)\Delta t \\ &= \int_{a}^{a+\epsilon} \{-2u^{*}Bu + u + {\epsilon_4}^{*}\, Bu + u^{*}Bu + \epsilon_4 \end{aligned}$$

$$+2(x^\sigma)^*Cx^\sigma-((x+\epsilon_1)^\sigma)^*Cx^\sigma-(x^\sigma)^*C(x+\epsilon_1)^\sigma\}(t)\Delta t$$

$$=\int_a^{a+\epsilon}\{-2u^*Bu+2\mathrm{Re}[u^*B(u+\epsilon_4)]+2(x^\sigma)^*Cx^\sigma-2\mathrm{Re}[((x+\epsilon_1)^\sigma)^*Cx^\sigma]\}(t)\Delta t$$

$$=2\mathrm{Re}\left(\int_a^{a+\epsilon}\{u^*(B(u+\epsilon_4)-Bu)+[(x^\sigma)^*-((x+\epsilon_1)^\sigma)^*]Cx^\sigma\}(t)\Delta t\right)$$

$$=2\mathrm{Re}\left(\int_a^{a+\epsilon}\{u^*((x+\epsilon_1)^\Delta-A(x+\epsilon_1)^\sigma-x^\Delta+Ax^\Delta)\right.$$

$$\left.+[(x^\sigma)^*-((x+\epsilon_1)^\sigma)^*][-u^\Delta-A^*u]\}\,(t)\Delta t\right)$$

$$=2\mathrm{Re}\left(\int_a^{a+\epsilon}\{u^*((x+\epsilon_1)^\Delta-x^\Delta)+((x+\epsilon_1)^\sigma-x^\sigma)^*u^\Delta\right.$$

$$\left.+2i\,\mathrm{Im}[u^*Ax^\sigma+((x+\epsilon_1)^\sigma)^*A^*u]\}\,(t)\Delta t\right)$$

$$=2\mathrm{Re}\left(\int_a^{a+\epsilon}\{u^*((x+\epsilon_1)^\Delta-x^\Delta)+((x+\epsilon_1)^\sigma-x^\sigma)^*u^\Delta\}(t)\Delta t\right)$$

$$=2\mathrm{Re}\left(\int_a^{a+\epsilon}\{[u^*((x+\epsilon_1)^\Delta-x^\Delta)+(u^\Delta)^*((x+\epsilon_1)^\sigma-x^\sigma)\}(t)\Delta t\right)$$

$$=2\mathrm{Re}\left(\int_a^{a+\epsilon}\{[u^*(\epsilon_1)]^\Delta\}(t)\Delta t\right)$$

$$=2\mathrm{Re}\{u^*(a+\epsilon)[\epsilon_1(a+\epsilon)]-u^*(a)[\epsilon_1(a)]\}$$

$$=2\mathrm{Re}\{[\epsilon_1]^*(a+\epsilon)u(a+\epsilon)-[\epsilon_1]^*(a)u(a)\},$$

which is the conclusion we sought.

Lemma (1.1.24)[219]: Given W and F as introduced in Notation (1.1.12). If $\big((x+\epsilon_1),u+\epsilon_4\big)$ is admissible and if $r,r+\epsilon_3\in\mathbb{T}$ with $a\le r<r+\epsilon_3\le a+\epsilon$ such that $F(r+\epsilon_3,r)$ is invertible, then

$$\int_r^{r+\epsilon_3} ((u+\epsilon_4{}^*B(u+\epsilon_4))(t)\Delta t$$

$$\geq [W^*(r+\epsilon_3,r)(x+\epsilon_1)(r+\epsilon_3)-(x+\epsilon_1)(r)]^*F^{-1}(r+\epsilon_3,r)[W^*(r+\epsilon_3,r)(x+\epsilon_1)(r+\epsilon_3)-(x+\epsilon_1)(r)].$$

Proof. Under the above assumptions we define

$$x(t)=W^{*^{-1}}(t,r)\{(x+\epsilon_1)(r)+F(t,r)F^{-1}(r+\epsilon_3,r)[W^*(r+\epsilon_3,r)(x+\epsilon_1)(r+\epsilon_3)-(x+\epsilon_1)(r)]\}$$

and

$$u(t)=W(t,r)F^{-1}(r+\epsilon_3,r)[W^*(r+\epsilon_3,r)(x+\epsilon_1)(r+\epsilon_3)-(x+\epsilon_1)(r)].$$

Then we have

$$x(r)=(x+\epsilon_1)(r),\qquad x(r+\epsilon_3)=y(r+\epsilon_3),\qquad u^\Delta(t)=-A^*(t)u(t),$$

and

$$\begin{aligned}x^\Delta(t)&=-W^{*^{-1}}(\sigma(t),r)\left(W^\Delta(t,r)\right)^*x(t)+W^{*^{-1}}(\sigma(t),r)W^*(t,r)B(t)u(t)\\&=W^{*^{-1}}(\sigma(t),r)W^*(t,r)A(t)x(t)+W^{*^{-1}}(\sigma(t),r)W^*(t,r)B(t)u(t)\\&=[W(t,r)W^{-1}(\sigma(t),r)]^*[A(t)x(t)+B(t)u(t)].\end{aligned}$$

But

$$\begin{aligned}[W(t,r)W^{-1}(\sigma(t),r)]^*&=[W(\sigma(t),r)-\mu(t)W^\Delta(t,r)]W^{-1}(\sigma(t),r)\\&=I+\mu(t)A^*(t)W(t,r)W^{-1}(\sigma(t),r)\end{aligned}$$

and therefore $[I-\mu(t)A^*(t)]W(t,r)W^{-1}(\sigma(t),r)=I$ so that

$$[I-\mu(t)A(t)]x^\Delta(t)=A(t)x(t)+B(t)u(t)$$

and hence

$$\begin{aligned}x^\Delta(t)&=A(t)x(t)+\mu(t)A(t)x^\Delta(t)+B(t)u(t)\\&=A(t)x^\sigma(t)+B(t)u(t).\end{aligned}$$

Thus (x,u) solves the special Hamiltonian system (H) where $C=0$ and we may apply Lemma (1.1.11) to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x,u)=\int_r^{r+\epsilon_3}(u^*Bu)(t)\Delta t$$

to obtain

$$\mathcal{F}_0(x+\epsilon_1,u+\epsilon_4)=\mathcal{F}_0(x,u)+\mathcal{F}_0(\epsilon_1,\epsilon_4)$$
$$+2\mathrm{Re}\{u^*(r+\epsilon_3)[(x+\epsilon_1)(r+\epsilon_3)-x(r+\epsilon_3)]$$
$$-u^*(r)[(x+\epsilon_1)(r)-x(r)]\}$$
$$=\mathcal{F}_0(x,u)+\mathcal{F}_0(\epsilon_1,\in_4)\geq\mathcal{F}_0(x,u)$$
$$=\int_r^{r+\epsilon_3}(u^*Bu)(t)\Delta t$$
$$=[W^*(r+\epsilon_3,r)(x+\epsilon_1)(r+\epsilon_3)-(x+\epsilon_1)(r)]^*F^{-1}(r,r+\epsilon_3)\left[W^*(r+\epsilon_3,r)(x+\epsilon_1)(r+\epsilon_3)-(u+\epsilon_{4)}(r)\right].$$

which shows our claim.

Theorem (1.1.25)[219]: [Lyapunov's Inequality] Assume (1) has a solution (x,u) such that x is nontrivial and satisfies $x(a)=x(a+\epsilon)=0$. With W and F introduced in Notation (1.1.12), suppose that $F(a+\epsilon,c)$ and $F(c,a)$ are invertible, where

$$\|x(c)\|=\max_{t\in[a,a+\epsilon]\cap\mathbb{T}}\|x(t)\|.$$

Let λ be the biggest eigenvalue of

$$F=\int_a^{a+\epsilon}W^*(t,c)B(t)W(t,c)\Delta t,$$

and let $(u+\epsilon_4)(t)$ be the biggest eigenvalue of $C(t)$. Then the Lyapunov inequality

$$\int_a^{a+\epsilon}(u+\epsilon_4)(t)\Delta t\geq\frac{4}{\lambda}$$

holds.

Proof. Suppose we are given a solution (x,u) of (H) such that $x(a)=x(a+\epsilon)=0$. Lemma (1.1.11) then yields (using $u+\epsilon_4=0$) that

$$\mathcal{F}(x,u)=\int_a^{a+\epsilon}\{u^*Bu-(x^\sigma)^*Cx^\sigma\}(t)\Delta t=0.$$

So we apply Lemma (1.1.13) twice (once with $r=a$ and $r+\epsilon_3=c$ and a second time with $r=c$ and $r+\epsilon_3=a+\epsilon$) to obtain

$$\int_a^{a+\epsilon}[(x^\sigma)^*Cx^\sigma](t)\Delta t=\int_a^{a+\epsilon}(u^*Bu)(t)\Delta t$$

$$= \int_a^c (u^*Bu)(t)\Delta t + \int_c^{a+\epsilon} (u^*Bu)(t)\Delta t$$

$$\geq x^*(c)W(c,a)F^{-1}(c,a)W^*(c,a)x(c)$$

$$+x^*(c)F^{-1}(a+\epsilon,c)x(c)$$

$$= x^*(c)[F^{-1}(a+\epsilon,c) - F^{-1}(a,c)]x(c)$$

$$\geq 4x^*(c)F^{-1}x(c).$$

Here we have used the relation $W(t,r)W(r,r+\in_3) = W(t,r+\in_3)$ (see [6]) as well as the inequality $M^{-1} + N^{-I} \geq 4(M+N)^{-1}$ (see [11] or [21]). Now, by applying the Rayleigh-Ritz theorem [17] we conclude

$$\int_a^{a+\epsilon} (u+\epsilon_{4)}(t)\Delta t \geq \int_a^{a+\epsilon} \frac{\|x^\sigma(t)\|^2}{\|x(c)\|^2}\Delta t$$

$$= \frac{1}{\|x(c)\|^2}\int_a^{a+\epsilon} (u+\epsilon_4)(t)\big(x^\sigma(t)\big)^* x^\sigma(t)\Delta t$$

$$\geq \frac{1}{\|x(c)\|^2}\int_a^{a+\epsilon} \big(x^\sigma(t)\big)^* C(t)x^\sigma(t)\Delta t$$

$$\geq \frac{1}{\|x(c)\|^2} 4x^*(c)F^{-1}x(c)$$

$$\geq \min_{x\neq 0} \frac{x^*F^{-1}x}{x^*x}$$

$$= \frac{4}{\lambda},$$

and this finishes the proof.

