Chapter 1

Transference and Spectral Decompositions

In this chapter every uniformly bounded one-parameter group on $L^p(\mu)$, is the Fourier-Stieltjes transform of a projection-valued mapping of $\mathbb{R}$,and that every hermitian-equivalent operator on $L^p(\mu)$, is well bounded of type (B). In the setting of an arbitrary Banach space X, power-bounded operators with a logarithm of the form iA with A well bounded of type (B) are studied. It is shown that if U is such an operator on X, then for every function f of bounded variation on the unit circle, $\sum_{n=-\infty}^{\infty} \hat{f}(n)U^n$ converges in the strong operator topology. This result, which formally is a transference by U of Stečkin's Theorem, makes it possible to calculate directly from U a (normalized) logarithm for U and the spectral projections for the logarithm.

Let Y be a closed subspace of $L^p(\mu)$, where $1 < p < \infty$. and μ is an arbitrary measure, and let V be any operator on Y which is power-bounded (i.e., V is invertible, and $\sup\{ \|V^n\|: n = 0, \pm 1, \pm 2, \dots\} < \infty$) .We show in Theorem $(1.34)(ii)$ that V can be written in the form e^{iA}, where A is a bounded operator on Y having a spectral "diagonalization" analogous to, though weaker than, that occurring in the spectral theorem for self-adjoint operators. In the precise terminology described and adopted in this Section , A is a well-bounded operator of type (B), and the conclusion of Theorem $(1.34)(ii)$ asserts that V is trigonometrically well-bounded. Also we develop an abstract Fourier-series analysis for Banach space operators which are simultaneously power-bounded and trigonometrically well-bounded. The abstract machinery of this Section automatically goes into effect for power-bounded operators on Y once Theorem $(1.34)(ii)$ is established. To illustrate the combined effects of this Section, we state here the following partial summary of Theorem $(1.34)(ii)$ in conjunction with Theorem $(1.25)(ii)$, postponing a description of spectral families of projections and their integration theory until in this Section.

Theorem(1.1) [178] : Let Y be a subspace of $L^p(\mu)$, where , μ is a measure and $1 < p < \infty$. Suppose that V is a power-bounded operator on Y. Then:

(i) V has a logarithm of the form $i \int_{[0,2\pi]}^{\oplus} \lambda dE(\lambda)$, where $E(.)$ is a spectral family of projections in Y;

(ii) for each complex-valued function f of bounded variation on the unit circle , $\{\sum_{n=-N}^{N} \hat{f}(n)V^n\}$ converges in the strong operator topology as$N \to +\infty$
Conclusion (i) of this Theorem is used to show (in Theorem (1.40)) that every uniformly bounded one-parameter group on Y is the Fourier Stieltjes transform of a spectral family of projections.

Without need of this Section , Theorem (1.34)(ii) has the direct consequence that every hermitian-equivalent operator on the space Y described above is well bounded of type (B) (see Theorem (1.36)).

To make the paper more self-contained, this section is denoted to a summary of the pertinent methodology from the theory of well-bounded operators. We conclude, in this Section, with some counterexamples which rule out various directions of generalization for Theorem (1.34)(ii). For instance, the space Y can not be replaced by an arbitrary reflexive Banach space (see Example (1.44)).
Our considerations merge three main themes: the transference notion of Coifman and Weiss [215]; well-bounded operators; and Stečkin's Theorem for the additive group $\mathbb{Z}$ of integers, which we take in the following form (see, e.g., [183, Theorem 20.7]).

Theorem(Stečkin**)(1.2) [178]:** Let $BV(\mathbb{T})$ denote the Banach algebra of all complex-valued functions of bounded variation on the unit circle $\mathbb{T}$, with the norm $\|.\|_{\mathbb{T}}$ given by

$$\|f\|_{\mathbb{T}} = |f(1)| + \mathrm{var}(f,\mathbb{T}),$$

where $\mathrm{var}(f,\mathbb{T})$ is the total variation of f .if $1 < p < \infty$, there is a constant C_p , such that for each $f \in BV(\mathbb{T})$ convolution by $\hat{f}$, the Fourier transform of f, is a bounded operator on $L^p(\mathbb{Z})$ whose norm does not exceed $C_p\|f\|_{\mathbb{T}}$.

Before describing the interplay of our main themes, it will be necessary to state in some detail the General Transference Theorem of Coifman and Weiss because of its central role. Suppose G is a locally compact, amenable group, m is a σ-finite measure space, $1 < p < \infty$, and S is a closed subspace of $L^p(m)$ Let $u \to R_u$, be a strongly continuous representation of G by bounded operators on S such that for $F \in S, (R_u, F)(x)$ is jointly measurable in $(u,x) \in G \times m$, and $c = \sup\{\|R_u\| : u \in G\} < \infty$. Let $\psi \in L^1(G)$ have compact support, and denote by Ψ the operator of convolution by ψ on $L^p(G)$.

Theorem (1.3) (general transference Theorem [215,Theorem 2.4]):**[178]** Put

$$(H_\psi, F)(x) = \int_G \psi(u)(R_{u^{-1}}F)(x)du \quad for\ F \in S, x \in m.$$

Then H_{ψ} , is a bounded linear mapping of S into S such that $\|H_{\psi}\| \leq c^2 \|\Psi\|$. (In should be mentioned that the General Transference Theorem is stated in [215] only for the case $S = L^p(m)$, but the proof therein covers the case $S \subseteq L^p(m)$, as asserted in Theorem (1.3) .When the group G in (1.3) is specialized to $\mathbb{Z}$, measure-theoretic complications associated with Fubini's Theorem are replaced by considerations with finite sums, and the proof of [215, Theorem 2.4] can be simplified so as to dispense with the σ-finiteness requirement on the measure space m (see Theorem (1.33)). The demonstration of Theorem (1.34)(ii) proceeds by applying this specialized version of Theorem (1.3) and Stečkin's Theorem to show that the power bounded operator V in the hypotheses has an $AC(\mathbb{T})$ −functional calculus, where $AC(\mathbb{T})$ denotes the Banach subalgebra of $BV(\mathbb{T})$ consisting of the absolutely continuous functions of $\mathbb{T}$. The proof is completed by invoking [213, Theorem (2.3)], which characterizes trigonometrically well-bounded operators on reflexive Banach spaces by the possession of an $AC(\mathbb{T})$ functional calculus. Thus Theorem (1.34)(ii) is a transference phenomenon. In contrast the abstract Fourier analysis in this Section is inspired by a purely formal simulation of the statement in the General Transference Theorem. Specifically, let U be a power-bounded, trigonometrically well bounded operator on a Banach space X, and let $f \in BV(\mathbb{T})$. Consider the bounded representation $\mathfrak{R}$ of $\mathbb{Z}$ in X defined by $\mathfrak{R}_n = U^n$ for $n \in \mathbb{Z}$. By Stečkin's Theorem (1.2) $\hat{f}$ determines a bounded convolution operator on $L^p(E)$, for $1 < p < \infty$. However, the General Transference Theorem (1.3) cannot possibly apply to the representation $\mathfrak{R}$ and the convlution kernel $\hat{f}$, since, in particular, $\hat{f}$ need not belong to $L^1(\mathbb{Z})$, and X need not even be reflexive.

Nevertheless, in the context at hand the conclusion of Theorem (1.3) states in a purely formal way that $\sum_{n=-\infty}^{\infty} \hat{f}(n)U^{-n}$ is a bounded operator on X. We show in Theorem (1.25)(ii) that this series does in fact converge in the strong operator topology, and thus an abstract transference of Stečkin's Theorem is valid for U. This result is deduced from a blend of classical Fourier series methods and an abstract type of Riemann-Stieltjes integration (with respect to a spectral family of projections), and provides formulas for direct calculation from U of its (normalized) logarithm and the corresponding spectral projections (Theorems (1.29) and (1.30)(ii)).

In this section we collect, in a convenient form, the known items we shall need from the theory of well-bounded operators.

Let $J = [a, b]$ be a compact interval of the real line $\mathbb{R}$. We denote by $BV(J)$ (resp. $AC(J)$) the Banach algebra of complex-valued functions having bounded variation on J (resp. absolutely continuous on J) with the norm $\|f\|_J = |f(b)| + \mathrm{var}(f, J)$. Let X be a Banach space, and $\mathcal{B}(X)$ the Banach algebra of bounded operators on X.

Definition(1.4)[178]: An $AC(J)$-functional calculus (resp. $A\,C(\mathbb{T})$-functional calculus) for an operator $T \in \mathcal{B}(X)$ is a norm-continuous homomorphism γ of $AC(J)$ (resp. $A\,C(\mathbb{T})$)-into $\mathcal{B}(X)$ such that γ sends the identity map to T and the function identically $1\ to\ I$, the identity operator of X. In either case we say γ is weakly compact provided that for each $.x \in X$, $\gamma(.)x$ is a weakly compact linear mapping of the domain of γ into X.

Definition(1.5) [178]: An operator $T \in \mathcal{B}(X)$ is called well bounded provided that for some compact interval J, T has an $AC(J)$-functional calculus.(Note that in this event, $\sigma(T)$, the spectrum of T, must be a subset of J). Well bounded operators were introduced by Ringrose [225,226] and Smart [227]. Without further specialization Definition (1.5) is too weak for our purposes in this Section , since a well bounded operator on the arbitrary Banach space X need not have a spectral decomposition in terms of projections acting in X (see, e.g., [183, Example 16.5]). The relevant notion for our considerations is that of type (B) well-bounded operator introduced in [212] (note especially [212, Theorem 4.2(ii]).

Definition(1.6) [178]: An operator $T \in \mathcal{B}(X)$ is said to be well bounded of type (B) provided that for some compact interval J, T has a weakly compact $AC(J)$-functional calculus. (Note that every well-bounded operator on a reflexive space is automatically of type (B).)

As will be seen presently (in Proposition (1.13)), well-bounded operators of type (B) can be characterized by an appropriate spectral decomposition expressed in terms of a "spectral family" of projections acting in the underlying Banach space.

Definition(1.7) [178]: A spectral family of projections in X is a uniformly bounded, projection-valued function $E(\,.\,): \mathbb{R} \to \mathcal{B}(X)$ which is right continuous on $\mathbb{R}$ in the strong operator topology, has a strong left-hand limit at each point of $\mathbb{R}$,and satisfies

(i) $E(s)E(t) = E(t)E(s) = E(\min\{s, t\})$, for $s, t \in \mathbb{R}$

(ii) $E(s) \to 0$ (resp. $E(s) \to I$) in the strong operator topology as $s \to -\infty$ (resp. $s \to +a$).

If there is a compact interval $[a,b]$ such that $E(s) = I$ for $s \geq b$ and $E(s) = 0$ for $s < a$, then we say that $E(\,.\,)$ is concentrated on $[a,b]$.

