Chapter 5

Stability of Standing Waves with Bound States

For the case when the frequency is equal to the critical frequency ω_c we show strong instability for all radially symmetric standing waves $e^{i\omega_c t}\phi(x)$. We show similar strong stability results for the Klein–Gordon–Zakharov system. We consider a Hamiltonian system which is invariant under a one-parameter unitary group. We give a criterion for the stability and instability of bound states for the degenerate case.

Section (5.1): Nonlinear Klein-Gordon Equation and Klein-Gordon-Zakharov System:

We study the strong instability of standing wave solutions $e^{i\omega t}\varphi(x)$ for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = |u|^{p-1}u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

Where $N \geq 2, 1 < p < 1 + 4/(N-2), -1 < \omega < 1$, and $\varphi \in H^1(\mathbb{R}^N)$ is anontrivial solution of

$$-\Delta\varphi + (1-\omega^2)\varphi - |\varphi|^{p-1}\varphi = 0, \qquad x \in \mathbb{R}^N. \tag{2}$$

We also study the same problem for the Klein-Gordon-Zakharov system

$$\partial_t^2 u - \Delta u + u + nu = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{3}$$

$$c_0^{-2}\partial_t^2 n - \Delta n = \Delta(|u|^2), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{4}$$

Where $N = 2, 3, c_0 > 0$ is a constant. The system (3)-(4) describes the interaction of a Langumiur wave and ion acoustic wave in a plasma. The complex valued function u denotes the fast time scale component of electric field raised by electrons, and the real valued function n denotes the deviation of ion density (see [101, 102, 103]).

From the result of Ginibre and Velo [104], the Cauchy problem for (1) is locally well-posed in the energy space $X := H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Thus, for any $(u_0, u_1) \in X$ there exists a unique solution $\vec{u} := (u, \partial_t u) \in C([0, T_{\max}); X)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ such that either $T_{\max} = \infty$(global existence) or $T_{\max} < \infty$and $\lim_{t\to T_{\max}} \|\vec{u}(t)\|_X = \infty$(finite time blowup). Moreover, the solution $u(t)$ satisfies the conservation laws of energy and charge:

$$E(\vec{u}(t)) = E(u_0, u_1), \quad Q(\vec{u}(t)) = Q(u_0, u_1), \qquad t \in [0, T_{\max}),$$

Where

$$E(u, v) = \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{5}$$

$$Q(u, v) = \operatorname{Im} \int_{\mathbb{R}^N} \bar{u} v dx. \tag{6}$$

Let $\phi_\omega \in H^1(\mathbb{R}^N)$ be the ground state (the least energy solution) of (2).We refer to [105, 106] for the existence of ϕ_ω, and to [107] for the uniqueness of ϕ_ω. The stability of standing waves $e^{i\omega t}\phi_\omega$ for (1) has been studied bymany authors. First, we consider the orbital stability of $e^{i\omega t}\phi_\omega$. Shatah [108] proves that $e^{i\omega t}\phi_\omega$is orbitally stable if $p < 1 + \frac{4}{N}$ and $\omega_c < |\omega| < 1$, where

$$\omega_c = \sqrt{\frac{p-1}{4 - (N-1)(p-1)}}. \tag{7}$$

Shatah and Strauss [109] prove that $e^{i\omega t}\phi_\omega$ is orbitally unstable when $p < 1 + \frac{4}{N}$ and $|\omega| < \omega_c$or when $p \geq 1 + \frac{4}{N}$ and $|\omega| < 1$. Here, we say that astanding wave solution $e^{i\omega t}\varphi$ is orbitally stable for (1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, u_1) \in X$ satisfies $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \delta$, then the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ exists globally and satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \left\| \vec{u}(t) - e^{i\theta}\big(\varphi(\cdot + y), i\mathbb{R}\varphi(\cdot + y)\big) \right\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\varphi$ is said to be orbitally unstable.

Next, we consider instability of $e^{i\omega t}\phi_\omega$ in a stronger sense. Berestycki and Cazenave [110] prove that the ground state standing wave $e^{i\omega t}\phi_\omega$ for (1) isvery strongly unstable (see Definition (5.1.1) below) when the frequency $\omega = 0$ (see also [111]). Shatah [112] proves that the ground state standing wave $e^{i\omega t}\phi_\omega$for nonlinear Klein-Gordon equations with general nonlinearity is strongly unstable (see Definition (5.1.2) below) when $\omega = 0$ and $N \geq 3$. Recently, the authors [113] prove that the ground state standing wave $e^{i\omega t}\phi_\omega$for (1) is very strongly unstable when $|\omega| \leq$

$\sqrt{(p-1)/(p+3)}$ and $N \geq 3$. Here, we give the definitions of very strong instability and strong instability.

Definition (5.1.1)[100]: (very strong instability) We say that $e^{i\omega t}\varphi$ is very strongly unstable for (1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$such that $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ blows up in finite time.

Definition (5.1.2)[100]: (strong instability) We say that $e^{i\omega t}\varphi$ is strongly unstable for (1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$such that $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ either blowsup in finite time or exists globally and satisfies $\limsup_{t\to\infty} \|\vec{u}(t)\|_X = \infty$.

Note that, by the definitions, if $e^{i\omega t}\varphi$ is very strongly unstable then it is strongly unstable, and that if $e^{i\omega t}\varphi$ is strongly unstable then it is orbitally unstable.

Before stating our main results, we recall instability results for the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \tag{8}$$

Let $\omega > 0$ and $\phi_\omega \in H^1(\mathbb{R}^N)$ be the ground state of

$$-\Delta\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \qquad x \in \mathbb{R}^N. \tag{9}$$

It is known that for any $\omega > 0$ the standing wave solution $e^{i\omega t}\phi_\omega$ for (8) is orbitally stable when $1 < p < 1 + 4/N$, and it is very strongly unstable when $1 + 4/N < p < 1 + 4/(N-2)$ (see [110, 114]). Moreover, for the critical case $p = 1 + 4/N$, for any $\omega > 0$ and any nontrivial solution $\varphi \in H^1(\mathbb{R}^N)$of (9), it is known that the standing wave $e^{i\omega t}\varphi$ is very strongly unstable for (8) (see [115]). For general theory of orbital stability and instability of solitary waves, we refer to Grillakis, Shatah and Strauss [116, 117].

Can we refine further this instability result? Namely, can we prove in certain cases that standing wave $e^{i\omega t}\phi_\omega$ for (1) is very strongly unstablein the sense of Definition (5.1.1)? The result of Cazenave [118] gives an answer of this question for the restricted range for the exponent p of nonlinearity $1 < p \leq 5$ for $N = 2$ and $1 < p \leq N/(N-2)$for $N \geq 3$. Cazenave proves that any global solution $u(t)$ of (1) is uniformly bounded in X, i.e.,$\sup_{t\geq 0}\|\vec{u}(t)\|_X < \infty$, if $1 < p \leq 5$ and $N = 2$, and if $1 < p \leq N/(N-2)$and $N \geq 3$. Therefore, for this range of the exponent p, Theorem (5.1.11) below together with the result of Cazenave gives us a very strongly instability result in the sense of Definition (5.1.1) for ground state standing waves $e^{i\omega t}\phi_\omega$of (1). Using an argument in Merle and Zaag [118], we can extend the result of Cazenave and prove the uniform boundedness of global solutions of (1) in $X$when $1 < p < 1 + \frac{4}{(N-1)}$ and $N \geq 2$. The following Lemma holds.

Corollary (5.1.3)[100]: In addition to the assumptions in Theorem (5.1.1), let $1 < p \leq 1 + 4/(N-1)$if $N = 2, 3$, and that $1 < p < 1 + 4/(N-1)$if $N \geq 4$. Then, the ground state standing wave $e^{i\omega t}\phi_\omega$ for (1) is very strongly unstable in the sense of Definition (5.1.1).

Remark (5.1.4)[100]: Let us mention that when the exponent p of nonlinearity is in the range $1 + 4/(N-1) < p < 1 + 4/(N-2)$ we were unable to give better instability results than those in Theorem (5.1.1) for ground state standing waves $e^{i\omega t}\phi_\omega$of (1) for large frequencies $|\omega| > \sqrt{(p-1)/(p+3)}$. The very strong instability result for small frequencies $|\omega| \leq \sqrt{(p-1)/(p+3)}$ and $N \geq 3$ is given in [113]. The following theorem is an important contribution of Kenji Nakanishi on the very strong instability in this area for large p and large frequencies ω.

This way, we have the entire picture for the very strong instability of ground state standing waves.

For the critical frequency $\omega = \omega_c$in the case $1 < p < 1 + 4/N$, we canprove a much more general instability result for standing waves which arenot necessarily related to the ground state.

For the existence of infinitely many radially symmetric solutions of (2), we refer to [119]. As mentioned above, a similar result of Theorem (5.1.14) below is known for the nonlinear Schrödinger equation (8) in the critical case $p = 1 + \frac{4}{N}$without assuming the radial symmetry of solution of (9) and the restrictionon space dimensions $N \geq 2$ (see [115]).

The proofs of Theorems (5.1.11) and (5.1.14) are based on using local versions of the virial type identities. To prove strong instability of the ground state for the case $\omega = 0$ and $N \geq 3$, Shatah in [112] considers a local version of the following identity

$$\frac{d}{dt}\mathrm{Re}\int_{\mathbb{R}^N} x \cdot \nabla u \partial_t \bar{u} dx = NK_1(\vec{u}(t)),$$

$$K_1(u,v) := -\frac{1}{2}\|v\|_2^2 + \left(\frac{1}{2} - \frac{1}{N}\right)\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \quad (10)$$

Since the integral in the left-hand side of (10) is not well-defined on theenergy space X, one needs to approximate the weight function x in (10)by suitable bounded functions. To control error terms by the approximation, initial perturbations are restricted to being radially symmetric and the decay estimate for radially symmetric functions in $H^1(\mathbb{R}^N)$:

$$\|w\|_{L^1(|x|\geq m)} \leq Cm^{-(N-1)/2}\|w\|_{H^1} (11)$$

(see [106]) is employed. The assumption $N \geq 2$ is needed here. In the case $N = 1$, we expect similar very strong instability results for the standing waves. This kind of approach has been also used for blowup problems of the nonlinear Schrödinger equation (8) (see, e.g., [120, 121, 122, 123, 124, 125, 126]).

In the proof of Theorem (5.1.11) for the case $p \geq 1 + 4/N$, we use a local version of the virial identity

$$-\frac{d}{dt}\mathrm{Re}\int_{\mathbb{R}^N} \{2x \cdot \nabla u + Nu\}\partial_t \bar{u} dx = P(u(t)), \quad (12)$$

Where

$$P(u) := 2\|\nabla u\|_2^2 - \frac{N(p-1)}{p+1}\|u\|_{p+1}^{p+1}. \quad (13)$$

Namely, instead of the left hand side of (12), which is not well defined inthe energy space X, we use (26) with conveniently chosen weights.

Note that (12) follows from (10) and

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2 = \frac{d}{dt}\mathrm{Re}\int_{\mathbb{R}^N} u\partial_t \bar{u} dx = -K_2(\vec{u}(t)),$$

$$K_2(u,v) = -\|v\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 - \|u\|_{p+1}^{p+1}, \quad (14)$$

and that the functional P appears in the virial identity for the nonlinear Schrödinger equation (8):

$$\frac{d^2}{dt^2}\|xu(t)\|_2^2 = 4P\big(u(t)\big). \tag{15}$$

The case $p < 1 + 4/N$is more delicate. Here we use a local version of the identity

$$-\frac{d}{dt}\mathrm{Re}\int_{\mathbb{R}^N}\{2x\cdot\nabla u + (N+\alpha)u\}\partial_t\bar{u}dx = K\big(\vec{u}(t)\big), \tag{16}$$

Where $\alpha := 4/(p-1) - N$and

$$K(u,v) := -\alpha\|v\|_2^2 + \alpha\|u\|_2^2 + (\alpha+2)\left\{\|\nabla u\|_2^2 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}\right\} \tag{17}$$

(cf. [109]). Note that

$$\begin{aligned} K(u,v) &= P(u) + \alpha K_2(u,v) \\ &= -2(\alpha+1)\|v - i\omega u\|_2^2 + 2(\alpha+2)(E-\omega Q)(u,v) \\ &\quad -2\alpha\omega Q(u,v) - 2\{1-(\alpha+1)\omega^2\}\|u\|_2^2, \end{aligned} \tag{18}$$

and that $1-(\alpha+1)\omega^2 > 0$ if $|\omega| > \omega_c$, and correspondingly $1-(\alpha+1)\omega^2 = 0$ if $|\omega| = \omega_c$. Again instead of the left hand side of (16) we use (27) with conveniently chosen weights.

Next, we consider the Klein-Gordon-Zakharov system (3)-(4). The well-posedness of the Cauchy problem for (3)-(4) in the energy space is studied by Ozawa, Tsutaya and Tsutsumi [127]. Here, the energy space $Y$is defined by $Y = H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)\times \dot{H}^{-1}(\mathbb{R}^N)$. When $N = 3$ and $c_0 \neq 1$, it is proved in [127] that for any $(u_0, u_1, n_0, n_1) \in Y$ there exists a unique solution $\mathbf{u} := (u, \partial_t u, n, \partial_t n) \in C([0, T_{\max})\,; Y)$ of (3)-(4)with initial data $\mathbf{u}(0) = (u_0, u_1, n_0, n_1)$ satisfying the conservation laws of the energy $H\big(\mathbf{u}(t)\big) = H\big(\mathbf{u}(0)\big)$ and the charge $Q\big(\mathbf{u}(t)\big) = Q\big(\mathbf{u}(0)\big)$ for all$t \in [0, T_{\max})$, where $Q$is defined by (6) and

$$\begin{aligned} H(u,v,n,\nu) = \frac{1}{2}\|v\|_2^2 + \frac{1}{4c_0^2}\|\nu\|_{\dot{H}^{-1}}^2 \\ + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\|n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}|u|^2 n dx. \end{aligned} \tag{19}$$

The case $N = 3$ and $c_0 = 1$ is treated in [128, 129], where the global small data solutions result is presented. For the case $N = 2$ by using the idea of Ozawa, Tsutaya and Tsutsumi [127] we can prove the local well -posedness of the Klein-Gordon-Zakharov system (3)-(4) in the energy space Y for all $c_0 > 0$.

We study instability of standing wave solutions

$$\left(u_\omega(t,x), n_\omega(t,x)\right) = \left(e^{i\omega t}\phi_\omega(x), -|\phi_\omega(x)|^2\right)$$

for (3)-(4), where $-1 < \omega < 1, N = 2, 3$, and $\phi_\omega \in H^1(\mathbb{R}^N)$ is the ground state of

$$-\Delta\varphi + (1-\omega^2)\varphi - |\varphi|^2\varphi = 0, \qquad x \in \mathbb{R}^N. \tag{20}$$

By a similar method as in the proof of Theorem (5.1.11) for the case $p \geq 1 + \frac{4}{N}$ together with an argument in Merle [124] for the Zakharov system, we have the following.

Remark (5.1.5)[100]: It is known (see [102]) that the negative initial energy $H\big(\mathbf{u}(0)\big)$ implies that the solution $\mathbf{u}(t)$ of (3)-(4) either blows up in finite time or blows up in infinite time, namely the solution exists globally and satisfies the asymptotic condition $\limsup_{t\to\infty}\|\mathbf{u}(t)\|_Y = \infty$. Since the energy

$$H(\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0) > 0$$

for λ close to 1, the result in [102] is not applicable to Theorem (5.1.1).

Next, we consider the very strong instability of $\left(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2\right)$ for (3)-(4). Since the second equation (4) of the KGZ system is massless, it seems difficult to obtain the uniform boundedness of global solutions for (3)-(4) similar to Lemma (5.1.15) below. Therefore, for the standing wave $\left(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2\right)$ we do not deduce a very strong instability similar to the instability result in Corollary (5.1.3) of Theorem (5.1.11) below. However, using the method in [113], we obtain the following very strong instability result for small frequencies.

Remark (5.1.6)[100]: In Theorem (5.1.18) below, the case $\omega = 0$ is proved by Gan and Zhang [130].

We prove Theorems (5.1.11) and (5.1.14) and Lemma (5.1.15) below for the nonlinear Klein-Gordon equation (1). The proof of Theorem (5.1.16) below is given. We devoted to applications to the Klein-Gordon-Zakharov system (3)-(4), and we prove Theorems (5.1.17) and (5.1.18) below.

We start with a convenient choice of the weight functions, as follows. Let $\Phi \in C^2([0,\infty))$ be a non-negative function such that

$$\Phi(r) = \begin{cases} N & \text{for } 0 \le r \le 1, \\ 0 & \text{for} r \ge 2, \end{cases} \Phi'(r) \le 0 \quad \text{for } 1 \le r \le 2.$$

For $m > 0$, we put

$$\Phi_m(r) = \Phi\left(\frac{r}{m}\right), \qquad \psi_m(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1} \Phi_m(s) ds. \tag{21}$$

Then, $\Phi_m$and ψ_m satisfy the following properties.

Lemma (5.1.7)[100]: For $m > 0$, we have

$$\Phi_m(r) = N, \qquad \psi_m(r) = r, \qquad 0 \le r \le m, \tag{22}$$

$$\psi_m'(r) + \frac{N-1}{r} \psi_m(r) = \Phi_m(r), \qquad r \ge 0, \tag{23}$$

$$\left|\Phi_m^{(k)}(r)\right| \le \frac{C}{m^k}, \qquad r \ge 0, \;\; k = 0,1,2, \tag{24}$$

$$\psi_m'(r) \le 1, \qquad r \ge 0. \tag{25}$$

Proof. Properties (22)-(24) follow from the definition (21). We show (25). Integrating by part implies

$$N r^{N-1} \psi_m(r) = \int_0^r N s^{N-1} \Phi_m(s) ds = r^N \Phi_m(r) - \int_0^r s^N \Phi_m'(s) ds\,.$$

Thus, by (23), we have

$$\psi_m'(r) = \Phi_m(r) - \frac{N-1}{r} \psi_m(r) = \frac{1}{N} \Phi_m(r) + \frac{N-1}{N r^N} \int_0^r s^N \Phi_m'(s) ds\,.$$

Since $\Phi_m(r) \le N$ and $\Phi_m'(r) \le 0$ for $r \ge 0$, we have (25).

Lemma (5.1.8)[100]: Let $u(t)$ be a radially symmetric solution of (1), and put

$$I_m^1(t) = 2\,\mathrm{Re} \int_{\mathbb{R}^N} \psi_m \partial_r u \partial_t \bar{u} dx + \mathrm{Re} \int_{\mathbb{R}^N} \Phi_m u \partial_t \bar{u} dx, \tag{26}$$

$$I_m^2(t) = I_m^1(t) + \alpha \mathrm{Re} \int_{\mathbb{R}^N} u \partial_t \bar{u} dx, \tag{27}$$

where $\alpha := 4/(p-1) - N$. Then, there exists a constant $C_0 > 0$ independent of m such that

$$-\frac{d}{dt} I_m^1(t) \leq P\big(u(t)\big) + \frac{N(p-1)}{p+1} \int_{|x|\geq m} |u(t,x)|^{p+1} dx + \frac{C_0}{m^2} \|u(t)\|_2^2, \tag{28}$$

$$-\frac{d}{dt} I_m^2(t) \leq K\big(\vec{u}(t)\big) + \frac{N(p-1)}{p+1} \int_{|x|\geq m} |u(t,x)|^{p+1} dx + \frac{C_0}{m^2} \|u(t)\|_2^2 \tag{29}$$

for all $t \in [0, T_{\max})$.

Proof. We multiply the equation (1) by $\psi_m \overline{\partial_r u}$ and by $\Phi_m \bar{u}$ respectively, and have

$$-\frac{d}{dt} 2\mathrm{Re} \int_{\mathbb{R}^N} \psi_m \partial_r u \partial_t \bar{u} dx = \int_{\mathbb{R}^N} \psi_m' + \frac{N-1}{r} \psi_m |\partial_t u|^2 dx$$

$$+ \int_{\mathbb{R}^N} \psi_m' - \frac{N-1}{r} \psi_m |\nabla u|^2 dx - \int_{\mathbb{R}^N} \psi_m' + \frac{N-1}{r} \psi_m |u|^2 dx$$

$$+ \frac{2}{p+1} \int_{\mathbb{R}^N} \psi_m' + \frac{N-1}{r} \psi_m |u|^{p+1} dx,$$

and

$$-\frac{d}{dt} \mathrm{Re} \int_{\mathbb{R}^N} \Phi_m u \partial_t \bar{u} dx = -\int_{\mathbb{R}^N} \Phi_m |\partial_t u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \Delta\Phi_m |u|^2 dx$$

$$+ \int_{\mathbb{R}^N} \Phi_m |\nabla u|^2 dx + \int_{\mathbb{R}^N} \Phi_m |u|^2 dx - \int_{\mathbb{R}^N} \Phi_m |u|^{p+1} dx.$$

By (23) in Lemma (5.1.7), we have the identity

$$-\frac{d}{dt} I_m^1(t) = 2 \int_{\mathbb{R}^N} \psi_m' |\nabla u|^2 dx - \frac{p-1}{p+1} \int_{\mathbb{R}^N} \Phi_m |u|^{p+1} dx - \frac{1}{2} \int_{\mathbb{R}^N} \Delta\Phi_m |u|^2 dx.$$

The inequality (28) follows from Lemma (5.1.7). Finally, (29) follows from (28),(14) and (18).

First, we consider the case $p \geq 1 + 4/N$. We define the functional

$$J_\omega(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1-\omega^2}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \quad (30)$$

and consider the following constrained minimization problem

$$d_\omega^1 = \inf\{J_\omega(u) : u \in H^1(\mathbb{R}^N)\setminus\{0\}, P(u) = 0\} \quad (31)$$

and the set

$$\mathcal{R}_\omega^1 = \{(u, v) \in X : (E - \omega Q)(u, v) < d_\omega^1, P(u) < 0\}, \quad (32)$$

Where E and Q are the energy and the charge respectively, and the functional P is defined by (13).

Note that

$$(E - \omega Q)(u, v) = J_\omega(u) + \frac{1}{2}\|v - i\omega u\|_2^2, \quad (33)$$

$$P(u) = 2\partial_\lambda J_\omega\left(\lambda^N/2u(\lambda\,\cdot)\right)\big|_{\lambda=1}. \quad (34)$$

Lemma (5.1.9)[100]: Let $N \geq 2, 1 + 4/N \leq p < 1 + 4/(N-2)$and $\omega \in (-1, 1)$. Then, we have the following.

(i) $J_\omega(u) - \frac{1}{N(p-1)}P(u) > d_\omega^1$for all $u \in H^1(\mathbb{R}^N)$ satisfying $P(u) < 0$.

(ii) The minimization problem (31) is attained at the ground state $\phi_\omega$of (2).

(iii) $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^1$ for all $\lambda > 1$.

Proof.(i) We put

$$d_\omega^1(u) \coloneqq J_\omega(u) - \frac{1}{N(p-1)}P(u)$$
$$= \left\{\frac{1}{2} - \frac{2}{N(p-1)}\right\}\|\nabla u\|_2^2 + \frac{1-\omega^2}{2}\|u\|_2^2. \quad (35)$$

Note that $1/2 - 2/N(p-1) \geq 0$ by the assumption $p \geq 1 + 4/N$. Let $u \in H^1(\mathbb{R}^N)$ satisfy $P(u) < 0$. Then, we have $u \neq 0$, and there exists $\lambda_1 \in (0, 1)$ such that $P(\lambda_1 u) = 0$. By (31), we have $d_\omega^1 \leq J_\omega(\lambda_1 u) = J_\omega^1(\lambda_1 u) < J_\omega^1(u)$.(ii) For the case $p > 1 + 4/N$, see [131], and for $p = 1 + 4/N$,see [125].(iii) By (33), we have

$$(E - \omega Q)\big(\lambda(\phi_\omega, i\omega\phi_\omega)\big) = J_\omega(\lambda\phi_\omega)$$
$$= \lambda^2\left(\frac{1}{2}\|\nabla\phi_\omega\|_2^2 + \frac{1-\omega^2}{2}\|\phi_\omega\|_2^2\right) - \frac{\lambda^{p+1}}{p+1}\|\phi_\omega\|_{p+1}^{p+1}.$$

Since $J_\omega(\phi_\omega) = d_\omega^1, \partial_\lambda J_\omega(\lambda\phi_\omega)|_{\lambda=1} = 0$and $\partial_\lambda^2 J_\omega(\lambda\phi_\omega)\big|_{\lambda=1} < 0$, we have $(E - \omega Q)\big(\lambda(\phi_\omega, i\omega\phi_\omega)\big) < d_\omega^1$for all $\lambda > 1$. Similarly, we have $P(\lambda\phi_\omega) < 0$for all $\lambda > 1$. Hence, we have $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^1$ for all $\lambda > 1$.

Lemma (5.1.10)[100]: Suppose that $N \geq 2, 1 + 4/N \leq p < 1 + 4/(N-2)$and $\omega \in (-1,1)$. If $(u_0, u_1) \in \mathcal{R}_\omega^1$, then the solution $u(t)$ of (1) with $\tilde{u}(0) = (u_0, u_1)$ satisfies

$$-\frac{1}{N(p-1)} P\big(u(t)\big) > d_\omega^1 - (E - \omega Q)(u_0, u_1), \qquad t \in [0, T_{\max}). \tag{36}$$

Proof. First, we show that $P\big(u(t)\big) < 0$ for all $t \in [0, T_{\max})$. Suppose that there exists $t_1 \in (0, T_{\max})$ such that $P\big(u(t_1)\big) = 0$ and $P\big(u(t)\big) < 0$ for $t \in [0, t_1)$. Then, by Lemma (5.1.9) (i) and (35), we have

$$\frac{1}{2} - \frac{2}{N(p-1)} \|\nabla u\|_2^2 + \frac{1-\omega^2}{2} \|u(t)\|_2^2 > d_\omega^1 > 0, \qquad t \in [0, t_1).$$

Thus, we have $u(t_1) \neq 0$. Therefore, by (31), we have $d_\omega^1 \leq J_\omega\big(u(t_1)\big)$.While, since $(u_0, u_1) \in \mathcal{R}_\omega^1, E$ and Q are conserved, and by (33), we have $J_\omega\big(u(t_1)\big) \leq (E - \omega Q)\big(\vec{u}(t_1)\big) < d_\omega^1$. This is a contradiction. Hence, we have $P\big(u(t)\big) < 0$ for all $t \in [0, T_{\max})$. From this fact, Lemma (5.1.9) (i) and (33), we obtain (36).

Theorem (5.1.11)[100]: Let $N \geq 2, 1 < p < 1 + 4/(N-2), \omega \in (-1,1)$and $\phi_\omega$be the ground state of (2). Assume that $|\omega| \leq \omega_c$if $p < 1 + 4/N$, where thecritical frequency $\omega_c$is given by (7). Then, the standing wave $e^{i\omega t}\phi_\omega$ for the nonlinear Klein-Gordon equation (1) is strongly unstable in the sense of Definition (5.1.2).