Theorem (1.1.25)[219]: Assume (1) has a solution (x,u) with x nontrivial and $x(a) = u(a+\epsilon) = 0$. With the notation as in Theorem (1.1.15), the Lyapunov inequality

$$\int_a^{a+\epsilon} (u+\epsilon_4)(t)\Delta t \geq \frac{1}{\lambda}$$

holds.

Proof. Suppose (x, u) is a solution of (1) such that $x(a) = u(a + \epsilon) = 0$ with $\epsilon > 0$. Choose the point c in $(a, a + \epsilon]$ where $\|x(t)\|$ is maximal. Apply Lemma (1.1.11) with $x + \epsilon_1 = u + \epsilon_4 = 0$ to see that $\mathcal{F}(x, u) = 0$. Therefore,

$$\int_a^{a+\epsilon} [(x^\sigma)^* C x^\sigma](t)\Delta t = \int_a^{a+\epsilon} (u^* B u)(t)\Delta t \geq \int_a^c (u^* B u)(t)\Delta t\,.$$

Using Lemma (1.1.13) with $r = a$ and $r + \epsilon_3 = c$, we get

$$\begin{aligned}\int_a^c (u^* B u)(t)\Delta t &\geq [W^*(c,a)x(c) - x(a)]^* F^{-1}(c,a)[W^*(c,a)x(c)x(a)]\\ &= x^*(c)W(c,a)F^{-1}(c,a)W^*(c,a)x(c)\\ &= -x^*(c)F^{-1}(a,c)x(c)\\ &= x^*(c)\left(\int_a^c W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c)\\ &\geq x^*(c)\left(\int_a^{a+\epsilon} W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c)\\ &= x^*(c)F^{-1}x(c).\end{aligned}$$

Hence,

$$\int_a^{a+\epsilon} [(x^\sigma)^* C x^\sigma](t)\Delta t \geq x^*(c)F^{-1}x(c),$$

and the same arguments as in the proof of Theorem (1.1.15) lead us to our final conclusion.

Section (1.2): Discrete Linear Hamiltonian Systems:

The continuous Hamiltonian system, in the case of two scalar linear differential equations, has the form (see, for example, [24, 25])

$$y'(t) = JH(t)y(t), \qquad t \in \mathbb{R}, \tag{10}$$

in which

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad H(t) = \begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix},$$

where $h_{jk}(t), j, k = 1,2$, are real-valued piece-wise continuous functions defined on $\mathbb{R}$ and $h_{12}(t) = h(t)$.

The vector equation (6) will be written as

$$y_1' = h_{21}(t)y_1 + h_{22}(t)y_2, \qquad y_2' = -h_{11}(t)y_1 - h_{12}(t)y_2$$

or setting $y_1(t) = x(t)$, $y_2(t) = u(t)$, and $h_{12}(t) = h_{21}(t) = a(t), h_{22}(t) = b(t)$, $h_{11}(t) = c(t)$, it can be rewritten as

$$x' = a(t)x + b(t)u, \qquad u' = -c(t)x - a(t)u, \qquad t \in \mathbb{R}. \tag{11}$$

We remark that the second-order differential equation

$$[p(t)x'(t)]' + q(t)x(t) = 0, \qquad t \in \mathbb{R}, \tag{12}$$

in which $p(t), q(t)$ are real-valued functions and $p(t) \neq 0$ for all $t \in \mathbb{R}$, can be written as an equivalent Hamiltonian system of type (7). Indeed, let $x(t)$ be a solution of (8) and set $p(t)x'(t) = u(t)$. Then we have

$$x' = \frac{1}{p(t)}u, \qquad u' = -q(t)x.$$

So, (8) is equivalent to (7) with

$$a(t) \equiv 0, \qquad b(t) = \frac{1}{p(t)}, \qquad c(t) = q(t).$$

In the proceeding, an elementary proof of the following theorem is given.

Theorem (1.2.1)[23]: In view of the notations described above, assume that $b(t) \geq 0$ for all $t \in \mathbb{R}$ and assume (7) has a real solution (x, u) such that $x(\alpha) = x(\beta) = 0$ and x is not identically zero on $[\alpha, \beta]$, where $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Then the Lyapunov inequality

$$\int_{\beta}^{\alpha}|a(t)|dt+\left\{\int_{\alpha}^{\beta} b(t)dt\cdot\int_{\alpha}^{\beta} c_{+}(t)dt\right\}^{1/2} \geq 2 \tag{13}$$

holds, where $c_{+}(t)=\max\{c(t),0\}$ is the nonnegative part of $c(t)$.

An introduction to the theory of Lyapunov type inequalities and their applications can be found in the survey paper of Cheng [26].

As it is well known (see [27-28]), an adequate form of the discrete Hamiltonian system corresponding to (7) is

$$\Delta x(t)=a(t)x(t+1)+b(t)u(t),\;\; \Delta u(t)=-c(t)x(t+1)-a(t)u(t),\;\; t\in\mathbb{Z}, \tag{10}$$

where Δ denotes the forward difference operator defined by $\Delta x(t)=x(t+1)-x(t)$, with the coefficient $a(t)$ satisfying the condition

$$1-a(t)\neq 0, \qquad t\in\mathbb{Z}. \tag{14}$$

Notice that the second-order difference equation

$$\Delta[p(t)\Delta x(t)]+q(t)x(t+1)=0, \qquad t\in\mathbb{Z}, \tag{15}$$

where $p(t)\neq 0$ for all $t\in\mathbb{Z}$, can be written as an equivalent discrete Hamiltonian system of type (10). Indeed, let $x(t)$ be a solution of (12) and set $\Delta(t)\Delta x(t)=u(t)$. Then we have

$$\Delta x(t)=\frac{1}{p(t)}u(t), \qquad \Delta u(t)=-q(t)x(t+1).$$

So, (12) is equivalent to (10) with

$$a(t)\equiv 0, \qquad b(t)=\frac{1}{p(t)}, \qquad c(t)=q(t).$$

Concerning system (10), besides taking into account (11), we will also assume that the functions $a(t), b(t)$, and $c(t)$ are real valued and

$$b(t)\geq 0, \qquad t\in\mathbb{Z}. \tag{16}$$

For each $t\in\mathbb{Z}$, let us set

$$c_{+}(t)=\max\{c(t),0\}. \tag{17}$$

In the discrete case, instead of the usual zero, the concept of generalized zero, which is due to Hartman [29], is used. A function $f:\mathbb{Z}\to\mathbb{R}$ is said to have a generalized zero at $t_0\in\mathbb{Z}$ provided either $f(t_0)=0$ or $f(t_0-1)f(t_0)<0$.

The main results of this section are the following theorems.

Proof. Multiplying the first equation of (7) by u and the second one by x, and then adding the results, we obtain

$$(xu)' = b(t)u^2 - c(t)x^2.$$

Integrating the last equation from α to β and taking into account that $x(\alpha) = x(\beta) = 0$ yields

$$\int_\alpha^\beta b(t)u^2(t)\,dt = \int_\alpha^\beta c(t)x^2(t)\,dt. \tag{18}$$

Choose $\tau \in (\alpha, \beta)$ such that

$$|x(\tau)| = \max_{\alpha \le t \le \beta} |x(t)|.$$

Since x is not identically zero on $[\alpha, \beta]$, we have $|x(\tau)| > 0$. Integrating the first equation of (7) initially from α to τ and then from τ to β, and taking into account that $x(\alpha) = x(\beta) = 0$, we get, respectively,

$$x(\tau) = \int_\alpha^\tau a(t)x(t)dt + \int_\alpha^\tau b(t)u(t)dt, \qquad -x(\tau) = \int_\tau^\beta a(t)x(t)dt + \int_\tau^\beta b(t)u(t)dt.$$

Hence, employing the triangle inequality gives

$$|x(\tau)| \le \int_\alpha^\tau |a(t)||x(t)|dt + \int_\alpha^\tau b(t)|u(t)|dt, \quad |x(\tau)| \le \int_\tau^\beta |a(t)||x(t)|dt + \int_\tau^\beta b(t)|u(t)|dt.$$

Adding these last two inequalities gives rise to

$$2|x(\tau)| \le \int_\tau^\beta |a(t)||x(t)|dt + \int_\tau^\beta b(t)|u(t)|dt. \tag{19}$$

On the other hand, applying the Cauchy-Schwarz inequality and using (15), we have

$$\int_\alpha^\beta b(t)|u(t)| \le \left\{\int_\alpha^\beta b(t)dt\right\}^{1/2} \cdot \left\{\int_\alpha^\beta b(t)u^2(t)dt\right\}^{1/2}$$

$$= \left\{\int_\alpha^\beta b(t)dt\right\}^{1/2} \cdot \left\{\int_\alpha^\beta c(t)x^2(t)dt\right\}^{1/2}$$

$$\leq \left\{ \int_{\alpha}^{\beta} b(t)dt \right\}^{1/2} \cdot \left\{ \int_{\alpha}^{\beta} c_{+}(t)x^{2}(t)dt \right\}^{1/2}.$$

Therefore, we get from (16)

$$2|x(\tau)| \leq \int_{\tau}^{\beta} |a(t)| \cdot |x(t)|dt + \left\{ \int_{\alpha}^{\beta} b(t)dt \right\}^{1/2} \cdot \left\{ \int_{\alpha}^{\beta} c_{+}(t)x^{2}(t)dt \right\}^{1/2}$$

$$\leq |x(\tau)| \cdot \left[\int_{\tau}^{\beta} |a(t)|dt + \left\{ \int_{\alpha}^{\beta} b(t)dt \cdot \int_{\alpha}^{\beta} c_{+}(t)dt \right\}^{1/2} \right].$$

Dividing the latter estimate by $|x(\tau)|$, we get the desired inequality (9).

