Chapter 5
Strong Convergence Theorems and Viscosity Approximation with Iterative Methods

In this chapter we consider the problem of finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping and the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping. It is shown that $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly-monotone mapping which solves some variational inequality. The explicit and implicit iterative algorithms are proposed by virtue of the general iterative method with strongly positive operators. Under two sets of quite mild conditions, we show the strong convergence of these explicit and implicit iterative algorithms to the unique common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the general variational inequality problem, respectively.

Sec (5.1): Strong Convergence Theorems for Nonexpansive Mappings and Inverse-Strongly Monotone Mappings

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping A of C into H is called monotone if for all $x, y \in C, \langle x - y, Ax - Ay \rangle \geq 0.$ The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0$$

for all $v \in C$; see [81,82,83,86,91]. The set of solutions of the variational inequality is denoted by $V\,I(C, A)$. A mapping A of C into H is called inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$; see [56,85,57,87]. For such a case, A is called α-inverse-strongly monotone. A mapping S of C into it self is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$; see [90,92,76] for the results of nonexpansive mappings. We denote by $F(S)$ the set of fixed points of S.

In this section , we introduce an iterative scheme for finding a common element of the set of fixed points of a non expansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a real Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. Using this result, we first obtain a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping. Further, we consider the problem of finding a common element of the set of fixed points of a

nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping.
Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\| . \|$, and let C be a closed convex subset of H.We write $x_n \rightharpoonup x$ to indicate that the sequence $\{ x_n \}$ converges weakly to x. $x_n \to x$ implies that $\{ x_n \}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C .We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \qquad (1)$$

For every $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x , P_C x - y\rangle \geq 0$ for all $y \in C$. In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \ \forall \lambda > 0. \qquad (2)$$

It is also known that H satisfies Opial's condition [78], i.e., for any sequence $\{ x_n \}$ with $x_n \rightharpoonup x$ the inequality

$$\lim_{n\to\infty} \inf\|x_n - x\| < \lim_{n\to\infty} \inf\|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

We state some examples for inverse-strongly monotone mappings. If $A = I - T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then A is $1/2$-inverse-strongly monotone and $VI(C, A) = F(T)$. A mapping A of C into H is called strongly monotone if there exists a positive real number η such that $\langle x - y, Ax - Ay \rangle \geq \eta\|x - y\|^2$ for all $x, y \in C$. In such a case, we say that A is η-strongly monotone. If A is η-strongly monotone and k-Lipschitz continuous, i.e., $\|Ax - Ay\| \leq k\|x - y\|$ for all $x, y \in C$, then A is η/k^2-inverse-strongly monotone.
If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $1/\alpha$-Lipschitz continuous.We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \qquad (3) \end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.
A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$, and define

$$T\,v = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in T\,v$ if and only if $v \in V\,I(C,A)$; see [88,89].

In this section, we prove a strong convergence theorem for nonexpansive mappings and inverse-strongly monotone mappings.

Theorem (5.1.1)[67]: Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap V\,I(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ *and* $\{x_n\}$ is given by

$$x_n + 1 = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ *is a* sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0\,, \sum_{n=1}^{\infty} \alpha_n = \infty\,, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \ \ and \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty\,,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)} \cdot x$.

Proof .

Put $y_n = P_C(x_n - \lambda_n A x_n)$ for every $n = 1, 2, \ldots$. Let $u \in F(S) \cap V\,I(C,A)$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n A u)$ from (2), we have

$$\begin{aligned} \|y_n - u\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\| \\ &\le \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \le \|x_n - u\| \end{aligned}$$

for every $n = 1, 2, \ldots$. Then we have

$$\begin{aligned} \|x_2 - u\| &= \|\alpha_1 x + (1 - \alpha_1) S y_1 - u\| \\ &\le \alpha_1 \|x - u\| + (1 - \alpha_1)\|S y_1 - u\| \\ &\le \alpha_1 \|x - u\| + (1 - \alpha_1)\|y_1 - u\| \\ &\le \alpha_1 \|x - u\| + (1 - \alpha_1)\|x - u\| \\ &= \|x - u\| . \end{aligned}$$

If $\|x_k - u\| \le \|x - u\|$ holds for some $k \in \boldsymbol{N}$, we can similarly show $\|x_{k+1} - u\| \le \|x - u\|$.

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{S y_n\}$ and $\{A x_n\}$ are also bounded. Since $I - \lambda_n A$ is nonexpansive,

we also have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|P_C(x_{n+1} - \lambda_{n+1} A x_{n+1}) - P_C(x_n - \lambda_n A x_n)\| \\ &\le \|(x_{n+1} - \lambda_{n+1} A x_{n+1}) - (x_n - \lambda_n A x_n)\| \\ &= \|(x_{n+1} - \lambda_{n+1} A x_{n+1}) - (x_n - \lambda_{n+1} A x_n) + (\lambda_n - \lambda_{n+1}) A x_n\| \\ &\le \|(x_{n+1} - \lambda_{n+1} A x_{n+1}) - (x_n - \lambda_{n+1} A x_n)\| + |\lambda_n - \lambda_{n+1}| \|A x_n\| \\ &\le \|x_{n+1} - x\| = n + |\lambda_n - \lambda_{n+1}| \|A x_n\| \qquad (4) \end{aligned}$$

for every $n = 1, 2, \ldots$. So, we obtain

$$\|x_{n+1} - x_n\| = \|(\alpha_n x + (1 - \alpha_n) S y_n) - (\alpha_{n-1} x + (1 - \alpha_{n-1}) S y_{n-1})\|$$

$$= \|(\alpha_n - \alpha_{n-1})(x - S y_{n-1}) + (1 - \alpha_n)(S y_n - S y_{n-1})\|$$

$\leq |\alpha_n - \alpha_{n-1}|\|x - Sy_{n-1}\| + (1 - \alpha_n)\|Sy_n - Sy_{n-1}\|$
$\leq |\alpha_n - \alpha_{n-1}|\|x - Sy_{n-1}\| + (1 - \alpha_n)\|y_n - y_{n-1}\|$
$\leq |\alpha_n - \alpha_{n-1}|\|x - Sy_{n-1}\| + (1 - \alpha_n)(\|x_n - x_{n-1}\|$
$+ |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\|)$
$\leq (1 - \alpha_n)\|x_n - x_{n-1}\| + M|\lambda_n - \lambda_{n-1}| + L|\alpha_n - \alpha_{n-1}|$

for every $n = 1, 2, \ldots$, where $L = \sup\{\|x - Sy_n\| : n \in \boldsymbol{N}\}$ and $M = \sup\{\|Ax_n\| : n \in \boldsymbol{N}\}$. By mathematical induction, we have

$$\|x_{n+m+1} - x_{n+m}\| \leq \prod_{k=m}^{n+m-1} (1 - \alpha_{k+1}) \|x_{m+1} - x_m\| + M \sum_{k=m}^{n+m-1} |\lambda_{k+1} - \lambda_k| + L \sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|$$

for every $n, m = 1, 2, \ldots$. So, we obtain

$$\lim_{n\to\infty} \sup\|x_{n+1} - x_n\| = \lim_{n\to\infty} \sup\|x_{n+m+1} - x_{n+m}\| \leq M \sum_{k=m}^{\infty} |\lambda_{k+1} - \lambda_k| + L \sum_{k=m}^{\infty} |\alpha_{k+1} - \alpha_k|$$

for every $m = 1, 2, \ldots$ Since $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ we obtain

$$\lim_{n\to\infty} \sup\|x_{n+1} - x_n\| \leq 0$$

and hence

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

From (4) and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, we also obtain $\|y_{n+1} - y_n\| \to 0$. Since

$$\|x_n - Sy_n\| \leq \|x_n - Sy_{n-1}\| + \|Sy_{n-1} - Sy_n\| \leq \alpha_{n-1}\|x - Sy_{n-1}\| + \|y_{n-1} - y_n\| ,$$

we have $\|x_n - Sy_n\| \to 0$. For $u \in F(S) \cap VI(C, A)$, from (3), we obtain

$$\|x_{n+1} - u\|^2 = \|\alpha_n x + (1 - \alpha_n)Sy_n - u\|^2$$
$$\leq \alpha_n\|x - u\|^2 + (1 - \alpha_n)\|Sy_n - u\|^2$$
$$\leq \alpha_n\|x - u\|^2 + (1 - \alpha_n)\|y_n - u\|^2$$
$$\leq \alpha_n\|x - u\|^2 + (1 - \alpha_n)\{\|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2\}$$
$$\leq \alpha_n\|x - u\|^2 + \|x_n - u\|^2 + (1 - \alpha_n)a(b - 2\alpha)\|Ax_n - Au\|^2.$$

Therefore, we have

$$-(1 - \alpha_n)a(b - 2\alpha)\|Ax_n - Au\|^2$$
$$\leq \alpha_n\|x - u\|^2 + \|x_n - u\|^2 - \|x_{n+1} - u\|^2$$
$$= \alpha_n\|x - u\|^2 + (\|x_n - u\| + \|x_{n+1} - u\|) \times (\|x_n - u\| - \|x_{n+1} - u\|)$$
$$\leq \alpha_n\|x - u\|^2 + (\|x_n - u\| - \|x_{n+1} - u\|) \times \|x_n - x_{n+1}\|.$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, we obtain $\|Ax_n - Au\| \to 0$. From (1), we have

$$\|y_n - u\|^2 = \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2$$
$$\leq \langle (x_n - \lambda_n Ax_n) - (u - \lambda_n Au), y_n - u\rangle$$
$$= \frac{1}{2}\{ \left\|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\right\|^2 + \|y_n - u\|^2$$
$$- \left\|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au) - (y_n - u)\right\|^2\}$$

$$\leq \frac{1}{2}\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|(x_n - y_n) - \lambda_n(Ax_n - Au)\|^2\}$$
$$= \frac{1}{2}\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|x_n - y_n\|^2$$
$$+ 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2 \}.$$

So, we obtain

$$\|y_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\langle x_n - y_n, Ax_n - Au\rangle$$
$$- \lambda_n^2\|Ax_n - Au\|^2$$

and hence

$$\|x_{n+1} - u\|^2 = \left\| \alpha_n x + (1 - \alpha_n)Sy_n - u\right\|^2$$
$$\leq \alpha_n\|x - u\|^2 + (1 - \alpha_n)\|Sy_n - u\|^2$$
$$\leq \alpha_n\|x - u\|^2 + (1 - \alpha_n)\|y_n - u\|^2$$
$$\leq \alpha_n\|x - u\|^2 + \|x_n - u\|^2 - (1 - \alpha_n)\|x_n - y_n\|^2$$
$$+ 2(1 - \alpha_n)\lambda_n\langle x_n - y_n, Ax_n - Au\rangle - (1 - \alpha_n)\lambda_n^2\|Ax_n - Au\|^2.$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\|Ax_n - Au\| \to 0$, we obtain $\|x_n - y_n\| \to 0$, Since $\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - y_n\|$, we obtain $\|Sy_n - y_n\| \to 0$.

Next we show that

$$\limsup_{n\to\infty} \langle x - z_0, Sy_n - z_0 \rangle \leq 0,$$

where $z_0 = P_{F(S)\cap VI(C,A)}x$ .To show it, choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle x - z_0, Sy_n - z_0 \rangle = \lim_{i\to\infty}\langle x - z_0, Sy_{n_i} - z_0 \rangle.$$

As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$converges weakly to z. We may assume without loss of generality that $y_{n_i} \rightharpoonup z$. Since $\|Sy_n - y_n\| \to 0$, we obtain $Sy_{n_i} \rightharpoonup z$. Then we can obtain $z \in F(S) \cap VI(C, A)$. In fact, let us first show that $z \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone . Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$,
we have

$$\langle v - y_n, w - Av\rangle \geq 0.$$

On the other hand, from $y_n = P_C(x_n - \lambda_n Ax_n)$, we have

$$\langle v - y_n, y_n - (x_n - \lambda_n A x_n)\rangle \geq 0,$$

and hence

$$\left\langle v - y_n, \frac{y_n - x_n}{\lambda_n} + A x_n \right\rangle \geq 0 .$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + A x_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, Av - A x_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - y_{n_i}, Av - A y_{n_i}\rangle + \langle v - y_{n_i}, A y_{n_i} - A x_{n_i}\rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - y_{n_i}, A y_{n_i} - A x_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle .
\end{aligned}$$

Hence we obtain $\langle v - z, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Let us show that $z \in F(S)$. Assume $z \notin F(S)$. From Opial's condition, we have

$$\begin{aligned}
\lim_{i\to\infty} \inf \|y_{n_i} - z\| &< \lim_{i\to\infty} \inf \|y_{n_i} - Sz\| \\
&= \lim_{i\to\infty} \inf \|y_{n_i} - S y_{n_i} + S y_{n_i} - Sz\| \\
&= \lim_{i\to\infty} \inf \|S y_{n_i} - Sz\| \leq \lim_{i\to\infty} \inf \|y_{n_i} - z\|.
\end{aligned}$$

This is a contradiction. Thus, we obtain $z \in F(S)$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \sup \langle x - z_0, S y_n - z_0\rangle &= \lim_{n\to\infty} \langle x - z_0, S y_{n_i} - z_0\rangle \\
&= \langle x - z_0, z - z_0\rangle \leq 0.
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $m \in \boldsymbol{N}$ such that

$$\langle x - z_0, S y_n - z_0\rangle \leq \varepsilon, \qquad \alpha_n \|x - z_0\|^2 \leq \varepsilon$$

for all $n \geq m$. For all $n \geq m$.we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n x + (1 - \alpha_n) S y_n - z_0\|^2 \\
&= \alpha_n^2 \|x - z_0\|^2 + 2\alpha_n (1 - \alpha_n)\langle x - z_0, S y_n - z_0\rangle \\
&\quad + (1 - \alpha_n)^2 \|S y_n - z_0\|^2 \\
&\leq \alpha_n \varepsilon + 2\alpha_n (1 - \alpha_n)\varepsilon + (1 - \alpha_n)\|S y_n - z_0\|^2 \\
&\leq 3\alpha_n \varepsilon + (1 - \alpha_n)\|S y_n - z_0\|^2 \\
&\leq 3\alpha_n \varepsilon + (1 - \alpha_n)\|x_n - z_0\|^2 \\
&= 3\varepsilon(1 - (1 - \alpha_n)) + (1 - \alpha_n)\|x_n - z_0\|^2.
\end{aligned}$$

By mathematical induction, we obtain

$$\|x_{n+1} - z_0\|^2 \leq 3\varepsilon\left(1 - \prod_{k=m}^{n}(1 - \alpha_k)\right) + \prod_{k=m}^{n}(1 - \alpha_k)\|x_m - z_0\|^2 .$$

Therefore, we have

$$\lim_{n\to\infty} \sup \|x_{n+1} - z_0\|^2 \leq 3\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we have $\limsup_{n\to\infty} \|x_{n+1} - z_0\|^2 \leq 0$ and hence $x_n \to z_0$. □

Remark(5.1.2)[67]: We obtain Wittmann's theorem [94] if $A = 0$ in Theorem (5.1.1); see also [84]. Takahashi and Toyoda [93] considered Mann's type iteration:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \end{cases}$$

and obtained that the sequence $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$.

As a direct consequence of Theorem (5.1.1), we obtain the following:

Corollary (5.1.3)[67]: Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C, A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) P_C (x_n - \lambda_n A x_n)$$

For every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ *is a* sequence in $[0, 2\alpha]$. *If* $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \ and \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty .$$

Then $\{x_n\}$ converges strongly to $P_{VI(C,A)} x$.

In this section, we prove two theorems in a real Hilbert space by using Theorem(5.1.1).A mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$. If $k = 0$, then T *i*s nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then A is $(1 - k)/2$-inverse-strongly monotone; see [56]. Actually, we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2 .$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-k}{2} \|Ax - Ay\|^2 .$$

Using Theorem (5.1.1), we first prove a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem (5.1.4)[67]: Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself and let T *be a* k-strictly pseudocontractive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S((1 - \lambda_n) x_n + \lambda_n T x_n)$$

For every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 1 - k]$. If $\{\alpha_n\}$ *and* $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 1 - k$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \ and \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}x$.

Proof. Put $A = I - T$. Then A is $(1 - k)/2$-inverse-strongly monotone. We have $F(T) = VI(C, A)$ and $P_C(x_n - \lambda_n Ax_n) = (1 - \lambda_n)x_n + \lambda_n T x_n$. So, by Theorem (5.1.1), we obtain the desired result. □

Using Theorem (5.1.1), we also have the following:

Theorem (5.1.5)[67]: Let H be a real Hilbert space. Let A be an α-inverse-strongly monotone mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Suppose $x_1 = x \in H$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)S(x_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ *and* $\{\lambda_n\}$ *is a* sequence in $[0, 2\alpha]$. *If* $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \ and \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$converges strongly to $P_{F(S)\cap A^{-1}0}x$.

Proof. We have $A^{-1}0 = VI(H, A)$. So, putting $P_H = I$, by Theorem (5.1.1), we obtain the desired result. □

Sec(5.2): Viscosity Approximation Methods for Nonexpansive Mappings and Monotone Mappings

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. Recall that a self-mapping $f : C \to C$ is a contraction on C if there is a constant $k \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \quad x, y \in C.$$

Π_C denotes the set of all contractions on C. Note that f has a unique fixed point in C.

A mapping A of C into H is called monotone if $\langle Au - Av, u - v\rangle \geq 0$, for all $u, v \in C$. The variational inequality problem is to find $u \in C$ such that $\langle Au, v - u\rangle \geq 0$ for all $v \in C$ (Refs. [56,75]). The set of solutions of the variational inequality is denoted by $VI(C, A)$. A mapping A of C to H is called inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$. For such a case, A is α-inverse-strongly monotone.