A theory of Riemann-Stieltjes integration with respect to spectral families of projections is described in detail in [183, Chapt. 17]. We outline here, for subsequent use, its main features. Let $J = [a,b]$ be a compact interval of $\mathbb{R}$, and let $E(\,.\,)$ be a spectral family of projections in X concentrated on J . For $\mathrm{g} \in BV(J)$ and $u = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ a partition of J, put

$$\varphi(\mathrm{g}, u) = \mathrm{g}(a)E(a) + \sum_{j=1}^{n} \mathrm{g}(\lambda_j)\{E(\lambda_j) - E(\lambda_{j-1})\} \tag{1}$$

Rearrangement of the terms on the right in the style of integration by parts gives

$$\varphi(\mathrm{g}, u) = \mathrm{g}(b)E(b) - \sum_{j=1}^{n} \{\mathrm{g}(\lambda_j) - \mathrm{g}(\lambda_{j-1})\}\, E(\lambda_{j-1}). \tag{2}$$

In particular, $\|\varphi(\mathrm{g},\mathrm{u})\| \leq \|\mathrm{g}\|_J \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\}$. Let $\mathfrak{B}$ be the set of all partitions of J partially ordered and directed by refinement. For $x \in X$ and $u \in \mathfrak{B}$ with $u = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ put

$$\omega(u,x) = \max_{1 \leq j \leq n} sup\{\, \|E(\lambda)x - E(\lambda_{j-1})x\| : \lambda \in [\lambda_{j-1}, \lambda_j)\}$$

Lemma (1.8) [178]: Given $x \in X, \lim_{u \in \mathfrak{B}} \omega(u,x) = 0.$

Proof **.** See [183, Lemma 17.2].

Lemma (1.9) [178]: Let $u, v \in \mathfrak{B}$ with $v \geq u$, and let $\mathrm{g} \in BV(J)$. Then for $x \in X$

$$\|\varphi(\mathrm{g},u)x - \varphi(\mathrm{g},v)x\| \leq \mathrm{var}(\mathrm{g},J)\omega(u,x).$$

Proof . Standard and elementary from (2).

It is evident from Lemmas (1.8) and (1.9) that for $\mathrm{g} \in BV(J)$

$$\int_J^{\oplus} \mathrm{g} dE = \lim_{u \in \mathfrak{B}} \varphi(\mathrm{g}, u)$$

exists in the strong operator topology.

Proposition (1.10) [178]: The mapping $\mathrm{g} \mapsto \int_J^{\oplus} \mathrm{g} dE$ is an identity-preserving algebra homomorphism of $BV(J)$ into $\mathcal{B}(X)$ satisfying

$$\left\| \int_J^{\oplus} \mathrm{g} dE \right\| \leq \|\mathrm{g}\|_J \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\}\, for\ \mathrm{g} \in BV(J). \tag{3}$$

Furthermore,

$$\left\|\left(\int_J^{\oplus} gdE\right)x - \varphi(g,u)\right\| \le \mathrm{var}(g,J)\omega(u,x) \ , for\ g \in BV(J), u \in \mathfrak{B}, x \in X. \quad (4)$$

Proof. The conclusion (4) is an immediate consequence of Lemma (1.9). The remaining assertions are obvious consequences of (1) and the defining properties of a spectral family of projections.

Remark (1.11) [178]: It is shown in [183, Theorem 17.4 and Theorem 17.8] that if $g \in BV(J)$ and g is continuous on J, then $\int_J^{\oplus} gdE$ is the strong limit of Riemann-Stieltjes sums obtained by using arbitrary intermediate points to evaluate g in (1), that is, sums of the form

$$g(a)E(a) + \sum_{j=1}^{n} g(\eta_j)\{E(\lambda_j) - E(\lambda_{j-1})\},$$

where $\eta_j \in [\lambda_{j-1}, \lambda_j]$ for $j = 1, 2, \ldots, n$. However, we shall not need this fact.

One further inequality will be useful. Given $f \in BV(J)$, $g \in BV(J)$, $x \in X$, $u \in \mathfrak{B}$, we have

$$\left\|\left\{\int_J^{\oplus} fdE - \int_J^{\oplus} gdE\right\}x\right\| \le \{\mathrm{var}(f,J) + \mathrm{var}(g,J)\}\omega(u,x) + \|\varphi(f,u)x - \varphi(g,u)x\| \ . \quad (5)$$

This comes from writing

$$\int_J^{\oplus} fdE - \int_J^{\oplus} gdE = \left\{\int_J^{\oplus} fdE - \varphi(f,u)\right\} - \left\{\int_J^{\oplus} gdE - \varphi(g,u)\right\} + \{\varphi(f,u) - \varphi(g,u)\} \ ,$$

and utilizing (4). As an immediate consequence of (5) and Lemma (1.8) we obtain the following proposition [183, Theorem 17.5], which will be of critical importance in this Section.

Proposition (1.12) [178]: Let $\{g_\alpha\}$ be a net in $BV(J)$, and let g be a cornplex-valued function on J such that

(i) $\sup_\alpha \mathrm{var}(g_\alpha, J) < \infty$,

(ii) $g_\alpha \to g$ pointwise on J.

Then $g \in BV(J)$ and $\left\{\int_J^{\oplus} g_\alpha dE\right\}$ converges to $\int_J^{\oplus} gdE$ in the strong operator topology.

We can now formulate the spectral decomposition characterization of type (B) well bounded operators alluded to earlier.

Proposition (1.13) [178]: Let X be a Banach space, and let $T \in \mathcal{B}(X)$. In order that T be well bounded of type (B) it is necessary and sufficient that there be a spectral family $E(.)$ of projections in X such that for some compact interval $J, E(.)$ is concentrated on J, and $T = \int_J^{\oplus} \lambda dE(\lambda)$. If this is the case, the spectral family $E(.)$ is uniquely determined (and called the spectral family of T).

Proof: Necessity and sufficiency are shown in [183, Theorem 17.14], while the uniqueness assertion follows from [183, Theorem 16.3(i)]. More direct methods of proof are available from the approach we have been following, and we shall indicate them briefly. An alternate necessity proof is described in the last paragraph in this Section of [213]. For sufficiency, let $\gamma(f) = \int_J^{\oplus} f dE$ for $f \in AC(J)$. By Proposition (1.10) above γ is an $AC(J)$ −functional calculus for T. The argument with absolutely convex hulls in[183, p. 347] shows that γ is indeed weakly compact. To verify the uniqueness assertion, suppose that

$$T = \int_{J_1}^{\oplus} \lambda dE_1(\lambda) = \int_{J_2}^{\oplus} \lambda dE_2(\lambda).$$

Let $\lambda_0 \in \mathbb{R}$, and choose $M > 0$ so that $J_1 \cup J_2 \cup \{\lambda_0\} \subseteq (-M, M)$. Then, in particular, $T = \int_{[-M,M]}^{\oplus} \lambda dE_k(\lambda)$ for $k = 1,2$. We can choose a sequence $\{f_n\}$ of polynomials uniformly bounded in $AC([-M, M])$ such that $\{f_n\}$ tends pointwise on $[-M, M]$ to the characteristic function of $[-M, \lambda_0]$. It follows from Proposition (1.10) and Proposition (1.12) that $\{f_n(T)\}$ converges in the strong operator topology to $E_k(\lambda_0)$ for $k = 1, 2$ (here $f_n(T)$has its elementary meaning).

An obvious corollary of the method in the uniqueness proof just concluded is stated in the next proposition (which is shown by a different method in [183, Theorem 16.3(ii)]).

Proposition (1.14) [178]: Let $T \in \mathcal{B}(X)$ be well bounded of type (B), and let $E(.)$ be its spectral family. Then an operator $S \in \mathcal{B}(X)$ commutes with T if and only if S commutes with $E(\lambda)$ for all $\lambda \in \mathbb{R}$.

The relationship between $E(.)$ and the resolvent set of $T, \rho(T)$, is spelled out in Proposition (1.15)(i).

Proposition (1.15) [178]: Under the hypotheses of Proposition (1.14) we have.

(i) an open interval φ is contained in $\rho(T)$, if and only if $E(.)$ is constant on φ.

(ii) for each , $\lambda \in \mathbb{R}$, $\{E(\lambda) - E(\lambda^-)\}$ is a projection operator and

$$\{E(\lambda) - E(\lambda^-)\}X = (x \in X: Tx = \lambda x\},$$

where $E(\lambda^-)$denotes the strong limit as $s \to \lambda^-$ of $E(s)$.

Proof. The assertion in (i) is a special case of a more general fact about well-bounded operators [211, Proposition (2.l)(iii)]. We sketch a more direct proof of

(1.15)(i) designed for type (B) operators. Suppose α, β are real numbers with $\alpha < \beta$. We can choose a compact interval J on which $E(.)$ is concentrated such that J contains $[\alpha, \beta]$ in its interior and satisfies

$T = \int_J^{\oplus} \lambda dE(\lambda)$. For $z \in \mathbb{C} \backslash [\alpha, \beta]$ let $f_z: J \to \mathbb{C}$ *be* given by $f_z(\lambda) = (z - \lambda)^{-1}$ for $\lambda \in (\alpha, \beta]$, $f_z(\lambda) = 0$ otherwise. Then

$$(z - T) \int_J^{\oplus} f_z dE = E(\beta) - E(\alpha).$$

Thus the spectrum of the restriction $T | \{E(\beta) - E(\alpha)\} X$ satisfies

$$\sigma(T | \{E(\beta) - E(\alpha)\} X) \subseteq [\alpha, \beta]. \tag{6}$$

Suppose now that φ is an open interval contained in $\rho(T)$. Without loss of generality we can assume φ is a bounded interval (r, s). Suppose α and β belong to (r, s) with $\alpha < \beta$. Since $\sigma(T) \subseteq \mathbb{R}$,

$$\sigma(T | \{E(\beta) - E(\alpha)\} X) \subseteq \sigma(T) . \tag{7}$$

In view of (6) and (7) $T | \{E(\beta) - E(\alpha)\} X$ has void spectrum. Hence $E(\alpha) = E(\beta)$. Conversely, suppose φ is an open interval, and $E(.)$ is constant on φ Once again we can assume that φ is a bounded open interval.

Let $\varphi = (\alpha, \beta)$ and pick J as in the discussion leading to (6). For $\lambda_0 \in \varphi$. define $g: J \to \mathbb{C}$ by setting $g(\lambda) = (\lambda_0, - \lambda)^{-1}$ for $\lambda \notin \varphi, g(\lambda) = 0$ for $\lambda \in \varphi$. Then

$$(\lambda_0 - T) \int_J^{\oplus} g dE = I + E(\alpha) - E(\beta^-)$$

Since $E(.)$ is constant on φ and $E(.)$ is strongly right continuous, it follows that $E(\alpha) = E(\beta^-)$. Hence $\lambda_0 \in \rho(T)$. Assertion (1.15)(ii) is the statement of [183, Theorem 17.15(iii)], and can be seen with the aid of Proposition (1.12) in conjunction with a sequence of polynomials pointwise convergent on an appropriate interval to the characteristic function of $\{\lambda\}$.

Corollary (1.16) [178]: Under the hypotheses of Proposition (1.14), if $J_0 = [a_0, b_0]$ contains $\sigma(T)$, then $E(.)$ is concentrated on J_0 and $T = \int_{J_0}^{\oplus} \lambda dE(\lambda)$.

Proof . Obvious from Proposition (1.15)(i).

Definition (1.17) [178]: Let X be a Banach space. An operator $U \in \mathcal{B}(X)$ is called trigonometrically well bounded provided there is a well bounded operator T of type (B) on X such that $U = e^{iT}$.