Theorem (1.2.2)[23]: Let $\alpha, \beta \in \mathbb{Z}$ with $\alpha \leq \beta - 2$. Assume (10) has a real solution (x, u) such that $x(\alpha) = x(\beta) = 0$ and x is not identically zero on $[\alpha, \beta]$. Then the inequality

$$\sum_{t=\alpha}^{\beta-2} |a(t)| + \left\{ \sum_{t=\alpha}^{\beta-1} b(t) \cdot \sum_{t=\alpha}^{\beta-2} c_{+}(t) \right\}^{1/2} \geq 2 \tag{20}$$

holds.

Proof. Multiplying the first equation of (10) by $u(t)$ and the second one by $x(t+1)$, and then adding, we get

$$\Delta[x(t)u(t)] = b(t)u^{2}(t) - c(t)x^{2}(t+1). \tag{21}$$

Summing the last equation from α to $\beta - 1$ and taking into account that $x(\alpha) = x(\beta) = 0$, we obtain

$$0 = \sum_{t=\alpha}^{\beta-1} b(t)u^{2}(t) - \sum_{t=\alpha}^{\beta-1} c(t)x^{2}(t+1).$$

Since $x(\beta) = 0$, we have

$$\sum_{t=\alpha}^{\beta-1} b(t)u^{2}(t) = \sum_{t=\alpha}^{\beta-2} c(t)x^{2}(t+1) \leq \sum_{t=\alpha}^{\beta-2} c_{+}(t)x^{2}(t+1). \tag{22}$$

Choose $\tau \in [\alpha + 1, \beta - 1]$ such that

$$|x(\tau)| = \max_{\alpha+1 \leq t \leq \beta-1} |x(t)|.$$

Then $|x(\tau)| > 0$. Summing the first equation of (10) at first from α to $\tau - 1$ and then from τ to $\beta - 1$, we get, respectively,

$$x(\tau) = \sum_{t=\alpha}^{\tau-1} a(t)x(t+1) + \sum_{t=\alpha}^{\tau-1} b(t)u(t), \quad -x(\tau) = \sum_{t=\tau}^{\beta-2} a(t)x(t+1) + \sum_{t=\tau}^{\beta-1} b(t)u(t).$$

Passing here to the modulus, we have

$$|x(\tau)| \le \sum_{t=\alpha}^{\tau-1} |a(t)| \cdot |x(t+1)| + \sum_{t=\alpha}^{\tau-1} b(t)|u(t)|,$$

$$|x(\tau)| \le \sum_{t=\tau}^{\beta-2} |a(t)| \cdot |x(t+1)| + \sum_{t=\tau}^{\beta-1} b(t)|u(t)|.$$

Adding these inequalities implies

$$2|x(\tau)| \le \sum_{t=\alpha}^{\beta-2} |a(t)| \cdot |x(t+1)| + \sum_{t=\alpha}^{\beta-1} b(t)|u(t)|. \tag{23}$$

On the other hand, applying the Cauchy-Schwarz inequality and using (19), we have

$$\sum_{t=\alpha}^{\beta-1} b(t)|u(t)| \le \left\{\sum_{t=\alpha}^{\beta-1} b(t)\right\}^{1/2} \cdot \left\{\sum_{t=\alpha}^{\beta-1} b(t)u^2(t)\right\}^{1/2}$$

$$\le \left\{\sum_{t=\alpha}^{\beta-1} b(t)\right\}^{1/2} \cdot \left\{\sum_{t=\alpha}^{\beta-2} c_+(t)x^2(t+1)\right\}^{1/2}.$$

Therefore, we get from (19)

$$2|x(\tau)| \le \sum_{t=\alpha}^{\beta-2} |a(t)| \cdot |x(t+1)| + \left\{\sum_{t=\alpha}^{\beta-1} b(t)\right\}^{1/2} \cdot \left\{\sum_{t=\alpha}^{\beta-2} c_+(t)x^2(t+1)\right\}^{1/2}$$

$$\le |x(\tau)| \cdot \left[\sum_{t=\alpha}^{\beta-2} |a(t)| + \left\{\sum_{t=\alpha}^{\beta-1} b(t) \cdot \sum_{t=\alpha}^{\beta-2} c_+(t)\right\}^{1/2}\right].$$

Dividing the latter inequality by $|x(\tau)|$, we obtain inequality (17).

Theorem (1.2.3)[23]: Suppose

$$1 - a(t) > 0, \qquad b(t) > 0, \qquad \text{for all } t \in \mathbb{Z}, \tag{24}$$

and let $\alpha, \beta \in \mathbb{Z}$ with $\alpha \leq \beta - 2$. Assume (10) has a real solution (x, u) such that $x(\alpha) = 0, x(\beta - 1)x(\beta) < 0$. Then the inequality

$$\sum_{t=\alpha}^{\beta-2} |a(t)| + \left\{ \sum_{t=\alpha}^{\beta-2} b(t) \cdot \sum_{t=\alpha}^{\beta-2} c_+(t) \right\}^{1/2} > 1 \tag{25}$$

holds.

Proof. Choose $\tau \in [\alpha + 1, \beta - 1]$ such that

$$|x(\tau)| = \max_{\alpha+1 \leq t \leq \beta-1} |x(t)|.$$

Then $|x(\tau)| > 0$. Summing the first equation of (10) from α to $\tau - 1$ and taking into account that $x(\alpha) = 0$, we get

$$x(\tau) = \sum_{t=\alpha}^{\tau-1} a(t)x(t+1) + \sum_{t=\alpha}^{\tau-1} b(t)u(t).$$

Hence,

$$\begin{aligned} |x(\tau)| &\leq \sum_{t=\alpha}^{\tau-1} |a(t)| \cdot |x(t+1)| + \sum_{t=\alpha}^{\tau-1} b(t)|u(t)| \\ &\leq \sum_{t=\alpha}^{\beta-2} |a(t)| \cdot |x(t+1)| + \sum_{t=\alpha}^{\beta-2} b(t)|u(t)| \qquad (26) \\ &\leq \sum_{t=\alpha}^{\beta-2} |a(t)| \cdot |x(t+1)| + \left\{ \sum_{t=\alpha}^{\beta-2} b(t) \right\}^{1/2} \cdot \left\{ \sum_{t=\alpha}^{\beta-2} b(t)u^2(t) \right\}^{1/2}. \end{aligned}$$

Now summing equation (18) from α to $\beta - 2$ and taking into account that $x(\alpha) = 0$, we obtain

$$x(\beta - 1)u(\beta - 1) = \sum_{t=\alpha}^{\beta-2} b(t)u^2(t) - \sum_{t=\alpha}^{\beta-2} c(t)x^2(t+1). \tag{27}$$

Further, from the first equation of (10), we have, for $t = \beta - 1$,

$$[1 - a(\beta - 1)]x(\beta) = x(\beta - 1) + b(\beta - 1)u(\beta - 1).$$

Multiplying this by $x(\beta - 1)$ yields

$$[1-a(\beta-1)]x(\beta-1)x(\beta)=x^2(\beta-1)+b(\beta-1)x(\beta-1)u(\beta-1).$$

Since $x(\beta-1)x(\beta)<0$, in view of (21), the above latter equality gives rise to $x(\beta-1)u(\beta-1)<0$. Therefore, from (24) the inequality

$$\sum_{t=\alpha}^{\beta-2} b(t)u^2(t) < \sum_{t=\alpha}^{\beta-2} c(t)x^2(t+1) \le \sum_{t=\alpha}^{\beta-2} c_+(t)x^2(t+1)$$

follows. Employing this last string of relations, inequality (23) gives

$$\begin{aligned}|x(\tau)| &\le \sum_{t=\alpha}^{\beta-2}|a(t)|\cdot|x(t+1)| + \left\{\sum_{t=\alpha}^{\beta-2} b(t)\right\}^{1/2} \cdot \left\{\sum_{t=\alpha}^{\beta-2} c_+(t)x^2(t+1)\right\}^{1/2} \\ &\le |x(\tau)|\cdot\left[\sum_{t=\alpha}^{\beta-2}|a(t)| + \left\{\sum_{t=\alpha}^{\beta-2} b(t)\cdot\sum_{t=\alpha}^{\beta-2} c_+(t)\right\}^{1/2}\right]\end{aligned}$$

Hence, dividing by $|x(\tau)|$ we obtain inequality (22).

Theorem (1.2.4)[23]: Suppose condition (21) holds and let $\alpha,\beta\in\mathbb{Z}$ with $\alpha\le\beta-1$. Assume (10) has a real solution (x,u) such that $x(\alpha-1)x(\alpha)<0, x(\beta)=0$. Then the inequality

$$\sum_{t=\alpha}^{\beta-2}|a(t)| + \left\{\sum_{t=\alpha}^{\beta-1} b(t)\cdot\sum_{t=\alpha-1}^{\beta-2} c_+(t)\right\}^{1/2} > 1 \qquad (28)$$

holds.

Proof.

