Chapter 4

Lyapunov Inequality for Linear Hamiltonian Systems

We state and prove some Lyapunov inequalities for linear Hamiltonian system on an arbitrary time scale, so that the well-known case of differential linear Hamiltonian systems when the time scale is a set of real and the recently developed case of discrete Hamiltonian systems when the time scale is a set of integers are unified. Applying these inequalities, a disconjugacy criterion for Hamiltonian systems on time scales is obtained.

Section (4.1): linear Hamiltonian Systems on Time Scales:

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unified continuous and discrete analysis (see [74]). A time scale is a closed subset of the real numbers. We denote a time scale by the symbol $\mathbb{T}$. For a function y defined on $\mathbb{T}$, we introduce a so-called deltaderivative y^{Δ}. This delta derivative is equal to y'(the usual derivative) if $\mathbb{T}=\mathbb{R}$ is the set of all real numbers, and it is equal to Δy(the usual forward difference) if $\mathbb{T}=\mathbb{Z}$ is the set of all integers. In recent years there has been much research activity concerning some different equations on time scales.

We would like to consider the Hamiltonian system (see [79,80,81]) which contain two scalar linear dynamic equations

$$x^{\Delta}(t)=a(t)x\big(\sigma(t)\big)+b(t)u(t), \qquad u^{\Delta}(t)=-c(t)x\big(\sigma(t)\big)-a(t)u(t) \quad (1)$$

on an arbitrary time scale $\mathbb{T}$, where a, b and $c$are real-valued rd-continuous functions on$\mathbb{T}$with the coefficient $a(t)$satisfying the condition

$$1-\mu(t)a(t)\neq 0, \qquad t\in\mathbb{T}. \quad (2)$$

Notice that the second order linear dynamic equation

$$[p(t)x^{\Delta}(t)]^{\Delta}+q(t)x\big(\sigma(t)\big)=0, \qquad t\in\mathbb{T}, \quad (3)$$

in which $p(t), q(t)$are real-valued rd-continuous function and $p(t)\neq 0$ for all $t\in\mathbb{T}$, can be written as an equivalent Hamiltonian system of type (1). Indeed, let $x(t)$be a solutionof (3) and set $p(t)x^{\Delta}(t)=u(t)$. Then we have

$$x^{\Delta}(t)=\frac{1}{p(t)}u(t), \qquad u^{\Delta}(t)=-q(t)x\big(\sigma(t)\big).$$

So, (3) is equivalent to (1) with

$$a(t) \equiv 0, \qquad b(t) = \frac{1}{p(t)}, \qquad c(t) = q(t).$$

We remark that system (1) cover the continuous Hamiltonian system (when $\mathbb{T} = \mathbb{R}$,see [82,83])

$$x'(t) = a(t)x(t) + b(t)u(t), \quad u'(t) = -c(t)x(t) - a(t)u(t), \qquad t \in \mathbb{R},$$

and the discrete Hamiltonian system (when $\mathbb{T} = \mathbb{Z}$, see [84,82])

$$x^{\Delta}(t) = a(t)x(t+1) + b(t)u(t), \quad u^{\Delta}(t) = -c(t)x(t+1) - a(t)u(t), \qquad t \in \mathbb{R}.$$

Furthermore, system (1) extends these classical cases to many cases in between as well, such as the so-called q-difference equations, where

$$\mathbb{T} = q^{\mathbb{Z}}: \{q^k | k \in \mathbb{Z}\} \cup \{0\} \text{for some } q > 1$$

and difference equations with constant step size, where

$$\mathbb{T} = h\mathbb{Z} \coloneqq \{hk | k \in \mathbb{Z}\} \text{for some } h > 0.$$

Particularly useful for the discretization aspect are time scales of the form

$$\mathbb{T} = \{t_k | k \in \mathbb{Z}\}, \ \text{ where} t_k \in \mathbb{R}, t_k < t_{k+1} \text{for all} k \in \mathbb{Z}.$$

Lyapunov inequalities have proved to be very useful in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications in the theory of differential and difference equation. An introduction to Lyapunov inequalities for continuous and discrete linear Hamiltonian system can be found in [82] by Guseinov. The main purpose of this section is to obtain Lyapunov inequalities for Hamiltonian system on time scales.

Concerning system (1) with (2), we will also assume that

$$b(t) \geq 0, \qquad t \in \mathbb{T}. \tag{4}$$

For each $t \in \mathbb{T}$, let us set

$$c_+(t) = \max\{c(t), 0\}. \tag{5}$$

Instead of the usual zero, the concept of generalized zero on time scales is given as follows.

Definition (4.1.1)[73]: Let $t \in \mathbb{T}$. A vector solution $(x, u)$of the system (1) has a generalized zero at $\sigma(t)$if one of the following two conditions is satisfied:

(i) t is dense and $x(t) = 0$;

(ii) t is right-scattered, and $x(t)x\big(\sigma(t)\big) < 0$ or $x\big(\sigma(t)\big) = 0$.

Note that under the condition (4) above, the definition of generalized zero, a special case of that given in [81], is consistent with what is used for the generalized zero in the discrete case [84,82]

The continuous and/or discrete versions of these results may be found in [82], but the following theorems have covered all of such results.

Corollary (4.1.2)[73]: *Suppose*

$$1 - \mu(t)a(t) > 0, \quad b(t) > 0, \quad c(t) > 0 \quad for\ all\ t \in \mathbb{T},$$

and let $\alpha, \beta \in \mathbb{T}^{\kappa}$ *with* $\sigma(\alpha) < \beta$. *Assume that* (1) *has a real solution* (x, u) *with generalized zeros in* $\sigma(\alpha)$ *and* $\sigma(\beta)$ *and* x *is not identically zero on* $[\sigma(\alpha), \beta]$. *Then the inequality*

$$\int_{\alpha}^{\sigma(\beta)} |a(t)|\Delta t + \left\{\int_{\alpha}^{\sigma(\beta)} b(t)\Delta t \cdot \int_{\alpha}^{\sigma(\beta)} c(t)\Delta t\right\}^{1/2} > 1$$

holds.

We state by introducing the following concepts related to the notion of time scales, which can be found in [79,80,81,74,86]. A time scale $\mathbb{T}$ is defined as a nonempty closed subset of the real numbers. The two most popular example are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Define the *forward jump operator* $\sigma(t): \mathbb{T} \to \mathbb{T}$ for $t \in \mathbb{T}$ by

$$\sigma(t) := \inf\{\tau > t\ |\tau \in \mathbb{T}\},$$

And *back jump operator* $\rho(t): \mathbb{T} \to \mathbb{T}$ for $t \in \mathbb{T}$ by

$$\rho(t) := \sup\{\tau < t|\tau \in \mathbb{T}\}$$

(supplemented by $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$). A point $t \in \mathbb{T}$ is called *right-scattered, right-dense, left-scattered, left-dense*, if $\sigma(t) > t, \sigma(t) = t, \rho(t) < t, \rho(t) = t$ holds, respectively.We define $\mathbb{T}^{\kappa} = \mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum $t_{\max}$; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}\backslash\{t_{\max}\}$. *The graininess function* $\mu: T \to [0, +\infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Hence the graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$ while it is constant 1 for $\mathbb{T} = \mathbb{Z}$.However, a time scale $\mathbb{T}$ could have no constant graininess.

Let X be a real Banach space. The function $f:\mathbb{T}\to X$is called (delta) differentiable at$t\in\mathbb{T}^{\kappa}$with (delta) derivative $f^{\Delta}(t)\in X$, if for any $\varepsilon>0$ there is a neighborhood $U$of t(i.e., $U=(t-\delta,t+\delta)\cap\mathbb{T}$for some $\delta>0$) such that

$$\left|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\right|\leq|\sigma(t)-s|\qquad for\ all\ s\in U.$$

The function f is differentiable on $\mathbb{T}^{\kappa}$provided $f^{\Delta}(t)$exists for all $t\in\mathbb{T}^{\kappa}$. The following lemma shows some important properties of f^{Δ}.

Lemma (4.1.3)[73]:[80,74]. *Let $f,g:\mathbb{T}\to\mathbb{X}$ be two functions, and let $t\in\mathbb{T}^{\kappa}$. Then we have*

(i) *If $f^{\Delta}(t)$and $g^{\Delta}(t)$exist, then $Af+Bg$is differentiable at $t$with $(Af+Bg)^{\Delta}(t)=Af^{\Delta}(t)+Bg^{\Delta}(t)$for any constants A,B.*

(ii) *If $f^{\Delta}(t)$exists, then $f$is continuous at t.*

(iii) *If t is right-scattered and $f$is continuous at t, then $f^{\Delta}(t)$exists and*

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}.$$

(iv) *If $f^{\Delta}(t)$ exists, then $f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$.*

(v) *If $f^{\Delta}(t)$ and $g^{\Delta}(t)$ exist, then $fg$is differentiable at t with*

$$(fg)^{\Delta}(t)=f(\sigma(t))g^{\Delta}(t)+f^{\Delta}(t)g(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)).$$

(vi) *Let $f,g:\mathbb{T}\to\mathbb{R}$ be such that $g(t)g(\sigma(t))\neq0$and $f^{\Delta}(t),g^{\Delta}(t)$exist. Then f/gisdifferentiable at t with*

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

A function $f:\mathbb{T}\to\mathbb{X}$ is called *rd-continuous* provided it is continuous at each right-dense point and has a left-sided limit at each point, which is at the same time right-scattered and left-dense. One can show, see [80,74], that if $f:\mathbb{T}\to\mathbb{X}$ is a rd-continuous function, then there exists an unique function (antiderivative) $F:\mathbb{T}\to\mathbb{X}$ with the properties $F^{\Delta}(t)=f(t)$for all $t\in\mathbb{T}^{\kappa}$and $F(\tau)=\eta$, where $\tau\in\mathbb{T}$and $\eta\in\mathbb{X}$. Then we define the *Cauchy integral* of $f$by$\int_a^b f(t)\Delta t=F(b)-F(a)$, where $a,b\in\mathbb{T}$. In the following lemma we present some properties of the integral that will be needed later.