Proposition (1.18) [178]: If U is a trigonometrically well bounded operator on the Banach space X, then there is a unique well bounded operator A of type (B) on

X such that: $U = e^{iA}$; $\sigma(A) \subseteq [0,2\pi]$; and the point spectrum of A does not contain 2π .

Proof : By [211, Proposition (3.15) and proof of Proposition (3.1l)].

Definition (1.19) [178]: The unique operator A in Proposition (1.18) will be denoted by arg U.

Proposition (1.20) [178]: Let U be a trigonometrically well bounded operator on the Banach space X. Then an operator $S \in \mathcal{B}(X)$ commutes with U if and only if S commutes with $arg\, U$.

Proof: The assertion here is [211, Proposition (3.14)(ii)].

We conclude this section with the following recent result from [213], which plays a crucial role in it.

Theorem (1.21) [178]: Let X be a Banach space, and let $U \in \mathcal{B}(X)$ .In order that U be trigonometrically well bounded it is necessary and sufficient that U have a weakly compact $AC(\mathbb{T})$-functional calculus. If this is the case, then

$$\sup\{\|E_0(\lambda)\| : \lambda \in \mathbb{R} \} \leq 3\|\gamma\| , \tag{8}$$

where γ is the unique $AC(\mathbb{T})$-functional calculus for U, and $E_0(\,.\,)$ is the spectral family of arg U.

Proof. The necessity and sufficiency assertion is the statement of [213, Theorem (2.3)]. Before taking up the proof of (8) we first observe that an operator can have at most one $AC(\mathbb{T})$-functional calculus because of the density in $AC(\mathbb{T})$ of the trigonometric polynomials. To demonstrate (8) we make use of the sufficiency proof in [213, Theorem (2.3)], where in a particular spectral family $E(.)$ is obtained so that $E(.)$ is concentrated on $[0,2\pi]$, and $U = e^{iA}$, where $A = \int_{[0,2\pi]}^{\oplus} \lambda dE(\lambda)$. Obviously A is a well bounded operator of type (B), $E(.)$ is its spectral family, and $\sigma(A) \in [0,2\pi]$. If $x \in X$, and $Ax = 2\pi x$, then $Ux = x$. Hence for every $f \in AC(\mathbb{T})$, $\gamma(f)\, x = f\,(1)\, x$. The specific construction of $E(.)$ in [213] now shows that $E(\lambda)\, x = x$ for $\lambda \in [0,2\pi]$. Thus

$$2\pi x = Ax = \int_{[0,2\pi]}^{\oplus} \lambda dE(\lambda)\, x = 0.$$

Hence $A = \arg U$. The construction of $E(\,.\,)$ further shows that $\|E(\lambda)\| \leq 3\|\gamma\|$ for all $\lambda \in \mathbb{R}$.Let $f \in BV(\mathbb{T})$, and put

$$F_1(t) = \lim_{s\to t^+} f\left(e^{is}\right), \qquad F_2(t) = \lim_{s\to t^-} f\left(e^{is}\right) \qquad for\ t \in \mathbb{R}\ . \tag{9}$$

Obviously F_1 , F_2 have period 2π . Moreover,

$$\mathrm{var}(F_j , [0,2\pi]) \leq \mathrm{var}(f,\mathbb{T}) \qquad for\ j = 1,2. \tag{10}$$

To see (10), put $F(t) = f(e^{it})$,for $t \in \mathbb{R}$, and note that for $\varepsilon > 0$ and $(t_0, t_1, \dots, t_n)$ a partition of $[0,2\pi]$,

$$\sum_{k=1}^{n} |F(t_k + \varepsilon) - F(t_{k-1} + \varepsilon)| \leq \operatorname{var}(f, \mathbb{T}).$$

Let $\varepsilon \to 0^+$ to get (10) for $j = 1$. The case $j = 2$ is similar. Thus for $j = I, 2$ $\|F_j\|_{[0,2\pi]}$ does not exceed $[\sup\{ |f(z)| : z \in \mathbb{T}\} + \operatorname{var}(f, \mathbb{T})]$, and so

$$\|F_j\|_{[0,2\pi]} \leq 2\|f\|_{\mathbb{T}} \qquad \text{for } j = 1,2. \tag{11}$$

For each $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to \mathbb{C}$ by

$$f_t(\lambda) = f(e^{it}\ e^{i\lambda}) \qquad \text{for } \lambda \in \mathbb{R}. \tag{12}$$

Obviously $f_t(\lambda + 2\pi) = f_t(\lambda)$ for $\lambda \in \mathbb{R}$, and $\operatorname{var}(f_t, [0,2\pi]) = \operatorname{var}(f, \mathbb{T})$. Hence

$$\|F_t\|_{[0,2\pi]} \leq 2\|f\|_{\mathbb{T}} \qquad \text{for } t \in \mathbb{R}. \tag{13}$$

The next theorem provides the technical underpinnings for the subsequent results of this section.

Theorem(1.22) [178]: Suppose X is a Banach space, and $U \in \mathcal{B}(X)$is a trigonometrically well bounded operator. Let $E(.)$ denote the spectral family of $\arg U$, and let $f \in BV(\mathbb{T})$. For each $t \in \mathbb{R}$ put

$$\Phi(t) = \int_{[0,2\pi]}^{\oplus} f_t(\lambda) dE(\lambda), \tag{14}$$

Where f_t is given by (12). Then

(i) $\Phi(t + 2\pi) = \Phi(t)$for $t \in \mathbb{R}$;

(ii) $\Phi(t) \to \int_{[0,2\pi]}^{\oplus} F_1(t_0 + \lambda) dE(\lambda)$ in the strong operator topology as $t \to t_0^+$ and $\Phi(t) \to \int_{[0,2\pi]}^{\oplus} F_2(t_0 + \lambda) dE(\lambda)$ in the strong operator topology as $t \to t_0^-$, where F_1, F_2 are given by (9); (iii) $\|\Phi(t)\| \leq 2\|f\|_{\mathbb{T}} \sup\{ \|E(\lambda)\| : \lambda \in \mathbb{R} >$ for $t \in \mathbb{R}$.

(iv) for each $x \in X$ $\Phi(.)\,x$ is an X-valued Lebesgue measurable function on $\mathbb{R}$.

(v) If for each $n \in \mathbb{Z}$, we define $\widehat{\Phi}(n) \in \mathcal{B}(X)$by setting

$$\widehat{\Phi}(n)x = (2\pi)^{-1} \int_0^{2\pi} e^{-int}\, \Phi(t)xdt \quad \text{for } x \in X, \tag{15}$$

then $\widehat{\Phi}(n) = \hat{f}(n)U^{\,n}\, for\ n \in \mathbb{Z}$.

Proof: Conclusion (i) is obvious since $f_{t+2\pi} = f_t$. Conclusion (ii) is immediate from (13) and Proposition (1.12). Conclusion (iii) follows at once from (13) and (14). To verify conclusion (iv), we first observe that by virtue of (4) we have for every partition u of $[0,2\pi]$ and every $t \in \mathbb{R}$,

$$\|\Phi(t)x - \varphi(f_t, u)x\| \leq \operatorname{var}(f, \mathbb{T})\omega(u, x) \tag{16}$$

Since for given $u, \varphi(f_t, u)\, x$ is a measurable function of t, conclusion (iv) now follows from Lemma (1.8). Note that if we replace $\Phi(t)$ in (15) by the right-hand side of (14) and formally interchange the order of integration, then the desired conclusion in (v) results. However, since $E(\,.\,)$ need not stem from a spectral measure, this procedure is purely heuristic, and we shall employ a technical alternative. Let $n \in \mathbb{Z}$.and fix a vector $x \in X$. For $t \in \mathbb{R}$ and $u = (\lambda_0, \lambda_1, \ldots, \lambda_m)$ a partition of $[0,2\pi]$ it follows from (16) that

$$\left\| e^{-int}\Phi(t)x - e^{-int}f\left(e^{it}\right)E(0)x - \sum_{k=1}^{m} e^{-int}f\left(e^{it}e^{i\lambda_k}\right)\{E(\lambda_k) - E(\lambda_{k-1})\}x \right\| \leq \operatorname{var}(f, \mathbb{T})\omega(u, x).$$

Hence from Lemma (1.8)

$$\widehat{\Phi}(n)x = \lim_{u} \left[\hat{f}(n)E(0)x + \sum_{k=1}^{m} \hat{f}(n)e^{in\lambda_k}\{E(\lambda_k) - E(\lambda_{k-1})\}x \right]$$

Thus for $n \in \mathbb{Z}$ and $x \in X$,

$$\widehat{\Phi}(n)x = \hat{f}(n) \int_{[0,2\pi]}^{\oplus} e^{in\lambda} dE(\lambda)x = \hat{f}(n)U^n x.$$

The stage is now set for our abstract transference theorem. Here and throughout, the relevant partial sums for a bilateral series $\sum_{n=-\infty}^{\infty} a_n$ will be the "balanced" *sums* $\sum_{n=-N}^{N} a_n$, for $N \geq 0$.

Theorem (1.23) [178]: Let U be a trigonometrically well-bounded operator on a Banach space X, and let $f \in BV(\mathbb{T})$. Let $E(.)$ denote the spectral family of arg U. Then

(i) $\sum_{n=-\infty}^{\infty} \hat{f}(n)U^n$ *is* $(C, 1)$-summable in the strong operator Topology to $\int_{[0,2\pi]}^{\oplus} 2^{-1}\{F_1(\lambda) + F_2(\lambda)\}dE(\lambda)$, where F_1, F_2 are defined in (9);

(ii) if in addition to the above hypotheses, $\sup\{\| U^n\|: n \in \mathbb{Z}\} < \infty$, Then $\sum_{n=-\infty}^{\infty} \hat{f}(n)U^n$ converges in the strong operator topology to

$$\int_{[0,2\pi]}^{\oplus} 2^{-1}\{F_1(\lambda) + F_2(\lambda)\}dE(\lambda).$$

Proof. We employ the notation of Theorem (1.22). For $x \in X$. it follows from Theorem (1.22)-(v) that the X-valued function $\Phi(t)x$ has Fourier series $\sum_{n=-\infty}^{\infty} e^{int} \hat{f}(n)U^n x$. Moreover, $\Phi(t)x$ has a right-hand limit and a left-hand limit at $t_0 = 0$, as described in Theorem (1.22)(ii). Applying the analogue for vector-valued functions of Fejér's Theorem [222, Theorem 1.3.1] to $\Phi(t)x$ at $t_0 = 0$, we obtain conclusion (1.23)(i). Under the power boundedness hypothesis of (1.23)(ii), we see from (1.22)-(v) that $\widehat{\Phi}(n)x$, the nth Fourier coefficient of $\Phi(t)x$, has norm $O(|n|^{-1})$, since $f \in BV(\mathbb{T})$. The vector-valued version of a simple Tauberian theorem of Hardy [222,Theorem II.2.2] now enables us to infer the convergence of the Fourier Series of $\Phi(t)x$ whenever the series is $(C, 1)$-summable. Application of (1.23)(i) completes the proof of (1.23)(ii).

Next, we consider the use of Theorem (1.23)(ii) for the direct calculation (from U) of arg U and its spectral family of projections.