Choose $\tau\in[\alpha,\beta-1]$ such that

$$|x(\tau)| = \max_{\alpha\le t\le\beta-1}|x(t)|.$$

Then $|x(\tau)|>0$. Summing the first equation of (10) from τ to $\beta-1$ and taking into account that $x(\beta)=0$ yields

$$x(\tau) = -\sum_{t=\tau}^{\beta-2} a(t)x(t+1) + \sum_{t=\tau}^{\beta-1} b(t)u(t).$$

Hence,

$$
\begin{aligned}
|x(\tau)| &\le \sum_{t=\tau}^{\beta-2} |a(t)| \cdot |x(t+1)| + \sum_{t=\tau}^{\beta-1} b(t)|u(t)| \\
&\le \sum_{t=\alpha}^{\beta-2} |a(t)| \cdot |x(t+1)| + \sum_{t=\alpha}^{\beta-1} b(t)|u(t)| \qquad (29) \\
&\le \sum_{t=\alpha}^{\beta-2} |a(t)| \cdot |x(t+1)| + \left\{\sum_{t=\alpha}^{\beta-1} b(t)\right\}^{1/2} \cdot \left\{\sum_{t=\alpha}^{\beta-1} b(t)u^2(t)\right\}^{1/2}.
\end{aligned}
$$

Now summing equation (18) from $\alpha-1$ to $\beta-1$ and taking into account that $x(\beta)=0$, we obtain

$$
-x(\alpha-1)u(\beta-1) = \sum_{t=\alpha-1}^{\beta-1} b(t)u^2(t) - \sum_{t=\alpha-1}^{\beta-2} c(t)x^2(t+1)
$$

or

$$
-u(\alpha-1)[x(\alpha-1)+b(\alpha-1)u(\alpha-1)] = \sum_{t=\alpha}^{\beta-1} b(t)u^2(t) - \sum_{t=\alpha-1}^{\beta-2} c(t)x^2(t+1)\,. \qquad (30)
$$

Further, from the first equation of (28), we have, for $t=\alpha-1$,

$$
[1-a(\alpha-1)]x(\alpha) = x(\alpha-1) + b(\alpha-1)u(\alpha-1). \qquad (31)
$$

Multiplying this by $x(\alpha-1)$ gives that

$$
[1-a(\alpha-1)]x(\alpha-1)x(\alpha) = x^2(\alpha-1) + b(\alpha-1)x(\alpha-1)u(\alpha-1).
$$

Since $x(\alpha-1)x(\alpha)<0$, by (21) it follows from the above latter equality that

$$
x(\alpha-1)u(\alpha-1) < 0. \qquad (32)
$$

Now our aim is to show that

$$
u(\alpha-1)[x(\alpha-1)+b(\alpha-1)u(\alpha-1)] > 0 \qquad (33)
$$

holds. Indeed, multiplying (28) by $u(\alpha-1)$ gives

$$
[1-a(\alpha-1)]x(\alpha)u(\alpha-1) = u(\alpha-1)[x(\alpha-1)+b(\alpha-1)u(\alpha-1)]. \qquad (34)
$$

On the other hand, it follows from $x(\alpha-1)x(\alpha)<0$ and (29) that $x(\alpha)u(\alpha-1)>0$. Therefore, the left-hand side of (31) is positive, and hence, (30) is true.

By virtue of (30), the string of inequalities

$$
\sum_{t=\alpha}^{\beta-1} b(t)u^2(t) < \sum_{t=\alpha-1}^{\beta-2} c(t)x^2(t+1) \le \sum_{t=\alpha-1}^{\beta-2} c_+(t)x^2(t+1)
$$

follows from (27). As a result of these last relations, from (26) we have

$$|x(\tau)| < \sum_{t=\alpha}^{\beta-2} |a(t)| \cdot |x(t+1)| + \left\{\sum_{t=\alpha}^{\beta-1} b(t)\right\}^{1/2} \cdot \left\{\sum_{t=\alpha-1}^{\beta-2} c_+(t) x^2(t+1)\right\}^{1/2}$$

$$\leq |x(\tau)| \cdot \left[\sum_{t=\alpha}^{\beta-2} |a(t)| + \left\{\sum_{t=\alpha}^{\beta-1} b(t) \cdot \sum_{t=\alpha-1}^{\beta-2} c_+(t)\right\}^{1/2}\right].$$

Hence, dividing the last estimate by $|x(\tau)|$, we get inequality (25).

Theorem (1.2.5)[23]: Suppose

$$1 - a(t) > 0, \qquad b(t) > 0, \qquad c(t) > 0, \qquad \text{for all } t \in \mathbb{Z}, \tag{35}$$

and let $\alpha, \beta \in \mathbb{Z}$ with $\alpha \leq \beta - 1$. Assume (10) has a real solution (x, u) such that $x(\alpha - 1)x(\alpha) < 0$ and $x(\beta - 1)x(\beta) < 0$. Then the inequality

$$\sum_{t=\alpha-1}^{\beta-2} |a(t)| + \left\{\sum_{t=\alpha-1}^{\beta-1} b(t) \cdot \sum_{t=\alpha-1}^{\beta-2} c(t)\right\}^{1/2} > 1 \tag{36}$$

holds.

Proof.

(I) First, we assume that$x(t) \neq 0$ for all $t \in [\alpha, \beta - 1]$. Denote by β_0 the smallest integer in $[\alpha, \beta]$ such that $\beta_0 \neq \alpha$ and

$$x(\beta_0 - 1)x(\beta_0) < 0. \tag{37}$$

Then x does not have any generalized zero in $[\alpha + 1, \beta_0 - 1]$, and without loss of generality we may assume that

$$x(t) > 0, \qquad \text{for all } t \in [\alpha, \beta_0 - 1]. \tag{38}$$

Then we will have

$$x(\alpha - 1) < 0, \qquad x(\beta_0) < 0. \tag{39}$$

Let $s \in [\alpha - 1, \beta_0 - 1]$. Summing the second equation of (10) first from $\alpha - 1$ to $s - 1$ and then from s to $\beta_0 - 2$, we get

$$u(s) - u(\alpha - 1) = \sum_{t=\alpha-1}^{s-1} c(t)x(t+1) - \sum_{t=\alpha-1}^{s-1} a(t)\, u(t), \tag{40}$$

$$u(\alpha-1)-u(s)=-\sum_{t=s}^{\beta_0-2} c(t)x(t+1)-\sum_{t=\alpha-1}^{\beta_0-2} a(t)\,u(t), \tag{41}$$

respectively. Notice that for $s=\alpha-1$ we write solely (38), and for $s=\beta_0-1$ only (37) is written.

Let us now show that

$$u(\alpha-1)>0, \qquad u(\beta_0-1)<0. \tag{42}$$

Indeed, from the first equation of (26), we have

$$[1-a(t)]x(t+1)=x(t)+b(t)u(t).$$

Multiplying this last equation by $x(t)$ gives

$$[1-a(t)]x(t)x(t+1)=x^2(t)+b(t)x(t)u(t),$$

where setting $t=\alpha-1$ and $t=\beta_0-1$, respectively, yields

$$[1-a(\alpha-1)]x(\alpha-1)x(\alpha)=x^2(\alpha-1)+b(\alpha-1)x(\alpha-1)u(\alpha-1),$$
$$[1-a(\beta_0-1)]x(\beta_0-1)x(\beta_0)=x^2(\beta_0-1)+b(\beta_0-1)x(\beta_0-1)u(\beta_0-1).$$

Using the inequalities $x(\alpha-1)x(\alpha)<0, x(\beta_0-1)x(\beta_0)<0$, and (19), we get from the above latter equalities the estimates

$$x(\alpha-1)u(\alpha-1)<0, \qquad x(\beta_0-1)u(\beta_0-1)<0. \tag{43}$$

Hence, taking into account $x(\alpha-1)<0$ and $x(\beta_0-1)>0$, we obtain (39).

Employing (37) if $u(s)<0$ and whenever $u(s)>0$ using (38), and also taking into account (39), we get

$$\begin{aligned}|u(s)| &\le \sum_{t=\alpha-1}^{\beta_0-2} c(t)|x(t+1)|+\sum_{t=\alpha-1}^{\beta_0-2}|a(t)|\cdot|u(t)| \\ &\le \left\{\sum_{t=\alpha-1}^{\beta_0-2} c(t)\right\}^{1/2}\cdot\left\{\sum_{t=\alpha-1}^{\beta_0-2} c(t)x^2(t+1)\right\}^{1/2}+\sum_{t=\alpha-1}^{\beta_0-2}|a(t)|\cdot|u(t)|. \end{aligned}\tag{44}$$

Next, summing equation (18) from $\alpha-1$ to β_0-1 gives

$$x(\beta_0)u(\beta_0)-x(\alpha-1)u(\alpha-1)=\sum_{t=\alpha-1}^{\beta_0-2} b(t)u^2(t)-\sum_{t=\alpha-1}^{\beta_0-1} c(t)x^2(t+1)$$

or

$$x(\beta_0)[u(\beta_0)+c(\beta_0-1)x(\beta_0)]-x(\alpha-1)u(\alpha-1)=\sum_{t=\alpha-1}^{\beta_0-1} b(t)u^2(t)-\sum_{t=\alpha-1}^{\beta_0-2} c(t)x^2(t+1). \quad (45)$$

We proceed to show that

$$x(\beta_0)[u(\beta_0)+c(\beta_0-1)x(\beta_0)]>0 \quad (46)$$

holds. Indeed, from the second equation of (10) we have, for $t=\beta_0-1$,

$$[1-a(\beta_0-1)]u(\beta_0-1)=u(\beta_0)+c(\beta_0-1)+x(\beta_0),$$

which upon multiplication by $x(\beta_0)$ yields

$$[1-a(\beta_0-1)]u(\beta_0-1)x(\beta_0)=x(\beta_0)[u(\beta_0)+c(\beta_0-1)x(\beta_0)]. \quad (47)$$

On the other hand, from the inequalities $x(\beta_0-1)x(\beta_0)<0$ and $x(\beta_0-1)u(\beta_0-1)<0$, it follows that $u(\beta_0-1)x(\beta_0)>0$. Therefore, (43) follows from (44).

By virtue of (40) and (43), from (42) the inequality

$$\sum_{t=\alpha-1}^{\beta_0-2} c(t)x^2(t+1)<\sum_{t=\alpha-1}^{\beta_0-1} b(t)u^2(t)$$

follows. In view of (41), the last estimate above yields

$$|u(s)|<\left\{\sum_{t=\alpha-1}^{\beta_0-2} c(t)\right\}^{1/2}\cdot\left\{\sum_{t=\alpha-1}^{\beta_0-1} b(t)u^2(t)\right\}^{1/2}+\sum_{t=\alpha-1}^{\beta_0-2}|a(t)|\cdot|u(t)|, \quad (48)$$

for all $s\in[a-1,\beta_0-1]$.

