

## Chapter 2

### Operators of Bounded and Unbounded Imaginary Powers

In this chapter we deal with sums of operators in  $\xi$ -convex spaces, and here the extensions of the Dore-Venni results are derived. We give an application to a Volterra equation in a Banach space, we consist of an operator of positive type in Hilbert space without bounded imaginary powers, and concerned with the closedness of the sum of two closed operators in a Hilbert space.

#### Sec(2.1): Operators with Bounded Imaginary Powers in Banach Spaces

Let  $X$  be a complex Banach space and let  $A, B$  be closed linear densely defined operators in  $X$  such that  $(-\infty, 0]$  is contained in the resolvent sets of both operators, such that their resolvents satisfy

$$|(t + A)^{-1}|, |(t + B)^{-1}| \leq M/(1 + t) \text{ for all } t \geq 0, \quad (1)$$

Then their purely imaginary powers are bounded, and

$$|A^{is}| \leq K e^{\theta_A |s|}, \quad |B^{is}| \leq K e^{\theta_B |s|} \text{ for all } s \in \mathbb{R} \quad (2)$$

holds. Recently, it has been shown by Dore and Venni [182] that the sum  $A + B$  with domain  $D(A + B) = D(A) \cap D(B)$  is closed, if in addition  $X$  is  $\xi$ -convex,  $A$  and  $B$  commute and  $\theta_A + \theta_B < \pi$ ; a brief explanation of the notion ' $\xi$ -convex Banach space' is given at the beginning of this Section. This result has important applications to the theory of partial differential operators since (1) but also (2) are known for large classes of such operators; cp. Seeley [206]. In another paper the authors also show that  $A + B$  then has properties (1) and (2) again, probably with different  $M, K$  but with  $\theta_{A+B} = \max(\theta_A, \theta_B) + \varepsilon$ , where  $\varepsilon > 0$  can be chosen arbitrarily small. This makes it possible to iterate the argument and to consider sums of finitely many operators  $A_i, i = 1, \dots, n$ , which are mutually commuting, and are subject to (1), (2), with exponents  $\theta_i$  such that  $\theta_i + \theta_j < \pi$  for all  $i \neq j$ .

In many cases, however, (1) is too strong and should be replaced by the weaker conditions  $(-\infty, 0) \subset \rho(A) \cap \rho(B)$  and

$$|(t + A)^{-1}|, |(t + B)^{-1}| \leq M/t \text{ for all } t > 0. \quad (3)$$

Examples for this generally come from differential operators on unbounded regions, like the Laplace operator or the Stokes operator on exterior domains; of Giga and Sohr [198]. In such situations one still has (3) as well as  $N(A) = 0$  and  $R(A)$  dense in  $X$ , but  $0 \in \sigma(A)$ . Therefore it is desirable to have also results for

this case available, similar to those for the somewhat simpler case considered by Dore and Venni [182]. A preliminary version of such a generalization was obtained by Giga and Sohr [199], who also gave an application to the Navier-Stokes equation on an exterior domain. It is the purpose of this section to study this extension to the case where only (3) holds thoroughly.

At first glance this seems to be an easy task; approximate  $A$  and  $B$  by  $\varepsilon + A$  and  $\varepsilon + B$ , use the Dore-Venni results and let  $\varepsilon \rightarrow 0$ . Actually, this approach works, however, it is not straightforward. This is due to the fact, that in case we have (3) only, the fractional powers  $A^z$  are in general unbounded, except for  $z \in i\mathbb{R}$ . For this reason it is not at all obvious whether  $\varepsilon + A$  has bounded imaginary powers and whether the crucial assumption (2) holds for  $\varepsilon + A$ . It turns out that this is indeed the case. For the proofs we use the functional calculus generated by the group  $A^{is}$ ; it is closely related to the inverse Mellin transform; cp. Titchmarsh [209]. Once this functional calculus is put to work it is possible to show that (3) also holds for  $\varepsilon + A$  with the same  $\theta$  and  $K$  uniformly in  $\varepsilon$  which is indispensable for the limiting process. By means of this method, it is also possible to improve the estimate on  $(A + B)^{is}$  derived by Dore and Venni [196]; we obtain

$$|(A + B)^{is}| \leq K \exp(\max(\theta_A, \theta_B)|s|) \quad \text{for all } s \in \mathbb{R}, \quad (4)$$

provided  $\theta_A \neq \theta_B$ , and an additional factor  $1 + |s|^{1/2}$  appears in case  $\theta_A = \theta_B$ .

Let  $X$  be a complex Banach space and let  $A$  denote a closed linear operator in  $X$  with dense domain  $D(A)$ ;  $N(A)$  and  $R(A)$  denote kernel and range of  $A$ , and we use the notation  $\rho(A)$  and  $\sigma(A)$  for resolvent set and spectrum of  $A$ .  $B(X)$  is the space of bounded linear operators in  $X$ . The basic assumption on  $A$  is

(H1)  $(-\infty, 0) \subset \rho(A)$ ,  $N(A) = 0$ ,  $R(A)$  is dense in  $X$ , and, for some constant  $M \geq 1$ , we have

$$|(t + A)^{-1}| \leq M/t \quad \text{for all } t > 0. \quad (5)$$

It is well known that operators  $A$  satisfying (H 1) admit not necessarily bounded fractional powers of any order  $z \in \mathbb{C}$ , and for  $|\operatorname{Re} z| \leq 1, z \neq 0$ , and  $x \in D(A) \cap R(A)$  we have the representation

$$A^z x = \frac{\operatorname{Sin} z \pi}{\pi} \left\{ z^{-1} x - (1 + z)^{-1} A^{-1} x + \int_0^1 t^{z+1} (t + A)^{-1} A^{-1} x dt + \int_1^\infty t^{z-1} (t + A)^{-1} A x dt \right\}; \quad (6)$$

cf. Krein [201] or Komatsu [184]. In particular, since  $\sin \pi z / \pi z$  is an entire function, it follows that  $A^z x$  is a holomorphic function of  $z$  for  $|\operatorname{Re} z| < 1$  on the set  $D(A) \cap R(A)$ ; the latter is easily seen to be dense in  $X$ . In fact, given  $x \in X$ , choose  $y_n \in D(A)$  such that  $A y_n \rightarrow x$ ; this is possible since  $R(A)$  is dense in  $X$ . Then we have  $x_n = n(n - A)^{-1} A y_n \in R(A) \cap D(A)$  and  $x_n \rightarrow x$ . Furthermore,  $A^z x$  satisfies the group property

$$A^{z_1} A^{z_2} x = A^{z_1 + z_2} x, \quad x \in D(A) \cap R(A), \quad \operatorname{Re} z_1, \operatorname{Re} z_2, \operatorname{Re}(z_1 + z_2) \in (-1, 1).$$

Therefore the following definition makes sense.

**Definition (2.1.1) [186]:** A closed linear densely defined operator  $A$  in  $X$  belongs to the class  $\text{BIP}(X, \theta)$ , where  $\theta \in [0, \pi)$ , if  $A$  satisfies (H 1) as well as the condition.

(H2) For all  $s \in \mathbb{R}$ ,  $A^{is} \in B(X)$ , and there is some  $K \geq 1$  such that

$$|A^{is}| \leq K e^{\theta |s|}, \quad s \in \mathbb{R} \quad (7)$$

In general, it is not quite simple to verify (H2); however there are a number of examples which underline the importance of this definition.

**Example (2.1.2) [186]:** (Normal operators in Hilbert space). Let  $X$  be a Hilbert space and  $A$  a normal operator in  $X$  with spectral family  $\{E_\lambda\}_{\lambda \in \mathbb{C}}$ . By the functional calculus for normal operators we have

$$f(A) = \int_{\sigma(A)} f(\lambda) dE_\lambda \in B(X),$$

for each Borel-measurable bounded  $f : \sigma(A) \rightarrow \mathbb{C}$ , and

$$|f(A)| = \sup \{|f(\lambda)| : \lambda \in \sigma(A)\} = \|f\|_0$$

holds. Let  $S_\alpha = \{\lambda \in \mathbb{C} : |\arg \lambda| < \alpha\}$ ; then we have

$$A \in \text{BIP}(X, \theta) \text{ iff } N(A) = 0 \text{ and } \sigma(A) \subset \bar{S}_\theta. \quad (8)$$

In fact, if  $N(A) = 0$  then  $\overline{R(A)} = X$  and with  $f(\lambda) = 1/(\lambda + t)$

$$\begin{aligned} |(t + A)^{-1}| &\leq \sup\{1/|\lambda + t| : \lambda \in \sigma(A)\} \\ &= 1/\operatorname{dist}(-t, \sigma(A)) \leq 1/(t \sin \theta), \end{aligned}$$

i.e (H 1) holds. Also, with  $f(\lambda) = \lambda^{is} = e^{is \log \lambda}$  we obtain

$$|A^{is}| = \sup \{|\lambda^{is}| : \lambda \in \sigma(A)\} = \sup \{e^{-s \arg \lambda} : \lambda \in \sigma(A)\} \leq e^{|s| \theta}.$$

From this the converse implication is also obvious.

**Example (2.1.3) [186]:** ( $m$ -accretive operators in Hilbert space). Suppose  $A$  is an  $m$ -accretive linear operator in a Hilbert space  $X$  such that  $N(A) = 0$ . Then we have  $R(A)$  dense in  $X$  and (H 1) holds with  $M = 1$ . Moreover, the functional calculus of  $A$  developed by Foias and Nagy [208] implies  $A \in \text{BIP}(X, \pi/2)$ , the constant  $K$  in (H 2) is 1.

**Example (2.1.4) [186]:** (Multiplication operators on  $L^p(\Omega, \mu)$ ). Let  $X = L^p(\Omega, \mu)$ ,  $1 \leq p < \infty$ , where  $(\Omega, \mu)$  denotes a  $\sigma$ -finite measure space, and consider a  $\mu$ -measurable function  $m(x)$  such that  $m(x) \neq 0$   $\mu$ -a.e.; let  $A$  be defined as

$$(Au)(x) = m(x)u(x), \quad x \in \Omega, \quad D(A) = \{u \in X: Au \in X\}.$$

This  $A$  is closed linear and densely defined,  $N(A) = 0$  and  $\overline{R(A)} = X$ . It is not difficult to see that

$$A \in \text{BIP}(X, \theta) \text{ iff } m(x) \in \overline{S_\theta} \text{ a.e.} \quad (9)$$

**Example (2.1.5) [186]:** ( $d/dt$  in  $L^p(\mathbb{R}_+; Y)$ ). Let  $Y$  denote another Banach space, and let  $X = L^p(\mathbb{R}_+; Y)$ , with  $1 < p < \infty$ . Define  $Au = du/dt$  for  $u \in D(A) = W_0^{1,p}(\mathbb{R}_+; Y)$ ; it is well known that  $A$  is closed linear densely defined, and that the adjoint  $A^*$  of  $A$  is given by  $A^*u^* = -du^*/dt$  for  $u^* \in D(A^*) = W^{1,q}(\mathbb{R}_+; Y^*)$ , in case  $Y$  is reflexive, and  $p^{-1} + q^{-1} = 1$ . Therefore we have  $N(A) = N(A^*) = 0$ , hence  $R(A)$  is dense in  $X$ . Furthermore,

$$(t + A)^{-1}f(x) = \int_0^x e^{-t(x-y)} f(y) dy, \quad t > 0, \quad f \in X,$$

hence (H 1) follows with  $M = 1$ .

It has been shown recently by Dore and Venni [182] that in case  $Y$  is  $\xi$ -convex, the imaginary powers of  $A$  satisfy the estimate

$$|A^{is}| \leq C(p, Y)(l + s^2)e^{\frac{\pi|s|}{2}} \quad s \in \mathbb{R}, \quad (10)$$

where the constant  $C(p, Y) > 0$  only depends on  $p$  and  $Y$ . Thus if  $Y$  is  $\xi$ -convex and  $1 < p < \infty$  then  $A \in \text{BIP}(X, \pi/2 + \varepsilon)$  for each  $\varepsilon > 0$ .

Actually, Dore and Venni proved this only for the case of a finite interval  $[0, T]$ , however, without any changes their proof carries over to the hairline case.

**Example (2.1.6) [186]:** (Diffusion semigroups). Suppose  $-A$  is the generator of a positive contraction semigroup  $T(t)$  in  $X = L^p(\Omega, \mu)$ ,  $1 \leq p \leq \infty$ , where as before  $(\Omega, \mu)$ , denotes a  $\sigma$ -finite measure space. Assume that  $T(t)$  is selfadjoint for  $p = 2$  and that  $T(t)1 = 1$  for  $t > 0$  in  $L^\infty(\Omega, \mu)$ , where  $1$  denotes the function which is constant 1. Stein [207] proved that then  $A \in \text{BIP}(X, \pi/2)$  holds, for any  $p \in (1, \infty)$ . This result covers elliptic boundary value problems of second order; the angle  $\pi/2$ , however is not best possible for this case, as the results of Seeley [206] show.

**Example (2.1.7) [186]:** (Stokes operator). Let  $\Omega \subset \mathbb{R}^n$  be a domain with compact smooth boundary, consider the space  $Y = L^p(\Omega; \mathbb{R}^n)$ , for  $1 < p < \infty$ ,  $n > 1$ , and let  $X = L_\sigma^p(\Omega; \mathbb{R}^n)$  denote the subspace of  $Y$  defined by the closure of  $C_{0,\sigma}^\infty(\Omega; \mathbb{R}^n) = \{u \in C_0^\infty(\Omega; \mathbb{R}^n): \text{div } u = 0\}$  in the norm of  $Y$ ;

here  $\operatorname{div} u$  means the divergence of the vector field  $u$ . Then for every  $f \in Y$  there exists the unique decomposition  $f = f_0 + \operatorname{grad} \varphi$  with  $f_0 \in X$  the Helmholtz decomposition;  $\operatorname{grad} \varphi$  is as usual the gradient of the scalar function  $\varphi$ . The operator  $P: Y \rightarrow X$  defined by  $Pf = f_0$  is a bounded linear projection in  $Y$  with  $R(P) = X$ . The Stokes operator  $B$  on  $X$  is then defined by  $Bu = -P\Delta$ ,  $D(B) = D(\Delta) \cap X$ ; here  $\Delta$  denotes the Laplacian on  $Y$  with zero boundary conditions, i.e.  $D(\Delta) = W_0^{1,p}(\Omega; \mathbb{R}^n) \cap W^{2,p}(\Omega; \mathbb{R}^n)$ . The Stokes operator represents the stationary linear part of the Navier-Stokes equation for the flow of an incompressible material with Newtonian viscosity.

It is known that (H1) holds for  $B$ ; cp. Borchers and Sohr [193]. Concerning (H2), it has been proved recently that for every  $\theta \in (0, \pi/2)$  there is a constant  $K = K(\theta, p)$  such that (H2) is satisfied. For the case of bounded domains this result is due to Giga [197], while for exterior domains this has been proved by Giga and Sohr [198]. Thus  $B \in \text{BIP}(X, \theta)$ , for any  $\theta > 0$ .

Furthermore,  $B$  is even selfadjoint in  $L_\sigma^p(\Omega; \mathbb{R}^n)$ . It should, however, be noted that  $B$  is not covered by Example (2.1.6), since the semigroup generated by  $B$  cannot be expected to be positive and it is an open question whether it is contractive for general  $p$ . Also, in the case of an exterior domain the Stokes operator is not invertible, hence (1) does not hold.

Note that the class  $\text{BIP}(X, \theta)$ , enjoys the symmetry property

$$A \in \text{BIP}(X, \theta) \text{ iff } A^{-1} \in \text{BIP}(X, \theta). \quad (11)$$

Let  $B$  denote the generator of the  $C_0$ -group  $A^{is}$ ; formally we obtain  $B = i \log A$ , and so we may use this relation as a definition of  $\log A$ .

**Definition (2.1.8) [186]:** Suppose  $A \in \text{BIP}(X, \theta)$  and let  $B$  be the generator of the  $C_0$ -group  $A^{is}$ . Then the logarithm of  $A$  is defined by

$$\log A = -iB. \quad (12)$$

Recall the Mellin transform defined by

$$F(\rho) = \int_0^\infty f(t)t^{\rho-1} dt ; \quad (13)$$

Mellin's inversion formula reads

$$f(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} F(\rho)t^{-\rho} d\rho. \quad (14)$$

(14) will serve for the construction of a functional calculus for operators of Class  $\text{BIP}(X, \theta)$ . For the convenience of the reader we now collect several well known transformation pairs and several useful properties of the Mellin transform.

$$f(t) \quad F(\rho) \\ 1/(1+t) \quad \pi/\sin z \rho \quad 0 < \operatorname{Re} \rho < 1 \quad (15)$$

$$e^{-t} \quad \Gamma(\rho) \quad 0 < \operatorname{Re} \rho \quad (16)$$

$$(1+t)^{-a} \quad \Gamma(\rho) \Gamma(a-\rho)/\Gamma(a) \quad 0 < \operatorname{Re} \rho < \operatorname{Re} a \quad (17)$$

$$f_1(t)f_2(t) \quad (1/2\pi i) \int_{c-\infty}^{c+\infty} F_1(\rho-\sigma) F_2(\sigma) d\sigma \quad (18)$$

$$f(\alpha t) \quad (1/\alpha)^\rho F(\rho) \quad (19)$$

$$-(d/dt)f(t) \quad (\rho-1)F(\rho-1) \quad (20)$$

$$t^a f(t) \quad F(\rho+a) \quad (21)$$

A detailed study of the Mellin transform can be found, e.g., in the classical monograph Titchmarsh [209].

