Dedication

To my family

Acknowledgment

At first my thanks to my God who gave me the health and age to conduct this study.

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Abstract

The information about the Hilbert transform is often scattered in books about signal processing. Their authors frequently use mathematical formulas without explaining them thoroughly to the reader.

The purpose of this research is to make a more stringent presentation of the Hilbert transform but still with the signal processing application in mind.

Everybody working in the field of singular integrals and integral equations will know that during the last few decades an entirely new mathematical field of Cauchy principal value integrals and hypersingular integral has developed.

Since this is a recent mathematical development, it is not always easy for readers including academics, engineers and researchers, to get a grap on.

Hilbert transforms deal with Calderon-Zygmund operators and the theory of Calderon-Zygmund operators in such a way that any body will be able to repeat the schedule use of Hilbert transforms.

المستخلص

المعلومات عن تحويلات هلبرت منتشرة كثيراً فى الكتب التى تتحدث عن معالجة الإشارات . ويستخدم كتابها الصيغ الرياضية دون شرحها لبقية القراء . والغرض من ذلك جعل العرض اكثر صرامة مع تحويل هلبرت . ولكن لا يزال مع تطبيق معالجة الإشارات فى الإعتبار , وكل من يعمل فى حقل التكامل المفرد والمعادلات التكاملية يرى انه خلال العقود القليلة الماضية قد وضِع حقل رياضى جديد تماماً من القيمة القياسية لتكملات كوشى بحيثاصبحت جزء لا يتجزأ منه. ومع التطورات الرياضية الأخيرة فليس من السهل دائماً للقراء والمهندسين والباحثين الأكاديميين الحصول على فهم واتفاق حول نظرية مشغلى كالديرون-سيقموند بطريقة تجعل كل فرد قادر على استخدام جداول تحويلات هلبرت.

Introduction

Today the signal processing is the fast growing area and a desired effectiveness in utilization of band width and energy makes the progress even faster. Special signal processors[96] have been developed to make it possible to implement the theoretical knowledge in an efficient way. Signal processors are nowadays frequently used in equipment for radio, transportation, medicine and production.

Areal function $f(t)$ and its Hilbert transform $\tilde{f}(t)$ are related to each other in such a way that they together create a so called strong analytic signal[. The strong analytic signal can be written with an amplitude and phase where the derivative of the phase can be identified as the instantaneous frequency.
It is easy to see that a function and its Hilbert transform also are orthogonal[84]. However, a function and its energy can be used to measure the calculation accuracy of the approximated Hilbert transform.

Whenever we write (P) in front of the integral we will mean that the Cauchy principal value[35] of that integral (when it exists).

Hilbert transform has the advantage of not requiring derivatives, but the serious disadvantage that it is not a bounded operator from L^1 to L^1. T o solve the problem,different approaches for gain-phase relationships in logarithmic frequency domain have been proposed. A suitable change of variable can give the bounded operator.

To solve the problem, different approaches for computing Hilbert transform have been proposed.

The goal of this research is to present a brief review of methods used to compute Hilbert transform when the signal is composed of discrete data, sampled at equidistant or arbitrarily instant.

Finally we study in this research Hilbert transform continuous and discrete with Properties in chapter one and in chapter tow we have show Singular integral equations with Cauchy's principal value[38] and Hilbert trans form. Then the study in chapter three is about Calderon-Zygmund operator. In chapter four we have show Some applications of the convolution theorem of the Hilbert transform. The aim of chapter five is to give information about Hilbert transform application. Lastly in chapter six have the numerical evaluation of hypersingular integral[35].

Contents

Chapter 1

Properties and Continuous of Discrete Hilbert Transforms

Sec.(1.1): Hilbert transform

The Hilbert transform defined in the time domain is a convolution between the Hilbert transform $1/\pi t$ and function $f(t)$ [7]

Definition (1.1.1) The Hilbert transform $\hat{f}(t) of$ a function $f(t)$ is defined for all (t) by

$$H(\hat{f}(t))=\hat{f}(t)=\frac{1}{\pi}p\int_{-\infty}^{\infty}\frac{f(\tau)}{t-\tau}d\tau,$$

when the integral exists.

It is normally not possible to calculate the Hilbert transform as an ordinary improper integral because of the pole $\tau=t.$ However, the ρ in front of the integral denotes the Cauchy principal value which expanding the class of functions for which the integral in

Definition (1.1.1) exist. It can be defined by the following definition [102].

Definition (1.1.2) Let [α, $\beta¿$ be a real interval and let f be a complex-value of function defined on $[\alpha,\beta]$. If f is unbounded near an interior point ε of $[\alpha,\beta]$, the integral of f over [α $,\beta¿$ does not always exist.

However, the two limits

$$\lim_{\varepsilon\to 0}\int_{\alpha}^{\xi-\varepsilon} f(x)\,dx \wedge \lim_{\varepsilon\to 0}\int_{\xi-\varepsilon}^{\beta} f(x)\,dx,$$

still may exist, and if they do their sum is called the improper integral of f over [α, $\beta¿$ and is denoted by the ordinary integration symbol

$$\int_{\alpha}^{\beta} f(x)\,dx.$$

Even if these two limits do not exist, it may happen that the "symmetric limit"

$$\lim_{\epsilon-0}\left(\int_{\alpha}^{\varepsilon-\epsilon} f(x)\,dx+\int_{\varepsilon+\epsilon}^{\beta} f(x)\,dx\right),$$

exists. If it does, it is called the principal value integral of ƒ from α to β and is denoted by the symbol

$$p\int_{\alpha}^{\beta} f(x)\,dx.$$

Example (1.1.3) An ordinary real function $\frac{1}{x}$ that is integrated from $-a¿a$ can be written as

$$\varepsilon \to 0+¿\int_{\varepsilon}^{a} \frac{1}{x}dx,$$

$$\delta \to 0+¿\int_{-a}^{-\delta} \frac{1}{x}dx+\lim_{¿} ¿$$

$$\int_{-a}^{a} \frac{1}{x}dx=\lim_{¿} ¿$$

and we see that it is not possible to calculate these integrals separately because of the pole in $x = 0$. However, if we apply the Cauchy principal value then δ and ϵ tend zero at the same speed , that is

$$\varepsilon \to 0+¿\left(\int_{-a}^{-\varepsilon} \frac{1}{x}dx+\int_{\varepsilon}^{a} \frac{1}{x}dx\right)=0,$$

$$p\int_{-a}^{a} \frac{1}{x}dx=\lim_{¿} ¿$$

and the integral converges.

(1.2.) Mathematical Motivations for Hilbert transform

In this chapter we motivate the Hilbert transform in three different ways. First we use the Cauchy integral in the complex plane and second we use the Fourier transform in the frequency domain and the third we look at the $\pm\pi/2$ phase-shift which is basic property of the Hilbert transform.

(1.2.1.) The Cauchy integral

The Cauchy integral is figurative way to motivate the Hilbert transform. The complex view helps us to relate the Hilbert transform to something more concrete and understandable.[152]

Consider an integral in the complex z -plane on the form

$$\oint_{\Gamma} \frac{f(z)}{z-a} dz,$$

Which is known as a Cauchy integral. If f is analytic and $\Gamma\, is$ piecewise smooth closed contour in an open domain then the Cauchy integral theorem is applicable as

$$\oint_{\Gamma} \frac{f(z)}{z-a} dz = \begin{cases} 2\pi \mathrm{if}(a)\, if\; a\, inside\, \Gamma \\ 0\; if\; a\, outside\, \Gamma \end{cases}$$

To get a result when a lies on Γ we have to create a new contour Γ'_{ε} where

$$\oint_{\Gamma} \frac{f(z)}{z-a} dz = 2\pi \mathrm{if}(a) (1.1)$$

If the radius ε of the semicircle γ tends to zero, the contribution from the semicircle Y_{ε} to the integral along Γ'_{ε} approaches $\pi if(a)$ according to lemma (1.2.4)[135] .

Lemma (1.2.4) If g has a simple pole at $z=a$ and is the circular arc defined by

$$\gamma_r : z = a + \gamma e^{i\theta} (\theta_1 \leq \theta \leq \theta_2).$$

From Lemma (1.2.4) and from the definition of Cauchy principal value[38,39], we see that

$$\lim_{\varepsilon-0} \oint_{\Gamma'_\varepsilon} \frac{f(z)}{z-a} dz = p\int_{\Gamma} \frac{f(z)}{z-a} dz + \lim_{\varepsilon-0} \int_{\gamma_\varepsilon} \frac{f(z)}{z-a} dz = 2\pi \text{if}(a),$$

and that the integral of Cauchy principal value is

$$p\int_{\Gamma} \frac{f(z)}{z-a} dz \equiv \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \frac{f(z)}{z-a} dz = \pi i f(a)(1.2)$$

where Γ_ε is a non closed contour with out the indentation γ_ε . By (2) we have generalized the definition of Cauchy principal value compared to definition (1.1.1).

If $f(z)$ is a function that is analytic in an open region that contain the upper half-plane and tends to zero at infinity in such a rate that the contribution from semicircle:

$\gamma R \; as \; R \to \infty, then \; we \; have$

$$p\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi = \pi i f(x)(1.3)$$

$$|f(z)| < \frac{C}{|z|},$$

for any positive constant $C.$ The same yields if

$$|f(z)| < C\left|e^{imz}\right|,$$

for positive m according to lemma (1.2.5)[7].

Lemma (1.2.5) (Jordan's lemma) if $m>0 \wedge P/Q$ is quotient of tow polynomials such that

Degree $Q \geq 1 + degree\, P$,

then

$$\lim_{p \to \infty} \int_{¿}^{C_p^{+¿}} \square \frac{P(z)}{Q(z)} e^{imz} dz = 0,$$

where $C_p^{+¿}$ is the upper half-circle with radius p.

If we express $f(x)$ as

$$f(x) = g(x) - ih(x)$$

on both sides of (3) with arguments on the real $x-¿$ axis and equating real and imaginary parts we obtain for the real part

$$g(x) = \frac{-1}{\pi} p \int_{-\infty}^{\infty} \frac{h(\xi)}{x-\xi} d\xi = -Hh(x),$$

and for the imaginary part

$$h(x) = \frac{1}{\pi} p \int_{-\infty}^{\infty} \frac{h(\xi)}{x-\xi} d\xi = Hg(x)(1.4)$$

From definition (1.1.1) we have that $h(x)$ is the Hilbert transform of $g(x)$ where H is the Hilbert transform operator. We also note that $g(x) = H^{-1} h(x)$ with H^{-1} as the inverse Hilbert transform operator. We see that $HRef(x) = Imf(x)$ which motivates the following definition.

Definition (1.2.5) A complex signal $f(x)$ that fulfills the conditions preceding is called a strong analytic signal.

Theorem (1.2.6) For strong analytic signal $f(x)$ we have that $HRe\, f(x) = \Im f(x)$.

(1.2.2) The Fourier transform [60,117]

The Fourier transform is important in theory of signal processing . When a function $f(t)$ is real , we only have to look on the positive frequency axis because it contains the complete information about the wave form in the time domain .

Therefore , we do not need negative frequency axis and the Hilbert transform can be used to remove it. This is explained below.

Let us define the Fourier transform $F(\omega)$ of a signal $f(t)$ by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (1.5)$$

This definition makes sense if $f \in L^1$ (r), that is if

$$\int_{-\infty}^{\infty} |f(t)| dt$$

exists. It is important to be able to recover the signal from its Fourier transform . To do that

$$\tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt$$

If both $f \wedge F$ belong to $L^1(R)$ then $f(t)$ is continuous and bounded for all real *t* and we have that $\tilde{f}(t)=f(t), that\,is$ [60]

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(t)e^{i\omega t}dt\,(1.6)$$

this result is a form of Fourier inversion theorem.

Another inversion theorem [2] is that if *f* belongs to $L^1(R), f$ is of bounded variation in a neighborhood of t and *f* of continuous at t

$$f(t)=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}F(t)e^{i\omega t}dt$$

This means that (1.6) is to be interpreted as a type of a Cauchy principal value**.[**102]

Further more general variants of the inversion theorem exist for $f\in L^1(R).$ There is also a theory for the Fourier transform when $f\in L^2(R).[60]$

In this case we define the Fourier transform as

$$F(\omega)=\lim_{N\to\infty}\int_{-N}^{N}f(t)e^{-i\omega t}dt$$

The mean limit

$$F(\omega)=\lim_{N\to\infty}F_N(\omega),$$

is to be interpreted as

$$\lim_{N\to\infty}\|F(\omega)-F_N(\omega)\|_2=0.$$

Theorem (1.2.7) If $f,g \wedge G, belong \text{¿} L^1(R) \vee if\, f \wedge g\, belong \text{¿} L^2(R)$ then

$$\int_{-\infty}^{\infty} f(t) g^{\text{¿}}(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G^{\text{¿}}(\omega)\, d\omega.$$

Proof For a proof refer to

If $f(t)$ a real function that can be represented by an inverse Fourier transform then we have the following relationship in the time domain

$$f(t) = f^{\text{¿}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

$$\text{¿} \frac{1}{2\pi} \int_{-\infty}^{\infty} F^{\text{¿}}(\omega) e^{-i\omega t}\, d\omega$$

$$\text{¿} \frac{1}{2\pi} \int_{-\infty}^{\infty} F^{\text{¿}}(-\omega) e^{i\omega t}\, d\omega.$$

This gives us the relation $F(\omega) = F^{\text{¿}}(-\omega) \vee F(\omega) = F^{\text{¿}}(\omega)$ in the frequency domain and we see that F for negative frequencies can be expressed by $F^{\text{¿}}$ for positive ones[1].

Theorem(1.2.8): If $f(t)$ is a real function then

$$\overset{F}{[\text{¿¿¿}(\omega) e^{-i\omega t} d\omega + F(\omega) e^{i\omega t} d\omega].}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{¿}$$

Proof :If we apply the Fourier transform of a real function $f(t)$ then

$$F$$
$$¿$$
$$¿$$
$$¿$$
$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{0} ¿$$

$$¿\frac{1}{2\pi}\int_{0}^{\infty} F(-\omega)e^{-i\omega t}d\omega+\frac{1}{2\pi}\int_{0}^{\infty} F(\omega)e^{i\omega t}d\omega.$$

$$¿\frac{1}{2\pi}\int_{-\infty}^{0} F^{¿}(\omega)e^{-i\omega t}d\omega+F(\omega)e^{i\omega t}d\omega$$

These means that positive frequency spectra is sufficient to represent a real signal.

Let us define a function $Z_f(\omega)$, that the zero for all negative frequencies and 2F(t) for all positive frequencies

$$Z_f(\omega)=F(\omega)+sgn(\omega)F(\omega),(1.7)$$

where

$$sng(\omega)=\begin{cases} 1 \, for \, \omega>0 \\ 0 \mathrm{for} \, \omega=0 \\ -1 \mathrm{for} \, \omega<0 \end{cases}$$

and $F(\omega)$ is the Fourier transform of the real function $f(t)$. We see the relation (1.7) between $F(\omega)$ and $Z_f(\omega)$. The inverse transform of $Z_f(\omega)$ is therefor written as

$$Z_f(\omega)=\frac{1}{2\pi}\int_{-\infty}^{0} F(\omega)e^{-i\omega t}d\omega=\frac{1}{\pi}\int_{0}^{\infty} F(\omega)e^{i\omega t}d\omega$$

where $Z_f(t)$ is complex function on the form

$$Z_f(t)=f(t)+ig(t) \quad (1.8)$$

we will show below that $g(t)$ is real and from (1.7) and (1.8) we have that

$$f(t)+ig(t) \underset{\Leftrightarrow}{F} F(\omega)(sgn)(\omega) \quad (1.9)$$

The definition of Fourier transform tells that $f(t) \underset{\Leftrightarrow}{F} F(\omega)$ and therefore know that $F(\omega) sgn(\omega)$ is the Fourier transform of $ig(t)$, thus

$$g(t) \underset{\Leftrightarrow}{F} F(\omega)(isgn(\omega))$$.

It is a standard result that the inverse transform of $-isgn(\omega)$ equals $1/(\pi t)$, that is

$$g(t)=f\frac{(t)*1}{\pi t}=\frac{1}{\pi}p\int_{-\infty}^{\infty}\frac{f(\tau)}{t-\tau}d\tau=Hf(t)=f\hat{}(t),$$

And we see that $g(t)$ can be written as $f\hat{}(t)$ which is known as Hilbert transform of $f(t).$ Further more $g(t)$ is real.

(1.2.3) The $\pm\pi/2$ phase shift

The phase shift is interpreted in frequency domain as a multiplication with the imaginary value $\pm i$, thus

$$H(\omega)=\begin{cases}-i=e^{-i\frac{\pi}{2}} & \omega>o \\ i=e^{i\frac{\pi}{2}} & \omega<o\end{cases}$$

$$\lim_{\sigma\to 0} G(\omega)=H(\omega)$$

Where $g(t) \to h(t)\, when\, \sigma \to 0$ and the inverse Fourier transform of the impulse response of $H(\omega)$ is

$$h(t) = \lim_{\sigma \to 0} g(t) = \lim_{\sigma \to 0} \frac{t}{\pi(\sigma^2 + t^2)} = \frac{1}{\pi t}$$

A convolution between $f(t)$ and the impulse response

$$\hat{f}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau ,$$

where $\hat{f}(t)$ is known as the Hilbert transform. Notice that this integral shall be considered as a principal-valued integral.

Sec.(1.3) Properties of Hilbert Transform:

In this chapter we look at some properties of the Hilbert transform. We assume that $F(\omega)$ does not contain any impulses for (ω) = 0 and that $f(t)$ is a real valued function.

So of the formulas are to be interpreted in a distributional sense[142,62].

(1.3.1) Linearity

The Hilbert transform that is a Cauchy principle-valued function, is expressed on the form

$$Hf(t) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{¿x-t \vee ¿\varepsilon}^{\square} \frac{f(\tau)}{t - \tau} d\tau .$$

If we write the function. $f(t)$ as $c_1 f_1(t) + c_2 f_2(t)$ where the Hilbert transform of $f_1(t)$ and $f_2(t)$ exists then

$$Hf(t)=H\left(\begin{matrix} t \\ c_1 f_1(¿) \\ t¿ \end{matrix} + c_2 f_2 ¿ \right.$$

$$\begin{matrix} \tau \\ c_1 f_1(¿) \\ ¿ \\ +c_2 f_2(\tau) \\ ¿ \\ ¿ \end{matrix}$$

$$¿\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon}^{\square} ¿$$

$$¿c_1 \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{¿x-t \vee ¿\varepsilon}^{\square} \frac{f_{1(t)}}{t-\tau} d\tau + c_2 \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{¿x-t \vee ¿\varepsilon}^{\square} \frac{f_{2(\tau)}}{t-\tau} d\tau$$

$$= c_1 H f_1(t) + c_2 H f_2(t)$$

this is the linearity property of the Hilbert transform.

(1.3.2) Multiple Hilbert Transform and Their Inverses

The Hilbert transform used twice on a real function but with altered sign

$$sgn(n)=\begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

$$since\, the - isgn = H$$

$$H=\begin{cases} -i \\ i \end{cases}$$

$$H=I$$

then H^2 = -1,

With 1 as the identity operator. The Hilbert transform used four times on the same real function gives us the original function back

$$H^2 H^2 \quad = \quad H^4 \quad = \; 1 \qquad .(1.11)$$

A more interesting property of multiple Hilbert transforms arises if we use the Hilbert transform 3 times, thus

$$H^3 H = 1 \Longrightarrow H^{-1} = H^3$$

This tell us that it is possible to use multiple Hilbert transform to calculate the inverse Hilbert transform.

As we seen before the Hilbert transform can be applied in the time domain by using the definition of the Hilbert transform. In the frequency domain we simply multiply the Hilbert transform operator –I sgn (ω) to the function $F(\omega)$.

By multiplying the Hilbert transform operator by itself we get an easy method to do multiple Hilbert transform, that is

$$H^n f(t) \underset{\Leftrightarrow}{F} \left(-i\, sgn(\omega)^n\right) \quad F(\omega),$$

where n is the number of Hilbert transform.

Example (1.3.7) We want to calculate the inverse Hilbert[97] transform of the function $f(t)$ by using multiple Hilbert transform in the frequency domain . First we have to Fourier transform the function $f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \, ,$$

and then use the Hilbert transform three times in the frequency domain, that is

$$H^3 = ¿ ¿ -i\, sgn(\omega) .$$

Finally we use the inverse Fourier transform

$$H^{-1} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H^3 F(\omega) e^{i\omega t}(\omega)\, d\omega.$$

From above we see that we only have to calculate two infinite integrals n the frequency domain compared to three infinite integrals in the time domain.

Another advantage in the frequency domain is that we formally can choose the number of times we want to use the Hilbert transform.

(1.3.3)Derivatives of the Hilbert Transform

Theorem (1.3.8) The Hilbert transform of the derivative of a function is equivalent to the derivative of the Hilbert transform of function, that is[61]

$$\hat{f}(t) \underset{\Leftrightarrow}{H} \frac{d}{dt} \hat{f}(t) (1.12)$$

proof. From Definition (1.1.1) we have that $\hat{f}(t)$

$$\hat{f}(t) = \frac{1}{\pi} p \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau.$$

If we substitute τ with $t-s$

$$\frac{d}{dt} \hat{f}(t) = \frac{1}{\pi} p \int_{-\infty}^{\infty} \frac{f(t-s)}{s} ds,$$

and then apply the derivative of t on both sides we get

$$\frac{d}{dt}\hat{f}(t)=\frac{1}{\pi}p\int_{-\infty}^{\infty}\frac{f'(t-s)}{s}ds.$$

The substitution $s=t-\tau$ gives us that

$$\frac{d}{dt}\hat{f}(t)=\frac{1}{\pi}p\int_{-\infty}^{\infty}\frac{f'(\tau)}{t-\tau}d\tau$$

And the relation in (1.12) is valid .

From the proof above we conclude that the relation can be used repeatedly. Let us look at an example where we also use of multiple Hilbert transforms,

Example (1.3.9)

By (1.3.2.) we may calculate the Hilbert transform of the delta function $\delta(t)$ and its derivatives. At the same time we get the Hilbert transform representation of the delta function. Consider the Hilbert transform of the delta function

$$H\delta(t)=\frac{1}{\pi t}(1.13)$$

.

The derivative of the delta function is calculated to

$$H\delta'(t)=\frac{-1}{\pi t^2}(1.14)$$

And if we apply the Hilbert transform on both sides then we get

$$\delta'(t)=H\left(\frac{1}{\pi t^2}\right).$$

The derivative (5) is

$$H\delta''(t)=\frac{2}{\pi t^3}$$

.

And when we apply the Hilbert transform on both sides we get

$$\delta''(t)=H\left(\frac{-2}{\pi t^3}\right)$$

.

This procedure can be continued.

(1.3.4) Orthogonally properties[112]

A symmetry about the Fourier transform $F(\omega)$ of a real function $f(t)$ leads us to the following definition[6]

Definition (1.3.10) A complex function is called Hermitian if its real part is even and its imaginary part is odd.

From this we have that the Fourier transform $F(\omega)$ of a real function $f(t)$ is Hermitian.

Theorem(1.3.11) A real function $f(t)$ and its Hilbert transform $\hat{f}(t)$ are orthogonal if $f, \hat{f}$ and F belonged to $L^1(R)$ or if $f \wedge \hat{f}$ belong to $L^2(R)$. [55]

Proof From Theorem (5) we have that

$$\int_{-\infty}^{\infty} f(t)\hat{f}(t)\,dt = ¿$$

$$¿\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left(-i\,sgn(\omega)F(\omega)\right)d\omega$$

$$¿\frac{i}{2\pi}\int_{-\infty}^{\infty} sgn(\omega)F(\omega)F^{¿}(\omega)d$$

$$sgn(\omega)¿F(\omega)\vee¿^2 d,$$

$$¿\frac{i}{2\pi}\int_{-\infty}^{\infty} ¿$$

Where sgn($\omega¿$ is an odd function and the fact that $F(\omega)$ is Hermitian gives us that $|F(\omega)|^2$ is an even function. We conclude that

$$\int_{-\infty}^{\infty} f(t)f^{\wedge}(t)dt=0,$$

and a real function and its Hilbert transform are orthogonal.

(1.3.5) Energy aspects of Hilbert transform

The energy of a function $f(t)$ is closely related to the energy of its Fourier transform $F(\omega)$. Theorem $f(t)=g(t)$ is called the Rayleigh theorem and it helps us define the energy of $f(t)\wedge F(\omega)$ as

$$¿f^{\wedge}(t)\vee¿^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d(\omega).(1.4)$$

$$E_f=\int_{-\infty}^{\infty} ¿$$

Here it is natural to assume that $f\in L^2(R)$ which means that E_f is finite. The same theorem is used to define the energy of Hilbert transform of $f(t) \wedge F(\omega)$ that is

$$¿ -i\, sng(\omega) F(\omega) \vee ¿^2 d\omega, (1.5)$$
$$f\hat{} = ¿ \int_{-\infty}^{\infty} |f\hat{}(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} ¿$$
$$E_{¿}$$

Where $\underset{¿}{¿ - i\, sgn(\omega) \vee ¿^2}$ = 1 except for $\omega = 0.$ But since $F(\omega)$ does not contain any impulses at the origin we get $\underset{E_{¿}}{f\hat{} = ¿\, E_{f.}}$

A consequence of (3.5) is that $f \in L^2(R)$ induces that $f\hat{} \in L^2(R).$ The accuracy of the approximated Hilbert transform operator can be measured by comparing the energy in (3.4) and (3.5). However, a minor difference in energy always exists in real applications due unavoidable

truncation errors.

(1.3.6)The Hilbert Transform of Strong Analytic Signal

From Section (1.3.2) we have that the Hilbert transform of two multiplied strong analytic signal $z(t)$ is

$$Hz(t) = H(f(t) + i\hat{f}(t)) = \hat{f}(t) - if(t) = -iz(t) (1.6)$$

From the follows the result of the Hilbert transform of two multiplied strong analytic signals.

Theorem (1.3.12)[55] The product of $H(z_1(t)z_2(t))$ is identical with the product of $z_1(t)H(z_2(t))$ if $z_1(t)\wedge z_2(t)$ are strong analytic signals.

Proof. Since $z_1(t)\wedge z_2(t)$ are strong analytic signals then

$$H(z_1(t))z_2(t) = (\hat{f}_1(t)-if_1(t)) \quad (f_2(t)+i\hat{f}_2(t))(1.7)$$

$$¿-i(f_1(t)+i\hat{f}_1(t)) \quad (f_2(t)+\hat{f}_2(t))$$

$$¿-iz_1(t)z_2(t)$$

$$¿(f_1(t)+i\hat{f}_1(t)) \quad (\hat{f}_2(t)-i\hat{f}_2(t))(1.8)$$

$$¿z_1(t)H(Z_2(t)) \quad , \qquad (1.9)$$

Where we make use of (1.6) in (1.7) and (1.8).

Theorem (1.3.14.)[97] The product of $z_1(t)z_2(t)$ is identical with the product $iH(z_1(t))z_2(t)=iz_1(t)H(z_2(t)) if\, z_2(t)\wedge z_1(t)$ are strong analytic signals.

Proof. Since $z_1(t)\wedge z_2(t)$ are strong analytic signals then

$$z_1(t)z_2(t)=(f_1(t)+i\hat{f}_1(t))(f_2(t)+i\hat{f}_2(t))$$

$$¿i(\hat{f}_1(t)-i\hat{f}_1(t))(f_2(t)+i\hat{f}_2(t))$$

$$z_2(t)$$
$$z_1(t)z_2(t)=iz_1(t)H(¿),$$
$$¿iH¿$$

And the theorem follows.

The Hilbert transform of the product of two strong analytic signals gives us the same result as in (3.9). To prove this we first need to show that the product of two strong analytic signals is strong analytic.

Theorem (1.3.15) The product of two strong analytic signals is strong analytic.

Proof. Let $z_1(t)$ and $z_2(t)$ be analytic signals of complex variable (t") = t+ it on the open upper half-plane. Then $z_1(t)$ and $z_2(t)$ is also analytic signal in the same region. Assume that $z_1(t)$ and $z_2(t)$ are decreasing in such rate at infinity that the discussion in Sec.(1.1) is true then $z_1(t)$ and $z_2(t)$ are strong analytic signals. If $z_1(t)$ and $z_2(t)$ are decreasing sufficiently rabid at infinity then $z_1(t)$ and $z_2(t)$ have two decrease faster than one of $z_1(t)$ and $z_2(t)$ that the decreasing with the last rate. From this we have that $\Re(z_1(t)\, z_2(t)) = \Im\, H(z_1(t)\, z_2(t))$ and that $(z_1(t)\, z_2(t))$ is strong analytic signal.

Theorem (1.3.16): $H\left(z_1(t)\ \{z\}\ \text{rsub}\ \{2\}\ (t\ )\right)=\text{¿}$ $-i\, z_1(t)\ \{z\}\ \text{rsub}\ \{2\}\ (t\ )$ if $\overset{t)}{z_1\text{¿}}$ and $\overset{t)}{z_2\text{¿}}$ are strong analytic signals.

Proof : Since $\overset{t)}{z_1\text{¿}}$ and $\overset{t)}{z_2\text{¿}}$ are strong analytic signals then

$$\begin{gathered} f_2(t)+i\hat{f}_2(t) \\ f_1(t)+i\hat{f}_1(t)\text{¿} \\ H\left(z_1(t)\ \{z\}\ \text{rsub}\ \{2\}\ (t\ )\right)=H\text{¿} \end{gathered}$$

$$\text{¿}\, H\left(f_1(t)f_2(t)-\hat{f}_1(t)\hat{f}_2(t)\right)$$

$$+i f_1(t)\hat{f}_2(t)+\hat{f}_1(t)f_2(t)$$

$$\text{¿}\, f_1(t)\hat{f}_2(t)-\hat{f}_1(t)f_2(t)$$

$$-f_1(t)f_2(t)-\hat{f}_1(t)\hat{f}_2(t)$$

$$\begin{gathered} f_1(t)f_2(t)-\hat{f}_1(t)\hat{f}_2(t) \\ \text{¿}-i\,\text{¿} \end{gathered}$$

$$+i\hat{f}_1(t)f_2(t)-\hat{f}_1(t)\hat{f}_2(t) \quad (1.10)$$

$$-i\left(f_1(t)+\hat{f}_1(t)\right)\left(f_2(t)+\hat{f}_2(t)\right)$$

$$\text{¿}-i\, z_1(t)\, z_2(t)$$

when we make use of theorem (1.8) in (1.10).

Consequently it possible to apply the Hilbert transform on product of two strong analytic signals in several different ways, thus

$$t$$
$$z_2(t)$$
$$z_1(t)z_2(t)=H(z_1(¿))z_2(t)=z_1(t)H(¿¿-iz_1(t)z_2(t).)$$
$$H¿$$

It does not matter on which strong analytic signal we apply the Hilbert transform. We conclude that the Hilbert transform of the product of n strong analytic signals from the equation

$$Hz^n(t)=H(z(t))z^{n-1}(t)=-iz(t)z^{n-1}(t)=-iz^n(t)$$

(1.3.7) Analytic signals in the time domain

The Hilbert transform can be used to create an analytic signal from areal signal. Instead of studying the signal in frequency to look at a rotating vector with an instantaneous phase $\varphi(t)$ and an instantaneous amplitude $A(t)$ in time domain, that is

$$z(t)=f(t)+i\hat{f}(t)=e^{i\varphi(t)}.$$

This notation is usually called the polar notation where

$$A(t)=\sqrt{f^2(t)\hat{f}^2(t)},$$

And

$$\varphi(t)=arcta\left(\frac{\hat{f}(t)}{f(t)}\right)$$

If we express the phase with Taylor series then

$$\varphi(t)=\varphi(t_0)+(t-t_0)\varphi'(t_0)+R,$$

Where R is small when t is close to (t_0). The analytic signal becomes

$$z(t)=A(t)e^{i\varphi(t)}=A(t)e^{i(\varphi(t_0)-t_0\varphi'(t_0))}e^{it\varphi'(t_0)}e^{iR}.$$

And we see that $\varphi'(t_0)$ has the role of frequency if R is neglected. This makes it natural to introduce the nation of instantaneous frequency, that is

$$\omega(t)=\frac{d\varphi(t)}{dt}.$$

Example (1.3.15) We have a real signal and its Hilbert transform

$$f(t)=\cos(\omega_0 t),$$

$$\hat{f}(t)=\sin(\omega_0 t),$$

Together they form an analytic signal where the instantaneous amplitude is

$$A(t)\sqrt{\cos^2(\omega_0 t)+\sin^2(\omega_0 t)}=1.$$

The instantaneous frequency is easy to calculate form the phase $\varphi(t)=\omega_0 t,$ that is

$$\omega(t)=\omega_{0.}$$

We see that in this particular case the instantaneous frequency is the same as the real frequency.

(1.4) Numerical calculations of the Hilbert transform

The purpose of this research is to study different type of numerical calculation methods for the Hilbert transform.

(1.4.1) Continuous

(1.4.1.1) Numerical integration.

Numerical integration works fine on smooth function that decrease rapidly at infinity. When we want to calculate the Hilbert transform by Definition (1.1.) we are facing some problems. In numerical integration we use finite intervals and it is therefore important to consider the integration region to control the calculation error This is the reason why a rapid decrease at infinity is an advantage. Another problem is that the integrand in Definition 1.1 is infinite when nominator vanishes. However, by using more integration grid point in the numerical integration close to this value we get a better approximation.

