Chapter 4
Maps Preserving the Harmonic Mean and Positive Operators

In this chapter let H be a complex Hilbert space. The symbol $A!B$ stands for the harmonic mean of the positive bounded linear operators A, B on H in the sense of Ando. We describe the general form of all automorphisms of the set of positive operators with respect to that operation. Let H be a complex Hilbert space. Denote by $B(H)^+$ the set of all positive bounded linear operators on H. A bijective map $\phi : B(H)^+ \to B(H)^+$ is said to preserve Lebesgue decompositions in both directions if for any quadruple A, B, C, D of positive operators, $B = C + D$ is an A-Lebesgue decomposition of B if and only if $\phi(B) = \phi(C) + \phi(D)$ is a $\phi(A)$-Lebesgue decomposition of $\phi(B)$.

Sec (4.1): Maps Preserving the Harmonic Mean or the Parallel Sum of Positive Operators

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$ Denote by $B(H)$ the algebra of all bounded linear operators on H. As usual, an operator $A \in B(H)$ is called positive if $\langle Ax, x\rangle \geq 0$ holds for every $x \in H$ and in that case we write $A \geq 0$. The set of all positive operators on H is denoted by $B(H)^+$.
In the recent paper [117] we described the structure of all bijective maps on $B(H)^+$ which preserve the geometric mean # introduced by Ando in [111]. It turned out that if dimH ≥ 2, any bijective map $\phi : B(H)^+ \to B(H)^+$ satisfying

$$\phi(A\#B) = \phi(A)\#\phi(B) \quad (A, B \in B(H)^+) \tag{1}$$

is necessarily of the form $\phi(A) = SAS^*$, $A \in B(H)^+$ with some invertible bounded linear or conjugate-linear operator S on H. The geometric mean is well-known to have important applications in operator theory but recently it has found serious applications in other areas, for example, in quantum information theory as well (see [117]). Since # is an operation which makes $B(H)^+$ an algebraic structure and there is general interest in the study of the automorphisms of algebraic structures, this has motivated us in [117] to determine the bijective maps satisfying (1).

The main aim of this section we consider the same problem for another important mean, namely for the harmonic mean of positive operators. This concept was introduced in [111] as follows. For arbitrary positive operators $A, B \in B(H)^+$, their harmonic mean $A!B$ is defined by

$$A!B = \max\left\{X \geq 0: \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\} .$$

Just as in [117], we recall that a general axiomatic theory of operator means was later developed by Kubo and Ando in [115].

For the rest of this section we list some important properties of the harmonic mean (see [111,115]). In what follows, for arbitrary self-adjoint operators $A, B \in B(H)$ we write $A \leq B$ if and only if $B - A \geq 0$.

The set of all nonnegative real numbers is denoted by $\mathbb{R}_+$. All operators appearing in the following list of properties are supposed to belong to $B(H)^+$ with the exception of S appearing in (iv).

(i) $A!B = B!A$.

(ii) For any $\lambda \in \mathbb{R}_+$, we have $(\lambda A)!(\lambda B) = \lambda(A!B)$.

(iii) If $A \leq C$ and $B \leq D$, then $A!B \leq C!D$.

(iv) (Transfer property), we have $S(A!B)S^* = (SAS^*)!(SBS^*)$ for every invertible bounded linear or conjugate-linear operator S on H.

(v) Suppose $A_1 \geq A_2 \geq \cdots \geq 0$, $B_1 \geq B_2 \geq \cdots \geq 0$ and $A_n \to A, B_n \to B$ strongly. Then we have that $A_n!B_n \to A!B$ strongly.

(vi) $A!A = A, I!A = 2A(I + A)^{-1}$ and $0!A = 0$.

(vii) $A!B = 2A(A + B)^{-1}B$ if A or B is invertible.

(viii) $A!B = 2(A^{-1} + B - 1)^{-1}$ if A and B are both invertible.

The transfer property shows that for an arbitrary invertible bounded linear or conjugate-linear operator S, the transformation $A \longmapsto SAS^*$ is a bijective map of $B(H)^+$ respecting the operation of the harmonic mean. The content of our main result is that the converse is also true: there is no other kind of transformations having this property.

Lemma (4.1.1)[104]: The operator $A \in B(H)^+$ is invertible if and only if

$$\{(\ldots((A!T_1)!T_2)!\ldots!T_n) : T_1, \ldots, T_n \in B(H)^+, n \in \mathbb{N}\} = B(H)^+. \qquad (2)$$

Proof. Let $A \in B(H)^+$ be invertible. In order to verify (2), observe that by the transfer property (iv) above there is no serious loss of generality in assuming that $A = I$. Define $f(t) = 2t/(1 + t), 0 \leq t \in \mathbb{R}$.

Then by (vi) we have

$$I!T1 = 2T_1(I + T_1)^{-1} = f(T_1) \qquad (T_1 \in B(H)^+).$$

As $f : [0, \infty[\to [0, 2[$ is a (strictly increasing) continuous bijective function, it follows that $I!T_1$ $(T_1 \in B(H)^+)$ runs through the set of all positive operators with spectrum contained in $[0, 2[$. In particular, for any real number $0 < \varepsilon < 2$ there exists a $T \in B(H)^+$ such that

$$\varepsilon I = I!T.$$

Now, from

$$(I!T)!T_2 = (\varepsilon I)!T_2 = \varepsilon(I!((1/\varepsilon)T_2))$$

it follows that $(I!T)!T_2$ runs through the set of all positive operators with spectrum contained in $[0, 2\varepsilon[$. Consequently, $(I!T_1)!T_2(T_1, T_2 \in B(H)^+)$ runs through the set of all positive operators with spectrum contained in $[0, 4[$. Continuing this process,we see that the operators

$(\ldots((A!T_1)!T_2)!\ldots!T_n)$ $(T_1,\ldots,T_n \in B(H)^+, n \in \mathbb{N})$ run through the whole set $B(H)^+$.

Next suppose that A is not invertible. We assert that in that case $A!T$ is non-invertible for every $T \in B(H)^+$. Indeed, if T is invertible, then by (vii) we have

$$A!T = 2A(A+T)^{-1}T$$

From which it is apparent that $A!T$ is non-invertible. If T is not invertible, then by (iii) for any positive $\lambda \in \mathbb{R}$,we have $A!T \leq A!(T+\lambda I)$. We already know that this latter operator is non-invertible. But this implies that $A!T$ is non-invertible, too. Consequently, we obtain that in the case when A is non-invertible, the operators $(\ldots((A!T_1)!T_2)!\ldots!T_n)(T_1,\ldots,T_n \in B(H)^+, n \in N)$ are all non-invertible. □

In the next lemma we compute the harmonic mean of an arbitrary positive operator $T \in B(H)^+$, and any rank-one projection P. To do so,we need the concept of the strength of a positive operator A along a ray represented by any unit vector in H. This concept was originally introduced by Busch and Gudder in[98] for the so-called Hilbert space effects in the place of positive operators. Effects play an important role in the mathematical foundations of the theory of quantum measurements. Mathematically, a Hilbert space effect is simply an operator $E \in B(H)$ which satisfies $0 \leq E \leq I$. Although in [98] the authors considered only effects, it is rather obvious that the following definition and result work also for arbitrary positive operators (the reason is simply that any positive operator can be multiplied by a positive scalar to obtain an effect). So, let $A \in B(H)^+$ be a positive operator, consider a unit vector φ in H and denote by P_φ the rank-one projection onto the subspace generated by φ. The quantity

$$\lambda(A,P_\varphi) = \sup\{\lambda \in \mathbb{R}_+ : \lambda P_\varphi \leq A\},$$

is called the strength of A along the ray represented by φ According to [98, Theorem 4] we have the following formula for the strength:

$$\lambda(A,P_\varphi) = \begin{cases} \left\|A^{-1/2}\varphi\right\|^{-2} & \text{if } \varphi \in \operatorname{rng}\left(A^{\frac{1}{2}}\right); \\ 0, & \text{else} \end{cases} \quad (3)$$

(The symbol rng denotes the range of operators and $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on its range).