In the sequel, we let  $\theta \in [0, \pi)$  be fixed and  $A$  denotes any element of  $BIP(X, \theta)$ . Define

$$M_\theta^1(\mathbb{R}) = \{\mu \in M^1(\mathbb{R}) : |\mu|_\theta = (1/2\pi) \int_{-\infty}^{\infty} e^{\theta|s|} |d\mu(s)| < \infty\},$$

the Banach space of all complex measures on  $\mathbb{R}$  which are finite w.r. to the weight  $e^{\theta|s|}$  normed by  $|\cdot|_\theta$ ;  $M_\theta^1(\mathbb{R})$  becomes a Banach algebra with unit, the convolution of measures, scaled by  $1/2\pi$ , being the multiplication. Evidently, the Dirac measure  $\delta_s$  with mass in  $s \in \mathbb{R}$  belongs to  $M_\theta^1(\mathbb{R})$ ;  $2\pi\delta_0$  is the unit of the algebra  $M_\theta^1(\mathbb{R})$ . For measures  $\mu \in M_\theta^1(\mathbb{R})$  we define

$$f(z) = (1/2\pi) \int_{-\infty}^{\infty} z^{-is} d\mu(s), \quad |\arg z| \leq \theta; \quad (22)$$

this map defines an algebra homomorphism from  $M_\theta^1(\mathbb{R})$  into the Banach algebra  $H_0(S_\theta)$  defined by

$$H_0(S_\theta) = \{f: \bar{S}_\theta \rightarrow \mathbb{C} \text{ continuous, holomorphic in } S_\theta\}$$

with norm  $\|f\|_0 = \sup\{|f(z)| : z \in \bar{S}_\theta\}$ , and pointwise multiplication. This gives rise to an algebra homomorphism from  $M_\theta^1(\mathbb{R})$  into  $B(X)$  defined by

$$f(A) = (1/2\pi) \int_{-\infty}^{\infty} A^{-is} d\mu(s), \quad (23)$$

where  $\mu$  and  $f$  are related by (22). Choosing  $\mu = 2\pi\delta_r$ , we obtain  $f(z) = z^{-ir}$  as well as  $f(A) = A^{-ir}$ ; in particular  $(2\pi\delta_0)(A) = I$ . Moreover,

$$(f_1 f_2)(A) = (1/2\pi) \int_{-\infty}^{\infty} A^{-is} d(\mu_1 * \mu_2)(s)$$

$$\begin{aligned}
&= (1/2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A^{-is} d\mu_1(s-t)d\mu_2(t) \\
&= (1/2\pi)^2 \int_{-\infty}^{\infty} A^{-it} d\mu_2(t) \int_{-\infty}^{\infty} A^{-is} d\mu_2(s) = f_1(A) f_2(A);
\end{aligned}$$

this proves that the map (23) is multiplicative.

**Theorem (2.1.9) [186]:** Let  $\theta \in [0, \pi)$  and  $A \in \text{BIP}(X, \theta)$ .

Then (23) defines an algebra homomorphism from  $M_{\theta}^1(\mathbb{R})$  into  $B(X)$  such that

$$f(z) = z^{-ir} \text{ implies } f(A) = A^{-ir}, \quad r \in \mathbb{R}. \quad (24)$$

Moreover, we have the estimate

$$|f(A)| \leq K|\mu|_{\theta} \quad (25)$$

with  $K$  from (H2). Here  $f$  and  $\mu$  are related by (22).

It is left to the reader to translate the properties (19)-(21) into the properties of the algebra homomorphism. However, let us state a consequence of (19) for future reference.

**Corollary (2.1.10) [186]:** Let  $\theta \in [0, \pi)$  and  $A \in \text{BIP}(X, \theta), \alpha > 0$ . Then  $\alpha A \in \text{BIP}(X, \theta)$  and we have

$$f(\alpha A) = f_{\alpha}(A) \quad (26)$$

Where  $f_{\alpha}(t) = f(\alpha t)$  and  $f$  is given by (22).

It is to be mentioned that this functional calculus is nothing else than the functional calculus of Phillips for the group  $A^{is}$ , after an exponential change of variable; cf. Hille and Phillips [200]. For our purposes, however it is more appropriate to have the Mellin-transform as a setting rather than the Laplace-transform.

Unfortunately, our functional calculus is not strong enough to recover the resolvent  $(\lambda - A)^{-1}$  of  $A$  from the group  $A^{is}$ . The reason for this is that the Mellin transform of  $1/(1 + t)$  has poles at  $\rho = 0, 1$ . We are going to remove this defect. Consider the transform pair (15); the inversion formula (14) then holds for each  $c \in (0, 1)$  since  $|\sin \pi \rho| \geq s h \pi |s|$ ,  $s = \text{Im} \rho$ . Let  $x \in D(A) \cap R(A)$ ; the vector-valued function  $A^{\rho} x, |\text{Re} \rho| < 1$ , is then holomorphic and we have the estimate

$$|A^{\rho} x| \leq C_{\varepsilon}(x) e^{\theta |s|}, \quad |\text{Re} \rho| \leq 1 - \varepsilon,$$

which easily follows from the representation (6) of  $A^{\rho} x$  and the group property. Therefore, the integral

$$Tx = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} (\pi / \sin \pi \rho) A^{-\rho} x d\rho, \quad 0 < c < 1, \quad (27)$$

Exists as an absolutely convergent integral and by Cauchy's Theorem it is independent of  $c$ . Applying  $(I + A)$  to (27) we obtain

$$(I + A)Tx = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} (\pi / \sin \pi \rho) A^{-\rho} x d\rho \\ + (1/2\pi i) \int_{c-i\infty}^{c+i\infty} (\pi / \sin \pi \rho) A^{1-\rho} x d\rho ;$$

Using Cauchy's Theorem again, we deform the path of integration in the first integral into the contour  $\Gamma_1$ , the contour consisting of the intervals  $(-i\infty, -i\varepsilon]$ ,  $[i\varepsilon, i\infty)$  connected by the positive halfcircle  $\Gamma_1^\varepsilon$  of radius  $\varepsilon > 0$ ; similarly, the path of integration in the second integral is deformed into  $\Gamma_2$ , the intervals  $(1-i\infty, 1-i\varepsilon]$  and  $[1+i\varepsilon, 1+i\infty)$  connected by the negative halfcircle  $\Gamma_2^\varepsilon$  of radius  $\varepsilon > 0$ . Since  $\sin \pi(1+\rho) = -\sin \pi\rho$  the contributions coming from the straight lines in  $\Gamma_i$  cancel each other, and therefore there remains

$$(I + A)Tx = (1/2\pi i) \int_{\Gamma_1^\varepsilon} (\pi / \sin \pi \rho) A^{-\rho} x d\rho \\ + (1/2\pi i) \int_{\Gamma_2^\varepsilon} (\pi / \sin \pi \rho) A^{1-\rho} x d\rho ;$$

it is easily seen that  $(I + A)Tx = x$  as  $\varepsilon \rightarrow 0$ , hence we obtain  $Tx = (I + A)^{-1}x$  for each  $x \in D(A) \cap R(A)$ . Shifting the contour to the imaginary axis in (27) and applying Corollary (2.1.10) we have shown

$$(I + \alpha A)^{-1}x = (1/2\pi i) PV \int_{-\infty}^{\infty} (\pi / sh \pi s) (\alpha A)^{-is} x ds + (1/2)x, \quad (28)$$

for each  $x \in D(A) \cap R(A)$ . and  $\alpha > 0$ ; here 'PV' indicates Cauchy's principal value.

Now consider  $\lambda = \alpha e^{i\varphi}$  with  $|\varphi| < \pi$ ; then (14) with  $f(t) = 1/(1 + \lambda t)$  yields

$$1/(1 + \lambda t) = 1/(1 + \alpha t) \\ + (1/2\pi i) \int_{-\infty}^{\infty} (\pi / sh \pi s) (\alpha t)^{-is} (e^{\varphi s} - 1) ds;$$

the measure  $\mu$  with density  $d\mu/ds = (\pi / sh \pi s) (e^{\varphi s} - 1) \alpha^{-is}$  belongs to  $M_\theta^1(\mathbb{R})$ , provided  $|\varphi| < \pi - \theta$ , and  $|\mu|_\theta \leq c/(\pi - \theta - |\varphi|)$  holds for some constant  $c$  which is independent of  $\varphi$  and  $\alpha$ . Thus by Theorem (2.1.9),  $1, (I + \lambda A)^{-1}$  exists for each  $\lambda \in \mathbb{C}$  with  $|\arg \lambda| < \pi - \theta$  and

$$|(I + \lambda A)^{-1}| \leq |(I + \alpha A)^{-1}| + C |\mu|_\theta \leq C.$$

On each sector  $\bar{S}_v$  with  $v < \pi - \theta$ . We have proved



**Theorem (2.1.11) [186]:** Let  $A \in \text{BIP}(X, \theta)$ ,  $0 \leq \theta < \pi$ . then  $\sigma(A) \subset \bar{S}_\theta$  and we have the estimate

$$|(\lambda + A)^{-1}| \leq C_v/|\lambda|, \quad \lambda \in S_v, \quad (29)$$

where  $v < \pi - \theta$ . In particular, if  $0 < \pi/2$  then  $-A$  generates a uniformly bounded analytic  $C_0$ -semigroup  $e^{-tA}$  in  $X$ .

By means of the transform pair (16) it is possible to obtain a representation of  $e^{-tA}$  in terms of the imaginary powers  $A^{i\rho}$ , but we will not do this here.

Suppose  $A$  satisfies (H1); it is then obvious that  $\varepsilon + A = A_\varepsilon$  also satisfies (H1) for each  $0 < \varepsilon < 1$ , and there holds the stronger estimate

$$|(t + A_\varepsilon)^{-1}| \leq M/(\varepsilon + t) \leq (M/\varepsilon)/(1 + t), \quad t > 0. \quad (30)$$

Therefore, the fractional powers  $A_\varepsilon^{-\alpha}$  exist and are bounded for  $\text{Re } \alpha > 0$ ; they even form an analytic semigroup. It is much less obvious whether this semigroup has boundary values in  $B(X)$  on the imaginary axis. However, this can be expected if  $A$  belongs to  $\text{BIP}(X, \theta)$ . In fact, we show that then  $A_\varepsilon \in \text{BIP}(X, \theta)$  for each  $\varepsilon > 0$ ; even more is true.

**Theorem (2.1.12) [186]:** Suppose  $A \in \text{BIP}(X, \theta)$  for some  $\theta \in (0, \pi)$ , and let  $A_\varepsilon = \varepsilon + A$ ,  $\varepsilon > 0$ . Then  $A_\varepsilon \in \text{BIP}(X, \theta)$  as well, and the constants  $M$  and  $K$  from (H1) and (H2), respectively, can be chosen uniformly w.r.t,  $\varepsilon > 0$ . Moreover, the group  $A_\varepsilon^{i\rho}$  converges strongly to the group  $A^{i\rho}$  as  $\varepsilon \rightarrow 0$ .

**Proof.** The proof is based on the functional calculus for operators of class  $\text{BIP}(X, \theta)$ . Let  $A, \theta, A_\varepsilon, \varepsilon$  be as in the theorem and let  $M$  and  $K$  denote the constants in (H1) and (H2) for  $A$ ; we first consider the case  $\varepsilon = 1$ . The transformation pair (17) clearly yields the complex powers of  $A_1$  with negative real part, however, this  $F(s)$  does not give rise to a measure of class  $M_\theta^1(\mathbb{R})$  since  $F(s)$  has a pole at  $s = 0$ ; also we are interested in the case  $\text{Re } a = 0$  and so we have to derive a corresponding formula of type (22).

First we use the Mellin inversion formula (14) for the pair (17),  $\text{Re } a > 0$ . Shifting the contour of integration to the imaginary axis yields

$$(1 + t)^{-a} = \frac{1}{2} + \left(\frac{1}{2\pi}\right) PV \int_{-\infty}^{\infty} \Gamma(is)\Gamma(a - is)\Gamma(a)^{-1} t^{-is} ds, \quad t > 0, \text{Re } a > 0, \quad (31)$$

where again 'PV' denotes Cauchy's principal value. To remove the pole at  $s = 0$ , we subtract from (31) the representation of  $(1 + \rho t)^{-1}$ ,  $\rho > 0$ , i.e. (31) with  $a = 1$ ; this gives

$$(1 + t)^{-a} = (1 + \rho t)^{-1} + (1/2\pi) \int_{-\infty}^{\infty} \left\{ \frac{\Gamma(is)\Gamma(a-is)}{\Gamma(a)} - \frac{\pi\rho^{-is}}{\sin(i\pi s)} \right\} t^{-is} ds. \quad (32)$$

Note that the integral now is absolutely convergent since the pole at  $s = 0$  has been removed. Next we want to let  $a \rightarrow i\rho$  in equation (32); then  $s = \rho$  becomes a singularity within the integrand. To avoid this we alter the contour of integration once more into  $\Gamma_\varepsilon$ , consisting of the two rays  $\{is: s \leq \rho - \varepsilon\}, \{is: s \geq \rho + \varepsilon\}$  and the left halfcircle with radius  $\varepsilon > 0$ . Now passage to the limit  $a \rightarrow i\rho$  can be carried out to the result

$$(1+t)^{-i\rho} = (1+\rho t)^{-1} + (1/2\pi i) \int_{\Gamma_\varepsilon} \{\Gamma(z)\Gamma(i\rho-z)/\Gamma(i\rho) - \pi\rho^{-z}/\sin \pi z\} t^{-z} dz. \quad (33)$$

Next we let  $\varepsilon \rightarrow 0$  and obtain

$$(1+t)^{-i\rho} = (1+\rho t)^{-1} + (1/2)t^{-i\rho} + (1/2\pi)PV \int_{-\infty}^{\infty} \left\{ \frac{\Gamma(is)\Gamma(i\rho-is)}{\Gamma(i\rho)} - \frac{\pi\rho^{-is}}{\sin \pi is} \right\} t^{-is} ds. \quad (34)$$

finally, to remove the singularity we add (use (15) and (21))

$$\rho t^{-i\rho}(\rho+t)^{-1} = (1/2)t^{-i\rho} + (1/2\pi)PV \int_{-\infty}^{\infty} \{\pi\rho^{is-i\rho}/\sin \pi i(s-\rho)\} t^{-is} ds \quad (35)$$

to (34)

$$(1+t)^{i\rho} = (1+\rho t)^{-1} + t^{-i\rho} - \rho t^{-i\rho}(\rho+t)^{-1} + (1/2\pi) \int_{-\infty}^{\infty} g_\rho(s)t^{-is} ds, \quad (36)$$

Where

$$g_\rho(s) = \{\Gamma(is)\Gamma(i\rho-is)/\Gamma(i\rho)\} - \{\pi\rho^{-is}/\sin \pi is\} - \{\pi\rho^{is-i\rho}/\sin \pi i(\rho-s)\}, \quad (37)$$

with  $s \in \mathbb{R}, \rho > 0$ .

It is not difficult to show that  $|g_\rho(s)| \leq C_\rho e^{-\pi|s|}, s \in \mathbb{R}$ , and so  $g_\rho$  gives rise to a measure  $\mu_\rho M_\theta^1(\mathbb{R})$ . This yields the representation

$$A_1^{-i\rho} = \rho^{-1} A A_{1/\rho}^{-1} + A^{-i\rho} - \rho A^{i\rho} A_\rho^{-1} + f_\rho(A), \quad (38)$$

where  $f_\rho$  is given by (22) with  $d\mu_\rho/ds = g_\rho$ , thanks to Theorem (2.1.9).

Thus the functional calculus developed in this Section shows that the imaginary powers of  $A_1$  exist and belong to  $B(X)$  for each fixed  $\rho > 0$ . We

now have to verify estimate (7). It is clear from (38) that it remains to derive the desired bound on  $f_\rho(A)$ , which means to prove

$$\int_{-\infty}^{\infty} |g_\rho(s)| e^{\theta|s|} ds \leq C e^{\theta\rho} \quad \rho > 0, \quad (39)$$

where  $C$  only depends on  $\theta$ , but not on  $\rho$ . The integral in (39) is broken up into five parts according to the intervals  $(-\infty, -\eta)$ ,  $(-\eta, \eta)$ ,  $(\eta, \rho - \eta)$ ,  $(\rho - \eta, \rho + \eta)$  and  $(\rho + \eta, \infty)$ , where  $\eta < \rho/2$  is fixed, to be chosen later on. These integrals will be named  $I_1, I_2, \dots, I_5$  and estimated separately. In the sequel we will use repeatedly the formula

$$|\Gamma(is)|^2 = \pi / (s \operatorname{sh} \pi s), \quad s \in \mathbb{R}, \quad (40)$$

see, e.g., Abramowitz and Stegun [191], p. 77, as well as the elementary estimate

$$e^{\pi r} / 2 \geq \operatorname{sh} \pi r \geq c_0 \eta e^{\pi r}, \quad r \geq \eta, \quad (41)$$

where  $c_0 > 0$  is independent of  $\eta > 0$ .