Lemma (4.1.4)[73]:[80,74]. *Let $f, g : \mathbb{T} \to \mathbb{X}$ are rd-continuous and $a, b \in \mathbb{T}$. Then*

(i) $\int_a^b [Af(t) + Bg(t)]\Delta t = A\int_a^b f(t)\Delta t + B\int_a^b g(t)\Delta t$, *where A, B are any constants.*

(ii) $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t)$ *for $t \in \mathbb{T}^\kappa$.*

(iii) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$.

(iv) $\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$(*integration by parts*).

(v) *If $|f(t)| \le g(t)$on $[a, b)$, then*

$$\left|\int_a^b f(t)\Delta t\right| \le \int_a^b g(t)\Delta t\,.$$

The notation $[a, b], [a, b), [a, +\infty)$, and so on, will denote time scales intervals, i.e.,for example, $[a, b) = \{t \in \mathbb{T} | a \le t < b\}$, where $a, b \in \mathbb{T}$. To prove our results, we will need the following auxiliary statement.

Lemma (4.1.5)[73]:(Cauchy–Schwarz inequality [85,80]). *Let $a, b \in \mathbb{T}$. For rd-continuous $f, g: [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \le \left\{\int_a^b f^2(t)\Delta t \cdot \int_a^b g^2(t)\Delta t\right\}^{1/2}.$$

Theorem (4.1.6)[73]:*Let $\alpha, \beta \in \mathbb{T}^\kappa$with $\sigma(\alpha) < \beta$. Assume that* (1) *has a real solution (x, u) such that $x(\sigma(\alpha)) = x(\sigma(\beta)) = 0$and $x$is not identically zero on $[\sigma(\alpha), \beta]$. Then the inequality*

$$\int_{\sigma(\alpha)}^{\beta} |a(t)|\Delta t + \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t \cdot \int_{\sigma(\alpha)}^{\beta} c_+(t)\Delta t\right\}^{1/2} \ge 2 \qquad (6)$$

holds.

Proof. Multiplying the first equation of (1) by $u(t)$ and the second one by $x(\sigma(t))$, and then adding the results, we obtain

$$(xu)^\Delta(t) = b(t)u^2(t) - c(t)x^2(\sigma(t)). \qquad (7)$$

Integrating the last equation from $\sigma(\alpha)$to $\sigma(\beta)$and noticing that $x\big(\sigma(\alpha)\big) = x\big(\sigma(\beta)\big) = 0$,we have

$$0 = \int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)u^2(t)\Delta t - \int_{\sigma(\alpha)}^{\sigma(\beta)} c(t)x^2\big(\sigma(t)\big)\Delta t.$$

Since $x\big(\sigma(\beta)\big) = 0$, by Lemma (4.1.4)(ii), (iii) we have

$$\begin{aligned}\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)u^2(t)\Delta t &= \int_{\sigma(\alpha)}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t + \int_{\sigma(\alpha)}^{\sigma(\beta)} c(t)x^2\big(\sigma(t)\big)\Delta t \\ &= \int_{\sigma(\alpha)}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t + \mu(\beta)c(\beta)x^2\big(\sigma(\beta)\big) \\ &= \int_{\sigma(\alpha)}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t. \qquad (8)\end{aligned}$$

Choose $\tau \in \big(\sigma(\alpha), \sigma(\beta)\big)$such that

$$|x(\tau)| = \max_{\sigma(\alpha)\le t\le\sigma(\beta)} \{|x(t)|\}.$$

Since x is not identically zero on $[\sigma(\alpha), \beta]$, we have $|x(\tau)| > 0$. Integrating the first equation of (1) initially from $\sigma(\alpha)$ to τ and then from $\tau$to $\sigma(\beta)$and observing that $x\big(\sigma(\alpha)\big) = x\big(\sigma(\beta)\big) = 0$, we get

$$x(\tau) = \int_{\sigma(\alpha)}^{\tau} a(t)x\big(\sigma(t)\big)\Delta t + \int_{\sigma(\alpha)}^{\tau} b(t)u(t)\Delta t,$$

$$\begin{aligned}-x(\tau) &= \int_{\tau}^{\sigma(\beta)} a(t)x\big(\sigma(t)\big)\Delta t + \int_{\tau}^{\sigma(\beta)} b(t)u(t)\Delta t \\ &= \int_{\tau}^{\beta} a(t)x\big(\sigma(t)\big)\Delta t + \int_{\tau}^{\sigma(\beta)} b(t)u(t)\Delta t,\end{aligned}$$

respectively, where for the second equal sign of the latter equation we have used Lemma (4.1.4)(ii) and (iii). Hence, employing the triangle inequality and Lemma (4.1.4)(v) gives

$$|x(\tau)| \leq \int_{\sigma(\alpha)}^{\tau} |a(t)|\left|x(\sigma(t))\right|\Delta t + \int_{\sigma(\alpha)}^{\tau} b(t)|u(t)|\Delta t,$$

$$|x(\tau)| \leq \int_{\tau}^{\beta} |a(t)|\left|x(\sigma(t))\right|\Delta t + \int_{\tau}^{\sigma(\beta)} b(t)|u(t)|\Delta t.$$

Adding these last two inequalities gives rise to

$$2|x(\tau)| \leq \int_{\sigma(\alpha)}^{\beta} |a(t)|\left|x(\sigma(t))\right|\Delta t + \int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)|u(t)|\Delta t. \quad (9)$$

Applying Cauchy–Schwarz inequality (Lemma (4.1.5)) and (8), we have

$$\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)|u(t)|\Delta t \leq \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)u^2(t)\Delta t\right\}^{1/2}$$

$$= \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\sigma(\alpha)}^{\beta} c(t)x^2(\sigma(t))\Delta t\right\}^{1/2}$$

$$\leq \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\sigma(\alpha)}^{\beta} c_+(t)x^2(\sigma(t))\Delta t\right\}^{1/2}.$$

Therefore, we get from (9)

$$2|x(\tau)| \leq \int_{\sigma(\alpha)}^{\beta} |a(t)|\left|x(\sigma(t))\right|\Delta t + \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\sigma(\alpha)}^{\beta} c_+(t)x^2(\sigma(t))\Delta t\right\}^{1/2}$$

$$\leq |x(\tau)| \cdot \left[\int_{\sigma(\alpha)}^{\beta} |a(t)|\Delta t + \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t \cdot \int_{\sigma(\alpha)}^{\beta} c_+(t)\Delta t\right\}^{1/2}\right].$$

Dividing the latter estimate by $|x(\tau)|$, we get the desired inequality (6).

Theorem (4.1.7)[73]: *Suppose*

$$1 - \mu(t)a(t) > 0, \qquad b(t) > 0 \qquad for\ all\ t \in \mathbb{T} \qquad (10)$$

and let $\alpha, \beta \in \mathbb{T}^{\kappa}$ *with* $\sigma(\alpha) < \beta$. *Assume that* (1) *has a real solution* (x, u) *such that* $x(\sigma(\alpha)) = 0, x(\beta)x(\sigma(\beta)) < 0$. *Then the inequality*

$$\int_{\sigma(\alpha)}^{\beta} |a(t)|\Delta t + \left\{ \int_{\sigma(\alpha)}^{\beta} b(t)\Delta t \cdot \int_{\sigma(\alpha)}^{\beta} c_+(t)\Delta t \right\}^{1/2} > 1 \tag{11}$$

holds.

Proof. Integrating (7) from $\sigma(\alpha)$ to β and observing that $x\big(\sigma(\alpha)\big) = 0$ we obtain

$$u(\beta)x(\beta) = \int_{\sigma(\alpha)}^{\beta} b(t)u^2(t)\Delta t - \int_{\sigma(\alpha)}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t. \tag{12}$$

Further, by using Lemma (4.1.3)(iv), we rewrite the first equation of (1) and get

$$[1 - \mu(t)a(t)]x\big(\sigma(t)\big) = x(t) + \mu(t)b(t)u(t). \tag{13}$$

Let $t = \beta$, we have

$$[1 - \mu(\beta)a(\beta)]x\big(\sigma(\beta)\big) = x(\beta) + \mu(\beta)b(\beta)u(\beta).$$

Multiplying this by $x(\beta)$ yields

$$[1 - \mu(\beta)a(\beta)]x(\beta)x\big(\sigma(\beta)\big) = x^2(\beta) + \mu(\beta)b(\beta)x(\beta)u(\beta).$$