Proof.for the case $p \geq 1 + 4/N$. Let $\lambda > 1$ be fixed and denote

$$\delta := \frac{N(p-1)}{2} \{d_\omega^1 - (E - \omega Q)\big(\lambda(\phi_\omega, i\omega\phi_\omega)\big)\}.$$

Then, by Lemma (5.1.9)(iii), we have $\delta > 0$. Suppose that the solution $u(t)$of (1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ exists for all $t \in [0,1)$ and is uniformly bounded in X, i.e.,

$$M_1 := \sup_{t \geq 0} \|\vec{u}(t)\|_X < \infty. \tag{37}$$

Since $u(t)$ is radially symmetric in x for all $t \geq 0$, we define $I_m^1(t)$ for $u(t)$ by(26). By (11) and (37), we have

$$\int_{|x|\geq m} |u(t,x)|^{p+1}dx \leq \|u(t)\|_{L^\infty(|x|\geq m)}^{p-1}\|u(t)\|_2^2$$

$$\leq Cm^{-(N-1)(p-1)/2}\|u(t)\|_{H^1}^{p+1} \leq CM_1^{p+1}m^{-(N-1)(p-1)/2}$$

for all $t \geq 0$ and $m > 0$. Note that we assume $N \geq 2$. Thus, there exists $m_0 > 0$ such that

$$\sup_{t\geq 0}\left(\frac{N(p-1)}{p+1}\int_{|x|\geq m_0} |u(t,x)|^{p+1}dx + \frac{C_0}{m_0^2}\|u(t)\|_2^2\right) < \delta.$$

Thus, by Lemmas (5.1.8) and (5.1.10), we have

$$\frac{d}{dt}I_{m_0}^1(t)$$

$$\geq -P\big(u(t)\big) - \left(\frac{N(p-1)}{p+1}\int_{|x|\geq m_0} |u(t,x)|^{p+1}dx + \frac{C_0}{m_0^2}\|u(t)\|_2^2\right)$$

$$\geq 2\delta - \delta = \delta$$

for all $t \geq 0$. Therefore, we have $\lim_{t\to\infty} I_{m_0}^1(t) = \infty$. On the other hand,there exists a constant $C = C(m_0) > 0$ such that $I_{m_0}^1(t) \leq C\|\vec{u}(t)\|_X^2 \leq CM_1^2$for all $t \geq 0$. This is a contradition. Hence, for any $\lambda > 1$, the solution $u(t)$of (1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ either blows up in finite time or exists forall $t \geq 0$ and $\lim\ \sup_{t\to\infty}\|\vec{u}(t)\|_X = \infty$. This completes the proof of Theorem (5.1.11) for the case $p \geq 1 + 4/N$. _

Next, we consider the case where $p < 1 + 4/N$. For this case, we needa different variational characterization of the ground state $\phi_\omega$of (2) from that for the case $p \geq 1 + 4/N$. We define the functional

$$K_\omega^0(u) = \alpha(1-\omega^2)\|u\|_2^2 + (\alpha+2)\left\{\|\nabla u\|_2^2 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}\right\},$$

and consider the constrained minimization problem

$$d_\omega^0 = \inf\{J_\omega(u) : u \in H^1(\mathbb{R}^N)\setminus\{0\}, K_\omega^0(u) = 0\} \tag{38}$$

and the set

$$\mathcal{R}_\omega^0 = \{(u,v) \in X : (E - \omega Q)(u,v) < d_\omega^0, K_\omega^0(u) < 0\}, \tag{39}$$

where $\alpha = 4/(p-1) - N > 0$. Note that

$$K_\omega^0(u) = 2\partial_\lambda J_\omega\left(\lambda^\beta u(\lambda\,\cdot)\right)\Big|_{\lambda=1}, \qquad \beta = \frac{\alpha+N}{2} = \frac{2}{p-1}. \tag{40}$$

for the case $p < 1 + 4/N$. Let $\lambda > 1$ be fixed and define

$$\delta_1 = (\alpha+2)\{d_\omega^0 - (E - \omega Q)(\lambda(\phi_\omega, i\omega\phi_\omega))\},$$

$$\delta_2 = \alpha\left\{\omega Q(\lambda(\phi_\omega, i\omega\phi_\omega)) - \frac{\omega^2(\alpha+2)}{1-\omega^2} d_\omega^0\right\},$$

and $\delta = \delta_1 + \delta_2$. Then, by Lemma (5.1.12)(iii) below, we have $\delta_1 > 0$. Moreover, by Lemma (5.1.12)(ii) below and (42), we have

$$\frac{\omega^2(\alpha+2)}{1-\omega^2} d_\omega^0 = \omega^2\|\phi_\omega\|_2^2 < \lambda^2\omega^2\|\phi_\omega\|_2^2 = \omega Q(\lambda(\phi_\omega, i\omega\phi_\omega)).$$

Thus, we have $\delta_2 > 0$and $\delta > 0$. Suppose that the solution $u(t)$ of (1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ exists for all $t \in [0,\infty)$ and is uniformly bounded in X.Since $u(t)$ is radially symmetric in x for all $t \geq 0$, we define $I_m^2(t)$ for $u(t)$ by(27). As in the proof of Theorem (5.1.11) for the case $p \geq 1 + 4/N$, there exists $m_0 > 0$ such that

$$\sup_{t\geq 0}\left(\frac{N(p-1)}{p+1}\int_{|x|\geq m_0} |u(t,x)|^{p+1}dx + \frac{C_0}{m_0^2}\|u(t)\|_2^2\right) < \delta.$$

Thus, by Lemma (5.1.8), we have

$$\frac{d}{dt} I_{m_0}^2(t) \geq -K(\vec{u}(t)) - \delta, \qquad t \geq 0.$$

Here, recall that we assume $|\omega| \leq \omega_c$, so we have $1 - (\alpha+1)\omega^2 \geq 0$. Thus,by (18) and Lemma (5.1.13) below, we have

$$\begin{aligned} &-K(\vec{u}(t)) \\ &\geq -2(\alpha+2)(E-\omega Q)(\vec{u}(t)) + 2\alpha\omega Q(\vec{u}(t)) + 2\{1-(\alpha+1)\omega^2\}\|u(t)\|_2^2 \\ &\geq -2(\alpha+2)(E-\omega Q)(\vec{u}(t)) + 2\alpha\omega Q(\vec{u}(t)) + 2\{1-\omega^2-\alpha\omega^2\}\frac{\alpha+2}{1-\omega^2} d_\omega^0 \\ &= 2\delta \end{aligned}$$

for all $t \geq 0$. Therefore, we have $(d/dt)I_{m_0}^2(t) \geq \delta$for all $t \geq 0$, and $\lim_{t\to\infty} I_{m_0}^2(t) = \infty$. The rest of the proof is the same as in the proof ofTheorem (5.1.11) for the case $p \geq 1 + 4/N$, and we omit the details.

Lemma (5.1.12)[100]: Let $N \geq 2, 1 < p < 1 + 4/N$and $\omega \in (-1,1)$. Then, we have the following.

(i) $\frac{1-\omega^2}{\alpha+2}\|u\|_2^2 > d_\omega^0$for all $u \in H^1(\mathbb{R}^N)$satisfying $K_\omega^0(u) < 0$.

(ii) The minimization problem (38) is attained at the ground state ϕ_ωOf(2).

(iii) $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^0$for all $\lambda > 1$.

Proof. First, we note that

$$J_\omega(u) - \frac{1}{2(\alpha+2)}K_\omega^0(u) = \frac{1-\omega^2}{\alpha+2}\|u\|_2^2, \tag{41}$$

$$\mathcal{R}_\omega^0 = \inf\left\{\frac{1-\omega^2}{\alpha+2}\|u\|_2^2 : u \in H^1(\mathbb{R}^N)\setminus\{0\}, K_\omega^0(u) = 0\right\}. \tag{42}$$

(i) Let $u \in H^1(\mathbb{R}^N)$ satisfy $K_\omega^0(u) < 0$. Then, we have $u \neq 0$, and there exists $\lambda_1 \in (0,1)$ such that $K_\omega^0(\lambda_1 u) = 0$. By (38), we have

$$\mathcal{R}_\omega^0 \leq \frac{1-\omega^2}{\alpha+2}\|\lambda_1 u\|_2^2 < \frac{1-\omega^2}{\alpha+2}\|u\|_2^2.$$

(ii) Note that $d_\omega^0 \geq 0$ by (42). Let $\{u_j\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for (38). By considering the Schwarz symmetrization of u_j, we can assume that $\{u_j\} \subset H_{rad}^1(\mathbb{R}^N)$. We refer to [105] for the definition and basic properties of the Schwarz symmetrization. By (42),we see that $\{u_j\}$is bounded in $L^2(\mathbb{R}^N)$. Moreover, by $K_\omega^0(u_j) = 0$ and theGagliardo-Nirenberg inequality, we have

$$\begin{aligned}&(\alpha+2)\|\nabla u_j\|_2^2 + \alpha(1-\omega^2)\|u_j\|_2^2 \\ &\qquad = \frac{2(\alpha+2)}{p+1}\|u_j\|_{p+1}^{p+1} \leq C\|u_j\|_2^{p+1-\theta}\|\nabla u_j\|_2^\theta,\end{aligned}$$

where $\theta = (p-1)N/2$. Since $p < 1 + 4/N$, we see that $\theta < 2$ and that $\{u_j\}$ is bounded in $H^1(\mathbb{R}^N)$. Therefore, there exist a subsequence of $\{u_j\}$(we still denote it by the same letter) and $w \in H_{rad}^1(\mathbb{R}^N)$ such that $u_j \rightharpoonup w$ weakly in$H^1(\mathbb{R}^N)$ and $u_j \rightharpoonup w$ strongly in $L^{p+1}(\mathbb{R}^N)$. Here, we used the fact that theembedding $H_{rad}^1(\mathbb{R}^N) \hookrightarrow L_{rad}^q(\mathbb{R}^N)$ is compact for $2 < q < 2 + 4/(N-2)$ (see[106]). Next, we show that $w \neq 0$. Suppose that $w = 0$. Then, by $K_\omega^0(u_j) = 0$and the strong convergence $u_j \to 0$in

$L^{p+1}(\mathbb{R}^N)$, we see that $u_j \to 0$ in $H^1(\mathbb{R}^N)$. On the other hand, by $K_\omega^0(u_j) = 0$ and the Sobolev inequality, wehave

$$(\alpha+2)\|\nabla u_j\|_2^2 + \alpha(1-\omega^2)\|u_j\|_2^2 = \frac{2(\alpha+2)}{p+1}\|u_j\|_{p+1}^{p+1}$$
$$\le C\left\{(\alpha+2)\|\nabla u_j\|_2^2 + \alpha(1-\omega^2)\|u_j\|_2^2\right\}^{(p+1)/2}.$$

Since $u_j \neq 0$, we have $\|u_j\|_{H^1} \ge C$ for some $C > 0$. This is a contradition.Thus, we see that $w \in H^1(\mathbb{R}^N)\setminus\{0\}$. Therefore, by (41) and (42), we have

$$d_\omega^0 \le \frac{1-\omega^2}{\alpha+2}\|w\|_2^2 \le \liminf_{j\to\infty}\frac{1-\omega^2}{\alpha+2}\|u_j\|_2^2 = \liminf_{j\to\infty} J_\omega(u_j) = d_\omega^0,$$

and $K_\omega^0(w) \le \liminf_{j\to\infty} K_\omega^0(u_j) = 0$. Moreover, by (i), we have $K_\omega^0(w) = 0$.Therefore, w attains (42) and (38). Since w attains (38), there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that

$$J'_\omega(w) = \frac{\eta}{2(\alpha+2)}(K_\omega^0)'(w). \tag{43}$$

That is, w satisfies

$$-(1-\eta)\Delta w + (1-\omega^2)\left(1-\frac{\alpha}{\alpha+2}\eta\right)w - (1-\eta)|w|^{p-1}w = 0 \tag{44}$$

in $H^{-1}(\mathbb{R}^N)$. First, we show that $\eta < 1$. Suppose that $\eta \ge 1$. Then, by (44) and $K_\omega^0(w) = 0$, we have

$$0 = (1-\eta)\|\nabla w\|_2^2 + (1-\omega^2)\left(1-\frac{\alpha}{\alpha+2}\eta\right)\|w\|_2^2 - (1-\eta)\|w\|_{p+1}^{p+1}$$
$$= \frac{(1-\eta)(p-1)}{2}\|\nabla w\|_2^2 + \frac{\alpha(p-1)(1-\omega^2)}{2(\alpha+2)}\left\{\eta - 1 + \frac{4}{\alpha(p-1)}\right\}\|w\|_2^2$$
$$\ge \frac{2(1-\omega^2)}{\alpha+2}\|w\|_2^2 > 0.$$

This is a contradition. Thus, we have $\eta < 1$. Since we have

$$1-\eta > 0, \qquad (1-\omega^2)\left(1-\frac{\alpha}{\alpha+2}\eta\right) > 0$$

in (44), by [131], we have $x\cdot\nabla w \in H^1(\mathbb{R}^N)$. Therefore, by(43), we have

$$0 = K_\omega^0(w) = 2\partial_\lambda J_\omega\left(\lambda^\beta w(\lambda\,\cdot)\right)\Big|_{\lambda=1} = 2\langle J'_\omega(w), x\cdot\nabla w + \beta w\rangle$$
$$= \frac{\eta}{\alpha+2}\langle (K_\omega^0)'(w), x\cdot\nabla w + \beta w\rangle = \frac{\eta}{\alpha+2}\partial_\lambda K_\omega^0\left(\lambda^\beta w(\lambda\,\cdot)\right)\Big|_{\lambda=1}$$

where $\beta = (\alpha + N)/2$. Moreover, by $K_\omega^0(w) = 0$, we have

$$\partial_\lambda K_\omega^0\left(\lambda^\beta w(\lambda\,\cdot)\right)\Big|_{\lambda=1}$$

$$= \alpha^2(1-\omega^2)\|w\|_2^2 + (\alpha+2)^2\left\{\|\nabla w\|_2^2 - \frac{2}{p+1}\|w\|_{p+1}^{p+1}\right\}$$

$$= -2\alpha(1-\omega^2)\|w\|_2^2 < 0.$$

Thus, we have $\eta = 0$. Therefore, w satisfies $J'(w) = 0$ and $K_\omega^2(w) = 0$, where

$$K_\omega^2(u) := \langle J_\omega'(u), u\rangle = \|\nabla u\|_2^2 + (1-\omega^2)\|u\|_2^2 - \|u\|_{p+1}^{p+1}.$$

Since $\phi_\omega$attains

$$\inf\{J_\omega(u) : u \in H^1(\mathbb{R}^N)\setminus\{0\}, K_\omega^2(u) = 0\}$$

(see, e.g., [113]), we have $J_\omega(\phi_\omega) \le J_\omega(w)$. On the other hand,$\phi_\omega$satisfies $K_\omega^2(\phi_\omega) = 0$, we have $d_\omega^0 = J_\omega(w) \le J_\omega(\phi_\omega)$. Hence, $\phi_\omega$attains(38).

(iii) The proof is similar to that of Lemma (5.1.9) (iii), and we omit it.

Lemma (5.1.13)[100]: Suppose that $N \ge 2, 1 < p < 1 + 4/N$and $\omega \in (-1,1)$. If$(u_0, u_1) \in \mathcal{R}_\omega^0$, then the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ satisfies

$$\frac{1-\omega^2}{\alpha+2}\|u(t)\|_2^2 > d_\omega^0, \qquad t \in [0, T_{\max}).$$

Proof. The proof is similar to that for Lemma (5.1.10). We omit the details.

Theorem (5.1.14)[100]: Let $N \ge 2, 1 < p < 1 + 4/N$and $\varphi \in H^1(\mathbb{R}^N)$be any nontrivial,radially symmetric solution of (2) with $\omega = \omega_c$. Then, the standing wave solution $e^{i\omega_c t}\varphi$ of (1) is very strongly unstable in the sense of Definition (5.1.1).The same assertion is true for $\omega = -\omega_c$.

Proof. Let us first note that identity (18) contains the reason that in Theorem (5.1.14) we can allow any radially symmetric solutions of (2), unlike the case of Theorem (5.1.11) where we can treat only the ground state of (2). Namely, when $\omega = \omega_c$we have $1 - (\alpha+1)\omega_c^2 = 0$, and therefore the identity (18) does not contain the norm $\|u\|_2^2$. Let us recall that inTheorem (5.1.11) we control this norm by using the variational characterization of the ground state.

Let $\varphi \in H^1(\mathbb{R}^N)\setminus\{0\}$be a radially symmetric solution of (2) with $\omega = \omega_c$. Let $\lambda > 1$ and put

$$\delta = \alpha\omega_c Q\big(\lambda(\varphi, i\omega_c\varphi)\big) - (\alpha+2)(E-\omega_c Q)\big(\lambda(\varphi, i\omega_c\varphi)\big).$$

Since $J'_{\omega_c}(\varphi) = 0$, we have $(E-\omega_c Q)\big(\lambda(\varphi, i\omega_c\varphi)\big) = J_{\omega_c}(\lambda\varphi) < J_{\omega_c}(\varphi)$ for$\lambda > 1$. Moreover, we have $\omega_c Q\big(\lambda(\varphi, i\omega_c\varphi)\big) = \omega_c^2\lambda^2\|\varphi\|_2^2 > \omega_c^2\|\varphi\|_2^2$for $\lambda > 1$.Thus, we have

$$\delta > \alpha\omega_c^2\|\varphi\|_2^2 - (\alpha+2)J_{\omega_c}(\varphi) = -\frac{1}{2}K^0_{\omega_c}(\varphi) - \{1-(\alpha+\ 1)\omega_c^2\}\|\varphi\|_2^2.$$

By [131], we have $x\cdot\nabla\varphi \in H^1(\mathbb{R}^N)$. Therefore, by (40) and by $J'_{\omega_c}(\varphi) = 0$, we have

$$K^0_{\omega_c}(\varphi) = 2\langle J'_{\omega_c}(\varphi), x\cdot\nabla\varphi + \beta\varphi\rangle = 0.$$

Moreover, since $(\alpha+1)\omega_c^2 = 1$, we have $\delta > 0$. Suppose that the solution $u(t)$ of (1) with $\vec{u}(0) = \lambda(\varphi, i\omega_c\varphi)$ exists for all $t \in [0,\infty)$ and is uniformly bounded in X. Since $u(t)$ is radially symmetric in x for all $t \geq 0$, we define $I_m^2(t)$ for $u(t)$ by (27). As in the proof of Theorem (5.1.11) for the case $p \geq 1 + 4/N$,there exists $m_0 > 0$ such that

$$\sup_{t\geq 0}\left(\frac{N(p-1)}{p+1}\int_{|x|\geq m_0}|u(t,x)|^{p+1}dx + \frac{C_0}{m_0^2}\|u(t)\|_2^2\right) < \delta.$$

Thus, by Lemma (5.1.8), we have

$$\frac{d}{dt}I^2_{m_0}(t) \geq -K\big(\vec{u}(t)\big) - \delta, \qquad t \geq 0.$$

Moreover, by (18) and $(\alpha+1)\omega_c^2 = 1$, we have

$$\begin{aligned} &-K\big(\vec{u}(t)\big) \\ &\geq -2(\alpha+2)(E-\omega_c Q)\big(\vec{u}(t)\big) + 2\alpha\omega_c Q\big(\vec{u}(t)\big) + 2\{1-(\alpha+1)\omega_c^2\}\|u(t)\|_2^2 \\ &\geq -2(\alpha+2)(E-\omega_c Q)\big(\vec{u}(0)\big) + 2\alpha\omega_c Q\big(\vec{u}(t)\big) = 2\delta \end{aligned}$$

for all $t \geq 0$. Therefore, we have $(d/dt)I^2_{m_0}(t) \geq \delta$for all $t \geq 0$, and $\lim_{t\to\infty} I^2_{m_0}(t) = \infty$. On the other hand, there exists a constant $C = C(m_0) > 0$ such that $I^2_{m_0}(t) \leq C\|\vec{u}(t)\|_X^2 \leq C$ for all $t \geq 0$. This is a contradiction. Therefore, for any $\lambda > 1$, the solution $u(t)$ of (1) with$u(0) = \lambda(\varphi, i\omega_c\varphi)$ either blows up in finite time or exists for all $t \geq 0$ and $\limsup_{t\to\infty}\|\vec{u}(t)\|_X = \infty$. Finally, by Lemma (5.1.15) below, if $u(t)$ exists for all $t \geq 0$,then $\sup_{t\geq 0}\|\vec{u}(t)\|_X < \infty$. Hence, $u(t)$ blows up in finite time. This completes the proof.

Lemma (5.1.15)[100]: Let $N \geq 2$ and $1 < p < 1 + 4/(N-1)$. If $\vec{u} \in C([0,1), X)$ is aglobal solution of (1), then $\sup_{t\geq 0}\|\vec{u}(t)\|_X < \infty$.

Proof. By Proposition (5.1.3) and Lemma (5.1.5) in [118], we have

$$\sup_{t\geq 0}\|\vec{u}(t)\|_2 < \infty, \tag{45}$$

$$\sup_{t\geq 0}\int_t^{t+1} \|\vec{u}(s)\|_X^2 ds < \infty. \tag{46}$$

By (46) and the conservation of energy E, we have

$$C_1 \coloneqq \sup_{t\geq 0}\int_t^{t+1} \|\vec{u}(s)\|_{p+1}^{p+1} ds < \infty. \tag{47}$$

Note that the estimates (45), (46) and (47) hold true for $1 < p < 1 + 4/(N-2)$. In what follows, we use an argument in Merle and Zaag [118].First, for $r = (p+3)/2$, we show

$$\sup_{t\geq 0}\|u(t)\|_r < \infty. \tag{48}$$

Indeed, by (47) and the mean value theorem, for any $t \geq 0$ there exists $\tau(t) \in [t, t+1]$ such that

$$\left\|u\big(\tau(t)\big)\right\|_{p+1}^{p+1} = \int_t^{t+1} \|u(s)\|_{p+1}^{p+1} ds \leq C_1. \tag{49}$$

Since $2 < r < p+1$, it follows from (45) and (49) that $\sup_{t\geq 0}\left\|u\big(\tau(t)\big)\right\|_r < \infty$. Moreover, for any $t \geq 0$, we have

$$\begin{aligned}\|u(t)\|_r^r - \left\|u\big(\tau(t)\big)\right\|_r^r &= \int_{\tau(t)}^{t} \frac{d}{ds}\|u(s)\|_r^r ds \\ &\leq C\int_t^{t+1}\int_{\mathbb{R}^N} |u(s,x)|^{r-1}|\partial_s u(s,x)| dx\, ds \\ &\leq C\int_t^{t+1}\left(\|u(s)\|_{2(r-1)}^{2(r-1)} + \|\partial_s u(s)\|_2^2\right) ds.\end{aligned}$$

Since $2(r-1)=p+1$, by (46), (47) and $\sup_{t\geq 0}\left\|u\big(\tau(t)\big)\right\|_r<\infty$, we have(48). Next, by the Gagliardo-Nirenberg inequality, we have

$$\|u(t)\|_{p+1}\leq C\|u(t)\|_r^{1-\theta}\|\nabla u(t)\|_2^{\theta},$$

where

$$\frac{1}{p+1}=\theta\left(\frac{1}{2}-\frac{1}{N}\right)+\frac{1-\theta}{r}.$$

Since we assume $p<1+4/(N-1)$, we have $(p+1)\theta<2$. Thus, by (48), there exists a constant $C_2>0$ such that

$$\frac{2}{p+1}\|u(t)\|_{p+1}^{p+1}\leq C_2+\frac{1}{2}\|\nabla u(t)\|_2^2,\qquad t\geq 0.$$

Moreover, by the conservation of energy E, for any $t\geq 0$ we have

$$\begin{aligned}\|\vec{u}(t)\|_X^2&=2E\big(\vec{u}(0)\big)+\frac{2}{p+1}\|u(t)\|_{p+1}^{p+1}\\&\leq 2E\big(\vec{u}(0)\big)+C_2+\frac{1}{2}\|\nabla u(t)\|_2^2,\end{aligned}$$

which implies $\|\vec{u}(t)\|_X^2\leq 4E\big(\vec{u}(0)\big)+2C_2$. This completes the proof.

We conclude this section with the proof of Theorem (5.1.16) below.

Theorem (5.1.16)[100]:(due to Kenji Nakanishi) Let $N\geq 2, 1+4/N\leq p<1+4/(N-2), |\omega|<1$and $\phi_\omega$be the ground state of (2). Then, the standing wave $e^{i\omega t}\phi_\omega$ for the nonlinear Klein-Gordon equation (1) is very strongly unstable in the sense of Definition (5.1.1).

Proof.(due to Kenji Nakanishi). Following the proof of Theorem (5.1.11), take the radially symmetric solution $u(t,r)(r=|x|)$ startingfrom $\big(u(0),\partial_t u(0)\big)=\lambda(\phi_\omega,i\omega\phi_\omega)$ with $\lambda>1$, and assume by contradiction that it exists for all $t\geq 0$. Then Cazenave's estimate (46) implies that there exists $M<1$ such that for all $T>0$

$$\int_T^{T+1}\int_{\mathbb{R}^N}|\partial_t u|^2+|\nabla u|^2+|u|^2dxdt\leq M. \tag{50}$$

Hence for any positive integer j, there exists $T_j\in[j-1,j]$ such that

$$\int_{\mathbb{R}^N} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 dx|_{t=T_j} \leq M.$$

By Lemmas (5.1.8), (5.1.9) and (5.1.10), there exists $\delta > 0$ such that for any $m > 1$ and $t > 0$ we have

$$\frac{d}{dt} I_m^1(t) \geq 2\delta - R_m(t), \qquad R_m(t) := \frac{N(p-1)}{p+1} \int_{|x|\geq m} |u|^{p+1} dx + \frac{C}{m^2} \|u(t)\|_2^2,$$

where $I_m^1$is defined by (26). Here and below C is a positive constant, which may depend only on p and N. Integrating in t, we get

$$I_m^1(T_{j+2}) - I_m^1(T_j) \geq 2\delta - \int_{T_j}^{T_{j+2}} R_m(t) dt,$$

since $T_{j+2} - T_j \geq 1$. Notice that (50) is enough to control the error term R_m uniformly in j. To see this, let $\chi(t,r) \in C^\infty(\mathbb{R}^2)$ satisfy $\chi(t,r) = 1$ when$|t| \leq 2$ and $|r| \geq 1$, and $\chi(t,r) = 0$ if $|t| \geq 4$ or $|r| \leq 1/2$. For any $m > 1$ and $T > 4$, let $v(t,r) = \chi(t-T, r/m)u(t,|r|)$. Then we have

$$\int_{\mathbb{R}^2} |\partial_t v|^2 + |\partial_r v|^2 + |v|^2 drdt$$

$$\leq Cm^{1-N} \int_{T-4}^{T+4} \int_{\mathbb{R}^N} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 dxdt \leq 8Cm^{1-N} M.$$

Hence the Sobolev embedding $H^1(\mathbb{R}^2) \subset L^{p+1}(\mathbb{R}^2)$ implies that

$$\int_{T-2}^{T+2} \int_{|x|\geq m} |u|^{p+1} dxdt \leq C \sum_{j=0}^{\infty} \int_{T-2}^{T+2} (2^j m)^{N-1} \int_{r\geq 2^j m} |u|^{p+1} drdt$$

$$\leq Cm^{-(p-1)(N-1)/2} M^{(p+1)/2}.$$

Therefore choosing m sufficiently large, we obtain

$$I_m^1(T_{j+2}) - I_m^1(T_j) \geq \delta$$

for all $j \geq 4$, which contradicts the global bound

$$I_m^1(T_j) \leq Cm \int_{\mathbb{R}^N} |\partial_t u|^2 + |\partial_r u|^2 + |u|^2 dx|_{t=T_j} \leq CmM.$$

Theorem (5.1.17)[100]: Let $N = 2, 3, \omega \in (-1, 1), \phi_\omega$ be the ground state of (20), and $c_0 \neq 1$ if $N = 3$. Then, the standing wave $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ of KGZ system (3)-(4) is strongly unstable in the following sense. For any $\lambda > 1$, the solution $u(t)$ of (3)-(4) with initial data $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ either blows up in finite time or exists globally and satisfies $\limsup_{t\to\infty} \|\mathbf{u}(t)\|_Y = \infty$.