(1.4.1.2) Hermite polynomials

The numerical integration is inefficient when a function decreases in a slow rate in infinity. It is sometimes better to use a series of orthogonal polynomials where the function does not have to decrease rapid at infinity. In this section we use the Hermite polynomials to calculate the Hilbert transform. First we need to take a look at the definition of Hermite polynomials.

The successive differentiation of the Gaussian pules e^{-t^2} generates the nth order Hermite polynomial which is defined by Rodrigues' formula as [111]

$$H_n(t)=(-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$$

It is also possible to calculate the Hermite polynomials by the recursion formula[95].

$$H_n(t)=2\mathrm{t}\mathrm{H}_{n-1}(t)-2(n-1)H_{n-2}(t),(1.4.1)$$

with n = I,2,3,...and the start condition

$H_0(t)=1$

Let us also define the weighted Hermite polynomials that is weighted by the generated function e^{-t^2} on the form[95]

$$g_n(t)=H_n(t)e^{-t^2}=(-1)^2\frac{d^n}{dt^n}e^{-t^2}$$

Theorem (1.4.16) If we assume that $f(t)\wedge f\hat{}(t)$ belong to L_1 then the Hilbert transform of $tf(t)$ is given by the equation

$$H(tf(t))=tf\hat{}(t)-\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\,d\tau.$$

The integral is a constant defined by the function $f(t).$ For odd constant equal zero.

Proof. Consider the Hilbert transform of $tf(t)$

$$H(tf(t))=\frac{1}{\pi}p\int_{-\infty}^{\infty}\frac{\tau f(\tau)}{t-\tau}d\tau.$$

The insertion of a new variable $s=t-\tau$ yields

$$H(tf(t))=\frac{1}{\pi}p\int_{-\infty}^{\infty}\frac{(t-s)f(t-s)}{s}ds$$

$$\frac{1}{\pi}p\int_{-\infty}^{\infty}\frac{tf(t-s)}{s}ds$$

$$tH(f(t))-\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)d\tau,$$

and theorem follows.

From Theorem (1.4.16) we have that

$$t$$
$$\eta$$
$$t\varphi_{n-1}(\text{¿})-\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_{n-1}(\text{¿}d\eta)$$
$$H(\varphi_n(t))=\varphi_n(t)=\sqrt{\frac{2(n-1)!}{n!}}\text{¿}$$

$$-(n-1)\sqrt{\frac{(n-2)!}{n!}}\varphi_{n-2}(t)$$

where n=1,2,3..... .The first term $\varphi_0(t)$ can be calculate by using the Fourier transform on equation that is

$$\varphi_0(t)=\pi^{\frac{-1}{4}}e^{\frac{-t^2}{2}}\underset{\Leftrightarrow}{F}\sqrt{2}\pi^{\frac{1}{4}}e^{\frac{-\omega^2}{2}}$$

In the frequency domain we multiply the Hermite function $\varphi_n(t)$ by the Hilbert transform $-isn(\omega)$ and finally we use the inverse Fourier transform to get the Hilbert transform of the Hermit function, that is

$$\varphi_0(t)=\hat{\varphi}_0(t)=\sqrt{2}\pi^{\frac{1}{4}}\int_{-\infty}^{\infty}e^{\frac{-\omega^2}{2}}(-isgn(\omega))e^{i\omega t}\,d\omega.$$
$$H\text{¿}$$

Since sgn($\underset{\omega\text{¿¿}}{e-\text{¿}^{\frac{\omega^2}{2}}}$ is odd we have

$$\hat{\varphi}_0(t)=2\sqrt{2}\pi^{\frac{1}{4}}\int_{0}^{\infty}e^{\frac{-\omega^2}{2}}\sin(\omega t)\,d\omega,$$

Which can be used to drive the rest of Hilbert transform.

It has been found that the error is large for higher order of the Hilbert transformed Hermit functions (e.g. $\hat{\varphi}_5(t)$) when we use the recursive formula .

It is therefore not suitable to use the recursive method to calculate the Hilbert transform for higher order Hermit functions.

Another method to calculate the Hilbert transform by the Hermit functions is to multiply the Hilbert transformer by the spectrum of the Hermit function and use the inverse Fourier transform. No infinite integrals is needed in the calculations of the Hermit functions. Therefore the error does not propagate as in (44).

The series expansion of $f(t)$ can be written as

$$f(t)=\sum_{n=0}^{\infty} a_n \varphi_n(t)$$

where

$$a_n=\int_{-\infty}^{\infty} f(t)\varphi_n(t)dt.$$

If the series expansion $f(t)$ is limited at infinity there will be an error $\varepsilon_N(t),$ that is

$$f(t)=\sum_{n=0}^{N-1} a_n\hat{\varphi}_n(t)+\varepsilon_N(t).$$

This series expansion can also be used to calculate the Hilbert transform

$$Hf(t)=\sum_{n-0}^{N-1} a_n\hat{\varphi}_n(t)+\hat{\varepsilon}_N(t)(1.4.4)$$

And we see that this kind of method is useful for functions where $\hat{\varepsilon}_N(t)$ is small.

To calculate the Hilbert transform of $\varphi_n(t)$ by using the inverse Fourier transform on the product of Hermit function spectra and the Hilbert transformer is a rather demanding method. However, the Hermit function functions never change and we therefore only have to calculate their Hilbert transforms once while $a_{n,}$ which depends on f, represent an easy integral to calculate.

(1.4.2) Discrete Fourier transform

To derive the discrete Hilbert transform we need the definition of the discrete Fourier transform (DFT), That is

$$F[k]=\sum_{n=0}^{n-1} f[n]e^{-i\frac{2\pi}{N}kn}, k=0,1,\ldots\ldots,N-1 \quad (1.4.5)$$

and the inversion formula

$$f[n]=\frac{1}{N}\sum_{k=0}^{N-1} F[k]e^{i\frac{2\pi}{N}kn}, n=0,1,\ldots\ldots N-1. \quad (1.4.6)$$

Where k is the discrete frequency and n is the discrete time. It is easy to prove (4.8) by inserting (4.8) into (4.7). Note that (4.8) defines a periodic function with period N. Let us expand (4.7) in its real and imaginary parts on both sides, thus

$$F[k]=F_{\Re}[k]+iF_{\Im}[k],$$

And

$$\sum_{n=0}^{N-1} f[n]e^{-i\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1} f[n]\cos\left(\frac{2\pi}{N}kn\right)-i\sum_{N=0}^{N-1} F[n]\sin\left(\frac{2\pi}{N}kn\right)$$

The real and imaginary part is now identical as

$$F_{\Re}[k]=\sum_{n=0}^{N-1} f[n]\cos\left(\frac{2\pi}{N}kn\right).$$

$$F_{\Im}[k]=\sum_{n=0}^{N-1} f[n]\sin\left(\frac{2\pi}{N}kn\right),$$

and we conclude that $F_{\Im}=0$ when $k=0\wedge k=N/2.$ As we have seen before the Hilbert transform of the delta pulse $\delta(t)$ give us the Hilbert transformer $1/(\pi t)$ and the Fourier transform of Hilbert transformer gives us the sign shift function

$$\delta(t)\underset{\Leftrightarrow}{H}\frac{1}{\pi t}\underset{\Leftrightarrow}{F}-isgn(\omega)(1.4.7)$$

The discrete analogue of Hilbert transform for even N is therefore given by

$$H[k]=\begin{cases}-i\,for\,k=1,2,\ldots,N/2-1\\ 0\,for\,k=0\wedge N/2\\ i\,for\,k=N/2+1,\ldots,N-2,N-1,\end{cases}$$

and $H[k]$ can be written on the form

$$H[k]=-isgn\left(\frac{N}{2}-k\right)sgn(k). \quad (1.4.8)$$

Here we have used the convention that sgn(0) = 0. The discrete frequency k is called positive in the interval $0<k<N/2$ and negative in the interval $N/2<k<N$ and alternate sign at $N/2.$

The discrete inverse Fourier transform of the discrete Hilbert transform in (4.10) gives us the discrete impulse response in the time domain, for even N, thus

$$h[n]=\frac{1}{N}\sum_{k=0}^{N-1} H[k]e^{i\frac{2\pi}{N}Kn}$$

$$¿\frac{1}{N}\sum_{k=0}^{N-1} -isgn\left(\frac{N}{2}-k\right)sgn(k)e^{i\frac{2\pi}{N}kn}$$

$$¿\frac{2}{N}\sum_{k=1}^{\frac{N}{2}-1} \sin\left(\frac{2\pi}{N}kn\right),(1.4.9)$$

And $h[n]$ can be expressed in closed form as

$$h[n]=\frac{2}{N}\sin^2\left(\frac{\pi n}{2}\right)\cot\left(\frac{\pi n}{n}\right).$$

The function is given by the cotangent function with every second sample $(n=0,2,4,\ldots)$ erased by $\sin^2(¿¿)\frac{2}{\pi n/¿}¿$, see Figure 4.2.

The same derivation for odd *N* is given by

$$H[k]=\begin{cases} -1\, for\; k=1,2,\ldots,(N-1) \\ 0\, for\; k=0 \\ i\, for\; k=N+1/2,\ldots.,N-2,N-1. \end{cases}$$

It is written on the same closed from as in the even case with the difference that there is no cancellation for sgn $(N/2-k)$ that is

$$H[k]=-isgn\left(\frac{N}{2}-k\right)sgn(k)$$

The discrete impulse response for odd N of the Hilbert transformer in (4.9.) is given by the discrete inverse Forier transform of $H[k]$ in (4.13), thus

$$\frac{2\pi}{N}kn$$

¿

¿

$\sin$ ¿

$$h[n]=\frac{2}{N}\sum_{k=1}^{\frac{N-1}{2}} ¿$$

where the closed form of $h[n]$ can be expressed as

$h[n]=\frac{1}{N}\left(\cot\left(\frac{\pi n}{N}\right)-\frac{\cos(\pi n)}{\sin\frac{\pi n}{N}}\right).$ As we mentioned before we do not have the same cancellation for odd N (4.11) as for even N (4.12), instead $h[n]$ is changing sign by odd and even values of n,

The discrete Hilbert transform of sequence $\hat{f}[n]$ is defined by convolution on the form

$$\hat{f}[n]=\sum_{m=0}^{N-1} h[n-m]f[m].$$

If we instead choose to use the DFT algorithm we then have the following relations

$$f[n]\underset{\Rightarrow}{DFT}F[k]\underset{\Rightarrow}{DFT}\hat{F}[k]=-sgn\left(\frac{N}{2}-k\right)sgn(k)F[k]\underset{\Rightarrow}{DFT}-1\hat{f}[n],$$

Where DFT denotes the discrete Fourier transform, DHT denotes the discrete Hilbert transform and DFT^{-1} denotes the discrete inverse Fourier transform. The discrete convolution algorithm (4.15) is faster than the DFT algorithm(4.16) because it involves only a single summation. However the DFT

algorithm may bere placed by a fast Fourier transform algorithm (FFT).

Example (1.4.1) Assume that we want to calculate the discrete Hilbert transform of a sine-wave $f(t)=\cos(2t) with N=10$ samples using the DFT algorithm according to (4.16) where the sampling frequency is $5/\pi$ and the signal frequency is $1/\pi.$ First we need to calculate the discrete Fourier transform (4.7) of $f(t)$. Then we need to use (4.11) (N even) to apply the inverse discrete Fourier transform on the product of discrete Hilbert transform operator $H[k]$ in (4.13) and discrete and the discrete Fourier transform of $f(t)$. The definition of the discrete Fourier transform of $f(t)$.

By using (4.11) we get the Hilbert transform $Hf[n]$ in the time domain ($Neven¿$

$$Hf[n]=\frac{2}{N}\sum_{k=1}^{\frac{N}{2}-1}\tilde{F}[k]\sin\left(\frac{2\pi}{N}kn\right)$$

$$¿\frac{1}{5}\sum_{k=1}^{4}\left(\sum_{n=0}^{9}\cos\left(\frac{2\pi}{5}n\right)e^{-i\frac{\pi}{5}kn}\right)\sin\left(\frac{\pi}{5}kn\right),$$

The fact that this is an harmonic periodic signal (of sin and cos) and that the sampling rate is more than double the signal frequency gives us the exact answer. The implementation in computer code is found in Appendix A.3.

Example (2.4.3) We want to calculate the Hilbert transform of $f(t)=1/(t^2+1)$ with $N=2$ terms of the Hermitian polynomial in (4.6) and use $n=20$ terms of the discrete Fourier transfor in (4.7) and (4.11).

First we need to calculate $\varphi_0(t),\ldots,\varphi_2(t)$ by (4.2) and then $\hat{\varphi}_0(t),\ldots.,\hat{\varphi}_2(t)$ as in e Example 4.2. a_n is possible to calculate with a numerical integration method because the integral converges sufficiently rapid at infinity. In Figure 4.6 we can see the function $f(t)$ and its approximated Hilbert transform $Hy(t)$ compared to the known Hilbert transform $Hf(T)=t/(t^2+1)$.

(1.4.3) Titchmarsh theorem

In analytic number theory, the Brun-Titchmarch theorem, is an upper bound on the distribution of prime numbers in arithmetic progression. It states that, if $\pi(x;q,a)$ counts the number of primes congruent to a modulo q with $p\le x$, then

$$\pi(x;q,a)\le \frac{2x}{\varphi(q)\log(x/q)}$$

For all $q<x$. The result was proven by sieve methods by Montgomery and Vauhan ; an earlier result of Brun and Titchmarsh obtained a weaker version of this inequality with an additional multiplicative factor of 1+ 0(1).If q is relatively small, e.g., $q\le x^{9/20}$, then there exists a better bound:

$$8$$
$$x/q^{3/¿}$$
$$¿$$
$$\varphi(q)\ln ¿$$
$$\pi(x;q,a)\leq\frac{(2+0(1))x}{¿}$$

This is due to Y . Motohashi (1973). He used a bilinear structure in the error term in the Solberg sieve, discovered by himself. Later this idea of exploiting structures in sieving errors developed into a major method in Analytic Number theory, due to H. Iwaniec's extension to combinatorial sieve.

By contrast, Dirichlet's theorem on arithmetic progressions gives an asymptotic result, which may be expressed in the form

$$\pi(x;q,a)=\frac{x}{\varphi(q)\log(x)}\left(1+0\left(\frac{1}{logx}\right)\right)$$

But this can only be proved to hold for the more restricted range $q<(\log x)^{c}$ for constant c: this is the Siegel-Walfisz theorem.

$F(x)$ is the limit as $z\rightarrow x$ of holomorphic function F(Z) such that

$$\int_{-\infty}^{\infty}|F(x+y)|^{2}dx<K$$

If f(z) is the holomorphic function such that for all y, then there is a complex

-valued function F(x) in $L^{p}(R)$ such that F(x+iy) $\rightarrow F(x)$ in the L^{p} norm as y $\rightarrow 0$. Furthermore.

$$F(x)=f(x)-ig(x).$$

Hilbert transform of f does converge almost everywhere to a finite function g such that

$$\int_{-\infty}^{\infty} \frac{|g(x)|^p}{1+x^2} dx < \infty .$$

(1.4.4) Rie mann-Hilbert problems[80]

One of the main boundary value problems of analytic function theory. It can be stated in the simplest case as follows. Let L be a simple smooth closed contour that splits the plane into a bounded interior domain $D^{+¿}$ and the domain $D^{-¿}$ complementary to it, containing the point at infinity . Let two functions $G(t) \wedge g(t)$ be given on $L,$ satisfying a (H-condition) with $G(t) \neq 0$ everywhere on $L.$ It is required to find two functions $\varnothing^{\pm}$ (z), analytic in $D^{\pm}.$ Respectively , continuous up to the contour except for finitely many points $t, k,$ where they may have discontinuities satisfying

$$¿ z - tk \vee ¿^{\alpha k}$$
$$|\varnothing(z)| < \frac{A}{¿}$$

And satisfying on L the boundary condition

$$-¿(t) + g(t),$$
$$+¿(t) = G(t)\varnothing^{¿}$$
$$\varnothing^{¿}$$

Where the function $G(t)$ is called the coefficient of the problem. The integer

$$k = ind\,G(t) = \frac{1}{2\pi}\int d\arg G(t) = \frac{1}{2\pi i}\int d \in G(t)$$

Is called index of the coefficient $G(t)$ and at the same time the index of the Riemann-Hilbert problem.

In the case when $G(t) \wedge g(t)$ are only continuous, but do not satisfy an *H*-condition, the results stated above remain valid, except that here the boundary values of the solutions exist only as the contour is approached along non-tangential paths, and they are not continuous, but $\varnothing^{\pm}(t) \in L_p$ for any $p > o\,; if\,G(t)$ is continuous and $g(t) \in L_{p,}$ then $\varnothing^{\pm}(t) \in L_{p.}$ The most general assumption for the coefficient $G(t)$ under which the Riemann-Hilbert problem has been solved is that it belongs to the class of measurable functions with an additional condition on the value of the jump of the argument; here also $g(t) \in L_{p.}$

Riemann-Hilbert problems with infinite index have been considered, in which simple smooth curves have been chosen for the contours with one or both ends going to infinity. The following cases have been investigated; (1)apolynomial order of growth, when as $|t| \to \infty$ the asymptotic equalities

$$\text{¿}t \vee \text{¿}^{p}$$
$$Ind\,G(t)\ \pm \text{¿}$$

Are satisfied $(0 < p < \infty)$ for the case one infinite end, $(0 < p < 1)$ for both ends infinite); and (2) a logarithmic order of groth, when as $|t| \to \infty$,

$$lnd\,G(t)\ \pm \ln^{\alpha}|t|, 0 < \alpha < \infty.$$

The solution of the Riemann-Hilbert problem on a Riemann surface, and the equivalent problem on fundamental domain of an automorphic function belonging to a group of permutation, has been investigated for automorphic functions of this class. The number of solutions or solvability conditions depends on the index, and in certain (singular) cases also on the genus of the surface or on the fundamental domain[80].

If in condition (*) G is a matrix and $\varnothing^{\pm}$ and gare (n-dimensional) vectors, then there arises the the Riemann-Hilbert problem for a component wise-analytic vector .This is significantly more complicated than the scalar case (n=1) considered above.

Chapter 2

Singular Integral Equations

Sec.(2.1) The Fredholm Integral Equation

The Fredholm integral equation [116,121] of the second kind for a function for a function $\varnothing(x)$ is an equation of the type

$$f(x)=\varnothing(x)-\int_a^b K(x,y)\varnothing(y)\,dy\,(a\le x\le b)$$

where the kernel is of type

$$K(x,y)=\frac{H(x,y)}{|y-x|^{\alpha}}$$

(0 $¿\alpha<1, H\, bounded ¿$

It is well known it can be transformed into a Ferdholm type with a bounded kernel, However, in the important case $\alpha=1$ (in which the integral of the equation must be considered as a Cauchy principal integral) the integral equation differs radically[1] from a Fredholm type with a bounded kernel. That is the kernel (with α =1) becomes infinite at an interior point $x=x_0$ of with the interval of integration ($a,b¿.$ Therefore we call this type the singular integral equations with Cauchy's principal value of an integral.

The purpose of this chapter is to consider this singular type and solve some general case with use of theory of analytic function, in particular, with the finite Hilbert transformations.

The theory of Hilbert transform

$$f(x) \equiv \frac{1}{\pi} p \int_{-\infty}^{\infty} \frac{\varnothing(y)}{y-x} dy = H,[\varnothing(y)]$$

Where *p* denotes the Cauchy’s principal value, discussed by Titchmarsh in his book on Fourier integral.

The finite Hilbert transform

$$f(x) \equiv \frac{1}{\pi} p \int_{-1}^{1} \frac{\varnothing(y)}{y-x} dy = [\varnothing(y)]$$

is less well-known, but it is studied by Tricomi, we quote some of his results which we use later. Consider

$$F(z) \int_{-1}^{1} \frac{\varnothing(t)}{t-z} dt$$

Where z is generally a complex number lying outside the segment (1,1) on the real axis. This transform changes a real function of class *L*, (*p* $¿1¿,$ where *L*, is Lebsgue class of *p*, into a single-valued analytic function which is regular in the whole z-plane cut along the segment (-1, 1) of the real axis, vanishes at infinity, and satisfies the condition

$$x-iy\vee¿^{p}dx<k$$
$$\varnothing ¿$$
$$¿¿$$
$$\int_{-\infty}^{\infty} ¿$$

(*p*>1)

For all values of y, k=cons,>0. Then it is shown that

$$\frac{1}{2}\left[\varnothing(x+i\varepsilon)+\varnothing(x-i\varepsilon)\right]=\frac{1}{2\mathrm{i}}\left[F(x+i\varepsilon)-F(x-i\varepsilon)\right].$$

In other words, since $F(x+i\varepsilon)\wedge F(x-i\varepsilon)$ are conjugate complex numbers for $\varnothing$ real, we can state that

$$x-i\varepsilon$$
$$\varnothing(x+i\varepsilon)+\varnothing ¿ \qquad (-1<x<1)=0$$
$$\Im F(x+i\varepsilon)=\frac{1}{2}¿$$

$$(x<-1\vee x>1),$$

That is we have discontinuities across the real axis $(-1<x<1)$.

From the above relation

$$\Re F(x+i\varepsilon)=H_{x}\left[\Im F(y+i\varepsilon)\right]=T_{x}\left[\varnothing^{¿}(y)\right]=T_{x}\left[\varnothing(y)\right]$$

Hence

$$F(x+i\varepsilon)=T_{x}\left[\varnothing(y)\right]+i\varnothing^{¿}(x)(-1<x<1),$$

where

$$\varnothing^{¿}(x)=\frac{1}{2}\left[\varnothing(x+i\epsilon)+\varnothing(x-i\varepsilon)\right],$$

$H_{x,}T_{x}$ are the operator of Hilbert transform, finite Hilbert transform respectively and we use the famous Hilbert relations between the real and imaginary part of an analytic function . From this there follows the formula which for $\varnothing$ continuous can be written

$$F(x+i\varepsilon)=\frac{1}{\pi}p\int_{-1}^{1}\frac{\varnothing(y)}{y-x}dy\pm i\varnothing(x)$$

In the operator form it is well-known formula

$$\frac{1}{x\pm i\varepsilon}=p\frac{1}{x}\pm ix\delta(x).$$

(2.1.2) Singular Integral Equation of second kind with singular kernel.

The singular kernel given as

$$K(x,y)=\frac{H(x,y)}{(x-y)}(2.1)$$

We can expand this in Taylor series about *x* as follows

$$H(x,y)=H(x,x)+(y-x)H_{,}^{'}(x,x)+\frac{1}{2!}(y-x)^{2}H_{,,}^{''}(x,x)+..$$

So that

$$K(x,y)=\frac{H(x,x)}{y-x}+K^{¿}(x,y)(2.2)$$

Where $K^{¿}(x,y)$ is bounded.

Consequently the main problem in studying integral equation with kernels of type (2) is solving the standard equation

$$a(x)\varnothing(x)\lambda p\int_{-1}^{1}\frac{\varnothing(y)}{y-x}dy=f(x)(2.3)$$

$$a(x)=\frac{1}{H(x,x)}$$

by means of Laplace transform. We use the finite Hilbert transform here.

If we put

$$F(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{\varnothing(y)}{y-z}dy$$

Then by result of the previous section, we have

$$F(x+i\varepsilon)-F(x-i\varepsilon)=2\mathrm{i}\varnothing(x)$$

$$F(x+i\varnothing)+F(x-i\varnothing)=\frac{2}{\pi}p\int_{-1}^{1}\frac{\varnothing(y)}{y-x}dy\}$$

Where we assumed that $\varnothing(x)$ is continuous in open interval (-1,1). Consequently equation (3) assumed the algebraic form

$$[a(x)-\lambda\tau i]F(x+i\varepsilon)-[a(x)+\lambda\tau i]F(x-i\varepsilon)=2\mathrm{if}(x).$$

This equation can be simplified by setting

$$F(z)=e^{\tau(z)}U(z)$$

Provided that the function $T(z)$ satisfies the condition

$$[a(x)-\lambda\tau i]e^{\tau(z+i\varepsilon)}=[a(x)+\lambda\tau i]e^{\tau(x-i\varepsilon)}$$

we obtain

$$U(x+i\varepsilon)-U(x-i\varepsilon)=\frac{2\mathrm{if}(x)}{a(x)-\lambda\tau i}e^{-\tau(x+i\varepsilon)}v=\frac{2\mathrm{if}(x)}{a(x)+\lambda\tau i}e^{-\tau(x-i\varepsilon)}$$

From which, if we consider the geometric mean of the two expressions for the difference on the left (which are equal), it follows that

$$U(x+i\varepsilon)-U(x-i\varepsilon)=\frac{2\mathrm{if}(x)}{[a^2(x)+\lambda^2\tau^2]\frac{1}{2}}\exp\left\{\frac{-1}{2}[T(x+i\varepsilon)+T(xi\varepsilon)]\right\}(2.4)$$

How, in order to determine the function T(z), we observe that from (2.4) we have

$$T(x+i\varepsilon)-T(x-i\varepsilon)=\log\frac{a(x)+\lambda\tau i}{a(x)-\lambda\tau i}=2\mathrm{i}\tan^{-1}\frac{\lambda\tau}{a(x)}.$$

Consequently, we can put

$$T(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{\theta(t)}{t-z}dt$$

with

$$\theta(x)=\tan^{-1}\frac{\lambda\tau}{a(x)}.$$

It follows from this that

$$\frac{1}{2\mathrm{i}}\left[T(x+i\varepsilon)-T(x-i\varepsilon)=\theta(x)=\tan^{-1}\frac{\lambda\tau}{a(x)}\right]$$

in accordance with (11).

On the other hand, we have

$$\frac{1}{2}[T(x+i\varepsilon)+T(x-i\varepsilon)]=\frac{1}{\pi}p\int_{-1}^{1}\frac{\theta(t)}{t-x}dt,$$

Hence, equation (10) becomes now

$$U(x+i\varepsilon)-U(x-i\varepsilon)=\frac{2\mathrm{i}}{\left[a^2(x)+\lambda^2\pi^2\right]\frac{1}{2}}e^{-\tau(x)}f(x)$$

where

$$\tau(x)=\frac{1}{\pi}p\int_{-1}^{1}\frac{\theta(t)}{t-x}dt$$

And can be satisfied by the function

$$U(z)=\frac{1}{\pi}\int_{-21}^{1}\frac{e^{-e(t)}f(t)}{\left[a^{2(x)+\lambda^2\pi}\right]\frac{1}{2}}\frac{t-z}{dt}$$

Finally, we determine $\varnothing$ by using the first equation view gives us

$$2\mathrm{i}\varnothing(x)=e^{\tau(x+i\varepsilon)}U(x+i\varepsilon)-e^{\tau(x-i\varepsilon)}U(x-i\varepsilon)$$

$$¿e^{\tau(x)+i\theta(x)}\left[\frac{1}{\pi}p\int_{-1}^{1}\frac{e^{-\tau(y)}f(y)}{\left[a^2(y)+\lambda^2\pi^2\right]\frac{1}{2}}\frac{dy}{y-x}+i\frac{e^{-\tau(x)}f(x)}{\left[a^2(x)+\lambda^2\pi^2\frac{1}{2}\right]}\right]$$

-

-

$$a^2(y)+\lambda^2\pi^2 \vee ¿$$
$$¿$$
$$\frac{1}{2}$$
$$[¿\frac{dy}{y-x}-i\frac{e^{-\tau(x)}f(x)}{[e^2(x)+\lambda^2\pi^2]\frac{1}{2}}¿]$$
$$\frac{e^{-\tau(y)}f(y)}{¿}$$
$$\frac{1}{\pi}p\int_{-1}^{1}¿$$
$$e^{\tau(x)+i\theta(x)}¿$$

After making some implications,

$$\varnothing(x)=\frac{a(x)f(x)}{a^2(x)+\lambda^2\pi^2}+\frac{\lambda e^{\tau(x)}}{[a^{2(x)}\lambda^2\pi^2]}p\int_{-1}^{1}\frac{e^{-\tau(y)}f(y)}{[a^2(y)+\lambda^2\pi^2]}\frac{dy}{y-x}$$

Where, according to (14) and (13),

$$\tau(x)=T_x[\theta(y),]\theta(y)=\tan^{-1}\frac{\lambda\pi}{a(y)}$$

(2.1.3) Singular Integral is an Integral Operator

In mathematics, singular integrals are central to harmonic analysis and intimately connected with the study of partial differential equations. Broadly speaking a singular integral is an integral operator[161]

$$T(f)(x)=\int K(x,y)f(y)dy,$$

whose kernel function $k:R^n\times R^n\to R^n$ is singular along the diagonal x = y.

(2.1.4) The Hilbert transform H $(f(x))$.

The archetypal singular integral operator is the Hilbert transform H. It is given by convolution against the kernel $k(x)=1/(\pi x)\, for\; x\in R$. More precisely.

$$H(f(x))=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{1}{x-y}f(y)\,dy.$$

The most straight forward higher dimension analogues of these are the Riesz transforms[4], which replace $k(x)=1/\pi$

With

$$k_i(x)=\frac{x_i}{|x|^{n+1}}$$

where $i=1,\ldots,n \wedge x_i$ is the i-th component of $x\in R^n$. All of these operators are bounded on L^p and satify weak-type (1,1) estimates.

(2.1.5) Singular Integral of Convolution Type

A singular integral of convolution type is an operator T defined by convolution with kernel K that is locally integrable on $R^n\setminus\{0\}$ in the sense that

$$T(f)(x)=\lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon}K(x-y)f(y)\,dy.$$

Suppose that the kernel satisfies:

(i)The size condition on the Fourier transform of K

$$\widehat{K}\in L^{\infty}(R^n)$$

(ii) The smoothness for some C >0

$$\int_{|X|<2|y|} |K(x-y)-K(x)|\,dx \le C.$$

Then be shown that T is bounded on $L^p(R^n)$ and satisfies a weak-type (1,1) estimate.

Property 1. Is needed to ensure that convolution (1) with the tempered distribution p.v. k. given by the principal value integral

$$p.v.\,K[\varnothing]=\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \varnothing(x)K(x)\,dx$$

Is a well-defined Fourier multiplier on $L^{2.}$ Neither of the properties 1.or 2. Is necessarily easy to verify, and variety of sufficient conditions exist. Typically in applications. One also has a cancellation condition

$$\int_{R_1<|x|<R2} K(x)\,dx=0,\forall R_1 R_2>0$$

Which is quite easy to check it is automatic, for instance, if in addition, one assumes 2 and the following size condition

$$\int_{R<|X|<2R} |K(x)|\,dx \le C,$$

Then it can be shown that 1. follows.

A condition of a kernel K can be used:

(i) $K \in C^1(R^n\{0\} ¿¿$

(ii) $|\nabla K(x)| \le \frac{C}{x^{n+1}}$

Observe that these conditions are satisfied for Hilbert and Riesz transforms.

Sec.(2.2) Finite Part Integral and Hypersingular Kernels

(2.2.1) Cauchy Type Singular Integral Equations

In general, the solution to the crack problems in the linear elastic fracture mechanics often leads to a system of Cauchy type singular integral equations

$$\frac{a_i}{\pi}\int_c^d \frac{\varnothing_i(x)}{t-x}dt+\sum_j^J \int_c^d k_{ij}(x,t)dt+b_i\varnothing_i(x)=p_{i(x)}$$

Where $c<x<d, a_i b_i (i=1,2,\ldots..)$ are real constants and the kernel $K_{ij}(x,t)$ are bounded in the closed domain $(x,t)\in[c,d]\times[c,d]$.

Function $p_{i(x)}$ is known and given by the boundary condition. Functions $\varnothing_i(x)$ are the unknowns of problems, also called by the density functions However, if the unknown density function is chosen to be the placement, say $\omega_i(x)$, then the order of singularity increases. Thus, a formulation of hypersingular integral equations is made.

Shear modulus $G(x)=G_0 e^{\beta x}$

then the governing a partial differential equation (PDE) in terms of the z component of the displacement victor $\omega¿^{x,y}$) is

$$\nabla^2\omega(x,y)+\beta\frac{\partial\omega(x,y)}{\partial x}=0 \quad (2.1)$$

With mixed boundary conditions

$$\omega(x,0)=0\, x\notin[c,d].$$

$$+¿ \; x,0^{¿} \; ¿ \; \sigma_{xy}¿ \quad (2.2)$$

Where $p(x)$ is the traction function along the crack surfaces (c, d). By a process of Fourier integral transform PDE can be reduced to hypersingular integral equation.

(2.2.2) Hadamard Finite Part (HFP)

Integral was first introduced by Jacques Hadamard to solve some linear partial differential equation (PDE) which can be considered as a generalization of Cauchy principal value (CPV) integral[2,14].

(2.2.3) HFP and Cauchy Principal value Integrals

HFP integral is a generalization of CPV integral, thus let us look at CPV integral first :

CPV is equations that involve integrals of the type

$$\int_c^d \frac{\varnothing(t)}{t-x} dt, |x|<1 (2.3)$$

Is not integrable in the ordinary (Riemann or Lebesgue) sense because of the Kernel $1/(t-x)$

Is not integrable over any interval that includes the point $t=x.$ Thus, it is regularized by CPV integral:

$$\frac{\varnothing(t)}{t-x}dt+¿$$

$$\int_{c}^{x+\varepsilon} ¿$$

$$¿$$
$$¿$$
$$¿$$

$$\frac{\varnothing(t)}{t-x}dt := \lim_{c\to 0} ¿$$

$$\int_{c}^{d} ¿$$

Where $c<x<d.$ Notice that the ε -neighborhood about the singular point $x-t$ must be symmetric, and it is how CPV works out for canceling off the singular(xrity).