Lemma (4.1.2)[104]: For an arbitrary positive operator $T \in B(H)^+$ and any rank-one projection P on H we have

$$T!P = \frac{2\lambda(T,P)}{\lambda(T,P)+1}P.$$

In particular, $T!P$ is nonzero if and only if $\lambda(T,P) \neq 0$ which is equivalent to $\operatorname{rng}P \subset \operatorname{rng}\sqrt{T}$.

Proof. Pick a scalar $1 \leq \lambda \in \mathbb{R}$ such that $T \leq \lambda I$. Using (ii), (iii) and (vi) one can see that

$$T!P \leq (\lambda I)!(\lambda P) = \lambda(I!P) = \lambda P.$$

As P is a rank-one operator, it follows that $T!P$ is a scalar multiple of P, i.e., we have

$$T!P = \varepsilon P \tag{4}$$

for some scalar $\varepsilon \in \mathbb{R}_+$.

We show that ε is necessarily strictly less than 2. Indeed, this follows from the inequality:

$$T!P \leq T!I = 2T(T + I)^{-1}$$

referring to the fact that the spectrum of $2T(T + 1)^{-1}$ is contained in $[0, 2[$ (see the proof of Lemma (4.1.1)).

Let $\delta \in \mathbb{R}_+$ be an arbitrary scalar such that $\delta P \leq T$. We assert that $2\delta/(1 + \delta)P \leq T!P$ holds true.

To see this, first observe that $(\delta P)!P \leq T!P$. We next compute $(\delta P)!P$. Applying (vii), for an arbitrarypositive $\lambda \in R$ we have

$$(\delta P)!(P + \lambda I) = 2(\delta P)((1 + \delta)P + \lambda I)^{-1}(P + \lambda I)$$
$$= 2(\delta P)((1 + \delta + \lambda)P + \lambda P^{\perp})^{-1}((1 + \lambda)P + \lambda P^{\perp}) = \frac{2\delta(1+\lambda)}{1+\delta+\lambda} P.$$

Here, $P^{\perp}$ denotes the projection $I - P$. Letting λ tend to 0, by (v) we infer that $(\delta P)!P = 2\delta/(1 + \delta)P$. Therefore, we have $2\delta/(1 + \delta)P \leq T!P$ as asserted.

Conversely, suppose now that $2\delta/(1 + \delta)P \leq T!P$. Then we have

$$2(\delta P)(\delta P + I)^{-1} = \frac{2\delta}{1 + \delta} P \leq T!P \leq T!I = 2T(T + I)^{-1}.$$

Setting $f(t) = 2t/(1 + t), 0 \leq t \in \mathbb{R}$,we can rewrite the above inequality as $f(\delta P) \leq f(T)$. As the inverse of the bijective function $f : [0, \infty[\to [0, 2[$ is the function $g : [0, 2[\to [0, \infty[$ defined by $g(s) = s/(2 - s) = -1 + 2/(2 - s), s \in [0, 2[$ which is clearly operator monotone, it follows that $\delta P \leq T$. Therefore, we have proved that for any $\delta \in \mathbb{R}_+$

$$\delta P \leq T \Leftrightarrow \frac{2\delta}{1 + \delta} P \leq T!P.$$

It now easily follows that for ε in (4) we have $\varepsilon = \frac{2\lambda(T,P)}{\lambda(T,P)+1}$. □

Let $B(H)_{-1}^{+}$ denote the set of all invertible positive operators on H. In the next lemma we describethe structure of all bijective maps on $B(H)_{-1}^{+}$ which preserve the arithmetic mean.

Lemma (4.1.3)[104]: Let $\psi : B(H)_{-1}^{+} \to B(H)_{-1}^{+}$ be a bijective map satisfying

$$\psi\left(\frac{A + B}{2}\right) = \frac{\psi(A) + \psi(B)}{2} \qquad (A, B \in B(H)_{-1}^{+}). \tag{5}$$

Then there exists a bounded invertible linear or conjugate-linear operator S on H such that

$$\psi(A) = SAS^{*} \qquad (A \in B(H)_{-1}^{+}).$$

Proof. We first recall that the functional equation (5) above is usually called Jensen equation. We learn from the paper [113] that every function from a nonempty $\mathbb{Q}$-

convex subset of a linear space X over $\mathbb{Q}$ into another linear space Y over $\mathbb{Q}$ satisfying the Jensen equation can be written in the form $x \longmapsto A_0 + A_1(x)$, where $A_0 \in Y$ and $A_1 : X \to Y$ is an additive function. (In fact, the main result in [113] concerns more general transformations.)

Let $B_s(H)$ denote the linear space of all self-adjoint operators in $B(H)$. By the above mentioned result it follows that there is an operator $X \in B_s(H)$ and an additive map $L : B_s(H) \to B_s(H)$ such that

$$\psi(A) = L(A) + X \quad (A \in B(H)^{+}_{-1}).$$

We assert that L is in fact a continuous linear transformation. First, we know that $L(B) \geq -X$ for every $B \in B(H)^{+}_{-1}$. It follows that for any operator $A \in B_s(H)$ with $\|A\| \leq 1/2$ we have $L(I + A) \geq -X$ implying that $L(A) \geq -L(I) - X$. Consequently, there is a negative constant $c \in \mathbb{R}$ such that

$$L(A) \geq cI$$

holds whenever $A \in B_s(H), \|A\| \leq 1/2$. Inserting $-A$ in the place of A, we get $L(-A) \geq cI$ which yields $L(A) \leq -cI$. Therefore, we obtain that

$$cI \leq L(A) \leq -cI$$

and hence $\|L(A)\| \leq |c|$ holds for every $A \in B_s(H), \|A\| \leq 1/2$. This clearly gives us that the additive map L is continuous and therefore linear.

We next prove that $X = 0$. $Let\, A \in B(H)^{+}_{-1}$ be arbitrary. For every $n \in N$ we have

$$nL(A) + X = L(nA) + X = \psi(nA) \geq 0$$

which gives us that $L(A) + (1/n)X \geq 0$. If n tends to infinity,we obtain $L(A) \geq 0$. Hencewe have $\psi(A) = L(A) + X \geq X$. Since the range of $\psi\, is\, B(H)^{+}_{-1}$, it follows that $0 \geq X$. On the other hand, by the continuity of L we deduce

$$X = X + L(0) = X + \lim_n L((1/n)I) = \lim_n \psi((1/n)I)$$

from which it follows that $X \geq 0$. Consequently, we have $X = 0$ as asserted. So, there is a continuous linear transformation $L : B_s(H) \to B_s(H)$ such that $\psi(A) = L(A), A \in B(H)^{+}_{-1}$. In the same manner there corresponds a continuous linear transformation $\acute{L} : B_s(H) \to B_s(H)$ to the transformation ψ^{-1}. Clearly,we have $\acute{L}(L(A)) = L(\acute{L}(A)) = A$ for every $A \in B(H)^{+}_{-1}$. Since $B(H)^{+}_{-1}$ linearly generates $B_s(H)$, it follows that $\acute{L}(L(A) = L(\acute{L}(A)) = A$ holds for every $A \in B_s(H)$. This shows that the transformation L is invertible and its inverse is $\acute{L}$. Next, it is easy to see that L is a bijective linear transformation of $B_s(H)$ which preserves the positive operators in both directions, i.e., $A \in B(H)^{+}$ if and only if $L(A) \in B(H)^{+}$. Indeed, as L coincides with ψ on $B(H)^{+}_{-1}$, it sends invertible positive operators to invertible positive operators. Using the continuity of L we obtain that L sends positive operators to positive operators. Applying the same argument for $\acute{L}$, it then follows that L preserves the positive operators in both directions. Now, by a well-known result of Kadison [114, Corollary 5] stating that every unital linear bijection between C*-algebras preserving positive elements in both directions is necessarily a Jordan $*$-isomorphism, we infer that L is of the form

$$L(A) = SAS^* \qquad (A \in B_s(H))$$

with some invertible bounded linear or conjugate-linear operator S on H. (We remark that a more general result concerning non-linear bijections of $B_s(H)$ preserving the order in both directions was obtained in [116].) This completes the proof of the lemma. □

Proposition(4.1.4)[104]: Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map satisfying

$$\phi(A \nabla B) = \phi(A) \nabla \phi(B) \qquad (A, B \in B(H)^+).$$

Then there exists an invertible bounded linear or conjugate-linear operator S on H such that ϕ is of the form

$$\phi(A) = SAS^* \quad (A \in B(H)^+).$$

Proof . A simplified version of the argument given above applies to verify the statement. □

Now we are in a position to prove the main result of this section.