A mapping S of C into itself is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$ (Ref. [76]). We denoted by $F(S)$ the set of fixed points of S.

The viscosity approximation method of selecting a particular fixed point of given nonexpansive mapping was proposed by Moudafi [62] who proved the following strong convergence of both the implicit and explicit methods in Hilbert space.

Theorem (5.2.1)[68]: In a Hilbert space define $\{x_n\}$ by implicit way

$$x_n = \frac{1}{1+\varepsilon_n} T x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n),$$

where ε_n is a sequence in $(0,1)$ tending to zero. Then $\{x_n\}$ converges strongly to the unique solution $\tilde{x} \in C$ of the variational inequality

$$\langle (I - f)\tilde{x}, \tilde{x} - x \rangle \leq 0 .$$

In other words, $\tilde{x}$ is the unique fixed point of $P_{Fix(T)} f$.

Theorem(5.2.2)[68]: In a Hilbert space define $\{x_n\}$ *by* ($x_0 \in C$ is arbitrary)

$$x_{n+1} = \frac{1}{1+\varepsilon_n} T x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n),$$

Suppose that $\{\varepsilon_n\}$ satisfies the conditions

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \sum_{n=1}^{\infty} \varepsilon_n = \infty ; \quad \lim_{n\to\infty} \left| \frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n-1}} \right| = 0.$$

Then $\{x_n\}$ converges strongly to the unique solution $\tilde{x} \in C$ of the variational inequality

$$\langle (I - f)\tilde{x}, \tilde{x} - x \rangle \leq 0 .$$

In other words, $\tilde{x}$ is the unique fixed point of $P_{Fix(T)} f$.

Very recently Xu [63] studied the viscosity approximation methods proposed by Moudafi [62] for a nonexpansive mapping in a Hilbert space. He proved the following theorems.

Theorem (5.2.3)[68]: (see Xu[63,theorem 3.1].) Let H be a Hilbert space, C *a* closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. *Let* $\{x_t\}$ be given by

$$x_t = tf(x_t) + (1 - t)T x_t, \quad t \in (0,1).$$

Then:

(i) $s - \lim_{t\to 0} x_t =: \tilde{x}$ exists;

(ii) $\tilde{x} = P_S f(\tilde{x})$, or equivalently, $\tilde{x}$ is the unique solution in $F(T)$ to the variational inequality

$$\langle (I - f)\tilde{x}, x - \tilde{x} \rangle \geq 0 , \; x \in S,$$

where $S = F(T)$ and P_S is the metric projection from H to S.

Theorem (5.2.4)[68]: (see Xu [63,theorem 3.2].) Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ a contraction.

Let $\{x_n\}$ be given by

$$x_0 \in C,\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T x_n,\quad n \geq 0.$$

Then under the following hypotheses

$(H1)$ $\alpha_n \to 0$;

$(H2)$ $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(H3)$ $either \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,

$x_n \to \tilde{x}$, where $\tilde{x}$ is the unique solution of the variational inequality

$$\langle (I - f)\tilde{x}, x - \tilde{x}\rangle \geq 0,\ x \in S.$$

In this section, we introduce an iterative scheme by viscosity approximation method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequalities for an inverse-strongly monotone mapping in a real Hilbert space. Then we show that the sequence converges strongly to a common element of two sets which solves some variational inequality. Using this results, we first obtain a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping. Further, we consider the problem finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping.

Let H be a real Hilbert space with inner product $\langle . , . \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|,$$

for all $y \in C$. P_C is called the metric projection of H to C. It is well known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 , \qquad (5)$$

for every $x, y \in H$, and P_C is characterized by the following properties:

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \qquad (6)$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 , \qquad (7)$$

for all $x \in H, y \in C$. In the context of the variational inequality problem, This implies

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u),\ \forall \lambda > 0. \qquad (8)$$

It is well known that H satisfies the Opial condition (Ref. [78]), i.e., for any sequence $\{x_n\}$with $x_n \rightharpoonup x$ the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. If A is an α-inverse-strongly monotone mapping of C to H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 = \|(x-y)-\lambda(Ax-Ay)\|^2$$
$$= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2$$
$$\le \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2.$$

So, if $\lambda \le 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x-y, f-g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A is an inverse-strongly monotone mapping of C to H and let $N_C v$ be normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H: \langle v-u, w\rangle \ge 0,\ \forall u \in C\}$, and define

$$T v = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases}$$

then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (Ref. [78]).

Lemma (5.2.5)[68]: (see Goebel and Kirk [79]) Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.

Lemma (5.2.6)[68]: (see Xu [80].) Let $\{s_n\}$ be a sequence of nonnegative real numbers such that:

$$s_{n+1} \le (1-\lambda_n)s_n + \beta_n, \quad n \ge 0,$$

where $\{\lambda_n\}, \{\beta_n\}$ satisfy the condition

(i) $\{\lambda_n\} \subset (0,1)$ *and* $\sum_{n=1}^{\infty}\lambda_n = \infty$,

(ii) $\lim_{n\to\infty}\sup\frac{\beta_n}{\lambda_n} \le 0$ or $\sum_{n=1}^{\infty}|\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Proposition (5.2.7)[68]: Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient k $(0 < k < 1)$, A an α-inverse-strongly monotone mapping of C *to* H and let S be a nonexpansive mapping *of* C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $\{x_n\}$ be sequences generated by

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n)$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ *and* $\{\alpha_n\}$ is a sequence in $(0,1)$. *If* $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty, \sum|\alpha_{n+1}-\alpha_n| < \infty, \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty,$$

then $\{x_n\}$ converges strongly to $q \in F(S) \cap VI(C,A)$, which is the unique solution in the $F(S) \cap VI(C,A)$ to the following variational inequality

$$\langle (I - f)q, q - p\rangle \le 0, \quad p \in F(S) \cap VI(C,A).$$

Proof. Put $y_n = P_C(x_n - \lambda_n Ax_n)$ for every $n = 0,1,2,\ldots$. Let $u \in F(S) \cap VI(C,A)$. We have

$$\begin{aligned}\|y_n - u\| &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\| \\ &\le \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\| \\ &\le \|x_n - u\|\end{aligned}$$

for every $n = 1,2,3,\ldots$. Then we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n f(x_n) + (1-\alpha_n)Sy_n - u\| \\ &\le \alpha_n\|f(x_n) - u\| + (1-\alpha_n)\|Sy_n - u\| \\ &\le \alpha_n\|f(x_n) - f(u)\| + \alpha_n\|f(u) - u\| + (1-\alpha_n)\|y_n - u\| \\ &\le \alpha_n k\|x_n - u\| + (1-\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\ &= (1 - (1-k)\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\ &\le \max\left\{\|x_n - u\|, \frac{1}{1-k}\|f(u) - u\|\right\}.\end{aligned}$$

By induction,

$$\|x_n - u\| \le \max\left\{\|x_0 - u\|, \frac{1}{1-k}\|f(u) - u\|\right\}, \quad n \ge 0.$$

Therefore, $\{x_n\}$ is bounded, $\{y_n\}$, $\{Sy_n\}$, $\{Ax_n\}$, $\{f(x_n)\}$ are also bounded. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n Au)$, we also have

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n A x_n)\| \\ &\le \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n)\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\| \\ &\le \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\|\end{aligned}$$

for every $n = 1,2,3,\ldots$. So we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)Sy_n \\ &\quad - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})Sy_{n-1}\| \\ &= \|(\alpha_n - \alpha_{n-1})(f(x_{n-1}) - Sy_{n-1}) + (1-\alpha_n)(Sy_n - Sy_{n-1}) \\ &\quad + \alpha_n(f(x_n) - f(x_{n-1}))\| \\ &\le |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - Sy_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| \\ &\quad + \alpha_n k\|x_n - x_{n-1}\| \\ &\le (1-\alpha_n)(\|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|Ax_{n-1}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - Sy_{n-1}\| + \alpha_n k\|x_n - x_{n-1}\| \\ &\le (1 - (1-k)\alpha_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}| + M|\alpha_n - \alpha_{n-1}|\end{aligned}$$

for every $n = 0,1,2,\ldots$, where $L = \sup\{\|f(x_n) - Sy_{n-1}\| : n \in N\}$, $M = \sup\{\|Ax_n\| : n \in N\}$, since $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty$, $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ in view of Lemma (5.2.6), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Then we also obtain $\|y_{n+1} - y_n\| \to 0$

$$\begin{aligned}\|x_n - Sy_n\| &\le \|x_n - Sy_{n-1}\| + \|Sy_{n-1} - Sy_n\| \\ &\le \alpha_{n-1}\|f(x_{n-1}) - Sy_{n-1}\| + \|y_{n-1} - y_n\|,\end{aligned}$$

we have $\|x_n - Sy_n\| \to 0$. For $u \in F(S) \cap VI(C,A)$,

$$\left\|x_{n+1}-u\right\|^2=\left\|\alpha_n f(x_n)+(1-\alpha_n)Sy_n-u\right\|^2$$
$$\le \alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)\|y_n-u\|^2$$
$$\le \alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)[\|x_n-u\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Au\|^2]$$

$$\le \alpha_n\|f(x_n)-u\|^2+\|x_n-u\|^2+(1-\alpha_n)a(b-2\alpha)\|Ax_n-Au\|^2.$$

So, we obtain

$$-(1-\alpha_n)a(b-2\alpha)\|Ax_n-Au\|^2$$
$$\le \alpha_n\|f(x_n)-u\|^2+(\|x_n-u\|+\|x_{n+1}-u\|)(\|x_n-u\|-\|x_{n+1}-u\|)$$
$$\le \alpha_n\|f(x_n)-u\|^2+(\|x_n-u\|+\|x_{n+1}-u\|)\|x_n-x_{n+1}\|.$$

Since $\alpha_n \to 0$ and $\|x_n-x_{n+1}\|\to 0$, then $\|Ax_n-Au\|\to 0, n\to\infty$. Further, from (5), we obtain

$$\|y_n-u\|^2=\|P_C(x_n-\lambda_n Ax_n)-P_C(u-\lambda_n Au)\|^2$$
$$\le \langle x_n-\lambda_n Ax_n-(u-\lambda_n Au), y_n-u\rangle$$
$$=\frac{1}{2}\left\{\left\|(x_n-\lambda_n Ax_n)-(u-\lambda_n Au)\right\|^2\right.$$
$$\left.+\|y_n-u\|^2-\|(x_n-\lambda_n Ax_n)-(u-\lambda_n Au)-(y_n-u)\|^2\right\}$$
$$\le\frac{1}{2}\{\|x_n-u\|^2+\|y_n-u\|^2-\|x_n-y_n\|^2$$
$$\left.+2\lambda_n\langle x_n-y_n, Ax_n-Au\rangle-\lambda_n{}^2\|Ax_n-Au\|^2\right\}.$$

So, we obtain

$$\left\|y_n-u\right\|^2\le\|x_n-u\|^2-\left\|x_n-y_n\right\|^2$$
$$+2\lambda_n\langle x_n-y_n, Ax_n-Au\rangle-\lambda_n{}^2\|Ax_n-Au\|^2.$$

And hence

$$\|x_{n+1}-u\|^2\le\alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)\left\|Sy_n-u\right\|^2$$
$$\le\alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)\|y_n-u\|^2$$
$$\le\alpha_n\|f(x_n)-u\|^2+\|x_n-u\|^2-(1-\alpha_n)\left\|x_n-y_n\right\|^2$$
$$+2(1-\alpha_n)\lambda_n\langle x_n-y_n, Ax_n-Au\rangle-(1-\alpha_n)\lambda_n{}^2\|Ax_n-Au\|^2.$$

Since $\alpha_n\to 0, \|x_{n+1}-x_n\|\to 0$ and $\|Ax_n-Au\|\to 0$, we obtain $\left\|x_n-y_n\right\|\to 0$. Choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\lim_{n\to\infty}\sup\langle f(q)-q, Sy_n-q\rangle=\lim_{i\to\infty}\langle f(q)-q, Sy_{n_i}-q\rangle$$

As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_ij}\}$of $\{y_{n_i}\}$converges weakly to z. We may assume without loss of generality that $y_{n_i}\rightharpoonup z$. Since $\|Sy_n-y_n\|\to 0$, we obtain $Sy_{n_i}\rightharpoonup z$. Then we can obtain $z\in F(S)\cap VI(C,A)$. In fact, let us first show that $z\in VI(C,A)$.

Let

$$Tv=\begin{cases}Av+N_C v, & v\in C,\\ \emptyset, & v\notin C,\end{cases}$$

Then T is maximal monotone. Let $(v,w)\in G(T)$. Since $w-Av\in N_C v$ and $y_n\in C$ we have

$$\langle v-y_n, w-Av\rangle\ge 0.$$

On the other hand, from $y_n = P_C(x_n - \lambda_n A x_n)$, we have $\langle v - y_n, y_n - (x_n - \lambda_n A x_n)\rangle \geq 0$ and hence

$$\langle v - y_n, \frac{y_n - x_n}{\lambda_n} + A x_n\rangle \geq 0 \quad .$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Av\rangle \\
&\geq \langle v - y_{n_i}, Av\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + A x_{n_i}\right\rangle \\
&= \left\langle v - y_{n_i}, Av - A x_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \langle v - y_{n_i}, Av - A y_{n_i}\rangle + \langle v - y_{n_i}, A y_{n_i} - A x_{n_i}\rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle v - y_{n_i}, A y_{n_i} - A x_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \quad .
\end{aligned}$$

Hence we have $\langle v - z, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\
&\leq \|x_n - Sy_n\| + \|x_n - y_n\|,
\end{aligned}$$

we have $\|x_n - Sx_n\| \to 0$. In view of Lemma (5.2.5), we obtain $z \in F(S)$

$$\begin{aligned}
\lim_{n\to\infty}\sup \langle f(q) - q, Sy_n - q\rangle &= \lim_{i\to\infty}\langle f(q) - q, Sy_{n_i} - q\rangle \\
&= \langle f(q) - q, z - q\rangle \leq 0,
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n f(x_n) + (1 - \alpha_n) Sy_n - q\|^2 \\
&= \alpha_n^2 \|f(x_n) - q\|^2 + 2\alpha_n(1 - \alpha_n)\langle f(x_n) - q, Sy_n - q\rangle \\
&\quad + (1 - \alpha_n)^2 \|Sy_n - q\|^2 \\
&\leq (1 - 2\alpha_n + \alpha_n^2)\|x_n - q\|^2 + \alpha_n^2\|f(x_n) - q\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x_n) - f(q), Sy_n - q\rangle \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(q) - q, Sy_n - q\rangle \\
&\leq \left[1 - 2\alpha_n + \alpha_n^2 + 2k\alpha_n(1 - \alpha_n)\right]\|x_n - q\|^2 \\
&\quad + \alpha_n^2\|f(x_n) - q\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(q) - q, Sy_n - q\rangle \\
&= (1 - \bar{\alpha}_n)\|x_n - q\|^2 + \bar{\alpha}_n \bar{\beta}_n ,
\end{aligned}$$

where

$$\bar{\alpha}_n = \alpha_n[2 - \alpha_n - 2k(1 - \alpha_n)],$$

$$\bar{\beta}_n = \frac{\alpha_n\|f(x_n) - q\|^2 + 2(1 - \alpha_n)\langle f(q) - q, Sy_n - q\rangle}{2 - \alpha_n - 2k(1 - \alpha_n)} \quad .$$

It is easily seen that $\bar{\alpha}_n \to 0$, $\sum_{n=1}^{\infty} \bar{\alpha}_n = \infty$, and $\limsup_{n\to\infty}\bar{\beta}_n \leq 0$, by Lemma (5.2.6) we obtain $x_n \to q$. This completes the proof. □

S is a nonexpansive mapping, A is an α-inverse strongly monotone, and $f \in \Pi_C$. Thus, by Banach contraction mapping principle, there exists a unique fixed point

$$z_n^f = \alpha_n f(z_n^f) + (1-\alpha_n)SP_C(z_n^f - \lambda_n Az_n^f), \qquad \alpha_n \in (0,1).$$

For simplicity we will write z_n for z_n^f provided no confusion occurs. Next we prove the convergence of $\{z_n\}$, while they claim the existence of the $q \in F(S) \cap VI(C,A)$ which solves the variational inequality

$$\langle (I-f)q, q-p\rangle \leq 0, \quad f \in \Pi_C,\ p \in F(S) \cap VI(C,A).$$

Theorem (5.2.8)[68]: Let C be a closed convex subset of a real Hilbert space H. Let $f: C \to C$ be a contraction with coefficient k $(0 < k < 1)$, A an α-inverse-strongly monotone mapping of C to H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $\{z_n\}$, be sequences generated by

$$z_n = \alpha_n f(z_n) + (1-\alpha_n)SP_C(z_n - \lambda_n Az_n), \quad \alpha_n \in (0,1),$$

where $\{\lambda_n\} \subset [a,b]$ and $\{\alpha_n\}$ is a sequence in $[0,1)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a,b *with* $0 < a < b < 2\alpha$, when $\lim_{n\to\infty} \alpha_n = 0$, z_n converges strongly to q, and such that the variational inequality

$$\langle (I-f)q, q-p\rangle \leq 0, \quad f \in \Pi_C,\ p \in F(S) \cap VI(C,A).$$

Proof. Put $y_n = P_C(z_n - \lambda_n Az_n)$ for every $n = 0,1,2,\dots$. Let $u \in F(S) \cap VI(C,A)$. We have

$$\begin{aligned}\|y_n - u\| &= \|P_C(z_n - \lambda_n Az_n) - P_C(u - \lambda_n Au)\|\\ &\leq \|(z_n - \lambda_n Az_n) - (u - \lambda_n Au)\|\\ &\leq \|z_n - u\|,\end{aligned}$$

for every $n = 1,2,3,\dots$. Then we have

$$\begin{aligned}\|z_n - u\| &= \|\alpha_n f(z_n) + (1-\alpha_n)Sy_n - u\|\\ &\leq \alpha_n\|f(z_n) - u\| + (1-\alpha_n)\|Sy_n - u\|\\ &\leq \alpha_n\|f(z_n) - f(u)\| + \alpha_n\|f(u) - u\| + (1-\alpha_n)\|y_n - u\|\\ &\leq \alpha_n k\|z_n - u\| + (1-\alpha_n)\|z_n - u\| + \alpha_n\|f(u) - u\|.\end{aligned}$$

