Chapter 6
Differential System of Krein and Triangular Factorization

In this chapter we show that broad classes of operators can be factorized. As a result, pure existence theorems in the well-known problems by Ringrose, Kadison and Singer are substituted by concrete examples.

Sec(6.1): Krein's Differential System and its Generalization

In the M.G. Krein's famous paper [47] a special class of differential systems (Krein's systems) was considered. M.G.Krein announced a number of fundamental facts of the direct and inverse spectral theory of this class. Unfortunately these important results where published without proof. In recent years we proved a part of the assertions stated in [47] and generalized them to a broad class of canonical differential systems (see [51,52,53]). In this article we continue our investigatation of Krein's systems and correct some assertions both in M.G. Krein's article [47] and in our earlier work [51,52,53]. In the last part of this section, we introduce the class of the matrix functions, which contains the Stummel class. Assuming that the coefficients of Krein's system belong to the introduced class, we prove some new results announced by M.G. Krein in [47].
We shall consider the operator

$$S_r f = f(x) + \int_0^r H(x-t) f(t) dt \quad , 0 < r < \infty. \tag{1}$$

Here we suppose that the operator S_r is positive and that the function $H(t)$ is continuous and satisfies the relation

$$H(t) = \overline{H(-t)} \quad , \quad -r \le t \le r. \tag{2}$$

In this case there exists a Hermitian resolvent $\Gamma_r(t,s) = \overline{\Gamma_r(s,t)}$ satisfying the relation

$$\Gamma_r(t,s) + \int_0^r H(t-u)\Gamma_r(u,s) du = H(t-s), \qquad 0 \le s,t \le r. \tag{3}$$

Following [47] we set

$$P(r,\lambda) = e^{ir\lambda}\left(1 - \int_0^r \Gamma_r(s,0)\, e^{-is\lambda}\, ds\right), \tag{4}$$

$$P_*(r,\lambda) = 1 - \int_0^r \Gamma_r(0,s)\, e^{is\lambda}\, ds \,. \tag{5}$$

M.G. Krein [47] deduced the differential system

$$\frac{dP(r,\lambda)}{dr} = i\lambda P(r,\lambda) - \overline{A(r)}\, P_*(r,\lambda), \frac{dP_*(r,\lambda)}{dr} = -A(r)P(r,\lambda)\,, \tag{6}$$

where

$$A(r) = \Gamma_r(0, r) \tag{7}$$

M.G.Krein proved that there exists a nondecreasing function $\sigma(\lambda)$ (spectral function) such that the operator

$$Uf = \int_0^\infty f(r)P(r,\lambda)dr \ , \quad -\infty < \lambda < \infty \tag{8}$$

isometrically maps $L_m^2(0,\infty)$ into $L_\sigma^2(-\infty,\infty)$. M.G.Krein formulated the following important results [47].

Theorem (6.1.1)[21]: The following propositions are equivalent:

1) The integral

$$K(z_0) = \int_0^\infty |P(r,z_0)|^2 dr \tag{9}$$

converges for at least one z_0, $Imz_0 > 0$.

2) The function $P_*(r,z_0), 0 \le r < \infty$ is bounded for at least one z_0, $Imz_0 > 0$

3) The integral $K(z)$ converges uniformly at any bounded closed set z of the open half-plane $\mathrm{Im}z > 0$.

4) There exists the limit

$$\Pi(z) = \lim P_*(r, z), \quad r \to \infty, \tag{10}$$

where the convergence is uniform on any bounded closed subset z of the open half-plane $Imz > 0$.

5) The integral

$$\int_{-\infty}^{\infty} \frac{Log\, \acute{\sigma}(\lambda)}{(1+\lambda^2)} d\lambda \tag{11}$$

is finite.

If conditions 1)-5) are fulfilled then $\Pi(z)$ can be represented in the form

$$\Pi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{1+tz}{(z-t)(1+t^2)}[\log\acute{\sigma}(t)]dt + i\alpha\right), \tag{12}$$

where $\alpha = \bar{\alpha}$.

Let us point out some inaccuracies of the article [47].

1. The condition of the continuity of $H(t)$ is omitted in [47]. Without this condition equality (7) does not make sense. It was Krein himself who wrote about this [48].
2. In formula (12) (see [47]) the expression $(t - z)$ is used instead of $(z - t)$.
3. The right part of (12) (see [47]) contains the multiplier $\exp(i\beta z)$, where $\beta \ge 0$. As it is shown (see [53]) this multiplier is equal to 1, i.e., $\beta = 0$.
4. M.G.Krein [47] writes that formula (12) shows that $\Pi(z)$ depends only on the absolutely continuous part $\sigma_a(\lambda)$ of the spectral function $\sigma(\lambda)$. This is true concerning the module $|\Pi(z)|$, but the question of the connection of α with the spectral function $\sigma(\lambda)$ remains unanswered.

However under some conditions it is possible to obtain the formula expressing α by $\sigma_a(\lambda)$ In a number of concrete examples (see [44,45,46]) the relations

$$\Pi(z) \to 1, \; z = ia, \; a \to +\infty, \tag{13}$$

$$\lim \acute{\sigma}(t) = \frac{1}{2\pi}, \quad t \to \infty \tag{14}$$

are fulfilled. From (12)– (14) it follows that

$$\alpha = \lim \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{a^2 t}{(t^2 + a^2)} \frac{\log[2\pi\acute{\sigma}(t)]}{(1 + t^2)} \, dt, \tag{15}$$

where $a \to +\infty$.

Thus in case when (13) and (14) are valid α is indeed defined by absolutely continuous part $\sigma_a(\lambda)$ of the spectral function $\sigma(\lambda)$. Now we shall find the conditions from which follows relation (13).

Proposition (6.1.2)[21]:

Suppose that for all $r > 0$ there exists $a \; \delta > 0$ such that

$$(S_r f, f) \geq \delta(f, f). \tag{16}$$

Relation (13) is valid if

$$\int_0^\infty |H(t)|^2 \, dt = M < \infty. \tag{17}$$

Proof. It follows from (16) that

$$S_r^{-1} \leq \frac{1}{\delta} I \quad . \tag{18}$$

From (3), (17) and (18) we deduce that

$$\int_0^r |\Gamma_r(t, 0)|^2 \, dt = \int_0^r |\Gamma_r(0, s)|^2 \, ds \leq M_\delta \quad , \tag{19}$$

where $M_\delta = M/\delta^2$. Let us estimate the integral

$$\left| \int_0^r \Gamma_r(0, s) \; e^{is\lambda} ds \right|^2 \leq M_\delta \int_0^r e^{-2as} ds \tag{20}$$

As $\int_0^\infty e^{-2as} ds \to 0$, when $\lambda = ia, \; a > 0, \; a \to \infty$, the assertion of the proposition follows from (5) and (20).

Corollary (6.1.3)[21]:

If relation (17) and inequality

$$\int_{-\infty}^{\infty} |H(t)| dt = q < 1 \tag{21}$$

are fulfilled , then condition (13) is valid.
Indeed from inequality (21) we deduce that

$$\|S_r - I\| \le q. \tag{22}$$

This implies that the conditions of Proposition (6.1.2) hold. Hence Corollary (6.1.3) follows.

Corollary (6.1.4)[21]: If conditions of Theorem(6.1.1) are fulfilled and coefficient $A(r)$ is real, then $\alpha = 0$.

Indeed in this case the function $P_*(r,i)$ is positive. Hence $\Pi(i)$ is positive as well. From formula (12) we obtain that

$$\Pi(i) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Log} \acute{\sigma}(t)}{(1 + t^2)} dt + i\alpha\right) \tag{23}$$

As $\Pi(\mathrm{i})$ is positive it follows from formula (23) that $\alpha = 0$.
Let us consider separately the case when

$$A(r) = 0, \quad r \ge R \tag{24}$$

In this case we have

$$\frac{dP_*}{dr} = 0 \quad ,r \ge R \tag{25}$$

Hence the following equality

$$\Pi(z) = P_*(R,z) \tag{26}$$

is true. From (5) and (26) we obtain the following assertion.

Corollary (6.1.5)[21]: If relation (24) is true, then relations (13) and (14) are true as well.

Let us note that there is no problem in defining the α value in the case of orthogonal polynomials (see [43]). It can be explained by a good choice of normalization. In the case of Krein's system such normalization is also possible. We shall introduce $\Pi(z)$ not with the help of relation (10), but with the help of the equality

$$\Pi(\mathrm{z}) = \lim\,[P_*(r,z)\exp(-i\gamma(r)], \quad r \to \infty, \tag{27}$$

where $\gamma(r) = \arg P_*(r,i)$. Then in view of (12) and (23) we have

$$\Pi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{(1+tz)\operatorname{Log} \acute{\sigma}(t)}{(z-t)(1 + t^2)} dt\right) \tag{28}$$

Theorem (6.1.1) was formulated by M.G.Krein without any proof. In our works [51,52,53] we gave the proof of this theorem but condition 4) of Theorem (6.1.1) must be replaced by the following condition:

$\acute{4}$) There exists a sequence $r_n \to \infty$ such that

$$\Pi(z) = \lim \mathrm{P}_*(r_n,z),\ r_n \to \infty,\ \Pi(z) \not\equiv \infty \tag{29}$$

at any bounded closed set z of the open half-plane $Imz > 0$.

Remark (6.1.6)[21]: A.Teplyaev called our attention to the necessity of replacing condition 4) by condition4́). In his article [54] Theorem (6.1.1) was partially proved (the equivalence of conditions 1), 2), 3) and 4́)).
The formula (12) doesn't follow from condition 4́). Further (see the next section) we shall prove that condition 4́) can be replaced by the stronger condition: $4(s)$). There exists a sequence $r_n \to \infty$ such that

$$\Pi(z) = \lim P_*(r_n, z), \ \lim P(r_n, z) = 0, \ r_n \to \infty, \tag{30}$$

at any bounded closed set z of the open half-plane $Imz > 0$.
In this case formula (12) is valid and conditions 1), 2), 3), 4(s)) and 5) are equivalent.
We show the generalized Krein systems (matrix case) .The matrix version of system (6) has the form

$$\frac{dP_1(x)}{dx} = izDP_1 + A_{11}(x)P_1 + A_{12}(x)P_2, \frac{dP_2(x)}{dx} = A_{21}(x)P_1, \quad x > 0 \tag{31}$$

where $A_{ij}(x)$ and $P_k(x,z)$ are $m \times m$ matrices and constant $m \times m$ matrix D has the form

$$D = diag[d_1,, d_2 \dots, d_m], \quad dk > 0 \ (k = 1,2,\dots,m). \tag{32}$$

We assume that the following conditions are fulfilled.

1. The matrices $A_{ij}(x)$ are continuous and
$$A_{11}(x) = -A_{11}^*(x), \ A_{21}(x) = A_{12}^*(x) \tag{33}$$
2. The matrix functions $P_1(x,z)$ and $P_2(x,z)$ satisfy the boundary conditions

$$P_1(0,z) = S_1, \ P_2(0,z) = S_2, \ detS_k \neq 0, \tag{34}$$

where S_1 and S_2 are constant $m \times m$ matrices such that

$$S_1^*S_1 = S_2^*S_2 \ . \tag{35}$$

We have proved the following theorem (see [51,52,53]):
Theorem (6.1.7) [21]: (Generalized Krein Theorem) The following propositions are equivalent:
1) The integral

$$K(z_0) = \int_0^\infty P_1^*(x,z_0)DP_1(x,z_0)dx \tag{36}$$

converges for at least one $z_0, Imz_0 > 0$.
2) The norm of matrix function $P_2(x,z_0)(0 \le x < \infty)$ is bounded for at least *one* $z_0, Imz_0 > 0$.
3) The integral $K(z)$ converges uniformly at any bounded closed set z of the open half-plane $Imz > 0$.
4) There exists a sequence $x_n \to \infty$ such that

$$\Pi(z) = \lim P_2(x_n, z), \quad x_n \to \infty, ||\Pi(z)|| \not\equiv \infty \tag{37}$$

at any bounded closed set z of the open half-plane Imz $>$ 0.

5) The integral

$$\int_{-\infty}^{\infty} \frac{\text{Log}\, det\, \acute{\sigma}(\lambda)}{\left(1 + \lambda^2\right)} d\lambda \tag{38}$$

is finite, where $\sigma(\lambda)$ is the spectral matrix function of system (31).
Now we shall prove that the condition 4) of Theorem (6.1.7) can be replaced by the stronger condition. We shall use the relation (see [51,52,53])

$$P_2^* (x,z) P_2(x,\xi) - P_1^* (x,z) P_1(x,\xi) = i(\bar{z} - \xi) \int_0^x P_1^* (x,z)\, D P_1(x,\xi) dx \tag{39}$$

In particular for $\xi = z$ we have

$$P_2^* (x,z) P_2(x,z) - P_1^* (x,z) P_1(x,z) = i(\bar{z} - z) \int_0^x P_1^* (x,z)\, D P_1(x,z) dx \tag{40}$$

There exists a sequence $R_k \to \infty$ such that (see [51,52,53])

$$\lim P_2(R_k, z) = \Pi(z), \lim P_1(R_k, z_0) = 0, \tag{41}$$

where $Imz0 > 0$. It follows from (40) that $||P_2(r,z)|| \geq ||P_1(r,z)||$.
Using this inequality we deduce that for a subsequence r_k of the sequence R_k there exist the limits

$$\lim P_2(r_k, z) = \Pi(z), \lim P_1(r_k, z) = \mathrm{Q(z)}, \tag{42}$$

where

$$Q(z_0) = 0. \tag{43}$$

Let us suppose that for another sequence $t_k \to \infty$ there exist some other limits

$$\lim P_2(t_k, z) = \Pi_1(z), \lim P_1(t_k, z) = \mathrm{Q}_1(\mathrm{z}). \tag{44}$$

It follows from condition 1) of Theorem (6.1.7) that there exists the limit of the right part of equality (39), when $R \to \infty$. Hence the following relation

$$\Pi_1^*(z)\Pi_1(\xi) - \mathrm{Q}_1^*(z)\mathrm{Q}_1(\xi) = \Pi^*(z)\Pi(\xi) - \mathrm{Q}^*(z)\mathrm{Q}(\xi) \tag{45}$$

is true. Under condition 5) of Theorem (6.1.7) the matrix $\acute{\sigma}(\lambda)$ is factorable , i.e.there exists an analytic maximal $m \times m$ matrix function $\Gamma(z), (Imz > 0)$ such that $det\, \Gamma(z) \neq 0$ and

$$\frac{1}{2\pi} \Gamma_+(\lambda)\, \Gamma_+^*(\lambda) = \acute{\sigma}(\lambda), \quad \lambda = \bar{\lambda}, \tag{46}$$

where $\Gamma_+(\lambda) = \lim \Gamma(\lambda + \epsilon)$, $\epsilon \to +0$. Following the argumentations of the [53] (Theorem 3.2) we obtain the assertion.