Theorem(4.1.5)[104]: Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map satisfying

$$\phi(A!B) = \phi(A)!\phi(B) \qquad (A, B \in B(H)^+).$$

Then there is an invertible bounded linear or conjugate-linear operator S on H such that ϕ is of the form

$$\phi(A) = SAS^* \qquad (A \in B(H)^+).$$

There is a concept closely related to the harmonic mean called parallel sum. For arbitrary positive operators $A, B \in B(H)^+$, their parallel sum $A : B$ is expressed as

$$A : B = \frac{1}{2}(A!B).$$

This notion originally defined by Anderson and Duffin [108] in a different way has many important applications in operator theory and in electrical network theory, too. The reason of these latter applications is the following: if A, B are impedance matrices of a resistive n-port network, then their parallel sum $A : B$ is just the impedance matrix of the parallel connection [107]. For the most classical results concerning this operation we refer to the papers [107–110]. As an easy corollary of our main result we shall obtain the following description of bijective maps preserving the parallel sum.

Proof . Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map preserving the harmonic mean .By Lemma (4.1.1), ϕ preserves the invertible operators in both directions. is invertible if and only if $\phi(A)$ is invertible. Therefore, $\phi(I)$ is an invertible positive operator. Considering the transformation

$$A \longmapsto \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2},$$

by the transfer property we obtain a bijective map on $B(H)^+$ which preserves the harmonic mean and sends I to I. Hence, there is no serious loss of generality in assuming that already ϕ has the property that $\phi(I) = I$.

We next prove that ϕ preserves the projections in both directions. To see this, consider the following simple characterization of projections. The operator $A \in B(H)^+$ is a projection if and only if $A!I = A$. Indeed, by (vi) we know that $I!A = 2(I + A)^{-1}A$. This implies that $I!A = A$ if and only if $2A = (I + A)A = A + A^2$ which is trivially equivalent to $A = A^2$. By the properties of ϕ we infer that ϕ indeed preserves the projections in both directions.

Proposition 2 in [112] tells us that for any projections P,Q we have $\mathrm{P}!\mathrm{Q} = \mathrm{P} \wedge \mathrm{Q}$. This implies that ϕ is a lattice-automorphism on the set of all projections on H. Consequently, ϕ sends 0 to 0 and it preserves the rank-one and also the corank-one projections in both directions.

Our next aim is to show that ϕ preserves the rank-one elements of $B(H)^+$ in both directions. Indeed, this follows from the following characterization of rank-one operators. The nonzero element $A \in B(H)^+$ is of rank-one if and only if there exists a corank-one projection Q such that $\mathrm{A}!\mathrm{Q} = 0$. To see this, first suppose that A is of rank-one. Then there is a rank-one projection P and a positive scalar λ such that $A = \lambda P$. By (ii) and Lemma (4.1.2), one can see that $A!(I - P) = 0$. Conversely, if $A!Q = 0$ for some projection Q of corank-one, then by (iii) we have $A!R = 0$ for every rank-one subprojection of Q. But by Lemma (4.1.2) again this implies that $\mathrm{rngQ} \cap \mathrm{rng}\sqrt{\mathrm{A}} = \{0\}$.We then infer that $\sqrt{\mathrm{A}}$ and hence A are both of rank-one.

Now let P be a rank-one projection. We assert that for every $\lambda \in \mathbb{R}_+$ there is a nonnegative scalar $f(\lambda)$ such that $\phi(\lambda P) = f(\lambda)\phi(P)$. In fact, we know that $\phi(\lambda P)$ is of rank-one. Moreover, by Lemma (4.1.2) we have $(\lambda P)!P \neq 0$ and hence

$$\phi(\lambda P)!\phi(P) = \phi((\lambda P)!P) \neq 0.$$

By Lemma (4.1.2) again, we deduce that the range of the rank-one projection $\phi(P)$ has nontrivial intersection with the range of the rank-one operator $\phi(\lambda P)$. This implies that $\phi(\lambda P)$ is a scalar multiple of $\phi(P)$ and hence there is a nonnegative scalar $f(\lambda)$ such that $\phi(\lambda P) = f(\lambda)\phi(P)$. As ϕ and ϕ^{-1} share the same properties, it is easy to verify that f is a bijection of $\mathbb{R}_+$sending 0 to 0.

Let λ and μ be positive real numbers. By Lemma (4.1.2) we compute

$$(\lambda P)!(\mu P) = \mu(((\lambda/\mu)P)!P) = \mu\frac{2\lambda/\mu}{\lambda/\mu + 1}P = \frac{2\lambda\mu}{\lambda + \mu}P.$$

Similarly, we obtain

$$(f(\lambda)\phi(P))!(f(\mu)\phi(P)) = \frac{2f(\lambda)f(\mu)}{f(\lambda) + f(\mu)}\phi(P).$$

As

$$\phi((\lambda P)!(\mu P)) = \phi(\lambda P)!\phi(\mu P) = (f(\lambda)\phi(P))!(f(\mu)\phi(P)),$$

It follows that

$$f\left(\frac{2\lambda\mu}{\lambda + \mu}\right)\phi(P) = \frac{2f(\lambda)f(\mu)}{f(\lambda) + f(\mu)}\phi(P).$$

Consequently, we obtain that f satisfies the functional equation

$$f\left(\frac{2\lambda\mu}{\lambda + \mu}\right) = \frac{2f(\lambda)f(\mu)}{f(\lambda) + f(\mu)} \qquad (0 < \lambda, \mu \in \mathbb{R}).$$

This means simply that f is a bijection of the set of all positive real numbers satisfying $f(\lambda ! \mu) = f(\lambda) ! f(\mu)$ for all $0 < \lambda, \mu$. Defining $\mathrm{g(t)} = 1/\mathrm{f}(1/\mathrm{t})$ for any $0 < t \in \mathbb{R}$, it is easy to check that g is a bijective function of the set of all positive real numbers satisfying the Jensen equation. As a very particular case of Lemma (4.1.3) we obtain that g as well as f is a positive scalar multiple of the identity. As we have $f(1) = 1$, it follows that f is in fact the identity on $\mathbb{R}_+$ .Hence we have proved that $\phi(\lambda P) = \lambda\phi(P)$ holds for every rank-one projection P and scalar $\lambda \in \mathbb{R}_+$.

After these preliminaries now the proof can be completed as follows. Similarly to the case of the scalar function f above, define a bijective transformations ψ on $B(H)_{-1}^{+}$ by

$$\psi(A) = \phi(A^{-1})^{-1} \qquad (A \in B(H)_{-1}^{+}).$$

Using the formula $A!B = 2(A^{-1} + B^{-1})^{-1}$ for invertible operators $A, B \in B(H)^{+}$, it is easy to verify that ψ satisfies the Jensen equation

$$\psi((A + B)/2) = (\psi(A) + \psi(B))/2 \qquad (A, B \in B(H)_{-1}^{+}).$$

Then Lemma (4.1.3) applies and we obtain that there exists an invertible bounded linear or conjugate-linear operator S on H such that

$$\psi(A) = SAS^{*} \qquad (A \in B(H)_{-1}^{+}).$$

As we have supposed that $\phi(I) = I$, it follows that $SS^{*} = I$, i.e., S is either a unitary or an antiunitary operator. Denote it by U.We easily obtain that

$$\phi(A) = UAU^{*} \qquad (A \in B(H)_{-1}^{+}).$$

Therefore, considering the transformation

$$A \mapsto U^{*}\phi(A)U,$$

We can further assume that $\phi(A) = A$ holds for every $A \in B(H)_{-1}^{+}$.