Since $x(\beta)x\big(\sigma(\beta)\big) < 0$, it is easy to see that $\mu(t) > 0$. In view of (10), the above latter equality gives rise to $u(\beta)x(\beta) < 0$. Therefore, from (12) the inequality

$$\int_{\sigma(\alpha)}^{\beta} b(t)u^2(t)\Delta t < \int_{\sigma(\alpha)}^{\beta} c(t)x^2\sigma(t)\Delta t \leq \int_{\sigma(\alpha)}^{\beta} c_+(t)x^2\big(\sigma(t)\big)\Delta t \tag{14}$$

follows. Choose $\tau \in [\sigma(\alpha), \beta]$such that

$$|x(\tau)| = \max_{\sigma(\alpha)\leq t\leq\beta} \{|x(t)|\}.$$

Then $|x(\tau)| > 0$. Integrating the first equation of (1) from $\sigma(\alpha)$ to τ and noticing that $x\big(\sigma(\alpha)\big) = 0$, we obtain

$$x(\tau) = \int_{\sigma(\alpha)}^{\tau} a(t)x\big(\sigma(t)\big)\Delta t + \int_{\sigma(\alpha)}^{\tau} b(t)u(t)\Delta t.$$

Hence, applying the Cauchy–Schwarz inequality and (14), it follows that

$$|x(\tau)| = \int_{\sigma(\alpha)}^{\tau} |a(t)|\big|x\big(\sigma(t)\big)\big|\Delta t + \int_{\sigma(\alpha)}^{\tau} b(t)|u(t)|\Delta t.$$

$$\leq \int_{\sigma(\alpha)}^{\beta} |a(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \int_{\sigma(\alpha)}^{\beta} b(t)|u(t)|\Delta t$$

$$\leq \int_{\sigma(\alpha)}^{\beta} |a(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \left\{\int_{\sigma(\alpha)}^{\beta} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\sigma(\alpha)}^{\beta} b(t)u^2(t)\Delta t\right\}^{1/2}$$

$$\leq \int_{\sigma(\alpha)}^{\beta} |a(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \left\{\int_{\sigma(\alpha)}^{\beta} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\sigma(\alpha)}^{\beta} c_+(t)x^2\big(\sigma(t)\big)\Delta t\right\}^{1/2}$$

$$\leq |x(\tau)| \cdot \left[\int_{\sigma(\alpha)}^{\beta} |a(t)|\Delta t + \left\{\int_{\sigma(\alpha)}^{\beta} b(t)\Delta t \cdot \int_{\sigma(\alpha)}^{\beta} c_+(t)\Delta t\right\}^{1/2}\right].$$

Therefore, dividing the latest estimate by $|x(\tau)|$we obtain inequality (11).

Theorem (4.1.8)[73]:*Suppose* (10)*holds and let* $\alpha, \beta \in \mathbb{T}^\kappa$*with* $\alpha < \beta$*. Assume that* (1) *has a real solution* (x, u)*such that* $x(\alpha)x\big(\sigma(\alpha)\big) < 0, x\big(\sigma(\beta)\big) = 0$*. Then the inequality*

$$\int_{\sigma(\alpha)}^{\beta} |a(t)|\Delta t + \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t \cdot \int_{\sigma(\alpha)}^{\beta} c_+(t)\Delta t\right\}^{1/2} > 1 \qquad (15)$$

holds.

Proof. Choose $\tau \in [\sigma(\alpha), \sigma(\beta)]$such that

$$|x(\tau)| = \max_{\sigma(\alpha)\leq t\leq\sigma(\beta)} \{|x(t)|\}.$$

Then $|x(\tau)| > 0$. Integrating the first equation of (1) from τ to $\sigma(\beta)$and taking into account$x\big(\sigma(\beta)\big) = 0$, we get

$$x(\tau) = -\int_{\tau}^{\sigma(\beta)} a(t)x\big(\sigma(t)\big)\Delta t - \int_{\tau}^{\sigma(\beta)} b(t)|u(t)|\Delta t$$

$$= -\int_{\tau}^{\beta} a(t)x\big(\sigma(t)\big)\Delta t - \int_{\beta}^{\sigma(\beta)} a(t)x\big(\sigma(t)\big)\Delta t - \int_{\tau}^{\sigma(\beta)} b(t)|u(t)|\Delta t$$

$$= -\int_{\tau}^{\beta} a(t)x\big(\sigma(t)\big)\Delta t - \mu(\beta)a(\beta)x\big(\sigma(\beta)\big) - \int_{\beta}^{\sigma(\beta)} b(t)|u(t)|\Delta t$$

$$= -\int_{\tau}^{\beta} a(t)x\big(\sigma(t)\big)\Delta t - \int_{\beta}^{\sigma(\beta)} b(t)|u(t)|\Delta t.$$

Therefore,

$$|x(\tau)| \le \int_{\tau}^{\beta} |a(t)|\big|x\big(\sigma(t)\big)\big|\Delta t + \int_{\tau}^{\sigma(\beta)} b(t)|u(t)|\Delta t$$

$$\le \int_{\sigma(\alpha)}^{\beta} |a(t)|\big|x\big(\sigma(t)\big)\big|\Delta t + \int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)|u(t)|\Delta t$$

$$\le \int_{\sigma(\alpha)}^{\beta} |a(t)|\big|x\big(\sigma(t)\big)\big|\Delta t$$

$$+\left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)u^2(t)\Delta t\right\}^{1/2}. \qquad (16)$$

Now integrating equation (7) from $\alpha$to $\sigma(\beta)$and taking into account that $x\big(\sigma(\beta)\big) = 0$, we get

$$-x(\alpha)u(\alpha) = \int_{\alpha}^{\sigma(\beta)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\sigma(\beta)} c(t)x^2\big(\sigma(t)\big)\Delta t$$

$$= \int_{\alpha}^{\sigma(\beta)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t - \int_{\beta}^{\sigma(\beta)} c(t)x^2\big(\sigma(t)\big)\Delta t$$

$$= \int_{\alpha}^{\sigma(\beta)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t.$$

Applying Lemma (4.1.4)(iii), we rewrite the above last equality as follows:

$$-x(\alpha)u(\alpha) - \int_{\alpha}^{\sigma(\alpha)} b(t)u^2(t)\Delta t = \int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t.$$

By Lemma (4.1.4)(ii), it follows that

$$-u(\alpha)[x(\alpha)+\mu(\alpha)b(\alpha)u(\alpha)] = \int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t. \quad (17)$$

Further, from (13), we have, for $t=\alpha$,

$$[1-\mu(\alpha)a(\alpha)]x\big(\sigma(\alpha)\big) = x(\alpha)+\mu(\alpha)b(\alpha)u(\alpha). \quad (18)$$

Multiplying this by $x(\alpha)$gives that

$$[1-\mu(\alpha)a(\alpha)]x(\alpha)x\big(\sigma(\alpha)\big) = x^2(\alpha)+\mu(\alpha)b(\alpha)x(\alpha)u(\alpha).$$

Since $x(\alpha)x\big(\sigma(\alpha)\big)<0$, it is easy to see that $\mu(\alpha)>0$ holds. By (10) and the above latter equality, we have

$$x(\alpha)u(\alpha)<0. \quad (19)$$

Now, we claim that

$$u(\alpha)[x(\alpha)+\mu(\alpha)b(\alpha)u(\alpha)]>0 \quad (20)$$

holds. Indeed, multiplying (18) by $u(\alpha)$gives

$$[1-\mu(\alpha)a(\alpha)]x\big(\sigma(\alpha)\big)u(\alpha) = u(\alpha)[x(\alpha)+\mu(\alpha)b(\alpha)u(\alpha)]. \quad (21)$$

On the other hand, it follows from $x(\alpha)x\big(\sigma(\alpha)\big)<0$ and (19) that $x\big(\sigma(\alpha)\big)u(\alpha)>0$.Therefore the left-hand side of (21) is positive, and hence, (20) is true.

By virtue of (20), the string of inequalities

$$\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)u^2(t)\Delta t < \int_{\alpha}^{\beta} c(t)x^2\big(\sigma(t)\big)\Delta t \le \int_{\alpha}^{\beta} c_+(t)x^2\big(\sigma(t)\big)\Delta t$$

follows from (17). As a result of these last relations, from (16), we obtain

$$\begin{aligned}|x(\tau)| &< \int_{\sigma(\alpha)}^{\beta} |a(t)|\big|x\big(\sigma(t)\big)\big|\Delta t \\ &\quad + \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\alpha}^{\beta} c_+(t)x^2\big(\sigma(t)\big)\Delta t\right\}^{1/2} \\ &\le |x(\tau)|\cdot\left[\int_{\sigma(\alpha)}^{\beta} |a(t)|\Delta t + \left\{\int_{\sigma(\alpha)}^{\sigma(\beta)} b(t)\Delta t \cdot \int_{\alpha}^{\beta} c_+(t)\Delta t\right\}^{1/2}\right].\end{aligned}$$

Hence, dividing the last estimate by $|x(\tau)|$, we get inequality (15).

Theorem (4.1.9)[73]: *Suppose*

$$1-\mu(t)a(t)>0, \quad b(t)>0, \quad c(t)>0 \quad for\ all\ t\in\mathbb{T}, \tag{22}$$

and let $\alpha,\beta\in\mathbb{T}^\kappa$ *with* $\alpha<\beta$. *Assume that* (1) *has a real solution* (x,u) *such that* $x(\alpha)x(\sigma(\alpha))<0, x(\beta)x(\sigma(\beta))<0$. *Then the inequality*

$$\int_\alpha^\beta |a(t)|\Delta t+\left\{\int_\alpha^{\sigma(\beta)} b(t)\Delta t\cdot\int_\alpha^\beta c(t)\Delta t\right\}^{1/2}>1 \tag{23}$$

holds.

Combining Theorems (4.1.6)–(4.1.9), is yield Corollary (4.1.6).