Proof. Let $\lambda > 1$ and put

$$\tilde{d}_\omega = (H - \omega Q)(\phi_\omega, i\omega\phi_\omega, -|\phi_\omega|^2, 0),$$
$$\delta = N\{\tilde{d}_\omega - (H - \omega Q)(\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)\},$$

Where H and Q are defined by (19) and (6), respectively. In the same way as in Lemma (5.1.9) (iii), we see that $\delta > 0$. Suppose that the solution $u(t)$ of (3)-(4) with $\mathrm{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ exists globally and satisfies $M \coloneqq \sup_{t\geq 0}\|\mathrm{u}(t)\|_Y < \infty$. Note that since the initial data is radially symmetric, the solution $\mathrm{u}(t)$ is also radially symmetric for all $t \geq 0$. Following Merle [124], we introduce the function $w(t) \coloneqq -(-\Delta)^{-1}\partial_t n(t)$, and for $m > 0$ we consider the function

$$\tilde{I}_m(t) = I_m^1(t) + \frac{1}{c_0^2}\int_{\mathbb{R}^N} \Psi_m n(t)\partial_r w(t) dx,$$

where $I_m^1(t)$ is defined by (26) and Φ_m and Ψ_m are given by (21). Note that since $\partial_t n(t) \in \dot{H}^{-1}(\mathbb{R}^N)$, we see that $w(t) \in \dot{H}^{-1}(\mathbb{R}^N)$ and $\|\partial_t n\|_{\dot{H}^{-1}} = \|\nabla w\|_2$. By the same computations as in Lemma (5.1.8), we have

$$-\frac{d}{dt}\tilde{I}_m(t) = 2\int_{\mathbb{R}^N} \Psi_m'|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \Phi_m(n^2 + 2|u|^2 n) dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^N} \Delta\Phi_m|u|^2 dx + \frac{1}{2c_0^2}\int_{\mathbb{R}^N}\left(\Psi_m' - \frac{N-1}{r}\Psi_m\right)|\nabla w|^2 dx.$$

By Lemma (5.1.7), we have

$$\int_{\mathbb{R}^N} \Psi_m' |\nabla u|^2 dx \leq \|\nabla u(t)\|_2^2,$$

$$-\frac{1}{2}\int_{\mathbb{R}^N} \Delta\Phi_m |u|^2 dx \leq \frac{C_1}{m_2}\|u(t)\|_2^2 \leq \frac{C_1 M^2}{m_2},$$

$$\int_{\mathbb{R}^N} \left(\Psi_m' - \frac{N-1}{r}\Psi_m\right)|\nabla w|^2 dx \leq \|\nabla w(t)\|_2^2 = \|\partial_t n(t)\|_{\dot{H}^{-1}}.$$

Moreover, we have

$$\int_{\mathbb{R}^N} \Phi_m(n^2 + 2|u|^2 n) dx$$

$$= Z\int_{\mathbb{R}^N} \Phi_m(n + |u|^2)^2 dx - \int_{\mathbb{R}^N} N|u|^4 dx + \int_{\mathbb{R}^N} (N - \Phi_m)|u|^4 dx$$

$$\leq N\|n + |u|^2\|_2^2 - N\|u\|_4^4 + \int_{|x|\geq m} (N - \Phi_m)|u|^4 dx,$$

and by (11) we have

$$\frac{1}{2}\int_{|x|\geq m} (N - \Phi_m)|u|^4 dx \leq C\|u(t)\|_{L^\infty(|x|\geq m)}^2 \|u(t)\|_2^2$$

$$\leq \frac{C_2}{m^{N-1}}\|u(t)\|_{H^1}^4 \leq \frac{C_2 M^4}{m^{N-1}}.$$

Therefore, we have

$$-\frac{d}{dt}\tilde{I}_m(t) \leq \tilde{P}\big(u(t)\big) + \frac{C_1 M^2}{m^2} + \frac{C_2 M^4}{m^{N-1}} \quad (51)$$

for all $t \geq 0$, where we put

$$\tilde{P}(u, v, n, \nu) = 2\|\nabla u\|_2^2 - \frac{N}{2}\|u\|_4^4 + \frac{N}{2}\|n + |u|^2\|_2^2 + \frac{1}{2c_0^2}\|\nu\|_{\dot{H}^{-1}}^2.$$

Note that

$$(H - \omega Q)(u, v, n, \nu) - \frac{1}{2N}\tilde{P}(u, v, n, \nu)$$

$$= \frac{1}{2}\|v - i\omega u\|_2^2 + \left(1 - \frac{1}{N}\right)\frac{1}{4c_0^2}\|\nu\|_{\dot{H}^{-1}}^2 + \left(\frac{1}{2} - \frac{1}{N}\right)\|\nabla u\|_2^2 + \frac{1-\omega^2}{2}\|u\|_2^2$$

$$\geq \left(\frac{1}{2} - \frac{1}{N}\right)\|\nabla u\|_2^2 + \frac{1-\omega^2}{2}\|u\|_2^2.$$

Using this inequality, in the same way as in Lemmas (5.1.9) and (5.1.10), we see that

$$-\tilde{P}\big(\mathrm{u}(t)\big) \geq 2N\{\tilde{d}_\omega - (H - \omega Q)\big(\mathrm{u}(0)\big)\} = 2\delta \tag{52}$$

holds for all $t \geq 0$. Therefore, taking $m_1 > 0$ such that

$$\frac{C_1 M_2}{m_1^2} + \frac{C_2 M^4}{m_1^{N-1}} < \delta,$$

by (51) and (52), we have $(d/dt)\tilde{I}_{m_1}(t) \geq \delta$for all $t \geq 0$, and $\lim_{t\to\infty} \tilde{I}_{m_1}(t) = \infty$. The rest of the proof is the same as in the proof of Theorem (5.1.11) for the case $p \geq 1 + 4/N$, and we omit the details.

Theorem (5.1.18)[100]: Let $N = 3, c_0 \neq 1, |\omega| < \frac{1}{\sqrt{3}}$and $\phi_\omega$be the ground state of(20). Then, the standing wave $\left(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2\right)$ of the KGZ system (3)-(4) is very strongly unstable in the following sense. For any $\lambda > 1$, the solution $\mathbf{u}(t)$ of (3)-(4) with the initial data $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ blows up in a finite time.

Proof. Let $\lambda > 1$. Suppose that the solution $u(t)$ of (3)-(4) with $\mathrm{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ exists globally. By the assumption$|\omega| < 1/\sqrt{3}$, we can take α such that $2\omega^2/(1-\omega^2) < \alpha < 1$. For such an α, we consider a function defined by

$$I_\alpha(t) = \frac{1}{2}\left\{\|u(t)\|_2^2 + \frac{\alpha}{c_0^2}\|n(t)\|_{\dot{H}^{-1}}^2\right\}.$$

Note that since $n(0) = -\lambda^2|\phi_\omega|^2 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \subset \dot{H}^{-1}(\mathbb{R}^3)$ and $\partial_t n \in C\left([0,\infty); \dot{H}^{-1}(\mathbb{R}^3)\right)$, we see that $n \in C^1\left([0,\infty); \dot{H}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)\right)$. Then, we have

$$\frac{d}{dt}I_\alpha(t) = \mathrm{Re}\langle u(t), \partial_t u(t)\rangle_{L^2} + \frac{\alpha}{c_0^2}\langle n(t), \partial_t n(t)\rangle_{\dot{H}^{-1}}$$

$$= \mathrm{Re}\langle u(t), \partial_t u(t) - i\omega u(t)\rangle_{L^2} + \frac{\alpha}{c_0^2}\langle n(t), \partial_t n(t)\rangle_{\dot{H}^{-1}},$$

and

$$\frac{d^2}{dt^2} I_\alpha(t) = \|\partial_t u(t)\|_2^2 + \frac{\alpha}{c_0^2}\|\partial_t n(t)\|_{\dot{H}^{-1}}^2 - \|\nabla u(t)\|_2^2 - \|u(t)\|_2^2$$

$$-\alpha\|n(t)\|_2^2 - (1+\alpha)\int_{\mathbb{R}^3} |u(t,x)|^2 n(t,x)dx\,.$$

Thus, we have

$$\frac{d^2}{dt^2} I_\alpha(t) + 2(1+\alpha)(H-\omega Q)\big(\mathrm{u}(0)\big) - 2\omega Q\big(\mathrm{u}(0)\big)$$

$$= (2+\alpha)\|\partial_t u(t) - i\omega u(t)\|_2^2 + \left(2 + \frac{1-\alpha}{2\alpha}\right)\frac{\alpha}{c_0^2}\|\partial_t n(t)\|_{\dot{H}^{-1}}^2$$

$$+K_{\omega,\alpha}\big(u(t), n(t)\big),$$

where we put

$$K_{\omega,\alpha}(u,n) = \alpha\left\{\|\nabla u\|_2^2 + \left(1 - \omega^2 - \frac{2}{\alpha}\omega^2\right)\|u\|_2^2 + \frac{1-\alpha}{2\alpha}\|n\|_2^2\right\}.$$

Here, we define

$$J_\omega(u,n) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1-\omega^2}{2}\|u\|_2^2 + \frac{1}{4}\|n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2 n(x)dx\,,$$

$$K_{\omega,\alpha}^1(u,n) = \partial_\lambda J_\omega(\lambda u, \lambda^{2\alpha} n)\Big|_{\lambda=1}$$

$$= \|\nabla u\|_2^2 + (1-\omega^2)\|u\|_2^2 + \alpha\|n\|_2^2 + (1+\alpha)\int_{\mathbb{R}^3} |u|^2 n dx\,,$$

$$K_{\omega,\alpha}^2(u,n) = 2\partial_\lambda J_\omega(\lambda^{(1-\alpha)/\alpha} u(\cdot/\lambda), n(\cdot/\lambda)\Big|_{\lambda=1}$$

$$= \frac{2-\alpha}{\alpha}\|\nabla u\|_2^2 \frac{2+\alpha}{\alpha}(1-\omega^2)\|u\|_2^2$$

$$+\frac{3}{2}\|n\|_2^2 \frac{2+\alpha}{\alpha}\int_{\mathbb{R}^3} |u|^2 n dx,$$

and put

$$J_{\omega,\alpha}^1(u,n) = J_\omega(u,n) - \frac{1}{2(1+\alpha)} K_{\omega,\alpha}^1(u,n)$$

$$= \frac{\alpha}{1+\alpha}\left\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{1-\omega^2}{2}\|u\|_2^2 + \frac{1-\alpha}{4\alpha}\|n\|_2^2\right\},$$

$$J_{\omega,\alpha}^{2}(u,n) = J_{\omega}(u,n) - \frac{\alpha}{2(2+\alpha)} K_{\omega,\alpha}^{2}(u,n)$$
$$= \frac{\alpha}{2+\alpha}\left\{\|\nabla u\|_2^2 + \frac{1-\alpha}{2\alpha}\|u\|_2^2\right\},$$
$$\theta = 1 - \frac{2\omega^2}{(1-\ \omega^2)\alpha}.$$

Then, we have $0 < \theta < 1$ and

$$K_{\omega,\alpha}(u,n) = 2(1+\alpha)\theta J_{\omega,\alpha}^{1}(u,n) + (2+\alpha)(1-\theta) J_{\omega,\alpha}^{2}(u,n).$$

Moreover, in a similar way as in Lemmas (5.1.3) and (5.1.4) in [113], we can prove that $J_{\omega,\alpha}^{j}\big(u(t),n(t)\big) \geq \tilde{d}_\omega$for all $t \geq 0$ and $j = 1,2$. Therefore, we have

$$K_{\omega,\alpha}\big(u(t),n(t)\big) \geq \{2(1+\alpha)\theta + (2+\alpha)(1-\theta)\}\tilde{d}_\omega$$
$$= 2\left(1+\alpha - \frac{\omega^2}{1-\omega^2}\right)\tilde{d}_\omega$$

for all $t \geq 0$. Moreover, since we have $\tilde{d}_\omega = (1-\omega^2)\|\phi_\omega\|_2^2$, putting $\beta = \min\{2+\alpha, 2+(1-\alpha)/2\alpha\}$, we have

$$\frac{d^2}{dt^2} I_\alpha(t) \geq \beta \left\{\|\partial_t u(t) - i\omega u(t)\|_2^2 + \frac{\alpha}{c_0^2}\|\partial_t n(t)\|_{\dot{H}^{-1}}^2\right\}$$
$$+2(1+\alpha)\{\tilde{d}_\omega - (H-\omega Q)(\mathrm{u}(0))\} + 2\omega Q(\mathrm{u}(0)) - 2\omega^2\|\phi_\omega\|_2^2$$

for all $t \geq 0$. Since $\beta > 2, (H-\omega Q)(\mathrm{u}(0)) < \tilde{d}_\omega$and $\omega Q(u(0)) > \omega^2\|\phi_\omega\|_2^2$ for all $\lambda > 1$, by the standard concavity argument, we see that there exists $T_1 \in (0,\infty)$ such that $\lim_{t\to T_1-0} I_\alpha(t) = \alpha$. This is a contradiction. Hence, for all $\lambda > 1$, the solution $\mathrm{u}(t)$ of (3)-(4) with $\mathrm{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ blows up in finite time.

This completes the proof.

Lemma (5.1.19)[219]: For $m > 0$, we have

$$\Phi_m(r) = \frac{2(1+\epsilon)}{\epsilon}, \qquad \psi_m(r) = r, \qquad 0 \leq r \leq m, \tag{53}$$

$$\psi_m'(r) + \frac{r+\epsilon}{r\epsilon}\psi_m(r) = \Phi_m(r), \qquad r \geq 0, \tag{54}$$

$$\left|\Phi_m^{(k)}(r)\right| \leq \frac{C}{m^k}, \qquad r \geq 0,\ \ k = 0,1,2, \tag{55}$$

$$\psi_m'(r) \leq 1, \qquad r \geq 0. \tag{56}$$

Proof. Properties (53)-(55) follow from the definition (52). We show (56). Integrating by part implies

$$\left(\frac{2(1+\epsilon)}{\epsilon}\right) r^{\frac{2+\epsilon}{\epsilon}} \psi_m(r) = \int_0^r \left(\frac{2(1+\epsilon)}{\epsilon}\right) s^{\frac{(2+\epsilon)}{\epsilon}} \Phi_m(s) ds$$

$$= r^{\frac{2(1+\epsilon)}{\epsilon}} \Phi_m(r) - \int_0^r s^{\frac{2(1+\epsilon)}{\epsilon}} \Phi_m'(s) ds.$$

Thus, by (54), we have

$$\psi_m'(r) = \Phi_m(r) - \frac{2+\epsilon}{r\epsilon} \psi_m(r)$$

$$= \frac{\epsilon}{2(1+\epsilon)} \Phi_m(r) + \frac{2+\epsilon}{2(1+\epsilon) r^{\frac{2(1+\epsilon)}{\epsilon}}} \int_0^r s^{\frac{2(1+\epsilon)}{\epsilon}} \Phi_m'(s) ds\,.$$

Since $\Phi_m(r) \leq \frac{2(1+\epsilon)}{\epsilon}$and $\Phi_m'(r) \leq 0$ for $r \geq 0$, we have (56).

Lemma (5.1.20)[219]: Let $u_n(t_j)$ be a radially symmetric sequence of solutions of (1), and put

$$I_m^1(t_j) = 2 \operatorname{Re} \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} \psi_m \partial_r u_n \partial_{t_j} \overline{u_n} dx_j + \operatorname{Re} \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} \Phi_m u_n \partial_{t_j} \overline{u_n} dx_j\,, \tag{57}$$

$$I_m^2(t_j) = I_m^1(t_j) + \alpha \operatorname{Re} \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} u_n \partial_{t_j} \overline{u_n} dx_j\,, \tag{58}$$

where $\alpha := \frac{-2\epsilon^2 4\epsilon+2}{\epsilon(\epsilon-1)}$. Then, there exists a constant $C_0 > 0$ independent of m such that

$$-\frac{d}{dt_j} I_m^1(t_j) \leq P\left(u_n(t_j)\right) + \frac{2(1+\epsilon)}{2+\epsilon} \int_{|x_j| \geq m} \left|u_n(t_j, x_j)\right|^{2+\epsilon} dx_j + \frac{C_0}{m^2} \left\|u_n(t_j)\right\|_2^2, \tag{59}$$

$$-\frac{d}{dt_j} I_m^2(t_j) \leq K\left(\overrightarrow{u_n}(t_j)\right) + \frac{2(1+\epsilon)}{2+\epsilon} \int_{|x_j| \geq m} \left|u_j(t_j, x_j)\right|^{2+\epsilon} dx_j + \frac{C_0}{m^2} \|u_n(t_n)\|_2^2 \tag{60}$$

for all $t_j \in [0, T_{\max})$.

Proof. We multiply the equation (1) by $\psi_m \overline{\partial_r u_n}$and by $\Phi_m \overline{u_n}$respectively, and have

$$-\frac{d}{dt_j}2\mathrm{Re}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\psi_m\partial_r u_n\partial_{t_j}\overline{u_n}dx_j = \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\psi'_m+\frac{2+\epsilon}{r\epsilon}\psi_m\left|\partial_{t_j}u_n\right|^2dx_j$$

$$+\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\psi'_m-\frac{2+\epsilon}{r\epsilon}\psi_m|\nabla u_n|^2dx_j-\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\psi'_m+\frac{2+\epsilon}{r\epsilon}\psi_m|u_n|^2dx_j$$

$$+\frac{2}{2+\epsilon}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\psi'_m+\frac{2+\epsilon}{r\epsilon}\psi_m|u_n|^{2+\epsilon}dx_j,$$

and

$$-\frac{d}{dt_j}\mathrm{Re}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m u_n\partial_{t_j}\overline{u_n}dx_j=-\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m\left|\partial_{t_j}u_n\right|^2dx_j-\frac{1}{2}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Delta\Phi_m|u_n|^2dx_j$$

$$+\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m|\nabla u_n|^2dx_j+\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m|u_n|^2dx_j-\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m|u_n|^{2+\epsilon}dx_j.$$

By (54) in Lemma (5.1.7), we have the identity

$$-\frac{d}{dt_j}I_m^1(t_j)=2\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\psi'_m|\nabla u_n|^2dx_j-\frac{\epsilon}{2+\epsilon}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m|u_n|^{2+\epsilon}dx_j-\frac{1}{2}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Delta\Phi_m|u_n|^2dx_j\,.$$

The inequality (59) follows from Lemma (5.1.7). Finally, (60) follows from (59),(14) and (18).

First, we consider the case $\epsilon \geq 1$. We define the sequence functional

$$J_{\omega_j}(u_n)=\frac{1}{2}\|\nabla u_n\|_2^2+\frac{1-\omega_j^2}{2}\|u_n\|_2^2-\frac{1}{2+\epsilon}\|u_n\|_{2+\epsilon}^{2+\epsilon}, \tag{61}$$

and consider the following constrained minimization problem

$$d_{\omega_j}^1=\inf\left\{J_{\omega_j}(u_n):u_n\in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)\setminus\{0\},P(u_n)=0\right\} \tag{62}$$

and the set

$$\mathcal{R}_{\omega_j}^1=\left\{(u_n,v_m)\in X:\left(E-\omega_jQ\right)(u_n,v_m)<d_{\omega_j}^1,P(u_n)<0\right\}, \tag{63}$$

where $E$and $Q$are the energy and the charge respectively, and the functional P is defined by (13).

Note that

$$\big(E-\omega_j Q\big)(u_n, v_m) = J_{\omega_j}(u_n) + \frac{1}{2}\left\|v_m - i\omega_j u_n\right\|_2^2, \tag{64}$$

$$P(u_n) = 2\partial_\lambda J_{\omega_j}\left(\lambda^{\frac{2(1+\epsilon)}{\epsilon}}/2u_n(\lambda\,\cdot)\right)\Big|_{\lambda=1}. \tag{65}$$

Lemma (5.1.21)[219]: Let $\epsilon > 0$ and $\omega_j \in (-1,1)$. Then, we have the following.

(iv) $J_{\omega_j}(u_n) - \frac{\epsilon}{2(1+\epsilon)(\epsilon-1)}P(u_n) > d^1_{\omega_j}$for all $u_n \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$ satisfying $P(u_n) < 0$.

(v) The minimization problem (62) is attained at the ground state of the sequence $\phi_{\omega_j}$of (2).

(vi) $\lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right) \in \mathcal{R}^1_{\omega_j}$for all $\lambda > 1$.

Proof.(i) We put

$$d^1_{\omega_j}(u_n) \coloneqq J_{\omega_j}(u_n) - \frac{1}{\frac{2(1+\epsilon)}{\epsilon}\epsilon}P(u_n)$$

$$= \left\{\frac{\in -1}{4(1+\in)}\right\}\|\nabla u_n\|_2^2 + \frac{1-\omega_j^2}{2}\|u_n\|_2^2. \tag{66}$$

By the assumption $\epsilon \geq 1$. Let $u_n \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$satisfy $P(u_n) < 0$. Then, we have $u_n \neq 0$, and there exists $\lambda_1 \in (0,1)$ such that $P(\lambda_1 u_n) = 0$. By (62), we have $d^1_{\omega_j} \leq J_{\omega_j}(\lambda_1 u_n) = J^1_{\omega_j}(\lambda_1 u_n) < J^1_{\omega_j}(u_n)$.

(ii) For the case $\epsilon \geq 1$, see [6] and see [19].

(iii) By (64), we have

$$\big(E-\omega_j Q\big)\left(\lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)\right) = J_{\omega_j}\left(\lambda\phi_{\omega_j}\right)$$

$$= \lambda^2\left(\frac{1}{2}\left\|\nabla\phi_{\omega_j}\right\|_2^2 + \frac{1-\omega_j^2}{2}\left\|\phi_{\omega_j}\right\|_2^2\right) - \frac{\lambda^{2+\epsilon}}{2+\epsilon}\left\|\phi_{\omega_j}\right\|_{2+\epsilon}^{2+\epsilon}.$$

Since $J_{\omega_j}\left(\phi_{\omega_j}\right) = d^1_{\omega_j}, \partial_\lambda J_{\omega_j}\left(\lambda\phi_{\omega_j}\right)\Big|_{\lambda=1} = 0$and $\partial^2_\lambda J_{\omega_j}\left(\lambda\phi_{\omega_j}\right)\Big|_{\lambda=1} < 0$, we have$\big(E-\omega_j Q\big)\left(\lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)\right) < d^1_{\omega_j}$for all $\lambda > 1$. Similarly, we have $P\left(\lambda\phi_{\omega_j}\right) < 0$ for all $\lambda > 1$. Hence, we have $\lambda(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}) \in \mathcal{R}^1_{\omega_j}$for all $\lambda > 1$.

Lemma (5.1.22)[219]: Suppose that $\epsilon > 0$and $\omega_j \in (-1,1)$. If $(u_{n-1}, u_{n+1}) \in \mathcal{R}^1_{\omega_j}$, then the sequence of solutions $u_n(t_j)$of (1)with $\widetilde{u_n}(0) = (u_{n-1}, u_{n+1})$ satisfies

$$-P\left(u_n(t_j)\right) > d^1_{\omega_j} - (E - \omega_j Q)(u_{n-1}, u_{n+1}), \quad t_j \in [0, T_{\max}). \qquad (67)$$

Proof. First, we show that $P\left(u_n(t_j)\right) < 0$ for all $t_j \in [0, T_{\max})$. Suppose that there exists $(t_j)_1 \in (0, T_{\max})$ such that $P\left(u_n(t_j)_1\right) = 0$ and $P\left(u_n(t_j)\right) < 0$ for $t_j \in \left[0, (t_j)_1\right)$. Then, by Lemma (5.1.9) (i) and (66), we have

$$\frac{\in -1}{4(1+\in)} \|\nabla u_n\|_2^2 + \frac{1-\omega_j^2}{2} \left\|u_n(t_j)\right\|_2^2 > d^1_{\omega_j} > 0, \qquad t_j \in \left[0, (t_j)_1\right).$$

Thus, we have $u_n(t_j)_1 \neq 0$. Therefore, by (62), we have $d^1_{\omega_j} \leq J_{\omega_j}\left(u_n(t_j)_1\right)$.While, since $(u_{n-1}, u_{n+1}) \in \mathcal{R}^1_{\omega_j}$, $E$and $Q$are conserved, and by (64), we have $J_{\omega_j}\left(u_n(t_j)_1\right) \leq (E - \omega_j Q)\left(\overrightarrow{u_n}(t_j)_1\right) < d^1_{\omega_j}$. This is a contradiction. Hence, we have$P\left(u_n(t_j)\right) < 0$ for all $t_j \in [0, T_{\max})$. From this fact, Lemma (5.1.9) (i) and (64), we obtain (67).

Lemma (5.1.23)[219]: Let $\epsilon > 0$ and $\omega_j \in (-1,1)$. Then, we have the following.

(i) $\frac{1-\omega_j^2}{\alpha+2} \|u_n\|_2^2 > d^0_{\omega_j}$for all $u_n \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$satisfying $K^0_{\omega_j}(u_n) < 0$.

(ii)The minimization problem (38) is attained at the ground state of the sequence $\phi_{\omega_j}$of (2).