$I_1$ : Here we have by (40) and (41)

$$|g_\rho(s)| \leq C_1 \eta^{-1} e^{-\pi|s|}, \quad s \leq -\eta$$

hence

$$|I_1| \leq \int_{-\infty}^{-\eta} C_1 \eta^{-1} e^{-\pi s} e^{\theta s} ds \leq C_1 \eta^{-1} (\pi - \theta)^{-1}. \quad (42)$$

$I_5$ : For this integral (40) and (41) yield

$$|g_\rho(s)| \leq C_5 \eta^{-1} e^{-\pi s} e^{\pi\rho}, \quad s \geq \rho + \eta$$

hence

$$|I_5| \leq C_5 \eta^{-1} \int_{\rho}^{\infty} e^{-(\pi-\theta)s} ds e^{\pi\rho} = C_5 \eta^{-1} (\pi - \theta)^{-1} e^{\theta\rho}. \quad (43)$$

$I_3$ : Similarly, here we obtain

$$|g_\rho(s)| \leq C_3 \eta^{-1} ([\rho/s(\rho - s)]^{1/2} + e^{-\pi s} + e^{-\pi(\rho-s)}), \quad \eta \leq s \leq \rho - \eta,$$

and so a simple calculation shows

$$\begin{aligned} |I_3| &\leq C_3 \eta^{-1} \int_0^{\rho} ([\rho/s(\rho - s)]^{1/2} + e^{-\pi s} + e^{-\pi(\rho-s)}) e^{\theta s} ds \\ &\leq C_3 \eta^{-1} \{\sqrt{\rho} e^{\theta\rho} \int_0^1 [t(1-t)]^{-1/2} e^{-\theta\rho t} dt \end{aligned}$$

$$\begin{aligned}
& +(\pi - \theta)^{-1} + (\pi + \theta)^{-1} e^{\theta\rho} \} \\
& \leq C_3 \eta^{-1} \{ (1/\sqrt{\theta}) e^{\theta\rho} + (\pi - \theta)^{-1} + (\pi + \theta)^{-1} e^{\theta\rho} \}. \tag{44}
\end{aligned}$$

Note that here we need  $\theta > 0$ !

$I_2$ : Since  $g_\rho(s)$  is continuous w.r. to  $\rho > 2\eta$  and  $s$  and the interval under consideration has length  $2\eta$  not dependent on  $\rho$  it is clear that  $|I_2(\rho)| \leq C$  as long as  $\rho$  is bounded. Therefore we may restrict our attention to large values of  $\rho$ . We have

$$|\pi \rho^{is-i\rho} / \sin i\pi(\rho - s)| \leq C_2 \eta^{-1} e^{-\pi(\rho-s)} \leq C_2 \eta^{-1}$$

and from the reflection formula of the gamma function

$$\Gamma(z) \Gamma(1 - z) = \pi / \sin \pi z .$$

We obtain for  $s \in [-\eta, \eta]$

$$|g_\rho(s)| \leq C_2 \eta^{-1} + |\Gamma(is)| | \Gamma(i(\rho - s)) / \Gamma(i\rho) - \rho^{is} \Gamma(1 - is) |.$$

Since  $\Gamma(is)$  has a simple pole at  $s = 0$  there is a constant  $c > 0$  such that

$$|s\Gamma(is)| \leq c, \quad |\Gamma(1 - is) - \Gamma(1)| \leq c|s|, \quad |s| \leq \eta;$$

note that  $\Gamma(1) = 1$ . Next we use Stirling's formula

$$\Gamma(z) \sim e^{-z} z^{z-1/2} \sqrt{2\pi}, \quad |z| \rightarrow \infty, \quad |\arg z| < \pi,$$

And for large values of  $\rho$  this yields

$$\begin{aligned}
|g_\rho(s)| & \leq C_2 \eta^{-1} + C_2 + c |e^{is}(1 - s/\rho)^{i(\rho-s)-1/2} e^{\pi s/2} - 1| / s \\
& \leq C_2 \eta^{-1}, \quad |s| \leq \eta,
\end{aligned}$$

hence

$$|I_2| \leq C_2 \eta^{-1} \int_{-\eta}^{\eta} e^{\theta|s|} ds \leq 2C_2 \eta^{-1} e^{\theta|\eta|} \leq C_2 \eta^{-1}.$$

$I_4$ : To treat  $I_4$  we use the symmetry  $g_\rho(\rho - s) = g_\rho(s)$ ; thus

$$|I_4| \leq \int_{\rho-\eta}^{\rho+\eta} |g_\rho(s)| e^{\theta|s|} ds \leq e^{\theta|s|} \int_{-\eta}^{\eta} |g_\rho(\rho - s)| e^{\theta|s|} ds \leq e^{\theta\rho} C_2 \eta^{-1}.$$

Thus we have shown that (39) holds for  $\rho \geq 2\eta$ , where  $\eta > 0$  is fixed. For  $\rho \in (0, 2\eta)$  we have to use a slightly different argument. This time we use the representation

$$(1+t)^{-i\rho} = (1+t)^{-1} + t^{-i\rho} - t^{-i\rho}(1+t)^{-1} + 1/2\pi \int_{-\infty}^{\infty} h_{\rho}(s)t^{-is}ds, \quad (45)$$

where

$$h_{\rho}(s) = \Gamma(is)\Gamma(i\rho - is)/\Gamma(i\rho) - \pi/\sin \pi is - \pi/\sin \pi i(\rho - s), \quad (46)$$

with  $\rho > 0$ ;

(45) is derived similarly to (36). This gives again

$$A_1^{-i\rho} = A_1^{-1} + A^{-i\rho} - A^{-i\rho}A_1^{-1} + f_{\rho}(A) \quad (47)$$

where  $f_{\rho}$  now is given by (22) with  $d\mu_{\rho}/ds = h_{\rho}$ . To obtain the desired estimate for  $A_1^{-i\rho}$ ,  $\rho < 2\eta$  we have to prove

$$\int_{-\infty}^{\infty} |h_{\rho}(s)|ds \leq C, \quad 0 < \rho < 2\eta. \quad (48)$$

We divide the integral in (48) into three parts according to the intervals  $(-\infty, 3\eta)$ ,  $(-3\eta, 3\eta)$ ,  $(3\eta, \infty)$ ; the corresponding integrals are named  $I_1, I_2, I_3$ , and are estimated separately.  $I_1, I_3$  can be treated as before, we obtain the same bounds as in (42) and (43), respectively. On the other hand,  $I_2$  is easily seen to be uniformly bounded, since the integrand  $h_{\rho}(s)$ , is continuous and bounded with respect to both variables

$$|s| \leq 3\eta, \quad 0 < \rho \leq 2\eta.$$

The case  $\rho < 0$  can be reduced to  $\rho > 0$  by taking complex conjugates in formulas (36) and (45).

Finally, let  $\varepsilon > 0$  be arbitrary and replace  $A$  by  $\varepsilon A$  in the above arguments. Since the constants  $M$  and  $K$  of (H 1), (H2) also apply to  $\varepsilon A$ , we obtain uniform bounds for  $(1 + \varepsilon A)^{-i\rho}$ . The strong convergence  $A_{\varepsilon}^{i\rho} \rightarrow A^{i\rho}$  follows from the Banach-Steinhaus theorem and from (6).  $\square$

Note that in Theorem (2.1.12) we had to exclude the case  $\theta = 0$ . We do not know whether this is essential or only due to the method of proof employed.

Recall that a Banach space  $X$  is said to be  $\xi$ -convex if there is a function  $\xi: X \times X \rightarrow \mathbb{R}$ , convex w.r.t, both variables, such that  $\xi(0, 0) > 0$  and

$$\xi(x, y) \leq |x + y| \quad \text{for all } x, y \in X \quad \text{with } |x| = |y| = 1.$$

Such spaces are of interest here since it is known that  $X$  is  $\xi$ -convex iff the Hilbert transform  $H$  on  $L^p(\mathbb{R}, X)$ ,  $1 < p < \infty$ , defined by

$$(Hf)(t) = (1/\pi i)PV \int_{\mathfrak{R}} f(t-s)ds/s, \quad t \in \mathbb{R},$$

is bounded. Hilbert spaces are  $\xi$ -convex (choose  $\xi(x, y) = 1 + (x, y)$  to see this), closed subspaces of  $\xi$ -convex spaces have this property again, and if  $X$  is  $\xi$ -convex then  $L^p(\Omega, \mu; X)$ ,  $1 < p < \infty$ , is  $\xi$ -convex, where  $(\Omega, \mu)$  denotes any  $\sigma$ -finite measure space. For the definition and these properties of  $\xi$ -convex spaces as well as others we refer to the survey article of Burkholder [194] and the references given there.

Now, let the Banach space  $X$  be  $\xi$ -convex, and suppose  $A \in \text{BIP}(X, \theta_A)$ ,  $B \in \text{BIP}(X, \theta_B)$ ,  $\theta_A + \theta_B < \pi$ , are resolvent commuting, i.e. there are

$\lambda \in \rho(A), \mu \in \rho(B)$  such that

$$(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}. \quad (49)$$

We want to extend the result of Dore and Venni [182], Theorem 2.1, to this more general setting.

**Theorem (2.1.13)[186]:** Let  $X$  be  $\xi$ -convex,  $A \in \text{BIP}(X, \theta_A)$ ,  $B \in \text{BIP}(X, \theta_B)$ , be resolvent commuting and assume  $\theta_A + \theta_B < \pi$ . Then the operator  $A + B$  with domain  $D(A + B) = D(A) \cap D(B)$  is closed and satisfies condition (H1). Moreover, there is a constant  $C > 0$  such that

$$|Au| + |Bu| \leq C|Au + Bu|, \quad u \in D(A) \cap D(B) \quad (50)$$

is satisfied;  $N(A + B) = 0$  and  $R(A + B)$  is dense in  $X$ .

**Proof.** Consider the approximations  $A_\varepsilon = \varepsilon + A$ ,  $B_\varepsilon = \varepsilon + B$  where  $\varepsilon > 0$ ; according to Theorem (2.1.12),  $A_\varepsilon \in \text{BIP}(X, \theta_A)$ ,  $B_\varepsilon \in \text{BIP}(X, \theta_B)$ , and the constants  $M_A, K_A$  and  $M_B, K_B$  appearing in (H1) and (H2) can be taken uniformly w.r. to  $\varepsilon > 0$ . By virtue of (30),  $A_\varepsilon, B_\varepsilon$  satisfy the assumptions of Theorem 2.1 in Dore and Venni [182], hence  $A_\varepsilon + B_\varepsilon$  with domain  $D(A) \cap D(B)$  is closed, and we have the representations

$$S_\varepsilon = (A_\varepsilon + B_\varepsilon)^{-1} = (1/2i) \int_{c-i\infty}^{c+i\infty} (A_\varepsilon^{-z} B_\varepsilon^{z-1}) / \sin(\pi z) dz, \quad 0 < c < 1, \quad (51)$$

as well as

$$A_\varepsilon S_\varepsilon x = (1/2i) PV \int_{-i\infty}^{i\infty} (A_\varepsilon^{-is} B_\varepsilon^{is} x) / \sin(\pi is) ds + (1/2)x, \quad x \in X; \quad (52)$$

Observe that the  $\xi$ -convexity of  $X$  is needed for the integral in (52) to exist for all  $x \in X$ . Since  $A_\varepsilon + B_\varepsilon = 2\varepsilon + A + B$  is closed with domain  $D(A + B) = D(A) \cap D(B)$  we see that  $A + B$  is closed as well. Further, the moment inequality yields with  $z = c + i\rho$

$$\begin{aligned} |A_\varepsilon^{-z}x| &\leq |A_\varepsilon^{-i\rho}| |A_\varepsilon^{-c}x| \leq K_A e^{\theta_A|\rho|} |A_\varepsilon^{-1}x|^c |x|^{1-c} \\ &\leq M_A^c K_A e^{\theta_A|\rho|} \varepsilon^{-c} |x|, \end{aligned}$$

and similarly for  $B_\varepsilon^{z-1}$ . By means of these estimates, (51) yields with  $c = 1/2$

$$\begin{aligned} |S_\varepsilon| &\leq M_A^c M_B^{1-c} K_A K_B \varepsilon^{-c} \varepsilon^{c-1} \int_{-\infty}^{\infty} e^{(\theta_A - \theta_B)|\rho|} e^{-\pi|\rho|} ds \\ &= M_{A+B}/(2\varepsilon), \end{aligned} \quad (53)$$

i.e. estimates (5) holds also for  $A + B$ . Since  $\xi$ -convex spaces are reflexive, from ergodic theory of linear operators (see, e.g., Hille and Phillips [200], chap. 18) we even obtain  $\varepsilon S_\varepsilon \rightarrow P$  strongly as  $\varepsilon \rightarrow 0$ , where  $P$  denotes the projection onto  $N(A + B)$ , and we have the decomposition

$$X = N(A + B) \oplus \overline{R(A + B)}. \quad (54)$$

As in Dore and Venni [182] we next use (52) to obtain a constant  $C$  independent of  $\varepsilon > 0$  such that

$$|A_\varepsilon S_\varepsilon| \leq C \text{ and } |B_\varepsilon S_\varepsilon| \leq C; \quad (55)$$

this follows from the fact that the Hilbert-transform is continuous on  $L^p(\mathbb{R}, X)$ ,  $1 < p < \infty$  whenever  $X$  is  $\xi$ -convex, and since the constants  $K$  for  $A_\varepsilon, B_\varepsilon$  are uniform in  $\varepsilon > 0$ , by Theorem (2.1.12); see Dore and Venni [182], p. 193, for details. From (55) we immediately get by (53)

$$|A S_\varepsilon| + |B S_\varepsilon| \leq C. \quad (56)$$

Let  $x \in D(A + B) = D(A) \cap D(B)$  and put

$y_\varepsilon = (\varepsilon + A + B)x$ ; then  $x = S_\varepsilon y_\varepsilon$  hence

$$|Ax| + |Bx| = |AS_\varepsilon y_\varepsilon| + |BS_\varepsilon y_\varepsilon| \leq C|(\varepsilon + A + B)x|,$$

and passing to the limit  $\varepsilon \rightarrow 0$  we obtain (50). Finally, inequality (50) shows that  $N(A + B) \subset N(A) \cap N(B) = 0$ , and so from (54) we also obtain density of  $R(A + B)$  in  $X$ , i.e.  $A + B$  satisfies (H 1).  $\square$

**Corollary (2.1.14) [186]:** Under the assumptions of Theorem (2.1.13) we have additionally that the operators  $A(A + B)^{-1}$  and  $B(A + B)^{-1}$  defined on the dense set  $R(A + B)$  are bounded, and so admit bounded extension to all of  $X$ .

A natural question arising in connection with Theorem (2.1.13) is whether the sum  $A + B$  is again of class  $BIP(X, \theta)$  for some  $\theta \in [0, \pi)$ .

A positive answer to this question would lead to the possibility to use an induction argument to treat sums  $\sum_1^n A_i$  of pairwise commuting operators of class  $BIP(X, \theta_i)$ . It was recently shown by Dore and Venni [196] that this is

indeed the case if both operators  $A$  and  $B$  are strongly positive in the sense that (5) is strengthened to

$$|(t + A)^{-1}| \leq M/(1 + t), \quad t > 0. \quad (57)$$

Their result, however, is not optimal, since they obtain  $A + B \in BIP(X, \theta + \varepsilon)$  where  $\theta = \max(\theta_A, \theta_B)$ . Our next theorem improves and extends Theorem 3.1 in Dore and Venni [196].

**Theorem (2.1.15) [186]:** Suppose  $X$  is  $\xi$ -convex,  $A \in BIP(X, \theta_A), B \in BIP(X, \theta_B)$ , with  $\theta_A + \theta_B < \pi$ , are resolvent commuting, and let  $\theta = \max(\theta_A, \theta_B)$ .  $\theta_A \neq \theta_B$ . Then  $A + B \in BIP(X, \theta)$ .