Proposition (6.1.8)[21]: Let condition 1) of Theorem (6.1.8)be fulfilled . Then Π^{-1} (z) is the maximal analytic matrix function satisfying the relation

$$\frac{1}{2\pi} \Pi_+^{-1} (\lambda) [\Pi_+^{-1} (\lambda)]^* = \acute{\sigma}(\lambda), \ \lambda = \bar{\lambda}, \tag{47}$$

where $\Pi_+^{-1} (\lambda) = \lim \Pi^{-1}(\lambda + \epsilon), \quad \epsilon \to +0.$

Remark (6.1.9)[21]: In paper [53] Proposition 2 is proved in the case that $z_0 = i$. It follows from (46) and (47) that

$$\Pi^{-1}(z) = \Gamma(z)U, \tag{48}$$

where U is a unitary constant $m \times m$ matrix. Using (40)–(43) and (48) we have

$$\Pi^*(z_0)\Pi(\xi) = [\Gamma^{-1}(z_0)]^*\Gamma^{-1}(\xi)$$
$$= i(\bar{z_0} - \xi)\int_0^\infty P_1^*(x, z_0)DP_1(x, \xi)dx \tag{49}$$

Theorem (6.1.10)[21]: Let condition 1) of Theorem (6.1.7) be fulfilled. If a sequence $R_k \to \infty$ is such that relation (41) is true then

$$Q(z) \equiv 0, \tag{50}$$

$$\Pi^*(z)\Pi(\xi) = [\Gamma^{-1}(z)]^* \Gamma^{-1}(\xi) = i(\bar{z} - \xi)\int_0^\infty P_1^*(x, z)DP_1(x, \xi)dx. \tag{51}$$

Proof. We can choose an arbitrary z_0, $(Imz_0 > 0)$. In this case the matrix function $\Pi(z)$ can change but not $\Gamma(z)$. Taking this fact into account we deduce from (48) and (49) relations (50) and (51). The theorem is proved.

Corollary (6.1.11)[21]: *Let* $\Pi_1(z)$ *and* $Q_1(z)$ be defined by relations (44). *Then* there exist constant $m \times m$ matrices A *and* B such that

$$\Pi_1(z) = A\Pi(z), \quad Q_1(z) = B\Pi(z), \tag{52}$$

where

$$A^* A - B^*B = I_m . \tag{53}$$

Proof. It follows from (45) and (50) that

$$\Pi_1^*(z)\Pi_1(\xi) - Q_1^*(z)Q_1(\xi) = \Pi^*(z)\Pi(\xi). \tag{54}$$

Relation (54) can be written in the form

$$Z^*jZ = \begin{bmatrix} I_m & I_m \\ I_m & I_m \end{bmatrix} \tag{55}$$

Where $j = diag[I_m, -I_m]$ and

$$Z = \begin{bmatrix} \Pi_2(z) & \Pi_2(\xi) \\ Q_2(z) & Q_2(\xi) \end{bmatrix} \tag{56}$$

Here matrix functions $\Pi_2(z)$ and $Q_2(z)$ are defined by the equalities

$$\Pi_2(z) = \Pi_1(z)\Pi^{-1}(z), \quad Q_2(z) = Q_1(z)\Pi^{-1}(z). \tag{57}$$

Relations of type (55) were investigated by V. Potapov ([49], Ch.2). Using Potapov's result we obtain the equality

$$ZT = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}, \tag{58}$$

where

$$T = \frac{1}{2}\begin{bmatrix} I_m & -I_m \\ I_m & I_m \end{bmatrix}. \tag{59}$$

We deduce from (58) that

$$\Pi_2(z) = \Pi_2(\xi) = \mathrm{A} = \mathrm{const}, \mathrm{Q}_2(z) = \mathrm{Q}_2(\xi) = \mathrm{B} = \mathrm{const}. \quad (60)$$

Hence the relations (52) and (53) are true.

Naw we show the generalized Krein systems in a particular case

Let us consider system (31), when

$$\mathrm{D} = I_m, \ \mathrm{A}_{11}(x) = 0, \ \mathrm{A}_{12}(x) = \mathrm{A}_{21}^*(x) = a(x). \quad (61)$$

We introduce the norm

$$||a||_p = [\sup \int_x^{x+1} \|a(t)\|^p dt]^{1/p}, \qquad x \geq 0. \quad (62)$$

Here $||a(x)||$ is the largest singular value of the $m \times m$ matrix $a(x)$. We assume that $p > 1$. When $p = 2$ the introduced norm coincides with the well-known Stummel norm (see [42]).

Theorem (6.1.12)[21]: *If condition* 1) of Theorem (6.1.7) is fulfilled and

$$||a||_p < \infty, \quad (p > 1), \quad (63)$$

then

$$\lim P_1(x,z) = 0, \quad x \to \infty, \ Imz > 0. \quad (64)$$

Proof. The system (31), (61) can be written in the form

$$\frac{dQ(x,ik)}{dx} = a(x)e^{kx}P_2(x,ik), \ \frac{dP_2(x,ik)}{dx} = a^*(x)e^{-kx}Q(x,ik), \quad (65)$$

where $Q(x,ik) = e^{kx}P_1(x,ik), \ z = ik$. From (65) we deduce that

$$e^{-kx}\int_0^x a(u)e^{ku}du = e^{-kx}\int_0^x \frac{dQ(u,ik)}{du}P_2^{-1}(u,ik)du. \quad (66)$$

It follows from (65) and (66) that

$$G(x,ik) = e^{-kx}\int_0^x e^{ku}Y(u,ik)a^*(u)Y(u,ik)du$$
$$+Y(x,ik) - e^{-kx}U, \quad (67)$$

where

$$G(x,ik) = e^{-kx}\int_0^x a(u)e^{ku}du, \quad (68)$$

$Y(x,ik) = P_1(x,ik)P_2^{-1}(x,ik), \ U = S_1S_2^{-1}$. In view of (35) the matrix U is unitary. Further we use the following inequality

$$\|G(x,ik)\| \leq$$
$$\|a\|_p e^{-kx}\left[\sum_{j=0}^{j=[x]-1}\left(\int_j^{j+1} e^{qku}du\right)^{\frac{1}{q}} + \left(\int_{[x]}^x e^{qku}du\right)^{\frac{1}{q}}\right], \quad (69)$$

where $[x]$ is the integer part of x and q is defined by the relation $1/p + 1/q = 1$. From (69) we deduce that

$$||G(x,ik)|| \leq \frac{\mathrm{C}}{k^{\frac{1}{q}}}e^{-kx}\left(\sum_{j=0}^{[x]} e^{kj} + e^{kx}\right) \leq \frac{\mathrm{C}_1}{k^{\frac{1}{q}}}. \quad (70)$$

It follows from relation (40) that

$$||Y(x, ik)|| \leq 1. \tag{71}$$

Inequalities (70) and (71) imply that

$$e^{-kx} \int_0^x \|a(u)\| e^{ku} \|Y(u, ik)\|^2 \, du \leq e^{-kx} \int_0^x \|a(u)\| e^{ku} du$$

$$\leq \frac{C_1}{k^{1/q}} \quad . \tag{72}$$

In view of (67), (70) and (71) we have

$$||Y(x, ik)|| \leq \frac{C_2}{k^{\frac{1}{q}}} + e^{-kx}. \tag{73}$$

There exists a sequence $x_k \to \infty$ such that

$$\lim P_2(x_k, z) = \Pi(z), \quad \lim P_1(x_k, z) = 0. \tag{74}$$

Let us assume that for another sequence $t_k \to \infty$ there exist some other limits

$$\lim P_2(t_k, z) = \Pi_1(z), \ \lim P_1(t_k, z) = Q_1(z). \tag{75}$$

Then according to Corollary (6.1.11) there exist constant $m \times m$ matrices A and B such that

$$\Pi_1(z) = A\Pi(z), \quad Q_1(z) = B\Pi(z). \tag{76}$$

It follows from (66) that

$$Q_1(z)\Pi_1^{-1}(z) = BA^{-1}. \tag{77}$$

Using inequality (73) we obtain that $B = 0$, i.e., $Q_1(z) = 0$. The theorem is proved. □

Theorem (6.1.13)[21]: If condition 1) of Theorem (6.1.7) is fulfilled and

$$\lim \int_x^{x+1} ||a(u)||du = 0, \qquad x \to \infty, \tag{78}$$

then

$$\lim P_1(x, z) = 0, \ x \to \infty, \ Imz > 0. \tag{79}$$

Proof. Let ε be an arbitrary positive number. There exists a natural number N such that

$$\int_x^{x+1} ||a(u)||du < \epsilon \quad , x \geq N. \tag{80}$$

Using notation (68) we have

$$\|G(x, ik)\| \leq e^{-kx} \int_0^N \|a(u)\| e^{ku} du + \varepsilon e^{-kx} \left(\sum_{j=N}^{[x]} e^{kj} + e^{kx} \right)$$

$$\leq e^{-kx} C_N + 4\varepsilon \ . \tag{81}$$

In view of (71) the inequality

$$e^{-kx} \int_0^x \|a(u)\| e^{ku} \|Y(u, ik)\|^2 du \leq e^{-kx} C_N + 4\varepsilon \tag{82}$$

is true. It follows from (67) and (81), (82) that

$$\|Y\ (x, ik)\| \le 2(e^{-kx}C_N + 4\varepsilon) + e^{-kx}. \quad (83)$$

Relations (74)–(77) are true in case $p = 1$ too. From (74)–(77) and estimation (83) we deduce the equality $B = 0$, i.e. $Q_1(z) = 0$. The theorem is proved. □

Corollary (6.1.14)[21]: If the conditions of either Theorem (6.1.12) or Theorem (6.1.13) are fulfilled, then

$$\Pi_1(z) = A\Pi(z), \quad (84)$$

where A is a constant unitary matrix.

Proposition (6.1.15)[21]: Let $a(x) \ge 0$ and let relation (79) be fulfilled. Then relation (78) is fulfilled too.

Proof. From (79) and inequality $a(x) \ge 0$ we obtain the relation

$$y(x, ik) \to 0,\ x \to \infty. \quad (85)$$

Using (76) , (77) and (85) we have that

$$e^{-kx} \int_0^x e^{ku} a(u) du \to 0. \quad (86)$$

It follows from (86) that

$$e^{-kx} \int_x^{x+1} e^{ku} a(u) du \to 0, \quad (87)$$

i.e. relation (78) is fulfilled. The proposition is proved. □

Corollary (6.1.16)[232]: Suppose that for all $r > 0$ and $\epsilon > 0$ such that

$$(S_r f, f) \ge (r+\epsilon)(f, f).$$

Relation (13) is valid if

$$\lim_{r\to\infty} \int_0^r |H(t)|^2\, dt = M_r < \infty.$$

Proof. It follows from (16) that

$$S_r^{-1} \le \frac{1}{(r+\epsilon)} I \quad .$$

From (3), (17) and (18) we deduce that

$$\int_0^r |\Gamma_r(t, 0)|^2\, dt = \int_0^r |\Gamma_r(0, s)|^2\, ds \le M_{(r+\epsilon)} \quad ,$$

where $M_{(r+\epsilon)} = M/(r+\epsilon)^2$. Let us estimate the integral

$$\left| \int_0^r \Gamma_r(0, s)\ e^{is\lambda} ds \right|^2 \le M_{(r+\epsilon)} \int_0^r e^{-2as} ds$$

As $\int_0^\infty e^{-2as} ds \to 0$, when $\lambda = ia,\ a > 0,\ a \to \infty$, the assertion of the Corollary follows from (5) and (20).

Corollary(6.1.17)[232]: If condition 1) of Theorem (6.1.7) is fulfilled and

$$\lim \int_{x}^{x+1} \sum_{r\in Z} \|a(u_r)\| \, du_r = 0, \qquad x \to \infty,$$

then

$$\lim P_1(x,z) = 0, \; x \to \infty, \; Imz > 0.$$

Proof. Let ε be an arbitrary positive number. There exists a natural number N such that

$$\int_x^{x+1} \sum_{r\in Z} \|a(u_r)\| \, du_r < \epsilon \quad , x \geq N.$$

Using notation (68) we have

$$\sum_{r\in Z} \|G(x,ik_r)\| \leq \sum_{r\in Z} e^{-k_r x} \int_0^N \|a(u_r)\| e^{k_r u_r} du_r + \sum_{r\in Z} \varepsilon e^{-k_r x} \left(\sum_{j=N}^{[x]} e^{k_r j} + e^{k_r x} \right)$$
$$\leq e^{-k_r x} C_N + 4\varepsilon \; .$$

In view of (71) the inequality

$$\sum_{r\in Z} e^{-k_r x} \int_0^x \|a(u_r)\| e^{k_r u_r} \|Y(u_r, ik_r)\|^2 du_r \leq \sum_{r\in Z} e^{-k_r x} \, \widetilde{C}_N + 4\varepsilon$$

is true. It follows from (67) and (81), (82) that

$$\sum_{r\in Z} \|Y(x,ik_r)\| \leq 2 \sum_{r\in Z} (e^{-k_r x} C_N + 4\varepsilon) + \sum_{r\in Z} e^{-k_r x} .$$

Relations (74)–(77) are true in case $p = 1$ too. From (74)–(77) and estimation (83) we deduce the equality $B = 0$, i.e. $Q_1(z) = 0$. The theorem is proved. □

Sec (6.2): Triangular Factorization of Positive Operators

In the Hilbert space $L_m^2(a,b)$ we define the orthogonal projectors $P_\xi f = f(x), \; a \leq x < \xi$ and $P_\xi f = 0, \; \xi < x \leq b$, where $f(x) \in L_m^2(a,b)$.

Definition (6.2.1)[20]: A bounded operator S_- on $L_m^2(a,b)$ is called lower triangular if for every ξ the relations

$$S_- Q_\xi = Q_\xi S_- Q_\xi , \qquad (88)$$

Are true, where $Q_\xi = I - P_\xi$.

Definition (6.2.2)[20]: A bounded operator S_+ on $L_m^2(a,b)$ is called upper triangular if for every ξ the relations

$$S_+ P_\xi = P_\xi S_+ P_\xi \qquad (89)$$

are true.

