Chapter 6

Transfer Principle with Measure invariance and Laplace Transform Identities

A new decomposition of the Ornstein-Uhlenbeck operator and a substructure of the standard Dirichlet structure on Wiener space, with applications to stochastic analysis on Poisson space and infinite-dimensional analysis for the exponential density are shown. In particular we show the measure invariance of transformations having a quasi-nilpotent covariant derivative via a Girsanov identity and an explicit formula for the expectation of Hermite polynomials in the Skorohod integral on path space.

Section (6.1): Wiener and Poisson Space:

The stochastic calculus of variations on the Wiener space, cf. [164], [165], makes use of the following ingredients: a gradient operator, its adjoint the divergence operator, and the Ornstein-Uhlenbeck operator which is obtained as the composition of the divergence with the gradient. The Ornstein-Uhlenbeck operator is a number operator on the Wiener chaotic decomposition and it allows to define Sobolev spaces and distributions on the Wiener space, cf. [166]. On the other hand, the connection with the Itô calculus is obtained via the divergence operator which extends the Itô integral,cf. [167]. An important tool in this analysis is the Meyer inequalities, cf. [168] which give equivalence between the norms defined with the gradient and the norms defined on Sobolev spaces with the Ornstein-Uhlenbeck operator. The question whether an analogous formalism exists on Poisson space has been investigated in e.g. [169],[170], [171]. In [170], a Fock space isomorphism using the Poisson and Wiener multiple stochastic integral is considered. This leads to a gradient defined by finite differences, which is not a derivation operator, and whose adjoint coincides with the compensated Poisson stochastic integral on square-integrable predictable processes. However, this isomorphism is not an isometry for the L^p norm, except for $p = 2$, and apparently it does not allow to transpose to the Poisson space case the analysis constructed on the Wiener space, in particular for $p \neq 2$. Another approach, initiated in [169] is to define a gradient by shifting the jump times of a standard Poisson process on the positive real line. The adjoint of this operator also extends the compensated Poisson stochastic integral. It has been shown in [172], [171] that there is a discrete chaotic decomposition on Poisson space on which the composition of this gradient with its adjoint acts as a number operator noted $\mathcal{L}$. In this approach, the trajectories of the Poisson process are considered as sequences of independent identically distributed exponential interjump times.

We consider the σ-algebra $\mathcal{F}$generated by a countable collection of independent identically distributed exponential random variables on the Wiener space, and call a Poisson functional any Wiener functional which is measurable with respect to $\mathcal{F}$. The Ornstein-Uhlenbeck operatoron the Wiener space appears to be an extension of the number operator $\mathcal{L}$defined in[171] for Poisson functionals. We deduce results in infinite dimensional analysis for the exponential density, such as the hypercontractivity of the semigroup associated to$\mathcal{L}$, the construction of distributions, and an algebra of test functions on the Poissonspace. We introduce a random unitary operator χ of the Cameron-Martinspace which allows to define a new gradient on Wiener space by composition with the Gross-Sobolev derivative. This gradient is related to the conditional gradient given $\mathcal{F}$on the Wiener space and to the derivative obtained by shifting the Poisson process jump times, and its adjoint extends the compensated Poisson stochastic integral. Several results in Malliavin calculus concerning the existence and smoothness of densities of Poisson functionals, as well as the Meyer inequalities, are derived on the Poisson space using the operators that are defined above. We devoted to the extension to higher orders of differentiation of the equivalence of norms obtained. We obtain in this way the continuity of the gradient and divergence operators on Sobolev spaces of Poisson functionals. We deal with the independence of Poisson functionals. From the existing criterion on Wiener space, cf. [173], we deduce necessary and sufficient conditions for the independence of discrete multiple Poisson stochastic integrals. Those integrals are defined with the Laguerre polynomials as stochastic integrals of deterministic discrete time kernels, and the conditions for independence are expressed in terms of the supports of those kernels. We study the infinite-dimensional diffusion process associated to the operator $\mathcal{L}$ and show that it gives another example of a process whose hitting probabilities can be estimated in terms of capacities.

The following definitions can be found in [166]. Let $(W; L^2(\mathbb{R}_+); \mu)$ be the classical Wiener space, and let $(h_k)_{k\geq 0}$ be an orthonormal basis of $L^2(\mathbb{R}_+)$, which will remain fixed throughout this work. We note respectively $\widehat{D}$and $\hat{\delta}$ the Gross-Sobolev derivative and its adjoint on the Wiener space. Recall that $\left(\hat{\delta}(h_k)\right)_{k\in\mathbb{N}}$is a system of independent gaussian normal random variables, and for $F = f\left(\hat{\delta}(h_0), \dots, \hat{\delta}(h_n)\right), f \in C_c^{\infty}(\mathbb{R}^{n+1}), \widehat{D}F \in L^2(W) \otimes L^2(\mathbb{R}_+)$ is defined as

$$\widehat{D}F = \sum_{k=0}^{k=n} \partial_k f\left(\hat{\delta}(h_0), \dots, \hat{\delta}(h_n)\right) h_k \,.$$

The Ornstein-Uhlenbeck operator on the Wiener space is denoted by $-\hat{\mathcal{L}} = 2$. It is self adjoint with respect to $\mu$and satisfies to $\hat{\mathcal{L}} = \hat{\delta}\widehat{D}$. Let $\hat{I}_n(g_n)$ represent the Wiener multiple stochastic integral of a symmetric function in the completed symmetric tensor product $L^2(\mathbb{R}_+)^{on}$. We have $\hat{\mathcal{L}}\hat{I}_n(g_n) = n\hat{I}_n(g_n) n \in \mathbb{N}$, and any square integrable functional Fon (W, μ) can be decomposed as a series

$$F = \sum_{n=0}^{\infty} \hat{I}_n(g_n) g_k \in L^2(\mathbb{R}_+)^{\text{o}k}, \qquad k \in \mathbb{N}\,.$$

Let P denote the algebra of polynomials in $\left(\hat{\delta}(h_k)\right)_{k\geq 0}$, which is dense in $L^2(W, \mu)$.

For $k \in \mathbb{N}$ and $p > 1$, let $\mathbb{D}_{p,k}$ be the completion of $\mathcal{P}$under the norm $\|F\|_{p,k} = \left\|\left(I + \hat{\mathcal{L}}\right)^{\frac{k}{2}} F\right\|_{L^p(W,\mu)}$, and let $\mathbb{D}_{p,-k}$be the dual space of $\mathbb{D}_{p,k}$. Let $\mathbb{D}_\infty = \bigcap_{p,k} \mathbb{D}_{p,k}$.The dual of $\mathbb{D}_\infty$is $\mathbb{D}_\infty = \bigcup_{p,k} \mathbb{D}_{p,k}$.

We shortly describe the method that will be used in the next sections. Let us write down the usual integration by parts formula on Wiener space:

$$E\big[F\hat{\delta}(u)\big] = E\left[\left(\widehat{D}F, u\right)_{L^2(\mathbb{R}_+)}\right],$$

for $u \in Dom(\hat{\delta})$ and $F \in Dom(\widehat{D})$. Consider also a random operator

$$\chi : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

such that χ is unitary,μ-a.s. This operator can be viewed as an isometry from $L^2(W) \otimes L^2(\mathbb{R}_+)$ to$L^2(W) \otimes L^2(\mathbb{R}_+)$. Let us apply the above integration by partsto F and χu, with $F \in Dom(\widehat{D})$ and $u \in L^2(W)L^2(\mathbb{R}_+)$ such that $\chi u \in Dom(\hat{\delta})$.

We have from the properties of χ:

$$\begin{aligned} E\left[F\hat{\delta} \circ \chi(u)\right] &= E\left[\left(\widehat{D}\, F, \chi u\right)_{L^2(\mathbb{R}_+)}\right] \\ &= E\left[\left(\chi^* \circ \widehat{D}F, u\right)L^2(\mathbb{R}_+)\right], \end{aligned}$$

χ^*being the adjoint of χ. We will show that it is possible to choose χsuch that $\widehat{\delta} \circ \chi$ extends the stochastic integral with respect to a compensated Poisson process defined on the Wiener space. It will appear that $\chi^* \circ \widehat{D}$ is closely related to a gradient defined on Poisson space by shifts of the Poisson process jump times, cf. [169], [171]. Moreover, we have

$$\left(\hat{\delta} \circ \chi\right) \circ \left(\chi^* \circ \widehat{D}\right) = \hat{\delta}\widehat{D} = \hat{\mathcal{L}},$$

and

$$\left\|\chi^* \circ \widehat{D}F\right\|_{L^2(\mathbb{R}_+)} = \left\|\widehat{D}F\right\|_{L^2(\mathbb{R}_+)} \mu - a.s.$$

from the fact that χ is a.s. unitary. As a consequence, any result in Malliavin calculus that involves the norm of the gradient $\widehat{D}$or the Ornstein-Uhlenbeck operator$\hat{\mathcal{L}}$will be valid on Poisson space and interpreted in terms of the stochastic calculus of variations for the Poisson process, using the compensated Poisson stochastic integral and the derivation with respect to shifts of the jump times.

A characterization of the standard Poisson process on the positive real line is that it is a jump process with jumps of fixed size 1 and independent identically distributed exponential interjump times. We intend here to construct a Poisson process, or equivalently countable collection of exponential random variables on the Wiener space. We will make use of the fact that the half sum of two independent normal random variables has a χ^2 law with 2 degrees of freedom, i.e. an exponential distribution.

Let

$$\tau_k = \frac{\hat{\delta}(h_{2k})^2 + \hat{\delta}(h_{2k+1})^2}{2} \qquad k \geq 0,$$

then $(\tau_k)_{k\geq 0}$ is a family of independent exponentially distributed random variables,hence it represents a Poisson process $(N_t)_{t\geq 0}$. This does not require the system$(h_k)_{k\geq 0}$to be complete in $L^2(\mathbb{R}_+)$. Let $T_k = \sum_{i=0}^{i=k-1} \tau_i$, $k \geq 0$, represent the k-th jump time of $(N_t)_{\geq 0}t$. We have

$$N_t = \sum_{k=1}^{\infty} 1_{[T_k,\infty[}(t)\,, \qquad\qquad t \in \mathbb{R}_+.$$

Note that this construction does not preserve the filtrations generated by the Poisson and Wiener processes, i.e. the filtrations generated by $(N_t)_{t\geq 0}$ and the Brownian motion $(B_t)_{t\geq 0}$on (W, μ) are not comparable. We define an application $\Xi : W \to \mathbb{R}^N$by

$$\Xi(\omega) = (\tau_k)_{k\in\mathbb{N}}\, \mu - a.s. \qquad (1)$$

Denote by Bthe range of Ξ, endowed with the largest σ-algebra that makes Ξmeasurable, and let P be the image measure of μ by Ξ:

$$P = \Xi_* \mu$$

and define an operator $\Theta : L^p(B, P) \to L^p(W, \mu)$ by

$$\Theta F = F \circ \Xi$$

where F is a polynomial functional on B, i.e. $F\big((x_k)k \in \mathbb{N}\big) = f(x_0, \ldots, x_n), n \in \mathbb{N}, f$polynomial. The operator Θ can be extended as an isometry from $L^p(B, P)$to$L^p(W, \mu), p > 1$. The dual of$\Theta : L^2(B, P) \to L^2(W, \mu)$is $\Theta^* : L^2(W, \mu) \to L^2(B, P)$,given by

$$\Theta^* F = \Theta^{-1} E[F|F], \qquad F \in L^2(W, \mu).$$

We call $\mathcal{F}$ the σ-algebra on W generated by Ξ. In the sequel, $L^p\big(W, \mathcal{F}, \mu_{|F}\big)$ will be identified with $L^p(B)$ for $p \geq 1$.

Definition (6.1.1)[163]:(Poisson space). The space $(W, \mathcal{F}, \mu_{|F})$ is called the Poisson space.We call a Poisson functional any random variable on $(W, \mathcal{F}, \mu_{|F})$. Let

$$\mathcal{P}_{\mathcal{F}} = \{f(\tau_0, \dots, \tau_n) : f \ polynomial, \ \ n \in \mathbb{N}\}$$

denote the set of polynomial Poisson functionals.

We recall, cf. [171] that $\mathcal{P}_{\mathcal{F}}$ is dense in $L^2(W, \mathcal{F}, \mu_{|F})$ and that there exists a discrete chaotic decomposition of the space $L^2(W, \mathcal{F}, \mu_{|F})$ of square-integrable Poisson functionals.This decomposition uses discrete multiple stochastic integrals defined with the Laguerre polynomials

$$L_k(x) = \sum_{i=0}^{i=k} \binom{k}{i} \frac{(-x)^i}{i!} \qquad x \in \mathbb{R}_+, \quad k \in \mathbb{N},$$

which are orthonormal with respect to the exponential density. Let $H = l^2(\mathbb{N})$ be the Hilbert space of square-summable sequences, and let $(e_k)_{k\in\mathbb{N}}$ denote the canonical basis of H. For $n \geq 1$, we define the discrete multiple stochastic integral of asymmetric function f^non $\mathbb{N}^n$as a linear mapping $I_n : H^{\circ n} \to L^2(W, \mathcal{F}, \mu_{|F})$, first on elementary functions:

$$I_n\left(e_{k_1}^{\circ n_1} \circ \dots \circ e_{k_d}^{\circ n_d}\right) = n_1! \dots n_d!\, L_{n_1}(\tau_{k_1}) \dots L_{n_d}(\tau_{k_d})$$

where $n_1 + \cdots + n_d = n, k_1 \neq \cdots \neq k_d \in \mathbb{N}$. The mapping $I_n$is extended to any element of the completed symmetric tensor product $H^{\circ n}$by density, since the linear functional I_n satisfies to an isometry formula, cf. [163]. Moreover, integrals of different orders are orthogonal. As a result, any Fin $L^2(W, \mathcal{F}, \mu_{|F})$ has the orthogonal decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n)\, f_k \in H^{\circ k}, \quad k \in \mathbb{N}$$

with the conventions $H^0 = \mathbb{R}$ and $I_0 = I_{\mathbb{R}}$. The following proposition says that the Poisson random variable $I_n(f_n)$ is a multiple Wiener integral of order $2n$, and givesits expression in the Wiener chaotic decomposition. For simplicity, the development is only written for $f_n = e_k^{\circ n}$. Let $C_n^k = n!/(k!\,(n-k)!), 0 \leq k\, n, n \in \mathbb{N}$.

Proposition (6.1.2)[163]: The Wiener chaos expansion of $I_n(e_k^{\circ n})$ is given by

$$I_n(e_k^{\circ n}) = \frac{(-1)^n}{2^n}\sum_{i=0}^{i=n}\hat{I}_{2n}\left(h_{2k}^{\circ 2i}\circ h_{2k+1}^{\circ(2n-2i)}\right)C_n^i/\left(C_{2n}^{2i}\right)^{1/2}.$$

Proof. The proof relies on the following relation between the Hermite and Laguerre polynomials, cf. [174]:

$$n!\,L_n\left(\frac{x^2+y^2}{2}\right) = \frac{(-1)^n}{2^n}\sum_{k=0}^{k=n}C_n^k H_{2k}(x)H_{2n-2k}(y)\sqrt{(2k)!\,(2n-2k)!}$$

and on the definition of the multiple Wiener integral with the Hermite polynomials,cf. [167]. Here, $H_k(x)$ is the k-th normalized Hermite polynomial, defined by the generating series

$$\sum_{n=0}^{\infty}\gamma^n\frac{H_n(x)}{\sqrt{n!}} = exp(\gamma x - \gamma^2/2) \qquad \gamma, x \in \mathbb{R}$$

Denote by $\mathcal{L}$ the number operator on the discrete chaotic decomposition, that is $\mathcal{L}$ is a linear operator with

$$\mathcal{L}I_n(g_n) = nI_n(g_n)g_n \in H^{\circ n}, \qquad n \in \mathbb{N},$$

so that the domain of $\mathcal{L}$ is made the following Poisson functionals:

$$Dom(\mathcal{L}) = \left\{\sum_{n=0}^{\infty}I_n(f_n) : \sum_{n=0}^{\infty}n^2\|I_n(f_n)\|_2^2 < \infty\right\},$$

and $\mathcal{L}$ leaves invariant the space $\mathcal{P}_{\mathcal{F}}$of polynomial Poisson functionals. The operator$\mathcal{L}$ is the infinite dimensional generalization of the operator $x\partial_x^2 + (1 - x)\partial_x \mathrm{on} C^{\infty}(\mathbb{R})$,whose eigenvectors are the Laguerre polynomials.

We now define Sobolev spaces of Poisson functionals. We call $\mathbb{D}_{p;k}^{\mathcal{F}}$the completion of the algebra $\mathcal{P}_{\mathcal{F}}$ of polynomial Poisson functionals under the norm

$$\|F\|_{\mathbb{D}_{p,k}^{\mathcal{F}}} = \left\|(I+\mathcal{L})^{k/2}F\right\|_{L^p(B)} \qquad F \in \mathcal{P}_{\mathcal{F}},$$

$p > 1, k \in \mathbb{Z}$, and let

$$\mathbb{D}_{\infty}^{\mathcal{F}} = \bigcap_{p,k}\mathbb{D}_{p,k}^{\mathcal{F}}, \qquad \mathbb{D}_{-\infty}^{\mathcal{F}} = \bigcap_{p,k}\mathbb{D}_{p,k}^{\mathcal{F}}.$$

The next proposition says that the σ-algebra Fgenerated by the Poisson functional is $\hat{\mathcal{L}}^{-1}$-stable. We refer to [175] for the notion of $\hat{\mathcal{L}}^{-1}$ -stable σ -algebra.

Proposition (6.1.3)[163]:The operators $\mathcal{L}$and $\hat{\mathcal{L}}/2$ commute with the conditional expectation with respect to $\mathcal{F}$:

$$E\left[\hat{\mathcal{L}}F|\mathcal{F}\right] = 2\mathcal{L}E[F|\mathcal{F}]F \in Dom(\mathcal{L}), \quad p > 1, k \in \mathbb{Z},$$

Hence $\hat{\mathcal{L}}/2$is an extension of $\mathcal{L}$. The norms $\|\cdot\|_{\mathbb{D}_{p,k}}$and $\|\cdot\|_{\mathbb{D}_{p,k}^{\mathcal{F}}}$are equivalent on $\mathcal{P}_{\mathcal{F}}$, and $\mathbb{D}_{\infty}^{\mathcal{F}}$is an algebra. Moreover,

$$E[\cdot|\mathcal{F}] : \mathbb{D}_{p,k} \to \mathbb{D}_{p,k}^{\mathcal{F}} \quad p \geq 1, k \in \mathbb{Z},$$

is continuous.

Proof: Proposition (6.1.2) gives

$$E\left[H_i\left(\hat{\delta}(h_{2k})\right) H_j\left(\hat{\delta}(h_{2k+1})\right)\middle| \mathcal{F}\right] = \begin{cases} (-1/2)^{(i+j)} \dfrac{(i!\,j!)^{1/2}}{(i/2)!\,(j/2)!} L_{(i+j)/2}(\tau_k) i \text{ and } j \text{ even}, \\ 0 \quad otherwise. \end{cases}$$

Hence

$$\begin{aligned} E\left[\hat{\mathcal{L}}\hat{I}_{i+j}\left(h_{2k}^{\circ i} \circ h_{2k+1}^{\circ j}\right)\middle|\tau_k\right] &= \sqrt{(i+j)!}\, E\left[\hat{\mathcal{L}}\left(H_i\left(\hat{\delta}(h_{2k})\right) H_j\left(\hat{\delta}(h_{2k+1})\right)\right)\middle|\tau_k\right] \\ &= (i+j)\sqrt{(i+j)!}\, E\left[H_i\left(\hat{\delta}(h_{2k})\right) H_j\left(\hat{\delta}(h_{2k+1})\right)\middle|\tau_k\right] \\ &= 2\sqrt{(i+j)!}\, \mathcal{L}E\left[H_i\left(\hat{\delta}(h_{2k})\right) H_j\left(\hat{\delta}(h_{2k+1})\right)\middle|\tau_k\right] \\ &= 2\mathcal{L}E\left[\hat{I}_{i+j}\left(h_{2k}^{\circ i} \circ h_{2k+1}^{\circ j}\right)\tau_k\right] \end{aligned}$$

for any $i, j, k \in \mathbb{N}$. It follows that if $i1, \dots, i_d, j_1, \dots, j_d \in \mathbb{N}$, $k_1 \neq \cdots \neq k_d$and$F = \prod_{l=1}^{l=d} \hat{I}_{i_l+j_l}\left(h_{2k_l}^{\circ i_l} \circ h_{2k_l+1}^{\circ i_l}\right)$,

$$E\left[\hat{\mathcal{L}}F|\mathcal{F}\right] = \sum_{p=1}^{p=d} E\left[\prod_{l\neq p} \hat{I}_{i_l+j_l}\left(h_{2k_l}^{\circ i_l} \circ h_{2k_l+1}^{\circ i_l}\right) E\left[\hat{\mathcal{L}}\hat{I}_{i_p+j_p}\left(h_{2k_p}^{\circ i_p} \circ h_{2k_p+1}^{\circ i_p}\right)\middle|\tau_p\right]\middle|\mathcal{F}\right]$$

$$= 2\sum_{p=1}^{p=d}\prod_{l\neq p} E\left[\hat{I}_{i_l+j_l}\left(h_{2k_l}^{\circ i_l}\circ h_{2k_l+1}^{\circ i_l}\right)\Big|\tau_l\right]\mathcal{L}E\left[\hat{I}_{i_p+j_p}\left(h_{2k_p}^{\circ i_p}\circ h_{2k_p+1}^{\circ i_p}\right)\Big|\tau_p\right]$$

$$= 2\mathcal{L}E\;[F|\mathcal{F}],$$

Hence

$$E\big[\hat{\mathcal{L}}\hat{I}_n(g_n)\big|\mathcal{F}\big] = 2\mathcal{L}E\big[\hat{I}_n(g_n)\big|\mathcal{F}\big], \qquad g_n \in L^2(\mathbb{R}_+)^{\circ n}.$$

This implies that $2\mathcal{L}I_n(f_n) = \hat{\mathcal{L}}I_n(f_n),\ f_n \in H^{\circ n}$.The equivalence of norms follows from the L^p-multiplier theorem, with the fact that $\mathbb{D}_\infty^{\mathcal{F}}$is an algebra, since for $p,q,r > 1$ such that $1/r = 1/p\ + 1/q$ and $k \in \mathbb{Z}$, there exists a constant $C_{p,q,k}$ such that

$$\left\|\left(I+\hat{\mathcal{L}}/2\right)^{k/2}(FG)\right\|_{L^r(W)}$$
$$\leq C_{p,q,k}\left\|\left(I+\hat{\mathcal{L}}/2\right)^{k/2}F\right\|_{L^p(W)}\left\|\left(I+\hat{\mathcal{L}}/2\right)^{k/2}F\right\|_{L^q(W)} F,G \in \mathcal{P}_{\mathcal{F}},$$

cf. [166]. The continuity of $E[\cdot|\mathcal{F}]$ can be established as follows. For $p > 1$and$k \in \mathbb{Z}$, there exists a constant $C_{p,k}$ such that

$$\begin{aligned}\|E[F|\mathcal{F}]\|_{\mathbb{D}_{p,k}^{\mathcal{F}}} &\leq C_{p,k}\|E[F|\mathcal{F}]\|_{\mathbb{D}_{p,k}}\\ &= C_{p,k}\left\|\left(I\ +\hat{\mathcal{L}}\right)^{k/2}E[F|\mathcal{F}]\right\|_{L^p(W)}\\ &= C_{p,k}\left\|E\left[\left(I+\hat{\mathcal{L}}\right)^{k/2}F|\mathcal{F}\right]\right\|_{L^p(W)}\\ &\leq C_{p,k}\|F\|_{\mathbb{D}_{p,k}} \quad F \in \mathcal{P}.\end{aligned}$$

Another consequence of this proposition is that $\mathcal{L}$ is self-adjoint with respect to $\mu_{|\mathcal{F}}$. Being the restriction of $\hat{\mathcal{L}}/2$ to Poisson functionals, $\mathcal{L}$ shares several properties with $\hat{\mathcal{L}}$. The theorem below can be interpreted as a result in infinite-dimensional analysis for the exponential density, since $\mathcal{L}$is the infinite dimensional generalization of the operator whose eigenvectors are the Laguerre polynomials, which form anorthonormal sequence for the measure $e^{-x}1_{\{x>0\}}dx$.

Theorem (6.1.4)[163]:(Hypercontractivity). Let $p > 1$ and $t > 0$. There exists $q > p$ such that

$$\|\exp(-t\mathcal{L})F\|_{L^q(B)} \leq \|F\|_{L^p(B)} \qquad F \in L^q(B).$$

Proof. Since $\exp(-t\mathcal{L}) = \exp\left(-t\hat{\mathcal{L}}/2\right)$on $L^2(W, F, \mu_{\mathcal{F}})$, we can apply to Poisson functionals the existing hypercontractivity theorem on Wiener space, which says that for any $t > 0$ there is $q > p$ such that

$$\left\|\exp\left(-t\hat{\mathcal{L}}/2\right) F\right\|_{L^q(W)} \leq \|F\|_{L^p(W)} F \in L^q(W).$$

Example of a generalized Wiener functionals which is a Poisson functional.

From [172], Proposition (6.1.2), we have the following Wiener chaos expansion for the distribution

$$T = 2\pi\delta\left(\int_0^\infty h_0(t)dB_t, \int_0^\infty h_1(s)dB_s\right) \in \mathbb{D}_{2,-r,} \qquad r > 1,$$

where δ is the Dirac distribution at 0 in $\mathbb{R}^2$:

$$T = \sum_{n\geq 0} \frac{1}{2n!} \hat{I}_{2n}\left(\left(\frac{-1}{2}\right)^n (2n)! \sum_{k=0}^{k=n} \frac{1}{k!\,(n-k!)} h_0^{\circ 2k} \circ h_1^{\circ 2n-2k}\right)$$

Hence from Proposition (6.1.2), $T$is the limit in $\mathbb{D}_{2,-r}^{\mathcal{F}}, r > 1$, of a sequence of polynomial Poisson functionals.

We end this section with two definitions. In [171], a gradient operator has been defined for Poisson functionals as a directional derivative in the directions of $H = l^2(\mathbb{N})$, or equivalently by shifts of the Poisson process jump times. We recall this definition with a different interpretation.

Definition (6.1.5)[163]: We define $D : L^2(W, \mu) \to L^2\left(W, \mathcal{F}, \mu_{|\mathcal{F}}\right) \otimes H$ by

$$(DF, h)_H = \lim_{\varepsilon \to 0} \frac{[\Theta^{-1}F](\Xi + \varepsilon h) - F}{\varepsilon} \quad h \in H, F \in \mathcal{P}_{\mathcal{F}}$$

If$F \in \mathcal{P}_{\mathcal{F}}$ with $F = f(\tau_0, \dots, \tau_n)$, then

$$DF = -\sum_{k=0}^{k=n} \partial_k f(\tau_0, \dots, \tau_n) 1_{\{k\}}.$$

The operator $D: L^2\left(W, \mathcal{F}, \mu_{|\mathcal{F}}\right) \to L^2\left(W, \mathcal{F}, \mu_{|\mathcal{F}}\right) \otimes H$ is closable and its expression in the discrete chaotic decomposition is written as follows, cf. [171]:

$$D_j I_n(f_n) = \sum_{k=0}^{k=n-1} \frac{n!}{k!} I_k\big(f_n(*, j, \dots, j)\big)\, j \in \mathbb{N}, \qquad f_n \in l^2(\mathbb{N})^{\circ n}.$$

Finally, we define for later use an operator i that turns a discrete-time process in to a continuous-time process, using the Poisson process itself.