Proof.(I) First, we assume that $x(t)\neq 0$ for all $t\in[\alpha,\beta]$. Denote by β_0 the smallest number in $(\alpha,\beta]$ such that

$$x(\beta_0)x(\sigma(\beta_0))<0. \tag{24}$$

Then x does not have any generalized zero in $[\sigma(\alpha),\beta_0]$. And without loss of generality we may assume that

$$x(t)>0 \quad \text{for all} t\in[\sigma(\alpha),\beta_0]. \tag{25}$$

Then we will have

$$x(\alpha)<0, \qquad x(\sigma(\beta_0))<0. \tag{26}$$

Let $s\in[\alpha,\sigma(\beta_0)]$. Integrating the second equality of (1) from α to s and then from s to β_0, we get

$$u(s)-u(\alpha)=-\int_\alpha^s c(t)x(\sigma(t))\Delta t-\int_\alpha^s a(t)u(t)\Delta t, \tag{27}$$

$$u(\beta_0)-u(s)=-\int_s^{\beta_0} c(t)x(\sigma(t))\Delta t-\int_s^{\beta_0} a(t)u(t)\Delta t, \tag{28}$$

respectively. Noticing that for $s=\alpha$ we write solely (28), and for $s=\beta_0$ only (27) is written.

Now, we aim to show that

$$u(\alpha)>0, \qquad u(\beta_0)<0. \tag{29}$$

Indeed, multiplying (13) by $x(t)$gives

$$[1-\mu(t)a(t)]x(t)x(\sigma(t))=x^2(t)+\mu(t)b(t)x(t)u(t),$$

where setting $t = \alpha$ and $t = \beta$, yields

$$[1 - \mu(\alpha)a(\alpha)]x(\alpha)x(\sigma(\alpha)) = x^2(\alpha) + \mu(\alpha)b(\alpha)x(\alpha)u(\alpha),$$

$$[1 - \mu(\beta_0)a(\beta_0)]x(\beta_0)x(\sigma(\beta_0)) = x^2(\beta_0) + \mu(\beta_0)b(\beta_0)x(\beta_0)u(\beta_0),$$

respectively. Using the inequalities $x(\alpha)x(\sigma(\alpha)) < 0$and $x(\beta_0)x(\sigma(\beta_0)) < 0$, $\mu(\alpha) > 0$and $\mu(\beta_0) > 0$ can be obtained easily. Combining (22) with the above equalities, we get the estimates

$$x(\alpha)u(\alpha) < 0, \qquad x(\beta_0)u(\beta_0) < 0. \tag{30}$$

Observing that $x(\alpha) < 0$ and $x(\beta_0) > 0$, we obtain (29).

Employing (27) if $u(s) < 0$ and using (28) whenever $u(s) > 0$, and also taking into account (29), we get

$$|u(s)| \leq \int_{\alpha}^{\beta_0} c(t)|x(\sigma(t))|\Delta t + \int_{\alpha}^{\beta_0} a(t)u(t)\Delta t$$

$$\leq \left\{\int_{\alpha}^{\beta_0} c(t)\Delta t\right\}^{1/2} \cdot \left\{\int_{\alpha}^{\beta_0} c(t)x^2(\sigma(t))\Delta t\right\}^{1/2} + \int_{\alpha}^{\beta_0} |a(t)||u(t)|\Delta t. \tag{31}$$

Next, integrating Eq. (7) from α to $\sigma(\beta_0)$gives

$$x(\sigma(\beta_0))u(\sigma(\beta_0)) - x(\alpha)u(\alpha) = \int_{\alpha}^{\sigma(\beta_0)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\sigma(\beta_0)} c(t)x^2(\sigma(t))\Delta t.$$

It follows from Lemma (4.1.4)(iii) that

$$x(\sigma(\beta_0))u(\sigma(\beta_0)) + \int_{\beta_0}^{\sigma(\beta_0)} c(t)x^2(\sigma(t))\Delta t - x(\alpha)u(\alpha)$$

$$= \int_{\alpha}^{\sigma(\beta_0)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\beta_0} c(t)x^2(\sigma(t))\Delta t.$$

Using Lemma (4.1.4)(ii), we have

$$x(\sigma(\beta_0))[u(\sigma(\beta_0)) + \mu(\beta_0)c(\beta_0)x(\sigma(\beta_0))] - x(\alpha)u(\alpha)$$

$$= \int_{\alpha}^{\sigma(\beta_0)} b(t)u^2(t)\Delta t - \int_{\alpha}^{\beta_0} c(t)x^2(\sigma(t))\Delta t. \tag{32}$$

We claim

$$x(\sigma(\beta_0))[\mu(\sigma(\beta_0)) + \mu(\beta_0)c(\beta_0)x(\sigma(\beta_0))] > 0 \qquad (33)$$

holds. Indeed, from the second equation of (1), we have, for $t = \beta_0$,

$$[1 - \mu(\beta_0)a(\beta_0)]u(\beta_0) = u(\sigma(\beta_0)) + \mu(\beta_0)c(\beta_0)x(\sigma(\beta_0)).$$

Multiplying the above equality by $x(\sigma(\beta_0))$yields

$$\begin{aligned}&[1 - \mu(\beta_0)a(\beta_0)]u(\beta_0)x(\sigma(\beta_0))\\ &= x(\sigma(\beta_0))[u(\sigma(\beta_0)) + \mu(\beta_0)c(\beta_0)x(\sigma(\beta_0))].\end{aligned} \qquad (34)$$

On the other hand, from (26) and (29), it follows that $u(\beta_0)x(\sigma(\beta_0)) > 0$. Therefore from $1 - \mu(\beta_0)a(\beta_0) > 0$ and (34), (33) follows.

In view of (30) and (33), the inequality, from (32),

$$\int_\alpha^{\beta_0} c(t)x^2(\sigma(t))\Delta t < \int_\alpha^{\sigma(\beta_0)} b(t)u^2(t)\Delta t$$

follows. By virtue of (31), the above estimate yields

$$|u(s)| < \left\{\int_\alpha^{\beta_0} c(t)\Delta t\right\}^{1/2} \cdot \left\{\int_\alpha^{\sigma(\beta_0)} b(t)u^2(t)\Delta t\right\}^{1/2} + \int_\alpha^{\beta_0} |a(t)||u(t)|\Delta t \qquad (35)$$

for all $s \in [\alpha, \sigma(\beta_0)]$. Choose $\tau \in [\alpha, \sigma(\beta_0)]$such that

$$|u(\tau)| = \max_{\alpha \le s \le \sigma(\beta_0)} \{|u(s)|\}.$$

Clearly, $|u(\tau)| > 0$. Then from (35), we have

$$|u(\tau)| < |u(\tau)| \cdot \left[\int_\alpha^{\beta_0} |a(t)|\Delta t + \left\{\int_\alpha^{\sigma(\beta_0)} b(t)\Delta t \cdot \int_\alpha^{\beta_0} c(t)\Delta t\right\}^{1/2}\right].$$

Hence, dividing this inequality by $|u(\tau)|$we get

$$\int_\alpha^{\beta_0} |a(t)|\Delta t + \left\{\int_\alpha^{\sigma(\beta_0)} b(t)\Delta t \cdot \int_\alpha^{\beta_0} c(t)\Delta t\right\}^{1/2} > 1.$$

Since $\beta_0 \le \beta$, (23) follows.

(II) Second, we consider the case when $x(t_0) = 0$ for some $t_0 \in (\sigma(\alpha), \beta)$. In this case, applying Theorem (4.1.7) to the points t_0 and β, we get the inequality

$$\int_{t_0}^{\beta}|a(t)|\Delta t+\left\{\int_{t_0}^{\beta}b(t)\Delta t\cdot\int_{t_0}^{\beta}c(t)\Delta t\right\}^{1/2}>1.$$

Therefore, inequality (23) holds in this case as well.

Let $\alpha,\beta\in\mathbb{T}^{\kappa}$ with $\sigma(\alpha)<\beta$. Consider the linear Hamiltonian dynamic system

$$x^{\Delta}(t)=a(t)x(\sigma(t))+b(t)u(t),$$
$$u^{\Delta}(t)=-c(t)x(\sigma(t))-a(t)u(t),\qquad t\in[\alpha,\beta]^{\kappa},\qquad(36)$$

where the coefficients $a(t), b(t)$and $c(t)$are real rd-continuous functions defined on $[\alpha,\beta]$ satisfying

$$1-\mu(t)a(t)>0,\qquad b(t)>0,\qquad c(t)>0\quad\text{for all} t\in[\alpha,\beta].\qquad(37)$$

Note that each solution (x,u) of system (36) will be a vector-valued function defined on$[\alpha,\sigma(\beta)]$.

Now we give the concept of a relatively generalized zero for the component x of areal solution (x,u) of system (36) and also the concept of disconjugacy of this system on$[\alpha,\sigma(\beta)]$. The definition is relative to the interval $[\alpha,\sigma(\beta)]$ and the left end-point α is treated separately.

Definition (4.1.10)[73]: The component x of a solution $(x,u)$of (36) has *a relatively generalized zero* at α if and only if $x(\alpha)=0$, while x has *a relatively generalized zero* at $\sigma(t_0)>\alpha$ provided (x,u) has a generalized zero at $\sigma(t_0)$. System (36) is called *disconjugacy* on$[\alpha,\sigma(\beta)]$if there is no real solution (x,u) of this system with x nontrivial and having two (or more) relatively generalized zeros in $[\alpha,\sigma(\beta)]$

Noting that when $\mathbb{T}=\mathbb{Z}$, the definitions of *relatively generalized zero* and that of *disconjugacy* are equivalent to those given in [84,82].