(iii)$\lambda\left(\phi_{\omega_j}, i\omega_j \phi_{\omega_j}\right) \in \mathcal{R}^0_{\omega_j}$for all $\lambda > 1$.

Proof. First, we note that

$$J_{\omega_j}(u_n) - \frac{1}{2(\alpha+2)} K^0_{\omega_j}(u_n) = \frac{1-\omega_j^2}{\alpha+2} \|u_n\|_2^2, \qquad (68)$$

$$\mathcal{R}^0_{\omega_j} = \inf\left\{\frac{1-\omega_j^2}{\alpha+2} \|u_n\|_2^2 : u_n \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right) \setminus \{0\}, K^0_{\omega_j}(u_n) = 0\right\}. \qquad (69)$$

(i) Let $u_n \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$ satisfy $K^0_{\omega_j}(u_n) < 0$. Then, we have $u_n \neq 0$, and there exists $\lambda_1 \in (0,1)$ such that $K^0_{\omega_j}(\lambda_1 u_n) = 0$. By (38), we have

$$\mathcal{R}^0_{\omega_j} \leq \frac{1-\omega_j^2}{\alpha+2} \|\lambda_1 u_n\|_2^2 < \frac{1-\omega_j^2}{\alpha+2} \|u_n\|_2^2.$$

(ii) Note that $d^0_{\omega_j} \geq 0$ by (69). Let $\{(u_n)_j\} \subset H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$ be a minimizing sequence for (38). By considering the Schwarz symmetrization of u_j, wecan assume that $\{(u_n)_j\} \subset H^1_{rad}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. We refer to [2] for the definition and basic properties of the Schwarz symmetrization. By (69), we see that the sequence$\{(u_n)_j\}$is bounded in $L^2\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. Moreover, by $K^0_\omega((u_n)_j) = 0$ and the Gagliardo-Nirenberg inequality, we have

$$(\alpha+2)\left\|\nabla(u_n)_j\right\|_2^2 + \alpha\left(1-\omega_j^2\right)\left\|(u_n)_j\right\|_2^2$$
$$= \frac{2(\alpha+2)}{2+\epsilon}\left\|(u_n)_j\right\|_{2+\epsilon}^{2+\epsilon} \leq C\left\|(u_n)_j\right\|_2^{2+\epsilon-\theta}\left\|\nabla(u_n)_j\right\|_2^{\theta},$$

where $\theta = 1+\epsilon$. Since $\epsilon \geq 1$, we see that $\theta < 2$ and that the sequence$\{(u_n)_j\}$ is bounded in $H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. Therefore, there exist a subsequence of $\{(u_n)_j\}$(we still denote it by the same letter) and $w \in H^1_{rad}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$ such that $(u_n)_j \rightharpoonup w$ weakly in$H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$ and $(u_n)_j \rightharpoonup w$strongly in $L^{2+\epsilon}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. Here, we used the fact that the embedding $H^1_{rad}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right) \hookrightarrow L^q_{rad}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$ is compact for $2 < q < 2(1+\epsilon)$ (see[30]). Next, we show that $w \neq 0$. Suppose that $w = 0$. Then, by $K^0_{\omega_j}((u_n)_j) = 0$ and the strong convergence $(u_n)_j \to 0$ in $L^{2+\epsilon}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$, we see that $(u_n)_j \to 0$ in$H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. On the other hand, by $K^0_{\omega_j}((u_n)_j) = 0$ and the Sobolev inequality, we have

$$(\alpha+2)\left\|\nabla(u_n)_j\right\|_2^2 + \alpha\left(1-\omega_j^2\right)\left\|(u_n)_j\right\|_2^2 = \frac{2(\alpha+2)}{2+\epsilon}\left\|(u_n)_j\right\|_{2+\epsilon}^{2+\epsilon}$$
$$\leq C\left\{(\alpha+2)\left\|\nabla(u_n)_j\right\|_2^2 + \alpha\left(1-\omega_j^2\right)\left\|(u_n)_j\right\|_2^2\right\}^{(2+\epsilon)/2}.$$

Since $(u_n)_j \neq 0$, we have $\left\|(u_n)_j\right\|_{H^1} \geq C$ for some $C > 0$. This is a contradition.Thus, we see that $w \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right) \setminus \{0\}$. Therefore, by (68) and (69), we have

$$d^0_{\omega_j} \leq \frac{1-\omega_j^2}{\alpha+2}\|w\|_2^2 \leq \liminf_{j\to\infty}\frac{1-\omega_j^2}{\alpha+2}\left\|(u_n)_j\right\|_2^2 = \liminf_{j\to\infty} J_{\omega_j}((u_n)_j) = d^0_{\omega_j},$$

and $K^0_{\omega_j}(w) \le \liminf_{j\to\infty} K^0_{\omega_j}\big((u_n)_j\big) = 0$. Moreover, by (i), we have $K^0_{\omega_j}(w) = 0$.Therefore, $w$attains (69) and (38). Since w attains (38), there exists aLagrange multiplier $\eta \in \mathbb{R}$ such that

$$J'_{\omega_j}(w) = \frac{\eta}{2(\alpha+2)}\left(K^0_{\omega_j}\right)'(w). \tag{70}$$

That is, w satisfies

$$-(1-\eta)\Delta w + \left(1-\omega_j^2\right)\left(1-\frac{\alpha}{\alpha+2}\eta\right)w - (1-\eta)|w|^{\epsilon}w = 0 \tag{71}$$

in $H^{-1}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. First, we show that $\eta < 1$. Suppose that $\eta \ge 1$. Then, by (71) and $K^0_{\omega_j}(w) = 0$, we have

$$\begin{aligned} 0 &= (1-\eta)\|\nabla w\|_2^2 + \left(1-\omega_j^2\right)\left(1-\frac{\alpha}{\alpha+2}\eta\right)\|w\|_2^2 - (1-\eta)\|w\|_{2+\epsilon}^{2+\epsilon} \\ &= \frac{(1-\eta)\epsilon}{2}\|\nabla w\|_2^2 + \frac{\alpha\epsilon\left(1-\omega_j^2\right)}{2(\alpha+2)}\left\{\eta - 1 + \frac{4}{\alpha\epsilon}\right\}\|w\|_2^2 \\ &\ge \frac{2\left(1-\omega_j^2\right)}{\alpha+2}\|w\|_2^2 > 0. \end{aligned}$$

This is a contradition. Thus, we have $\eta < 1$. Since we have

$$1-\eta > 0, \qquad \left(1-\omega_j^2\right)\left(1-\frac{\alpha}{\alpha+2}\eta\right) > 0$$

in (71), by [6], we have $x_j \cdot \nabla w \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. Therefore, by(70), we have

$$\begin{aligned} 0 = K^0_{\omega_j}(w) &= 2\partial_\lambda J_{\omega_j}\left(\lambda^\beta w(\lambda\,\cdot)\right)\Big|_{\lambda=1} = 2\left\langle J'_{\omega_j}(w), x_j\cdot\nabla w + \beta w\right\rangle \\ &= \frac{\eta}{\alpha+2}\left\langle\left(K^0_{\omega_j}\right)'(w), x_j\cdot\nabla w + \beta w\right\rangle = \frac{\eta}{\alpha+2}\partial_\lambda K^0_{\omega_j}\left(\lambda^\beta w(\lambda\,\cdot)\right)\Big|_{\lambda=1} \end{aligned}$$

where $\beta = \left(\alpha + \frac{2(1+\epsilon)}{\epsilon}\right)/2$. Moreover, by $K^0_{\omega_j}(w) = 0$, we have

$$\begin{aligned} &\partial_\lambda K^0_{\omega_j}\left(\lambda^\beta w(\lambda\,\cdot)\right)\Big|_{\lambda=1} \\ &= \alpha^2\left(1-\omega_j^2\right)\|w\|_2^2 + (\alpha+2)^2\left\{\|\nabla w\|_2^2 - \frac{2}{2+\epsilon}\|w\|_{2+\epsilon}^{2+\epsilon}\right\} \\ &= -2\alpha\left(1-\omega_j^2\right)\|w\|_2^2 < 0. \end{aligned}$$

Thus, we have $\eta = 0$. Therefore, w satisfies $J'(w) = 0$ and $K^2_{\omega_j}(w) = 0$, where

$$K^2_{\omega_j}(u_n) \coloneqq \left\langle J'_{\omega_j}(u_n), u_n\right\rangle = \|\nabla u_n\|_2^2 + \left(1-\omega_j^2\right)\|u_n\|_2^2 - \|u_n\|_{2+\epsilon}^{2+\epsilon}.$$

Since ϕ_ω attains

$$\inf\left\{J_{\omega_j}(u_n) : u_n \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right) \setminus \{0\}, K_{\omega_j}^2(u_n) = 0\right\}$$

(see, e.g., [23]), we have $J_{\omega_j}\left(\phi_{\omega_j}\right) \le J_{\omega_j}(w)$. On the other hand, the sequence ϕ_{ω_j} satisfies $K_{\omega_j}^2\left(\phi_{\omega_j}\right) = 0$, we have $d_{\omega_j}^0 = J_{\omega_j}(w) \le J_{\omega_j}\left(\phi_{\omega_j}\right)$. Hence, the sequence ϕ_{ω_j} attains (38).

(iii) The proof is similar to that of Lemma (5.1.9) (iii), and we omit it.

Lemma (5.1.24)[219]: Suppose that $\epsilon > 0$ and $\omega_j \in (-1, 1)$. If$(u_{n-1}, u_{n+1}) \in \mathcal{R}_{\omega_j}^0$, then the sequence of the solutions $u_n(t_j)$of (1) with $\overrightarrow{u_n}(0) = (u_{n-1}, u_{n+1})$ satisfies

$$\frac{1-\omega_j^2}{\alpha+2}\left\|u_n(t_j)\right\|_2^2 > d_{\omega_j}^0, \qquad t_j \in [0, T_{\max}).$$

Proof of Theorem 1 for the case $\epsilon \ge 1$. Let $\lambda > 1$ be fixed and denote

$$\delta \coloneqq \frac{(\epsilon^2-1)}{\epsilon}\left\{d_{\omega_j}^1 - \left(E-\omega_j Q\right)\left(\lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)\right)\right\}.$$

Then, by Lemma (5.1.9)(iii), we have $\delta > 0$. Suppose that the sequence of solutions $u_n(t_j)$ of (1) with $\overrightarrow{u_n}(0) = \lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)$ exists for all $t_j \in [0,1)$ and is uniformly bounded in X, i.e.,

$$M_1 \coloneqq \sup_{t_j \ge 0}\left\|\overrightarrow{u_n}(t_j)\right\|_X < \infty. \tag{72}$$

Since the sequence $u_n(t_j)$ is radially symmetric in x_j for all $t_j \ge 0$, we define $I_m^1(t_j)$ for $u_n(t_j)$ by(57). By (11) and (72), we have

$$\int_{|x_j|\ge m}\left|u_n(t_j,x_j)\right|^{2+\epsilon}dx_j \le \|u_n(t_k)\|_{L^\infty(|x_k|\ge m)}^{\epsilon-1}\left\|u_n(t_j)\right\|_2^2$$

$$\le Cm^{-\frac{2+\epsilon}{2}}\left\|u_n(t_j)\right\|_{H^1}^{2+\epsilon} \le CM_1^{2+\epsilon}m^{-\left(\frac{2+\epsilon}{2}\right)(\zeta-1)/2}$$

for all $t_j \ge 0$ and $m > 0$. there exists $m_0 > 0$ such that

$$\sup_{t\ge 0}\left(\frac{2(1+\epsilon)}{2+\epsilon}\int_{|x_j|\ge m_0}\left|u_n(t_j,x_j)\right|^{2+\epsilon}dx_j + \frac{C_0}{m_0^2}\left\|u_n(t_j)\right\|_2^2\right) < \delta.$$

Thus, by Lemmas (5.1.8) and (5.1.10), we have

$$\frac{d}{dt_j} I^1_{m_0}(t_j)$$

$$\geq -P\big(u_n(t_k)\big) - \left(\frac{2(1+\epsilon)}{2+\epsilon} \int_{|x_j|\geq m_0} \left|u_n(t_j, x_j)\right|^{2+\epsilon} dx_j + \frac{C_0}{m_0^2}\left\|u_n(t_j)\right\|_2^2\right)$$

$$\geq 2\delta - \delta = \delta$$

for all $t_j \geq 0$. Therefore, we have $\lim_{t_j\to\infty} I^1_{m_0}(t_j) = \infty$. On the other hand, there exists a constant $C = C(m_0) > 0$ such that $I^1_{m_0}(t_j) \leq C\left\|\overrightarrow{u_n}(t_j)\right\|_X^2 \leq CM_1^2$ for all $t_j \geq 0$. This is a contradition. Hence, for any $\lambda > 1$, the sequence of solutions $u_n(t_j)$of (1) with $\overrightarrow{u_n}(0) = \lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)$ either blows up in finite time or exists forall $t_j \geq 0$ and $\lim\ \sup_{t_j\to\infty}\left\|\overrightarrow{u_n}(t_j)\right\|_X = \infty$. This completes the proof of Theorem(5.1.1) for the case $\epsilon \geq 1$.

Next, we consider the case where $\epsilon \geq 1$. For this case, we needa different variational characterization of the ground state of the sequence $\phi_{\omega_j}$of (2) from that for the case $\epsilon \geq 1$. We define the sequence of the functional

$$K^0_{\omega_j}(u_n) = \alpha\left(1-\omega_j^2\right)\left\|u_j\right\|_2^2 + (\alpha+2)\left\{\|\nabla u_n\|_2^2 - \frac{2}{2+\epsilon}\|u_n\|_{2+\epsilon}^{2+\epsilon}\right\},$$

and consider the constrained minimization problem

$$d^0_{\omega_j} = \inf\left\{J_{\omega_j}(u_n) : u_n \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)\setminus\{0\}, K^0_{\omega_j}(u_n) = 0\right\} \quad (73)$$

and the set

$$\mathcal{R}^0_{\omega_j} = \left\{(u_n, v_m) \in X : \left(E - \omega_j Q\right)(u_n, v_m) < d^0_{\omega_j}, K^0_{\omega_j}(u_n) < 0\right\}, \quad (74)$$

where $\alpha = 2(1-\epsilon) > 0$. Note that

$$K^0_{\omega_j}(u_n) = 2\partial_\lambda J_{\omega_j}\left(\lambda^\beta u_n(\lambda\,\cdot)\right)\Big|_{\lambda=1}, \qquad \beta = \frac{\alpha + \left(\frac{2(1+\epsilon)}{\epsilon}\right)}{2} = \frac{2}{\epsilon}. \quad (75)$$

Proofof Theorem 1 for the case $p < \frac{1+3\epsilon}{1+\epsilon}$. Let $\lambda > 1$ be fixed and define

$$\delta_1 = (\alpha+2)\left\{d^0_{\omega_j} - \left(E - \omega_j Q\right)\left(\lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)\right)\right\},$$

$$\delta_2 = \alpha\left\{\omega_j Q\left(\lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)\right) - \frac{\omega_j^2(\alpha+2)}{1-\omega_j^2} d^0_{\omega_j}\right\},$$

and $\delta = \delta_1 + \delta_2$. Then, by Lemma (5.1.12)(iii) below, we have $\delta_1 > 0$. Moreover, by Lemma (5.1.12)(ii) below and (69), we have

$$\frac{\omega_j^2(\alpha+2)}{1-\omega_j^2} d_{\omega_j}^0 = \omega_j^2 \left\|\phi_{\omega_j}\right\|_2^2 < \lambda^2\omega_j^2 \left\|\phi_{\omega_j}\right\|_2^2 = \omega_j Q\left(\lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)\right).$$

Thus, we have $\delta_2 > 0$ and $\delta > 0$. Suppose that the sequence of solutions $u_n(t_j)$ of (1) with $\overrightarrow{u_n}(0) = \lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)$ exists for all $t_j \in [0,\infty)$ and is uniformly bounded in X.Since the sequence $u_n(t_j)$ is radially symmetric in x_j for all $t_j \ge 0$, we define $I_m^2(t_j)$ for $u_n(t_j)$ by (58). As in the proof of Theorem (5.1.1) for the case $\epsilon \ge 1$, there exists $m_0 > 0$ such that

$$\sup_{t_j\ge 0}\left(\frac{2(1+\epsilon)}{2+\epsilon}\int_{|x_j|\ge m_0}\left|u_n(t_j,x_j)\right|^{2+\epsilon}dx_j + \frac{C_0}{m_0^2}\left\|u_n(t_j)\right\|_2^2\right) < \delta.$$

Thus, by Lemma (5.1.8), we have

$$\frac{d}{dt_j} I_{m_0}^2(t_j) \ge -K\left(\overrightarrow{u_n}(t_j)\right) - \delta, \qquad t_j \ge 0.$$

Here, recall that we assume $|\omega_j| \le (\omega_j)_c$, so we have $1-(\alpha+1)\omega_j^2 \ge 0$. Thus, by (18) and Lemma (5.1.13) below, we have

$$\begin{aligned}
&-K\left(\overrightarrow{u_n}(t_j)\right)\\
&\quad \ge -2(\alpha+2)(E-\omega_j Q)\left(\overrightarrow{u_n}(t_j)\right) + 2\alpha\omega_j Q\left(\overrightarrow{u_n}(t_j)\right) + 2\{1-(\alpha+1)\omega_j^2\}\left\|u_n(t_j)\right\|_2^2\\
&\quad \ge -2(\alpha+2)(E-\omega_j Q)\left(\overrightarrow{u_n}(t_j)\right) + 2\alpha\omega_j Q\left(\overrightarrow{u_n}(t_j)\right) + 2\{1-\omega_j^2-\alpha\omega_j^2\}\frac{\alpha+2}{1-\omega_j^2}d_{\omega_j}^0\\
&\quad = 2\delta
\end{aligned}$$

for all $t_j \ge 0$. Therefore, we have $(d/dt_j)I_{m_0}^2(t_j) \ge \delta$for all $t_j \ge 0$, and$\lim_{t_j\to\infty} I_{m_0}^2(t_j) = \infty$. The rest of the proof is the same as in the proof of Theorem (5.1.11) for the case $\epsilon \ge 1$, and we omit the details.

Theorem (5.1.25)[219]: Let $\epsilon > 2$and $\varphi \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$be any nontrivial,radially symmetric solution of (2) with$\omega_j = (\omega_j)_c$. Then, the sequence of the standing wave solutions $e^{i\left(\sum_{j=1}^{n}(\omega_j)_c t_j\right)}\varphi$ of (1) are very strongly unstable in the sense of Definition (5.1.1).The same assertion is true for$\omega_j = -\left(\omega_j\right)_c$.

Proof. We know that identity (18) contains the reason that in Theorem (5.1.14) we can allow any radially symmetric solutions of (2), unlike the case of Theorem (5.1.1) where we can treat only the ground state of (2). Namely, when the sequence $\omega_j = \left(\omega_j\right)_c$we have $1-(\alpha+1)\left(\omega_j\right)_c^2 = 0$, and therefore the identity (18) does not contain the sequence of norms $\|u_n\|_2^2$. Let us recall that inTheorem (5.1.11) we control this norms by using the variational characterization of the ground state.

Let $\varphi \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)\setminus\{0\}$ be a radially symmetric solution of (2) with the sequence $\omega_j = \left(\omega_j\right)_c$. Let $\lambda > 1$ and put

$$\delta = \alpha\left(\omega_j\right)_c Q\left(\lambda\left(\varphi, i\left(\omega_j\right)_c\varphi\right)\right) - (\alpha+2)\left(E-\left(\omega_j\right)_c Q\right)\left(\lambda\left(\varphi, i\left(\omega_j\right)_c\varphi\right)\right).$$

Since $J'_{(\omega_j)_c}(\varphi) = 0$, we have $\left(E-\left(\omega_j\right)_c Q\right)\left(\lambda\left(\varphi, i\left(\omega_j\right)_c\varphi\right)\right) = J_{(\omega_j)_c}(\lambda\varphi) < J_{(\omega_j)_c}(\varphi)$ for$\lambda > 1$. Moreover, we have $\left(\omega_j\right)_c Q\left(\lambda\left(\varphi, i\left(\omega_j\right)_c\varphi\right)\right) = \left(\omega_j\right)_c^2\lambda^2\|\varphi\|_2^2 > \left(\omega_j\right)_c^2\|\varphi\|_2^2$for $\lambda > 1$.Thus, we have

$$\delta > \alpha\left(\omega_j\right)_c^2\|\varphi\|_2^2 - (\alpha+2)J_{(\omega_j)_c}(\varphi) = -\frac{1}{2}K^0_{(\omega_j)_c}(\varphi) - \left\{1-(\alpha+\ 1)\left(\omega_j\right)_c^2\right\}\|\varphi\|_2^2.$$

By [6], we have $x_j\cdot\nabla\varphi \in H^1\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$. Therefore, by (40) and by $J'_{(\omega_j)_c}(\varphi) = 0$, we have

$$K^0_{(\omega_j)_c}(\varphi) = 2\left\langle J'_{(\omega_j)_c}(\varphi), x_j\cdot\nabla\varphi + \beta\varphi\right\rangle = 0.$$

Moreover, since $(\alpha+1)\left(\omega_j\right)_c^2 = 1$, we have $\delta > 0$. Suppose that the sequence of solutions $u_n(t_j)$ of (1) with $\overrightarrow{u_n}(0) = \lambda\left(\varphi, i\left(\omega_j\right)_c\varphi\right)$ exists for all $t_j \in [0,\infty)$ and is uniformly bounded in X. Since the sequence $u_n(t_j)$ is radially symmetric in x_j for all $t_j \geq 0$, we define$I_m^2(t_j)$ for $u_n(t_j)$ by (58). As in the proof of Theorem (5.1.11) for the case $\epsilon \geq 1$,there exists $m_0 > 0$ such that

$$\sup_{t_j\geq 0}\left(\frac{2(1+\epsilon)}{2+\epsilon}\int_{|x_j|\geq m_0}|u_n(t_j,x_j)|^{2+\epsilon}dx_j+\frac{C_0}{m_0^2}\|u_n(t_j)\|_2^2\right)<\delta.$$

Thus, by Lemma (5.1.8), we have

$$\frac{d}{dt_j}I_{m_0}^2(t_j)\geq -K\left(\overrightarrow{u_n}(t_j)\right)-\delta,\qquad t_j\geq 0.$$

Moreover, by (18) and $(\alpha+1)(\omega_j)_c^2=1$, we have

$$-K\left(\overrightarrow{u_n}(t_j)\right)$$
$$\geq -2(\alpha+2)\left(E-(\omega_j)_cQ\right)\left(\overrightarrow{u_n}(t_j)\right)+2\alpha(\omega_j)_cQ\left(\overrightarrow{u_n}(t_j)\right)+2\left\{1-(\alpha+1)(\omega_j)_c^2\right\}\|u_n(t_j)\|_2^2$$
$$\geq -2(\alpha+2)\left(E-(\omega_j)_cQ\right)\left(\overrightarrow{u_n}(0)\right)+2\alpha(\omega_j)_cQ\left(\overrightarrow{u_n}(t_j)\right)=2\delta$$

for all $t_j\geq 0$. Therefore, we have $(d/dt_j)I_{m_0}^2(t_j)\geq\delta$for all $t_j\geq 0$, and $\lim_{t_j\to\infty}I_{m_0}^2(t_j)=\infty$. On the other hand, there exists a constant $C=C(m_0)>0$ such that $I_{m_0}^2(t_j)\leq C\|\overrightarrow{u_n}(t_j)\|_X^2\leq C$for all $t_j\geq 0$. This is a contradiction. Therefore, for any $\lambda>1$, the sequence of the solutions $u_n(t_j)$ of (1) with $u_n(0)=\lambda\left(\varphi,i(\omega_j)_c\varphi\right)$ either blows up in finite time or exists for all $t_j\geq 0$ and $\limsup_{t_j\to\infty}\|\overrightarrow{u_n}(t_j)\|_X=\infty$. Finally, by Lemma (5.1.15) below, if the sequence $u_n(t_j)$ exists for all $t_j\geq 0$,then $\sup_{t_j\geq 0}\|\overrightarrow{u_n}(t_j)\|_X<\infty$. Hence, the sequence $u_n(t_j)$ blows up in finite time. This completes the proof.

Lemma (5.1.26)[219]: Let$0<\epsilon<\frac{4(2+\epsilon)}{\epsilon}$. If $\overrightarrow{u_n}\in C([0,1),X)$is aglobal solution of (1), then $\sup_{t_j\geq 0}\|\overrightarrow{u_n}(t)\|_X<\infty$.

We have the following (see[35]).

Proof. By Proposition (5.1.3) and Lemma (5.1.5) [118], we have

$$\sup_{t_j\geq 0}\|\overrightarrow{u_n}(t_j)\|_2<\infty, \tag{76}$$

$$\sup_{t_j\geq 0}\int_{t_j}^{t_j+1}\|\overrightarrow{u_n}(s)\|_X^2ds<\infty. \tag{77}$$

By (77) and the conservation of energy E, we have

$$C_1 \coloneqq \sup_{t_j \geq 0} \int_{t_j}^{t_j+1} \|\overrightarrow{u_n}(s)\|_{2+\epsilon}^{2+\epsilon} ds < \infty. \tag{78}$$

Note that the estimates (76), (77) and (78) hold true. In what follows, we use an argument in Merle and Zaag [118].First, for $r = (4+\epsilon)/2$, we show

$$\sup_{t_j \geq 0} \|u_n(t_j)\|_r < \infty. \tag{79}$$

Indeed, by (78) and the mean value theorem, for any $t_j \geq 0$ there exists $\tau(t_j) \in [t_j, t_j+1]$ such that

$$\left\|u\left(\tau(t_j)\right)\right\|_{2+\epsilon}^{2+\epsilon} = \int_{t_j}^{t_j+1} \|u_n(s)\|_{2+\epsilon}^{2+\epsilon} ds \leq C_1. \tag{80}$$

Since $2 < r < 2+\epsilon$, it follows from (76) and (80) that $\sup_{t_j \geq 0} \left\|u\left(\tau(t_j)\right)\right\|_r < \infty$. Moreover, for any $t_j \geq 0$, we have

$$\|u_n(t_j)\|_r^r - \left\|u_n\left(\tau(t_j)\right)\right\|_r^r = \int_{\tau(t_j)}^{t_j} \frac{d}{ds} \|u_n(s)\|_r^r ds$$

$$\leq C \int_{t_j}^{t_j+1} \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} |u_n(s,x_j)|^{r-1} |\partial_s u_n(s,x_j)| dx_j \, ds$$

$$\leq C \int_{t}^{t_j+1} \left(\|u_n(s)\|_{2(r-1)}^{2(r-1)} + \|\partial_s u_n(s)\|_2^2\right) ds.$$

By (77), (78) and $\sup_{t_j \geq 0} \left\|u_n\left(\tau(t_j)\right)\right\|_{\frac{4+\epsilon}{2}} < \infty$, we have(79). Next, by the Gagliardo-Nirenberg inequality, we have

$$\|u_n(t_j)\|_{2+\epsilon} \leq C \|u_n(t_j)\|_r^{1-\theta} \|\nabla u_n(t_j)\|_2^{\theta},$$

where

$$\theta = \frac{2(1+\epsilon)}{2+\epsilon}.$$

Since we assume $\epsilon < 2$, we have $\theta < \frac{2}{2+\epsilon}$. Thus, by (79), there exists a constant $C_2 > 0$ such that

$$\frac{2}{2+\epsilon}\left\|u_n(t_j)\right\|_{2+\epsilon}^{2+\epsilon} \le C_2 + \frac{1}{2}\left\|\nabla u_n(t_j)\right\|_2^2, \qquad t_j \ge 0.$$

Moreover, by the conservation of energy E, for any $t_j \ge 0$ we have

$$\begin{aligned}\left\|\overrightarrow{u_n}(t_j)\right\|_X^2 &= 2E\left(\overrightarrow{u_n}(0)\right) + \frac{2}{2+\epsilon}\left\|u_n(t_j)\right\|_{2+\epsilon}^{2+\epsilon} \\ &\le 2E\left(\overrightarrow{u_n}(0)\right) + C_2 + \frac{1}{2}\left\|u_n(t_j)\right\|_{2+\epsilon}^{2+\epsilon},\end{aligned}$$

which implies $\left\|\overrightarrow{u_n}(t_j)\right\|_X^2 \le 4E\left(\overrightarrow{u_n}(0)\right) + 2C_2$. This completes the proof.