Definition (6.1.6)[163]: If $f : \mathbb{N}^d \to \mathbb{R}^n$ is a function of discrete variable, we define a d-parameter process $i(f)$ by

$$i(f)(t_1, \dots, t_d) = f\big(N_{t_1^-}, \dots, N_{t_d^-}\big) t_1, \dots, t_d \in \mathbb{R}_+.$$

The operator $i$is easily extended to stochastic processes of discrete d-dimensional parameter. If $n = d = 1$, let $j : L^2(W) \otimes L^2(\mathbb{R}_+) \to L^2(W) \otimes l^2(\mathbb{N})$ denote the dualof $i : L^2(W) \otimes l^2(\mathbb{N}) \to L^2(W) \otimes L^2(\mathbb{R}_+)$, i.e. j is a random operator such that

$$(i(u), v)_{L^2(\mathbb{R}_+)} = \big(u, j(v)\big)_{l^2(\mathbb{N})} \quad \mu - a.s.$$

for $u \in L^2(W) \otimes l^2(\mathbb{N}), v \in L^2(W) \otimes L^2(\mathbb{R}_+)$. We have explicitely

$$j(v) = \sum_{k \geq 0} 1_{\{k\}} \int_{T_k}^{T_{k+1}} v(s) ds\,.$$

In this section, we define an extension to Wiener functionals of the above gradient operator D, taking into account the conditional gradient given F, cf. [172] for this notion. This new gradient has the following properties: its adjoint coincides with the compensated Poisson stochastic integral under certain conditions, and by composition with its adjoint it yields the Ornstein-Uhlenbeck operator on the Wiener space. It is expressed by composition of the Gross-Sobolev derivative $\widehat{D}$on the Wiener space with a random unitary operator which is defined below. The n-th jump time of the Poisson process $(N_t)_{t \in \mathbb{R}_+}$defined on (W, μ) is denoted by $T_n = \sum_{k=0}^{k=n-1} \tau_k\, , n \geq 0$.

Definition (6.1.7)[163]: For μ-a.s.ω, we define an operator $\chi : L^2(\mathbb{R}_+, \mathbb{R}^2) \to L^2(\mathbb{R}_+)$ by

$$\chi_u = \frac{1}{\sqrt{2}} \sum_{k \geq 0} \frac{\hat{\delta}(h_{2k}) h_{2k} + \hat{\delta}(h_{2k+1}) h_{2k+1}}{\tau_k} \int_{T_k}^{T_{k+1}} u^{(1)}(s) ds$$

$$+ \frac{\hat{\delta}(h_{2k}) h_{2k+1} - \hat{\delta}(h_{2k+1}) h_{2k}}{\tau_k} \int_{T_k}^{T_{k+1}} u^{(2)}(s) ds \;\; u = \big(u^{(1)}, u^{(2)}\big) \in C_c^\infty(\mathbb{R}_+, \mathbb{R}^2).$$

We are going to show that μ-a.s., $\chi$is unitary from a certain random sub space $\widetilde{H}$ of $L^2(\mathbb{R}_+,\mathbb{R}^2)$ in to $L^2(\mathbb{R}_+)$.

Definition (6.1.8)[163]: For μ-a.s. $\omega \in W$, we define $\widetilde{H}$ to be the random sub space of $L^2(\mathbb{R}_+,\mathbb{R}^2)$of the form

$$\widetilde{H} = \left\{i\big((f,g)\big) : \ (f,g) \in l^2(\mathbb{N},\mathbb{R}^2)\right\}.$$

The operator i was introduced in Definition (6.1.6).

Proposition (6.1.9)[163]: The operator χ is unitary from $\widetilde{H}$ in to $L^2(\mathbb{R}_+)$:

$$\chi^*\chi = I_{\widetilde{H}} and \quad \chi\chi^* = I_{L^2(\mathbb{R}_+)} \ \mu - a{:}\, s{:}$$

and its adjoint is $\chi^*: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+,\mathbb{R}^2)$, given by

$$\chi^* v = -\frac{1}{\sqrt{2}}\sum_{k\geq 0}^{\infty}\frac{1}{\tau_k}1_{[T_k,T_{k+1}[}((v,h_{2k})L^2(\mathbb{R}_+)\hat{\delta}(h_{2k}) + (v,h_{2k+1})L^2(\mathbb{R}_+)\hat{\delta}(h_{2k+1})\,,$$

$$(v,h_{2k+1})L^2(\mathbb{R}_+)\hat{\delta}(h_{2k}) - (v,h_{2k})L^2(\mathbb{R}_+)\hat{\delta}(h_{2k+1}) \quad \mu - a{:}\, s{:}$$

Proof. We have if $u = \big(u^{(1)},u^{(2)}\big) \in C_c^\infty(\mathbb{R}_+,\mathbb{R}^2)$ and $v \in C_c^\infty(\mathbb{R}_+)$:

$$(\chi u, v)_{L^2(\mathbb{R}_+)}$$

$$= \frac{1}{\sqrt{2}}\sum_{k\geq 0}\int_{T_k}^{T_{k+1}} u^{(1)}(s)ds\,\frac{\hat{\delta}(h_{2k})(h_{2k},v)L^2(\mathbb{R}_+) + \hat{\delta}(h_{2k+1})(h_{2k}+1,v)_{L^2(\mathbb{R}_+)}}{\tau_k}$$

$$+\int_{T_k}^{T_{k+1}} u^{(1)}(s)ds\,\frac{\hat{\delta}(h_{2k+1})(h_{2k+1},v)L^2(\mathbb{R}_+) - \hat{\delta}(h_{2k+1})(h_{2k}+1,v)_{L^2(\mathbb{R}_+)}}{\tau_k}$$

$$= (u,\chi^* v)L^2(\mathbb{R}_+,\mathbb{R}^2) \ \mu - a{:}\, s{:}$$

Hence χ and $\chi^*$are adjoint μ –a:s: It is easy to check that $\chi^*\chi = I_{\widetilde{H}}$and $\chi\chi^* = I_{L^2(\mathbb{R}_+)}$ μ –a:s:, and the fact that $\chi : \widetilde{H} \to L^2(\mathbb{R}_+)$ is unitary follows.

Again, χ is easily extended to two dimensional stochastic processes as an isometry $\chi : L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2) \to L^2(W)\otimes L^2(\mathbb{R}_+)$, with the properties that

$$\chi\chi^* = I_{L^2(W)\otimes L^2(\mathbb{R}_+)} and \ (\chi u, v)_{L^2(\mathbb{R}_+)} = (u,\chi^* v)L^2(\mathbb{R}_+,\mathbb{R}^2)\ \mu - a.s.,$$

$u \in L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2), v \in L^2(W)\otimes L^2(\mathbb{R}_+)$. We now define a gradient $\widetilde{D}$ by composition of $\widehat{D}$ with χ^*.

Definition (6.1.10)[163]: We define an operator $\widetilde{D}: L^2(W) \to L^2(W) \otimes L^2(\mathbb{R}_+, \mathbb{R}^2)$ by

$$\widetilde{D}F = \frac{1}{\sqrt{2}} \chi^* \circ \widehat{D}F \qquad F \in \mathcal{P}.$$

Then $\widehat{D}F = \sqrt{2}\chi \circ \widetilde{D}F, F \in \mathcal{P}$. According to this definition, $\widetilde{D}$ is a derivation operator on the Wiener space.

Proposition (6.1.11)[163]: As a direct consequence of the fact that χ is unitary, we have:

- The operators $\widetilde{D}$and $\widehat{D}$can be extended to the same domains. More precisely,

$$2\|\widetilde{D}F\|^2_{L_2(\mathbb{R}_+,\mathbb{R}^2)} = \|\widehat{D}F\|^2_{L^2(\mathbb{R}_+)} F \in \mathbb{D}_{2,1}, \mu - a.s.,$$

  hence the operator $\widetilde{D}$is closable and local.

- Let $-\hat{\mathcal{L}}/2$ denote the Ornstein-Uhlenbeck operator on the Wiener space. Wehave the following decomposition of $\hat{\mathcal{L}}$:

$$\frac{\hat{\mathcal{L}}}{2} = \tilde{\delta}\widetilde{D}.$$

Note that the usual decomposition of the Ornstein-Uhlenbeck operator is given by $\hat{\mathcal{L}} = \tilde{\delta}\widetilde{D}$.

We now show that for $F \in Dom(\widehat{D})$, the second component $\widetilde{D}^{(2)}F$ of $\widetilde{D}F$ is related to the conditional gradient of Fgiven $\mathcal{F}$, cf. [172], whereas its first component $\widetilde{D}^{(1)}F$is expressed with the operator Ddefined in Definition (6.1.5) by shifts of the Poisson process jump times. Denote by $\mathcal{H}$ the orthogonal subspace in $L^2(W) \otimes L^2(\mathbb{R}_+)$ of the set

$$\{Z\widehat{D}U : U \in \mathbb{D}^{\mathcal{F}}_{2,1}, Z \in L^\infty(W,\mu)\}.$$

Let $P^{\mathcal{H}}$ be the orthogonal projection on $\mathcal{H}$in $L^2(W) \otimes L^2(\mathbb{R}_+)$. Recall that the conditional gradient given $\mathcal{F}$of $F \in \mathbb{D}_{2,1}$is defined as $\widehat{D}^{\mathcal{F}}F = P^{\mathcal{H}}\widehat{D}F, F \in \mathbb{D}_{2,1}$,cf. [172].

Proposition (6.1.12)[163]: The conditional gradient $\widehat{D}^{\mathcal{F}}F$of $F \in \mathbb{D}_{2,1}$given $\mathcal{F}$ is

$$\widehat{D}^{\mathcal{F}}F = \sqrt{2}\chi\big(0, \widetilde{D}^{(2)}F\big)\mu - a.s.$$

Let $F \in \mathcal{P}_{\mathcal{F}}$be a polynomial Poisson functional. We have

$$\widetilde{D}F = i\big((DF, 0)\big)\mu - a.s.$$

Proof. Let $F = f\left(\hat{\delta}(h_0), \dots, \hat{\delta}(h_n)\right)$ with $f \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$. We have

$$\widehat{D}F-\sqrt{2}\chi\big(0,\widetilde{D}^{(2)}F\big)=\sqrt{2}\chi\big(\widetilde{D}^{(1)}F,0\big),$$

hence $\widehat{D}F-\sqrt{2}\chi\big(0,\widetilde{D}^{(2)}F\big)\in\{Z\widehat{D}U : U\in\mathbb{D}_{2,1}^{\mathcal{F}}, Z\in L^\infty(W,\mu)\}$. We also have

$$E\left[Z\big(\chi\big(0,\widetilde{D}^{(2)}F\big),\widehat{D}U\big)_{L^2(\mathbb{R}_+)}\right]=0\quad U\in\mathcal{P}_{\mathcal{F}},\ \ Z\in L^\infty(W,\mu),$$

hence $\sqrt{2}\chi\big(0,\widetilde{D}^{(2)}F\big)=P^{\mathcal{H}}\widehat{D}F$. The result is obtained by density. For the second part, we notice that the conditional gradient of a Poisson functional given $\mathcal{F}$ is 0 and that a simple calculation yields $\chi^*\widehat{D}U=\sqrt{2}i(DU,0),\ U\in\mathcal{P}_{\mathcal{F}}$.

The following definition gives the adjoint of $\widetilde{D}$. Let $\mathcal{V}$ be the class of processes defined by

$$\mathcal{V}=\{u\in L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2):$$
$$u=\left(f\left(\cdot,\hat{\delta}(h_0),\ldots,\hat{\delta}(h_n)\right),g\left(\cdot,\hat{\delta}(h_0),\ldots,\hat{\delta}(h_n)\right)\right),f,g\in\mathcal{C}_c^\infty(\mathbb{R}_+^{n+1}),n\in\mathbb{N}\}.$$

Definition (6.1.13)[163]: We define the operator $\hat{\delta}: L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2)\longrightarrow L^2(W)$ by

$$\hat{\delta}(v)=\frac{1}{\sqrt{2}}\hat{\delta}\circ\chi(v),\qquad v\in\mathcal{V}.$$

We have the following commutative diagram:

$$\begin{array}{ccccccc}
L^2(W) & \overset{\sqrt{2}\widehat{D}}{\longrightarrow} & L^2(W)\otimes L^2(\mathbb{R}_+) & \qquad & L^2(W)\otimes L^2(\mathbb{R}_+) & \overset{\sqrt{2}\widehat{\delta}}{\longrightarrow} & L^2(W)\\
\updownarrow & & \chi^*\downarrow & & \chi\uparrow & & \updownarrow\\
L^2(W) & \overset{\widehat{D}}{\rightarrow} & L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2) & & L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2) & \overset{\delta}{\rightarrow} & L^2(W).
\end{array}$$

Proposition (6.1.14)[163]: The operator $\hat{\delta}$ is closable, adjoint of $\widetilde{D}$ and satisfies to

$$\hat{\delta}(u)=\int_0^\infty u^{(1)}(s)d(N_s-s)-trace\big(\widetilde{D}u\big),\quad u\in\mathcal{V},$$

where $trace\big(\widetilde{D}u\big)=\int_0^\infty\widetilde{D}_s^{(1)}u^{(1)}(s)ds+\int_0^\infty\widetilde{D}_s^{(2)}u^{(1)}(s)ds.$

Let $Dom\big(\tilde{\delta}\big)$ denote the domain of the closed extension of $\tilde{\delta}$.

Proof. Recall that by definition, cf. [167], [166], if $v=\sum_{i=0}^{i=n}h_if_i\left(\hat{\delta}(h_0),\ldots,\hat{\delta}(h_n)\right)$ with $f_i\in\mathcal{C}_c^\infty(\mathbb{R}^{n+1}), i=0,\ldots,n$, then

$$\hat{\delta}(v)=\sum_{i=0}^{i=n}\hat{\delta}(h_i)f_i\left(\hat{\delta}(h_0),\ldots,(h_n)\right)-\partial_if_i\left(\hat{\delta}(h_0),\ldots,(h_n)\right).$$

Applying the above formula to $\chi u, u \in \mathcal{V}$, we obtain:

$$\hat{\delta}(u) = \hat{\delta}(\chi u)$$

$$= \sum_{k\geq 0} u^{(1)}(T_{k+1}) \frac{\hat{\delta}(h_{2k})^2 + \hat{\delta}(h_{2k+1})^2}{2\tau_k} + u^{(2)}(T_{k+1}) \frac{\hat{\delta}(h_{2k})\hat{\delta}(h_{2k+1}) - \hat{\delta}(h_{2k+1})\hat{\delta}(h_{2k})}{2\tau_k}$$

$$+ \frac{1}{2\tau_k} \int_{T_k}^{T_{k+1}} \left(\left(\widehat{D}u^{(1)}(s), h_{2k} \right) \hat{\delta}(h_{2k}) + \left(\widehat{D}u^{(2)}(s), h_{2k+1} \right) \hat{\delta}(h_{2k+1}) \right) ds$$

$$+ \frac{1}{2\tau_k} \int_{T_k}^{T_{k+1}} \left(\left(\widehat{D}u^{(2)}(s), h_{2k+1} \right) \hat{\delta}(h_{2k+1}) - \left(\widehat{D}u^{(1)}(s), h_{2k} \right) \hat{\delta}(h_{2k}) \right) ds$$

$$+ \frac{1}{\tau_k} \int_{T_k}^{T_{k+1}} u^{(1)}(s) - \int_{T_k}^{T_{k+1}} u^{(1)}(s) \frac{\hat{\delta}(h_{2k})^2 + \hat{\delta}(h_{2k+1})^2}{2\tau_k^2} - \int_{T_k}^{T_{k+1}} u^{(1)}(s) ds$$

$$= \int_0^\infty u^{(1)}(s) d(N_s - s) - \int_0^\infty \widetilde{D}_2^{(1)} u^{(1)}(s) ds - \int_0^\infty \widetilde{D}_s^{(2)} u^{(2)}(s) ds.$$

The operator $\hat{\delta}$is adjoint of $\widetilde{D}$and closable since χ and $\chi^*$are adjoint and the domain of $\widetilde{D}$is dense in $L^2(W)$.

Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(N_t)_{t\geq 0}$on (W, μ).

Corollary (6.1.15)[163]: If $u = \left(u^{(1)}, u^{(2)}\right) \in L^2(W) \otimes L^2(\mathbb{R}_+, \mathbb{R}^2)$is $(\mathcal{F}_t)$-predictable, then $\hat{\delta}(u)$ coincides with the compensated Poisson stochastic integral of $u^{(1)}$:

$$\hat{\delta}(u) = \int_0^\infty u^{(1)}(s) d(N_s - s),$$

and any Poisson stochastic integral has a representation as an anticipative Wiener-Skorohod integral:

$$\int_0^\infty u^{(1)}(s) d(N_s - s) = \hat{\delta}\left(\chi\left(u^{(1)}, 0\right)\right) / \sqrt{2}.$$

Proof. The conditional gradient given $\mathcal{F}$ of a Poisson functional is 0, cf. Proposition (6.1.12),hence from Proposition (6.1.14) the first part of this statement is identical to the Poisson space result that can be found in [172], [169], [49]. The representation property comes from the relation $\hat{\delta} = \frac{1}{\sqrt{2}}\hat{\delta} \circ \chi$.

The above coincidence can occur under weaker conditions, for instance without predictability requirements. For example, it is sufficient to have $(u^{(1)}, 0) \in \mathcal{V}$ with $u^{(1)} \in L^2(W, \mathcal{F}, \mu|_{\mathcal{F}}) \otimes L^2(\mathbb{R}_+)$ and

$$u^{(2)}(t) = -\sum_{k\geq 0} 1_{[T_k, T_{k+1}]}(t) \arctan\left(\frac{\hat{\delta}(h_{2k})}{\hat{\delta}(h_{2k+1})}\right) \widetilde{D}_t^{(1)} u^{(1)}(t)\ t \in \mathbb{R}_+.$$

In this case, $\widetilde{D}^{(1)}u^{(1)} + \widetilde{D}^{(2)}u^{(2)} = 0\ \mu \otimes dt$-a.e., and the trace term in (6) vanishes. The representation property for Poisson stochastic integrals as Wiener-Skorohod integrals also extends to anticipative integrands in Dom$(\hat{\delta})$. This result differs from the result obtained via the Clark formula, cf. [176], in that the process $\frac{1}{\sqrt{2}}\chi u$ that we obtain is not adapted and its expression is easier to compute.

The first consequence of the above propositions is that the Meyer inequalities on Poisson space hold for the operators $\widetilde{D}$and $\mathcal{L}$, given that they are verified for $\widehat{D}$and $\hat{\mathcal{L}}$. The spaces $L^p(B, P)$ and $L^p(W, \mathcal{F}, \mu|_{\mathcal{F}})$ are identified via the operator Θ for$p \geq 1$.

Theorem (6.1.16)[163]: For any $p > 1$, there exist $A_p, B_p > 0$ such that for any Poisson polynomial functional $F \in \mathcal{P}_{\mathcal{F}}$,

$$A_p \left\|\widetilde{D}F\right\|_{L^p\left(B, L^2(\mathbb{R}_+)\right)}$$
$$\leq \left\|(I + \mathcal{L})^{1/2}\right\|_{FL^p(B)} \leq B_p\left(\left\|\widetilde{D}F\right\|_{L^p\left(B, L^2(\mathbb{R}_+)\right)} + \|F\|_{L^p(B)}\right).$$

Proof. We write the Meyer inequalities, cf. [168], on the Wiener space and make use of the facts that χ is unitary from $\widetilde{H}$ to $L^2(\mathbb{R}_+)$, μ-a:s: and $\hat{\mathcal{L}}$is an extension of $2\mathcal{L}$.

The difference between this result and the Meyer inequalities on the Wiener space comes from the fact that on Poisson functionals, $\widetilde{D}$ is defined by shifting the jump times of the Poisson process, and its adjoint extends the compensated Poisson stochastic integral, whereas $\widehat{D}$is defined by shifts of the Wiener process trajectories and its adjoint extends the Itô-Wiener stochastic integral.

We can also define the composition of a Schwartz distribution with a Poisson functional as a distribution in $\mathbb{D}^{\mathcal{F}}_{-\infty}$. Let $S_{2k}, k \in \mathbb{Z}$, be the completion of the Schwartz space $S(\mathbb{R}^d)$ under the norm $\|\phi\|_{S_{2k}} = \|(1+|x|^2+\Delta)^k \phi\|_\infty$.

Theorem (6.1.17)[163]: Let $F_1, \dots, F \in \mathbb{D}^{\mathcal{F}}_{-\infty}$such thatdet$\left(\left(\left(\widetilde{D}F_i, \widetilde{D}F_j\right)_{L^2(\mathbb{R}_+,\mathbb{R}^2)}\right)_{1\le i,j\le d}\right)^{-1} \in \bigcap_{p>1} L^p(B,P)$. Then for $k \in \mathbb{Z}$and $p > 1$, there exists $C_{p,k} > 0$ such that

$$\|\phi \circ F\|_{p,2k} \le C_{p,k} \|\phi\|_{S_{2k}} \phi \in S(\mathbb{R}^d).$$

This implies that if $T \in S_{2k}, T \circ F$is well defined in $\mathbb{D}_{p,k}, p > 1, k \in \mathbb{Z}$.

The proof relies again on the fact that χ is unitary and $\mathbb{D}^{\mathcal{F}}_\infty \subset \mathbb{D}_\infty$, given the Wiener space result in [166]. In the same way, we obtain:

Theorem (6.1.18)[163]: Under the hypothesis of the preceding theorem, the Poisson functional $F = (F_1, \dots, F_d)$ has a C^∞density on $\mathbb{R}^d$.

The hypothesis is expressed by perturbations of the Poisson process trajectories. The following exponential integrability criterion comes from [177] and [178] for the Gaussian case. It is proved in the same way as Theorem (6.1.16) and (6.1.17).

Theorem (6.1.19)[163]: If $F \in \mathbb{D}^{\mathcal{F}}_{p,1}, p > 1$, is such that $\left\|\widetilde{D}F\right\|_{L^\infty\left(W,L^2(\mathbb{R}_+,\mathbb{R}^2)\right)} < \infty$, then there exists $\lambda > 0$ such that

$$E[\exp(\lambda F^2)] < \infty.$$

Denote by $\left(W, \mu, \mathbb{D}_{2,1}, \epsilon\right)$ the standard Dirichlet structure on Wiener space, cf. [172].The Dirichlet form ϵ is defined as $\epsilon(F,G) = -\frac{1}{2} E\left[F\hat{\mathcal{L}}G\right], F, G \in Dom\left(\widehat{D}\right)$. It admits a carré du champ operator Γ defined by $\Gamma(F,G) = \left(\widehat{D}F, \widehat{D}G\right)_{L^2(\mathbb{R}_+)}$. Proposition (6.1.11)shows that this structure admits $\sqrt{2}\widetilde{D}$as well as $\widehat{D}$as a gradient, i.e. $\Gamma(F,G) = 2\left(\widehat{D}F, \widehat{D}G\right)_{L^2(\mathbb{R}_+)}$. Moreover, $\left(W, \mathcal{F}, \mu|_{\mathcal{F}}, \mathbb{D}^{\mathcal{F}}_{2,1}, \epsilon \,|\mathbb{D}^{\mathcal{F}}_{2,1}\right)$ is the Dirichlet substructure generated by $(\tau_k)_{k\in\mathbb{N}}$, cf. [172]. As a substructure, $\left(W, \mathcal{F}, \mu|_{\mathcal{F}}, \mathbb{D}^{\mathcal{F}}_{2,1}, \epsilon \,|\mathbb{D}^{\mathcal{F}}_{2,1}\right)$ is local, admits a carré du champ operator, and satisfies the energy image density property:

Theorem (6.1.20)[163]: If $F_1,\dots,F_d \in \mathbb{D}_{2,1}^{\mathcal{F}}$ with $det\left(\left(\left(\widetilde{D}F_i,\widetilde{D}F_j\right)_{L^2(\mathbb{R}_+,\mathbb{R}^2)}\right)_{1\le i,j\le d}\right) > 0$ μ-a.s., then the law of $F=(F_1,\dots,F_d)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

We give a version of the Meyer inequalities for higher orders of differentiation, and extend the operators $j\circ i\circ D$ and $\tilde{\delta}\circ i$ to Sobolev spaces of H-valued functionals. Let $\mathcal{P}_{\mathcal{F}}^*$ denote the set of functions $u:\mathbb{N}\to\mathcal{P}$ such that u has a finite
support in $\mathbb{N}$. This set is dense in $L^2(B)\otimes l^2(\mathbb{N})$.

Lemma (6.1.21)[163]: Define the operator $P_t^{(1)}: L^2(B)\otimes l^2(\mathbb{N})\to L^2(B)\otimes l^2(\mathbb{N}), t\in \mathbb{R}_+$ by $P_t^{(1)}u=\left(\left(P_t^{(1)}u\right)_k\right)_{k\ge 0}$, where

$$\left(P_t^{(1)}u\right)_k=(e^{-t}-1)D_kP_tu_k+e^{-t}P_tu_k,\ \ k\in\mathbb{N},\ \ u\in\mathcal{P}_{\mathcal{F}}^*.$$

Then $\left(P_t^{(1)}\right)_{t\in\mathbb{R}_+}$ is a semi-group, and we have the relation

$$P_tF=P_t^{(1)}DF \qquad F\in\mathcal{P}_{\mathcal{F}},\ \ t\in\mathbb{R}_+.$$

Proof. Let $F=I_n(f_n), n\ge 1$ and $f_n\in l^2(\mathbb{N})^{\circ n}$. We have from the expression of D as an annihilation operator, cf. [171]:

$$D_kP_tF=e^{-nt}\sum_{l=0}^{l=n-1} I_l\big(f_n(*,k,\dots,k)\big)$$

and

$$e^{-t}P_tD_kF=e^{-t}\sum_{l=0}^{l=n-1} e^{-lt}I_l\big(f_n(*,k,\dots,k)\big).$$

Hence

$$(e^{-t}-1)D_kP_tD_kF=\sum_{p=1}^{p=n-1}\left(e^{-(p+1)t}-e^{-pt}\right)\sum_{l=0}^{l=p-1} I_l\big(f_n(*,k,\dots,k)\big)$$
$$=D_kP_tF-e^{-t}P_tD_kF,$$

or $\left(P_t^{(1)}DF\right)_k = D_kP_tF\ F \in \mathcal{P}$.From the following equalities, $\left(P_t^{(1)}\right)_{t\in\mathbb{R}_+}$ is asemigroup.Let $u \in \mathcal{P}_{\mathcal{F}}^*, k \in \mathbb{N}$, and choose $F_k \in \mathcal{P}$ such that $u_k = D_kF_k$. We have for $s, t > 0$:

$$\left(P_{t+s}^{(1)}u\right)_k = \left(P_{t+s}^{(1)}DF_k\right)_k = D_kP_{t+s}F_kD_kP_tP_sF_k$$

$$= \left(P_t^{(1)}DP_sF_k\right)_k = \left(P_t^{(1)}P_s^{(1)}DF_k\right)_k = \left(P_t^{(1)}P_s^{(1)}u\right)_k k \in \mathbb{N}.$$

Hence $P_{t+s}^{(1)} = P_t^{(1)}P_s^{(1)}$, for $s, t > 0$.