Theorem (4.1.11)[73]:*Assume* (37) *holds. If*

$$\int_{\alpha}^{\sigma(\beta)}|a(t)|\Delta t+\left\{\int_{\alpha}^{\sigma(\beta)}b(t)\Delta t\cdot\int_{\alpha}^{\sigma(\beta)}c(t)\Delta t\right\}^{1/2}\leq 1.\qquad(38)$$

Then (36) *is disconjugate on* $[\alpha,\sigma(\beta)]$.

Proof. Suppose, on the contrary, that system (36) is not disconjugate on $[\alpha, \sigma(\beta)]$. By Definition (4.1.10), there exists a real solution $(x, u)$of (36) with x nontrivial and such that x has at least two relatively generalized zeros in $[\alpha, \sigma(\beta)]$. Now, we have the following two cases to consider.

(I) One of the two relatively generalized zeros is at the end-point α, i.e., $x(\alpha) = 0$, the other is at $\sigma(\beta_0) \in (\alpha, \sigma(\beta)]$. Therefore, applying Theorem (4.1.6) or Theorem (4.1.7), we get

$$\int_{\alpha}^{\sigma(\beta_0)} |a(t)|\Delta t + \left\{ \int_{\alpha}^{\sigma(\beta_0)} b(t)\Delta t \cdot \int_{\alpha}^{\sigma(\beta_0)} c(t)\Delta t \right\}^{1/2} > 1.$$

This contradicts with condition (38) of the theorem.

(II) None of the two relatively generalized zeros is at α, that is, x have two generalized zeros $\sigma(\alpha_0), \sigma(\beta_0) \in (\alpha, \sigma(\beta)]\big(\sigma(\alpha_0) < \sigma(\beta_0)\big)$. Therefore, applying Corollary (4.1.2), we have

$$\int_{\alpha_0}^{\sigma(\beta_0)} |a(t)|\Delta t + \left\{ \int_{\alpha_0}^{\sigma(\beta_0)} b(t)\Delta t \cdot \int_{\alpha_0}^{\sigma(\beta_0)} c(t)\Delta t \right\}^{1/2} > 1,$$

which is contrary to condition (38) of the theorem.

The proof of Theorem (4.1.11) is now completed by combining cases (I) and (II).

Section (4.2): Inequalities of Lyapunov for Linear Hamiltonian Systems:

In recent years, the theory of time scales (or measure chains) has been developed by several authors with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case).A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Throughout this section, we assume that $\mathbb{T}$ is a time scale and $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{T}$. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. In the next section, we'll briefly introduce the time scale calculus and some relatedbasic concepts of Hilger [88–89] and refer to the books of Kaymakcalan et al. [90] and Bohner and Peterson [91] for further details.

Consider a linear Hamiltonian system

$$x^{\Delta}(t) = \alpha(t)x\big(\sigma(t)\big) + \beta(t)y(t), \quad y^{\Delta}(t) = -\gamma(t)x\big(\sigma(t)\big) - \alpha(t)y(t), \quad (39)$$

on an arbitrary time scale $\mathbb{T}$, where $\alpha(t), \beta(t)$and $\gamma(t)$are real-valued rd-continuous functions defined on $\mathbb{T}$.We always assume that

$$\beta(t) \geq 0, \qquad \forall t \in \mathbb{T}. \qquad (40)$$

For the second-order linear dynamic equation

$$[p(t)x^{\Delta}(t)]^{\Delta} + q(t)x\big(\sigma(t)\big) = 0, \qquad t \in \mathbb{T}, \qquad (41)$$

where $p(t) > 0$, and $p(t), q(t)$are real-valued rd-continuous functions defined on $\mathbb{T}$. If we let $y(t) = p(t)x^{\Delta}(t)$, then (41) can be written as an equivalent Hamiltonian system of type (48):

$$x^{\Delta}(t) = \frac{1}{p(t)}y(t), \quad y^{\Delta}(t) = -q(t)x\big(\sigma(t)\big), \qquad (42)$$

where

$$\alpha(t) = 0, \qquad \beta(t) = \frac{1}{p(t)}, \qquad \gamma(t) = q(t).$$

It is obvious that system (39) covers the continuous Hamiltonian system and discrete Hamiltonian system respectively when $\mathbb{T} = \mathbb{R}$and $\mathbb{T} = \mathbb{Z}$, i.e.,

$$x'(t) = \alpha(t)x(t) + \beta(t)y(t), \qquad y'(t) = -\gamma(t)x(t) - \alpha(t)y(t), \ \ t \in \mathbb{R},$$

$$\Delta x(n) = \alpha(n)x(n+1) + \beta(n)y(n), \ \ \Delta y(n) = -\gamma(n)x(n+1) - \alpha(n)y(n), \ \ n \in \mathbb{Z}.$$

Furthermore, system (39) extends the above classical cases to some cases in between as well, such as the so-called q difference equations, where

$$\mathbb{T} = q^{\mathbb{Z}} \coloneqq \{q^k | k \in \mathbb{Z}\} \cup \{0\}$$

for some $q > 1$, and difference equations with constant step size, where

$$\mathbb{T} = h\mathbb{Z} \coloneqq \{hk | k \in \mathbb{Z}\}$$

for some $h > 0$. Particularly useful for the discretization aspect are time scales of the form

$$\mathbb{T} = \{t_k \in \mathbb{R} | t_k < t_{k+1}, k \in \mathbb{Z}\}.$$

It is a classical topic for us to study Lyapunov type inequalities which have proved to be very useful in oscillation theory, disconjugacy, eigenvalue problems and numerous other applications in the theory of differential and difference equations.There are many literatures which improved and extended the classical Lyapunov inequality for the Hamiltonian systems including continuous and discrete cases. We refer to [92,93–94,95]. Recently, there has been much attention paid to Lyapunov type inequality for linear Hamiltonian systems on time scales and some authors including Agarwal [96], He [97], Jiang [98] and Saker [99] have contributed the above results. Our motivation comes from the recent sections by Guseinov and Kaymakcalan[72] and Jiang and Zhou [98]. In [98], Jiang has obtained some interesting Lyapunov-type inequalities and these results have almost covered the corresponding continuous and discrete versions that may be found in [94].

Theorem (4.2.1)[87]:*(See [98].) Suppose*

$$1-\mu(t)\alpha(t)>0, \quad \beta(t)>0, \quad \gamma(t)>0, \qquad \forall t\in\mathbb{T}, \tag{43}$$

and let $a,b\in\mathbb{T}^k$ *with* $\sigma(a)<b$. *Assume that* (39) *has a real solution* $\big(x(t),y(t)\big)$*such that* $x(a)x\big(\sigma(a)\big)<0, x(b)x\big(\sigma(b)\big)<0$. *Then the inequality*

$$\int_a^b |\alpha(t)|\Delta(t)+\left[\int_a^{\sigma(b)}\beta(t)\Delta(t)\int_a^b\gamma(t)\Delta(t)\right]^{1/2}>1 \tag{44}$$

holds, where and in the sequel

$$\gamma^+(t)=\max\{\gamma(t),0\}. \tag{45}$$

Theorem (4.2.2)[87]:*(See [98].) Suppose*

$$1-\mu(t)\alpha(t)>0, \quad \beta(t)>0, \quad for\ all\ t\in\mathbb{T}, \tag{46}$$

and let $a,b\in\mathbb{T}^k$ *with* $\sigma(a)<b$. *Assume that* (39) *has a real solution* $\big(x(t),y(t)\big)$*such that* $x(a)x\big(\sigma(a)\big)<0, x\big(\sigma(b)\big)=0$. *Then the inequality*

$$\int_{\sigma(a)}^b |\alpha(t)|\Delta(t)+\left[\int_{\sigma(a)}^{\sigma(b)}\beta(t)\Delta(t)\int_a^b\gamma^+(t)\Delta(t)\right]^{1/2}>1 \tag{47}$$

holds.

In this section, by using some simpler methods different from [98], we obtain several better Lyapunov-type inequalities than (44) and (47)

$$\int_a^b |\alpha(t)|\Delta(t) + \left[\int_a^{\sigma(b)} \beta(t)\Delta(t) \int_a^b \gamma^+(t)\Delta(t)\right]^{1/2} \geq 2, \qquad (48)$$

and

$$\int_a^{\rho(b)} |\alpha(t)|\Delta(t) + \left[\int_a^b \beta(t)\Delta(t) \int_a^b \gamma^+(t)\Delta(t)\right]^{1/2} \geq 2, \qquad (49)$$

only under the assumption

$$1 - \mu(t)\alpha(t) > 0, \qquad \forall t \in \mathbb{T}. \qquad (50)$$

Our results not only cover the corresponding continuous versions, but also improve greatly discrete versions that may be found in [94]. In addition, when the end-point b satisfies some general conditions (see Theorem (4.2.16) below), it is not necessarily a generalized zero, we also establish a better Lyapunov-type inequality than (48)

$$\int_a^b |\alpha(t)|\Delta(t) + \left[\int_a^b \beta(t)\Delta(t) \int_a^b \gamma^+(t)\Delta(t)\right]^{1/2} \geq 2. \qquad (51)$$

Instead of the usual zero, we adopt the following concept of generalized zero on time scales.