We conclude with the proof of Theorem (5.1.16).

Theorem (5.1.27)[219]:(due to Kenji Nakanishi) Let $\epsilon > 2, \left|\omega_j\right| < 1$and ϕ_{ω_j}be the ground state of (2). Then, the sequence of the standing waves $e^{i\left(\sum_{j=1}^{n}\omega_j t_j\right)}\phi_{\omega_j}$for the nonlinear Klein-Gordon equation (1) are very strongly unstable in the sense of Definition (5.1.1).

For the critical sequence of frequency $\omega_j = \left(\omega_j\right)_c$in the case$1 < 1+\epsilon < 1 + 4/\frac{2(1+\epsilon)}{\epsilon}$, we can show a much more general instability result for the sequence of the standing waves which are not necessarily related to the ground state[35].

Proof.(due to Kenji Nakanishi). Following the proof of Theorem (5.1.11), take the radially symmetric the sequence of solutions $u_n\left(t_j, \left|x_j\right|\right)$ starting from $\left(u_n(0), \partial_{t_j}u_n(0)\right) = \lambda\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}\right)$ with $\lambda > 1$, and assume by contradiction that it exists for all $t_j \ge 0$. Then Cazenave's estimate (77) implies that there exists $M < 1$such that for all $T > 0$

$$\int_T^{T+1}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\left|\partial_{t_j}u_n\right|^2 + |\nabla u_n|^2 + |u_n|^2 dx_j dt_j \le M. \tag{81}$$

Hence for any positive integer j, there exists $T_j \in [j-1, j]$ such that

$$\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} \left|\partial_{t_j} u_n\right|^2 + |\nabla u_n|^2 + |u_n|^2 dx_j\Big|_{t_j=T_j} \le M.$$

By Lemmas (5.1.8), (5.1.9)and (5.1.10), there exists $\delta > 0$ such that for any $m > 1$ and$t_j > 0$ we have

$$\frac{d}{dt_j} I_m^1(t_j) \ge 2\delta - R_m(t_j), \qquad R_m(t_j) := \frac{2(1+\epsilon)}{2+\epsilon} \int_{|x_j|\ge m} |u_n|^{2+\epsilon} dx_j + \frac{C}{m^2} \left\|u_n(t_j)\right\|_2^2,$$

where I_m^1 is defined by (57). Here and below C is a positive constant, which may depend only on $(1+\epsilon)$and $\frac{2(1+\epsilon)}{\epsilon}$. Integrating in t_j, we get

$$I_m^1(T_{j+2}) - I_m^1(T_j) \ge 2\delta - \int_{T_j}^{T_{j+2}} R_m(t_j) dt_j\,,$$

since $T_{j+2} - T_j \ge 1$. Notice that (81) is enough to control the error term R_m uniformly in j. To see this, let $\chi(t_j, r) \in C^\infty(\mathbb{R}^2)$ satisfy $\chi(t_j, r) = 1$ when$|t_j| \le 2$ and $|r| \ge 1$, and $\chi(t_j, r) = 0$ if $|t_j| \ge 4$ or $|r| \le 1/2$. For any $m > 1$ and $T > 4$, let $v_m(t_j, r) = \chi(t_j - T,\ r/m) u_n(t_j, |r|)$. Then we have

$$\int_{\mathbb{R}^2} \left|\partial_{t_j} v_m\right|^2 + |\partial_r v_m|^2 + |v_m|^2 dr dt_j$$

$$\le C m^{-\frac{(2+\epsilon)}{2}} \int_{T-4}^{T+4} \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} \left|\partial_{t_j} u_n\right|^2 + |\nabla u_n|^2 + |u_n|^2 dx_j dt_j \le 8C m^{-\frac{(2+\epsilon)}{2}} M.$$

Hence the Sobolev embedding $H^1(\mathbb{R}^2) \subset L^{2+\epsilon}(\mathbb{R}^2)$ implies that

$$\int_{T-2}^{T+2} \int_{|x_j|\ge m} |u_n|^{2+\epsilon} dx_j dt_j \le C \sum_{j=0}^{\infty} \int_{T-2}^{T+2} \left(2^j m\right)^{\frac{2(1+\epsilon)}{\epsilon} - 1} \int_{r \ge 2^j m} |u_n|^{2+\epsilon} dr dt_j$$

$$\le C m^{-\epsilon\left(\frac{2(1+\epsilon)}{\epsilon} - 1\right)/2} M^{(2+\epsilon)/2}.$$

Therefore choosing m sufficiently large, we obtain

$$I_m^1(T_{j+2}) - I_m^1(T_j) \ge \delta$$

for all $j \ge 4$, which contradicts the global bound

$$I_m^1(T_j) \le Cm \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} \left|\partial_{t_j} u_n\right|^2 + |\partial_r u_n|^2 + |u_n|^2 dx_j \Big|_{t_j=T_j} \le CmM.$$

Theorem (5.1.28)[219]: Let $\epsilon = 2, \omega_j \in (-1, 1)$, the sequence ϕ_{ω_j}be the ground state of (20), and$c_0 \neq 1$if $\epsilon = 2$,. Then, the sequence of the standing waves$\left(e^{i\left(\sum_{j=1}^n \omega_j t_j\right)}\phi_{\omega_j}, -\left|\phi_{\omega_j}\right|^2\right)$ of KGZ system (3)-(4) are strongly unstable in the following sense. For any $\lambda > 1$, the sequence of solutions $u_j(t_j)$of (3)-(4) with initial data $u_n(0) = \left(\lambda\phi_{\omega_j}, \lambda i\omega_j\phi_{\omega_j}, -\lambda^2\left|\phi_{\omega_j}\right|^2, 0\right)$either blows up in finite time or exists globally and satisfies $\limsup_{t_j\to\infty}\left\|u_n(t_j)\right\|_Y = \infty$ (see [35],Remark)

Since the energy

$$H\left(\lambda\phi_{\omega_j}, \lambda i\omega_j\phi_{\omega_j}, -\lambda^2\left|\phi_{\omega_j}\right|^2, 0\right) > 0$$

for λ close to 1, the result in [4] is not applicable to Theorem (5.1.1).

Now, we consider the very strong instability of the sequence $\left(e^{i\left(\sum_{j=1}^n \omega_j t_j\right)}\phi_{\omega_j}, -\left|\phi_{\omega_j}\right|^2\right)$ for (3)-(4).(see[35]) Since the second equation (4) of the KGZ system is massless, it seems difficult to obtain the uniform boundedness of global solutions for (3)-(4) similar to Lemma (5.1.2) below. Therefore, for the sequence of the standing waves $\left(e^{i\left(\sum_{j=1}^n \omega_j t_j\right)}\phi_{\omega_j}, -\left|\phi_{\omega_j}\right|^2\right)$we do not deduce a very strong instability similar to the instability result in Corollary (5.1.3) of Theorem (5.1.11) below. Using the method in [23], we obtain the following very strong instability result for small frequencies [35].

Proof. Let $\lambda > 1$ and put

$$\tilde{d}_{\omega_j} = (H - \omega_j Q)\left(\phi_{\omega_j}, i\omega_j\phi_{\omega_j}, -\left|\phi_{\omega_j}\right|^2, 0\right),$$

$$\delta = \frac{2(1+\epsilon)}{\epsilon}\left\{\tilde{d}_{\omega_j} - (H - \omega_j Q)\left(\lambda\phi_j \;, \lambda i\omega_j\phi_{\omega_j}, -\lambda^2\left|\phi_{\omega_j}\right|^2, 0\right)\right\},$$

where $H$and $Q$are defined by (19) and (6), respectively. In the same way as in Lemma (5.1.9) (iii), we see that $\delta > 0$. Suppose that the sequence of solutions $u_n(t_j)$ of (3)-(4)

with $\mathbf{u}(0)=\left(\lambda\phi_{\omega_j},\lambda i\omega_j\phi_{\omega_j},-\lambda^2\left|\phi_{\omega_j}\right|^2,0\right)$ exists globally and satisfies $M:=\sup_{t_j\geq 0}\left\|\mathbf{u}(t_j)\right\|_Y<\infty$. Note that since the initial data is radially symmetric, the solution $\mathbf{u}(t_j)$ is also radially symmetric for all $t_j\geq 0$. Following Merle [17], we introduce the function $w(t_j):=-(-\Delta)^{-1}\partial_{t_j}n(t_j)$, and for $m>0$ we consider the function

$$\tilde{I}_m(t_j)=I_m^1(t_j)+\frac{1}{c_0^2}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Psi_m n(t_j)\partial_r w(t_j)dx_j\,,$$

where $I_m^1(t_j)$ is defined by (57) and $\Phi_m$and $\Psi_m$are given by (21). Note that since $\partial_{t_j}n(t_j)\in\dot{H}^{-1}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$, we see that $w(t_j)\in\dot{H}^{-1}\left(\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}\right)$ and $\left\|\partial_{t_j}n\right\|_{\dot{H}^{-1}}=\|\nabla w\|_2$.By the same computations as in Lemma (5.1.8), we have

$$-\frac{d}{dt_j}\tilde{I}_m(t_j)=2\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Psi_m'|\nabla u_n|^2dx_j+\frac{1}{2}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m(n^2+2|u_n|^2n)dx_j$$

$$-\frac{1}{2}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Delta\Phi_m|u_n|^2dx_j+\frac{1}{2c_0^2}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\left(\Psi_m'-\frac{2+\epsilon}{r\epsilon}\Psi_m\right)|\nabla w|^2dx_j\,.$$

By Lemma (5.1. 7), we have

$$\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Psi_m'|\nabla u_n|^2dx_j\leq\left\|\nabla u_n(t_j)\right\|_2^2,$$

$$-\frac{1}{2}\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Delta\Phi_m|u_n|^2dx_j\leq\frac{C_1}{m_2}\left\|u_n(t_j)\right\|_2^2\leq\frac{C_1M^2}{m_2},$$

$$\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\left(\Psi_m'-\frac{2+\epsilon}{r\epsilon}\Psi_m\right)|\nabla w|^2dx_j\leq\left\|\nabla w(t_j)\right\|_2^2=\left\|\partial_{t_j}n(t_j)\right\|_{\dot{H}^{-1}}.$$

Moreover, we have

$$\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}}\Phi_m(n^2+2|u_n|^2n)dx_j$$

$$= Z\int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} \Phi_m(n+|u_n|^2)^2 dx_j - \int_{\mathbb{R}^{\frac{2(1+\epsilon)}{\epsilon}}} \frac{2(1+\epsilon)}{\epsilon}|u_n|^4 dx_j + \int_{\mathbb{R}^N}\left(\frac{2(1+\epsilon)}{\epsilon} - \Phi_m\right)|u_n|^4 dx_j$$

$$\leq \frac{2(1+\epsilon)}{\epsilon}\|n+|u_n|^2\|_2^2 - \frac{2(1+\epsilon)}{\epsilon}\|u_n\|_4^4 + \int_{|x_j|\geq m}\left(\frac{2(1+\epsilon)}{\epsilon} - \Phi_m\right)|u_n|^4 dx_j,$$

and by (11) we have

$$\frac{1}{2}\int_{|x_j|\geq m}\left(\frac{2(1+\epsilon)}{\epsilon} - \Phi_m\right)|u_n|^4 dx_j \leq C\left\|u_n(t_j)\right\|_{L^\infty(|x_j|\geq m)}^2 \left\|u_n(t_j)\right\|_2^2$$

$$\leq \frac{C_2}{m^{\frac{2+\epsilon}{2}}}\left\|u_n(t_j)\right\|_{H^1}^4 \leq \frac{C_2 M^4}{m^{\frac{2+\epsilon}{2}}}.$$

Therefore, we have

$$-\frac{d}{dt_j}\tilde{I}_m(t_j) \leq \tilde{P}\left(u_n(t_j)\right) + \frac{C_1 M^2}{m^2} + \frac{C_2 M^4}{m^{\frac{2+\epsilon}{2}}} \quad (82)$$

for all $t_j \geq 0$, where we put

$$\tilde{P}(u_n, v_m, n, v_m) = 2\|\nabla u_n\|_2^2 - \frac{1+\epsilon}{\epsilon}\|u_n\|_4^4 + \frac{1+\epsilon}{\epsilon}\|n+|u_n|^2\|_2^2 + \frac{1}{2c_0^2}\|v_m\|_{\dot{H}^{-1}}^2.$$

Note that

$$(H-\omega_j Q)(u_n, v_m, n, v_m) - \frac{2+\epsilon}{2(1+\epsilon)}\tilde{P}(u_n, v_m, n, v_m)$$

$$= \frac{1}{2}\left\|v_m - i\omega_j u_n\right\|_2^2 + \left(\frac{2+\epsilon}{2(1+\epsilon)}\right)\frac{1}{4c_0^2}\|v_m\|_{\dot{H}^{-1}}^2 + \left(\frac{1}{2(1+\epsilon)}\right)\|\nabla u_n\|_2^2 + \frac{1-\omega_j^2}{2}\|u_n\|_2^2$$

$$\geq \left(\frac{1}{2(1+\epsilon)}\right)\|\nabla u_n\|_2^2 + \frac{1-\omega_j^2}{2}\|u_n\|_2^2.$$

Using this inequality, in the same way as in Lemmas (5.1.9) and (5.1.10), we see that

$$-\tilde{P}\left(u_{\mathrm{n}}(t_j)\right) \geq 2N\left\{\tilde{d}_{\omega_j} - (H-\omega_j Q)\left(u_{\mathrm{n}}(0)\right)\right\} = 2\delta \quad (83)$$

holds for all $t_j \geq 0$. Therefore, taking $m_1 > 0$ such that

$$\frac{C_1 M_2}{m_1^2} + \frac{C_2 M^4}{m_1^{\frac{2+\epsilon}{\epsilon}}} < \delta,$$

by (82) and (83), we have $(d/dt_j)\tilde{I}_{m_1}(t_j) \geq \delta$for all $t_j \geq 0$, and $\lim_{t_j\to\infty} \tilde{I}_{m_1}(t_j) = \infty$. The rest of the proof is the same as in the proof of Theorem (5.1.11) for the case $\epsilon \geq 1$, and we omit the details.

Theorem (5.1.27)[219]: Let $\epsilon = 2, c_0 \neq 1, |\omega_j| < 1/\sqrt{3}$ and the sequence ϕ_{ω_j}be the ground state of(20). Then, the sequence of the standing waves$\left(e^{i\left(\sum_{j=1}^{n}\omega_j t_j\right)}\phi_{\omega_j}, -\left|\phi_{\omega_j}\right|^2\right)$ of the KGZ system (3)-(4) are very strongly unstable in the following sense. For any $\lambda > 1$, the sequence of solutions $u_n(t_j)$of (3)-(4) with the initial data $u_n(0) = \left(\lambda\phi_{\omega_j}, \lambda i\omega_j\phi_{\omega_j}, -\lambda^2\left|\phi_{\omega_j}\right|^2, 0\right)$blows up in a finite time.

Proof. Let $\lambda > 1$. Suppose that the sequence of solutions $u_n(t_j)$ of (3)-(4) with $u_{\mathrm{n}}(0) = \left(\lambda\phi_{\omega_j}, \lambda i\omega_j\phi_{\omega_j}, -\lambda^2\left|\phi_{\omega_j}\right|^2, 0\right)$ exists globally. By the assumption $|\omega_j| < 1/\sqrt{3}$, we can take α such that $2\omega_j^2/(1-\omega_j^2) < \alpha < 1$. For such an α, we consider a function defined by

$$I_\alpha(t_j) = \frac{1}{2}\left\{\left\|u_n(t_j)\right\|_2^2 + \frac{\alpha}{c_0^2}\left\|n(t_j)\right\|_{\dot{H}^{-1}}^2\right\}.$$

Note that since $n(0) = -\lambda^2\left|\phi_{\omega_j}\right|^2 \in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3) \subset \dot{H}^{-1}(\mathbb{R}^3)$ and $\partial_{t_j}n \in C\left([0,\infty); \dot{H}^{-1}(\mathbb{R}^3)\right)$, we see that $n \in C^1\left([0,\infty); \dot{H}^{-1}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)\right)$. Then, we have

$$\begin{aligned}\frac{d}{dt_j}I_\alpha(t_j) &= \operatorname{Re}\langle u_n(t_j), \partial_{t_j}u_n(t_j)\rangle_{L^2} + \frac{\alpha}{c_0^2}\langle n(t_j), \partial_{t_j}n(t_j)\rangle_{\dot{H}^{-1}} \\ &= \operatorname{Re}\langle u_n(t_j), \partial_{t_j}u_n(t_j) - i\omega_j u_n(t_j)\rangle_{L^2} + \frac{\alpha}{c_0^2}\langle n(t_j), \partial_{t_j}n(t_j)\rangle_{\dot{H}^{-1}},\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt_j^2}I_\alpha(t_j) &= \left\|\partial_{t_j}u_n(t_j)\right\|_2^2 + \frac{\alpha}{c_0^2}\left\|\partial_{t_j}n(t_j)\right\|_{\dot{H}^{-1}}^2 - \left\|\nabla u_n(t_j)\right\|_2^2 - \left\|u_n(t_j)\right\|_2^2 \\ &\quad -\alpha\left\|n(t_j)\right\|_2^2 - (1+\alpha)\int_{\mathbb{R}^3}\left|u_n(t_j,x_j)\right|^2 n(t_j,x_j)\,dx_j.\end{aligned}$$

Thus, we have

$$\frac{d^2}{dt_j^2} I_\alpha(t_j) + 2(1+\alpha)(H - \omega_j Q)(u_n(0)) - 2\omega_j Q(u_n(0))$$

$$= (2+\alpha)\left\|\partial_{t_j} u_n(t_j) - i\omega_j u_n(t_j)\right\|_2^2 + \left(\frac{3\alpha+1}{2c_0^2}\right)\frac{\alpha}{c_0^2}\left\|\partial_{t_j} n(t_j)\right\|_{\dot{H}^{-1}}^2$$

$$+K_{\omega_j,\alpha}\left(u_n(t_j), n(t_j)\right),$$

where we put

$$K_{\omega_j,\alpha}(u_n, n) = \alpha\left\{\|\nabla u_n\|_2^2 + \left(1 - \omega_j^2 - \frac{2}{\alpha}\omega_j^2\right)\|u_n\|_2^2 + \frac{1-\alpha}{2\alpha}\|n\|_2^2\right\}.$$

Here, we define

$$J_{\omega_j}(u_n, n) = \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1-\omega_j^2}{2}\|u_n\|_2^2 + \frac{1}{4}\|n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} \left|u_n(x_j)\right|^2 n(x_j)\,dx_j\,,$$

$$K_{\omega_j,\alpha}^1(u_n, n) = \partial_\lambda J_{\omega_j}(\lambda u_n, \lambda^{2\alpha} n)\Big|_{\lambda=1}$$

$$= \|\nabla u_n\|_2^2 + (1-\omega_j^2)\|u_n\|_2^2 + \alpha\|n\|_2^2 + (1+\alpha)\int_{\mathbb{R}^3} |u_n|^2 n\,dx_j\,,$$

$$K_{\omega_j,\alpha}^2(u_n, n) = 2\,\partial_\lambda J_{\omega_j}(\lambda^{(1-\alpha)/\alpha} u_n(\cdot/\lambda), n(\cdot/\lambda)\Big|_{\lambda=1}$$

$$= \frac{2-\alpha}{\alpha}\|\nabla u_n\|_2^2 \frac{2+\alpha}{\alpha}(1-\omega_j^2)\|u_n\|_2^2$$

$$+\frac{3}{2}\|n\|_2^2 \frac{2+\alpha}{\alpha}\int_{\mathbb{R}^3} |u_n|^2 n\,dx_j,$$

and put

$$J_{\omega_j,\alpha}^1(u_n, n) = J_{\omega_j}(u_n, n) - \frac{1}{2(1+\alpha)} K_{\omega_j,\alpha}^1(u_n, n)$$

$$= \frac{\alpha}{1+\alpha}\left\{\frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1-\omega_j^2}{2}\|u_n\|_2^2 + \frac{1-\alpha}{4\alpha}\|n\|_2^2\right\},$$

$$J_{\omega_j,\alpha}^2(u_n, n) = J_{\omega_j}(u_n, n) - \frac{\alpha}{2(2+\alpha)} K_{\omega_j,\alpha}^2(u_n, n)$$

$$= \frac{\alpha}{2+\alpha}\left\{\|\nabla u_n\|_2^2 + \frac{1-\alpha}{2\alpha}\|u_n\|_2^2\right\},$$

$$\theta = 1 - \frac{2\omega_j^2}{(1-\omega_j^2)\alpha}.$$

Then, we have $0 < \theta < 1$ and

$$K_{\omega_j,\alpha}(u_n, n) = 2(1+\alpha)\theta J^1_{\omega_j,\alpha}(u_n, n) + (2+\alpha)(1-\theta)J^2_{\omega_j,\alpha}(u_n, n).$$

Moreover, in a similar way as in Lemmas (5.1.3) and (5.1. 4) in [23], we can prove that $J^j_{\omega_j,\alpha}\left(u(t_j), n(t_j)\right) \geq \tilde{d}_{\omega_j}$for all $t_j \geq 0$ and $j = 1, 2$. Therefore, we have

$$\begin{aligned} K_{\omega_j,\alpha}\left(u_n(t_j), n(t_j)\right) &\geq \{2(1+\alpha)\theta + (2+\alpha)(1-\theta)\}\tilde{d}_{\omega_j} \\ &= 2\left(1+\alpha - \frac{\omega_j^2}{1-\omega_j^2}\right)\tilde{d}_{\omega_j} \end{aligned}$$

for all $t_j \geq 0$. Moreover, since we have $\tilde{d}_{\omega_j} = (1-\omega_j^2)\left\|\phi_{\omega_j}\right\|_2^2$, putting $\beta = \min\left\{\frac{3\alpha+1}{2\alpha}\right\}$, we have

$$\begin{aligned} \frac{d^2}{dt_j^2} I_\alpha(t_j) \geq \beta &\left\{\left\|\partial_{t_j} u_n(t_j) - i\omega_j u_n(t_j)\right\|_2^2 + \frac{\alpha}{c_0^2}\left\|\partial_{t_j} n(t_j)\right\|_{\dot{H}^{-1}}^2\right\} \\ &+2(1+\alpha)\left\{\tilde{d}_{\omega_j} - (H-\omega_j Q)(u_n(0))\right\} + 2\omega_j Q(u_n(0)) - 2\omega_j^2\left\|\phi_{\omega_j}\right\|_2^2 \end{aligned}$$

for all $t_j \geq 0$. Since $\beta > 2$, $(H-\omega_j Q)(u_n(0)) < \tilde{d}_{\omega_n}$and $\omega_n Q(u_n(0)) > \omega_j^2\left\|\phi_{\omega_j}\right\|_2^2$for all $\lambda > 1$, by the standard concavity argument, we see that there exists $T_1 \in (0, \infty)$ such that $\lim_{t_j \to T_1 - 0} I_\alpha(t_j) = \alpha$. This is a contradiction. Hence, for all$\lambda > 1$, the sequence of solutions $u_n(t_j)$ of (3)-(4) with $u_n(0) = \left(\lambda\phi_{\omega_j}, \lambda i\omega_j\phi_{\omega_j}, -\lambda^2\left|\phi_{\omega_j}\right|^2, 0\right)$blows up in finite time.

This completes the proof.

Section (5.2): One-Dimensional Nonlinear Schrödinger Equation with Multiple-Power Nonlinearity:

We consider the stability and instability of standing waves for the following nonlinear Schrödinger equation:

$$iu_t + u_{xx} + f(u) = 0, \qquad t \geq 0, x \in \mathrm{R}, \tag{84}$$

Where $f(u) = \sum_{j=1}^m a_j|u|^{p_j-1}u$ with $a_j \in \mathrm{R}$ and $1 < p_1 < \cdots < p_m < \infty$. Equation (84) arises in various regions of mathematical physics.

The unique local existence of (84) is well known. That is for any $u_0 \in H^1\mathrm{R}$, there exists a positive constant $T$and a unique local solution $u \in C\big([0,T);\ H^1(\mathrm{R})\big) \cap C^1\big([0,T); H^{-1}(\mathrm{R})\big)$ of (53) with $u(0) = u_0$. Furthermore, $u(t)$ satisfies the two conservation laws $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and $E\big(u(t)\big) = E(u_0)$, where $E(v) := \frac{1}{2}\|v_x\|_{L^2} - \int_{\mathrm{R}} F(|v(x)|)dx$and $F(s) = \int_0^s f(\sigma)d\sigma$. For details, see, e.g., [133], [134] and [135].

We say that the solution of (84) is a standing wave if it has a form $u(t,x) = e^{i\omega t}\varphi_\omega(x)$, where $\omega > 0$. Here $\varphi_\omega$is a solution of the following equation:

$$\varphi_{xx} - \omega\varphi + f(\varphi) = 0, \qquad x \in \mathrm{R}, \varphi \in H^1(\mathrm{R})\backslash\{0\}. \tag{85}$$

The existence and uniqueness of the solution of (54) is well known: Set

$$\omega^* = \sup\left\{\omega > 0\text{: there exists } s > 0, \quad \text{s.t.}\frac{1}{2}\omega s^2 - F(s) < 0\right\},$$

then for any $\omega \in (0,\omega^*)$, there exists a solution $\varphi_\omega$of (54). Further the solution is unique up to a translation and a phase change ([136]). We study how the stability of standing waves depends on frequency ω in the multiple power nonlinearity case.