Theorem(1.24) [178]: Suppose X is a Banach space, and let $U \in \mathcal{B}(X)$ be a trigonometrically well-bounded operator such that $\sup\{ \|U^n\| : n \in \mathbb{Z} \} < \infty$. Let $E(.)$ be the spectral family of arg U. Then

$$\arg U = \pi\{I - E(0)\} + i \sum_{n=-\infty}^{\infty} n^{-1} U^n , \tag{17}$$

where the prime superscript in the series on the right indicates omission of $n = 0$ as a summation index, and the series converges in the strong operator topology.

Proof. Let $g_0 : \mathbb{T} \to \mathbb{C}$ be defined by $g_0(e^{it}) = i(\pi - t)$ for $0 < t < 2\pi$, $g_0(1) = 0$. It is elementary that $\hat{g}_0(1) = 0$ and $\hat{g}_0(n) = n^{-1}$ for $n \neq 0$. Applying Theorem (1.23)(ii) to U and $g_0 \in BV(\mathbb{T})$, we find that

$$\sum_{n=-\infty}^{\infty} n^{-1}\, U^n = \int_{[0,2\pi]}^{\oplus} g_0(e^{i\lambda}) dE(\lambda). \qquad (18)$$

With the aid of Remark (2.18)[178], the right-hand side of (18) is easily calculated to get

$$\sum_{n=-\infty}^{\infty} n^{-1}\, U^n \; = i\pi\,\{I - E(0)\} - i(\arg U)\,.$$

The desired conclusion is now evident.

Next we proceed to consider a method for calculating $E(\,.)$ from U under the hypotheses of Theorem (1.24). In particular, we shall obtain a concrete formula for $E(0)$ in (17). We begin with a companion theorem to Proposition (1.18)(ii).

Theorem (1.25) [178]: Let U_0 be a trigonometrically well-bounded operator on a Banach space X, and let $E(.)$ be the spectral family of $\arg U_0$. Then for $0 \le \lambda < 2\pi$:

(i) $\{E(\lambda) - E(\lambda^-)\}\, X = \{x \in X\colon U_0 x = e^{i\lambda} x\}$;

(ii) $[I - \{E(\lambda) - E(\lambda^-)\}]\, X = \overline{(e^{i\lambda} - U_0\,)X}$, where the bar superscript denotes closure

Proof. If $x \in \{E(\lambda) - E(\lambda^-)\}X$, then $(\arg U_0\,)x = \lambda x$, and so $U_0\, x = e^{i\lambda} x$. Conversely, if $U_0\, x = e^{i\lambda} x$, then we choose a sequence $\{Q_n\}$ of trigonometric polynomials (i.e., linear combinations, with complex coefficients, of continuous characters of $\mathbb{T}$) such that $\{Q_n\}$is bounded in $AC(\mathbb{T})$ and $\{Q_n\}$tends pointwise on $\mathbb{T}$ to $\mathcal{X}_{\{e^{i\lambda}\}}$, the characteristic function, relative to $\mathbb{T}$, of the singleton set $\{e^{i\lambda}\}$ Thus $Q_n\,(U_0)x = \int_{[0,2\pi]}^{\oplus} Q_n\,(e^{it}) dE(t)x.$ Approaches $\int_{[0,2\pi]}^{\oplus} \mathcal{X}_{\{e^{i\lambda}\}}\,(e^{it}) dE(t)x.$ as $n \to +\infty$. However, $Q_n\,(U_0)x = Q_n\,(e^{i\lambda})x$ for all n, and so

$$x = \int_{[0,2\pi]}^{\oplus} \mathcal{X}_{\{e^{i\lambda}\}}\,(e^{it}) dE(t)x = \{E(\lambda) - E(\lambda^-)\}x,$$

where in the second equality the cases $0 < \lambda < 2\pi$ and $\lambda = 0$ are considered separately, and the equalities $E((2\pi)^-) = I$, $E(0^-) = 0$ are taken into account in the latter case. Thus (1.25)(i) is established, and we proceed to (1.25)(ii), taking up the case $0 < \lambda < 2\pi$ first. Suppose that $\{E(\lambda) - E(\lambda^-)\}x = 0$. Let φ_Δ denote the characteristic function, relative to $[0,2\pi]$, of the arbitrary subset Δ of $[0,2\pi]$. Obviously we have been given that

$$\int_{[0,2\pi]}^{\oplus} \varphi_{\{\lambda\}}(t)dE(t)x = 0 \,. \qquad (19)$$

For small positive δ, let $\Gamma_\delta = [0, \lambda - \delta] \cup [\lambda + \delta, 2\pi]$ .Clearly $\|\varphi_{\Gamma_\delta}\|_{[0,2\pi]} = 3$ and $\varphi_{\Gamma_\delta} \to 1 - \varphi_{\{\lambda\}}$ pointwise on $[0,2\pi]$ as $\delta \to 0^+$. Making use of Proposition (1.12) and taking account of (19) we see that

$$\int_{[0,2\pi]}^{\oplus} \varphi_{\Gamma_\delta}(t)dE(t)x \to x \quad \text{as} \quad \delta \to 0^+ \,. \qquad (20)$$

For each δ, let $h_\delta : [0,2\pi] \to \mathbb{C}$ be defined by putting $h_\delta(t) = (e^{i\lambda} - e^{it})^{-1}$ For $t \in \Gamma_\delta$, $h_\delta(t) = 0$ for $t \in [0,2\pi] \setminus \Gamma_\delta$. It is evident that

$$\left(e^{i\lambda} - \mathrm{U}_0\right) \int_{[0,2\pi]}^{\oplus} h_\delta(t)dE(t)x = \int_{[0,2\pi]}^{\oplus} \varphi_{\Gamma_\delta}(t)dE(t)x \,.$$

In particular $\int_{[0,2\pi]}^{\oplus} \varphi_{\Gamma_\delta}(t)dE(t)x \in \left(e^{i\lambda} - \mathrm{U}_0\right)X$. Hence from (20) , $x \in \overline{(e^{i\lambda} - \mathrm{U}_0)X}$ .To prove the converse suppose first that $y \in X, z \in X$ with $y = \left(e^{i\lambda} - \mathrm{U}_0\right)z$. Put $z_\lambda = \{E(\lambda) - E(\lambda^-)\}z$. *By* (1.25)$(i)$, $\mathrm{U}_0 z_\lambda = e^{i\lambda} z_\lambda$. Hence

$$\{E(\lambda) - E(\lambda^-)\}y = \left(e^{i\lambda} - \mathrm{U}_0\right)z_\lambda = 0 \,.$$

Thus $\left(e^{i\lambda} - \mathrm{U}_0\right)X$ is contained in the kernel of $\{E(\lambda) - E(\lambda^-)\}$, and we have established (1.25)(ii) for the case $0 \leq \lambda < 2\pi$. The proof of (1.25)(ii) when $\lambda = 0$ is entirely analogous. One uses the intervals $(\delta, 2\pi - \delta]$ in place of the sets Γ_δ. The proof of Theorem (1.25) is complete.

Although Theorem (1.25) lies outside of ergodic theory, it allows us to deduce directly, in the following corollary, a result of the discrete-averages variety (cf. [216, Corollary VIII.53 and Corollary VIII.5.2]).

Corollary (1.26) [178]: Suppose that, in addition to the hypotheses of Theorem (1.25), sup{ ($\|U_0^n\|: n \in \mathbb{Z}, n \geq 1\} < \infty$. Then for $0 \leq \lambda < 2\pi$,

$$n^{-1} \sum_{k=0}^{n-1} e^{-ik\lambda} U_0^k \to E(\lambda) - E(\lambda^-) \qquad as\, n \to +\infty,$$

in the strong operator topology,

Proof: Put $\mathcal{A}(n) = n^{-1} \sum_{k=0}^{n-1} e^{-ik\lambda} U_0^k$ for each positive integer n (λ will be fixed in the range $0 \leq \lambda < 2\pi$). If $x \in \{E(\lambda) - E(\lambda^-)\} X$, then, by Theorem (1.25) - (i), $\mathcal{A}(n)x = x$ for all $n \geq 1$. If $y \in (e^{i\lambda} - U_0)X$, put $y = (I - e^{-i\lambda} U_0)\, z$, and observe that

$$\mathcal{A}(n)y = n^{-1}(I - e^{-in\lambda} U_0^n)\, z \to 0 \quad as \quad n \to +\infty,$$

since $\sup\{ (\|U_0^n\| : n \geq 1\} < \infty$. However, $\sup\{ \|\mathcal{A}(n)\| : n \geq 1\}$ is also finite. Hence we see with the aid of Theorem (1.25)(ii) that $\{\mathcal{A}(n)\}$ tends to zero pointwise on the kernel of $\{E(\lambda) - E(\lambda^-)\}$and the proof of the corollary is complete.

Using Corollary (1.26) for the case $\lambda = 0$, we obtain the following restatement of Theorem (1.24).

Theorem (1.27) [178]: Let U be a trigonometrically well-bounded operator on a Banach space X such that $sup\{ (\|U^n\| : n \in \mathbb{Z} \} < \infty$. Then

$$\arg U = \pi I - \pi \underset{n \to +\infty}{\text{st. lim}} \left\{ n^{-1} \sum_{k=0}^{n-1} U^k \right\} + i \underset{n \to +\infty}{\text{st. lim}} \sum_{k=-n}^{n} k^{-1} U^k , \tag{21}$$

where st. lim denotes "limit in the strong operator topology" .We now turn to the explicit calculation of the spectral family $E(.)$ occurring in Theorem (1.24). Since $E(.)$ is concentrated on $[0,2\pi]$, we need only consider $E(\lambda)$ for $0 \leq \lambda < 2\pi$.

Theorem (1.28) [178]: Under the hypotheses of Theorem (1.24), we have for $0 \leq \lambda < 2\pi$:

(i) With series convergence in the strong operator topology,

$$\sum_{k=-\infty}^{\infty} \hat{G}_\lambda(k)\, U^k = 2^{-1} \{E(\lambda^-) + E(\lambda) - E(0)\},$$

Where $G_\lambda \in BV(\mathbb{T})$ is the characteristic function relative to $\mathbb{T}$ of $\{e^{it} : 0 \leq t \leq \lambda\}$.

(ii) $E(\lambda) = \underset{n \to +\infty}{\text{st. lim}} \sum_{k=-n}^{n} \hat{G}_\lambda(k)\, U^k + \underset{n \to +\infty}{\text{st. lim}} (2n)^{-1} \sum_{k=0}^{n-1} e^{-ik\lambda} U^k$

$+ \underset{n \to +\infty}{\text{st. lim}} (2n)^{-1} \sum_{k=0}^{n-1} U^k$.

Proof . Conclusion (1.28)(i) is an immediate consequence of Theorem (1.23)(ii) for $0 < \lambda < 2\pi$, and is trivial for $\lambda = 0$. Conclusion (1.28)(ii) is easily obtained by combining (1.28)(i) with Corollary (1.26).

We conclude this section with consideration of a norm estimate.