Proposition (6.1.22)[163]: Let $\mathcal{L}^{(1)}$denote the generator of $\left(P_t^{(1)}\right)_{t\geq 0}$. For $\in \mathcal{P}_{\mathcal{F}}^*$, we have$\mathcal{L}^{(1)}u = \left(\left(\mathcal{L}^{(1)}u\right)_k\right)_{k\in\mathbb{N}}$with

$$\left(\mathcal{L}^{(1)}u\right)_k = (\mathcal{L} + I + D_k)u_k, \quad k \in \mathbb{N}.$$

The duality relation

$$\left(i(u), i\left(\mathcal{L}^{(1)}v\right)\right)_{L^2(B)\otimes L^2(\mathbb{R}_+)} = \left(i\left(\mathcal{L}^{(1)}u\right), i(v)\right)_{L^2(B)\otimes L^2(\mathbb{R}_+)} u, v \in \mathcal{P}_{\mathcal{F}}^*,$$

holds, and we have the commutation relation

$$\mathcal{L}^{(1)}D = D\mathcal{L} \quad on\ \mathcal{P}_{\mathcal{F}}.$$

Proof. This is a consequence of the above proposition. The duality relation comes from the equality

$$E\left[\tau_k u_k\left(\mathcal{L}^{(1)}v\right)_k\right] = E\left[\tau_k\left(\mathcal{L}^{(1)}u\right)_k v_k\right] u, v \in \mathcal{P}_{\mathcal{F}}^*, \quad k \in \mathbb{N},$$

that can be checked using the explicit expression of $\mathcal{L}^{(1)}$:

$$E[\tau_k u_k(\mathcal{L} + I + D_k)v_k]$$

$$= E[v_k\mathcal{L}(\tau_k u_k) + \tau_k u_k v_k + \tau_k u_k D_k v_k]$$

$$= E\left[v_k\tau_k\mathcal{L}u_k + v_k u_k\mathcal{L}\tau_k - 2\left(\widetilde{D}\tau_k, \widetilde{D}u_k\right) + \tau_k u_k v_k + \tau_k u_k D_k v_k\right]$$

$$= E[v_k\tau_k\mathcal{L}u_k - v_k u_k + \tau_k v_k u_k + 2v_k\tau_k D_k u_k + u_k D_k(\tau_k v_k)]$$

$$= E[\tau_k v_k\mathcal{L}u_k + v_k\tau_k D_k u_k + \tau_k u_k v_k]$$

$$= E[\tau_k v_k(\mathcal{L} + I + D_k)u_k], u, v \in \mathcal{P}_{\mathcal{F}}^*, \quad k \in \mathbb{N}.$$

We used here the relation $\mathcal{L}(FG) = F\mathcal{L}G + G\mathcal{L}F - 2\left(\widetilde{D}F, \widetilde{D}G\right)_{L^2(\mathbb{R}_+,\mathbb{R}^2)} F, G \in \mathcal{P}_{\mathcal{F}}$,cf. [171], and the fact that $I + D_k$is adjoint of $D_k, k \in \mathbb{N}$ with respect to P.

We now aim to construct Sobolev spaces of H-valued functionals, in order to extendthe Poisson gradient and divergence operators to distributions.

Definition (6.1.23)[163]: We define the norm $\|\cdot\|_{\mathbb{D}_{p,k}(H)}$on $\mathcal{P}_{\mathcal{F}}^*$ by

$$\|u\|_{\mathbb{D}_{p,k}(H)} = \left\| i\left(\left(I_H + \mathcal{L}^{(1)}\right)^{k/2} u\right)\right\|_{L^p\left(B,L^2(\mathbb{R}_+)\right)}.$$

The space$\mathbb{D}_{p,k}(H)$ is defined to be the completion of $\mathcal{P}_{\mathcal{F}}^*$ with respect to the norm$\|\cdot\|_{\mathbb{D}_{p,k}(H)}$.

The following extension of Theorem (6.1.16) holds:

Theorem (6.1.24)[163]: For $p > 1$ and $k \in \mathbb{Z}$, there exists two constants $A_{p,k}, B_{p,k} > 0$ such that for any Poisson polynomial functional $F \in \mathcal{P}_{\mathcal{F}}$:

$$A_{p,k}\|DF\|_{\mathbb{D}_{p,k}(H)} \leq \|F\|_{\mathbb{D}_{p,k+1}^{\mathcal{F}}} \leq B_{p,k}\left(\|DF\|_{\mathbb{D}_{p,k}(H)} + \|F\|_{L^p(B)}\right).$$

Proof. We have $\left(I_H + \mathcal{L}^{(1)}\right)^{k/2} DF = D(I + \mathcal{L})^{k/2}F, F \in \mathcal{P}_{\mathcal{F}}$. Hence

$$\begin{aligned}\|DF\|_{\mathbb{D}_{p,k}(H)} &= \left\| i\left(\left(I_H + \mathcal{L}^{(1)}\right)^{k/2} DF\right)\right\|_{L^p\left(B,L^2(\mathbb{R}_+)\right)} \\ &= \left\| i\left(D(I + \mathcal{L})^{k/2}F\right)\right\|_{L^p\left(B,L^2(\mathbb{R}_+)\right)} \\ &= \left\| D(I + \mathcal{L})^{k/2}F\right\|_{\mathbb{D}_{p,0}(H)} \\ &= \left\| \widetilde{D}(I + \mathcal{L})^{k/2}F\right\|_{L^p\left(B,L^2(\mathbb{R}_+)\right)} k \in \mathbb{Z},\ \ p > 1.\end{aligned}$$

It remains to apply Theorem (6.1.16) to $(I + \mathcal{L})^{k/2}F$.

Corollary (6.1.25)[163]: The operator $j \circ i \circ D$ can be extended as a continuous operator

$$j \circ i \circ D : \mathbb{D}_{p,k}^{\mathcal{F}} \longrightarrow \mathbb{D}_{p,k-1}(H) k \in \mathbb{Z},\ \ p > 1.$$

The operator $\tilde{\delta} \circ i(\cdot\,,0)$can be extended as a continuous operator

$$\tilde{\delta} \circ i \circ (\cdot\,,0) : \mathbb{D}_{p,k}(H) \longrightarrow \mathbb{D}_{p,k-1}^{\mathcal{F}} k \in \mathbb{Z},\ \ p > 1.$$

Proof. We have for $u \in \mathcal{P}_{\mathcal{F}}^*$ and $F \in \mathcal{P}_{\mathcal{F}}$:

$$\begin{aligned}\left|E\left[F\tilde{\delta} \circ i \circ (u,0)\right]\right| &= \left|E\left[\left(i(u,0), \widetilde{D}F\right)_{L^2\left(\mathbb{R}_+,\mathbb{R}^2\right)}\right]\right| \\ &= \left|E\left[\left(i(u), i(DF)\right)_{L^2(\mathbb{R}_+)}\right]\right| \\ &= E\left(i\left(\left(I_H + \mathcal{L}^{(1)}\right)^{k/2} u\right), i\left(\left(I_H + \mathcal{L}^{(1)}\right)^{-k/2} DF\right)\right)_{L^2(\mathbb{R}_+)}\end{aligned}$$

$$\leq \|u\|_{\mathbb{D}_{p,k}(H)} \|DF\|_{\mathbb{D}_{q,-k}(H)}$$
$$\leq C_{p,k} \|u\|_{\mathbb{D}_{p,k}(H)} \|F\|_{\mathbb{D}_{q,-k+1}}$$

from Theorem (6.1.24) and Proposition (6.1.22), where $p, q > 1$ are such that $1/p + 1/q = 1$ and C_p, k is aconstant. Hence $\|\tilde{\delta} \circ i(u)\|_{\mathbb{D}^{\mathcal{F}}_{p,k-1}} \leq C_{p,k} \|u\|_{\mathbb{D}_{p,k}(H)}$. For the second relation, we have

$$\begin{aligned}(j \circ i \circ DF, u)_{L^2(B) \otimes l^2(\mathbb{N})} &= E\big[F \tilde{\delta} \circ i(u)\big] \\ &= E\big[(I + \mathcal{L})^{k/2} F (I + \mathcal{L})^{-k/2} \tilde{\delta} \circ i(u)\big] \\ &\leq \|F\|_{\mathbb{D}^{\mathcal{F}}_{p,k}} \|\tilde{\delta} \circ i(u)\|_{\mathbb{D}^{\mathcal{F}}_{q,-k}} \\ &\leq C_{p,k} \|F\|_{\mathbb{D}^{\mathcal{F}}_{p,k}} \|u\|_{\mathbb{D}^{\mathcal{F}}_{q,-k+1}} u \in \mathcal{P}^*_{\mathcal{F}}, \ \ F \in \mathcal{P}_{\mathcal{F}}.\end{aligned}$$

Hence $\|j \circ i \circ DF\|_{\mathbb{D}^{\mathcal{F}}_{p,k-1}} \leq C_{p,k} \|F\|_{\mathbb{D}^{\mathcal{F}}_{p,k}}, F \in \mathcal{P}_{\mathcal{F}}$.

The main problem that we encounter in the extension of the Meyer inequalities to the case of higher derivatives lies with the definition of the iterated gradient $\widetilde{D}\widetilde{D}F$.In fact, even for $F \in \mathcal{P}_{\mathcal{F}}$, e.g. $F = \tau_0, \widetilde{D}F$ is a random indicator function and $\widetilde{D}\widetilde{D}F$ can not make sense as a random variable. To circumvent this difficulty, we choose to take

$$\|i \circ D^k F\|_{L^2(B) \otimes L^2(\mathbb{R}^k_+)},$$

where $D^k: L^2(B) \longrightarrow L^2(B) \otimes H^{\circ k}$is the k-th iteration of D, for the norm of the iterated gradient of $F \in \mathcal{P}_{\mathcal{F}}$. We are going to give an equivalence of norms betweenthe norm $\|\cdot\|_{\mathbb{D}^{\mathcal{F}}_{2,k}}$and the norm defined with $i \circ D^k$, for $p = 2$ and $k \geq 0$.

Lemma (6.1.26)[163]: Let $F \in \mathcal{P}_{\mathcal{F}}$. We have for $n \geq 1$ and $k_1, \dots, k_n \in \mathbb{N}$:

$$\begin{aligned}D_{k_1} \cdots D_{k_n} P_t F &= e^{-nt} P_t D_{k_1} \cdots D_{k_n} F \\ &\quad + (e^{-t} - 1) \sum_{j=1}^{j=n} e^{-jt} D_{k_1} \cdots D_{k_{n-j}} P_t D_{k_{n-j+1}} \cdots D_{k_n} D_{k_j} F\ .\end{aligned}$$

Proof. By induction. From Lemma (6.1.21), the result is true for $n = 1$. Assume that the relation is verified at the order $n \geq 1$. We have for $k_1, \dots, k_{n+1} \in \mathbb{N}$:

$$D_{k_1} \cdots D_{k_{n+1}} F = e^{-nt} D_{k_1} P_t D_{k_2} \cdots D_{k_n} F$$

$$+D_{k_1}\sum_{j=2}^{j=n} e^{-jt} D_{k_2}\cdots D_{k_{n-j-1}} P_t D_{k_{n-j}}\cdots D_{k_{n+1}} D_{k_j} F$$

$$= e^{-nt} D_{k_1} P_t D_{k_2}\cdots D_{k_n} F (e^{-t}-1) e^{-nt} D_{k_1} P_t D_{k_1}\cdots D_{k_{n+1}} F$$

$$+(e^{-t}-1)\sum_{j=2}^{j=n} e^{-jt} D_{k_1}\cdots D_{k_{n-j-1}} P_t D_{k_{n-j}}\cdots D_{k_{n+1}} D_{k_j} F$$

$$= e^{-nt} D_{k_1} P_t D_{k_2}\cdots D_{k_{n+1}} F$$

$$+(e^{-t}-1)\sum_{j=2}^{j=n} e^{-jt} D_{k_1}\cdots D_{k_{n-j-1}} P_t D_{k_{n-j}}\cdots D_{k_{n+1}} D_{k_j} F\,.$$

This shows that the equality is satisfied for any $n \geq 1$.

Proposition (6.1.27)[163]: For $k \in \mathbb{N}$, there exists $A_k, B_k > 0$such that for any Poisson polynomial functional $F \in \mathcal{P}_{\mathcal{F}}$,

$$A_k \|i \circ D^k F\|^2_{L^2(B)\otimes L^2(\mathbb{R}^{+k})}$$

$$\leq \|F\|^2_{\mathbb{D}^{\mathcal{F}}_{2,k}} \leq B_k \|i \circ D^k F\|^2_{L^2(B)\otimes L^2(\mathbb{R}^{+k})} + \|F\|^2_{L^2(B)}.$$

Proof. Let us write the discrete chaotic decomposition of F:

$$F = \sum_{n\geq 0} I_n(f_n)\,,$$

which gives

$$\|F\|^2_{\mathbb{D}^{\mathcal{F}}_{2,k}} = E[F(I+\mathcal{L})^k F] = \sum_{n\geq 0} (1+n)^k \|I_n(f_n)\|_2^2\,.$$

Taking $A_k = 1/((k+1)^k)$and $B_k = 1$, we have

$$A_k(k+1)^k \leq 1 + n(n-1)\cdots(n-k) \leq B_k(k+1)^k n > k.$$

Hence $\|F\|_{\mathbb{D}^{\mathcal{F}}_{2,k}}$is equivalent to

$$F \to (E[F\mathcal{L}(\mathcal{L}-I)\cdots(\mathcal{L}-(k-1)I)(\mathcal{L}-kI)F] + E[F^2])^{\frac{1}{2}}.$$

It remains to show that

$$E[F\mathcal{L}(\mathcal{L}-I)\cdots(\mathcal{L}-nI)F] = \|i\circ D^{n+1}F\|^2_{L^2(B)\otimes L^2(\mathbb{R}^{n+1}_+)}, \qquad n \geq 0.$$

We know that this statement is true for $n = 0$. Suppose that it is true at the rank n, and let us show that then it is also true at the rank $n+1$.

$$E[F\mathcal{L}(\mathcal{L}-I)\cdots(\mathcal{L}-(n+1)I)F]$$

$$= E\left[(i \circ D^n F, i \circ D^n (\mathcal{L} - (n+1)I)F)_{L^2(\mathbb{R}_+^n)}\right]$$

$$= E\left[\sum_{k_1,\dots,k_n} \tau_{k_1} \cdots \tau_{k_n} D_{k_1} \cdots D_{k_n} F D_{k_1} \cdots D_{k_n} (\mathcal{L} - (n+1)I)F\right]$$

$$= E\left[\sum_{k_1,\dots,k_n} \tau_{k_1} \cdots \tau_{k_n} D_{k_1} \cdots D_{k_n} F \left(\sum_{i=1}^{i=n} D_{k_i} + \mathcal{L}\right) D_{k_1} \cdots D_{k_n} F\right]$$

$$= E\left[\sum_{k_1,\dots,k_n} \tau_{k_1} \cdots \tau_{k_n} D_{k_1} \cdots D_{k_n} F \left(\sum_{i=1}^{i=n} D_{k_i}\right) D_{k_1} \cdots D_{k_n} F\right]$$

$$+ E\left[\sum_{k_1,\dots,k_n} \tau_{k_{n+1}} D_{k_{n+1}} \tau_{k_1} \cdots \tau_{k_n} D_{k_1} \cdots D_{k_n} F D_{k_1} \cdots D_{k_{n+1}} F\right]$$

$$= E\left[\sum_{k_1,\dots,k_{n+1}} \tau_{k_1} \cdots \tau_{k_{n+1}} D_{k_1} \cdots D_{k_{n+1}} F D_{k_1} \cdots D_{k_{n+1}} F\right]$$

$$= E\left[(i \circ D^{n+1} F, i \circ D^{n+1} F)_{L^2(\mathbb{R}_+^{n+1})}\right] \quad F \in \mathcal{P}_{\mathcal{F}},$$

where we used the relation

$$D_{k_1} \cdots D_{k_n} (\mathcal{L} - (n+1)I)F = \left(\sum_{i=1}^{i=n} D_{k_i} + \mathcal{L}\right) D_{k_1} \cdots D_{k_n} F,$$

Obtained by differentiating the result of Lemma (6.1.26).

We apply the criterion given in [173] for the independence of multiple Wiener integral in order to obtain similar results for discrete multiple Poisson stochastic integrals of the type $I_n(f_n), f_n \in l^2(\mathbb{N})^{\circ n}$. The following result allows to characterize the independence of discrete multiple Poisson stochastic integrals interms of the supports of their discrete time kernels.

Theorem (6.1.28)[163]: Let $f_n \in H^{\circ n}$and $g_m \in H^{\circ m}, m \geq n$. The Poisson functionals $I_n(f_n)$ and $I_m(g_m)$are independent if and only if

$$f_n(k_1, \dots, k_n) g_m(k_1, k_{n+1}, \dots, k_{n+m-1}) = 0, \quad \forall k_1, \dots, k_{n+m-1} \in \mathbb{N}$$

Proof. We have the following orthogonal decompositions for f_n and g_m:

$$f_n = \sum_{\substack{k_1 \neq \cdots \neq k_d \\ n_1+\cdots+n_d=n}} \alpha_{k_1,\dots,k_d}^{n_1,\dots,n_d} e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_d}^{n_d}$$

$$g_m = \sum_{\substack{l_1 \neq \cdots \neq l_p \\ m_1+\cdots+m_p=m}} \beta_{l_1,\dots,l_p}^{m_1,\dots,m_p} e_{l_1}^{\circ m_1} \circ \cdots \circ e_{l_p}^{m_p} .$$

From Proposition (6.1.2), the random variables $I_n(f_n)$ and $I_m(g_m)$ belong respectively to the$2n$-th and $2m$-th Wiener chaos. Denote by $\hat{f}_{2n}$and $\hat{g}_{2m}$the corresponding kernels.We have $I_n(f_n) = \hat{I}_{2n}(\hat{f}_{2n})$ and $I_m(g_m) = \hat{I}_{2m}(\hat{g}_{2m})$,i.e.

$$\hat{f}_{2n} = \sum_{\substack{k_1 \neq \cdots \neq k_d \\ n_1+\cdots+n_d=n}} \alpha_{k_1,\dots,k_d}^{n_1,\dots,n_d} f_{k_1,\dots,k_d}^{n_1,\dots,n_d}$$

and

$$\hat{g}_{2m} = \sum_{\substack{l_1 \neq \cdots \neq l_p \\ m_1+\cdots+m_p=m}} \beta_{l_1,\dots,l_p}^{m_1,\dots,m_p} g_{l_1,\dots,l_p}^{m_1,\dots,m_p} .$$

with

$$\hat{I}_{2n}\left(f_{k_1,\dots,k_d}^{n_1,\dots,n_d}\right) = I_n\left(e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_d}^{n_d}\right), \qquad \hat{I}_{2m}\left(g_{l_1,\dots,l_p}^{m_1,\dots,m_p}\right) = I_n\left(e_{l_1}^{\circ m_1} \circ \cdots \circ e_{l_p}^{m_p}\right).$$

From Proposition(6.1.2), we find explicitly

$$f_{k_1,\dots,k_d}^{n_1,\dots,n_d} = \sum_{\substack{0 \le i_1 \le n_1 \\ \cdots \\ 0 \le i_d \le n_d}} \frac{(-1)^n C_{n_1}^{i_1} \cdots C_{n_d}^{i_d}}{2^n \left(C_{2n_1}^{2i_1} \cdots C_{2n_d}^{2i_d}\right)^{1/2}} h_{\circ 2k_1}^{\circ 2i_1} \circ h_{\circ 2k_1+1}^{\circ 2n_1-2i_1} \circ \cdots \circ h_{\circ 2k_d}^{\circ 2i_d} \circ h_{\circ 2k_d+1}^{\circ 2n_d-2i_d}$$

and

$$g_{l_1,\dots,l_p}^{m_1,\dots,m_p} = \sum_{\substack{0 \le j_1 \le m_1 \\ \cdots \\ 0 \le j_p \le m_p}} \frac{(-1)^m C_{m_1}^{i_1} \cdots C_{m_p}^{j_p}}{2^m \left(C_{2m_1}^{2j_1} \cdots C_{2m_p}^{2j_p}\right)^{1/2}} h_{\circ 2l_1}^{\circ 2j_1} \circ h_{\circ 2l_1+1}^{\circ 2m_1-2j_1} \circ \cdots \circ h_{\circ 2l_p}^{\circ 2j_p} \circ h_{\circ 2l_p+1}^{\circ 2m_p-2j_p} .$$

From [173], $I_n(f_n)$ is independent of $I_m(g_m)$ if and only if $\hat{f}_{2n} \otimes_1 \hat{g}_{2n} = 0$ a.s., i.e.

$$\sum \alpha_{k_1,\dots,k_d}^{n_1,\dots,n_d} \beta_{l_1,\dots,l_p}^{m_1,\dots,m_p} f_{k_1,\dots,k_d}^{n_1,\dots,n_d} \otimes_1 g_{l_1,\dots,l_p}^{m_1,\dots,m_p} = 0,$$

which means

$$\alpha_{k_1,\dots,k_d}^{n_1,\dots,n_d} \beta_{l_1,\dots,l_p}^{m_1,\dots,m_p} f_{k_1,\dots,k_d}^{n_1,\dots,n_d} \otimes_1 g_{l_1,\dots,l_p}^{m_1,\dots,m_p} = 0$$

for $k_1 \neq \cdots \neq k_d$and $l_1 \neq \cdots \neq l_p$, since

$$\left\{f_{k_1,\dots,k_d}^{n_1,\dots,n_d} \otimes_1 g_{l_1,\dots,l_p}^{m_1,\dots,m_p} : k_1 \neq \cdots \neq k_d \ \ and \ \ l_1 \neq \cdots \neq l_p\right\}$$

is orthogonal in $L^2(\mathbb{R}_+)^{\circ n+m-2}$, due to the particular form of $f_{k_1,\dots,k_d}^{n_1,\dots,n_d}$and $g_{l_1,\dots,l_p}^{m_1,\dots,m_p}$.This condition is equivalent to $\alpha_{k_1,\dots,k_d}^{n_1,\dots,n_d}\beta_{l_1,\dots,l_p}^{m_1,\dots,m_p} = 0$if $\{k_1,\dots,k_d\} \cap \{l_1,\dots,l_p\} \neq \emptyset$,or

$$f_n(k_1,\dots,k_n)g_m(k_1,k_{n+1},\dots,k_{n+m-1}) = 0 \ \forall k_1,\dots,k_{n+m-1} \in \mathbb{N}.$$

We study the diffusion process associated with$-\mathcal{L}$, and show that itgives another example of a process whose hitting probabilities of open sets can be estimateds in terms of capacities, cf. [179], [180]. We start by introducing capacities on the Poisson space. The space $B$is endowed with the largest topology that makes $O \subset B$open in B if $\Xi^{-1}(O)$ is open in W. We can define the capacities $c_{r,p}$on Bas follows:

$$c_{r,p}(O) = \inf\left\{\|u\|_{\mathbb{D}_{p,r}^{\mathcal{F}}} : \Theta^{-1}u \geq 1_{\{O\}}P \ a.s.\right\}$$

for O open in B, and

$$c_{r,p}(A) = \inf\left\{c_{r,p}(O) : O \ open \ and \ A \subset O\right\}$$

for any subset A of B.

Let $\left(X_t^{(n)}\right)_{t\in\mathbb{R}_+^n}$denote the W-valued n-parameter Ornstein-Uhlenbeck process, i.e.

$$X_t^{(n)} = e^{-(t_1+\cdots+t_n)/2}W_{e^{t_1},\dots,e^{t_n}}^{(n+1)},$$

where $W^{(n+1)}$is the $(n+1)$-parameter Brownian sheet defined on a probability space (Ω, A, Q), cf. [181].

Proposition (6.1.29)[163]: Let $Y_t^{(n)} = \Xi\left(X_t^{(n)}\right), t \in \mathbb{R}_+^n$. The process $Y^{(n)}$is a B-valued P-symmetric n-parameter process with continuous paths. Its transition semi-groups aregiven by

$$P_t^i = \Theta^{-1}\exp(-t\mathcal{L})\Theta\,, \ \ t \in \mathbb{R}_+, \ \ i = 1,\dots,n.$$

Proof. We refer to the definitions in [179]. We know that $X^{(n)}$is a μ-symmetric n-parameter process. Let $\left(\mathcal{F}_t^i\right)_{t\in\mathbb{R}_+}, i = 1,\dots,n$ denote its associated filtrations. We have

(a) For any $t \in \mathbb{R}_+^n, Y_t^{(n)} \in \bigcup_{1\leq i\leq n}\mathcal{F}_{t_i}^i$ since $X_t^{(n)} \in \bigcup_{1\leq i\leq n}\mathcal{F}_{t_i}^i$, and the law of $Y_t^{(n)}$is P since $P = \Xi_*\mu$and $X_t^{(n)}$ has law μ.

(b) For any $1 \leq i \leq n$ and $F \in L^2(B,P)$, we have for $u \in \mathbb{R}_+^n$ and $a \in \mathbb{R}_+$:

$$E\left[F\left(Y^{(n)}_{u_1,\dots,u_i+a,\dots,u_n}\right)\middle|\mathcal{F}^i_{u_i}\right] = E\left[\Theta F\left(X^{(n)}_{u_1,\dots,u_i+a,\dots,u_n}\right)\middle|\mathcal{F}^i_{u_i}\right]$$
$$= e^{-a\hat{\mathcal{L}}}\Theta F\left(X^{(n)}_u\right)$$
$$= \Theta^{-1}e^{-a\hat{\mathcal{L}}}\Theta F\left(Y^{(n)}_u\right).$$

Applying the result of [179], [180], we obtain that the process $\left(Y^{(n)}_t\right)_{t\in\mathbb{R}_+^n}$ is another example of a process whose hitting probabilities can be estimated in terms of capacities:

Theorem (6.1.30)[163]: Let O be an open set in B. For $t \in \mathbb{R}_+^n$, there exists two constants $K_1, K_2 > 0$ depending only on t and $n \in \mathbb{N}$ such that

$$K_1 c_{n,2}(O) \leq Q\left(\exists s \in [0,t] : Y^{(n)}_s \in O\right) \leq K_2 c_{n,2}(O).$$

Proof. From [180], there exists $\widehat{K}_1, \widehat{K}_2 > 0$ such that

$$\widehat{K}_1 \hat{c}_{n,2}\left(\Xi^{-1}(O)\right) \leq Q\left(\exists s \in [0,t] : Y^{(n)}_s \in O\right) \leq \widehat{K}_2 \hat{c}_{n,2}\left(\Xi^{-1}(O)\right),$$

where $\hat{c}_{n,2}\left(\Xi^{-1}(O)\right)$ is the usual capacity on Wiener space, defined as

$$\hat{c}_{n,2}\left(\Xi^{-1}(O)\right) = \inf\left\{\|u\|_{\mathbb{D}_{2,n}} : u \geq 1_O \circ \Xi \ \ \mu - a.s.\right\}.$$

We need to show that $c_{n,2}(O)$ can be estimated in terms of $\hat{c}_{n,2}\left(\Xi^{-1}(O)\right)$. We have

$$c_{n,2}(O) \geq \inf\left\{\|u\|_{\mathbb{D}_{2,n}} : u \geq 1_O \circ \Xi \ \ \mu - a.s.\right\}$$
$$\geq \hat{c}_{n,2}\left(\Xi^{-1}(O)\right)$$
$$\geq K \inf\left\{\|u\|_{\mathbb{D}_{2,n}} : E[u|\mathcal{F}] \geq 1_O \circ \Xi \ \ \mu - a.s.\right\}$$
$$\geq K \inf\left\{\|E[u|\mathcal{F}]\|_{\mathbb{D}^{\mathcal{F}}_{2,n}} : u \in \mathbb{D}_{2,n} \ and \ E[u|\mathcal{F}] \geq 1_O \circ \Xi \ \ \mu - a.s.\right\}$$
$$= K\ c_{n,2}(O),$$

with $K > 0$. The last inequality comes from the continuity of $E[\cdot|\mathcal{F}]$ from $\mathbb{D}_{p,k}$ to $\mathbb{D}^{\mathcal{F}}_{p,k}$, cf. Proposition (6.1.3).