For the existence of the CPV Integral, the function $\varnothing(x)$ needs to be at least Holder continuous on (c,d) that is $\varnothing(x)\in C^{0,\alpha}(c,d).$ This requirement of regularity can be easily checked by following manipulation:

$$\int_{c}^{d}\frac{\varnothing(t)}{t-x}dt=\lim_{c\to o}\left\{\int_{|t-x|\geq\varepsilon}^{\square}\frac{\varnothing(t)-\varnothing(x)}{t-x}dt+\varnothing(x)\right\}$$

$$¿\int_{c}^{d}\frac{\varnothing(t)-\varnothing(x)}{t-x}dt+\varnothing(x)\int_{c}^{d}\frac{dt}{t-x}(2.5)$$

Thus, for any $\varnothing\in C^{a,\propto},\propto>0,$ the first integral on the right side is an ordinary Riemann integral and the second integral is

$$\int_{c}^{d}\frac{dt}{t-s}=\log\frac{d-t}{x-c},c<x<d.$$

Although Cauchy principal value integral is defined for an interior point in (c,d) above, it can be evaluated separately on both sides of the end point:

$$\int_c^x \frac{\varnothing(t)}{t-x} := \lim_{\varepsilon \to 0}\left\{\int_c^{x-\varepsilon} \frac{\varnothing(t)}{t-x}\,dt = \varnothing(x)\,ln\varepsilon\right\}$$

where $x>c,$ and

$$\int_x^d \frac{\varnothing(t)}{t-x}\,dt := \lim_{\varepsilon \to 0}\left\{\frac{\varnothing(t)}{t-x}\,dt + \varnothing(x)\,ln\varepsilon\right\}.$$

`**And CPV** integral does not work for a higher singularity. For instance, consider $\varnothing(t)=1 \wedge x=0 \in ¿$

$$t-x$$
$$¿$$
$$¿$$
$$¿2$$
$$¿$$
$$\frac{\varnothing(t)}{¿}$$
$$\int_c^d ¿$$

That is ,

$$\int_c^d \frac{dt}{t^2} c<0<d.$$

$This\,integral$ is not convergent, neither does the principal value exist, since

$$\overset{¿}{\overset{/\,c,d \vee -c,c}{\int_{¿} ¿¿ \square \frac{dt}{t^2}}} = \lim_{c \to 0}\left(\frac{1}{c} - \frac{1}{d} + \frac{2}{\varepsilon}\right)$$

is not finite. Hadamard finite part integral is defined by disregarding the finite part, $2/\varepsilon,$ and keeping the finite part[8], i.e

$$\int_c^d \frac{dt}{t^2}=\frac{1}{c}-\frac{1}{d}(2.7)$$

D efinition (2.2.1)[(Hadamard Finite Part Integral)

Let $c>0, \wedge denote$

$$F(c,x)=\int_{\text{¿}}^{|c,d|(x-\varepsilon,x+\varepsilon\text{¿}} \text{¿¿¿❑} f(t,x)\,dt, c<x<d.$$

Where the singularity at the point $t=x. If\ F(\varepsilon,x)$ is decomposed into

$$F(\varepsilon,x)=F_0(\varepsilon,x)+F_1(\varepsilon,x),$$

And

$$\lim_{\varepsilon\to 0} F_0(\varepsilon,x)<\infty. \lim_{\varepsilon\to 0} F_1(E,x)<\infty.$$

Then the finite part integral is defined by keeping the finite part i.e

$$\int_c^d f(t,x)\,dt=\lim_{\varepsilon\to 0} F_o(\varepsilon,x)$$

Notice that HFP integral can be considered as generalization of the CVP integral in the sense that if the principal value integral exists, then they give the same result.

Definition (2.2.2) If $\varnothing(x)\in C^{1,\alpha}(c,d)$, then

$$\int_c^d \frac{\varnothing(t)}{(t-x)^2}dt$$

$$¿\lim_{\varepsilon\to 0}\left[\int_c^{x-\varepsilon}\frac{\varnothing(x)}{(t-x)^2}dt+\int_{x+\varepsilon}^{d}\frac{\varnothing(t)}{(t-x)^2}dt-\frac{2\varnothing(x)}{\varepsilon}\right](2.8)$$

Following observation may help to understand Definition 2 for HFP. By a step of integration by-parts , the first integral under the limit $\varepsilon\to 0$ (8) can be written as

$$t-x$$

$$\int_c^{x-\varepsilon}\frac{\varnothing(t)}{(t-x)^2}dt=\frac{\varnothing(x-\varepsilon)}{\varepsilon}-\frac{c}{c-x}+\int_c^{x-\varepsilon}\frac{\varnothing'(t)}{¿}dt.¿$$

Similarly,

$$\int_{x+\varepsilon}^{d}\frac{\varnothing(t)}{(t-x)^2}dt=\frac{\varnothing(x+\varepsilon)}{\varepsilon}-\frac{d}{d-x}+\int_{x+\varepsilon}^{d}\frac{\varnothing'(t)}{(t-x)}$$

Thus, the term $-2\varnothing(x)/\varepsilon$ in (8) will kill the singularity [$\varnothing(x-\varepsilon)+\varnothing(x+\varepsilon)¿/\varepsilon,$ and under the assumption that $\varnothing(x)\in C^{1,\alpha}(c,d)$ Definition 2 indeed takes the finite part of the integral according to Definition 1.

Another direction of viewing to Definition 2 is by taking direct differentiation d/dx to(5) with Leibnitz's rule, i.e

$$\frac{\varnothing(t)}{t-¿}$$

$$\int_c^d ¿$$

$$¿\lim_{\varepsilon \to o} \frac{d}{dx}\left[\int_{c}^{x-\varepsilon} \frac{\varnothing(t)}{t-x} dt + \int_{x+\varepsilon}^{d} \frac{\varnothing(t)}{t-x}\right] = \lim_{\varepsilon \to 0}\left[\int_{c}^{x-\varepsilon} \frac{\varnothing(t)}{(t-x)^2} dt + \int_{x+\varepsilon}^{d} \frac{\varnothing(t)}{(t-x)^2} dt - \frac{\varnothing(x-\varepsilon)+\varnothing(x+\varepsilon)}{\varepsilon}\right] (2.9)$$

Proposition (2.2.3) $if\ \varnothing(x) \in C^{1,\propto}(c,d),$ then

$$\int_{c}^{d} \frac{\varnothing(t)}{(t-x)^2} dt - \frac{dy}{dx}\int_{c}^{d} \frac{\varnothing(t)}{(t-x)} dt (2.10)$$

Alternatively. One can define finite part integrals[28,30] $by\ requation(10) \wedge deduce\ Definition\ 2\ as\ property.$

For general HFP integrals can be defined recursively as follws.

Definition (2.2.4) (Finite part integral) Let

$$1+¿=¿\, p>1\, L^{P}[c,d]$$
$$L^{¿}$$

For any $1+¿, c<x<d, \wedge n=1,2,3\ldots.$ $\varnothing \in C^{n,\propto}(c,d) \cap L^{¿}$

$$\int_{c}^{d} \frac{\varnothing(t)}{(t-x)^{n+1}} dt = \frac{1}{n}\frac{d}{dt}\int_{c}^{d} \frac{\varnothing(t)}{(t-x)^2} dt (2.11)$$

with

$$\int_{c}^{d} \frac{\varnothing(t)}{t-x} dt \coloneqq \int_{c}^{d} \frac{\varnothing(t)}{t-x} dt$$

By means of (2.5) and the definition of finite part integrals,

(2.2.4) Hypersingular Kernels

For the derivation of hypersingular kernels, we use three basic ingredients (i)Finite part integrals

(ii)Identity

$$i^n \frac{d^n}{dy^n}\left[\frac{1}{y-i(t-x)}\right]=\frac{d^n}{d^n}\left[\frac{1}{y-i(t-x)}\right](2.8)$$

(iii)Plemelj formulas

$$\frac{\varnothing(t)}{t-x}dt+\pi i\,\varnothing(x),\varnothing\in L^{1+¿}$$

$$\lim_{\varepsilon\to 0}\int_c^d \frac{\varnothing(t)}{(t-x)+i\varepsilon}dt=\int_c^d ¿$$

The key point of identity, (8) is that it allows one to switch the differentiation from $\frac{dy}{d/¿d/dy¿}$, and vice versa, HFP integral has been defined and addressed in previous section, for the sake of completeness , we shall briefly address Plemelj formulas.

(iii)Plemelj Formulas

In general, the Cauchy principal value type of integrals

$$\int_c^d \frac{\varnothing(t)}{t-x}c<x<d$$

Is evaluated indirectly by to using complex Function theory [7, 9]. Define

$$\varnothing(z)=\int_c^d \frac{\varnothing(t)}{t-z}dt.$$

With z not on the integration contour. The principal value is then recovered by sending z to the point x on the interval (c, d), and the result is different as $z\to x$ from above and below. Say, define

$$\varnothing(x+i|y|).\varnothing^{-¿(x)=\lim_{y\to 0}\varnothing(x-i|y|).}$$
$$+¿(x)=\lim_{y\to 0}¿$$
$$\varnothing^{¿}$$

Then the limits are

$$+¿(x)=\int_c^d \frac{\varnothing(t)}{t-x}dt+i\pi\varnothing(x)(2.9)$$

$\varnothing^{¿}$

and

$$-¿(x)=\int_c^d \frac{\varnothing(t)}{t-x}dt-i\varnothing\pi(x)(2.10)$$

$\varnothing^{¿}$

Equations above sometimes called Sokhertski formulas. It is (2.9) that we will be using in the derivation of hypersingular kernels. Notice that $\varnothing(x)$ can be recovered from Plemelj formulas. i.e

$$+¿(x)-\frac{\varnothing^{-¿(x)}}{2\pi i}$$

$\varnothing^{¿}$

$\varnothing(x)=¿$

(2.2.4.1) Rising of Hypersingular Kernels

To demonstrate how the hypersingular kernels arise, we go through Fourier transform $\omega(x,y)$

To be expressed as

$$\omega(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\propto(\xi)e^{\lambda(\xi)y}\right]e^{ix\xi}d\xi(2.11)$$

where

$$[\lambda(\xi)]^2=\xi^2+i\beta\xi, \qquad (2.12)$$

to satisfy the far field boundary condition

$\lim_{y\to\infty}\omega(x,y)=0. we\,choose\,the\,root\,\lambda(\xi)$ to be the non-positive real part,

$$\sqrt{\xi^{4+¿\beta^2\xi^2}+\xi^2-\frac{1}{\sqrt{2}}sgn(\beta\xi)}\sqrt{\sqrt{\xi^4}+\beta^2\xi^2-\xi^2}(2.13)$$

$$\frac{-1}{\sqrt{2}}\sqrt{¿}$$

Where the signum function sgn(.) is defined as

$$sgn(\eta)=\begin{cases}1,\eta>0\\ o,\eta=0\\ -1,\eta<0\end{cases} \qquad (2.14).$$

As the limit of $\begin{matrix}+¿ is taken,\\ y\rightarrow 0^{¿}\end{matrix}$

$$\begin{matrix}+¿\\ x,0^{¿}\\ ¿\\ \omega¿\end{matrix}$$

$$¿\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\alpha(\xi)e^{-ix\xi}d\xi(2.15)$$

That is, $\begin{matrix}+¿\\ x,0^{¿}\\ ¿\\ \omega¿\end{matrix}$ inverse Fourier transform of $\alpha(\xi)$. By inverting Fourier transform one optains

$$\begin{matrix}x,0^{+¿}e^{-ix\xi}dx\\ \omega¿\\ \alpha(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}¿\end{matrix}$$

$$\begin{matrix}+¿\\ t,0^{¿}\\ ¿\\ \omega¿\\ ¿\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}¿\end{matrix}$$

Where the first boundary condition and a change of dummy variable $(x \to t)$ have been applied.

Defining

$$K(\xi, y) = \lambda(\xi) e^{\lambda(\xi) y} (2.17)$$

and using the second boundary condition in 2 one reaches that for $c < x < d$

$$+¿$$
$$t, 0^{¿}$$
$$¿$$
$$\omega ¿$$
$$y \to 0^{+¿} \frac{G(x)}{2\pi} \int_{c}^{d} ¿$$
$$\lim_{¿} ¿$$

$$+¿$$
$$t - 0^{¿}$$
$$¿$$
$$\omega ¿$$
$$y \to 0^{+¿} \frac{G(x)}{2\pi} \int_{c}^{d} ¿$$
$$¿ \lim_{¿} ¿$$

$$y \to 0^{+¿} \int_{-\infty}^{\infty} k(\xi, y) e^{(t-x)\xi} d\xi (2.19)$$
$$¿ \lim_{¿} ¿$$

Let $\begin{matrix} +¿ \\ \xi, 0^{¿} \\ ¿ \\ K(\xi) \equiv K ¿ \end{matrix}$ and by a step of decomposition

$$K(\xi) = [K(\xi) - K_{\infty}(\xi)] + K_{\infty}(\xi). \qquad (2.20)$$

One obtains a closed form expressions of

$$K_{\infty}(\xi) = -|\xi| - \frac{i\beta}{2} \frac{|\xi|}{\xi} \qquad (2.21)$$

This $K_\infty(\xi)$ gives rise to quadratic hypersingular and Cauchy singular kernels by the following

$$[|\xi|e^{-|\xi|y}]e^{i(t-x)\xi}d\xi\ y \underset{\rightarrow}{\rightarrow} 0^{+¿}\frac{-2}{(t-x)^2}(2.22)$$

$$\int_{-\infty}^{\infty} ¿$$

$$i\frac{|\xi|}{\xi}e^{-|i(t-x)\xi|}d\xi\ y \underset{\rightarrow}{\rightarrow} 0^{+¿}\frac{-2}{t-x}(2.23)$$

$$\int_{-\infty}^{\infty} ¿$$

(2.3.4) Higher Order Hypersingular Kernels For more general and higher order of hypersingular kernels, they can be derived by observing that

$$k_n(t-x,y)\ :=\frac{1}{2\pi}\int_{-\infty}^{\infty} i^n|\xi|^n\frac{|\xi|}{i\xi}e^{-|\xi|y+i(t-x)\xi}d\xi$$

$$y-i(y-i(t-x))^{-1}$$

$$\frac{d^n}{dy^n}¿$$

$$¿\sqrt{\frac{2}{\pi}}(-i)^n\Im ¿$$

$$¿(-1)^n\sqrt{\frac{2}{\pi}}\Im\frac{d^n}{dx^n}(y-i(t-x))^{-1}$$

$$t-x+iy$$

$$\frac{d^n}{dx^n}¿$$

$$¿(-1)^n\sqrt{\frac{2}{\pi}}\Re[¿¿-1]$$

thus

$$y \to 0^{+¿} \int_{-1}^{1} k_n(t-x, y)\varnothing(t)\,dt$$
$$\lim_{¿} ¿$$

$$t-x+iy$$
$$\frac{d^n}{dx^n} ¿$$
$$y \to 0^{+¿}(-1)^n \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \Re[¿¿-1]\varnothing(t)\,dt$$
$$¿\lim_{¿} ¿$$

$$y \to 0^{+¿} \int_{-1}^{1} (t-x+iy)^{-1} \varnothing(t)\,dt$$
$$\frac{d^n}{dx^n} \lim_{¿} ¿$$
$$¿(-1)^n \sqrt{\frac{2}{\pi}} \Re ¿$$

$$¿-1^n \sqrt{\frac{2}{\pi}} \frac{d^n}{dx^n} \int_{-1}^{1} \frac{\varnothing(t)}{(t-x)^{n+1}}\,dt$$

$$¿n!(-1)^n \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \frac{\varnothing(t)}{(t-x)^n}\,dt,$$

where the Plemelj formula and the definition of finite part integrals have been used. Note that, when n is an odd integer,

$$i^n \xi^n \frac{|\xi|}{i\xi} e^{-|\xi| y + i(t-x)\xi}\,d\xi$$
$$¿$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ¿$$

$$¿-\sqrt{\frac{2}{\pi}} \Im\left[\frac{d^n}{dx^n}(t-x+iy)^{-1}\right]$$

Thus we have

$$y \to 0^{+¿} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i^n \xi^n \frac{|\xi|}{i\xi} e^{-|\xi| y + i(t-x)\xi} d\xi$$

$$¿$$

$$dt \varnothing(t) \lim_{¿} ¿$$

$$\int_{-1}^{1} ¿$$

$$¿ - \sqrt{\frac{2}{\pi}} \Im \left[\frac{d^n}{dx^n} (t - x + iy)^{-1} \right] - \sqrt{2\pi} \frac{d^n}{dx^n} \varnothing(x),$$

where the Plemelj formula is used again.

Chapter 3

Calderon-Zygmund Operators

Sec.(3.1): General Calderon-Zygmund Operators

This section study of general Calderon-Zygmund Operators, that operators given formally as[76]:

$$T(f)=\int K(x,y)f(y)dy$$

For an appropriate kernel k let us quickly review what we used in order to show that the Hilbert transform H is of weak type (1.1) and strong type (2.2).

First of all we essentially used the fact that the linear operator H is defined on L^2

This information of linear operator *H* was used in two different ways. First of

all the fact that $H \, is \, defined \, on \, L^2$ means that it is defined on a dense sub space of L^p for every $1 \leq p < +\infty$.

Furthermore, the boundedness of the Hilbert transform on L^2 allowed us to treat the set $\{|H(g)| > \lambda\}$ where g is the 'good part' in the Calderon-Zygmund decomposition of function *f*.[1]

Secondly, we used the fact that there is a specific representation of the operator H of the form

$$H(f)(x)=\int K(x.y)f(y)dy$$

Whenever $f \in L^2$ and has compact support and $x \notin supp(f)$. For the Hilbert transform we had that, the kernel k is given as:[3]

$$k(x.y)=\frac{1}{x-y}$$

We used the previous representation and the formula of k to prove a sort of restricted L^1 boundedness of H on functions which are localized and have mean zero. This, in turn, allowed us to treat the 'bad part' of the Calderon – Zegmund decomposition of $f.$ That what we really need for k is a Holder type condition. Note as well that for the Hilbert transform we first proved the L^p bounds for $1<p<2$ and then the corresponding boundedness for $2<p<\infty$ follwoed by the fact that H is essentially self-adjoint.

Sec. (3.2) Singular kernels and Calderon-Zygmund Operators

We will now define the class of Calderon-Zygmund operators in such a wa that we will be able to repeat the schedule used for the Hilbert transform. We begin by defining an appropriate class of kernels $k,$ name the singular kernels[4].

Definition(3.1.1).[4] **(Singular or Standard Kernels)** A singular or standard kernel is a function $k: R^{\} \times \{R\} ^ \{} \to C,$ defined away from the diagonal $x=y,$ which satisfies the decay estimate

$$|K(x,y)| \lesssim_n |x-y| ¿^{-¿^n} \quad (3.1.1)$$

For $x \neq y$ and the holder-type regularity estimates

$$\left|K(x,y_1)-K(x,y)\right| \lesssim_{n.\sigma} \frac{|y-y_1|^{\sigma}}{|x-y|^{n+\sigma}} \; if \; |y-y_1| < \frac{1}{2}|x-y| \quad (3.1.2)$$

and

$$\left|K(x_1,y)-K(x,y)\right| \lesssim_{n.\sigma} \frac{|x-x_1|^{\sigma}}{|x-y|^{n+\sigma}} \; if \; |x-x_1| < \frac{1}{2}|x-y| \quad (3.1.3)$$

for some Holder exponent $0<\sigma\leq 1.$

Example (3.1.2): let $K: R\times R\to R$ be given as: $K(x,y)=(x-y)^{-1}$ for $x,y\in R$ with $x\neq y$. Then K is a singular kernel. Observe that

K is the singular kernel associated with the Hilbert transform.

Example (3.1.3): Let $K: R^n\times R^n\to R$ be given as

$$\frac{x-y}{|x-y|} ¿ |x-y|^{-n}$$
$$K(x,y)=\Omega ¿$$

Where $\Omega: S^{n-1}\to C$ is a Holder continuous function :

$$\left|\Omega(x')-(y')\right| \lesssim_{n,\sigma} |x'-y'|^{\sigma},$$

for some $0<\sigma\leq 1.$ Then K is a singular kernel.

Example (3.1.4): Let $K: R^n\times R^n\to C$ satisfy the size estimate

$$y \vee ¿^{-n},$$
$$|K(x,y)| ¿ n \vee x - ¿$$

and the regularity estimates:

$$y \vee ¿^{-(n+1)},$$
$$y \vee ¿^{-(n+1)}, \left|\nabla_y K(x, y) \lesssim_n \right| x - ¿$$
$$\left|\nabla_x K(x, y) \lesssim_n \right| x - ¿$$

away from the diagonal $x - y,$ then K is a singular kernel. In particular, the kernel $K: R^n \to R^n \to C$ given as:

$$y \vee ¿^{-n}$$
$$K(x, y) = ¿\, x - ¿$$

is a singular kernel since the gradient of K is of the order

$$y \vee ¿^{-(n+1)}.$$
$$¿\, x - ¿$$

Thus the estimates (3.1.2) and (3.1.3) are consistent with (3.1.1) but of course do not follow from it.

Remark (3.1.5): The constant $\frac{1}{2}$ appearing in (3.1.2), (3.1.3) is inessential. The condition are equivalent with the corresponding conditions where $\frac{1}{2}$ is replaced by any constant between zero and one.

We are now ready to define Calderon-Zygmund operators.

Definition (3.1.6) (Calderon-Zygmund operators) A Calderon-Zygmund operator (in short CZO) is a linear operator $T: L^2(R^n) \to L^2(R^n)$ which is bounded on $L^2(R)^n$:

$$\|T(f)\|_{L^2 R^n} \lesssim_{T.n} \|f\|_{L^2(R)^n} \quad \text{for all} \quad \overset{R^n}{f \in L^2 ¿} \;)$$

and such that there exist a singular kernel K for which we have

$$T(f)(x)=\int_{R^n} k(x,y) f(y)\, dy.$$

for all $f \in L^2(R^n)$ with compact support and $x \notin$ supp (f) .

Remark(3.1.7) : Note that the integral **:**

$$\int k(x,y) f(y)$$

converge absolutely whenever $f \in L^2(R^n)$ has compact support and x lies outside the support of f indeed,

$$|k(x,y)||f(y)|\, dy \le \left(\int_{y \notin supp(f)} |k(x,y)|^2 dy \right)^{\frac{1}{2}} \| f \|_{L^2(¿¿n)}$$

R

$\int_{R^n} ¿$

$$\le \left(\int_{|x-y| \ge \delta} \frac{1}{|x-y|^{2n}} dy \right)^{\frac{1}{2}} \| f \|_{L^2(R^n)}$$

By (1), for some $\delta > 0.$ observe that the integral in the last estimate converges.

Remark (3.1.8): for any singular kernel *k* one can define T_k by means of

$$T(f)(x)=\int_{R^n} K(x,y) f(y)\, dy.$$

For $f \in L^2(R^n)$ with compact support, and $x \notin supp(f)$. It is not necessary however that T_k is a CZO since it might fail to be bounded on $L^2(R^n)$.

Remark (3.1.9): It is not hard to see that (T) uniquely determines the kernel *K*. that is if

$$T(f)(x)\int k(x,y)f(y)dy=\int_{R^n} K_1(,y)f(y)dy.$$

for all $f \in L^2(R^n)$ with compact support then $K=K_1$ almost everywhere . The opposite is not true. Indeed, for any bounded function $b \in L^\infty(R^n)$ the operator defined as $T(f)(x)=b(x)f(x)$ is a Calderon-Zygmund kernel zero. A more specific example is the identify operator which also falls in the previous class, and is CZO with kernel o. however, this is the only ambiguity.

If *T* is a CZO, the definition already contains the fact that *T* is defined and bounded on $L^2(R^n)$, so we don't need to worry about that. The next step is to establish the restricted L^1 boundless for L^1 functions with means zero **Lemma(3.1.10)** : let *B=B(z, R)* be a Euclidean ball in R^n and denote by $B^¿$ the ball with the same center and twice the radius, that is $B^¿=B(z,2R)$. Let $f \in L^1(B)$ have mean zero, that is $\int_B f=0.$ then we have that

$$|T(f)(x)| \lesssim_{n,\sigma} \frac{R^{\sigma}}{|x-z|^{n+\sigma}} \int_B \vee f(y) \vee dy$$

for all $x \notin B¿$ we conclude that

$$\| T(f) \| L^2_{(R^n \setminus B¿)} \lesssim_{n,\sigma} \| f \|_{L^2 B}.$$

Proof using the fact *f* has zero mean on *B*, for $x \notin B¿$ we can estimate

$$¿T(f)(x) \vee \leq \int_B \vee K(x,y) - K(x,z) \vee ¿ f(y)\, dy \leq \int_B \frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}} \vee f(y) \vee dy$$

$$\lesssim_{n,\sigma} \frac{R^{\sigma}}{|x-z|^{n+\sigma}} \int_B |f(y)|\, dy.$$

Integrating through $R^n \{ B¿$ we also get the second estimate in the lemma.

The only thing missing in order to conclude the proof of the L^p bounds for CZOs is the fact that they are self adjoint as a class. In particular, we need the following.

Lemma(3.1.11): let *T* be a CZO. Consider the adjoint $T¿$ defined by means of

$$\int T(f)\bar{g} = \int \overline{fT(g)} (3.1.4)$$

for all f, g in L^2 . Then $T¿$ is a CZO.

Proof: it is immediate from (4) and the fact that *T* is bounded on L^2 that $T¿$ is also bounded on L^2 with the same norm. Now let $f, g \notin L^2(R^n)$ have disjoint compact supports. We have

$$\int T(f)\bar{g}$$

$$¿\iint K(x,y)f(y)dy\,\bar{g}(x)dx$$

$$¿\int \overline{f(y)\int \overline{K(x,y)\,y(x)dxdy}}(3.1.5)$$

Now let $z \notin supp(g)$ and $\varnothing \in C_c^{\infty}(R^n)$ have support inside *B(0,1)* with

$$\int \varnothing = 1$$

for $\in > 0$, the functions $\varnothing_{\in}(y-z)$ are supported in $B(z,\in)$ so, for small enough, the support of $\varnothing_{\in}$ is disjoint from the support of g. By (5) we conclude that

$$\int \varnothing_{\in}(z-y)\overline{T^{¿}(g)(y)}dy = \int \varnothing_{\in}(z-y)\overline{\int \overline{K(x,y)}\,g(x)dxd}$$

Letting $\in \to 0$ we get

$$T^{¿}(g)(z) = \int \overline{K(x,z)}\,g(x)dx.$$

for almost every $z \notin supp(g).$ Since the conditions defining singular kernals are symmetric in the variable ($x, y¿$, the kernel

$$S(x,y) := \overline{K(y,x)}$$

is again a singular kernel, so we are done.

The discussion above leads the main theorem for CZO:

Theorem (3.1.12)[4,5]: Let *T* be a Calderon-Zygmund operator. Then *T* extends to a linear operator which is of weak type *(1,1)* and of strong type *(p,p)* for all $1 < p + \infty$

where the corresponding norms depend only on (n)

and($\sigma¿$ and (p) .

Sec.(3.2) Pointwise Convergence and Maximal Truncations

Let T be CZO. The example of the Hilbert transform suggests that we should have the almost everywhere convergence.

$$T(f)(x)=\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} K(x,y)f(y)\,dy,$$

at last for nice functions $f\in S(R^n)$ because of (3.1).

However, the limit

$$\lim_{\epsilon\to 0} T_\epsilon(f)(x)$$

need not even exist and be different from $T(f)(x).$ Here we can use the trivial

example of the of the operator $T(f)(x)=bf(x)$. As we have already observed this is a CZO operator with kernel 0.

Thus $T_\epsilon(f)(x)=0$ *for all* $\epsilon>0$ but clearly $T(f)$ $\neq 0$ in general.

Lemma (3.1.13): the limit

$$\lim_{\epsilon\to 0} T_\epsilon(f)(x)$$

exists almost everywhere for all $f\in S(R^n)$ if and only if the limit

$$\lim_{\epsilon\to 0}\int_{\epsilon|x-y|<1} K(x,y)f(y)\,dy,$$

exists almost everywhere.

proof : First suppose that the limit $\lim_{\epsilon\to 0} T_\epsilon(f)(x)$

exist for all $f \in S(R^n)$, and let ($(\varnothing)\in S(R^n)$ $with \varnothing \equiv 1 \text{on} \, b(0.1)$,

then

$$\lim_{\epsilon\to 0} T_\epsilon(\varnothing)(x) = \lim_{\epsilon\to 0} \int_{\epsilon|x-y|<1} K(x,y) f(y)\,dy + \int_{|x-y|>1} k(x,y)\varnothing(y)\,dy.$$

Observe that by (3.1) the second integral on the right hands side converges absolutely. Since the limit on the left hand side exists we conclude that the limit on the right hand side exists as well. Conversely, suppose that the limit

$$\lim ¿ \; {}_{\epsilon\to 0}^{¿} Y \int_{\epsilon<¿x-y\vee¿1} K(x,y)\,dy = L$$

Exists and let $f \in S(R^n)$. We have that

$$T,(f) = \int_{¿\vee¿} 1¿¿ \; K(x,y) f(y)\,dy + \int_{|x-y|>1} K(x,y) f(y)\,dy \quad \epsilon<¿x-y$$

$$¿ \int_{¿\vee¿} 1¿¿ \; K(x,y) f(y)\,dy - f(x)\,dy + f(x) \int_{\epsilon<¿x-y\vee¿1} K(x,y)\,dy \quad \epsilon<¿x-y$$

$$K(x,y) f(y)\,dy =: 1_1(\epsilon) + 1_2(\epsilon) + ¿ 1_3$$

$$+ \int_{¿x-y\vee¿1} ¿$$

By the same considerations are before $¿ 1_3 \vee ¿$ is a positive number that doesn't depend on ϵ . By the hypotheses we also have that

$\lim ¿_{\epsilon\to 0} 1_2(\epsilon\epsilon) = Lf(x). \; ¿$ Finally, for $1_1(\epsilon)$ observe that we have

$$\int_{0<¿x-y\vee¿1} |K(x,y)| \vee x-y \vee dy \lesssim_n \int_{¿x-y\vee¿1} ¿ \; ¿x-y\vee¿^{-(n-1)} dy.$$

Dominated convergence implies that $\lim_{¿} ¿_{\in\to 0} 1_1(\epsilon)$ exists as well.

Thus, for specific kernel k one has an easy criterion to establish whether the limit $\lim_{¿} ¿_{\epsilon\to 0} T_\epsilon(f)$ exists a.e for nice functions f . for example, for the kernel $k(x,y)=¿ \; ¿ \; x-¿ \; y$ of the Hilbert transform, the existence of the limit

$$\lim ¿ \; ¿ \int_{\epsilon\to 0\epsilon<¿x-y\vee¿1} \frac{1}{x-y} dy = 0$$

Is obvious. In order to extend the almost everywhere convergence to the class $L^p(R^n)$ we need to consider the corresponding maximal function.

Definition (3.2.14): Let T be a CZO and define the truncations of T as before

$$T_\epsilon(f)(x) := \int_{|x-y|\vee¿\epsilon} K(x,y) f(y) dy, x \epsilon R^n, f \epsilon S(R^n)$$

The maximal truncations of T is the sublinear operator defined as

$$T_¿(f)(x) = {}_{\epsilon<0}^{¿} |T_\in(f)(x)|, x\epsilon R^n$$

The maximal truncation of a CZO has the same continuity properties as T itself.

Theorem (3.2.15): let T be a CZO and $T_{¿}$ denote its maximal truncation. Then $T_{¿}$ is of weak type *(1.1)* and strong type *(p,p)* for $1<p<\infty.$

The proof of theorem 8 depends on the following two results.

Lemma (3.2.16): Let S be an operator for weak type (1,1) and $v\epsilon(0,1).$ Then for every set $Ec\,R^n$ with $0<|E|<+x$ we have that

$$
\begin{array}{c}
x \\
¿ \\
¿ \\
¿S(f)¿ \\
\int_E ¿
\end{array}
$$

The proof of this lemma is a simple application of the representation of the L^v norm in term of level sets and is left as an exercise.