Stability and instability of standing waves is defined as follows.

Definition (5.2.1)[132]:A standing wave $u_\omega(t) = e^{i\omega t}\varphi_\omega$ is said to be stable if for all $\varepsilon > 0$ there exists $\delta > 0$ with the following property; for any $u_0 \in H^1(\mathrm{R})$ satisfying $\|u_0 - \varphi_\omega\|_{H^1(\mathrm{R})} < \delta$, the solution $u(t)$of (84) with $u(0) = u_0$ can be continued to a solution in $0 \le t < \infty$ and it satisfies the following condition

$$\sup_{0\le t<\infty}\ \inf_{0,y\in\mathrm{R}}\left\|u(t) - e^{i\theta}\varphi_\omega(\cdot - y)\right\|_{H^1(\mathrm{R})} < \varepsilon.$$

Otherwise, $u_\omega$is called unstable.

Remark (5.2.2)[132]: We note that the conception of stronger stability which does not involve the translation

$$\sup_{0\le t<\infty}\ \inf_{0\in\mathrm{R}}\left\|u(t) - e^{i\theta}\varphi_\omega\right\|_{H^1(\mathrm{R})},$$

Cannot be used in studying the stability of (84). It is because if $u(x,t) = e^{i\omega t}\varphi_\omega(x)$ is a solution of (84), then by a simple calculation, we observe that $u_c(x,t) = e^{i(\omega t - cx - c^2 t)}\varphi_\omega(x - 2ct)$ is also a solution. It is not hard to see that

$u_c(x,0)$ can be taken arbitrary near $u(x,0)$by taking c small, and if c is not zero, then $u_c(t)$always goes away from $u(t)$(see [137]).

Recently, many authors have been studying the problem of stability and instability of standing waves for nonlinear Schrödinger equations (see, e.g., [138, 136,139, 140, 141, 142, 143, 144, 137, 145, 146]).

At first, we will introduce the results in the single power case $f(u) = a|u|^{p-1}u$ with $a > 0$ and $p > 1$. For this case, if $1 < p < 5$, then u_ω is stable for every $\omega \in (0,\infty)$, and if $5 \le p$, then $u_\omega$is unstable for every $\omega \in (0,\infty)$(see [138], [147] and[146]). For the single power case, (84) has scaling invariance, and using it, one can verify the stability. Note that the stability of standing waves is independent of the frequency o in the single power case. Although it is not the case with the double power nonlinearity. In this case, there is no scaling invariance in (84),so the problem to investigate the stability of standing waves becomes more complicated.

Although, when $f(u) = a_1|u|^{p_1-1}u + a_2|u|^{p_2-1}u$, Ohta [148] proved the following theorem.

Theorem (5.2.3)[132]: (Ohta [148]). Let $1 < p_1 < p_2$.

(I) Let $a_1, a_2 > 0$.

(I.1) If $p_2 \le 5$, then $u_\omega$is stable for any $\omega \in (0,\infty)$.

(I.2) If $p_1 \le 5$, then $u_\omega$is unstable for any $\omega \in (0,\infty)$.

(I.3) If $p_1 < 5 < p_2$, then there exist positive constants $\omega_1$and $\omega_2$such that $u_\omega$is stable for any $\omega \in (0,\omega_1)$, and unstable for any $\omega \in (\omega_2,\infty)$.

(II) Let $a_1 > 0, a_2 < 0$.

(II.1) If $p_1 \le 5$, then $u_\omega$is stable for any $\omega \in (0,\omega^*)$.

(II.2) If $p_1 > 5$, then there exist positive constants ω_3 and ω_4 such that u_ω is unstable for any $\omega \in (0,\omega_3)$, and stable for any $\omega \in (\omega_4,\omega^*)$.

(III) Let $a_1 < 0, a_2 > 0$.

(III.1) If $p_2 \ge 5$, then $u_\omega$is unstable for any $\omega \in (0,\infty)$.

(III.2) If $p_2 < 5$, then there exists a positive constant ω_5 such that $u_\omega$is stable for any $\omega \in (\omega_5, \infty)$. Furthermore if $p_1 + p_2 > 6$, then there exists a positive constant $\omega_6$such that $u_\omega$is unstable for any $\omega \in (0, \omega_6)$.

Theorem (5.2.3) shows that, in the double power case, the stability of standing waves can change when the frequency ω varies. In Theorem (5.2.3), there are gaps in(I.3) (for $\omega \in [\omega_1, \omega_2]$), (II.2) (for $\omega \in [\omega_3, \omega_4]$) and (III.2) (for $\omega \in [0, \omega_5]$in the case of $p_1 + p_2 < 6$ and for $\omega \in [\omega_6, \omega_5]$in the case of $p_1 + p_2 > 6$). It seems difficult to verify whether the standing wave $u_\omega$is stable or not if the equationdoes not have scaling invariance. Our first target is to fill these gaps.

Our main results are the following.

Remark (5.2.4)[132]: Since $a_2 > 0$ for the cases (84) and (86), we observe that $\omega^* = \infty$.On the other hand, since $a_2 < 0$ for the case (54), we have $\omega^* < \infty$.

Remark (5.2.5)[132]: There are still gaps in the cases of Theorem (5.2.12 (iii). However Ohta [148] showed, when $a_1 < 0, a_2 > 0$ and $p_1 = 2, p_2 = 3, u_\omega$is stable for any $\omega \in (0, \infty)$. So, in Theorem (5.2.12)(iii), the condition $p_1 + p_2 > 6$ is needed, althoughit may be not optimal.

In the single power case, the stability of standing waves does not change by ω,and in the double power case, stability of standing waves change at most once. So,the natural question arises: if the equation has more powers, then could we getstanding waves that change its stability more than once? The next theorem gives examples of standing waves that change its stability, by ω, two and three times.

Remark (5.2.6)[132]: The conditions in Theorem (5.2.13) are for only technical reasons. Our motivation was to show there are equations whose standing waves change its stability several times when the frequency ωvaries.

We first summarize three lemmas needed for the proof of Theorems (5.2.12) and (5.2.13).

Lemma (5.2.7)[132]: (Grillakis, Shatah and Strauss [144]). Set

$$I(\omega) = \|\varphi_\omega\|_2^2 = \int_0^\infty |\varphi_\omega(x)|^2 dx\,.$$

If $I'(\omega) > 0$, then $u_\omega(t) = e^{i\omega t}\varphi_\omega$ is stable, and if $I'(\omega) < 0$, then u_ω is unstable.

For the case $I'(0) = 0$, Comech and Pelinovski proved the following theorem.

Theorem (5.2.8)[132]: (Comech and Pelinovski [149]). Let $e^{i\omega t}\varphi_\omega$be the standing wave solution of (84). Assume that $I'(\omega_*) = 0$and $I''(\omega_*) \neq 0$for some $\omega_* \in (0, \omega^*)$.Then there is a positive number ε such that for any $\delta > 0$, there exists $t_1 = t_1(\delta, \varepsilon) < \infty$ and a pair of functions $(\omega, \rho) \in C^1\big([0, t_1]; (0, \omega^*)\big) \times C^1\big([0, t_1]; H^1(\mathrm{R})\big)$,such that $u(t) = e^{i\int_0^t \omega(t')dt'}\left(\varphi_{\omega(t)} + \rho(t)\right)$is a solution to (84) and such that

$|\omega(0) - \omega_*| < \delta, \|\rho(t)\|_{H^1(\mathrm{R})} \leq 0$ and $|\omega(t_1) - \omega_*| > \varepsilon$.

The following lemma is a direct consequence of Theorem (5.2.8).

Lemma (5.2.9)[132]: If $I'(\omega_*) = 0$and $I''(\omega_*) \neq 0$, then $u_{\omega_*}$is unstable.

Proof. Because $\varphi_\omega$is an even real valued function, $\partial_\omega\varphi_\omega$is an even real valued function and $\partial_x\varphi_\omega$is an odd real valued function. It follows that $\partial_\omega\varphi_\omega \perp \partial_x\varphi_\omega$and $\partial_\omega\varphi_\omega \perp i\varphi_\omega$in $H^1(\mathrm{R})$. Now, note that the tangent space of the orbit $\{e^{is}\varphi_\omega(\cdot + y) | s, y \in \mathrm{R}\}$is spanned by $\partial_\omega\varphi_{\omega_*}$and $i\varphi_{\omega_*}$. So by Theorem (5.2.8),

$$u(t) = \exp\left(i\int_0^t \omega(\tau)d\tau\right)\left(\varphi_{\omega(t)} + \rho(t)\right) \sim \varphi_{\omega_*} + t\partial\varphi_{\omega_*}\,.$$

Therefore $u(t)$, which was initially close to the orbit, leaves the ε-tubular neighborhood of the orbit in finite time.

Lemma (5.2.10)[132]: (Iliev and Kirchev [137]). Suppose $f(u) = \sum_{j=1}^m a_j|u|^{p_j-1}u$, then we have

$$I'(\omega) = \frac{1}{W'(h)}\int_0^h \frac{\sum_{j=1}^m c_j(h^{q_j} - s^{q_j})}{\left(\sum_{j=1}^m d_j(h^{q_j} - s^{q_j})\right)^{3/2}}ds\,,$$

where $q_j = \frac{p_j-1}{2}, c_j = \frac{a_j(5-p_j)}{p_j+1}, d_j = \frac{2a_j}{p_j+1}, W(s) = \omega s - \sum_{j-1}^m \frac{2a_j}{p_j+1}s^{(p_j+1)/2}$and $h = h(\omega)$ is a positive number satisfying $W(h) = 0, W'(h) < 0$ and $W(s) > 0$ for all $s \in (0, h)$.

Remark (5.2.11)[132]: Function $h(\omega)$ can be defined as

$$h(\omega) \coloneqq \sup\{h > 0 | W(s) > 0 \text{ for all} s \in (0, h)\}.$$

Recall the definition of ω^*. Since $\omega \in (0, \omega^*)$, we have $W\big(h(\omega)\big) = 0$. Further,by

$$W(s) > 0 \Leftrightarrow \omega > V(s) \coloneqq \sum_{j=1}^{m} \frac{2a_j}{p_j + 1} s^{(p_j - 1)/2},$$

we see that

$$h(\omega) = \sup\{h > 0 | \omega > V(s) \text{for all} s \in (0, h)\}. \tag{86}$$

So we have that $h(\omega)$is a monotone increasing function. Furthermore by (3), for $a_1 > 0,\ h(0) = 0$ and for $a_1 < 0, h(0) > 0$. Also for $a_m > 0, \lim_{\omega\to\infty} h(\omega) = \infty$ and for $a_m < 0, \lim_{\omega\to\omega^*} h(\omega) < \infty$.

Theorem (5.2.12)[132]: Let $f(u) = a_1|u|^{p_1-1}u + a_2|u|^{p_2-1}u$.

(i) Suppose $a_1, a_2 > 0$ and $1 < p_1 < 5 < p_2$. Then there exists $\omega_1 > 0$ such that for $\omega \in (0, \omega_1), u_\omega$is stable, and for $\omega \in [\omega_1, \infty), u_\omega$is unstable.

(ii) Suppose $a_1 > 0, a_2 < 0$and $5 < p_1 < p_2$. Then there exists $\omega_2 > 0$ such that for $\omega \in (0, \omega_2], u_\omega$is unstable, and for $\omega \in (\omega_2, \omega^*), u_\omega$is stable.

(iii) Suppose $a_1 < 0, a_2 > 0, \frac{7}{3} < p_1 < p_2 < 5$ and $p_1 + p_2 > 6$. Then there exists $\omega_3 > 0$ such that for $\omega \in (0, \omega_3], u_\omega$is unstable, and for $\omega \in (\omega_3, \infty)$ then $u_\omega$is stable.

Proof. By Lemmas (5.2.7) and (5.2.9), we have only to check the sign of $I'(\omega)$ given by lemma (5.2.10).

In the case $m = 2, I'(\omega)$ can be written as

$$I'(\omega) = \frac{h^{(5-p_1)/4}}{2W'(h)} \int_0^1 \frac{H(h, s)}{\big(d_1(1 - s^{q_1}) + d_2(1 - s^{q_2})h^{q_2-q_1}\big)^{3/2}} ds,$$

where $(h, s) \coloneqq c_1(1 - s^{q_1}) + c_2(1 - s^{q_2})h^{q_2-q_1}$. Because $-h^{(5-p_1)/4}/2W'(h)$is always positive and we only care about the sign of I', it suffices to consider

$$F(h) = \int_0^1 \frac{H(h, s)}{\big(d_1(1 - s^{q_1}) + d_2(1 - s^{q_2})h^{q_2-q_1}\big)^{3/2}} ds.$$

By a simple calculation, we have

$$F'(h) = \frac{a_2(p_2 - p_1)}{2(p_2+1)} h^{q_2-q_1-1} \times \int_0^1 \frac{(1-s^{q_2})\widetilde{H}(h,s)}{\left(d_1(1-s^{q_1}) + d_2(1-s^{q_2})h^{q_2-q_1}\right)^{5/2}} ds\,,$$

where $\widetilde{H}(h,s) := -r(1-s^{q_1}) - c_2(1-s^{q_1})h^{q_2-q_1}$ and $r = c_1 + 2d_1(q_2 - q_1)$.

Now define

$$l(s) = \frac{1-s^{q_1}}{1-s^{q_2}}.$$

Then, $l(s)$ is a monotone decreasing function in $(0,1)$ and $l(s)$ satisfies

$$\frac{q_1}{q_2} < l(s) < 1, \qquad \forall s \in (0,1). \tag{87}$$

Part (i). In this case, we have $c_1 > 0, c_2 < 0$ and $r > 0$. Put

$$A_{p_1,p_2} := \left(-\frac{c_1 q_1}{c_2 q_2}\right)^{1/(q_2-q_1)}, \qquad B_{p_1,p_2} := \left(-\frac{r q_1}{c_2 q_2}\right)^{1/(q_2-q_1)}.$$

Taking $h = A_{p_1,p_2}\alpha^{1/(q_2-q_1)}$ for $a > 0, H(h,s)$can be rewritten as $H(h,s) = c_1(1-s^{q_2})\{l(s) - \alpha q_1/q_2\}$. From (56), if $a < 1$, i.e. $h < A_{p_1,p_2}$, then $F(h) > 0$, and if $\alpha > q_1/q_2$, i.e. $h > A_{p_1,p_2}(q_1/q_2)^{1/(q_2-q_1)}$, then $F(h) < 0$. In the same way, we see that if $h < B_{p_1,p_2}$, we have $F'(h) < 0$.

Now, $A_{p_1,p_2} < A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}$ always holds since $q_1 < q_2$. Also if $7/3 \le p_1$, then by a simple calculation we have $A_{p_1,p_2} < A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} \le B_{p_1,p_2}$. Since $F(A_{p_1,p_2}) > 0, F\left(A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}\right) < 0$ and $F$is a monotone decreasing function at $\left(A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}\right)$, there exists an $\omega_1 > 0$ such that if $\omega \in (0,\omega_1)$, then $\partial_\omega \|\varphi_\omega\|^2 < 0$. So, by lemmas (5.2.7) and (5.2.9), we have the conclusion.

For the case $1 < p_1 < 7/3$, it suffices to prove that, if $B_{p_1,p_2} \le h$, then $F(h) < 0$. If $h \in \left(A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}\right)$, there exists a solution of $l(s^*) - \alpha q_2/q_1 = 0$, since $\alpha \in (1, q_2/q_1)$and $l(s)$ decreases from 1 to q_2/q_1. Furthermore if $s \in (0,s^*)$, $H(h,s)$is positive and if $s \in (s^*,1)$, it is negative. Also because the denominator of the integrand of F is monotonically decreasing function, we see that $\int_0^1 H(h,s)ds < 0$ implies $F(h) < 0$. Now we note that

$$\int_0^1 H(h,s)ds = \frac{c_1 q_1}{q_1+1} + \frac{c_2 q_2}{q_2+1} h^{q_2-q_1},$$

and

$$\frac{c_1 q_1}{q_1+1} + \frac{c_2 q_2}{q_2+1} h^{q_2-q_1} < 0 \Leftrightarrow \left(-\frac{c_1 q_1 (q_2+1)}{c_2 q_2 (q_1+1)}\right)^{1/(q_2-q_1)} < h.$$

Therefore, since

$$\left(-\frac{c_1 q_1 (q_2+1)}{c_2 q_2 (q_1+1)}\right)^{1/(q_2-q_1)} < B_{p_1,p_2} \Leftrightarrow q_1 < q_2,$$

we see that $F(h) < 0$ for $B_{p_1,p_2} \leq h$.

Part (ii). In this case we have $c_1 < 0, c_2 > 0$ and $h(0) = 0, h(\omega^*) < \infty$.Since the signs of c_1, c_2 are opposite from part (i), we see that if $A_{p_1,p_2} > h$, then $F(h) < 0$, and if $A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < h$, then $F(h) > 0$. Also by a simple calculation we see that $A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < h(\omega^*)$.

First if $0 \leq r := c_1 + 2d_1(q_2 - q_1)$, then $\widetilde{H}(h,s)$ will be always negative. Consequently, since $a_2 < 0, F' > 0$ will always hold. So we have the conclusionin this case.

Next if $r < 0$, then we see that

$$\widetilde{H}(h,s) < 0 \Leftrightarrow -rl(s) < c_2 h^{q_2-q_1}.$$

Since $l(s) < 1$, if $h > (-r/c_2)^{1/(q_2-q_1)}$, then $\widetilde{H}(h,s) < 0$ and $F'(h) > 0$ follows.By a simple calculation, we see that $(-r/c_2)^{1/(q_2-q_1)} < A_{p_1,p_2}$. So, for $h \in \left(0, A_{p_1,p_2}\right], F$is negative, and for $h \in \left[A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}\right], F'$is positive,and for $h \in \left[A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}), h(\omega^*)\right)$, $F$is positive. From this, we have the conclusion.

Part (iii). In this case we have $c_1 < 0, c_2 > 0, r < 0$, and $h(0) > 0$. Since thesigns of c_1, c_2 are the same as in part (ii), we see that if $A_{p_1,p_2} > h$, then $F(h) < 0$,and if $A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < h$, then $F(h) > 0$. Now, since $h(0) > 0$, we wish to make A_{p_1,p_2} larger than $h(0)$. By a simple calculation we see that if $q_1 + q_2 > 2$,i.e. $p_1 + p_2 > 6$, then $h(0) < A_{p_1,p_2}$ holds.

Next, observing $\widetilde{H}(h,s)$, we see that if $h < (-rq_1/c_2q_2)^{1/(q_2-q_1)}$, then $F'(h) > 0$. Now, if $p_1 > 7/3$, by a simple calculation we see that

$$A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < (-rq_1/c_2q_2)^{1/(q_2-q_1)}.$$

So, for $h \in \left(0, A_{p_1,p_2}\right]$, $F$is negative, and for $h \in \left[A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}\right]$, $F'$is positive, and for $h \in \left[A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}), h(\omega^*)\right)$, $F$is positive. This gives us the conclusion.

Theorem (5.2.13)[132]:

(a) Let $f(u) = a_1|u|^2u + |u|^6u - |u|^8u$, let $a_1 > 0$ be sufficiently small. Then there exist five real numbers $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5$ such that for $\omega \in (0,\omega_1) \cup (\omega_4,\omega_5)$, u_ω is stable, and for $\omega \in (\omega_2,\omega_3)$, u_ω is unstable.

(b) Let $f(u) = a_1|u|^2u + |u|^6u - a_3|u|^8u + |u|^{10}u$, let $a_1 > 0$ be sufficiently small and $a_3 > 0$ sufficiently large. Then there exist six real numbers $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5 < \omega_6$ such that for $\omega \in (0,\omega_1) \cup (\omega_4,\omega_5)$, $u_\omega$is stable, and for $\omega \in (\omega_2,\omega_3) \cup (\omega_6,\infty)$, u_ω is unstable.

Proof. We will not consider the point where $I'(\omega) = 0$, sowe will only use Lemmas (5.2.7) and (5.2.10), and will not use Lemma (5.2.9).

Part (a). Since $a_1 > 0$and $a_3 := -1 < 0$, we have $h(0) = 0$ and $h(\omega^*) < 0$.Furthermore, calculating $h(\omega^*)$from the definition, we see that $h(\omega^*) > 1$.

By Lemma (5.2.10),

$$I'(\omega) = \int_0^h \frac{\frac{1}{2}a_1(h-s) - \frac{1}{4}(h^3 - s^3) + \frac{2}{5}(h^4 - s^4)}{\left(\frac{1}{2}a_1(h-s) - \frac{1}{4}(h^3 - s^3) + \frac{1}{5}(h^4 - s^4)\right)^{3/2}} ds\,.$$

Set

$$H(h,s) := \frac{1}{2}a_1(h-s) - \frac{1}{4}(h^3 - s^3) + \frac{2}{5}(h^4 - s^4).$$

Then we have

$$H(h,s) > 0 \Leftrightarrow \frac{a_1}{2} - \frac{1}{4}(h^2 - hs + s^2) + \frac{2}{5}(h^3 + h^2s + hs^2 + s^3) > 0.$$

So, by setting

$$G := \frac{a_1}{2} - \frac{3}{4}h^2 + \frac{2}{5}h^3,$$

$$\tilde{G}(h) := \frac{a_1}{2} - \frac{3}{4}h^2 + \frac{8}{5}h^3,$$

we see that $G \leq H \leq \tilde{G}$. Therefore if $G(h) > 0$, then $H(h, s) > 0$ for $\forall s \in (0, h)$,and if $\tilde{G}(h) < 0$, then $H(h, s) < 0$ for $\forall s \in (0, h)$.

Now, $G(h)$is positive near $h = 0$ and $\tilde{G}(h)$ takes negative values for some$h \in (0,1)$when a_1 is small. So, we see that there exist $h_1 < h_2 < h_3 < 1$ such that for $h \in (0, h_1), I' > 0$ and for $h \in (h_2, h_3), I' < 0$.

Next, we will show that for $h = 1, I' > 0$.

$$\begin{aligned} H(1, s) &= \frac{a_1}{2}(1 - s) - \frac{1}{4}(1 - s^3) + \frac{2}{5}(1 - s^4) \\ &= \frac{a_1}{2}(1 - s) + \frac{3}{20} + s^3\left(\frac{1}{4} - \frac{2}{5}s\right) \\ &> 0. \end{aligned}$$

So, there exist two numbers h_4 and h_5 such that $h_3 < h_4 < 1 < h_5 < h(\omega^*)$ and $I'(h) > 0$ for $h \in (h_4, h_5)$.

Part (b). In this case, we only have to calculate $G$and $\tilde{G}$as in Part (a).

Remark (5.2.14)[132]: We can make an example of standing waves that change its stability exactly $2m - 1$ times when the frequency ωvaries, by considering $2m$-power nonlinearity. In fact, by taking

$$f(u) = a_1|u|^2u + \sum_{j=2}^{2m}(-1)^j N^{-\left((2m-j)(2m+1-j)/2\right)}|u|^{2j+2}u\,,$$

and if $N$is sufficiently large and a_1 is sufficiently small, we see that $F(h)$changesits sign as frequency ωvaries in the same way as Theorem (5.2.12) and Theorem (5.2.13).This is possible because we can set $a_1 > 0$and $a_{2m} > 0$, so that $h(0) = 0$ and$h(\infty) = \infty$. This makes computation simpler because we do not have to consider the situations like in the proof of part (iii) of Theorem (5.2.12) (for example, the situation $h(0) < A_{p_1,p_2}$).

Section (5.3): Hamiltonian PDEs in the Degenerate Cases:

In this section, following a celebrated in [151] by Grillakis, Shatah and Strauss, we consider the abstract Hamiltonian system of the form

$$\frac{du}{dt}(t) = JE'(u), \tag{88}$$

Where E is the energy functional on a real Hilbert space X, and $J : X^* \to Y^*$is a skew-symmetric operator. Here, $Y$is another real Hilbert space and $u \in C(I, X) \cap C^1(I, Y^*)$for some interval I. equation (88) can be considered as a generalization of nonlinear Schrödinger equations(NLS) and nonlinear Klein–Gordon equations (NLKG). We assume that $E$is invariant under aone-parameter unitary group $\{T(s)\}_{s\in\mathbb{R}}$. We consider the stability and instability of bound state solutions $T(\omega t)_{\phi_\omega}$of (88), where $\omega \in \mathbb{R}$and $\phi_\omega \in X$.We assume that the linearized Hamiltonian

$$S''_\omega(\phi_\omega) := E(\phi_\omega) - \omega Q''(\phi_\omega)$$

has one negative eigenvalue, where $Q$is the invariant quantity which comes out from the Noether's principal due to the symmetry $T(s)$.

In [151], it is proved that if $d''(\omega) > 0$ (resp. < 0), then the bound state $T(\omega t)_{\phi_\omega}$is stable (resp.unstable), where

$$d(\omega) \coloneqq E(\phi_\omega) - \omega Q(\phi_\omega).$$

Further, Theorem (5.3.2) of [151] claims that "bound states are stable if and only if $d$is strictly convex in a neighborhood of ω". However, as pointed out by Comech and Pelinovsky [152], their argument seems to be not correct for the case $d''(\omega) = 0$. Our aim of this section is to recover this criterion,i.e. investigate the stability and instability for the case $d''(\omega) = 0$.

For the case $d''(\omega) = 0$, Comech and Pelinovsky [152] showed that if $d''(\widetilde{\omega}) \leq 0$ in a one-sidedopen neighborhood of ω, then the bound state $T(\omega t)_{\phi_\omega}$is unstable. Their proof is based on the observation that in the case $d''(\omega) = 0$, the linearized operator $JS''_\omega(\phi_\omega)$has a degenerate zero eigenvalue which leads to a polynomial growth of perturbations. They showed the instability by considering (88) as a perturbation of the linearized equation around ϕ_ω. Recently, Ohta [153] gave another proof for the instability of bound states for the case $d''(\omega) = 0, d'''(\omega) \neq 0$. His proof is based on [151] and [154] which uses a Lyapunov functional to "push out" the solutions from the neighborhood of the bound states. However, [153] assumes $T'(0) = J$and this assumption preven this result to apply to the NLKG equations.

We follow the work of [151,154,153] and extend the results of [151] and [153]. We show that, if $d''(\omega)$is strictly convex in a neighborhood of ω, then the bound is stable and if $d(\widetilde{\omega}) - d(\omega) - (\widetilde{\omega} - \omega)d'(\omega) < 0$ in $\omega < \widetilde{\omega} < \omega + \varepsilon$or $\omega - \varepsilon < \widetilde{\omega} < \omega$ for some $\varepsilon > 0$, then the bound state is unstable. For the meaning of assumption "$d(\widetilde{\omega}) - d(\omega) - (\widetilde{\omega} - \omega)d'(\omega) < 0$", consider the following three conditions.