**Proof .** Let  $A, B, \theta_A, \theta_B$  and  $\theta$  be as in the theorem, w.l.o.g.  $\theta_A < \theta_B$  and let  $\varepsilon > 0$ . We claim that  $A(\varepsilon + B)^{-1} \in BIP(X, \theta_A + \theta_B)$ . In fact, for  $t > 0$  we have

$$\begin{aligned} |(t + A(\varepsilon + B)^{-1})^{-1}| &= |(\varepsilon + B)(t\varepsilon + tB + A)^{-1}| \\ &\leq \varepsilon|(t\varepsilon + tB + A)^{-1}| + |B(t\varepsilon + tB + A)^{-1}| \leq \varepsilon M/(t\varepsilon) + C/t, \end{aligned}$$

since  $A$  and  $tB$  satisfy the assumptions of Theorem (2.1.13); here  $M$  and  $C$  are from (53) and (55). On the other hand, the groups  $A^{i\rho}$  and  $(\varepsilon + B)^{-i\rho}$  commute and we have

$$|(A(\varepsilon + B)^{-1})^{i\rho}| \leq |A^{i\rho}| |(\varepsilon + B)^{-i\rho}| \leq K_A K_B e^{(\theta_A + \theta_B)|\rho|}, \quad \rho \in \mathbb{R}.$$

Next, using the function  $g_\rho(s)$  introduced in (37) in the proof of Theorem (2.1.12) we have the representation

$$\begin{aligned} (1 + A(\varepsilon + B)^{-1})^{-i\rho} &= (1 + \rho A(\varepsilon + B)^{-1})^{-1} - \rho(A(\varepsilon + B)^{-1})^{-i\rho} \\ &\quad \times (\rho + A(\varepsilon + B)^{-1})^{-1} + (A(\varepsilon + B)^{-1})^{-i\rho} \\ &\quad + (1/2\pi) \int_{-\infty}^{\infty} g_\rho(s) (A(\varepsilon + B)^{-1})^{-is} ds, \quad \rho > 0. \end{aligned}$$

Multiplying this equation by  $(\varepsilon + B)^{-i\rho}$  we obtain for

$$x \in D(A) \cap D(B) \cap R(A + B)$$

$$\begin{aligned} (\varepsilon + A + B)^{-i\rho} x &= (\varepsilon + B)^{-i\rho} (1 + A(\varepsilon + B)^{-1})^{-i\rho} x \\ &= (\varepsilon + B)^{-i\rho} (\varepsilon + B)(\varepsilon + \rho A + B)^{-1} x + A^{-i\rho} \\ &\quad - A^{-i\rho} \rho (\varepsilon + B)(\rho\varepsilon + A + \rho B)^{-1} x \end{aligned}$$

$$+ (1/2\pi) \int_{-\infty}^{\infty} g_\rho(s) A^{-is} (\varepsilon + B)^{-i(\rho-s)} x ds, \quad \rho > 0.$$

Passing to the limit  $\varepsilon \rightarrow 0$  for such  $x$  we arrive at the representation

$$(A + B)^{-i\rho} x = B^{-i\rho} B(\rho A + B)^{-1} x + A^{-i\rho} A(A + \rho B)^{-1} x$$

$$+ (1/2\pi) \int_{-\infty}^{\infty} g_\rho(s) A^{-is} B^{i(s-\rho)} x ds, \quad \rho > 0. \quad (58)$$

It is therefore sufficient to estimate the integral

$$\int_{-\infty}^{\infty} |g_\rho(s)| e^{\theta_A |s|} e^{\theta_B |s-\rho|} ds \leq C e^{\theta |\rho|}, \quad \rho > 0. \quad (59)$$



By Corollary (2.1.14) this implies the desired estimate for  $(A + B)^{-i\rho}$ , i.e. we obtain  $A + B \in \text{BIP}(X, \theta)$ . For this purpose we use the estimates on  $|g_\rho(s)|$  obtained in the proof of Theorem (2.1.12). We divide the integral into  $I_j$ ,  $j = 1, \dots, 5$  as there and get

$$\begin{aligned}
|I_1| &\leq C_1 \eta^{-1} \int_\eta^\infty e^{-\pi s} e^{\theta_A s} e^{\theta_B(s+\rho)} ds \leq C_1 \eta^{-1} e^{\theta_B \rho} (\pi - \theta_B - \theta_A)^{-1}; \\
|I_5| &\leq C_5 \eta^{-1} \int_\rho^\infty e^{-\pi s} e^{\pi \rho} e^{\theta_A s} e^{\theta_B(s-\rho)} ds \leq C_5 \eta^{-1} e^{\theta_A \rho} (\pi - \theta_A - \theta_B)^{-1}; \\
|I_3| &\leq C_3 \eta^{-1} \int_0^\rho ([\rho/s(\rho-s)]^{1/2} + e^{-\pi s} + e^{-\pi(\rho-s)}) e^{\theta_A s} e^{\theta_B(\rho-s)} ds \\
&\leq C_3 \eta^{-1} \{ e^{\theta_B \rho} \sqrt{\rho} \int_0^1 [t/(1-t)]^{1/2} e^{(\theta_A - \theta_B)\rho t} dt \\
&\quad + (\pi - \theta_B - \theta_A)^{-1} e^{\theta_B \rho} + (\pi + \theta_A - \theta_B)^{-1} e^{\theta_A \rho} \} \\
&\leq C_3 \eta^{-1} \{ e^{\theta_A \rho} / (\theta_B - \theta_A)^{1/2} + (\pi + \theta_B - \theta_A)^{-1} e^{\theta_B \rho} \\
&\quad + (\pi + \theta_A - \theta_B)^{-1} e^{\theta_A \rho} \}; \\
|I_2| &\leq C_2 \int_{-\eta}^\eta e^{\theta_A |s|} e^{\theta_B(\rho+|s|)} ds \leq 2C_2 e^{\theta_B \rho} e^{\pi \eta},
\end{aligned}$$

and finally by symmetry

$$|I_4| \leq 2C_2 e^{\theta_A \rho} e^{\pi \eta}.$$

Thus the estimate (59) follows for large  $\rho$ ; for small  $\rho$  use  $h_\rho(s)$  instead of  $g_\rho(s)$ . The theorem is proved.  $\square$

There are two interesting corollaries to this result; the first one deals with products of operators of class  $\text{BIP}(X, \theta)$ .

**Corollary (2.1.16)[186]:** Let  $X$  be  $\xi$ -convex,  $A \in \text{BIP}(X, \theta_A), B \in \text{BIP}(X, \theta_B)$  with  $0 \leq \theta_A + \theta_B < \pi$  be resolvent commuting. Define the product  $AB$  of  $A$  and  $B$  by means of

$$(AB)x = ABx, \quad D(AB) = \{x \in D(B): Bx \in D(A)\}.$$

Then  $AB$  is closable and its closure  $\overline{AB}$  belongs to  $\text{BIP}(X, \theta_A + \theta_B)$ . If in addition  $A$  is invertible then  $AB$  is closed.

**Proof:** Since  $B \in \text{BIP}(X, \theta_B)$  implies  $B^{-1} \in \text{BIP}(X, \theta_B)$ , by Theorem (2.1.13) we know that  $A + B^{-1}$  with domain  $D(A) \cap R(B)$  is closed,  $N(A + B^{-1}) = 0$  and  $|Ax| + |B^{-1}x| \leq C|Ax + B^{-1}x|$  on  $D(A) \cap R(B)$ . Suppose  $x_n \in D(AB) = B^{-1}D(A), x_n \rightarrow 0$  and  $ABx_n \rightarrow y$ . Since  $A$  and  $B$  commute with  $(I + B)^{-1}$  we obtain

$$AB(I + B)^{-1}x_n \rightarrow z, \quad B(I + B)^{-1}x_n \rightarrow 0, \quad z = (I + B)^{-1}y$$

Hence  $(I + B)^{-1}y = 0$ , by closedness of  $A$ , and so  $y = 0$ . This shows that  $AB$  is closable. Since  $A$  and  $B$  are resolvent commuting, it is also easy to see that  $AB$  is densely defined, has dense range and is also injective.

Next we obtain

$$|(t + AB)^{-1}| = |B^{-1}(tB^{-1} + A)^{-1}| \leq C/t.$$

By Corollary (2.1.14), hence  $\overline{AB}$  satisfies (H 1). Finally, the relation

$$(\overline{AB})^{i\rho} x = A^{i\rho} B^{i\rho} x, \quad x \in D(A) \cap R(A) \cap D(B) \cap R(B)$$

shows the estimate

$$|(\overline{AB})^{i\rho}| \leq |A^{i\rho}| |B^{i\rho}| \leq K_A e^{\theta_A |\rho|} K_B e^{\theta_B |\rho|}, \quad \rho \in \mathbb{R},$$

hence (H 2) holds and  $\overline{AB}$  belongs to  $BIP(X, \theta_A + \theta_B)$ .

To see that  $AB$  is already closed in case  $A$  is invertible, let  $(x_n) \subset D(AB)$ ,  $x_n \rightarrow x$ , and  $ABx_n \rightarrow z$ . Then  $Bx_n \rightarrow A^{-1}z$  since  $A^{-1}$  is bounded, hence  $x \in D(B)$  and  $Bx = A^{-1}z$  by closedness of  $B$ ; but this in turn implies  $Bx \in D(A)$  and  $z = AA^{-1}z = ABx$ , closedness of  $A$ . Hence  $AB$  is closed.  $\square$

The next corollary deals with sums of  $n$  commuting operators.

**Corollary (2.1.17)[186]:** Suppose  $X$  is  $\xi$ -convex,  $A_i \in BIP(X, \theta_i)$ ,  $i = 1, \dots, n$ , such that, for each pair  $i \neq j$ ,  $A_i$  and  $A_j$  are resolvent commuting and satisfy  $\theta_i + \theta_j < \pi$ . Let  $\theta = \max \theta_i$  and assume that there is only one  $i$  with  $\theta = \theta_i$ .

Then  $A = \sum_1^n A_i$  with domain  $D(A) = \bigcap_1^n D(A_i)$  is closed and belongs to the class  $BIP(X, \theta)$ . Moreover, there is a constant  $c > 0$  such that

$$\sum_1^n |A_i x| \leq C |Ax|, \quad x \in D(A) \tag{60}$$

is satisfied. In particular,  $N(A) = 0$  and  $R(A)$  is dense in  $X$ . Corollary (2.1.17) follows by induction from Theorems (2.1.13) and (2.1.15). Before we conclude this section we want to make another remark. Suppose we are in the situation of Theorem (2.1.13) or more generally of Corollary (2.1.17). If one of the operators  $A_i$  is invertible then we obtain from (60) the estimate

$$|x| \leq C |Ax|, \quad x \in D(A);$$

in other words the range of  $A$  is closed. Since  $R(A)$  is dense in  $X$  this implies that  $A$  itself is invertible.

Will show the applications. Let  $Y$  be a  $\xi$ -convex Banach space,  $B_0$  a closed linear densely defined operator in  $Y$ , and  $a \in BV_{loc}(\mathbb{R}_+)$ , i.e. a scalar-valued function of bounded variation on each interval  $[0, T]$ . As an application of the theory developed above, we consider the abstract Volterra equation of convolution type

$$(d/d t) u (t) + \int_0^t B_0 u (t - \tau) da (\tau) = \int_0^t g(t - \tau) da(\tau), \quad t \in J$$

$$u(0) = 0 \quad (61)$$

where  $J = [0, T]$  or  $J = \mathbb{R}_+$ , and  $g: J \rightarrow Y$  is measurable; in the sequel convolution of the functions  $f$  and  $g$  will be denoted by  $f * g$ . Given  $g \in L^p(J; Y)$ , a continuous function  $u: J \rightarrow Y$  is called a strong solution of (61) if  $u(t) \in D(B_0)$  for a.e.  $t \in J$ ,  $B_0 u(\cdot) \in L^p(J; Y)$ ,  $u \in W_{loc}^{1,p}(J; Y)$ , and (61) is satisfied almost everywhere on  $J$ .

(61) arises naturally in the theory of linear incompressible viscoelastic materials; there  $B_0$  is the Stokes operator introduced in Example (2.1.7) and  $Y = L^p_\sigma(\Omega; \mathbb{R}^3)$ . The kernel  $da$  is called the stress relaxation modulus and is in general of the form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau) d\tau, \quad t \geq 0, \quad (62)$$

where  $a_0 \geq 0$  is a Newtonian viscosity,  $a_\infty \geq 0$  the stationary elasticity modulus, and the relaxation function  $a_1(t)$  is nonnegative, nonincreasing, of positive type, and  $a_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ . We refer to Pipkin [202] for the physical background and to Priiss [203-205] for a detailed study of the properties of (61), as well as to the references given there.

In virtue of the properties of the Stokes operator our main assumption on  $B_0$  is

$$(V1) \quad B_0 \in \text{BIP}(Y, \theta_B) \text{ for some } \theta_B \in [0, \pi/2);$$

concerning the kernel we assume

$$(V2) \quad a(t) \text{ is of the form (62) with } a_0, a_\infty \geq 0 \text{ and } a_1(t) \text{ completely monotonic on } (0, \infty), a_1(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

In the following we shall denote the class of kernels  $a(t)$  satisfying (V2) by **CM**. The assumption on the kernel (V2) could be relaxed to some extent, however, we will not do this here since on the one hand complete monotonicity of  $a_1$  is a quite reasonable assumption which holds for many materials (if not for all), and on the other hand, we want to keep our treatment of (61) as simple as possible, and still obtain significant results. Note that the case of an ordinary Cauchy problem as studied in Giga and Sohr [199] is contained in (61) by choosing  $a(t) \equiv 1$ .

We want to study (61) in the space  $X = L^p(J; Y)$ ,  $1 < p < \infty$  which is again  $\xi$ -convex; cp. the remarks at the beginning of this Section. For this purpose we first introduce an operator  $B$  in  $X$  by means of the definition

$$(Bu)(t) = B_0 u(t), \quad t \in J, \quad D(B) = \{u \in X: u(t) \in D(B_0) \text{ a. e.} \\ \text{on } J, Bu \in X\}; \quad (63)$$

it is easy to verify that  $B$  is a closed linear densely defined operator in  $X$  which belongs to  $BIP(X, \theta_B)$ . The latter follows from (V1) and the fact that  $B_0$  is independent of  $t$ ; the constants  $M, K, \theta$  for  $B$  in  $X$  are in fact the same as those for  $B_0$  in  $Y$ . To obtain a reformulation of (61) to which Theorem (2.1.13) can be applied we have to invert the convolution with the kernel  $da$ . If (V2) holds, this can be done since then there is a kernel  $k(t)$  of class **CM** such that

$$\int_0^t k(t - \tau) da(\tau) = t, \quad t \geq 0, \quad (64)$$

holds; this is a theorem which basically is due to Reuter; cp. Clement and Prüss [195] for the reference and a discussion of this result. In viscoelasticity the function  $k(t)$  is called the creep compliance of the material. Now suppose  $u$  is a strong solution of (61); convolving (61) with  $dk$  and differentiating we then obtain the equation

$$D(dk * Du)(t) + B_0 u(t) = g(t), \quad t \in J, \\ u(0) = 0, \quad (dk * Du)(0) = 0; \quad (65)$$

here we used  $D = d/dt$  for short. On the other hand, if  $u \in D(B) \cap W_{loc}^{1,p}(J; Y)$  is such that  $dk * Du \in W_{loc}^{1,p}(J; Y)$  and (65) holds almost everywhere on  $J$ , convolving (65) with  $da$  we see that  $u$  is a strong solution of (61). Thus (61) and (65) are completely equivalent. We therefore define an operator  $A$  in  $X$  by means of

$$(Au)(t) = D(dk * Du)(t), \quad t \in J, \\ D(A) = \{u \in L^p(J; Y): u, dk * Du \in W_{loc}^{1,p}(J; Y), Au \in X, \\ u(0) = (dk * Du)(0) = 0\}. \quad (66)$$

$A$  is a densely defined linear operator in  $X$  which is also closed. In fact, let  $u_n \rightarrow u$ ,  $u_n \in D(A)$ , and  $Au_n \rightarrow z$  in  $X$ ; put  $w_n = dk * Du_n$ . Then convolving  $Au_n$  with  $da$  we obtain with (64) and  $w_n(0) = 0$  the convergence  $Du_n \rightarrow v$  in  $L_{loc}^p(J; Y)$ , for some  $v \in L_{loc}^p(J; Y)$ , hence  $u \in W_{loc}^{1,p}(J; Y)$  and  $v = Du$ , by closedness of  $D$ . Therefore, we get  $w_n \rightarrow w = dk * Du$  in  $L_{loc}^p(J; Y)$  and since  $Au_n = Dw_n \rightarrow z$ , the closedness of  $D$  yields  $w \in W_{loc}^{1,p}(J; Y)$  and  $z = Dw = D(dk * Du) = Au$ , i.e.  $u \in D(A)$ . It is also easy to see that  $N(A) = 0$ , this follows from the initial conditions  $u(0) = (dk * u)(0) = 0$ .

Convolving the equation  $Au = v$  with  $da$  we derive  $A^{-1}v = a * v$ , for each  $v \in R(A) = D(A^{-1}) = \{v \in X: a * v \in X\}$ ; in particular,  $A$  is invertible iff  $J$  is bounded since  $a(t)$  is never integrable on  $\mathbb{R}_+$ . Equation (61) can now be rewritten in abstract form in the Banach space  $X$  as

$$Au + Bu = g. \quad (67)$$

To prove  $A \in \text{BIP}(X, \theta_A)$  for some  $\theta_A > 0$ , we will need the following lemma which in simpler form was derived in Prüss [203, 205]; for the sake of completeness a proof is included here.

**Lemma (2.1.18) [186]:** Suppose  $a \in L^1_{loc}(\mathbb{R}_+)$ , satisfies (V2). Then there is a function  $c \in \mathbf{CM}$  with  $c_0 = 0$  such that  $a = Dc * Dc$  holds.