The second result we need is the following lemma that gives a pointwise control of the maximal truncations of the CZO T by an expression that involves the maximal function of f and the maximal function of *T(f).*

Lemma (3.2.17): Let T be a CZO and $0<v\leq 1.$ Then for all $f\in C_{\epsilon}^{\infty}(R^n)$ we have that

$$
\begin{array}{c}
f\vee ¿^v \\
T\vee ¿(x) \\
M¿ \\
¿ \\
¿ \\
T_{¿}(f)(x)\lesssim_{v,n,\sigma} ¿
\end{array}
$$

Proof: Let us fix a function $f \in S¿\,(¿¿n)\,R$ and $\epsilon > 0$ and consider the balls $B = B(x, \epsilon/2)$ and its double $B^{¿} = B(x, \epsilon)$. We decompose f in the form

$$f = fx\,B^{¿} + f\,(1 - x\,B^{¿}) =: f_1 + f_2$$

Since $supp\,(f_2) \cap B = \theta$ and obviously $f_2 \in L^2(R^n)$ has compact support we can write

$$T(f_2)(x) = \int_{R^n} k(x, y)\, f_2(y)\, dy = \int_{|x-y|>\epsilon} K(x, y)\, f(y)\, dy = T_\epsilon(f)(x). \quad (3.6)$$

Also every $w \in B$ is not contained in the support of f_2 thus

$$|T(f_2)(w) - T(f_2)(x)| = \left| \int_{|x-y|>\epsilon} [K(x, y) - K(w, y)]\, f_2(y)\, dy \right|$$

$$¿\,x - \frac{w \vee ¿^{\sigma}}{¿\,x - y \vee ¿^{n+\sigma}} \vee f(y) \vee dy$$

$$¿$$

$$\leq \int_{|x-y|>\epsilon} ¿$$

By (3.6), since $¿\,x - w \vee ¿\,\frac{1}{2} \vee x - y \vee ¿$ for y in the area of integration above. By this estimate we get that

$$\epsilon$$
$$2^k ¿$$
$$¿$$
$$¿ n+\sigma$$
$$¿$$
$$¿$$
$$¿ f(y) \vee \frac{¿}{¿}$$
$$¿$$
$$\int_{2^k \epsilon < ¿ x-y \vee ¿ 2^{k+1}\epsilon} ¿$$
$$¿ T_\epsilon (f_2)(w) - T(f_2)(x) \vee \lesssim_\sigma \epsilon^\sigma \sum_{k=0}^{\infty} ¿$$

$$¿ x-y \vee ¿ 2^{k+¿ 1_\epsilon}$$
$$¿ f(y) \vee ¿ dy$$
$$\lesssim_\sigma \sum_{k=0}^{\infty} \frac{1}{\epsilon^n} \frac{1}{2^{k(n+\sigma)}} \int_{¿} ¿$$

$$\lesssim_{\sigma,n} \sum_{k=0}^{\infty} \frac{1}{2^{k\sigma}} (M)(f)(x) \lesssim_{n,\sigma} M(f)(x) dy$$

Combining the previous estimates we conclude that for any $w \in B$.

$$w$$
$$¿ \vee ¿_1$$
$$¿ T_\epsilon (f)(x) \leq AM(f)(x) + T(f_2)(x) + ¿ T(f_1) ¿ \quad (3.7)$$

For some constant *A* depending only on *n* and σ .

If $T_\epsilon (f)(x) = 0$ then we are done. If $¿ T_\epsilon (f)(x) \vee ¿ 0$ then there is $\lambda > 0$ such that $¿ T_\epsilon (f)(x) \vee ¿ \lambda .$ Let $B_1 = \{ w \in B : |Tf(w)| > \lambda/3 \}$.

$$B_2 = \{ w \in B : |T f_1 (w)| > \lambda/3 \}$$.

and

$$B_3 = \begin{cases} \varnothing, if M(f)(x) \leq A^{-1}\lambda/3, \\ B_1 if M(f)(x) > A^{-1}\lambda/3, \end{cases}.$$

Let $w \in B.$ Then either $w \in B_1$ or $w \in B_2$ or $AM(f)(x) > \lambda/3.$ In the last case $B_3 = B$ so in every case we conclude that $w \in B_1 \cup B_2 \cup B_3$ thus $B \subset B_1 \cup B_2 \cup B_3$. However, we have that

$$|B_1| ¿ \frac{1}{\lambda} \int_B |T(f)(y)| dy \leq \frac{|B|}{\lambda} M(Tf)(x).$$

Also, by the *(1.1)* type of *T* we get

$$|B_2| ¿ \frac{1}{\lambda} \|f_1\| L^1(R^n) = \frac{1}{\lambda} \int_B |f(y)| dy \leq \frac{|B|}{\lambda} M(f)(x).$$

Finally, if $B_3 = B$ then $\lambda \lesssim_{n,\sigma} M(f)(x).$ Otherwise $B_3 = \theta$ so,

$$|B| \leq |B_1| + |B_2| \lesssim_{n,\sigma} \frac{B}{\lambda} (M(Tf)(x) + M(f)(x)).$$

Thus in every case we get that

$$\lambda \lesssim_{n,\sigma} M(Tf)(x) + M(f)(x),$$

Since the previous estimate is true for any $\lambda < T_\epsilon(f)(x)$ we conclude that

$$T_\epsilon(f)(x) \lesssim_{n,\sigma} M(Tf)(x) + M(f)(x),$$

Which gives the desired estimate in the case *v=1.*

For *v<1* estimate (3.7) implies that

x
¿
¿
ω
¿
ω
¿
¿ $T_\epsilon(f)$ ¿

And integration in $w \in B$ to get

$$x$$
$$¿$$
$$¿$$
$$x$$
$$w$$
$$¿$$
$$¿$$
$$¿v$$
$$¿$$
$$w$$
$$¿$$
$$¿$$
$$¿v$$
$$¿T(f)¿$$
$$¿\vee¿^{v}+\frac{1}{|B|}\int_{B}¿$$
$$¿T_{\in}(f)¿$$

And thus

$$\infty$$
$$¿$$
$$¿$$
$$x$$
$$w$$
$$¿\vee¿^{v}$$
$$¿$$
$$w$$
$$¿\vee¿^{v}$$
$$¿$$
$$T(f)(dw¿)^{\frac{1}{v}}_{\square}$$
$$¿B\vee¿\int_{B}¿$$
$$\frac{1}{¿}$$
$$T(f)(dw¿)^{\frac{1}{v}}¿$$
$$¿B\vee¿\int_{B}¿$$
$$\frac{1}{¿}$$
$$¿\vee¿^{v}\lesssim_{n,\sigma}+¿$$
$$¿T_{\in}(f)¿$$

Note that

$$w$$
$$¿\vee¿^{v}$$
$$¿$$
$$f\vee¿^{v}$$
$$¿T¿(x)]$$
$$M¿^{\frac{1}{v}}.$$
$$T(f)(dw¿)\leq¿$$
$$¿B\vee¿\int_{B}¿$$
$$\frac{1}{¿}$$
$$¿$$

And by lemma 9 the last term is controlled by

$$w$$
$$¿\vee¿^{v}$$
$$¿$$
$$T(f)(dw¿)^{\frac{1}{v}}\leq\frac{1}{|B|}\vee|f_{1}|\vee 1\leq M(f)(x)$$
$$¿B\vee¿\int_{B}¿$$
$$\frac{1}{¿}$$
$$¿$$

Since is of weak type *(1.1).* gathering these estimates we get

$$x$$
$$¿$$
$$¿$$
$$f\vee¿^{v}(x)$$
$$M¿$$
$$¿$$
$$¿$$
$$T_{\epsilon}(f)¿$$

as we want to show.

We can now give the proof of the fact that maximal truncation of a CZO is of weak type *(1.1)* and strong type (*p*,*p*) for $1<p<\infty$.

Proof: proof of theorem 8. By lemma 10 for *v=1* we immediately get that $T_{¿}$ is of strong type *(p, p)* for $1<p<\infty$ since both *M* and *T* are.

In order to show that $T_{¿}$ is of weak type (1, 1) we argue as follows. By lemma 10 we have that

$$\left|\{x\in R^n : T_{¿}(f)(x) > \lambda\}\right| \lesssim_{n,\sigma} \vee \left\{x\in R^n : M(f)(x) > \lambda \frac{}{2}\right\} \vee ¿$$

$$\begin{gathered} f \vee ¿^{v} \\ ¿T¿(x)(x) \\ x\in R^n : \left[M ¿^{\frac{1}{v}} ¿ \lambda \frac{}{2} \right\} \vee ¿ \\ + ¿ ¿ \end{gathered}$$

$$\begin{gathered} f \vee ¿^{v}(x) \\ x\in R^n : \left[M ¿^{\frac{1}{v}} ¿ \lambda \frac{}{2} \right\} \vee ¿ \\ \lesssim \frac{1}{\lambda} \vee ¿ f \|_{L^1 R^n} + ¿ \end{gathered}$$

Thus the proof will be complete if we show that

$$\begin{gathered} | \\ f \vee ¿^{v} \\ \left\{ x\in R^n : \left[M \left(T ¿¿(x)\right]^{\frac{1}{v}} ¿ \lambda \frac{}{2} \right\} \right| ¿ \frac{1}{\lambda} \vee ¿ f \vee ¿_1 \\ ¿ \end{gathered}$$

As we have seen we have that

$$¿\{x\in R^n : M(g)(x) > 4^n \lambda\} \vee \leq 2^n \vee \{x\in R^n : M_{\Delta}(g)(x) > \lambda\} \vee ¿ \quad .$$

Where M_{Δ} is the dyadic maximal function. Furthermore, using the calderon-Zygmund decomposition it is not hard to see that

$$¿\{x\in R^n : M_{\Delta}(g)(x) > \lambda\} \vee ^{¿} \frac{1}{\lambda} \int_{\{M_{\Delta}(g)(x) > \lambda\}} ¿ g(x)\, dx$$

Applying the last estimate to $\begin{gathered} f \vee ¿^{v} \\ ¿T¿(x) \\ M ¿^{\frac{1}{v}} \\ g(x) = ¿ \end{gathered}$ we get

$$|$$
$$f \vee ¿^{v}$$
$$f \vee ¿^{v}$$
$$x$$
$$¿$$
$$¿$$
$$¿ \frac{1}{v} > \searrow / 2\}$$
$$¿$$
$$f(x) \vee ¿^{v} dx$$
$$¿ T ¿ ¿$$
$$\{ \vee M_{\Delta} ¿$$
$$\left\{ x \in R^{n} : \left[M (T ¿ ¿ (x)) \right]^{\frac{1}{v}} > 4^{n} \searrow \frac{}{2} \right\} \Big|^{¿} n, v \frac{1}{\searrow v} \int_{¿} ¿$$
$$¿$$

For $f \in C_{c}^{\infty}(R^{n})$ the set $x\ ¿\ ¿\ f \vee ¿^{v} ¿\ T ¿\ M_{\Delta} ¿\ \{ ¿$ has finite measure. Thus by lemma 9

we conclude that

$$x$$
$$¿$$
$$¿$$
$$M_{\Delta}$$
$$¿$$
$$f \vee ¿^{v} (x)^{\frac{1}{v}}$$
$$f \vee ¿^{v} ¿$$
$$T ¿$$
$$M_{\Delta} ¿$$
$$¿ \{ x \in R^{n} : ¿$$

And thus by (8) that

$$f \vee ¿^{v}$$
$$x \in R^{n} : [M ¿) \left(|x|^{v} > \searrow \frac{}{2} \right\} \vee^{¿} n, v \frac{1}{\searrow v} \vee ¿ f \vee ¿_{L^{1}}^{v} R^{n}$$
$$¿ ¿$$

This conclude the proof.

(3.2.2) Singular integral operator on L^{∞} and BMO.

The theory of Calderon-Zygmund operators developed so far is pretty satisfactory except for one point, the action of a CZO on L^∞ .. Furthermore, it is at the moment unclear how to define the action of *T* on a general bounded function or even on a dense subset of L^∞ . With a little effort however this can be achieved.

Let us first fix a function $f \in L^\infty(R^n)$ and look at the formula

$$T(f)(x)=\int K(x,y)f(y)dy.$$

As we have already mentioned several times, such a formula is not meaningful through R^n . Indeed the integral above need not converge, both close to the diagonal $x=y$ since *K* is singular, as well as at infinity since *K* only decay like $|x-y|^{-n}$, not fast enough to make the integral above
absolutely convergent. The first problem we have dealt with so far by considering functions with compact support and requiring the validity of (3.6) only for $x \notin supp(f)$. A similar solution could work now but we still have a problem at infinity. Note that we didn't run into this problem yet since we only considered functions in

$\overset{R}{(¿¿n)}\underset{}{L^p ¿}$ which necessarily possess decay at infinity. This is not necessarily the case for bounded functions. However, looking at the difference of the values of *T*($f ¿$ at two points $x_1 x$ with $x_1 \neq x_2$, we can formally write

$$T(f)(x_1) - T(f)(x_2) = \int [\, K(x_1, y)\, K(x_2, y)] f(y)\, dy.$$

Using the regularity condition (3) we see that

$$¿ K(x_1, y) - K(x_2, y) \lesssim_{n,\sigma} ¿ \frac{¿ x_1 - \frac{x_2 \vee ¿^{\sigma}}{¿}}{¿ x - y \vee ¿^{n+\sigma}}$$

when $y \to \infty$. This is enough to assure intgrability in the previous integral, as long as $x_1, x_3 \notin supp(f)$. Motivated by this heuristic we define for $f \in L^{\infty}(R^n)$:

$$T(f)(x) = T(f x_B)(x) + \int_{¿}^{\overset{¿}{R^n\}}} [K(x, y) - K(0, y)] f(y)\, dy. (3.7)$$

For some Euclidean ball *B* so that $0, y \in B$. First of all it is easy to see that the integrals above make sense. Indeed, $T(\overset{f}{\underset{¿}{¿¿}} XB)$ is well define since f_{XB} is in $L^2(R^n)$. On the other hand, the integral in the second summand converges absolutely since we integrate away from $B \ni 0, y, f$ is bounded and $K(x, y) - K ¿ 0, y ¿$ behaves like

$n+\sigma$ $-¿$ $y \vee ¿^{¿}$ $¿¿$ for $|y| \to +\infty$. However, (7) only defines *T(f)* up to a constant. Indeed it is easy to see that if $B.B^{'}$ are two different balls containing 0,y the difference in the two definitions is equal to

$$\int_{B \Delta B^{'}} K(0,y) f(y)\, dy$$

which is constant independent of $x.$. Thus we only define $T(f)$ modulo constants. This definition of *T* gives a linear operator which extends our previous definition on R $(¿¿n)$ $L^2 ¿$ or $S(R^n)$. To deal with the ambiguity in the definition, we have to define the appropriate space.

Definition (3.2.19)[8**]**: We say that two functions $f, g \in R^n$ are equivalent modulo a constant c $\in C$ such that $f(x) - g(x) = c$ almost everywhere on $R^n.$ This is an equivalence relationship $f = g \, mod \, a$ we will identify an equivalence class with a representative of the class, much like we do with measureable functions.

Theorem (3.2.20)[8]: Let *T* be a CZO. Then for every $f \in L^{\infty}(R^n)$ we have that:

$$\| T(f) \|_{BMO(R^n)} \lesssim_{n,\sigma} \| f \|_{L^\infty(R^n)}.$$

Proof: let *B* = *(z,r)* be some ball in R^n. We need to show that

$$f \text{ ¿ ¿ } T(f) - T \text{ ¿ ¿ ¿ } \text{¿} B \vee \text{¿} \int_B \text{¿} \frac{1}{\text{¿}}$$

And denote $B^{¿} = B¿$ $z, 2\sqrt{nr}$). We set

$$f = f_{X B^{¿}} + f_{X R^n} \{ B ¿^{¿} =: f_1 + f_2$$

Since *T* is of strong type we have

$$\| T(f_1) \|_{L^2(R^n)} \lesssim_{n,\sigma} \| f_1 \|_{L^2(R^n)} |B^{¿}|^{\frac{1}{2}}$$

Thus by Cauchy-Schwartz we have

$$f_1 \quad ¿ B \vee ¿ \| T(f_1) \|_{L^2 R^n} |B|^{\frac{1}{2}} \lesssim_{n,\sigma} \| f \|_{L^\infty} \quad ¿ T(¿) \vee \leq \frac{1}{¿} \quad \frac{1}{B} \int_B ¿$$

On other hand for $x \in B,$ the ball $B^{¿}$ certainly contains both x and *z* so

$$¿ \quad R^{¿} K(x,y) - K(z,y) \quad T(f_2)(x) = T(f_{2 X B^{¿}}) + \int_{¿} f_2(y)\, dy$$

$$\leq \| f \|_{L^\infty(R^n)} \int_{|y-z| \geq 2r} \vee K(x,y) - K(z,y)\, dy$$

$$¿x-y\vee¿^{n+\sigma}dy$$
$$¿x-\frac{z\vee¿^{\sigma}}{¿}$$
$$\leq\|f\|_{L^{\infty}(R^{n})}\int_{|y-z|\geq 2r}¿$$

$$\frac{1}{¿a-y\vee¿^{n+\sigma}}dy.$$
$$\|f\|_{L^{\infty}(R^{n})r^{\sigma}}\int_{||\infty-y|>r|}¿$$

$$R$$
$$L^{\infty}(¿¿n)$$
$$\lesssim_{n,\sigma}\|f\|_{¿}$$

Remembering that only defines *T* up to a constant *(B)* we get

$$¿T(f)(x)-c(B)\vee¿\lesssim_{n,\sigma}\|f\|_{L^{\infty}(R^{n})}$$
$$¿B\vee¿\int_{B}¿$$
$$\frac{1}{¿}$$

By preposition (3.2.13) this proves the theorem[9].

We will now see that although the space BMO contains unbounded functions like log $|x|$, this in sense the maximum possible growth for BMO functions. Although such a claim is not precise in apointwise sense, it can be rigorously proved in the sense of level sets. Indeed, assuming $\|f\|_{BMO}=1$

$$\frac{1}{|B|}\int_{B}|f-f_{B}|\leq 1.$$

For all ball B. Using Chebyshev's inequality this implies

$$|\{x\in B:|f(x)-f_{B}|¿\lambda\}|\leq\frac{|B|}{\lambda}$$

Theorem (3.2.21)

Let $f \in BMO(R^n)$. Then for any Euclidean cube Q we have that

$$\left|\{x \in Q : |f(x) - f_Q| ¿ \lambda\}\right| \lesssim_n e^{-c_n \lambda / \|f\|_{BMO}} |Q|$$

for all $\lambda \in 0$ where the constant $c_n > 0$ depends only on the dimension n.

Which is of course quite far from the desired estimate.

$c(\lambda)^{¿} n, c^{-en\lambda}$

This will be achieved by iterating a local Calderon-Zygmund decomtion as follows.

Let us fix a cube Q_o and consider the family $B_1 of\, 2^n$ cubes inside Q_o which are formed by bisecting each side of Q. Then define the second generation B_2 by bisecting the sides of each cube in B_1 and so on. The family of all cubes in all generation will denoted by B'. For a level $\wedge > 1$ to be chosen later let B''. Be the bad cubes in B', that is the cubes $Q \in B'$ such that

$$¿ Q \vee ¿ \int_Q F(w)\, dw > A.$$
$$\frac{1}{¿}$$

Where $F(w) = |f(w) - f_{Q_o}|$

Finally, let B be the family of maximal bad cubes. Since

$¿ Q_{o \vee ¿} \int_{Q_o} F(w)\, dw \leq 1 < A$ $\frac{1}{¿}$ for the original cube Q_o, every bad cube is contained in a maximal bad cube. As in the global Calderon-Zyfgmund decomposition we conclude that

$$¿Q\vee¿\int_Q F(w)\,dw\leq r_n A$$

$$A\leq\frac{1}{¿}$$

For each cube $Q\in B$ where the constant r_n depends only on the dimension n. We also conclude that

$$F(w)\leq A$$

If $\infty\notin\cup_{Q\in B}Q$ by the dyadic maximal theorem. Remembering the initial normalization $¿\vee f||_{BMO}=1$ we get

$$\sum_{Q\in B}¿Q\vee\leq\frac{1}{A}\sum_{Q\in B}\int_Q F(w)\,dw\leq\frac{1}{A}\vee Q_o\vee¿$$

And for $Q\in B$

$$|f_Q-f_{Q_o}|=¿\frac{1}{Q}\int_Q[f-f_{Q_o}]\vee\leq\frac{1}{Q}\int_Q F(w)\,sw\leq r_n A$$

Now consider $\gamma>r_n A$ we have

$$¿\{\propto\in Q_o:\left|(f-f_{Q_o}(\propto)\right|¿\gamma\}\Big|=¿\{\propto\in\cup_{Q\in B}Q:|f(\propto)-f_{Q_o}|>\gamma\}\vee¿$$

$$¿\Big|\{\propto\in\cup_{Q\in B}Q:|f(\propto)-f_Q|>\gamma\}\Big|f_Q-f_{Q_o}\vee\}\vee¿$$

$$¿\{\propto\in Q:F(\propto)>\gamma r_n A\}\vee¿$$

$$\sum_{Q\in B}¿$$

$$c(\gamma-r_n A)\sum_{Q\in B}|Q|$$

$$c(\gamma-r_n A)\frac{1}{A}Q_o$$

However that

$$c(\leftthreetimes) \leq \frac{c(\leftthreetimes - r_n A)}{A}$$

Whenever $\leftthreetimes - r_n A$ suppose that $N r_n A < \leftthreetimes \leq (n+1) r_n A$. Since $c(\leftthreetimes)$ is non-increasing and the trivial estimate $c(\leftthreetimes) \leq 1$ we get

$$r_n A - 1$$

$$\frac{A}{\text{¿}}$$

$$c(\leftthreetimes) \leq c(N r_n A) \leq \frac{c(N r_n A)}{A^N} \leq c^{-N \vee nA} \leq c^{-(\text{¿}-1) inA \text{¿}} c^{-en\leftthreetimes}$$

For A=c (say) and $\leftthreetimes > r_n c.$ On the other hand, for $\leftthreetimes < r_n$ we have

$$c(\leftthreetimes) \leq 1^{\text{¿}} n c^{-on\leftthreetimes}$$

So the proof is complete.

Corollary () Consider the L^p version of the BMO norm

$$p$$

$$\frac{1}{|B|} \int_B \text{¿} f - f_B \vee \text{¿}$$

$$\text{¿}$$

$$\text{¿} f - a \vee \text{¿}^p$$

$$\frac{1}{B} \int_B \text{¿}^{\frac{1}{p}}$$

$$\|f\|_{BMO,p} := \text{¿}_B \text{¿}$$

$$p$$

$$\frac{1}{|Q|} \int_Q \text{¿} f - f_Q \vee \text{¿}$$

$$\text{¿}$$

$$\text{¿} f - a \vee \text{¿}^p$$

$$\frac{1}{Q} \int_Q \text{¿}^{\frac{1}{p}}$$

$$Q \text{¿}$$

$$\lesssim_{n,p} \text{¿}_{\text{¿}}$$

Then

$$\|f\|_{BMO} \lesssim_{n,p} \|f\|_{BMO,p}$$

Finally, we show how we can use the space *BMO* R^n as a different endpoint in the log-convexity estimates for the L^p norms.

Lemma 18: let $0<p<q<\infty$ and $f \in L^p\left(R^n\right) \cap BMO\left(R^n\right).$ Then $f \in L^p\left(R^n\right)$ and

$$n$$
$$R^{¿}$$
$$¿$$
$$¿$$
$$p¿$$
$$L^{¿}$$
$$\left\| |f||_{L^q(R^n)} {}^{¿} p,q.d \right\| f||_{¿}$$

Proof: Obviously it is enough to assume that $¿ \vee f||_{BMO} \neq 0$ otherwise there is nothing to prove. Also by homogeneity we can normalize so that $¿ \vee f||_{BMO} = 1$. Now from the Calderon-Zygmund decomposition of $¿ f \vee ¿^p \, ¿$ at level *1* and denote by B the family of bad cubes as usual.

For each cube $Q \in B$ we then have

$$¿ f \vee ¿^p \lesssim_n = 1$$
$$f \vee \leq \frac{1}{¿ Q \vee ¿} \int_Q ¿$$
$$¿ Q \vee ¿ \int_Q ¿$$
$$\frac{1}{¿}$$

From the John-Nirenberg inequality we conclude that

$$xQ : \left| f(x) - f_Q \right| > \lambda - ¿ f_Q \vee ¿$$
$$(¿| \lesssim_n c^{-cn\lambda} |Q|$$
$$\left| \left\{ xQ : \left| f(x) \right| > \lambda \right\} \right| \leq ¿$$

For all the bad cubes $Q \in B$ since we have that $|f(x)|<1$ for $x \notin \cup_{QEB} Q$ we get

$$\left|\{x \in R^n : |f(x)| > \lambda\}\right| \lesssim_n c^{-cn\lambda} \vee ¿ f \vee ¿_{L^p}^{p} \quad (11)$$

For all $\lambda > 1$. On the other hand, since $f \in L^p$ we have

$$\left|\{\propto \in R^n : |f(x)| > \lambda\}\right| \leq \frac{¿ \vee f \|_{L^p}^{p}}{\lambda^p} \quad (12)$$

We conclude the proof by using the description of the L^p norm in terms of level sets and using (3.2.12) for $\lambda > 1$ and (3.2.11) for $\lambda > 1$.

Sec.(3.3) Calderon-Zygmund Kernels and Operators[64,65]

We denote the diagonal of $R^n \times R^n \, by \, \triangle = \{(x, x) : x \in R^n\}$.

Definition (3.3.21) Let $0 < \alpha \leq 1$. A Calderon-Zygmund Kernel of order $\alpha \, is \, a \, contenuous \, function \, K : {}^c\Delta \rightarrow k \, such \, that \, there \, exist \, a \, C > 0 \wedge satisfes:$

i. $for \, all (x, y) \in {}^c\Delta$

$$|K(x,y)| \le \frac{C}{|x-y|^{n,}}$$

ii. $for\ all\ x, y, y' \in R^n\ satisfing |x-y'| \le \frac{1}{2}|x-y|\ when\ x \ne y$

$$|K(x,y)-K(x,y')| \le C\left(\frac{|y-y'|}{|x-y|}\right)^{\alpha} \frac{1}{|x-y|^{n,}}$$

iii. $for\ all\ x, x'\ y' \in R^n\ satisfing |x-x'| \le \frac{1}{2}|x-y|\ when\ x \ne y,$

$$|K(x,y)-K(x'\ y)| \le C\left(\frac{|x-x'|}{|x-y|}\right)^{\alpha} \frac{1}{|x-y|^{n,}}$$

We write $K \in CZ\,K_{\alpha} \wedge norm\,it\ via |K|_{\alpha} = inf\,\{C:(i)¿(iii)\,hold\}.$

Remark(3.1.2) $(i)\, the\ constant\ \frac{1}{2}\, can\ be\ replaced\ by\ any\ \theta \in (0,1).$

Then the constant C changes.

(ii) $The\ Euclidean\ norm |.|\ can\ be\ changed$ ¿ $any\ other$

$norm. \quad Again, C\ changes.$

(iii) When $\alpha = 1, \nabla_y K(x,y) \exists\, almost$ everywhere and satisfies :

$$|\nabla_y K(x,y)| \le \frac{C'}{|x-y|^{n+1}}$$

$for\ all\,(x,y) \in {}^c\Delta$.

$(iv)\ when\ \alpha = 1, \nabla_x K(x,y) \exists\, almost\ everywhere \wedge satisfies:$

$$\left|\nabla_x K(x,y)\right| \le \frac{C'}{|x-y|^{n+1}}$$

$for\ all\ (x,y) \in {}^c\Delta.$

(v) $Define\ K^{¿}(x,y)=K(y,x).Then\ K\in CZK_{\alpha}\ implies\ K^{¿}\in CZK_{\alpha}.$

Definition (3.3.23) (Kernel associated to an operator) $Let\ T\in L¿\ L^2(R^n).$

$We\ say\ that\ a\ Kernel\ K: {}^c\Delta \to K\ is\ associated\ ¿\ T\ if\ for\ all\ f\in L^2(R^n), with\ spt\ f\ compact,$

$$Tf(x)=\int_{R^n} K(x,y)f(y)\,dy(y)$$

$for\ almost\ every\ x\in {}^c¿\ spt\ f$)[70].

Remark(3.3.24) This integral is a Lebesgue integral for all $xC^c(spt\ f).$

$Moreover, this\ says\ that\ Tf\ can$ be $represented\ by\ this\ integral$ away from the support of $f.$

Definition (3.3.25) (Calderon-Zygmund operator) .

A Calderon-zygmund operator of order α is an operator T $L¿\ L^2(R^2)$ that is associated to a K $\in CZK_{\alpha}. we\ define\ CZO_{\alpha}¿\ be\ the\ collection\ of\ all$

$Calderon-Zygmund\ operators\ of\ order\ \alpha. Also,$

$¿ L^2(R^n) + \| K \|_\alpha$

$L¿$

$\| T \|_{CZO_\alpha} = \| T \|_¿$

Remark(3.3.26): (i) $T \in CZO_\alpha if \wedge only if T^¿ \in CZO_\alpha$

$L^2(R^n) with spt f.spt g compact \wedge spt f \cap spt g \varnothing .then,$

$also, let f, g \in ¿$

$$(T^¿ g, f) = (g, Tf) = \int_{R^n} g(x) \left(\int_{R^n} K(x, y') f(y') dy(y) \right) dy(x)$$

$$\int_{R^n} K(x, y') g(x)$$

$f(y)¿$

$¿ \int_{R^n} ¿$

$¿ since f was arbitrary, T^¿ g(y) = \int_{R^n} K(x, y') g(x) dy(x) for almost every y \in^c (sptg). that$

$is, T^¿ has associated Kernel K^¿.$ (ii)

$T \in CZO_\alpha if \wedge only if T^{tr} CZO_\alpha, where T^{tr} is the real transpose of T.$

$the associated kernel ¿$ فـ $ted = ¿$

$$T^{tr} is K^{tr}(x, y) = K(x, y).$$

(iii) the map

$CZK_\alpha the associated kernel, is not injective Consequantly,. one cannot$

$define a CZO_\alpha uniquly given a \in CZO_\alpha.$

The following is an important illustration.

$Let\, m \in L^{\infty}(R^n) \wedge let\, T_m\, be\, the\, map \mapsto mf. it\, is\, easy$ ¿ $\check{t}hat\, this\, is\, a\, bounded\, operator\, on:$

$L^2\, let\, K = 0\, on^c\, \Delta . \wedge$ ¿

$Let\, f \in L^{\infty}(R^n)\, with\, spt\, f\, compact. then, whenever\, x \notin spt\, f, T_m f(x) = m(x) f(x) = 0.$Therefore

$T_m f(x) = \int_{R^n} K(x, y) f(y) d\, y(y) d\, y\, whenever\, x \in^c (sptf), which\, shows\, the$

associated kernel to T_m is 0.

(3.3.1)Calderon-Zygmundoperat0r in One-Parameter Settings

In this chapter I will start my study of Caldron-Zygmund operators in one-parameter setting. The canonical example of such an operator is the Hilbert transform, which is given by

$$Hf(x) = \frac{p.v}{\pi} \int_R \frac{f(y)}{x-y} dy.$$

In the case of several variables, the canonical example becomes Riesz transforms, which is given by

$$R_j f(x) = \frac{p.v}{C(n)} \int_{R^n} \frac{x_{j-yj}}{|x-y|^{n+1}} f(y) dy \,\forall_j = 1, \ldots\ldots, n$$

Note that these generate operators that are of convolution type, T(f) =f * K for some appropriate kernel function K. However, the obvious estimates on the kernel give that

$$|k_j(x)| \lesssim \frac{1}{|x|^n}$$

And so these kernels are not integrable. However it is easy to see that these kernels

Have an additional property some cancellation, which we will make more precise momentarily. These two properties together will imply that the operator are in fact bounded on bounded $L^p(\mathbb{R}^n)$ when $1<p<\infty$. Our goal in this section is to flesh out the details behind this fact.

Definition(3.3.27) We will consider Caldron- Zygmund operators of the following forms. We will have a convolution kernel K(x) that satisfies the following conditions

(a) $|k(x)| \lesssim |x|^{-n}$

(b) $\int_{r<|x|<R} k(x)\,dx = 0 \text{ for all } 0<r<R<\infty;$

(c) $\int_{|x|>2|y|} |K(x-y)-K(x)|\,dx \lesssim 1 \text{ when } |y|>0$

$|x|^{-n-1}$ for condition. It is easy to see that the kernels for the Hilbert transforms satisfy these conditions. Our goal is to prove the following theorem.

Theorem(3.3.28): Suppose that the operator T given by:

$$Tf(x) = p.v.\int_{R^n} K(x-y) f(y)\,dy,$$

has a kernel k that satisfies (a),(c) above. Then for $1<p<\infty$, we have that $Tf: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with the operator norm controlled by the constants appearing in the definition of kernel and the dimension.

Since the ideas that we need are contained in weaker statement, we will also look at the following theorem:
Theorem(3.3.29) Suppose that the theorem given

$$Tf(x) = p.v.\int_{R^n} \widehat{K}(x-y)f(y)dy,$$

has a kernel k that satisfies conitions(a),(b) above. Then for $1<p<\infty$, we have that $|k| \lesssim 1.$ Then for $1<p<0$ we have that $\begin{matrix} R \\ R \\ (¿¿n) \\ (¿¿n) \to L^p ¿ \\ Tf : L^p ¿ \end{matrix}$ with the operator norm controlled by the constants appearing in the definition of kernel and the dimension.

In this statement of the theorem $|\hat{k}|$ denote Fourier transform of the kernel k. By the imposing the condition that $|\widehat{K}| \lesssim 1$, we are supposing that the operator is in fact priory bounded on $\begin{matrix} R \\ (¿¿n) \\ L^p ¿ \end{matrix}$ as can easy be seen. In fact it is a good to show that the conditions on a kernel imply that for the function

$$K_\epsilon(x) = \begin{cases} K(x) : |x| \geq \epsilon \\ 0 : |x| < \epsilon. \end{cases}$$

Weak Type Estimates for Calderon-Zygmund Operators

In this section we prove a very useful decomposition theorem for functions.