(A) $\exists \varepsilon > 0$ s.t. $\forall \lambda \in (0, \varepsilon)$(resp. $\forall \lambda \in (-\varepsilon, 0)$), $d''(\omega + \lambda) < 0$.

(B) $\exists \varepsilon > 0$ s.t. $\forall \lambda \in (0, \varepsilon)\big(\forall \lambda \in (-\varepsilon, 0)\big), d(\omega + \lambda) - d(\omega) + \lambda d'(\omega) < 0$.

(C) $\exists \{\lambda_n\}$s.t.$\lambda_n \to 0$ and $d''(\omega + \lambda_n) < 0$.

Then, we have (A)⇒(B)⇒(C) and (C) is equivalent to "d is not convex in the neighborhood of ω". Therefore, our assumption, which is condition (B), do not cover the case "d is not convexin the neighborhood of ω", but the gap can considered to be small. If $d''(\omega) = 0$and $d'''(\omega) \neq 0$,then we have (A). So, our result covers the result of [153]. The only natural case which we cannot treat in our theorem is the case d is linear in a one-sided open neighborhood of ω. In this sense we have almost proved the criterion "bound states are stable if and only if $d(\omega)$ is strictly convex".

The proof is based on a purely variational argument. We note that our result almost covers the result of [152] but not completely. The case d is linear in the neighborhood of ω is excluded by our theorem, which in this case can be covered by [152]. However, our proof requires less regularity for E, which is $E \in C^2$ and does not need an assumption for nonlinearity .

We give an application of our theorem for the single power NLKG equations and double power nonlinear Schrödinger equations. For the one dimensional NLKG with $|u|^{p-1}u, 1 < p < 2$, ourresult seems to be new. Further, we remark our result covers all dimensions in a unified way.

We formulate our assumptions and the main results in a precise manner. We prepare some notations and lemmas for the proof of the main results. In particular, we construct a curve $\Psi(\lambda)$ on the hyper-surface $\mathcal{M} = \{Q(u) = Q(\phi_\omega)\}$, which crosses the set of the bound state. Then, we calculate $S_\omega(\Psi(\lambda))$and $P(\Psi(\lambda))$, where P is a functional which we will use for the instability. This curve $\Psi(\lambda)$ corresponds to the degenerate direction of the energy functional E in the hyper-surface $\mathcal{M}$. We calculate S_ω and P for general u in a neighborhood of ϕ_ω under some restrictions on the value of S_ω. The restrictions give us a good estimate for the "nondegenerate" directions and enables us to use the results. We give an applications of the main theorem for NLKG and NLS equations.

Let $X, Y$and H be real Hilbert spaces such that

$$X \hookrightarrow H \simeq H^* \hookrightarrow X^*, Y \hookrightarrow H \simeq H^* \hookrightarrow Y^*,$$

where all the embeddings are densely continuous. We identify $H$with H^*naturally. We denotethe inner product of H, the coupling between $X$and $X^*$and the coupling between $Y$and Y^*allby $\langle\cdot,\cdot\rangle$. The norms of $X$and $H$are denoted as $\|\cdot\|_X$and $\|\cdot\|_H$, respectively. Let $J : H \to H$ be a skew-symmetric operator in such a sense that

$$\langle Ju, v\rangle = -\langle u, Jv\rangle, \;\; u, v \in H.$$

Further, we assume $J|_X: X \to Y$and $J|_Y: Y \to X$are bijective and bounded. The operator Jcan be naturally extended to $\tilde{J}: X^* \to Y^*$by

$$\langle \tilde{J}u, v\rangle := -\langle u, Jv\rangle, \quad u \in X^*, v \in Y.$$

Let $T(s)$be a one parameter unitary group on $X$and let $T'(0)$is the generator of $T(s)$. We denote the domain of $T'(0)$by $D(T'(0)) \subset X$. As J, we cannaturally extend $T(s)$to $\tilde{T}(s): X^* \to X^*$by

$$\langle \tilde{T}(s)u, v\rangle := \langle u, T(-s)v\rangle, \qquad u \in X^*, \;\; v \in X.$$

We assume $\tilde{T}(s)(Y) \subset Y$for all $s \in \mathbb{R}$. For simplicity, we just denote $\tilde{T}(s)$as $T(s)$. We further assume that $J$and $T(s)$commute.

Let $E \in C^2(X, \mathbb{R})$. We consider the following Hamiltonian PDE.

$$\frac{du}{dt}(t) = \tilde{J}E'u(t), \tag{89}$$

where $E'$is the Fréchet derivative of E. We say that $u(t)$is a solution of (58) in an interval Iif $u \in C(I, X) \cap C^1(I, Y^*)$and satisfies (89) in $Y^*$for all $t \in I$. We assume that $E$is invariant under T, that is,

$$E(T(s)u) = E(u), \qquad s \in \mathbb{R}, \ u \in X.$$

We assume that there is a bounded operator $B : X \to X^*$such that $B^* = B$and the operator B is an extension of $J^{-1}T'(0)$. We define $Q: X \to \mathbb{R}$by

$$Q(u) := \frac{1}{2}\langle Bu, u\rangle, \qquad u \in X. \tag{90}$$

Then, we have $Q(T(s)u) = Q(u)$for $u \in X$. Indeed, for $u \in D\big(T'(0)\big)$, we have

$$\frac{d}{ds}Q(T(s)u) = \langle BT(s)u, T'(0)T(s)u\rangle$$
$$= \langle BT(s)u, JBT(s)u\rangle = 0.$$

For general $u \in X$, we only have to take a sequence $u_n \in D\big(T'(0)\big), u_n \to u$in X. Further, formally Q conserves under the flow of (58). Indeed,

$$\frac{d}{dt}Q\big(u(t)\big) = \langle Bu(t), J^{-1}E'\big(u(t)\big)\rangle$$
$$= \langle T'(0)u(t), E'\big(u(t)\big)\rangle$$
$$= \frac{d}{ds}\Big|_{s=0} E\big(T(s)u(t)\big) = 0.$$

We now assume that the Cauchy problem of (581) is well-posed in X.

Assumption (5.3.1)[150]:(*Existence of solutions*). Let $\mu > 0$. Then, there exists $T(\mu) > 0$ such that for all$u_0 \in X$ with $\|u_0\|_X \le \mu$, we have a solution $u$of (89)in $\big[0, T(\mu)\big)$with $u(0) = u_0$. Further,u satisfies $E\big(u(t)\big) = E(u_0)$and $Q\big(u(t)\big) = Q(u_0)$for $t \in \big(0, T(\mu)\big)$.

We next define the bound state, which is a stationary solution modulo symmetry $T(s)$.

Definition (5.3.2)[150]:(*Bound state*). By a bound state we mean a solution of (89)in $\mathbb{R}$with the form

$$u(t) = T(\omega t)\phi,$$

where $\omega \in \mathbb{R}$and $\phi \in X$.

Remark (5.3.3)[150]: If $T(\omega t)\phi$ is a bound state and $\phi \in D\big(T(0)\big)$, then it satisfies

$$\omega T(\omega t)T(0)\phi = JE'(T(\omega t)\phi).$$

Thus, by $E(T'(s)u) = T(s)E'(u)$ and the definition of Q, we have

$$E'(\varphi) - \omega Q'(\phi) = 0. \tag{91}$$

On the other hand, if $\phi \in X$ satisfies (60), then $T(\omega t)\phi$ is a bound state.

Definition (5.3.4)[150]:(*Stability of bound states*). We say the bound state $T(\omega t)\phi$ is stable if for all $\varepsilon > 0$there exists $\delta > 0$ with the following property. If $\|u_0 - \phi\|_X < \delta$and $u(t)$is a solution of (58) given in Assumption (5.3.1), then $u(t)$ can be continued to a solution in $[0, \infty)$and

$$\sup_{0<t} \inf_{s\in\mathbb{R}} \|u(t) - T(s)\phi\|_X < \varepsilon.$$

Otherwise the bound state $T(\omega t)\phi$ is said to be unstable.

Assumption (5.3.5)[150]:(*Existence of bound states*). Let $\omega_1 < \omega_2$. We assume that there exists a C^3 map$(\omega_1, \omega_2) \to X, \omega \mapsto \phi_\omega$ such that

(i) $T(\omega t)\phi_\omega$is a bound state.

(ii) $\phi_\omega \in D(T'(0)^3), \partial_\omega\phi_\omega \in D(T'(0)^2), \partial_\omega^2\phi_\omega \in D\big(T'(0)\big)$,
$T'(0)\phi_\omega, \partial_\omega\phi_\omega,\ T'(0)\partial_\omega\phi_\omega, \partial_\omega^2\phi_\omega \in Y$.

(iii) $T'(0)\phi_\omega \neq 0, \partial_\omega\phi_\omega \neq 0$ and $\langle T'(0)\phi_\omega, \partial_\omega\phi_\omega\rangle = 0$.

Remark (5.3.6)[150]: By the fact that $T'(0)\phi_\omega \in Y$, we have $B\phi_\omega = J^{-1}T'(0)\phi_\omega \in X$.

Remark (5.3.7)[150]: In $\langle T'(0)\phi_\omega, \partial_\omega\phi_\omega\rangle = 0$is actually not an assumption. Indeed, suppose $\omega \mapsto \phi_\omega$ does not satisfy $\langle T'(0)\phi_\omega, \partial_\omega\phi_\omega\rangle = 0$. Then, set $\tilde{\phi}_\omega = T\big(s(\omega)\big)\phi_\omega$, where

$$s(\omega) = -\int_0^\omega \frac{\langle T'(0)\phi_\mu, \partial_\mu\phi_\mu\rangle}{\|T'(0)\phi_\mu\|_H^2} d\mu.$$

Then, $\tilde{\phi}_\omega$satisfies $\langle T'(0)\tilde{\phi}_\omega, \partial_\omega\tilde{\phi}_\omega\rangle = 0$.

Set

$$S_\omega(u) \coloneqq E(u) - \omega Q(u), \quad u \in X,$$
$$d(\omega) \coloneqq S_\omega(\phi_\omega), \tag{92}$$

where $\phi_\omega$is given in Assumption (5.3.5).

Remark (5.3.8)[150]: Condition (91)is equivalent to $S'_\omega(\phi) = 0$.

We further assume that the linearized Hamiltonian $S''_\omega(\phi_\omega)$ satisfies the following spectral condition.

Assumption (5.3.9)[150]:(*Spectral conditions for the bound states*). For $\omega \in (\omega_1, \omega_2)$, we assume the following.

(i) $\ker S''_\omega(\phi_\omega) = \mathrm{span}\{T'(0)\phi_\omega\}$,

(ii) $S_\omega(\phi_\omega)$ has only one simple negative eigenvalue $-\mu < 0$,

(iii) $\inf\{s > 0 | s \in \sigma(S''_\omega(\phi_\omega))\} > 0$,

where $\sigma(S''_\omega(\phi_\omega)) \subset \mathbb{R}$is the spectrum of $S''_\omega(\phi_\omega)$.

Grillakis, Shatah and Strauss [151] proved the following theorem.

Theorem (5.3.10)[150]:*Let Assumptions* (5.3.1)*,* (5.3.5)*and* (5.3.9) *be satisfied. Then, if* $d''(\omega) > 0$*, the bound state* $T(\omega t)\phi_\omega$ *is stable and if* $d''(\omega) < 0$*, the bound state* $T(\omega t)\phi_\omega$*is unstable.*

We investigate the case $d''(\omega) = 0$.

We denote $f(\lambda) \sim g(\lambda)$if $f$and $g$satisfy

$$0 < \liminf_{|\lambda|\to 0} f(\lambda)/g(\lambda) \le \limsup_{|\lambda|\to 0} f(\lambda)/g(\lambda) < \infty. \tag{93}$$

We assume

$$d(\omega+\lambda) - d(\omega) - \lambda d'(\omega) \sim \lambda\big(d'(\omega+\lambda) - d'(\omega)\big). \tag{94}$$

This is a technical assumption which we need in the proof.

Remark (5.3.11)[150]: If $d \in C^n$and $d^{(m)}(\omega) \neq 0$ for some $2 < m \le n$, then the assumption (94)is satisfied. Let $d(\omega+\lambda) = e^{-1/|\lambda|}$, then d does not satisfy 94). However, this assumption seems to be natural.

We now state our main results.

Remark (3.3.12)[150]: For Theorem (5.3.2) below, we can remove the condition $J|_X, J|_Y$ are bijective and bounded.Further, we only need $\omega \mapsto \phi_\omega$to be C^2. We only use these conditions for Theorem (5.3.3) below, which is concerned with the instability. Therefore, we can treat the case $J = \partial_x$, which appears for KdV type equations and BBM type equations.

Remark (5.3.13)[150]: If $d''(\omega) > 0$ (resp. < 0), then the assumption of Theorem (5.3.2)below (resp. Theorem (5.3.3) is satisfied. Therefore, Theorems (5.3.2) and (5.3.3) below are extension of Theorem (5.3.10).

Remark (5.3.14)[150]: The assumption $\langle \phi_{\omega+\lambda}, J^{-1}\partial_\omega^2 \phi_{\omega+\lambda} \rangle = 0$ is technical. However, for the NLS and $NLKG$ cases, this is satisfied when as far as the real-valued standing waves are concerned.

Corollary (5.3.15)[150]:*Let Assumptions* (5.3.1)*,* (5.3.5)*and* (5.3.9)*be satisfied. Let* $n \geq 4$ *be an even integer. Assume that*$d \in C^n$*in an open neighborhood of* ω *and assume*

$$d(\omega) = \cdots = d^{(n-1)}(\omega) = 0, \qquad d^{(n)}(\omega) > 0.$$

Then $T(\omega t)\phi_\omega$*is stable.*

Corollary (5.3.16)[150]:*Let Assumptions* (5.3.1)*,* (5.3.5)*and* (5.3.9)*be satisfied. Further, assume there exists* $\varepsilon > 0$ *such that* $\langle \phi_{\omega+\lambda}, J^{-1}\partial_\omega^2 \phi_{\omega+\lambda} \rangle = 0$ *for* $|\lambda| < \varepsilon$*. Let* $n \geq 3$ *be an integer. Assume that* $d \in C^n$*in an open neighborhood of* ω *and*

$$d''(\omega) = \cdots = d^{(n-1)}(\omega) = 0,$$

$$d^{(n)}(\omega) < 0 \ (n : even), \ \ d^{(n)}(\omega) \neq 0 \ (n : odd).$$

Then $T(\omega t)\phi_\omega$*is unstable.*

We assume Assumptions (5.3.1)*,* (5.3.5), (5.3.9), (94) and $d''(\omega) = 0$. Note that by differentiating (92) with respect to ω, we have

$$d'(\omega) = S'_\omega(\phi_\omega) - Q(\phi_\omega) = -Q(\phi_\omega), \tag{95}$$

$$d''(\omega) = -\langle B\phi_\omega, \partial_\omega \phi_\omega \rangle. \tag{96}$$

Further, differentiating the equation $S'_\omega(\phi_\omega) = 0$ with respect to ω, we have

$$S''_\omega(\phi_\omega)\partial_\omega \phi_\omega = B\phi_\omega. \tag{97}$$

We will use these relations in the following. Set

$$\eta_1(\lambda) \coloneqq d(\omega + \lambda) - d(\omega) - \lambda d'(\omega), \tag{98}$$

$$\eta_2(\lambda) \coloneqq d'(\omega + \lambda) - d(\omega). \tag{99}$$

Recall that in (94), we have assumed $\eta_1(\lambda) \sim \lambda \eta_2(\lambda)$. Further, since we are assuming $d''(\omega) = 0$,we have $\eta_2(\lambda) = o(\lambda)$as $\lambda \to 0$.

Lemma (5.3.17)[150]:*Let* $\varepsilon > 0$ *sufficiently small. Then, there exists* $\sigma(\lambda): (-\varepsilon, \varepsilon) \to \mathbb{R}$ *such that* $\sigma(\lambda) \sim \eta_2(\lambda)$ *and*

$$Q(\phi_{\omega+\lambda} + \sigma(\lambda)B\phi_{\omega+\lambda}) = Q(\phi_\omega), \tag{100}$$

for $|\lambda| < \varepsilon$*, where we have used "$\sim$" in the sense of* (93).

Proof. Set

$$F(\sigma,\lambda) = Q(\phi_{\omega+\lambda} + \sigma B\phi_{\omega+\lambda}).$$

Then, $F(0,0) = Q(\phi_\omega)$ and $\partial_\sigma F|_{\sigma=\lambda=0}(\sigma,\lambda) = \|B\phi_\omega\|_H^2 = 0$ by Remark (5.3.6). Therefore, by the implicit function theorem, there exist $\varepsilon > 0, \delta > 0$ and $\sigma : (-\varepsilon,\varepsilon) \to (-\delta,\delta)$ such that $\sigma(\lambda)$satisfies (100) for $|\lambda| < \varepsilon$. Further, by (100), we have

$$\begin{aligned}\sigma(\lambda)\big(\|B\phi_{\omega+\lambda}\|_H^2 + \sigma(\lambda)Q(B\phi_{\omega+\lambda})\big) &= -Q(\phi_{\omega+\lambda}) + Q(\phi_\omega)\\ &= d'(\omega+\lambda) - d'(\omega)\\ &= \eta_2(\lambda),\end{aligned}$$

where we have used (95) and (99). Since

$$\sigma(\lambda)\big(\|B\phi_{\omega+\lambda}\|_H^2 + \sigma(\lambda)Q(B\phi_{\omega+\lambda})\big) = \sigma(\lambda)\big(\|B\phi_\omega\|_H^2 + o(1)\big) \text{as} \lambda \to 0,$$

we have the conclusion.

We now define a curve on the neighborhood of ϕ_ω. Let $\varepsilon > 0$ as in Lemma (5.3.17). For $|\lambda| < \varepsilon$, set

$$\Psi(\lambda) \coloneqq \phi_{\omega+\lambda} + \sigma(\lambda)B\phi_{\omega+\lambda}.$$

We next calculate the value of $S_\omega\big(\Psi(\lambda)\big)$.

Lemma (5.3.18)[150]:*Let* $\varepsilon > 0$ *as in Lemma* (5.3.17)*. Then for* $|\lambda| < \varepsilon$*, we have*

$$S_\omega\big(\Psi(\lambda)\big) - S_\omega(\phi_\omega) = \eta_1(\lambda) + o\big(\eta_1(\lambda)\big), \qquad \lambda \to 0.$$

Proof. By the definition of S_ω, we have $S_\omega = S_{\omega+\lambda} + \lambda Q$. Using this and the Taylor expansion,we have

$$\begin{aligned}S_\omega\big(\Psi(\lambda)\big) &= S_{\omega+\lambda}\big(\Psi(\lambda)\big) + \lambda Q\big(\Psi(\lambda)\big)\\ &= S_{\omega+\lambda}(\phi_{\omega+\lambda} + \sigma(\lambda)B\phi_{\omega+\lambda}) + \lambda Q(\phi_\omega)\\ &= S_{\omega+\lambda}(\phi_{\omega+\lambda}) + \lambda Q(\phi_\omega) + O(\sigma(\lambda)^2)\\ &= d(\omega+\lambda) - \lambda d'(\omega) + o\big(\eta_1(\lambda)\big), \qquad \lambda \to 0.\end{aligned}$$

Here, we have used $Q\big(\Psi(\lambda)\big) = Q(\phi_\omega)$for the second equality,$S'_{\omega+\lambda}(\phi_{\omega+\lambda}) = 0$ for the third equality and $\sigma(\lambda) = o(\lambda), O\big(\lambda\sigma(\lambda)\big) = O\big(\lambda\eta_2(\lambda)\big) = O\big(\eta_1(\lambda)\big)$as $\lambda \to 0$ for the last equality. Therefore, by (98), we have the conclusion.

We define a tubular neighborhood of ϕ_ω. Set

$$N_\varepsilon := \left\{u \in X \middle| \inf_{s\in\mathbb{R}} \|u - T(s)\phi_\omega\|_X < \varepsilon\right\},$$

$$N_\varepsilon^0 := \{u \in N_\varepsilon | Q(u) = Q(\phi_\omega)\}.$$

Lemma (5.3.19)[150]: *Let $\varepsilon > 0$ sufficiently small. Then for $u \in N_\varepsilon$, there exist $\theta(u), \Lambda(u), w(u)$ and $\alpha(u)$ such that*

$$T(\theta(u))u = \Psi(\Lambda(u)) + w(u) + \alpha(u)B\phi_{\omega+\Lambda(u)},$$

and

$$\langle w(u), T'(0)\phi_{\omega+\Lambda(u)}\rangle = \langle w(u), \partial_\omega\phi_{\omega+\Lambda(u)}\rangle = \langle w(u), B\phi_{\omega+\Lambda(u)}\rangle = 0$$

Further, Λ and θ are C^2.

Proof. Set

$$G(u,\theta,\Lambda) = \begin{pmatrix} \langle T(\theta)u - \Psi(\Lambda), T'(0)\phi_{\omega+\Lambda}\rangle \\ \langle T(\theta)u - \Psi(\Lambda), \partial_\omega\phi_{\omega+\Lambda}\rangle \end{pmatrix}.$$

Then, we have $G(\phi_\omega, 0, 0) = 0$ and

$$\frac{\partial G}{\partial(\theta,\Lambda)} = \left(G_{ij}(u,\theta,\Lambda)\right)_{i,j=1,2}, \tag{101}$$

where

$$G_{11}(u,\theta,\Lambda) = \langle T'(0)T(\theta)u, T'(0)\phi_{\omega+\Lambda}\rangle,$$

$$G_{12}(u,\theta,\Lambda) = -\langle \partial_\lambda\Psi(\Lambda), T'(0)\phi_{\omega+\Lambda}\rangle + \langle T(\theta)u - \Psi(\Lambda), T'(0)\partial_\omega\phi_{\omega+\Lambda}\rangle,$$

$$G_{21}(u,\theta,\Lambda) = \langle T'(0)T(\theta)u, \partial_\omega\phi_{\omega+\Lambda}\rangle,$$

$$G_{22}(u,\theta,\Lambda) = -\langle \partial_\lambda\Psi(\Lambda), \partial_\omega\phi_{\omega+\Lambda}\rangle + \langle T(\theta)u - \Psi(\Lambda), \partial_\omega^2\varphi_{\omega+\Lambda}\rangle.$$

Therefore,

$$\left.\frac{\partial G}{\partial(\theta,\Lambda)}\right|_{u=\phi_\omega,\theta=\Lambda=0} = \begin{pmatrix} \|T'(0)\phi_\omega\|_H^2 & 0 \\ 0 & -\|\partial_\omega\phi_\omega\|_H^2 \end{pmatrix},$$

is invertible. Thus, there exist functionals $\theta(u)$ and $\Lambda(u)$ defined in the neighborhood of ϕ_ω such that $G(u, \theta(u), \Lambda(u)) = 0$. Since, $\omega' \mapsto \phi_{\omega'}$ is a C^3 map, we see that G is C^2. Therefore, Λ and θ are C^2. For $u \in N_\varepsilon$, define $\theta(T(s)u) = \theta(u) - s$ and $\Lambda(T(s)u) = \Lambda(u)$. Finally, define

$$\alpha(u) = \langle T(\theta(u))u - \Psi(\Lambda(u)), B\phi_{\omega+\Lambda(u)}\rangle \left\|B\phi_{\omega+\Lambda(u)}\right\|_H^{-2},$$

$$w(u) = T(\theta(u))u - \Psi(\Lambda(u)) - \alpha(u)B\phi_{\omega+\Lambda(u)}.$$

Therefore, we have the conclusion.

Let $\varepsilon > 0$ as in Lemma (5.3.19). Set

$$M(u) := T\big(\theta(u)\big)u, \quad u \in N_\varepsilon.$$

Remark (5.3.20)[150]: By the uniqueness of the solution of $G = 0$, we have

$$\theta\big(\Psi(\lambda)\big) = 0, \ \ \Lambda\big(\Psi(\lambda)\big) = \lambda,$$
$$w\big(\Psi(\lambda)\big) = 0, \ \ \alpha\big(\Psi(\lambda)\big) = 0.$$

We next show that the Fréchet derivatives of θ and $\Lambda$are in Y.

Lemma (5.3.21)[150]:*Let $\varepsilon > 0$ sufficiently small. Let $u \in N_\varepsilon$. Then, $\theta'(u), \Lambda'(u) \in Y$.*

Proof. By differentiating $G\big(u, \theta(u), \Lambda(u)\big) = 0$ with respect to u, we have

$$H(u)\begin{pmatrix}\theta'(u)\\ \Lambda'(u)\end{pmatrix} = -\begin{pmatrix}T\big(-\theta(u)\big)T'(0)\phi_{\omega+\Lambda(u)}\\ T\big(-\theta(u)\big)\partial_\omega\phi_{\omega+\Lambda(u)}\end{pmatrix}, \tag{102}$$

where $H(u) = \Big(G_{i,j}\big(u, \theta(u), \Lambda(u)\big)\Big)_{i,j=1,2}$. Since $H(u)$is invertible in N_ε for sufficiently small $\varepsilon > 0$ and $T'(0)\phi_{\omega+\Lambda(u)} \in Y, \partial_\omega\phi_{\omega+\Lambda(u)} \in Y$by Assumption (5.3.5), we have the conclusion.

Remark (5.3.22)[150]: As the proof of Lemma (5.3.21), by differentiating *(*102*)* with respect to u, we see that $\theta''(u)w \in Y$and $\Lambda''(u)w \in Y$for $u \in N_\varepsilon$ and $w \in X$.

Let $\varepsilon > 0$ sufficiently small. We now introduce the following functionals A and P defined in N_ε, which we use to show the instability theorem.

$$A(u) := \langle M(u), J^{-1}\partial_\omega\phi_{\omega+\Lambda(u)}\rangle,$$
$$P(u) := \langle S'_{\omega+\Lambda(u)}(u), JA'(u)\rangle.$$

Remark (5.3.23)[150]:A and $P$are well-defined in N_ε for sufficiently small $\varepsilon > 0$. Indeed,

$$A'(u) = J^{-1}T\big(-\theta(u)\big)\partial_\omega\phi_{\omega+\Lambda(u)} + \langle T'(0)M(u), J^{-1}\partial_\omega\phi_{\omega+\Lambda(u)}\rangle\theta'(u)$$
$$+\langle M(u), J^{-1}\partial_\omega^2\phi_{\omega+\Lambda(u)}\rangle\Lambda'(u). \tag{103}$$

So, by Assumption (5.3.5) and Lemma (5.3.21), we have $A'(u) \in Y$and $JA'(u) \in X$. Therefore, the definitionof Pmakes sense.