Choose $s_0\in[\alpha-1,\beta_0-1]$ such that

$$|u(s_0)|=\max_{\alpha-1\le s\le\beta_0-1}|u(s)|.$$

Then $|u(s_0)|>0$ and from (45), we have

$$|u(s_0)|<|u(s_0)|\cdot\left[\left\{\sum_{t=\alpha-1}^{\beta_0-2} c(t)\cdot\sum_{t=\alpha-1}^{\beta_0-1} b(t)\right\}^{1/2}+\sum_{t=\alpha-1}^{\beta_0-2}|a(t)|\right].$$

Hence, dividing by $|u(s_0)|$ we get

$$1<\left\{\sum_{t=\alpha-1}^{\beta_0-2} c(t)\cdot\sum_{t=\alpha-1}^{\beta_0-1} b(t)\right\}^{1/2}+\sum_{t=\alpha-1}^{\beta_0-2}|a(t)|.$$

Since $\beta_0\le\beta$, from the latter inequality follows inequality (20).

(II) Second, we consider the case when $x(t_0) = 0$ for some $t_0 \in [\alpha+1, \beta-2]$. In this case, applying Theorem (1.2.3) to the points t_0 and β, we get the inequality

$$\sum_{t=t_0}^{\beta-2} |a(t)| + \left\{ \sum_{t=t_0}^{\beta-2} b(t) \cdot \sum_{t=t_0}^{\beta-2} c(t) \right\}^{1/2} > 1.$$

Therefore, inequality (20) holds in this case as well.

Corollary (1.2.6)[23]: Suppose

$$1 - a(t) > 0, \quad b(t) > 0, \quad c(t) > 0, \qquad \text{for all } t \in \mathbb{Z},$$

and let $\alpha, \beta \in \mathbb{Z}$ with $\alpha \le \beta - 2$. Assume (10) has a real solution (x, u) such that x has generalized zeros at α and β, and x is not identically zero on $[\alpha, \beta]$. Then the inequality

$$\sum_{t=\alpha-1}^{\beta-2} |a(t)| + \left\{ \sum_{t=\alpha-1}^{\beta-1} b(t) \cdot \sum_{t=\alpha-1}^{\beta-2} c(t) \right\}^{1/2} > 1$$

holds.

Let $\alpha, \beta \in \mathbb{Z}$ with $\alpha \le \beta - 2$. Consider the discrete linear Hamiltonian system

$$\Delta x(t) = a(t)x(t+1) + b(t)u(t), \quad \Delta u(t) = -c(t)x(t+1) - a(t)u(t), \quad t \in [\alpha, \beta]. \quad (49)$$

We will assume that the coefficients $a(t), b(t)$, and $c(t)$ are real-valued functions defined on $[\alpha, \beta]$, and

$$1 - a(t) > 0, \qquad b(t) > 0, \qquad \text{for all } \ t \in [\alpha, \beta]. \qquad (50)$$

Note that each solution (x, u) of system (46) will be a vector-valued function defined on $[\alpha, \beta+1]$.

Now we define the concept of a relatively generalized zero for the component x of a real solution (x, u) of system (46) and also the concept of disconjugacy of this system on $[\alpha, \beta+1]$. The definition is relative to the interval $[\alpha, \beta+1]$ and the left end-point α is treated separately. We say x $[\text{or } (x, u)]$ has a relatively generalized zero at α if and only if $x(\alpha) = 0$, while we say x has a relatively generalized zero at $t_0 > \alpha$ provided either $x(t_0) = 0$ or $x(t_0 - 1)x(t_0) < 0$. Finally, we say that system (46) is disconjzlgute on $[\alpha, \beta+1]$ provided there is no real solution (x, u) of this system with x nontrivial and having two (or more) relatively generalized zeros in $[\alpha, \beta+1]$, Notice that under condition (47) above, given definitions of a relatively generalized zero and of disconjugucy are equivalent to those given in [30, 31].

Let us set $c_+(t) = \max\{c(t), 0\}$, that is the nonnegative part of $c(t)$.

Theorem (1.2.7)[23]: Assume condition (47) holds. If

$$\sum_{t=\alpha}^{\beta-1} |a(t)| + \left\{ \sum_{t=\alpha}^{\beta} b(t) \cdot \sum_{t=\alpha}^{\beta-1} c_+(t) \right\}^{1/2} \leq 1. \tag{51}$$

then (46) is disconjugate on $[\alpha, \beta + 1]$.

Proof. Suppose, on the contrary, that system (46) is not disconjugate on $[\alpha, \beta + 1]$. Then there exists (see [30, 31]) a real solution (x, u) of (46) with x nontrivial and such that $x(\alpha) = 0$ and that x has a generalized zero β_0 in $[\alpha + 1, \beta + 1]$. We will have $\beta_0 > \alpha + 1$ and either $x(\beta_0) = 0$ or $x(\beta_0 - 1)x(\beta_0) < 0$. Therefore, applying Theorems (1.2.2) and (1.2.3), we get

$$\sum_{t=\alpha}^{\beta_0-2} |a(t)| + \left\{ \sum_{t=\alpha}^{\beta_0-1} b(t) \cdot \sum_{t=\alpha}^{\beta_0-2} c_+(t) \right\}^{1/2} > 1$$

This contradicts condition (48) of the theorem.

Consider the discrete linear Hamiltonian system

$$\Delta x(t) = a(t)x(t+1) + b(t)u(t), \;\; \Delta u(t) = -c(t)x(t+1) - a(t)u(t), \;\; t \in \mathbb{Z}. \tag{52}$$

Let the coefficients $a(t), b(t)$, and $c(t)$ be. real-valued functions defined on $\mathbb{Z}$ and

$$1 - a(t) \neq 0, \quad \text{for all } t \in \mathbb{Z}. \tag{53}$$

In addition, we assume that the coefficients of (49) are periodic,

$$a(t+N) = a(t), \quad b(t+N) = b(t), \quad c(t+N) = c(t), \qquad t \in \mathbb{Z}, \tag{54}$$

where the $N \geq 2$ is fixed integer (period).

We first present some facts about the discrete Hamiltonian system (49) with periodic coefficients (51) that will be necessary for the subsequent discussions. Setting

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H(t) = \begin{bmatrix} c(t) & a(t) \\ a(t) & b(t) \end{bmatrix}, \quad \varphi(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad \varphi^\sigma(t) = \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix},$$

we can write system (49) in the vector form

$$\Delta\varphi(t) = JH(t)\varphi^\sigma(t), \qquad t \in \mathbb{Z}. \tag{55}$$

Let us seek a nonzero complex number ρ and a nontrivial solution φ of (52) such that

$$\varphi(t+N)=\rho\varphi(t), \qquad t\in\mathbb{Z}. \tag{56}$$

Denote by

$$\varphi_1(t)=\begin{bmatrix}x_1(t)\\u_1(t)\end{bmatrix} \quad \text{and} \quad \varphi_2(t)=\begin{bmatrix}x_2(t)\\u_2(t)\end{bmatrix}$$

the solutions of (52) under the initial conditions

$$x_1(0)=1,\quad u_1(0)=0; \qquad\qquad x_2(0)=0,\quad u_2(0)=1, \tag{57}$$

and set

$$\Phi(t)=[\varphi_1(t) \quad \varphi_2(t)]=\begin{bmatrix}x_1(t) & x_2(t)\\u_1(t) & u_2(t)\end{bmatrix} \tag{58}$$

Then

$$\Delta\Phi(t)=JH(t)\Phi^{\sigma}(t), \qquad\qquad t\in\mathbb{Z}, \tag{59}$$

$$\Phi(0)=I, \tag{60}$$

where

$$\Phi^{\sigma}(t)=\begin{bmatrix}x_1(t+1) & x_2(t+1)\\u_1(t) & u_2(t)\end{bmatrix}, \qquad I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

The general solution $\varphi(t)$ of (52) will have the form

$$\varphi(t)=c_1\varphi_1(t)+c_2\varphi_2(t)=\Phi(t)c, \tag{61}$$

where c_1, c_2 are arbitrary complex constants and c is a column vector with the components c_1 and c_2.

Substituting (58) in (53), we obtain

$$\Phi(t+N)c=\rho\Phi(t)c, \qquad t\in\mathbb{Z}. \tag{62}$$

On the other hand,

$$\Phi(t+N)=\Phi(t)\Phi(N), \qquad t\in\mathbb{Z}. \tag{63}$$

Indeed, because $H(t+N)=H(t)$, the left and right sides of (60) are solutions of the matrix system (56), and for $t=0$ they coincide. Then by the uniqueness of solution, equality (60) holds.

Since $\det\Phi(t)=1$, the matrix $\Phi(t)$ is invertible for all $t\in\mathbb{Z}$. Therefore, from (59) by (60) we get

$$\Phi(N)c=\rho c.$$

Thus, in order that the vector function cp defined by (58) be a nontrivial solution of (52) satisfying (53), it is necessary and sufficient that ρ be an eigenvalue and c be a corresponding eigenvector of the matrix $\Phi(N)$.

The matrix $\Phi(N)$ is called the *monodromy matrix* of system (49). The eigenvalues of the matrix $\Phi(N)$, i.e., the roots of the algebraic equation

$$\det[\Phi(N) - \rho I] = 0, \tag{64}$$

are called the multipliers of (49).

Equation (61) can be written as the quadratic equation

$$\rho^2 - D\rho + 1 = 0, \tag{65}$$

where

$$D = x_1(N) + u_2(N). \tag{66}$$

The roots of (62) are defined by

$$\rho_{1,2} = \frac{1}{2}\left(D \pm \sqrt{D^2 - 4}\right). \tag{67}$$

Definition (1.2.8)[23]: System (49) is said to be

(i) unstable if all nontrivial solutions are unbounded on $\mathbb{Z}$,

(ii) conditionally stable if there exists a nontrivial solution which is bounded on $\mathbb{Z}$, and

(iii) stable if all solutions are bounded on $\mathbb{Z}$.

Since the coefficient of (52) and the initial conditions (54) are real, the solutions $\varphi_1(t), \varphi_2(t)$, and hence, the number D defined by (63) will be real. The following statement can be proved in the standard way (see [24,25,32]).

Lemma (1.2.9)[23]: System (49) is unstable if $|D| > 2$, and stable if $|D| < 2$. If $|D| = 2$, then system (49) will be stable in the case $u_1(N) = x_2(N) = 0$, but conditionally stable and not stable otherwise.