Theorem(3.3.30). Let $f \in \begin{matrix} R \\ (¿¿n) \\ L^p ¿ \end{matrix}$ and $\lambda > 0$ be given, then there exists functions (g) and (b) such that

$(i) f = g + b;$

$(ii) \| g \|_{L^1(R^n)} \leq \| f \|_{L^1(R^n) \wedge \| g \|_{L^\infty(R^n)}} \leq 2^n \lambda;$

$(iii) b = \sum_j bi$ *were each* b_i *is supported on a dyadic cube* $Q_{j.} \wedge the$ *collecdiction*

of dyadic cubes $\{Q_j\}$ *are disjoint ;*

(Iv) $\int_{Q_i} b_j(x)\,dx = 0;$

(v) $\| b_j \|_{L^1(R^n)} \leq 2^{n+1} \lambda |Q_j|;$

$(vi) \sum_j |Q_j| \leq \frac{\| f \|_{L^1(R^n)}}{\lambda}.$

Note that if $Q_j \in S$ that $Q'_j \notin S,$ so we have the opposite equality

$$\frac{1}{|Q|} \int_Q |f(x)|\,dx > \lambda$$

$$b_j(x) = \left(\frac{-1}{|Q_j|} \int_{Qj} f(y)\,dy \right) x Q_j(x).$$

Now using these observations we have

$$\| b_i \|_{L^1(R^n)} \leq 2 \int_{Q_i} |f(x)|\,dx$$

$$¿ 2 \frac{|Q'_j|}{|Q'_j|} \int_{Q_j} |f(x)|\,dx$$

$$\leq 2^{n+1}|Q_j|\frac{1}{Q'_j}\int_{Q'_j}|f(x)|\,dx$$

¿ $2^{n+1}|Q_i|\lambda$

¿

$$f(x) : x \in R^n(\cup_j Q_j \frac{1}{|Q'_j|}\int_{Q_j}|f(x)|\,dx : x \in Q_j$$

$g(x) = \{$ ¿

$$|g(x)| = \left|\frac{1}{|Q_j|}\int_{Q_j}|f(x)|\,dx\right| \leq 2^n \frac{1}{Q'_j}\int_{Q'_j}|f(x)|\,dx \leq 2^n\lambda .$$

$$\left|\frac{1}{|Q|_k}\int_{Q_k} f(x)\,dx\right| \leq \frac{1}{|Q_k|}\int_{Q_k}|f(x)|\,dx \leq \lambda .$$

By the Lebesgue Differentiation theorem we then have that

¿

$|f(x)| \leq \lambda\, x \in R^n(\cup_j Q_j$ ¿¿

Combining these tow estimates we see that

$$\|g\|_{L^\infty(R^n)} \leq 2^n\lambda .$$

Finally, observe that

$$\sum_j |Q_j| \leq \frac{1}{\lambda}\sum_j \int_{Q_i}|f(x)|\,dx$$

¿ $\frac{1}{\lambda}\int_{\cup_i Q_i}|f(x)|\,dx$

$L^1(¿R^n).$
$\leq \frac{1}{\lambda} \| f \|_{¿}$

We now turn to show how to use this Theorem to deduce the following result.

Theorem(3.3.31)

suppose that K is a calderon − zygmuund kernel as defiend above ∈ theorem then for all f $\epsilon L^1(R^n) \wedge$ *any*

$\lambda > 0$ *we have*

$$L^1(¿R^n).$$
$$|\{x \epsilon R^n : |Tf(x)| ¿ \lambda\}| \lesssim \frac{1}{\lambda} \| f \|_{¿}$$

proof.Fix $\lambda \wedge f \epsilon L^1(R^n)$. *Apply the Calderon − Zygmund decomposition ∈ Theorem*

¿ obtain function g , b so that f = g + b.Now observe that

$$\{x \epsilon R^n : |Tf| > \lambda\} \subset \left\{x \epsilon R^n : |Tg| > \frac{\lambda}{2}\right\} \cup \left\{x \epsilon R^n : |Tb| > \frac{\lambda}{2}\right\}$$

Now observe that

$$|\{x \epsilon R^n : |Tf| ¿ \lambda\}| \leq \left|\left\{x \epsilon R^n : |Tg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \epsilon R^n : |Tb| > \frac{\lambda}{2}\right\}\right|,$$

we need to find estimates on each of these terms. The estimate on the good function is easy since we have

$$\left|\left\{x \epsilon R^n : |Tg| > \frac{\lambda}{2}\right\}\right| \lesssim \frac{\| Tg \|^2_{L^2(R^n)}}{\lambda^2}$$

$$\lesssim \frac{1}{\lambda^2} \| g \|^2_{L^2(R^n)}$$

$$\lesssim \frac{1}{\lambda^2} \| f \|_{L^1(R^n)}$$

$$¿ \frac{1}{\lambda} \| f \|_{L^1(R^n)}$$

Here, we have used that $T : L^2(R^n) \rightarrow L^2(R^n)$

3.3.31

is bounded.In the estimate we have used property (ii) ∈ theorem ¿

We now turn to understanding the estimate on the bad function. Let $\{Q_j\}$ be the cubes obtained in theorem (3.3.31)Let $Q_j^{¿}$ denote the cube concentric with Q_j and having side length $\sqrt[2]{n}$ times the side length of Q_j. Then we have that

$$\left| \left\{ x \epsilon R^n : |Tb| > \frac{\lambda}{2} \right\} \right| = \left| \left(\cup Q_j^{¿} \right) \cap \left\{ x \epsilon R^n : |Tb| > \frac{\lambda}{2} \right\} \right| + \left| \left(\cup Q_j^{¿} \right)^c \cap \left\{ x \epsilon R^n : |Tb| > \frac{\lambda}{2} \right\} \right|.$$

Consider now the first term above, we then have that

$$\left| \cup Q_j^{¿} \cap \left\{ x \epsilon R^n : |Tb| > \frac{\lambda}{2} \right\} \right| \leq \left| \cup Q_j^{¿} \right|$$

$$\leq \sum_j \left| Q_j^{¿} \right|$$

$$\lesssim \sum_j \left| Q_j \right|$$

$$\lesssim \frac{1}{\lambda} \| f \|_{L^1} (R^n)$$

It only remains to handle the term

$$\left| \left(\cup Q_j^{¿} \right)^c \cap \left\{ x \epsilon R^n : |Tb| > \frac{\lambda}{2} \right\} \right|$$

And for this one we will use the properties of the function b. Note that by simple estimates we have

$$\left|\left(\cup Q_j^{¿}\right)^c \cap \left\{x \in R^n : |Tb| > \frac{\lambda}{2}\right\}\right| \lesssim \frac{1}{\lambda} \int_{\left(\cup Q_j^{¿}\right)^c} \left|T(b)(x)\right| dx$$

$$\leq \frac{1}{\lambda} \sum_j \int_{\left(\cup Q_j^{¿}\right)^c} \left|T b_j(x)\right| dx$$

Suppose for the moment that we prove

$$\int_{\left(\cup Q_j^{¿}\right)} \left|T b_j(x)\right| dx \int_{Q_j} \left|b_j(x)\right| dx$$

Then we could continue the sum to find

$$\left|\left(\cup Q_j^{¿}\right)^c \cap \left\{x \in R^n : |Tb| > \frac{\lambda}{2}\right\}\right| \lesssim \frac{1}{\lambda} \sum_j \int_{\left(\cup Q_j^{¿}\right)^c} \left|\left|T b_j(x)\right| dx\right|$$

$$\lesssim \frac{1}{\lambda} \sum_j \int_{Q_i} \left|b_j(x)\right| dx$$

$$¿ \frac{1}{\lambda} \sum_j \|b_j\|_{L^1(R^n)}$$

$$\lesssim \sum_j \left|Q_j\right|$$

$$\lesssim \frac{1}{\lambda} \sum_j \|f\|_{L^1(R^n)}$$

We now turn to proving (2.2). Here we will use the fact that b_j has mean value zero to introduce some cancellation into the integral. Let c_j denote the center of the cube Q_j. Observe that

$$\int_{\left(\cup Q_j^{¿}\right)^c} \left|T b_j(x)\right| dx = \int_{\left(\cup Q_j^{¿}\right)^c} \left|b_j(y) K(x-y)\, dx\right| dx$$

$$¿ \int_{(\cup Q_j^{¿})^c} \left| b_j(y) K(x-y) - K(x-C_j) dy \right| dx$$

$$x - C_j$$
$$K(x-y) - K ¿ dx$$
$$\int_{(\cup Q_j^{¿})^c} |¿) dy$$
$$\left| b_j(y) \right| ¿$$
$$\leq \int_{Q_j} ¿$$

Focus on the inner integral now,

$$\int_{(\cup Q_j^{¿})^c} \left| K(x-y) - K(x-C_j) \right| dx$$

And inspection reveals that this is very similar to what appears in condition (C) on the Calderon-Zygmund Kernel. A change of variable, and simple estimates allow one to show

$$\int_{(\cup Q_j^{¿})^c} \left| K(x-y) - K(x-C_j) \right| dx \leq \int_{|x| \geq 2|y-C_j|} \left| K(x-(y-C_j)) - K(x) \right| dx \lesssim 1.$$

This then completes the proof of the Theorem (3.3.31) With Theorem 2.2 at our

disposal, it is very easy now to conclude the proof of Theorem (3.3.30) Proof of

Theorem 1.2.The hypothesis of the Theorem give that

$$T : L^2(R^n) L^2(R^n) T : is\, bounded.$$

We have proved that in Theorem(3.3.30) that the operator

$$T:L^1(R^n)\to L^{1,\infty}(R^n)$$

is bounded too. Now, we apply the Marcinkiewicz interpolation Theorem (3.3.30) conclude that

$T:L^p(R^n)\to L^p(R^n) when 1<p<2.$

$¿ obtain the range\ 2<p<\infty$

one simply considers the argument given, but now for the

adjoint operator.

It is easy ¿ see that the kernel of the adjoin will still be a Calderon-Zygmund Kernel and so everything we have said so far applies again.

(3.3.3) BEHAVIOR NEAR L¹ AND L^∞

As we have seen, the convolution-type Calderon-Zygumund operators are bounded on $L^p(R^n)$ when $1<p<\infty$. We have also see that the operators satisfy a weak-typ bound when p=1. It turns out that we can have them be actually bounded if we change the target and domains.

Theorem (3.3.32) Let T be a Calderon-Zygmund operator as defined above, then we have:

$T:H^1(R^n)\to L^1(R^n)$

And

$T\,L^\infty(R^n)\to BMO(R^n)$

While we haven't introduced the function spaces of $H^1(R^n)$ and $BMO(R^n)$ (though we likely will), its useful to at least have this theorem in mind.

(3.3.4) The T(b) Theorem

The T(b) theorem provides sufficient conditions for a singular integral operators to be Calderon-Zygmund operators, that is for singular integral operator associated to Calderon-Zygmund kernel to be bounded on L^2. In order to state the result we must first define some terms.

A normalize bump is a smooth function φ on R^n supported in a ball of radius 10 and centred at the origin such that $|\partial^\alpha \varphi(x)| \le 1$, for all multi-indices $|\alpha| \le n+2$. Denote by $\tau^r(\varphi)(y) = \varphi(y-x) \wedge \varphi_r(x) = r^{-n}(x/r)$ for all x in R^n

and $r > 0$. An operator is said to be weakly bounded if there is a constant C such that

$$\left| \int T\left(\tau^x(\varphi_r)\right)(y)\, \tau^x(\Psi_r)(y)\, dy \le C_r^{-n} \right.$$

for all normalized bumps φ and $\Psi.$ A function is said to be accretive if there is a constant $C > 0$ such that Re $(b)(x) \ge C$ for all x in R. Denote by M_b the operator given by a function b.

the T(b) theorem states that a singular integral operator T associated to a Calderon-Zygmund kernel is bounded on L^2 if it satisfies all of the following three conditions for some bounded accretive functions b_1 and b_2.

(a) $M_{b1} T M_{b2}$ is weakly bounded.

(b) T_{b1} is in BMO.

(c) $T^1(b_1)$ is in BMO, where T is the transpose operator of T.

Sec.(3.4)The Hilbert Transform, Riesz Transforms, and The Cauchy Operator

We discuss three important examples that have motivated the theory .

(3.4.1) The Hilbert Transform

Definition (3.4.33). We define a map $H: y(R^n) \to y'(R^n) by$

$$H(\varphi) = p.v.\left(\frac{1}{\pi t}\right) * \varphi.$$

That is

$$\frac{1}{n\pi} * \varphi$$

$$p.v(¿)(y) = \lim_{\epsilon \to 0} \int_{\{x:|x|>\epsilon\}}^{\cdot} \frac{1}{\pi x} \varphi(y-x) d\, y(x)$$

¿

Proposition (3.3.34) H extends to a bounded operator on $L^2(R)$.

Proof. We can analyze this convolution via the Fourier Transform. For a function $\varphi \epsilon\, y(R^n), the\, Fourier\, transform\, is\, given\, by$

$$\varphi'(\xi) = \int_{R^n}^{\cdot} e^{-ix.\xi d} d\, y(x).$$

We can extend this naturally to $T \in Y'(R^n)\, by\, definig$

$T' \, via \langle T', \varphi \rangle = \langle T, \varphi' \rangle \, for \, every \, \varphi \in y(R^n). \, so, when \, \varphi \in y(R),$

$$\left\langle p.v.\left(\frac{1}{\pi x}\right), \varphi \right\rangle = \left\langle p.v. \frac{1}{\pi x}, \varphi' \right\rangle$$

$$¿\lim_{\epsilon \to 0} \int_{\{x:|x|>\epsilon\}} \frac{1}{\pi x} \varphi'(x) \, d\, y(x)$$

$$¿\lim_{\epsilon \to 0} \int_{\{x:\epsilon^{-1}|>|x|>\in} \frac{1}{\pi x} \varphi(x) \, d\, y(x)$$

$$\varphi(¿E)\left(\int_{\{x:\in^{-1}|>|x|>\in} \frac{1}{\pi x} e^{ix} d\, y(x)\right) d\, y(\xi)$$

$$¿\lim_{\in \to 0} \int_{R^n} ¿$$

Now fix $\xi \in R^n. Then$

$$\int_{\{x:\epsilon^{-1}>|x|>\in\}} \frac{1}{\pi x} e^{ix.\xi} d\, y(x) = -c \int_{\{x:\epsilon^{-1}|>|x|>\epsilon} \frac{1}{\pi x} \sin(x.\xi) \, d\, y(x)$$

$$¿-2i \int_{\{x:\epsilon^{-1}|>|x|>\epsilon} \sin(x.\xi) \, d\, y(x)$$

$$¿-2i \int \cdot_{\{x:\epsilon^{-1}|>|x|>\epsilon\}} \sin(x|\xi|) \, sgn(c) \, d\, y(x)$$

$$¿-\frac{2i}{\pi} seg(\xi) \int_{\frac{\varepsilon}{|\xi|}}^{\frac{1}{\epsilon|\xi|}} \frac{\sin u}{u} d\, y(u)$$

.

The integral appearing on the right hand side is uniformly bounded on $\epsilon and\, \varepsilon.$ thus by Dominated Convergence,

$$\left\langle p.v.\left(\frac{1}{\pi x}\right)¿,\varphi\right\rangle = \int_R -i\, sgn(\xi)\varphi(\xi)\, d\, y(\xi) \wedge also\, for\, all\, \varphi \epsilon\, y(R), \widehat{H\varphi}(\xi) = -i\, sgn(\xi)\varphi^{¿}(\xi).\, since\, the\, Fourie$$

$is\, bounded\, on\, L^2(R), we\, extend\, H\, ¿\, the\, whole\, of \quad L^2(R)\, by$

$$Hf(¿\xi) = -i\, sgn(\xi) f^{¿}(\xi)$$

$defined\, \hat{¿}$

$almost\, everywhere \in R.\, then, this\, extension\, agrees\, on$

$y(R) \wedge by Plancherel\, Theorem, \| Hf \|_2 = \| f \|_2.$

Proposition (3.4.35). $H \in CZO_1.$

$proof.\, Let\, K \in CZK_1\, be\, defined\, by$

$$K(x,y) = \frac{1}{\pi(x-y)},$$

$when\, x \neq y. Fix\, f \in L^2(R)\, with\, spt\, f\, compact. Then,$

$fix\, x \epsilon^c(spt\, f) \wedge ¿\, \varnothing.\, so, there\, \exists\, a\, sequence$

$\varphi_n \epsilon\, C_c^\infty(R)\, such\, that\, spt\, \varphi_n \cap B(x,r) = \varnothing \wedge \varphi_n \to f \in L^2(R^n).\, then\, for\, every$

$z \in B(x,r),$

$$H\varphi_n(z) = \int_R K(z,y)\varphi_n(y)\, d\, y(y)$$

$¿\, H\varphi_n \to Hf \in L^2(R^n).\, Covering^c(spt\, f)\, with countably\, many\, such\, balls$

$we\, conclude\, that$

$$K(x,y)f(y)\,dy(y)$$
$$Hf(x)=\int_{\cdot R}¿$$

$^{c}_{\cdot}(sptf).therefor\ H\in CZO_{1}.$
$almost\ every\ where\ x\in ¿$

The Hilbert Transform comes from Complex Analysis. Let $f\in C_c^{\infty}(R)$ and take the Cauchy extension $f¿\overset{¿}{C}\ is$

$$f(z)=\frac{1}{2\pi i}\int\frac{f(t)}{z-t}\,dy(t)$$

Where $z=x+iy,\neq 0.\text{It}\ is\,an\,easy\,that!\,F\,is\,holomorophic\,on\overset{¿}{C}$ $R.$

but $\overset{¿}{C}$ $R\,is\,not\,connected,so,$

$$\lim_{¿}\overset{x\pm}{y\to 0^{+¿}}f¿\quad iy¿=\frac{1}{2}(f(x)\pm i\,Hf(x))$$

$$Hf(x)=\lim_{¿}\overset{y\to 0^{+¿}}{}\frac{F(x+iy)-F(x-iy)}{i}¿$$

And

$$f(x)=\lim_{¿}\overset{y\to 0^{+¿}}{}(x+iy)+F(x-iy)¿$$

We have the following Theorem of M.Riesz:

Theorem (3.4.36)[79] (Boundedness of the Hilbert Transform).H has a bounded extension to

$$L^p(R)\, for\, 1<p<\infty$$

$$y \to 0^{+¿} + F(x \pm i\, y).$$

$$Corollary\,(3.4.37).\, Let\, F\pm(x)=\lim_{¿} ¿$$

then the decomposition

$$-¿\, is\, topological \in L^p(R)$$

$$+¿+F_{¿}$$

$$f=F_{¿}$$

that is

$$p$$

$$p$$

$$\|F_{¿}$$

$$\|F_{¿}¿$$

$$\|f\|_p \simeq ¿$$

Remark(3.4.38)

$$+¿.$$

$$when\, f\, is\, real\, valued, \frac{1}{2} Hf\, is\, the\, imaginary\, part\, of\, F_{¿}$$

(3.4.2) Riesz Transforms

Motivated by the symbol side of the Hilbert Transform, we define operators R_j for

J = 1…, n on R^n .

Definition (3.4.39) (Riesz Transform).define

$$R_j : \quad L^2(R^n) \to L^2(R^n)\, by$$

$$(R_j f)^{¿}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi)$$

For j = 1….,n

We note that by Plancherels Theorem, R_j is well defined and in particular

$$\| R_j f \|_2 \leq \| f \|_2 .$$

Proposition(3.4.40) $R_j \in CZO_1$.

Proof. Consider

$$K_j(x) = p.v.\, c_n \frac{x_j}{|x|^{n+1}}$$

For some $c_n > 0$. Then $K_j \in y'(R^n)$. *If we can show that for appropriate* c_n

$$\widehat{K}_j(\xi) = \frac{-\xi_j}{|\xi|}$$

In $y'(R^n)$, *by the same argument as for the Hilbert Transform*,

$$R_j f = c_n \int_{R^n} \frac{x_{j-yi}}{|x-y|^{n+1}} f(y)\, d\, y(y)$$

For all $f \in L^2(R^n)$ *with spt f compact* $\wedge$ *for almost every* $x \in^c (spt\, f)$.

We compute the Fourier Transform of K_j. *Fix* $\varphi \in y(R^n)$. *then*

$$\langle \widehat{K}_j, \varphi \rangle = \langle K_j, \varphi ¿ \rangle = \lim_{\epsilon \to 0} C_n \int_{\{x : \epsilon < |x| < \epsilon^{-1}\}} \frac{x_j}{|x|^{n+1}} \int_{R^n} e^{-ix.\xi} \varphi(\xi)\, d\, y(\xi)\, d\, y(x).$$

For $\in \neq 0$, *let*

$$I_\epsilon = C_n \int_{\{x : \epsilon < |x| < \epsilon^{-1}\}} \frac{x_j}{|x|^{n+1}} e^{-ix.\xi} \varphi(\xi)\, d\, y(x)$$

As before, we show that $|I_\epsilon|$ *is uniforly bounded* $\in \xi$ *and* $\epsilon \wedge$ *that*

$$I_\epsilon \to -i\frac{\xi_j}{|\xi|}$$

$as\, \epsilon \to 0$

As previously, write $e^{-ix.\xi} = \cos(x.\xi) - i\sin(x.\xi)$. *we only need ¿ regard the*

Imaginary part . by a change of variables, let w = $\frac{\xi}{\xi} \wedge x = |\xi|\, y.then,$

$$I_\epsilon = -i\, C_n \int_{\left\{x : \frac{\epsilon}{|\xi|} < |x| < \frac{I}{|\xi|\,\epsilon}\right\}} \frac{yi}{|y|^{n+1}} \sin(y.w)\, d\, y(y)$$

Since the Jacobian factor of the change of variables is cancelled by the homogeneity of $\frac{x_j}{|x|^{n+1}}$.

We change variables again, this time to polar coordinates.

Let $y = r\theta$, *for* $r > ¿\, \theta \in S^{n-1}$. *then,*

$$r\theta.w$$
$$¿$$
$$\sin(¿\, r\, dr ¿¿)\, d\sigma(\theta)$$
$$¿$$
$$\int_{\frac{\xi}{|\xi|}}^{\frac{1}{\epsilon|E|}} ¿$$
$$I_\epsilon = -i\, C_n \int_{S^{n-1}} \theta_j ¿$$

Where $d\sigma$ *is the surface measure on* S^{n-1}. *So* , $|I_\epsilon|$ *is uniformly* *bounded since*

$$\left| \int_{\frac{\xi}{|\xi|}}^{\frac{1}{\epsilon|\xi|}} \frac{\sin(r\theta.\omega)}{r} dr \right|$$

Is uniformly bounded in $\epsilon, |\xi|$ and $\theta. Furthermore,$

$$\int_{\frac{E}{|E|}}^{\frac{1}{\epsilon|\xi|}} \frac{\sin(r\theta.\omega)}{r} dr \to sgn(\theta.\omega)$$

As $\epsilon \to 0 \wedge so$

$$I_\epsilon \to -i c_n \frac{\pi}{2} \int_{S^{n-1}} \theta_j \, sgn(\omega.\omega) d\sigma(\theta).$$

Write

$$a_j = \int_{S^{n-1}} \theta_j \, sgn(\theta.\omega) d\sigma(\theta).$$

And let

$$a = (a_1, \ldots\ldots a_n) = i c_n \frac{\pi}{2} \int_{S^{n-1}} ((\theta - (\theta.\omega)\omega) + (\theta.\omega)\omega) sgn(\theta.\omega) d\sigma(\theta).$$

$$= \begin{matrix} \int \\ ¿ \\ ¿ \\ -i c_n \frac{\pi}{2} ¿ \end{matrix}$$

Because $(\theta(\theta.\omega)\omega) sgn$

$(\theta.\omega)$ *is odd* $\in$ *the symmetry with respect ¿ the*

Hyperplane $\{\omega\}^{\perp} \wedge S^{n-1}$ *is invariant under this symmetry.By rotational*

Invariance,

$$\int_{S^{n-1}} |\theta.w| \, d\sigma(\theta) = \int_{S^{n-1}} |\theta_1| \, d\sigma(\theta).$$

And so we define C_n by

$$C_n \frac{\pi}{2} \int_{S^{n-1}} |\theta| \, d\sigma(\theta) = 1.$$

Then, it follows that

$$a_j = -i\omega_j = -i\frac{\xi_j}{|\xi|}$$

And the proof is complete.

Theorem(3.4.41) R_j *is bounded on* $L^p(R^n)$ *whenever* $1 < p < \infty$.

Corollary (3.4.42)(Application to PD_s).let

$\varphi \in y(R^n)$. *Then*

$$, \partial_i \partial_j \varphi = -R_i R_j \Delta\varphi \wedge ¿$$

$$\| \partial_i \partial_j \varphi \|_p \leq C(n, p) \| \Delta\varphi \|_p$$

Proof. We note that for all $(\xi) = (-i\xi_i)(-i\xi_j)\hat{\varphi}(\xi)$

$$= \left(-i\frac{i\xi_i}{|\xi|}\right)\left(-i\frac{i\xi_i}{|\xi|}\right)|\xi|^2 \hat{\varphi}(\xi)$$

$$= \left(-i\frac{i\xi_i}{|\xi|}\right)\left(-i\frac{i\xi_i}{|\xi|}\right)\left(\sum_{j=1}^{n} \xi_j^{\,2} \hat{\varphi}(\xi)\right)$$

$$¿\left(-i\frac{i\xi_i}{|\xi|}\right)\left(-i\frac{i\xi_i}{|\xi|}\right)(-\Delta\varphi)^{¿}(\omega)$$

And by application of the theorem, the proof is complete.

(3.4.3) Cauchy Operator

The Cauchy operator is an example of an operator that is not of the convolution type.

$identify\ R^n \simeq C. Let \varphi : R \to R\ be\ a\ Lipschitz\ map. that\ is, there \exists an$

$M > 0\ such\ that \left|\varphi(t) - \varphi(s)\right| \le M \left|t - s\right|. By\ Rademachers\ Theorem$

$\left|Fed96, Theorem\ 3.1.6\right|, \varphi\ is\ diffrentiable\ almost\ every\ where$

$¿\varphi' \in L^\infty(R)\ with \infty \le M. Now, let\ \Gamma = \{t + i\varphi(t) : t \in R\}$

$\subset C. if\ f\ is\ smooth \in a\ neighbourhood\ of\ \Gamma \wedge has\ compact$

$support, then\ whenever \simeq \notin \Gamma, define$

$$F(z) = \frac{1}{2\pi_i} \int \frac{f(\omega)}{z - \omega} d\omega = \int_R \frac{f(s + i\varphi(s))}{z - (s + i\varphi(s))} (1 + i\varphi'(s)) ds$$

$where\ z = Z(t) + i\alpha \wedge Z(t) = t + i\varphi(t). Fix\ t. Then$

$$\lim_{\epsilon \to 0^{\mp}} ¿ F(Z(t) + i\alpha) = \frac{1}{2} f(z(t)) + Cf(z(t))$$

(which are the Plemelj formulas-details
$where\ the\ Cauchy\ operator\ is\ given$

by

$$z(t) = \frac{\lim\limits_{\epsilon \to 0^-} 1}{2\pi i} \int_{\left|s : |z(t)| - z(s) > \epsilon\right|} \frac{f(z(s))}{z(t) - z(s)} z'(s) ds = \frac{\lim\limits_{\epsilon \to 0^-} 1}{2\pi i} \int_{\left|\omega \in \Gamma : |z - \omega| > \epsilon\right|} \frac{f(\omega)}{z - \omega} d\omega.$$

$Cf ¿$

$Let\ \hat{f}(s) = f(z(s)) z'(s),$

$$then\, cf(z(t)) = p.v. \int_{\{s:|z(t)-z(s)|>\epsilon\}} \frac{1}{z(t)-z(s)} \hat{f}(s)\, ds = T\hat{f}(t).$$

Theorem (3.4.43)(Coifman, McIntosh, Meyer(1982).

$T \in CZO_1$. $with\, kernel$

$$p.v \frac{1}{z(t)-z(s)} \in CZK_1.$$

$the\, hard\, part\, of\, the\, theorem\, is$ ¿ $show$ $\|T\hat{f}\|_2 \le C\|\hat{f}\|_2$. $As\, a\, consequence$,

Corollary (3.4.44)

$(I)\, C\, is\, bounded\, on\, L^2(\Gamma, |dw|)\, where\, |dw|\, is\, the\, arclength\, measure$,

$(ii)\, the$ Decomposition

$$f(x) = ¿ \quad \begin{matrix} z(t)+i\alpha \\ \alpha \to 0^{-¿} F ¿ \\ \alpha \to 0^{+¿} F(z(t)+i\alpha) + \lim_{¿} ¿ \\ \lim_{¿} ¿ \end{matrix}$$

Is topological in $L^p(\Gamma, |dw|)$.

These results have important applications in boundary value problems, geometric measure theory and partial differential equations.

Remark (3.4.45) We emphasize that this operators C is not of convolution type . Unlike in the previous two examples, we cannot employ the Fourier transform or simple techniques.

Sec.(3.5) L^p boundedness of CZOα operators

The L^2 boundedness of CZOα operators comes for free by definition .It is an interesting question to ask when

$T \epsilon CZO\alpha\, is\, a\, bounded\, map$ ¿ $L^q(R^n)$ ¿

$L^p(R^n)$. *But first, we have the following proposition which shows that at least*

For Hilbert transform, p = q.

Proposition (3.4.46)

suppose the Hilbert transform $H: L^q(R^n) \to L^p(R^\cdot)$ *for some* $p, q > 1$ *is bounded. Then* $p = q$.

Proof. Let $f \in L^q(R^n) \wedge$ *consider the function* $g(x) = f(\lambda x)$ *for* $\lambda < 0$.

Then,

$$\lambda^\alpha \| Hf \|_q \leq C \lambda^\beta \| f \|_p$$

And so $\alpha = \frac{-1}{q}, \beta = \frac{-1}{p} \wedge \alpha = \beta$ *which implies* $p = q$.

As a heuristic, we cannot hope to prove L^q to L^p boundedness unless p= q.

Definition (3.4.47) (Hormander kernel).

$$\Delta$$
$$¿\wedge \text{ suppose there}$$
$$¿$$
$$Let\ K \in L^1_{Ioc}¿$$

$\exists C_H > 0$ *such that*

$$\underset{(y,y')\in R^{2n}}{esssup} \int_{|x:|x-y|\geq 2|y-y'||} K ¿ d\, y(x) \leq C_H .$$

Then, K is called a Hormander kernel .

Remark (3.4.48) the number 2 appearing in the set of integration is irrelevant. This can be replaced by any A˃1 at the cost of changing C_H

Lemma(3.4,49)

(i) Every CZK_α kernel is Hormander.

(ii) the adjoint of a CZK_α kernel is Hormander.

Proof. The proof of (ii) follows easily from (i) observing that $K \in CZK_\alpha\, implies$

$K^{¿} \in CZK_\alpha.$

We prove (i). Let $K \in CZK_\alpha \wedge so\, we\, have\, that$

$$|K(x,y)-K(x,y')| \le C_\alpha \left(\frac{|y-y'|}{|x-y|}\right)^\alpha \frac{1}{|x-y|^n}$$

Whenever $|y-y'| \le \frac{1}{2}|x-y| \wedge x \ne y.so,$

$$\int_{\{x:|x-y|\ge 2|y-y'|\}} \left(\frac{|y-y'|}{|x-y|}\right)^\alpha \frac{1}{|x-y|^n} d\,y(x)$$

=

$$\sum_{j=o}^{\infty} \int_{\{x:2i2|y-y|\le|x-y||\le 2^{i+1}2||y-y'|\}} \left(\frac{|y-y'|}{|x-y|}\right)^\alpha \frac{1}{|x-y|^n} d\,y(x)$$

$$\le \frac{|y-y'|^\alpha}{|x-y'|^{\alpha+n}} \sum_{j=0}^{\infty} y\left(B\left(y,2^{i+1}|y-y'|\right)\right)$$

We now present the following important and main lemma.

Lemma(3.4.50) (**Calderon-zygmund decomposition**).

$let\, f \in L^1(R^n) \wedge \searrow > 0.$

$then\, there\, \exists\, a\, C(n) > 0 \wedge a\, decomposition\, of\, f = g + b\, almost$

$everywhere\, on$

$R^n\, where\, g \epsilon L^{\infty}(R^n)\, with\, \| g \|_{\infty} \le C(n) \lambda, \wedge b = \sum_{i=1}^{\infty} b_i\, where$

$(i)\, spt\, b_i \subset B_i\, with\, B_i\, a\, ball,$

$(ii) \int_{B_i} |bi|\, d\, y \le C(n) \lambda\, y(B_i),$

$(iii) \int_{R^n} bi = 0,$

$(iv) \{B_i\}\, have\, the\, bounded\, overlap\, property$

$\sum_{i=1}^{\infty} x_{B_i} \le C(n),$

$y(¿\, B_i) \le C(n) \frac{1}{\lambda} \| f \|_1.$

$(v) \sum_{i=1}^{\infty} ¿$

Remark (3.4.51):

(i) the constant C(n) depends only on the dimension n.