Theorem (1.29) [178]: Let U he a trigonometrically well-bounded operator on a Banach space X such that $\sup\{ \|U^n\|: n \in \mathbb{Z} \} < \infty$, and let $E(.)$ be the spectral family of arg U. Suppose that $f \in BV(\mathbb{T})$. Then for each nonnegative integer n we have

$$\left\| \sum_{k=-n}^{n} \hat{f}(k)\, U^k \right\| \leq \pi^{-1} K_1 \mathrm{var}(f, \mathbb{T}) + 2K_2 \|f\|_{\mathbb{T}} \quad , \tag{22}$$

where $K_1 = \sup\{ \|U^m\|: m \in \mathbb{Z} \}$ and $K_2 = \sup \{\|E(\lambda)\|: \lambda \in \mathbb{R}\}$.

Proof. We employ the notation of Theorem (1.22). Let $x \in X$. Standard considerations with the Cesáro means for the Fourier series of $\Phi(t)x$ (in conjunction with Theorem (1.22)(v)) show that

$$\sum_{k=-n}^{n} \left(1 - \frac{|K|}{n+1}\right) \hat{f}(k)\, U^k\, x = (2\pi)^{-1} \int_0^{2\pi} k_n(t)\, \Phi(t) x dt \,, \tag{23}$$

where k_n is the nth term of the Fejér kernel. Applying Theorem (1.22) (iii) to (23) we see that

$$\left\| \sum_{k=-n}^{n} \left(1 - \frac{|K|}{n+1}\right) \hat{f}(k)\, U^k \right\| \leq 2K_2 \|f\|_{\mathbb{T}} \,. \tag{24}$$

Moreover,

$$\sum_{k=-n}^{n} \hat{f}(k)\, U^k - \sum_{k=-n}^{n} \left(1 - \frac{|K|}{n+1}\right) \hat{f}(k)\, U^k = \sum_{k=-n}^{n} \frac{|K|}{n+1} \hat{f}(k)\, U^k \quad . \tag{25}$$

Since $f \in BV(\mathbb{T})$, it is elementary that $|\hat{f}(k)| \leq (2\pi|k|)^{-1}$ var $(f, \mathbb{T})$ for $k \in \mathbb{Z}$, $k \neq 0$. Using this fact with (25), we get easily

$$\left\| \sum_{k=-n}^{n} \hat{f}(k)\, U^k - \sum_{k=-n}^{n} \left(1 - \frac{|K|}{n+1}\right) \hat{f}(k)\, U^k \right\| \leq \pi^{-1} K_1 \mathrm{var}\,(f, \mathbb{T}) \tag{26}$$

The desired conclusion now follows at once from (24) and (26).

Remarks (1.30) [178]: We make two observations under the hypotheses of Theorem (1.29).

(a) The purpose of the estimate in (22) is to provide an explicit bound, intrinsic to f and U, for the sequence $\{\left\|\sum_{k=-n}^{n} \hat{f}(k)\, U^k\right\|\}$ .It is already clear, without calculations, from Theorem (1.23)(ii) and the Banach-Steinhaus Theorem that this sequence is bounded.

(b) It is obvious from Proposition (1.10) that

$$\sum_{k=-n}^{n} \hat{f}(k)\, U^k = \int_{[0,2\pi]}^{\oplus} \left(\sum_{k=-n}^{n} \hat{f}(k)\, e^{ikt} \right) dE(t) \ .$$

Thus, from (3) we get

$$\left\| \sum_{k=-n}^{n} \hat{f}(k)\, U^k \right\| \leq \|S_n(f,.)\|_{\mathbb{T}} \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\}, \tag{27}$$

where $S_n(f,z) = \sum_{k=-n}^{n} \hat{f}(k)\, z^k$ for all $z \in \mathbb{T}$. Hence, if the sequence of partial sums for the Fourier series of f is bounded in $BV(\mathbb{T})$, then (27) can be utilized to get a bound for the sequence $\{ \left\|\sum_{k=-n}^{n} \hat{f}(k)\, U^k\right\| \}$. However, not every function in $BV(\mathbb{T})$has a Fourier series with this property. For example, it is easy to see for the function g_0 in the proof of Theorem (1.24) that

$$\operatorname{var}(S_n\,(g_0,.), \mathbb{T}) \geq 2\pi(L_n - 1),$$

where L_n is the nth Lebesgue constant [231, p. 67]. Since $L_n \to \infty$ as $n \to +\infty, \|S_n(g_0,.)\|_{\mathbb{T}} \to +\infty$.

We begin this section with an adaptation to the special case $G = \mathbb{Z}$ of the Coifman-Weiss technique for proving their General Transference Theorem. As mentioned in it, the simplifications available for this special case allow us to eliminate measure-theoretic technicalities from the hypotheses and the proof. For $\xi = \{\xi_n\} \in L^1(\mathbb{Z})$, we denote by $K_{\xi,p}$ the operator of convolution by ξ on $L^p(\mathbb{Z})$, for $1 < p < \infty$. Notice that

$$\|K_{\xi,p}\| = \|K_{\zeta,p}\| \quad , \tag{28}$$

where $\zeta = \{\xi_{-n}\}$.

Theorem (1.31) [178]: Suppose $(\mathcal{M}, \mu)$ is an arbitrary measure space, and $1 < p < \infty$. Let Y be a closed subspace of $L^p(\mu)$, , and let $V: Y \to Y$ be an invertible bounded linear operator such that $\sup\{\|V^n\|: n \in \mathbb{Z}\} < \infty$. Then for every trigonometric polynomial Q

$$\|Q\,(V)\| \le c^2 \left\|K_{\hat{Q},p}\right\|, \tag{29}$$

where $c = \sup\{\|V^n\|: n \in \mathbb{Z}\}$.

Proof. In view of (28) the desired conclusion is clearly equivalent to the assertion that for every trigonometric polynomial Q

$$\|Q\,(V^{-1})\| \le c^2 \left\|K_{\hat{Q},p}\right\|, \tag{30}$$

We show (30) for an arbitrary trigonometric polynomial Q . For notational convenience, put $\hat{Q} = \{a_n\}$, and choose N sothat $a_n = 0$ for $|n| > N$. Fix an element $f \in Y$, and for each $n \in \mathbb{Z}$, treat $V^n f$ as a function on $\mathcal{M}$, rather than as an equivalence class of functions. Let M be an arbitrary positive integer. Obviously

$$\|g\|_p \le c\,\|V^m g\|_p \qquad \text{for } m \in \mathbb{Z}, g \in Y,$$

Where we denote the norm of $L^p(\mu)$, by $\|\,.\,\|_p$. It follows that

$$\|g\|_p^p \le c^p(2M+1)^{-1} \sum_{m=-M}^{M} \|V^m g\|_p^p, \quad \text{for } g \in Y. \tag{31}$$

Taking g in (31) to be $Q\,(V^{-1})f$ we easily obtain

$$\|Q\,(V^{-1})f\|_p^p \le c^p(2M+1)^{-1} \int_{\mathcal{M}} \left\{ \sum_{m=-M}^{M} \left| \sum_{n=-N}^{N} a_n (V^{m-n}f)(x) \right|^p \right\} d\mu(x) \tag{32}$$

Fix $x \in \mathcal{M}$ temporarily. Let k be the characteristic function relative to $\mathbb{Z}$ of $\{n \in \mathbb{Z}: |n| \le M + N\}$. We have

$$\sum_{m=-M}^{M} \left| \sum_{n=-N}^{N} a_n (V^{m-n}f)(x) \right|^p = \sum_{m=-M}^{M} \left| \sum_{n=-\infty}^{\infty} a_n K_{m-n} (V^{m-n}f)(x) \right|^p$$

$$\le \left\|K_{\hat{Q},p}\right\|^p \sum_{n=-M-N}^{M+N} |(V^n f)(x)|^p.$$

Using this inequality in (32) .

We see that

$$\|Q\,(V^{-1})f\|_p^p \le c^p(2M+1)^{-1}\left\|K_{\hat{Q},p}\,\right\|^p \sum_{n=-M-N}^{M+N} \|V^n f\|_p^p \,.$$

Hence

$$\|Q\,(V^{-1})f\|_p^p \le c^{2p}(2M+1)^{-1}\left\|K_{\hat{Q},p}\,\right\|^p (2M+2N+1)\|f\|_p^p \qquad (33)$$

Letting $M \to +\infty$ in (33) we obtain

$$\|Q\,(V^{-1})f\|_p^p \le c^{2p}\left\|K_{\hat{Q},p}\,\right\|^p \|f\|_p^p \quad ,$$

for $f \in Y$. This shows (30), which completes the proof of the theorem.

Theorem (1.32) [178]: Under the hypotheses of Theorem (1.31), we have,

(i) for every trigonometric polynomial Q,

$$\|Q\,(V)\| \le c^2 C_p\,\|Q\|_{\mathbb{T}}\,,$$

where C_p , is the constant occurring in Stečkin's Theorem (Theorem (1.2), and $c = \sup\{\|V^n\|: n \in \mathbb{Z}\}$;

(ii) V is trigonometrically well-bounded;

(iii) $\sup\{\|E(\lambda)\|: \lambda \in \mathbb{R}\} \le 3\,c^2 C_p$, where $E(.)$ is the spectral family of $\arg V$.

Proof. Conclusion (1.32)(i) is an immediate consequence of Theorem (1.31) and Stečkin's Theorem for $\mathbb{Z}$. Since the trigonometric polynomials are dense in $AC(\mathbb{T})$, it follows from conclusion (1.32)(i), by continuous extension, that V has an $AC(\mathbb{T})$-functional calculus whose norm does not exceed $c^2 C_p$. Conclusions (1.32)(ii) and (1.32)(iii) are now apparent by virtue of Theorem (1.21) in conjunction with Remark (2.22)[178]. This completes the demonstration of the theorem.

Remarks (1.33) [178]: It is well known (and easy to see from Theorem (1.2)) that if $1 < p < \infty$ and $f \in BV(\mathbb{T})$, then $f \in M_p(\mathbb{T})$, the space of p-multipliers for $L^p(\mathbb{Z})$, and convolution by $\hat{f}$ on $L^p(\mathbb{Z})$has norm equal to $\|f\|_{M_p(\mathbb{T})}$, the p-multiplier norm of f . Hence in the special case of $p = 2$ and V unitary, Theorem (1.31) becomes a form of the spectral theorem for unitary operators. Specifically, in this context, Theorem (1.31) asserts that V has a norm-decreasing $C(\mathbb{T})$-functional calculus, where $C(\mathbb{T})$ is the Banach algebra of complex-valued continuous functions on $\mathbb{T}$ with the usual "sup" norm. (Apart from the σ-finiteness restriction on $\mathcal{M}$ for Theorem (1.3) the General Transference Theorem likewise contains the spectral theorem for unitary operators [215]. Thus our results stemming from Theorems (1.31) and (1.21) (specifically, Theorems (1.32), (1.35), (1.36), and (1.39)) can be viewed as generalizing the spectral theorem from Hilbert space to arbitrary reflexive L^p-spaces. In the special case $p = 2$, these results are immediate consequences of B. Sz.-Nagy's similarity theorems [228, Theorems I and II], which they also generalize to reflexive L^p-spaces. Theorem (1.32)(ii) has a surprising consequence for hermitian-equivalent operators (Theorem (1.35)). First, let us recall some basic facts about hermiticity. A bounded operator T on a Banach space X is said to be hermitian (in the sense of Lumer and Vidav [223.229]) provided $\|e^{itT}\| = 1$,for all $t \in \mathbb{R}$. The operator T is said to be hermitian-equivalent provided T can be made hermitian by equivalent renorming of X. It is shown in [185, Theorem 6] that T is hermitian-equivalent if and only if $\sup\{ \|e^{itT}\| : t \in \mathbb{R}\} < \infty$. We shall also require a lemma of independent interest.