Definition (6.2.3)[20]: A bounded, positive and invertible operator S on $L_m^2(a,b)$ is said to admit the right triangular factorization if it can be represented in the form

$$S = S_+ S_+^* . \tag{90}$$

where S_+ and S_+^{-1} are upper triangular, bounded operators.

Definition (6.2.4)[20]: A bounded, positive and invertible operator S on $L_m^2(a,b)$ is said to admit the left triangular factorization if it can be represented in the form

$$S = S_- S_-^* , \tag{91}$$

where S_- and S_-^{-1} are lower triangular, bounded operators.

Gohberg and M.G. Krein [29] studied the problem of factorization under the assumption

$$S - I \in \gamma_\infty, \tag{92}$$

where γ_∞ is the set of compact operators. The operators $S_-$and S_+were assumed to have the form $S_+ = I + X_+$, $S_- = I + X_-$; $X_+, X_- \in \gamma_\infty$. The factorization method plays an important role in a number of analysis problems (for instance integral equations [39], spectral theory [40], nonlinear integrable equations). Giving up condition (92) and considering more general triangular operators would essentially widen the scope of the factorization method. D. Larson proved in his famous work [33] the existence of positive non-factorable operators. In this Section we formulate the necessary and sufficient conditions under which the positive operator S admits a triangular factorization. The factorizing operator $V = S_-^{-1}$ is constructed in an explicit form, also we consider the class of positive operators S which satisfy the operator identity

$$AS - SA^* = \Pi J \Pi^* . \tag{93}$$

For operators of this class, the factorization conditions have a simpler form. The general results of this Section are applied to operators with difference kernels

$$Sf = \frac{d}{dx}\int_0^a f(t)s(x - t)dt. \tag{94}$$

and to operators with sum-difference kernels,

$$Sf = \frac{d^2}{dx^2}\int_0^b [s_1(x - t) + s_2(x + t)]f(t)dt, \tag{95}$$

where $f(t) \in L^2(0,b)$. In particular, we prove that the Dixon operator [28], [32], [41]

$$Sf = f(x) - \frac{\lambda}{\pi}\int_0^1 \frac{f(t)}{x + t}dt = g(x). \tag{96}$$

where $f(x) \in L^2(0,1)$ and $\lambda < 1$, admits a left triangular factorization. We note that the operators of the forms (94) and (95) play an important role in theoretical and applied problems (inverse problems, stationary processes, prediction theory). also we investigate the case when

$$Af = i\int_0^x f(t)dt, \ \text{rank}(AS - SA^*) = 1. \tag{97}$$

In this case the factorizing operator S_- has the special form

$$S_-f = \frac{d}{dx}\int_0^x f(t)\phi(x - t)dt. \tag{98}$$

In this Section we consider a class of operators of the form

$$SF = F(x) - \int_0^1 F(y)k\left(\frac{y}{x}\right)\frac{1}{x}dy = G(x), \tag{99}$$

Where $F(x) \in L^2(0,1)$. The Dixon operator belongs to this class.

Remark(6.2.5)[20]: We consider triangular operators in the space $L_m^2(a,b)$ with the special set of projectors P_ξ. A general theory of triangular operators is constructed in the works [26], [27], [31], [33]–[36].

Let S be a linear, bounded and invertible operator S on $L_m^2(a,b)$.We introduce the notation

$$S_\xi = P_\xi S P_\xi, \ (f,g)_\xi = \int_a^\xi g^*(x)f(x)dx, \tag{100}$$

where $f(x), g(x) \in L_m^2(a,b)$.

Theorem (6.2.6)[20]: Let the bounded and invertible operator S on $L_m^2(a,b)$.be positive. For the operator S to admit the left triangular factorization it is necessary and sufficient that the following assertions are true.

1. There exists an $m \times m$ matrix function $F_0(x)$ such that

$$Tr\int_a^b F_0^*(x)F_0(x)dx < \infty, \tag{101}$$

that the $m \times m$ matrix function

$$M(\xi) = \left(F_0(x), S_\xi^{-1} F_0(x)\right)_\xi \tag{102}$$

is absolutely continuous, and almost everywhere

$$det\acute{M}(\xi) \neq 0. \tag{103}$$

2. The vector functions

$$\int_a^x v^*(x,t)f(t)dt \tag{104}$$

are absolutely continuous. Here $f(x) \in L_m^2(a,b)$ and

$$v(\xi,t) = S_\xi^{-1} P_\xi F_0(x), \tag{105}$$

(In (102) the operator S_ξ^{-1} transforms the matrix column of the original into the corresponding column of the image.)

3. The operator

$$V\,f \;=\; [R^{*}(x)]^{-1}\,\frac{d}{dx}\int_{0}^{x} v^{*}(x,t)f(t)dt \qquad (106)$$

is bounded, invertible and lower triangular with its inverse V^{-1}. Here $R(x)$ *is an* $m\times m$ matrix function such that

$$R^{*}(x)R(x) \;=\; \acute{M}(x). \qquad (107)$$

Proof. Necessity. We suppose that the operator S admits the left triangular factorization (91). Let $F_0(x)\in L_m^2(a,b)$ be a fixed $m\times m$ matrix function satisfying relation (101). We introduce the $m\times m$ matrix function

$$R(x) \;= V\,F_0(x), \qquad (108)$$

where $V=S_{-}^{-1}$. We can choose $F_0(x)$ in such a way that almost everywhere thenequality

$$\det R(x)\;\neq\;0 \qquad (109)$$

is true. From relations (91), (102) and (108) we have

$$M(\xi)\;=\;\int_{a}^{\xi} R^{*}(x)R(x)dx. \qquad (110)$$

Hence the function $M(\xi)$ is absolutely continuous and

$$\acute{M}(x)\;=\;R^{*}(x)R(x). \qquad (111)$$

Now we use the equality

$$(f,\,S_{\xi}^{-1}\,F_0)_{\xi}\;=\;(Vf,VF_0)_{\xi}. \qquad (112)$$

Relations (108) and (112) imply that

$$\frac{d}{dx}\int_{a}^{x} v^{*}(x,t)f(t)dt\;=\;R^{*}(x)(V\,f). \qquad (113)$$

The necessity is proved.

Sufficiency. Let the conditions 1–3 of Theorem (6.2.6) be fulfilled. It follows from (105)–(107) that

$$V\,F_{\,0}=\;R(x). \qquad (114)$$

From relations (105), (106) and (214) we deduce that $(V\,f,V\,F_0)_{\xi}\;=\;(f,S_{\xi}^{-1}\,P_{\xi}F_0)_{\xi}$, *i.e.*,

$$V^{*}P_{\xi}V\,P_{\xi}F_0\;=\;S_{\xi}^{-1}P_{\xi}F_0. \qquad (115)$$

We define $v(\xi,t)$ in the domain $\xi\;\leq\;t\;\leq\;b$ by the equality $v(\xi,t)\;=\;0$. It follows from the triangular structure of the operators V and V^{-1} that

$$P_{\xi}V^{-1}P_{\xi}V\,P_{\xi}\;=\;P_{\xi}. \qquad (116)$$

Hence in view of (105) and (115) we have

$$P_{\xi}V^{-1}[V^{*}]^{-1}v(\xi,t)\;=\;P_{\xi}F_0. \qquad (117)$$

It is easy to see that $P_{\xi}Sv(\xi,t)\;=\;P_{\xi}F_0$. Thus according to relations (116) and(117), the equality

$$(V^{-1}[V^{*}]^{-1}v(\xi,t),v(\mu,t))\;=\;(Sv(\xi,t),v(\mu,t)) \qquad (118)$$

is true. If there exists such a vector function $f_0(x) \in L_m^2(a,b)$ that $(f_0, v(\xi,t)) = 0$,then due to (106) the relation

$$V f_0 = 0 \tag{119}$$

is valid. The operator V is invertible. Hence from (119) we deduce that $f_0 = 0$. This means that $v(\xi,t)$ is a complete system in $L_m^2(a,b)$ Using this fact and relation (118) we obtain the desired equality

$$S = V^{-1}[V^*]^{-1}. \tag{120}$$

The theorem is proved. □

Corollary (6.2.7)[20]: If the conditions of Theorem (6.2.6) are fulfilled, then the corresponding operator S^{-1} can be represented in the form

$$S^{-1} = V^*V. \tag{121}$$

We introduce the notation

$$C_\xi = Q_\xi S Q_\xi, \qquad [f,g]_\xi = \int_\xi^b g^*(x)f(x)dx. \tag{122}$$

In the same way as Theorem (6.2.6) we deduce the following result.

Theorem (6.2.8)[20]: Let the bounded and invertible operator S on $L_m^2(a,b)$ *be* positive. For the operator S to admit the right triangular factorization it is necessary and sufficient that the following assertions are true.

1. There exists an $m \times m$ matrix function $F_0(x)$ such that

$$Tr\int_a^b F_0^*(x)F_0(x)dx < \infty, \tag{123}$$

that the $m \times m$ matrix function

$$N(\xi) = \left[F_0(x), C_\xi^{-1} F_0(x)\right]_\xi \tag{124}$$

is absolutely continuous, and almost everywhere

$$det\acute{N}(\xi) \neq 0. \tag{125}$$

2. The vector functions

$$\int_x^b u^*(x,t)f(t)dt \quad , \tag{126}$$

are absolutely continuous. Here $f(x) \in L^2(a,b)$ and

$$u(\xi,t) = C_\xi^{-1} Q_\xi F_0. \tag{127}$$

3. The operator

$$Uf = -[Q^*(x)]^{-1}\frac{d}{dx}\int_x^b u^*(x,t)f(t)dt \tag{128}$$

is bounded, upper triangular and invertible together with its inverse U^{-1}. Here

$$Q^*(x)Q(x) = -\acute{N}(x). \tag{129}$$

Corollary (6.2.9)[20]: If the conditions of Theorem (6.2.8) are fulfilled, then the corresponding operator S^{-1} can be represented in the form

$$S^{-1} = U^*U. \tag{130}$$

Remark (6.2.10)[20]: Formulas (105), (106) and (127), (128) give the right and left factorization of the operator $T = S^{-1}$. It can be useful for solving operator equations of the form $Sf = g$. Using the notation

$$T = S^{-1},\ T_{\xi} = Q_{\xi}TQ_{\xi},\ w(\xi,t) = T_{\xi}^{-1}\,Q_{\xi}TF_0. \tag{131}$$

We introduce the operator

$$Wf = -[R^*(x)]^{-1}\frac{d}{dx}\int_x^b w^*(x,t)f(t)dt. \tag{132}$$

The connection between the operators V and W is given by the following assertion.

Proposition (6.2.11)[20]: Let the operator V defined by formula (106) be bounded. Then the operator W defined by formula (132) is also bounded and

$$WT = V. \tag{133}$$

Proof. It can be proved by linear algebra methods that (see [40], p. 41)

$$TQ_{\xi}T_{\xi}^{-1}\,Q_{\xi}T = T - S_{\xi}^{-1}\,P_{\xi}. \tag{134}$$

From relations (105), (131) and (134) we have

$$Tw(\xi,t) = TF_0 - v(\xi,t). \tag{135}$$

Hence the equality

$$[Tf,w(\xi,t)]_{\xi} = (Tf,F_0) - \big(f,v(\xi,t)\big)_{\xi} \tag{136}$$

is true. From formulas (106) , (132) and (136) we obtain relation (133). The proposition is proved. □

Using Proposition (6.2.11) we deduce the following important assertion.

Proposition (6.2.12)[20]: Let S be a bounded, positive, invertible operator and let the operator V defined by formula (106) be bounded. If the relations

$$V F_0 = R(x), \tag{137}$$

and

$$V f \neq 0,\ \ \|f\| \neq 0 \tag{138}$$

are true, then the operator V is invertible, the operator V^{-1} is lower triangular, and

$$T = V^*V. \tag{139}$$

(Thus the operator T admits the right triangular factorization.)

Proof. It follows from the boundedness of the operator V and relation (133) that the operator W is also bounded. Let us consider

$$(Wf,R) = \int_a^b w^*(a,t)f(t)dt = (f,F_0), \tag{140}$$

i.e.,

$$W^*R = F_0. \tag{141}$$

Due to (137) and (141) we have

$$VW^*R = R. \tag{142}$$

From (133) we deduce that

$$WTW^* = VW^*. \tag{143}$$

Using (143) we see that the operator VW^* is selfadjoint and lower triangular. It means that the operator VW^* has the form

$$VW^*f = L(x)f, \tag{144}$$

where $L(x)$ is an $m \times m$ matrix function. Taking into account equality (142) we have $L(x) = I_m$, i.e.,

$$VW^* = I, \ WV^* = I. \tag{145}$$

Let us introduce the notation $H = W^* L_m^2(a,b)$. If for all $h \in H$ the relation $(g,h) = 0$ is true, then $Wg = 0$. Hence in view of relation (133) we obtain that

$$Vf = 0 \qquad (f = T^{-1}g). \tag{146}$$

From condition (138) we deduce that $g = 0$.
Then the equality

$$H = L_m^2(a,b) \tag{147}$$

is valid. Due to (145) and (147) the operator W^* maps $L_m^2(a,b)$ onto $L_m^2(a,b)$ one-to-one. According to the classical Banach theorem [25] the operator W^* is invertible. It follows from (145) that the operator V is also invertible and

$$V^{-1} = W^*, \tag{148}$$

and

$$V^*W = I. \tag{149}$$

From (133) and (149) we directly obtain that $T = V^*V$. The proposition is proved. □

Example (6.2.13) : Let us consider the operator

$$Sf = f(x) + \frac{i}{\pi} V.P. \int_a^b f(t) \frac{c(t)c(x)}{x - t} dt,$$

$$-\infty < a < b < \infty, \tag{150}$$

where $0 < m < c(t) < 1$. The operator (150) does not satisfy condition (92) but admits the left triangular factorization (see [14]).

We consider the operators A, S, Π and J satisfying the operator identity

$$AS - SA^* = i\Pi J \Pi^*. \tag{151}$$

We suppose that the operators A and S act on the Hilbert space $L_m^2(0,b)$, the operator Π maps G (dim$G = n < \infty$) into $L_m^2(0,b)$, the operator J acts on G, and $J = J^*$, and $J^2 = I_n$. We note that the operator Π has the form $\Pi g = [\phi_1(x), \phi_2(x), \dots, \phi_n(x)]g$, where $\phi_k(x)$ are $m \times 1$ vector functions, $g = col[g_1, g_2, \dots, g_n]$, $\phi_k(x) \in L_m^2(0,b)$, Relation (151) is fulfilled for the operators S which play an important role in the spectral theory of the canonical differential systems (see [40]). We shall use the following result ([40], Ch. 4).