Hence,

$$\|z_n - u\| \leq \frac{1}{1-k}\|f(u) - u\|,$$

and $\{z_n\}$ is bounded, $\{y_n\}, \{Sy_n\}, \{Az_n\}$ and $\{f(z_n)\}$ are also bounded.

$$\begin{aligned}&\|z_n - u\|^2 = \|\alpha_n f(z_n) + (1-\alpha_n)Sy_n - u\|^2\\ &\leq \alpha_n\|f(z_n) - u\|^2 + (1-\alpha_n)\|y_n - u\|^2\\ &\leq \alpha_n\|f(z_n) - u\|^2 + (1-\alpha_n)\left[\|z_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Az_n - Au\|^2\right]\\ &\leq \alpha_n\|f(z_n) - u\|^2 + (1-\alpha_n)\|z_n - u\|^2\\ &+(1-\alpha_n)a(b-2\alpha)\|Az_n - Au\|^2.\end{aligned}$$

Therefore, we have

$$-(1-\alpha_n)a(b-2\alpha)\|Az_n - Au\|^2 \leq \alpha_n(\|f(z_n) - u\|^2 + \|z_n - u\|^2).$$

Since $\alpha_n \to 0$ $(n \to \infty)$, and $\{f(z_n)\}, \{z_n\}$ are bounded, we obtain

$$\|Az_n - Au\| \to 0 \quad (n \to \infty).$$

From (5) we have

$$\begin{aligned}\|y_n - u\|^2 &= \|P_C(z_n - \lambda_n Az_n) - P_C(u - \lambda_n Au)\|^2 \\ &\leq \langle z_n - \lambda_n Az_n - (u - \lambda_n Au), y_n - u\rangle \\ &= \frac{1}{2}\Big\{\big\|(z_n - \lambda_n Az_n) - (u - \lambda_n Au)\big\|^2 + \|y_n - u\|^2 \\ &- \|(z_n - \lambda_n Az_n) - (u - \lambda_n Au) - (y_n - u)\|^2\Big\} \\ &\leq \frac{1}{2}\{\|z_n - u\|^2 + \|y_n - u\|^2 - \|z_n - y_n\|^2 \\ &+ 2\lambda_n\langle z_n - y_n, Az_n - Au\rangle - {\lambda_n}^2\|Az_n - Au\|^2\}.\end{aligned}$$

So, we obtain

$$\begin{aligned}\|y_n - u\|^2 &\leq \|z_n - u\|^2 - \|z_n - y_n\|^2 \\ &+ 2\lambda_n\langle z_n - y_n, Az_n - Au\rangle - {\lambda_n}^2\|Az_n - Au\|^2 .\end{aligned}$$

So we have

$$\begin{aligned}\|z_n - u\|^2 &\leq \alpha_n\|f(z_n) - u\|^2 + (1 - \alpha_n)\|Sy_n - u\|^2 \\ &\leq \alpha_n\|f(z_n) - u\|^2 + (1 - \alpha_n)\|y_n - u\|^2 \\ &\leq \alpha_n\|f(z_n) - u\|^2 + (1 - \alpha_n)\|z_n - u\|^2 - (1 - \alpha_n)\|z_n - y_n\|^2 \\ &+ 2(1 - \alpha_n)\lambda_n\langle z_n - y_n, Az_n - Au\rangle - (1 - \alpha_n){\lambda_n}^2\|Az_n - Au\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}(1 - \alpha_n)\|z_n - y_n\|^2 &\leq \alpha_n\|f(z_n) - u\|^2 - \alpha_n\|z_n - u\|^2 \\ &+ 2(1 - \alpha_n)\lambda_n\langle z_n - y_n, Az_n - Au\rangle - {\lambda_n}^2\|Az_n - Au\|^2.\end{aligned}$$

Since $\alpha_n \to 0, \|Az_n - Au\| \to 0$, we obtain $\|z_n - y_n\| \to 0$ $(n \to \infty)$. By the proof of Proposition (5.2.7), we have $y_{n_i} \rightharpoonup q$ and $q \in F(S) \cap VI(C, A)$, so $z_{n_i} \rightharpoonup q$

$$\begin{aligned}\|z_{n_i} - q\|^2 &= \|\alpha_{n_i} f(z_{n_i}) + (1 - \alpha_{n_i})Sy_{n_i} - q\|^2 \\ &= \langle \alpha_{n_i}(f(z_{n_i}) - q) + (1 - \alpha_{n_i})(Sy_{n_i} - q), z_{n_i} - q\rangle \\ &= \alpha_{n_i}\langle f(z_{n_i}) - q, z_{n_i} - q\rangle + (1 - \alpha_{n_i})\langle Sy_{n_i} - q, z_{n_i} - q\rangle \\ &\leq (1 - \alpha_{n_i})\|z_{n_i} - q\|^2 + \alpha_{n_i}\langle f(z_{n_i}) - q, z_{n_i} - q\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\|z_{n_i} - q\|^2 &\leq \langle f(z_{n_i}) - q, z_{n_i} - q\rangle \\ &= \langle f(z_{n_i}) - f(q), z_{n_i} - q\rangle + \langle f(q) - q, z_{n_i} - q\rangle \\ &\leq k\|z_{n_i} - q\|^2 + \langle f(q) - q, z_n - q\rangle .\end{aligned}$$

This implies that

$$\|z_{n_i} - q\|^2 \leq \frac{1}{1-k}\langle z_{n_i} - q, f(q) - q\rangle .$$

But $z_{n_i} \rightharpoonup q$, it follows that $z_{n_i} \to q$. Now we show that q solves the variational inequality

$$\langle (I - f)q, q - p\rangle \leq 0, \quad f \in \Pi_C, p \in F(S) \cap VI(C, A).$$

Because

$$z_n - f(z_n) = -\frac{1-\alpha_n}{\alpha_n}(z_n - Sy_n),$$

for any $p \in F(S) \cap VI(C,A)$ and notice $p = P_C(p - \lambda_n Ap)$, we infer that

$$\langle z_n - f(z_n), z_n - p\rangle = -\frac{1-\alpha_n}{\alpha_n}\langle z_n - SP_C(z_n - \lambda_n Az_n), z_n - q\rangle$$

$$= -\frac{1-\alpha_n}{\alpha_n}\langle z_n - SP_C(z_n - \lambda_n Az_n) - (p - SP_C(p - \lambda_n Ap)), z_n - p\rangle$$

$$\leq 0,$$

since $I - SP_C(I - \lambda_n A)$ is strong monotone. Let $i \to \infty$, we have

$$\langle q - f(q), q - p\rangle \leq 0. \quad (9)$$

Assume that there exists another subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \to q^*$, so $q^* \in F(S) \cap VI(C,A)$, and from $\langle z_n - f(z_n), z_n - p\rangle \leq 0$, let $j \to \infty$ We have

$$\langle q^* - f(q^*), q^* - p\rangle \leq 0, \ p \in F(S) \cap VI(C,A). \quad (10)$$

Setting $p = q^*$ in (9), we have

$$\langle q - f(q), q - q^*\rangle \leq 0, \quad (11)$$

and setting $p = q$ in (10), we obtain

$$\langle q^* - f(q^*), q^* - q\rangle \leq 0. \quad (12)$$

Inequality (11) and (12) yield

$$\|q - q^*\|^2 \leq \langle f(q) - f(q^*), q - q^*\rangle \leq k\|q - q^*\|^2,$$

which implies that $q = q^*$, since $k \in (0,1)$. Thus, $z_n \to q$ as $n \to \infty$ and $q \in F(S) \cap VI(C,A)$ is unique. And q is the unique solution of variational inequality

$$\langle q - f(q), q - p\rangle \leq 0, \ p \in F(S) \cap VI(C,A).$$

This completes the proof. □

In this section we prove two theorems in a Hilbert space by using Proposition (5.2.7) and Theorem (5.2.8).

A mapping $T : C \to C$ is called strictly pseudocontractive if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for every $x, y \in C$. If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then A is $\frac{(1-k)}{2}$-inverse-strongly monotone. Actually,

we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2$$

$$= \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-k}{2}\|Ax - Ay\|^2.$$

Using Proposition (5.2.7) and Theorem (5.2.8), we first prove a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem (5.2.10)[68]: Let C be a closed convex subset of a real Hilbert space H. Let f be a contractive mapping of C into itself with coefficient $k \in (0,1)$, S be a nonexpansive mapping of C into itself and let T be a strictly pseudocontractive mapping of C into itself with α, such that $F(S) \cap F(T) \neq \emptyset$ Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)S((1-\lambda_n)x_n + \lambda_n T x_n)$$

for every $n = 1, 2, \ldots,$ *where* $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0, 1-\alpha)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 1 - \alpha$,

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty, \sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, such that

$$\langle f(q) - q, q - p\rangle \leq 0, \quad p \in F(S) \cap F(T).$$

Proof. Put $A = I - T$. Then A is $\frac{1-\alpha}{2}$ -inverse-strongly monotone. We have $F(T) = VI(C,A)$ and $P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n$. So by Proposition (5.2.7) and Theorem (5.2.8), we obtain the desired result. □

Theorem (5.2.11)[68]: Let H be a real Hilbert space H. Let f be a contractive mapping of H into itself with coefficient $k \in (0,1)$, S be a nonexpansive mapping of H into itself and let A be *a* α-inverse strongly monotone mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)S(x_n - \lambda_n A x_n),$$

for every $n = 1, 2, \ldots,$ where $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ *is a* sequence in $[0, 2\alpha)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty, \sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $q \in F(S) \cap A^{-1}0$, such that

$$\langle f(q) - q, q - p\rangle, \quad p \in F(S) \cap A^{-1}0.$$

Proof. We have $A^{-1}0 = VI(C,A)$. So putting $P_H = I$, by Proposition (5.2.7) and Theorem (5.2.8), we obtain the desired result.

Corollary(5.2.12)[232]: Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $(1-\epsilon)$ $(0 < \epsilon < 1)$, A an $\frac{\lambda+\epsilon}{2}$-inverse-strongly monotone sequence of mapping of C to H and let S be a

nonexpansive sequence of mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $\{(u_m)_n\}$ be sequences generated by $(u_m)_0 \in C$,
$(u_m)_{n+1} = \left(\frac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n) SP_C((u_m)_n - \lambda_n A(u_m)_n)$,
for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ and $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$ is a sequence in $(0,1)$. If $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < (\lambda + \epsilon)$,

$$\lim_{n\to\infty} \left(\frac{\lambda+\epsilon}{2}\right)_n = 0, \sum_{n=1}^{\infty} \left(\frac{\lambda+\epsilon}{2}\right)_n = \infty, \sum \left| \left(\frac{\lambda+\epsilon}{2}\right)_{n+1} - \left(\frac{\lambda+\epsilon}{2}\right)_n \right| < \infty, \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

then $\{(u_m)_n\}$ converges strongly to $q \in F(S) \cap VI(C, A)$, which is the unique solution in the $F(S) \cap VI(C, A)$ to the following variational inequality

$$\langle (I - f)q, \epsilon \rangle \leq 0, \quad (q - \epsilon) \in F(S) \cap VI(C, A).$$

Proof. Put $(u_{m+1})_n = P_C((u_m)_n - \lambda_n A(u_m)_n)$ for every $n = 0, 1, 2, \ldots$. Let $u_m \in F(S) \cap VI(C, A)$. We have

$$\begin{aligned} \|(u_{m+1})_n - u_m\| &= \|P_C((u_m)_n - \lambda_n A(u_m)_n) - P_C(u_m - \lambda_n A u_m)\| \\ &\leq \|((u_m)_n - \lambda_n A(u_m)_n) - (u_m - \lambda_n A u_m)\| \\ &\leq \|(u_m)_n - u_m\| \end{aligned}$$

for every $n = 1, 2, 3, \ldots$. Then we have

$$\begin{aligned} \|(u_m)_{n+1} - u_m\| &= \left\|\left(\frac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n - u_m\right\| \\ &\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\| + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) \|S(u_{m+1})_n - u_m\| \\ &\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - f(u_m)\| \\ &\quad + \left(\frac{\lambda+\epsilon}{2}\right)_n \|f(u_m) - u_m\| + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n) \|(u_{m+1})_n - u_m\| \\ &\leq \left(\frac{\lambda+\epsilon}{2}\right)_n (1 - \epsilon) \|(u_m)_n - u_m\| + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) \|(u_m)_n - u_m\| \\ &\quad + \left(\frac{\lambda+\epsilon}{2}\right)_n \|f(u_m) - u_m\| \\ &= \left(1 - \epsilon \left(\frac{\lambda+\epsilon}{2}\right)_n\right) \|(u_m)_n - u_m\| + \left(\frac{\lambda+\epsilon}{2}\right)_n \|f(u_m) - u_m\| \\ &\leq \max\left\{\|(u_m)_n - u_m\|, \frac{1}{\epsilon} \|f(u_m) - u_m\|\right\}. \end{aligned}$$

By induction,

$$\|(u_m)_n - u_m\| \leq \max\left\{\|(u_m)_0 - u_m\|, \frac{1}{\epsilon}\|f(u_m) - u_m\|\right\}, \ n \geq 0.$$

Therefore, $\{(u_m)_n\}$ is bounded, $\{(u_{m+1})_n\}$, $\{S(u_{m+1})_n\}, \{A(u_m)_n\}, \{f((u_m)_n)\}$ are also bounded. Since $I - \lambda_n A$ is nonexpansive of sequence and $u_m = P_C(u_m - \lambda_n A u_m)$, we also have

$$\|(u_{m+1})_{n+1} - (u_{m+1})_n\|$$
$$\leq \|((u_m)_{n+1} - \lambda_{n+1}A(u_m)_{n+1}) - ((u_m)_n - \lambda_n A(u_m)_n)\|$$
$$\leq \left\|\left((u_m)_{n+1} - \lambda_{n+1}A(u_m)_{n+1}\right) - \left((u_m)_n - \lambda_{n+1}A(u_m)_n\right)\right\|$$
$$+ |\lambda_n - \lambda_{n+1}|\|A(u_m)_n\|$$
$$\leq \|(u_m)_{n+1} - (u_m)_n\| + |\lambda_n - \lambda_{n+1}|\|A(u_m)_n\|$$

for every $n = 1, 2, 3, \ldots$. So we obtain

$$\|(u_m)_{n+1} - (u_m)_n\| = \left\|\left(\frac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1} f((u_m)_{n-1}) - (1 - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}) S(u_{m+1})_{n-1}\right\|$$

$$= \left\|\left(\left(\frac{\lambda+\epsilon}{2}\right)_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right)(f((u_m)_{n-1}) - S(u_{m+1})_{n-1}) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)(S(u_{m+1})_n - S(u_{m+1})_{n-1}) + \left(\frac{\lambda+\epsilon}{2}\right)_n \left(f((u_m)_n) - f((u_m)_{n-1})\right)\right\|$$

$$\leq \left|\left(\frac{\lambda+\epsilon}{2}\right)_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right| \|f((u_m)_{n-1}) - S(u_{m+1})_{n-1}\|$$
$$+ (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - (u_{m+1})_{n-1}\|$$
$$+ \left(\frac{\lambda+\epsilon}{2}\right)_n (1-\epsilon)\|(u_m)_n - (u_m)_{n-1}\|$$

$$\leq \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\left(\|(u_m)_n - (u_m)_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|A(u_m)_{n-1}\|\right)$$
$$+ \left|\left(\frac{\lambda+\epsilon}{2}\right)_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right| \|f((u_m)_{n-1}) - S(u_{m+1})_{n-1}\|$$
$$+ \left(\frac{\lambda+\epsilon}{2}\right)_n (1-\epsilon)\|(u_m)_n - (u_m)_{n-1}\|$$
$$\leq \left(1 - \epsilon\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_m)_n - (u_m)_{n-1}\|$$
$$+ L|\lambda_n - \lambda_{n-1}| + M\left|\left(\frac{\lambda+\epsilon}{2}\right)_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right|$$

For every $n = 0, 1, 2, \ldots$, where $L = \sup\{\|f((u_m)_n) - S(u_{m+1})_{n-1}\| : n \in N\}$, $M = \sup\{\|A(u_m)_n\| : n \in N\}$, since $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty$, $\sum_{n=1}^{\infty}\left|\left(\frac{\lambda+\epsilon}{2}\right)_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right| < \infty$ in view of Lemma 2, we have $\lim_{n\to\infty}\|(u_m)_{n+1} - (u_m)_n\| = 0$. then we also obtain $\|(u_{m+1})_{n+1} - (u_{m+1})_n\| \to 0$

$$\|(u_m)_n - S(u_{m+1})_n\| \leq \|(u_m)_n - S(u_{m+1})_{n-1}\|$$
$$+ \|S(u_{m+1})_{n-1} - S(u_{m+1})_n\|$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_{n-1} \|f((u_m)_{n-1}) - S(u_{m+1})_{n-1}\| + \|(u_{m+1})_{n-1} - (u_{m+1})_n\|,$$

we have $\|(u_m)_n - S(u_{m+1})_n\| \to 0.$ For $u_m \in F(S) \cap VI(C, A)$,

$$\left\|(u_m)_{n+1} - u_m\right\|^2$$
$$= \left\|\left(\frac{\lambda+\epsilon}{2}\right)_n f\left((u_m)_n\right) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n - u_m\right\|^2$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|^2$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2$$
$$+\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)[\|(u_m)_n - u_m\|^2 + \lambda_n(\lambda_n - (\lambda+\epsilon))\|A(u_m)_n - Au_m\|^2]$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + \left\|(u_m)_n - u_m\right\|^2$$
$$+\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) a\big(b - (\lambda+\epsilon)\big)\|A(u_m)_n - Au_m\|^2.$$