Lemma (1.34) [178]: Suppose X is a Banach space, $A \in \mathcal{B}(X)$ and e^{iA} is trigonometrically well-bounded. Then A is well-bounded of type (B).

Proof . We adapt the demonstration of [183, Theorem 20.28] to the general Banach space setting. Let $A_0 = \arg(e^{iA})$. By Proposition (1.20), A and A_0 commute. Thus $e^{i(A-A_0)} = I$. By [183, Proposition 10.6] $A - A_0$ has the form

$$A - A_0 = \sum_{k=1}^{n} \lambda_k F_k \ , \tag{34}$$

where $\sigma(A - A_0) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$; the operators $F_1, F_2, \dots, F_n$ are projections satisfying $\sum_{k=1}^{n} F_k = I$, and $F_j F_k = 0$ for $j \neq k$; $\{A, A_0, F_1, F_2, \dots, F_n\}$ is a commutative set of operators. Clearly $\{\lambda_1, \lambda_2, \dots, \lambda_n\} = \sigma(A - A_0)$ is a subset of $\mathbb{R}$. For $k = 1, 2, \dots, n$, $X_k = F_k X$ is invariant under A_0, and hence from (34) we see that A has the direct sum representation

$$A = \oplus_{k=1}^{n} [(A_0 + \lambda_k) | X_k]. \tag{35}$$

The proof of the lemma is easily completed by applying to (35) the following facts, readily deducible from the definition of type (B) well-bounded operator in this Section . if T is a type (B) well-bounded operator, then the restriction of T to an invariant subspace and any sum of the form $T + \lambda$ with λ real are type (B) well-bounded operators; a finite direct sum of type (B) well-bounded operators is a type (B) well-bounded operator.

Theorem (1.35) [178]: Let Y be a closed subspace of $L^p(\mu)$, where $,\mu$ is an arbitrary measure and $1 < p < \infty$. If $A \in \mathcal{B}(Y)$ is hermitian-equivalent, then A is well bounded of type (B).

Proof. Since $\sup\{\|e^{itA}\| : t \in \mathbb{R}\} < \infty$, e^{iA} is power bounded. The desired conclusion follows at once from Theorem (1.32)(ii) and Lemma (1.34).

It follows from Theorem (1.32)(ii) that if V is a power-bounded operator on a closed subspace Y of an L^p-space, where $1 < p < \infty$, then the machinery of this Section applies to V (notably Theorems (1.23)(ii), (1.24), (1.27), (1.28), (1.29)). In particular, by Theorem (1.29) (alternatively by Theorem (1.23)(ii), (3), and (11)) there is a constant $\mathcal{K}$ (depending on V) such that

$$\left\|\sum_{n=-\infty}^{\infty} \hat{f}(n)\, V^n\right\| \leq \mathcal{K}\|f\|_{\mathbb{T}} \quad for\ f \in BV(\mathbb{T}). \tag{36}$$

However, by using the actual transference result (29) rather than the purely abstract operator-theoretic methods of this Section , we can improve on the estimate in (36) and generalize (29) from trigonometric polynomials to all of $BV(\mathbb{T})$. Specifically, the following inequality can be obtained for the p-multiplier norm.

Theorem (1.36) [178]: Under the hypotheses of Theorem (1.31) we have

$$\left\|\sum_{n=-\infty}^{\infty} \hat{f}(n)\, V^n\right\| \leq c^2\|f\|_{M_p(\mathbb{T})} \quad \text{for} \qquad f \in BV(\mathbb{T}).$$

where $c = \sup\{\|V^n\| : n \in \mathbb{Z}\}$.

Proof. Let $f \in BV(\mathbb{T})$, and for N a nonnegative integer let $\sigma_N(f, V)$ be the Nth Cesáro mean for the series $\sum_{k=-\infty}^{\infty} \hat{f}(k)\, V^k$. Thus, if $\{\mathcal{K}_n\}_{n=0}^{\infty}$ denotes the Fejér kernel, then $\sigma_N(f, V) = Q_N(V)$, where Q_N is the trigonometric polynomial $\mathcal{K}_n * f$. By Theorem (1.31),

$$\|\sigma_N(f, V)\| \leq c^2\|\mathcal{K}_N * f\|_{M_p(\mathbb{T})} .$$

It is a well-known and easy consequence of Parseval's formula and Hölder's inequality that if G is a locally compact abelian group with dual group Γ , $1 < p < \infty$, $\phi \in L^1(\Gamma)$, and $\psi \in M_p(\Gamma)$, then

$$\|\phi * \psi\|_{M_p(\Gamma)} \leq \|\phi\|_{L^1(\Gamma)}\|\psi\|_{M_p(\Gamma)} .$$

Thus,

$$\|\mathcal{K}_N * f\|_{M_p(\mathbb{T})} \leq \|f\|_{M_p(\mathbb{T})} , \tag{37}$$

and so

$$\|\sigma_N(f, V)\| \leq c^2\|f\|_{M_p(\mathbb{T})} .$$

The proof is now complete, since $\sigma_N(f,V) \to \sum_{n=-\infty}^{\infty} \hat{f}(n)\, V^n \quad as \quad N \to +\infty$, in the strong operator topology.

Remarks (1.37) [178]: We shall make some observations which show that the estimate in Theorem (1.36) improves (36). The argument in [231,Theorem III.(3.7)] demonstrates that if $f \in BV(\mathbb{T})$, then

$$\|S_n(f,.)\|_\infty \le \pi^{-1}\,\mathrm{var}(f,\mathbb{T}) + \|f\|_\infty \text{ for each nonnegative integer } n, \tag{38}$$

where $S_n(f,.)$ is as in (27) and, for each $g : \mathbb{T} \to \mathbb{C}$, $\|g\|_\infty = \sup\{\,|g(z)| : z \in \mathbb{T}\,\}$. The argument used to establish (38) can easily be modified to give, for $f \in BV(\mathbb{T})$ and $1 < p < \infty$,

$$\|S_n(f,.)\|_{M_p(\mathbb{T})} \le \pi^{-1}\,\mathrm{var}(f,\mathbb{T}) + \|f\|_{M_p(\mathbb{T})} \text{ for } n \ge 0. \tag{39}$$

To obtain (39) one need only begin with (37) and then replace $\|\,.\,\|_\infty$ by $\|\,.\,\|_{M_p(\mathbb{T})}$ throughout the demonstration of (38). Next, note that, by Stečkin’s Theorem, for $1 < p < \infty$,

$$\|f\|_{M_p(\mathbb{T})} \le C_p \|f\|_{\mathbb{T}} \quad \text{for}, \quad f \in BV(\mathbb{T})\,.$$

And hence Theorem (1.36) implies (36). However, there does not exist a constant B_p, such that

$$\|f\|_{\mathbb{T}} \le B_p \|f\|_{M_p(\mathbb{T})} \quad \text{for} \quad f \in BV(\mathbb{T})\,.$$

In fact, the function $g_0 \in BV(\mathbb{T})$ discussed in (1.30)(b) satisfies $\|S_n(g_0,.)\|_{\mathbb{T}} \to +\infty \; as \; n \to +\infty$, where as from (39),$\left\{\|S_n(g_0,.)\|_{M_p(\mathbb{T})}\right\}$ is a bounded sequence. Thus the inequality (36) has a larger order of magnitude occurring in its majorant than does Theorem (1.36) and consequently (36) provides a weaker estimate than Theorem (1.36).

We next take up an application of Theorem (1.32) to one-parameter groups. We shall require the following theorem from [211].

Theorem (1.38) [178]: (Generalized Stone's Theorem). Let $\{W_t\}, t \in \mathbb{R}$, be a strongly continuous one-parameter group of trigonometrically well-bounded operators acting on a Banach space X. For each $t \in \mathbb{R}$, let $E_t(\,.\,)$ hethe spectral family of $\arg W_t$, and suppose that

$$K \equiv \sup\{\, \|E_t(\lambda)\| : t \in \mathbb{R}, \lambda \in \mathbb{R}\} < \infty. \tag{40}$$

Then:

(i) there is a unique spectral family $\mathfrak{F}(\,.\,)$ in X (called the Stone-type spectral family of $\{W_t\}$) such that

$$W_t\, x = \lim_{a\to+\infty} \int_{-a}^{a} e^{it\lambda}\, d\mathfrak{F}(\lambda)x \qquad \text{for } t \in \mathbb{R}, x \in X, \tag{41}$$

where the integral on the right in (41) exists as a strong limit of Riemann- Stieltjes sums;

(ii) $\{ W_t : t \in \mathbb{R} \}$ and $\{\mathfrak{F}(\lambda): \lambda \in \mathbb{R} \}$ have the same commutants;

(iii) $\sup\{ \|\mathfrak{F}(\lambda)\| : \lambda \in \mathbb{R}\} \le 24K^3$.

Conclusions (1.38) (i), (ii) are contained in the statement of [211, Theorem (4.20)], while conclusion (1.40) (iii) can be seen from an examination of the proof for [211, Theorem (4.20)].

Theorem (1.39) [178]: Let Y be a closed subspace of $L^p(\mu)$ where μ is a measure and $1 < p < \infty$, and let $\{\mathcal{W}_t\}$ be a strongly continuous one-parameter group of continuous linear operators on Y such that

$$s = \sup\{ \|\mathcal{W}_t\| : t \in \mathbb{R}\} < \infty.$$

Then the group $\{\mathcal{W}_t\}, t \in \mathbb{R}$, satisfies the hypotheses of the Generalized Stone's Theorem (1.38) on Y, and its Stone-type spectral family $\mathfrak{F}(.)$ satisfies

$$\sup\{ \|\mathfrak{F}(\lambda)\| : \lambda \in \mathbb{R}\} \le (648) s^6 C_p^3 ,$$

where C_p is the constant in Stečkin's Theorem (1.2).

Proof. For each $t \in \mathbb{R}$, $\sup\{ \|\mathcal{W}_t^n\| : n \in \mathbb{Z}\} \le s$. By Theorem (1.32)(ii), (1.32)(iii) $\mathcal{W}_t$ is trigonometrically well-bounded, and

$$\sup\{ \|E_t(\lambda)\| : \lambda \in \mathbb{R}\} \le 3s^2 C_p ,$$

where $E_t(\,.\,)$is the spectral family of arg $\mathcal{W}_t$. It is clear that the group $\{\mathcal{W}_t\}, t \in \mathbb{R}$, satisfies the hypotheses of Theorem (1.38) on Y, with K in (40) satisfying $K \le 3s^2 C_p$. Application of (1.38)(iii) completes the proof.

The spectral decomposition, afforded by its Stone-type spectral family, for the one-parameter group $\{\mathcal{W}_t\}$ in Theorem (1.39) was obtained by D. Fife in the special case where $Y = L^p(\mu), \mu$ is σ-finite, and the operators $\mathcal{W}_t, t \in \mathbb{R}$, are induced by a one-parameter group of measure-preserving transformations (of the underlying measure space) satisfying appropriate measurability and continuity conditions (see [218, Theorem 1 and p. 139], or, for more details, [217, especially Theorem 12]). Thus Theorem (1.39) extends in various ways Fife's spectral decomposition for ergodic flows.