(ii) Note that

B

$(¿¿\, i) \le C(n)^2 \| f \|_1$

$\sum_{i=1}^{\infty} \| bi \|_1 \le C(n) \lambda \sum_{i=1}^{\infty} y\, ¿$

$which\, shows\, that$

$\sum_{i=1}^{\infty} bi\, converges \in L^1.\, Hence, b \in L^1(R^n)\, with\, \| b \|_1 \le C(n)^n \| f \|_{1.}$

(iii) *the that!* $g \in L^{\infty}(R^n)$ *implies* $g \in L^p(R^n)$ *for all* $\in |1,\infty|. \in$ *the*

case of $p=2,$

$$\| g \|_2 \leq \sqrt{\| g \|_1 \| g \|_\infty} \leq \sqrt{(1+C(n))^2 C(n)} \sqrt{\lambda \| f \|_1}.$$

proof of the Calderon − Zygmund decomposition, Recall that $M' f$ *is the*

uncentred maximal function of f *on balls of* R^n

. We know that the set Ω_λ *set*

$\{x \in R^n : M' f(x) > \lambda\}$ *is open* $\wedge$ *constsnt of finite measure by the*

Maximal Theorem:

$$y(\Omega_\lambda) \leq \frac{C}{\lambda} \| f \|_1 .$$

Also $\Omega_\lambda \neq R^n$. *Let* $\in$ *be a whitney covering of* Ω_λ. *Set* $\{B_i = c_1 \hat{B}_i : \hat{B}_i \in \varepsilon\}$

Where c_1 is the constant in the whiney covering

Lemma (2.3.20) Then, (iv) is proved and

$$y(¿ B_i) = \int \sum_{i=1}^{\infty} x_{B_i} d y \leq \int C(n) x_{\Omega_\lambda} d y \leq C \frac{C(n)}{\lambda} \| f \|_1$$

$$\sum_{i=1}^{\infty} ¿$$

Which proves (v)?

We can now take

$c \in (0,1)$ $(say, c = c_1^{-1})$ $\wedge$ *so* $\{c B_i\}$ *are mutually disjoint.*

Then construct a partition of unity

$\varphi_1\, so\ that \sum_i \varphi_i = 1\, on\, \Omega_{\lambda} \wedge \varphi_i = 1\, on$

$c\, B_i$. Explicitly,

$$\varphi_i = \frac{x_{B_i}}{\sum_j x_{B_i}}.$$

Now set,

$$b_i = \begin{cases} f\,\varphi_i - f_{B_i} f\,\varphi_i\, d\,y\, on\, B_i \\ 0\, otheerwise \end{cases}$$

Since we allow B_i to be closed we (i) is proved and (iii) is apparent from the construction of b_i.

Now, to prove (ii), we note that

$\int_{B_i} |b_i|\, d\,y \le 2 \int_{B_i} |f|\, d\,y \wedge ¿$

$$4\,B_i \cap^c \Omega_{\lambda} = 4\,c_1 \widehat{B}_i \cap^c \Omega \ne \varnothing\,.$$

$then, \int_{4B_i} |f|\, d\,y \le M' f(z)\, y(4\,B_i)\, for\, all\, z \in 4\,B_i.\, Choosing\, z \in^c \Omega_{\lambda}\, we$

$observe\, that\, M' f(z) \le \lambda \wedge so$

$$\int_{B_i} |b_i|\, d\,y \le 2 \lambda\, y(4\,B_i) \le 2\, 4^n \lambda\, y(B_i)$$

Which establish (ii).

Define:

$$g = \begin{cases} f\, on.^c\, \Omega_{\lambda} \\ \sum_i \left(f_{B_i} f\,\varphi_i\, d\,y\right) x_{B_i}\, on\, \Omega_{\lambda} \end{cases}$$

$then, on.^c\, \Omega_{\lambda}, f \le M' f \le \lambda\, almosteverywhere.\, On\, \Omega_{\lambda}, by\, invoking\, the$

Bounded overlap property,

$$\left|\sum_i \left(\int_{B_i} f\,\varphi_i\,dy\right) x_{B_i}\right| \le C(n)\,su_i\,p\left|\int_{B_i} f\,\varphi_i\,dy\right| \le C(n)\,su_i\,p\int_{B_i}|f|\,dy \le C(n)\,4^n \lambda$$

This completes the proof.

Theorem $(3.4.52)$

Every $T \in CZO_\alpha$ *is of weak type* $(1,1)$.

We have the following immediate consequence .

Corollary $(3.4.53)$ *let* $T \in CZO_\alpha$. *then , for all* $p \in (1,\infty)$, *T is strong type* (p,p).

proof . Since T is weak type $(1,1)$ *by the theorem* $\wedge$ ¿

strong type $(2,2)$ *by*

definition , we have that T is strong type (p,p) *for* $p \in (1,2)$.

Now , note that $T \in CZO_\alpha$ *implies that* T¿ $CZO_\alpha \wedge$ *so* T¿ *has abounded*

extension ¿ $L^p(R^n)$ *for* $2 < p < \infty$.

Theorem $(3.4.54)$ *let* $f \in L^1(R^n) \cap L^2(R^n) \wedge$ *fix* $\lambda > 0$.

We show that :

$$y\left(\{x \in R^n : |Tf(x)| \ge \lambda\}\right) \le \frac{C}{\lambda}\|f\|_1$$

With C independent of f and λ . Since we only know Tf(x) when $x \notin spt\, f$, *we use* the Calderon-Zygmund decomposition to localize .Let f = g +p this decomposition at level λ with the properties of g and b from Lemma (3.3.5.)Since f, g

$\in L^2(R^n)$, we also have $b \in L^2(R^n)$.

$Since\ b=\sum_{i=1}^{\infty} b_i\ with\ b_i=(f\varphi_i - m b_i f \varphi_i) x_{B_i}$ we have that this series convegres ¿ $L^2(R^n)$. So, $Tf=Tg+Tb$

And we estimate by Markovs inequality

$$A = y\left(\left\{x \in R^n : |Tg(x)| > \frac{\lambda}{2}\right\}\right) \leq \frac{4}{\lambda^2} \int_{R^n} |Tg|^2 dy \leq \frac{4}{\lambda^2} \| T \|_{¿}$$

$L^2(R^n)$ ¿ L ¿

b_i $\sum_{i=1}^{\infty}$ ¿ ¿

$T(b_i)$ with the series on the

$Now, T(b) = T$ ¿

¿ $L^2(R^n) \wedge |T(b)| \leq \sum_i T|(b_i)|$ almost everywhere. So, with

$c>1$ ¿ be chosen later

$$B = y\left(\left\{x \in R^n : |Tb(x)| > \frac{\lambda}{2}\right\}\right) \leq y\left(\left\{x \in R^n : \sum_{i=1}^{\infty} |Tb_i(x)| > \frac{\lambda}{2}\right\}\right)$$

∪

∪

$$(¿¿ j c B_i) + y\left(\left\{x \in R^n \{(¿¿ j c B_i) : \sum_{j=1}^{\infty} |Tb_i(x)| > \frac{\lambda}{2}\right\}\right)$$

$\leq y$ ¿

$$\bigcup$$
$$c B_i + \frac{1}{\lambda} \int_{R^n \setminus (\text{¿¿} j c B_i)}$$
$$\text{¿}$$
$$y \text{¿}$$
$$\le \sum_{j=1}^{\infty} \text{¿}$$

$$\bigcup$$
$$\le c^n \frac{C(n)}{\lambda} \| f \|_1 + \frac{1}{\lambda} \int_{R^n \setminus (\text{¿¿} j c B_i)} \sum_{i=1}^{\infty} |Tb_i(x)| d y$$

Consequently, it is enough to prove that

$$\text{¿}$$
$$i$$
$$su_i p \int_{\text{¿}} |Tb_i| d y \le C(T) \| b_i \|_1$$

Since $\| b_i \|_1 \le C(n) \lambda y(B_i)$ which gives

$$\sum_{i=1}^{\infty} \frac{1}{\lambda} \| b_i \|_1 \le C(n) \sum_{i=1}^{\infty} y(B_i) \le \frac{C(n)^2}{\lambda} \| f \|_1 .$$

We note that for almost every where

$$\text{¿}$$
$$x \in R^n \setminus \{ B \text{¿}_i , T b_i(x) = \int_{B_i} K(x, y) b_i(y) dy(y) .$$

Let y_i be the centre of the ball B_i .Since

$$\text{¿}$$
$$\int_{R^n} b_i d y = 0, almost\ all\ x \in R^n \setminus \{ B \text{¿}_i ,$$

$$T b_i(x) = \int_{B_i} (K(x-y) - K(x-y_i)) b_i(y) d y(y) .$$

We choose c = 2 since 2 $|y - y_i| \le 2 rad B_i \le |x - y| \wedge \text{¿}$

$$¿$$
$$i$$
$$x, y_i$$
$$\int_{\{x:|x-y|\geq 2|y-y_i|\}} |K(x,y) - K ¿ d\,y(x)| d\,y(y)$$
$$\int_{¿} |T\,b_i| d\,y \leq \int_{y\in B_i} |b_i(y)| ¿$$

$$\leq \int_{R^n} |b_i(y)| C_H(K)\, d\,y(y) \leq C_H(K) \| b_i \|_1$$

Where $C_H(K)$ is the Hormander constant associated with K. Taking an infimum on the right hand side, we have

$$¿$$
$$i$$
$$\int_{¿} |T\,b_i| d\,y \leq \| K \|_{CZO_\alpha} \| b_i \|_1 .$$

The sum A + B gives us the desired conclusion with constant

$$L(L^2(¿R^n)) + \| K \|_{CZO_\alpha} = C(n) \| T \|_{CZO_\alpha} .$$
$$\| T \|_{¿}$$
$$C \leq C(n) ¿$$

for a general $f \in L^1(R^n)$ *let* $f_k \to f$ *be a sequence which converges* $\in L^1(R^n)$

with each $f_k \in L^2(R^n)$. *Without loss of generality, assume that* $f_k \to f$

Almost everywhere (since we can pass to a subsequence).The weak type (1,1) condition gives that T f_k is Cauchy in measure and call $\hat{T} f$ *the limit.This* $\exists$

Almost everywhere and $\hat{T} f \quad \in L^{1,\infty}(R^n)$. *Furthermore ,*

$$\hat{T}(f)(x) = \int_{R^n} K(x,y) f(y) dy(y)$$

For almost every $x \in^c$ (spt f) with spt f compact.

Remark (3.4.55). It would also suffice to prove for general f in the previous Theorem by noting

$L^1(R^n) \cap L^2(R^n)$ *is dense* $\in L^1(R^n) \wedge$ *that* $T : L^1(R^n) \cap L^2(R^n) \to L^{1,\infty}(R^n)$ *is bounded.Since* $L^{1,\infty}(R^n)$ *is compl*

T *extends* ¿ *a bounded map* $\hat{T} : L^1(R^n) \to L^{1,\infty}(R^n)$.

Example (3.4.56). We note that

$$H\left(x_{|0,1|}\right)(x) = \frac{-1}{\pi} \in \left|\frac{x-1}{x}\right|$$

Whenever $x \notin |0,1|$.

this example is of importance because H *is a CZO*, $x_{|0,1|} \in L^1(R^{\cdot})$ *but* $H\left(x_{|0,1|}\right) \notin L^1(R^{\cdot})$.

Sec.(3.4) CZO and H^1

A natural question to ask is: what subspace of L^1 should we choose so that a CZO_α *maps that space back into* L^1.

Theorem (3.4.57) *let* $T \in CZO_\alpha$ *.then ,* T *induces a bounded operator* $H^1 \to L^1(R^n)$.

Corollary (3.4.57) *let* $T \in CZO_\alpha$ *. Then* T *extends* ¿ *a bounded operator* ¿

$L^\infty(R^n)$ ¿ BM

proof . Let $f \in L^{1,\infty}(R^n) \wedge g \in H^1$ *. Then* $L_g = \langle f, T^{tr} g \rangle$ *is a linear functional*

on H^1 *satisfing*

$$\underset{L\,¿}{\overset{H^1,\,L^1(R^n)}{¿}}$$

$$\left|\langle f, T^{tr} g \rangle\right| = \left|\int_{R^n} f\, T^{tr} g\, dy\right| \le \|f\|_\infty \|T^{tr}\|_{¿}$$

By duality , there $\exists$ *a* $\beta \in BMO$ *such that* $L = L_\beta$ *. Define* $Tf = \beta$ *, with* β

$idetified\ with\ L_\beta$

Remark (3.4.58) (i) this was originally proved directly, without alluding to duality .

(ii) we apply Tf ¿ $p-atoms$. Let $a \in \wp^p$. Then,

$$\langle T, f, \alpha\rangle = L_\beta(\alpha) = \int_{R^n} \beta\alpha \quad dy.$$

Let $B = B(yB, rB) = spt\ \alpha$. Then,

$$\langle T, f, \alpha\rangle = \int_{R^n} f T^{tr}(\alpha)\, dy.$$

¿

$R^n\} f T^{tr}(\alpha)\, dy.$

¿ $\int_{2B} f T^{tr}(\alpha)\, dy + \int$ ¿

¿

$R^n\} f(y)\left(\int_{R^n} K(x,y)\alpha(x)\, dy(x)\right) dy(y)$

¿ $b \int_{R^n} T(f x_{2B})\alpha\, dy + \int$ ¿

¿

$R^n\} f(y)(K(x,y) - K(yB,y))\, dy(y)$

$\int$ ❑ ¿

¿

¿ $\int_{R^n} T(f x_{2B})\alpha\, dy + \int_{R^n}$ ¿

by the application of Fubini. So, on B there $\exists\, a$

constant C_B such that $\beta(y) =$ ¿

¿

$R^n\} f(y)(K(x,y) - K(yB,y))\, dy(y) + C_B.$

$+\int$ ¿

proof of Theorem 3.4.1. We show that whenever $\alpha \in \wp^\infty$, then $T_\alpha \in L^1(R^n)$

with $\|T_\alpha\|_1 \le C(n,T)$. We automaticlly have $T_\alpha \in L^2(R^n)$ since $\alpha \in L^\infty(R^n)$

$¿ spt\, \alpha \subset B\, a\, ball. Then, since \|a\|_2 \leq \frac{1}{y(B)^{\frac{1}{2}}},$

$$L^2(R^n)$$
$$¿$$
$$L^2(R^n)$$
$$L¿$$
$$L¿$$
$$2(B)^{\frac{1}{2}} \|T\|_{¿}$$
$$2(B)^{\frac{1}{2}} \|T\alpha\|_{L^2(2B)} \leq y¿$$
$$\int_{2B} |T\alpha|\, dy \leq y¿$$

As in the proof of Theorem (3.3.25)

$$R^n\}|T\alpha|\, dy \leq C(n)\|K\|_{CZO_\alpha} \|a\|_1 .$$
$$\int_{¿}$$

$Since\ H^1 \subset L^1(R^n), Tf \in L^\infty(R^n)$ $for\ every\ f \in H^1. So, fix\ f \in H^1 \wedge pick\ a$

$representation: f = \sum_{j=1}^{\infty} \lambda_j a_j\ where \sum_{j=1}^{\infty} |\lambda_j| \leq 2\|f\|_{H^1}\ with\ a_j \in \wp^\infty. This$

$series\ converges\ almost\ everywhere \in L^1 ¿ f.$

$$Thus, T\left(\sum_{j=1}^{\infty} \lambda_j a_j\right) = Tf$$

Almost everywhere . Also

$$\sum_{j=1}^{\infty} \|\lambda_j T a_j\|_1 \leq \sum_{j=1}^{\infty} |\lambda_j| C(n, T) \leq 2\|f\|_{H^1} C(n, T).$$

$Thus, \sum_{j=1}^{\infty} \lambda_j T a_j converges \in L^1 \wedge hence,$

$\lambda_j T a_j = ¿ \left(\sum_{j=1}^{\infty} \lambda_j a_j \right)$
$\sum_{j=1}^{\infty} ¿$

$Hence\ Tf \in L^1(R^n) \wedge \| Tf \|_1 \leq 2C(n,T) \| f \|_{H^1}.$

Propositio(3.4.59
$¿ let\ T \in CZO_\alpha. Then, T1 is defined as a BMO function.$

$proof. Follows easily ¿ the that ! 1 \in L^\infty(R^n), \wedge T : L^\infty(R^n) \to BMO$

$is bounded.$

Remark (3.4.60)
$¿ compute\ T1, use the formula for Tf on each ball Bfor$

$f=1.$

Corollary (3.4.61 $¿$ $letT \in CZO_\alpha. Then T maps H^1 ¿ H^1 if \wedge ¿$

$only if T^{tr} 1 = 0$ in BMO .

Before we prove this corollary, we need the following lemmas.

Lemma($3.4.62 ¿ let\ T \in CZO_\alpha with associated kernel K \in CZK_\alpha \wedge a \in \wp^\infty$

$with spt a \subset B = B(yB, rB). For each j \in N with j \geq 1, let C_j(B) = 2^{j+1} B \{ 2 ¿^j B. Then, for all x \in C_j(B)$

$$|Ta(x)| \leq \| K \|_{CZK_\alpha} 2^{-j(n+a)} r_B^{-n}.$$

Proof. We compute and use the α $regularity of K,$

$$\left|Ta(x)\right| \leq \| K \|_{CZK_{\alpha}} \int_{y \in B} \left(\frac{\left|y - y_B\right|}{\left|x - y_B\right|} \right)^{\alpha} \frac{1}{\left|x - y_B\right|^{n}} \left|f(y)\right| d\, y(y)$$

$$\leq \| K \|_{CZK_{\alpha}} r_B^{\alpha} \frac{1}{(2^j r_B)^{n+1}} \int_{y \in B} \left|a(y)\right| d\, y(y)$$

And the result follows since

$$\int_{y \in B} \left|a(y)\right| d\, y(y) \leq 1.$$

Lemma (3.4.63) $let\, m : R^n \rightarrow C \wedge B = B(yB, rB)\, a\, ball\, such\, that$

1. $\int_{2B} |m|^2 d\, y \leq \frac{c}{y(B)},$

2.for $every\ j \in N, j \geq 1, \wedge x \in C_j(B) = 2^{j+1} B \{ 2¿^j B\, we\, have$

$\left|m(x)\right| \leq \| K \|_{CZK_{\alpha}} 2^{-j(n+\alpha)} r_B^{-n}.$

$then, m \in H^1 \wedge \| m \|_{H^1}\, dose\, not\, exceed\, a\, constsnt\, depending\, on\, n,$

$\| K \|_{CZK_{\alpha}} \wedge \alpha > 0$

The proof is left as an exercise.

Chapter 4

Convolution Theorem of the Hilbert Transform

For the Hilbert transform

$$\hat{f}(x)=\frac{1}{\pi}\int_{R}\frac{f(t)}{x-t}dt$$

A new proof of the convolution formula is given. This convolution formula is then applied to calculate some Cauchy's integrals and to solve a nonlinear singular integral equation.

Applications of the convolution formulae of Fourier, Laplace and Millen transforms are well-known. Recently some applications of the convolutions formulae for Hanklel. Stieltjes transforms are given for the Hilbert transform

$$H|f|(x)=\hat{f}(x)=\frac{1}{\pi}\int_{R}\frac{f(t)}{x-t}dt\,,(4.1)$$

The convolution theorem has been established in L_p spaces is missing in modern text books on integral transforms. In this research we give an another proof of this theorem and then apply this result to calculate some Cauchy integrals of special functions and to obtain explicit solutions of a nonlinear singular equation.

Sec.(4.1) Convolution Theorem

Let f,g be defined on R and belong corresponding to $L_p(R), \wedge L_q(R), 1<p,q<\infty, p^{-1}+q^{-1}<1.$

Then Hilbert transform $\tilde{f}\wedge\tilde{g}$ of $f\wedge g$ exit and belong to $L_p(R)\wedge L_q(R),$ too. Furthermore $fg\in L_r(R)$ with $r^{-1}=p^{-1}+q^{-1}$. Consequently, the Hilbert transform $\dot{fg}$ of fg exists and belongs to $L_r(R).$ Therefore, if we put

$$h(x)=(f\otimes g)(x)=\frac{1}{\pi}\int_R (f(x)g(t)+g(x)f(t)-f(t)g(t))\frac{dt}{x-t}(4.1.2)$$

then h exists and belongs to $L_r(R).$ Our main result in this paragraph is a new proof of the following.

Theorem (4.1.1) The Hilbert transform of h is the product of the Hilbert transforms of $f\wedge g$

$$\hat{h}(x)=\hat{f}(x)\hat{g}(x) \quad . \tag{4.1.3}$$

Proof. Let $f\wedge g$ belongs to S, the space of infinitely differentiable functions which , together with their derivatives, approach zero more rabidly than any power of $|x|^{-1} as |x|\rightarrow\infty.$ Applying the Hilbert transform to the function $h(x)$ we obtain

$$\hat{h}=\hat{f}\,\hat{g}+\hat{g}\hat{f}+fg.(4.1.4)$$

Applying now the Fourier transform

$$F|f|(x)=\int_R f(t)\exp(-ixt)\,dt,(4.1.5)$$

To using the properties

$$F|\hat{f}|=-isgnxF|f|(4.1.6)$$

And 2 $\pi F|fg| = F|f| \odot F|g|,$ where

$$f \odot g = \int_R f(t) g(x-t) dt \quad (4.1.7)$$

is the Fourier convolution, we get

$$2\pi[\hat{h}] = 2\pi F[\hat{f}\hat{g} + \hat{f}\hat{g} + fg]$$

= -2 $\pi i \operatorname{sgn} x F[f\hat{g} + \hat{f}g] + F[fg]$

= -isgn

$x\left(F|f| \odot F|\hat{g}| + F|\hat{f}| \odot F|g|\right) + F|f| \odot F|g|$

= -sgn

$\operatorname{sgn} x F|f| \odot F|g|$
$F|f| \odot (\operatorname{sgn} x F|g|) + ¿ + F|f| \odot F|g|$
$x¿$

=

$$(-i \operatorname{sgn} x F|f|) \odot (-i \operatorname{sgn} x F|g|) = (F|\hat{f}|) \odot (F|\hat{g}|)$$

consequently

$$\hat{h} = \hat{f}\hat{g},$$

That means h is the convolution of the Hilbert transform.

Since the space S is dense in $L_p(R)$ and $L_q(R)$,where Hilbert transform is bounded, formula, first proved to be valid on dense subspaces of $L_p(R)$ and $L_q(R)$ still holds for all $f \in L_p(R) \wedge g \in L_q(R).$ Thus Theorem is proved.

Sec.(4.2) Evaluation of some Cauchy Integral

Let $g = \hat{f}.$ Then formula becomes

$$h = -f^2 + \hat{f}^2 - \hat{f}\hat{f}. \quad (4.1.8)$$

But $\hat{h} = \hat{f}\hat{g} = -f\hat{\hat{f}}$. Therefore, $h = \tilde{f}\tilde{f}$. Consequently, we have

$$\hat{f}^2(x) - f^2(x) = \frac{2}{\pi}\int_R \frac{f(t)\hat{f}(t)}{x-t}dt$$

The upper formula can be applied to evaluate new Hilbert transforms. Namely, if the Hilbert transform of f is known, then Hilbert transform of $f\hat{f}$ is $\frac{1}{2}(\hat{f}^2 - f^2)$ For example, let $f(x) = \exp(-|x|)I_0(x) \in L_{p(R)}$. Then $\hat{f}(x) = 2\sinh(x)K_0(x)$. Therefore,

$$-\int_R \frac{\exp(-|t|)\sinh(t)K_0(t)I_0(t)}{x-t}dt = \sinh^2(x)K_0^2(x) - \frac{1}{4}\exp(-2|x|)I_0^2(x).$$

Using tables of Hilbert transform one can calculate new Cauchy integrals by this method.

(4.2.1) A Nonlinear Singular Integral Equation

Consider now a nonlinear singular integral equation

$$\lambda f(x) + \frac{2}{\pi}f(x)\int_R \frac{f(t)}{x-t}dt - \frac{1}{\pi}\int_R \frac{f^2(x)}{x-t}dt = g(x). (4.1.11)$$

This equation can be rewritten in the equivalent form

$$\lambda f(x) + (f \otimes f)(x) = g(x)$$

Applying now the Hilbert transform using Theorem we have

$$\lambda \hat{f} + \hat{f}^2 = \hat{g}$$

Solving this equation we obtain

$$\hat{f}(x) = \frac{-\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} + \hat{g}(x)}$$

Here $\sqrt{\frac{\lambda^2}{4} + \hat{g}(x)}$ is a branch of the square such that $\mathfrak{R}$ $\left\{\sqrt{\frac{\lambda^2}{4} + \hat{g}(x)}\right\} \geq 0.$ Let $\lambda = 0.$ If $f \in L_p(R).$ Then $g \in L_{p/2}(R)$ and therefore , $\hat{g} \in L_{p/2}(R).$ We have

$$\hat{f}(x) = \pm\sqrt{\hat{g}(x)}$$.

Taking

$\hat{f}_\Omega(x) = ¿$ $\sqrt{\hat{g}(x)}$ if $x \in \Omega - \sqrt{\hat{g}(x)},$

Otherwise,

where Ω is any measurable subset of R. It is not difficult to see that f_Ω consist all of solutions of the equation.

Let $\lambda \neq 0.$ We choose

$\hat{f}_\Omega(x) = \{ -\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \hat{g}(x)}$ if $x \in \Omega$ (4.1.12)

$$\frac{-\lambda}{2} - \sqrt{\frac{\lambda^2}{4} + \hat{g}(x)}$$

otherwise it is easy to see that if *f* is a solution of (4.1.12) , then its Hilbert transform has the form (20). But not every f_{Ω} belongs to $L_p(R)$. We show that $\hat{f}_{\Omega} \in L_p(R)$ if and only if

$$|\Omega| < \infty \, if \, R\lambda < 0$$

$$|R/\Omega| < \infty \quad \text{otherwise} \qquad (4.1.13)$$

where $|\Omega|$ is the measure of Ω. Indeed, let $\Re\lambda$ ¿0. Then

$$\|\hat{f}_{\Omega}\|^{p} \geq \int_{\Omega} \left| \frac{-\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \hat{g}(x)} \right| dx \geq \left|\frac{\lambda}{2}\right|^{p} |\Omega|$$

Therefore, if $\hat{f}_{\Omega} \in L_p(R)$, then $|\Omega|$ ¿∞. We prove that this condition is not only necessary, but also sufficient. We have

$$\|\hat{f}_{\Omega}\|_p^p =$$

$$\|\tilde{f}_{\Omega}\|^p_{L_p(\Omega)} + \left|\frac{2}{\lambda}\right|^p \int_{R/\Omega} \left| \frac{\hat{g}(x)}{1+\sqrt{1+4\lambda^{-2}\hat{g}(x)}} \right|^p dx$$

$$\leq \|\hat{f}_{\Omega}\|^p_{L_p(\Omega)} + \left|\frac{2}{\lambda}\right|^p \|\hat{g}\|_p < \infty.$$

Analogously for the case $\Re\lambda \geq 0$.

Therefore all solutions of the equation (14) are Hilbert transforms of $-\hat{f}_{\Omega}$ having form (20) with the condition(21).

(4.2.2)Singular integral operators of convolution

In mathematics, Singular integral operators of convolution type are the singular integral operators that arise on R^n and T^n through convolution by distributions; equivalently they are the singular integral operators that commute with translations. The classical examples in harmonic analysis are the harmonic conjugation operator on the circle, the Hilbert transform on the circle and the real line, the Beurling transform in the complex plane and the Riesz transform in Euclidean space. The continuity of these operators on L^2 is evident because the fourier transform converts them into multiplication operators. Continuity on L^p spaces was first established by Marcel Risez. The classical techniques include the use of poisson integrals, interpolation theory and they hardy-Littewood maximal function.

(4.2.3) Hilbert transform on the circle

See also: Harmonic conjugate

The theory for L^2 functions is particularly simple on the circle.

Then it has a fourier series expansion

$$f(\theta)=\sum_{n\varepsilon Z} a_n z^{in\theta}$$

Hardy space H^2 (**T**) consists of the functions for which the negative coefficients vanish, a = 0 for n $¿o$. These are precisely the square-integrable functions that arise as boundary values of holomorphic functions in the open unit disk. Indeed **f** is the boundary value of the

function

$F(z)=\sum_{n\geq 0} a_n z^n$,

In the sense that the functions

$f_r(\theta)=F(\Re^{i\theta}),$

Defined by the restriction of F to the concentric circles

$|z|=r, satisfy$

$\|f_r-f\|_2 \to 0.$

$the\ orthogonal\ projection\ P\ of\ L^2(T)\ onto\ H^2(T)\ is\ called$

$the\ Szego\ projection$

$it\ is\ a\ bounnded\ operator\ on\ L^2(T)\ with\ operator\ norm$

$1: By\ Cauchys\ theorem$

$$\frac{f(\varsigma)}{\varsigma-z}d\varsigma=¿\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(\theta)}{1-e^{-i\theta}z}d\theta.$$

$$F(x)=\frac{1}{2\pi}\int_{|\varsigma|=1}^{\pi}¿$$

Thus

$$F(re^{i\varphi})=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(\varphi-\theta)}{1-re^{i\varphi}}d\theta.$$

$where\ r=1, the\ integrand\ on\ the\ ¿\ hand\ side\ has\ a$

$singuularity\ at\ \theta=0.$The $truncated\ Hilbert\ transform\ is$

$defined\ by$

$$H_{\varepsilon} f(\varphi)=\frac{i}{\pi}\int_{\varepsilon\le|\theta|\le\pi}\frac{f(\varphi-\theta)}{1-r\,e^{i\varphi}}\,d\theta=\frac{1}{\pi}\int_{|\varsigma-e^{i\theta}|\ge\delta}\frac{f(\varsigma)}{\varsigma-e^{i\varphi}}\,d\varsigma\,,$$

$where\ \delta=\left|1-\theta^{i\varepsilon}\right|.\ Since\ is\ defined\ as\ convolution\ with\ a$

$bounded\ function,\ it\ is\ a\ bounded\ operator\ on\ L^2(T).\ Now$

$$\int_{\varepsilon}^{\pi} 2R(1-¿\,e^{i\theta})^{-1}\,d\theta=\frac{i}{\pi}\int_{\varepsilon}^{\pi}1\,d\theta=i-\frac{i\varepsilon}{\pi}.$$

$$H_{\varepsilon} I=\frac{i}{\pi}¿$$

If f is polynomial in z then

$$H_{\varepsilon} f(z)-\frac{i(1-\varepsilon)}{\pi}f(z)=\frac{1}{\pi i}\int_{|\varsigma-z|\ge\delta}\frac{f(\varsigma)-f(z)}{\varsigma-z}\,d\varsigma.$$

By Cauchy theorem the right hand side tends to 0 uniformly as ε and hence $\delta tends\ ¿\ 0\ so$

$$H_{\varepsilon} f\rightarrow if$$

Uniformly for polynomials. On the other hand if u(z) =zit is immediate that

$$H_{\varepsilon}' f=-u^{-1}H_{\varepsilon}(uf')$$

$Thus\ if\ f\ is\ a\ polynomial\in z^{-1}\ without\ constsnt term$

$$H_{\varepsilon} f \qquad \rightarrow if\ uniformaly\,.$$

Define the Hilbert transform on the circle by

$$H=i(2\mathrm{P}-1).$$

Thus if f is a polynomial in trigonometric polynomial

$$H_\varepsilon f \quad \to if\ uniformly$$

It follows that if f is any L^2 function

$$H_\varepsilon f \quad \to if$$ In the L^2 norm.

This is an immediate consequence of the result for trigonometric polynomial once it is established that the operators H_ε are uniformly bounded in operator norm. But on $[-\pi,\pi]$

$$\begin{array}{c}(1-e^{i\theta})\\ ¿[¿¿-1-i\theta^{-1}]+i\theta^{-1}.\\ (1-e^{i\theta})^{-1}¿\end{array}$$

The first term is bounded on the whole of $[-\pi,\pi]$ so it suffices to show that the convolution operators S_ε defined by

$$S_\varepsilon f(\varphi)=\int_{\epsilon\le|\theta|\le\pi} f(\varphi-\theta)\theta^{-1}d\theta$$

Are uniformly bounded. With respect to the orthonormal basis e^{ine}

Convolution operators are diagonal and their operator norms are given by taking the supremum of the moduli of the Fourier coefficients. Direct computation shows that these all have the form

$$\frac{1}{\pi}\left|\int_a^b \frac{\sin t}{t}dt\right|$$

With 0 ¿ a ¿ b.These integrals are well-known to be uniformly bounded.

It also follows that, for a continuous function f on the circle, H_ε f converges uniformly to Hf, so in particular pointwise. The pointwise limit is a Cauchy principal value, written

$$Hf = P.V. \frac{1}{\pi} \int \frac{f(\zeta)}{\zeta - e^{i\varphi}} d\zeta.$$

If f is just in L^2 then H_ε f converges to Hf pointwise almost everywhere. In fact define the poisson operators on L^2 functions by

$$T_r\left(\sum a_n e^{i\varphi}\right) = \sum r^{|n|} \sum a_n e^{i\varphi},$$

to f in L^2 as r increases to 1. Moreover, as lebesgue proved, T_r^f also tends pointwise to f at each lebesgue point of f. on the other hand, it is also known that T_r^f - H_1 – r f tends to zero at each lebesgue point of f. Hence H_1 – r f tends pointwise to f on the common lebesgue points of f and Hf and there for almost everywhere.