So, we obtain

$$-(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)a(b - (\lambda+\epsilon))\|A(u_m)_n - Au_m\|^2$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2$$
$$+(\left\|(u_m)_n - u_m\right\| + \|(u_m)_{n+1} - u_m\|)(\|(u_m)_n - u_m\| - \|(u_m)_{n+1} - u_m\|)$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2$$
$$+(\left\|(u_m)_n - u_m\right\| + \|(u_m)_{n+1} - u_m\|)\|(u_m)_n - (u_m)_{n+1}\|.$$

Since $\left(\frac{\lambda+\epsilon}{2}\right)_n \to 0$ and $\|(u_m)_n - (u_m)_{n+1}\| \to 0$, then $\|A(u_m)_n - Au_m\| \to 0, n \to \infty$. Further, from (1), we obtain

$$\|(u_{m+1})_n - u_m\|^2 = \|P_C((u_m)_n - \lambda_n A(u_m)_n) - P_C(u_m - \lambda_n Au_m)\|^2$$
$$\leq \langle (u_m)_n - \lambda_n A(u_m)_n - (u_m - \lambda_n Au_m), (u_{m+1})_n - u_m\rangle$$
$$= \frac{1}{2}\left\{\left\|\left((u_m)_n - \lambda_n A(u_m)_n\right) - (u_m - \lambda_n Au_m)\right\|^2\right.$$
$$+\|(u_{m+1})_n - u_m\|^2$$
$$- \|((u_m)_n - \lambda_n A(u_m)_n) - (u_m - \lambda_n Au_m) - ((u_{m+1})_n - u_m)\|^2\}$$
$$\leq \frac{1}{2}\{\|(u_m)_n - u_m\|^2 + \|(u_{m+1})_n - u_m\|^2 - \|(u_m)_n - (u_{m+1})_n\|^2$$
$$\left.+2\lambda_n\langle (u_m)_n - (u_{m+1})_n, A(u_m)_n - Au_m\rangle - {\lambda_n}^2\|A(u_m)_n - Au_m\|^2\right\}.$$

So, we obtain

$$\left\|(u_{m+1})_n - u_m\right\|^2 \leq \|(u_m)_n - u_m\|^2 - \left\|(u_m)_n - (u_{m+1})_n\right\|^2$$
$$+2\lambda_n\langle (u_m)_n - (u_{m+1})_n, A(u_m)_n - Au_m\rangle - {\lambda_n}^2\|A(u_m)_n - Au_m\|^2.$$

And hence

$$\|(u_m)_{n+1} - u_m\|^2 \leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2$$
$$+\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\left\|S(u_{m+1})_n - u_m\right\|^2$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|^2$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + \|(u_m)_n - u_m\|^2$$
$$-(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\left\|(u_m)_n - (u_{m+1})_n\right\|^2$$

$+2\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n\langle(u_m)_n-(u_{m+1})_n, A(u_m)_n-Au_m\rangle$
$-\left(1\ -\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n{}^2\|A(u_m)_n\ -\ Au_m\|^2.$

Since $\left(\frac{\lambda+\epsilon}{2}\right)_n \to 0, \|(u_m)_{n+1}-(u_m)_n\| \to 0$ and $\|A(u_m)_n\ -Au_m\| \to 0$, we obtain $\|(u_m)_n-(u_{m+1})_n\| \to 0$. Choose a subsequence $\{(u_{m+1})_{n_i}\}$ of $\{(u_{m+1})_n\}$ such that

$$\lim_{n\to\infty}\sup\langle f\ (q)-q, S(u_{m+1})_n\ -q\rangle = \lim_{i\to\infty}\langle f\ (q)-\ q, S(u_{m+1})_{n_i}-\ q\rangle$$

As $\{(u_{m+1})_{n_i}\}$ is bounded, we have that a subsequence $\{(u_{m+1})_{n_ij}\}$of $\{(u_{m+1})_{n_i}\}$ converges weakly to (u_{m+3}). We may assume without loss of generality that $(u_{m+1})_{n_i}\ \rightharpoonup (u_{m+3})$.

Since $\|S(u_{m+1})_n\ -\ (u_{m+1})_n\| \to 0$, we obtain $S(u_{m+1})_{n_i}\ \rightharpoonup (u_{m+3})$. Then we can obtain $u_{m+3}\ \in\ F(S)\ \cap\ VI(C,A)$. In fact, let us first show that $u_{m+3}\ \in\ VI(C,A)$. Let

$$T\,u_{m+1}=\begin{cases}Au_{m+1}\ +\ N_Cu_{m+1}, & u_{m+1}\ \in\ C\ ,\\ \quad\emptyset, & u_{m+1}\notin\ C\ ,\end{cases}$$

Then T is maximal monotone sequence. Let $(u_{m+1},u_{m+2})\ \in\ G(T\ )$. Since $u_{m+2}\ -\ Au_{m+1}\ \in\ N_Cu_{m+1}$ and $(u_{m+1})_n\in\ C$ we have

$$\langle u_{m+1}\ -(u_{m+1})_n\ ,u_{m+2}\ -Au_{m+1}\rangle\ \geq 0.$$

On the other hand, from $(u_{m+1})_n\ =\ P_C((u_m)_n-\ \lambda_nA(u_m)_n)$, we have $\langle u_{m+1}\ -\ (u_{m+1})_n,(u_{m+1})_n\ -\ ((u_m)_n\ -\ \lambda_nA(u_m)_n)\rangle \geq 0$ and hence

$$\langle u_{m+1}\ -(u_{m+1})_n,\frac{(u_{m+1})\ n\ -(u_m)_n}{\lambda_n}+A(u_m)_n\rangle \geq 0\ \ .$$

Therefore, we have

$$\begin{aligned}\langle u_{m+1}\ -(u_{m+1})_{n_i},u_{m+2}\rangle &\geq \langle u_{m+1}\ -(u_{m+1})_{n_i},Au_{m+1}\rangle\\ &\geq \langle u_{m+1}-(u_{m+1})_{n_i},Au_{m+1}\rangle\\ &-\left\langle u_{m+1}\ -(u_{m+1})_{n_i},\frac{(u_{m+1})_{n_i}-(u_m)_{n_i}}{\lambda_{n_i}}+A(u_m)_{n_i}\right\rangle\\ &=\left\langle u_{m+1}\ -\ (u_{m+1})_{n_i},Au_{m+1}\ -A(u_m)_{n_i}-\frac{(u_{m+1})_{n_i}-(u_m)_{n_i}}{\lambda_{n_i}}\right\rangle\\ &=\ \langle u_{m+1}-(u_{m+1})_{n_i},Au_{m+1}-\ A(u_{m+1})_{n_i}\rangle\\ &+\langle u_{m+1}-\ (u_{m+1})_{n_i},A(u_{m+1})_{n_i}-\ A(u_m)_{n_i}\rangle\\ &-\left\langle u_{m+1}-\ (u_{m+1})_{n_i},\frac{(u_{m+1})\ n_i\ -(u_m)\ n_i}{\lambda_{n_i}}\right\rangle\\ &\geq \langle u_{m+1}-\ (u_{m+1})_{n_i}\ ,A(u_{m+1})_{n_i}-\ A(u_m)_{n_i}\rangle\\ &-\left\langle u_{m+1}\ -\ (u_{m+1})\ n_i\ ,\frac{(u_{m+1})_{n_i}-(u_m)_{n_i}}{\lambda_{n_i}}\right\rangle\ .\end{aligned}$$

Hence we have $\langle u_{m+1}-u_{m+3},u_{m+2}\rangle \geq\ 0$ as $i \to \infty$. since T is maximal monotone sequence, we have $u_{m+3}\ \in\ T^{-1}0$ and hence $u_{m+3}\in\ VI(C,A)$

$$\|(u_m)_n - S(u_m)_n\| \le \|(u_m)_n - S(u_{m+1})_n\| + \|S(u_{m+1})_n - S(u_m)_n\|$$
$$\le \|(u_m)_n - S(u_{m+1})_n\| + \|(u_m)_n - (u_{m+1})_n\|,$$

we have $\|(u_m)_n - S(u_m)_n\| \to 0$. In view of Lemma 1, we obtain $u_{m+3} \in F(S)$

$$\limsup_{n\to\infty} \langle f(q) - q, S(u_{m+1})_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, S(u_{m+1})_{n_i} - q\rangle$$
$$= \langle f(q) - q, u_{m+3} - q\rangle \le 0,$$

$$\|(u_m)_{n+1} - q\|^2 = \left\|\left(\frac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n - q\right\|^2$$
$$= \left(\frac{\lambda+\epsilon}{2}\right)_n^2 \|f((u_m)_n) - q\|^2$$
$$+2\left(\frac{\lambda+\epsilon}{2}\right)_n \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\langle f((u_m)_n) - q, S(u_{m+1})_n - q\rangle$$
$$+\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)^2\|S(u_{m+1})_n - q\|^2$$
$$\le \left(1 - 2\left(\frac{\lambda+\epsilon}{2}\right)_n + \left(\frac{\lambda+\epsilon}{2}\right)_n^2\right)\|(u_m)_n - q\|^2$$
$$+\left(\frac{\lambda+\epsilon}{2}\right)_n^2 \|f((u_m)_n) - q\|^2$$
$$+2\left(\frac{\lambda+\epsilon}{2}\right)_n \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\langle f((u_m)_n) - f(q), S(u_{m+1})_n - q\rangle$$
$$+2\left(\frac{\lambda+\epsilon}{2}\right)_n \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\langle f(q) - q, S(u_{m+1})_n - q\rangle$$
$$\le \left[1 - 2\left(\frac{\lambda+\epsilon}{2}\right)_n + \left(\frac{\lambda+\epsilon}{2}\right)_n^2 + 2(1-\epsilon)\left(\frac{\lambda+\epsilon}{2}\right)_n\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\right]\|(u_m)_n - q\|^2$$
$$+\left(\frac{\lambda+\epsilon}{2}\right)_n^2 \|f((u_m)_n) - q\|^2$$
$$+2\left(\frac{\lambda+\epsilon}{2}\right)_n \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\langle f(q) - q, S(u_{m+1})_n - q\rangle$$
$$= \left(1 - \overline{\left(\frac{\lambda+\epsilon}{2}\right)_n}\right)\|(u_m)_n - q\|^2 + \overline{\left(\frac{\lambda+\epsilon}{2}\right)_n}\bar{\beta}_n,$$

where

$$\overline{\left(\frac{\lambda+\epsilon}{2}\right)_n} = \left(\frac{\lambda+\epsilon}{2}\right)_n\left[2 - \left(\frac{\lambda+\epsilon}{2}\right)_n - 2(1-\epsilon)\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\right],$$

$$\bar{\beta}_n = \frac{\left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - q\|^2 + 2\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\langle f(q) - q, S(u_{m+1})_n - q\rangle}{2 - \left(\frac{\lambda+\epsilon}{2}\right)_n - 2(1-\epsilon)\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)}.$$

It is easily seen that $\overline{\left(\frac{\lambda+\epsilon}{2}\right)_n} \to 0$, $\sum_{n=1}^{\infty}\overline{\left(\frac{\lambda+\epsilon}{2}\right)_n} = \infty$, and $\limsup_{n\to\infty}\bar{\beta}_n \le 0$, by Lemma 2 we obtain $(u_m)_n \to q$. This completes the proof. □

Corollary(5.2.13)[232]: Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $(1-\epsilon)(0 < \epsilon < 1)$, A an $\frac{\lambda+\epsilon}{2}$-inverse-strongly monotone sequence of mapping of C to H and let S be a nonexpansive sequence of mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $\{(u_{m+3})_n\}$, be sequences generated by

$$(u_{m+3})_n = \left(\frac{\lambda+\epsilon}{2}\right)_n f((u_{m+3})_n) + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right) SP_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n), \quad \left(\frac{\lambda+\epsilon}{2}\right)_n \in (0,1),$$

where $\{\lambda_n\} \subset [a,b]$ and $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$ is a sequence in $[0,1)$. If $\{\left(\frac{\lambda+\epsilon}{2}\right)_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < (\lambda+\epsilon)$, when $\lim_{n\to\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = 0$, $(u_{m+3})_n$ converges strongly to q, and such that the variational inequality

$$\langle (I-f)q, \epsilon\rangle \le 0, \quad f \in \Pi_C, \quad (q-\epsilon) \in F(S) \cap VI(C,A).$$

Proof. Put $(u_{m+1})_n = P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n)$ for every $n = 0,1,2,\dots$. Let $u_m \in F(S) \cap VI(C,A)$. We have

$$\begin{aligned}\|(u_{m+1})_n - u_m\| &= \|P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - P_C(u_m - \lambda_n A u_m)\| \\ &\le \|((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - (u_m - \lambda_n A u_m)\| \\ &\le \|(u_{m+3})_n - u_m\|\end{aligned}$$

for every $n = 1,2,3,\dots$. Then we have

$$\begin{aligned}\|(u_{m+3})_n - u_m\| &= \left\|\left(\frac{\lambda+\epsilon}{2}\right)_n f((u_{m+3})_n) + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n - u_m\right\| \\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\| + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|S(u_{m+1})_n - u_m\| \\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - f(u_m)\| \\ &\quad + \left(\frac{\lambda+\epsilon}{2}\right)_n \|f(u_m) - u_m\| + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+1})_n - u_m\| \\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n (1-\epsilon)\|(u_{m+3})_n - u_m\| + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+3})_n - u_m\| \\ &\quad + \left(\frac{\lambda+\epsilon}{2}\right)_n \|f(u_m) - u_m\|.\end{aligned}$$

Hence,

$$\|(u_{m+3})_n - u_m\| \le \frac{1}{\epsilon}\|f(u_m) - u_m\|$$

and $\{(u_{m+3})_n\}$ is bounded, $\{(u_{m+1})_n\}, \{S(u_{m+1})_n\}, \{A(u_{m+3})_n\}$ and $\{f((u_{m+3})_n)\}$ are also bounded.

$$\begin{aligned}&\|(u_{m+3})_n - u_m\|^2 \\ &= \left\|\left(\frac{\lambda+\epsilon}{2}\right)_n f((u_{m+3})_n) + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n - u_m\right\|^2 \\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2 + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+1})_n - u_m\|^2 \\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2\end{aligned}$$

$$+\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\left[\left\|(u_{m+3})_n-u_m\right\|^2+\lambda_n\left(\lambda_n-(\lambda+\epsilon)\right)\left\|A(u_{m+3})_n-Au_m\right\|^2\right]$$
$$\leq\left(\frac{\lambda+\epsilon}{2}\right)_n\left\|f\left((u_{m+3})_n\right)-u_m\right\|^2+\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\left\|(u_{m+3})_n-u_m\right\|^2$$
$$+\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)a(b-(\lambda+\epsilon))\left\|A(u_{m+3})_n-Au_m\right\|^2.$$
Therefore, we have
$$-\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)a\left(b-(\lambda+\epsilon)\right)\left\|A(u_{m+3})_n-Au_m\right\|^2$$
$$\leq\left(\frac{\lambda+\epsilon}{2}\right)_n\left(\left\|f\left((u_{m+3})_n\right)-u_m\right\|^2+\|(u_{m+3})_n-u_m\|^2\right).$$
Since $\left(\frac{\lambda+\epsilon}{2}\right)_n\to 0\ (n\to\infty)$, and $\{f\left((u_{m+3})_n\right)\},\{(u_{m+3})_n\}$ are bounded, we obtain
$$\left\|A(u_{m+3})_n-Au_m\right\|\to 0\qquad(n\to\infty).$$
From (1) we have
$$\|(u_{m+1})_n-u_m\|^2=\|P_C((u_{m+3})_n-\lambda_nA(u_{m+3})_n)-P_C(u_m-\lambda_nAu_m)\|^2$$
$$\leq\langle(u_{m+3})_n-\lambda_nA(u_{m+3})_n-(u_m-\lambda_nAu_m),(u_{m+1})_n-u_m\rangle$$
$$=$$
$$\frac{1}{2}\left\{\left\|\left((u_{m+3})_n-\lambda_nA(u_{m+3})_n\right)-(u_m-\lambda_nAu_m)\right\|^2+\|(u_{m+1})_n-u_m\|^2-\|((u_{m+3})_n-\lambda_nA(u_{m+3})_n)-(u_m-\lambda_nAu_m)-((u_{m+1})_n-u_m)\|^2\right\}$$
$$\leq\frac{1}{2}\{\|(u_{m+3})_n-u_m\|^2+\|(u_{m+1})_n-u_m\|^2-\|(u_{m+3})_n-(u_{m+1})_n\|^2$$
$$+2\lambda_n\langle(u_{m+3})_n-(u_{m+1})_n,A(u_{m+3})_n-Au_m\rangle-{\lambda_n}^2\|A(u_{m+3})_n-Au_m\|^2\}.$$

So, we obtain
$$\|(u_{m+1})_n-u_m\|^2\leq\|(u_{m+3})_n-u_m\|^2-\|(u_{m+3})_n-(u_{m+1})_n\|^2$$
$$+2\lambda_n\langle(u_{m+3})_n-(u_{m+1})_n,A(u_{m+3})_n-Au_m\rangle$$
$$-{\lambda_n}^2\|A(u_{m+3})_n-Au_m\|^2.$$