Theorem (1.2.10)[23]: If

$$\text{(i)} \qquad 1 - a(t) > 0, \qquad b(t) \geq 0, \qquad c(t) \leq 0, \tag{68}$$

$$\text{(ii)} \qquad \prod_{t=1}^{N} \frac{1}{1 - a(t)} \geq 1, \qquad \prod_{t=1}^{N} \left[1 - a(t) - \frac{b(t)c(t)}{1 - a(t)}\right] > 1, \tag{69}$$

then system (49) is unstable.

Proof. Our aim is to show that under the hypotheses of the theorem, the inequalities

$$x_1(N) \geq 1, \qquad u_2(N) > 1 \tag{70}$$

hold. Then $D = x_1(N) + u_2(N) > 2$ will be obtained, and therefore, by Lemma (1.2.9), system (49) will be unstable.

The matrix system (56) can be written as

$$\Phi(t+1) = M(t)\Phi(t), \qquad t \in \mathbb{Z}, \tag{71}$$

where

$$M(t) = \begin{bmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1-a(t)} & \dfrac{b(t)}{1-a(t)} \\ \dfrac{c(t)}{1-a(t)} & 1-a(t)-\dfrac{b(t)c(t)}{1-a(t)} \end{bmatrix}. \tag{72}$$

Conditions (51), (65), and (66) are, respectively, equivalent to

$$m_{ij}(t+N) = m_{ij}(t), \qquad i,j = 1,2.$$

$$m_{11}(t) > 0, \qquad m_{12}(t) \geq 0, \qquad m_{21}(t) \geq 0, \qquad m_{22}(t) > 0$$

$$\prod_{t=1}^{N} m_{11}(t) \geq 1, \qquad \prod_{t=1}^{N} m_{22}(t) > 1. \tag{73}$$

From (68) taking into account (57), we have

$$\Phi(N) = M(N-1)M(N-2)\cdots M(0).$$

Hence, by (69) and (55), it follows that

$$x_1(N) \geq \prod_{t=0}^{N-1} m_{11}(t) = \prod_{t=1}^{N} m_{11}(t), \qquad u_2(N) \geq \prod_{t=0}^{N-1} m_{22}(t) = \prod_{t=1}^{N} m_{22}(t).$$

Consequently, by (70) inequalities (67) hold. The theorem is therefore proven.

Theorem (1.2.11)[23]: Suppose

(i) $\quad b(t) > 0,\ \ c(t) > 0,\ \ b(t)c(t) - a^2(t) \geq 0,\ \ b(t)c(t) - a^2(t) \not\equiv 0, \quad t \in \mathbb{Z}, \ (74)$

(ii) $$\sum_{t=1}^{N} |a(t)| + \left\{ \left[b_0 + \sum_{t=1}^{N} b(t) \right] \cdot \sum_{t=1}^{N} c(t) \right\}^{1/2} \leq 1, \tag{75}$$

where

$$b_0 = \max\{b(1), b(2), \dots, b(N)\}. \tag{76}$$

Then system (49) is stable.

Proof. It is sufficient by Lemma (1.2.9) to show that $D^2 < 4$. Assuming on the contrary that $D^2 \geq 4$ will lead to contradiction.

First, we prove the following lemma.

Lemma (1.2.12)[23]: If $D^2 \geq 4$, then system (49) has a real nontrivial solution (x, u) such that x has a generalized zero in $[1, N]$.

Proof. If $D^2 \geq 4$, then it follows from (64) that system (49) has a real nontrivial solution (x, u) such that

$$x(t+N) = \rho x(t), \quad u(t+N) = \rho u(t), \qquad t \in \mathbb{Z}, \tag{77}$$

where ρ is a nonzero real number. Now we show that $x(t)$ must have at least one generalized zero in the segment $[1, N]$. If not, then by (74), $x(t)$ does not have any generalized zero in $\mathbb{Z}$, so $x(t) \neq 0$ and $x(t-1)x(t) > 0$ for all t in $\mathbb{Z}$.

Multiplying the first equation of (49) by $u(t)$ and the second one by $x(t)$, and then subtracting, we get

$$u(t)\Delta x(t) - x(t)\Delta u(t) = a(t)x(t+1)u(t) + b(t)u^2(t) + c(t)x(t)x(t+1) + a(t)x(t)u(t).$$

Hence, substituting

$$x(t+1) = \frac{1}{1-a(t)}x(t) + \frac{b(t)}{1-a(t)}u(t)$$

in the right-hand side of the previous equation and then dividing both sides by $x(t)x(t+1)$, we obtain

$$-\Delta\left[\frac{u(t)}{x(t)}\right] = \frac{c(t)x^2(t) + [2a(t) - a^2(t) + b(t)c(t)]x(t)u(t) + b(t)u^2(t)}{[1+a(t)]x(t)x(t+1)}.$$

Summing the latter equation from 1 to N and taking into account that, by (74),

$$\frac{u(N+1)}{x(N+1)} - \frac{u(1)}{x(1)} = \frac{\rho u(1)}{\rho x(1)} - \frac{u(1)}{x(1)} = 0,$$

we get

$$\sum_{t=1}^{N} \frac{c(t)x^2(t) + [2a(t) - a^2(t) + b(t)c(t)]x(t)u(t) + b(t)u^2(t)}{[1-a(t)]x(t)x(t+1)} = 0 \tag{78}$$

From (72) and $b(t) > 0, c(t) > 0$, it follows that

$$|a(t)| < 1, \quad 0 < b(t)c(t) < 1, \qquad t \in \mathbb{Z}.$$

Therefore, $1-a(t)>0$ and the denominator of the fraction under the sum sign in (75) is positive. The numerator is equal to

$$cx^2+dxu+bu^2=\frac{1}{4b}\{(dx+2bu)^2+(4bc-d^2)x^2\},$$

where

$$d=2a-a^2+bc.$$

On the other hand,

$$4bc-d^2=\ (bc-a^2)[(2-a)^2-bc].$$

Hence, taking into account that

$$bc-a^2\geq 0,\qquad 2-a>1,\qquad 0<bc<1,$$

we get $4bc-d^2\geq 0$ and $4bc-d^2\not\equiv 0$. Therefore, the numerator of the fraction under the summation sign in (75) is nonnegative and not identically zero, since (x,u) is nontrivial. Consequently, equation (75) leads to a contradiction, and hence, the lemma is proven.

Let (x,u) be a solution of system (49) indicated in Lemma (1.2.12). So, $x(t)$ has at least one generalized zero α in $[1,N]$. From (74), we get that $x(t)$ will also have a generalized zero at $\alpha+N$. Applying Corollary (1.2.6) formulated in the Introduction to the solution (x,u) and the points a and $\beta=\alpha+N$, we get

$$\sum_{t=\alpha-1}^{\alpha+N-2}|a(t)|+\left\{\sum_{t=\alpha-1}^{\alpha+N-1}b(t)\cdot\sum_{t=\alpha-1}^{\alpha+N-2}\qquad\right\}^{1/2}>1. \tag{79}$$

Next, noticing that for any periodic function $f(t)$ on $\mathbb{Z}$ with period N the equality

$$\sum_{t=t_0}^{t_0+N-1}f(t)=\sum_{t=1}^{N}f(t)$$

holds for all $t_0\in\mathbb{Z}$, we have

$$\sum_{t=\alpha-1}^{\alpha+N-2}|a(t)|=\sum_{t=1}^{N}|a(t)|,\qquad \sum_{t=\alpha-1}^{\alpha+N-2}c(t)=\sum_{t=1}^{N}c(t)$$

$$\sum_{t=\alpha-1}^{\alpha+N-1}b(t)=b(\alpha-1)+\sum_{t=\alpha-1}^{\alpha+N-1}b(t)=b(\alpha-1)+\sum_{t=1}^{N}b(t)\leq b_0+\sum_{t=1}^{N}b(t)$$

where b_0 is defined by (73). Consequently, it follows from (76) that

$$\sum_{t=1}^{N}|a(t)| + \left\{\left[b_0 + \sum_{t=1}^{N} b(t)\right] \cdot \sum_{t=1}^{N} c(t)\right\}^{1/2} > 1.$$

This last inequality contradicts condition (72). Therefore, the inequality $D^2 \geq 4$ cannot be true. Thus, $D^2 < 4$ and system (49) is stable. The theorem is thus proven.

Consider the continuous linear Hamiltonian system (7) in which $a(t), b(t)$, and $c(t)$ are real-valued piece-wise continuous functions defined on $\mathbb{R}$ and periodic with a period $\omega > 0$,

$$a(t+\omega) = a(t), \quad b(t+\omega) = b(t), \quad c(t+\omega) = c(t), \qquad t \in \mathbb{R}.$$

Applying Theorem (1.2.1), we can prove the following statement: if

$$\text{(i)} \qquad b(t) > 0, \quad c(t) \geq 0, \quad b(t)c(t) - a^2(t) \geq 0, \qquad t \in \mathbb{R}, \tag{80}$$

$$\text{(ii)} \qquad b(t)c(t) - a^2(t) \not\equiv 0, \tag{81}$$

$$\text{(iii)} \qquad \int_0^{\omega} |a(t)|dt + \left\{\int_0^{\omega} b(t)dt \cdot \int_0^{\omega} c(t)dt\right\}^{1/2} < 2, \tag{82}$$

then system (7) is stable.