Definition (4.2.3)[87]:A function $f : \mathbb{T} \to \mathbb{R}$ is said to have a generalized zero at $t_0 \in \mathbb{T}$ provided either $f(t_0) = 0$ or$f(t_0)f\big(\sigma(t_0)\big) < 0$.

Now, we introduce the basic notions connected to time scales. We start by defining the forward and backward jump operators.

Definition (4.2.4)[87]:(See [91].) Let $t \in \mathbb{T}$. We define *the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T}: s > t\} \text{ for all} t \in \mathbb{T},$$

While *the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$by

$$\rho(t) := \sup\{s \in \mathbb{T}: s < t\} \text{ for all} t \in \mathbb{T}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$(i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$(i.e., $\rho(m) = m$if $\mathbb{T}$ has a minimum m), where $\emptyset$ denotes the empty set. If

$\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$and $\rho(t) = t$, then t is called *left-dense*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Points that are right-dense and left-dense at the same time are called *dense*. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$,otherwise $\mathbb{T}^k = \mathbb{T}$. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) \coloneqq \sigma(t) - t, \qquad \forall t \in \mathbb{T}.$$

We consider a function $f : \mathbb{T} \to \mathbb{R}$ and define so-called delta (or Hilger) derivative of $f$at a point $t \in \mathbb{T}^k$.

Definition (4.2.5)[87]:(See [91].) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$to be the number (providedit exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$of t(i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$for some $\delta > 0$)such that

$$\left|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \le \varepsilon|\sigma(t) - s|, \qquad \forall s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t.

Lemma (4.2.6)[87]:*(See [91].) Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differential at $t \in \mathbb{T}^k$. Then*

(i) *For any constant a and b, the sum $af + bg : \mathbb{T} \to \mathbb{R}$ is differential at t with*

$$(af + bg)^\Delta(t) = af^\Delta(t) + bg^\Delta(t).$$

(ii) *If $f^\Delta(t)$exists, then f is continuous at t.*

(iii) *If $f^\Delta(t)$exists, then $f\big(\sigma(t)\big) = f(t) + \mu(t)f^\Delta(t)$.*

(iv) *The product $fg : \mathbb{T} \to \mathbb{R}$ is differential at t with*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f\big(\sigma(t)\big)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g\big(\sigma(t)\big).$$

(v) *If $g(t)g\big(\sigma(t)\big) \neq 0$, then f/g is differential at t and*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g\big(\sigma(t)\big)}.$$

Definition (4.2.7)[87]:(See [91].) A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$and left-sided limits exist (finite) at left-dense points in $\mathbb{T}$and denotes by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}C(\mathbb{T}, \mathbb{R})$.

Definition (4.2.8)[87]:(See [91].) A function $F:\mathbb{T}\to\mathbb{R}$ is called an antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$holds for all $t\in\mathbb{T}^{k}$. We define the Cauchy integral by

$$\int_{\tau}^{s} f(t)\Delta t = F(s)-F(\tau), \qquad \forall s,\tau\in\mathbb{T}.$$

The following lemma gives several elementary properties of the delta integral.

Lemma (4.2.9)[87]:*(See [91].) If $a,b,c\in\mathbb{T}, k\in\mathbb{R}$ and f , $g\in C_{rd}$, then*

(i) $\int_a^b [f(t)+g(t)]\Delta(t)=\int_a^b f(t)\Delta(t)+\int_a^b g(t)\Delta(t)$;

(ii) $\int_a^b (kf)(t)\Delta(t)=k\int_a^b f(t)\Delta(t)$;

(iii) $\int_a^b f(t)\Delta(t)=\int_a^c f(t)\Delta(t)+\int_c^b f(t)\Delta(t)$;

(iv) $\int_a^b f\big(\sigma(t)\big)g^{\Delta}(t)\Delta(t)=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g(t)\Delta(t)$;

(v) $\int_t^{\sigma(t)} f(s\Delta)(s)=\mu(t)f(t)$*for* $t\in\mathbb{T}^{k}$;

(vi) *if* $|f(t)|\le g(t)$*on* $[a,b)$, *then*

$$\int_a^b f(t)\Delta(t)\le\int_a^b g(t)\Delta(t).$$

The notation $[a,b],[a,b)$and $[a,+\infty)$will denote time scales intervals. For example, $[a,b)=\{t\in\mathbb{T}|a\le t<b\}$. To proveour results, we present the following lemma.

Lemma (4.2.10)[87]:*(Cauchy–Schwarz inequality). (See [91].) Let $a,b\in\mathbb{T}$. For $f,g\in C_{rd}$ we have*

$$\int_a^b f(t)g(t)\Delta(t)\le\left\{\int_a^b f^2(t)\Delta(t)\cdot\int_a^b g^2(t)\Delta(t)\right\}^{\frac{1}{2}}.$$

Lemma (4.2.11)[87]:*(See [91].) Let*

$$A=\{t\in\mathbb{T}|t \text{ is left-dense and right-scattered}\},$$
$$B=\{t\in\mathbb{T}|t \text{ is right-dense and left-scattered}\}.$$

Then

$$\sigma\rho(t)=t,\quad\forall t\in\mathbb{T}\backslash A;\qquad \rho\sigma(t)=t,\quad\forall t\in\mathbb{T}\backslash B.$$

In this section, we establish some new Lyapunov type inequalities on time scales $\mathbb{T}$.

Theorem (4.2.12)[87]: *Suppose that* (50) *holds and let $a, b \in \mathbb{T}^k$ with $\sigma(a) \leq b$. Assume* (39) *has a real solution $(x(t), y(t))$ such that $x(t)$ has generalized zeroes at end-points a and b and $x(t)$ is not identically zero on $[a, b]$, i.e.,*

$$x(a) = 0 \ \text{ or } \ x(a)x(\sigma(a)) < 0; \ x(b) = 0 \ \text{ or } \ x(b)x(\sigma(b)) < 0;$$
$$\max_{a \le t \le b} |x(t)| > 0. \tag{52}$$

Then one has the following inequality

$$\int_a^b |\alpha(t)|\Delta(t) + \left[\int_a^{\sigma(b)} \beta(t)\Delta(t) \int_a^b \gamma^+(t)\Delta(t)\right]^{1/2} \geq 2. \tag{53}$$

Proof. It follows from (52) that there exist $\xi, \eta \in [0, 1)$ such that

$$(1-\xi)x(a) + \xi x(\sigma(a)) = 0, \tag{54}$$

and

$$(1-\eta)x(b) + \eta x(\sigma(b)) = 0. \tag{55}$$

Multiplying the first equation of (39) by $y(t)$ and the second one by $x(\sigma(t))$, and then adding, we get

$$[x(t)y(t)]^\Delta = \beta(t)y^2(t) - \gamma(t)x^2(\sigma(t)). \tag{56}$$

Integrating equation (56) from a to b, we can obtain

$$x(b)y(b) - x(a)y(a) = \int_a^b \beta(t)y^2(t)\Delta t - \int_a^b \gamma(t)x^2(\sigma(t))\Delta t. \tag{57}$$

From the first equation of (39) and using Lemma (4.2.6)(iii), we have

$$[1 - \mu(t)\alpha(t)]x(\sigma(t)) = x(t) + \mu(t)\beta(t)y(t). \tag{58}$$

Combining (58) with (54), we have

$$x(a) = -\frac{\xi\mu(a)\beta(a)}{1-(1-\xi)\mu(a)\alpha(a)}y(a). \tag{59}$$

Similarly, it follows from (58) and (55) that

$$x(b) = -\frac{\eta\mu(b)\beta(b)}{1-(1-\eta)\mu(b)\alpha(b)}y(b). \tag{60}$$

Substituting (59) and (60) into (57), we have

$$\int_a^b \beta(t)y^2(t)\Delta t - \int_a^b \gamma(t)x^2(\sigma(t))\Delta t =$$

$$-\frac{\eta\mu(b)\beta(b)}{1-(1-\eta)\mu(b)\alpha(b)}y^2(b) + \frac{\xi\mu(a)\beta(a)}{1-(1-\xi)\mu(a)\alpha(a)}y^2(a),$$

by using Lemma (4.2.9)(v), we get

$$\frac{(1-\xi)[1-\mu(a)\alpha(a)]}{1-(1-\xi)\alpha(a)\mu(a)}\mu(a)\beta(a)y^2(a) + \int_{\sigma(a)}^b \beta(t)y^2(t)\Delta t + \frac{\eta\mu(b)\beta(b)}{1-(1-\eta)\mu(b)\alpha(b)}y^2(b)$$
$$= \int_a^b \gamma(t)x^2(\sigma(t))\Delta t. \tag{61}$$

Denote that

$$\tilde{\beta}(a) = \frac{(1-\xi)[1-\mu(a)\alpha(a)]}{1-(1-\xi)\mu(a)\alpha(a)}\beta(a), \tag{62}$$

$$\tilde{\beta}(b) = \frac{\eta}{1-(1-\eta)\mu(b)\alpha(b)}\beta(b), \tag{63}$$

and

$$\tilde{\beta}(t) = \beta(t), \qquad \sigma(a) \le t \le \rho(b). \tag{64}$$

Then we can rewrite (61) as

$$\int_a^{\sigma(b)} \tilde{\beta}(t)y^2(t)\Delta t = \int_a^b \gamma(t)x^2(\sigma(t))\Delta t. \tag{65}$$