Proposition (1.40) [219]: Let G be a locally compact abelian group with dual group Γ, let $x \in G$, and suppose that $1 < p < \infty$. Let R_x be the translation operator on $L^p(G)$ corresponding to x. Then R_x is trigonometrically well-bounded, and the spectral family $E_x(.)$ of arg R_x satisfies $\sup\{ \|E_x(\lambda)\| : \lambda \in \mathbb{R}\} \le \Omega_p$,where Ω_p is a constant depending only on p and not on x or G. Furthermore,

(i) the function $\phi_x : \Gamma \to [0, 2\pi)$ defined by

$$\phi_x(\alpha) = \operatorname{Arg}(\alpha(x)) \qquad \text{for } \alpha \in \Gamma.$$

Is an $L^p(G)$-multiplier whose corresponding multiplier transform is arg R_x;

(ii) for each $\lambda \in [0, 2\pi)$ the function $\psi_{x,\lambda} : \Gamma \to \{0,1\}$ defined by

$$\psi_{x,\lambda}(\alpha) = K_\lambda(\alpha(x)) \qquad \text{for } \alpha \in \Gamma$$

is an $L^p(G)$ -multiplier whose corresponding multiplier transform is $E_x(\lambda)$.

First, we observe that by virtue of Theorem (1.32)(iii) Ω_p can be taken to be $3C_p$ Next, consider conclusion (1.40)(i). For f in $L^p(G) \cap L^2(G)$, let $f_n = n^{-1} \sum_{k=0}^{n-1} R_x^k f$ for $n \ge 1$. Thus $f_n \in L^p(G) \cap L^2(G)$ and $\hat{f}_n = n^{-1} \sum_{k=0}^{n-1} \tilde{x}^k \hat{f}$, where $\tilde{x} : \Gamma \to \mathbb{T}$ is given by $\tilde{x}(\alpha) = \alpha(x)$ for $\alpha \in \Gamma$. By Theorem (1.27)

$$(\arg R_x) f = \pi f - \pi \lim_n f_n + i \lim_n \sum_{k=-n}^{n} K^{-1} R_x^k f \,, \tag{42}$$

the limits being taken in $L^p(G)$. Let h be the characteristic function relative to Γ of $\{\alpha \in \Gamma : \tilde{x}(\alpha) = 1\}$. It is elementary that the sequence $\{n^{-1} \sum_{k=0}^{n-1} \tilde{x}^k\}$ is uniformly bounded on Γ and tends pointwise to h. Thus $\hat{f}_n \to h\hat{f}$, in $L^2(\Gamma)$. Taking inverse Fourier transforms, we see that $\{f_n\}$ converges in $L^2(G)$ to$(h\hat{f})$. Since $\{f_n\}$ converges [mean ^{p}], we infer that

$$(h\hat{f}) \in L^p(G) \quad \text{and} \quad f_n \to (h\hat{f}) \quad \text{in } L^p(G). \tag{43}$$

Let $F_n = \sum_{k=-n}^{n} K^{-1} R_x^k f$ for $n \ge 1$. Clearly $F_n \in L^p(G) \cap L^2(G)$ and $\hat{F}_n = \sum_{k=-n}^{n} K^{-1} \tilde{x}^k \hat{f}$. Let g_0 be the function employed in the proof of Theorem (1.21). Since $g_0 \in BV(\mathbb{T})$, the sequence $\{S_n(g_0, .)\}$ is uniformly bounded (due to (38)) and pointwise convergent on $\mathbb{T}$ to g_0 (by [222,Corollary II.2.2]). It now follows that $\{\sum_{k=-n}^{n} K^{-1} \tilde{x}^k\}$ is uniformly bounded on Γ and pointwise convergent to $g_0(\tilde{x}(.))$. Hence $\hat{F}_N \to g_0(\tilde{x}(.))\hat{f}$, in $L^2(\Gamma)$ Similar reasoning to that just used to establish (43) now gives

$$[g_0(\tilde{x}(.))\hat{f}\ ] \in L^p(G) \quad \text{and} \quad F_n \to [g_0(\tilde{x}(.))\hat{f}\ ] \quad \text{in } L^p(G).$$

Using this and (43) in (42) we find that $(\arg R_x)f$ belongs to $L^p(G) \cap L^2(G)$, and has Fourier transform given by

$$\pi(1-h)\hat{f} + i\, g_0(\tilde{x}(.))\hat{f}\ .$$

Since it is easy to see that

$$g_0(\tilde{x}(.)) = i(1-h)[\pi - \mathrm{Arg}(\tilde{x}(.))]\,,$$

the conclusion (1.40)(i) follows at once. Similar reasoning based on Theorem (1.28)(ii) readily demonstrates conclusion (1.40) (ii). Alternatively, (1.40)(ii) can be deduced from (1.40)(i) and the following result of Ralph [224, Theorem 3.2.4].

Proposition (1.41) [178]: Let G be a locally compact abelian group with dual group Γ Suppose that $1 < p < \infty$, and ϕ is a real-valued function belonging to $M_p(\Gamma)$. Then the multiplier transform T_ϕ corresponding to ϕ is well bounded of type (B) if and only if for all $\lambda \in \mathbb{R}$ the characteristic function of $\phi^{-1}((-\infty, \lambda])$ belongs to $M_p(\Gamma)$, and, for the corresponding multiplier transforms $E(\lambda)$, $\lambda \in \mathbb{R}$, we have $\sup\{\ \|E(\lambda)\| : \lambda \in \mathbb{R}\} < \infty$. If this is the case, then $E(\,.\,)$ is the spectral family of T_ϕ .

This section exhibits a few counterexamples which preclude various potential extensions of Theorems (1.32) and (1.35).

Example (1.42) [178]: There exist a reflexive Banach space X and a power-bounded Operator $U_0 \in \mathcal{B}(X)$ such that U_0has no logarithm in $\mathcal{B}(X)$, in particular Theorem (1.32)(ii) fails if we replace Y by an arbitrary reflexive Banach space . In fact, let $\{\,P_k\}_{k=1}^{\infty}$, be a sequence in the interval $(1, +\infty)$ such that $P_k \to 1$, and take

$$X = l^2 - \oplus_{k=1}^{\infty}\ X_k\ ,$$

where $X_k = \{f \in L^{pk}(\mathbb{T}) \colon \hat{f}(0) = 0\}$ for $k \geq 1$. Thus X is a reflexive space. Denote by $\mathbb{N}$ the set of positive integers. Let $\{\alpha_k\}_{k=1}^{\infty}$, be a sequence of distinct irrational numbers in $(0,1)$ such that: (i) for $j,k \in \mathbb{N}$ with $j \neq k$, the set $\{\alpha_j\,, \alpha_k\,, 1\}$ is linearly independent over the rational field, and (ii) $\{\alpha_k\}$ converges to an irrational number α . (An easy way to construct such a sequence is to take $\alpha \in (0,1)$ transcendental and $\alpha_k = \alpha + s_k^{-\frac{1}{2}}$ for $k \in \mathbb{N}$, where $\{s_k\}_{k=1}^{\infty}$, is a strictly increasing sequence of positive primes such that $\alpha + s_k^{-\frac{1}{2}} < 1$ for $k \in \mathbb{N}$). Let T_k be the translation operator on X_k, corresponding to $e^{2\pi i \alpha_k}$ for $k \in \mathbb{N}$. Regarding each X_k, as a subspace of X, put $U_0 = \oplus_{k=1}^{\infty}\ T_k$. Thus U_0 is a surjective isometry of X. We show that there does not exist $A \in \mathcal{B}(X)$ such that $U_0 = e^{iA}$.

Suppose, to the contrary, that $U_0 = e^{iA}$. For $n \in \mathbb{Z}\backslash\{0\}$ and $k \in \mathbb{N}$, let $\phi_n^{(k)} \in X_k$, be defined by $\phi_n^{(k)}(z) = z^n$ for all $z \in \mathbb{T}$. Thus the linear span of $\left\{ \phi_n^{(k)} : n \in \mathbb{Z}\backslash \{ 0\}, k \in \mathbb{N}\right\}$ is dense in X, and $U_0\phi_n^{(k)} = e^{\,2\pi n i\alpha_k}\phi_n^{(k)}$ for $n \in \mathbb{Z}\backslash \{ 0\}$, and $k \in \mathbb{N}$. We claim that for $n \in \mathbb{Z}\backslash \{ 0\}$, and $k \in \mathbb{N}$, $\phi_n^{(k)}$ spans the kernel of $(e^{\,2\pi in\alpha_k}I - U_0)$. For $f \in \ker\left(e^{\,2\pi in\alpha_k}I - U_0\right)$, let $f = \sum_{j=1}^{\infty} f_j$ with $f_j \in X_j$ for all $j \in \mathbb{N}$. Thus, for each $j \in \mathbb{N}$, we have $T_j\, f_j = e^{\,2\pi in\alpha_k} f_j$ Equating the Fourier coefficients from both sides of this equation, we see that

$$e^{\,2\pi im\alpha_j}\, \hat{f}_j\,(m) = e^{\,2\pi in\alpha_k}\, \hat{f}_j\,(m) \quad for\ \ m \in \mathbb{Z}\,, j \in \mathbb{N}.$$

From property (i) of the sequence of $\alpha\,'S$,we have that $e^{\,2\pi im\alpha_j} = e^{\,2\pi in\alpha_k}$ implies $j = k$ and $m = n$. Thus $f = \hat{f}_k(n)\phi_n^{(k)}$ and the claim is established. Since A commutes with U_0 it follows from the claim that for $n \in \mathbb{Z}\backslash \{ 0\}$, and $k \in \mathbb{N}$, $A\phi_n^{(k)} = \lambda_n^{(k)}\phi_n^{(k)}$ for some $\lambda_n^{(k)} \in \mathbb{C}$ .Hence $U_0\phi_n^{(k)} = \left[\exp\,(i\lambda_n^{(k)})\right]\phi_n^{(k)}$, and so

$$\exp(2\pi in\alpha_k) = \exp\left(i\lambda_n^{(k)}\right) \quad \text{for } n \in \mathbb{Z}\backslash \{ 0\}\,, k \in \mathbb{N}. \tag{44}$$

Applying Tychonoff's Theorem to the sequence $\{A_k\}_{k=1}^{\infty}$ in the compact metric space $\prod_{n \in \mathbb{Z}\backslash \{ 0\}} D_n$, where $D_n = \{z \in \mathbb{C} : |z| \le \|A\|\}$ for all $n \in \mathbb{Z}\backslash \{ 0\}$, and $A_k = \left\{\lambda_n^{(k)}\right\}_{n \in \mathbb{Z}\backslash \{ 0\}}$ for all $k \in \mathbb{N}$, we obtain a strictly increasing sequence $\{K_j\}_{j=1}^{\infty}$ of positive integers, and an element $A = \{\lambda_n\}$ of $\prod_{n \in \mathbb{Z}\backslash \{ 0\}} D_n$ such that for each no $n \in \mathbb{Z}\backslash \{ 0\}\,, \lambda_n^{(k_j)} \to \lambda_n$ Define $\lambda_0 = 0$ .Now fix a trigonometric polynomial Q and let $Q_0 = Q - \widehat{Q}(0)$. Fix $M \in \mathbb{N}$ so that $[-M, M]$ contains the support of $\widehat{Q}$. For $j \in \mathbb{N}$ regard Q_0 as an element of X_{k_j} and, denoting the norm of L^{pk_j} ($\mathbb{T}$) by $|\,.\,|_j$ observe that

$$|\, A(Q_0)|_j \le \|A\|\, |\,(Q_0)|_j \ .$$

In other words,

$$\left|\sum_{n=-M}^{M} \lambda_n^{(k_j)}\ \widehat{Q}(n)\, z^n\right|_j \le \|A\|\, |\,(Q_0)|_j \quad \text{for } j \in \mathbb{N}\,. \tag{45}$$