For $n = 1$, Proposition (6.1.29) shows that the diffusion process associated to $-\mathcal{L}$ is the B-valued process $Y = \left(\Xi\left(X^{(1)}_t\right)\right)_{t\geq 0}$. The coordinates of $(Y_t)_{t\in\mathbb{R}_+}$ are the square norms of independent two-dimensional Ornstein-Uhlenbeck processes, hence they satisfy the stochastic differential equation

$$dV_t = \sqrt{2V_t}dW_t + (1 - V_t)dt,$$

where $(W_t)_{t\in\mathbb{R}_+}$is a brownian motion. In the usual Poisson space interpretation, thetrajectories of $(Y_t)_{t\geq 0}$ take their values in a space of step functions whose interjumptimes move according to the square norms of independent 2-dimensional Ornstein-Uhlenbeck processes.

Section (6.2): Lie-Wiener Path Space:

The Wiener measure is known to be invariant under random isometries whose Malliavin gradient satisfies a quasi-nilpotence condition, cf. [183]. In particular, the Skorohod integral $\delta(Rh)$is known to have a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$and $R$is a random isometry of Hsuch that DRh is a.s. a quasi-nilpotent operator.Such results can be proved using the Skorohod integral operator $\delta$and its adjoint the Malliavin derivative Don the Wiener space, and have been recently recovered under simple conditions and with short proofs in [184] using moment identities and in[185] via an exact formula for the expectation of random Hermite polynomials. Indeed it is well known that the Hermite polynomial defined by its generating function

$$e^{xt-t^2\mu^2/2} = \sum_{n=0}^{\infty} \frac{t^2}{n!} H_n(x,m), \qquad x,t \in \mathbb{R},$$

satisfies the identity

$$E[H_n(X,\sigma^2)] = 0, \tag{2}$$

when $X \simeq \mathcal{N}(0,\sigma^2)$is a centered Gaussian random variable with variance $\sigma^2 \geq 0$,and that the generating function can be used to characterize the gaussianity of X.In [185], conditions on the process $(u_t)_{t\in\mathbb{R}_+}$have been deduced for the expectation$E[H_n(\delta(u), \|u\|^2)], n \geq 1$, to vanish. Such conditions cover the quasi-nilpotencecondition of [183] and include the adaptedness of $(u_t)_{t\in\mathbb{R}_+}$, which recovers the aboveinvariance result using the characteristic function of $\delta(u)$.

On the other hand, the Skorohod integral and Malliavin gradient can also be defined on the path space over a Lie group, cf. [186], [187], [188]. We prove an extension of (2) to the path space case, by computing in Theorem (6.2.11) the expectation

$$E[H_n(\delta(u), \|u\|^2)], \qquad n \geq 1,$$

of the random Hermite polynomial $H_n(\delta(u), \|u\|^2)$, where $\delta(u)$is the Skorohod integralof a possibly anticipating process $(u_t)_{t\in\mathbb{R}_+}$. This result also recovers the above conditions for the invariance of the path space measure, and extends the results of[185] and [184] to path spaces over Lie group.

In Corollaries (6.2.14) and (6.2.15) below we summarize our results in the derivation formula

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right] = \lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\mathrm{trace}(\nabla u)(I-\nabla u)^{-1}(Du)\right]$$
$$-\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle (I-\nabla u)^{-1}u, D\log\det{}_2(I-\nabla u)\rangle\right], \qquad (3)$$

for λ in a neighborhood of 0, in which D, ∇ respectively denote the Malliavin gradient and covariant derivative on path space, and $\det_2(I-\lambda\nabla u)$ denotes the Carleman-Fredholm determinant of $I-\lambda\nabla u$. When $\nabla u$is quasi-nilpotent in the sense of (17) below we have $\det_2(I-\lambda\nabla u)=1$, cf. Theorem 3.6.1 of [189], or [190], and the derivative(3) vanishes, which yields

$$E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right] = 1,$$

for λ in a neighborhood of 0, cf. Corollary (6.2.12). If in addition $\langle u,u\rangle$is a.s. constant, this implies

$$E\left[e^{\lambda\delta(u)}\right] = e^{-\frac{\lambda^2}{2}\|u\|^2}, \lambda \in \mathbb{R},$$

Showing that $\delta(u)$is centered Gaussian with variance $\|u\|^2$.

We review some notation on closable gradient and divergence operators, and associated commutation relations. We derive moment identities for the Skorohod integral on path spaces. We consider the expectation of Hermite polynomials, we derive Girsanov identities on path space.

We recall some notation on the Lie-Wiener path space, cf. [186], [187],[188], [29], and we prove some auxiliary results. Let Gdenote either $\mathbb{R}^d$or a compact connected d-dimensional Lie group with associated Lie algebra $\mathscr{G}$ identified to $\mathbb{R}^d$and equipped with an Ad-invariant scalar product on $\mathbb{R}^d \simeq \mathscr{G}$, also denoted by$(\cdot,\cdot)$. The commutator in $\mathscr{G}$is denoted by $[\cdot,\cdot]$. Let ad $(u)v = [u, v], u, v \in \mathscr{G}$, with $\mathrm{Ad} e^u = e^{\mathrm{ad} u}, u \in \mathscr{G}$.

The Brownian motion $(\gamma(t))_{t\in\mathbb{R}_+}$on $\mathscr{G}$ with paths in $\mathscr{C}_0(\mathbb{R}_+,\mathscr{G})$is constructed from $(B_t)_{t\in\mathbb{R}_+}$via the Stratonovich differential equation

$$\begin{cases} d\gamma(t) = \gamma(t) \odot dB_t \\ \gamma(0) = e, \end{cases}$$

where e is the identity element in G. Let $\mathbb{P}(\mathrm{G}) = \mathscr{C}_0(\mathbb{R}_+,\mathscr{G})$ denote the space of continuous G-valued paths starting at e, with the image measure of the Wiener measureby $I : (B_t)_{t\in\mathbb{R}_+} \mapsto (\gamma(t))_{t\in\mathbb{R}_+}$. Here we take

$$\mathscr{S} = \left\{F = f\big(\gamma(t_1), \dots, \gamma(t_n)\big):\ f \in \mathscr{C}_b^\infty(\mathrm{G}^n)\right\},$$

and

$$\mathscr{U} = \left\{\sum_{i=1}^{n} u_i F_i:\ \ F_i \in \mathscr{S};\ u_i \in L^2(\mathbb{R}_+;\mathscr{G}), i = 1, \dots, n, n \geq 1\right\}.$$

Next is the definition of the right derivative operator D.

Definition (6.2.1)[182]: For $F = f\big(\gamma(t_1), \dots, \gamma(t_n)\big) \in \mathscr{S}, f \in \mathscr{C}_b^\infty(\mathrm{G}^n)$, we let $DF \in L^2(\Omega \times \mathbb{R}_+;\mathscr{G})$ be defined as

$$\langle DF, v\rangle = \frac{d}{d\varepsilon} f\left(\gamma(t_1)e^{\varepsilon\int_0^{t_1} v_s ds}, \dots, \gamma(t_n)e^{\varepsilon\int_0^{t_n} v_s ds}\right)_{|\varepsilon=0}, \quad v \in L^2(\mathbb{R}_+,\mathscr{G}).$$

For $F \in \mathscr{S}$ of the form $F = f\big(\gamma(t_1), \dots, \gamma(t_n)\big)$ we also have

$$D_t F = \sum_{i=1}^{n} \partial_i f\big(\gamma(t_1), \dots, (t_n)\big)\mathbf{1}_{[0,t_i]}(t), \qquad t \geq 0.$$

The operator $D$is known to be closable and to admit an adjoint δ that satisfies

$$E[F\delta(v)] = E[\langle DF, v\rangle], \qquad F \in \mathscr{S};\ v \in \mathscr{U}, \tag{4}$$

cf. e.g. [186]. Let $\mathbb{D}_{p,k}(X), k \geq 1$, denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p(W, X\otimes H^{\otimes l})}, \qquad p \in [1, \infty],$$

where $H = L^2(\mathbb{R}_+,\mathscr{G})$, and $X \otimes H$denotes the completed symmetric tensor product of X and H. We also let $\mathbb{D}_{p,k} = \mathbb{D}_{p,k}(\mathbb{R}), p \in [1,\infty], k \geq 1$.

Next we turn to the definition of the covariant derivative on the path space $\mathbb{P}(\mathrm{G})$,cf. [186].

Definition (6.2.2)[182]: Let the operator ∇be defined on $u \in \mathbb{D}_{2,1}(H)$as

$$\nabla_s u_t = D_s u_t + \mathbf{1}_{[0,t]}(s)\mathrm{ad}u_t \in \mathscr{G} \otimes \mathscr{G}, \qquad s,t \in \mathbb{R}_+. \tag{5}$$

When $h \in H$we have

$$\nabla_s h_t = \mathbf{1}_{[0,t]}(s)\mathrm{ad}h_t, \qquad s,t \in \mathbb{R}_+,$$

and $ad\ v \in \mathscr{G} \otimes \mathscr{G}, v \in \mathscr{G}$, is the matrix

$$\left(\langle e_j,\mathrm{ad}(e_i)v\rangle\right)_{1\leq i,j\leq d} = \left(\langle e_j,[e_i,v]\rangle\right)_{1\leq i,j\leq d}.$$

The operator ad $(v)$is antisymmetric on $\mathscr{G}$ because$(\cdot,\cdot)$is Ad-invariant. In addition if $u = hF, h \in H, F \in \mathbb{D}_{2,1}$, we have

$$D_s u_t = D_s F \otimes h(t), \quad \mathrm{ad}u_t = F\mathrm{ad}h(t), \qquad s,t \in \mathbb{R}_+,$$

and

$$\begin{aligned}\langle e_i \otimes e_j, \nabla_s u_t\rangle &= \langle e_i \otimes e_j, \nabla_s (hF)(t)\rangle \\ &= \langle e_i \otimes e_j, D_s F \otimes h(t)\rangle + \mathbf{1}_{[0,t]}(s)F\langle e_i \otimes e_j,\mathrm{ad}h(t)\rangle \\ &= \langle h(t), e_j\rangle\langle e_i, D_s F\rangle + \mathbf{1}_{[0,t]}(s)F\langle e_j,\mathrm{ad}(e_i)h(t)\rangle \\ &= \langle h(t), e_j\rangle\langle e_i, D_s F\rangle + \mathbf{1}_{[0,t]}(s)F\langle e_j,[e_i,h(t)]\rangle,\end{aligned}$$

$i,j = 1,\dots,d$. In the commutative case we have ad $(v) = 0, v \in \mathscr{G}$, hence $\nabla = D$.

By (4) we have

$$(\nabla_v u)(t) := (\nabla u)v_t = \int_0^t (\nabla_s u_t)v_s ds\,, \qquad t \in \mathbb{R}_+,$$

is the covariant derivative of $u \in \mathscr{U}$in the direction $v \in L^2(\mathbb{R}_+;\mathscr{G})$, with $\nabla_v u \in L^2(\mathbb{R}_+;\mathscr{G})$, cf. [186] and Lemma (6.2.4) in [192].

It is known that $D$and ∇ satisfy the commutation relation

$$D\delta(u) = u + \delta(\nabla^* u), \tag{6}$$

for $u \in \mathbb{D}_{2,1}(H)$such that $\nabla^* u \in \mathbb{D}_{2,1}(H \otimes H)$, cf. e.g. [186]. On the other hand, the commutation relation (6) shows that the Skorohod isometry [193]

$$E[\delta(u)\delta(v)] = E[\langle u, v\rangle] + E\,[\mathrm{trace}(\nabla u)(\nabla v)], \qquad u, v \in \mathbb{D}_{2,1}(H), (7)$$

holds as a consequence of (6), cf. [186] and Theorem (6.2.3) in [192], where

$$\mathrm{trace}(\nabla u)(\nabla v) = \langle \nabla u, \nabla^* v\rangle_{H\otimes H} = \int_0^\infty \int_0^\infty \langle \nabla_s u_t, \nabla_t^\dagger v_s\rangle_{\mathbb{R}^d\otimes\mathbb{R}^d} ds dt\,,$$

and $\nabla_t^\dagger v_s$ denotes the transpose of the matrix $\nabla_t v_s, s, t \in \mathbb{R}_+$. Note also that we have

$$\nabla_s u_t = D_s u_t, \qquad s > t, \tag{8}$$

Note that for $u \in \mathbb{D}_{2,1}(H)$and $v \in H$ we have

$$(\nabla u)^k v(t) = \int_0^\infty \cdots \int_0^\infty \left(\nabla_{t_k} u_t \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_1} u_{t_2}\right) v_{t_1} dt_1 \cdots dt_k\,, \qquad t \in \mathbb{R}_+,$$

and

$$\mathrm{trace}(\nabla u)^k = \langle \nabla^\dagger u, (\nabla u)^{k-1}\rangle$$
$$= \int_0^\infty \cdots \int_0^\infty \langle \nabla_t^\dagger u_{t_1}, \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_1} u_{t_2}\rangle dt_1 \cdots dt_k,$$

$k \geq 2$.

In addition we have the following lemma, which will be used to apply our invariance results to adapted processes.

Lemma (6.2.3)[182]: Assume that the process $u \in \mathbb{D}_{2,1}(H)$ is adapted with respect to the Brownian filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Then we have

$$\mathrm{trace}(\nabla u)^k = \mathrm{trace}(\nabla u)^{k-1}(Du) = 0, \qquad k \geq 2.$$

Proof. For almost all $t_1, \ldots, t_{k+1} \in \mathbb{R}_+$there exists $i \in \{1, \ldots, k+1\}$such that $t_i > t_{i+1 \bmod k+1}$, and (7) yields

$$\nabla_{t_i} u_{t_{i+1 \bmod k+1}} = D_{t_i} u_{t_{i+1 \bmod k+1}} + \mathbf{1}_{\{0, t_{i+1 \bmod k+1}\}}(t_i)$$
$$= D_{t_i} u_{t_{i+1 \bmod k+1}}$$
$$= 0,$$

since $(u_t)_{t\in\mathbb{R}_+}$is adapted.

We close this section with three lemmas that will be used in the sequel.

Lemma (6.2.4)[182]: For any $u \in \mathbb{D}_{2,1}(H)$we have

$$\langle (\nabla u)v, u\rangle = \frac{1}{2}\langle v, D\langle u, u\rangle\rangle, \qquad v \in H.$$

Proof. We have

$$(\nabla^* u)u_t = \int_0^\infty (\nabla_t u_s)^\dagger u_s ds$$

$$= \int_0^\infty (D_t u_s)^\dagger u_s ds + \int_0^\infty \mathbf{1}_{[0,s]}(t)(\mathrm{ad}u_s)^\dagger u_s ds$$

$$= \int_0^\infty (D_t u_s)^\dagger u_s ds - \int_0^\infty \mathbf{1}_{[0,s]}(t)\mathrm{ad}\,(u_s)u_s ds$$

$$= \int_0^\infty (D_t u_s)^\dagger u_s ds$$

$$= (D^* u)u_t.$$

Next, the relation $D\langle u,u\rangle = 2(D^*u)u$ shows that

$$\langle(\nabla u)v,u\rangle = \langle(\nabla^* u)u,v\rangle$$

$$= \langle(\nabla^* u)u,v\rangle$$

$$= \frac{1}{2}\langle v, D\langle u,u\rangle\rangle.$$

Lemma (6.2.5)[182]: For all $u \in \mathbb{D}_{2,2}(H)$and $v \in \mathbb{D}_{2,1}(H)$ we have

$$\langle\nabla^* u, D((\nabla u)^k v)\rangle = \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=2}^{k+1} \frac{1}{i}\langle(\nabla u)^{k+1-i} v, D\mathrm{trace}(\nabla u)^i\rangle, k \in \mathbb{N}.$$

Proof. Note that we have the commutation relation $\nabla D = D\nabla$, and as a consequence for all $1 \le k \le n$ we have

$$\langle\nabla^* u, D((\nabla u)^k v)\rangle = \int_0^\infty \cdots \int_0^\infty \langle\nabla_{t_k}^\dagger u_{t_{k+1}}, D_{t_{k+1}}\big(\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} v_{t_0}\big)\rangle dt_0 \cdots dt_{k+1}$$

$$= \int_0^\infty \cdots \int_0^\infty \langle\nabla_{t_k}^\dagger u_{t_{k+1}}, \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_{k+1}$$

$$+ \int_0^\infty \cdots \int_0^\infty \langle\nabla_{t_k}^\dagger u_{t_{k+1}}, D_{t_{k+1}}\big(\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1}\big) v_{t_0}\rangle dt_0 \cdots dt_{k+1}$$

$$= \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \int_0^\infty \cdots \int_0^\infty$$

$$\langle \nabla_{t_k}^\dagger u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} \big(\nabla_{t_i} D_{t_{k+1}} u_{t_{i+1}}\big) \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0} \rangle dt_0 \cdots dt_{k+1}$$

$$= \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \int_0^\infty \cdots \int_0^\infty$$

$$\langle \nabla_{t_i} \langle \nabla_{t_k}^\dagger u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} D_{t_{k+1}} u_{t_{i+1}} \rangle, \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0} \rangle dt_0 \cdots dt_{k+1}$$

$$= \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (\nabla u)^i v, D\mathrm{trace}(\nabla u)^{k+1-i} \rangle.$$

Lemma (6.2.6)[182]: For all $u \in \mathbb{D}_{2,2}(H)$and $v \in \mathbb{D}_{2,1}(H)$ such that $\|\nabla u\|_{L^\infty(\Omega;H\otimes H)} < 1$we have

$$\langle \nabla^* u, D((I - \nabla u)^{-1} v) \rangle$$
$$= \mathrm{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) - \langle (I - \nabla u)^{-1} v, D \log \det{}_2 (I - \nabla u) \rangle$$

Proof. By Lemma (6.2.5) we have

$$\langle \nabla^* u, D((I - \nabla u)^{-1} v) \rangle = \sum_{n=0}^{\infty} \langle \nabla^* u, D((\nabla u)^n v)) \rangle$$

$$= \sum_{n=0}^{\infty} \mathrm{trace}((\nabla u)^{n+1} Dv) + \sum_{n=0}^{\infty} \sum_{i=2}^{n+1} \langle (\nabla u)^{n+1-i} v, D\mathrm{trace}(\nabla u)^i \rangle$$

$$= \mathrm{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) + \sum_{i=2}^{\infty} \frac{1}{i} \sum_{n=0}^{\infty} \langle (\nabla u)^n v, D\mathrm{trace}(\nabla u)^i \rangle$$

$$= \mathrm{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) + \sum_{i=2}^{\infty} \frac{1}{i} \langle (I - \nabla u)^{-1} v, D\mathrm{trace}(\nabla u)^i \rangle$$

$$= \mathrm{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) - \langle (I - \nabla u)^{-1} u, D \log \det{}_2 (I - \nabla u) \rangle,$$

Since $\det_2(I - \lambda \nabla u)$ satisfies

$$\det{}_2(I - \lambda \nabla u) = \exp\left(- \sum_{i=2}^{\infty} \frac{\lambda^i}{i} \mathrm{trace}(\nabla u)^i \right), (9)$$

cf. [194], which shows that

$$D\log\det_2(I-\lambda\nabla u) = -\sum_{i=2}^{\infty}\frac{\lambda^i}{i}\operatorname{trace}(\nabla u)^i.$$

The following moment identity extends Theorem (6.2.1) of [184] to the path space setting. The Wiener case is obtained by taking $\nabla = D$.

Corollary (6.2.7)[182]: Let $u \in \mathbb{D}_{\infty,2}(H)$ such that $\langle u, u\rangle$ is deterministic and

$$\operatorname{trace}(\nabla u)^k = \operatorname{trace}(\nabla u)^{k-1}(Du) = 0, \quad a.s., \quad k \geq 2. \tag{10}$$

Then $\delta(u)$ has a centered Gaussian distribution with variance $\langle u, u\rangle$.

In particular, under the conditions of Corollary (6.2.1), $\delta(Rh)$ has a centered Gaussian distributionwith variance $\langle h, h\rangle$ when $u = Rh, h \in H$, and R is a random mapping with values in the isometries of H, such that $Rh \in \bigcap_{p>1}\mathbb{D}_{p,2}(H)$and $\operatorname{trace}(DRh)^k = 0,\ k \geq 2$. In theWiener case this recovers Theorem 2.1-b) of [183], cf. also Corollary (6.2.2)of [62].

In addition, Lemma (6.2.3) shows that Condition (10) holds when the process u is adapted with respect to the Brownian filtration.

Next we prove Proposition (6.2.1) based on Lemmas (6.2.4), (6.2.5) and Lemma (6.2.9) below.

Proposition (6.2.8)[182]: For any $n \geq 1$and $u \in \mathbb{D}_{n+1,2}(H), v \in \mathbb{D}_{n+1,1}(H)$ we have

$$E[\delta(u)^n\delta(v)] = nE[\delta(u)^{n-1}\langle u, v\rangle]$$
$$+\frac{1}{2}\sum_{k=2}^{n}\frac{n!}{(n-k)!}E[\delta(u)^{n-k}\langle(\nabla u)^{k-2}v, D\langle u,u\rangle\rangle]$$
$$+\sum_{k=1}^{n}\frac{n!}{(n-k)!}E\left[\delta(u)^{n-k}\left(\operatorname{trace}\big((\nabla u)^{k+1}Dv\big)+\sum_{i=2}^{k}\frac{1}{i}\langle(\nabla u)^{k-i}v, D\operatorname{trace}(\nabla u)^i\rangle\right)\right]. \tag{11}$$

For $n = 1$ the above identity (11) coincides with the Skorohod isometry (7).

When $\langle u, u\rangle$ is deterministic, $u \in \mathbb{D}_{2,1}(H)$, and trace $(\nabla u)^k = 0$ a.s., $k \geq 2$, Proposition (6.2.8) yields

$$E[\delta(u)^{n+1}] = n\langle u, u\rangle E[\delta(u)^{n-1}], \qquad n \geq 1,$$

and by induction we have

$$E[\delta(u)^{2m}] = \frac{(2m)!}{2^m m!}\langle u,u\rangle^m, 0 \le 2m \le n+1,$$

and $E[\delta(u)^{2m+1}] = 0, 0 \le 2m \le n$, while $E[\delta(u)] = 0$ for all $u \in \mathbb{D}_{2,1}(H)$, hence the following corollary of Proposition (6.2.8).

Proof. Let $n \ge 1$and $u \in \mathbb{D}_{n+1,2}(H)$. We show that for any $n \ge 1$ and $u \in \mathbb{D}_{n+1,2}(H), v \in \mathbb{D}_{n+1,1}(H)$, we have

$$E[\delta(u)^n\delta(v)] = \sum_{k=1}^{n}\frac{n!}{(n-k)!}E[\delta(u)^{n-k}(\langle(\nabla u)^{k-1}v,u\rangle + \langle\nabla^* u, D((\nabla u)^{k-1}v)\rangle)]. (12)$$

We have $(\nabla u)^{k-1}v \in \mathbb{D}_{(n+1)/k,1}(H), \delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}$, and by Lemma (6.2.9) below applied to $F = 1$ we get

$$\begin{aligned} &E\left[\delta(u)^l\langle(\nabla u)^i v, D\delta(u)\rangle\right] - lE\left[\delta(u)^{l-1}\langle(\nabla u)^{i+1}v, D\delta(u)\rangle\right] \\ &= E\left[\delta(u)^l\langle(\nabla u)^i v,u\rangle\right] + E\left[\delta(u)^l\langle(\nabla u)^i v,\delta(\nabla^* u)\rangle\right] \\ &\quad -lE\left[\delta(u)^{l-1}\langle(\nabla u)^{i+1}v,u\rangle\right] - lE\left[\delta(u)^{l-1}\langle(\nabla u)^{i+1}v,\delta(\nabla^* u)\rangle\right] \\ &= E\left[\delta(u)^l\langle(\nabla u)^i v,u\rangle\right] + E\left[\delta(u)^l\langle\nabla^* u, D\big((\nabla u)^i v\big)\rangle\right], \end{aligned}$$

and applying this formula to $l = n-k$ and $i = k-1$ yields

$$\begin{aligned} &E[\delta(u)^n\delta(v)] = E[\langle v, D\delta(u)^n\rangle] = nE[\delta(u)^{n-1}\langle v, D\delta(u)\rangle] \\ &= \sum_{k=1}^{n}\frac{n!}{(n-k)!}\big(E[\delta(u)^{n-k}\langle(\nabla u)^{k-1}v, D\delta(u)\rangle] - (n-k)E[\delta(u)^{n-k-1}\langle(\nabla u)^k v, D\delta(u)\rangle]\big) \\ &= \sum_{k=1}^{n}\frac{n!}{(n-k)!}E([\delta(u)^{n-k}\langle(\nabla u)^{k-1}v,u\rangle] + E[\delta(u)^{n-k}\langle\nabla^* u, D((\nabla u)^{k-1}v)\rangle]). \end{aligned}$$

We conclude by applying Lemmas (6.2.4) and (6.2.5). The next lemma extends the argument of Lemma (6.2.3) in [184] to the path space case, including an additional random variable $F \in \mathbb{D}_{2,1}$.

Lemma (6.2.9)[182]: Let $F \in \mathbb{D}_{2,1}, u \in \mathbb{D}_{n+1,2}(H)$, and $v \in \mathbb{D}_{n+1,1}(H)$. For all $k, i \ge 0$ we have

$$\begin{aligned} &E\left[F\delta(u)^k\langle(\nabla u)^i v,\delta(\nabla^* u)\rangle\right] - kE\left[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1}v,\delta(\nabla^* u)\rangle\right] \\ &= kE\left[F\delta(u)^{k-1}\langle(\nabla u)^{i+1}v,u\rangle\right] + E\left[\delta(u)^k\langle(\nabla u)^{i+1}v,DF\rangle\right] + E\left[F\delta(u)^k\langle\nabla^* u, D\big((\nabla u)^i v\big)\rangle\right]. \end{aligned}$$

Proof. We have

$$E\left[F\delta(u)^k\langle(\nabla u)^i v,\delta(\nabla^* u)\rangle\right] - iE\left[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1}v,\delta(\nabla^* u)\rangle\right]$$

$$= E\big[\langle \nabla^* u, D\big(F\delta(u)^k (\nabla u)^i v\big)\rangle\big] - kE\big[F\delta(u)^{k-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle\big]$$

$$= kE\big[F\delta(u)^{k-1}\langle \nabla^* u, (\nabla u)^i v \otimes D\delta(u)\rangle\big] - kE\big[F\delta(u)^{k-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle\big]$$

$$+E\big[\delta(u)^k \langle \nabla^* u, D\big(F(\nabla u)^i v\big)\rangle\big]$$

$$= kE\big[F\delta(u)^{k-1}\langle \nabla^* u, (\nabla u)^i v \otimes u\rangle\big] + kE\big[F\delta(u)^{k-1}\langle \nabla^* u, (\nabla u)^i v \delta(\nabla^* u)\rangle\big]$$

$$-kE\big[F\delta(u)^{k-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle\big] + E\big[\delta(u)^k \langle \nabla^* u, D\big(F(\nabla u)^i v\big)\rangle\big]$$

$$= kE\big[F\delta(u)^{k-1}\langle (\nabla u)^{i+1} v, u\rangle\big] + E\big[\delta(u)^k \langle (\nabla u)^{i+1} v, DF\rangle\big]$$

$$+E\big[F\delta(u)^k \langle \nabla^* u, D\big((\nabla u)^i v\big)\rangle\big],$$

where we used the commutation relation (6).