Theorem (6.2.14)[20] : Let the following conditions be fulfilled.

1. The operator S is bounded, positive and invertible.

2. The relations

$$A^*P_\xi = P_\xi A^* P_\xi, \quad 0 \le \xi \le b \tag{152}$$

are true.

3. The spectrum of the operator A is concentrated at the origin and there is a constant $M > 0$ such that

$$\|(P_{\xi+\Delta\xi} - P_\xi)A(P_{\xi+\Delta\xi} - P_\xi)\| \le M|\Delta\xi|, \ 0 \le \xi \le b. \tag{153}$$

Then the $n \times n$ matrix function

$$W(\xi,z) = I_n + izJ\Pi^* S_\xi^{-1}(I - zA)^{-1}P_\xi\Pi \tag{154}$$

satisfies the matrix integral equation

$$W(x,z) = I_n + izJ\int_0^x [dB(t)]W(t,z), \tag{155}$$

where

$$B(\xi) = \Pi^* S_\xi^{-1} P_\xi \Pi. \tag{156}$$

From relations (91) and (156) we obtain the necessary conditions for the operator S to admit the left triangular factorization.

Proposition (6.2.15)[20]: Let the operator S satisfy the relation (151) and let the conditions of Theorem (6.2.14) be fulfilled. If the operator S admits the left triangular factorization, then the matrix function $B(x)$ is absolutely continuous and

$$\frac{d}{dx}B(x) = H(x) = \beta^*(x)\beta(x), \tag{157}$$

where

$$\beta(x) = [h_1(x), h_2(x), \dots, h_n(x)], \ h_k(x) = V\phi_k(x), V = S_-^{-1}. \tag{158}$$

Using relations (155) and (157) we obtain that

$$\frac{d}{dx}W(x,z) = izJH(x)W(x,z). \tag{159}$$

Lemma (6.2.16)[20]: Let the conditions of Proposition (6.2.15) be fulfilled and let the m × 1 vector functions

$$F_j(x,z) = (I - Az)^{-1}\phi_j, \quad 1 \le j \le n \tag{160}$$

form a complete system in $L_m^2(a,b)$.

Then we have the equality

$$\text{mes}E = 0, \tag{161}$$

where the set E is defined by the relation

$$x \in E \qquad if \ \ H(x) = 0. \tag{162}$$

Proof. We use the following relation (see [40], Ch. 4):

$$\frac{J - W^*(\xi,\mu)JW(\xi,\lambda)}{i(\bar{\mu} - \lambda)} = \Pi^*(I - \bar{\mu}A^*)^{-1}S_\xi^{-1}(I - \lambda A^{-1}P_\xi\Pi). \tag{163}$$

Formula (163) implies that

$(S_\xi^{-1} F_j(x,\lambda), F_\ell(x,\mu))_\xi$

$$= \frac{i[Y_\ell^*(\xi,\mu)JY_j(\xi,\lambda) - Y_\ell^*(0,\mu)JY_j(0,\lambda)]}{\bar{\mu} - \lambda}, \quad (164)$$

Where

$Y_j(x,\lambda) = col[W_{1,j}(x,\lambda), W_{2,j}(x,\lambda), \ldots W_{n,j}(x,\lambda)]$. Here $W_{i,j}(x,\lambda)$ are entries of $W(x,\lambda)$. In view of (159) and (164) we have

$$\frac{d}{d\xi}(S_\xi^{-1}F_j(x,\lambda), F_\ell(x,\mu))_\xi = 0, \ \xi \in E. \quad (165)$$

From (162) and (165) it follows that

$$\frac{d}{d\xi}(V F_j(x,\lambda), V F_\ell(x,\mu))_\xi = 0, \quad \xi \in E, \quad (166)$$

i.e., the relation

$$[V F_j](x,\lambda) = 0 \quad , x \in E, \quad 1 \le j \le n, \quad (167)$$

is true. As the operator V is invertible and the system of functions $F_j(x,\lambda)$ is complete in $L_m^2(0,b)$, the system of the functions $V F_j(x,\lambda)$ is also complete in $L_m^2(0,b)$The assertion of the lemma follows from this fact and equality (167). □

Further we suppose that the $n \times n$ matrix function $B(x)$ is absolutely continuous and that relations (157), (158) are true.

Let us introduce the $m \times m$ matrix functions

$$R(x) = h_1(x)\alpha_1 + h_2(x)\alpha_2 + \cdots + h_n(x)\alpha_n, \quad (168)$$

$$F_0(x) = \phi_1(x)\alpha_1 + \phi_2(x)\alpha_2 + \cdots + \phi_n(x)\alpha_n, \quad (169)$$

$$v(\xi,x) = S_\xi^{-1} P_\xi F_0(x), \quad (170)$$

where α_k are constant $1 \times m$ matrices. From Proposition (6.2.15) we deduce:

Corollary (6.2.17)[20]: Let the conditions of Theorem (6.2.14) and Lemma (6.2.16) be fulfilled. If $m = 1$, then there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ *such* that almost everywhere we have the inequality

$$R(x) \ne 0. \quad (171)$$

Now we can formulate the main result of this section.

Theorem (6.2.18)[20]: Let the following conditions be fulfilled.

1. The operator S satisfies relation (151).
2. The conditions of Theorem (6.2.14) are valid.
3. The matrix function $B(x)$ is absolutely continuous and formulas (157) and (158) are true.
4. The vector functions $F_j(x,\lambda)$ $(1 \le j \le n)$ form a complete system in $L_m^2(a,b)$.
5. Almost everywhere the inequality

$$\det R(x) \ne 0 \quad (172)$$

holds.

Then the operator $T = S^{-1}$ admits the right triangular factorization

Proof. We introduce the operator

$$V f = [R^*(x)]^{-1}\frac{d}{dx}\int_0^x v^*(x,t)f(t)dt \ . \tag{173}$$

From (154), (172) and (173) we deduce the equality

$$V F_j = [h_1(x),\dots,h_n(x)]Y_j(x,z). \tag{174}$$

Relation (174) implies that

$$\left(\mathrm{V}\,F_j(x,\lambda),\mathrm{V}\,\mathrm{F}_\ell(x,\mu)\right) = \int_0^b \mathrm{Y}_\ell^*(x,\mu)\mathrm{H}(x)\mathrm{Y_j}(x,\lambda)\mathrm{d}x \ . \tag{175}$$

Using equality (174) and relation

$$\frac{d}{dx}\mathrm{Y_j}(x,z) = izJH(x)\mathrm{Y_j}(x,z) \tag{176}$$

we have

$$\left(V\,F_j(x,\lambda),VF_\ell(x,\mu)\right) = \frac{i\left[Y_\ell^*(b,\mu)J\mathrm{Y_j}(b,\lambda) - Y_\ell^*(0,\mu)J\mathrm{Y_j}(0,\lambda)\right]}{\bar{\mu}-\lambda} \tag{177}$$

Comparing formulas (164) and (177) we obtain the equality

$$T = V^*V. \tag{178}$$

This means that the introduced operator V is bounded, $V f \neq 0$, and $\|f\| \neq 0$. Taking into account (168), (169) and (174) when $z = 0$ we obtain the relation

$$V F_0 = R. \tag{179}$$

Thus all conditions of Proposition (6.2.12) are fulfilled. The assertion of the theorem follows from Proposition (6.2.12). □

Proposition (6.2.19)[20]: Let the following conditions be fulfilled.

1. Conditions 1–3 of Theorem (6.2.18) are valid.
2. The $m \times m$ blocks $b_{1,j}(x)$ $(1 \le j \le n)$ of the matrix $B(x)$ are absolutely continuous and

$$b_{1,j}(x) = h_1^*(x)h_j(x). \tag{180}$$

3. All the entries of the matrices $h_j(x)$ belong to $L^2(a,b)$.
4. Almost everywhere the inequality (172) holds. Here $R(x) = h_1(x)$. Then the operator V defined by formula (173) and the equality

$$v(\xi,x) = S_\xi^{-1}P_\xi\varphi_1(x) \tag{181}$$

are bounded.

Proof. We introduce the matrix $H(x) = \beta^*(x)\beta(x)$, where $\beta(x) = [h_1(x),h_2(x),\dots,h_n(x)]$. Relations (173)–(175) remain true. We use the formula

$$\int_0^b Y_\ell^*(x,\mu)[dB(x)]Y_j(x,\lambda)dx = \frac{i\left[Y_\ell^*(b,\mu)JY_j(b,\lambda) - Y_\ell^*(0,\mu)JY_j(0,\lambda)\right]}{\bar{\mu}-\lambda} \tag{182}$$

and the inequality $H(x)dx \le dB(x)$. From formulas (164), (175) and (182) we deduce that

$$V^*V \le T. \tag{183}$$

The proposition is proved. □

Let us consider the bounded, positive and invertible operator S with the difference kernel

$$Sf = \frac{d}{dx}\int_0^a f(t)s(x - t)dt. \qquad (184)$$

Let us put

$$Af = i\int_0^x f(t)dt, \qquad f \in L^2(0,a). \qquad (185)$$

Equality (151) is valid (see [39], Ch. 1), if

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad (186)$$

$$\phi_1(x) = M(x), \phi_2(x) = 1, \qquad (187)$$

where $M(x) = s(x)$, $0 \le x \le a$. In the case under consideration the matrix $B(\xi)$ has the form

$$B(\xi) = \begin{bmatrix} (S_\xi^{-1}M, M) & (S_\xi^{-1}1, M) \\ (S_\xi^{-1} M, 1) & (S_\xi^{-1} 1, 1) \end{bmatrix}. \qquad (188)$$

The corresponding function $F(x,\lambda)$ has the form

$$F(x,\lambda) = e^{ix\lambda}. \qquad (189)$$

The operator A defined by formula (185) satisfies all the conditions of Theorem (6.2.14). The following fact is useful here.

Theorem (6. 2.20)[20]: Let the operator S be bounded, positive, invertible and have the form (184). If the matrix function $B(x)$ is absolutely continuous and

$$\acute{B}(x) = \beta^*(x)\beta(x), \ \beta(x) = [h_1(x), h_2(x)], \qquad (190)$$

Then the equality

$$h_1(x)\overline{h_2(x)} + h_2(x)\overline{h_1(x)} = 1 \qquad (191)$$

is true almost everywhere.

Proof. Let us consider the expression

$$i_\xi = (S_\xi^{-1} P_\xi M, 1) + (1, S_\xi^{-1} P_\xi M). \qquad (192)$$

Setting

$$N_1(x,\xi) = S_\xi^{-1} P_\xi M, \qquad (193)$$

we rewrite formula (192) in the form $i_\xi = \int_0^\xi [N_1(x,\xi) + \overline{N_1(x,\xi)}]dx$, i.e.,

$$i_\xi = \int_0^\xi \left[N_1(x,\xi) + \overline{N_1(\xi - x,\xi)}\right]dx . \qquad (194)$$

We use the relation (see [39], Ch. 1)

$$N_1(x,\xi) + \overline{N_1(\xi - x,\xi)} = 1. \qquad (195)$$

In view of (194) and (195) .

We obtain the equality

$$i_\xi \;=\; \xi. \qquad (196)$$

Taking into consideration Equalities (100), (158), (184) and (192) we deduce that

$$i_\xi \;=\int_0^\xi [h_1(x)\overline{h_2(x)} \;+\; h_2(x)\overline{h_1(x)}]dx\;. \qquad (197)$$

Relation (191) follows from (196) and (197). The theorem is proved. □
From equality (191) we have

$$h_2(x) \;\neq\; 0, \qquad 0 \;\leq\; x \;\leq\; a. \qquad (198)$$

Remark (6.2.21)[20]: The operators of the form

$$Sf \;=\; f(x) \;+\int_0^a f(t)k(x - t)dt, \qquad (199)$$

where $k(x) \;\in\; L(-a,a)$, belong to class (184). For this case inequality (198) was deduced by M.G. Krein by another method (see [29], Ch. 4). The main result of this section follows directly from Proposition (6.2.15), Theorem (6.2.18) and Inequality (198).

Theorem (6.2.22)[20]: Let the operator S be positive, invertible and have the form (184). Then the operator S admits the left triangular factorization if and only if the matrix $B(x)$ is absolutely continuous and relation (190) is valid.

Example (6.2.23)[20]: Let us consider the operator S_β of the form

$$S_\beta f \;=\; f \;+\frac{i\beta}{\pi}V.P.\int_0^b \frac{f(t)}{x-\,t}dt, \qquad (200)$$

where $-1 \;<\; \beta \;<\; 1$. This operator with a difference kernel is bounded, invertible and positive (see [14]). The operator S_β does not satisfy condition (91). Nevertheless S_β admits the left triangular factorization $S_\beta \;=\; W_\alpha W_\alpha^*$, where

$$W_\alpha f \;=\frac{x^{i\alpha}}{\sqrt{ch(\pi\alpha)}\;\Gamma(i\alpha - 1)}\frac{d}{dx}\int_0^x f(t)(x \;-\; t)^{-i\alpha}dt. \qquad (201)$$

Here $\alpha \;=\; \frac{1}{\pi}$ arcth β, and $\Gamma(z)$ is the gamma function.

Let us consider the following class of bounded and positive operators which can be represented in the form $((+,-) -$ class):

$$Sf \;=\frac{d^2}{dx^2}\int_0^b [s_1(x \;-\; t) \;+\; s_2(x \;+\; t)]f(t)dt\;, \qquad (202)$$

where $f(t) \;\in\; L^2(0,b)$. We introduce the operator

$$Af \;=\int_0^x (t \;-\; x)f(t)dt. \qquad (203)$$

Then the operator identity (151) is valid. Here the 4×4 matrix J is defined by the relation

$$J = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \quad , \tag{204}$$

and the operator Π has the form

$$\Pi = [\Phi_1, \Phi_2], \tag{205}$$

the operators Φ_1 and Φ_2 are defined by the relations

$$\Phi_1 g = -iM(x)g_1 - iM_0(x)g_2, \tag{206}$$

$$\Phi_2 g = g_1 + xg_2, \tag{207}$$

where

$$M(x) = -[s_1(x) + s_2(x)], \; M_0(x) = \acute{s}_1(x) - \acute{s}_2(x), \tag{208}$$

and a constant 2×1 vector g has the form $g = \mathrm{col}[g_1, g_2]$. The main result of this section follows directly from Proposition (6.2.15), Lemma (6.2.16) and Theorem (6.2.18).