We next prove that ϕ is the identity on the rank-one projections. Let P be a rank-one projection. Pick an arbitrary operator $A \in B(H)_{-1}^{+}$. By Lemma (4.1.2) we have

$$\frac{2\lambda(A,P)}{\lambda(A,P)+1}\phi(P) = \phi\left(\frac{2\lambda(A,P)}{\lambda(A,P)+1}P\right)$$

$$= \phi(A!P) = \phi(A)!\phi(P) = A!\phi(P) = \frac{2\lambda(A,\phi(P))}{\lambda(A,\phi(P))+1}\phi(P).$$

It follows that

$$\lambda(A,P) = \lambda(A,\phi(P)) \qquad (6)$$

holds for every invertible operator $A \in B(H)^{+}_{-1}$. Moreover, for such operator A and arbitrary unit vector $\varphi \in H$, applying (3) we compute

$$\lambda(A, P_{\varphi}) = \left\| A^{-\frac{1}{2}}\varphi \right\|^{-2} = \frac{1}{\langle A^{-\frac{1}{2}}\varphi, A^{-\frac{1}{2}}\varphi \rangle} = \frac{1}{\langle A^{-1}\varphi, \varphi \rangle} = \frac{1}{\mathrm{tr}A^{-1}P_{\varphi}} .$$

Therefore, from (6) we deduce that

$$\mathrm{tr}A^{-1}P = \mathrm{tr}A^{-1}\phi(P)$$

holds for every $A \in B(H)^{+}_{-1}$. As $B(H)^{+}_{-1}$ linearly generates $B(H)$, we obtain the equality

$$\mathrm{tr}TP = \mathrm{tr}T\phi(P)$$

for every bounded operator T on H. Inserting rank-one projections into the place of T we conclude that

$$\langle x, Px \rangle = \langle x, \phi(P)x \rangle$$

holds for every unit vector x in H. This gives us that $\phi(P) = P$ is valid for any rank-one projection P implying that ϕ is the identity on the set of all rank-one operators in $B(H)^{+}$.

It remains to show that $\phi(A) = A$ holds for every $A \in B(H)^{+}$ To see this, let $A \in B(H)^{+}$ be arbitrary.

Take any rank-one projection P on H. Applying Lemma (4.1.2), from the equalities

$$A!P = \phi(A!P) = \phi(A)!\phi(P) = \phi(A)!P$$

We infer that

$$\frac{2\lambda(A,P)}{\lambda(A,P) + 1}P = \frac{2\lambda(\phi(A),P)}{\lambda(\phi(A),P) + 1}P.$$

This implies that $\lambda(A,P) = \lambda(\phi(A),P)$ holds for every rank-one projection P on H. since every positive operator is uniquely determined by its strength function [98, Corollary 1], we obtain that $\phi(A) = A$. This completes the proof of the theorem.

We conclude this section with the simple proof of the corollary.

Corollary (4.1.6)[104]: Let $\phi : B(H)^{+} \rightarrow B(H)^{+}$ be a bijective map satisfying

$$\phi(A : B) = \phi(A) : \phi(B) \qquad (A, B \in B(H)^{+}).$$

Then ϕ respects the operation of the harmonic mean . Consequently, there exists an invertible bounded linear or conjugate-linear operator S on H such that ϕ is of the form

$$\phi(A) = SAS^{*} \qquad (A \in B(H)^{+}).$$

Finally, to make our investigation more complete we conclude with the following rather simple result concerning the structure of bijective maps of $B(H)^{+}$ preserving the arithmetic mean ∇. This operation is defined by

$$A\nabla B = \frac{1}{2}(A + B)$$

for all $A, B \in B(H)^{+}$.

Proof . Let $A \in B(H)^{+}$. Using (vi) we get

$$\phi((1/2)A) = \phi(A : A) = \phi(A) : \phi(A) = (1/2)\phi(A).$$

This implies that $\phi(2A) = 2\phi(A)$ for every $A \in B(H)^{+}$. Therefore, we have

$\phi(A!B) = \phi(2(A : B)) = 2\phi(A : B) = 2(\phi(A) : \phi(B)) = \phi(A)!\phi(B)$ for all $A, B \in B(H)^+$. This shows that ϕ preserves the harmonic mean. An application of Theorem completes the proof. □

Corollary(4.1.7)[232]: Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map satisfying

$$\phi\big((A_1 + A_2 + \cdots + A_n) : B\big) = \phi\big((A_1 + A_2 + \cdots + A_n)\big) : \phi(B) \qquad ((A_1 + A_2 + \cdots + A_n), B \in B(H)^+).$$

Then ϕ respects the operation of the harmonicmean. Consequently, there exists an invertible bounded linear or conjugate-linear operator S on H such that ϕ is of the form

$$\phi(A_1 + A_2 + \cdots + A_n) = S(A_1 + A_2 + \cdots + A_n)S^* \qquad ((A_1 + A_2 + \cdots + A_n) \in B(H)^+).$$

Finally, to make our investigation more complete we conclude with the following rather simple result concerning the structure of bijective maps of $B(H)^+$ preserving the arithmetic mean ∇. This operation is defined by

$$(A_1 + A_2 + \cdots + A_n)\nabla B = \frac{1}{2}((A_1 + A_2 + \cdots + A_n) + B)$$

for all $(A_1 + A_2 + \cdots + A_n), B \in B(H)^+$.

Proof . Let $(A_1 + A_2 + \cdots + A_n) \in B(H)^+$. Using (vi) we get

$$\begin{aligned}\phi\left(\left(\tfrac{1}{2}\right)(A_1 + A_2 + \cdots + A_n)\right) &= \phi\big((A_1 + A_2 + \cdots + A_n) : (A_1 + A_2 + \cdots + A_n)\big)\\ &= \phi(A_1 + A_2 + \cdots + A_n) : \phi(A_1 + A_2 + \cdots + A_n)\\ &= (1/2)\phi(A_1 + A_2 + \cdots + A_n).\end{aligned}$$

This implies that $\phi(2(A_1 + A_2 + \cdots + A_n)) = 2\phi((A_1 + A_2 + \cdots + A_n))$ for every $(A_1 + A_2 + \cdots + A_n) \in B(H)^+$. Therefore, we have

$$\begin{aligned}\phi\big((A_1 + A_2 + \cdots + A_n)!B\big) &= \phi\left(2\big((A_1 + A_2 + \cdots + A_n) : B\big)\right)\\ &= 2\phi\big((A_1 + A_2 + \cdots + A_n) : B\big) = 2\big(\phi(A_1 + A_2 + \cdots + A_n) : \phi(B)\big)\\ &= \phi(A_1 + A_2 + \cdots + A_n)!\phi(B)\end{aligned}$$

for all $(A_1 + A_2 + \cdots + A_n), B \in B(H)^+$. This shows that ϕ preserves the harmonic mean. An application of Theorem completes the proof. □

Corollary(4.1.8)[232]: For an arbitrary self-adjoint positive operator $T^* \in B(H)^+$ and any rank-one projection P^n on H we have

$$T^*!P^n = \frac{2\lambda(T^*, P^n)}{\lambda(T^*, P^n) + 1}P^n.$$

In particular, $T^*! P^n$ is nonzero if and only if $\lambda(T^*, P^n) \neq 0$ which is equivalent to rng$P^n \subset$ rng$\sqrt{T}$.