**Proof .** Let  $a \in \mathbf{CM}$  and put  $f(\lambda) = \lambda \hat{a}(\lambda)$ ,  $\lambda > 0$ , where the hat indicates Laplace transform. Define operators  $L_k, k = 0, 1, 2, \dots$ , by means of

$$\begin{aligned} (L_k f)(\lambda) &= (-1)^{k-1} (d/d\lambda)^{2k-1} [\lambda^k f(\lambda)], \quad \lambda > 0, \quad k = 1, 2, \dots, \\ (L_0 f)(\lambda) &= f(\lambda), \quad \lambda > 0; \end{aligned} \quad (68)$$

Then  $a \in \mathbf{CM}$  is characterized by  $f \in C^\infty(0, \infty)$  and  $L_k f(\lambda) \geq 0$  for all  $k \geq 0$ ,  $\lambda > 0$ . This is a kind of Bernstein's theorem for the Stieltjes transform; cp. the monograph of Widder [210], Theorems 18b, 14b.

Let  $a_s(t) = a(t) + st, t, s > 0$ , and let  $k_s \in \mathbf{CM}$  denote the solution of (64) with  $a(t)$  replaced by  $a_s(t)$ ; the convolution theorem for the Laplace transform and (64) imply the relation

$$g_s(\lambda) = \lambda \hat{k}_s(\lambda) = [\lambda f(\lambda) + s]^{-1}, \quad \lambda > 0.$$

Note that  $g_s$  satisfies (68) for each  $s > 0$  since  $k_s \in \mathbf{CM}$ . Define  $h \in C^\infty(0, \infty)$  by  $h(\lambda) = \lambda^{-1} \hat{a}(\lambda)^{-1/2}$ ; the formula

$$r^{-1/2} = \pi^{-1} \int_0^\infty (r+s)^{-1} s^{-1/2} ds, \quad r > 0,$$

Then yields the representation

$$\begin{aligned} h(\lambda) &= \pi^{-1} \int_0^\infty (\lambda s + f(\lambda))^{-1} s^{-1/2} ds \\ &= \pi^{-1} \int_0^\infty (r + \lambda f(\lambda))^{-1} r^{-1/2} dr, \end{aligned} \quad (69)$$

where the change of variables  $s = r/\lambda^2$  has been used. Applying the operators  $L_k$  to (69) and interchanging differentiation with integration, we obtain

$$(L_k h)(\lambda) = \pi^{-1} \int_0^{\infty} L_k g_s(\lambda) s^{-\frac{1}{2}} ds, \quad \lambda > 0, k = 0, 1, \dots; \quad (70)$$

note that all integrals are absolutely convergent. Since  $g_s$  satisfies (68) for each  $s > 0$ , (70) shows that  $h$  also has this property hence there is a  $b \in \mathbf{CM}$  such that  $h(\lambda) = \lambda \hat{b}(\lambda)$  for  $\lambda > 0$ . Finally, let  $c \in \mathbf{CM}$  denote the solution of (64) with  $a(t)$  replaced by  $b(t)$ ; then

$$\lambda \hat{c}(\lambda) = \left( \lambda^2 \hat{b}(\lambda) \right)^{-1} = \left( \lambda h(\lambda) \right)^{-1} = \hat{a}(\lambda)^{1/2}, \quad \lambda > 0,$$

and  $c_0 = \lim_{\lambda \rightarrow \infty} \lambda \hat{c}(\lambda) = 0$ . This shows that the function  $Dc(t) = c_\infty + c_1(t)$  satisfies  $Dc * Dc = a$ , by the convolution theorem of the Laplace transform.  $\square$  Let  $C$  denote the closed linear operator in  $X$  defined by means of

$$(Cu)(t) = (Dc * u)(t), t > 0, D(C) = \{u \in X : Cu \in X\}; \quad (71)$$

in Clement and Prüss [195] it has been shown that  $C \in \text{BIP}(X, \theta_c + \varepsilon)$  for any  $\varepsilon > 0$ , where

$$\theta_c = \sup \{|\arg \lambda \hat{c}(\lambda)| : \text{Re } \lambda > 0\} \leq \pi/2, \quad (72)$$

and the constants  $M_c$  and  $K_c$  can be chosen independent of  $J$ , since  $L^p(J; Y)$  are closed subspaces of  $L^p(\mathbb{R}_+; Y)$ ; however, they do depend on  $\varepsilon > 0$ .

Obviously, since  $Dc * Dc = a$  holds,  $A^{-1}$  is a closed linear extension of  $C^2$ ; note that  $C^2$  is always closed since  $\rho(C) \supset (-\infty, 0)$ . We show next that even  $A^{-1} = C^2$  holds; for the case  $J = [0, T]$  this is trivial since  $c \in W^{1,p}(J)$ . For the case  $J = \mathbb{R}_+$ , we have to show that  $u \in X$  and  $a * u \in X$  imply  $Cu = Dc * u \in X$ ; this, however, follows from the identity

$$Cu = Dc * u = (\lambda + C)^{-1}(a * u - \lambda^2 u), \quad \lambda > 0, u \in R(A), \quad (73)$$

which evidently is true for each finite interval  $J = [0, T]$ , but with  $T \rightarrow \infty$  also for  $\mathbb{R}_+$ . If in addition

$$\theta_A = \sup \{|\arg \hat{a}(\lambda)| : \text{Re } \lambda > 0\} = 2\theta_c < \pi \quad (74)$$

is satisfied then by Corollary (2.1.16),  $A$  as well as  $A^{-1}$  belong to  $\text{BIP}(X, \theta_A + \varepsilon)$ . From Theorem (2.1.13) we can now derive

**Theorem (2.1.19) [186]:** Let  $Y$  be a  $\xi$ -convex Banach space,  $p \in (1, \infty)$ ,  $B_0 \in \text{BIP}(Y, \theta_B)$ ,  $a \in \mathbf{CM}$ , and let  $\theta_A + \theta_B < \pi$  hold, where  $\theta_A$  is defined by (74). Then, for every  $g \in L^p_{loc}(\mathbb{R}_+; Y)$  there exists a unique function  $u$

$W_{loc}^{1,p}(\mathbb{R}_+; Y) \cap L_{loc}^p(\mathbb{R}_+; D(B_0))$  which is a strong solution of (61) on each finite interval  $J = [0, T]$  (here  $D(B_0)$  is equipped with the graph norm of  $B_0$ ), and for each  $T > 0$  there is a constant  $c(T) > 0$  such that for each  $g \in L_{loc}^p(\mathbb{R}_+; Y)$ , we have the estimate

$$\int_0^T |u(t)|^p dt + \int_0^T |Ddk * Du(t)|^p dt + \int_0^T |B_0 u(t)|^p dt \leq c(T) \int_0^T |g(t)|^p dt. \quad (75)$$

If, in addition,  $B_0$  is invertible, then  $c(T)$  can be chosen independently of  $T > 0$ . Moreover, if  $g \in L^p(\mathbb{R}_+; Y)$  then  $Ddk * Du, B_0 u \in L^p(\mathbb{R}_+; Y)$  as well, and there is a constant  $c > 0$ , independent of  $T > 0$ , such that

$$\int_0^T |(Ddk * Du)(t)|^p dt + \int_0^T |B_0 u(t)|^p dt \leq c \int_0^T |g(t)|^p dt \text{ for all } 0 < T \leq \infty. \quad (76)$$

**Proof .** Let  $J = [0, T]$  be finite, first. Then  $A$  and  $B$  satisfy the assumptions of Theorem (2.1.13) and  $A$  is invertible; consequently  $A + B$  is again invertible, i.e. for each  $g \in X = L^p(J; Y)$  there exists a unique solution  $u \in D(A) \cap D(B)$  of (67), i.e. of (65) which in turn is equivalent to (61), as we have seen above. Furthermore, we have the estimates

$$|u| + |Au| + |Bu| \leq c_0(T)|g| \text{ and } |Au| + |Bu| \leq c_1|g|, \quad \text{where}$$

$c_1$  is independent of  $T$ ; the latter follows from the fact that the constants  $M_A, M_B, K_A, K_B$  as well as  $\theta_A, \theta_B$  are independent of  $T$  and therefore the constant  $C$  in Theorem (2.1.13) is also independent of  $T$ . Now consider  $g \in L_{loc}^p(\mathbb{R}_+; Y)$  then  $g \in L^p(J; Y)$  for each  $J = [0, T]$  hence there is a unique strong solution  $u$  of (61) on  $J$ . Since the restriction of a solution on  $J$  to a smaller interval  $J_0 = [0, T_0]$  is again a strong solution on this smaller interval, by uniqueness we obtain a unique  $u \in W_{loc}^{1,p}(\mathbb{R}_+; Y) \cap L_{loc}^p(\mathbb{R}_+; D(B_0))$  which satisfies  $dk * Du \in W_{loc}^{1,p}(\mathbb{R}_+; Y)$  and (61) on  $\mathbb{R}_+$ . This proves the local part of Theorem (2.1.19) as well as estimate (75). To prove the second part, let  $g \in L^p(\mathbb{R}_+; Y)$ ; Since  $c_1$  is independent of  $T$  we conclude that  $u$  is then even a strong solution on  $\mathbb{R}_+$  and that estimate (76) holds. The last assertion is also clear since in case  $B_0$  is invertible in  $Y, B$  is so in  $X$  and  $|B^{-1}| = |B_0^{-1}|$  for any interval  $J = [0, Y], 0 < T \leq \infty$ .  $\square$

In the case where  $B_0$  is the Stokes operator from Example (2.1.7) we may choose  $\theta_B > 0$  as small as we want. To apply Theorem (2.1.19) in this case we

thus only need  $\theta_A < \pi$ . Note that  $\theta_A \leq \pi$  always holds, even more is true, namely  $|\arg \hat{a}(\lambda)| < \pi$  for each  $\lambda \in \mathbb{C}, \lambda \neq 0, \operatorname{Re} \lambda \geq 0$ , provided  $a(t) \neq a_\infty t$ ; this excludes the purely elastic case, only. Thus we have to study the limits  $\theta_\infty = \lim_{|\lambda| \rightarrow \infty} \sup |\arg \hat{a}(\lambda)|$  and  $\theta_0 = \lim_{|\lambda| \rightarrow 0} \sup |\arg \hat{a}(\lambda)|$  in the half plane  $\operatorname{Re} \lambda > 0$ , but it is enough to do this for  $\lambda = i\rho, \rho > 0$ , since the Laplace transform of  $a(t)$  is holomorphic on  $\mathbb{C} \setminus (-\infty, 0]$  and  $a(t)$  is real.

Define  $\mathcal{X}: (0, \infty) \rightarrow \mathbb{R}_+$  by means of  $\mathcal{X}(\rho) = \operatorname{Im} \hat{a}(i\rho) / \operatorname{Re} \hat{a}(i\rho)$ ; note that  $\hat{a}(i\rho)$  belongs to the third quadrant for  $\rho > 0$ .

We then have  $\operatorname{tg} \theta_\infty = \lim_{\rho \rightarrow \infty} \inf \mathcal{X}(\rho)$ , i.e.

$$\begin{aligned} \theta_A < \pi \quad \text{iff} \quad v_\infty = \lim_{\rho \rightarrow \infty} \inf \mathcal{X}(\rho) > 0 \quad \text{and} \\ v_0 = \lim_{\rho \rightarrow 0} \inf \mathcal{X}(\rho) > 0. \end{aligned} \tag{77}$$

The following estimate is taken from Prüss [204].

$$\begin{aligned} c_1 \mathcal{X}(\rho) &\leq \left[ a_0 - \int_0^{1/\rho} t D a_1(t) dt \right] / \left[ a_\infty / 2\rho + \rho \int_0^{1/\rho} t a_1(t) dt \right] \\ &\leq c_2 \mathcal{X}(\rho), \quad \rho > 0. \end{aligned} \tag{78}$$

Passing to the limits  $\rho \rightarrow \infty, 0$ , it becomes apparent that  $v_\infty > 0$  implies  $a_0 > 0$  or  $a_1(0+) = \infty$  and  $v_0 > 0$  yields  $a_\infty = 0$ ; thus these conditions are necessary for  $\theta_A < \pi$ . On the other hand, if  $a_0 > 0$  then  $v_\infty = \infty$ , and if  $a_0 = 0$  but  $-\lim_{t \rightarrow 0} t D a_1(t) / a_1(t) > 0$  then  $v_\infty > 0$  by the rule of de l'Hospital; similarly, if  $a_\infty = 0$  and  $-\lim_{t \rightarrow \infty} t D a_1(t) / a_1(t) > 0$  then  $v_0 > 0$ . Obviously, these conditions are satisfied if  $a_1(t)$  behaves like  $t^{-\alpha}$  for  $t \rightarrow \infty$  and like  $t^{-\beta}$  for  $t \rightarrow 0$ , for some  $\alpha, \beta \in (0, 1)$ . Note also that  $a_\infty = 0$  and  $a_1 \in L^1(\mathbb{R}_+)$  imply  $v_0 > 0$  since then  $\hat{a}(\lambda) \sim (a_0 + \int_0^\infty a_1(t) dt) / \lambda$  as  $\lambda \rightarrow 0$ .

In case  $B_0$  is also invertible (i.e. if the domain  $\Omega$  in Example(2.1.7) is bounded) or if the interval  $J$  under consideration is finite then the behavior of  $\hat{a}(\lambda)$  near  $\lambda = 0$  is of no importance. In fact, if  $B \in \text{BIP}(X, \theta_B)$  is invertible then  $B_1 = B - \eta^2 \in \text{BIP}(X, \theta_B + \varepsilon)$  for  $\eta^2 > 0$  sufficiently small; this can be shown by a simple Neumann-series argument. Thus we may replace  $B$  by  $B_1$  and  $A$  by  $A + \eta^2$  in (67), in particular both operators are invertible. For the Laplace transform of  $A + \eta^2$  we obtain the symbol  $\hat{a}(\lambda)^{-1} + \eta^2$ , hence the analog of (74) becomes

$$\theta(A + \eta^2) = \sup \{ \arg (\hat{a}(\lambda)^{-1} + \eta^2) : \operatorname{Re} \lambda > 0 \} < \pi. \tag{79}$$

Since  $\hat{a}(\lambda)$  is never negative real unless  $a_1(t) = a_0 = 0$  and  $\hat{a}(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow 0, \operatorname{Re} \lambda \geq 0$ , similar to the derivation above, we obtain  $\theta(A + \eta^2) <$



$\pi$  iff  $v_\infty > 0$ ; thus in this case the behavior of  $a(t)$  at zero alone determines whether Theorem (2.1.19) is applicable.

For the case of a finite interval  $J = [0, T]$ , we observe that by the change of variables  $v(t) = u(t) e^{-\beta t}$  the kernel  $a(t)$  is transformed into  $a(t) e^{-\beta t}$ ; the Laplace transform of this kernel is given by  $\hat{a}(\lambda + \beta)$ ,  $\text{Re } \lambda > 0$ .

Thus (74) is changed into

$$\theta_A = \sup \{ \arg (\hat{a}(\lambda)): \text{Re } \lambda > \beta \} < \pi; \quad (80)$$

it is clear from this that  $\theta_A < \pi$  iff  $v_\infty > 0$ , and so in this case the behavior of  $a(t)$  at zero alone is important for applicability of Theorem (2.1.19). Let us summarize this as

**Corollary (2.1.20) [186]:** Let  $\Omega \subset \mathbb{R}^n$  be an open domain with smooth and compact boundary  $\partial \Omega$  for  $p \in (1, \infty)$  let  $B_0 \in \text{BIP}(Y, \varepsilon)$  denote the Stokes operator in  $Y = L^p_\sigma(\Omega; \mathbb{R}^n)$ , and let  $a \in \mathbf{CM}$  be such that either  $a_0 > 0$  or  $-\lim_{t \rightarrow 0} tDa_1(t) / a_1(t) > 0$  holds. Then, for every  $g \in L^p_{\text{Log}}(\mathbb{R}_+; Y)$  there exists a unique function  $u \in W^{1,p}_{\text{Log}}(\mathbb{R}_+; Y) \cap L^p_{\text{Loc}}(\mathbb{R}_+; D(B_0))$  which is a strong solution of (61) on each finite interval  $J = [0, T]$ , and for each  $T > 0$  there is a constant  $c(T) > 0$  such that Estimate (75) holds. If in addition  $\Omega$  is bounded then  $c(T)$  can be chosen independently of  $t > 0$ . If  $\Omega$  is unbounded but  $a_\infty = 0$  and  $-\lim_{t \rightarrow 0} tDa_1(t) / a_1(t) > 0$  or  $a_1 \in L^1(\mathbb{R}_+)$  as well as  $g \in L^p(\mathbb{R}_+; Y)$ , then  $Ddk * Du$  and  $B_0 u \in L^p(\mathbb{R}_+; Y)$ , and there is a constant  $c > 0$ , independent of  $T > 0$ , such that Estimate (76) is satisfied. Theorem (2.1.19) and Corollary (2.1.20) generalize recent results of Giga and Sohr [199] who considered the case of a purely Newtonian fluid  $a(t) = a_0 = 1$ , i.e.  $a_\infty = a_1(t) = 0$ . Note that the conditions on  $a(t)$  at  $t = 0$  mean physically that a sufficiently strong viscosity must be present while the condition on  $a(t)$  at  $t = \infty$  prohibits the presence of a stationary elasticity modulus; the case where  $a_\infty = 0$  and  $a_1 \in L^1(\mathbb{R}_+)$  corresponds to a fluid while the material is called solid otherwise; cf. Pipkin [202].