Theorem (6.2.24)[20]: Let the operator S be positive, invertible and have the form (202). The operator S admits the left triangular factorization if and only if the matrix $B(x)$ is absolutely continuous and

$$\acute{B}(x) = \beta^*(x)\beta(x), \; \beta(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]. \tag{209}$$

Example (6.2.25)[20]: Let us consider the equation

$$Sf = f(x) + \frac{i\mu}{\pi} V.P. \int_0^1 \frac{f(t)}{x - t}\, dt - \frac{\lambda}{\pi} \int_0^1 \frac{f(t)}{x + t} dt = g(x), \tag{210}$$

where $f(x) \in L^2(0,1)$, $\lambda = \bar{\lambda}$, $\mu = \overline{\mu}$, *and* $|\lambda| + |\mu| < 1$. *It is well* known ([29], Ch. 9) that the operator S is bounded, positive and invertible, i.e., the operator S belongs to the $(+,-)$ class . We introduce the functions

$$v(x, \lambda, \mu) = S^{-1}1, \alpha(\lambda, \mu) = \int_0^1 v(x, \lambda, \mu)dx = (S^{-1}1, 1) > 0. \tag{211}$$

In view of (210) and (211) the relations

$$S_\xi^{-1} P_\xi 1 = v\left(\frac{x}{\xi}, \lambda, \mu\right), \quad \left(S_\xi^{-1} P_\xi 1\,, 1\right)_\xi = \xi\alpha(\lambda, \mu) \tag{212}$$

are true. We introduce the operator

$$V f = \frac{1}{\sqrt{\alpha(\lambda, \mu)}} \frac{d}{dx} \int_0^x f(t) v(\frac{t}{x}, \lambda, \mu)dt. \tag{213}$$

Using Proposition (6.2.19) we deduce that the operator V is bounded and $S^{-1} \geq V^* V$.

Open problem . Prove that

$$V f \neq 0, \quad \text{when } \|f\| \neq 0. \tag{214}$$

Remark (6.2.26)[20]: If relation (214) is true, then $S^{-1} = V^* V$ and the operator S admits the left triangular factorization

$$S = V^{-1}[V^*]^{-1}. \tag{215}$$

Remark (6.2.27)[20]: Relation (214) *is valid when* $\lambda = 0$ (see Example (6.2.23)). Now we consider separately the case when $\mu = 0$, i.e., the case of the Dixon equation [28],[32], [41]:

$$Sf = f(x) - \frac{\lambda}{\pi}\int_0^1 \frac{f(t)}{x+t}dt = g(x). \tag{216}$$

where $f(x) \in L^2(0,1)$, *and* $\lambda < 1$. M.G. Krein deduced the formula for the Dixon equation resolvent (see [32], Ch. 4). This formula can be written in the following way: $S^{-1} = V^*V$. Thus we obtain:

Proposition (6.2.28)[20]: The Dixon operator S defined by (216) admits the left triangular factorization $S = V^{-1}[V^*]^{-1}$, where the operator V has the form (213).

Let us consider the integral operators

$$Af = i\int_0^x f(t)dt, \quad A^*f = -i\int_x^b f(t)dt, \tag{217}$$

where $f(x) \in L^2(0,b)$.

Definition (6.2.29)[20]: We say that the linear bounded operator S acting in the Hilbert space $L^2(0,b)$. belongs to the class R_1 (rank 1) if the following conditions are fulfilled:

1) $m(f,f) \le (Sf,f) \le M(f,f), \qquad 0 < m < M < \infty.$ (218)

2) $\operatorname{rank}(AS - SA^*) = 1$, i.e.,

$$(AS - SA^*)f = i(f,\phi)\phi, \quad \phi(x) \in L^2(0,b). \tag{219}$$

We associate with the operator S the operator

$$S_{-}f = \frac{d}{dx}\int_0^x f(t)\phi(x-t)dt. \tag{220}$$

It is easy to see that

$$S_{-}1 = \phi. \tag{221}$$

Lemma (6.2.30)[20]: Let the bounded operator S satisfy relation (219). If the corresponding operator S_{-} is bounded, then the representation

$$S = S_{-}S_{-}^* \tag{222}$$

is true.

Proof. We consider the operator

$$X = S_{-}S_{-}^*. \tag{223}$$

Using formula (219) and relation $AS_{-} = S_{-}A$ we deduce the equality

$$AX - XA^* = S_{-}(A - A^*)S_{-}^* = AS - SA^*. \tag{224}$$

The equation $AX - XA^{*} = F$ has no more than one solution X (see [39], Ch. 1).Hence we deduce from (224) *that* $S = X$. The lemma is proved. □

Lemma (6.2.31)[20]: If the bounded operator S satisfies the relation (219), then this operator can be represented in the form (222), where the operator S_{-} is defined by formula (220).

Proof. To prove that the operator S_{-} is bounded we introduce the operator

$$X_{-}f = A S_{-}f = i\int_{0}^{x} f(t)\phi(x - t)dt. \tag{225}$$

We note that

$$X_{-}^{*}f = S_{-}^{*}A^{*}f = -i\int_{x}^{b} f(t)\overline{\phi(t - x)}dt \tag{226}$$

where the operator S_{-}^{*} has the form

$$S_{-}^{*}f = -\frac{d}{dx}\int_{x}^{b} f(t)\overline{\phi(t - x)}dt . \tag{227}$$

According to Lemma (6.2.30) we have

$$ASA^{*} = X_{-}X_{-}^{*}. \tag{228}$$

It follows from relations (225) and (228) that $S = S_{-}S_{-}^{*}$. Hence the operator S_{-} is bounded. The lemma is proved . □

Now we shall deduce the main result of this section.

Theorem (6.2.32)[20]: If the operator S belongs to the class R_1 , then this operator admits the left triangular factorization.

Proof. We suppose that for some $f_0(x) \in L^2(0,b)$ the relation

$$S_{-}f_0 = 0 \qquad (\|f_0\| \neq 0) \tag{229}$$

is true. In view of the well-known Titchmarsh theorem (see [41], Ch. 11) and (229) we have

$$\phi(x) = 0, \qquad 0 \leq x \leq \delta. \tag{230}$$

Using (219) and (230) we deduce that

$$A_{\delta}S_{\delta} - S_{\delta}A_{\delta}^{*} = 0, \tag{231}$$

where $A_{\delta}f = i\int_{0}^{x} f(t)dt$, $0 \leq x \leq \delta$, and $S_{\delta} = P_{\delta}SP_{\delta}$. Operator equation (231) has only the trivial solution $S_{\delta} = 0$ (see [39], Ch. 1). The last equality contradicts relation (218). It means that equality (229) is impossible when $\|f_0\| \neq 0$. Hence in view of (222) the operator S_{-} *maps* $L^2(0,b)$ *one-to-one* onto $L^2(0,b)$. This fact according to the classical Banach theorem [25] implies that the operator S_{-} is invertible. The operator S_{-}^{-1} is defined by formula (see [39], Ch. 1)

$$S_{-}^{-1} f = \frac{d}{dx}\int_0^x f(t)N(x - t)dt, \qquad (232)$$

Where $N(x) = S_{-}^{-1} 1$. Thus the operators S_{-} *and* S_{-}^{-1} are bounded and lower triangular . The assertion of the theorem now follows directly from Definition (6.2.4). □

Example (6.2.33)[20]: We consider the case when

$$\phi(x) = \log(b - x). \qquad (233)$$

In this case we have

$$S_{-} f = \frac{d}{dx}\int_0^x f(t)\log(b - x + t)dt = f(x)\log b - \int_0^x \frac{f(t)}{b - x + t}\, dt. \qquad (234)$$

Let us introduce the operator

$$Kf = \int_0^x \frac{f(t)}{b - x + t} dt. \qquad (235)$$

It is well known (see [41], Ch. 11) that $\|K\| \le \pi$. Hence the operator S_{-} defined by (234) and the operator S_{-}^{-1} are bounded, when $\log b > \pi$. From Lemma (6.2.31) we obtain the assertion.

Proposition (6.2.34)[20]: If $\log(b) > \pi$, then the operator S defined by relations (219) and (233) admits the left triangular factorization (222) where the operator S_{-} has the form (234).

In this section we consider operators of the form

$$SF = F(x) - \int_0^1 F(y)k\left(\frac{y}{x}\right)\frac{1}{x}dy = G(x), \qquad (236)$$

where $F(x) \in L^2(0,1)$ and

$$k\left(\frac{y}{x}\right)\frac{1}{x} = \overline{k\left(\frac{x}{y}\right)}\frac{1}{y}. \qquad (237)$$

We assume that

$$A = 2\int_0^1 \left|k\left(\frac{1}{x}\right)\right| x^{-\frac{3}{2}}\ dx < \infty. \qquad (238)$$

It follows from condition (237) that the operator S is selfadjoint. From condition (238) we deduce that the operator

$$KF = \int_0^1 F(y)k\left(\frac{y}{x}\right)\frac{1}{x}\ dy \qquad (239)$$

Is bounded and (see [29], Ch. 9)

$$\|k\| \le A. \qquad (240)$$

Theorem (6.2.35)[20]: Let conditions (237) and (238) be fulfilled and let the corresponding operator S be positive and invertible, then the operator S admits the left triangular factorization.

Proof. We introduce the change of variables $x = e^{-u}$ and $y = e^{-v}$. Hence equation (236) takes the form

$$Lf = f(u) - \int_0^\infty f(v)H(u - v)dv = g(u). \quad (241)$$

where

$$f(u) = F(e^{-u})e^{-\frac{u}{2}},\ g(u) = G(e^{-u})e^{-\frac{u}{2}}. \quad (242)$$

$$H(u) = \overline{H(-u)} = k(e^{u})e^{\frac{u}{2}}, \quad u \geq 0. \quad (243)$$

It follows from relation (238) that

$$\int_{-\infty}^{\infty} |H(u)|du = A. \quad (244)$$

We denote by $\gamma(u)$ the solution of Equation (241) when $g(u) = H(u)$. In the theory of equations (241) the following function plays an important role (see [32], Ch. 2):

$$G_+(\lambda) = 1 + \int_0^\infty \gamma(u)e^{it\lambda}dt, \quad \text{Im}\lambda \geq 0.$$

Let us consider the solution $\gamma_\xi(u)$ of equation (241) when $g(u) = e^{iu\xi}$ and $\text{Im}\xi \geq 0$.

We use the formula (see [32], Ch. 2)

$$\gamma_\xi(u) = \overline{G_+(-\bar{\xi})}\ [1 + \int_0^u \gamma(r)e^{-ir\xi}dr\ ]e^{iu\xi}\ . \quad (245)$$

Further we need the particular case of $\gamma_\xi(u)$ when $\xi = i/2$. In this case we have

$$\gamma_{\frac{i}{2}}(u) = \beta[1 + \int_0^u \gamma(r)e^{\frac{r}{2}}dr]e^{-\frac{u}{2}}\ , \quad (246)$$

Where

$$\beta = \overline{G_+(\imath/2)}. \quad (247)$$

Let us introduce the function $v(x)$, which satisfies Equation (236) when $G(x) = 1$. It is easy to see that

$$v(e^{-u}) = \gamma_{\frac{i}{2}}(u)e^{\frac{u}{2}}. \quad (248)$$

From (246) and (248) we deduce that

$$\acute{v}(x)x^2 = -\beta\gamma(t)e^{-\frac{t}{2}}, \quad (249)$$

and

$$v(1) = \beta. \tag{250}$$

Using relations (246) and (248) we can calculate the integral

$$\alpha = \int_0^1 v(x)dx = \beta\left[1 + \int_0^1 \int_0^{-\log x} \gamma(r)e^{\frac{r}{2}}drdx\right].$$

Hence the equalities

$$\alpha = \beta[1 + \int_0^\infty \gamma(r)e^{-\frac{r}{2}}drdx] = \beta\bar{\beta} \tag{251}$$

are true. The operator V in (236) has the form

$$V f = \frac{1}{\beta}\frac{d}{dx}\int_0^x f(t)v\left(\frac{t}{x}\right)dt . \tag{252}$$

In view of (249) and (250) we can represent the operator V in the form

$$V f = f(x) + \int_0^x f(t)L\left(\frac{t}{x}\right)\frac{1}{t}dt, \tag{253}$$

where

$$L(x) = \gamma(t)e^{-\frac{t}{2}}. \tag{254}$$

Now the assertion of the theorem follows from Proposition (6.2.12). □

Corollary (6.2.36)[20]: Let the conditions of Theorem (6.2.35) be fulfilled. Then we have the equality

$$S^{-1} = V^*V, \tag{255}$$

where the operator V is defined by relations (253) and (254).

Example (6.2.37)[20]: We obtain an interesting example when

$$k(u) = \frac{\lambda}{|1 - u|^\alpha(1 + u)^\beta} . \tag{256}$$

where $\lambda = \bar{\lambda}, \alpha \geq 0, \beta > 0,$ and $\alpha + \beta = 1$. We note that $k(u)$ satisfies conditions (237) and (238). Equations (236) and (256) coincide with the Dixon equation when $\alpha = 0$.

Corollary (6.2.38)[232]: Let the bounded and invertible operator S_+^2 on $L_m^2(a, a + \epsilon_2)$ be positive. For the self-adjoint operator S_+^2 to admit the left triangular factorization it is necessary and sufficient that the following assertions are true.

1. There exists *an* $m \times m$ matrix function $F_0(x)$ such that

$$Tr \int_a^{a+\epsilon_2} |F_0^*(x)|^2dx < \infty,$$

that the $m \times m$ matrix function

$$M(x+\epsilon) = \left(F_0^*(x), [S^*]_{(x+\epsilon)}^{-1} F_0^*(x)\right)_{(x+\epsilon)}$$
is absolutely continuous, and almost everywhere
$$det\acute{M}(x+\epsilon) \neq 0.$$
2. The vector functions
$$\int_a^x v^*(x,t) f(t) dt \quad ,$$
are absolutely continuous. Here $f(x) \in L_m^2(a, a+\epsilon_2)$ and
$$v((x+\epsilon), t) = [S^*]_{(x+\epsilon)}^{-1} P_{(x+\epsilon)} F_0^*(x),$$
(In (102) the self -adjoint operator $[S^*]_{(x+\epsilon)}^{-1}$ transforms the matrix column of the original into the corresponding column of the image.)
3. The operator
$$V^* f = [R^*(x)]^{-1} \frac{d}{dx} \int_0^x v^*(x,t) f(t) dt$$
is bounded, invertible and lower triangular with its inverse $[V^*]^{-1}$. Here $R^*(x)$ is an $m \times m$ matrix function such that
$$[R^*(x)]^2 = \acute{M}(x).$$
Proof. Necessity. We suppose that the self-adjoint operator S^* admits the left triangular factorization (91). Let $F_0^*(x) \in L_m^2(a, a+\epsilon_2)$ be a fixed $m \times m$ matrix function satisfying relation (101). We introduce the $m \times m$ matrix function
$$R^*(x) = V^* F_0^*(x),$$
where $V^* = [S^*]^{-2}$. We can choose $F_0^*(x)$ in such a way that almost everywhere then equality
$$\det R^*(x) \neq 0$$
is true. From relations (91), (102) and (108) we have
$$M(x+\epsilon) = \int_a^{(x+\epsilon)} |R^*(x)|^2 dx.$$
Hence the function $M(x+\epsilon)$ is absolutely continuous and
$$\acute{M}(x) = [R^*(x)]^2.$$
Now we use the equality
$$(f, [S^*]_{(x+\epsilon)}^{-1} F_0^*)_{(x+\epsilon)} = (V^* f, V^* F_0^*)_{(x+\epsilon)}.$$
Relations (108) and (112) imply that
$$\frac{d}{dx} \int_a^x v^*(x,t) f(t) dt = R^*(x)(V^* f).$$
The necessity is proved.