On the other hand, integrating the first equation of (39) from a to τ and using (59), (62), (64) and Lemma (4.2.9)(v), we obtain

$$x(\tau) = x(a) + \int_a^\tau \alpha(t)x(\sigma(t))\Delta t + \int_a^\tau \beta(t)y(t)\Delta t$$

$$= -\frac{\xi\mu(a)\beta(a)}{1-(1-\xi)\mu(a)\alpha(a)}y(a) + \int_a^\tau \alpha(t)x\sigma(t)\Delta t + \int_a^\tau \beta(t)y(t)\Delta t$$

$$= \int_a^\tau \alpha(t)x(\sigma(t))\Delta t + \frac{(1-\xi)[1-\mu(a)\alpha(a)]}{1-(1-\xi)\mu(a)\alpha(a)}\mu(a)\beta(a)y(a) + \int_{\sigma(a)}^\tau \beta(t)y(t)\Delta t$$

$$= \int_a^\tau \alpha(t)x\big(\sigma(t)\big)\Delta t + \int_a^\tau \tilde{\beta}(t)y(t)\Delta t\,, \;\; \sigma(a) \le \tau \le b. \qquad (66)$$

Similarly, integrating the first equation of (39) from τ to b and using (60), (63), (64) and Lemma (4.2.9)(v), we have

$$x(\tau) = x(b) - \int_\tau^b \alpha(t)x\big(\sigma(t)\big)\Delta t - \int_\tau^b \beta(t)y(t)\Delta t$$

$$= -\frac{\eta\mu(b)\beta(b)}{1-(1-\eta)\mu(b)\alpha(b)}y(b) - \int_\tau^b \alpha(t)x\big(\sigma(t)\big)\Delta t - \int_\tau^b \beta(t)y(t)\Delta t$$

$$= -\int_\tau^b \alpha(t)x\sigma(t)\Delta t - \int_\tau^{\sigma(b)} \tilde{\beta}(t)y(t)\Delta t\,, \quad \sigma(a) \le \tau \le b. \qquad (67)$$

It follows from (66), (67) and Lemma (4.2.9) that

$$|x(\tau)| \le \int_a^\tau |\alpha(t)|\big|x\big(\sigma(t)\big)\big|\Delta t + \int_a^\tau \tilde{\beta}(t)|y(t)|\Delta t\,, \quad \sigma(a) \le \tau \le b,$$

and

$$|x(\tau)| \le \int_\tau^b |\alpha(t)|\big|x\big(\sigma(t)\big)\big|\Delta t + \int_\tau^{\sigma(b)} \tilde{\beta}(t)|y(t)|\Delta t\,, \quad \sigma(a) \le \tau \le b.$$

Adding the above two inequalities, we have

$$2|x(\tau)| \le \int_a^b |\alpha(t)|\big|x\big(\sigma(t)\big)\big|\Delta t + \int_a^{\sigma(b)} \tilde{\beta}(t)|y(t)|\Delta t\,, \;\; \sigma(a) \le \tau \le b. \qquad (68)$$

Let $|x(\tau^*)| = \max_{\sigma(a)\le t\le b}|x(t)|$. There are two possible cases:

(a) end-point b is left-scattered;

(b) end-point b is left-dense.

Case (1). In this case, it follows from Lemma (4.2.11) that $\sigma\big(\rho(b)\big) = b$. Hence,

$$\int_a^b |\alpha(t)|\big|x\big(\sigma(t)\big)\big|\Delta t = \int_a^{\rho(b)} |\alpha(t)|\big|x\big(\sigma(t)\big)\big|\Delta t + \int_{\rho(b)}^b |\alpha(t)|\big|x\big(\sigma(t)\big)\big|\Delta t\, \alpha(t)$$

$$= \int_a^{\rho(b)} |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \int_{\rho(b)}^{\sigma(\rho(b))} |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t$$

$$= \int_a^{\rho(b)} |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \mu\big(\rho(b)\big)\left|\alpha\big(\rho(b)\big)\right|\left|x\left(\sigma\big(\rho(b)\big)\right)\right|$$

$$= \int_a^{\rho(b)} |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \mu\big(\rho(b)\big)\left|\alpha\big(\rho(b)\big)\right||x(b)|$$

$$\leq |x(\tau^*)|\left[\int_a^{\rho(b)} |\alpha(t)|\Delta t + \mu\big(\rho(b)\big)\left|\alpha\big(\rho(b)\big)\right|\right]$$

$$\leq |x(\tau^*)|\int_a^b |\alpha(t)|\Delta t. \qquad (69)$$

That is

$$\int_a^b |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t \leq |x(\tau^*)|\int_a^b |\alpha(t)|\Delta t. \qquad (70)$$

Similarly, we have

$$\int_a^b \gamma^+(t)x^2\big(\sigma(t)\big)\Delta t \leq |x(\tau^*)|^2\int_a^b \gamma^+(t)\Delta t. \qquad (71)$$

Case (2). In this case, there exists a sequence $\{b_n\}$ of $\mathbb{T}$ such that

$$a < b_1 < b_2 < b_3 < \cdots < b_n < \cdots < b, \qquad \lim_{n\to\infty} b_n = b.$$

Hence

$$\int_a^b |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t = \int_a^{b_n} |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \int_b^{b_n} |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t$$

$$\leq |x(\tau^*)|\int_a^{b_n} |\alpha(t)|\Delta t + \int_{b_n}^b |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t$$

$$\leq |x(\tau^*)|\int_a^b |\alpha(t)|\Delta t + \int_{b_n}^b |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t$$

$$\le |x(\tau^*)| \int_a^b |\alpha(t)|\Delta t\,, \qquad n \to \infty,$$

which implies that (70) holds. Similarly, we can prove that (71) holds as well. Applying Lemma (4.2.10) and using (65),(70) and (71), we have

$$2|x(\tau^*)| \le \int_a^b |\alpha(t)|\left|x\big(\sigma(t)\big)\right|\Delta t + \int_a^{\sigma(b)} \tilde{\beta}(t)|y(t)|\Delta t$$

$$\le |x(\tau^*)| \int_a^b |\alpha(t)|\Delta t + \left[\int_a^{\sigma(b)} \tilde{\beta}(t)\Delta t \int_a^{\sigma(b)} \tilde{\beta}(t) y^2(t)\Delta t\right]^{1/2}$$

$$= |x(\tau^*)| \int_a^b |\alpha(t)|\Delta t + \left[\int_a^{\sigma(b)} \tilde{\beta}(t)\Delta t \int_a^{\sigma(b)} \gamma(t) x^2\big(\sigma(t)\big)\Delta t\right]^{1/2}$$

$$\le |x(\tau^*)| \left\{\int_a^b |\alpha(t)|\Delta t + \left[\int_a^{\sigma(b)} \tilde{\beta}(t)\Delta t \int_a^b \gamma^+(t)\Delta t\right]^{1/2}\right\}. \qquad (72)$$

Dividing the latter inequality of (72) by $|x(\tau^*)|$, we obtain

$$\int_a^b |\alpha(t)|\Delta t + \left[\int_a^{\sigma(b)} \tilde{\beta}(t)\Delta t \int_a^b \gamma^+(t)\Delta t\right]^{1/2} \ge 2. \qquad (73)$$

Since $\mu(t) \ge 0, 1 - \mu(t)\alpha(t) > 0$ and $\xi, \eta \in [0,1)$, we have

$$\tilde{\beta}(t) \le \beta(t), \qquad a \le t < \sigma(b),$$

then it follows from (73) that (53).

Remark (4.2.13)[87]: It is obvious that the Lyapunov type inequality (53) of Theorem (4.2.12) is better than (44) of Theorem (4.2.1) for the bound 2 in the right side of (53) is better than that of (44). Furthermore, the assumptions of the former is weaker than the ones of the latter.

In case $x(b) = 0$, i.e. $\eta = 0$, then we have the following equation

$$\int_a^b \tilde{\beta}(t) y^2(t)\Delta t = \int_a^b \gamma(t) x^2(\sigma(t))\Delta t \quad (74)$$

and inequality

$$2|x(\tau)| \le \int_a^{\rho(b)} |\alpha(t)||x(\sigma(t))|\Delta t + \int_a^b \tilde{\beta}(t)|y(t)|\Delta t, \quad \sigma(a) \le \tau \le b, \qquad (75)$$

instead of (65) and (68), respectively. It is easy to see that (74) holds because $\tilde{\beta}(b) = 0$. Next, we prove (75) is true.If b is left-dense, then $\rho(b) = b$, and so (75) holds. If b is left-scattered, then it follows from Lemma (4.2.11) that $\sigma(\rho(b)) = b$.Thus, it follows from Lemma (4.2.9)(iii) and (v) and the assumption $x(b) = 0$ that

$$\int_a^b |\alpha(t)||x(\sigma(t))|\Delta t = \int_a^{\rho(b)} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\rho(b)}^b |\alpha(t)||x(\sigma(t))|\Delta t$$

$$= \int_a^{\rho(b)} |\alpha(t)||x(\sigma(t))|\Delta t + \mu(\rho(b))|\alpha(\rho(b))||x(b)|$$

$$= \int_a^{\rho(b)} |\alpha(t)||x(\sigma(t))|\Delta t,$$

which, together with (68) and the fact that $\tilde{\beta}(b) = 0$, implies that (75) holds. Similar to the proof of (73), we have

$$\int_a^{\rho(b)} |\alpha(t)|\Delta t + \left[\int_a^b \tilde{\beta}(t)\Delta t \int_a^b \gamma^+(t)\Delta t\right]^{1/2} \ge 2. \qquad (76)$$

Since $\tilde{\beta}(t) \le \beta(t)$for $a \le t \le b$, it follows that

$$\int_a^{\rho(b)} |\alpha(t)|\Delta t + \left[\int_a^b \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t\right]^{1/2} \ge 2. \qquad (77)$$

Therefore, we can obtain the following theorem.