Proof. Pick a scalar $1 \leq \lambda \in \mathbb{R}$ such that $T^* \leq \lambda I$. Using (ii), (iii) and (vi) one can see that

$$T^*! P^n \leq (\lambda I)! (\lambda P^n) = \lambda(I! P^n) = \lambda P^n.$$

As P^n is a rank-one operator, it follows that $T^*! P^n$ is a scalar multiple of P^n, i.e., we have

$$T^*! P^n = \varepsilon P^n$$

for some scalar $\varepsilon \in \mathbb{R}_+$.

We show that ε is necessarily strictly less than 2. Indeed, this follows from the inequality:

$$T^*! P^n \leq T^*! I = 2T(T^* + I)^{-1}$$

referring to the fact that the spectrum of $2T^*(T^* + 1)^{-1}$ is contained in $[0, 2[$ (see the proof of Lemma (4.1.1)).

Let $\delta \in \mathbb{R}_+$ be an arbitrary scalar such that $\delta P^n \leq T^*$. We assert that $2\delta/(1 + \delta)P^n \leq T^*! P^n$ holds true.

To see this, first observe that $(\delta P^n)! P^n \leq T^*! P^n$. We next compute $(\delta P^n)! P^n$. Applying (vii), for an arbitrarypositive $\lambda \in R$ we have

$$(\delta P^n)! (P^n + \lambda I) = 2(\delta P^n)((1 + \delta)P^n + \lambda I)^{-1}(P^n + \lambda I)$$
$$= 2(\delta P^n)((1 + \delta + \lambda)P^n + \lambda(P^n)^{\perp})^{-1}((1 + \lambda)P^n + \lambda(P^n)^{\perp}) = \frac{2\delta(1+\lambda)}{1+\delta+\lambda} P^n.$$

Here, $(P^n)^{\perp}$ denotes the projection $I - P^n$. Letting λ tend to 0, by (v) we infer that $(\delta P^n)! P^n = 2\delta/(1 + \delta)P^n$. Therefore, we have $2\delta/(1 + \delta)P^n \leq T^*! P^n$ as asserted.

Conversely, suppose now that $2\delta/(1 + \delta)P^n \leq T^*! P^n$. Then we have

$$2(\delta P^n)(\delta P^n + I)^{-1} = \frac{2\delta}{1 + \delta} P^n \leq T^*! P^n \leq T^*! I = 2T^*(T^* + I)^{-1}.$$

Setting $f(t) = 2t/(1 + t), 0 \leq t \in \mathbb{R}$,we can rewrite the above inequality as $f(\delta P^n) \leq f(T^*)$. As the inverse of the bijective function $f : [0, \infty[\to [0, 2[$ is the function $\mathrm{g} : [0, 2[\to [0, \infty[$ defined by $\mathrm{g(s)} = \mathrm{s}/(2 - \mathrm{s}) = -1 + 2/(2 - \mathrm{s}), \mathrm{s} \in [0, 2[$ which is clearly operator monotone, it follows that $\delta P^n \leq T^*$. Therefore, we have proved that for any $\delta \in \mathbb{R}_+$

$$\delta P^n \leq T^* \Leftrightarrow \frac{2\delta}{1 + \delta} P^n \leq T^*! P^n.$$

It now easily follows that for ε in (4) we have $\varepsilon = \frac{2\lambda(T^*, P^n)}{\lambda(T^*, P^n)+1}$. □

Sec (4.2): Maps on Positive Operators Preserving Lebesgue Decompositions

In what follows, H denotes a complex Hilbert space with inner product $\langle .,. \rangle$and $B(H)$ stands for the algebra ofall bounded linear operators on H. The space of all self-adjoint elements of $B(H)$ is denoted by $B_s(H)$. An operator $A \in B(H)$ is called positive if $\langle Ax, x\rangle \geq 0$ holds for every $x \in H$ in which case we write $A \geq 0$. (Observe that we use the expression "positive" in the operator algebraic sense. For matrices, this is the same as positive semi-definiteness.) The set of all positive elements of $B(H)$ is denoted by $B(H)^+$.
The usual order among self-adjoint operators is defined by means of positivity as follows. For any $T, S \in B_s(H)$, we write $T \leq S$ if $S - T \geq 0$.

In analogy with the Lebesgue decomposition of positive measures, in [97], Ando defined a Lebesgue-type decomposition of positive operators, a concept which has proved to be very useful in operator theory. To explain that decomposition we need the following notions (for details, see [97]).

Given a positive operator $A \in B(H)^+$, the positive operator $\mathrm{C} \in B(H)^+$is said to be A-absolutely continuous if there is a sequence $(\mathrm{C_n})$ of positive operators and a sequence (α_n) of nonnegative real numbers such that $\mathrm{C_n} \uparrow \mathrm{C}$ and $\mathrm{C_n} \leq \alpha_n A$ for every n. Here, $\mathrm{C_n} \uparrow C$ means that the sequence $(\mathrm{C_n})$ is monotone increasing with respect to the usual order and it strongly converges to C. A positive operator C is called A-singular if for any $D \in B(H)^+$, the inequalities $D \leq A$ and $D \leq C$ imply $D = 0$. Now, for any pair $A, B \in B(H)^+$ of positive operators, by an A-Lebesgue decomposition of B we mean a decomposition $B = C + D$ where C, D are positive operators, C is A-absolutely continuous and D is A-singular. Ando proved in [97] that such decomposition exists for every pair A, B of positive operators.

In this section, we study the problem of characterizing maps on positive operators which preserve Lebesgue decompositions. Investigations of this kind, i.e., the study of maps on different structures preserving important operations, quantities, relations, etc. corresponding to the underlying structures belong to the gradually enlarging field of so-called preserver problems. For important surveys on preservers in the classical sense, we refer to [99,101,102,106]. For recent results concerning preservers in extended sense and defined on more general domains (especially on operator structures), we refer to [103] and its bibliography.

We say that the bijective map $\phi : B(H)^+ \to B(H)^+$ preserves Lebesgue decompositions in both directions if it has the following property. For any quadruple A, B, C, D of positive operators, $B = C + D$ is an A-Lebesgue decomposition of B if and only if $\phi(B) = \phi(C) + \phi(D)$ is a $\phi(A)$-Lebesgue decomposition of $\phi(B)$. It is rather clear from the definitions that any transformation of the form $A \longmapsto SAS^*$ for some invertible bounded linear or conjugate-linear operator S on H preserves Lebesgue decompositions in both directions. The aim of this section we show that the reverse statement is also true: only transformations of this form have the above preserver property.

Theorem (4.2.1)[96]: Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map preserving Lebesgue decompositions in both directions. Then there is an invertible bounded linear or conjugate-linear operator S on H such that ϕ is of the form

$$\phi(A) = SAS^* \qquad (A \in B(H)^+).$$

Proof. This section is devoted to the proof of the theorem. First we recall some of the results in [97] that we shall use in our arguments. In what follows, rng stands for the range of operators. Let $A, B \in B(H)^+$.

(A1) The operator B is A-singular if and only if $\text{rng}A^{1/2} \cap \text{rng}B^{1/2} = \{0\}$ (see [97, p. 256]).

(A2) The operator B is A-absolutely continuous if and only if the subspace $\{x \in H : B^{1/2}x \in \text{rng}A^{1/2}\}$ is dense in H (see [97, Theorem 5]).

(A3) The operator B has an A-Lebesgue decomposition, which can be constructed in the following way. Define $[A]B = \lim_n (nA) : B$. Here : denotes the operation of parallel sum of positive operators. The sequence $((nA) : B)_n$ of positive operators is monotone increasing and bounded by B from above. Hence, $[A]B$ is a well-defined positive bounded linear operator on H. Now, according to Theorem 2 in [97], we have that

$$B = [A]B + (B - [A]B),$$

is an A-Lebesgue decomposition of B and, further, $[A]B$ is the maximum of all A-absolutely continuous positive operators C with $C \leq B$.