Remark (5.3.24)[150]: Let u be the solution of *(*58*)*, then

$$\frac{d}{dt}A\big(u(t)\big) = -P\big(u(t)\big). \tag{104}$$

Indeed, first, since $A(T(s)u) = A(u)$, for $u \in D\big(T(0)\big)$,

$$0 = \langle A'(u), T'(0)u\rangle = -\langle Bu, JA'(u)\rangle.$$

Therefore, formally, we have

$$\frac{d}{dt}A\big(u(t)\big) = \langle A'(u), u_t\rangle = \langle A'(u), \tilde{J}E'(u)\rangle = -\langle E'(u), JA'(u)\rangle = -P(u).$$

By Lemma (5.3.4) of [151], we have $A \circ u \in C^1$ for $u \in C(I, X) \cap C^1(I, Y^*)$. Therefore, the formal calculation is justified.

Remark (5.3.25)[150]: A and P are invariant under T, that is

$$A(T(s)u) = A(u),$$

$$P(T(s)u) = P(u).$$

Indeed, the invariance of A follows from the invariance of M and Λ. The invariance of P follows from the invariance of S and A. More precisely, since $A(T(s)u + h) = A(u + T(-s)h)$, we have $A'(T(s)u) = T(s)A'(u)$. So, we have

$$P(T(s)u) = \langle S'(T(s)u), JA'(T(s)u)\rangle = T(s)S'(u), JT(s)A'(u) = P(u),$$

where we have used the fact J and $T(s)$ commutes.

We now calculate the value of P along the curve Ψ.

Lemma (5.3.26)[150]: *Let $\varepsilon > 0$ sufficiently small. Assume $\langle \phi_\omega, J^{-1}\partial_\omega^2\phi_\omega\rangle = 0$. Then, for $|\lambda| < \varepsilon$, we have*

$$P\big(\Psi(\lambda)\big) = \eta_2(\lambda) + o\big(\eta_2(\lambda)\big), \qquad \lambda \to 0.$$

Proof. First, we calculate $S'_{\omega+\Lambda(\Psi(\lambda))}\big(\Psi(\lambda)\big)$.

$$S'_{\omega+\Lambda(\Psi(\lambda))}\big(\Psi(\lambda)\big) = S'_{\omega+\lambda}(\phi_{\omega+\lambda} + \sigma(\lambda)B\phi_{\omega+\lambda})$$

$$= \sigma(\lambda)S''_{\omega+\lambda}(\phi_{\omega+\lambda})B\phi_{\omega+\lambda} + o\big(\sigma(\lambda)\big).$$

Next, we calculate $JA'\big(\Psi(\lambda)\big)$. Recall that $M\big(\Psi(\lambda)\big) = \Psi(\lambda) = \phi_{\omega+\lambda} + \sigma(\lambda)B\phi_{\omega+\lambda}$ and we assumed

$$-\langle B\phi_\omega, \partial_\omega\phi_\omega\rangle = d''(\omega) = 0,$$

and $\langle \phi_\omega, J^{-1}\partial_\omega^2\phi_\omega\rangle = 0$. So, we have

$$\langle T'(0)M\big(\Psi(\lambda)\big), J^{-1}\partial_\omega\phi_{\omega+\lambda}\rangle = o(1), \;\; \lambda \to 0,$$

$$M\big(\Psi(\lambda)\big), J^{-1}\partial_\omega^2\phi_{\omega+\lambda} = o(1), \;\; \lambda \to 0.$$

Therefore, by (103), we have

$$JA'\big(\Psi(\lambda)\big) = \partial_\omega \phi_{\omega+\lambda} + o(1), \quad \lambda \to 0.$$

Combining these calculations, we have

$$\begin{aligned} P\big(\Psi(\lambda)\big) &= \sigma(\lambda)\langle S''_{\omega+\lambda}(\phi_{\omega+\lambda})B\phi_{\omega+\lambda}, \partial_\omega \phi_{\omega+\lambda}\rangle + o\big(\sigma(\lambda)\big) \\ &= \sigma(\lambda)\|B\phi_{\omega+\lambda}\|_H + o\big(\sigma(\lambda)\big) \\ &= \eta_2(\lambda) + o\big(\eta_2(\lambda)\big), \ \ \lambda \to 0, \end{aligned}$$

where we have used the relation $S''_{\omega+\lambda}(\phi_{\omega+\lambda})\partial_\omega \phi_{\omega+\lambda} = B\phi_{\omega+\lambda}$.

The following lemma is well known. For example see [153].

Lemma (5.3.27)[150]: *There exists $k_0 > 0$ such that if $w \in X$ satisfies $\langle w, T'(0)\phi_\omega\rangle = \langle w, \partial_\omega \phi_\omega\rangle = \langle w, B\phi_\omega\rangle = 0$, then $\langle S''_\omega(\phi_\omega)w, w\rangle \geq k_0\|w\|_X^2$.*

By a continuity argument and Lemma (5.3.27), we can show the following lemma.

Lemma (5.3.28)[150]: *There exists $\varepsilon_0 > 0$ such that for $|\lambda| < \varepsilon_0$, if $w \in X$ satisfies $\langle w, T'(0)\phi_{\omega+\lambda}\rangle = \langle w, \partial_\omega \phi_{\omega+\lambda}\rangle = \langle w, B\phi_{\omega+\lambda}\rangle = 0$, then $\langle S''_\omega(\phi_\omega)w, w\rangle \geq \frac{1}{2}k_0\|w\|_X^2$.*

We assume Assumptions (5.3.1), (5.3.5), (5.3.9),(63) and $d''(\omega) = 0$. We first estimate $\alpha(u)$ which is given in Lemma (5.3.19).

Lemma (5.3.29)[150]: *Let $\varepsilon > 0$ sufficiently small. Let $\in N_\varepsilon^0$. Let $\sigma(\lambda)$ as in Lemma* (5.3.1) *and $\alpha(u), w(u)$ and Λ as in Lemma* (5.3.19). *Then, there exists a constant $C > 0$ such that*

$$|\alpha(u)| \leq C\big(\sigma\big(\Lambda(u)\big)\|w(u)\|_X + \|w(u)\|_X^2\big).$$

Proof. We first calculate $Q(u)$. By Lemma (5.3.19) and (59) (definition of Q), we have

$$\begin{aligned} Q(\phi_\omega) &= Q(u) \\ &= Q\big(\Psi\big(\Lambda(u)\big) + w(u) + \alpha(u)B\phi_{\omega+\Lambda(u)}\big) \\ &= Q\left(\Psi\big(\Lambda(u)\big)\right) + Q\big(w(u) + \alpha(u)B\phi_{\omega+\Lambda(u)}\big) \\ &\quad + \langle B\phi_{\omega+\Lambda(u)} + \sigma\big(\Lambda(u)\big)B^2\phi_{\omega+\Lambda(u)}, w(u) + \alpha(u)B\phi_{\omega+\Lambda(u)}\rangle \\ &= Q(\phi_\omega) + \alpha(u)\big\|B\phi_{\omega+\Lambda(u)}\big\|_H^2 + \sigma\big(\Lambda(u)\big)\langle B^2\phi_{\omega+\Lambda(u)}, w(u)\rangle \\ &\quad + \alpha(u)\sigma\big(\Lambda(u)\big)\langle B^2\phi_{\omega+\Lambda(u)}, B\phi_{\omega+\Lambda(u)}\rangle + Q\big(w(u)\big) \\ &\quad + \alpha(u)\langle Bw(u), B\phi_{\omega+\Lambda(u)}\rangle + \alpha(u)^2 Q\big(B\phi_{\omega+\Lambda(u)}\big), \end{aligned}$$

Therefore, we have

$$-\alpha(u)\left(\|B\phi_\omega\|_H^2 + o(1)\right) = \sigma\left(\Lambda(u)\right)\langle B^2\phi_{\omega+\Lambda(u)}, w(u)\rangle + Q\left(w(u)\right), \Lambda(u) \to 0.$$

Thus, we have the conclusion.

Next, we show that under a restriction of the value of $S_\omega$we get a good estimate for $w(u)$ and $\alpha(u)$.

Lemma (5.3.30)[150]:*Let $\varepsilon > 0$ sufficiently small. Let $a \in \mathbb{R}$. Suppose $u \in N_\varepsilon^0$and*

$$S_\omega(u) - S_\omega(\phi_\omega) \le a\eta_1\left(\Lambda(u)\right) + \frac{k_0}{10}\|w(u)\|_X^2,$$

where $k_0$is given as in Lemma (5.3.27)*. Then, $\|w(u)\|_X^2 = O\left(\eta_1\left(\Lambda(u)\right)\right)$ as $\Lambda(u) \to 0$. In particular, $\alpha(u) = O\left(\eta_1\left(\Lambda(u)\right)\right)$as $\Lambda(u) \to 0$.*

Proof. Suppose there exists $u_n \in N_\varepsilon^0, u_n \to \phi_\omega$in X, s.t.

$$S_\omega(u_n) - S_\omega(\phi_\omega) \le a\eta_1(\Lambda_n) + \frac{k_0}{10}\|w_u\|_X^2,$$

and $\|w_n\|_X^2 = C_n\eta_1(\Lambda_n)$, where $w_n = w(u_n), \Lambda_n = \Lambda(u_n), \alpha_n = \alpha(u_n)$and $C_n \to \infty$. Then, we have $\eta_1(\Lambda_n) = o(\|w_n\|_X^2)$. Further, by Lemma (5.3.17), (94) and assumption of contradiction, we have

$$\sigma(\Lambda_n) \sim \eta_2(\Lambda_n) \sim \frac{\eta_1(\Lambda_n)}{\Lambda_n} = \frac{\|w_n\|_X^2}{\Lambda_n C_n} = \frac{\eta_1^{1/2}(\Lambda_n)}{\Lambda_n C_n^{1/2}}\|w_n\|_X = o(\|w_n\|_X), \quad n \to \infty,$$

where we have used "∼" in the sense of (93). Thus, by Lemma (5.3.8), $\alpha_n = O(\|w_n\|_X^2)$. Now, by Taylor expansion and Lemma (5.3.19),

$$\begin{aligned} S_\omega(u_n) - S_\omega(\phi_\omega) &= S_\omega\left(\Psi(\Lambda_n) + w_n + \alpha_n B\phi_{\omega+\Lambda_n}\right) - S_\omega(\phi_\omega) \\ &= S_\omega\left(\Psi(\Lambda_n)\right) - S_\omega(\phi_\omega) + \langle S_\omega'\left(\Psi(\Lambda_n)\right), w_n + \alpha_n B\phi_{\omega+\Lambda_n}\rangle \\ &\quad + \frac{1}{2}\langle S_\omega''\left(\Psi(\Lambda_n)\right)w_n, w_n\rangle + o(\|w_n\|_X^2), \quad n \to \infty. \end{aligned}$$

Further, by Lemma (5.3.18) and $S_\omega'(\phi_\omega) = 0$, we have

$$S_\omega\left(\Psi(\Lambda_n)\right) - S_\omega(\phi_\omega) = O\left(\eta_1(\Lambda_n)\right) = o(\|w_n\|_X^2),$$

$$\langle S_\omega'\left(\Psi(\Lambda_n)\right), \alpha_n B\phi_{\omega+\Lambda_n}\rangle = o(\|w_n\|_X^2), \quad n \to \infty,$$

and by $S_\omega' = S_{\omega+\lambda}' + \lambda B, \langle B\phi_{\omega+\Lambda_n}, w_n\rangle = 0$ and $\sigma(\Lambda_n) = o(\|w_n\|_X)$as $n \to \infty$, we have

$$\langle S_\omega'\left(\Psi(\Lambda_n)\right), w_n\rangle = \langle S_{\omega+\Lambda_n}'\left(\Psi(\Lambda_n)\right) + B\Psi(\Lambda_n), w_n\rangle$$

$$= \langle S'_{\omega+\Lambda_n}\big(\Psi(\Lambda_n)\big) + \sigma(\Lambda_n)B\phi_{\omega+\Lambda_n}, w_n\rangle$$
$$= o(\|w_n\|_X^2), \quad n \to \infty.$$

Therefore, by Lemma (5.3.28), we have

$$S_\omega(u_n) - S_\omega(\phi_\omega) = \frac{1}{2}\langle S''_\omega\big(\Psi(\Lambda_n)\big)w_n, w_n\rangle + o(\|w_n\|_X^2)$$
$$\geq \frac{k_0}{4}\|w_n\|_X^2 + o(\|w_n\|_X^2)$$
$$\geq \frac{k_0}{8}\|w_n\|_X^2,$$

for sufficiently large n. This contradicts to the assumption. Therefore, we have the conclusion.

Theorem (5.3.31)[150]: *Let Assumptions* (5.3.1), (5.3.5), (5.3.9) *and* (94) *be satisfied. Assume that d is strictly convex in anopen neighborhood of ω. Then $T(\omega t)\phi_\omega$ is stable.*

Proof. Let $u \in N_\varepsilon^0$. Suppose, $S_\omega(u) - S_\omega(\phi_\omega) < \eta_1\big(\Lambda(u)\big) + \frac{k_0}{10}\|w(u)\|_X^2$. Then, by Lemma (5.3.30), we have $\|w(u)\|_X^2 = O\big(\eta_1(u)\big)$ as $\Lambda(u) \to 0$. Now,

$$S_\omega(u) - S_\omega(\phi_\omega) = S_\omega\big(\Psi\big(\Lambda(u)\big) + w(u) + \alpha(u)B\phi_{\omega+\Lambda(u)}\big)$$
$$= \eta_1\Lambda(u) + \langle S'_\omega\left(\Psi\big(\Lambda(u)\big)\right), w(u)\rangle + \frac{1}{2}\langle S''_\omega(\phi_\omega)w(u), w(u)\rangle$$
$$+ o\left(\eta_1\big(\Lambda(u)\big)\right).$$

Using $S'_\omega = S''_{\omega+\lambda} + B, \sigma\big(\Lambda(u)\big) = O\left(\eta_2\big(\Lambda(u)\big)\right)$ and $\|w(u)\|_X = O\left(\eta_1\big(\Lambda(u)\big)^{1/2}\right)$,

$$\langle S'_\omega\left(\Psi\big(\Lambda(u)\big)\right), w(u)\rangle = \langle S'_{\omega+\Lambda(u)}\left(\Psi\big(\Lambda(u)\big)\right) + \Lambda(u)B\Psi\big(\Lambda(u)\big), w(u)\rangle$$
$$= \sigma\Lambda(u)\langle S''_\omega\big(\phi_{\omega+\Lambda(u)}\big)B\phi_{\omega+\Lambda(u)}, w(u)\rangle$$
$$+ \Lambda(u)\sigma\big(\Lambda(u)\big)\langle B^2\phi_\omega, w(u)\rangle$$
$$= o\left(\eta_1\big(\Lambda(u)\big)\right).$$

Since we have assumed that d is strictly convex in an open neighborhood of $\omega, \eta_2(\lambda)$ is strictly increasing in an open neighborhood of 0 (if $\eta_2(\lambda)$ is not increasing, then d would not be convex, if $\eta_2(\lambda)$ is constant, then d would not be strictly convex). So, we have

$$S_\omega(u) - S_\omega(\phi_\omega) \geq c\Lambda(u)\eta_2\big(\Lambda(u)\big) + \frac{k_0}{4}\|w\|_X^2,$$

for a constant $c > 0$.

Now, suppose that there exist a sequence of solutions u_n and $t_n > 0$ s.t. $u_n \to \phi_\omega$in $X$and $\inf_{s\in\mathbb{R}}\|u_n(t_n) - T(s)\phi_\omega\|_X = \varepsilon_0/10$. Take

$$vn := \sqrt{Q(\phi_\omega)/Q(u_n)}u_n(t_n).$$

Since $\sqrt{Q(\phi_\omega)/Q(u_n)} \to 1$, we have $\|v_n - u_n(t_n)\|_X \to 0$ and $S_\omega(v_n) - S_\omega(\phi_\omega) \to 0$. Thus,$\Lambda(v_n), w(v_n)$ and $\alpha(v_n)$ converge to zero. This implies

$$\inf_{s\in\mathbb{R}}\|u_n(t_n) - T(s)\phi_\omega\|_X \to 0.$$

This is a contradiction.

We next show Theorem (5.3.32). We first calculate P.

Lemma (5.3.32)[150]:*Let $\varepsilon > 0$, sufficiently small. Let $u \in N_\varepsilon^0$ and $S_\omega(u) - S_\omega(\phi_\omega) < 0$. Further, assume $\langle \partial_\omega \phi_{\omega+\Lambda(u)}, J^{-1}\partial_\omega \phi_{\omega+\Lambda(u)}\rangle = 0$. Then*

$$P(u) = \eta_2(\Lambda(u)) + o\left(\eta_2(\Lambda(u))\right).$$

Proof. By Taylor expansion,

$$\begin{aligned} P(u) &= P\left(\Psi(\Lambda(u)) + w(u)\right) + o\left(\eta_2(\Lambda(u))\right) \\ &= \eta_2(\Lambda(u)) + \langle S''_{\omega+\Lambda(u)}\left(\Psi(\Lambda(u))\right)w(u), JA'\left(\Psi(\Lambda(u))\right)\rangle \\ &\quad + \langle S'_{\omega+\Lambda(u)}\left(\Psi(\Lambda(u))\right), JA''\left(\Psi(\Lambda(u))\right)w(u)\rangle + o\left(\eta_2(\Lambda(u))\right) \\ &= \eta_2(\Lambda(u)) + \langle S''_{\omega+\Lambda(u)}\left(\Psi(\Lambda(u))\right)w(u), JA'\left(\Psi(\Lambda(u))\right)\rangle \\ &\quad + o\left(\eta_2(\Lambda(u))\right), \ \ \Lambda(u) \to 0, \end{aligned}$$

where we have used $\|w(u)\|_X^2 = o\left(\eta_2(\Lambda(u))\right)$ and $S'_{\omega+\Lambda(u)}\left(\Psi(\Lambda(u))\right) = \left(\eta_2(\Lambda(u))\right)$.

Now, by (72),

$$\begin{aligned} JA'\left(\Psi(\Lambda(u))\right) &= \partial_\omega \phi_{\omega+\Lambda(u)} - \langle B\phi_{\omega+\Lambda(u)}, \partial_\omega \phi_{\omega+\Lambda(u)}\rangle \theta'\left(\Psi(\Lambda(u))\right) \\ &\quad + \langle \partial_\omega \phi_{\omega+\Lambda(u)}, J^{-1}\partial_\omega \phi_{\omega+\Lambda(u)}\rangle \Lambda'\left(\Psi(\Lambda(u))\right) \\ &\quad + O\left(\eta_2(\Lambda(u))\right), \ \ \Lambda(u) \to 0, \end{aligned}$$

where we have used Lemma (5.3.17). Now, by$\langle \partial_\omega \phi_{\omega+\Lambda(u)}, J^{-1}\partial_\omega \phi_{\omega+\Lambda(u)}\rangle = 0, \langle w(u),\ B\phi_{\omega+\Lambda(u)}\rangle = 0$ and (97), we have

$$\langle S''_{\omega+\Lambda(u)}\left(\Psi(\Lambda(u))\right)w(u), JA'\left(\Psi(\Lambda(u))\right)\rangle = \langle S''_{\omega+\Lambda(u)}(\phi_{\omega+\Lambda(u)})w(u), \partial_\omega\phi_{\omega+\Lambda(u)}\rangle$$

$$-\langle B\phi_{\omega+\Lambda(u)}, \partial_\omega\phi_{\omega+\Lambda(u)}\rangle \langle w(u), S''_{\omega+\Lambda(u)}(\phi_{\omega+\Lambda(u)})\theta'\left(\Psi(\Lambda(u))\right)\rangle + o\left(\eta_2(\Lambda(u))\right)$$

$$= o\left(\eta_2(\Lambda(u))\right), \quad \Lambda(u) \to 0,$$

where we have used $\|w\|_X = O\left(\eta_1(\Lambda(u))^{1/2}\right)$and $\theta'\left(\Psi(\Lambda(u))\right)$is a linear combination of $\partial_\omega\phi_{\omega+\Lambda(u)}$ and $T'(0)\phi_{\omega+\Lambda(u)}$ because of (102). Therefore, we have the conclusion.

Theorem (5.3.33)[299]:*Let Assumptions* (5.3.1)*,* (5.3.5)*,* (5.3.9) *and* (94) *be satisfied. Assume there exists* $\varepsilon > 0$ *such that* $d(\omega+\lambda) - d(\omega) - \lambda d'(\omega) < 0$*in* $0 < \lambda < \varepsilon$ *or* $-\varepsilon < \lambda < 0$*. Further, assume*$\langle \phi_{\omega+\lambda}, J^{-1}\partial_\omega^2\phi_{\omega+\lambda}\rangle = 0$*. Then* $T(\omega t)\phi_\omega$*is unstable.*

Proof. By the assumption of Theorem (5.3.33), we have $\eta_1(\lambda) < 0$ in a one-sided open neighborhood of 0. Therefore, by Lemma (5.3.18), we can take the initial data from $\Psi(\lambda_n)$, where$S(\Psi(\lambda_n)) < S(\phi_\omega)$ and $\lambda_n \to 0$. Suppose, $u_n$stays in N_ε^0. By the conservation of $E$and Q, we have

$$S_\omega(u_n(t)) - S_\omega(\phi_\omega) = \eta_1\left(\Lambda(u_n(t))\right) + o\left(\eta_1\left(\Lambda(u_n(t))\right)\right),$$

and by Lemma (5.3.32),

$$P(u(t)) = \eta_2\left(\Lambda(u_n(t))\right) + o\left(\eta_2\left(\Lambda(u_n(t))\right)\right).$$

Then, since $\lambda\eta_2(\lambda) \sim \eta_1(\lambda)$, we have

$$S_\omega(\phi_\omega) - S_\omega(u_n(t)) \le C\left|\Lambda(u_n(t))P(u_n(t))\right|,$$

for some constant $C > 0$. Thus, we have $0 < \delta < \left|P(u_n(t))\right|$ for arbitrary t. So, P has the same sign. Suppose $P > 0$. Then, $\frac{dA}{dt}(u_n(t)) > P(u_n(t)) > \delta$. Thus, A is unbounded. However, this is contradiction. For the case $P < 0$ we have the same conclusion.

We consider the following single power nonlinear Klein–Gordon (NLKG) equation.

$$u_{tt} - \Delta u + u - |u|^{p-1}u = 0, (x,t) \in \mathbb{R}^d, \tag{105}$$

where $d \geq 1$ and $1 < p < \infty$ for $d = 1, 2$ and $1 < p < 1 + 4/(d-2)$ for $d \geq 3$. To put (74) onto our setting, set $X = H_r^1(\mathbb{R}^d) \times L_r^2(\mathbb{R}^d), Y = L_r^2(\mathbb{R}^d) \times H_r^1(\mathbb{R}^d)$ and $H = \left(L_r^2(\mathbb{R}^d)\right)^2$, where H_r^1 and L_r^2 are subspaces of H^1 and L^2 which consist with radial functions. Then define J and E as

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$E(U) = \frac{1}{2}\int |v|^2 + |\nabla u|^2 + |u|^2 - \frac{1}{p+1}\int |u|^{p+1}.$$

Then, $J : H \to H$ is skew symmetric, and $J|_X : X \to Y, J|_Y : Y \to X$ are bounded and bijective.Further, E is C^2. Let $U = (u, v)^t$, where t means transposition. Then NLKG equation is rewritten as

$$\frac{d}{dt} U = JE'(U)$$

in Y^*. Further, in this case, we take $(s) = e^{is} I$, where I is the identity matrix. So, we have$(u) = \mathrm{Im} \int \bar{u} u_t$. From the results of Ginibre and Velo [155], it is known that NLKG equation is locally well-posed and E and Q are conserved (i.e. Assumption (5.3.1) is satisfied). For, $\omega^2 < 1$, let ϕ_ω be the unique positive radial solution of

$$0 = -\Delta\phi_\omega + (1 - \omega^2)\phi_\omega - \phi_\omega^p.$$

Then, $e^{i\omega t}\phi_\omega$ is the solution of (105). It is well known that $\phi \in S(\mathbb{R}^d)$, where $S(\mathbb{R}^d)$ is the Schwartz space (see for example Chapter B of [156]). Further, by scaling, we have $\phi_\omega = (1 - \omega^2)^{1/(p-1)} \phi_0\left((1 - \omega^2)^{1/2}) x\right)$. Therefore, it is easy to check $\omega \mapsto \phi_\omega$ satisfies Assumption (5.3.5).Further, Assumption (5.3.9) is also well known to be satisfied (see for example [157]).

Now, since $(1 - \omega^2)^{1/(p-1)} \phi_0\left((1 - \omega^2)^{1/2}) x\right)$, we can calculate d directly. Since $Q(\phi_\omega) = \omega \int \phi_\omega^2$, we have

$$d''(\omega) = -\left(1 - \left(1 + \frac{4}{p-1} - d\right)\omega^2\right)(1 - \omega^2)^{\frac{2}{p-1} - \frac{d}{2} - 1} \int_{\mathbb{R}^d} \phi_0^2.$$

So, we see that for the case $p > 1 + 4/d$, then $d''(\omega) < 0$ for all $\omega \in (-1, 1)$and for the case $1 < p < 1 + 4/d$, there exists

$$0 < \omega_* = \sqrt{\frac{p-1}{4-(d-1)(p-1)}} < 1,$$

such that if $|\omega| < \omega_*$, then $d''(\omega) < 0$ and if $|\omega| > \omega_*$, then $d''(\omega) > 0$. Therefore, in these cases we know the stability and instability. These are the results by [158] and [159].

For the case $\omega = \pm\omega_*$, we can show $d'''(\omega_*) \neq 0$, so by Corollary (5.3.16), we see that in this case, we have the instability.

We have to remark that for the case $d \geq 2$, this result was proved by Ohta and Todorova [160] and for the case $d = 1, p \geq 2$, one can prove this result by applying Comech and Pelinovsky's result[152] (for the case $1 < p < 2$, Therefore, for $1 < p < 2, d = 1$, this result seems to be new. Further, our proof, the proof of [160] and the proof of [152] are completely different from each other and our proof gives a simple and unified proof for the critical case.

We next consider the double power nonlinear Schrödinger equations.

$$iu_t + \partial_{xx}u + a_1|u|^{p_1-1}u + a_2|u|^{p_2-1}u, (t,x) \in \mathbb{R}^2,$$

where $a_1, a_2 \in \mathbb{R}$ and $1 < p_1 < p_2 < \infty$. In this case, let $X = Y = H_r^1(\mathbb{R}), H = L_r^2(\mathbb{R})$, $J = i, T(s) = e^{is}$ and

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 dx - \frac{a_1}{p_1+1}\int_{\mathbb{R}} |u|^{p_1+1} dx - \frac{a_1}{p_1+1}\int_{\mathbb{R}} |u|^{p_1+1} dx\,.$$

Then, we are on the setting of our theory. In this case, by the combination of a_1, a_2, it is known that there exists some $\omega > 0$ such that $d''(\omega) = 0$ and $d'''(\omega) \neq 0$ (see [161]). So, for such $\omega > 0$,we can show the instability.