Letting j → ∞ in (45) we obtain

$$\left|\sum_{n=-M}^{M} \lambda_n \; \widehat{Q}(n)\, z^n \right|_{L^1(\mathbb{T})} \leq \|A\| \, \| Q_0\|_{L^1(\mathbb{T})} \quad .$$

Recalling that $\lambda_0 = 0$,we see that

$$\left|\sum_{n=-M}^{M} \lambda_n \; \widehat{Q}(n)\, z^n \right|_{L^1(\mathbb{T})} \leq 2\|A\| \, \| Q\|_{L^1(\mathbb{T})} \; .$$

This shows that the sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ is a multiplier sequence for $L^1(\mathbb{T})$. Let B denote the corresponding bounded operator on $L^1(\mathbb{T})$. Replacing k by k_j in (44) and letting $j \to \infty$, we infer that $e^{2\pi i n\alpha} = e^{i\lambda_n}$ for all $n \in \mathbb{Z}$ (the case $n = 0$ being trivial). It follows that e^{iB} is the translation operator on $L^1(\mathbb{T})$ corresponding to $e^{2\pi i\alpha}$. However, this translation operator on $L^1(\mathbb{T})$ has no logarithm [221], and we have reached a contradiction (cf. Example(1.45)).

Example (1.43) [178]: There are a reflexive Banach space X, and a hermitian operator A on X such that A is not a well-bounded operator of type (B). In particular, Theorem (1.35) fails if Y is replaced by an arbitrary reflexive Banach space. For $n \in \mathbb{Z}$ let $E_n : \mathbb{T} \to \mathbb{T}$ be defined by $E_n(z) = z^n$ for all $z \in \mathbb{T}$. Let $C(\mathbb{T})$ be the Banach space of all complex-valued continuous functions on $\mathbb{T}$ with the norm $\|\,.\,\|_\infty$, where $\| f\|_\infty = \sup\{ |f(z)| : z \in \mathbb{T}\}$ for $f \in C(\mathbb{T})$ For each k $\in \mathbb{N}$, let X_k be the linear span in $C(\mathbb{T})$ of $\{E_m : -k \leq m \leq k\}, X_k$ being equipped with $\|\,.\,\|_\infty$. Define $A_k : X_k \to X_k$ as the linear operator such that $A_k(E_m) = mk^{-1} E_m$ for $-k \leq m \leq k$. Thus for each $f \in X_k$, and all $s \in \mathbb{R}$, $(ik)^{-1}(df(e^{it})/dt) = (A_k f)(e^{it})$. Using Bernstein's Inequality $[231,, ,p.11]$, we see that $\|A_k\| = 1$. Note that for all $s \in \mathbb{R}$ and all $f \in X_k$,

$$(e^{isA_k} f)(z) = f(e^{is/k} z) \text{ for all } z \in \mathbb{T} .$$

Thus A_k is a hermitian operator on X_k for all $k \in \mathbb{N}$. Now let

$$X = l^2 - \bigoplus_{k=1}^{\infty} X_k \qquad and \quad A = \bigoplus_{k=1}^{\infty} A_k \quad .$$

Then X is reflexive, and A is hermitian. (The construction of X and A for this example was inspired by [214, Example 1, p. 69], which deals with the differentiation operator on a space of almost periodic functions). For each Positive integer N we have

$$\sum_{n=-N}^{N} n^{-1} e^{inA} = \bigoplus_{k=1}^{\infty} \left(\sum_{n=-N}^{N} n^{-1} e^{inA_k} \right).$$

Suppose that $k \in \mathbb{N}$ and $f \in X_k$. For all $j \in \mathbb{Z}$, $N \in \mathbb{N}$,

$$\left(\sum_{n=-N}^{N} \widehat{n^{-1} e^{inA} f} \right)(j) = \sum_{n=N}^{N} n^{-1} e^{inj/k} \hat{f}(j). \tag{46}$$

Suppose that the sequence $\{\sum_{n=-N}^{N} n^{-1} e^{inA}\}_{N=1}^{\infty}$, converges in the strong operator topology to $B \in \mathcal{B}(X)$. For $k \in \mathbb{N}$, $f \in X_k$, it follows by letting $N \to \infty$ in (46) that

$$(\widehat{Bf})(j) = \sum_{n=-\infty}^{\infty} n^{-1} e^{inj/k} \hat{f}(j) \quad \text{for all } j \in \mathbb{Z}. \tag{47}$$

Let $g_0 \in BV(\mathbb{T})$ be the function employed in the proof of Theorem (1.24). As noted in the discussion immediately following (43), $g_0(z) = \sum_{n=-\infty}^{\infty} n^{-1} z^n$ for all $z \in \mathbb{T}$. Using this in (47), we get

$$(\widehat{Bf})(j) = g_0(e^{ij/k}) \hat{f}(j) \qquad \text{for } k \in \mathbb{N}, f \in X_k, j \in \mathbb{Z}. \tag{48}$$

Fix a trigonometric polynomial Q, and let Q have degree k_0 . By considering Q as an element of X_k for $k \geq k_0$, we see from (48) that

$$\left\| \sum_{j=-k_0}^{k_0} g_0(e^{ij/k}) \hat{Q}(j) E_j \right\|_{\infty} \leq \|B\| \, \|Q\|_{\infty} \text{ for } k \geq k_0. \tag{49}$$

Letting $k \to +\infty$ in (49), while taking into account the definition of g_0 we obtain

$$\left\| \pi \sum_{j=-k_0}^{k_0} [\hat{Q}(j) E_j - \hat{Q}(-j) E_{-j}] \right\|_{\infty} \leq \|B\| \, \|Q\|_{\infty} . \tag{50}$$

Since Q is an arbitrary trigonometric polynomial, (50) implies that for every $f \in C(\mathbb{T})$ the conjugate Fourier series for f, $\sum_{j=-\infty}^{\infty} (-i)(\operatorname{sgn} j) \hat{f}(j) E_j$, is the Fourier series of a function $\tilde{f} \in C(\mathbb{T})$.This conclusion is well known to be false (see, e.g., [231,, VI1.(2.3)]). Hence $\sum_{n=-\infty}^{\infty} n^{-1} e^{inA}$ does not converge in the strong operator topology. Put $U = e^{iA}$. Since A is hermitian, U is power bounded. If U were trigonometrically well-bounded, then (17) would produce a contradiction. So e^{iA} is not trigonometrically well-bounded, and hence A cannot be a well-bounded operator of type (B). We observe that the present operator e^{iA}, like the operator U_0 of Example (1.42) is power bounded on a reflexive Banach space, but not trigonometrically well-bounded. However, e^{iA}does have a logarithm in contrast to U_0.

Example (1.44) [178]: There is a well-bounded operator T of type (B) on $L^2(\mathbb{N})$ which is not hermitian-equivalent. Hence already in the Hilbert space case $(p = 2)$ the converses of Theorems (1.32)(ii) and (1.35) fail. By the construction of a suitable conditional basis for $L^2(\mathbb{N})$ it is shown in [220] (see [183, Chap. 18] for a discussion) that there is a sequence $\{P_n\}_{n=1}^{\infty}$ of projection operators defined on $L^2(\mathbb{N})$ such that:

(i) $P_n P_m = 0$ for $n \neq m$;

(ii) $\sum_{n=1}^{\infty} P_n$ converges to I in the strong operator topology;

(iii) $\|\sum_{j=1}^{\infty} P_{2j}\| \to \infty \ as \ n \to \infty$; and

(iv) for every bounded, strictly decreasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, $\sum_{n=1}^{\infty} \lambda_n P_n$ converges to a well-bounded operator of type (B) in the strong operator topology. Choose a sequence $\{\lambda_n\}$ as in (iv), and let T be the well-bounded operator of type (B) given by the series in (iv). If T were hermitian-equivalent, then it follows from [185, Theorem 6] that T can be made self-adjoint after an appropriate equivalent Hilbert space renorming of $L^2(\mathbb{N})$ .Let T then have spectral measure $\mathfrak{F}$. Thus, for each $k \in \mathbb{N}$, $\mathfrak{F}(\{\lambda_k\})$ has range equal to the kernel of $(T - \lambda_k)$. By properties (i) and (ii) for the sequence $\{P_n\}_{n=1}^{\infty}$, P_k likewise has range equal to the kernel of $(T - \lambda_k)$. Since P_k commutes with T, it commutes with $\mathfrak{F}(\{\lambda_k\})$ .Thus $P_k = \mathfrak{F}(\{\lambda_k\})$ for $k \in \mathbb{N}$. But this implies that $\left\{\sum_{j=1}^{n} P_{2j}\right\}_{n=1}^{\infty}$ is strongly convergent, hence uniformly bounded. We have reached a contradiction to (iii), and so T is not hermitian-equivalent. It follows from this that $\sup\left\{\|e^{inT}\| : n \in \mathbb{Z}\right\} = +\infty$.

Example (1.45) [178]: For the index $p = 1$, Theorem (1.32)(ii) fails. Let R_{-1} , denote the translation operator on $L^1(\mathbb{Z})$ corresponding to (-1). Thus R_{-1} is a surjective isometry. However, R_{-1} is not trigonometrically well bounded for each of the following reasons:

(i) R_{-1} does not have a logarithm in $\mathcal{B}(L^1(\mathbb{Z}))$

(ii) $\sum_{n=-\infty}^{\infty} n^{-1} R_{-1}^{n}$ does not converge in the strong operator topology;

(iii) R_{-1} does not have an $AC(\mathbb{T})$-functional calculus. The assertion (i) is shown in [183, Example 20.1]. Let δ_0 be the characteristic function relative to $\mathbb{Z}$ of $\{0\}$ For each $N \in \mathbb{N}$ it is straight forward that

$$\left\| \sum_{n=-N}^{N} n^{-1} R_{-1}^{n} \delta_0 \right\| = 2 \sum_{n=1}^{N} n^{-1} \ .$$

This shows (ii). If R_{-1} had an $AC(\mathbb{T})$-functional calculus, then there would be a constant M such that for every trigonometric polynomial Q ,

$$\| Q(R_{-1})\delta_0 \| \leq M \|Q\|_{\mathbb{T}} \ . \tag{51}$$

But $\| Q(R_{-1})\delta_0 \| = \sum_{n=-\infty}^{\infty} \left| \hat{Q}(n) \right|$. Hence by (51), every function in $AC(\mathbb{T})$ would have an absolutely convergent Fourier series. This conclusion is well known to be false [231, VI.(3.7)], and so assertion (iii) is established.