(A4) A-Lebesgue decomposition is not unique in general. Namely, according to [97, Corollary 7], for a given $A \in B(H)^+$, every positive operator admits a unique A-Lebesgue decomposition if and only if $\text{rng}A$ is closed.

We begin the route leading to the proof of the theorem with the following simple lemma.

Lemma (4.2.2)[96]: Let $A \in B(H)^+$. The range $\text{rng}A$ of A is closed if and only if $\text{rng}A^{1/2}$ is closed, and in this case, we have $\text{rng}A = \text{rng}A^{1/2}$.

Proof. It is clear that

$$\text{rng}A \subset \text{rng}A^{1/2} \subset \overline{\text{rng}A^{1/2}} = \overline{\text{rng}\,A},$$

where the last equality follows from the easy fact that $\ker A^{1/2} = \ker A$. Therefore, we see that if $\text{rng}A$ is closed, then so is $\text{rng}A^{1/2}$ and they coincide. Conversely, if $\text{rng}A^{1/2}$ is closed then we have

$$A(H) = A^{1/2}(A^{1/2}(H)) = A^{1/2}\overline{(\text{rng}A^{1/2})} = A^{1/2}((\ker A^{1/2})^{\perp}) = \text{rng}A^{1/2}.$$

The proof is complete. □

By (A2), we immediately have the following.

Corollary (4.2.3)[96]: Let $A, B \in B(H)^+$ be operators with closed ranges. Then B is A-absolutely continuous if and only if rngB $\subset$ rngA. Therefore, we have rngB $=$ rngA if and only if B is A-absolutely continuous and A is B-absolutely continuous.

In the proof of our theorem, we need the following additional corollary which gives a characterization of invertibility of positive operators.

Corollary (4.2.4)[96]: Let $A \in B(H)^+$. Then A is invertible if and only if rngA is closed and for every $B \in B(H)^+$ with closed range, we have that B is A-absolutely continuous.

In the next lemma, we compute the Lebesgue decomposition of an arbitrary positive operator with respect to a rank-one element of $B(H)^+$ (recall that by $(A4)$, in this case, we have unique Lebesgue decomposition). To do so, we need the concept of the strength of a positive operator A along a ray represented by a unit vector in H. This concept was originally introduced by Busch and Gudder in [98] for the so-called Hilbert space effects in the place of positive operators. Effects play a basic role in the mathematical foundations of the theory of quantum measurements. Mathematically, a Hilbert space effect is simply an operator $E \in B(H)$ that satisfies $0 \leq E \leq I$. Although in [98] the authors considered only effects, it is rather obvious that the following definition and result work also for arbitrary positive operators (the reason is simply that any positive operator can be multiplied by a positive scalar to obtain an effect). So, let $A \in B(H)^+$, consider a unit vector φ in H and denote by P_φ the rank-one projection onto the subspace generated by φ The quantity

$$\lambda(A, P_\varphi) = \sup\{\lambda \in \mathbb{R}_+ : \lambda P_\varphi \leq A\}$$

is called the strength of A along the ray represented by φ. ($\mathbb{R}_+$ stands for the set of all non-negative real numbers.) According to [98, Theorem 4], we have the following formula for the strength:

$$\lambda(A, P_\varphi) = \begin{cases} \left\| A^{-\frac{1}{2}}\varphi \right\|^{-2}, \text{if } \varphi \in \operatorname{rng}\left(A^{-\frac{1}{2}}\right) \\ 0, \quad \text{else} \end{cases} \qquad (7)$$

where $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on its range.

Clearly, every positive rank-one operator can be written in the form μP, where P is a rank-one projection and μ is a positive real number.

Lemma (4.2.5)[96]: Let P be a rank-one projection, μ a positive real number and B an arbitrary positive operator. Then we have

$$[\mu P]B = \lambda(B, P)P.$$

Therefore, the (μP)-Lebesgue decomposition of B is

$$B = \lambda(B, P)P + (B - \lambda(B, P)P).$$

In particular, the (μP)-Lebesgue decomposition of I is

$$I = P + (I - P).$$

Proof. In paper [104], we presented structural results for the automorphisms of $B(H)^+$ with respect to the operation of the harmonic mean or that of the parallel sum. We recall that the harmonic mean $T\,!\,S$ of the positive operators T, S is the double of their parallel sum $T : S$. In [104, Lemma 2] we proved that for any $T \in B(H)^+$ and rank-one projection P, we hav e

$$T\,!\,P = \frac{2\lambda(T,P)}{\lambda(T,P) + 1}P.$$

Using this, we compute

$$[\mu P]B = \lim_n (n\mu P) : B = \lim_n \frac{(n\mu P)!\,B}{2}$$

$$= \lim_n \frac{B!\,(n\mu P)}{2} = \lim_n n\mu \frac{(B/(n\mu))!\,P}{2} = \lim_n n\mu \frac{\lambda(B/(n\mu),P)}{\lambda(B/(n\mu),P) + 1}P \qquad (8)$$

$$= \lim_n n\mu \frac{(1/(n\mu))\lambda(B,P)}{(1/(n\mu))\lambda(B,P) + 1}P = \lambda(B,P)P.$$

Here, we use the following properties of the harmonic mean and the strength function: for any $T, S \in B(H)^+$, rank-one projection P, and nonnegative number α, we have

$$T\,!\,S = S!\,T,\ (\alpha T)!\,(\alpha S) = \alpha(T\,!\,S),\ \lambda(\alpha T,P) = \alpha\lambda(T,P). \quad \square$$

In the proof of our theorem, the solution of the following functional equation will play an important role.

Lemma (4.2.6)[96]: Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a bijective function with $f(0) = 0$, and $\varphi : [0,1] \to [0,1]$ be a function such that

$$f\left(\frac{1}{(1/\lambda)\alpha + (1/\mu)(1-\alpha)}\right) = \frac{1}{(1/f(\lambda))\varphi(\alpha) + (1/f(\mu))(1-\varphi(\alpha))}. \qquad (9)$$

holds for every $0 < \lambda, \mu \in \mathbb{R}$ and $\alpha \in [0,1]$. If $f(1) = 1$, then f, φ are the identities on their domains.

Proof . First choose $\alpha = 1/2$. For $\acute{\alpha} = \varphi\left(\frac{1}{2}\right)$ *and* $\acute{\beta} = 1 - \acute{\alpha}$, we have

$$f\left(\frac{2}{(1/\lambda) + (1/\mu)}\right) = \frac{1}{(1/f(\lambda))\acute{\alpha} + (1/f(\mu))\acute{\beta}}. \qquad (10)$$

Define $g(\lambda) = 1/f(1/\lambda)$ for every positive λ. Then g is a bijection of the set of all positive real numbers, and (10) turns into

$$\frac{1}{g\left(\frac{(1/\lambda) + (1/\mu)}{2}\right)} = \frac{1}{g(1/\lambda)\acute{\alpha} + g(1/\mu)\acute{\beta}}.$$

Therefore, we have that

$$g\left(\frac{\lambda + \mu}{2}\right)2 = g(\lambda)\acute{\alpha} + g(\mu)\acute{\beta}$$

holds for all positive numbers λ, μ. Interchanging λ and μ, we get

$$g(\mu)\acute{\alpha} + g(\lambda)\acute{\beta} = g(\lambda)\acute{\alpha} + g(\mu)\acute{\beta}$$

$0 < \lambda, \mu$. Since g is injective, we infer that $\acute{\alpha} = \acute{\beta}$ and then it follows that $\acute{\alpha} = \acute{\beta} = 1/2$. Thus, we obtain that g satisfies the so-called Jensen equation

$$g\left(\frac{\lambda + \mu}{2}\right) = \frac{g(\lambda) + g(\mu)}{2}$$

on the set of all positive real numbers. From [100] we learn that every real-valued function defined on a convex subset of $\mathbb{R}^n$ with nonempty interior which satisfies the Jensen equation can be written as the sum of a real-valued additive function defined on the whole $\mathbb{R}^n$ and a real constant. This gives us that there exist an additive function $a : \mathbb{R} \to \mathbb{R}$ and a constant $d \in \mathbb{R}$ such that $g(\lambda) = a(\lambda) + d$ holds for every positive λ. As g takes only positive values, it follows that a is bounded from below on the set of positive real numbers. It is a classical result of Ostrowski from 1929 [105] that any additive function of $\mathbb{R}$ that is bounded from one side on a set of positive measure is necessarily a constant multiple of the identity. Hence, we have a constant $c \in \mathbb{R}$ such that $a(\lambda) = c\lambda$ for every $\lambda \in \mathbb{R}$. As g is a self-bijection of the set of all positive numbers with $g(1) = 1$, one can easily verify that $c = 1$ and $d = 0$. Clearly, this implies that f is the identity on $\mathbb{R}_+$. Finally, it immediately follows from (9) that φ is the identity on $[0, 1]$. □

After this preparation, we are now in a position to prove Theorem (4.2.1).