Result of this kind on pointwise convergence are proved more generally below for L^p functions using the poisson operators and the Hardy-Littlewood maximal function of f.

The Hilbert transform has a natural compatibility with orientation-presrving diffeomophisms of the circle. Thus if H is a diffeomorphism on the circle with

$$H\left(e^{i\varphi}\right)=e^{ih(\theta)},h(\theta+2\pi)=h(\theta)+2\pi ,$$

Then the operators

$$H_{\varepsilon}^{h}f\left(e^{i\varphi}\right)=\frac{1}{\pi}\int_{\left|e^{ih}-e^{i\varphi}\right|\geq\varepsilon}\frac{f\left(e^{i\theta}\right)}{e^{i\theta}-e^{i\varphi}}d\theta ,$$

Are uniformly bounded and tend in the strong operator topology to H. Moreover if Vf(z) = f(H(z)), then VH V^{-1} _H is an operator with smooth kernel, so a Hilbert-Schmidt operator.

In fact if G is the inverse of H with corresponding function

$$\left(V H_{\varepsilon}^{h} V^{-1}-H_{\varepsilon}\right)=\frac{1}{\pi}\int_{\left|e^{i\theta}-e^{i\varphi}\right|\geq\varepsilon}\left[\frac{g'(\theta)e^{ig(\theta)}}{e^{ig(\theta)}-e^{ig(\varphi)}}-\frac{e^{i\theta}}{e^{i\theta}-e^{i\varphi}}\right]f\left(e^{i\theta}\right)d\theta.$$

Since the kernel on the right hand side is smooth on **T** $\times T$, it follows that the operators on the right hand side are uniformly bounded and hence so too are the operators H_{ε}^{h} **.** To see that they tend strongly to H, it suffices to check this on trigonometric polynomials. In the case

$$H_{\varepsilon}^{h}f(\zeta)=\frac{1}{\pi i}\int_{|H(z)-H(\zeta)|\geq\varepsilon}\frac{f(z)}{z-\zeta}dz=\frac{1}{\pi i}\int_{|H(z)-H(\zeta)|\geq\varepsilon}\frac{f(z)-f(\zeta)}{z-\zeta}dz+\frac{f(\zeta)}{\pi i}\int_{|H(z)-H(\zeta)|\geq\varepsilon}\frac{dz}{z-\zeta}$$

In the first integral the integral is a trigonometric polynomial in z and ζ

And so the integral is a trigonometric polynomial ζ It tends in L^2 to the trigonometric polynomial

$$\frac{1}{\pi i}\int\frac{f(z)-f(\zeta)}{z-\zeta}dz.$$

The integral in the second term can be calculated by the principle of the argument. It tends in L^2 to the constant function 1, so that

$$\lim_{\varepsilon\to 0} H_\varepsilon^h f(\zeta)=f(\zeta)+\frac{1}{\pi i}\int\frac{f(z)-f(\zeta)}{z-\zeta}dz,$$

Where the limit is in L^2 . On the other hand the right hand side is independent of the diffeomorphism . since for the identity diffeomorphism, the left hand side equals Hf, (this can also be checked directly if f is atrigonometric poly nomial). Finally, letting $\varepsilon\to 0,$

$$(VHV^{-1}-H)f(e^{i\varphi})=\frac{1}{\pi}\int\left[\frac{g'(\theta)e^{ig(\theta)}}{e^{ig(\theta)}-e^{ig(\varphi)}}-\frac{e^{i\theta}}{e^{i\theta}-e^{i\varphi}}\right]f(e^{i\theta})d\theta.$$

The direct method of evaluating Fourier coefficients to prove the uniform boundedness of the operator H^ε dose not generalize directly to L^P spaces with $1>o>\infty.$ Instead a direct comparison of $H^\varepsilon f$ with the poisson integral of the Hilbert transform is used classically to prove this . If f has Fourier series

$$f\left(e^{i\theta}\right)=\sum_{n\in z} a_n e^{in\theta},$$

It is Poisson integral is defined by

$$P_r f\left(e^{i\theta}\right)=\sum_{n\in z} a_n r^{|n|} \cdot =\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(1-r^2\right)f\left(e^{i\theta}\right)}{1-2r\cos\theta+r^2}d\theta=K_r \star f\left(e^{i\theta}\right),$$

Where the Poisson kernel K_r is given by

$$K_r\left(e^{i\theta}\right)=\sum_{n\in z} r^{|n|} \cdot =\frac{1-r^2}{1-2r\cos\theta+r^2}.$$

$¿ f\ is \in L^p(T)\ then\ the\ operators\ P_r\ satisfy$

$$K_r(¿\, e^{i\theta})\,d\theta=1.$$

$$\|K_r\|_1=\frac{1}{2\pi}\int_0^{2\pi} ¿$$

Thus operators P_r have operator norm bounded by 1 on L^p . The convergence statement above follows by continuity from the result for trigonometric polynomials, where it is an immediate consequence of the formula for the Fourier coefficients of K_r .

The uniform boundedness of the operator norm of H_ε follows because HP_r –H_{1-r} is given as convolution by the function Ψ_r where [7].

$$\psi_r\left(e^{i\theta}\right)=1+\frac{1-r}{1+r}\cot\left(\frac{\theta}{2}\right)K_r\left(e^{i\theta}\right)\leq 1+\frac{1-r}{1+r}\cot\left(\frac{1-r}{2}\right)K_r\left(e^{i\theta}\right)$$

$$for\ 1-r\leq|\theta|\leq\pi,\wedge,for\ |\theta|<1-r,$$

$$\psi_r(e^{i\theta}) = 1 + \frac{2\mathrm{r}\sin\theta}{1 - 2\mathrm{r}\, cos\theta + r^2}.$$

These estimate show that the L^1 norms $|\Psi_r|$ are uniformly bounded. Since H a bounded operator, it follows that the operators H_ε are uniformly bounded in operator norm on L^2(T). The same argument can be used on L^p(T) once it is known that the Hilbert transform H is bounded in operator norm on L^p(T).

(4.2.3) Hilbert transform on the real line

See also: Hilbert transform

As in case of the circle, the theory for L^2 functions is particularly easy to develop. In fact, as observed by Rosenblum and Devinatz, the two Hilbert transform can be related using the Cayley transform.[8].

The Hilbert transform H$_R$ on L^2(R) is defined by

$$\widehat{H_R f} = (i X_{[0,\infty]} - i X_{[-\infty,0]})\hat{f}$$

Where the fourier transform is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx.$$

Define the hardy space H^2 **(R)** to be the closed subspace of L^2 **(R)** consisting of functions for which the fourier transform vanishes on negative part of the real axis. Its orthogonal complement is given by functions for which the fourier transform vanishes on the positive part of the real axis. It is the complex conjugate of H^2 **(R).**if

p_R is the orthogonal projection onto H^2 **(R),** then

$H_R = i(2P_R - 1).$

The cayley transform

$$C(x) = \frac{x-i}{x+i}$$

Carries the extended real line onto the circle, sending the point at ∞ to 1, And the upper halfplane onto the unit disk.

Define the unitary operator from L^2 **(T)** onto L^2 **(R)** by

$$Uf(x) = \pi^{\frac{-1}{2}}(x+i)^{-1} f(C(x)).$$

The operator carries the Hardy space of the circle H^2 **(T)** onto H^2 **(R).**in fact for $|w| <¿$ 1, the linear span of the functions

$$f_w(z) = \frac{1}{1-wz}$$

Is dense in H^2(T). Moreover

$$U f_w(x) = \frac{1}{\sqrt{\pi}} \frac{1}{(1-w)(x-\acute{z})}$$

Where

$z = C^{-1}(\acute{w})$.

On the other hand, for z $\in H,$ the linear span of the functions

$$g_{\dot{z}}(t) = e^{it\dot{z}} x_{0,\infty}(t)$$

Γthey are the Fourier transforms of

$$h_z(x) = \widehat{g_{\square}}(-x) = \frac{i}{\sqrt{2\pi}}(x+z)^{-1}$$

So the linear span of these functions is dense in H^2(R).Since U carries the $f_{\dot{w}}$ s onto multiple of the $h_{\dot{z}}$ s, it follows that U carries H^2(T) onto H^2(R). thus

$$U H_T U^{\dot{\iota}} = H_{R.}$$

In Nikolski (1986), part of the L^2 theory on the real line and the upper halfplane is developed by transferring the results from the circle and the unit disk**.** the natural replacements for concentric circles in the disk are lines parallel to the real axis in H. under the Cauchy transform these correspond to circles in the disk that are tangent to the unit circle at the point one. The behavior of functions in H^2 (T) on these circles is part of the theory of Carleton measures. The theory of singular integrals, however, can be developed more easily by working directly on R.

H^2 (R) consists exactly of L^2 functions f that arise of boundary values of holomorphic functions on **H** in the following sense **:**

Is in H^2 provided that there is a holomorphic function F(z) on **H** such that the functions $f_y(x)=f(x+iy)\, for\, y>0\, are \in L^2 \wedge f_y\, tends ¿ f \in L^2\, as\, y \to 0$. In this cases F is necessary unique and given by Cauchy integral formula:

$$F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds.$$

$¿ identifing!\, H^2\, with\, L^2(0,\infty)\, via\, the\, Fourier\, transform, for$

$y>0\, multiplication\, by\, e^{-yt}$

$On\, L^2(0,\infty)\, includes\, a\, contraction$ semi group V_y on H^2 . Hence for in L^2.

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(s)\hat{g};(s)ds=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(s)g;(s)ds=V_y Pf(x).$$

$if\, f\, is \in H^2, F(z)\, is\, holomorphic\, for\, \Im z>0, since\, the\, family\, of$

$L^2\, function\, g_z depends\, holomorphiclly\, on\, z. Moreever\, f_y=V_y f$

$tends ¿ f \in H^2\, since\, this\, is\, true\, for\, the\, Fourier\, trans$

$forms.$

Conversely if such an F exists, by Cauchy's integral theorem and the above identify applied to f_y

$$f_{y+t} = V_t P f_y$$

$$for\ t > 0, Letting\ t\ tend ¿ 0, it\ follow\ that\ P f_y = f_y\ lies \in H^2 .$$

$$But\ then\ so\ too\ dose\ the\ limit\ f. Since$$

$$V_t f_y = f_{y+t} = V_y f_t ,$$

Uniqueness of F follows from

$$f_t = \lim_{y \to 0} f_{y+t} = ¿ \qquad \lim V_t f_y \underset{y \to 0}{=} V_t f.$$

Foe F in L^2, the truncated Hilbert transforms are defined by

$$H_{\varepsilon,R} f(x) = \frac{1}{\pi} \int_{\varepsilon \le |y-x| \le R} \frac{f(y)}{x-y} dy = \frac{1}{\pi} \int_{\varepsilon \le |y| \le R} \frac{f(x-y)}{y} dy$$

$$H_\varepsilon f(x) = \frac{1}{\pi} \int_{|y-x| \ge \varepsilon} \frac{f(y)}{x-y} dy = \frac{1}{\pi} \int_{|y| \varepsilon} \frac{f(x-y)}{y} dy$$

The operator $H_{\varepsilon,R}$ are convolution by bounded functions of compact support, so their operator norms are given by the uniform norm of their Fourier transforms. As before the absolute values have the form

$$\frac{1}{\sqrt{2\pi}} \left| \int_a^b \frac{2 \sin t}{t} dt \right| .$$

With 0 < a < b, so the operators $H_{\varepsilon,R}$ are uniformly bounded in operator norm. Since $H_{\varepsilon,R}$ f tends $H_\varepsilon f \in ¿$ L^2 for f with compact support, and hence for arbitrary f, the operators H_ε are also uniformly bounded in operator norm.

To prove that H_ε f tends to Hf as ε tends to zero, it suffices to check this on a dense set of functions . On the other hand,

$$H'_\varepsilon f = -H_\varepsilon(\hat{f}),$$

So it suffices to prove that H_ε f tends to if for a dense set of function in H^2 (R), for example the Fourier transforms of smooth functions g with compact support in (0, ∞). But the Fourier transform f extends to an entire function F on C, which is bounded on Im(z)≥0. The same is true of the derivatives of g. Up to a scalar these correspond to multiplying F(z) by powers of z. Thus F satisfies a payley –Wiener estimate for Im(z)≥0.[10]

$$\left|F^{(m)}(z)\right| \le K_{V,m}(1+|z|)^{-N}$$

For any m, N ≥ 0. In particular, the integral define $H_\varepsilon f(x)$ can be computed by taking a standard semicircle contour centered on x, it consists of a large semicircle with radius R and a small circle radius ε with the two portions of the real axis between them. By Cauchy's theorem, the integral round the contour is zero. The integral round the large contour tends to zero by the Payley-Wiener estimate. The integral on the real axis is the limit sought. It is therefore given as minus the limit on the small semicircular contour. But this is the limit of

$$\frac{1}{\pi}\int_\Gamma \frac{F(z)}{z-x}\,dz.$$

Where Γ is small semicircular contour, oriented anticlockwise. By the usual techniques of contour integration, this limit equals if(x).[11] In this case, it is easy to check that the convergence is dominated in L^2 since

$$H_\varepsilon f(x) = ¿ \quad \frac{1}{\pi} \quad \int_{|y-x|\geq\varepsilon} \frac{f(y)-f(x)}{y-x} dy = \frac{1}{\pi} \int_{|y-x|\geq\varepsilon} \int_0^1 f'(x+t(y-x))\,dt\,dy$$

So that convergence is dominated by

$$G(x) = \frac{1}{2\pi} \int_0^1 \int_{-\infty}^{\infty} |f'(x+ty)|\,dy$$

Which is in L^2 by the Payley-Wiener estimate.

It follows that for f on $L^2(R)$

$$H_\varepsilon f \to Hf.$$

This can also be deducted directly because, after passing to Fourier transform, H_ε and H become multiplication operators by uniformly bounded functions. The multipliers for H_ε tend pointwise almost everywhere to the multiplier for H, so the statement above follows from the dominated convergence theorem applied to the Fourier transforms.

As for the Hilbert transform on the circle, H_ε f tends to Hf pointwise almost everywhere if f is an L^2 function. In fact, define the poisson operators on L^2 function by

$$T_y f(x) = \int_{-\infty}^{\infty} P_y(x-t) f(t)\,dt,$$

Where the Poisson kernel is given by

$$P_y(x) = \frac{y}{\pi(x^2+y^2)}.$$

For y > 0 Its Fourier transform is

$$\hat{P}_y(t) = e^{-y|t|,},$$

From which it is easy to see that T_y f tends to f in L^2 as y increases to 0. Moreover, as Lebesgue proved, T_y f also tends pointwise to f at each Lebesgue point of f. On the other hand, it is also known that T_y H f –H_y f tends to zero at each Lebesgue point of f. Hence H_ε f tends pointwise to f on the common Lebesgue points of f and Hf and therefore almost everywhere.[12][13]. The absolute values of the functions T_yf- f and T_y H f–H_y f can be bounded pointwise by multiples of the maximal function of f.[14].

As for the Hilbert transform on the circle, the uniform boundedness of the operator norms of H_ε follows from that of the T_ε if H is known to be bounded, since H T_ε - H_ε is convolution operator by the function

$$g_\varepsilon(x)=\begin{cases} \dfrac{x}{\pi(x^2+\varepsilon^2)} & |x|>\varepsilon \\ \dfrac{x}{\pi(x^2+\varepsilon^2)}-\dfrac{1}{\pi x \pi} & |x|>\varepsilon \end{cases}$$

The L^1 norms of these functions are uniformly bounded.

(4.2.5) Convolution: The Hilbert transform can be realized as a convolution with tempered distribution

$$h(t)=p.v,\frac{1}{\pi t}$$

Thus formally.

$$H\vee(u)=h*u$$

Alternatively, one may use the fact that $h(t)$ is the distributional derivative of the function $\log\frac{|t|}{\pi}$

$$H(u)(t) = \frac{d}{dt}\left(\frac{1}{\pi}(u * \log|.|)(t)\right)$$

For most operational purposes the Hilbert transform can be treated as a convolution, the convolution of Hilbert transform of either vector is

$$H(u*v) = H(u)*v = u*H(v)$$

This rigorously true if $u \wedge v$ are compactly supported distributions cines, in that case,

$$h*(u*v) = (h*u)*v = u*(h*v)$$

by passing to an appropriate limit, it is thus also true if $u \in L^p$ and $v \in L^r$ provided

$$1 < \frac{1}{p} + \frac{1}{r}$$

(4.2.6) Conjugate functions: The Hilbert transform can be understood in terms of a pair of functions $f(x) \wedge g(x)$ such that the function

$$F(x) = f(x) + ig(x)$$

Is the boundary value of a holomorphic function $F(z)$. Under these circumstances if $f and\, g$ are sufficiently integrable then one is the Hilbert transform of the other. Suppose that $f \in L^p(R)$.

Then by the theory of the Poisson integral, *f* admits a unique harmonic extension in to the upper half-plane, and the extension given by

$x-s$

¿

¿

¿ $2+y$

¿

$f(s)\frac{y}{¿}$

$$u(x+iy)=y(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty} ¿$$

which the convolution of *f* with Poisson kernel

$$P(x,y)=\frac{1}{\pi}\frac{y}{x^2+y^2}$$

thus

$$v(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\frac{x-s}{x-s^2+y^2}\,ds$$

.

Chapter 5

Hilbert Transform and Applications

Sec.(5.1) Mathematical foundations of Hilbert transform

The desire to construct the Hilbert transform stemmed from simple quest: Given a real-valued function $f: R \to R,$ can we find an imaginary part ig such that $f_c = f + ig$ can be analytically extended? For example, $if f(x) = \cos(x),$ then by inspection we can find $g(x) = \sin(x) such that f_c(x) = f + ig = \exp(ix).$ This function can obviously be extended analytically to the entire complex plane by replacing the real variable x with the complex[153] variable z in the expression, the result is $f_{ext}(z) = \exp(iz)$ and we have

$$\Re\{f_{ext}(z)\} ¿_{z=x} = f(x),$$

Which states that real part of the extended function is equal to the original given function $f(x)$ on the real line. The companion function $g(x)$ is called the Hilbert transform of $f(x)$.

(5.1.2)Hilbert transform as a boundary-value problem

To establish the uniqueness of the companion function, we first note that any analytic function $f_{ext}(x) = f_R(z) + i f_I(z)$ defined on the complex plane $z = x + iy$ must satisfy Cauchy-Riemann equations,

$$\frac{\partial^2 f_R}{\partial x} = \frac{\partial f_I}{\partial y},$$

$$\frac{\partial f_I}{\partial x^2} = \frac{\partial f_R}{\partial y}.$$

Consequently, both $f_R \wedge f_I$ satisfy Laplace's equation,

$$\frac{\partial F_R}{\partial x^2} + \frac{\partial^2 F_R}{\partial y^2} = 0,$$

$$\frac{\partial^2 F_R}{\partial x^2} + \frac{\partial^2 F_I}{\partial y^2} = 0$$

Over the region where $f_{ext}(z)$ is analytic. Conventionally, by requiring $f_{exr}(z)$ to be analytic in the upper half-plane, the quest of finding the Hilbert transform for any given function $f(x)$ can be formulated as boundary value problem . By specifying the boundary conditions that

$(i) f_R(x,0) = f(x),$ and that

(ii) $f_R(x,y) = 0 \, as \, x \to \pm\infty \vee y \to \infty.$

$f_g(x,y)$ can be uniquely determined by solving Laplace's equation in the upper half plane. Thus $g(x) = F_I(X,0)$ is the Hilbert transform of the given function $f(x).$

(5.1.2) Calculation through improper integrals[49]

The above formulation of Hilbert transform as a boundary-value problem is hardly mentioned in recent texts. Instead, Hilbert transform is commonly introduced and defined through an improper [Hahn 96]

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \frac{1}{x-u} du \quad (5.1.4)$$

Here, note that the convolution kernel function $h(x) = 1/\pi x$ is singular at $x = 0$ therefore, the integral in Eq. 4 is improper in the sense of Cauchy's principal value:

$$g(x) = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty} f(u) . h(x-u) du \right) (5.1.5)$$

To be convinced that Eq. 4 indeed produces the Hilbert transform,[106] we need to think about the effects of Hilbert transform in the frequency domain. First, for any frequency k, note that the Hilbert transform of

$f_k(x) = \cos(kx) . is\ g_k(x) = \sin(kx).$ So, we can understand Hilbert transform as a phase shifter which gives every sinusoidal function -90degrees of phase shift.

Therefore, in the frequency domain, we have

$$G(k) = F(k) . (-i. sgn(k)) \quad (5.1.6)$$

Where G(k) and F(k) are the Fourier transform of $g(x) and f(x),$ respectively, and sgn(x) is the sign function $(i.e., sgn(k) = 1 \text{if } k > 0 \wedge sgn(k) = -1 \text{if } k > 0.)$ Therefore, if we think of $H(k) = -i.sn(k)$ as the transfer function of a phase-shift kernel $h(x),$ the kernel can be written as the inverse Fourier transform of the transfer function ; that is,

$$h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} H(k)e^{ikx}dk.(5.1.7)$$

Note that $H(k)=-i\, for\, k>o. \wedge H(k)=i\, for\, k<0.$ Therefore, $H(k)'s$ first derivative with respect to k is

$$\frac{\partial H}{\partial k}=2i\delta(k),(5.1.8)$$

where $\delta(k)$ is the Dirac delta function.

(5.1.3)The notion of Hilbert transform "pairs"

The phase-shift interpretation of Hilbert transform leads to the fact that if $f(x)'s$ Hilbert transform is $g(x),$ then $g(x),s$ Hilbert transform is $-f(x);$ in this sense, $f(x)\wedge g(x)$ form a Hilbert transform pair.

This symmetric property can be understood as follows. Note that the $H^2(k)$ = -1 for all k since H(k) = $\pm i.$ This means that if we take the Hilbert transform twice, the result would be the original function with a negative sign.

(5.1.4)The convolution kernel h(x) **as the Hilbert transform of $\delta(x)$ [106,126,148]**

Therefore, $h(x)$ must be regarded as the Hilbert transform of the impulse function $\delta(x).$ Then it is of our interest to check that

$f_c(x)=\delta(x)+ih(x)$

Can be regarded as an analytic function. To see it consider a family of complex analytic functions $f(x)=i/\pi(z+i\eta)$ parameterized by a variable $\eta>0.$ Since the only singularity of $f(z) is at z=-\eta, f(z)$ is analytic . Therefore, the real part and imaginary part of $f(z)$ from a Hilbert transform pair on the real line $x \in R.$ With a little algebra, the real and imaginary parts can be written as

$$f(x)=\frac{i}{\pi(x+i\eta)}=f_R(x)+i f_I ¿ (x)$$

where

$$f_R(x)=\frac{\eta}{\pi(x^2+\eta^2)}$$

And

$$f_I(x)=\frac{x}{\pi(x^2+\eta^2)}$$

Form a Hilbert transform pair for any $\eta>0.$

(5.1.5) The Discrete-time Hilbert Transform and Hilbert Transformers

Recall that the Hilbert transform introduce 90-degree phase shift to all sinusoidal components. In the discrete-time periodic-frequency domain, the transfer function of Hilbert transform is specified as follows,

$$H(j\omega)=\begin{cases}-j, 0<\omega<\pi \\ j, -\pi<\omega<0\end{cases}$$

The convolution kernel for $H(j\omega)$ can be calculated through inverse Fourier transform

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} H(j\omega)e^{j\omega n}d\omega \frac{\sin^2(\pi n)}{n},$$

$$¿\begin{cases}\frac{2}{\pi}\frac{\sin^2(\pi n)}{n}, \wedge n\neq 0 \\ 0, \wedge n=0\end{cases}$$

Note that h[n] has a infinite support form $n=-\infty¿\infty.$ In practice, the entire function can not be stored digitally. To circumvent this difficulty, we now discuss two major method for calculating the discrete-time Hilbert transform.

Sec.(5.2)Application in system identification

Hilbert transform relates the real part and the imaginary part of transfer function of any physically viable linear time-invariant system. By "physical viability" we mean a system should be stable and causal. Stability requires the systems to produce bounded output if the input is bounded. Causality prohibits the system from producing responses before any stimulus comes in. Denote the impulse response as h(t) and its Laplace transform as H(s). The above conditions requires that

- $h(t)=0 \, for \, all \, t<0$ (causality)

-All singularities of $H(s)$ are located in the left half-plane (stability).

The tow conditions above ensure that H(s) converges and analytic in entire half-plane, and in particular on the imaginary axis $s=j\omega.$ Therefore, the real and imaginary part of $H(j\omega)=H_R(\omega)+jH_I(\omega)$ are inter-dependent in term of the Kramers-Kronig relations

$$H_I(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_R(u)\,du}{\omega-u}$$

$$H_R(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_I(u)\,du}{\omega-u}$$

Which is basically Hilbert transform in its time-frequency dual form.

To a certain extent, the concept that the real and the imaginary parts are inter-dependent similarly applies to the magnitude and phase of transfer functions of a physically viable system. Note that any transfer function $H(j\omega)$ can be decomposed logarithmically into magnitude and phase

$$\log H(j\omega) = \log\left|H(j\omega)\right| + j < H(j\omega).$$

This shows that the log-magnitude and the phase are real and imaginary parts of the log-spectrum, respectively. It might appear that they must satisfy the Kramers-Krong relations. Unfortunately, this is a wishful thinking since apparently $H(j\omega) = \exp(-j\omega\tau)H(j\omega), where \tau is$ a constant, would have the same magnitude as $H(j\omega)$ but a different phase

It turns out that, for any given magnitude response, the uniqueness of phase response can be established if the transfer function satisfies a minimum-phase criterion, requires that all zeros and poles of the transfer function $H(s)$ to be located in the left-half plane. This criterion that all the singularities of log $H(s)$ are located in the left-half plane so the real and imaginary parts of log $H(s)$ become a Hilbert transform pair. Otherwise, any transfer function can be uniquely factorized as a product of a minimum-phase function $M(j\omega)$ and an all-pass function $p(j\omega)$. It is noteworthy that the system whose transfer function is $M(j\omega)$ has the minimal energy delay among all linear time-invariant systems of the same magnitude response.

Chapter 6

Numerical Evaluation of Hypersingular Integrals

In this chapter we will consider only a subclass which is of interest in boundary integral equation applications. For instance, in the one-dimensional case we have to deal mainly with integrals of the form

$$\int_a^b K(x;t)\, g(t)\, dt, a \le x \le b, (6.1)$$

where the kernel K(x;t) has only a pole of order $p+1\, at\, t=x, i.e.,$ it can be expanded in the form

$$\sum_{k=0}^{p} \frac{f_k(t)}{(t-x)^{k+1}} + h(x;t), (6.2)$$

with $f_k(t)$ and $h(x;t)$ smooth. We also assume g(t) either smooth or of the form g(*t)* = $w(t)$ g $_1(t)$, where $w(t)$ is a weight function containing integrable endpoint singularities and $g_1(t)$ is smooth. For the numerical evaluation of (6.1) it will then be sufficient to construct a quadrature rule for the term in (6.2) which contains the strongest singularity $(k-p)$, since the same rule will integrate with comparable accuracy also the remaining terms.

The importance of these integrals springs from the increasing number of there successful applications to solve many two- and three-dimensional problems in applied mechanics and in aerodynamics; see, [1,11,13,].

We will recall definitions and basic properties of such integral, review some numerical rules that have been proposed for their evaluation, including convergence results, and present some new formulas and estimates.

Sec.(6.1): One-dimensional Finite-part Integrals

(6.1.1): Basic definitions and properties

The concept of finite-part integral seems to have been first introduction and examined by Hadamard [] in 1923. However, in spite of its relatively early appearance, the use of it in applications came much later.

Several boundary value problems are expressed as integral equations containing integral of this form.

To introduce the concept of finite-part integral, let us consider first the integrals

$$\int_x^b \frac{dt}{t-x}, \int_a^b \frac{dt}{t-x}, a<x<b,$$

and define them as the finite components of the corresponding

Divergent integrals, as follows.

Definition(6.1.2)

$$\int_x^b \frac{dt}{t-x} + \lim_{\varepsilon \to 0}\left[\int_{x+\varepsilon}^b \frac{dt}{t-x} + \log \varepsilon\right] = \log(b-x), (6.1.3)$$

$$\int_a^b \frac{dt}{t-x} = \int_a^x \frac{dt}{t-x} + \int_x^b \frac{dt}{t-x} = \log \frac{b-x}{x-a}. (6.1.4)$$

Analogously we define the following

Definition (6.1.3) For any real $p>0$

$$\int_x^b \frac{dt}{(t-x)^{p+1}} = \lim_{\varepsilon \to 0}\left[\int_{x+\varepsilon}^b \frac{dt}{(t-x)^{p+1}} - \frac{1}{p\varepsilon^p}\right] = \frac{-1}{p(b-x)^p}$$

furthermore, if p is an integer and $a<x<b$,

$$\int_a^b \frac{dt}{(t-x)^{p+1}} = \int_b^x + \int_x^b \frac{dt}{(t-x)^{p+1}} - \frac{1}{p}\left[\frac{1}{(b-x)^p}\right] (6.1.5)$$

Notice that in all cases above we have

$$\frac{d}{dt}\int_a^b \frac{dt}{(t-x)^p} = p\int_a^b \frac{dt}{(t-x)^{p+1}}.$$

In a more general situation, given a Riemann-integral function $f(t)$ of class $C^r, r=[p],$ in a neighborhood of the singularity $x, a \le x \le b\,(p \, integer \, if \; a<x<b),$ with $f^{(r)}(t)$ Holder continuous when p is an integer, we consider the expressio

$$\int_{I\varepsilon} \frac{f(t) - \sum_{k-0}^{r} f^k(x)(t-x)^k / k!}{(t-x)^{p+1}} dt + \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} \int \frac{dt}{(t-x)^{p+1-k}},$$

xpre $(x+\varepsilon, b)\, if x=b.$ By examining the behavior of this expression as $\varepsilon \to 0,$ we discard the divergent terms, and, recalling the previous definitions of finite-part integrals, we define the following.

Definition (6.1.6)

$$\int_a^b \frac{f(t)}{(t-x)^{p+1}} dt = \int_a^b \frac{f(t) - \sum_{k=0}^{r} f^{(k)}(x)(t-x)^k / k!}{(t-x)^{p+1}} dt$$

$$+\sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} \int_a^b \frac{dt}{(t-x)^{p+1-k}}. (6.2.5)$$

Remark(6.1.7). In the last sum present in (6.2.5), when *p* is an integer, we have the term

$$\frac{f^{(p)}(x)}{p!}dt=\int_a^b \frac{dt}{t-x}.$$

When $a<x<b,$ we may interpret the integral in the Cauchy principal value sense. However, as already pointed out in [] we could also generalize its definition as follows:

$$\int_a^b \frac{dt}{t-x}=\lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon_{1(\varepsilon)}} + \int_{x+\varepsilon_2(\varepsilon)}^b \square\right]=\log\frac{b-x}{x-a}+\lim_{\epsilon\to 0}\log\frac{\varepsilon_1(\varepsilon)}{\varepsilon_2(\varepsilon)}.$$

If we assume

$$\lim_{\varepsilon\to 0}\frac{\varepsilon_1(\varepsilon)}{\varepsilon_2(\varepsilon)}=e^c,$$

then we have

$$\int_a^b \frac{dt}{t-x}=\int_a^b \frac{dt}{t-x}+c,$$

where $\int$ denotes the standard Cauchy principal value integral. This definition leads to a corresponding generalization for the definition of (2.5) when a $¿x<b$ and $I_\varepsilon=(a,x-\varepsilon_1(\varepsilon))\cup(x+\varepsilon_2(\varepsilon),b).$

From the previous definitions it follows immediately that

$$\frac{\alpha f(t)+\beta g(t)}{(t-x)^{p+1}}dt=\alpha\int_a^b \frac{f(t)}{(t-x)^{p+1}+¿\beta\int_a^b \frac{g(t)}{(t-x)^{p+1}}dt.}$$

$$\int_a^b ¿$$

Property (6.1.5.) When

$$x=a \wedge p \, is \, not \, an integer\,, \vee a<x<b \; we \, have \frac{d}{dx}\int_a^b \frac{f(t)}{(t-x)^p}dt = p\int_a^b \frac{f(t)}{(t-x)^{p+1}}dt\,;(6.1.6)$$

$$\int_a^b \frac{f(t)}{(t-x)^{p+1}}dt = \frac{1}{p!}\frac{d^p}{dx^p}\int_a^b \frac{f(t)}{t-x}dt\,(6.1.7)$$

Definition (6.1.6) guarantees also that integration-by-parts rule still remains valid when $a<x<b$:

$$\int_a^b \frac{f(t)}{(t-x)^{p+1}}dt = \frac{-1}{p}\left[\frac{f(b)}{(b-x)^p}-\frac{f(a)}{(a-x)^p}\right]+\frac{1}{p}\int_a^b \frac{f'(t)}{(t-x)^p}dt$$

The use of this formula may be of interest. For instance, if we consider the well-known Pandtl's integral-differential equation

$$c(x)\,\Gamma(x)+d\int_{-1}^{1}\frac{\Gamma'(t)}{t-x}d(x)=\alpha(x),-1<x<1,$$

$$\text{g} \quad c(x)\,\Gamma(x)+d\int_{-1}^{1}\frac{\Gamma(t)}{(t-x)^2}dt=\alpha(x).$$

From the definition of finite-part integral it also follows that the standard linear change-of-variable rule is always permit if p is mot an integer, the rule is valid only if $a<x<b,$ while when x coincides with one of the end points, let us say $x=a,$ this is not allowed. For example, we have

$$\int_a^b \frac{f(t)}{(t-a)^{p+1}}dt = \left(\frac{2}{b-a}\right)^p \int_{-1}^{1}\frac{g(u)}{(u+1)^{p+1}}du + \frac{f^{(p)}(a)}{p!}\,lo\,g\left\{\frac{1}{2}(b-a)\right\},$$

where $t=\frac{1}{2}(b-a)u+\frac{1}{2}(b+a) \wedge g(u)=f\left(\frac{1}{2}(b-a)u+\frac{1}{2}(b+a)\right).$

Finally we have the following property

Property(6.1.6) $For\, c \geq 0 (p \geq 1)\, we\, have$

$-1+1$

$h^{¿},$

$¿$

$if\, c=1 \wedge p\, is\, even$

$O¿$

$\frac{f(t)}{(t-x)^{p+1}} dt¿$

$\int_{x-ch}^{x+h} ¿$

Hence the above integral, which is well defined for h fixed, tends to infinity as $h \rightarrow 0$ (obviously, except for the case $f^{(k)}(x)=0, k=0, \ldots.. p¿.$

A few quadrature rule have been proposed for the numerical evaluation of finite-part integral of the form

$$If=\int_{a}^{b} \frac{f(t)}{(t-x)^{p+1}} dt.$$

Remark (6.1.7) If in (6.2.5) we use the Gauss-Radau or the Gauss-Lobatto rule, then we obtain corresponding formulas for our finite-part integral.