The case of the left derivative D^L defined as

$$\langle D^L F, v\rangle = \frac{d}{d\varepsilon} f\left(e^{\varepsilon \int_0^{t_1} v_s ds}\gamma(t_1), \cdots, e^{\varepsilon \int_0^{t_n} v_s ds}\gamma(t_n)\right)_{|\varepsilon=0}, \qquad v \in L^2(\mathbb{R}_+, \mathscr{G}),$$

for $F = f\big(\gamma(t_1), \cdots, \gamma(t_n)\big) \in \mathscr{S}, f \in \mathscr{C}_b^\infty(\mathrm{G}^n)$, can be dealt with by application of the existing results on the flat Wiener space, using the expression of its adjoint the left divergence δ^L which can be written as

$$\delta^L(u) = \hat{\delta}(\mathrm{Ad}\gamma.u.)$$

using the Skorohod integral operator $\hat{\delta}$ on the flat space $\mathbb{R}^d$, cf. [187], [188], and §13.1of [191].

We extend the results of [185] on the expectation of Hermite polynomials to the path space framework. This also allows us to recover the invariance results in Corollary (6.2.10) and to derive a Girsanov identity in Corollary (6.2.12) as a consequence of the derivation formula stated in Proposition (6.2.13).

It is well known that the Gaussianity of $X$is not required for $E[H_n(X, \sigma^2)]$ to vanish when σ^2 is allowed to be random. Indeed, such an identity also holds in the random adapted case under the form

$$E\left[H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right)\right] = 0, \tag{13}$$

where $(u_t)_{t\in\mathbb{R}_+}$is a square-integrable process adapted to the filtration generated by$(B_t)_{t\in\mathbb{R}_+}$, since $H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right)$is the n-th order iterated multiple stochastic integral of $u_{t_1}\cdots u_{t_n}$with respect to $(B_t)_{t\in\mathbb{R}_+}$, cf. [195] and [196].

In Theorem (6.2.11) below we extend Relations (2) and (13) by computing the expectation of the random Hermite polynomial $H_n(\delta(u),\|u\|^2)$in the Skorohod integral $\delta(u), n\geq 1$. This also extends Theorem (6.2.3) of [185] to the setting of path spaces over Lie groups.

Corollary (6.2.10)[182]: Let $u\in\mathbb{D}_{n,2}(H)$such that $\nabla u : H\to H$ is a.s. quasi-nilpotent in the sense that

$$\operatorname{trace}(\nabla u)^k = \operatorname{trace}(\nabla u)^{k-1}(Du) = 0, \qquad k\geq 2, \tag{14}$$

or more generally that (15) holds. Then for any $n\geq 1$we have

$$E[H_n(\delta(u),\|u\|^2)] = 0.$$

As above, Lemma (6.2.3) shows that Corollary (6.2.10) holds when the process $(u_t)_{t\in\mathbb{R}_+}$isadapted with respect to the Brownian filtration, and this shows that (13) holds for the stochastic integral $\delta(u)$on path space when the process $(u_t)_{t\in\mathbb{R}_+}$is adapted.

Theorem (6.2.11)[182]: For any $n\geq 0$and $u\in\mathbb{D}_{n+1,2}(H)$ we have

$$E[H_{n+1}(\delta(u),\|u\|^2)] = \sum_{l=0}^{n-1}\frac{n!}{l!}E\left[\delta(u)^l\sum_{0\leq 2k\leq n-1-l}\frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k}\langle\nabla^* u, D((\nabla u)^{n-2k-l-1}u)\rangle\right].$$

Clearly, it follows from Theorem (6.2.1) that if $u\in\mathbb{D}_{n,2}(H)$and

$$\langle\nabla^* u, D((\nabla u)^k u)\rangle = 0, \qquad 0\leq k\leq n-2, \tag{15}$$

then we have

$$E[H_n(\delta(u),\|u\|^2)] = 0, \qquad n\geq 1, \tag{16}$$

which extends Relation (14) to the anticipating case. In addition, from Theorem (6.2.11) and Lemma (6.2.4) we have

$E[H_{n+1}(\delta(u),\|u\|^2)]$

$$= \sum_{l=0}^{n-1} \frac{n!}{l!} E\left[\delta(u)^l \sum_{0\le 2k\le n-1-l} \frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k} \text{trace}\Big((\nabla u)^{n-2k-l}(Du)\Big)\right]$$

$$+\sum_{l=0}^{n-1} \frac{n!}{l!} E\left[\delta(u)^l \sum_{0\le 2k\le n-1-l} \frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k} \sum_{i=2}^{n-2k-l} \frac{1}{i}\langle(\nabla u)^{n-2k-l-i}u, D\text{trace}(\nabla u)^i\rangle\right]. \quad (17)$$

As a consequence, Lemma (6.2.3) leads to the following corollary of Theorem (6.2.11), which extends Corollary (6.3.3) of [185] to the path space setting.

Proof. Step 1. We show that for any $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have

$$E[H_{nn+1}(\delta(u), \|u\|^2)] = \sum_{0\le 2k\le n-1} \frac{(-1)^k n!}{k!\,2^k(n-2k-1)!} E[\delta(u)^{n-2k-1}\langle u,u\rangle^k \langle u, \delta(\nabla^* u)\rangle]$$

$$+\sum_{0\le 2k\le n} \frac{(-1)^k n!}{k!\,2^k(n-2k-1)!} E[\delta(u)^{n-2k}\langle u, D\langle u,u\rangle^k\rangle]. \quad (18)$$

For $F \in \mathbb{D}_{2,1}$ and $k, l \geq 1$ we have

$$E[F\delta(u)^{l+1}] = \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[F\delta(u)^{l+1}]$$

$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[\langle u, D(\delta(u)^l F)\rangle]$$

$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l(l+1)}{2k} E[F\delta(u)^{l-1}\langle u, D\delta(u)\rangle] \frac{l+1}{2k} E[\delta(u)^l\langle u, DF\rangle]$$

$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l(l+1)}{2k} E[F\delta(u)^{l-1}\langle u,u\rangle]$$

$$-\frac{l(l+1)}{2k} E[F\delta(u)^{l-1}\langle u, \delta(\nabla^* u)\rangle] - \frac{l+1}{2k} E[\delta(u)^l\langle u, DF\rangle],$$

i.e.

$$E[F\delta(u)^{n-2k+1}] + \frac{(n-2k)(n-2k+1)}{2k} E[F\delta(u)^{n-2k-1}\langle u,u\rangle]$$

$$= \frac{n+1}{2k} E[F\delta(u)^{n-2k+1}] - \frac{(n-2k)(n-2k+1)}{2k} E[F\delta(u)^{n-2k-1}\langle u, \delta(\nabla^* u)\rangle]$$

$$-\frac{n-2k+1}{2k} E[\delta(u)^{n-2k}\langle u, DF\rangle].$$

Hence, taking $F = \langle u,u\rangle^k$, we get

$$E[\delta(u)^{n+1}] = E[\langle u, D\delta(u)^n\rangle]$$

$$= nE[\delta(u)^{n-1}\langle u, D\delta(u)\rangle]$$

$$= nE[\delta(u)^{n-1}\langle u,u\rangle] + nE[\delta(u)^{n-1}\langle u, \delta(\nabla^* u)\rangle]$$

$$= nE[\delta(u)^{n-1}\langle u, \delta(\nabla^* u)\rangle]$$

$$- \sum_{1\le 2k\le n+1} \frac{(-1)^k n!}{(k-1)!\,2^{k-1}(n+1-2k)!}\Big(E[\delta(u)^{n-2k+1}\langle u,u\rangle^k]$$

$$+ \frac{(n-2k+1)(n-2k)}{2k} E[\delta(u)^{n-2k-1}\langle u,u\rangle^{k+1}]\Big)$$

$$= nE[\delta(u)^{n-1}\langle u, \delta(\nabla^* u)\rangle]$$

$$- \sum_{1\le 2k\le n+1} \frac{(-1)^k n!}{(k-1)!\,2^{k-1}(n+1-2k)!}\Big(\frac{n+1}{2k} E[\delta(u)^{n-2k+1}\langle u,u\rangle^k]$$

$$- \frac{(n-2k+1)(n-2k)}{2k} E[\delta(u)^{n-2k-1}\langle u,u\rangle^k \langle u, \delta(\nabla^* u)\rangle]$$

$$- \frac{n-2k+1}{2k} E[\delta(u)^{n-2k} u, D\langle u,u\rangle^k]\Big)$$

$$= - \sum_{1\le 2k\le n+1} \frac{(-1)^k (n+1)!}{k!\,2^k(n+1-2k)!} E[\delta(u)^{n-2k+1}\langle u,u\rangle^k]$$

$$+ \sum_{0\le 2k\le n-1} \frac{(-1)^k n!}{k!\,2^k(n-2k-1)!} E[\delta(u)^{n-2k-1}\langle u,u\rangle^k u, \delta(\nabla^* u)]$$

$$+ \sum_{1\le 2k\le n} \frac{(-1)^k n!}{k!\,2^k(n-2k)!} E[\delta(u)^{n-2k} u, D\langle u,u\rangle^k]\,,$$

which yields (18) after using the identity (20).

Step 2.For $F \in \mathbb{D}_{2,1}$and $k, i \ge 0$, by Lemma (6.2.9) we have

$$E\big[F\delta(u)^k\langle(\nabla u)^i u, \delta(\nabla^* u)\rangle\big] - kE\big[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1} u, \delta(\nabla^* u)\rangle\big]$$

$$= kE\big[F\delta(u)^{k-1}\langle(\nabla u)^{i+1} u, u\rangle\big] + E\big[\delta(u)^k\langle(\nabla u)^{i+1} u, DF\rangle\big] + E\big[F\delta(u)^k\langle\nabla^* u, D\big((\nabla u)^i u\big)\rangle\big].$$

Hence, replacing k above with $l - i$, we get

$$E[F\delta(u)^l\langle u, \delta(\nabla^* u)\rangle] = l!\,E[F\langle(\nabla u)^l u, \delta(\nabla^* u)\rangle]$$

$$+ \sum_{i=0}^{l-1} \frac{l!}{(l-i)!}\Big(E\big[Fd\delta(u)^{l-i}\langle(\nabla u)^i u, \delta(\nabla^* u)\rangle\big] - (l-i)E\big[F\delta(u)^{l-i-1}\langle(\nabla^* u)^{i+1} u, \delta(\nabla^* u)\rangle\big]\Big)$$

$$= l!\,E[F\langle(\nabla u)^l u, \delta(\nabla^* u)\rangle] + \sum_{i=0}^{l-1} \frac{l!}{(l-i-1)!} E\big[F\delta(u)^{l-i-1}(\nabla u)^{i+1}\langle u,u\rangle\big]$$

$$+ \sum_{i=0}^{l-1} \frac{l!}{(l-i)!} E\big[\delta(u)^{l-i}\langle(\nabla u)^{i+1} u, DF\rangle\big] + \sum_{i=0}^{l-1} \frac{l!}{(l-i)!} E\big[F\delta(u)^{l-i}\langle\nabla^* u, D\big((\nabla u)^i u\big)\rangle\big]$$

$$= l!\,E[\langle(\nabla u)^{l+1}u, DF\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E\big[F\delta(u)^{l-i-1}(\nabla u)^{i+1}\langle u,u\rangle\big]$$

$$+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E\big[\delta(u)^{l-i}\langle(\nabla u)^{i+1}u, DF\rangle\big] + \sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E\big[F\delta(u)^{l-i}\langle\nabla^* u, D\big((\nabla u)^i u\big)\rangle\big]$$

$$= l!\,E[\langle(\nabla u)^{l+1}u, DF\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E\big[F\delta(u)^{l-i-1}\langle(\nabla u)^{i+1}u,u\rangle\big]$$

$$+\sum_{i=1}^{l}\frac{l!}{(l-i+1)!}E\big[\delta(u)^{l-i+1}\langle(\nabla u)^{i}u, DF\rangle\big] + \sum_{i=0}^{l}E\big[F\delta(u)^{l-i}\langle\nabla^* u, D\big((\nabla u)^i u\big)\rangle\big],$$

thus, letting $F=\langle u,u\rangle^k$and $l=n-2k-1$ above, and using (18) in Step 1, we get

$$E[H_{n+1}(\delta(u),\|u\|^2)] = \sum_{0\le 2k\le n-1}\frac{(-1)^k n!}{k!\,2^k(n-2k-1)!}E[\delta(u)^{n-2k-1}\langle u,u\rangle^k\langle u,\delta(\nabla^* u)\rangle]$$

$$+\sum_{1\le 2k\le n}\frac{(-1)^k n!}{k!\,2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u, D\langle u,u\rangle^k\rangle]$$

$$=\sum_{1\le 2k\le n}(-1)^k\frac{n!}{k!\,2^k}E[\langle(\nabla u)^{n-2k}u, D\langle u,u\rangle^k\rangle]$$

$$+\sum_{0\le 2k\le n-2}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E\big[\langle u,u\rangle^k\delta(u)^{n-2(k+1)-i}\langle(\nabla u)^{i+1}u,u\rangle\big]$$

$$+\sum_{1\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-i)!}E\big[\delta(u)^{n-2k-i}\langle(\nabla u)^{i}u, D\langle u,u\rangle^k\rangle\big]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E\big[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^* u, D\big((\nabla u)^i u\big)\rangle\big]$$

$$+\sum_{1\le 2k\le n-1}\frac{(-1)^k n!}{k!\,2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u, D\langle u,u\rangle^k\rangle]$$

$$=\sum_{1\le 2k\le n}(-1)^k\frac{n!}{k!\,2^k}E[\langle(\nabla u)^{n-2k}u, D\langle u,u\rangle^k\rangle]$$

$$-\sum_{0\le 2k\le n-2}\frac{(-1)^{k+1}}{(k+1)!\,2^{k+1}}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E\big[\delta(u)^{n-2(k+1)-i}\langle(\nabla u)^{i}u, D\langle u,u\rangle^{k+1}\rangle\big]$$

$$+\sum_{1\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-i)!}E\big[\delta(u)^{n-2k-i}\langle(\nabla u)^{i}u, D\langle u,u\rangle^k\rangle\big]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E\left[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^*u,D\left((\nabla u)^iu\right)\rangle\right]$$

$$=\sum_{0\le 2k\le n-2}\frac{(-1)^{k+1}}{(k+1)!\,2^{k+1}}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E\left[\delta(u)^{n-2(k+1)-i}(\nabla u)^iu,D\langle u,u\rangle^{k+1}\right]$$

$$+\sum_{1\le 2k\le n}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k}\frac{n!}{(n-2k-i)!}E\left[\delta(u)^{n-2k-i}\langle(\nabla u)^iu,D\langle u,u\rangle^k\rangle\right]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E\left[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^*u,D\left((\nabla u)^iu\right)\rangle\right]$$

$$=\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E\left[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^*u,D\left((\nabla u)^iu\right)\rangle\right],$$

where we applied Lemma (6.2.4) with $v=(\nabla u)^iu$, which shows that

$$\langle u,u\rangle^k\langle(\nabla u)^{i+1}u,u\rangle=\frac{1}{2}\langle u,u\rangle^k\langle(\nabla u)^iu,D\langle u,u\rangle\rangle=\frac{1}{2(k+1)}\langle(\nabla u)^iu,D\langle u,u\rangle^{k+1}\rangle.$$

In the sequel we let $\mathbb{D}_{\infty,2}(H)=\bigcap_{n\ge 1}\mathbb{D}_{n,2}(H)$. The next result follows from Theorem (6.2.11) and extends Corollary (6.2.4) of [185] with the same proof, which is omitted here.

Corollary (6.2.12)[182]: Let $u\in\mathbb{D}_{\infty,2}(H)$ with $E\left[e^{|\delta(u)|+\|u\|^2/2}\right]<\infty$, and such that $\nabla u:H\to H$ is a.s. quasi-nilpotent in the sense of (17) or more generally that (15) holds. Then we have

$$E\left[\exp\left(\delta(u)-\frac{1}{2}\|u\|^2\right)\right]=1. \tag{19}$$

Again, Relation (19) shows in particular that if $u\in\mathbb{D}_{\infty,2}(H)$ is such that $\|u\|$ is deterministic and (17) or more generally (15) holds, then we have

$$E\left[e^{\delta(u)}\right]=e^{-\frac{1}{2}\|u\|^2},$$

i.e. $\delta(u)$ has a centered Gaussian distribution with variance $\|u\|^2$.

As a consequence of Theorem (6.2.11) we also have the following derivation formula.

Proposition (6.2.13)[182]: Let $u \in \mathbb{D}_{\infty,2}(H)$ such that $E\left[e^{a|\delta(u)|+a^2\|u\|^2}\right] < \infty$ for some $a > 0$.Then we have

$$\frac{\partial}{\partial\lambda} E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right] = \lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle\nabla^* u, D((I-\lambda\nabla u)^{-1}u)\rangle\right],$$

for all $\lambda \in (-a/2, a/2)$such that $|\lambda| < \|\nabla u\|^{-1}_{L^\infty(\Omega;H\otimes H)}$.

Proof. From the identity

$$H_n(x,\mu) = \sum_{0\le 2k\le n} \frac{n!\,(-\mu/2)^k}{k!\,(n-2k)!} x^{n-2k}, \qquad x,\mu \in \mathbb{R}, \tag{20}$$

we get the bound

$$|H_n(x,\sigma^2)| \le \sum_{0\le 2k\le n} \frac{(-1)^k}{k!\,2^k} \frac{n!}{(n-2k)!} |x|^{n-2k}(-\sigma^2)^k = H_n(|x|,-\sigma^2),$$

hence

$$\begin{aligned} E\left[\sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} |H_{n+1}(\delta(u),\|u\|^2)|\right] &\le E\left[\sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} H_{n+1}(\delta(u),-\|u\|^2)\right] \\ &= E\left[(|\delta(u)| + \lambda\|u\|^2)e^{|\lambda\delta(u)|+\lambda^2\|u\|^2/2}\right] \\ &= E\left[e^{2\lambda\delta(u)+4\lambda^2\|u\|^2}\right] \\ &< \infty, \end{aligned}$$

hence by the Fubini theorem we can exchange the infinite sum and the expectation to obtain

$$\begin{aligned} \frac{\partial}{\partial\lambda} E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right] &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E[H_{n+1}(\delta(u),-\|u\|^2)] \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{l=0}^{n-1} \frac{n!}{l!} E\left[\delta(u)^l \sum_{0\le 2k\le n-1-l} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \langle\nabla^* u, D((\nabla u)^{n-2k-l-1}u)\rangle\right] \\ &= \lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle\nabla^* u, D((I-\lambda\nabla u)^{-1}u)\rangle\right]. \end{aligned}$$

In addition, Relation (17) yields the following result, in which $\det_2(I-\lambda\nabla u)$ denotes the Carleman-Fredholm determinant of $I-\lambda\nabla u$.

Corollary (6.2.14)[182]: Let $u \in \mathbb{D}_{\infty,2}(H)$such that $E\left[e^{a|\delta(u)|+a^2\|u\|^2}\right] < \infty$ for some $a > 0$. Then we have

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right]=\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\mathrm{trace}(\nabla u)(I-\lambda\nabla u)^{-1}(Du)\right]$$

$$-\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle (I-\lambda\nabla u)^{-1}u, D\log\det{}_2(I-\lambda\nabla u)\rangle\right],$$

for all $\lambda\in(a/2,a/2)$such that $|\lambda|<\|\nabla u\|^{-1}_{L^\infty(\Omega;H\otimes H)}$.

Proof. From Lemma (6.2.6) we have

$\lambda\langle\nabla^* u, D((I-\lambda\nabla u)^{-1}v)\rangle$

$$=\lambda\,\mathrm{trace}(\nabla u)(I-\lambda\nabla u)^{-1}(Du)-\lambda\langle (I-\lambda\nabla u)^{-1}u, D\log\det{}_2(I-\lambda\nabla u)\rangle.$$

When (17) or more generally (15) holds, Proposition (6.2.13) and Corollary (6.2.14) show that

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right]=0,$$

for λ in a neighborhood of 0, which recovers the result of Corollary (6.2.12).

On the Wiener space we have $\nabla=D$and we obtain the following corollary.

Corollary (6.2.15)[182]: Let $u\in\mathbb{D}_{\infty,2}(H)$ such that $E\left[e^{a|\delta(u)|+a^2\|u\|^2}\right]<\infty$ for some $a>0$. Then we have

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right]=-E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\frac{\partial}{\partial\lambda}\log\det{}_2(I-\lambda Du)\right]$$

$$-\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle (I-\lambda Du)^{-1}u, D\log\det{}_2(I-\lambda Du)\rangle\right],$$

for all $\lambda\in(-a/2,a/2)$such that $|\lambda|<\|Du\|^{-1}_{L^\infty(\Omega;H\otimes H)}$.

Proof. We note that (9) shows that

$$\begin{aligned}\frac{\partial}{\partial\lambda}\log\det{}_2(I-\lambda Du)&=-\sum_{n=2}^{\infty}\lambda^{n-1}\mathrm{trace}(Du)^n\\&=-\lambda\sum_{n=2}^{\infty}\lambda^n\langle D^*u,(Du)^{n+1}\rangle\\&=-\lambda\langle D^*u,(I-\lambda Du)^{-1}Du\rangle\\&=-\lambda\,\mathrm{trace}(Du)(I-\lambda Du)^{-1}(Du),\end{aligned}$$

and apply Corollary (6.2.14).

Section (6.3): Measure-Preserving Transformations on the Lie-Wiener-Poisson Spaces:

We work in the general framework of an arbitrary probability space$(\Omega, \mathcal{F}, \mu)$. We consider a linear space Sdense in $L^2(\Omega, \mathcal{F}, \mu)$, and a closable linearoperator

$$D : S \longmapsto L^2(\Omega; H),$$

with closed domain $\mathrm{Dom}(D)$ containing S, where $H = L^2(\mathbb{R}_+; \mathbb{R}^d)$ for some $d \geq 1$.We assume that

(H1)there exists a closable divergence (or Skorohod integral) operator

$$\delta : S \otimes H \longmapsto L^2(\Omega),$$

acting on stochastic processes, adjoint of D, with the duality relation

$$E[\langle DF, u\rangle_H] = E[F\delta(u)], \quad F \in \mathrm{Dom}(D), \quad u \in \mathrm{Dom}(\delta), \tag{21}$$

where $\mathrm{Dom}(\delta)$ is the domain of the closure of δ,

and we are interested in characterizing the distribution of $\delta(u)$ under a given choiceof covariance derivative operator ∇ associated to $D$and δ, cf. (24) below.

The canonical example for this setting is when (Ω, μ) is the d-dimensional Wiener space with the Wiener measure μ, which is known to be invariant under random isometries whose Malliavin gradient Dsatisfies a quasi-nilpotence condition, cf. [198],[199], and Corollary (6.3.7) and Relation (57) below. This property is an anticipating extension of the classical invariance of Brownian motion under adapted isometries.

In addition to the Wiener space, the general framework of this section covers both the Lie-Wiener space, for which the operators $D$and δ can be defined on the path space over a Lie group, cf. [200], [201], [202], and the discrete probability space of the Poisson process, cf. [203], [204], [205]. In those settings the distribution of $\delta(h)$ is given by

$$E\left[e^{i\delta(h)}\right] = \exp\left(\int_0^\infty \Psi\big(ih(t)\big)dt\right), \qquad h \in H = L^2(\mathbb{R}_+; \mathbb{R}^d),$$

where the characteristic exponent Ψ is $\Psi(z) = \|z\|^2/2$ on the Lie-Wiener space, and

$$\Psi(z) = e^z - z - 1, \qquad z \in \mathbb{C}, \tag{22}$$

in the Poisson case with $d = 1$.

In order to state our main results we make the following additional assumptions.

(H2) The operator D satisfies the chain rule of derivation

$$D_{tg}(F) = g'(F)D_tF, \qquad t \in \mathbb{R}_+, \quad g \in \mathcal{C}_b^1(\mathbb{R}), \quad F \in \mathrm{Dom}(D), \tag{23}$$

where $D_tF = (DF)(t), t \in \mathbb{R}_+$.

(H3) There exists a covariant derivative operator

$$\nabla: L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^d) \to L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^d \otimes \mathbb{R}^d)$$

with domain $\mathrm{Dom}(\nabla)$ such that $D, \delta$and r satisfy the commutation relation

$$D_t\delta(u) = u_t + \delta\left(\nabla_t^\dagger u\right), \tag{24}$$

for $u \in \mathrm{Dom}(\nabla)$ such that $\nabla_t^\dagger u \in \mathrm{Dom}(\delta), t \in \mathbb{R}_+$, where †denote matrix transposition in $\mathbb{R}^d \otimes \mathbb{R}^d$.

In this general framework we prove in Proposition (6.3.1) below the Laplace transform identity

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = \lambda E\left[e^{\lambda\delta(u)}\langle(I-\lambda\nabla u)^{-1}u, u\rangle\right] + \lambda E\left[e^{\lambda\delta(u)}\langle\nabla^* u, D((I-\lambda\nabla u)^{-1}u)\rangle\right],$$

for λ in a neighborhood on 0, without any requirement on the probability measure μ.As a consequence of Proposition (6.3.1), we derive in Propositions (6.3.6), (6.3.12) and (6.3.18) below a family of Laplace transform identities of the form

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = E\left[e^{\lambda\delta(u)}\langle\Psi'(\lambda u), u\rangle\right] + E\left[e^{\lambda\delta(u)}\,\langle(I-\lambda\nabla u)^{-1}u, D\int_0^\infty \Psi(\lambda u_t)dt\rangle\right]$$
$$+\lambda E\left[e^{\lambda\delta(u)}\langle\nabla^* u, D((I-\lambda\nabla u)^{-1}u)\rangle\right], \tag{25}$$

which hold on both the Lie-Wiener and Poisson spaces. These identities are obtained inductively from the integration by parts (21), by removing all occurrences of the stochastic integral operator δin factor of the exponential $e^{\lambda\delta(u)}$. We will study the relations between such identities and quasi-nilpotence and measure invariance in Corollaries (6.3.7), (6.3.13) and (6.3.15).

On the Lie-Wiener path spaces as well as on the Poisson space, Relation (25) involves a covariant derivative operator ∇, which appears in the commutation relation (24)of Condition (H3) above between $D$and δ, and the series

$$(I-\nabla u)^{-1}=\sum_{n=0}^{\infty}(\nabla u)^{n}, \qquad \|\nabla u\|_{L^2(\mathbb{R}_+^2)}<1, \tag{26}$$

cf. (40) below for the definition of the operator $(\nabla u)^n$on H. The proof of (25) relies on the relation

$$\langle (I-\nabla u)^{-1}v,u\rangle=\langle \Psi'(u),v\rangle+\langle (I-\nabla u)^{-1}v,D\int_0^\infty \Psi(u_t)dt\rangle, \tag{27}$$

$u\in \mathrm{Dom}(\nabla), v\in H$, cf. Lemmas (6.3.10) and (6.3.16) below, which holds on both the path space and the Poisson space, respectively for $\Psi(z)$ of the form $\Psi(z)=\|z\|^2/2$ or (22).