So we have

$$\left\|(u_{m+3})_n-u_m\right\|^2\leq\left(\frac{\lambda+\epsilon}{2}\right)_n\left\|f\left((u_{m+3})_n\right)-u_m\right\|^2$$
$$+(1-\left(\frac{\lambda+\epsilon}{2}\right)_n)\|S(u_{m+1})_n-u_m\|^2$$
$$\leq\left(\frac{\lambda+\epsilon}{2}\right)_n\|f\left((u_{m+3})_n\right)-u_m\|^2+(1-\left(\frac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n-u_m\|^2$$
$$\leq\left(\frac{\lambda+\epsilon}{2}\right)_n\left\|f\left((u_{m+3})_n\right)-u_m\right\|^2+\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\left\|(u_{m+3})_n-u_m\right\|^2$$
$$-\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+3})_n-(u_{m+1})_n\|^2$$
$$+2\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n\langle(u_{m+3})_n-(u_{m+1})_n,A(u_{m+3})_n-Au_m\rangle$$
$$-\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right){\lambda_n}^2\|A(u_{m+3})_n-Au_m\|^2.$$

Hence,

$$\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\left\|(u_{m+3})_n - (u_{m+1})_n\right\|^2$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f\ ((u_{m+3})_n)\ - u_m\|^2 - \left(\frac{\lambda+\epsilon}{2}\right)_n \|(u_{m+3})_n\ - u_m\|^2$$
$$+2\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n\langle(u_{m+3})_n - (u_{m+1})_n\ , A(u_{m+3})_n - Au_m\rangle$$
$$-\lambda_n{}^2\|A(u_{m+3})_n\ -\ Au_m\|^2.$$

Since $\left(\frac{\lambda+\epsilon}{2}\right)_n \to 0, \|A(u_{m+3})_n\ -\ Au_m\| \to 0,$ we obtain $\|(u_{m+3})_n - (u_{m+1})_n\| \to 0\ (n \to \infty)$. By the proof of Proposition 3.1we have $(u_{m+1})_{n_i} \rightharpoonup q$ and $q\ \in\ F(S)\ \cap\ VI(C, A)$, so $(u_{m+3})_{n_i}\ \rightharpoonup q$

$$\left\|(u_{m+3})_{n_i}\ - q\right\|^2\ = \left\|\left(\frac{\lambda+\epsilon}{2}\right)_{n_i} f\ ((u_{m+3})_{n_i})\ + (1\ - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i})S(u_{m+1})_{n_i}\ -\ q\right\|^2$$
$$= \langle\left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\left(f\ ((u_{m+3})_{n_i}) - q\right) + \left(1\ - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\right)\left(S(u_{m+1})_{n_i} - q\right), (u_{m+3})_{n_i} - q\rangle$$
$$= \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\langle f\ \left((u_{m+3})_{n_i}\right) - q, (u_{m+3})_{n_i} - q\rangle$$
$$+\left(1\ - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\right)\langle S(u_{m+1})_{n_i} - q, (u_{m+3})_{n_i}\ - q\rangle$$
$$\leq \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\right)\left\|(u_{m+3})_{n_i} - q\right\|^2$$
$$+\left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\langle f\ ((u_{m+3})_{n_i}) - q, (u_{m+3})_{n_i}\ - q\rangle.$$

Hence

$$\left\|(u_{m+3})_{n_i} - q\right\|^2 \leq \langle f\ \left((u_{m+3})_{n_i}\right) - q, (u_{m+3})_{n_i} - q\rangle$$
$$= \langle f\ \left((u_{m+3})_{n_i}\right) - f(q), (u_{m+3})_{n_i} - q\rangle$$
$$+\langle f\ (q) - q, (u_{m+3})_{n_i} - q\rangle \leq (1 - \epsilon)\left\|(u_{m+3})_{n_i} - q\right\|^2$$
$$+\langle f\ (q) - q, (u_{m+3})_n\ - q\rangle\ .$$

This implies that

$$\left\|(u_{m+3})_{n_i}\ - q\right\|^2 \leq \frac{1}{\epsilon}\langle(u_{m+3})_{n_i} - q, f\ (q)\ - q\rangle\ .$$

But $(u_{m+3})_{n_i} \rightharpoonup q$, it follows that $(u_{m+3})_{n_i}\ \to q$. Now we show that q solves the variational inequality

$$\langle(I\ -\ f\ )q, \epsilon\ \rangle \leq 0, \quad f \in \Pi_C\ , (q - \epsilon)\ \in\ F(S)\ \cap\ VI(C, A).$$

Because

$$(u_{m+3})_n - f\ ((u_{m+3})_n) = -\frac{1-\left(\frac{\lambda+\epsilon}{2}\right)_n}{\left(\frac{\lambda+\epsilon}{2}\right)_n}((u_{m+3})_n\ -\ S(u_{m+1})_n),$$

For any $(q - \epsilon) \in F(S)\ \cap\ VI(C, A)$ and notice $(q - \epsilon)\ =\ P_C((q - \epsilon)\ - \lambda_n A(q - \epsilon))$, we infer that

$$\langle(u_{m+3})_n - f\ ((u_{m+3})_n), (u_{m+3})_n - (q - \epsilon)\rangle$$

$$= -\frac{1-\left(\frac{\lambda+\epsilon}{2}\right)_n}{\left(\frac{\lambda+\epsilon}{2}\right)_n}\langle (u_{m+3})_n - SP_C\left((u_{m+3})_n - \lambda_n A(u_{m+3})_n\right), (u_{m+3})_n - q\rangle$$

$$= -\frac{1-\left(\frac{\lambda+\epsilon}{2}\right)_n}{\left(\frac{\lambda+\epsilon}{2}\right)_n}\langle (u_{m+3})_n - SP_C\left((u_{m+3})_n - \lambda_n A(u_{m+3})_n\right) -$$
$$\left((q-\epsilon) - SP_C\left((q-\epsilon) - \lambda_n A(q-\epsilon)\right)\right), (u_{m+3})_n - (q-\epsilon)\rangle \le 0,$$

Since $I - SP_C(I - \lambda_n A)$ is strong monotone sequence. Let $i \to \infty$, we have

$$\langle q - f(q), \epsilon\rangle \le 0.$$

Assume that there exists another subsequence $\{(u_{m+3})_{n_j}\}$ of $\{(u_{m+3})_n\}$ such that $(u_{m+3})_{n_j} \to q^*$, so $q^* \in F(S) \cap VI(C,A)$,and from$\langle (u_{m+3})_n - f((u_{m+3})_n), (u_{m+3})_n - (q-\epsilon)\rangle \le 0$, let $j \to \infty$ We have

$$\langle q^* - f(q^*), q^* - (q-\epsilon)\rangle \le 0, \qquad (q-\epsilon) \in F(S) \cap VI(C,A).$$

Setting $(q-\epsilon) = q^*$ in (5), we have

$$\langle q - f(q), q - q^*\rangle \le 0,$$

and setting $\epsilon = 0$ in (6), we obtain

$$\langle q^* - f(q^*), q^* - q\rangle \le 0.$$

Inequality (7) and (8) yield

$$\|q - q^*\|^2 \le \langle f(q) - f(q^*), q - q^*\rangle \le (1-\epsilon)\|q - q^*\|^2,$$

Which implies that $q = q^*$,since $0 < \epsilon < 1$Thus, $(u_{m+3})_n \to q$ as $n \to \infty$ and $q \in F(S) \cap VI(C,A)$ is unique. And q is the unique solution of variational inequality

$$\langle q - f(q), \epsilon\rangle \le 0, \ (q-\epsilon) \in F(S) \cap VI(C,A).$$

This completes the proof. □

Corollary(5.2.14)[232]: Let C be a closed convex subset of a real Hilbert space H. Let f be a contractive mapping of C into itself with coefficient $0 < \epsilon < 1$,S be a nonexpansive sequence of mapping of C into itself and let T^2 be a strictly pseudocontractive and projection mapping of C into itself with $\left(\frac{\lambda+\epsilon}{2}\right)$, such that $F(S^2) \cap F(T^2) \ne \emptyset$ .Suppose $(u_m)_1 = u_m \in C$ *and* $\{(u_m)_n\}$ is given by

$$(u_m)_{n+1} = \left(\frac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) S^2((1-\lambda_n)(u_m)_n + \lambda_n T^2(u_m)_n)$$

For every $n = 1, 2, \ldots$, where $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ *is a* sequence in $[0, \frac{2-(\lambda+\epsilon)}{2})$. If $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < 2a < 2b < 2 - (\lambda+\epsilon)$,

$$\lim_{n\to\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = 0, \sum_{n=1}^{\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = \infty, \sum \left|\left(\frac{\lambda+\epsilon}{2}\right)_{n+1} - \left(\frac{\lambda+\epsilon}{2}\right)_n\right| < \infty, \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{(u_m)_n\}$ converges strongly to $q \in F(S^2) \cap F(T^2)$, such that

$$\langle f(q) - q, \epsilon\rangle \le 0, \qquad (q-\epsilon) \in F(S^2) \cap F(T^2).$$

Proof. Put $A^2 = I - T^2$. Then A^2 is $\frac{2-(\lambda+\epsilon)}{4}$ -inverse-strongly monotone sequence. We have$F(T^2) = VI(C, A^2)$and $P_C((u_m)_n - \lambda_n A^2(u_m)_n) = (1-\lambda_n)(u_m)_n + \lambda_n T^2 (u_m)_n$.So by Proposition 3.1and Theorem 3.1, we obtain the desired result. □

Corollary(5.2.15)[232]: Let H be a real Hilbert space H. Let f be a contractive mapping of H into itself with coefficient $0 < \epsilon < 1$, S^2 be a nonexpansive sequence mapping of H into itself and let A^2 *be a* contraction and projection of $a\left(\frac{\lambda+\epsilon}{2}\right)$-inverse strongly monotone sequence of mappings of H into itself such that $F(S^2) \cap (A^2)^{-1}0 \neq \emptyset$. Suppose $(u_m)_1 = (u_m) \in C$ and $\{(u_m)_n\}$ *is* given by

$$(u_m)_{n+1} = \left(\frac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n) S^2((u_m)_n - \lambda_n A^2(u_m)_n)$$

for every $n = 1, 2, \ldots$, where $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0, \lambda+\epsilon)$. *If* $\{\left(\frac{\lambda+\epsilon}{2}\right)_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < (\lambda+\epsilon)$,

$$\lim_{n\to\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = 0, \sum_{n=1}^{\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = \infty, \sum_{n=1}^{\infty}\left|\left(\frac{\lambda+\epsilon}{2}\right)_{n+1} - \left(\frac{\lambda+\epsilon}{2}\right)_n\right| < \infty, \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{(u_m)_n\}$ converges strongly to $q \in F(S^2) \cap (A^2)^{-1}0$, such that

$\langle f(q) - q, \epsilon\rangle$, $(q-\epsilon) \in F(S^2) \cap (A^2)^{-1}0$.

Proof. We have $(A^2)^{-1}0 = VI(C, A^2)$. so putting $P_H = I$, by Proposition 3.1 and Theorem 3.1, we obtain the desired result. □

Sec (5.3): A General Iterative Method with Strongly Positive Operators for General Variational Inequalities

Let H be a real Hilbert space with norm $\|.\|$and inner product$\langle .,. \rangle$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. Recall that S is called Lipschitz continuous if there exists a constant $L > 0$ such that $\|Sx - Sy\| \le L\|x - y\|$ for all $x; y \in C$. Whenever $0 < L < 1$, S is a contraction on C; whenever $L = 1$, S is a nonexpansive mapping on C. We denote by Fix(S) the set of fixed points of $S.\Pi_C$ denotes the set of all contractions on C. Note that each $f \in \Pi_C$ has a unique fixed point in C.

Recall that a mapping $T : C \to H$ is called monotone if $\langle Tx - Ty, x - y\rangle \ge 0$ for all $x; y \in C$. A mapping $T : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|Tx - Ty\|^2, \quad \forall x, y \in \mathrm{C}.$$

In this case, it is clear that T is monotone and Lipschitz continuous. Moreover, every mapping $\mathrm{g} : C \to H$, which is both δ-strongly monotone (i.e., $\langle \mathrm{g}(x) - \mathrm{g}(y), x - y\rangle \ge \delta\|x - y\|^2, \forall x, y \in \mathrm{C}$, for some $\delta > 0$) and σ-Lipschitz continuous $(i.e., \|\mathrm{g}(x) - \mathrm{g}(y)\| \le \sigma\|x - y\|, \forall x, y \in \mathrm{C}$, for some $\sigma >$

$0), is\ \delta/\sigma^2$-inverse-strongly monotone. Recall that the classical variational inequality problem is to find an $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0, \ \forall x \in C \ ; \qquad (13a)$$

see [56,57]. The set of solutions of the variational inequality (13a) is denoted $by\ VI(C, T)$.

In this section , we consider the following problem of finding $x^* \in C$ such that $\mathrm{g}(x^*) \in C$ and

$$\langle Tx^*, x - \mathrm{g}(x^*) \rangle \geq 0, \ \forall x \in C , \qquad (13b)$$

which is called a general variational inequality problem. The set of solutions of the general variational inequality (13b) is denoted by $GVI(C, \mathrm{g}, T)$. The general variational inequality problem (13b) was introduced and studied by Noor [58] and Isac [59]. Subsequently, Zeng and others (see, e.g., [60]) further considered iterative algorithms for finding its solutions and established some convergence results for iterative algorithms. Whenever $\mathrm{g}(x) = x$ for all $x \in C$, the general variational inequality problem (13b) reduces to the variational inequality problem (13a).

The iterative methods for nonexpansive mappings have been extensively studied and recently applied to solving convex minimization problems and other problems; see, e.g., [61,73] and the references therein. A typical problem is to minimize a quadratic function over the fixed point set of a nonexpansive mapping on H:

$$\min_{x \in \mathrm{Fix(T)}} \frac{1}{2}\langle Ax, x \rangle - (x - b),$$

where $\mathrm{Fix}(T)$ denotes the fixed point set of a nonexpansive mapping T on H, and b is a given point in H. Assume that A is strongly positive; that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma}\ \|x\|^2, \quad \forall x \in H.$$

We assume that $\mathrm{Fix}(T) \neq \emptyset$. It is well known that $\mathrm{Fix}(T)$is closed and convex (cf. [74]). In [65], it was proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad \forall n \geq 0,$$

converges strongly to the unique solution of the minimization problem as above provided the sequence $\{\alpha_n\}$ satisfies certain suitable conditions.

Furthermore, Moudafi [62] introduced the viscosity approximation method for nonexpansive mappings (see [63] for further development in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \quad \forall n \geq 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It was proved in [62,63] that under certain appropriate conditions imposed in $\{\sigma_n\}$, the sequence $\{x_n\}$ strongly converges to the unique solution $\tilde{x}$ in $\mathrm{Fix}(T)$ to the variational inequality

$$\langle (I - f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

Very recently, Marino and Xu [66] combined the iterative method in [65] with the viscosity approximation method in [62,63] and introduced the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 0, \qquad (14)$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (14) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

which is the optimality condition for the minimization problem

$$\min_{x\in \mathrm{Fix}(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $\acute{h}(x) = \gamma f(x)$ for all $x \in H$). On the other hand, let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$, let $T : C \to H$ be an α-inverse-strongly monotone mapping and let S be a nonexpansive self-mapping on C such that $\mathrm{Fix}(S) \cap VI(C,T) \neq \emptyset$. Chen, Zhang and Fan [68] introduced the explicit and implicit iterative schemes by the viscosity approximation method.

(I) Explicit iterative scheme [68]: define a sequence $\{x_n\}$ by

$$x_0 \in C, \; x_{n+1} = (I - \alpha_n)SP_C(x_n - \lambda_n Tx_n) + \alpha_n f(x_n), \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$.

(II) Implicit iterative scheme [68]: define a sequence $\{z_n\}$ by

$$z_n = (I - \alpha_n)SP_C(z_n - \lambda_n Tz_n) + \alpha_n f(z_n), \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$ and $\{\alpha_n\}$ is a sequence in $[0,2)$. Under some very mild conditions, they proved that the sequences $\{x_n\}$ and $\{z_n\}$ generated by algorithms (I) and (II), respectively, converge strongly to $q \in \mathrm{Fix}(S) \cap VI(C,T)$, which is the unique solution in the $\mathrm{Fix}(S) \cap VI(C,T)$ to the following variational inequality

$$\langle (I - f)q, q - p\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(S) \cap VI(C,T).$$

In this section, motivated and inspired by the iterative algorithms (14), (I) and (II), we suggest and analyze a more general iterative method with strongly positive operators for finding solutions of the general variational inequality problem (13b) in a real Hilbert space. The explicit and implicit iterative algorithms are proposed by virtue of the general iterative method with strongly positive operators. Let S be a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H, f be a contraction on C with coefficient $k \in (0,1)$ and $A, B : H \to H$ be two strongly positive linear bounded operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$. For an arbitrary initial $x_0 \in C$, we define a sequence $\{x_n\}$ via the explicit iterative scheme

$$\begin{cases} y_n = P_C\,[x_n - g(x_n) + P_C\,(g(x_n) - \lambda_n T x_n)]\ , \\ x_{n+1} = P_C\{(I - \alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))]\},\ \forall n \geq 0 \end{cases}$$

where $\{\alpha_n\} \in (0,1], \{\beta_n\} \subset (0\,,\min\{1, \|B\|^{-1}\}], \{\lambda_n\} \subset (0,2\alpha), g: C \to H$ is both δ-strongly monotone and σ -Lipschitz continuous, and $T - I : C \to H$ is an inverse-strongly monotone mapping of C into H. Furthermore, we also define a sequence $\{z_n\}$ via the implicit iterative scheme

$$z_n = P_C\{(I - \alpha_n A)SP_C\,[z_n - g(z_n) + P_C(g(z_n) - \lambda_n T z_n)] + \alpha_n[SP_C\,[z_n - g(z_n) + P_C(g(z_n) - \lambda_n T z_n)] - \beta_n(BSP_C[\,z_n - g(z_n) + P_C(g(z_n) - \lambda_n T z_n)] - \gamma f\,(z_n))]\}.$$

It is shown that under appropriate conditions the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to a unique common element of the set of fixed points of the nonexpansive mapping S and the set of solutions of the general variational inequality (13b) in a Hilbert space. The results presented in this section may be viewed as the improvement, extension and development of some earlier and recent results in the literature including, for instances, the corresponding results of Marino and Xu [66], Iiduka and Takahashi [67], Chen, Zhang and Fan [68].