**Corollary(2.1.21)[232]:** Let  $X$  be  $\xi$ -convex,  $A \in \text{BIP}(X, \theta_A)$ ,  $(A + \varepsilon) \in \text{BIP}(X, \theta_{(A+\varepsilon)})$  with  $0 \leq \theta_A + \theta_{(A+\varepsilon)} < \pi$  be resolvent commuting. Define the product  $AB$  of  $A$  and  $(A + \varepsilon)$  by means of

$$\begin{aligned} (A(A + \varepsilon))x &= A(A + \varepsilon)x, \quad D(A(A + \varepsilon)) \\ &= \{x \in D((A + \varepsilon)): (A + \varepsilon)x \in D(A)\}. \end{aligned}$$

Then  $A(A + \varepsilon)$  is closable and its closure  $\overline{A(A + \varepsilon)}$  belongs to  $\text{BIP}(X, \theta_A + \theta_{(A+\varepsilon)})$ . If in addition  $A$  is invertible then  $A(A + \varepsilon)$  is closed.

**Proof:** Since  $(A + \epsilon) \in \text{BIP}(X, \theta_{(A+\epsilon)})$  implies  $(A + \epsilon)^{-1} \in \text{BIP}(X, \theta_{(A+\epsilon)})$ , by Theorem (2.1.13) we know that  $A + (A + \epsilon)^{-1}$  with domain  $D(A) \cap R(A + \epsilon)$  is closed,  $N(A + (A + \epsilon)^{-1}) = 0$  and  $|Ax| + |(A + \epsilon)^{-1}x| \leq C|Ax + (A + \epsilon)^{-1}x|$  on  $D(A) \cap R(A + \epsilon)$ . Suppose  $x_n \in D(A(A + \epsilon)) = (A + \epsilon)^{-1}D(A)$ ,  $x_n \rightarrow 0$  and  $A(A + \epsilon)x_n \rightarrow y$ . Since  $A$  and  $(A + \epsilon)$  commute with  $(I + (A + \epsilon))^{-1}$  we obtain

$$\begin{aligned} A(A + \epsilon)(I + (A + \epsilon))^{-1}x_n &\rightarrow z, & (A + \epsilon)(I + (A + \epsilon))^{-1}x_n &\rightarrow 0, \\ z &= (I + (A + \epsilon))^{-1}y \end{aligned}$$

hence  $(I + (A + \epsilon))^{-1}y = 0$ , by closedness of  $A$ , and so  $y = 0$ . This shows that  $A(A + \epsilon)$  is closable. Since  $A$  and  $(A + \epsilon)$  are resolvent commuting, it is also easy to see that  $A(A + \epsilon)$  is densely defined, has dense range and is also injective.

Next we obtain

$$|(t + A(A + \epsilon))^{-1}| = |(A + \epsilon)^{-1}(t(A + \epsilon)^{-1} + A)^{-1}| \leq C/t$$

by Corollary (2.1.14), hence  $\overline{A(A + \epsilon)}$  satisfies (H 1). Finally, the relation

$$(\overline{A(A + \epsilon)})^{i\rho}x = A^{i\rho}(A + \epsilon)^{i\rho}x, \quad x \in D(A) \cap R(A) \cap D(A + \epsilon) \cap R(A + \epsilon)$$

shows the estimate

$$|(\overline{A(A + \epsilon)})^{i\rho}| \leq |A^{i\rho}| |(A + \epsilon)^{i\rho}| \leq K_A e^{\theta_A|\rho|} K_{(A+\epsilon)} e^{\theta_{(A+\epsilon)}|\rho|}, \quad \rho \in \mathbb{R},$$

hence (H 2) holds and  $\overline{A(A + \epsilon)}$  belongs to  $\text{BIP}(X, \theta_A + \theta_{(A+\epsilon)})$ .

To see that  $A(A + \epsilon)$  is already closed in case  $A$  is invertible, let  $(x_n) \subset D(A(A + \epsilon))$ ,  $x_n \rightarrow x$ , and  $A(A + \epsilon)x_n \rightarrow z$ . Then  $(A + \epsilon)x_n \rightarrow A^{-1}z$  since  $A^{-1}$  is bounded, hence  $x \in D(A + \epsilon)$  and  $(A + \epsilon)x = A^{-1}z$  by closedness of  $B$ ; but this in turn implies  $(A + \epsilon)x \in D(A)$  and  $z = AA^{-1}z = A(A + \epsilon)x$ , closedness of  $A$ . Hence  $A(A + \epsilon)$  is closed.  $\square$

## Sec (2.2): Examples of Unbounded Imaginary Powers of Operators

In a recent section, Dore and Venni [182] have used imaginary powers of operators in connection with the problem of the closedness of the sum of two operators. Roughly speaking, if  $A$  and  $B$  are two commuting closed operators in a UMD-space, then their sum is closed provided that the following condition holds:

$$\begin{cases} \|A^{is}\| \leq M e^{\omega_A|s|} \text{ and } \|B^{is}\| \leq M e^{\omega_B|s|}, s \in \mathbb{R} \\ \text{with } \omega_A + \omega_B < \pi. \end{cases} \quad (81)$$

The UMD-spaces are precisely the Banach spaces  $X$  for which the vector valued Hilbert transform is bounded in  $L^2(\mathbb{R}; X)$  [179,180]. In particular, the Hilbert spaces and  $L^p$ -spaces,  $1 < p < \infty$ , are UMD-spaces.

The growth condition (81) implies that the spectrum of  $A$  (resp.  $B$ ) lies in a sector of “angle”  $\omega_A$  (resp.  $\omega_B$ ).

In [182], the question was raised whether the converse is true. The Example (2.2. 1) below shows that this is not the case, even in a Hilbert space.

However, in a Hilbert space, the conditions for the closedness of the sum can be weakened, as shown again by Dore and Venni [182]. Based on a characterization of the domain of fractional powers together with an earlier result of Da Prato and Grisvard [181], they proved the following result.

If  $A^{is}$  is a  $c_0$ -group of bounded operators (without any assumption on  $B^{is}$ ), then  $A + B$  is closed provided that the sum of the “angles”  $\omega_A$  and  $\omega_B$  is less than  $\pi$ .

In Example (2.2.2) we give two operators  $A$  and  $B$  in a Hilbert space which satisfy the “angle condition” such that  $A + B$  is not closed. This shows again that  $A^{is}$  and  $B^{is}$  are not  $c_0$ -groups of bounded operators. Moreover this implies that some extra condition is needed for the closedness of the sum.

In this Section, we state the main results. also , we introduce the main tools for the examples, in particular the notion of spectral family [178,183], also we construct the Example (2.2.1) inspired by Example 5.10,p. 168, of Berkson and Gillespie [ 187].

Finally we give Example (2.2.2) we are convinced that the method used to can lead to other examples.

Let  $(X, \| \cdot \|)$  be a complex Banach space, and let  $A : D(A) \subset X \rightarrow X$  be a closed and densely defined operator with domain  $D(A)$  and range  $R(A)$ . As usual, we denote the resolvent set of  $A$  by  $\rho(A)$  and its spectrum by  $\sigma(A)$ . The operator  $A$  is called positive [182,190] if

- (i)  $(-\infty, 0) \subset \rho(A)$
- (ii) there exists  $M \geq 1$  such that  $\|(I - t A)^{-1}\| \leq M$ , for every  $t > 0$ .

In particular, if  $M = 1$ , then  $A$  is called  $m$ -accretive.

For  $\theta \in [0, \pi)$ , we define the sector  $\Sigma_\theta$  as

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\}; |\arg z| \leq \theta\}.$$

The operator  $A$  is said to be of type  $(\omega, M)$  [ 189], if there exist  $0 < \omega < \pi$  and  $M \geq 1$  such that

- (i)  $\sigma(A) \subset \Sigma_\omega \cup \{0\}$ ;

(ii) for every  $\theta \in [0, \pi - \omega)$ , there exists  $M(\theta) \geq 1$  with  $M(0) = M$ , such that  $\|(I + zA)^{-1}\| \leq M(\theta)$  for any  $z \in \Sigma_\theta$ .

We recall that if the operator  $A$  is positive, then there exist  $\theta \in (0, \pi)$  and  $M \geq 1$  such that  $A$  is of type  $(\theta, M)$  [190].

We also recall that if  $A$  is  $m$ -accretive, then  $A$  is of type  $(\pi/2, 1)$  [189]. Moreover if  $A$  is of type  $(\omega, M)$  for some  $\omega \in (0, \pi/2)$  and  $M \geq 1$ , then  $-A$  generates an analytic semigroup on the space  $X$ .

If  $A$  is a bounded positive operator with  $0 \in \rho(A)$ , then the fractional powers of  $A$  denoted by  $A^z$  with  $z \in \mathbb{C}$  are usually defined by the Dunford integral

$$A^z = \frac{1}{2i\pi} \int_{\Gamma} \lambda^z (\lambda - A)^{-1} d\lambda,$$

where the contour  $\Gamma$  does not meet  $(-\infty, 0]$  and contains the spectrum of  $A$ . Then for  $z \in \mathbb{C}$ ,  $A^z$  is a bounded operator satisfying the group property

$$A^{z_1+z_2} = A^{z_1}A^{z_2}, \quad z_1, z_2 \in \mathbb{C}, \text{ with } A^0 = I \text{ and } A^1 = A.$$

The function  $z \mapsto A^z$  is also holomorphic. Moreover, one has the other representations of  $A^z$  [186],

$$A^z x = \frac{\sin \pi z}{\pi} \left\{ z^{-1}x - (1+z)^{-1}A^{-1}x + \int_0^1 t^{z+1}(t+A)^{-1}A^{-1}x dt + \int_1^\infty t^{z-1}(t+A)^{-1}Ax dt \right\} \quad \text{for } |\operatorname{Re} z| < 1, z \neq 0, \quad (82)$$

$$A^0 x = x.$$

Or equivalently

$$A^z x = \frac{\sin \pi z}{\pi} \left\{ z^{-1}x - (1+z)^{-1}A^{-1}x + (1-z)^{-1}Ax + \int_0^1 t^z(1+t^{-1}A)^{-1}A^{-1}x dt - \int_0^1 t^{-z}(1+t^{-1}+A^{-1})^{-1}Ax dt \right\} \quad \text{for } |\operatorname{Re} z| < 1, z \neq 0, \quad (83)$$

$$A^0 x = x.$$

If the positive operator  $A$  satisfies only  $N(A) = \{0\}$  and  $R(A)$  dense in  $X$ ,

then for every  $x \in D(A) \cap R(A)$ , which is dense in  $X$ , the function  $z \mapsto A^z x$  defined by (82) or (83) is holomorphic and satisfies the group property  $A^{z_1+z_2}x = A^{z_1}A^{z_2}x = A^{z_2}A^{z_1}x$  for every  $x \in D(A^2) \cap R(A^2)$  and  $[|\operatorname{Re} z_1|, |\operatorname{Re} z_2|, |\operatorname{Re}(z_1 + z_2)|] < 1$  [186].

For  $s \in \mathbb{R} \setminus \{0\}$ , we say that  $A^{is}$  is bounded if the operator  $A^{is}$  defined by (82) (or (83)) is bounded on  $D(A) \cap R(A)$ . Then it can be uniquely extended to  $X$ , as a bounded operator.

Following Prüb and Sohr [186], the operator  $A$  is said to belong to the class  $BIP(X, \theta)$  for some  $\theta \in [0, \pi)$  if:

- (i)  $A$  is positive;
- (ii)  $N(A) = \{0\}$  and  $R(A)$  dense in  $X$ ;
- (iii)  $A^{is} \in B(X)$  for every  $s \in \mathbb{R}$  and there exists  $M > 0$  such that  $\|A^{is}\| \leq M e^{\theta|s|}$ ,  $s \in \mathbb{R}$ .

In the case where  $A$  is positive,  $N(A) = \{0\}$  implies the density of  $R(A)$  in  $X$  if  $X$  is a reflexive Banach space (a Hilbert space, for example).

It is proven in [186], that if  $A \in BIP(X, \theta)$  then  $A$  is of type  $(\theta, M)$  for some  $M \geq 1$ . In Example (2.2.1), we show in particular that the converse is not true even if the space  $X$  is a Hilbert space.

**Example (2.2.1)[177]:** There exists an operator  $A$  in a Hilbert space which is of type  $(\omega, M)$  for some  $M > 1$  and for all  $\omega \in (0, \pi)$  and such that the imaginary powers  $A^{is}$  are not bounded for all  $s \in \mathbb{R} \setminus \{0\}$ .

Let  $A$  and  $B$  be two positive operators in a Banach space  $(X, \|\cdot\|)$ . The operators  $A$  and  $B$  are called resolvent commuting if  $(I + tA)^{-1}$  and  $(I + sB)^{-1}$  commute for some  $t$  and  $s > 0$  (equivalently for all  $t$  and  $s > 0$ ).

Building upon results of Dore and Venni [182], Prüb and Sohr [186] have proven that if  $A_i \in BIP(X, \theta_i)$ ,  $i = 1, 2$  with  $\theta_1 \neq \theta_2$ ,  $\theta_1 + \theta_2 < \pi$ , are resolvent commuting and if  $X$  is a UMD-space, then  $A_1 + A_2 \in BIP(X, \theta)$  where  $\theta = \max(\theta_1, \theta_2)$ .

Da Prato and Grisvard [181] have proved that if  $A_i$  are of type  $(\theta_i, M_i)$ ,  $i = 1, 2$ ,  $\theta_1 + \theta_2 < \pi$ , resolvent commuting (hence  $A_1 + A_2$  closable) then the closure of  $A_1 + A_2$  is of type  $(\theta, M)$  with  $\theta = \max(\theta_1, \theta_2)$  for some  $M \geq 1$ .

Therefore a natural question is to know whether the sum of two operators  $A$  and  $B$  satisfying the assumptions of Da Prato and Grisvard in a UMD-space is closed. In the Hilbert space, Da Prato and Grisvard [181] gave a sufficient condition for this to be the case, namely if the interpolation spaces  $D_A(\theta, 2)$  and  $D_{A^*}(\theta, 2)$  are equal for some  $\theta \in (0, 1)$ . For the definition of these spaces, we refer the reader to the original paper [181]. Since  $A + B$  is closed if and only if  $I + A + B$  is closed, we may assume without loss of generality that  $0 \in \rho(A)$

and  $0 \in \rho(B)$ . Under these assumptions Dore and Venni [182, p. 194], have shown that if the imaginary powers  $A^{is}$  are uniformly bounded for  $s \in [-1, 1]$ , then  $A + B$  is closed. We have:

**Example (2.2.2)[177]:** There exist two resolvent commuting operators  $A$  and  $B$  in a Hilbert space which are of type  $(\omega, M)$  for some  $M > 1$  and for every  $\omega \in (0, \pi)$  such that  $A + B$  is not closed.

**Remark (2.2.3)[177]:** (i) It follows from Da Prato and Grisvard [181] that  $D_A(\theta, 2) \neq D_{A^*}(\theta, 2)$  and  $D_B(\theta, 2) \neq D_{B^*}(\theta, 2)$  for every  $\theta \in (0, 1)$ .

(ii) It follows from Dore and Venni [182] that both  $A^{is}$  and  $B^{is}$  are not uniformly bounded on  $[-1, 1]$ .

We recall the notion of spectral family of projections in a Hilbert space  $H$  [178, 183].

**Definition (2.2.4)[177]:** A spectral family of projections in  $H$  is a uniformly bounded projection-valued function  $F: \mathbb{R} \rightarrow B(H)$  (the algebra of bounded linear operators in  $H$ ) such that:

- (i)  $F$  is right-continuous on  $\mathbb{R}$  in the strong operator topology,
- (ii)  $F$  has a strong left-hand limit at each  $s \in \mathbb{R}$ ,
- (iii)  $F(s)F(t) = F(t)F(s) = F(s)$  for  $s \leq t$ ,
- (iv)  $F(s) \rightarrow 0$  (resp.  $F(s) \rightarrow I$ ) in the strong operator topology as  $s \rightarrow -\infty$  (resp. as  $s \rightarrow +\infty$ ).

If there is a compact interval  $[a, b]$  such that  $F(s) = 0$  for  $s < a$  and  $F(s) = I$  for  $s \geq b$ , then we say that  $F$  is concentrated on  $[a, b]$ . Following [178, 183], if  $F$  is a spectral family concentrated on  $[a, b]$ , each complex valued function  $f \in C[a, b] \cap BV[a, b]$  defines a bounded operator  $A$  in  $H$  ( $BV$  stands for bounded variation):

$$Ax = \int_{[a,b]} f(\lambda) dF(\lambda)x, \quad x \in H, \quad (84)$$

by means of convergence of Riemann-Stieltjes sums. Moreover the norm of  $A$  can be estimated by

$$\|A\| \leq |f(b)| + (|f(a)| + \text{Var}[f; [a, b]]). \|F\|, \quad (85)$$

where

$$\|f\| := \sup_{\lambda \in \mathfrak{R}} \|F(\lambda)\|. \quad (86)$$

If  $F$  is concentrated on  $[0, \infty)$  and  $f \in C[0, \infty) \cap BV[0, \infty)$ , then  $s$ - $\lim_{N \rightarrow \infty} \int_{[0, N]} f(\lambda) dF(\lambda)$  exists. This limit defines a bounded operator  $A$  in  $H$  satisfying

$$\|A\| \leq |f(\infty)| + (|f(0)| + \text{Var}[f; [0, \infty)]). \|F\|, \quad (87)$$

where  $\|F\|$  is defined by (86) and  $f(\infty) = \lim_{\lambda \rightarrow \infty} f(\lambda)$  which exists since  $f \in BV[0, \infty)$ .