Sufficiency. Let the conditions 1–3 of Theorem (6.2.6) be fulfilled. It follows from (105)–(107) that

$$V^*F_0^* = R^*(x).$$

We can write $\acute{M}(x) = [V^*F_0^*]^2$.

From relations (105), (106) and (214) we deduce that

$$(V^*f, V^*F_0^* =)_{(x+\epsilon)} = (f, [S^*]_{(x+\epsilon)}^{-1} P_{(x+\epsilon)}F_0^*)_{(x+\epsilon)}, \quad i.e.,$$

$$V^*P_{(x+\epsilon)}V^* P_{(x+\epsilon)}F_0 = [S^*]_{(x+\epsilon)}^{-1} P_{(x+\epsilon)}F_0^*.$$

We define $v((x-\epsilon_1), t)$ in the domain $(x-\epsilon_1) \le t \le a+\epsilon_2$ by the equality $v((x-\epsilon_1), t) = 0$. It follows from the triangular structure of the self - adjoint operators V^* and $[V^*]^{-1}$ that

$$P_{(x-\epsilon_1)}[V^*]^{-1} P_{(x-\epsilon_1)}V^* P_{(x-\epsilon_1)} = P_{(x-\epsilon_1)}.$$

Hence in view of (105) and (115) we have

$$P_{(x-\epsilon_1)}[V^*]^{-2}v((x-\epsilon_1), t) = P_{(x-\epsilon_1)}F_0^*.$$

It is easy to see that $P_{(x-\epsilon_1)}S^*v((x-\epsilon_1), t) = P_{(x-\epsilon_1)}F_0^*$. Thus according to relations (116) and (117), the equality

$$([V^*]^{-2}v((x-\epsilon_1), t), v(\mu, t)) = (S^*v((x-\epsilon_1), t), v(\mu, t))$$

is true.

If there exists such a vector function $f_0(x) \in L_m^2(a, a+\epsilon_2)$ that $(f_0, v((x-\epsilon_1), t)) = 0$,then due to (106) the relation

$$V^*f_0 = 0$$

is valid. The self -adjoint operator V^* is invertible. Hence from (119) we deduce that $f_0 = 0$. This means that $v((x-\epsilon_1), t)$ is a complete system in $L_m^2(a, a+\epsilon_2)$ Using this fact and relation (118) we obtain the desired equality

$$S^* = [V^*]^{-2}.$$

The Corollary is proved. □

Corollary (6.2.39)[232]: Let the self-adjoint operator V^* defined by formula (106) be bounded. Then the operator W defined by formula (132) is also bounded and

$$WT^* = V^*.$$

Proof. It can be proved by linear algebra methods that (see [40], p. 41)

$$T^*Q_{(x-\epsilon_1)}[T^*]_{(x-\epsilon_1)}^{-1} Q_{(x-\epsilon_1)}T^* = T^* - [S^*]_{(x-\epsilon_1)}^{-1} P_{(x-\epsilon_1)}.$$

From relations (105), (131) and (134) we have

$$T^*w((x-\epsilon_1), t) = T^*F_0^* - v((x-\epsilon_1), t).$$

Hence the equality

$$[T^*f, w((x-\epsilon_1), t)]_{(x-\epsilon_1)} = (T^*f, F_0^*) - \Big(f, v\big((x-\epsilon_1), t\big)\Big)_{(x-\epsilon_1)}$$

is true. From formulas (106), (132) and (136) we obtain relation (133). The corollary is proved. □

Corollary (6.2.40)[232]: Let S^* be a bounded, positive, self-adjoint and invertible operator and let the operator V^* defined by formula (106) be bounded. If the relations

$$V^* F_0^* = R^*(x),$$

and

$$V^* f \neq 0, \ \|f\| \neq 0$$

are true, then the self-adjoint operator V^* is invertible, the operator $[V^*]^{-1}$ *is* lower triangular, and

$$T^* = [V^*]^2.$$

(Thus the self-adjoint operator T^* admits the right triangular factorization.)

Proof. It follows from the boundedness of the self-adjoint operator V^* and relation (133) that the operator W is also bounded. Let us consider

$$(Wf, R^*) = \int_a^{a+\epsilon_2} w^*(a,t) f(t) dt = (f, F_0^*),$$

i.e.,

$$W^* R^* = F_0^*.$$

Due to (137) and (141) we have

$$V^* W^* R^* = R^*.$$

From (133) we deduce that

$$W T^* W^* = V^* W^*.$$

Using (143) we see that the self-adjoint operator $V^* W^*$ is lower triangular. It means that the operator $V^* W^*$ has the form

$$V^* W^* f = L(x) f,$$

where $L(x)$ is an $m \times m$ matrix function. Taking into account equality (142) we have $L(x) = I_m$, i.e.,

$$V^* W^* = I, \ W V^* = I.$$

Let us introduce the notation $H = W^* L_m^2(a, a+\epsilon_2)$. If for all $h \in H$ the relation $(g, h) = 0$ is true, then $Wg = 0$. Hence in view of relation (133) we obtain that

$$V^* f = 0 \qquad (f = [T^*]^{-1} g).$$

From condition (138) we deduce that $g = 0$. Then the equality

$$H = L_m^2(a, a+\epsilon_2)$$

is valid. Due to (145) and (147) the operator self-adjoint W^* maps $L_m^2(a, a+\epsilon_2)$ onto $L_m^2(a, a+\epsilon_2)$ one-to-one. According to the classical Banach theorem [25] the operator W^* is invertible. It follows from (145) that the self-adjoint operator V^* is also invertible and

$$[V^*]^{-1} = W^*,$$

and

$$V^* W^* = I.$$

From (133) and (149) we directly obtain that $T^* = [V^*]^2$. The proposition is proved. □

Now we can deduce the following results.

Corollary (6.2.41)[232]: Suppose the hypothesis of Propositions (6.2.11) and (6.2.12) are satisfied

(i) $WT^*F_0^* = R^*(x)$.

$(ii) W[U^*U]F_0^* = R^*(x)$ and hence $T^* = [U^*]^2$.

(iii) $[WT^*F_0^*]^2 = \acute{M}(x)$.

Proof: (i) since $WT^* = V^*, WT^*F_0^* = V^*F_0^* = R^*(x)$.

(ii) $V^*F_0^* = WT^*F_0^* = W[S^*]^{-1}F_0^* = W[U^*U]F_0^*$, which implied that $T^* = [U^*]^2$.

(iii) Since $\acute{M}(x) = [R^*(x)]^2 = [V^*F_0^*]^2 = [WT^*F_0^*]^2$.

Corollary (6.2.42)[232]: Let the following conditions be fulfilled.

1. The self-adjoint operator S^* satisfies relation (151).
2. The conditions of Theorem (6.2.14) are valid.
3. The matrix function $B(x)$ is absolutely continuous and formulas (157) and (158) are true.
4. The vector functions $F_j(x,\lambda)$ $(1 \le j \le n)$ form a complete system in $L_m^2(a, a+\epsilon_2)$.
5. Almost everywhere the inequality

$$\det R^*(x) \neq 0$$

holds. Then the self-adjoint operator $T^* = [S^*]^{-1}$ admits the right triangular factorization

Proof. We introduce the self-adjoint operator

$$V^* f = [R^*(x)]^{-1}\frac{d}{dx}\int_0^x v^*(x,t)f(t)dt .$$

From (154), (172) and (173) we deduce the equality

$$V^* F_j = [h_1(x),\dots,h_n(x)]Y_j(x,z).$$

Relation (174) implies that

$$\left(V^* F_j(x,\lambda), V^* \mathrm{F}_\ell(x,\mu)\right) = \int_0^{a+\epsilon_2} \mathrm{Y}_\ell^*(x,\mu)\mathrm{H}(x)\mathrm{Y}_\mathrm{j}(x,\lambda)\mathrm{d}x .$$

Using equality (174) and relation

$$\frac{d}{dx}\mathrm{Y}_\mathrm{j}(x,z) = izJH(x)\mathrm{Y}_\mathrm{j}(x,z) \quad ,$$

we have

$$\left(V^* F_j(x,\lambda), V^*F_\ell(x,\mu)\right) = \frac{i\left[Y_\ell^*(a+\epsilon_2,\mu)J\mathrm{Y}_\mathrm{j}(a+\epsilon_2,\lambda) - Y_\ell^*(0,\mu)J\mathrm{Y}_\mathrm{j}(0,\lambda)\right]}{\bar{\mu}-\lambda}$$

Comparing formulas (164) and (177) we obtain the equality

$$T^* = [V^*]^2.$$

This means that the introduced self-adjoint operator V^* is bounded, $V^*f \neq 0$, and $\|f\| \neq 0$. Taking into account (168), (169) and (174) when $z = 0$ we obtain the relation

$$V^* F_0^* = R^*.$$

Thus all conditions of Proposition (6.2.12) are fulfilled. The assertion of the theorem follows from Proposition (6.2.12). □

Corollary (6.2.43)[232]: Let the following conditions be fulfilled.

1. Conditions 1–3 of Theorem (6.2.18) are valid.
2. The $m \times m$ blocks $b_{1,j}(x)$ $(1 \leq j \leq n)$ of the matrix $B(x)$ are absolutely continuous and

$$b_{1,j}(x) = h_1^*(x)h_j(x).$$

3. All the entries of the matrices $h_j(x)$ belong to $L^2(a, a + \epsilon_2)$.
4. Almost everywhere the inequality (172) holds. Here $R^*(x) = h_1(x)$. Then the self-adjoint operator V^* defined by formula (173) and the equality

$$v((x - \epsilon_1), x) = [S^*]_{(x-\epsilon_1)}^{-1} P_{(x-\epsilon_1)}\varphi_1(x)$$

are bounded.

Proof. We introduce the matrix $H(x) = [\beta^*(x)]^2$ where $\beta^*(x) = [h_1(x), h_2(x), \dots, h_n(x)]$. Relations (173)–(175) remain true. We use the formula

$$\int_0^{a+\epsilon_2} Y_\ell^*(x,\mu)[dB(x)]Y_j(x,\lambda)dx = \frac{i[Y_\ell^*(a+\epsilon_2,\mu)JY_j(a+\epsilon_2,\lambda) - Y_\ell^*(0,\mu)JY_j(0,\lambda)]}{\bar{\mu} - \lambda}$$

And the inequality $H(x)dx \leq dB(x)$. From formulas (164), (175) and (182) we deduce that

$$[V^*]^2 \leq T.$$

The Corollary is proved. □

Corollary (6.2.44)[232]: Let the self-adjoint operator S^* be bounded, positive, invertible and have the form (184). If the matrix function $B(x)$ is absolutely continuous and

$$\acute{B}(x) = \beta^*(x)\beta(x), \ \beta(x) = [h_1(x), h_2(x)],$$

Then the equality

$$h_1(x)\overline{h_2(x)} + h_2(x)\overline{h_1(x)} = 1 \quad ,$$

is true almost everywhere.

Proof. Let us consider the expression

$$i_{(x+\epsilon)} = ([S^*]_{(x+\epsilon)}^{-1} P_{(x+\epsilon)}M, 1) + (1, [S^*]_{(x+\epsilon)}^{-1} P_{(x+\epsilon)}M).$$

Setting

$$N_1(x, (x+\epsilon)) = [S^*]_{(x+\epsilon)}^{-1} P_{(x+\epsilon)}M,$$

we rewrite formula (192) in the form $i_{(x+\epsilon)} = \int_0^{(x+\epsilon)}[N_1(x,(x+\epsilon)) + \overline{N_1(x,(x+\epsilon))}]dx$, i.e.,

$$i_{(x+\epsilon)} = \int_0^{(x+\epsilon)} \left[N_1\big(x,(x+\epsilon)\big) + \overline{N_1\big((x+\epsilon)-x,(x+\epsilon)\big)}\right] dx \quad .$$

We use the relation (see [39], Ch. 1)

$$N_1\big(x,(x+\epsilon)\big) + \overline{N_1\Big(\big((x+\epsilon)-x,(x+\epsilon)\big)\Big)} = 1.$$

In view of (194) and (195) we obtain the equality

$$i_{(x+\epsilon)} = (x+\epsilon).$$

Taking into consideration Equalities (100), (158), (184) and (192) we deduce that

$$i_{(x+\epsilon)} = \int_0^{(x+\epsilon)} [h_1(x)\overline{h_2(x)} + h_2(x)\overline{h_1(x)}]dx \,.$$

Relation (191) follows from (196) and (197). The Corollary is proved. □

Corollary (6.2.45)[232]: Let the bounded self-adjoint operator S^* satisfy relation (219). If the corresponding operator S_-^* is bounded, then the representation

$$S^* = [S_-^*]^2$$

is true.

Proof. We consider the operator

$$X = [S_-^*]^2.$$

Using formula (219) and relation $A^*S_-^* = S_-^*A^*$ we deduce the equality

$$A^*X - XA^* = S_-^*(A^* - A^*)S_-^* = A^*S^* - S^*A^*.$$

The equation $A^*X - XA^* = F^*$ has no more than one solution X (see [39], Ch. 1). We can deduce that $A^*X = XA^*$ and $F^* = 0$. Hence we deduce from (224) *that* $S^* = X$.

The lemma is proved . □

Corollary (6.2.46)[232]: If the bounded self-adjoint operator S^* satisfies the relation (219), then this operator can be represented in the form (222), where the operator S_-^* is defined by formula (220).