Under the condition

$$\langle \nabla^* u, D((\nabla u)^n u)\rangle=0, \qquad n\in\mathbb{N}, \tag{28}$$

Relation (25) reads

$$\frac{\partial}{\partial\lambda}E\big[e^{\lambda\delta(u)}\big]=E\big[e^{\lambda\delta(u)}\langle\Psi'(\lambda u),u\rangle\big]+E\left[e^{\lambda\delta(u)}\,\langle (I-\lambda\nabla u)^{-1}u,D\int_0^\infty \Psi(\lambda u_t)dt\rangle\right], \tag{29}$$

for λ in a neighborhood of zero. This is true in particular when $(u_t)_{t\in\mathbb{R}_+}$is adapted with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$generated by the underlying process, cf. Lemmas(6.3.8) and (6.3.14) below, in which case $\delta(u)$ is known to coincide with the forward Itô-Wiener, resp. Itô-Poisson, stochastic integral of $(u_t)_{t\in\mathbb{R}_+}$as recalled.

In Corollaries (6.3.7) and (6.3.15) we apply (29) to obtain sufficient conditions for the invariance of Gaussian and infinitely divisible distributions on the Lie-Wiener path spaces and on the Poisson space. In particular, whenever the exponent$\int_0^\infty \Psi(\lambda u_t)dt$ is deterministic, $\lambda\in\mathbb{R}$, and ∇u satisfies (28), Relation (29) shows that we have

$$\frac{\partial}{\partial\lambda}E\big[e^{\lambda\delta(u)}\big]=E\big[e^{\lambda\delta(u)}\big]\int_0^\infty \langle u_t,\Psi'(\lambda u_t)\rangle dt$$

$$= E\left[e^{\lambda\delta(u)}\right]\frac{\partial}{\partial\lambda}\int_0^\infty \Psi(\lambda u_t)dt\,, \qquad \lambda\in\mathbb{R},$$

which yields

$$E\left[e^{\delta(u)}\right] = \exp\left(\int_0^\infty \Psi(u_t)dt\right), \tag{30}$$

i.e. $\delta(u)$ is infinitely divisible with Lévy exponent $\int_0^\infty \Psi(u_t)dt$. Taking $u \in \bigcap_{p\geq 1} L^p(\mathbb{R}_+)$ to be a deterministic function, this also shows that the duality relation (21) in Hypothesis **(H1)** above and the definition of the gradient ∇ characterize the infinitely divisible law of $\delta(u)$.

In the Lie-Wiener case we also find the commutation relation

$$\langle\nabla^* v, (I-\nabla v)^{-1}Du - D((I-\nabla v)^{-1}u)\rangle = \langle (I-\nabla v)^{-1}, D\log\det{}_2(I-\nabla v)\rangle, \tag{31}$$

cf. Lemma (6.3.9) below, where

$$\det{}_2(I-\nabla u) = \exp\left(-\sum_{n=2}^\infty \frac{1}{n}\mathrm{trace}(\nabla u)^n\right) \tag{32}$$

is the Carleman-Fredholm determinant of $I-\nabla u$, cf. e.g. in [206].

In this case, Relations (31) and (32) allow us to rewrite (25) as

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = \lambda E\left[e^{\lambda\delta(u)}\langle u,u\rangle\right] + \frac{1}{2}\lambda^2 E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\rangle\right]$$
$$+\lambda E\left[e^{\lambda\delta(u)}\langle\nabla^* u, (I-\lambda\nabla u)^{-1}Du\rangle\right] - \lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, \log\det{}_2(I-\nabla v)\rangle\right], \tag{33}$$

cf. Proposition (6.3.6), which becomes

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = \lambda E\left[e^{\lambda\delta(u)}\langle u,u\rangle\right] + \frac{1}{2}\lambda^2 E\left[e^{\lambda\delta(u)}\langle (I-\lambda Du)^{-1}u, D\langle u,u\rangle\rangle\right]$$
$$-E\left[e^{\lambda\delta(u)}\frac{\partial}{\partial\lambda}\log\det{}_2(I-\lambda Du)\right] - \lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda Du)^{-1}u, D\log\det{}_2(I-\lambda Du)\rangle\right], \tag{34}$$

on the Wiener space, cf. Proposition (6.3.18), in which case we have $\nabla = D$.

As was noted in [207] the Carleman-Fredholm $\det_2(I-\lambda Du)$ is equal to 1 when thetrace

$$\mathrm{trace}(\nabla u)^n = \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_n} u_{t_1}, \nabla_{t_{n-1}} u_{t_n} \cdots \nabla_{t_1} u_{t_2}\rangle_{\mathbb{R}^d\otimes\mathbb{R}^d} dt_1 \cdots dt_n \tag{35}$$

vanishes for all $n \geq 2$, and Condition (28) can be replaced by assuming quasi-nilpotence condition

$$\operatorname{trace}(\nabla u)^n = \operatorname{trace}(\nabla u)^{n-1} Du = 0, \qquad n \geq 2, \tag{36}$$

cf. Corollary (6.3.7).

In this way, and by a direct argument, (30) extends to the Lie-Wiener space the sufficient conditions found in [198] for the Skorohod integral $\delta(Rh)$ on the Wiener spaceto have a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and $R$is a random isometry of $H$with quasi-nilpotent gradient, cf. Theorem (6.3.1) of [198]. Such results hold in particular when the process $(u_t)_{t\in\mathbb{R}_+}$is adapted with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$by Lemmas (6.3.8) and (6.3.14), and we extend them to the Lie-Wiener space. An example of anticipating process u satisfying (36) is also provided in (59) below on the Lie-Wiener space.

The results of this section also admit various finite-dimensional interpretations. For such an interpretation, let us restrict ourselves to the 1-dimensional Wiener space, consider an orthonormal family $e = (e_1, \dots, e_n)$ in $H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and the sequence

$$X_k = \int_0^\infty e_k(t)dt, \qquad k = 1, \dots, n,$$

of independent standard Gaussian random variables. We define u to be the process

$$u_t = \sum_{k=1}^{n} e_k(t) f_k(X_1, \dots, X_n), \qquad t \in \mathbb{R}_+,$$

where $f_k \in \mathcal{C}_b^1(\mathbb{R}^n), k = 1, \dots, n$. In that case, from (49) below we have

$$D_s u_t = \sum_{k=1}^{n}\sum_{l=1}^{n} e_k(t) e_l(s) \partial_l f_k(X_1, \dots, X_n) = \langle e(t), (\partial f) e^\dagger(s) \rangle_{\mathbb{R}^n}, \qquad s,t \in \mathbb{R}_+,$$

where

$$\partial f = (\partial_l f_k)_{k,l=1,\dots,n}$$

denotes the usual matrix gradient of the column vector $f = (f_1, \dots, f_n)^\dagger$on $\mathbb{R}^n$. We assume in addition that $\partial_l f_k = 0, 1 \leq k \leq l \leq n$, i.e. $\partial f$is strictly lower triangular and thus nilpotent. The divergence operator δ is then given by standard Gaussian integration by parts on $\mathbb{R}^n$as

$$\delta(u)=\sum_{k=1}^{n}X_k f_k(X_1,\dots,X_n)-\sum_{k=1}^{n}\partial_k f_k(X_1,\dots,X_n)=\sum_{k=1}^{n}X_k f_k(X_1,\dots,X_n)\,.$$

In that case, (28) and (36) are satisfied and (29) reads, letting $\bar{x}_n=(x_1,\dots,x_n)$,

$$\frac{\partial}{\partial\lambda}\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}(x_1^2+\cdots+x_n^2)+\lambda\sum_{k=1}^{n}x_k f_k(\bar{x}_n)\right)dx_1\cdots dx_n$$

$$=\lambda\int_{\mathbb{R}^n}\left(\sum_{k=1}^{n}|f_k(\bar{x}_n)|^2+\frac{\lambda}{2}\sum_{k,l=1}^{n}\sum_{p=0}^{n-1}\lambda^p((\partial f)^p f)_k(\bar{x}_n)\partial_k f_l^2(\bar{x}_n)\right)$$

$$\times\exp\left(-\frac{1}{2}(x_1^2+\cdots+x_n^2)+\lambda\sum_{k=1}^{n}x_k f_k(\bar{x}_n)\right)dx_1\cdots dx_n\,.$$

More complicated finite-dimensional identities can be obtained from (34) when ∂f is not quasi-nilpotent. On the other hand, simplifying to the extreme, if $n=2$ ande.g. $f_1=0$ and $f_2(x_1,x_2)=x_1$, we explicitly recover the calculus result

$$\frac{\partial}{\partial\lambda}\int_{\mathbb{R}^2}\exp\left(\lambda x_1x_2-\frac{1}{2}(x_1^2+x_2^2)\right)dx_1dx_2=2\pi\frac{\partial}{\partial\lambda}\frac{1}{\sqrt{1-\lambda^2}}$$

$$=\lambda\int_{\mathbb{R}^2}x_1^2\exp\left(\lambda x_1x_2-\frac{1}{2}(x_1^2+x_2^2)\right)dx_1dx_2=\frac{2\pi\lambda}{(1-\lambda^2)^{3/2}},\qquad\lambda\in(-1,1),$$

The path space setting is less suitable for finite-dimensional examples as the Lie-group valued Brownian motion is inherently infinite-dimensional with respect to the underlying $\mathbb{R}^d$-valued Wiener process. To some extent, the same is true of Poisson stochastic integrals, as they naturally depend on an infinity of jump times.

Indeed, this geometric framework also covers the Poisson distribution the use of a covariant derivative operator on the Poisson space, showing that the derivation property of the gradient operators is on the Lie-Wiener space is not characteristic of the Gaussianity of the underlying distribution. The results of this section can also be applied to the computation of moments for the Itô-Wiener and Poisson stochastic integrals [208]. A different family of identities has been obtained for Hermite polynomials and stochastic exponentials in $\delta(u)$ in [209] on the Wiener space and in[210] on the path space, see also [211] for the use of finite difference operators on the Poisson space.

We organized as follows. This section ends with a review of some notation on closable gradient and divergence operators and their associated commutation relations. We derive a general moment identity of the type (25), and we consider the setting of path spaces over Lie groups, which includes the Wiener space as a special case. We show that the general results also apply on the Poisson space. Finally we prove (34)and recover some classical Laplace identities for second order Wiener functionals in Proposition (6.3.19).

We close this introduction with some additional notation.

Given X a real separable Hilbert space, the definition of $D$is naturally extended to$X$-valued random variables by letting

$$DF = \sum_{k=1}^{n} x_i \otimes DF_i \ (37)$$

for$F \in X \otimes S \subset L^2(\Omega; X)$ of the form

$$F = \sum_{k=1}^{n} x_i \otimes F_i$$

$x_1, \dots, x_n \in X, F_1, \dots, F_n \in S$. When D maps S to $S \otimes H$, as on the Lie-Wiener space, iterations of this definition starting with $X = \mathbb{R}$, then $X = H$, and successively replacing $X$with $X \otimes H$at each step, allow one to define

$$D^n: X \otimes S \mapsto L^2\left(\Omega; X \widehat{\otimes} H^{\widehat{\otimes} n}\right)$$

for all $n \geq 1$, where $\widehat{\otimes}$ denotes the completed symmetric tensor product of Hilbert spaces. In that case we let $\mathbb{D}_{p,k}(X)$ denote the completion of the space $X \otimes S$ of X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p\left(\Omega; X \widehat{\otimes} H^{\widehat{\otimes} l}\right)}, \qquad p \geq 1, \qquad (38)$$

with

$$\mathbb{D}_{\infty,k}(X) = \bigcap_{k \geq 1} \mathbb{D}_{p,k}(X),$$

and $\mathbb{D}_{p,k} = \mathbb{D}_{p,k}(\mathbb{R}), p \in [1, \infty], k \geq 1$. Note that for all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$and $k \geq 1$, the gradient operator D is continuous from $\mathbb{D}_{p,k}(X)$ into

$\mathbb{D}_{q,k-1}(X \widehat{\otimes} H)$ and the Skorohod integral operator δ adjoint of D is continuous from $\mathbb{D}_{p,k}(H)$ in to $\mathbb{D}_{q,k-1}$.

Given $u \in \mathbb{D}_{2,1}(H)$ we also identify

$$\nabla u = \big((s,t) \longmapsto \nabla_t u_s\big)_{s,t\in\mathbb{R}_+} \in X \widehat{\otimes} H$$

to the random operator

$$\nabla u : H \longmapsto H$$
$$u \longmapsto (\nabla u)v = \big((\nabla u)v_s\big)_{s\in\mathbb{R}_+},$$

almost surely defined by

$$(\nabla u)v_s := \int_0^\infty (\nabla_t u_s) v_t dt\,, \qquad s \in \mathbb{R}_+,\ \ v \in H, \tag{39}$$

in which $a \otimes b \in X \widehat{\otimes} H$is identified to a linear operator from $a \otimes b : H \longmapsto X$ via

$$(a \otimes b)c = a\langle b, c\rangle_H, \qquad a \otimes b \in X \widehat{\otimes} H, \quad c \in H.$$

More generally, for $u \in \mathbb{D}_{2,1}(H)$ and $v \in H$ we have

$$(\nabla u)^k v_s = \int_0^\infty \cdots \int_0^\infty \big(\nabla_{t_k} u_s \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_1} u_{t_2}\big) v_{t_1} dt_1 \cdots dt_k\,, \qquad s \in \mathbb{R}_+, \tag{40}$$

i.e.

$$(\nabla u)^k = \left((s,t) \longmapsto \int_0^\infty \cdots \int_0^\infty \big(\nabla_{t_k} u_s \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_t u_{t_2}\big) dt_2 \cdots dt_k,\right)_{s,t\in\mathbb{R}_+} \in H \widehat{\otimes} H,$$

$k \geq 1$. We also define

$$\nabla^* u := \big((s,t) \longmapsto \nabla_s^\dagger u_t\big)_{s,t\in\mathbb{R}_+} \in H \widehat{\otimes} H$$

where $\nabla_s^\dagger u_t$ denotes the transpose matrix of $\nabla_s u_t$in $\mathbb{R}^d \otimes \mathbb{R}^d, s,t \in \mathbb{R}_+$, and we identify $\nabla^* u$ to the adjoint of ∇u on Hwhich satisfies

$$\langle (\nabla u)v, h\rangle_H = \langle v, (\nabla^* u)h\rangle_H, \qquad v,h \in H,$$

and is given by

$$(\nabla^* u)v_s = \int_0^\infty \big(\nabla_s^\dagger u_t\big) v_t dt\,, \qquad s \in \mathbb{R}_+,\ \ v \in L^2(W;H). \tag{41}$$

Although D is originally defined for scalar random variables, its definition extends pointwise to $u \in \mathbb{D}_{2,1}(H)$ by (37), i.e.

$$D(u) := \big((s,t) \mapsto D_t u_s\big)_{t,s\in\mathbb{R}_+} \in H \,\widehat{\otimes}\, H, \tag{42}$$

and the operators $Du$and D^*u are constructed in the same way as ∇u and $\nabla^* u$ in(39) and (41).

The commutation relation (24) shows that the Skorohod [212] isometry

$$E[\delta(u)^2] = E[\langle u,u\rangle_H] + E[\mathrm{trace}(\nabla u)^2], \qquad u \in \mathcal{U}, \tag{43}$$

holds, with

$$\mathrm{trace}(\nabla u)^k = \langle \nabla^* u, (\nabla u)^{k-1}\rangle, \qquad k \geq 2.$$

As will be recalled, such operators D, ∇, δcan be constructed inat least three instances, i.e. on the Wiener space, on the path space over a Lie group, and on the Poisson space for $k = 1$. In the sequel, all scalar products will be simply denoted by $\langle\cdot,\cdot\rangle$.

The results of this section rely on the following general Laplace identity (44) for the Skorohod integral operator δ, obtained in Proposition (6.3.1) using the adjoint gradient D and the covariant derivative ∇under Conditions **(H1)**, **(H2)**and **(H3)** above. Here we do not specify any underlying probability measure on Ω, so that the characteristic exponent $\Psi(z)$ plays no role in this section.

Proposition (6.3.1)[197]: Let $u \in \mathbb{D}_{\infty,1}(H)$such that for some $a > 0$ we have $E\left[e^{a|\delta(u)|}\right] < \infty$, and the power series (26) of $(I - \lambda\nabla u)^{-1}u$ converges in $\mathbb{D}_{2,1}(H)$ for all $\lambda \in (-a/2, a/2)$. Then we have

$$\frac{\partial}{\partial\lambda} E\left[e^{\lambda\delta(u)}\right] = \lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, u\rangle\right] + \lambda E\left[e^{\lambda\delta(u)}\langle \nabla^* u, D((I-\lambda\nabla u)^{-1})\rangle\right], \tag{44}$$

for all $\lambda \in (-a/2, a/2)$.

Proof. We start by showing that for any $u, v \in \mathbb{D}_{\infty,1}(H)$ such that the power series of $(I - \nabla v)^{-1}u$ converges in $\mathbb{D}_{2,1}(H)$and $E\left[e^{2|\delta(v)|}\right] < \infty$, we have

$$E\left[\delta(u)e^{\delta(v)}\right] = E\left[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, v\rangle\right] + E\left[e^{\delta(v)}\langle \nabla^* v, D((I-\nabla v)^{-1}u)\rangle\right]. \tag{45}$$

Indeed, Lemma (6.3.2) below shows that

$$E\left[\delta(u)e^{\delta(v)}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} E[\delta(v)^n \delta(u)]$$

$$= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{1}{(n-k)!}E[\delta(v)^{n-k}(\langle(\nabla v)^{k-1}u, v\rangle + \langle\nabla^* v, D((\nabla v)^{k-1}u)\rangle)]$$

$$= \sum_{k=1}^{\infty}E\left[e^{\delta(v)}(\langle(\nabla v)^{k-1}u, v\rangle + \langle\nabla^* v, D((\nabla v)^{k-1}u)\rangle)\right]$$

$$= E\left[e^{\delta(v)}\langle(I-\nabla v)^{-1}u, v\rangle\right] + E\left[e^{\delta(v)}\langle\nabla^* v, D((I-\nabla v)^{-1}u)\rangle.\right]$$

Hence, applying (45) for $u = v$ we get

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = E\left[\delta(u)e^{\lambda\delta(u)}\right]$$

$$= \lambda E\left[e^{\lambda\delta(u)}\langle(I-\lambda\nabla u)^{-1}u, u\rangle\right] + E\left[e^{\lambda(u)}\langle\nabla^* u, D((I-\lambda\nabla u)^{-1}u)\rangle\right], \quad (46)$$

$\lambda \in (a/2, a/2)$.

Finally we prove the next Lemma (6.3.3) which has been used in the proof of Lemma (6.3.2) below and extends Lemma (6.3.1) in [213] to include a random variable $F \in \mathbb{D}_{2,1}$.

We will describe the applications of Proposition (6.3.1) successively on the Lie-Wiener path space, on the Wiener space, and on the Poisson space. In order to prove Proposition (6.3.1) we will need the moment identity proved in the next Lemma (6.3.2).

Lemma (6.3.2)[197]: For any $n \in \mathbb{N}$and $F \in \mathbb{D}_{2,1}, u \in \mathbb{D}_{n+1,2}(H), v \in \mathbb{D}_{n+1,1}(H)$, we have

$$E[F\delta(u)^n\delta(v)] = \sum_{k=1}^{n}\frac{n!}{(n-k)!}E[F\delta(u)^{n-k}(\langle(\nabla u)^{k-1}v, u\rangle + \langle\nabla^* u, D((\nabla u)^{k-1}v)\rangle)]$$

$$+\sum_{k=0}^{n}\frac{n!}{(n-k)!}E[\delta(u)^{n-k}\langle(\nabla u)^k v, DF\rangle]. \quad (47)$$

Proof. We have $(\nabla u)^{k-1}v \in \mathbb{D}_{(n+1)/k,1}(H), \delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}$, and by Lemma (6.3.3) below we get

$$E\left[F\delta(u)^l\langle(\nabla u)^i v, D\delta(u)\rangle\right] - lE\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1}v, D\delta(u)\rangle\right]$$

$$= E\left[F\delta(u)^l\langle(\nabla u)^i v, u\rangle\right] + E\left[F\delta(u)^l\langle(\nabla u)^i v, (\nabla^* u)\rangle\right]$$

$$-lE\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1}v, u\rangle\right] - lE\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1}v, \delta(\nabla^* u)\rangle\right]$$

$$= E\left[F\delta(u)^l\langle(\nabla u)^i v, u\rangle\right] + E\left[F\delta(u)^l\langle\nabla^* u, D\big((\nabla u)^i v\big)\rangle\right] + E\left[\delta(u)^l\langle(\nabla u)^{i+1}v, DF\rangle\right],$$

and applying this formula to $l = n - k$and $i = k - 1$ via a telescoping sum yields

$$E[F\delta(u)^n\delta(v)] = E[F\langle v, D\delta(u)^n\rangle] + E[\delta(u)^n\langle v, DF\rangle]$$
$$= nE[F\delta(u)^{n-1}\langle v, D\delta(u)\rangle] + E[\delta(u)^n\langle v, DF\rangle]$$
$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!}\left(E[F\delta(u)^{n-k}\langle(\nabla u)^{k-1}v, D\delta(u)\rangle] - (n-k)E[F\delta(u)^{n-k-1}\langle(\nabla u)^k v, D\delta(u)\rangle]\right)$$
$$+E[\delta(u)^n\langle v, DF\rangle]$$
$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!}\left(E[F\delta(u)^{n-k}\langle(\nabla u)^{k-1}v, u\rangle] + E[F\delta(u)^{n-k}\langle\nabla^* u, D((\nabla u)^{k-1}v)\rangle]\right)$$
$$+\sum_{k=0}^{n} \frac{n!}{(n-k)!} E[\delta(u)^{n-k}\langle(\nabla u)^k v, DF\rangle].$$

Lemma (6.3.2) coincides with the Skorohod isometry (43) when $n = 1$.

Lemma (6.3.3)[197]: Let $F \in \mathbb{D}_{2,1}, u \in \mathbb{D}_{n+1,2}(H)$, and $v \in \mathbb{D}_{n+1,1}(H)$. For all $i, l \in \mathbb{N}$we have

$$E\left[F\delta(u)^l\langle(\nabla u)^i v, \delta(\nabla^* u)\rangle\right] - lE\left[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1}v, \delta(\nabla^* u)\rangle\right]$$
$$= lE\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1}v, u\rangle\right] + E\left[\delta(u)^l\langle(\nabla u)^{i+1}v, DF\rangle\right] + E\left[F\delta(u)^l\langle\nabla^* u, D\left((\nabla u)^i v\right)\rangle\right].$$

Proof. Using the duality (21) between $D$and δ, the chain rule of derivation (23) and the commutation relation (24), we have

$$E\left[F\delta(u)^l\langle(\nabla u)^i v, \delta(\nabla^* u)\rangle\right] - lE\left[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1}v, \delta(\nabla^* u)\rangle\right]$$
$$= E\left[\langle\nabla^* u, D\left(F\delta(u)^l(\nabla u)^i v\right)\rangle\right] - lE\left[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1}v, (\nabla^* u)\rangle\right]$$
$$= lE\left[F\delta(u)^{l-1}\langle\nabla^* u, (\nabla u)^i v \otimes D\delta(u)\rangle\right] - lE\left[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1}v, (\nabla^* u)\rangle\right]$$
$$+E\left[\delta(u)^l\nabla^* u, D\left(F(\nabla u)^i v\right)\right]$$
$$= lE\left[F\delta(u)^{l-1}\langle\nabla^* u, (\nabla u)^i v \otimes u\rangle\right] + lE\left[F\delta(u)^{l-1}\langle\nabla^* u, (\nabla u)^i v \otimes \delta(\nabla^* u)\rangle\right]$$
$$-lE\left[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1}v, (\nabla^* u)\rangle\right] + E\left[\delta(u)^l\langle\nabla^* u, D\left(F(\nabla u)^i v\right)\rangle\right]$$
$$= lE\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1}v, u\rangle\right] + E\left[\delta(u)^l\langle(\nabla u)^{i+1}v, DF\rangle\right]$$
$$+E\left[F\delta(u)^l\langle\nabla^* u, D\left((\nabla u)^i v\right)\rangle\right].$$

In the following sections we will reconsider Proposition (6.3.1) and its consequences in the Lie-Wiener and Poisson frameworks.

Let G denote either $\mathbb{R}^d$ or a compact connected d-dimensional Lie group with associated Lie algebra $\mathcal{G}$ identified to $\mathbb{R}^d$ and equipped with an Ad-invariant scalar producton $\mathbb{R}^d \simeq \mathcal{G}$, also denoted by $\langle\cdot,\cdot\rangle$, with $H = L^2(\mathbb{R}_+;\mathcal{G})$. The commutator in $\mathcal{G}$ isdenoted by $[\cdot,\cdot]$ and we let $\mathrm{ad}(u)v = [u, v], u, v \in \mathcal{G}$, with $\mathrm{Ad}e^u = e^{\mathrm{ad}u}, u \in \mathcal{G}$. Here, $\Psi(x) = \|x\|^2/2$.

The Brownian motion $\left(\gamma(t)\right)_{t\in\mathbb{R}_+}$ on G is constructed from a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ via the Stratonovich diffierential equation

$$\begin{cases} d\gamma(t) = \gamma(t)\odot dB_t \\ \gamma(0) = \mathrm{e}, \end{cases}$$

where e is the identity element in G. Let $\mathbb{P}(\mathrm{G})$ denote the space of continuous G-valued paths starting at e, endowed with the image of the Wiener measure by the mapping I : $(B_t)_{t\in\mathbb{R}_+} \longmapsto \left(\gamma(t)\right)_{t\in\mathbb{R}_+}$. Here we take

$$S = \left\{F = f\left(\gamma(t_1), \dots, \gamma(t_n)\right) \quad : \quad f \in \mathcal{C}_b^\infty(\mathrm{G}^n)\right\},$$

and

$$\mathcal{U} = S \otimes H = \left\{\sum_{i=1}^{n} u_i F_i : \quad F_i \in S,\;\; u_i \in L^2(\mathbb{R}_+;\mathcal{G}), i = 1, \dots, n, n \geq 1\right\}.$$

Next is the definition of the right derivative operator D, which satisfies Condition **(H2)**.

Definition (6.3.4)[197]: For F of the form

$$F = f\left(\gamma(t_1), \dots, (t_n)\right) \in S, \qquad f \in \mathcal{C}_b^\infty(\mathrm{G}^n), \tag{48}$$

we let $DF \in L^2(\Omega \times \mathbb{R}_+;\mathcal{G}) \simeq L^2(\Omega; H)$ be defined by

$$\langle DF, v\rangle = \frac{d}{d\varepsilon} f\left(\gamma(t_1)e^{\varepsilon\int_0^{t_1} v_s ds}, \dots, \gamma(t_n)e^{\varepsilon\int_0^{t_n} v_s ds}\right)_{|\varepsilon=0}, \;\; v \in L^2(\mathbb{R}_+,\mathcal{G}).$$

For F of the form (48) we also have

$$D_t F = \sum_{i=1}^{n} \partial_i f\left(\gamma(t_1), \dots, (t_n)\right)\mathbf{1}_{[0,t_i]}(t), \quad t \geq 0. \tag{49}$$

The operator D is known to admit an adjoint δ that satisfies Condition **(H1)**, i.e.

$$E[F\delta(v)] = E[\langle DF, v\rangle], \qquad F \in S, \;\; v \in L^2(\mathbb{R}_+,\mathcal{G}), \tag{50}$$

cf. e.g. [200]. The operator D is linked to the Malliavin derivative $\widehat{D}$ with respect to the underlying linear Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, cf. (51) below, and to its adjoint $\hat{\delta}$,via the relations

$$\langle h, DF\rangle = \langle h, \widehat{D}F\rangle + \hat{\delta}\left(\int_0^{\cdot} \mathrm{ad}\big(h(s)\big)ds\widehat{D}.F\right), \quad h \in H, \tag{51}$$

cf. e.g. Lemma (6.3.4) of [214], and

$$\delta(hF) = \hat{\delta}(hF) - \hat{\delta}\left(\int_0^{\cdot} \mathrm{ad}\big(h(s)\big)ds\widehat{D}.F\right), \ \ h \in H,$$

which follows from (51) by duality. When $(u_t)_{t\in\mathbb{R}_+}$is square-integrable and adapted with respect to the Brownian filtration, $\delta(u)$ coincides with the Itô integral of $u \in L^2(\Omega; H)$ with respect to the underlying Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, i.e.

$$\delta(u) = \int_0^{\infty} u_t dB_t\,.$$

Definition (6.3.5)[197]: The operator $\nabla\colon \mathbb{D}_{2,1}(H) \longmapsto L^2\big(\Omega; H \widehat{\otimes} H\big)$is defined as

$$\nabla_s u_t = D_s u_t + \mathbf{1}_{[0,t]}(s)\mathrm{ad}u_t \in \mathcal{G} \otimes \mathcal{G}, \qquad s, t \in \mathbb{R}_+, \tag{52}$$

$u \in \mathbb{D}_{2,1}(H)$.