Theorem (6.1.8) When

$w(t)=(b-t)^{\alpha}(t-a)^{\beta}, with -1<\beta \leq 0, \wedge x=a, we\, have$

$$\frac{\begin{matrix} t \\ h_i \end{matrix}}{(¿¿ i-a)^{p+1}} = \begin{cases} O(\log n), if p=0, \\ O(n^{2(p-g\beta)}) if\ p \geq 1. \end{cases}$$

$$\sum_{i=1}^{n} ¿$$

This bound implies that

$$|v_k(a)| + ¿ \sum_{i=1}^{n} |w^G(a)| = \begin{cases} O(\log n), if p=0, \\ O(n^{2(p-\beta)}) if\ p \geq 1. \end{cases}$$

$$\sum_{k=0}^{p} ¿$$

To prove the next theorem, we need to use the following lemma.

Lemma (6.1.9). Let $g \in C^q[a,b].\ q \geq 1. For\ every\ integer\ m \geq 2q+1$ there exists a polynomial $q_m(t)$ of degree m such that for all $t \in [a,b]$.

$$|g^{(k)}(t) - q_m^{(k)}(t)| \leq c \left(\frac{\sqrt{(b-t)(t-a)}}{m} \right)^{q-k} \omega ¿ \quad g^{(q)}; m^{-1}$$

$where\ c\ is\ constant\ independent\ of\ m \wedge t, \wedge \omega(g^{(q)};.)$

$denotes\ the\ modulus\ of\ continuity\ of\ g^{(q)} \in [a,b].$

Theorem (6.1.10) $When\ w(t) = ¿ \quad ¿ \quad ¿ \quad b-t \vee ¿$

$$R_n^G(f;a) = \begin{cases} O(¿)\,\omega(f^{(q)}; n^{-1}), if p+1 \leq q \leq 2p \\ O(n^q)\,\omega(f^{(q)}; n^{-1}) if q \geq 2p+1 \end{cases} \quad n^{¿} \quad -2(q-p)+\uparrow\gamma$$

Where $\gamma = max¿ \; ¿ \; ¿. \; 1, -2\beta$

Proof For the polynomial $p_n(t)$ of degree n defined in Lemma 2.9 we have $f^{(k)}(x) = q_n^{(k)}(x), k = 0,1\ldots,p;$

Thus

$$R_n^G(f - q_n; x) = \int_x^b w(t) \frac{f(t) - q_n(t)}{(t-x)^{p+1}} dt - \sum_{i=1}^n \frac{h_i}{(t-x)^{p+1}} dt [f(t_i)]$$

$$-q_n(t_i).$$

To estimate the behavior of the integral in (), we proceed as follows. When $q \geq 2p+1,$ by applying () with $k=0$ we optain the bound O($n^{-q} ¿ \omega(f^{(q)}; n^{-1}). When\ p+1 \leq q \leq 2p, we\ write$

$$f(t) - q_n(t) = (t-x)^k [f^k(\xi_i) - q_n^{(k)}(\xi_i)],$$

With $k = 2p+1-q, \wedge apply: we\ obtain O(n^{2q+2p+1})\omega$

$(f^{(q)}; f^{(q)}; n^{-1}).$

In this section we have mainly considered quadrature rules of interpolatory type, i.e., obtained by approximately the function *f (x)* by interpolation polynomials (based on the zeros of Jacobi polynomials). Of course this is not the only possible approach ; indeed, in this same section we have also mentioned a couple of alternatives which are based on piecewise polynomial interpolation. When *a<x <b* and *p* is an integer, given any quadrature formula for Cauchy principal value integrals, by means of (6.2.7)

we can derive a corresponding rule for finite-part integrals. For instance, the rules recently presented in (6.2,16) , which we have not described here, are derived exactly in this way. Actually, following this approach, we could have obtained most of the quadratures of this section.

Sec.(6.2.) Two-dimensional finite-part integrals

(6.2.1.) Basic definitions and properties

While the one-dimensional Cauchy principal value integral concept is well know and often used in applications, the two-dimensional analogue dose not seem to be equally known. Furthermore, the description of this latter is of some help to understand what happens when we consider two-dimensional finite-part integrals. For this reason we start section 3 by illustrating the definition of two-dimensional Cauchy principal value integrals on bounded domains R^2 .

The definition and some properties of the two-dimensional Cauchy principal value integral were explicitly given by Tricomi (56) in 1928. We recall that at the end of his paper, he states that the same concept had already been used by Petrini [47, 48] in 1908 and 1909 and by Muntz [41] in 1910.

Let $F(U_0;U)$ be integrable on a bounded domain $T \subset R^2$, expect at the point U_0 . Furthermore, denoting by $r\theta$ the polar coordinates with origin at U_0 , we assume that in a neighborhood of U_0 we can write

$$F(U_0;U)=\frac{f(U_0;\theta)}{r^2}+F_1(U_0;U),$$

Where $r^2=|U-U_0|^2$ and $F_1(U_0;U)$ may still become infinite at U_0, but with order less than 2. Let σ denote a neighborhood of U_0 with contour C_1 given by $r(\theta)=\alpha(\epsilon,\theta)$, where ϵ is the radius of the smallest circle containing $\sigma. Let\, C_2$ be the contour of *T*, given by $r(\theta)=A(\theta)$. We consider first.

$$f(¿U_0;\theta)\left[\overset{¿}{\int_{\alpha(\epsilon,\theta)}^{A(\theta)}}\frac{1}{r}dr\right]d\theta.$$

$$F_1(U_0;\theta)dv+\int_0^{2\pi}¿$$

$$F(¿U_0;\theta)dv=\int_{r-\sigma}^{.}¿$$

$$\int_{r-\sigma}^{.}¿$$

Taking the limit as $\epsilon\to 0,$ we obtain

$$\overset{U}{¿}$$

$$f(¿U_0;\theta)\log A(\theta)d\theta-\lim_{\epsilon\to 0}\int_0^{2\pi}f(U_0;\theta)\log\alpha(\epsilon,\theta)d\theta.(6.2.1)$$

$$F_1(¿U_0;\theta)dv+\int_0^{2\pi}¿$$

$$(¿¿0;\theta)dv=\int_T^{.}¿$$

$$\overset{¿}{\lim_{\epsilon\to\infty}}F¿$$

In particular, if we let σ be a circle with center U_0 and radius $\epsilon,$ the last integral becomes

$$f\left(¿U_0;\theta\right)d\theta$$
$$\epsilon\int_0^{2\pi} ¿$$
$$\log ¿$$

,

Which gives rise to the following theorem?

Theorem(6. 2.11). *A necessary and sufficient condition for the existence of the limit in (6.3.2) is*

$$¿\left(U_0;\theta\right)d\theta=0.$$
$$f ¿$$
$$\int_0^{2\pi} ¿$$

In this case we define the following.

Definition (6.2.13). For any $F\left(U_0;U\right)$ of type (6.3.1) satisfying condition (6.2.3), when σ is a circle, we define

$$f\left(¿U_0;\theta\right)\log A\left(\theta\right)d\theta.$$
$$\oint_r F\left(U_0;U\right)dv=\int_T F_1\left(U_0;U\right)dv+\int_0^{2\pi} ¿$$

If σ is not a circle , but nevertheless $\alpha\left(\epsilon,\theta\right)$ is such that

$$\lim_{\epsilon\to 0}\frac{\alpha\left(\epsilon,\theta\right)}{\epsilon}=\alpha_0\left(\theta\right),$$

Then

$$f(¿U_0;\theta)\left[\log\frac{A(\theta)}{\alpha_0(\theta)}\right]d\theta$$

$$⨏_T F(U_0;U)\,dv=\int_T F_1(U_0;U)\,dv+\int_0^{2\pi}¿$$

A definition analogous to (6.5) has already been introduced in the one –dimensional case; see Remark 2.4.

Definition(6.2.15) if condition (6.3) does not hold, then we can define the integral of $F(U_0;U)$ only in the finite-part sense. In this latter case in the previous expression we discard the term containing the factor $\log\epsilon$ and use the second members of (6.2.4) and (6.5) to define the corresponding finite part integrals.

The concepts we have already presented in this section can be generalized to functions with stronger singularities and with a source point U_0 that may even lie on the boundary $C_2.$ Here we consider integrals of the form

$$⨎_T K_p(U_0;\theta)\phi(U)\,dv,\ U_0\epsilon\, T\subset R^2,$$

estimate on the good function is easy since we have

$$\left|\left\{x\epsilon\, R^2;|Tg|>\frac{\lambda}{2}\right\}\right|\lesssim\frac{\|Tg\|^2_{L^2(R^2)}}{\lambda^2}$$

$$\lesssim\frac{1}{\lambda^2}\|g\|^2_{L^2(R^2)}$$

$$\lesssim \frac{1}{\lambda^2} \lambda \| f \|_{L^1(R^2)}$$

$$= \quad \frac{1}{\lambda} \| f \|_{L^1(R^2)} \quad .$$

Here, we have used that *T*: $L^2(R^2) \to L^2(R^2)$ is bounded. In the final estimate we have used property (ii) in Theorem (6.2.1)

We now turn to understanding the estimate on the bad function. Let $\{Q_j\}$ be the cubes obtained in Theorem(6.2.1) Let $Q_j^{\text{¿}}$ denote the cube concentric with Q_j and having side length $\sqrt[2]{n}$ times the side length of Q_j . Then we have that

$$\left|\left\{x \in R^2 : |Tb| > \frac{\lambda}{2}\right\}\right| = \left|\left(\cup Q_j^{\text{¿}}\right) \cap \left\{x \in R^2 : |Tb| > \frac{\lambda}{2}\right\}\right| + \left|\left(\cup Q_j^{\text{¿}}\right)^c \cap \left\{x \in R^2 : |Tb| > \frac{\lambda}{2}\right\}\right|.$$

Consider now the first term above we then have that

$$\left|\cup Q_j^{\text{¿}} \cap \left\{x \in R^2 : |Tb| > \frac{\lambda}{2}\right\}\right| \leq \left|\cup Q_j^{\text{¿}}\right|$$

$$\leq \sum_j \left|\cup Q_j^{\text{¿}}\right|$$

$$\lesssim \sum_j |Q_j|$$

$$\lesssim \frac{1}{\lambda} \| f \|_{L^1(R^2).}$$

It only remains to handle the term

$$\left|\left(\cup Q_j^{¿}\right)^c \cap \left\{x \epsilon R^2 : |Tb| > \frac{\lambda}{2}\right\}\right|$$

And for this one we will use the properties of the function *b* Note that by simple estimates we have

$$\left|\left(\cup Q_j^{¿}\right)^c \cap \left\{x \epsilon R^2 : |Tb| > \frac{\lambda}{2}\right\}\right| \lesssim \frac{\lambda}{2} \int_{\left(\cup Q_j^{¿}\right)^c} |T(b)(x)| dx$$

$$\leq \frac{\lambda}{2} \sum_j \int_{\left(\cup Q_j^{¿}\right)^c} |Tb_j(x)| dx$$

Suppose for the moment that we proved

$$\int_{\left(\cup Q_j^{¿}\right)^c} |Tb_j(x)| dx \lesssim \int_{Q_j} |b_j(x)| dx \, (6.1.9)$$

Where the Kernel K_p admits the expansion

$$K_p(U_0; U) = \sum_{I=0}^{p+1} \frac{f_{p-1}(U_0; \theta)}{r^{p+2-1}} + K_p^{¿}(U_0; r, \theta). \qquad (6.1.10)$$

Where $f_p(U_0;\theta)$ and $\phi(u)$ are smooth.

We set $\sigma = \{u \epsilon T : |u - u_0| \leq \epsilon\}.$ as before in the case examined by Tricomi, a different choice of σ would introduce changes in the values of some of the integrals we are going to define.

Analogous to the one-dimensional case, we preliminarily define finite-part integrals of the simpler form

$$\oint_{r} \frac{f_p(U_0;\theta)}{r^{p+2}} dv.$$

To this end, we consider first the regular integral

$$\int_{T-\sigma} \frac{g(U_0;\theta)}{r^{p+2}} dv = \int_0^w g(U_0;\theta) \left[\int_\epsilon^{A(\theta)} \frac{dr}{r^{p+1}} \right] d\theta = \int_0^w g(U_0;\theta) [h_0(\theta) - e_0(\epsilon)] d\theta, (6.1.11)$$

Where $0 < w \leq 2\pi$,

$$\theta = \begin{cases} \log A(\theta), if\, P = 0, \\ \frac{-1}{P[A(\theta)]^{P,}} if\, P > 0, \end{cases}$$

h_0¿

And

$$e_0(\epsilon) = \begin{cases} \log \epsilon, if\, P = 0, \\ \frac{-1}{P\epsilon^{P,}} if\, P > 0. \end{cases}$$

$if\, w = 2\pi \wedge$¿

$$\int_0^{2\pi} g(U_0;\theta) d\theta = 0,$$

Then the limit of (6.10), as $\epsilon \to 0,$ exist and we define it as the Cauchy principal value of (6.9).otherwise, we neglect the term $e_0(\epsilon)$ in (6.10); thus we have the following definition.

By this definition of g we immediately see that $\|g\|_{L^1(R^n)} \leq \|f\|_{L^1(R^n)}$. to see the L^∞ estimate, note that if $x \epsilon Q_j$ then we have

$$|g(x)|=\left|\frac{1}{Q_j}\int_{Q_j}|f(x)|dx\right|\leq 2^n\frac{1}{|Q_j'|}\int_{Q_j'}|f(x)|dx\leq 2^n\lambda .$$

On the other hand, if $x \epsilon R^n \{U \text{¿}_j ; Q_j \text{¿},$ then the exist a sequence of non-selected cubes Q_k that converge to x. we then have that

$$\left|\frac{1}{|Q|_k}\int_{Q_k} f(x)\,dx\right|\leq\frac{1}{|Q_k|}|f(x)|dx\leq\lambda .$$

By the Lebesgue Differentiation Theorem we then have that

$$|f(x)|\leq\lambda\, x \epsilon R^n\{U \text{¿}_j ; Q_j \text{¿} \quad .$$

Combining these two estimates we see that $\|g\|_{L^1(R^n)}\leq 2^n\lambda .$ finally, observe that

$$\sum_j |Q_j|\leq\frac{1}{\lambda}\sum_j\int_{Q_i}|f(x)|dx$$

$$\text{¿}\frac{1}{\lambda}\int_{U_j ; Q_j}|f(x)|dx$$

$$\leq\frac{1}{\lambda}\|f\|_{L^1(R^n)} .$$

We now turn to show how to use this Theorem to deduce the following result.

Theorem (.6.2.12) Suppose that K is a Calderon –Zygmund kernel as defined above in Theorem 1.2. then for all $f \epsilon L^1(R^n)$ and any $\lambda > 0$ we have

$$\left|\{x \in R^n : |Tf(x)| ¿ \lambda\}\right| \lesssim \frac{1}{\lambda} \|f\|_{L^1(R^n)}$$

$proof. Fix \lambda \wedge f \in L^1(R^n).$ Apply the Calderon-Zygumund decomposition in the Theorem 2.1 to obtain functions g, b so that *f* = *g* + *b*. now observe that

$$\{x \in R^n : |Tf| > \lambda\} \subset \left\{x \in R^n : |Tg| > \frac{\lambda}{2}\right\} \cup \left\{x \in R^n : |Tb| > \frac{\lambda}{2}\right\}$$

And so, we have

$$\left|\{x \in R^n : |Tf| > \lambda\}\right| \leq \left|\left\{x \in R^n : |Tg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in R^n : |Tb| > \frac{\lambda}{2}\right\}\right|$$

Definition (6.3.14) we define

$$⨏ T \frac{g(\cup_0, \theta)}{r^{p+2}} dv = \int_0^w g(\cup_0, \theta) h_0, (\theta) d\theta. \qquad (6.2.11)$$

In the more general case of (3.8) we consider the Taylor expansion of $\varnothing(v)$ arround $v = v_0$ we write

$$¿ \int_{T-\sigma} \frac{f_p(\cup_0, \theta)}{r^{p+2}} \varnothing(v) dv$$

$$I_\epsilon \varnothing + \int_0^w \int_\epsilon^{A(\theta)} \frac{f_p(\cup_0, \theta)}{r^{p+1}} \left[\sum_{|K| \leq p} \frac{1}{K!} D^K \varnothing(\cup_0) r^{|K|} \cos^{K_1}\theta \sin^{K_2}\theta \right] dr\, d\theta,$$

Where $|K| = K_1 + K_2, K_1 \geq 0, \wedge I_\epsilon \varnothing$ denote the regulapart of the integral whose limit exist as $\epsilon \to 0.$ From (6.12) we obtain

$$¿ \int_{T-\sigma} \frac{f_p(\cup_0; \theta)}{r^{p+2}} \varnothing(v) dv$$

$$I_\epsilon \varnothing + \sum_{|K|\le p} \frac{1}{K!} D^K \varnothing(\cup_0) \int_0^w f_p(\cup_0, \theta) \cos^{K_1}\theta \sin^{K_2}\theta \left[h_{|K|}(\theta) - e_{|K|}(\epsilon)\right] d\theta$$

.

where

$$h_{|K|}(\theta) = \begin{cases} \log A(\theta) \, if \, |K| = P, \\ \frac{1}{(|K| - P) A(\theta)^{P-|K|}}, if \, |K| < P \end{cases} \qquad (3.14)$$

And

$$e_{|K|}(\epsilon) = \begin{cases} \log \epsilon \, if \, |K| = P \\ \frac{1}{(|K| - P) A(\theta)^{P-|K|}} \, if \, |K| < P \end{cases}$$

$if\, w = 2\pi$ and, furthermore,

$$\cup_0$$
$$f_P(¿, \theta) \cos^{K_1}\theta \sin^{K_2}\theta \, d\theta = 0, |K| \le P,$$
$$\int_0^{2\pi} ¿$$

Then the limit of (6.13) exist and we define it as the Cauchy principal value of (6.8).notice that (6.15) is equivalent to

$$\cup_0$$
$$\cup_0$$
$$m\theta \, d\theta = \int_0^{2\pi} f_P(¿, \theta) \sin m\theta \, d\theta = 0.$$
$$f_P(¿, \theta) \cos ¿$$
$$\int_0^{2\pi} ¿$$

For $m = 0,1,.....p$.

If condition (6.15) are violated then in (3.13) we discard the divergent terms $e_{|K|}(\epsilon)$ and, letting $\epsilon \to 0$, we obtain the finite-part value of the integral, as follows.

Definition(6.1.5) for any $\varnothing \in C^{P+1}$ ¿ ¿ ¿ T

$$⨏_T \frac{f_P(\cup_0;\theta)}{r^{P+2}} \varnothing(v)\,dv \int_T \frac{f_P(\cup_0;\theta)}{r^{P+2}} \left[\varnothing(v) - \sum_{|K|\le p} \frac{1}{K!} D^K \varnothing(\cup_0) r^{|K|} \cos^{K_1}\theta \sin^{K_2}\theta\right] dv$$

+

$$\sum_{|K|\le p} \frac{1}{K!} D^K \varnothing(\cup_0) \int_0^w f_P(\cup_0;\theta) \cos^{K_1}\theta \sin^{K_2}\theta\, h_{|K|}(\theta)\, d\theta,$$

$where\, h_{|K|}(\theta)$ is given in (6.2.14).

As in the one-dimensional case, a change of variable in (6.3.6), or in (6.3.8), in general introduces additional point functional (at $v = v_0$). Explicit representations for the coefficients of these extra point functional were given in [51].

Remark (6.6). when we take an integration region T of size $h \to 0,$ the behavior of integral (3.8) is of type O([log h]) if p =0, and $O(h^{-P})\, if\, P \ge 1.$

In boundary element method applications, very often one has to deal with surface integrals of the type

$$I = ⨏_S \acute{K}_P(\acute{\cup}_0; \acute{\cup} - \acute{\cup}_0)\, \dot{\phi}(\acute{\cup})\, dS_{\dot{v}}, P\, integer \qquad (6.3.16)$$

$where\, S \subset R^3$ has an analytic parametric representations the Kernel $\acute{K}_P$ is homogenous of degree $-P-2$ in the second argument and has a pole of order *P + 2* at $\acute{U}_0 ; \acute{U},$ and $\dot{\phi}(\acute{v})$ is a smooth function. These integrals are defined as the limit, as $\epsilon \to 0,$ of the finite part of the expansion of

$$I(\epsilon) = \dot{\int}_{S-B,} \acute{K}_P(\acute{\cup}_0 ; \acute{\cup} - \acute{\cup}_0) \acute{\phi}(\acute{\cup}) dS_{\dot{v}}$$

Where $B_\epsilon = \{ \acute{u} \in R^3 : |\acute{u} - \acute{u}_0| < \epsilon \}, \epsilon > 0.$ that is having derived an expansion of the form

$$I(\epsilon) \backsim I_0(\epsilon) + I_{-1} \log \epsilon + \sum_{j=1} I_j \epsilon^{-j}$$

$$I = \lim_{\epsilon \to 0} I_0(\epsilon).$$

As illustrated in [51,52], using the representation of *S* in those applications, one obtains corresponding integrals of form (3.6), with the Kernel satisfying (3.7), Plus additional point functional (at $\acute{u} - \acute{u}_0 ¿$ whose coefficients vanish whenever $\acute{u}_0$ lies in the interior of *S*. thus also in the more general situation of (6.2.16) one van deal with integrals of the form (6.2.16).

(6.2.12) *Cubature formulas*

Using the polar coordinates, we express (3.6) in the form

$$\int_0^w ⨏_0^{A(\theta)} \frac{f_P(u_0;\theta)}{r^{P+1}} \varnothing(u)\, dr\, d\theta, P\, integrer, (6.3.17)$$

Where $\varnothing(u) = \varnothing¿$ $r\cos\theta, r\sin¿.$ θ Further, we assume that the function A($\theta¿$ is analytic on $[0, w]$. If A($\theta¿$ is only piecewise analytic in its domain of definition, we subdivide this latter into subintervals where A($\theta¿$ is analytic and treat separately each of the subintervals.

From Definition(6 2.3) it is straightforward to see that if in (6.2.17) we assume $f_P \in C^q[0, w]$ and $\varnothing \in C^{q+P+1}(\acute{T}), we\, have$

$$g_P(\theta) = f_P(u_0;\theta) \oint_0^{A(\theta)} ¿ \quad \varnothing¿ \quad ¿ \quad r\cos\theta, r\sin¿. \quad \theta$$

To approximate (3.17) we generalize the approach recently proposed in [39] (see also [52]). In particular, we evaluate the outer integral by an m-point Gauss-Legender –Lobatto rule , and the inner one using the formulas presented in Section 2.2.But before deciding which formula to use for the inner integral, we need function to recall that the $f_P(u_0;\theta)$ is usually not known explicitly as we have pointed out in (3.7), in general the term $f_P(u_0;\theta)/¿$ r^{P+2} arises from a Laurent expansion of the given Kernel function and it contains the strongest singularity. Since we will apply our final cubature to (3.6), and not to (3.17) , we need a quadrature which uses only function values, not derivatives.

To this end, when *P=0*, we choose rule (2.10) or its analogue of Lobatto type, because of the simplicity of its coefficients and its higher performance (see tables

1 and 2). When $P \geq 1,$ we are forced to use rules of type (2.16) here we choose the one with w(t) = 1and (t_1) nodes of the corresponding Gauss-Legender (or Gauss-Radau or Gauss-Lobatto) rule.

Denote by (h_1), { $\theta\}$ the coefficients and the nodes of the m-point Gauss-Legender or Gauss-Lobatto rule

$$\int_0^w g_P(\theta)\, d\theta = \sum_{i=1}^{m} h_i g_P(\theta_i) + R_m^{GL}(g_P).$$

For $g_P \in C^q[0,w]$ we have $R_m^{GL}(g_P) = O(m^{-q}) w(g_P^{(q)}; m^{-1}).$ Then use (2.10) if *p = 0* and (2.16) if $P \geq 1, ¿\, approximate$

$$\theta$$
$$r\, cos\theta, r \sin ¿.$$
$$¿$$
$$\varnothing ¿$$
$$\oint_0^{A(\theta)} ¿$$

That is,

$$r_j \cos\theta_i$$
$$w_j^G(0) \varnothing (¿, r_j \sin\theta_i) + R_n^G(\varnothing), if\ P = 0,$$
$$u_0(0) \varnothing (0{,}0) + \sum_{j=1}^{n} ¿$$

$$\sum_{j=1}^{n} w_j^I(0) \varnothing (r_j \cos\theta_i r_j \sin\theta_i) + R_n^I(\varnothing), if\ P \geq 1.$$

Notice that

$$\theta$$
$$(¿¿ i; 0), w_j^G(0) = w_j^G(\theta_i; 0), r_j = r_j(\theta_i), w_j^I(0) = w_j^I(\theta_i; 0), R_n^G(\varnothing) = R_n^G(\theta_i; \varnothing), R_n^I(\varnothing) = R_n^I(\theta_i; \varnothing).$$
$$u_0(0) = u_0 ¿$$

We remark that under our assumption on A($\theta ¿$ the bounds

$$\theta \; (¿¿ i;0) \; u_0 ¿ \; ¿ \; \theta \; (¿¿ i;0) \; n \; see(6.2.11) \wedge ¿ \; ¿ \; \theta \; (¿¿ i;0) \; w_j^G ¿ \; ¿ \; see(6.2.13) \; ¿ \; \log ¿¿ \; w_j^G ¿ = O ¿ \; ¿ \; ¿$$

hold uniformly with respect to i.

The remainder term of the final cubature formula is given by

$$u_0; \; \theta \; (¿¿ i;\Phi),(6.18) \; h_i f_P(¿\theta_i) R_n ¿ \; R_{m.n}(f_P \Phi) = R_m^{GL}(g_P) + \sum_{i=1}^{m} ¿$$

With

$$\theta \; \theta \; \theta \; \theta \; (¿¿ i;\Phi) \, if \, P \geq 1. \; (¿¿ i;\Phi) = R_n^1 ¿ \; (¿¿ i;\Phi) \, if \, P = 0, \wedge R_n ¿ \; (¿¿ i;\Phi) = R_n^G ¿ \; R_n ¿$$

Before deriving a convergence result for our cubature formula, we recall the definition of the space

$H_{\mu,\lambda}^{(q)}(D)$. we say that $f(x,y) \in H_{\mu,\lambda}^{(q)}(D), q \geq 0,$ if *f(x ,y)* and all its partial derivatives of order *j = 0,...q* exist and are continuous in D, and each derivative of order q satisfies the Holder condition

$g(x_1, y_1) - g(x_2, y_2) \leq A|x_1 - x_2|^{\mu} + B|y_1 - y_2|^{\lambda},$ where A, B are constants.

To estimate the behavior of (3.14) we proceed as follows. Consider the Taylor expansion of $\Phi(r\cos\theta, r\sin\theta)$ with respect to the variable r, around r = 0; denote by $T_P(r,\theta)$ the associated polynomial of degree p . define

$$\Phi(r\cos\theta, r\sin\theta) = \Phi(r,\theta) + r^{P+1}\Phi_P(r,\theta),$$

And consider the best (uniform) approximation polynomial $P_{n-P-2.m}(r,\theta)$ of degree $n-P-2 \in r \wedge m \in \theta$ associated with $\Phi_P(r,\theta)$. if assume $\phi \in H_{\mu,\mu}^{(q+P+1)}(\acute{T})$, then we have $\Phi_P \in H_{\mu,\mu}^{(q)}(\acute{T})$. next form the function

$$P_{n.m}(r,\theta) = T_P(r,\theta) + r^{P+1}P_{n-p-2,m}(r,\theta),$$

Which for fixed $\theta,$ is a polynomial of degree n – 1 with respect to r. since it is known (see[32, p.90]) that for $\Phi_P \in H_{\mu,\mu}^{(q)}(\acute{T}),$

$$\|\Phi_P - P_{n-P-2,m}\|_\infty = 0\left(n^{-q-\mu} + m^{-q-\mu}\right),$$

And , furthermore, $R_n(\theta_i;\Phi) = R_n(\theta_i;\Phi - P_{n.m}^{¿}),$ we can state the following.

Theorem (6.7). if in (3.17) we assume $f_P(u_0;\theta) \in C^q[0,w] \wedge \varnothing(v) \in H_{\mu,\mu}^{(q+P+1)}(\acute{T}),$ for the remainder term (3.18) we have

$$R_{m,n}(f_P\Phi) = \begin{cases} O\left(m^{-q-\mu} + n^{-q-\mu}\right)\log n, if\ P=0, \\ O\left(m^{-q-\mu} + n^{-q-\mu}\right) n^{2p+1/2} if P \geq 1, \end{cases} \qquad (6.2.19)$$

Remark (6.8) if we are interested in the construction of a cubature rule for integral (3.6), then recalling expansion (6.2.17), in the case $P \geq 1$ for example we immediately derive

$$⨏_T K_P(u_0;u)\varnothing(v) = \sum_{i=1}^{m} h_i \sum_{j=1}^{n} w_j^I(0) r_j^{P+2} K_p(u_0;u_{ij})\varnothing(u_{ij}) + R_{m.n}(\varnothing),$$

Where we have set $\begin{matrix} \theta_i \\ \cos\theta_i, \sin ¿ \\ ¿ \\ ¿ \\ u_{ij} = u_0 + r_j ¿ \end{matrix}$ for the new remainder $R_{m,n}(\varnothing)$ a bound of type (3.19) holds provided $K_P^{¿}$ in (3.7) is of class C^{q+P+1}, with respect to the variable r,θ in the closure of the domain of integration.

In theorem 3.7 our goal is to approximate integral (3.17) to the desired accuracy. In particular, if we think of the boundary element method application, since the convergence results for these methods rely on the exact evaluation of the integrals over each element, our goal is the rate of convergence of our cubature formulas as $m,n \to \infty$.

In the authors examine the approximation[7] of the integrals from a different point of view.

They consider the reference triangle T with vertices (0, 0), (h, 0), (h, h), and a cubature formula obtained by integrating the outer integral in (3.17) by an m-point Gaussian rule and the inner one by an n-point rule of type (2.16), with m, n fixed. By assuming $f_P¿^{r,\theta}$) and $\varnothing(v)$ analytic, they obtain an error estimate, as $h \to 0,$ of the form.

$$O¿ h^{-P} \langle ¿, \left(1+\delta_{0.P}|\log h|\right) e^{-cm}+h^{n}\rangle \tag{3.20}$$

Where c is a constant and $\delta_{i.j}$ represents the Kronecker symbol. That is, for a given cubature with fixed number of nodes, they examine the behavior of remainder term as the size of each element of the surface triangulation tends to zero. Recall that under the assumptions made in [52], in the estimates we have derived in this section we would have $q=\infty$ and for fixed h, the remainder terms would decay, as $m,n \to \infty$. Faster than any negative power of m and n, Incidentally we notice that in the above situation the contribution of the integral itself is (see Remark 3.6) of order $O¿ h^{-P}\langle\left(1+\delta_{0.P}|\log h|\right),$ hence the error bound (3.20) guarantees that for m , n fixed the behavior of the cubature rule, as $h \to 0,$ is similar to that of the integral it is applied to.

In boundary element applications we have the same cancellation problems already observed in the one-dimensional case. This cancellation phenomenon arises in the summation of the contributions given by the neighbor elements of the singular point. Only if this sum is small with respect to the contribution of all the remaining elements, numerical cancellation will have little effect on the final accuracy. Otherwise, there will be a h_0 such that for $h<h_0,$ accuracy will decrease. Finally we mention that in [12,13,52] alternative approaches (for constructing cubature for hypersingular integrals) to the one described here are prod not consider them.

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