Proof . Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map which preserves Lebesgue decompositions in both directions.

First we show that ϕ sends 0 to 0. Indeed, $0 = 0 + 0$ is a 0-Lebesgue decomposition of 0. This implies that $\phi(0) = \phi(0) + \phi(0)$ is a $\phi(0)$-Lebesgue decomposition of $\phi(0)$. We have $\phi(0) = 0$.

Next, we assert that ϕ preserves absolutely continuity in both directions. This means that for any pair A, B of positive operators, B is A-absolutely continuous if and only if $\phi(B)$ is $\phi(A)$-absolutely continuous. In fact, this follows from the preserver property of ϕ and from the easy fact that B is A-absolutely continuous if and only if $B = B + 0$ is an A-Lebesgue decomposition of B. In a similar way, one can check that ϕ preserves singularity in both directions.

By the criterion (A4) of uniqueness of Lebesgue decompositions, ϕ preserves the elements of $B(H)^+$ with closed range in both directions. This means that the operator $A \in B(H)^+$ has closed range if and only if ϕ has closed range.

Now, by Corollary (4.2.3), for arbitrary operators $A, B \in B(H)^+$with closed ranges, we have

$$\operatorname{rng} B \subset \operatorname{rng} A \Leftrightarrow \operatorname{rng} \phi(B) \subset \operatorname{rng} \phi(A),$$

and hence,

$$\text{rng}B = \text{rng}A \Leftrightarrow \text{rng}\,\phi(B) = \text{rng}\phi(A).$$

We next prove that ϕ preserves the rank of finite rank operators. In fact, this follows from the following characterization of the rank. The positive operator A is of rank n $(1 \leq n \in N)$ if and only if it has closed range, there exists a sequence $A_0, A_1, \ldots, A_{n-1}$ of positive operators with closed range of length n such that

$$\text{rng}A_0 \subsetneq \text{rng}A_1 \subsetneq \ldots \not\subseteq \text{rng}A_{n-1} \subsetneq \text{rng}A$$

and there is no similar sequence of length $n + 1$. The already verified properties of ϕ imply that ϕ preserves the rank.

As ϕ preserves the positive operators with closed range in both directions, by Corollary (4.2.4), we obtain that ϕ preserves the invertible elements of $B(H)^+$ in both directions. Therefore, $\phi\,(I)$ is an invertible positive operator. Consider the transformation

$$A \longmapsto \phi\,(I)^{-1/2}\phi\,(A)\phi\,(I)^{-1/2}\,. \qquad (11)$$

Referring to the already mentioned fact that any transformation of the form $A \longmapsto SAS^*$ with some invertible bounded linear or conjugate-linear operator S on H preserves Lebesgue decompositions in both directions, we see that the transformation in (11) is a bijective map on $B(H)^+$ which has the same preserver property and, in addition, it sends I to I. Hence, there is no serious loss of generality in assuming that already ϕ satisfies $\phi(I) = I$.

We prove that ϕ preserves the rank-one projections in both directions. Let P be a rank-one projection. By Lemma (4.2.5), the P-Lebesgue decomposition of I is $I = P + (I - P)$. As ϕ preserves the rank, $\phi(P)$ is a rank-one operator. Hence, we have $\phi(P) = \mu Q$ with some rank-one projection Q and positive number μ. Now, on one hand, by the original preserver property of ϕ, the $\phi(P)$-Lebesgue decomposition of $\phi(I)$ is

$$I = \phi(I) = \phi(P) + \phi(I - P).$$

But on the other hand, by Lemma (4.2.5), the (μQ)-Lebesgue decomposition of I is

$$I = Q + (I - Q).$$

Using the uniqueness of the Lebesgue decomposition with respect to positive operators having closed range, we obtain that $\phi(P) = Q$ and $\phi(I - P) = I - Q = I - \phi(P)$. Consequently, $\phi(P)$ is a rank-one projection, and we also have $\phi(I - P) = I - \phi(P)$.

We show that ϕ preserves the orthogonality among rank-one projections. Let P, Q be orthogonal rank-one projections. It is easy to see that the Q-Lebesgue decomposition of $(I - P)$ *is* $I - P = Q + (I - P - Q)$. Therefore, we have $\phi(I - P) = \phi(Q) + \phi(I - P - Q) \geq \phi(Q)$. But from the previous paragraph of the proof we know that $\phi(I - P) = I - \phi(P)$. Therefore, we obtain $I - \phi(P) \geq \phi(Q)$, which means that the projections $\phi(P)$ and $\phi(Q)$ are orthogonal to each other.

We assert that for any rank-one projection P, we have a bijective function f_P on $\mathbb{R}_+$such that $\phi(\lambda P) = f_P(\lambda)\phi(P)$. This follows from the fact that for any positive λ, the ranges of the rank-one operators $\phi(\lambda P)$ and $\phi(P)$ coincide which is a consequence of $\operatorname{rng} \lambda P = \operatorname{rng} P$.

We next prove that the functions f_P are all the same. In order to verify this, first consider an arbitrary rank-one projection P. By Lemma (4.2.5), for any positive λ, the P-decomposition of λI is

$$\lambda I = \lambda P + \lambda(I - P).$$

Therefore, we obtain

$$\phi(\lambda I) = f_P(\lambda)\phi(P) + \phi\big(\lambda(I - P)\big). \tag{12}$$

The range of $\lambda(I - P)$ is equal to the range of $I - P$, and hence, we obtain that

$$\operatorname{rng} \phi\big(\lambda(\mathrm{I} - \mathrm{P})\big) = \operatorname{rng} \phi(\mathrm{I} - \mathrm{P}) = \operatorname{rng}\big(\mathrm{I} - \phi(\mathrm{P})\big) = \operatorname{rng} \phi(\mathrm{P})^{\perp}.$$

Consequently, the operators on the right hand side of (12) act on orthogonal subspaces. This means that the range of the rank-one projection $\phi(P)$ is an eigensubspace of $\phi(\lambda I)$. As P is an arbitrary rank-one projection, and hence, $\phi(P)$ runs through the set of all rank-one projections, we infer that $\phi(\lambda I)$ is a scalar operator. Again by (12), we see that this scalar is $f_P(\lambda)$. So, we have $\phi(\lambda I) = f_P(\lambda)I$. This shows that the bijection f_P of $\mathbb{R}_+$ in fact does not depend on P. We conclude that there is a bijective function f on $\mathbb{R}_+$such that for every rank-one projection P and nonnegative real number λ, we have $\phi(\lambda P) = f(\lambda)\phi(P)$.