Theorem (4.2.14)[87]:*Suppose that* (50) *holds and let* $a, b \in \mathbb{T}^k$ *with* $\sigma(a) \le \rho(b)$. *Assume* (39) *has a real solution* $(x(t), y(t))$ *such that* $x(t)$ *has a generalized zero at end-point* a *but a usual zero at end-point* b *and* $x(t)$*is not identically zero on* $[a, b]$, *i.e.,*

$$x(a)=0 \quad or \quad x(a)x\big(\sigma(a)\big)<0; \quad x(b)=0; \quad \max_{a\le t\le b}|x(t)|>0.$$

Then inequality (77) *holds.*

Remark (4.2.15)[87]: In view of the proof of Theorem (4.2.14), in case both end-points a and b are usual zeros, i.e. $x(a)=x(b)=0$, then assumption (50) can be dropped in Theorem (4.2.15).

While the end-point b is not necessarily a generalized zero of $x(t)$, we still can establish the following more general theorem.

Theorem (4.2.16)[87]:*Suppose that* (50) *holds and let* $a,b\in\mathbb{T}^k$ *with* $\sigma(a)\le b$. *Assume* (39) *has a real solution* $\big(x(t),y(t)\big)$*such that* $x(t)$ *has a generalized zero at end-point* a *and* $\big(x(b),y(b)\big)=\big(\lambda_1 x(a),\lambda_2 y(a)\big)$*with* $0<\lambda_1^2\le\lambda_1\lambda_2\le 1$*and* $x(t)$*is not identically zero on*$[a,b]$. *Then one has the following inequality*

$$\int_a^b|\alpha(t)|\Delta t+\left[\int_a^b\beta(t)\Delta t\int_a^b\gamma^+(t)\Delta t\right]^{1/2}\ge 2. \tag{78}$$

Proof. It follows from the assumption $x(t)=0$ or $x(t)x\big(\sigma(t)\big)<0$ that there exists $\xi\in[0,1)$such that (54) holds. Further, by the proof of Theorem (4.2.12), (56)–(59) hold. Since $\big(x(b),y(b)\big)=\big(\lambda_1 x(a),\lambda_2 y(a)\big)$, then by (57), we have

$$(\lambda_1\lambda_2-1)x(a)y(a)=\int_a^b\beta(t)y^2(t)\Delta t-\int_a^b\gamma(t)x^2\big(\sigma(t)\big)\Delta t. \tag{79}$$

Substituting (59) into (79), we have

$$\int_a^b\beta(t)y^2(t)\Delta t-\int_a^b\gamma(t)x^2\big(\sigma(t)\big)\Delta t=\frac{(1-\lambda_1\lambda_2)\xi\mu(a)\beta(a)}{1-(1-\xi)\mu(a)\alpha(a)}y^2(a),$$

which implies that

$$\kappa_1\mu(a)\beta(a)y^2(a)+\int_{\sigma(a)}^b\beta(t)y^2(t)\Delta t=\int_a^b\gamma(t)x^2\big(\sigma(t)\big)\Delta t, \tag{80}$$

where

$$\kappa_1=\frac{(1-\xi)\big(1-\mu(a)\alpha(a)\big)+\lambda_1\lambda_2\xi}{1-(1-\xi)\mu(a)\alpha(a)}. \tag{81}$$

On the other hand, integrating the first equation of (39) from a to τ and using (59) and Lemma (4.2.9)(v), we can obtain the following inequality which is similar to (66):

$$x(\tau) = \frac{(1-\xi)[1-\mu(a)\alpha(a)]}{1-(1-\xi)\mu(a)\alpha(a)}\mu(a)\beta(a)y(a) + \int_a^\tau \alpha(t)x(\sigma(t))\Delta t$$

$$+ \int_{\sigma(a)}^\tau \beta(t)y(t)\Delta t\,, \;\; \sigma(a) \le \tau \le b. \qquad (82)$$

Similarly, integrating the first equation of (39) from τ to b and using (59), Lemma (4.2.9)(v) and the fact that $x(b) = \lambda_1 x(a)$,we have

$$x(\tau) = x(b) - \int_\tau^b \alpha(t)x(\sigma(t))\Delta t - \int_\tau^b \beta(t)y(t)\Delta t$$

$$= \lambda_1 x(a) - \int_\tau^b \alpha(t)x(\sigma(t))\Delta t - \int_\tau^b \beta(t)y(t)\Delta t$$

$$= \frac{-\lambda_1\xi\mu(a)}{1-(1-\xi)\mu(a)\alpha(a)}\beta(a)y(a) - \int_\tau^b \alpha(t)x(\sigma(t))\Delta t - \int_\tau^b \beta(t)y(t)\Delta t\,, \;\; \sigma(a) \le \tau \le b. \, (83)$$

From (82) and (83), we obtain

$$|x(\tau)| \le \frac{(1-\xi)[1-\mu(a)\alpha(a)]}{1-(1-\xi)\mu(a)\alpha(a)}\mu(a)\beta(a)|y(a)|$$

$$+ \int_a^\tau |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\sigma(a)}^\tau \beta(t)|y(t)|\Delta t\,, \;\; \sigma(a) \le \tau \le b$$

and

$$|x(\tau)| \le \frac{|\lambda_1|\xi}{1-(1-\xi)\mu(a)\alpha(a)}\mu(a)\beta(a)|y(a)|$$

$$+ \int_\tau^b |\alpha(t)||x(\sigma(t))|\Delta t + \int_\tau^b \beta(t)|y(t)|\Delta t\,, \;\; \sigma(a) \le \tau \le b,$$

Adding the above two inequalities, we have

$$2|x(\tau)| \le \kappa_2\mu(a)\beta(a)|y(a)| + \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\sigma(a)}^b \beta(t)|y(t)|\Delta t\,, \sigma(a) \le \tau \le b, (84)$$

where

$$\kappa_2 = \frac{(1-\xi)[1-\mu(a)\alpha(a)] + |\lambda_1|\xi}{1-(1-\xi)\mu(a)\alpha(a)}. \tag{85}$$

Let $|x(\tau^*)| = \max_{\sigma(a)\le\tau\le b}|x(\tau)|$. Applying Lemmas (4.2.9) and (4.2.10) and using (70), (71), (80) and (84), we have

$$\begin{aligned}
2|x(\tau^*)| &\le \kappa_2\mu(a)\beta(a)|y(a)| + \int_a^b |\alpha(t)|\left|x(\sigma(t))\right|\Delta t + \int_{\sigma(a)}^b \beta(t)|y(t)|\Delta t \\
&\le |x(\tau^*)|\int_a^b |\alpha(t)|\Delta t \\
&\quad + \left\{\left[\frac{\kappa_2^2}{\kappa_1}\mu(a)\beta(a) + \int_{\sigma(a)}^b \beta(t)\Delta t\right]\left[\kappa_1\mu(a)\beta(a)y^2(a) + \int_{\sigma(a)}^b \beta(t)y^2(t)\Delta t\right]\right\}^{1/2} \\
&= |x(\tau^*)|\int_a^b |\alpha(t)|\Delta t + \left\{\left[\frac{\kappa_2^2}{\kappa_1}\mu(a)\beta(a) + \int_{\sigma(a)}^b \beta(t)\Delta t\right]\int_a^b \gamma(t)x^2(\sigma(t))\Delta t\right\}^{1/2} \\
&\le |x(\tau^*)|\left\{\int_a^b |\alpha(t)|\Delta t\left[\left(\frac{\kappa_2^2}{\kappa_1}\mu(a)\beta(a) + \int_{\sigma(a)}^b \beta(t)\Delta t\right)\int_a^b \gamma^+(t)\Delta t\right]^{1/2}\right\}.
\end{aligned} \tag{86}$$

Dividing the latter inequality of (86) by $|x(\tau^*)|$, we obtain

$$\int_a^b |\alpha(t)|\Delta t + \left[\left(\frac{\kappa_2^2}{\kappa_1}\mu(a)\beta(a) + \int_{\sigma(a)}^b \beta(t)\Delta t\right)\int_a^b \gamma^+(t)\Delta t\right]^{1/2} \ge 2. \tag{87}$$

Set $d = 1-(1-\xi)\mu(a)\alpha(a)$. Since $(1-\xi)[1-\mu(a)\alpha(a)] > 0$, it follows that $d > \xi \ge 0$, and so

$$[d-1-|\lambda_1|\xi]^2 \le d[d-(1-\lambda_1\lambda_2)\xi].$$

This, together with (81) and (85), implies that

$$\frac{\kappa_2^2}{\kappa_1} = \frac{\left[\frac{(1-\xi)(1-\mu(a)\alpha(a))+|\lambda_1|\xi}{1-(1-\xi)\mu(a)\alpha(a)}\right]^2}{\frac{(1-\xi)(1-\mu(a)\alpha(a))+\lambda_1\lambda_2\xi}{1-(1-\xi)\mu(a)\alpha(a)}} = \frac{[d-(1-|\lambda_1|)\xi]^2}{d[d-(1-\lambda_1\lambda_2)\xi]} \le 1.$$

Substituting this into (87), we obtain (78).