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|\,.\,\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. The notation $x_n \to x$ means that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|\ , \quad \forall x \in C.$$

P_C is called the metric projection of H to C. It is well known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H, \qquad (15)$$

And P_C is characterized by the following properties:

$$\langle x - P_C x\,, P_C x - y\rangle \geq 0\ ,$$
$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2,$$

For all $x \in H, y \in C$. In the context of the variational inequality problem (13a), this implies

$$x^* \in VI(C,T\,) \Leftrightarrow x^* = P_C(x^* - \lambda T x^*), \qquad \forall\,\lambda > 0. \qquad (16a)$$

Further, in the context of the general variational inequality problem (13b), this also implies

$$x^* \in \mathrm{GVI}(C, g\,, T\,) \Leftrightarrow g(x^*) = P_C(g(x^*) - \lambda T x^*), \quad \forall\,\lambda > 0. \qquad (16b)$$

Proposition (5.3.1)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a mapping such that $T - I : C \to H$ be α-inverse-strongly monotone, and let $g : C \to H$ be both δ-strongly monotone and σ-Lipschitz continuous. If $2\sqrt{1 - 2\delta + \sigma^2} < \lambda < 2\alpha/(1 + 2\alpha)$, then for each $x, y \in C$

$$\|P_C[x-g(x)+P_C(g(x)-\lambda Tx)]-P_C[y-g(y)+P_C(g(y)-\lambda Ty)]\|$$

$$\leq [1-(\lambda-2\sqrt{1-2\delta+\sigma^2})]\|x-y\|.$$

Proof. Utilizing the δ-strong monotonicity and σ-Lipschitz continuity of $g: C \to H$, we have

$$\|x-g(x)-(y-g(x))\| \leq \sqrt{1-2\delta+\sigma^2}\,\|x-y\|, \ \forall x,y \in C.$$

Since $2\sqrt{1-2\delta+\sigma^2} < \lambda < 2\alpha/(1+2\alpha)$, and $T-I: C \to H$ is α-inverse-strongly monotone, so we obtain $\lambda - 2\alpha(1-\lambda) < 0$ and

$$\begin{aligned}&\|(1-\lambda)(x-y)-\lambda[(T-I)x-(T-I)y]\|^2\\ &= (1-\lambda)^2\|x-y\|^2 - 2\lambda(1-\lambda)\langle (T-I)x-(T-I)y, x-y\rangle\\ &+\lambda^2\|(T-I)x-(T-I)y\|^2\\ &\leq (1-\lambda)^2\|x-y\|^2+\lambda(\lambda-2\alpha(1-\lambda))\|(T-I)x-(T-I)y\|^2\\ &\leq (1-\lambda)^2\|x-y\|^2,\end{aligned}$$

which implies that

$$\|(1-\lambda)(x-y)-\lambda[(T-I)x-(T-I)y]\| \leq (1-\lambda)\|x-y\|, \qquad \forall x,y \in C.$$

Therefore, we get for each $x,y \in C$.

$$\begin{aligned}&\|P_C[x-g(x)+P_C(g(x)-\lambda Tx)]-P_C[y-g(y)+P_C(g(y)-Ty)]\|\\ &\leq \|x-g(x)+P_C(g(x)-\lambda Tx)-[y-g(y)+P_C(g(y)-\lambda Ty)]\|\\ &\leq 2\|x-g(x)-(y-g(y))\|\\ &+\|(1-\lambda)(x-y)-\lambda[(T-I)x-(T-I)y]\|\\ &\leq 2\sqrt{1-2\delta+\sigma^2}\,\|x-y\|+(1-\lambda)\|x-y\|\\ &= [1-(\lambda-2\sqrt{1-2\delta+\sigma^2})]\|x-y\|.\end{aligned}$$

This completes the proof. □

The following lemmas will be used for the proof of our main results in what follows.

Lemma (5.3.2)[55]: (see [64,lemma 2.1].)Let $\{S_n\}$ be a sequence of nonnegative numbers satisfying the condition

$$S_{n+1} \leq (1-\alpha_n)S_n + \alpha_n\beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$and $\sum_{n=0}^{\infty}\alpha_n = \infty$, or equavalently, $\prod_{n=0}^{\infty}(1-\alpha_n)=0$;

(ii) $\limsup_{n\to\infty}\beta_n \leq 0$ or

(ii)\ $\sum_{n=0}^{\infty}a_n\beta_n$ is convergent.

Then, $\lim_{n\to\infty}S_n = 0$.

Lemma (5.3.3)[55]: (see Goebel and kirk [64].) Demiclosedness Principle. Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.

Lemma (5.3.4)[55]: (see [66,lemma 2.3]) .Let C be a closed convex subset of a Hilbert space $H, f : C \to C$ be a contraction with coefficient $k \in (0,1)$, and B be a strongly positive linear bounded operator with coefficient $\beta > 0$. Then, for $0 < \gamma < \frac{\beta}{k}$

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\beta - \gamma k)\|x - y\|^2, \forall x, y \in C.$$

That is, $B - \gamma f$ is strongly monotone with coefficient $\beta - \gamma k$.

Lemma (5.3.5)[55]: (see [66,lemma 2.5]) Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.

Throughout the rest of this section, we always assume that $f : C \to C$ is a contraction on C with coefficient $k \in (0,1)$, and A, B are two strong positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$, and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$ and $\lim_{n\to\infty} \beta_n = \eta \in (\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k})$. Then, we may assume without loss of generality that there exists $c \in (\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k})$ such that

$$\frac{1-\bar{\gamma}}{\beta-\gamma k} < c \leq \beta_n < \frac{2-\bar{\gamma}}{\beta-\gamma k}, \quad \forall n \geq 0. \tag{17}$$

Let $T : C \to H$ be a mapping such that $T - I: C \to H$ is α-inverse-strongly monotone, and let $g : C \to H$ be both δ-strongly monotone and σ-Lipschitz continuous. Let S be a nonexpansive self-mapping on C. Let $2\sqrt{1 - 2\delta + \sigma^2} < \lambda_n < \frac{2\alpha}{1+2\alpha}$, $\{\alpha_n\} \subset (0, \min\{1, \|A\|^{-1}\}]$ and $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$. For each $n \geq 0$, consider a mapping $V_n : C \to C$ defined by

$$\begin{aligned} V_n x = {} & P_C\{ (I - \alpha_n A) S P_C [x - g(x) + P_C(g(x) - \lambda_n T x)] \\ & + \alpha_n [S P_C [x - g(x) + P_C(g(x) - \lambda_n T x)] \\ & - \beta_n (B S P_C [x - g(x) + P_C (g(x) - \lambda_n T x)] - \gamma f(x))]\}, \end{aligned} \tag{18}$$

for all $x \in C$. Indeed, by Proposition (5.3.1) and Lemma (5.3.5) we have

$$\begin{aligned} \|V_n x - V_n y\| = {} & \| P_C\{ (I - \alpha_n A) S P_C [x - g(x) + P_C(g(x) - \lambda_n T x)] \\ & + \alpha_n [S P_C [x - g(x) + P_C(g(x) - \lambda_n T x)] \\ & - \beta_n (B S P_C [x - g(x) + P_C (g(x) - \lambda_n T x)] - \gamma f x)]\} \\ & - P_C\{ (I - \alpha_n A) S P_C [y - g(y) + P_C(g(y) - \lambda_n T y)] \\ & + \alpha_n [S P_C [y - g(y) + P_C(g(y) - \lambda_n T y)] \\ & - \beta_n (B S P_C [y - g(y) + P_C (g(y) - \lambda_n T y)] - \gamma f y)]\} \end{aligned}$$

$$\leq \big\|\{ (I-\alpha_n A)S\, P_C\,[x-\ g(x)+\ P_C(\, g(x)-\lambda_n Tx)]$$
$$+\alpha_n\big[S\,P_C[x - g(x)$$
$$+\ P_C(\, g(x)-\ \lambda_n Tx)]-\beta_n\big(BS\,P_C\,[x-\ g(x)$$
$$+\ P_C\,(g(x)-\ \lambda_n Tx)]-\gamma f(x)\big)\big]\big\}$$
$$-\{(I-\alpha_n A)S\,P_C\,[y-\ g(y)+\ P_C(\, g(y)-\lambda_n Ty)]$$
$$+\ \alpha_n[S\,P_C[y\ -g(y)+\ P_C(\, g(y)-\ \lambda_n Ty)]-\beta_n(BS\,P_C\,[y$$
$$-\ g(y)+\ P_C\,(g(y)-\ \lambda_n Ty)]-\gamma f(y))]\}\big\|$$
$$\leq \|(I-\alpha_n A)S\,P_C\,[x-\ g(x)+\ P_C(\, g(x)-\lambda_n Tx)]$$
$$-(I-\alpha_n A)S\,P_C\,[y-\ g(y)+\ P_C(\, g(y)-\lambda_n Ty)]\|$$
$$+\big\|\alpha_n\big[S\,P_C\big[x - g(x)+\ P_C(\, g(x)-\ \lambda_n Tx)\big]-\beta_n\big(BS\,P_C\,[x-\ g(x)+ P_C\,(g(x)-\ \lambda_n Tx)]-\gamma f(x)\big)\big]-\alpha_n\big[S\,P_C[y\ -g(y)+\ P_C(\, g(y)-\lambda_n Ty)]-\beta_n\big(BS\,P_C\,[y-\ g(y)+\ P_C\,(g(y)-\ \lambda_n Ty)]-\gamma f(y)\big)\big]\big\|$$
$$\leq \|I-\alpha_n A\|\|S\,P_C[x\ -g(x)+\ P_C(\, g(x)-\ \lambda_n Tx)]$$
$$-S\,P_C[y\ -g(y)+\ P_C(\, g(y)-\ \lambda_n Ty)]\|$$
$$+\alpha_n\|(I-\beta_n B)(S\,P_C\,[x-\ g(x)+\ P_C(\, g(x)-\lambda_n Tx)]\quad -S\,P_C\,[y-g(y)+\ P_C(\, g(y)-\lambda_n Ty)]+\beta_n\gamma(f(x)-f(y))\|$$
$$\leq (1-\alpha_n\bar{\gamma})[1-(\lambda_n-2\sqrt{1-\ 2\delta+\sigma^2})]\|x-y\|$$
$$+\alpha_n[(1-\beta_n B)\left[1-\left(\lambda_n-2\sqrt{1-\ 2\delta+\sigma^2}\right)\right]\|x-y\|+\beta_n\gamma k\|x-y\|]$$
$$=[1-\alpha_n(\bar{\gamma}-1+\beta_n(\beta-\gamma k))]\|x-y\|$$
$$=(1-\alpha_n\tau_n)\|x-y\|\ ,$$
where $\tau_n := \bar{\gamma}-1+\beta_n(\beta-\gamma\alpha)$Since $c\in\left(\frac{1-\bar{\gamma}}{\beta-\gamma k},\frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$ .we have
$\tau_n := \bar{\gamma}-1+c(\beta-\gamma\alpha)\in(0,1)$ and

$$\tau_n=\bar{\gamma}-1+\beta_n(\beta-\gamma k)\geq\bar{\gamma}-1+c(\beta-\gamma k)=\tau\,.$$

Hence we get

$$\|V_n x-V_n y\|\leq(1-\alpha_n\tau)\|x-y\|\,. \tag{19}$$

This shows that V_n is a contraction. Therefore, by the Banach contraction principle, V_n has a unique fixed point $z_n\in\ C$ such that

$$z_n=\ P_C\big\{ (I-\alpha_n A)S\,P_C\,[z_n-\ g(z_n)+\ P_C(\, g(z_n)-\lambda_n Tz_n)]$$
$$+\ \alpha_n\big[S\,P_C[z_n\ -g(z_n)$$
$$+\ P_C(\, g(z_n)-\ \lambda_n Tz_n)]-\beta_n\big(BS\,P_C\,[z_n-\ g(z_n)$$
$$+\ P_C\,(g(z_n)-\ \lambda_n Tz_n)]-\gamma f(z_n)\big)\big]\big\}.$$

Note that z_n indeed depends on f as well, but we will suppress this dependence of z_n on f for simplicity of notation throughout the rest of this section. We will also always use γ to mean a number in $\left(0,\frac{\beta}{k}\right)$.

In this section, we first prove a strong convergence result on the explicit iterative algorithm for the general variational inequality problem (13b).

Theorem (5.3.6)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$, let $T : C \to H$ be a mapping such that $T - I : C \to H$ is α-inverse-strongly monotone, and let $g : C \to H$ be both δ-strongly monotone and σ-Lipschitz continuous. Let S be a nonexpansive self-mapping on C such that $\text{Fix}(S) \cap \text{GVI}(C, g, T) \neq \emptyset$. Let A, B be two strong positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$. Assume that $\{x_n\}$ and $\{y_n\}$ are sequences in C generated by $x_0 \in C$ and

$$\begin{cases} y_n = P_C[x_n - g(x_n) + P_C(g(x_n) - \lambda_n T x_n)], \\ x_{n+1} = P_C\{(I - \alpha_n A)S y_n + \alpha_n[S y_n - \beta_n(BS\, y_n - \gamma f(x_n))]\}, \quad \forall n \geq 0 \end{cases}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $2\sqrt{1 - 2\delta + \sigma^2} + \xi \leq \lambda_n < 2\alpha/(1 + 2\alpha)$ for some $\xi > 0$. Suppose that there hold the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta - \gamma k}, \frac{2-\bar{\gamma}}{\beta - \gamma k}\right)$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1}\beta_{n+1} - \alpha_n \beta_n| < \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then both $\{x_n\}$and $\{y_n\}$ converge strongly to the unique element of $\text{Fix}(S) \cap \text{GVI}(C, g, T)$.

Proof. First, we may assume that $\alpha_n < \|A\|^{-1}$ due to $\lim_{n\to\infty} \alpha_n = 0$. By Lemma (5.3.5), we obtain $\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}$. Also, Since $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta - \gamma k}, \frac{2-\bar{\gamma}}{\beta - \gamma k}\right)$,we may assume that for some constant $c \in \left(\frac{1-\bar{\gamma}}{\beta - \gamma k}, \frac{2-\bar{\gamma}}{\beta - \gamma k}\right)$.

$$\frac{1-\bar{\gamma}}{\beta - \gamma k} < c \leq \beta_n < \frac{2-\bar{\gamma}}{\beta - \gamma k}, \quad \forall n \geq 0.$$

Let $p \in \text{Fix}(S) \cap \text{GVI}(C, g, T)$. Then $p = Sp$ and p is a solution of the general variational inequality (13b). Hence utilizing (16b) we have

$$P = P_C[p - g(p) + P_C(g(p) - \lambda_n T p)], \ \forall n \geq 0.$$

Thus utilizing Proposition (5.3.1) we obtain

$$\begin{aligned} \|y_n - p\| &= \|P_C[x_n - g(x_n) + P_C(g(x_n) - \lambda_n T x_n)] - P_C[p - g(p) + P_C(g(p) \\ &\quad - \lambda_n T p)]\| \\ &\leq [1 - (\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2})]\|x_n - p\| \\ &\leq \|x_n - p\| \end{aligned}$$

for every $n = 0, 1, 2, \ldots$. Observe that

$$\begin{aligned} V_n p &= P_C\{(I - \alpha_n A)S\, P_C[p - g(p) + P_C(g(p) - \lambda_n T p)] \\ &\quad + \alpha_n[S\, P_C[p - g(p) + P_C(g(p) - \lambda_n T p)] \\ &\quad - \beta_n(BS\, P_C[p - g(p) + P_C(g(p) - \lambda_n T p)] - \gamma f(p))]\} \\ &= P_C\{(I - \alpha_n A)p + \alpha_n[p - \beta_n(Bp - \gamma f(p))]\}. \end{aligned}$$

Then from (19) we have

$$\|x_{n+1} - p\| = \|V_n x_n - V_n p + V_n p - p\|$$

$\leq \|V_n x_n - V_n p\| + \|V_n p - p\|$

$\leq (1 - \alpha_n \tau)\|x_n - p\|$

$+\left\|P_C\{ (I - \alpha_n A)p + \alpha_n \left[p - \beta_n\left(Bp - \gamma f(p)\right)\right]\} - P_C p\right\|$

$\leq (1 - \alpha_n \tau)\|x_n - p\| + \left\|(I - \alpha_n A)p + \alpha_n\left[p - \beta_n\left(Bp - \gamma f(p)\right)\right] - p\right\|$

$\leq (1 - \alpha_n \tau)\|x_n - p\| + \alpha_n\left\|-Ap + p - \beta_n\left(Bp - \gamma f(p)\right)\right\|$

$\leq (1 - \alpha_n \tau)\|x_n - p\| + \alpha_n[\|A - I\|\|p\| + \|B\|\|p\| + \gamma\|f(p)\|],$

which hence implies that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|A - I\|\|p\| + \|B\|\|p\| + \gamma\|f(p)\|}{\tau}\right\}, \forall n \geq 0 .$$

So, $\{x_n\}$ is bounded and we also obtain that $\{y_n\}$, $\{Sy_n\}$, $\{Tx_n\}$ and $\{f(x_n)\}$ are bounded. Since each $P_C[x - g(x) + P_C(g(x) - \lambda_n Tx)]$ is nonexpansive according to Proposition (5.3.1), we also have