In other words we have

$$\langle e_i \otimes e_j, \nabla_s(uF)(t)\rangle = \langle u_t, e_j\rangle\langle e_i, D_s F\rangle + \mathbf{1}_{[0,t]}(s)F\langle e_j,\mathrm{ad}(e_i)u_t\rangle,$$

$i, j = 1, \ldots, d$, where $(e_i)_{i=1,\ldots,d}$is an orthonormal basis of $\mathcal{G}$ and $\mathrm{ad}u \in \mathcal{G} \otimes \mathcal{G}, u \in \mathcal{G}$,is the matrix

$$\langle\big(e_j,\mathrm{ad}(e_i)u_i\big)\rangle_{1\le i,j\le d} = \big(\langle e_j, [e_i, u]\rangle\big)_{1\le i,j\le d}.$$

The operator ad(u) is antisymmetric on $\mathcal{G}$ because $\langle\cdot,\cdot\rangle$is Ad-invariant. By (52),

$$(\nabla u)v_t = \int_0^{\infty} (\nabla_s u_t)v_s ds\,, \qquad t \in \mathbb{R}_+,$$

is the covariant derivative of $u \in \mathcal{U}$ in the direction $v \in L^2(\mathbb{R}_+; \mathcal{G})$, with $\nabla_v u \in L^2(\mathbb{R}_+; \mathcal{G})$, cf. [200]. Note that if $u_t$is $\mathcal{F}_t$-measurable we have

$$\nabla_t u_t = D_s u_t + \mathbf{1}_{[0,t]}(s)\mathrm{ad}u_t = D_s u_t = 0, \qquad s > t. \tag{53}$$

It is known that $D$and ∇ satisfy Condition (H3) and the commutation relation (24), as well as the Skorohod isometry (43) as a consequence, cf. [200]. Proposition (6.3.6) below is a corollary of Proposition (6.3.1) and it yields (33) on the Lie-Wiener path space.

Proposition (6.3.6)[197]: Let $u \in \mathbb{D}_{1,2}(H)$such that $E\left[e^{a|\delta(u)|}\right] < \infty$ for some $a > 0$. We have

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = \lambda E\left[\langle u,u\rangle e^{\lambda\delta(u)}\right] + \frac{1}{2}\lambda^2 E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\rangle\right]$$
$$+\lambda E\left[e^{\lambda\delta(u)}\langle \nabla^* u, (I-\lambda\nabla u)^{-1}Du\rangle\right] - \lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\log\det{}_2(I-\lambda\nabla u)\rangle\right], \quad (54)$$

for$\lambda \in (a/2, a/2)$such that $|\lambda| < \|\nabla u\|^{-1}_{\mathbb{D}_{\infty,1}(H)}$.

Proof. Let $u \in \mathbb{D}_{\infty,1}(H)$ and $v \in \mathbb{D}_{1,2}(H)$ such that $\|\nabla v\|_{\mathbb{D}_{\infty,1}(H)} < 1$, and$E\left[e^{2|\delta(u)|}\right] < \infty$. From Relation (45) above and Lemma (6.3.10) below we have

$$E\left[\delta(u)e^{\delta(v)}\right] = E\left[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, v\rangle\right] + E\left[e^{\delta(v)}\langle \nabla^* v, D((I-\nabla v)^{-1}u)\rangle\right]$$
$$= E\left[\langle u,v\rangle e^{\delta(v)}\right] + \frac{1}{2}E\left[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, D\langle v,v\rangle\rangle\right] + E\left[e^{\delta(v)}\langle \nabla^* v, D((I-\nabla v)^{-1}u)\rangle\right].$$

As a consequence of Lemma (6.3.9) below, this yields

$$E\left[\delta(u)e^{\delta(v)}\right] = E\left[\langle u,v\rangle e^{\delta(v)}\right] + \frac{1}{2}E\left[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, D\langle v,v\rangle\rangle\right]$$
$$+E\left[e^{\delta(v)}\langle \nabla^* v, (I-\nabla v)^{-1}Du\rangle\right] - E\left[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, D\log\det{}_2(I-\nabla v)\rangle\right]. \quad (55)$$

Next, taking $v = \lambda u$with $|\lambda| < \|\nabla u\|^{-1}_{\mathbb{D}_{\infty,1}(H)}$in (55), we get

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = E\left[\delta(u)e^{\lambda\delta(u)}\right]$$
$$= \lambda E\left[\langle u,u\rangle e^{\lambda\delta(u)}\right] + \frac{1}{2}\lambda^2 E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\rangle\right]$$
$$+\lambda E\left[e^{\lambda\delta(u)}\langle \nabla^* u, (I-\lambda\nabla u)^{-1}Du\rangle\right]$$
$$-\lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\log\det{}_2(I-\lambda\nabla u)\rangle\right],$$

which yields (54).

When the operator $\nabla u : H \longmapsto H$ is quasi-nilpotent in the sense of (36), Proposition(6.3.6) shows that

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)}\right] = \lambda E\left[\langle u,u\rangle e^{\lambda\delta(u)}\right] + \frac{1}{2}\lambda^2 E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\rangle\right], \quad (56)$$

which is (29) with $\Psi(x) = \|x\|^2/2$.

In particular we have the following result, cf. Theorem (6.3.1) of [198] on the commutative Wiener space.

Corollary (6.3.7)[197]: Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u \rangle$ is deterministic and

$$\operatorname{trace}(\nabla u)^k = \operatorname{trace}(\nabla u)^{k-1}(Du) = 0, \qquad a.s., \quad k \geq 2. \tag{57}$$

Then $\delta(u)$ has a centered Gaussian distribution with variance $\langle u, u\rangle$.

Proof. Proposition (6.3.6) and Relation (56) show that when $\langle u, u\rangle$ is deterministic and $u \in \mathbb{D}_{2,1}(H)$,

$$\frac{\partial}{\partial \lambda} E\left[e^{\lambda \delta(u)}\right] = \lambda \langle u, u\rangle E\left[e^{\lambda\delta(u)}\right], \qquad \lambda \in \mathbb{R},$$

under Condition (57), which implies

$$E\left[e^{\lambda\delta(u)}\right] = e^{\lambda^2 \langle u,u\rangle/2}, \qquad \lambda \in \mathbb{R},$$

from which the conclusion follows.

Condition (57) holds in particular when the process $(u_t)_{t\in\mathbb{R}_+}$ is adapted, according to the next Lemma (6.3.8) which follows from (53).

Lemma (6.3.8)[197]: Assume that the process $u \in \mathbb{D}_{2,1}(H)$ is adapted with respect to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Then we have

$$\operatorname{trace}(\nabla u)^k = \operatorname{trace}(\nabla u)^{k-1}(Du) = 0, \qquad k \geq 2. \tag{58}$$

Proof. For almost all $t_1, \dots, t_{k+1} \in \mathbb{R}_+$ there exists $i \in \{f_1, \dots, k+1\}$ such that $t_i > t_{i+1} \operatorname{mod} k + 1$, and (53) yields

$$\begin{aligned}\nabla_{t_i} u_{t_{i+1} \operatorname{mod} k+1} &= D_{t_i} u_{t_{i+1} \operatorname{mod} k+1} + \mathbf{1}_{[0, t_{i+1} \operatorname{mod} k+1]}(t_i) \\ &= D_{t_i} u_{t_{i+1} \operatorname{mod} k+1} \\ &= 0,\end{aligned}$$

since $u_{t_{i+1} \operatorname{mod} k+1}$ is $\mathcal{F}_{t_{i+1} \operatorname{mod} k+1}$-measurable because $(u_t)_{t\in\mathbb{R}_+}$ is Ft-adapted, and this implies (58) by (35).

An anticipating example for Corollary (6.3.7) can be constructed by considering two orthonormal sequences $(e_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ that are also mutually orthogonal in H, and such that the sequence $(e_k)_{k\in\mathbb{N}} \subset E$ is made of commuting elements in $\mathcal{G}$, while $(e_k)_{k\in\mathbb{N}}$ may not commute with $(f_k)_{k\in\mathbb{N}}$. We let

$$u = \sum_{k=0}^{\infty} A_k e_k \in \mathbb{D}_{2,1}(H) \tag{59}$$

where $(A_k)_{k\in\mathbb{N}}$ is a sequence of $\sigma(\delta(f_k) : k \in \mathbb{N})$-measurable scalar random variables, satisfying

$$\sum_{k=0}^{\infty} |A_k|^2 = 1, \qquad a.s.$$

Then we have $\|u\|_H = 1$, a.s.,

$$\nabla_{t_2} u_{t_3} = \sum_{k=0}^{\infty} e_k(t_3) \otimes D_{t_2} A_k + \mathbf{1}_{[0,t_3]}(t_2) \sum_{k=0}^{\infty} A_k \mathrm{ad} e_k(t_3)$$

$$= \sum_{k,l=0}^{\infty} \langle DA_k, f_l \rangle e_k(t_3) \otimes f_l(t_2) + \mathbf{1}_{[0,t_3]}(t_2) \sum_{k=0}^{\infty} A_k \mathrm{ad} e_k(t_3)\,, t_2, t_3 \in \mathbb{R}_+,$$

and

$$D_{t_1} u_{t_2} = \sum_{k=0}^{\infty} e_k(t_2) \otimes D_{t_1} A_k = \sum_{k,l==0}^{\infty} \langle DA_k, f_l \rangle e_k(t_2) \otimes f_l(t_1)\,, \qquad t_1, t_2 \in \mathbb{R}_+,$$

hence

$$\nabla u_{t_3} \nabla_{t_1} u = \int_0^{\infty} \nabla_{t_2} u_{t_3} \nabla_{t_1} u_{t_2} dt_2$$

$$= \int_0^{\infty} \left(D_{t_2} u_{t_3} + \mathbf{1}_{[0,t_3]}(t_2) \mathrm{ad}(t_2) \mathrm{ad} u_{t_3}\right)\left(D_{t_1} u_{t_2} + \mathbf{1}_{[0,t_2]}(t_1) \mathrm{ad} u_{t_2}\right) dt_2$$

$$= \int_0^{\infty} D_{t_2} u_{t_3} D_{t_1} u_{t_2} dt_2$$

$$= \sum_{p,q,k,l=0}^{\infty} \langle DA_k, f_l \rangle \langle DA_p, f_q \rangle \langle f_q, e_k \rangle e_p(t_3) \otimes f_l(t_1)$$

$$= 0, \qquad t_1, t_3 \in \mathbb{R}_+,$$

since $\left[u_{t_2}, u_{t_3}\right] = 0, t_2, t_3 \in \mathbb{R}_+$. Similarly we have $\nabla_{u_{t_3}} D_{t_1} u = 0, t_1, t_3 \in \mathbb{R}_+$, and this shows that (58) holds.

Next we state and prove Lemma (6.3.9) which has been used in the proof of Proposition (6.3.6) and corresponds to the commutation relation (31).

Lemma (6.3.9)[197]: Let $u \in \mathbb{D}_{\infty,1}(H)$ and $v \in \mathbb{D}_{1,2}(H)$ such that $\|\nabla v\|_{\mathbb{D}_{\infty,1}(H)} < 1$. Then we have

$$\langle \nabla^* v, D((I-\nabla v)^{-1}u)\rangle = \langle \nabla^* v, (I-\nabla v)^{-1}Du\rangle - \langle (I-\nabla v)^{-1}, D\log\det_2(I-\nabla v)\rangle. \quad (60)$$

Proof. By the commutation relation $\nabla_s D_t = D_t\nabla_s, s,t \in \mathbb{R}_+$, for all $1 \le k \le n$ we have

$$\langle \nabla^* u, D((\nabla u)^k v)\rangle = \int_0^\infty \cdots \int_0^\infty \langle \nabla_{t_k}^\dagger u_{t_{k+1}}, D_{t_{k+1}}(\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} v_{t_0})\rangle dt_0 \cdots d_{t_{k+1}}$$

$$= \int_0^\infty \cdots \int_0^\infty \langle \nabla_{t_k}^\dagger u_{t_{k+1}}, \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} D_{t_{k+1}} v_{t_0}\rangle dt_0 \cdots d_{t_{k+1}}$$

$$+ \int_0^\infty \cdots \int_0^\infty \langle \nabla_{t_k}^\dagger u_{t_{k+1}}, D_{t_{k+1}}(\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1}) v_{t_0}\rangle dt_0 \cdots dt_{k+1}$$

$$= \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \int_0^\infty \cdots \int_0^\infty$$

$$\langle \nabla_{t_k}^\dagger u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} (\nabla_{t_i} D_{t_{k+1}} u_{t_{i+1}}) \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_{k+1}$$

$$= \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \int_0^\infty \cdots \int_0^\infty$$

$$\langle \nabla_{t_i} \langle \nabla_{t_k}^\dagger u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} D_{t_{k+1}} u_{t_{i+1}}\rangle, \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_{k+1}$$

$$= \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (\nabla u)^i v, D\mathrm{trace}(\nabla u)^{k+1-i}\rangle,$$

which shows that

$$\langle \nabla^* u, D((\nabla u)^k v)\rangle = \mathrm{trace}((\nabla u)^{k+1} Dv) + \sum_{i=2}^{k+1} \frac{1}{i} \langle (\nabla u)^{k+1-i} v, D\mathrm{trace}(\nabla u)^i\rangle,$$

$k \in \mathbb{N}$. This yields

$$\langle \nabla^* u, D((I-\nabla v)^{-1}u)\rangle = \sum_{k=0}^{\infty} \langle \nabla^* v, D(\nabla v)^k u\rangle$$

$$= \sum_{k=0}^{\infty} \mathrm{trace}((\nabla v)^{k+1} Du) + \sum_{k=0}^{\infty} \sum_{n=2}^{k+1} \frac{1}{n} \langle (\nabla v)^{k+1-n} u, D\mathrm{trace}(\nabla v)^n\rangle$$

$$= \frac{1}{2}\langle (I-\nabla v)^{-1}u, D\langle v,v\rangle\rangle$$

$$+\langle \nabla^* v, (I-\nabla v)^{-1}Du\rangle + \sum_{n=2}^{\infty}\frac{1}{n}\langle (I-\nabla v)^{-1}u, D\mathrm{trace}(\nabla v)^n\rangle$$

$$= \langle \nabla^* v; (I-\nabla v)^{-1}Du\rangle - \langle (I-\nabla v)^{-1}u, D\log\det{}_2(I-\nabla v)\rangle, \quad (61)$$

where we used the relation

$$D\log\det{}_2(I-\nabla v) = -\sum_{n=2}^{\infty}\frac{1}{n}D\mathrm{trace}(\nabla v)^n,$$

that follows from (32).

Next we prove Lemma (6.3.10) which has been used in the proof of Proposition (6.3.6), and corresponds to (27) on the path space with $\Psi(x) = \|x\|^2/2$.

Lemma (6.3.10)[197]: For any $u \in \mathbb{D}_{2,1}(H)$ with $\|\nabla u\|_{L^2(\mathbb{R}_+^2)} < 1$ a.s., we have

$$\langle (I-\nabla u)^{-1}v, u\rangle = \langle u,v\rangle + \frac{1}{2}\langle (I-\nabla u)^{-1}v, D\langle u,u\rangle\rangle, \qquad v \in H.$$

Proof. We first show that

$$\langle (\nabla u)v, u\rangle = \frac{1}{2}\langle v, D\langle u,u\rangle\rangle, \qquad u \in \mathbb{D}_{2,1}, \ \ v \in H. \quad (62)$$

Indeed we have

$$(\nabla^* u)u_t = \int_0^\infty (\nabla_t u_s)^\dagger u_s ds$$

$$= \int_0^\infty (D_t u_s)^\dagger u_s ds + \int_0^\infty \mathbf{1}_{[0,s]}(t)(\mathrm{ad}u_s)^\dagger u_s ds$$

$$= \int_0^\infty (D_t u_s)^\dagger u_s ds - \int_0^\infty \mathbf{1}_{[0,s]}(t)\mathrm{ad}\,(u_s)u_s ds$$

$$= \int_0^\infty (D_t u_s)^\dagger u_s ds$$

$$= (D^* u)u_t,$$

hence by the relation

$$D_t\langle u,u\rangle = \int_0^\infty D_t\langle u_s,u_s\rangle_{\mathbb{R}^d}ds$$

$$= 2\int_0^\infty \left(D_t^\dagger u_s\right)u_s ds$$

$$= 2(D^*u)u_t,$$

we get

$$(\nabla^* u)u_t = \frac{1}{2}D_t\langle u,u\rangle, \qquad t\in\mathbb{R}_+, \tag{36}$$

and by integration against $v(t)dt$ we find that

$$\langle(\nabla u)v,u\rangle = \langle(\nabla^* u)u,v\rangle = \langle(D^*u)u,v\rangle = \frac{1}{2}\langle v, D\langle u,u\rangle\rangle. \tag{64}$$

In addition, (62) easily extends to all powers of ∇u as

$$\langle(\nabla u)^n v,u\rangle = \frac{1}{2}\langle(\nabla u)^{n-1}v, D\langle u,u\rangle\rangle, \qquad n\geq 1. \tag{65}$$

Hence for any $u\in\mathbb{D}_{2,1}(H)$ such that $\|\nabla u\|_{\mathbb{D}_{\infty,1}(H)} < 1$ we have

$$\langle(I-\nabla u)^{-1}v,u\rangle = \sum_{n=0}^{\infty}\langle(\nabla u)^n v,u\rangle$$

$$= \langle u,v\rangle + \frac{1}{2}\sum_{n=1}^{\infty}\langle(\nabla u)^{n-1}v, D\langle u,u\rangle\rangle$$

$$= \langle u,v\rangle + \frac{1}{2}\sum_{n=0}^{\infty}\langle(\nabla u)^{n}v, D\langle u,u\rangle\rangle$$

$$= \langle u,v\rangle + \frac{1}{2}\langle(I-\nabla u)^{-1}v, D\langle u,u\rangle\rangle.$$

Conditions for the Skorohod integral on path space to have a Gaussian distribution have been obtained from (47) and Corollary (6.3.7). In this section we show that the general framework also includes other infinitely divisible distributions as we apply it to the Poisson space on $\mathbb{R}_+$, with $\Psi(x) = e^x - x - 1$.

Let $(N_t)_{t\in\mathbb{R}_+}$be a standard Poisson process with jump times $(T_k)_{k\geq 1}$, and $T_0 = 0$,generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$on a probability space $(\Omega,\mathcal{F}_\infty,P)$. The gradient $\widetilde{D}$ defined as

$$\widetilde{D}_t F = -\sum_{k=1}^{n} \mathbf{1}_{[0,T_k]}(t) \frac{\partial f}{\partial x_k}(T_1, \dots, T_n), \qquad (66)$$

for

$$F \in S := \{F = f(T_1, \dots, T_n) : f \in \mathcal{C}_b^\infty(\mathbb{R}^n)\},$$

has the derivation property and therefore satisfies Condition **(H2)**. In addition, the operator $\widetilde{D}$ has an adjoint $\tilde{\delta}$ that satisfies (21) and Condition **(H1)**, cf. [203], [204], [205],and §7.2 of [215]. Moreover, $\tilde{\delta}$ coincides with the compensated Poisson stochastic integral on square-integrable adapted processes $(u_t)_{t\in\mathbb{R}_+}$, i.e.

$$\tilde{\delta}(u) = \int_0^\infty u_t d(N_t - t),$$

The next definition of the covariant derivative ∇ in the jump case, cf. [216], is the counterpart of Definition (6.3.5). Here we let

$$\mathcal{U} = \left\{ \sum_{i=1}^{n} u_i F_i : \; F_i \in S, u_i \in \mathcal{C}_c^1(\mathbb{R}_+), i = 1, \dots, n, \;\; n \geq 1 \right\}.$$

Definition (6.3.11)[197]: Let the operator $\widetilde{\nabla}$ be defined as

$$\widetilde{\nabla}_s u_t := \widetilde{D}_s u_t - \dot{u}_t \mathbf{1}_{[0,t]}(s), \qquad s, t \in \mathbb{R}_+, \qquad u \in \mathcal{U}, \qquad (67)$$

where $\dot{u}_t$ denotes the derivative of $t \mapsto u_t$ with respect to t.

In other words, given a vector field $u \in \mathcal{U}$ of the form $u = \sum_{i=1}^n F_i h_i$ we have

$$\widetilde{\nabla}_s u_t := \sum_{i=1}^{n} h_i(t) \widetilde{D}_s F_i - F_i \dot{h}_i(t) \mathbf{1}_{[0,t]}(s), \qquad s, t \in \mathbb{R}_+,$$

and

$$(\widetilde{\nabla} u) v_t = \int_0^\infty v_s \widetilde{\nabla}_s u_t ds, \qquad t \in \mathbb{R}_+,$$

is the covariant derivative of $u \in \mathcal{U}$ in the direction $v \in L^2(\mathbb{R}_+)$, cf. [216]. The operator $\widetilde{D}$ defines the Sobolev spaces $\widetilde{\mathbb{D}}_{p,1}$ and $\widetilde{\mathbb{D}}_{p,1}(H), p \in [1, \infty]$, respectively by the Sobolev norms

$$\|F\|_{\widetilde{\mathbb{D}}_{p,1}} = \|F\|_{L^p(\Omega)} + \left\|\widetilde{D}F\right\|_{L^p(\Omega,H)}, \qquad F \in S,$$

and

$$\|u\|_{\widetilde{\mathbb{D}}_{p,1}(H)} = \|u\|_{L^p(\Omega,H)} + \left\|\widetilde{D}u\right\|_{L^p(\Omega,H\widehat{\otimes}H)} + E\left[\left(\int_0^\infty t|\dot{u}_t|^2 dt\right)^{p/2}\right]^{1/p},$$

$u \in \mathcal{U}$, with

$$\widetilde{\mathbb{D}}_{\infty,1}(H) = \bigcap_{p\geq 1} \widetilde{\mathbb{D}}_{p,1}(H).$$

In addition, the operators $\widetilde{\nabla}, \tilde{\delta}$and $\widetilde{D}$ satisfy the Skorohod isometry (43) under the form

$$E\left[\tilde{\delta}(u)^2\right] = E[\langle u,u\rangle_H] + E\left[\int_0^\infty \int_0^\infty \widetilde{\nabla}_s u_t \widetilde{\nabla}_t u_s ds dt\right], \qquad u \in \widetilde{\mathbb{D}}_{2,1}(H),$$

and the commutation relation

$$\widetilde{D}_t \tilde{\delta}(u) = u_t + \tilde{\delta}\big(\widetilde{\nabla}_t u\big), \qquad t \in \mathbb{R}_+,$$

which is the commutation relation (24) in Condition (H3), for $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ such that $\widetilde{\nabla}_t u \in \widetilde{\mathbb{D}}_{2,1}(H), t \in \mathbb{R}_+$, cf. Relation (53) and Proposition (6.3.3)in [216].

As a consequence of Proposition (6.3.1) we have the following result, which yields (25) in the Poisson case with $\Psi(x) = e^x - x - 1$.

Proposition (6.3.12)[197]: Let $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ such that the power series $\big(I - \lambda\widetilde{\nabla}u\big)^{-1}u$ converges in $\mathbb{D}_{2,1}(H)$, and

$$\sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int_0^\infty \|u_t^n\|_{\mathbb{D}_{2,1}} dt < \infty, \qquad \lambda \in (-a/2, a/2),$$

and $E\left[e^{a|\tilde{\delta}(u)|}\right] < \infty$for some $a > 0$. Then we have

$$\frac{\partial}{\partial\lambda} E\left[e^{\lambda\tilde{\delta}(u)}\right] = E\left[e^{\lambda\tilde{\delta}(u)}\langle e^{\lambda u} - 1, u\rangle\right]$$

$$+E\left[e^{\lambda\tilde{\delta}(u)}\left\langle\left(I - \lambda\widetilde{\nabla}u\right)^{-1}u, \widetilde{D}\int_0^\infty \left(e^{\lambda u_t} - \lambda u_t - 1\right)dt\right\rangle\right]$$

$$+\lambda E\left[e^{\lambda\tilde{\delta}(u)}\left\langle\widetilde{\nabla}^* u, \widetilde{D}\left(\left(I - \lambda\widetilde{\nabla}u\right)^{-1}u\right)\right\rangle\right],$$

for$\lambda \in (-a/2, a/2)$such that $|\lambda| < \left\|\widetilde{\nabla}u\right\|^{-1}_{L^\infty(\Omega;H\widehat{\otimes}H)}$.

Proof. We apply Proposition (6.3.1) and Lemma (6.3.16) below.

As a consequence of Lemma (6.3.14) below, when $(u_t)_{t\in\mathbb{R}_+}$is adapted with respect to the Poisson filtration$(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, we have

$$\langle\widetilde{\nabla}^* u, \widetilde{D}\left(\left(I-\lambda\widetilde{\nabla}u\right)^{-1}u\right)\rangle = 0,$$

in which case Proposition (6.3.12) yields

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\widetilde{\delta}(u)}\right] = E\left[e^{\lambda\widetilde{\delta}(u)}\langle e^{\lambda u}-1,u\rangle\right]$$

$$+E\left[e^{\lambda\widetilde{\delta}(u)}\langle\left(I-\lambda\widetilde{\nabla}u\right)^{-1}u,\widetilde{D}\int_0^\infty\left(e^{\lambda u_t}-\lambda u_t-1\right)dt\rangle\right], \qquad (68)$$

which is (29) on the Poisson space.

The next consequence of Proposition (6.3.12) is the Poisson analog of Corollary (6.3.7). It applies in particular to adapted process by Lemma (6.3.14) below.

Corollary (6.3.13)[197]: Let $(u_t)_{t\in\mathbb{R}_+}$be a process in $\widetilde{\mathbb{D}}_{\infty,1}(H)$ that satisfies Condition (69) below,i.e. $\langle\widetilde{\nabla}^* u, \widetilde{D}\left(\left(\widetilde{\nabla}u\right)^k u\right)\rangle = 0, k\geq 1$, and assume that$\int_0^\infty (u_t)^i dt$ is deterministic for all$i\geq 1$ and such that

$$\sum_{n=2}^\infty \frac{\lambda^n}{n!}\int_0^\infty \|u_t^n\|_{\mathbb{D}_{2,1}}dt < \infty,$$

$\lambda\in(-a/2,a/2)$, and $E\left[e^{a\left|\widetilde{\delta}(u)\right|}\right]<\infty$ for some $a>0$. Then $\tilde{\delta}(u)$ has an infinitely divisible distribution with cumulants $\left\{0,\int_0^\infty (u_t)^i dt\,, i\geq 2\right\}$.