Proof. To prove that the self-adjoint operator S_-^* is bounded we introduce the operator

$$X_-f = A^*S_-^*f = i\int_0^x f(t)\phi(x-t)dt.$$

We note that

$$X_-^*f = S_-^*A^*f = -i\int_x^{a+\epsilon_2} f(t)\overline{\phi(t-x)}dt$$

where the operator S_-^* has the form

$$S_-^* f = -\frac{d}{dx}\int_x^{a+\epsilon_2} f(t)\overline{\phi(t-x)}dt .$$

According to Lemma (6.2.31) we have

$$S^*[A^*]^2 = X_- X_-^*.$$

It follows from relations (225) and (228) that $S^* = [S_-^*]^2$. Hence the operator S_-^* is bounded. The lemma is proved. □

Corollary (6.2.47)[232]: If the self-adjoint operator S^* belongs to the class R_1^*, then this operator admits the left triangular factorization.

Proof. We suppose that for some $f_0(x) \in L^2(0, a+\epsilon_2)$ the relation

$$S_-^* f_0 = 0 \qquad (\|f_0\| \neq 0)$$

is true . In view of the well-known Titchmarsh theorem (see [41], Ch. 11) and (229) we have

$$\phi(x) = 0, \qquad 0 \le x \le \delta.$$

Using (219) and (230) we deduce that

$$A_\delta^* S_\delta^* - S_\delta^* A_\delta^* = 0,$$

where $A_\delta^* f = i\int_0^x f(t)dt$, $0 \le x \le \delta$, *and* $S_\delta^* = P_\delta S^* P_\delta$. Operator equation (231) has only the trivial solution $S_\delta^* = 0$ (see [39], Ch. 1). The last equality contradicts relation (218). It means that equality (229) is impossible when $\|f_0\| \neq 0$. Hence in view of (222) the self-adjoint operator S_-^* *maps* $L^2(0,b)$ one-to-one onto $L^2(0, a+\epsilon_2)$. This fact according to the classical Banach theorem [25] implies that the self-adjoint operator S_-^* is invertible. The self-adjoint operator $[S^*]_-^{-1}$ is defined by formula (see [39], Ch. 1)

$$[S^*]_-^{-1} f = \frac{d}{dx}\int_0^x f(t)N(x-t)dt,$$

Where $N(x) = [S^*]_-^{-1} 1$. Thus the self-adjoint operators S_-^* *and* $[S^*]_-^{-1}$ are bounded and lower triangular .The assertion of the theorem now follows directly from Definition (6.2.4).□

Sec (6.3): Effective Construction of a Class of Positive Operators in Hilbert Space, which do not Admit Triangular Factorization

To introduce the main notions of the triangular factorization (see [4,6,8,14,15, 20]) consider a Hilbert space $L^2(a,b)$ $(-\infty \le a < b \le \infty)$. The orthogonal projectors P_ξ in $L^2(a,b)$ are defined by the relations

$$(P_\xi f)(x) = f(x) \text{ for } a < x < \xi, \qquad (P_\xi f)(x) = 0 \text{ for } \xi < x < b \quad \left(f \in L^2(a,b)\right).$$

Denote the identity operator by I.

Definition (6.3.1)[1]: A bounded operator S_-on $L^2(a,b)$is called lower triangular if for every ξ the relations

$$S_-Q_\xi = Q_\xi S_- Q_\xi, \tag{257}$$

where $Q_\xi = I - P_\xi$, are true. The operator S_-^* is called upper triangular.

Definition (6.3.2)[1]: A bounded, positive definite and invertible operator S on $L^2(a,b)$ is said to admit a left (right) triangular factorization if it can be represented in the form

$$S = S_- S_-^* \quad (S = S_-^* S_-), \tag{258}$$

where S_- and S_-^{-1} are bounded and lower triangular operators. Further, we often write factorization meaning a left triangular factorization.

In paper [20] (see p. 291) we formulated necessary and sufficient conditions under which the positive definite operator S admits a triangular factorization. The factorizing operator S_-^{-1} was constructed in the explicit form. We proved that a wide class of operators admits a triangular factorization [20].

D. Larson proved [8] the existence of positive definite and invertible but non-factorable operators. In the present article we construct concrete examples of such operators. In particular, the following operator

$$Sf = f(x) - \mu \int_0^\infty \frac{\sin\pi(x - t)}{\pi(x - t)} f(t)dt , \qquad f(x) \in L^2(0,\infty), 0 < \mu < 1 \tag{259}$$

is positive definite and invertible but non-factorable. Using positive definite and invertible but non-factorable operators we have managed to substitute pure existence theorems [8] by concrete examples in the well-known problems posed by J.R. Ringrose [13], R.V. Kadison and I.M. Singer [6]. We note that Kadison-Singer problem was posed independently by I. Gohberg and M.G. Krein [5].

The non-factorable operator S, which is defined by formula (259), is used in a number of theoretical and applied problems (in optics [22], random matrices [24], generalized stationary processes [11, 12], and Bose gas theory[10]). The results obtained in this section are interesting from this point of view too.

In this section we consider operators S of the form

$$Sf = f(x) - \mu \int_0^\infty h(x - t) f(t)dt , \qquad f(x) \in L^2(0,\infty), \tag{260}$$

where $\mu = \bar{\mu}$ and $h(x)$ admits representation

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} \rho(\lambda) d\lambda . \tag{261}$$

We suppose that the function $\rho(\lambda)$ satisfies the following conditions

1. The function $\rho(\lambda)$ is real and bounded

$$|\rho(\lambda)| \le U^2, \quad U > 0 \ (-\infty < \lambda < \infty). \tag{262}$$

2. $\rho(\lambda) = \rho(-\lambda) \in L(-\infty, \infty)$.
Hence, the function $h(x)$ $(-\infty < x < \infty)$ is continuous and real. The corresponding operator

$$Hf = \int_0^\infty h(x - t)f(t)dt \tag{263}$$

is self-adjoint and bounded, where $\|H\| \leq U$. We introduce the operators

$$S_\xi f = f(x) - \mu \int_0^\xi h(x - t)f(t)dt, \qquad f(x) \in L^2(0, \xi), 0 < \xi < \infty. \tag{264}$$

The following statement is true.

Proposition (6.3.3)[1]: If $-1/U < \mu < 1/U$, then the operator S_ξ, which is defined by formula (264), is positive definite, bounded and invertible.

Hence, we have

$$S_\xi^{-1} f = f(x) + \int_0^\xi R_\xi(x, t, \mu)f(t)dt. \tag{265}$$

The function $R_\xi(x, t, \mu)$ is jointly continuous in x, t, ξ, μ. M.G. Krein (see [5],Ch. IV, Section 7) proved that

$$S_b^{-1} = (I + V_+)(I + V_-), \qquad 0 < b < \infty, \tag{266}$$

where the operators V_+ and V_- are defined in $L^2(0, b)$ by the relations

$$(V_+^* f)(x) = (V_- f)(x) = \int_0^x R_x(x, t, \mu)f(t)dt. \tag{267}$$

The Krein's formula (266) is true for the Fredholm class of operators. The operator S_b belongs to this class. The kernel of the operator V_- does not depend of b. Hence, if the operator S admits the factorization, then formula (266) holds for the case $b = \infty$ too, i.e.

$$S^{-1} = (I + V_+)(I + V_-). \tag{268}$$

Remark (6.3.4)[1]: Relation (268) also follows from Theorem 2.1 in the paper [20]. Let us introduce the function

$$q_1(x) = 1 + \int_0^x R_x(x, t, \mu)dt. \tag{269}$$

Using the relation $R_x(x, t, \mu) = R_x(x - t, 0, \mu)$ (see [5], formula (8.12)), we obtain

$$q_1(x) = 1 + \int_0^x R_x(u, 0, \mu)du. \tag{270}$$

According to the well-known Krein's formula ([5], Ch. IV, formulas (8.3) and (8.14)) we have

$$q_1(x) = \exp\left\{\int_0^x R_t(t,0,\mu)dt\right\}. \tag{271}$$

Together with $q_1(x)$ we shall consider the function

$$q_2(x) = M(x) + \int_0^x M(t)R_x(x,t,\mu)dt , \tag{272}$$

where

$$M(x) = \frac{1}{2} - \mu \int_0^x h(s)ds . \tag{273}$$

The functions $q_1(x)$ and $q_2(x)$ generate the 2×2 differential system

$$\frac{dW}{dx} = izJH(x)W, \qquad W(0,z) = I_2. \tag{274}$$

Here $W(x,z)$ and $H(x)$ are 2×2 matrix functions and J is a 2×2 matrix :

$$H(x) = \begin{bmatrix} q_2^2(x) & \frac{1}{2} \\ \frac{1}{2} & q_1^2(x) \end{bmatrix} , \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{275}$$

Note that according to [19] (see formulas (53) and (56) therein) we have:

$$q_1(x)q_2(x) = \frac{1}{2} \quad . \tag{276}$$

It is easy to see that

$$JH(x) = T(x)PT^{-1}(x), \tag{277}$$

where

$$T(x) = \begin{bmatrix} q_1(x) & -q_1(x) \\ q_2(x) & q_2(x) \end{bmatrix} , \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{278}$$

Consider the matrix function

$$V(x,z) = e^{-\frac{ixz}{2}} \; T^{-1}(x)W(x,z)T(0). \tag{279}$$

Due to (274)-(279) we get

$$\frac{dV}{dx} = (iz/2)jV + \Gamma(x)V, \qquad V(0) = I_2, \tag{280}$$

where

$$\Gamma(x) = \begin{bmatrix} 0 & B(x) \\ B(x) & 0 \end{bmatrix} , \; j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{281}$$

$$B(x) = \frac{q_1'(x)}{q_1(x)} = R_x(x,0,\mu). \quad (282)$$

Let us introduce the functions

$$\Phi_n(x,z) = v_{1n}(x,z) + v_{2n}(x,z) \quad (n = 1,2), \quad (283)$$

$$\Psi_n(x,z) = i[v_{1n}(x,z) - v_{2n}(x,z)] \quad (n = 1,2), \quad (284)$$

where $v_{in}(x,z)$ are elements of the matrix function $V(x,z)$. It follows from (280) that

$$\frac{d\Phi_n}{dx} = \left(\frac{z}{2}\right)\Psi_n - B(x)\Phi_n, \quad \Phi_1(0,z) = \Phi_2,(0,z) = 1, \quad (285)$$

$$\frac{d\Psi_n}{dx} = -\left(\frac{z}{2}\right)\Phi_n + B(x)\Psi_n, \Psi_1(0,z) = -\Psi_2(0,z) = i. \quad (286)$$

Consider again the differential system (274) and the solution $W(x,z)$ of this system. The element $w_{1,2}(\xi,z)$ of the matrix function $W(\xi,z)$ can be represented in the form (see [17], p. 54, formula (2.6))

$$w_{1,2}(\xi,z) = iz\left(\left(I - zA\right)^{-1} 1, S_\xi^{-1} 1\right)_\xi \quad , \quad (287)$$

where the operator A has the form

$$Af = i\int_0^x f(t)dt. \quad (288)$$

It is well-known that

$$(I - zA)^{-1} 1 = e^{izx} \quad . \quad (289)$$

We can obtain a representation of $W(\xi,z)$ without using the operator S_ξ^{-1}. Indeed, it follows from (279), (283), and (284) that

$$W(x,z) = (1/2)e^{\frac{ixz}{2}}T(x)\begin{bmatrix}\Phi_1 - i\Psi_1 & \Phi_2 - i\Psi_2 \\ \Phi_1 + i\Psi_1 & \Phi_2 + i\Psi_2\end{bmatrix} T^{-1}(0). \quad (290)$$

According to equality (270) we have $q_1(0) = 1$. Due to (278) we infer

$$T(0) = \begin{bmatrix}1 & -1 \\ 1/2 & 1/2\end{bmatrix}, \qquad T^{-1}(0) = \begin{bmatrix}1/2 & 1 \\ -1/2 & 1\end{bmatrix}. \quad (291)$$

Further we plan to use a Krein's result from [7]. For that purpose we intro- duce the functions

$$P(x,z) = e^{\frac{ixz}{2}}[\Phi(x,z) - i\Psi(x,z)]/2, \quad (292)$$

$$P_*(x,z) = e^{\frac{ixz}{2}}[\Phi(x,z) + i\Psi(x,z)]/2, \tag{293}$$

where

$$\Phi(x,z) = \Phi_1(x,z) + \Phi_2(x,z),\ \Psi(x,z) = \Psi_1(x,z) + \Psi_2(x,z). \tag{294}$$

Using (285), (286) and (292), (293) we see that the pair $P(x,z)$ and $P_*(x,z)$ is a solution of the following Krein system

$$\frac{dP}{dx} = izP - B(x)P_* \ , \qquad \frac{dP_*}{dx} = -B(x)P, \tag{295}$$

where

$$P(0,z) = P_*(0,z) = 1. \tag{296}$$

It follows from (292) and (293) that

$$P(x,z) - P_*(x,z) = -ie^{\frac{ixz}{2}}\Psi(x,z). \tag{297}$$

We assume that the following relation is true:

$$M(x) = (1 - \mu)/2 + q(x),\ \ q(x) \in L^2(0,\infty), \tag{298}$$

where the function $M(x)$ is defined by (273). Condition (298) can be rewritten in an equivalent form:

$$\int_0^\infty h(x)dx = \frac{1}{2} \quad , \int_x^\infty h(x)dx \in L^2(0,\infty). \tag{299}$$

Now, we need the relations (see [16], Ch. 1, formulas (1.37) and (1.44)):

$$S_\xi 1 = M(x) + M(\xi - x),\ \ S_\xi = U_\xi S_\xi U_\xi, \tag{300}$$

where $U_\xi f(x) = \overline{f(\xi - x)},\ \ 0 \le x \le \xi$. It follows from (298) and (300) that

$$S_\xi 1 = 1 - \mu + q(x) + U_\xi q(x). \tag{301}$$

Hence the relation

$$S_\xi^{-1} 1 = \frac{1}{(1-\mu)}[1 - r_\xi(x) - U_\xi r_\xi(x)] \tag{302}$$

is true. Here $r_\xi(x) = S_\xi^{-1} q(x)$. Using formulas (287), (298), and (302), we obtain the following representation of $w_{1,2}(\xi,z)$.