If  $f, g \in C[0, \infty) \cap BV[0, \infty)$  and

$$Ax = \int_{[0, \infty)} f(\lambda) dF(\lambda)x, \quad Bx = \int_{[0, \infty)} g(\lambda) dF(\lambda)x, \quad x \in H,$$

$$\text{then } (A + B)x = \int_{[0, \infty)} (f(\lambda) + g(\lambda))dF(\lambda)x.$$

If moreover  $f, g \in BV[0, \infty)$ , then

$$ABx = BAx = \int_{[0, \infty)} f(\lambda)g(\lambda)dF(\lambda)x.$$

If  $f(\lambda) \neq 0$ , for every  $\lambda \geq 0$  and  $\lambda \mapsto f(\lambda)^{-1}$  belongs to  $BV[0, \infty)$ , then  $0 \in \rho(A)$  and

$$A^{-1}x = \int_{[0, \infty)} f(\lambda)^{-1} dF(\lambda)x.$$

For the construction of a spectral family in  $\ell^2(\mathbb{N})$  which is not a spectral measure, we shall use, as in [178], a conditional basis which can be found in Singer [187]. For the sake of completeness, we give it here explicitly. Conditional Bases in  $\ell^2(\mathbb{N})$ . The sequences  $\{f_n\}_{n \geq 1}$  and  $\{h_n\}_{n \geq 1}$  in  $\ell^2(\mathbb{N})$  defined by

$$f_{2n-1} = e_{2n-1} + \sum_{i=n}^{\infty} \alpha_{i-n+1} e_{2i}, \quad f_{2n} = e_{2n}, \quad (n = 1, 2, \dots) \quad (88)$$

$$h_{2n-1} = e_{2n-1}, \quad h_{2n} = - \sum_{i=1}^n \alpha_{i-n+1} e_{2i-1} + e_{2n}, \quad (n = 1, 2, \dots) \quad (89)$$

where  $\{e_n\}_{n \geq 1}$  is the canonical basis of  $\ell^2(\mathbb{N})$  and  $\alpha_n \geq 0$ ,  $n = 1, 2, \dots$ ,  $\sum_{j=1}^{\infty} j \alpha_j^2 < \infty$ ,  $\sum_{j=1}^{\infty} \alpha_j = +\infty$  (e.g., one can take  $\alpha_n = 1/n \log(n+1)$ ) are biorthogonal conditional bases of  $\ell^2(\mathbb{N})$ . Defining  $P_n \in B(\ell^2(\mathbb{N}))$  by

$$P_n x = (x, h_n) f_n, \quad x \in \ell^2(\mathbb{N}), \quad n = 1, 2, \dots,$$

where  $(\cdot, \cdot)$  is the scalar product, then each  $P_n$  is a projection with  $P_n P_m = 0$  for  $m \neq n$  satisfying

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} P_j x = x, \quad x \in \ell^2(\mathbb{N}). \quad (90)$$

Moreover

$$\sup \left\| \sum_{j=1}^n P_{2j} \right\| = \infty. \quad (91)$$

For the proofs of (90)-(91), see Singer [187].

In the Example (2.2.1) We construct an example of a positive operator  $A$  in a Hilbert space  $H$  such that the imaginary powers  $A^{is}$  are not bounded for  $s \in \mathbb{R} \setminus \{0\}$ , although  $A$  is of type  $(\omega, M)$  for some  $M > 1$  and for every  $\omega \in (0, \pi)$ . In order to do that, we construct the operator  $A$  on a Hilbert product.

Let  $\{H_k, \|\cdot\|_k\}_{k \in \mathbb{Z}}$  be a family of complex Hilbert spaces. Let  $(H, \|\cdot\|)$  be the Hilbert product

$$H = \left( \prod_{k \in \mathbb{Z}} H_k \right)_2 = \left\{ x = (x_k), x_k \in H_k, \|x\|^2 = \sum_{k \in \mathbb{Z}} \|x_k\|_k^2 < \infty \right\}.$$

The family  $\{A_k\}_{k \in \mathbb{Z}}$  of bounded operators on  $H_k$ , defines the following closed densely defined operator  $A$  on  $H$ :

$$D(A) := \left\{ x = (x_k), x_k \in H_k, \sum_{k \in \mathbb{Z}} \|A_k x_k\|_k^2 < \infty \right\} \quad (92)$$

$$(Ax)_k := A_k x_k, \quad k \in \mathbb{Z} \quad \text{for } x = (x_k) \in D(A).$$

Moreover  $A$  is bounded if and only if  $\text{Sup}_{k \in \mathbb{Z}} \|A_k\|_k < \infty$  and if this is the case  $\|A\| = \text{Sup}_{k \in \mathbb{Z}} \|A_k\|$ .

We say that the family of positive operators  $\{A_k\}_{k \in \mathbb{Z}}$  satisfies Property (P) if :

- (i)  $\sigma(A_k) \subset [0, \infty)$ ;
- (ii) for every  $\theta \in [0, \pi)$ , there is  $M(\theta)$  independent of  $k$ , such that  $\|(I + zA_k)^{-1}\|_k \leq M(\theta)$  for every  $k \in \mathbb{Z}$  and every  $z \in \Sigma_\theta$ .

We have

**Lemma (2.2.5)[177]:** Let  $\{A_k\}_{k \in \mathbb{Z}}$  be a family of bounded positive operators on  $H_k, k \in \mathbb{Z}$ , satisfying Property (P). Then there exists  $M \geq 1$ , such that the operator  $A$  defined by (92), is of type  $(\omega, M)$  for every  $\omega \in (0, \pi)$ .

Moreover if  $N(A) = \{0\}$ , then for every  $x = (x_k) \in D(A) \cap R(A)$ , and  $s \in \mathbb{R} \setminus \{0\}$ , we have  $x_k \in D(A_k) \cap R(A_k)$ , and  $(A^{is}x)_k = (A_k)^{is}x_k, k \in \mathbb{Z}$ .

**Proof** (i) Let  $z \in \mathbb{C} \setminus (-\infty, 0]$  and let  $\theta = \arg z$ . Let  $y = (y_k) \in H$ . Since  $A$  satisfies Property (P),  $-z^{-1} \notin \sigma(A_k)$  and there exists  $x_k \in H_k, k \in \mathbb{Z}$  such that

$$(I + zA_k)x_k = y_k, \quad k \in \mathbb{Z}.$$



Since  $\|x_k\| \leq M(\theta)\|y_k\|_k$ , we have  $x = (x_k) \in D(A)$  and  $\|x\| \leq M(\theta)\|y\|$ . Moreover since  $N(I + zA_k) = \{0\}$ , we have  $N(I + zA) = \{0\}$ ,  $-z^{-1} \in \rho(A)$ , and  $\|(I + zA)^{-1}\| \leq M(\theta)$ . This implies that  $A$  is of type  $(\omega, M)$  with  $M = M(0)$ , for every  $\omega \in (0, \pi)$ .

(ii) Assume  $N(A) = \{0\}$ , then  $N(A_k) = \{0\}$  for every  $k \in \mathbb{Z}$ . Let  $x = (x_k) \in D(A) \cap R(A)$ . Then clearly,  $x_k \in D(A_k) = H_k$ . Since  $x = Ay$  for some  $y \in D(A)$ , we have  $x_k = A_k y_k$  hence  $x_k \in R(A_k)$ . Therefore  $A^{is}x$  and  $(A_k)^{is}x_k$  are well-defined by (82), for

$s \in \mathbb{R} \setminus \{0\}$ , Since  $((I + tA)^{-1}x)_k = (I + tA_k)^{-1}x_k$ ,  $t > 0$ ,  $x = (x_k) \in H$ , we obtain  $(A^{is}x)_k = (A_k)^{is}x_k$ ,  $k \in \mathbb{Z}$ . This completes the proof of Lemma(2.2.5).

Next, we construct a family of bounded positive operators  $\{A_k\}_{k \in \mathbb{Z}}$  in  $\ell^2(\mathbb{N})$ , such that  $0 \in \rho(A_k)$  and satisfying Property (P). Notice that the imaginary powers  $A_k^{is}$ ,  $s \in \mathbb{R}$ , are then bounded. We give a necessary condition for  $\sup_{k \in \mathbb{Z}} \|A_k^{is}\|$  to be finite for some  $s \in \mathbb{R} \setminus \{0\}$ .

**Lemma (2.2.6)[177]:** Let  $\{f_n\}_{n \geq 1}$  be a (Schauder) basis of  $\ell^2(\mathbb{N})$ , with corresponding projections  $\{P_n\}_{n \geq 1}$ .

Let  $F: \mathbb{R} + B(\ell^2(\mathbb{N}), \cdot)$  be the spectralfamily concentrated on  $[0, 1]$  defined by

$$F(\lambda) = 0 \quad \text{for } \lambda < 1/2$$

$$F(\lambda) = \sum_{k=1}^n P_k \quad \text{for } \frac{n}{n+1} \leq \lambda < \frac{n+1}{n+2} \quad \text{for } n = 1, 2, \dots$$

$$F(\lambda) = I \quad \text{for } \lambda \geq 1.$$

Then for every  $k \in \mathbb{Z}$  and every  $x \in \ell^2(\mathbb{N})$ ,

$$A_k x = \int_{[0,1]} e^{k\lambda} dF(\lambda)x \quad \text{is well defined}$$

and

(i) The family of operators  $\{A_k\}_{k \in \mathbb{Z}}$  satisfies Property (P) and  $0 \in \rho(A_k)$ ,  $k \in \mathbb{Z}$ .

(ii) For every  $s \in \mathbb{R}$ , the imaginary power  $A_k^{is}$  is bounded and  $A_k^{is}x = \int_{[0,1]} e^{isk\lambda} dF(\lambda)x$ ,  $x \in \ell^2(\mathbb{N})$ ,  $k \in \mathbb{Z}$ . Moreover  $A_k^{is} = A_1^{iks}$ .

(iii) If for Some  $s \in \mathbb{R} \setminus \{0\}$ ,  $\sup_{k \in \mathbb{Z}} \|A_k^{is}\| < \infty$ , then the basis  $\{f_n\}_{n \geq 1}$  is unconditional.

(iv) If the basis  $\{f_n\}_{n \geq 1}$  is unconditional, then for all  $s \in \mathbb{R}$ ,  $\sup_{k \in \mathbb{Z}} \|A_k^{is}\| < \infty$ .

**Proof.** (i) For every  $k \in \mathbb{Z}$ , the function  $\lambda \mapsto \exp\{k\lambda\}$  is continuous, bounded, increasing, hence of bounded variation on  $[0, 1]$ . Therefore  $A_k$  is well-defined and bounded on  $\ell^2(\mathbb{N})$ , as well as  $A_k^{-1}x$ . Moreover  $A_k = A_j^k$ .

Let  $z \in \mathbb{C} \setminus (\infty, 0]$  and  $\theta = \arg z$ . Then the function  $\lambda \rightarrow a(\lambda; k, z) := (1 + z \exp(k\lambda))^{-1}$  is continuous, bounded, and of bounded variation on  $[0, 1]$ . Indeed  $|1 + z e^{k\lambda}|^{-1} = |1 + |z|e^{i\theta} e^{k\lambda}|^{-1}$ , then  $|a(\lambda; z, z)| \leq m_1(\theta)$ , where

$$m_1(\theta) \leq \begin{cases} 1 & \text{when } 0 \leq |\theta| \leq \frac{\pi}{2} \\ \frac{1}{\sin|\theta|} & \text{when } |\theta| > \frac{\pi}{2} \end{cases} .$$

Moreover

$$\begin{aligned} \text{Var}_{\lambda \in [0,1]} [a(\lambda; k, z)] &= \int_{[0,1]} \left| \frac{d}{d\lambda} a(\lambda, k, z) \right| d\lambda = \int_{[0,1]} \frac{|kz| e^{k\lambda}}{|a(\lambda; k, z)|^2} d\lambda \\ &= \int_0^{|k|} \frac{|z| e^{(\text{sign}k)\lambda}}{|a((\text{sign}k)\lambda, 1, z)|^2} d\lambda \leq \int_0^\infty \frac{|z| e^{(\text{sign}k)\lambda}}{|1 + |z| e^{i\theta} e^{(\text{sign}k)\lambda}|^2} \\ &\leq \int_0^\infty \frac{dt}{|1 + t e^{i\theta}|^2} = m_2(\theta) \quad \text{with} \end{aligned}$$

$$m_2(\theta) = \begin{cases} 1 & \text{if } \theta = 0 \\ \frac{\theta}{\sin \theta} & \text{if } 0 < |\theta| < \pi \end{cases} .$$

Let  $M(\theta) = m_1(\theta) + (m_1(\theta) + m_2(\theta)) \cdot \|F\|$ . We observe that  $M(-\theta) = M(\theta)$  and  $M(\theta)$  increases on  $0 \leq \theta < \pi$ .

Therefore  $-z^{-1} \in \rho(A_k)$  and  $\|(I + zA_k)^{-1}\| \leq M(\theta)$ , which implies that the family  $\{A_k\}_{k \in \mathbb{Z}}$  satisfies Property (P).

(ii) Let  $b(\lambda; k, s) := \exp(isk\lambda)$  for  $\lambda \in [0, 1], k \in \mathbb{Z}$ , and  $s \in \mathbb{R}$ . Then  $|b(\lambda; k, s)| \leq 1$  and

$$\text{Var}_{\lambda \in [0, \infty)} b(\lambda; k, s) = \int_0^1 \left| \frac{db}{d\lambda}(\lambda; k, s) \right| d\lambda = |sk| .$$

Hence  $\int_{[0,1]} e^{isk\lambda} dF(\lambda)$  defines a bounded operator  $C_{k,s}$  in  $\ell^2(\mathbb{N})$ , for every  $s \in \mathbb{R}$  and  $k \in \mathbb{Z}$ . For  $x = (x_k) \in c_{00}$  (finite sequences in  $\ell^2(\mathbb{N})$ ), we have

$$C_{k,s}x = \sum_{l=-m}^m \exp(iskl) P_l x \quad \text{for some } m \in \mathbb{N} \text{ depending on } x.$$

By using the Dunford integral for the imaginary power  $A_k^{is}x$ , we obtain

$$\begin{aligned}
A_k^{is}x &= \frac{1}{2i\pi} \int_{\Gamma} \lambda^{is} (\lambda - A_k)^{-1} x \, d\lambda \\
&= \frac{1}{2i\pi} \int_{\Gamma} \lambda^{is} \sum_{l=-m}^m (\lambda - \exp(kl))^{-1} P_l x \, d\lambda \\
&= \sum_{l=-m}^m \frac{1}{2i\pi} \int_{\Gamma} \lambda^{is} (\lambda - \exp(kl))^{-1} P_l x \, d\lambda \\
&= C_{k,s}x.
\end{aligned}$$

Since both  $A_k^{is}$  and  $C_{k,s}$  are bounded on  $\ell^2(\mathbb{N})$  and  $c_{00}$  is dense in  $\ell^2(\mathbb{N})$ , we have  $C_{k,s} = A_k^{is}$ . We also have  $A_k^{is} = A_1^{iks}$ .

(iii) If  $\sup_{k \in \mathbb{Z}} \|A_k^{is}\| < \infty$  for some  $s \in \mathbb{R} \setminus \{0\}$ , then  $\sup_{k \in \mathbb{Z}} \|A_1^{iks}\| < \infty$  and without loss of generality, we may assume  $s > 0$ . We also have  $A_1^{iks} = (A_1^{is})^k$ . By using a result of Nagy [185,188], there exists an equivalent Hilbertian norm  $\|\cdot\|$  on  $H$  such that  $\|A_1^{iks}\| = 1$ , for every  $k \in \mathbb{Z}$ . (Take, e.g.,  $\|x\| = \lim_{n \rightarrow \infty} \|A_1^{isn}x\|^2)^{1/2}$  where  $\text{Lim}$  is a Banach limit in  $\mathbb{N}$ ) Then  $A_1^{is}$  is unitary in  $(H, \|\cdot\|)$  and  $\{f_n\}_{n \geq 1}$  are eigenvectors corresponding to the eigenvalues

$$\mu_n = e^{isn/(n+1)}, \quad n = 1, 2, \dots$$

Then for  $m, n > s/2\pi$ ,  $m \neq n$ , we have  $\mu_m \neq \mu_n$  Therefore  $\{f_n\}_{n > s/2\pi}$  is an orthogonal system in  $(H, \|\cdot\|)$ , hence  $\{f_n\}_{n \geq 1}$  is an unconditional basis in  $(H, \|\cdot\|)$  and also in  $(H, \|\cdot\|)$ .

(iv) Suppose the basis  $\{f_n\}_{n \geq 1}$  is unconditional. By using a characterization of unconditional bases, see, e.g., [187, Theorem 17.1.6], there exists a constant  $C > 0$  such that  $\|\sum_{i=1}^n \alpha_i f_i\| \leq C \|\sum_{i=1}^n |\alpha_i| f_i\|$  for every  $n \in \mathbb{N}$  and every finite scalar sequence  $\{\alpha_i\}$ .