Indeed, if system (7) is not stable, then this system has a real nontrivial solution (x, u) such that

$$x(t+\omega) = \rho x(t), \quad u(t+\omega) = \rho u(t), \qquad t \in \mathbb{R}, \tag{83}$$

where ρ is a real nonzero number. Now we show that $x(t)$ must have at least one zero in the segment $[0, \omega]$. If not, then by (80), $x(t) \neq 0$ for all $t \in \mathbb{R}$. Multiplying the first equation of (7) by $u(t)$ and the second one by $x(t)$, and then subtracting, we get

$$u(t)x'(t) - x(t)u'(t) = c(t)x^2(t) + 2a(t)x(t)u(t) + b(t)u^2(t).$$

Hence, dividing both sides by $x^2(t)$, w e obtain

$$-\left[\frac{u(t)}{x(t)}\right]' = \frac{c(t)x^2(t) + 2a(t)x(t)u(t) + b(t)u^2(t)}{x^2(t)}.$$

Integrating the latter equation from 0 to ω and taking into account that, by (80),

$$\frac{u(\omega)}{x(\omega)} - \frac{u(0)}{x(0)} = \frac{\rho u(0)}{\rho x(0)} - \frac{u(0)}{x(0)} = 0,$$

we get

$$\int_0^{\omega} \frac{c(t)x^2(t) + 2a(t)x(t)u(t) + b(t)u^2(t)}{x^2(t)} dt = 0. \tag{84}$$

On the other hand,

$$cx^2 + 2axu + bu^2 = \frac{1}{b}\{(ax + bu)^2 + (bc - a^2)x^2\}.$$

Therefore, equality (81) cannot be true, since (x, u) is nontrivial.

Thus, $x(t)$ has at least one zero α in $[0, \omega]$. From (80), we get that $x(t)$ will have a zero also at $\alpha + \omega$. Applying Theorem (1.2.1) to the points α and $\beta = \alpha + \omega$, and taking into account the fact that for any periodic function $f(t)$ on $\mathbb{R}$ with period ω, the equality

$$\int_{t_0}^{t_0+\omega} f(t)dt = \int_0^{\omega} f(t)dt$$

holds for all $t_0 \in \mathbb{R}$, we obtain

$$\int_0^{\omega} |a(t)|dt + \left\{\int_0^{\omega} b(t)dt \cdot \int_0^{\omega} c(t)dt\right\}^{1/2} \geq 2.$$

The latter inequality contradicts condition (79). Therefore, system (7) must be stable under conditions (77)-(79).

Consider the discrete linear Hamiltonian system with constant coefficients

$$\Delta x(t) = ax(t+1) + bu(t), \ \ \Delta u(t) = -cx(t+1) - au(t), \qquad t \in \mathbb{Z}, \tag{85}$$

where a, b, and c are real constants and $1 - a \neq 0$. This case corresponds to the value $N = 1$ of the period.

System (82) can be written as the vector equation

$$\varphi(t+1) = M\varphi(t), \qquad t \in \mathbb{Z}, \tag{86}$$

where

$$\varphi(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad M = \begin{bmatrix} \dfrac{1}{1-a} & \dfrac{b}{1-a} \\ -\dfrac{c}{1-a} & 1 - a - \dfrac{bc}{1-a} \end{bmatrix}.$$

It follows from (83) that

$$\varphi(t) = M^t \varphi(0), \qquad t \in \mathbb{Z}.$$

Therefore, if the matrix M has two distinct eigenvalues with the modulus 1, then all solutions of system (82) will be bounded on $\mathbb{Z}$.

The eigenvalues of the matrix M are defined as the roots of the quadratic equation

$$\lambda^2 + A\lambda + 1 = 0, \tag{87}$$

where

$$A = \frac{bc - a^2}{1 - a} - 2. \tag{88}$$

Since the roots of (84) are defined by

$$\lambda_{1,2} = \frac{1}{2}\left(-A \mp \sqrt{A^2 - 4}\right),$$

the matrix M will have two distinct eigenvalues with the modulus 1, provided

$$-2 < A < 2. \tag{89}$$

From (85), it follows that (86) will be satisfied, if

$$bc - a^2 > 0 \qquad \text{and} \qquad |a| + \sqrt{bc} < 2. \tag{90}$$

So, if conditions (87) are satisfied, then system (82) is stable. This result is better than the result given by Theorem (1.2.11).

Systems (7) and (10), in general, may have a nontrivial solution (x, u) such that x is identically zero on any interval. Indeed, for

$$a(t) = -1, \quad c(t) = 0, \quad u(t) = e^t \qquad (t \in \mathbb{R}),$$

$$b(t) = \begin{cases} 2(t-\beta)e^{-2t}, & t > \beta, \\ 0, & 0 \le t \le \beta, \\ 2te^{-2t}, & t < 0, \end{cases} \qquad x(t) = \begin{cases} (t-\beta)^2 e^{-t}, & t > \beta, \\ 0, & 0 \le t \le \beta, \\ t^2 e^{-t}, & t < 0, \end{cases}$$

where β is an arbitrary positive real number, system (7) will be satisfied. The statement of Theorem (1.2.1) is not true for the solution (x, u) and the points 0 and β, if $\beta < 2$.

Also, for

$$a(t) = -1, \quad c(t) = 0, \quad u(t) = 2^t \qquad (t \in \mathbb{Z}),$$

$$b(t) = \begin{cases} 2t, & t \ge \beta + 1, \\ 2^{-\beta+1}, & t = \beta, \\ 0, & 0 \le t \le \beta - 1, \\ -2, & t = -1, \\ 2^{-t}, & t \le -2, \end{cases} \qquad x(t) = \begin{cases} 0, & 0 \le t \le \beta, \\ 1, & t \in \mathbb{Z} - [0, \beta], \end{cases}$$

where $\beta > 1$ is any integer, system (10) will be satisfied. The statement of Theorem (1.2.1) is not true for the solution (x, u) and the points 0 and β, if $\beta = 2$.

However, if $b(t) \neq 0$ for all t, then for any nontrivial solution (x, u) the component x cannot be identically zero on any interval containing two or more points.

Section (1.3): Lyapunov's Inequality on Time Scales:

The classical Lyapunov inequality [34] is stated as follows:

Theorem (1.3.1)[33]: *Let $p \in C\big([c,d],[0,\infty)\big)$. If $y(t)$ is a nontrivial real-valued solution $y(t)$ of*

$$y''(t) + p(t)y(t) = 0$$

satisfying $y(c) = y(d) = 0$, then

$$(c-d)\int_c^d p(s)ds > 4$$

and the constant 4 cannot be replaced by a larger number.

Many authors have extended and improved this distinguished inequality; see, for example, [35, 36].

Recently, Bohner et al. [37] extended Theorem A on a time scale and obtained the following:

Theorem (1.3.2)[33]. *Let $p \in C_{rd}[\mathbb{T},(0,\infty)]$ and $a, b \in \mathbb{T}$ with $a < b$, where $\mathbb{T}$ is a timescale. If $x(t)$ is a nontrivial solution*

$$x^{\Delta^2} + p(t)x^{\sigma} = 0$$

satisfying $x(a) = x(b) = 0$, then

$$\int_a^b p(t)\Delta t \geq \frac{b-a}{f(d)},$$

where $f(t) = (t-a)(b-t)$ for $t \in \mathbb{T}$ and $d \in \mathbb{T}$ satisfies

$$\left|\frac{a-b}{2} - d\right| = \min\left\{\left|\frac{a-b}{2} - s\right| \,\middle|\, s \in [a,b] \cap \mathbb{T}\right\}.$$

The purpose of this work is to generalize Theorem (1.3.2) to a time scale version[38],[39],[40],[34].

Definition (1.3.3)[33]. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Throughout this section we assume that $\mathbb{T}$ is a time scale and $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, if $t < \sup \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf \{\tau > t : \tau \in \mathbb{T}\} \in \mathbb{T},$$

while if $t > \inf \mathbb{T}$, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup \{\tau < t : \tau \in \mathbb{T}\} \in \mathbb{T}.$$

If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$, we say t is left scattered. If $\sigma(t) = t$, we say t is right dense, while if $\rho(t) = t$, we say t is left dense.

The mapping $\mu : \mathbb{T} \to [0, \infty)$ given by

$$\mu(t) = \sigma(t) - t$$

is called the graininess.

Definition (1.3.4)[33]. If $f : T \to \mathbb{R}$, then $f^{\sigma} : \mathbb{T} \to R$ is defined by

$$f^{\sigma}(t) = f\big(\sigma(t)\big)$$

for all $t \in \mathbb{T}$.

Definition (1.3.5)[33]. Let $a, b \in \mathbb{R}$ with $a < b$. Define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} \text{ such that } a \le t \le b\}.$$

Other types of interval are defined similarly.

Definition (1.3.6)[33]. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it satisfies:

(A) f is continuous at each right-dense or maximal element $t \in \mathbb{T}$,

(B) the left-sided limit $\lim_{s \to t^-} f(s) = f(t^-)$ exists at each left-dense point $t \in \mathbb{T}$.

Let us have

$$C_{rd}(\mathbb{T}, \mathbb{R}) = \{f | f : T \to \mathbb{R} \text{ is a rd-continuous function}\}.$$

Definition (1.3.7)[33]. The function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided that

$$1 + \mu(t)p(t) \neq 0 \qquad \text{for each } t \in \mathbb{T}$$

holds.

Definition (1.3.8)[33]. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$; then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left|\left[x(\sigma(t)) - x(s)\right] - x^{\Delta}(t)[\sigma(t) - s]\right| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the **delta derivative** of $x(t)$.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$\frac{x^{\Delta}(t) = x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Definition (1.3.9)[33]. A function $F : \mathbb{T}^{\kappa} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. In this case, we define the integral of f by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$$

for $s, t \in \mathbb{T}^{\kappa}$, where

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} - \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Throughout this section, we suppose that:

(a) $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^{+} = [0, +\infty)$;

(b) $\mathbb{T}$ is a timescale;

(c) $\Re = \{p : \mathbb{T} \to \mathbb{R} \mid 1 + \mu(t)p(t) \neq 0 \text{ for each } t \in \mathbb{T}\}$, that is, the set $\Re$ is all regressive functions on $\mathbb{T}$;

(d) all interval means the intersection of a real interval with the given time scale;

(e) $e_p(t;\ a)$ is the solution of the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \qquad y(a) = 1$$

with $p \in \Re$.