$\|y_{n+1} - y_n\| \leq \|P_C[x_{n+1} - g(x_{n+1}) + P_C g(x_{n+1}) - \lambda_{n+1} Tx_{n+1})] - P_C[x_n - g(x_n) + P_C(g(x_n) - \lambda_n Tx_n)]\|$

$\leq \|P_C[x_{n+1} - g(x_{n+1}) + P_C(g(x_{n+1}) - \lambda_{n+1} Tx_{n+1})] - P_C[x_n - g(x_n) + P_C(g(x_n) - \lambda_{n+1} Tx_n)]\|$

$+\|P_C[x_n - g(x_n) + P_C(g(x_n) - \lambda_{n+1} Tx_n)] - P_C[x_n - g(x_n) + P_C(g(x_n) - \lambda_n Tx_n)]\|$

$\leq \|x_{n+1} - x_n\| + \|P_C(g(x_n) - \lambda_{n+1} Tx_n) - P_C(g(x_n) - \lambda_n Tx_n)\|$

$\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Tx_n\|$

for every $n = 0,1,2, \ldots$. Thus it follows that

$\left\|P_C\{ (I - \alpha_n A)Sy_n + \alpha_n\left[Sy_n - \beta_n\left(BSy_n - \gamma f(x_n)\right)\right]\} - P_C\{ (I - \alpha_n A)Sy_{n-1} + \alpha_n\left[Sy_{n-1} - \beta_n\left(BSy_{n-1} - \gamma f(x_{n-1})\right)\right]\}\right\|$

$\leq \left\|\{ (I - \alpha_n A)Sy_n + \alpha_n\left[Sy_n - \beta_n\left(BSy_n - \gamma f(x_n)\right)\right]\} - \{ (I - \alpha_n A)Sy_{n-1} + \alpha_n\left[Sy_{n-1} - \beta_n\left(BSy_{n-1} - \gamma f(x_{n-1})\right)\right]\}\right\|$

$= \|(I - \alpha_n A)Sy_n - (I - \alpha_n A)Sy_{n-1} + \alpha_n[(I - \beta_n B)Sy_n - (I - \beta_n B)Sy_{n-1} + \beta_n\gamma(f(x_n) - f(x_{n-1}))]\|$

$\leq \|I - \alpha_n A\|\|Sy_n - Sy_{n-1}\|$

$+\alpha_n[\|I - \beta_n B\|\|Sy_n - Sy_{n-1}\| + \beta_n\gamma\|f(x_n) - f(x_{n-1})\|]$

$\leq (1 - \alpha_n \bar{\gamma})\|y_n - y_{n-1}\| + \alpha_n(1 - \beta_n B)\|y_n - y_{n-1}\|$

$+\alpha_n\beta_n\gamma k\|x_n - x_{n-1}\|$

$\leq [(1 - \alpha_n \bar{\gamma}) + \alpha_n(1 - \beta_n B)][\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Tx_{n-1}\|]$

$+\alpha_n\beta_n\gamma k\|x_n - x_{n-1}\|$

$= \left[1 - \alpha_n\left(\bar{\gamma} - 1 + \beta_n(\beta - \gamma k)\right)\right]\|x_n - x_{n-1}\|$

$+[(1 - \alpha_n(\bar{\gamma} - 1 + \beta_n B))]|\lambda_n - \lambda_{n-1}|\|Tx_{n-1}\|$

$\leq \left[1 - \alpha_n\left(\bar{\gamma} - 1 + \beta_n(\beta - \gamma k)\right)\right](\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Tx_{n-1}\|)$

$= (1 - \alpha_n \tau_n)(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Tx_{n-1}\|)$

$\leq (1 - \alpha_n \tau)\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Tx_{n-1}\|.$ (20)

Furthermore, note that

$\|P_C\{ (I-\alpha_n A)Sy_{n-1} + \alpha_n[Sy_{n-1} - \beta_n(BSy_{n-1} - \gamma f(x_{n-1}))]\} - P_C\{ (I-\alpha_{n-1} A)Sy_{n-1} + \alpha_{n-1}[Sy_{n-1} - \beta_{n-1}(BSy_{n-1} - \gamma f(x_{n-1}))]\}\|$
$\le \|\{ (I-\alpha_n A)Sy_{n-1} + \alpha_n[Sy_n - \beta_n(BSy_{n-1} - \gamma f(x_{n-1}))]\} - \{ (I-\alpha_{n-1} A)Sy_{n-1} + \alpha_{n-1}[Sy_{n-1} - \beta_{n-1}(BSy_{n-1} - \gamma f(x_{n-1}))]\}\|$
$= \|(I-\alpha_n(A-I))Sy_{n-1} - \alpha_n\beta_n BSy_{n-1} + \alpha_n\beta_n\gamma f(x_{n-1}) - (I-\alpha_{n-1}(A-I))Sy_{n-1} + \alpha_{n-1}\beta_{n-1}BSy_{n-1} - \alpha_{n-1}\beta_{n-1}\gamma f(x_{n-1})\|$
$\le |\alpha_n - \alpha_{n-1}|\|(A-I)Sy_{n-1}\| + |\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|\|BSy_{n-1}\|$
$+ |\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|\,\gamma\|f(x_{n-1})\|$
$\le M|\alpha_n - \alpha_{n-1}| + M|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|$, (21)

where M is a positive constant such that $M \ge \|(A-I)Sy_n\| + \|BSy_n\| + \gamma\|f(x_n)\|$ for every $n = 0,1,2,\ldots$. So from (20) *and* (21) we derive

$\|x_{n+1} - x_n\| = \|P_C\{ (I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))]\} - P_C\{ (I-\alpha_n A)Sy_{n-1} + \alpha_n[Sy_{n-1} - \beta_n(BSy_{n-1} - \gamma f(x_{n-1}))]\}\|$
$+ \|P_C\{ (I-\alpha_n A)Sy_{n-1} + \alpha_n[Sy_{n-1} - \beta_n(BSy_{n-1} - \gamma f(x_{n-1}))]\} - P_C\{ (I-\alpha_{n-1} A)Sy_{n-1} + \alpha_{n-1}[Sy_{n-1} - \beta_{n-1}(BSy_{n-1} - \gamma f(x_{n-1}))]\}\|$
$\le (1-\alpha_n\tau)\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Tx_{n-1}\|$
$+ M|\alpha_n - \alpha_{n-1}| + M|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|$
$\le (1-\alpha_n\tau)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}|$
$+ M|\alpha_n - \alpha_{n-1}| + M|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|$

for every $n = 0,1,2,\ldots$, where L is a positive constant such that $L \ge \|Tx_n\|$ for every $n = 0,1,2,\ldots$. Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=0}^{\infty}|\alpha_{n+1}\beta_{n+1} - \alpha_n\beta_n| < \infty$, in view of Lemma (5.3.2) we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Then we also obtain $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.

Since by Proposition (5.3.1) we have for each $n \ge 0$

$$\|y_n - p\| \le \left(1 - \left(\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2}\right)\right)\|x_n - p\|,$$

we deduce that for $p \in \mathrm{Fix}(S) \cap \mathrm{GVI}(C,g,T)$,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|P_C\{ (I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))]\} - p\|^2 \\
&\le \|(I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))] - p\|^2 \\
&= \|(I-\alpha_n A)(Sy_n - p) + \alpha_n[Sy_n - \beta_n(BSy_n - \gamma f(x_n))] - Ap\|^2 \\
&= \|(I-\alpha_n A)(Sy_n - p)\|^2 + \alpha_n{}^2\|Sy_n - \beta_n(BSy_n - \gamma f(x_n)) - Ap\|^2 \\
&\quad + 2\alpha_n\langle (I-\alpha_n A)(Sy_n - p), Sy_n - \beta_n(BSy_n - \gamma f(x_n)) - Ap\rangle \\
&\le (1-\alpha_n\bar{\gamma})^2\|y_n - p\|^2 + \alpha_n{}^2\|(I-\beta_n B)Sy_n + \beta_n\gamma f(x_n) - Ap\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n\bar{\gamma})\|y_n - p\|\|(I-\beta_n B)Sy_n + \beta_n\gamma f(x_n) - Ap\| \\
&\le \|y_n - p\|^2 + \alpha_n{}^2\|(I-\beta_n B)Sy_n + \beta_n\gamma f(x_n) - Ap\|^2 \\
&\quad + 2\alpha_n\|y_n - p\|\|(I-\beta_n B)Sy_n + \beta_n\gamma f(x_n) - Ap\| \\
&\le \|y_n - p\|^2 + \alpha_n{}^2[(1-\beta_n B)\|Sy_n\| + \beta_n\gamma\|f(x_n)\| + \|Ap\|]^2 \\
&\quad + 2\alpha_n\|y_n - p\|[(1-B\beta_n)\|Sy_n\| + \beta_n\gamma\|f(x_n)\| + \|Ap\|] \\
&\le \|y_n - p\|^2 + \alpha_n{}^2[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]^2
\end{aligned}$$

$$+2\alpha_n\|y_n - p\|[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]$$
$$\le \left[\left(1 - \left(\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2}\right)\right)\|x_n - p\|\right]^2$$
$$+\alpha_n{}^2\left[\left\|Sy_n\right\| + \gamma\|f(x_n)\| + \|Ap\|\right]^2$$
$$+2\alpha_n\|y_n - p\|[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]$$
$$\le \left(1 - \left(\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2}\right)\right)\|x_n - p\|^2$$
$$+\alpha_n{}^2[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]^2$$

$$+2\alpha_n\|y_n - p\|[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]\,. \qquad (22)$$

Thus we obtain

$$\xi\|x_n - p\|^2 \le \left(\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2}\right)\|x_n - p\|^2$$
$$\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n{}^2[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]^2$$
$$+2\alpha_n\|y_n - p\|[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]$$
$$\le (\|x_n - p\| - \|x_{n+1} - p\|)\|x_n - x_{n+1}\|$$
$$+\alpha_n{}^2[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|]^2$$
$$+2\alpha_n\|y_n - p\|[\|Sy_n\| + \gamma\|f(x_n)\| + \|Ap\|].$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, and since $\{x_n\}$, $\{y_n\}$,$\{Sy_n\}$ and $\{f(x_n)\}$ are bounded, so we know that $\|x_n - p\| \to 0$ as $n \to \infty$. Note that $\|y_n - p\| \le \|x_n - p\|$ for all $n \ge 0$. Consequently, $\|y_n - p\| \to 0$ as $n \to \infty$. Moreover, there is no doubt that $\mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)} = \{p\}$. This completes the proof. □

S is a nonexpansive mapping, $T - I$ is α-inverse-strongly monotone, g is both δ-strongly monotone and σ-Lipschitz continuous, $f \in \Pi_{\mathrm{C}}$, and A, B are two strongly positive bounded linear operators. Thus, in terms of (19), the Banach contraction principle guarantees that there exists a unique fixed point

$$z_n^f = P_C\{(I - \alpha_n A)SP_C\left[z_n^f - \mathrm{g}(z_n^f) + P_C(\mathrm{g}(z_n^f) - \lambda_n T z_n^f\right]$$
$$+\alpha_n[SP_C[z_n^f - \mathrm{g}(z_n^f) + P_C(\mathrm{g}(z_n^f) - \lambda_n T z_n^f)]$$
$$-\beta_n(BSP_C[z_n^f - \mathrm{g}(z_n^f) + P_C(\mathrm{g}(z_n^f) - \lambda_n T z_n^f)] - \gamma f(z_n^f))]\},$$

Where $\{\alpha_n\} \subset (0, \min\{1, \|A\|^{-1}\}]$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$, $2\sqrt{1 - 2\delta + \sigma^2} < \lambda_n < 2\alpha/(1 + 2\alpha)$ and $\frac{1-\bar{\gamma}}{\beta-\gamma k} < c \le \beta_n < \frac{2-\bar{\gamma}}{\beta-\gamma k}$. For simplicity we will write z_n for z_n^f provided no confusion occurs. Next we will prove the strong convergence of $\{z_n\}$.

Remark (5.3.7)[55]: According to the definition of strongly positive operator, A is strongly positive, that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \ge \bar{\gamma}\,\|x\|^2 \quad \text{for all } x \in H.$$

Beyond question, we may assume without loss of generality that $\bar{\gamma} < 1$. Consequently, whenever $0 < \gamma < \frac{\bar{\gamma}}{k}$, $B = I$ and $\beta = 1$,

Then we have

$$\frac{1-\bar{\gamma}}{\beta-\gamma k}=\frac{1-\bar{\gamma}}{1-\gamma k}<1<\frac{2-\bar{\gamma}}{1-\gamma k}=\frac{2-\bar{\gamma}}{\beta-\gamma k}.$$

Thus, we can pick $\beta_n = 1$ for all $n \geq 0$ and so, as an immediate consequence of Theorem (5.3.6) we obtain

Corollary(5.3.8)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$, let $T : C \to H$ be a mapping such that $T - I : C \to H$ is α-inverse-strongly monotone, and let $g : C \to H$ be both δ-strongly monotone and σ-Lipschitz continuous. Let S be a nonexpansive self-mapping on C such that $\mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)} \neq \emptyset$.

Let A be a strong positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $0 < \gamma < \frac{\bar{\gamma}}{k}$. Assume that $\{x_n\}$ and $\{y_n\}$ are sequences in C generated by $x_0 \in C$ and

$$\begin{cases} y_n = P_C\,[x_n - g(x_n) + P_C(g(x_n) - \lambda_n T x_n)], \\ x_{n+1} = P_C\{(I-\alpha_n A)Sy_n + \alpha_n \gamma f(x_n)\}, \ \forall n \geq 0 \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $2\sqrt{1-2\delta+\sigma^2}+\xi \leq \lambda_n < 2\alpha/(1+2\alpha)$ for some $\xi > 0$. Suppose that there hold the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique element of $\mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)}$.

Corollary (5.3.9)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$, let $T : C \to H$ be a mapping such that $T - I : C \to H$ is both μ-strongly monotone and ν-Lipschitz continuous, and let $g: C \to H$ be both δ-strongly monotone and σ-Lipschitz continuous. Let S be a nonexpansive self-mapping on C such that $\mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)} \neq \emptyset$. Let A, B be two strong positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$. Assume that $\{x_n\}$ and $\{y_n\}$ are sequences in C generated by $x_0 \in C$ and

$$\begin{cases} y_n = P_C\,[x_n - g(x_n) + P_C(g(x_n) - \lambda_n T x_n)], \\ x_{n+1} = P_C\{(I-\alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BS\,y_n - \gamma f(x_n)]\}, \ \forall n \geq 0 \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $2\sqrt{1-2\delta+\sigma^2}+\xi \leq \lambda_n < \frac{2\mu/\nu^2}{1+2\mu/\nu^2}$ for some $\xi > 0$. Suppose that there hold the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1}\beta_{n+1} - \alpha_n\beta_n| < \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique element of $\mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)}$.

Proof. Note that $T - I : C \to H$ is ν-Lipschitz continuous, that is,

$$\|(T-I)x-(T-I)y\| \le \nu\|x-y\|, \qquad \forall x, y \in C .$$

Since $T - I : C \to H$ is also μ-strongly monotone, we have

$$\langle (T-I)x-(T-I)y, x-y\rangle \ge \mu\|x-y\|^2 \ge \frac{\mu}{\nu^2}\|(T-I)x-(T-I)y\|^2, \quad \forall x, y \in C.$$

This implies that $T - I : C \to H$ is μ/ν^2-inverse-strongly monotone. Hence, all conditions in Theorem (5.3.6) are satisfied. Therefore the conclusion follows immediately from Theorem (5.3.6). □

Theorem (5.3.10)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$, let $T : C \to H$ be a mapping such that $T - I : C \to H$ is α-inverse-strongly monotone, and let $g : C \to H$ be both δ-strongly monotone and σ-Lipschitz continuous. Let S be a nonexpansive self-mapping on C such that $\mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)} \neq \emptyset$.

Let A, B be two strong positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$ .Assume that $\{z_n\}$ is a sequence in C generated by

$$z_n = P_C\{ (I-\alpha_n A)S\, P_C\,[z_n - g(z_n) + P_C(g(z_n)-\lambda_n T z_n)] + \alpha_n[S\, P_C[z_n - g(z_n) + P_C(g(z_n)-\lambda_n T z_n)] - \beta_n(BS\, P_C\,[z_n - g(z_n) + P_C\,(g(z_n)-\lambda_n T z_n)] - \gamma f(z_n))]\}$$

where $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $2\sqrt{1-2\delta+\sigma^2} + \xi \le \lambda_n < 2\alpha/(1+2\alpha)$ for some $\xi > 0$.

If $\lim_{n\to\infty}\alpha_n = 0$, and $\lim_{n\to\infty}\beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$. Then $\{z_n\}$ converges strongly to the unique element of $\mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)}$.