Let P, Q be orthogonal rank-one projections and λ, μ positive real numbers. Set $B = \lambda P + \mu Q$. The P-Lebesgue decomposition of B is $B = \lambda P + \mu Q$. Therefore, we have

$$\phi(\lambda P + \mu Q) = \phi(B) = \phi(\lambda P) + \phi(\mu Q) = f(\lambda)\phi(P) + f(\mu)\phi(Q).$$

In particular, we obtain that $\phi(P + Q) = \phi(P) + \phi(Q)$. Next, let R be an arbitrary rank-one subprojection of $P + Q$. Then we infer that $\phi(R)$ is a subprojection of $\phi(P) + \phi(Q)$ (ϕ preserves the inclusion of ranges of operators with closed range). As we have seen in Lemma (4.2.5), the absolutely continuous part in the R-Lebesgue decomposition of B is $\lambda(B, R)R$. We compute the quantity $\lambda(B, R)$ in the following way. Let r be a unit vector in the range of R. By (7), we have

$$\lambda(B, R) = \left\|B^{-1/2}r\right\|^{-2} = \frac{1}{\langle B^{-1}r, r\rangle} = \frac{1}{\langle((1/\lambda)P + (1/\mu)Q)r, r\rangle}$$

$$= \frac{1}{(1/\lambda)\langle Pr, r\rangle + (1/\mu)\langle Qr, r\rangle} = \frac{1}{(1/\lambda)\operatorname{tr}PR + (1/\mu)\operatorname{tr}QR} .$$

Therefore, we obtain

$$\lambda(B,R)R = \frac{1}{(1/\lambda)\,\mathrm{tr}PR + (1/\mu)\,\mathrm{tr}QR}R.$$

Similarly, the absolutely continuous part in the $\phi(R)$-Lebesgue decomposition of $\phi(B) = f(\lambda)\phi(P) + f(\mu)\phi(Q)$ is

$$\lambda\big(\phi(B),\phi(R)\big)\phi(R) = \frac{1}{(1/f(\lambda))\,\mathrm{tr}\,\phi(P)\phi(R) + (1/f(\mu))\,\mathrm{tr}\,\phi(Q)\phi(R)}\phi(R).$$

As ϕ preserves Lebesgue decompositions, it follows that

$$\phi(\lambda(B,R)R) = \lambda(\phi(B),\phi(R))\phi(R).$$

Hence, using $\phi(\lambda(B,R)R) = f(\lambda(B,R))\phi(R)$, we have the following functional equation:

$$f\left(\frac{1}{(1/\lambda)\,\mathrm{tr}PR + (1/\mu)\,\mathrm{tr}QR}\right) = \frac{1}{(1/f(\lambda))\,\mathrm{tr}\,\phi(P)\phi(R) + (1/f(\mu))\,\mathrm{tr}\,\phi(Q)\phi(R)},$$

which can be rewritten as

$$f\left(\frac{1}{(1/\lambda)\alpha + (1/\mu)(1-\alpha)}\right) = \frac{1}{(1/f(\lambda))\acute{\alpha} + (1/f(\mu))(1-\acute{\alpha})}, \qquad (13)$$

where λ,μ are arbitrary positive numbers, $\alpha \in [0,1]$ is also arbitrary and $\acute{\alpha} \in [0,1]$. It is clear from the discussion above that $\acute{\alpha}$ does not depend on λ,μ, and thus, by (13), it depends only on α. Hence, we can write (13) into the following form

$$f\left(\frac{1}{(1/\lambda)\alpha + (1/\mu)(1-\alpha)}\right) = \frac{1}{(1/f(\lambda))\varphi(\alpha) + (1/f(\mu))(1-\varphi(\alpha))}$$

$(\lambda,\mu > 0, \alpha \in [0,1])$. Here, f is a bijective map on $\mathbb{R}_+$ sending 0 to 0 and 1 to 1, and $\varphi : [0,1] \to [0,1]$ is a function. We apply Lemma (4.2.6) and conclude that f,φ are the identities on their domains. What concerns φ, this gives us that

$$\mathrm{tr}PQ = \mathrm{tr}\,\phi(P)\phi(Q).$$

This means that the transformation ϕ, when restricted onto the set of all rank-one projections, is a bijective map preserving the trace of products. This latter quantity appears in the mathematical foundations of quantum mechanics and is usually called there transition probability. Transformations on the set of rank-one projections which preserve the transition probability are holding the name quantum mechanical symmetry transformations, and they play a fundamental role in the probabilistic aspects of quantum mechanics. A famous theorem of Wigner describes the structure of those transformations. It says that every such map is implemented by a unitary or antiunitary operator on the underlying Hilbert space. This means that we have a unitary or antiunitary operator U on H such that

$$\phi(P) = UPU^*$$

holds for every rank-one projection P on H. (For generalizations of Wigner's theorem concerning different structures, we refer to this Section 2.1, 2.3 of [103]; see also the references therein.) Therefore, considering the transformation $A \longmapsto U^*\phi(A)U$ if necessary, we can further assume without serious loss of generality that $\phi(P) = P$ holds for every rank-one projection P.

We complete the proof by showing that $\phi(B) = B$ holds for every positive operator B. Indeed, we already know that for an arbitrary rank-one projection P, the absolutely continuous part in the P-Lebesgue decomposition of B is $\lambda(B,P)P$. As ϕ preserves Lebesgue decompositions, we obtain that the absolutely continuous part in the $\phi(P)$ −Lebesgue decomposition of $\phi(B)$ is $\phi(\lambda(B,P)P)$. Since f is the identity on $\mathbb{R}_+$ and $\phi(P) = P$, we have $\phi(\lambda(B,P)P) = \lambda(B,P)P$. On the other hand, the absolutely continuous part in the P-Lebesgue decomposition of $\phi(B)$ is $\lambda(\phi(B),P)P$. Therefore, we have

$$\lambda(\phi(B),P)P = \phi(\lambda(B,P)P) = \lambda(B,P)P.$$

This gives us that

$$\lambda(B,P) = \lambda(\phi(B),P)$$

holds for every rank-one projection P. Since according to [98, Corollary 1], every positive operator is uniquely determined by its strength function, we obtain that $\phi(B) = B$. This completes the proof of the theorem. □

Corollary(4.2.7)[232]: Let P^n be a rank-one projection, μ a positive real number and B an arbitrary positive operator. Then we have

$$[\mu P^n]B = \lambda(B,P^n)P^n.$$

Therefore, the (μP^n)-Lebesgue decomposition of B is

$$B = \lambda(B,P^n)P^n + (B - \lambda(B,P^n)P^n).$$

In particular, the (μP^n)-Lebesgue decomposition of I is

$$I = P^n + (I - P^n).$$

Proof. In paper [104], we presented structural results for the automorphisms of $B(H)^+$ with respect to the operation of the harmonic mean or that of the parallel sum. We recall that the harmonic mean $T \,!\, S$ of the positive operators T, S is the double of their parallel sum $T : S$. In [104, Lemma 2] we proved that for any $T \in B(H)^+$ and rank-one projection P^n, we hav e

$$T \,!\, P^n = \frac{2\lambda(T,P^n)}{\lambda(T,P^n) + 1} P^n.$$

Using this, we compute

$$[\mu P^n]B = \lim_n(n\mu P^n) : B = \lim_n \frac{(n\mu P^n)!B}{2}$$

$$= \lim_n \frac{B!(n\mu P^n)}{2} = \lim_n n\mu \frac{(B/(n\mu))!P^n}{2} = \lim_n n\mu \frac{\lambda(B/(n\mu),P^n)}{\lambda(B/(n\mu),P^n)+1} P^n$$

$$= \lim_n n\mu \frac{(1/(n\mu))\lambda(B,P^n)}{(1/(n\mu))\lambda(B,P^n)+1} P^n = \lambda(B,P^n)P^n.$$

Here, we use the following properties of the harmonic mean and the strength function: for any $T, S \in B(H)^+$, rank-one projection P^n, and nonnegative number α, we have

$T\,!\,S = S!\,T,\ (\alpha T)!\,(\alpha S) = \alpha(T\,!\,S),\ \lambda(\alpha T, P^n) = \alpha\lambda(T, P^n).$ □