Lemma (6.3.5)[1]: The function $w_{1,2}(\xi,z)$. has the form

$$w_{1,2}(\xi,z) = e^{iz\xi}G(\xi,z) - \overline{G(\xi,\bar{z})}, \tag{303}$$

where

$$G(\xi, z) = \frac{1}{(1 - \mu)} \left[1 - iz \int_0^{\xi} e^{-izx} r_{\xi}(x)dx\right]. \tag{304}$$

Note that the operator S is positive definite, bounded and invertible. According to (266) we have

$$Q(x) = (I + V_{-})q(x) \in L^2(0, \infty). \tag{305}$$

Hence, there exists a sequence x_n such that

$$Q(x_n) \to 0, \qquad x_n \to \infty. \tag{306}$$

Now, we prove the following statement.

Lemma (6.3.6)[1]: Let relation (306) be true. Then we have

$$\lim_{x_n \to \infty} q_1(x_n) = \frac{1}{\sqrt{1 - \mu}} \quad . \tag{307}$$

Proof. In view of (269), (272), and (298) we get

$$q_2(x) = q_1(x)(1 - \mu)/2 + Q(x). \tag{308}$$

Taking into account the relation $q_1(x)q_2(x) = 1/2$ (see [19], formulas (53) and (56)), we obtain the equality

$$1/2 = q_1^2(x)(1 - \mu)/2 + q_1(x)Q(x). \tag{309}$$

Formula (307) follows directly from (306), (309), and inequality $q_1(x) > 0$. □

It follows from (278) and (307) that

$$T(x_n) \to \begin{bmatrix} C & -C \\ 1/2C & 1/2C \end{bmatrix}, \quad x_n \to \infty, \ C = 1/\sqrt{(1 - \mu)}. \tag{310}$$

Hence, in view of (291), (292), (294), and (310) the following assertion is true.

Lemma (6.3.7)[1]: Let x_n tend to ∞ . Then, $w_{1,2}$ has the following asymptotics

$$w_{1,2}(x_n, z) = -iCe^{\frac{ix_n z}{2}} \Psi(x_n, z)\big(1 + o(1)\big). \tag{311}$$

Lemma (6.3.8)[1]: Suppose that the operator S admits a factorization. Then we have

$$\lim_{\xi \to \infty} e^{-iz\xi} w_{1,2}(\xi, z) = G(z), \quad \Im z < 0, \tag{312}$$

$$\lim_{\xi \to \infty} w_{1,2}(\xi, z) = -\overline{G(\bar{z})}, \qquad \Im z > 0. \tag{313}$$

where

$$G(z) = \frac{1}{(1 - \mu)}\left[1 - iz\int_{0}^{\infty} e^{-izx}r(x)dx\right], r(x) = S^{-1}q(x). \qquad (314)$$

Proof . According to (268) we *have* $S_{-}^{-1} = I + V_{-}$, where V_{-} is defined by (267). Hence, the operator function S_{ξ}^{-1} strongly converges to the operator S^{-1} when $\xi \to \infty$. Then the function $r_{\xi}(x) = S_{\xi}^{-1}\, q(x)$ strongly converges to $r(x) = S^{-1}q(x)$, when $\xi \to \infty$. and $r(x) \in L^2(0,\infty)$. Using (303) and (304) we obtain relations (312) and (313). The lemma is proved. □
From Lemma (6.3.8) we derive the following important assertion.

Proposition (6.3.9)[1]: If at least one of the equalities (312) and (313) is not true, then the corresponding operator S does not admit factorization.

Note that a new approach to the notion of the limit of a function was used in Lemma (6.3.6). Namely, we introduce a continuous function $F(x)$, which belongs to $L(0,\infty)$, and consider sequences $x_n \to \infty$, such that

$$F(x_n) \to 0. \qquad (315)$$

Definition (6.3.10)[1]: We say that the function $f(x)$ tends to A almost sure (a. s.) if relation (315) implies

$$f(x_n) \to A, \qquad x_n \to \infty. \qquad (316)$$

Equality (307) can be written in the form

$$\lim_{x\to\infty} q_1(x) = \frac{1}{\sqrt{1-\mu}} \quad , \text{a.s.} \qquad (317)$$

Remark (6.3.11)[1]: From heuristic point of view "almost all" sequences $x_n \to \infty$ satisfy relation (315). This is the reason of using the probabilistic term" almost sure".

Introduce a partition

$$0 = a_0 < a_1 < \ldots < a_n = a, \qquad (318)$$

and consider the function $\rho(\lambda) = \rho(-\lambda)$ such that

$$\rho(\lambda) = \begin{cases} 0\,, & a \le \lambda, \\ b_{k-1}, & a_{k-1} \le \lambda < a_k, \end{cases} \qquad (319)$$

where

$$b_0 = 1; \ -1 \le b_k \le 1 \quad (0 < k \le n-1). \qquad (320)$$

In the case of ρ given by (319) and (320) we can put $U = 1$ in (262). Further we investigate the operators S, which are defined by formulas (260), (261), and (319). The spectral function $\sigma(\lambda)$ of the corresponding system (295) is absolutely continuous and such that (see [7]):

$$\acute{\sigma}(\lambda) = \frac{[1 - \mu\rho(\lambda)]}{2\pi}. \qquad (321)$$

Remark (6.3.12)[1]: The operators S, which are defined by formulas (260), (261), and (319), appear in the theory of generalized stationary processes of white noise type (see [11,12]). If $n = 1$ and $a_1 = \pi$, then the corresponding operator S has the form (259).

It follows from (261) and (319) that

$$h(x) = \frac{1}{\pi}\sum_{k=1}^{n} b_k \frac{\sin a_k x - \sin a_{k-1} x}{x}. \qquad (322)$$

According to (321) we have

$$\int_{-\infty}^{\infty} \frac{\log \acute{\sigma}(u)}{1 + u^2} du < \infty. \qquad (323)$$

It follows from (324) (see [7]) that

$$\int_{0}^{\infty} |P(x, z_0)|^2 dx < \infty, \qquad \Im z_0 > 0. \qquad (324)$$

Hence, there exists a sequence x_n such that

$$|P(x_n, z_0)|^2 \to 0, \qquad x_n \to \infty. \qquad (325)$$

Now, we use the corrected form of Krein's theorem (see [7, 21]):

Proposition (6.3.13)[1]: There exists the limit

$$\Pi(z) = \lim_{x_n \to \infty} P_*(x_n, z), \qquad (326)$$

where the convergence is uniform at any bounded closed set of the upper half-plane $\Im z > 0$.

2) The function $\Pi(z)$ can be represented in the form

$$\Pi(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{1 + tz}{(z - t)(1 + t^2)} (\log \acute{\sigma}(t))dt + i\alpha\right\}, \qquad (327)$$

where $\alpha = \bar{\alpha}$. Here σ is the spectral function of system (295), which corresponds to ρ given by (319) and (320), that is, this σ is defined by (321).

Remark (6.3.14)[1]: The function $|Q(x)|^2 + |P(x, z_0)|^2$ belongs to the space $L(0, \infty)$. Hence, there exists a sequence x_n such that relations (306) and (325) are true simultaneously.

If (322) holds, then the following conditions are fulfilled:

$$0 < \delta \leq \|S\| \leq \Delta < \infty, \qquad \int_{0}^{\infty} |h(x)|^2 dx < \infty. \qquad (328)$$

Therefore, in formula (327) we have (see [19], Proposition 1):

$$\alpha = 0. \qquad (329)$$

One can easily see that

$$\frac{-1}{2i\pi}\int_{-\infty}^{\infty}\frac{1\,+\,tz}{(z-t)(1\,+\,t^2)}\log(2\pi)dt\;=\frac{1}{2}\log(2\pi). \qquad (330)$$

It follows from (327), (329), and (330) that $\Pi(z)$ has the form

$$\Pi(z)\;=\prod_{k=0}^{n-1}\left[\left(\frac{a_{\,k+1}+\,z}{a_{k+1}-\,z}\right)\left(\frac{a_k\;-\,z}{a_k\;+\,z}\right)\right]^{\frac{\log(1-b_k\mu)}{2\mathrm{i}\pi}},\Im z>\;0. \qquad (331)$$

Next, we prove the main result of this section.

Theorem (6.3.15)[1]: The bounded positive definite and invertible operator S, whichis defined by formulas (260) and (322), does not admit a left triangular factorization.

Proof . Taking into account Lemma (6.3.7) and relations (297), (325), and (326) we have

$$\lim_{x_n\to\infty} w_{1,2}(x_n,z)=\;-C\Pi(z),\qquad \Im z\;>\;0,C\;=\;1/\sqrt{(1-\mu)}. \qquad (332)$$

Now, we use the following relations

$$\lim_{y\to+0}\left(\frac{a_{k+1}-\,iy}{a_{k+1}\;+\,iy}\right)\left(\frac{a_k+\,iy}{a_k\;-\,iy}\right)\;=\;1,\qquad k\;>\;0, \qquad (333)$$

$$\lim_{y\to+0}\left(\frac{a_{k+1}-\,iy}{a_{k+1}\;+\,iy}\right)\left(\frac{a_k+\,iy}{a_k\;-\,iy}\right)\;=\;-1,\qquad k\;=\;0. \qquad (334)$$

Formulas (331), (333), and (334) imply that

$$\lim_{y\to+0}\Pi(iy)\;=\;\sqrt{(1-\mu)}. \qquad (335)$$

Suppose that the operator S admits a factorization. It follows from the asymptotics of sinus integral (see [3], Ch. 9, formulas (2) and (10)), that the kernel $h(x)$, defined by formula (322), satisfies conditions (299). Hence, the conditions of Lemma (6.3.8) are fulfilled. Comparing formulas (313) and (332), we see that

$$-\lim_{y\to+0}\overline{G(-\imath y)}\;=\;-1/(1\;-\;\mu)\neq\;-C\lim_{y\to+0}\Pi(iy)\;=\;-1. \qquad (336)$$

Hence, the relation (313) is not true. According to Proposition (6.3.9) the operator S does not admit a factorization. The theorem is proved. □

Let the nest N be the family of subspaces $Q_{\xi}L^2(0,\infty)$. The corresponding nest algebra $Alg(N)$ is the algebra of all linear bounded operators in the space $L^2(0,\infty)$ for which every subspace from N is an invariant subspace. Put $D_N = Alg(N) \cap Alg(N)^*$. The set N has multiplicity one if the diagonal D_N is abelian, that is, D_N is a commutative algebra. We can see that the lower triangular operators S_- form the algebra $Alg(N)$, the corresponding diagonal D_N is abelian, and it consists of the commutative operators

$$T_{\varphi}f = \varphi(x)f , \qquad f \in L^2(0,\infty), \tag{337}$$

where $\varphi(x)$ is bounded. Hence, the introduced nest N has the multiplicity

Ringrose Problem. Let N be a multiplicity one nest and T be a bounded invertible operator. Is TN necessarily multiplicity one nest?

We obtain a concrete counterexample to Ringrose's hypothesis.

Proposition (6.3.16)[1]: Let the positive definite, invertible operator S be defined by the relations (260) and (322). The set $S^{1/2}N$ fails to have multiplicity 1.

Proof . We use the well-known result (see [4], p. 169): The following assertions are equivalent:

1. The positive definite, invertible operator T admits factorization.
2. $T^{1/2}$ preserves the multiplicity of N.

We stress that in our case the set $N = Q_{\xi}L^2(0,\infty)$ is fixed.) The operator S does not admit the factorization. Therefore, the set $S^{1/2}N$ fails to have multiplicity 1. The proposition is proved. □

Next, consider the operator

$$Vf = \int_0^x e^{-(x+y)}f(y)dy , \qquad f(x) \in L^2(0,\infty). \tag{338}$$

An operator is said to be hyperintransitive if its lattice of invariant subspaces contains a multiplicity one nest. Note that the lattice of invariant subspaces of the operator V coincides with N, see [9] and [23] (Ch. 11, Theorem 150). Hence we deduce the answer to Kadison-Singer [6] and to Gohberg-Krein [5] question.

Corollary (6.3.17)[1]: The operator $W = S^{1/2}V\,S^{-1/2}$ is a non-hyperintransitive compact operator.

Indeed, the lattice of the invariant subspaces of the operator W coincides With $S^{1/2}N$.

Corollary (6.3.18)[232]: Let relation (306) be true. Then we have

$$\lim_{x_n\to\infty}\sum_{m\in Z} q_{m+1}(x_n) = \frac{1}{\sqrt{1-\mu_m}} .$$

Proof. In view of (269), (272), and (298) we get

$$\textstyle\sum_{m\in Z} q_{m+2}(x) = \sum_{m\in Z}(q_{m+1}(x)\,(1-\mu_m)/2 + Q_m(x)).$$

Taking into account the relation $q_{m+1}(x)q_{m+2}(x) = 1/2$ (see [19], formulas (53) and (56)), we obtain the equality

$$1/2 = \sum_{m\in Z} q_{m+1}^2(x)\,(1 - \mu_m)/2 + \sum_{m\in Z} q_{m+1}(x)Q_m(x).$$

Formula (307) follows directly from (306), (309), and inequality $q_{m+1}(x) > 0$. □

It follows from (278) and (307) that

$$\sum_{m\in Z} T^m(x_n) \to \sum_{m\in Z} \begin{bmatrix} C_m & -C_m \\ 1/2C_m & 1/2C_m \end{bmatrix}, \quad x_n \to \infty, \quad \sum_{m\in Z} C_m = 1/\sqrt{(1 - \mu_m)}.$$

Hence, in view of (291), (292), (294), and (310) the following assertion is true.

Corollary (6.3.19)[232]: Suppose that the operator S admits a factorization. Then we have

$$\lim_{\xi_j\to\infty} e^{-iz_j\xi_j} w_{1,2}(\xi_j, z_j) = G(z_j), \quad \Im z_j < 0,$$

$$\lim_{\xi_j\to\infty} w_{1,2}(\xi_j, z_j) = -\overline{G(\overline{z_j})}, \qquad \Im z_j > 0.$$

Where

$$G(z) = \frac{1}{(1 - \mu)}\left[1 - iz_j \int_0^\infty e^{-iz_jx} r_j(x)dx\right], r_j(x) = S^{-1}q_j(x).$$

Proof . According to (268) we $have\ S_-^{-1} = I + V_-$, where V_- is defined by (267). Hence, the operator function $S_{\xi_j}^{-1}$ strongly converges to the operator S^{-1} when $\xi_j \to \infty$. Then the function $r_{j_{\xi_j}}(x) = S_{\xi_j}^{-1} q_j(x)$ strongly converges to $r_j(x) = S^{-1}q_j(x)$, when $\xi_j \to \infty$ and $r(x) \in L^2(0,\infty)$.