Proof. First, we may assume that $\alpha_n < \|A\|^{-1}$ due to $\lim_{n\to\infty}\alpha_n = 0$, By Lemma (5.3.5), we obtain $\|I-\alpha_n A\| \le 1-\alpha_n\bar{\gamma}$. Also, since $\lim_{n\to\infty}\beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$, we may assume that for some constant $c \in \left(\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$.

$$\frac{1-\bar{\gamma}}{\beta-\gamma k} < c \le \beta_n < \frac{2-\bar{\gamma}}{\beta-\gamma k}, \qquad \forall n \ge 0 .$$

Put $y_n = P_C\,[z_n - g(z_n) + P_C(g(z_n)-\lambda_n T z_n)]$ for every $n = 0,1,2,\ldots$. Let $p \in \mathrm{Fix(S)} \cap \mathrm{GVI(C, g, T)}$. Then utilizing Proposition (5.3.1) we obtain

$$\begin{aligned}\|y_n - p\| &= \|P_C\,[z_n - g(z_n) + P_C(g(z_n)-\lambda_n T z_n)] - P_C\,[p - g(p) + P_C(g(p) - \lambda_n Tp)]\| \\ &\le [1-(\lambda_n - 2\sqrt{1-2\delta+\sigma^2})]\|z_n - p\| \\ &\le \|z_n - p\|\end{aligned}$$

for every $n = 0,1,2, \ldots$. Observe that

$$\begin{aligned} V_n p = & P_C\{ (I-\alpha_n A) S P_C [p - g(p) + P_C(g(p) - \lambda_n T p)] \\ & + \alpha_n [S P_C [p - g(p) + P_C(g(p) - \lambda_n T p)] \\ & - \beta_n (B S P_C [p - g(p) + P_C(g(p) - \lambda_n T p)] - \gamma f(p)])\} \\ = & P_C\{ (I-\alpha_n A)p + \alpha_n [p - \beta_n (Bp - \gamma f(p))]\}. \end{aligned}$$

Then from (19) we have

$$\begin{aligned} \|z_n - p\| &= \|V_n z_n - V_n p + V_n p - p\| \\ &\le \|V_n z_n - V_n p\| + \|V_n p - p\| \\ &\le (1-\alpha_n \tau)\|z_n - p\| \\ &+ \|P_C\{ (I-\alpha_n A)p + \alpha_n [p - \beta_n (Bp - \gamma f(p))]\} - P_C p\| \\ &\le (1-\alpha_n \tau)\|z_n - p\| + \|(I-\alpha_n A)p + \alpha_n [p - \beta_n (Bp - \gamma f(p))] - p\| \\ &\le (1-\alpha_n \tau)\|z_n - p\| + \alpha_n \|-Ap + p - \beta_n (Bp - \gamma f(p))\| \\ &\le (1-\alpha_n \tau)\|z_n - p\| + \alpha_n [\|A - I\|\|p\| + \|B\|\|p\| + \gamma \|f(p)\|]. \end{aligned}$$

Hence,

$$\|z_n - p\| \le \frac{1}{\tau} [\|A - I\|\|p\| + \|B\|\|p\| + \gamma \|f(p)\|].$$

This implies that $\{z_n\}$ is bounded, and so are $\{y_n\}$ $\{Sy_n\}, \{T(z_n)\}$ and $\{f(z_n)\}$. For $p \in \mathrm{Fix}(S) \cap \mathrm{GVI}(C, g, T)$.

$$\begin{aligned} \|z_n - p\|^2 &= \|P_C\{ (I-\alpha_n A)Sy_n + \alpha_n [Sy_n - \beta_n (BSy_n - \gamma f(z_n))]\} - p\|^2 \\ &\le \|(I-\alpha_n A)Sy_n + \alpha_n [Sy_n - \beta_n (BSy_n - \gamma f(z_n))] - p\|^2 \\ &= \|(I-\alpha_n A)(Sy_n - p) + \alpha_n [Sy_n - \beta_n (BSy_n - \gamma f(z_n)) - Ap]\|^2 \\ &= \|(I-\alpha_n A)(Sy_n - p)\|^2 + {\alpha_n}^2 \|Sy_n - \beta_n (BSy_n - \gamma f(z_n)) - Ap\|^2 \\ &+ 2\alpha_n \langle (I-\alpha_n A)(Sy_n - p), Sy_n - \beta_n (BSy_n - \gamma f(z_n)) - Ap \rangle \\ &\le (1-\alpha_n \bar{\gamma})^2 \|y_n - p\|^2 + {\alpha_n}^2 \|(I - \beta_n B)Sy_n + \beta_n \gamma f(z_n) - Ap\|^2 \\ &+ 2\alpha_n (1-\alpha_n \bar{\gamma}) \|y_n - p\| \|(I - \beta_n B)Sy_n + \beta_n \gamma f(z_n) - Ap\| \\ &\le \|y_n - p\|^2 + {\alpha_n}^2 \|(I - \beta_n B)Sy_n + \beta_n \gamma f(z_n) - Ap\|^2 \\ &+ 2\alpha_n \|y_n - p\| \|(I - \beta_n B)Sy_n + \beta_n \gamma f(z_n) - Ap\| \\ &\le \|y_n - p\|^2 + {\alpha_n}^2 [(1-\beta_n B)\|Sy_n\| + \beta_n \gamma \|f(z_n)\| + \|Ap\|]^2 \\ &+ 2\alpha_n \|y_n - p\| [(1-\beta_n B)\|Sy_n\| + \beta_n \gamma \|f(z_n)\| + \|Ap\|] \\ &\le \|y_n - p\|^2 + {\alpha_n}^2 [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|]^2 \\ &+ 2\alpha_n \|y_n - p\| [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|] \\ &\le [\left(1 - \left(\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2}\right)\right) \|z_n - p\|]^2 \\ &+ {\alpha_n}^2 [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|]^2 \\ &+ 2\alpha_n \|y_n - p\| [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|] \\ &\le \left(1 - \left(\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2}\right)\right) \|z_n - p\|^2 \\ &+ {\alpha_n}^2 [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|]^2 \\ &+ 2\alpha_n \|y_n - p\| [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|]. \end{aligned} \tag{23}$$

So, we obtain

$$\begin{aligned} \xi \|z_n - p\|^2 &\le \left(\lambda_n - 2\sqrt{1 - 2\delta + \sigma^2}\right) \|z_n - p\|^2 \\ &\le {\alpha_n}^2 [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|]^2 \\ &+ 2\alpha_n \|y_n - p\| [\|Sy_n\| + \gamma \|f(z_n)\| + \|Ap\|]. \end{aligned}$$

Since $\alpha_n \to 0\ (n \to \infty)$,and $\{y_n\}$, $\{Sy_n\}$ and $\{f(z_n)\}$ are bounded, we derive $\|z_n - p\| \to 0\ as\ n \to \infty$. Moreover, there is no doubt that Fix(S) ∩ GVI(C, g , T) = {p}. This completes the proof. □

In terms of Remark (5.3.7) we can take $B = I, \beta = 1$ and $\beta_n = 1, \forall n \geq 0$ in Theorem (5.3.10). Then we get

Corollary (5.3.11)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient k ∈ (0,1), let T: C → H be a mapping such that $T - I : C \to H$ is α-inverse-strongly monotone, and let g: C → H be both δ-strongly monotone and σ-Lipschitz continuous. Let S be a nonexpansive self-mapping on C such that Fix(S) ∩ GVI(C, g , T) ≠ ∅.

Let A be a strong positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Let $0 < \gamma < \frac{\bar{\gamma}}{k}$ .Assume that $\{z_n\}$, is a sequence in C generated by

$$z_n = P_C\{ (I - \alpha_n A)S\, P_C\, [z_n - g(z_n) + P_C(g(z_n) - \lambda_n T z_n)] + \alpha_n \gamma f(z_n)\}$$

where $\{\alpha_n\} \subset [0,1)$ and $2\sqrt{1 - 2\delta + \sigma^2} + \xi \leq \lambda_n < 2\alpha/(1 + 2\alpha)$ for some $\xi > 0$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{z_n\}$, converges strongly to the unique element of Fix(S) ∩ GVI(C, g , T).

Corollary (5.3.12)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient k ∈ (0,1), let T : C → H be a mapping such that $T - I : C \to H$ is both μ-strongly monotone and ν-Lipschitz continuous, and let g : C → H be both δ-strongly monotone and σ-Lipschitz continuous. Let S be a nonexpansive self-mapping on C such that Fix(S) ∩ GVI(C, g , T) ≠ ∅. Let A, B be two strong positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$. Assume that $\{z_n\}$, is a sequence in C generated by

$$\begin{aligned} z_n = {} & P_C\{ (I - \alpha_n A)S\, P_C\, [z_n - g(z_n) + P_C(g(z_n) - \lambda_n T z_n)] \\ & + \alpha_n[S\, P_C[z_n - g(z_n) + P_C(g(z_n) - \lambda_n T z_n)] \\ & -\beta_n(BS\, P_C\, [z_n - g(z_n) + P_C\, (g(z_n) - \lambda_n T z_n)] - \gamma f(z_n))]\} \end{aligned}$$

Where $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $2\sqrt{1 - 2\delta + \sigma^2} + \xi \leq \lambda_n < \frac{2\mu/\nu^2}{1+2\mu/\nu^2}$ for some $\xi > 0$. if $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta - \gamma k}, \frac{2-\bar{\gamma}}{\beta - \gamma k}\right)$, Then $\{z_n\}$ converges strongly to the unique element of Fix(S) ∩ GVI(C, g, T).

Proof. Observe that $T - I : C \to H$ is μ/ν^2-inverse-strongly monotone. Moreover, it is easy to see that all conditions in Theorem (5.3.10) are satisfied. Therefore the conclusion follows immediately from Theorem (5.3.10). □

A mapping V : C → C is called strictly pseudocontractive if there exists $k \in [0,1)$ such that

$\|Vx - Vy\|^2 \le \|x - y\|^2 + k\|(I - V)x - (I - V)y\|^2, \quad \forall x, y \in C.$
If $k = 0$, then V is nonexpansive. Put $T = 2I - V$, where $V : C \to C$ is a strictly pseudocontractive mapping with k. Then $T - I$ is $\frac{1-k}{2}$ -inverse-strongly monotone. Actually, we have

$$\|(2I - T)x - (2I - T)y\|^2 \le \|x - y\|^2 + k\|(T - I)x - (T - I)y\|^2 \quad, \forall x, y \in C.$$

On the other hand, since H is a real Hilbert space, we have for all $x, y \in C$

$$\|(2I - T)x - (2I - T)y\|^2 = \|x - y\|^2 + \|(T - I)x - (T - I)y\|^2 - 2\langle x - y, (T - I)x - (T - I)y\rangle .$$

Hence we have

$$\langle x - y, (T - I)x - (T - I)y\rangle \ge \frac{1-k}{2} \|(T - I)x - (T - I)y\|^2, \qquad \forall x, y \in C.$$

Utilizing Theorem (5.3.6) we first establish a strong convergence theorem for finding a fixed point of mapping $\frac{1}{2} V$ where $V : C \to C$ is strictly pseudocontractive.

Theorem (5.3.13)[55]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$, let S be a nonexpansive self-mapping on C and let $V : C \to C$ be a strictly pseudocontractive self-mapping on C with, such that $\mathrm{Fix}(S) \cap \mathrm{Fix}(\frac{1}{2} V) \ne \emptyset$. Let A, B be two strong positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively.Let $0 < \gamma < \frac{\beta}{k}$. Suppose that $\{x_n\}$and $\{y_n\}$ are sequences in C generated by $x_0 \in C$ and

$$\begin{cases} y_n = P_C [(1 - \lambda_n)x_n + \lambda_n(V - I)x_n], \\ x_{n+1} = P_C\{(I - \alpha_n A)Sy_n + \alpha_n[Sy_n - \beta_n(BS\, y_n - \gamma f(x_n)]\}, \ \forall n \ge 0 \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $\xi \le \lambda_n < (1 - \alpha)/(2 - \alpha)$ for some $\xi > 0$. Suppose that there hold the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta-\gamma k}, \frac{2-\bar{\gamma}}{\beta-\gamma k}\right)$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1}\beta_{n+1} - \alpha_n\beta_n| < \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then both $\{x_n\}$, and$\{y_n\}$ converge strongly to the unique element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\frac{1}{2} V)$.

Proof. Put $g = I$ and $T = 2I - V$. Then $\delta = \sigma = 1$ and $T - I$ is $\frac{1-\alpha}{2}$ -inverse-strongly monotone. In this case, the condition $\xi \le \lambda_n < \frac{1-\alpha}{2-\alpha}$ is equivalent to the one $2\sqrt{1 - 2\delta + \sigma^2} + \xi \le \lambda_n < \frac{2.\frac{1-\alpha}{2}}{1+2.\frac{1-\alpha}{2}}$ .Moreover , $y_n = P_C(x_n - \lambda_n T x_n) = P_C[(1 - \lambda_n)x_n - \lambda_n(V - I)x_n]$. Note that $\mathrm{VI}(C, T) =$

$\mathrm{Fix}(\frac{1}{2}V)$ So by Theorem (5.3.6), we obtain the desired result. Utilizing Theorem (5.3.6), we also establish another strong convergence theorem for finding a zero of mapping $\mathrm{T}: \mathrm{H} \to \mathrm{H}$ with the property that $T - I$ is α-inverse-strongly monotone.

Theorem (5.3.14)[55]: Let $f : H \to H$ be a contraction with coefficient $k \in (0,1)$, let $\mathrm{T}: \mathrm{H} \to \mathrm{H}$ be a mapping such that $T - I$ is an α-inverse-strongly monotone mapping and let $S : H \to H$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap T^{-1}0 \neq \emptyset$. Let A, B be two strong positive bounded linear operators with coefficients $\bar{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Let $0 < \gamma < \frac{\beta}{k}$ .Suppose that $x_0 \in H$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = x_n - \lambda_n T x_n, \\ x_{n+1} = (I - \alpha_n A) S y_n + \alpha_n [S y_n - \beta_n (BS\, y_n - \gamma f(x_n)], \ \forall n \geq 0 \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0, \min\{1, \|B\|^{-1}\}]$ and $\xi \leq \lambda_n < 2\alpha/(1 + 2\alpha)$ for some $\xi > 0$. Assume that there hold the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\lim_{n\to\infty} \beta_n = \eta \in \left(\frac{1-\bar{\gamma}}{\beta - \gamma k}, \frac{2-\bar{\gamma}}{\beta - \gamma k}\right)$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1}\beta_{n+1} - \alpha_n \beta_n| < \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to the unique element of $\mathrm{Fix}(S) \cap T^{-1}0$.

Proof . We have $\mathrm{T}^{-1}0 = \mathrm{VI}(\mathrm{C}, \mathrm{T})$. So putting $P_H = I$, by Theorem (5.3.6),, we obtain the desired result. □

Corollary(5.3.15)[232]: Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : \mathrm{C} \to H$ be a mapping such that $T - I : \mathrm{C} \to H$ be α-inverse-strongly monotone, and let $\mathrm{g} : \mathrm{C} \to H$ be both δ-strongly monotone and σ-Lipschitz continuous. If $2\sqrt{1 - 2\delta + \sigma^2} < \lambda < 2\alpha/(1 + 2\alpha)$, then for each $x, x + \epsilon \in C$

$$\|P_C\,[x - \mathrm{g}(x) + P_C(\mathrm{g}(x) - \lambda T x)] - P_C\,[(x + \epsilon) - \mathrm{g}(x + \epsilon) + P_C(\mathrm{g}(x + \epsilon) - \lambda T x + \epsilon)]\| \leq \left[1 - \left(\lambda - 2\sqrt{1 - 2\delta + \sigma^2}\right)\right]\epsilon.$$

Proof. Utilizing the δ-strong monotonicity and σ-Lipschitz continuity of $\mathrm{g} : C \to H$, we have

$$\|x - \mathrm{g}(x) - ((x + \epsilon) - \mathrm{g}(x))\| \leq \sqrt{1 - 2\delta + \sigma^2}\,\epsilon\,, \ \forall x, x + \epsilon \in \mathrm{C}.$$

Since $2\sqrt{1 - 2\delta + \sigma^2} < \lambda < 2\alpha/(1 + 2\alpha)$, and $T - I : \mathrm{C} \to H$ is α-inverse-strongly monotone, so we obtain $\lambda - 2\alpha(1 - \lambda) < 0$ and

$$\begin{aligned} &\|(\lambda - 1)\epsilon - \lambda[(T - I)x - (T - I)(x + \epsilon)]\|^2 \\ &= (1 - \lambda)^2 \epsilon - 2\lambda(1 - \lambda)\langle (T - I)x - (T - I)(x + \epsilon), \epsilon\rangle \\ &+ \lambda^2 \|(T - I)x - (T - I)(x + \epsilon)\|^2 \\ &\leq (1 - \lambda)^2 \epsilon + \lambda\big(\lambda - 2\alpha(1 - \lambda)\big)\|(T - I)x - (T - I)(x + \epsilon)\|^2 \\ &\leq (1 - \lambda)^2 \epsilon\,, \end{aligned}$$

which implies that

$$\|(\lambda-1)\epsilon-\lambda[(T- \ I)x-(T- \ I)(x+\epsilon)]\| \leq (1-\lambda)\epsilon, \qquad \forall x,(x+\epsilon) \in \mathrm{C}.$$

Therefore, we get for each $x,(x+\epsilon) \in C$.

$$\begin{aligned}
&\|P_C\,[x- \mathrm{g}(x)+ P_C(\mathrm{g}(x)-\lambda Tx)]- P_C[(x+\epsilon)-\mathrm{g}(x+\epsilon)+ P_C(\mathrm{g}(x+\epsilon)-T(x+\epsilon))]\| \\
&\quad \leq \|x- \mathrm{g}(x)+P_C(\mathrm{g}(x)-\lambda Tx)-[(x+\epsilon)- \mathrm{g}(x+\epsilon)+ P_C(\mathrm{g}(x+\epsilon)-\lambda T(x+\epsilon))]\| \\
&\qquad\qquad \leq 2\|x- \mathrm{g}(x)-((x+\epsilon)- \mathrm{g}(x+\epsilon))\| \\
&\qquad\qquad +\|(\lambda-1)\epsilon-\lambda[(T- \ I)x \ -(T- \ I)(x+\epsilon)]\| \\
&\qquad\qquad \leq 2\sqrt{1- \ 2\delta+\sigma^2}\,\epsilon+(1-\lambda)\epsilon \\
&\qquad\qquad = \left[1-\left(\lambda-2\sqrt{1- \ 2\delta+\sigma^2}\right)\right]\epsilon
\end{aligned}$$

This completes the proof. □