Proof. Proposition (6.3.12) and Relation (68) show that

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\widetilde{\delta}(u)}\right] = \langle e^{\lambda u}-1,u\rangle E\left[e^{\lambda\widetilde{\delta}(u)}\right],$$

as n goes to infinity, which yields

$$E\left[e^{\lambda\widetilde{\delta}(u)}\right] = \exp\left(\int_0^\infty\left(e^{\lambda u_t}-\lambda u_t-1\right)dt\right), \qquad \lambda\in(-a/2,a/2)$$

from which the conclusion follows.

The next lemma is the Poisson analog of Lemma (6.3.9) on the Lie-Wiener space.

Lemma (6.3.14)[197]: Let $u, v \in \widetilde{\mathbb{D}}_{\infty,1}(H)$ be two processes adapted with respect to the Poisson filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, such that $\left(\widetilde{\nabla}u\right)^n u \in \mathbb{D}_{2,1}(H), n \geq 1$. Then we have

$$\langle \widetilde{\nabla}^* u, \widetilde{D}\left(\left(\widetilde{\nabla}u\right)^k v\right)\rangle = 0, \qquad k \in \mathbb{N}. \tag{69}$$

Proof. The proof of this lemma differs from the argument of Lemma (6.3.9) due to the fact that here, $\widetilde{D}_s u_t$ and $\widetilde{\nabla}_s u_t$ defined by (66) and (67) no longer belong to $\widetilde{\mathbb{D}}_{2,1}$, and $\widetilde{D}$ does not commute with $\widetilde{\nabla}$. We have

$$\widetilde{\nabla}_s u_t = \widetilde{D}_s u_t = 0, \qquad s \geq t,$$

since $(u_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted. Hence for all $k \geq 1$ we have, with the convention $\int_a^b f(s)ds = 0$ for $a > b$,

$$\begin{aligned}
&\langle \widetilde{\nabla}^* u, \widetilde{D}\left(\left(\widetilde{\nabla}u\right)^{k-1} v\right)\rangle \\
&\qquad= \int_0^\infty \int_0^\infty \widetilde{\nabla}_{t_{k-1}} u_{t_k} \widetilde{D}_{t_k} \int_0^\infty \cdots \int_0^\infty v_{t_0} \widetilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \widetilde{\nabla}_{t_0} u_{t_1} dt_0 dt_1 \cdots dt_k \\
&\qquad= \int_0^\infty \int_0^{t_k} \widetilde{\nabla}_{t_{k-1}} u_{t_k} \int_0^{t_{k-1}} \cdots \int_0^{t_1} \widetilde{D}_{t_k} v_{t_0} \widetilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \widetilde{\nabla}_{t_0} u_{t_1} dt_0 dt_1 \cdots dt_k \\
&\qquad\quad+ \int_0^\infty \int_0^{t_k} v_{t_0} \widetilde{\nabla}_{t_{k-1}} u_{t_k} \widetilde{D}_{t_k} \int_0^{t_{k-1}} \cdots \int_0^{t_1} \widetilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \widetilde{\nabla}_{t_0} u_{t_1} dt_0 dt_1 \cdots dt_k \\
&= \int_0^\infty \int_0^{t_0} \left(\widetilde{D}_{t_k} v_{t_0}\right) \int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \widetilde{\nabla}_{t_{k-1}} u_{t_k} \cdots \widetilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_k dt_0 \\
&\quad+ \int_0^\infty v_{t_0} \int_0^\infty \int_{t_0}^{t_k} \widetilde{\nabla}_{t_{k-1}} u_{t_k} \widetilde{D}_{t_k} \int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \widetilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \widetilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_k dt_0 \\
&\quad= \int_0^\infty v_{t_0} \int_0^\infty \int_{t_0}^{t_k} \widetilde{\nabla}_{t_{k-1}} u_{t_k} \widetilde{D}_{t_k} \int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \widetilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \widetilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_k dt_0 \\
&= 0,
\end{aligned}$$

since

$$\int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \widetilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \widetilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_{k-1}$$

is$\mathcal{F}_{t_k}$-measurable, cf. e.g. Lemma (6.3.2)in [215].

Examples of processes satisfying the conditions of Corollary (6.3.13) can be constructed by composition of a function of $\mathbb{R}_+$with an adapted process of measure-preserving transformations, as in the next consequence of Corollary (6.3.13), cf. also (70) below.

Corollary (6.3.15)[197]: Let $T > 0$ and $\tau : [0, T] \longmapsto [0, T]$ be an adapted process of measure-preserving transformations, such that $\tau_t \in \widetilde{\mathbb{D}}_{\infty,1}, t \in \mathbb{R}_+$, with

$$\sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int_0^T \|\tau_t^n\|_{L^2(\Omega)} dt < \infty,$$

$\lambda \in (-a/2, a/2)$, for some $a > 0$. Then for all $f \in \mathcal{C}_c^1([0, T]), \tilde{\delta}(f \circ \tau)$ has same distribution as the Poisson stochastic integral $\tilde{\delta}(f)$.

Proof. We check that $f \circ \tau \in \widetilde{\mathbb{D}}_{\infty,1}(H)$ by (23) and

$$\sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int_0^T \|f^n(\tau_t)\|_{\mathbb{D}_{2,1}} dt = \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int_0^T \|f'\|_\infty^n \int_0^T \|\tau_t^n\|_{L^2(\Omega)} dt$$

$$+\|f'\|_\infty \int_0^T \left\|\widetilde{D}\tau_t\right\|_{\mathbb{D}_{2,1}} dt \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \|f\|_\infty^{n-1}$$

$$< \infty,$$

$\lambda \in (-a/2, a/2)$, hence Corollary (6.3.13) can be applied as the condition $E\left[e^{a|\tilde{\delta}(u)|}\right] < \infty$ follows from $\left|\tilde{\delta}(f \circ \tau)\right| \leq \|f\|_\infty (T + N_T)$.

As a consequence of Corollary (6.3.15) the mapping $T_n \longmapsto \tau(T_n)$ preserves the Poissonmeasure, see e.g. Theorem (6.3.10.) of [217] for the classical version of the Girsanov theorem for adapted transformations of jump processes.

As an example of a process to which Corollary (6.3.15) can be applied, we can take

$$\tau_t = t\mathbf{1}_{[0,T_1]}(t) + (2t - T_1)\mathbf{1}_{(T_1, T_1/2+T/2]}(t)$$

$$+(2T+T_1-2t)\mathbf{1}_{(T_1/2+T/2,T]}(t)+t\mathbf{1}_{(T\vee T_1,\infty)}(t), \qquad (70)$$

$t\in\mathbb{R}$, for some $T>0$.

The following Lemma (6.3.16) has been used in the proof of Proposition (6.3.12), is the analog of Lemma (6.3.10) in the Lie-Wiener case and corresponds to the Poisson space version of the general identity (27). We note that

$$\begin{aligned}(\tilde{\nabla}^* u)u_t &= \int_0^\infty u_s \tilde{\nabla}_t u_s ds \\ &= \int_0^\infty u_s \tilde{D}_t u_s ds - \int_t^\infty u_s \dot{u}_s ds \\ &= \frac{1}{2}u_t^2 + \frac{1}{2}\int_0^\infty \tilde{D}_t u_s^2 ds \\ &= \frac{1}{2}u_t^2 + \frac{1}{2}\tilde{D}_t\langle u,u\rangle,\end{aligned}$$

for all $u\in\tilde{\mathbb{D}}_{2,1}(H)$, which corresponds to (63) on the Lie-Wiener space, and implies

$$\langle(\tilde{\nabla}u)v,u\rangle = \frac{1}{2}\langle v,u^2\rangle + \frac{1}{2}\langle v, D\langle u,u\rangle\rangle, \qquad v\in H,$$

Provided $u\in L^2(\mathbb{R}_+)\cap L^4(\mathbb{R}_+)$ a.s., cf. (62) on the Lie-Wiener space. In the next lemma we show that this relation can be extended to all powers of $\tilde{\nabla}u$as in (72) below, although the extension is more complex to obtain than (65) in the path space case.

Lemma (6.3.16)[197]: Let $u\in\tilde{\mathbb{D}}_{2,1}(H)$ such that

$$\sum_{n=2}^{\infty}\frac{1}{n!}\int_0^\infty \|u_t^n\|_{\tilde{\mathbb{D}}_{2,1}} dt < \infty,$$

and $\|\tilde{\nabla}u\|_{L^\infty(\Omega;H\widehat{\otimes}H)} < 1$. We have

$$\langle(I-\tilde{\nabla}u)^{-1}v,u\rangle = \langle e^u-1,v\rangle + \langle(I-\tilde{\nabla}u)^{-1}v, \tilde{D}\int_0^\infty (e^{u_t}-u_t-1)dt\rangle, \qquad (71)$$

$v\in H$.

Proof. We begin by showing that for all $n\in\mathbb{N}$ and $u\in\tilde{\mathbb{D}}_{2,1}(H)$ such that $u\in\bigcap_{k=1}^{2n+2}L^k(\mathbb{R}_+)$ a.s. we have

$$\langle(\tilde{\nabla} u)^n v, u\rangle = \frac{1}{(n+1)!}\int_0^\infty u_s^{n+1} v_s ds + \sum_{i=2}^{n+1} \frac{1}{i!}\langle(\tilde{\nabla} u)^{n+1-i} v, \tilde{D}\int_0^\infty u_t^i dt\rangle, \quad v \in H. \quad (72)$$

For all $n \geq 1$ we have

$$(\tilde{\nabla}^* u)^n u_{t_0} = \int_0^\infty \cdots \int_0^\infty u_{t_n} \tilde{\nabla}_{t_0} u_{t_1} \tilde{\nabla}_{t_1} u_{t_2} \cdots \tilde{\nabla}_{t_{n-1}} u_{t_n} dt_1 \cdots dt_n, \quad (73)$$

and we will show by induction on $1 \leq k \leq n+1$ that we have

$$(\tilde{\nabla}^* u)^n u_{t_0} = \sum_{i=2}^{k} \frac{1}{i!}\int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u_{t_{n+2-i}}^i dt_1 \cdots dt_{n+1-i}$$

$$+ \frac{1}{k!}\int_0^\infty \cdots \int_0^\infty u_{t_{n+1-k}}^k \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k}} u_{t_{n+1-k}} dt_1 \cdots dt_{n+1-k}, \quad (74)$$

which holds for $k = 1$ by (73), and yields the desired identity for $k = n+1$. Next, assuming that the identity (74) holds for some $k \in \{1, \ldots, n\}$, and using the relation

$$\tilde{\nabla}_{t_{n-k}} u_{t_{n+1-k}} = \tilde{D}_{t_{n-k}} u_{t_{n+1-k}} - \mathbf{1}_{[0, t_{n+1-k}]}(t_{n-k}) \dot{u}_{t_{n+1-k}}, \qquad t_{n-k}, t_{n+1-k} \in \mathbb{R}_+,$$

we have

$$(\tilde{\nabla}^* u)^n u_{t_0} = \sum_{i=2}^{k} \frac{1}{i!}\int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u_{t_{n+2-i}}^i dt_1 \cdots dt_{n+2-i}$$

$$+ \frac{1}{k!}\int_0^\infty \cdots \int_0^\infty u_{t_{n+1-k}}^k \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k}} u_{t_{n+1-k}} dt_1 \cdots dt_{n+1-k}$$

$$= \sum_{i=2}^{k} \frac{1}{i!}\int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u_{t_{n+2-i}}^i dt_1 \cdots dt_{n+2-i}$$

$$+ \frac{1}{k!}\int_0^\infty \cdots \int_0^\infty u_{t_{n+1-k}}^k \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} \tilde{D}_{t_{n-k}} u_{t_{n+1-k}} dt_1 \cdots dt_{n+1-k}$$

$$- \frac{1}{k!}\int_0^\infty \cdots \int_0^\infty \int_{t_{n-k}}^\infty \dot{u}_{t_{n+1-k}} u_{t_{n+1-k}}^k \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-1-k}} u_{t_{n-k}} dt_1 \cdots dt_{n+1-k}$$

$$= \sum_{i=2}^{k} \frac{1}{i!}\int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u_{t_{n+2-i}}^i dt_1 \cdots dt_{n+2-i}$$

$$+\frac{1}{(k+1)!}\int_0^\infty \cdots \int_0^\infty \widetilde{\nabla}_{t_0} u_{t_1} \cdots \widetilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} \widetilde{D}_{t_{n-k}} u_{t_{n+1-k}}^{k+1} dt_1 \cdots dt_{n+1-k}$$

$$-\frac{1}{(k+1)!}\int_0^\infty \cdots \int_0^\infty \widetilde{\nabla}_{t_0} u_{t_1} \cdots \widetilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} \int_{t_{n-k}}^\infty \left(u_t^{k+1}\right)' dt dt_1 \cdots dt_{n-k}$$

$$=\sum_{i=2}^{k+1} \frac{1}{i!}\int_0^\infty \cdots \int_0^\infty \widetilde{\nabla}_{t_0} u_{t_1} \cdots \widetilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \widetilde{D}_{t_{n+1-i}} u_{t_{n+2-i}}^{i} dt_1 \cdots dt_{n+2-i}$$

$$+\frac{1}{(k+1)!}\int_0^\infty \cdots \int_0^\infty u_{t_{n-k}}^{k+1} \widetilde{\nabla}_{t_0} u_{t_1} \cdots \widetilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} dt_1 \cdots dt_{n-k}$$

$$=\sum_{i=2}^{k+1} \frac{1}{i!}\left(\widetilde{\nabla}^* u\right)^{n+1-i} \widetilde{D}_t \int_0^\infty u_s^i ds + \frac{1}{(k+1)!}\left(\widetilde{\nabla}^* u\right)^{n-k} u_t^{k+1},$$

which shows by induction for $k = n$ that

$$\left(\widetilde{\nabla}^* u\right)^n u_t = \frac{1}{(n+1)!} u_t^{n+1} + \sum_{i=2}^{n+1} \frac{1}{i!}\left(\widetilde{\nabla}^* u\right)^{n+1-i} \widetilde{D}_t \int_0^\infty u_s^i ds, \qquad t \in \mathbb{R}_+, \qquad (75)$$

and (72) follows by integration with respect to $t \in \mathbb{R}_+$.

Next, by (72), for all $u \in \mathbb{D}_{2,1}(H)$, $v \in H$and $n \in \mathbb{N}$, by (75) we have

$$\langle \left(I - \widetilde{\nabla} u\right)^{-1} v, u \rangle = \sum_{n=0}^\infty \langle \left(\widetilde{\nabla} u\right)^n v, u \rangle$$

$$= \sum_{n=0}^\infty \frac{1}{(n+1)!}\int_0^\infty u_s^{n+1} v_s ds + \sum_{n=0}^\infty \sum_{i=2}^{n+1} \frac{1}{i!} \langle \left(\widetilde{\nabla} u\right)^{n+1-i} v, \widetilde{D} \int_0^\infty u_t^i dt \rangle$$

$$= \langle e^u - 1, v \rangle + \sum_{i=2}^\infty \frac{1}{i!} \langle \left(I - \widetilde{\nabla} u\right)^{-1} v, \widetilde{D} \int_0^\infty u_t^i dt \rangle$$

$$= \langle e^u - 1, v \rangle + \langle \left(I - \widetilde{\nabla} u\right)^{-1} v, \widetilde{D} \int_0^\infty (e^{u_t} - u_t - 1) dt \rangle,$$

Which shows (1).

We also have the following moment identity, which is the Poisson analog of Proposition (6.3.1) in [210], cf. also Lemma 1 of [211] for another version using finite difference operators.

Corollary (6.3.17)[197]: For any $n \geq 1, u, v \in \mathbb{D}_{n+1,2}(H)$and $F \in \widetilde{\mathbb{D}}_{2,1}$ we have

$$E\left[F\tilde{\delta}(u)^n\tilde{\delta}(v)\right] = \sum_{k=1}^{n}\binom{n}{k}E\left[F\tilde{\delta}(u)^{n-k}\int_0^\infty u_s^k v_s ds\right]$$
$$+\sum_{k=2}^{n}\frac{n!}{(n-k)!}\sum_{i=2}^{k}\frac{1}{i!}E\left[F\tilde{\delta}(u)^{n-k}\langle\left(\widetilde{\nabla}u\right)^{k-i}v,\widetilde{D}\int_0^\infty u_s^i ds\rangle\right]$$
$$+\sum_{k=1}^{n}\frac{n!}{(n-k)!}E\left[F\tilde{\delta}(u)^{n-k}\langle\widetilde{\nabla}^*u,\widetilde{D}\left(\left(\widetilde{\nabla}u\right)^{k-1}v\right)\rangle\right]$$
$$+\sum_{k=0}^{n}\frac{n!}{(n-k)!}E\left[\tilde{\delta}(u)^{n-k}\langle\left(\widetilde{\nabla}u\right)^{k}v,\widetilde{D}F\rangle\right].$$

Proof. This result is a consequence of Lemma (6.3.2) associated to Relation (75).

We consider the case where $\mathrm{G} = \mathbb{R}^d$and $\left(\gamma(t)\right)_{t\in\mathbb{R}_+} = (B_t)_{t\in\mathbb{R}_+}$is astandard $\mathbb{R}^d$-valued Brownian motion on the Wiener space $W = \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^d)$, in whichcase $\nabla$is equal to the Malliavin derivative $\widehat{D}$ defined by

$$\widehat{D}_t F = \sum_{i=1}^{n}\mathbf{1}_{[0,t_i]}(t)\partial_i f\left(B_{t_1}, \dots, B_{t_n}\right), \qquad t \in \mathbb{R}_+, \tag{76}$$

for $F$of the form

$$F = f\left(B_{t_1}, \dots, B_{t_n}\right), \tag{77}$$

$f \in \mathcal{C}_b^\infty(\mathbb{R}^n, X), t_1, \dots, t_n \in \mathbb{R}_+, n \geq 1$. Let $\hat{\delta}$ denote the Skorohod integral operator adjointof $\widehat{D}$, which coincides with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.

$$\hat{\delta}(u) = \int_0^\infty u_t dB_t\,,$$

When u is square-integrable and adapted with respect to the Brownian filtration. As a consequence of Proposition (6.3.6) we obtain the following derivation formula, which yields (34).

Proposition (6.3.18)[197]: Let $u \in \mathbb{D}_{\infty,2}(H)$such that $E\left[e^{a|\delta(u)|}\right] < \infty$for some $a > 0$. We have

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\hat{\delta}(u)}\right]=\lambda E\left[\langle u,u\rangle e^{\lambda\hat{\delta}(u)}\right]+\frac{1}{2}\lambda^2E\left[e^{\lambda\hat{\delta}(u)}\,\langle\left(I-\lambda\hat{D}u\right)^{-1}u,\hat{D}\langle u,u\rangle\rangle\right]$$
$$-\lambda E\left[e^{\lambda\hat{\delta}(u)}\frac{\partial}{\partial\lambda}\log\det{}_2\left(I-\lambda\hat{D}u\right)\right]$$
$$-\lambda E\left[e^{\lambda\hat{\delta}(u)}\left(I-\lambda\hat{D}u\right)^{-1}u,\hat{D}\log\det{}_2\left(I-\lambda\hat{D}u\right)\right], \quad (78)$$

for$u\in\mathbb{D}_{\infty,2}(H)$such that $E\left[e^{a|\delta(u)|}\right]<\infty$ for some $a>0$and $\lambda\in(-a/2,a/2)$ such that $|\lambda|<\left\|\hat{D}u\right\|_{\mathbb{D}_{\infty,1}(H)}^{-1}$,

Proof. We apply Proposition (6.3.6) with $\nabla=\hat{D}$, and use the equality

$$\frac{\partial}{\partial\lambda}\log\det{}_2\left(I-\lambda\hat{D}u\right)=-\sum_{n=2}^{\infty}\lambda^{n-1}\mathrm{trace}\left(\hat{D}u\right)^n$$
$$=-\sum_{n=2}^{\infty}\lambda^{n-1}\,\langle\hat{D}^*u,\left(\hat{D}u\right)^{n-1}\rangle$$
$$=-\lambda\,\langle\hat{D}^*u,\left(I-\lambda\hat{D}u\right)^{-1}\hat{D}u\rangle\,,\qquad\lambda\in(-a,a), \qquad (79)$$

that follows from (32),

Next we show how (8) can be used to recover some known results on the Laplace transform of second order Wiener functionals of the form

$$\hat{\delta}(\psi)+\hat{\delta}\left(\hat{\delta}(\phi)\right)$$

where $\psi\in L^2(\mathbb{R}_+)$ and $\phi\in L^2(\mathbb{R}_+^2)$, cf. e.g. [218].

Proposition (6.3.19)[197]: Let $\psi\in L^2(\mathbb{R}_+)$and $\phi\in L^2(\mathbb{R}_+^2)$ such that $\|\phi\|_{L^2(\mathbb{R}_+^2)}<1$. We have

$$E\left[e^{\hat{\delta}(\psi)+\frac{1}{2}\hat{\delta}\left(\hat{\delta}(\phi)\right)}\right]=\frac{1}{\sqrt{\det_2(I-\phi)}}e^{\frac{1}{2}\langle\psi,(I-\phi)^{-1}\rangle}. \qquad (80)$$

Proof. We let $u_t=\frac{1}{2}\hat{\delta}\left(\phi(\cdot,t)\right),t\in\mathbb{R}_+$, and we start by showing that

$$E\left[e^{\delta(u)}\right]=\frac{1}{\sqrt{\det_2\left(I-2\hat{D}u\right)}}. \qquad (81)$$

Since $\hat{D}u=\phi/2$ is deterministic, by Proposition (6.3.6), Relation (78) we have

$$E\left[\hat{\delta}(u)e^{\lambda\hat{\delta}(u)}\right]=\lambda E\left[\langle u,u\rangle e^{\lambda\hat{\delta}(u)}\right]+\frac{\lambda^2}{2}E\left[e^{\lambda\hat{\delta}(u)}\,\langle\left(I-\lambda\hat{D}u\right)^{-1}u,\hat{D}\langle u,u\rangle\rangle\right]$$

$$+\lambda E\left[e^{\lambda\hat{\delta}(u)}\right]\langle\hat{D}^*u,\left(I-\lambda\hat{D}u\right)^{-1}\hat{D}u\rangle$$

$$=\lambda E\left[\langle\left(I-\lambda\hat{D}u\right)^{-1}\left(I-\lambda\hat{D}u\right)u,u\rangle\, e^{\lambda\hat{\delta}(u)}\right]+\lambda^2 E\left[e^{\lambda\hat{\delta}(u)}\langle\left(I-\lambda\hat{D}u\right)^{-1}u,\left(\hat{D}u\right)u\rangle\right]$$

$$+\lambda E\left[e^{\lambda\hat{\delta}(u)}\right]\langle\hat{D}^*u,\left(I-\lambda\hat{D}u\right)^{-1}\hat{D}u\rangle$$

$$=\lambda E\left[\langle\left(I-\lambda\hat{D}u\right)^{-1}u,u\rangle\, e^{\lambda\hat{\delta}(u)}\right]+\lambda E\left[e^{\lambda\hat{\delta}(u)}\right]\langle\hat{D}^*u,\left(I-\lambda\hat{D}u\right)^{-1}\hat{D}u\rangle$$

$$=\lambda E\left[\langle(I-\lambda u)^{-1}\hat{D}u,\hat{\delta}(u)\rangle e^{\lambda\hat{\delta}(u)}\right]+2\lambda E\left[e^{\lambda\hat{\delta}(u)}\right]\langle\left(I-\lambda\hat{D}u\right)^{-1}\hat{D}u,\hat{D}u\rangle, \qquad (82)$$

Since

$$u_s u_t=\hat{\delta}\left(\hat{D}u_s\right)\hat{\delta}\left(\hat{D}u_t\right)=\hat{\delta}\left(\hat{D}u_s\hat{\delta}\left(\hat{D}u_t\right)\right)+\langle\hat{D}u_s,\hat{D}u_t\rangle=\hat{D}u_s\hat{\delta}(u_t)+\langle\hat{D}u_s,\hat{D}u_t\rangle.$$

Hence by repeated application of (82) we get

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\hat{\delta}(u)}\right]=E\left[\hat{\delta}(u)e^{\lambda\hat{\delta}(u)}\right]$$

$$=2\lambda E\left[e^{\lambda\hat{\delta}(u)}\right]\sum_{n=0}^{\infty}\langle\hat{D}^*u,\left(\left(I-\lambda\hat{D}u\right)^{-1}\hat{D}u\right)^n\left(\left(I-\lambda\hat{D}u\right)^{-1}\hat{D}u\right)\rangle$$

$$=2\lambda E\left[e^{\lambda\hat{\delta}(u)}\right]\langle\hat{D}^*u,\left(I-2\lambda\hat{D}u\right)^{-1}\hat{D}u\rangle$$

$$=-\frac{1}{2}\frac{\partial}{\partial\lambda}\log\det{}_2\left(I-2\lambda\hat{D}u\right),$$

and (81) holds. Next, since $\hat{D}u\in L^2(\mathbb{R}_+^2)$is deterministic and $u=\hat{\delta}\left(\hat{D}u\right)$, from(61) we have, for $\psi\in L^2(\mathbb{R}_+)$,

$$E\left[\hat{\delta}(\psi)e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]=E\left[\langle\lambda\psi+u,\psi\rangle e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]$$

$$+\frac{1}{2}E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\langle\left(I-\hat{D}u\right)^{-1}\psi,\hat{D}\langle\lambda\psi+u,\lambda\psi+u\rangle\rangle\right]$$

$$=\lambda\langle\psi,\psi\rangle E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]+E\left[\langle u,\psi\rangle e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]+\lambda E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\langle(I-\hat{D}u)^{-1}\psi,\hat{D}\langle\psi,u\rangle\rangle\right]$$

$$+\frac{1}{2}E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\langle\left(I-\hat{D}u\right)^{-1}\psi,\hat{D}\langle u,u\rangle\rangle\right]$$

$$=\lambda\langle\psi,\psi\rangle E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]+\lambda\langle\left(I-\hat{D}u\right)^{-1}\psi,\left(\hat{D}u\right)\rangle E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]$$

$$+E\left[\langle u,\psi\rangle e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]+\ E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\langle\left(I-\hat{D}u\right)^{-1}\psi,\left(\hat{D}u\right)u\rangle\right]$$

$$=\lambda\langle\psi,\left(I-\hat{D}u\right)^{-1}\psi\rangle E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]+E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\langle\left(I-\hat{D}u\right)^{-1}\psi\rangle\right]$$

$$=\lambda\langle\psi,\left(I-\hat{D}u\right)^{-1}\psi\rangle E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]+E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\langle\left(I-\hat{D}u\right)^{-1}\psi,\hat{\delta}\left(\hat{D}u\right)\rangle\right],$$

hence by induction on $n\geq 1$,

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right] = \lambda E\left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right]\sum_{n=0}^{\infty}(-1)^n \langle\psi, \left(I-\hat{D}u\right)^{-1}\left(\left(I-\hat{D}u\right)^{-1}\hat{D}u\right)^n \psi\rangle$$

$$= \lambda\, \langle\psi, \left(I-2\hat{D}u\right)^{-1}\psi\rangle \left[e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}\right],$$

which yields (80).

Finally we remark that the formulas can be applied to the Skorohod integral $\hat{\delta}$ on the Wiener space when it is used to represent the Poisson stochastic integral $\hat{\delta}(u)$ of a deterministic function by Proposition (6.3.1) of [205].