For  $x \in H_0$  (the linear dense subspace spanned by  $\{f_n\}_{n \geq 1}$ ),  $k \in \mathbb{Z}$ ,  $s \in \mathbb{R}$ .

We have  $A_1^{iks}x = \sum_{n \geq 1} \exp(iskn/(n+1))P_n x$ , the sum is finite. Hence  $\|A_1^{iks}x\| \leq C \|\sum_{n \geq 1} \exp(iskn/(n+1))P_n x\| = C\|x\|$ . Then  $\|A_1^{iks}x\| \leq C$ .

After these preparations, we can easily construct the operator  $A$ . Construction of  $A$ . Let  $H_k = \ell_2(\mathbb{N})$ ,  $k \in \mathbb{Z}$ , and let  $\{f_n\}_{n \geq 1}$  be a conditional basis of  $\ell^2(\mathbb{N})$  for example, the basis defined in (88). Define  $A_k$ , like in Lemma (2.2.6), then for every  $s \in \mathbb{R} \setminus \{0\}$ ,  $\sup_{k \in \mathbb{Z}} \|A_k^{is}\| = \infty$ . Then define the operator  $A$ , like in

Lemma (2.2.5). The operator  $A$  is of type  $(\omega, M)$  for some  $M \geq 1$  and for every  $\omega \in (0, \pi)$ . Moreover for  $s \in \mathbb{R} \setminus \{0\}$ ,  $A^{is}$  cannot be bounded, otherwise  $\sup_{k \in \mathbb{Z}} \|A_k^{is}\|$  would be finite. Therefore the operator  $A$  satisfies all the required properties.

In this section, we construct an example of two resolvent commuting, closed operators  $A$  and  $B$ , in a Hilbert space  $H$  such that  $A$  and  $B$  are of type  $(\omega, M)$  for some  $M > 1$  and every  $\omega \in (0, \pi)$ , with  $A + B$  not closed. Let  $H = \ell^2(\mathbb{N})$ ,  $\{f_n\}_{n \geq 1}$  be a (Schauder) basis in  $\ell^2(\mathbb{N})$ , and  $\{P_n\}_{n \geq 1}$  be the associated projections.

We shall denote by  $H_0$  the linear dense subspace spanned by  $\{f_n\}_{n \geq 1}$ . Let  $F: \mathbb{R} \rightarrow B(H)$  be the spectral family defined by

$$F(\lambda) = 0 \quad \text{for } \lambda < 1$$

$$F(\lambda) = \sum_{k=1}^{[\lambda]} P_k, \quad \text{where } [\lambda] \text{ denotes the greatest integer } \leq \lambda.$$

We define  $\|F\| = \sup_{\lambda \geq 0} \|F(\lambda)\| < \infty$ .

**Lemma (2.2.7)[177]:** Let  $H, H_0$ , and  $F$  be as above. Let  $h: [0, \infty) \rightarrow [1, \infty)$  be a continuous and increasing function. For any  $x \in H_0$ , let

$$T_0 x = \sum_{n=1}^{\infty} h(n) P_n x, \quad (\text{the sum is finite}). \quad (93)$$

Then, for every  $\theta \in (-\pi, \pi)$ , there exists  $M(\theta) > 0$  such that for every  $z \in \Sigma_{\theta}$ ,  $I + zT_0$  is a bijection in  $H_0$  and

$$\|(I + zT_0)^{-1}x\| \leq M(\theta)\|x\| \quad \text{holds for every } x \in H_0. \quad (94)$$

Moreover  $T_0$  is closable and its closure  $T$  is of type  $(\omega, M)$  for some  $M > 1$ , for every  $\omega \in (0, \pi)$  and satisfies  $0 \in \rho(T)$ .

**Proof (i).** Proof of (94). For every  $z \in \mathbb{C} \setminus (-\infty, 0]$ , we define  $S_0 x = \sum_{n=1}^{\infty} (1/(1 + zh(n))) P_n x$ ,  $x \in H_0$ . We get  $(I + zT_0)S_0 = S_0(I + zT_0) = I|_{H_0}$ . The spectral representation of  $S_0$  is given by

$$S_0 x = \int_{[0, \infty)} \frac{1}{1 + zh(\lambda)} dF(\lambda)x, \quad x \in H_0.$$

By using (87), we have

$$\|S_0 x\| \leq \left( \frac{1}{|1 + zh(\infty)|} + \frac{1}{|1 + zh(0)|} \|F\| + \text{Var}_{[0, \infty)} \left[ \frac{1}{1 + zh(\cdot)} \right] \cdot \|F\| \right) \|x\|,$$

for every  $x \in H_0$ ,  $h(\infty) = \lim_{\lambda \rightarrow \infty} h(\lambda) = \sup_{\lambda \geq 0} h(\lambda)$ , which may be infinite.

$$\text{Var}_{[0,\infty)} \left[ \frac{1}{1 + zh(\cdot)} \right] \leq \int_0^\infty \frac{dt}{|1 + e^{i\theta}t|} \leq \infty \quad \text{with } z = |z|e^{i\theta}.$$

Then we get (94).

(ii) Closure of  $T_0$ . It is known, see, e.g., [181], that (94) implies that  $T_0$  is closable and that its closure  $T$  satisfies the same inequality. For the sake of completeness, we prove that  $T_0$  is closable.

Let  $x_n \in H_0$  be such that  $x_n \rightarrow 0$  and  $T_0 x_n \rightarrow y$  for some  $y \in H$ . We have to prove  $y = 0$ . Let  $v \in H_0$ , then for  $t > 0$ , we have  $\|x_n + tv\| \leq M\|x_n + tv + tT_0(x_n + tv)\|$  and  $\|tv\| \leq M\|(t(v + y) + t^2T_0 v)\|$  by taking the limit. Hence  $\|v\| \leq M\|x + y + tT_0 v\|$  and  $\|v\| \leq M\|v + y\|$  by letting  $t \downarrow 0$  for every  $v \in H_0$ . Since  $H_0$  is dense in  $H$ ,  $y = 0$ .

(iii) Type of  $T$ . From (94), we get  $\|y\| \leq M(\theta) \|(I + zT)y\|$  for every  $y \in D(T)$  and  $z \in \Sigma_\theta$ , which implies that  $I + zT$  is injective and that  $R(I + zT)$  is closed, hence  $R(I + zT) \supset \overline{H_0} = H$ . Therefore  $z^{-1} \in \rho(T)$  and  $\|(I + zT)^{-1}x\| \leq M(\theta)\|x\|$  holds for every  $x \in H$ .

(iv)  $0 \in \rho(T)$  Let  $L_0x = \sum_{n=1}^\infty (1/h(n)) P_n x$  for  $x \in H_0$ .  $L_0$  is the inverse of  $T_0$ . By using (87), we get

$$\|L_0x\| \leq \left( \frac{1}{h(\infty)} + \left( 2 - \frac{1}{h(\infty)} \right) \|F\| \right) \|x\| \quad \text{for every } v \in H_0 .$$

Then  $L_0$  is bounded and densely defined. This implies that the closure of  $L_0$  is the inverse of  $T$ .

Next, we consider properties of two operators  $A_0$  and  $B_0$  of the form given by Lemma (2.2.7).

**Lemma (2.2.8)[177]:** Let  $f$  and  $g$  be two continuous, increasing functions from  $[0, \infty)$  into  $[1, \infty)$ . Let  $A_0$  and  $B_0$  be the corresponding operators in  $H_0$  defined by  $A_0 x = \sum_{n=1}^\infty f(n) P_n x$  and  $B_0 x = \sum_{n=1}^\infty g(n) P_n x$  for every  $x \in H_0$ .

Let  $A$  and  $B$  be their closure in  $H$ . Then, we have:

- (i)  $A_0 (A_0 + B_0)^{-1} = (A_0 + B_0)^{-1} A_0$  on  $H_0$ ;
- (ii)  $A$  and  $B$  are resolvent commuting;
- (iii)  $A + B$  is closable and  $\overline{A + B} = \overline{A_0 + B_0}$ .

**Proof.** (i) We have  $A_0 B_0 x = (\sum_n f(n) P_n)(\sum_m g(m) P_m x) = \sum_n f(n) g(n) P_n x = B_0 A_0 x$  for every  $x \in H_0$ . Since  $A_0 + B_0$  is a bijection on  $H_0$  it follows that  $A_0$  and  $(A_0 + B_0)^{-1}$  commute.

(ii) As is well known, it suffices to prove  $(I + A)^{-1}(I + B)^{-1} = (I + B)^{-1}(I + A)^{-1}$ . But this is a consequence of the commutativity of  $(I + A_0)^{-1}$  and  $(I + B_0)^{-1}$  on  $H_0$ . together with their boundedness.

(iii) First we prove that  $A + B$  is closable. Let  $x_n \in D(A) \cap D(B)$  be such that  $x_n \rightarrow 0$  and  $y_n := (A + B)x_n \rightarrow y$  with  $y \in H$ . Then

$$(I + A)^{-1}(I + B)^{-1}y_n = (I + A)^{-1}(I + B)^{-1}Bx_n \\ + (I + B)^{-1}(I + A)^{-1}Ax_n$$

$$= (I + A)^{-1}[I - (I + B)^{-1}]x_n + (I + B)^{-1}[I - (I + A)^{-1}]x_n \rightarrow 0.$$

Hence  $(I + A)^{-1}(I + B)^{-1}y = 0$ , and  $y = 0$ .

Since the closure of  $A_0 + B_0$  is contained in the closure of  $A + B$ , we only have to prove  $\overline{A + B} \subset \overline{A_0 + B_0}$  or  $A + B \subset \overline{A_0 + B_0}$ . Let  $x \in D(A) \cap D(B) = D(A + B)$ . Then there are two sequences  $x_n, \acute{x}_n \rightarrow x$  and  $A_0x_n \rightarrow Ax$  and  $B_0\acute{x}_n \rightarrow Bx$ . Set  $h_n = \acute{x}_n - x_n$ . We have

$$x_n = (A_0 + B_0)^{-1}(A_0x_n + B_0\acute{x}_n) - B_0(A_0 + B_0)^{-1}h_n \quad (95)$$

by using part (i). Since  $(A_0 + B_0)^{-1}$  is bounded by Lemma (2.2.7), we obtain that the sequence  $B_0(A_0 + B_0)^{-1}h_n$  converges to some  $v \in H$ . Moreover  $(A_0 + B_0)^{-1}h_n \rightarrow 0$ , then  $v = 0$  since  $B_0$  is closable by Lemma (2.2.7). Rewriting (95), we get

$$(A_0 + B_0)(x_n + B_0(A_0 + B_0)^{-1}h_n) = A_0x_n + B_0\acute{x}_n$$

Which implies by passing to the limit

$$x \in \overline{D(A_0 + B_0)} \quad \text{and} \quad \overline{(A_0 + B_0)x} = Ax + Bx.$$

Now we give a lemma which characterizes the closedness of  $A + B$ .

**Lemma (2.2.9)[177]:** Let the operators  $A$  and  $B$  be defined as in Lemma (2.2.8). Then  $A + B$  is not closed if and only if there exists a sequence  $x_n$  in  $H_0$  such that

$$\|x_n\| \leq 1 \text{ and } \text{Sup}_{n \geq 1} \|A_0(A_0 + B_0)^{-1}x_n\| = \infty. \quad (96)$$

**Proof.** (i) Let  $E = D(A) \cap D(B)$ . We define two norms on  $E$ :

$$\|x\|_1 := \|x\| + \|Ax\| + \|Bx\| \quad \text{and}$$

$$\|x\|_2 = \|x\| + \|(A + B)x\|, \quad x \in E.$$

Clearly  $\|x\|_2 \leq \|x\|_1$  for  $x \in E$ . Since  $A$  and  $B$  are closed,  $E$  is complete with respect to the norm  $\|\cdot\|_1$ . Moreover  $E$  is complete with respect to  $\|\cdot\|_2$  if and only if  $A + B$  is closed, By using the open mapping theorem (for one implication), one has  $A + B$  is closed if and only if there exists  $C > 0$  such that

$$\|x\|_1 \leq C\|x\|_2 \quad \text{for every } x \in E. \quad (97)$$

(ii) Let  $x_n \in H_0$  be such that  $\|x_n\| \leq 1$  and  $y_n = (A_0 + B_0)^{-1}x_n$  with  $\text{Sup}_{n \geq 1} \|A_0 y_n\| = +\infty$ . Then (97) cannot hold. Indeed, we have

$$\begin{aligned}\|y_n\|_2 &= \|y_n\| + \|(A_0 + B_0)y_n\| \\ &= \|(A_0 + B_0)^{-1}x_n\| + \|x_n\| \leq \|(A_0 + B_0)^{-1}\| + 1\end{aligned}$$

and

$$\|y_n\|_1 \geq \|A_0 y_n\| \quad \text{which is unbounded.}$$

Hence  $A + B$  is not closed.

(iii) Assume  $C_A = \text{Sup}\{\|A_0(A_0 + B_0)^{-1}y\|, \|y\| \leq 1, y \in H_0\} < \infty$ . By triangular inequality, there is  $C_B > 0$  such that

$$\|B_0(A_0 + B_0)^{-1}y\| \leq C_B \|y\|, \quad \text{for every } y \in H_0.$$

Then if  $x = (A_0 + B_0)y$ , we have

$$\begin{aligned}\|y\|_1 &= \|y\| + \|A_0 y\| + \|B_0 y\| \\ &= \|y\| + \|A_0(A_0 + B_0)^{-1}x\| + \|B_0(A_0 + B_0)^{-1}x\|\end{aligned}$$

$$\leq \|y\| + (C_A + C_B)\|x\| \leq (1 + C_A + C_B)\|y\|_2 \quad \text{for every } y \in H_0.$$

Then the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent on  $H_0$ . Observe that  $H_0 = D(A_0 + B_0)$  which is dense in  $\overline{D(A_0 + B_0)}$  with respect to the norm

$\|x\|_3 := \|x\| + \|\overline{(A_0 + B_0)}x\|$ ,  $x \in \overline{D(A_0 + B_0)}$ . Notice that  $E = D(A + B) \subset \overline{D(A_0 + B_0)} = \overline{D(A + B)}$ . Hence  $H_0$  is dense on  $E$  with respect to  $\|\cdot\|_3$  for  $x \in E$ , there exists  $x_n \in H_0$  such that  $\|x - x_n\|_3 \rightarrow 0$  and  $\|x\|_3 = \lim_{n \rightarrow \infty} \|x_n\|_3 = \lim_{n \rightarrow \infty} \|x_n\|_2 = \|x\|_2$ , by using the continuity of  $\|\cdot\|_2$  on  $E$ . It follows that the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent on  $E$ .

Construction of the Example (2.2.2). It is enough to choose  $A$  and  $B$  as in Lemma (2.2.7) and (2.2.8) such that condition (96) of Lemma (2.2.9) is satisfied, i.e., to find two functions  $f$  and  $g$  as in Lemma (2.2.7) such that

$$\sup \left\{ \left\| \sum_{n=1}^{\infty} \frac{f(n)}{f(n) + g(n)} P_n x \right\|, x \in H_0, \|x\| \leq 1 \right\} = \infty. \quad (98)$$

We show that this is possible.

First we choose for  $\{f_n\}_{n \geq 1}$  the conditional basis of example (88) which satisfies

$$\sup_{m \geq 1} \left\| \sum_{n=1}^m P_{2n} \right\| = +\infty.$$

If we impose the following conditions on  $f$  and  $g$ ,

$$\frac{f(n)}{f(n) + g(n)} = \begin{cases} \frac{1}{4} & \text{for } n \text{ odd} \\ \frac{3}{4} & \text{for } n \text{ even} \end{cases} \quad (99)$$

Then  $\sum_{n=1}^{2m} \left( \frac{f(n)}{f(n)+g(n)} \right) P_n x = \left( \frac{1}{4} \right) \sum_{n=1}^{2m} P_n x + (1/2) \sum_{n=1}^m P_{2n} x$ , which satisfies (98).

Finally, we give one possible choice of functions  $f$  and  $g$  satisfying the hypothesis of Lemma (2.2.7) and condition (99).

Set  $h(t) = \frac{1}{2} + \frac{1}{4} \cos(\pi t)$ ,  $t \geq 0$ .

We construct  $f$  and  $g$  by induction:

$$f(0) = 3 \quad \text{and} \quad g(0) = 1.$$

Suppose we know the functions between  $[0, 2n]$ ,  $n = 0, 1, 2, \dots$  then we define for  $t \in (2n, 2n + 1]$

$$f(t) = f(2n) \quad \text{and} \quad g(t) = f(2n) \left( \frac{1}{h(t)} - 1 \right).$$

And for  $t \in (2n + 1, 2n + 2]$

$$f(t) = g(2n + 1) \frac{h(t)}{1 - h(t)} \quad \text{and} \quad g(t) = g(2n + 1).$$

Then,  $f, g$  are continuous on  $[0, \infty)$  nondecreasing, not less than one with  $f(t)/(f(t) + g(t)) = h(t)$ . ■