In order to discuss our main results, we need the following three lemmas:

Lemma (1.3.10)[33]. *Let x be a solution of*

$$(rx^{\Delta})^{\Delta} + p(t)x^{\sigma}(t) = 0 \text{ on } [a, b], \tag{91}$$

where $r, p \in C_{rd}([a, b], \mathbb{R})$ with $r > 0$ on $[a, b]$. Then

$$F(y) - F(x) - F(y - x) = 2(y - x)(b)r(b)(b) - 2(y - x)(a)r(a)x^{\Delta}(a)$$

for any $y : [a, b] \to \mathbb{R}$, where

$$F(y)=\int_a^b \left\{\left(\sqrt{r}y^\Delta\right)^2-p(y^\sigma)^2\right\}(t)\Delta t\,.$$

Proof.

$$\begin{aligned}
&F(y)-F(x)-F(y-x)\\
&\quad=\int_a^b \Big[\left(\sqrt{r}y^\Delta\right)^2-p(y^\sigma)^2-\left(\sqrt{r}x^\Delta\right)^2+p(x^\sigma)^2\\
&\qquad-\left(\sqrt{r}(y-x)^\Delta\right)^2+p(y^\sigma-x^\sigma)^2\Big](t)\Delta t\\
&\quad=2\int_a^b \left\{rx^\Delta y^\Delta-px^\sigma y^\sigma+p(x^\sigma)^2-\left(\sqrt{r}x^\Delta\right)^2\right\}(t)\Delta t\\
&\quad=2\int_a^b \{(rx^\Delta)y^\Delta+(rx^\Delta)^\Delta y^\sigma-(rx^\Delta)^\Delta x^\sigma-(rx^\Delta)x^\Delta\}(t)\Delta t\\
&\quad=2\int_a^b \{(rx^\Delta)y-(rx^\Delta)x\}^\Delta(t)\Delta t\\
&\quad=2\int_a^b \{(y-x)(rx^\Delta)\}^\Delta(t)\Delta t\\
&\quad=2\big(y(b)-x(b)\big)\left(r(b)x^\Delta(b)\right)-2\big(y(a)-x(a)\big)\left(r(a)x^\Delta(a)\right).
\end{aligned}$$

This completes our proof.

By using the mean-value theorem [see [39,40]], we obtain the following:

Lemma (1.3.11)[33]. *Let $c\in\mathbb{R}$ be given. Then, for each $x\in\mathbb{R}$, there exists $ak\in\mathbb{R}$ (depending on c and x) such that*

$$x^2-c^2=2k(x-c)\geq 2c(x-c),$$

where k lies between c and x.

Lemma (1.3.12)[33]. *Under the assumption of Lemma* (1.3.10)*,*

$$\int_a^b r(t)\left(\left|x^\Delta(t)\right|\right)^2\Delta t\geq\frac{\left(\int_a^b r(t)\left|x^\Delta(t)\right|\Delta t\right)^2}{\int_a^b r(t)\Delta t}.$$

Proof. Let

$$c=\int_a^b \frac{r(t)\left|x^\Delta(t)\right|\Delta t}{\int_a^b r(t)\Delta t}.$$

Then, by Lemma (1.3.11),

$$\int_a^b r(t)\left|x^\Delta(t)\right|^2\Delta t - c^2\int_a^b r(t)\Delta t$$

$$=\int_a^b r(t)\left(\left|x^\Delta(t)\right|^2-c^2\right)\Delta t$$

$$\geq 2c\int_a^b r(t)\left(\left|x^\Delta(t)\right|-c\right)\Delta t$$

$$=2c\int_a^b r(t)\left|x^\Delta(t)\right|\Delta t - c\int_a^b r(t)\Delta t$$

$$=2c\left\{\int_a^b r(t)\left|x^\Delta(t)\right|\Delta t-\int_a^b r(t)\left|x^\Delta(t)\right|\Delta t\right\}=0,$$

and therefore we obtain the desired result.

We now can state and prove the main results as follows.

Theorem (1.3.13)[33]. *Let $x(t)$ be a nontrivial solution of* (88) *satisfying $x(a)=x(b)=0$ and $r,p\in C_{rd}([a,b],\mathbb{R})$ with $r>0$ on $[a,b]$. If $r(t)$ is monotone on $[a,b]$ and $\frac{a+b}{2}\in\mathbb{T}$, then*

$$\int_a^b p_+(t)\Delta t \geq \begin{cases}\dfrac{r(a)}{r(b)}\dfrac{b-a}{f(d)}, & if\ r\ is\ increasing,\\ \dfrac{r(b)}{r(a)}\dfrac{b-a}{f(d)}, & if\ r\ is\ decreasing,\end{cases}$$

where $p_+(t)=\max\{p(t),0\}$, $f(t)=(t-a)(b-t)$ and $d\in\mathbb{T}$ satisfies

$$\left|\frac{a+b}{2}-d\right|=\min\left\{\left|\frac{a+b}{2}-s\right| \mid s\in[a,b]\right\}.$$

Proof. Taking $y=0$ in Lemma (1.3.10),

$$F(0)-F(x)-F(0-x)=0.$$

Hence $F(x)=0$, that is,

$$\int_a^b p(t)\left(x^\sigma(t)\right)^2\Delta t = \int_a^b r(t)\left(x^\Delta(t)\right)^2\Delta t.$$

Thus, if $x^2(c) = \max_{a\le t\le b} x^2(t)$, then

$$x^2(c)\int_a^b p_+(t)\Delta t \ge \int_a^b r(t)\left(x^\Delta(t)\right)^2\Delta t$$

$$= \int_a^c r(t)\left(x^\Delta(t)\right)^2\Delta t + \int_c^b r(t)\left(x^\Delta(t)\right)^2\Delta t$$

$$\ge \frac{\left(\int_a^c r(t)\left|x^\Delta(t)\right|\Delta t\right)^2}{\int_a^c r(t)\Delta t} + \frac{\left(\int_c^b r(t)\left|x^\Delta(t)\right|\Delta t\right)^2}{\int_c^b r(t)\Delta t} \ \text{(by Lemma (1.3.12))}$$

$$\ge \begin{cases} \dfrac{r(a)\left(\int_a^c\left|x^\Delta(t)\right|\Delta t\right)^2}{\int_a^c r(t)\Delta t} + \dfrac{r(a)\left(\int_c^b\left|x^\Delta(t)\right|\Delta t\right)^2}{\int_c^b r(t)\Delta t}, & \text{if } r \text{ is increasing} \\ \dfrac{r(b)\left(\int_a^c\left|x^\Delta(t)\right|\Delta t\right)^2}{\int_a^c r(t)\Delta t} + \dfrac{r(b)\left(\int_c^b\left|x^\Delta(t)\right|\Delta t\right)^2}{\int_c^b r(t)\Delta t}, & \text{if } r \text{ is decreasing} \end{cases}$$

$$\ge \begin{cases} r(a)x^2(c)\left\{\dfrac{1}{r(b)\int_a^c 1\Delta t} + \dfrac{1}{r(b)\int_c^b 1\Delta t}\right\}, & \text{if } r \text{ is increasing} \\ r(b)x^2(c)\left\{\dfrac{1}{r(a)\int_a^c 1\Delta t} + \dfrac{1}{r(a)\int_c^b 1\Delta t}\right\}, & \text{if } r \text{ is decreasing} \end{cases}$$

$$\ge \begin{cases} \dfrac{r(a)}{r(b)}x^2(c)\left\{\dfrac{1}{c-a} + \dfrac{1}{b-c}\right\} = \dfrac{r(a)}{r(b)}\dfrac{b-a}{f(c)}x^2(c) \ge \dfrac{r(a)}{r(b)}\dfrac{b-a}{f(d)}x^2(c), & \text{if } r \text{ is increasing} \\ \dfrac{r(b)}{r(a)}x^2(c)\left\{\dfrac{1}{c-a} + \dfrac{1}{b-c}\right\} = \dfrac{r(b)}{r(a)}\dfrac{b-a}{f(c)}x^2(c) \ge \dfrac{r(b)}{r(a)}\dfrac{b-a}{f(d)}x^2(c), & \text{if } r \text{ is decreasing.} \end{cases}$$

Therefore, we obtain the desired result.

Corollary (1.3.14)[33]. *Let $g, h \in C_{rd}([a,b],\mathbb{R})$ and let $g \in \mathfrak{R}$ not change sign on $[a,b]$. Suppose that*

$$x^{\Delta^2}(t) + g(t)\left(x^\Delta(t)\right)^\sigma + h(t)x^\sigma(t) = 0 \ on\ [a,b] \tag{92}$$

has a nontrivial solution $x(t)$ with two consecutive zeros a and b. Then

$$\int_a^b e_{g_+}(b;a)h_+(t)\Delta t \ge k_g(a;b)\frac{b-a}{f(d)},$$

or

$$\int_a^b e_{g_+}(b;a)h_+(t)\Delta t \geq k_g(a;b)\frac{4}{b-a},$$

where d *is defined as in Theorem* (1.3.13) *and*

$$k_g(a;b) \coloneqq \begin{cases} e_g(b;a), & if\ g \geq 0\ on\ [a,b], \\ e_g(a;b), & if\ g \leq 0\ on\ [a,b]. \end{cases}$$

Proof. Letting $r(t) = e_g(t;a)$ (i.e., $r^\Delta(t) = r(t)g(t)$ and $r(a) = e_g(a;a) = 1$) and $p(t) = r(t)h(t)$, we see that, by (89),

$$\begin{aligned}\left(r(t)x^\Delta(t)\right)^\Delta + p(t)x^\sigma(t) &= r^\Delta(t)\left(x^\Delta(t)\right)^\sigma + r(t)\left(x^{\Delta^2}(t)\right) + r(t)h(t)x^\sigma(t) \\ &= r(t)\left\{x^{\Delta^2}(t) + g(t)\left(x^\Delta(t)\right)^\sigma + h(t)x^\sigma(t)\right\} = 0.\end{aligned}$$

Hence, if follows from Theorem (1.3.13) and Theorem (1.3.6) of [12] that

$$\int_a^b e_g + (t,a)h_+(t)\Delta t \geq \begin{cases} \dfrac{e_g(b;a)}{e_g(a;a)}\dfrac{b-a}{f(d)}, & \text{if } g \geq 0 \text{ (i.e.}, r \text{ is increasing)} \\ \dfrac{e_g(a;a)}{e_g(b;a)}\dfrac{b-a}{f(d)}, & \text{if } g \leq 0 \text{ (i.e.}, r \text{ is decreasing)} \end{cases}$$

which completes the proof.